IL NUOVO CIMENTO Online First
DOI 10.1393/ncb/i2010-10940-1

BASIC TOPICS: Fundamental Physics

Preliminarily group classification of a class of 2D nonlinear heat equations

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(ricevuto il 14 Aprile 2010; revisionato l'8 Novembre 2010; approvato il 9 Novembre 2010; pubblicato online il 16 Dicembre 2010)

Summary. — A preliminary group classification of the class 2D nonlinear heat equations $u_t = f(x, y, u, u_x, u_y)(u_{xx}+u_{yy})$, where f is the arbitrary smooth function of the variables x, y, u, u_x and u_y is given. Furthermore, we have proved that an optimal system of one-dimensional Lie subalgebras of this equation is generated by $\langle Z^1, \dots, Z^{12}\rangle$ and we obtain Z^is in Theorem 4.1. Also we take their optimal system's projections on the space (x, y, u, u_x, u_y, f). The paper is one of the few applications of an algebraic approach to the problem of group classification using Lie method that is called the method of preliminary group classification.

PACS 02.20.Sv – Lie algebras of Lie groups.
PACS 02.30.Jr – Partial differential equations.

1. – Introduction

It is well known that the symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie. The first paper on this subject is [1], where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity). He computed the maximal invariance group of the one-dimensional heat conductivity equation and utilized this symmetry to construct its explicit solutions. The theory of Lie systems [2, 3] deals with non-autonomous systems of first-order ordinary differential equations [4] and then for partial differential equations [5] such that all their solutions can be written in terms of generic sets of particular solutions and some constants, by means of a time-independent function. Such functions are called *superposition rules* and

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the systems admitting this mathematical property are called *Lie systems*. Lie succeeded in characterizing systems admitting a superposition rule. Saying it the modern way, he performed symmetry reduction of the heat equation. Nowadays symmetry reduction is one of the most powerful tools for solving nonlinear partial differential equations (PDEs). Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [6] developed the method of partially invariant solutions. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study. In [7], symmetry reduction for the some equation is applied using a loop algebra.

In an attempt to study nonlinear effects Saied and Hussain [8] gave some new similarity solutions of the (1+1)-nonlinear heat equation. Later Clarkson and Mansfield [9] studied classical and nonclassical symmetries of the (1+1)-heat equation and gave new reductions for the linear heat equation and a catalogue of closed-form solutions for a special choice of the function $f(x, y, u, u_x, u_y)$ that appears in their model. In higher dimensions Servo [10] gave some conditional symmetries for a nonlinear heat equation while Goard *et al.* [11] studied the nonlinear heat equation in the degenerate case. Nonlinear heat equations in one or higher dimensions are also studied in literature by using both symmetry as well as other methods [12, 13].

There are a number of papers to study (1+1)-nonlinear heat equations from the point of view of Lie symmetries method. In the paper [14], Basarab-Horwath *et al.* solve completely the problem of the group classification of nonlinear heat-conductivity equations of the form

$$u_t = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x). \tag{1}$$

There are some papers in which group classification of particular equations of the form (1) is presented in table I. The present paper solves the problem of the preliminarily group classification of two-dimensional nonlinear heat equations of the form

$$u_t = f(x, y, u, u_x, u_y)(u_{xx} + u_{yy}). \tag{2}$$

In a particular case in eq. (2), for $f = f(u)$, the (2+1)-dimensional nonlinear heat equation $u_t = f(u)(u_{xx} + u_{yy})$ is investigated by using Lie symmetry method in [15].

2. – Symmetry methods

Let a partial differential equation contain p independent variables and q dependent variables. The one-parameter Lie group of transformations

$$x_i \longmapsto x_i + \epsilon\xi^i(x, u) + O(\epsilon^2); \qquad u_\alpha \longmapsto u_\alpha + \epsilon\varphi^\alpha(x, u) + O(\epsilon^2), \tag{3}$$

where $i = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. The action of the Lie group can be recovered from that of its associated infinitesimal generators. We consider the general vector field

$$X = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u)\frac{\partial}{\partial u_\alpha} \tag{4}$$

TABLE I. – *Some papers in which group classification of particular equations of the form* (1) *has been carried out.*

Authors	Functions of F and G	Reference
Ovsjannikov (1959)	$F = F(u), \quad G = \frac{dF}{du} u_x^2$	[16]
Akhatov *et al.* (1987)	$F = F(u_x), \quad G = 0$	[17]
Dorodnitsyn (1982)	$F = F(u), \quad G = \frac{dF}{du} u_x^2 + g(u)$	[18]
Oron and Rosenau (1986) Edwards (1994)	$F = F(u), \quad G = \frac{dF}{du} u_x^2 + f(u) u_x$	[19, 20]
Gandarias (1996)	$F = u^n, \quad G = \frac{dF}{du} u_x^2 + g(x) u^m u_x + f(x) u^s$	[21]
Cherniha and Serov (1998)	$F = F(u), \quad G = \frac{dF}{du} u_x^2 + f(u) u_x + g(u)$	[22]
Zhdanov and Lahno (1999)	$F = 1, \quad G = G(t, x, u, u_x)$	[23]

on the space of independent and dependent variables. Therefore, the characteristic of the vector field X given by (4) is the function

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial u^\alpha}{\partial x_i}, \quad \alpha = 1, \ldots, q. \tag{5}$$

Theorem 2.1. *Let X be a vector field given by* (4)*, and let $Q = (Q^1, \ldots, Q^q)$ be its characteristic, as in* (5)*. The n-th prolongation of X is given explicitly by*

$$X^{(n)} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}, \tag{6}$$

with coefficients

$$\varphi_{J,i}^\alpha = D_i \varphi_J^\alpha - \sum_{j=1}^{p} D_i \xi^j u_{J,j}^\alpha. \tag{7}$$

Here, $J = (j_1, \ldots, j_k)$, with $1 \leq k \leq p$ is a multi-index, and D_i represents a total derivative and subscripts of u are the derivatives with respect to the respective coordinates [24].

