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SYMMETRY ANALYSIS OF TELEGRAPH EQUATION

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Lie symmetry group method is applied to study the telegraph equation. The symmetry group and one-parameter group associated to the symmetries with the structure of the Lie algebra symmetries are determined. The reduced version of equation and its one-dimensional optimal system are given.

Keywords: Lie group analysis; symmetry group; optimal system; invariant solution; telegraph equation.

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1. Introduction

The telegrapher's equations (or just telegraph equations) are a pair of linear differential equations which describe the voltage and current on an electrical transmission line with distance and time. The equations come from *Oliver Heaviside* who developed the transmission line model. The model demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line. The cylindrical telegrapher's equations [2]

$$u_{tt} + ku_t = a^2\Big[\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}\Big], \tag{1.1}$$

can be understood as a simplified case of Maxwell's equations. In a more practical approach, one assumes that the conductors are composed of an infinite series of twoport elementary components, each representing an infinitesimally short segment of the transmission line.

2. Lie Symmetries of the Equation

A PDE with $p-$independent and $q-$dependent variables has a Lie point transformations

$$\widetilde{x}_i = x_i + \varepsilon\xi_i(x,u) + \mathcal{O}(\varepsilon^2), \qquad \widetilde{u}_\alpha = u_\alpha + \varepsilon\varphi_\alpha(x,u) + \mathcal{O}(\varepsilon^2)$$

where $\xi_i = \dfrac{\partial\widetilde{x}_i}{\partial\varepsilon}\Big|_{\varepsilon=0}$ for $i = 1, ..., p$ and $\varphi_\alpha = \dfrac{\partial\widetilde{u}_\alpha}{\partial\varepsilon}\Big|_{\varepsilon=0}$ for $\alpha = 1, ..., q$. The action of the Lie group can be considered by its associated infinitesimal generator

$$\mathbf{v} = \sum_{i=1}^{p}\xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\varphi_\alpha(x,u)\frac{\partial}{\partial u_\alpha} \tag{2.1}$$

on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the vector field (2.1) is given by

$$Q^\alpha(x,u^{(1)}) = \varphi_\alpha(x,u) - \sum_{i=1}^{p}\xi_i(x,u)\frac{\partial u^\alpha}{\partial x_i},$$

and its $n-$th prolongation is determined by

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p}\xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\sum_{\sharp J=j=0}^{n}\varphi_\alpha^J(x,u^{(j)})\frac{\partial}{\partial u_J^\alpha},$$

where $\varphi_\alpha^J = D_J Q^\alpha + \sum_{i=1}^{p}\xi_i u_{J,i}^\alpha$. ($D_J$ is the total derivative operator describes in (2.2).)

The aim is to analysis the point symmetry structure of equation (1.1), which is where u is a smooth function of (r,θ,z,t).

Let us consider a one-parameter Lie group of infinitesimal transformations (r,θ,y,t,u) given by

$$\begin{aligned}
&\widetilde{r} = r + \varepsilon\xi_1(r,\theta,z,t,u) + \mathcal{O}(\varepsilon^2), \ \widetilde{\theta} = \theta + \varepsilon\xi_2(r,\theta,z,t,u) + \mathcal{O}(\varepsilon^2),\\
&\widetilde{z} = z + \varepsilon\xi_3(r,\theta,z,t,u) + \mathcal{O}(\varepsilon^2), \ \widetilde{t} = t + \varepsilon\xi_4(r,\theta,z,t,u) + \mathcal{O}(\varepsilon^2),\\
&\widetilde{u} = u + \varepsilon\eta(r,\theta,z,t,u) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of equation (1.1). This yields to the linear system of equations for the infinitesimals $\xi_1(r,\theta,z,t,u)$, $\xi_2(r,\theta,z,t,u)$, $\xi_3(r,\theta,z,t,u)$, $\xi_4(r,\theta,z,t,u)$ and $\eta(r,\theta,z,t,u)$. The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of $\mathbf{v} = \xi_1\partial_r + \xi_2\partial_\theta + \xi_3\partial_z + \xi_4\partial_t + \eta\partial_u$. This vector field has the second prolongation

