Symmetry analysis of wave equation on hyperbolic space

Mehdi Nadjafikhah* Amir Hesam Zaeim†

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Abstract

Lie symmetry group method is applied to study the wave equation on hyperbolic space. Using Lie symmetry algebra of the equation, an optimal system of subalgebras is presented.

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1 Introduction

It is well known that the symmetry group method plays an important role in the analysis of differential equations. One of useful applications is to construct new solutions from known ones. To do this, a particular linear combinations of infinitesimals must be considered and their corresponding invariants must be determined.

Inspired by Galois' theory of, Sophus Lie developed an analogous theory of symmetry for differential equations. Lie's theory led to an algorithmic way to find special explicit solutions to differential equations with symmetry. These special solutions are called group invariant solutions and they constitute practically every known explicit solution to the systems of non-linear

*Corresponding author. School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 1684613114, Iran. *E-mail:* m_nadjafikhah@iust.ac.ir

†Department of Complementary Education, Payame noor University, Tehran, Iran. *E-mail:* zaeim@phd.pnu.ac.ir

partial differential equations arising in mathematical physics, differential geometry and other areas.

These group-invariant solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original system. For example, the solutions to a partial differential equation in two independent variables which are invariant under a given one-parameter symmetry group are all found by solving a system of ordinary differential equations. Today the search for group invariant solutions is still a common approach to explicitly solving non-linear partial differential equations. [9].

Symmetry and reductions for a lot of equations with the flat background metric are considered [5, 6]. Also the classifying problem has been completely expressed for some wave equations on flat spaces [6, 1, 3, 4, 8, 11]. Symmetry and reductions of wave equation with $ds^2 = dt^2 - dx^2 - \sin^2(x)dy^2$ background metric on the space $\mathbb{S}^2 \times \mathbb{R}$ was studied in [2]. In the present paper we find symmetries of the wave equation on the space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ equipped with the metric of constant negative sectional curvature -1. Such space named as hyperbolic space. Next we present an optimal system of subalgebras related to the Lie algebra of symmetries. The wave differential equation is:

$$t(u_{xx} + u_{yy} + u_{tt}) = u_t. \tag{1}$$

For classifying all subalgebras of the symmetry Lie algebra, we classify subalgebras in to conjugate classes by the adjoint action of the symmetry group [12]. The wave equation on the hyperbolic space is considered in section 2 and Symmetry algebra of equation (1) has been presented in section 3. In section 4 we study complete classification of subalgebras for the symmetry algebra, obtained from section 3.

2 Wave equation on the hyperbolic space

Wave equation on the hyperbolic space is a special type of the equation $\Delta_g u + f(u) = 0$,

$$\begin{aligned} \Delta_g u &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\frac{\partial u}{\partial x^j}) \\ &= g^{ij}\nabla_i\nabla_j u = \nabla^j\nabla_j u = \nabla_i\nabla^i u \end{aligned} \tag{2}$$

where ∇_g is the Laplace-Beltrami operator on an arbitrary (pseudo) Riemannian manifold (M^n, g) and ∇_i is the covariant derivative corresponding to the Levi-Civita connection, the Einstein summation convention over repeated indices is understood and $f(u)$ is a smooth function on the background manifold [7].

Now if we set the metric $ds^2 = \frac{1}{t^2}(dx^2 + dy^2 + dt^2)$ on the space $\mathbb{H}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ and write the Laplace-Beltrami equation where $f(u) = 0$, the wave equation (1) is obtained on the hyperbolic space.

3 Lie symmetries of the equation

How to find the symmetry Lie algebra of a differential equation is studied in many books such as [10].

A PDE with p, independent and q, dependent variables has a Lie point transformations

$$\tilde{x}_i = x_i + \epsilon\xi_i(x,u) + O(\epsilon^2), \qquad \tilde{u}_\alpha = u_\alpha + \epsilon\phi_\alpha(x,u) + O(\epsilon^2) \tag{3}$$

where $\xi_i = \frac{\partial \tilde{x}_i}{\partial \epsilon}|_{\epsilon=0}$ for $i = 1, ..., p$ and $\phi_\alpha = \frac{\partial \tilde{u}_\alpha}{\partial \epsilon}|_{\epsilon=0}$ for $\alpha = 1, ..., q$ The action of the Lie group can be considered by its associated infinitesimal generator

$$V = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u_\alpha} \tag{4}$$

on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the above vector field is given by

$$Q^\alpha(x,u^{(1)}) = \phi_\alpha(x,u) - \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial u^\alpha}{\partial x_i} \tag{5}$$

and its n-th prolongation is determined by

$$V^{[n]} = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\sum_{\sharp J=j=0}^{n} \phi_\alpha^J(x,u^{(j)})\frac{\partial}{\partial u_J^\alpha} \tag{6}$$

where $\phi_\alpha^J = D_J Q^\alpha + \sum_{i=1}^{p} \xi_i u_{J,i}^\alpha$. ($D_J$ is the total derivative operator described in [10]).