The symmetry generator associated with (4) is given by

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u}, \tag{8}$$

where ξ^1, ξ^2, ξ^3, φ are real functions with respect to x, y, t, u variables. The second prolongation of X is denoted by the vector field

$$X^{(2)} = X + \varphi^x \frac{\partial}{\partial u_x} + \varphi^y \frac{\partial}{\partial u_y} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xy} \frac{\partial}{\partial u_{xy}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{yy} \frac{\partial}{\partial u_{yy}} + \varphi^{yt} \frac{\partial}{\partial u_{yt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}}, \tag{9}$$

its coefficients are obtained with the following formulas with coefficients

$$\varphi^\iota = D_\iota Q + \xi^1 u_{x\iota} + \xi^2 u_{y\iota} + \xi^3 u_{t\iota}, \tag{10}$$

$$\varphi^{\iota\jmath} = D_\iota(D_\jmath Q) + \xi^1 u_{x\iota\jmath} + \xi^2 u_{y\iota\jmath} + \xi^3 u_{t\iota\jmath}, \tag{11}$$

where $Q = \varphi - \xi^1 u_x - \xi^2 u_y - \xi^3 u_t$ is the characteristic of the vector field X and the operators D_ι, $D_\jmath$ denote the total derivatives with respect to ι and $\jmath$ where $\iota, \jmath \in \{x, y, t\}$.

Theorem 2.2. *A connected group of transformations G is a symmetry group of a differential equation $\Delta = 0$ if and only if the classical infinitesimal symmetry condition*

$$X^{(n)}(\Delta) = 0 \quad \text{whenever} \quad \Delta = 0, \tag{12}$$

holds for every infinitesimal generator $X \in \mathfrak{g}$ of G [24].

Therefore we have $X^{(2)}[u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy})]|_{(2)} = 0$ whenever $u_t - f(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) = 0$. We obtain the following determining function:

$$\varphi^t - (f_x\xi^1 + f_y\xi^2 + f_u\varphi + f_{u_x}\varphi^x + f_{u_y}\varphi^y)(u_{xx} + u_{yy}) - f(\varphi^{xx} + \varphi^{yy}) = 0. \tag{13}$$

In the case of arbitrary f it follows

$$\xi^1 = \xi^2 = \varphi = 0, \quad \xi^3 = C. \tag{14}$$

Therefore, for arbitrary $f(x, y, u, u_x, u_y)$ eq. (2) admits the one-dimensional Lie algebra $\mathfrak{g}_1$, with the basis

$$X_1 = \frac{\partial}{\partial t}. \tag{15}$$

$\mathfrak{g}_1$ is called the principle Lie algebra for eq. (2). So, the remaining part of the group classification is to specify the coefficient f such that eq. (2) admits an extension of the principal algebra $\mathfrak{g}_1$. Usually, the group classification is obtained by inspecting the determining equation. But in our case the complete solution of the determining eq. (13) is a wasteful venture. Therefore, we do not solve the determining equation but, instead we obtain a partial group classification of eq. (2) via the so-called method of preliminary group classification. This method was suggested in [6] and applied when an equivalence group is generated by a finite-dimensional Lie algebra $\mathfrak{g}_\mathcal{E}$. The essential part of the method is the classification of all nonsimilar subalgebras of $\mathfrak{g}_\mathcal{E}$. Actually, the application of the method is simple and effective when the classification is based on the finite-dimensional equivalence algebra $\mathfrak{g}_\mathcal{E}$.

3. – Equivalence transformations

An equivalence transformation is a nondegenerate change of the variables t, x, y, u taking any equation of the form (2) into an equation of the same form, generally speaking, with different $f(x,y,u,u_x,u_y)$. The set of all equivalence transformations forms an equivalence group $\mathcal{E}$. We shall find a continuous subgroup $\mathcal{E}_C$ of it making use of the infinitesimal method.

We consider an operator of the group $\mathcal{E}_C$ in the form

$$Y = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u} + \mu(x,y,t,u,u_x,u_y,u_t,f)\frac{\partial}{\partial f}, \tag{16}$$

from the invariance conditions of eq. (2) written as the system

$$u_t - f(x,y,u,u_x,u_y)(u_{xx}+u_{yy}) = 0, \tag{17}$$

$$f_t = f_{u_t} = 0, \tag{18}$$

where u and f are considered as differential variables: u on the space (x,y,t) and f on the extended space (x,y,t,u,u_x,u_y).