$$\mathbf{v}^{(2)} = \mathbf{v} + \varphi^r\partial_r + \varphi^\theta\partial_\theta + \varphi^z\partial_z + \varphi^t\partial_t + \cdots + \varphi^{zz}\partial_{u_{zz}} + \varphi^{zt}\partial_{u_{zt}} + \varphi^{tt}\partial_{tt}$$

with the coefficients

$$
\begin{aligned}
\varphi^{r} &= D_rQ + \xi_1 u_{rr} + \xi_2 u_{r\theta} + \xi_3 u_{rz} + \xi_4 u_{rt},\\
\varphi^{\theta} &= D_\theta Q + \xi_1 u_{r\theta} + \xi_2 u_{\theta\theta} + \xi_3 u_{\theta z} + \xi_4 u_{\theta t},\\
\varphi^{z} &= D_zQ + \xi_1 u_{rz} + \xi_2 u_{\theta z} + \xi_3 u_{zz} + \xi_4 u_{zt},\\
\varphi^{t} &= D_tQ + \xi_1 u_{rt} + \xi_2 u_{\theta t} + \xi_3 u_{\theta t} + \xi_4 u_{tt},\\
\varphi^{rr} &= D_r^2Q + \xi_1 u_{rrr} + \xi_2 u_{rr\theta} + \xi_3 u_{rrz} + \xi_4 u_{rrt},\\
\varphi^{rx} &= D_rD_\theta Q + \xi_1 u_{r\theta r} + \xi_2 u_{r\theta\theta} + \xi_3 u_{r\theta z} + \xi_4 u_{r\theta t},\\
\varphi^{rz} &= D_rD_zQ + \xi_1 u_{rzr} + \xi_2 u_{r\theta z} + \xi_3 u_{rzz} + \xi_4 u_{rzt},\\
\varphi^{rt} &= D_rD_tQ + \xi_1 u_{rtr} + \xi_2 u_{r\theta t} + \xi_3 u_{rzt} + \xi_4 u_{rtt},\\
\varphi^{\theta\theta} &= D_\theta^2Q + \xi_1 u_{\theta\theta r} + \xi_2 u_{\theta\theta\theta} + \xi_3 u_{\theta\theta z} + \xi_4 u_{\theta\theta t},\\
\varphi^{\theta z} &= D_\theta D_zQ + \xi_1 u_{\theta zr} + \xi_2 u_{\theta\theta z} + \xi_3 u_{\theta zz} + \xi_4 u_{\theta zt},\\
\varphi^{\theta t} &= D_\theta D_tQ + \xi_1 u_{\theta rt} + \xi_2 u_{\theta\theta t} + \xi_3 u_{\theta zt} + \xi_4 u_{\theta tt},\\
\varphi^{zz} &= D_z^2Q + \xi_1 u_{rzz} + \xi_2 u_{\theta zz} + \xi_3 u_{zzz} + \xi_4 u_{zzt},\\
\varphi^{zt} &= D_zD_tQ + \xi_1 u_{rzt} + \xi_2 u_{\theta zt} + \xi_3 u_{zzt} + \xi_4 u_{ztt},\\
\varphi^{tt} &= D_t^2Q + \xi_1 u_{rtt} + \xi_2 u_{\theta tt} + \xi_3 u_{ztt} + \xi_4 u_{ttt},
\end{aligned}
$$

where the operators D_r, D_θ, D_z and D_t denote the total derivative with respect to r, θ, z and t:

$$
\begin{aligned}
D_r &= \partial_r + u_r\partial_u + u_{rr}\partial_{u_r} + u_{r\theta}\partial_{u_\theta} + \cdots\,,\\
D_\theta &= \partial_\theta + u_\theta\partial_u + u_{\theta\theta}\partial_{u_\theta} + u_{r\theta}\partial_{u_r} + \cdots\,,\\
D_z &= \partial_z + u_z\partial_u + u_{zz}\partial_{u_z} + u_{rz}\partial_{u_r} + \cdots\,,\\
D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{rt}\partial_{u_r} + \cdots\,.
\end{aligned}
\tag{2.2}
$$