For specifying the symmetry algebra, we think the algebra generator as following:

$$V = \xi_1(x,y,t,u)\partial_x + \xi_2(x,y,t,u)\partial_y + \xi_3(x,y,t,u)\partial_t + \varphi(x,y,t,u)\partial_u. \tag{7}$$

Because of the differential equation (1) can be considered as a function on the second jet, we prolong the vector field V to the second order. By affecting $V^{[2]}$ on the differential equation (1) and vanishing, where u is the solution of (1), we find the following equations:

$$\begin{gathered}
\partial_u\xi_1 = 0, \quad \partial_u\xi_2 = 0, \quad \partial_u\xi_3 = 0, \quad \partial_{u^2}\varphi = 0, \\
\partial_y\xi_1 + \partial_x\xi_2 = 0, \quad \partial_t\xi_2 + \partial_y\xi_3 = 0, \quad \partial_x\xi_3 + \partial_t\xi_1 = 0, \\
t\partial_{t^2}\varphi + t\partial_{x^2}\varphi + t\partial_{y^2}\varphi - \partial_t\varphi = 0, \\
-t\partial_{x^2}\xi_1 + 2\,t\partial_{xu}\varphi - t\partial_{y^2}\xi_1 + \partial_t\xi_1 - t\partial_{t^2}\xi_1 = 0, \\
-t\partial_{x^2}\xi_2 - t\partial_{t^2}\xi_2 + \partial_t\xi_2 - t\partial_{y^2}\xi_2 + 2\,t\partial_{yu}\varphi = 0, \\
-t^2\partial_{x^2}\xi_3 - t^2\partial_{t^2}\xi_3 - t^2\partial_{y^2}\xi_3 + \xi_3 - t\partial_t\xi_3 + 2\,t^2\partial_{tu}\varphi = 0, \\
-2\,t\partial_x\xi_1 - t^2\partial_{y^2}\xi_3 + 2\,t^2\partial_{tu}\varphi - \partial_{t^2}\xi_3 + \xi_3 - t^2\partial_{x^2}\xi_3 + t\partial_t\xi_3 = 0, \\
-t^2\partial_{y^2}\xi_3 + t\partial_t\xi_3 - t^2\partial_{t^2}\xi_3 + 2\,t^2\partial_{tu}\varphi + \xi_3 - 2\,t\partial_y\xi_2 - t^2\partial_{x^2}\xi_3 = 0.
\end{gathered} \tag{8}$$

After solving the above system of PDEs we have:

$$\begin{aligned}
\xi_1(x,y,t,u) &= C_3(-t^2 + x^2 - y^2) + (2C_1y + C_2)x + C_4y + C_5, \\
\xi_2(x,y,t,u) &= C_1(-t^2 - x^2 + y^2) + (2C_3y - C_4)x + C_2y + C_6, \\
\xi_3(x,y,t,u) &= (2C_3x + 2C_1y + C_2)t, \quad \varphi(x,y,t,u) = C_7u + f(x,y,t).
\end{aligned} \tag{9}$$

Where $f(x,y,t)$ is a function that satisfies equation (1). So the symmetry algebra generators are:

$$\begin{gathered}
X_1 = xy\partial_x - \tfrac{1}{2}(t^2 - y^2 + x^2)\partial_y + yt\partial_t, \quad X_2 = x\partial_x + y\partial_y + t\partial_t, \\
X_3 = xy\partial_y - \tfrac{1}{2}(t^2 - x^2 + y^2)\partial_x + xt\partial_t, \quad X_4 = -y\partial_x + x\partial_y, \\
X_5 = \partial_y, \quad X_6 = \partial_x, \quad X_7 = u\partial_u, \quad X_f = f\partial_u.
\end{gathered} \tag{10}$$

Table 1: Commutation table of symmetry algebra of the equation (1)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	$-X_1$	0	$-X_3$	$-X_2$	X_4	0
X_2	X_1	0	X_3	0	$-X_5$	$-X_6$	0
X_3	0	$-X_3$	0	X_1	$-X_4$	$-X_2$	0
X_4	X_3	0	$-X_1$	0	X_6	$-X_5$	0
X_5	X_2	X_5	X_4	$-X_6$	0	0	0
X_6	$-X_4$	X_6	X_2	X_5	0	0	0
X_7	0	0	0	0	0	0	0

4 Classifying subalgebras of the symmetry algebra

Attending to table 1, we understand that the center of algebra is $Z = \langle X_7 \rangle$ and then $\mathcal{L}^7 = \langle X_7 \rangle \oplus \langle X_1, \cdots, X_6 \rangle$.