The invariance conditions of the system (17) and (18) are

$$Y^{(2)}[u_t - f(x,y,u,u_x,u_y)(u_{xx}+u_{yy})] = 0, \tag{19}$$

$$Y^{(2)}[f_t] = Y^{(2)}[f_{u_t}] = 0, \tag{20}$$

where $Y^{(2)}$ is the 2th prolongation of the operator Y

$$Y^{(2)} = Y + \varphi^x\frac{\partial}{\partial u_x} + \varphi^y\frac{\partial}{\partial u_y} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{xy}\frac{\partial}{\partial u_{xy}} + \varphi^{xt}\frac{\partial}{\partial u_{xt}} + \varphi^{yy}\frac{\partial}{\partial u_{yy}} + \varphi^{yt}\frac{\partial}{\partial u_{yt}} + \varphi^{tt}\frac{\partial}{\partial u_{tt}} + \mu^t\frac{\partial}{\partial f_t} + \mu^{u_t}\frac{\partial}{\partial f_{u_t}}. \tag{21}$$

The coefficients φ^x, φ^y, φ^t, φ^{xx}, φ^{xy}, φ^{xt}, φ^{yy}, φ^{yt}, φ^{tt} are given in (10) and (11). The other coefficients of (21) are obtained by applying the prolongation procedure to differential variables f with independent variables (x,y,t,u,u_x,u_y,u_t). We have

$$\mu^t = \widetilde{D}_t(\mu) - f_x\widetilde{D}_t(\xi^1) - f_y\widetilde{D}_t(\xi^2) - f_u\widetilde{D}_t(\varphi) - f_{u_x}\widetilde{D}_t(\varphi^x) - f_{u_y}\widetilde{D}_t(\varphi^y),$$

$$\mu^{u_t} = \widetilde{D}_{u_t}(\mu) - f_x\widetilde{D}_{u_t}(\xi^1) - f_y\widetilde{D}_{u_t}(\xi^2) - f_u\widetilde{D}_{u_t}(\varphi) - f_{u_x}\widetilde{D}_{u_t}(\varphi^x) - f_{u_y}\widetilde{D}_{u_t}(\varphi^y),$$

where

$$\widetilde{D}_t = \frac{\partial}{\partial t}, \qquad \widetilde{D}_{u_t} = \frac{\partial}{\partial u_t}. \tag{22}$$

So, we have the following prolongation formulas:

$$\mu^t = \mu_t - f_x\xi^1_t - f_y\xi^2_t - f_u\varphi_t - f_{u_x}(\varphi^x)_t - f_{u_y}(\varphi^y)_t, \tag{23}$$

$$\mu^{u_t} = \mu_{u_t} - f_{u_x}(\varphi^x)_{u_t} - f_{u_y}(\varphi^y)_{u_t}. \tag{24}$$

By the invariance conditions (19) and (20) give rise to

$$\mu^t = \mu^{u_t} = 0, \tag{25}$$

that holds for every f. Substituting (25) into (23) and (24), we obtain

$$\begin{aligned} \mu_t &= \mu_{u_t} = 0, \\ \xi^1_t &= \xi^2_t = \varphi_t = 0, \\ (\varphi^x)_t &= (\varphi^x)_{u_t} = (\varphi^y)_t = (\varphi^y)_{u_t} = 0. \end{aligned} \tag{26}$$

Moreover with substituting (21) into (19) we have

$$\varphi^t - f(x, y, u, u_x, u_y)(\varphi^{xx} + \varphi^{yy}) - \mu(u_{xx} + u_{yy}) = 0. \tag{27}$$

We are left with a polynomial equation involving the various derivatives of $u(x, y, t)$ whose coefficients are certain derivatives of ξ^1, ξ^2, ξ^3 and φ. Since ξ^1, ξ^2, ξ^3, φ only depend on x, y, t, u we can equate the individual coefficients to zero, leading to the complete set of the determining equations:

$$\begin{aligned} \xi^1 &= \xi^1(x, y), \\ \xi^2 &= \xi^2(y), \\ \xi^3 &= \xi^3(t), \\ \varphi_u &= \xi^1_x = \xi^2_y, \\ \varphi_{uu} &= \varphi_{xu} = \varphi_{yu} = 0, \\ \mu &= (\xi^1_x - \xi^3_t)f. \end{aligned}$$

So, we find that

$$\begin{aligned} \xi^1 &= c_1x + c_2y + c_3, \\ \xi^2 &= c_1y + c_4, \\ \xi^3 &= a(t), \\ \varphi &= c_1u + \beta(x, y), \\ \mu &= (c_1 - a'(t))f, \end{aligned}$$

with constants c_1, c_2, c_3 and c_4, also we have $\beta_{xx} = -\beta_{yy}$.

The class of eq. (2) has an infinite continuous group of equivalence transformations generated by the following infinitesimal operators:

$$\begin{aligned}(28)\qquad Y = (c_1x+c_2y+c_3)\frac{\partial}{\partial x} + (c_1y+c_4)\frac{\partial}{\partial y} + a(t)\frac{\partial}{\partial t} + (c_1u+\beta(x,y))\frac{\partial}{\partial u}\\ +(c_1-a'(t))f\frac{\partial}{\partial f}\,.\end{aligned}$$

Therefore the symmetry algebra of eq. (2) is spanned by the vector fields

$$\begin{aligned}(29)\qquad &Y_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}, \quad Y_2 = y\frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial x},\\ &Y_4 = \frac{\partial}{\partial y}, \quad Y_5 = a(t)\frac{\partial}{\partial t} - a'(t)f\frac{\partial}{\partial f}, \quad Y_\beta = \beta(x,y)\frac{\partial}{\partial u}.\end{aligned}$$