Using the invariance condition, i.e., vanishes the second prolongation $\mathbf{v}^{(2)}$ applied to equation (1.1), the following system of 27 determining equations are obtained:

$$
\begin{array}{lll}
\xi_{2u} = 0, & \xi_{2zz} = 0, & \xi_{3z} = 0,\\
\xi_{3u} = 0, & \xi_{4t} = 0, & \xi_{4u} = 0,\\
\xi_{4rr} = 0, & \xi_{4\theta z} = 0, & \xi_{4zz} = 0,\\
\xi_{4rz} = 0, & \eta_{tu} = 0, & \eta_{uu} = 0,\\
k\xi_{4z} + 2\eta_{ru} = 0, & \xi_1 + r\xi_{2\theta} = 0, & \xi_{2\theta} + r\xi_{2r\theta} = 0,\\
\xi_{2z} + r\xi_{2rz} = 0, & \xi_{2\theta\theta} - r\xi_{2r} = 0, & k\xi_{4z} + 2\eta_{zu} = 0,\\
2\xi_{2r} + r\xi_{2rr} = 0, & \xi_{3r} - r\xi_{2\theta z} = 0, & \\
\xi_{3\theta} + r^2\xi_{2z} = 0, & \xi_{3t} - a^2\xi_{4z} = 0, & \xi_{4\theta} - r\xi_{4r\theta} = 0,\\
\xi_{4\theta\theta} + r\xi_{4r} = 0, & r^2\xi_{2t} - a^2\xi_{4\theta} = 0, & k\xi_{4\theta} + 2\eta_{u\theta} = 0,
\end{array}
$$

$$
a^2r^2\eta_{rr} - kr^2\eta_t + a^2r\eta_r + a^2r^2\eta_{zz} - r^2\eta_{tt} + a^2\eta_{\theta\theta} = 0.
$$

The solution of the above system gives the following coefficients of the vector

Table 1. Commutation relations of $\mathcal{G}$

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$	$\mathbf{v}_{11}$
$\mathbf{v}_1$	0	0	0	0	0	$-\mathbf{v}_7$	$\mathbf{v}_6$	$-\mathbf{v}_9$	$\mathbf{v}_8$	$-\mathbf{v}_{11}$	$\mathbf{v}_{10}$
$\mathbf{v}_2$	$*$	0	0	0	$2a^2\mathbf{v}_3 - k\mathbf{v}_4$	0	0	$\mathbf{v}_7$	$-\mathbf{v}_6$	0	0
$\mathbf{v}_3$	$*$	$*$	0	0	$\mathbf{v}_2$	0	0	0	0	$\mathbf{v}_6$	$\mathbf{v}_7$
$\mathbf{v}_4$	$*$	$*$	$*$	0	0	0	0	0	0	0	0
$\mathbf{v}_5$	$*$	$*$	$*$	$*$	0	0	0	$\mathbf{v}_{11}$	$-\mathbf{v}_{10}$	$-\mathbf{v}_9$	$\mathbf{v}_8$
$\mathbf{v}_6$	$*$	$*$	$*$	$*$	$*$	0	0	$\mathbf{v}_7$	$-\mathbf{v}_6$	0	0
$\mathbf{v}_7$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-\mathbf{v}_2$	0	0	$2a^2\mathbf{v}_3 - k\mathbf{v}_4$
$\mathbf{v}_8$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-\mathbf{v}_1$	0	$\mathbf{v}_5$
$\mathbf{v}_9$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-\mathbf{v}_5$	0
$\mathbf{v}_{10}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	$a^2\mathbf{v}_1$
$\mathbf{v}_{11}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

field $\mathbf{v}$:

$$\begin{aligned}
\xi_1 &= c_6 \sin\theta - c_7\cos\theta - c_8 z\cos\theta - c_9 z\sin\theta + 2c_{10}a^2 t\sin\theta - 2c_{11}a^2 t\cos\theta,\\
\xi_2 &= c_1 + c_6 r^{-1}\cos\theta + c_7 r^{-1}\sin\theta + c_8 z r^{-1}\sin\theta - c_9 z r^{-1}\sin\theta\\
&\qquad\qquad +2c_{10}a^2 t r^{-1}\cos\theta - 2c_{11}a^2 t r^{-1}\sin\theta,\\
\xi_3 &= c_2 + 2c_5 a^2 t + c_8 r\cos\theta + c_9 r\sin\theta,\\
\xi_4 &= c_3 + 2c_5 a t + 2c_{10} r\sin\theta - 2c_{11} r\cos\theta,\\
\eta &= c_4 u - c_5 k z u - c_{10} k r u\sin\theta + c_{11} k r u\cos\theta,
\end{aligned}$$

where $c_1, ..., c_{11}$ are arbitrary constants, thus the Lie algebra $\mathcal{G}$ of the telegraph equation is spanned by the eleven vector fields

$$\begin{aligned}
&\mathbf{v}_1 = \partial_\theta, \quad \mathbf{v}_2 = \partial_z, \quad \mathbf{v}_3 = \partial_t, \quad \mathbf{v}_4 = u\partial_u,\\
&\mathbf{v}_5 = 2a^2 t\partial_z + 2z\partial_t - kzu\partial_u, \quad \mathbf{v}_6 = \sin\theta\partial_r + r^{-1}\cos\theta\partial_\theta,\\
&\mathbf{v}_7 = -\cos\theta\partial_r + r^{-1}\sin\theta\partial_\theta, \quad \mathbf{v}_8 = -z\cos\theta\partial_r + r^{-1}z\sin\theta\partial_\theta + r\cos\theta\partial_z,\\
&\mathbf{v}_9 = -z\sin\theta\partial_r - r^{-1}z\cos\theta\partial_\theta + r\sin\theta\partial_z,\\
&\mathbf{v}_{10} = 2a^2 t\sin\theta\partial_r + 2a^2 t r^{-1}\cos\theta\partial_\theta + 2r\sin\theta\partial_t - kru\sin\theta\partial_u,\\
&\mathbf{v}_{11} = -2a^2 t\cos\theta\partial_r + 2a^2 t r^{-1}\sin\theta\partial_\theta - 2r\cos\theta\partial_t + kru\cos\theta\partial_u,
\end{aligned}$$

which $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are translation on θ, z and t, $\mathbf{v}_4$ is scaling on u, $\mathbf{v}_5$ is scaling on z, t and u simultaneously, $\mathbf{v}_6$ and $\mathbf{v}_7$ are rotation on r and θ and $\mathbf{v}_8$ and $\mathbf{v}_9$ are rotation on r, θ and z and also $\mathbf{v}_{10}$ and $\mathbf{v}_{11}$ are rotation on r, θ, t and u. The commutation relations between these vector fields is given by Table 1, where entry in row i and column j representing $[\mathbf{v}_i, \mathbf{v}_j]$. The one-parameter groups G_i generated by the base of $\mathcal{G}$ are given in the following table.