Moreover, in the group of equivalence transformations there are included also discrete transformations, *i.e.* the reflections

$$(30)\qquad t\longrightarrow -t, \qquad x\longrightarrow -x, \qquad y\longrightarrow -y, \qquad u\longrightarrow -u, \qquad f\longrightarrow -f.$$

4. – Preliminary group classification

It is possible to observe in many applications of group analysis that most of the extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $\mathfrak{g}_{\mathcal{E}}$, these extensions are called *$\mathcal{E}$-extensions of the principal Lie algebra*. The classification of all nonequivalent equations with respect to a given equivalence group $G_{\mathcal{E}}$, admitting $\mathcal{E}$-extensions of the principal Lie algebra is called a *preliminary group classification*. Note that $G_{\mathcal{E}}$ is not necessarily the largest equivalence group but it can be any subgroup of the group of all equivalence transformations.

Therefore, we can take any finite-dimensional subalgebra (desirable as large as possible) of an infinite-dimensional algebra with basis (29) and use it for a preliminary group classification. We select the subalgebra $\mathfrak{g}_6$ spanned on the following operators:

$$\begin{aligned}(31)\qquad &Y_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}\,, \quad Y_2 = y\frac{\partial}{\partial x}\,, \quad Y_3 = \frac{\partial}{\partial x}\,,\\ &Y_4 = \frac{\partial}{\partial y}\,, \quad Y_5 = \frac{\partial}{\partial t}\,, \quad Y_6 = \frac{\partial}{\partial u}\,.\end{aligned}$$

The communication relations of (31) are shown in table II. For each s-parameter subgroup there corresponds a family of group-invariant solutions. Generally, it is quite impossible to determine all possible group-invariant solutions of a PDE. For minimizing this search, we construct the optimal system of solutions. It is well known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras [6, 25]. Here, we want to construct the optimal system of subalgebras of $\mathfrak{g}_6$.

Let G be a Lie group with the corresponding Lie algebra $\mathfrak{g}$. There is an inner automorphism $T_a \longmapsto TT_aT^{-1}$ of the group G for each $T \in G$. Every automorphism of the group G induces an automorphism of $\mathfrak{g}$. The set of all these automorphism is a Lie group

TABLE II. – *Commutation relations satisfied by infinitesimal generators in* (33).

$[\,,\,]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	0	0	$-Y_3$	$-Y_4$	$-Y_5$	$-Y_6$
Y_2	0	0	0	$-Y_3$	0	0
Y_3	Y_3	0	0	0	0	0
Y_4	Y_4	Y_3	0	0	0	0
Y_5	Y_5	0	0	0	0	0
Y_6	Y_6	0	0	0	0	0

called *the adjoint group* G^A. The corresponding Lie algebra of the Lie group G^A is the adjoint algebra of $\mathfrak{g}$, defined as follows.

Let us give two infinitesimal generators $X, Y \in \mathfrak{g}$. The linear mapping $\mathrm{Ad}X : Y \longrightarrow [X, Y]$ is an automorphism of $\mathfrak{g}$, called *the inner derivation of* $\mathfrak{g}$. The set of all inner derivations $\mathrm{ad}X(Y)$ where $X, Y \in \mathfrak{g}$, together with the Lie bracket $[\mathrm{Ad}X, \mathrm{Ad}Y] = \mathrm{Ad}[X, Y]$ is a Lie algebra $\mathfrak{g}^A$ called the *adjoint algebra of* $\mathfrak{g}$. Clearly $\mathfrak{g}^A$ is the Lie algebra of G^A. Two subalgebras in $\mathfrak{g}$ are *conjugate* if there is a transformation of G^A which takes one subalgebra into the other. The collection of pairwise non-conjugate s-dimensional subalgebras is the optimal system of subalgebras of order s. The construction of the one-dimensional optimal system of subalgebras can be carried out by using a global matrix of the adjoint transformations as suggested by Ovsiannikov [6]. The latter problem tends to determine a list (that is called an *optimal system*) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation, *i.e.* $\overline{\mathfrak{h}}\,\mathrm{Ad}(g)\,\mathfrak{h}$ for some g of a considered Lie group.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp[s\,Y_i])Y_j = Y_j - s\,[Y_i, Y_j] + \frac{s^2}{2}\,[Y_i, [Y_i, Y_j]] - \cdots, \tag{32}$$

where s is a parameter and $i, j = 1, \cdots, 6$. The adjoint representations of $\mathfrak{g}_6$ are listed in table III, it consists of the separate adjoint actions of each element of $\mathfrak{g}_6$ on all other elements.

TABLE III. – *Adjoint relations satisfied by infinitesimal generators in* (33).

$[\,,\,]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	Y_1	Y_2	e^sY_3	e^sY_4	e^sY_5	e^sY_6
Y_2	Y_1	Y_2	Y_3	$Y_4 + sY_3$	Y_5	Y_6
Y_3	$Y_1 - sY_3$	Y_2	Y_3	Y_4	Y_5	Y_6
Y_4	$Y_1 - sY_4$	$Y_2 - sY_3$	Y_3	Y_4	Y_5	Y_6
Y_5	$Y_1 - sY_5$	Y_2	Y_3	Y_4	Y_5	Y_6
Y_6	$Y_1 - sY_6$	Y_2	Y_3	Y_4	Y_5	Y_6

Theorem 4.1. *The optimal system of one-dimensional Lie subalgebras of eq.* (2) *is provided by those generated by*

1) $Y^1 = Y_1 = x\partial_x + y\partial_y + u\partial_u + f\partial_f,$
2) $Y^2 = Y_2 = y\partial_x,$
3) $Y^3 = -Y_4 = -\partial_y,$
4) $Y^4 = Y_1 + Y_5 = x\partial_x + y\partial_y + \partial_t + u\partial_u + f\partial_f,$
5) $Y^5 = Y_1 - Y_2 = (x-y)\partial_x + y\partial_y + u\partial_u + f\partial_f,$
6) $Y^6 = Y_2 - Y_4 = y\partial_x - \partial_y,$
7) $Y^7 = -Y_4 + Y_6 = -\partial_y + \partial_u,$
8) $Y^8 = -Y_4 - Y_6 = -\partial_y - \partial_u,$
9) $Y^9 = Y_2 + Y_5 = y\partial_x + \partial_t,$
10) $Y^{10} = Y_2 - Y_5 = y\partial_x - \partial_t,$
11) $Y^{11} = Y_2 + Y_6 = y\partial_x + \partial_u,$
12) $Y^{12} = Y_2 - Y_6 = y\partial_x - \partial_u,$
13) $Y^{13} = Y_1 + Y_2 = (x+y)\partial_x + y\partial_y + u\partial_u + f\partial_f,$
14) $Y^{14} = -Y_4 + Y_5 + Y_6 = -\partial_y + \partial_t + \partial_u,$
15) $Y^{15} = Y_2 - Y_4 - Y_5 + Y_6 = y\partial_x - \partial_y - \partial_t + \partial_u,$
16) $Y^{16} = Y_2 - Y_4 + Y_6 = y\partial_x - \partial_y + \partial_u,$
17) $Y^{17} = Y_2 - Y_4 + Y_5 - Y_6 = y\partial_x - \partial_y + \partial_t - \partial_u,$
18) $Y^{18} = Y_2 - Y_4 - Y_6 = y\partial_x - \partial_y - \partial_u,$
19) $Y^{19} = Y_1 + Y_2 + Y_5 = (x+y)\partial_x + y\partial_y + \partial_t + u\partial_u + f\partial_f,$
20) $Y^{20} = Y_2 + Y_5 + Y_6 = y\partial_x + \partial_t + \partial_u,$
21) $Y^{21} = Y_2 + Y_5 - Y_6 = y\partial_x + \partial_t - \partial_u,$
22) $Y^{22} = Y_2 - Y_5 - Y_6 = y\partial_x - \partial_t - \partial_u,$
23) $Y^{23} = Y_2 - Y_5 + Y_6 = y\partial_x - \partial_t + \partial_u,$
24) $Y^{24} = -Y_4 - Y_5 - Y_6 = -\partial_y - \partial_t - \partial_u,$
25) $Y^{25} = -Y_4 - Y_5 + Y_6 = -\partial_y - \partial_t + \partial_u,$
26) $Y^{26} = -Y_4 + Y_5 - Y_6 = -\partial_y + \partial_t - \partial_u,$
27) $Y^{27} = Y_2 - Y_4 + Y_5 + Y_6 = y\partial_x - \partial_y + \partial_t + \partial_u,$
28) $Y^{28} = Y_1 + Y_2 - Y_5 = (x+y)\partial_x + y\partial_y - \partial_t + u\partial_u + f\partial_f,$
29) $Y^{29} = Y_1 - Y_2 - Y_5 = (x-y)\partial_x + y\partial_y - \partial_t + u\partial_u + f\partial_f,$
30) $Y^{30} = Y_1 - Y_2 + Y_5 = (x-y)\partial_x + y\partial_y + \partial_t + u\partial_u + f\partial_f,$
31) $Y^{31} = Y_1 - Y_5 = x\partial_x + y\partial_y - \partial_t + u\partial_u + f\partial_f$
32) $Y^{32} = Y_2 - Y_4 - Y_5 - Y_6 = y\partial_x - \partial_y - \partial_t - \partial_u.$

Proof. Let $\mathfrak{g}_6$ be the symmetry algebra of eq. (2) with the adjoint representation determined in table III and

$$Y = a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5 + a_6Y_6 \tag{33}$$

is a nonzero vector field of $\mathfrak{g}_6$. We will simplify as many of the coefficients $a_i, i = 1, \dots, 6$, as possible through proper adjoint applications on Y. We follow our aim in the following easy cases:

Case 1)

At first, assume that $a_1 \neq 0$. Scaling Y if necessary, we can assume that $a_1 = 1$ and so we get

$$Y = Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5 + a_6Y_6. \tag{34}$$

Using the table of adjoint (table III), if we act on Y with $\mathrm{Ad}(\exp[a_3Y_3])$, the coefficient of Y_3 can be vanished:

$$Y' = Y_1 + a_2Y_2 + a_4Y_4 + a_5Y_5 + a_6Y_6. \tag{35}$$

Then we apply $\mathrm{Ad}(\exp[a_4Y_4])$ on Y' to cancel the coefficient of Y_4:

$$Y'' = Y_1 + a_2Y_2 + a_5Y_5 + a_6Y_6. \tag{36}$$

At last, we apply $\mathrm{Ad}(\exp[a_6Y_6])$ on Y'' to cancel the coefficient of Y_6:

$$Y''' = Y_1 + a_2Y_2 + a_5Y_5. \tag{37}$$

Case 1a)

If a_2, $a_5 \neq 0$ then we can make the coefficient of Y_2 and Y_5 either $+1$ or -1. Thus any one-dimensional subalgebra generated by Y with a_2, $a_5 \neq 0$ is equivalent to one generated by $Y_1 \pm Y_2 \pm Y_5$ which introduces parts 19), 28), 29) and 30) of the theorem.