$$g_1 : (r,\theta,z,t,u) \longmapsto (r,\theta+\varepsilon,z,t,u), \qquad g_2 : (r,\theta,z,t,u) \longmapsto (r,\theta,z+\varepsilon,t,u),$$
$$g_3 : (r,\theta,z,t,u) \longmapsto (r,\theta,z,t+\varepsilon,u), \qquad g_4 : (r,\theta,z,t,u) \longmapsto (r,\theta,z,t,ue^{\varepsilon}),$$
$$g_5 : (r,\theta,z,t,u) \longmapsto \Big(r,\theta,z+\varepsilon t,\frac{\varepsilon z}{a^2}+t,-\frac{\varepsilon kru}{2a^2}+u\Big),$$
$$g_6 : (r,\theta,z,t,u) \longmapsto \Big(\varepsilon\sin\theta+r,\frac{\varepsilon}{r}\cos\theta+\theta,z,t,u\Big),$$
$$g_7 : (r,\theta,z,t,u) \longmapsto \Big(-\varepsilon\cos\theta+r,\frac{\varepsilon}{r}\sin\theta+\theta,z,t,u\Big),$$
$$g_8 : (r,\theta,z,t,u) \longmapsto \Big(-\varepsilon z\cos\theta+r,\frac{\varepsilon z}{r}\sin\theta+\theta,\varepsilon r\cos\theta+z,t,u\Big),$$
$$g_9 : (r,\theta,z,t,u) \longmapsto \Big(-\varepsilon z\sin\theta+r,-\frac{\varepsilon z}{r}\cos\theta+\theta,\varepsilon r\sin\theta+z,t,u\Big),$$
$$g_{10} : (r,\theta,z,t,u) \longmapsto \Big(\varepsilon t\sin\theta+r,\frac{\varepsilon t}{r}\cos\theta+\theta,z,\frac{\varepsilon r}{a^2}\sin\theta+t,\frac{\varepsilon kru}{2a^2}\sin\theta+u\Big),$$
$$g_{11} : (r,\theta,z,t,u) \longmapsto \Big(-\varepsilon t\cos\theta+r,\frac{\varepsilon t}{r}\sin\theta+\theta,z,-\frac{\varepsilon r}{a^2}\cos\theta+t,\frac{\varepsilon kru}{2a^2}\cos\theta+u\Big).$$

Since each group G_i is a symmetry group and if $u = f(r,\theta,z,t)$ is a solution of the Telegraph equation, so are the functions

$$u^1 = U(r,\theta+\varepsilon,z,t), \qquad u^2 = U(r,\theta,z+\varepsilon,t), \qquad u^3 = U(r,\theta,z,t+\varepsilon),$$
$$u^4 = e^{-\varepsilon}U(r,\theta,z,t), \qquad u^5 = \Big(\frac{2a^2}{2a^2-\varepsilon kz}\Big)U\Big(r,\theta,z+\varepsilon t,\frac{1}{a^2}\varepsilon z+t\Big),$$
$$u^6 = U\Big(\varepsilon\sin\theta+r,\frac{\varepsilon}{r}\cos\theta+\theta,z,t\Big), \quad u^7 = U\Big(-\varepsilon\cos\theta+r,\frac{\varepsilon}{r}\sin\theta+\theta,z,t\Big),$$
$$u^8 = U\Big(-\varepsilon z\cos\theta+r,\frac{\varepsilon}{r}z\sin\theta+\theta,\varepsilon r\cos\theta+z,t\Big),$$
$$u^9 = U\Big(-\varepsilon z\sin\theta+r,-\frac{\varepsilon}{r}z\cos\theta+\theta,\varepsilon r\sin\theta+z,t\Big),$$
$$u^{10} = \Big(\frac{2a^2}{2a^2-\varepsilon kr\sin\theta}\Big)U\Big(\varepsilon t\sin\theta+r,\frac{\varepsilon}{r}t\cos\theta+\theta,z,\frac{\varepsilon}{a^2}r\sin\theta+t\Big),$$
$$u^{11} = \Big(\frac{2a^2}{2a^2+\varepsilon kr\cos\theta}\Big)U\Big(-\varepsilon t\cos\theta+r,\frac{\varepsilon}{r}t\sin\theta+\theta,z,-\frac{\varepsilon}{a^2}r\cos\theta+t\Big),$$

where ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_1\mathbf{v}_{11}$ of the given vector fields. In particular if g is the action of the symmetry group near the identity, it can be represented in the form $g = \exp(\varepsilon_{11}\mathbf{v}_{11}) \circ \cdots \circ \exp(\varepsilon_1\mathbf{v}_1)$.