Case 1b)

For $a_2 = 0$, $a_5 \neq 0$ we can see that each one-dimensional subalgebra generated by Y is equivalent to one generated by $Y_1 \pm Y_5$ which introduces parts 4) and 31) of the theorem.

Case 1c)

For $a_2 \neq 0$, $a_5 = 0$, each one-dimensional subalgebra generated by Y is equivalent to one generated by $Y_1 \pm Y_2$ which introduces parts 5) and 13) of the theorem.

Case 1d)

For $a_2 = 0$, $a_5 = 0$, each one-dimensional subalgebra generated by Y is equivalent to one generated by Y_1 which introduces part 1) of the theorem.

Case 2)

The remaining one-dimensional subalgebras are spanned by vector fields of the form Y with $a_1 = 0$.

Case 2a)

If $a_4 \neq 0$ then by scaling Y, we can assume that $a_4 = -1$. Now by the action of $\mathrm{Ad}(\exp[a_3Y_3])$ on Y, we can cancel the coefficient of Y_3:

$$\overline{Y} = a_2Y_2 - Y_4 + a_5Y_5 + a_6Y_6. \tag{38}$$

Let $a_2 \neq 0$ then by scaling Y, we can assume that $a_2 = 1$, and we have

$$\overline{Y}' = Y_2 - Y_4 + a_5Y_5 + a_6Y_6. \tag{39}$$

Case 2a-1)

Suppose $a_5 = a_6 = 0$, then the one-dimensional subalgebra generated by Y is equivalent to one generated by $Y_2 - Y_4$ which introduces part 6).

Case 2a-2)

Suppose $a_5 = 0$, $a_6 \neq 0$, all of the one-dimensional subalgebra generated by Y is equivalent to one generated by $Y_2 - Y_4 \pm Y_6$ which introduces parts 16) and 18).

Case 2a-3)

Suppose $a_5 \neq 0$, $a_6 \neq 0$, all of the one-dimensional subalgebra generated by Y is equivalent to one generated by $Y_2 - Y_4 \pm Y_5 \pm Y_6$ which introduces parts 15), 17), 27), and 32).

Now if $a_2 = 0$, we have

$$\overline{Y}'' = -Y_4 + a_5 Y_5 + a_6 Y_6. \tag{40}$$

Case 2a-4)

Suppose $a_5 = a_6 = 0$, then the one-dimensional subalgebra generated by Y is equivalent to one generated by $-Y_4$ which introduces part 3).

Case 2a-5)

Suppose $a_5 = 0$, $a_6 \neq 0$, all of the one-dimensional subalgebra generated by Y is equivalent to one generated by $-Y_4 \pm Y_6$ which introduces parts 7) and 8).

Case 2a-6)

Suppose $a_5 \neq 0$, $a_6 \neq 0$, all of the one-dimensional subalgebra generated by Y is equivalent to one generated by $-Y_4 \pm Y_5 \pm Y_6$ which introduces parts 14), 24), 25) and 26).

Case 2b)

Let $a_4 = 0$ then Y is in the form

$$\widehat{Y} = a_2 Y_2 + a_5 Y_5 + a_6 Y_6. \tag{41}$$

Suppose that $a_2 \neq 0$ then, if necessary, we can let it equal to 1 and we obtain

$$\widehat{Y}' = Y_2 + a_5 Y_5 + a_6 Y_6. \tag{42}$$

Case 2b-1)

Let $a_5 = a_6 = 0$, then Y_2 remains and we find 2) section of the theorem.

Case 2b-2)

If $a_5 \neq 0$, $a_6 \neq 0$, then $\widehat{Y}'$ is equal to $Y_2 \pm Y_5 \pm Y_6$. Hence this case suggests parts 20), 21), 22) and 23).

Case 2b-3)

If $a_5 \neq 0$, $a_6 = 0$, then $\widehat{Y}' = Y_2 \pm Y_5$. Hence this case suggests parts 9) and 10).

Case 2b-4)

If $a_5 = 0$, $a_6 \neq 0$, then $Y_2 \pm Y_6$ is obtained. So, this case suggests part 11) and 12).

There is not any more possible case for studying and the proof is complete.