2.1. *Reduction of the equation*

In this part, we determine the reduced form of equation (1.1) by using the Lie-Bianchi theorem.

The Lie algebra $\mathcal{G}$ has a *Levi decomposition* in the form of $\mathcal{G} = \mathcal{R} \ltimes \mathcal{H}$, where $\mathcal{R} = \langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_6, \mathbf{v}_7 \rangle$ is the radical (the large solvable ideal) of $\mathcal{G}$ which is a

Table 2. Commutation relations of $\mathcal{G}_1$

$[\,,\,]$	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$	$\mathbf{w}_5$	$\mathbf{w}_6$
$\mathbf{w}_1$	0	0	$-\mathbf{w}_4$	$\mathbf{w}_3$	$-\mathbf{w}_6$	$\mathbf{w}_5$
$\mathbf{w}_2$	0	0	$\mathbf{w}_6$	$-\mathbf{w}_5$	$-\mathbf{w}_4$	$\mathbf{w}_3$
$\mathbf{w}_3$	$\mathbf{w}_4$	$-\mathbf{w}_6$	0	$-\mathbf{w}_1$	0	$\mathbf{w}_2$
$\mathbf{w}_4$	$-\mathbf{w}_3$	$\mathbf{w}_5$	$\mathbf{w}_1$	0	$-\mathbf{w}_2$	0
$\mathbf{w}_5$	$\mathbf{w}_6$	$\mathbf{w}_4$	0	$\mathbf{w}_2$	0	$a^2\mathbf{w}_1$
$\mathbf{w}_6$	$-\mathbf{w}_5$	$-\mathbf{w}_3$	$-\mathbf{w}_2$	0	$-a^2\mathbf{w}_1$	0

nilpotent nilradical of $\mathcal{G}$ and $\mathcal{H} = \langle \mathbf{v}_1, \mathbf{v}_5, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11} \rangle$ is a subalgebra of $\mathcal{G}$ with centralizer $\langle \mathbf{v}_4 \rangle$ containing in the minimal ideal $\langle \mathbf{v}_1, \mathbf{v}_2, 2a^2\mathbf{v}_3 - k\mathbf{v}_4, \mathbf{v}_5, ..., \mathbf{v}_{11} \rangle$.

We can construct the quotient algebra by $\mathcal{G}$ and $\mathcal{R}$ denoted

$$\mathcal{G}_1 = \mathcal{G}/\mathcal{R}, \tag{2.3}$$

with commutators Table 2, where $\mathbf{w}_1 = \mathbf{v}_1 + \mathcal{R}$ and $\mathbf{w}_i = \mathbf{v}_{i+5} + \mathcal{R}$ for $i = 2, ..., 6$ are members of $\mathcal{G}_1$. Structure (2.3) helps us to reduce differential equations. If we want to integrate an evolutive distribution, the process decomposes into two steps:

- integration of the evolutive distribution with symmetry Lie algebra $\mathcal{G}/\mathcal{R}$, and
- integration on integral manifolds with symmetry algebra $\mathcal{R}$.

First, applying this procedure to the radical $\mathcal{R}$ we decompose the integration problem into two parts: the integration of the Lie algebra $\mathcal{G}/\mathcal{R}$, then integration of the solvable symmetry algebra $\mathcal{R}$.

The last step can be performed by quadratures. Moreover, every semisimple Lie algebra $\mathcal{G}/\mathcal{R}$ is a direct sum of simple ones which are ideal in $\mathcal{G}/\mathcal{R}$. Thus, the Lie-Bianchi theorem [8] states that any differential equation with solvable symmetry Lie algebra is integrable by quadratures, reduces the integration problem to subalgebra R of G. We can use the similarity transformations corresponding to R and find the reduced forms of (1.1). These similarities and new equations are given in Tables 3 and 4. The reduced form corresponding to $\mathbf{v}_2$ and $\mathbf{v}_3$ are two-dimensional telegraph equation in polar coordinate and homogeneous wave equation in polar coordinate respectively.

3. One-dimensional Optimal System of Telegraph Equation

As is well known, the theoretical Lie group method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential

Table 3. Similarity variables

symmetries	similarities
$\mathbf{v}_2$	$r=\tilde{r},\theta=\tilde{\theta},z=\tilde{z},t=\tilde{t},u=\tilde{u}(\tilde{r},\tilde{\theta},\tilde{t})$
$\mathbf{v}_3$	$r=\tilde{r},\theta=\tilde{\theta},z=\tilde{z},t=\tilde{t},u=\tilde{u}(\tilde{r},\tilde{\theta},\tilde{z})$
$\mathbf{v}_4$	There is no any non-trivial similarity.
$\mathbf{v}_6$	$r=\dfrac{\tilde{r}}{\sin\theta},\theta=\arccos\dfrac{1}{r\tilde{\theta}},z=\tilde{z},t=\tilde{t},u=\tilde{u}(\tilde{\theta},\tilde{z},\tilde{t})$
$\mathbf{v}_7$	$r=-\dfrac{\tilde{r}}{\cos\theta},\theta=\arcsin\dfrac{1}{r\tilde{\theta}},z=\tilde{z},t=\tilde{t},u=\tilde{u}(\tilde{\theta},\tilde{z},\tilde{t})$

Table 4. Reduced equations

symmetries	Reduced forms
$\mathbf{v}_2$	$\tilde{u}_{\tilde{t}\tilde{t}}+k\tilde{u}_{\tilde{t}}=a^2\left(\dfrac{\tilde{u}_{\tilde{r}}}{\tilde{r}}+\tilde{u}_{\tilde{r}\tilde{r}}+\dfrac{\tilde{u}_{\tilde{\theta}\tilde{\theta}}}{\tilde{\theta}^2}\right)$
$\mathbf{v}_3$	$\dfrac{\tilde{u}_{\tilde{r}}}{\tilde{r}}+\tilde{u}_{\tilde{r}\tilde{r}}+\dfrac{\tilde{u}_{\tilde{\theta}\tilde{\theta}}}{\tilde{r}^2}+\tilde{u}_{\tilde{z}\tilde{z}}=0$
$\mathbf{v}_6$	$\tilde{u}_{\tilde{t}\tilde{t}}+k\tilde{u}_{\tilde{t}}=a^2\left(\tilde{r}^4\tilde{u}_{\tilde{r}\tilde{r}}+2\tilde{r}^3\tilde{u}_{\tilde{r}}+\tilde{u}_{\tilde{\theta}\tilde{\theta}}\right)$
$\mathbf{v}_7$	$\tilde{u}_{\tilde{t}\tilde{t}}+k\tilde{u}_{\tilde{t}}=a^2\left(\tilde{r}^4\tilde{u}_{\tilde{r}\tilde{r}}+2\tilde{r}^3\tilde{u}_{\tilde{r}}+\tilde{u}_{\tilde{\theta}\tilde{\theta}}\right)$

equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system [6]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation for classifying group-invariant solutions is due to [7, 8, 9].

Optimal system of a Lie algebra is equivalent to find nonessentially different invariant solutions which are not in the same orbit of adjoint actions. It means that the problem of finding invariant solutions under a group action G or its subgroups is reduced to problem of constructing optimal system of its corresponding subalgebras. As a result if the optimal system of subalgebras is found this optimality is meant that it is the smallest subspace in the set of solutions possessing the following property: Any invariant solution which can be found from any subgroup of G; is contained in one of the orbits of G-action, i.e.; optimal system.