The coefficients f of eq. (2) depend on the variables x, y, u, u_x, u_y. Therefore, we take their optimal system's projections on the space (x, y, u, u_x, u_y, f). We have

$$
\begin{array}{rl}
1) & Z^1 = Y^1 = Y^4 = x\partial_x + y\partial_y + u\partial_u + f\partial_f,\\
2) & Z^2 = Y^2 = Y^9 = Y^{10} = y\partial_x,\\
3) & Z^3 = Y^3 = -\partial_y,\\
4) & Z^4 = Y^5 = Y^{29} = Y^{30} = Y^{31} = (x-y)\partial_x + y\partial_y + u\partial_u + f\partial_f,\\
5) & Z^5 = Y^6 = y\partial_x - \partial_y,\\
6) & Z^6 = Y^7 = Y^{25} = Y^{14} = -\partial_y + \partial_u,\\
7) & Z^7 = Y^8 = Y^{24} = Y^{26} = -\partial_y - \partial_u,\\
8) & Z^8 = Y^{11} = Y^{20} = Y^{23} = y\partial_x + \partial_u,\\
9) & Z^9 = Y^{12} = Y^{21} = Y^{22} = y\partial_x - \partial_u,\\
10) & Z^{10} = Y^{13} = Y^{19} = Y^{28} = (x+y)\partial_x + y\partial_y + u\partial_u + f\partial_f,\\
11) & Z^{11} = Y^{15} = Y^{16} = Y^{27} = y\partial_x - \partial_y + \partial_u,\\
12) & Z^{12} = Y^{17} = Y^{18} = Y^{32} = y\partial_x - \partial_y - \partial_u,
\end{array}
$$

Proposition 4.2. *Let $\mathfrak{g}_m := \langle Y_1, \ldots, Y_m\rangle$ be an m-dimensional algebra of infinite-dimensional algebra $\mathfrak{g}$. Denote by $Y^i (i = 1, \ldots, r, 0 < r \leq m, r \in \mathbb{N})$ an optimal system of one-dimensional subalgebras of $\mathfrak{g}_m$ and by $Z^i\ (i = 1, \cdots, t, 0 < t \leq r, t \in \mathbb{N})$ the projections of Y^i, i.e. $Z^i = \mathrm{pr}(Y^i)$. If equations*

$$f = \Phi(x, y, u, u_x, u_y) \tag{43}$$

are invariant with respect to the optimal system Z^i, then the equation

$$u_t = \Phi(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) \tag{44}$$

admits the operators X^i = projection of Y^i on (t, x, y, u, u_x, u_y).

Proposition 4.3. *Let eq. (44) and the equation*

$$u_t = \Phi'(x, y, u, u_x, u_y)(u_{xx} + u_{yy}) \tag{45}$$

be constructed according to Proposition 4.2 via optimal systems Z^i and $Z^{i'}$, respectively. If the subalgebras spanned on the optimal systems Z^i and $Z^{i'}$, respectively, are similar in $\mathfrak{g}_m$, then eqs. (44) and (45) are equivalent with respect to the equivalence group G_m, generated by $\mathfrak{g}_m$.

Now we apply Proposition 4.2 and Proposition 4.3 to the optimal system (19) and obtain all nonequivalent equation (2) admitting $\mathcal{E}$-extensions of the principal Lie algebra $\mathfrak{g}$, by one dimension, *i.e.* equations of the form (2) such that they admit, together with the one basic operators (22) of $\mathfrak{g}$, also a second operator $X^{(2)}$. For every case, when this extension occurs, we indicate the corresponding coefficients f and the additional operator $X^{(2)}$.

We perform the algorithm passing from operators $Z^i\ (i = 1, \cdots, 12)$ to f and $X^{(2)}$ via the following example.

TABLE IV. – *The result of the classification.*

N	Z	Invariant λ	Function f	Additional operator $X^{(2)}$
1	Z^1	$u - y\alpha(\frac{x}{y})$	$\Phi(\lambda) - y\beta(\frac{x}{y})$	$x\partial_x + y\partial_y + t\partial_t + u\partial_u$
2	Z^2	u	$\Phi(\lambda)$	$y\partial_x,\ y\partial_x - \partial_t,\ y\partial_x + \partial_t$
3	Z^3	u	$\Phi(\lambda)$	$-\partial_y$
4	Z^4	$u - y\gamma\big(\frac{x-y}{y}\big)$	$\Phi(\lambda) - y\delta\big(\frac{x-y}{y}\big)$	$(x-y)\partial_x + y\partial_y + u\partial_u$, $(x-y)\partial_x + y\partial_y - \partial_t + u\partial_u$, $(x-y)\partial_x + y\partial_y + \partial_t + u\partial_u$
5	Z^5	$y + \frac{x}{y}$	$\Phi(\lambda)$	$y\partial_x - \partial_y$
6	Z^6	$u + y$	$\Phi(\lambda)$	$-\partial_y + \partial_u,\ -\partial_y - \partial_t + \partial_u,\ -\partial_y + \partial_t + \partial_u$
7	Z^7	$u - y$	$\Phi(\lambda)$	$-\partial_y - \partial_u,\ -\partial_y - \partial_t - \partial_u,\ -\partial_y + \partial_t - \partial_u$
8	Z^8	$u - \frac{x}{y}$	$\Phi(\lambda)$	$y\partial_x + \partial_u,\ y\partial_x - \partial_t + \partial_u,\ y\partial_x + \partial_t + \partial_u$
9	Z^9	$u + \frac{x}{y}$	$\Phi(\lambda)$	$y\partial_x - \partial_u,\ y\partial_x - \partial_t - \partial_u,\ y\partial_x + \partial_t - \partial_u$
10	Z^{10}	$u - y\zeta(\frac{x+y}{y})$	$\Phi(\lambda) - y\eta(\frac{x+y}{y})$	$(x+y)\partial_x + y\partial_y + u\partial_u$, $(x+y)\partial_x + y\partial_y + \partial_t + u\partial_u$, $(x+y)\partial_x + y\partial_y - \partial_t + u\partial_u$
11	Z^{11}	$u - y - \omega(u - \frac{x}{y})$	$\Phi(\lambda)$	$y\partial_x - \partial_y + \partial_u$, $y\partial_x - \partial_y - \partial_t + \partial_u$, $y\partial_x - \partial_y + \partial_t + \partial_u$
12	Z^{12}	$u - y - \pi(u + \frac{x}{y})$	$\Phi(\lambda)$	$y\partial_x - \partial_y - \partial_u$, $y\partial_x - \partial_y - \partial_t - \partial_u$, $y\partial_x - \partial_y + \partial_t - \partial_u$