The problem of finding optimal system is divided into two methods; a) using the generators of Lie algebra directly and b) using the ideal of Lie algebra. The first method is processing in the sequel. But second method has two levels. Finding optimal system from ideals and extend it on the whole Lie algebra. It means that suppose $\mathcal{G}$ is a subalgebra spanned by n vector fields $X_1, ..., X_n$ with m ideals $\mathcal{G}_1, ..., \mathcal{G}_m$. This method starts by selecting $X_1 \in \mathcal{G}_1$ and we introduce it a member of the optimal system. Next by selecting $X_2 \in \mathcal{G}_2$ where X_2 is not in $\mathcal{G}_1$. Thus, $X_1 + aX_2$ is another member of optimal system and if it is possible we can normalize the coefficient a by adjoint action. This procedure will continue until the last ideal. Finally we normalize the linear combination of obtained members with other generators of $\mathcal{G}$ that are not in any ideals with adjoint action. The last level gives us the one-dimensional optimal system.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \tag{3.1}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 11$. Let $F_i^\varepsilon : \mathcal{G} \to \mathcal{G}$ defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1, \cdots, 11$. The two of eleven matrices M_i^ε of F_i^ε, $i = 1, \cdots, 11$, with respect to basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_{11}\}$ are given by

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\varepsilon & \sin\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin\varepsilon & \cos\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\varepsilon & \sin\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\varepsilon & \cos\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix},$$

$$M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2a^2\varepsilon & k\varepsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

by acting these matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathcal{G}$ is given by

$$
\begin{aligned}
X_1 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + a_4\mathbf{v}_8,\\
X_2 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + a_4\mathbf{v}_5 - a_6\mathbf{v}_9\\
X_3 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_6\mathbf{v}_8,\\
X_4 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + a_4\mathbf{v}_5 - a_6\mathbf{v}_8\\
X_5 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_3 + a_3\mathbf{v}_5 + a_4\mathbf{v}_6 + a_6\mathbf{v}_8\\
X_6 &= a_1\mathbf{v}_1 + \mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + a_4(\mathbf{v}_6 - \mathbf{v}_{10})\\
X_7 &= a_1\mathbf{v}_1 + \mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + \mathbf{v}_7 - \mathbf{v}_{11},\\
X_8 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_6\mathbf{v}_6,\\
X_9 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 - (2a^2 - k)\mathbf{v}_5 + \mathbf{v}_7,\\
X_{10} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \mathbf{v}_3 + a_3\mathbf{v}_4 - (2a^2 - k)\mathbf{v}_5 + \mathbf{v}_6,\\
X_{11} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_6\mathbf{v}_6 + a_7\mathbf{v}_{11},\\
X_{12} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_6\mathbf{v}_6 + a_7\mathbf{v}_7 + a_8\mathbf{v}_{11},\\
X_{13} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_6\mathbf{v}_6 - a_7\mathbf{v}_7 - a_8\mathbf{v}_9,\\
X_{14} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_4\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_6\mathbf{v}_6 + \mathbf{v}_7 - (2a + k)\mathbf{v}_{11},\\
X_{15} &= \tfrac{1}{2a^2-k}(\mathbf{v}_1 + \mathbf{v}_8) + a_1\mathbf{v}_2 + a_2\mathbf{v}_3 + a_3\mathbf{v}_4 + a_5\mathbf{v}_5 + \mathbf{v}_6 + \mathbf{v}_7,\\
X_{16} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 - (2a^2 - k - 1)\mathbf{v}_5 + a_3\mathbf{v}_6 + a_4\mathbf{v}_7 + a_5\mathbf{v}_8 + a_6\mathbf{v}_9.
\end{aligned}
$$

The process above beings with the selection of vector $X = a_i\mathbf{v}_i$ and its image under M_i^ε, obtained by the adjoint automorphism. If M_i^ε is the matrix of the automorphism M^ε in the basis $\{\mathbf{v}_i\}$, then the components on the image of X in the basis are given by an equation in the form of

$$M^{\varepsilon}_{i\ j} a_j \quad (j = 1, ..., 11). \tag{3.2}$$

The next step is the selection of values of the parameter M_i^ε, on which this automorphism depends, to achieve the maximum possible simplification of the set of equations (3.2). This permits the choice of simplest representative of class of similar algebras to which the element X belongs. Usually, this means choosing the maximum possible number of null values for these components.

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