Let us consider the vector field $Z^1 = x\partial_x + y\partial_y + u\partial_u + f\partial_f$. Then the characteristic equation corresponding to Z^1 is

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{u} = \frac{\mathrm{d}f}{f}\,, \tag{46}$$

and can be taken in the form $I_1 = u - y\alpha(\frac{x}{y})$; $I_2 = f - y\beta(\frac{x}{y})$, where g, h are arbitrary smooth functions. From the invariance equations, we can write $I_2 = \Phi(I_1)$, it follows that $f = \Phi(\lambda) + y\beta(\frac{x}{y})$, where $\lambda = I_1$. From Proposition 4.3 applied to the operator Z^1 it is obtained the additional operator $X^{(2)}$ is equal to $y\partial_x + \partial_y + \partial_t + \partial_u$.

After similar calculations applied to all operators (19) we obtain the results (table IV) for the preliminary group classification of eq. (2) admitting an extension $\mathfrak{g}_3$ of the principal Lie algebra $\mathfrak{g}_1$.

5. – Conclusions and discussions

Summarizing the results of our group classification of nonlinear 2D heat equations of the form (2) we conclude that:

1) In the present paper the preliminary group classification for the class of heat equation (2) is obtained and the algebraic structure of the symmetry groups for this equation is investigated.

2) The classification is obtained by constructing an optimal system with the aid of Propositions 4.2 and 4.3. We have proved that an optimal system of one-dimensional Lie subalgebras of eq. (2) is provided by those generated by $\langle Z^1, \dots, Z^{12} \rangle$ and we obtain Z^is, then we take their optimal system's projections on the space (x, y, u, u_x, u_y, f).

3) The result of the work is summarized in table IV. Of course it is also possible to obtain the corresponding reduced equations for all the cases in the classification reported in it.

REFERENCES

[1] Lie S., *Arch Math.*, **6** (1881) 328.
[2] Lie S. and Scheffers G., *Vorlesungen uber continuierliche Gruppen mit geometrischen und anderen Anwendungen* (Teubner, Leipzig) 1893.
[3] Winternitz P., *Lect. Not. Phys.*, **189** (1983) 263.
[4] Carinena J. F., Grabowski J. and Lucas, *J. Phys. A: Math. Theor.*, **43** (2010) 201.
[5] Carinena J. F., Grabowski J. and Marmo G., *Rep. Math. Phys.*, **60** (2007) 237.
[6] Ovsiannikov L. V., *Group Analysis of Differential Equations* (Academic Press, New York) 1982.
[7] David D., Kamran N., Levi D. and Winternitz P., *Phys. Rev. Lett.*, **55** (1985) 2111.
[8] Saied E. A. and Hussain M. M., *J. Nonlinear Math. Phys.*, **3** (1996) 219.
[9] Clarkson P. A. and Mansfield E. L., *Phys. D*, **70** (1993) 250.
[10] Servo M. I., *J. Nonlinear Math. Phys.*, **3** (1996) 63.
[11] Goard V., Broadbridge P. and Arrigo D. J., *Math. Phys.*, **47** (1996) 926.
[12] Estevez P. G., Qu C. and Zhang S. L., *J. Math. Anal. Appl.*, **275** (2002) 44.
[13] Doyle P. W. and Vassiliou P. J., *J. Non-Linear Mech.*, **33** (2002) 315.
[14] Basarab-Horwath P., Lahno V. and Zhdanov R., *Acta Appl. Math.*, **69** (2001) 43.
[15] Ahmad A., Bokhari Ashfaque H., Kara A. H. and Zaman F. D., *J. Math. Annal. Appl.*, **339** (2008) 175.
[16] Ovsiannikov L. V., *Dokl. Akad. Nauk SSSR*, **125** (1959) 492 (in Russian).
[17] Akhatov I. S., Gazizov R. K. and Ibragimov N. K., *Proc. Acad. Sci. USSR*, **293** (1987) 1033 (in Russian).
[18] Dorodnitsyn V. A., *Zh. Vychisl. Mat. Mat. Fiz.*, **22** (1982) 1393 (in Russian).
[19] Oron A. and Rosenau P., *Phys. Lett. A*, **118** (1986) 172.
[20] Edwards M. P., *Phys. Lett. A*, **190** (1994) 149.
[21] Gandarias M. L., *J. Phys. A: Math. Gen.*, **29** (1996) 607.
[22] Cherniha R. and Serov M., *Eur. J. Appl. Math.*, **9** (1998) 527.
[23] Zhdanov R. Z. and Lahno V. I., *J. Phys. A: Math. Gen.*, **32** (1999) 7405.
[24] Olver P. J., *Equivalence, Invariants, and Symmetry* (Cambridge University Press, Cambridge) 1995.
[25] Nadjafikhah M. and Bakhshandeh-Chamazkoti R., *Commun. Nonlinear Sci. Num. Simul.*, **15** (2010) 2303.