Because of the subalgebra $\langle X_1, \cdots, X_6 \rangle$ is semi-simple, $R = \langle X_7 \rangle$ is the radical and $\mathcal{L}^6 = \langle X_1, \cdots, X_6 \rangle$ is the Levi factor [12]. The subalgebra $\langle X_7 \rangle$ is the center of algebra, so it is enough to specify the subalgebras of $\mathcal{L}^6$.

Keep in mind vectors $\{X_1, \cdots, X_6\}$ as a basis for $\mathcal{L}^6$. For an unspecified vector $V = v_1 X_1 + \cdots + v_6 X_6$ the adjoint map $\mathrm{ad}(V)\langle x \rangle = [x, V]$ is defined as follows:

$$\mathrm{ad}(V) = \begin{bmatrix} v_2 & -v_1 & -v_4 & v_3 & 0 & 0 \\ v_5 & 0 & v_6 & 0 & -v_1 & -v_3 \\ v_4 & -v_3 & v_2 & -v_1 & 0 & 0 \\ -v_6 & 0 & v_5 & 0 & -v_3 & v_1 \\ 0 & v_5 & 0 & v_6 & -v_2 & -v_4 \\ 0 & v_6 & 0 & -v_5 & v_4 & -v_2 \end{bmatrix} \tag{11}$$

So the Killing form $K\langle V, W \rangle = \mathrm{tr}(\mathrm{ad}(V) \circ \mathrm{ad}(W))$ has the form:

$$K(V, W) := 4(v_2 w_2 - v_1 w_5 - v_4 w_4 - v_3 w_6 - v_5 w_1 - v_6 w_3), \tag{12}$$

where $W = w_1 X_1 + \cdots + w_6 X_6$; Thus the matrix of Killing form is:

$$K = 4 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

Because of the above matrix is non-degenerate, the subalgebra $\mathcal{L}^6$ is semi-simple.

We classify one dimensional subalgebras of $\mathcal{L}^6$ by specifying a list of non-equivalent subalgebras under the conjugate relation of subalgebras. So any one dimensional subalgebra is equivalent with some element of the list. It means $\bar{\mathfrak{h}} = \mathrm{ad}(g)\mathfrak{h}$ for some g in the Lie group G.

If $V \in \mathfrak{g}$ be an optional element, the conjugate action related to V, named M_V, is obtained by solving the system $\partial_t M_V = \mathrm{ad}(V) \circ M_V$, where $M_V(0) = I$.

If the action of each base element of the Lie algebra $\mathfrak{g}$ be specified, we can obtain the general conjugate action $\mathrm{ad}(g)$ as the combination of base elements actions.

Theorem. *One dimensional subalgebras of* $\mathcal{L}^7$ *are:*

$$\begin{array}{llll}
\mathcal{A}_1^1 : X_3 + aX_5 + bX_6 + cX_7, & \mathcal{A}_1^2 : X_1 + aX_5 + bX_6 + cX_7, \\
\mathcal{A}_1^3 : X_4 + aX_2 + bX_5 + cX_7, & \mathcal{A}_1^4 : X_2 + aX_7, \\
\mathcal{A}_1^5 : X_6 + aX_7, & \mathcal{A}_1^6 : X_5 + aX_7, \\
\mathcal{A}_1^8 : X_7. &
\end{array}$$

Proof: As mentioned above, to specify subalgebras of $\mathcal{L}^7$, we find subalgebras of $\mathcal{L}^6$ at first and next we add the vector X_7. Now we characterize the conjugate action related to each base element of the algebra $\mathcal{L}^6$. If $F_i^s : \mathcal{L}^6 \to \mathcal{L}^6$ be the conjugate map related to e_i, $(i = 1, 2, \cdots, 6)$, we show the matrix of F_i^s in the base $\{e_i\}_{i=1}^6$ by M_i^s. So we have:

$$M_1(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s & 1 & 0 & 0 \\ s^2/2 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & -s^2/2 & -s & 0 & 1 \end{bmatrix}, \quad M_2(s) = \begin{bmatrix} 1/s & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & s \end{bmatrix},$$

$$M_3(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & s & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -s & 0 & 0 & 1 & 0 & 0 \\ -s^2/2 & 0 & 0 & s & 1 & 0 \\ 0 & s & s^2/2 & 0 & 0 & 1 \end{bmatrix}, \quad M_4(s) = \begin{bmatrix} \cos s & 0 & -\sin s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin s & 0 & \cos s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos s & -\sin s \\ 0 & 0 & 0 & 0 & \sin s & \cos s \end{bmatrix},$$

$$M_5(s) = \begin{bmatrix} 1 & -s & 0 & 0 & s^2/2 & 0 \\ 0 & 1 & 0 & 0 & -s & 0 \\ 0 & 0 & 1 & -s & 0 & -s^2/2 \\ 0 & 0 & 0 & 1 & 0 & s \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_6(s) = \begin{bmatrix} 1 & 0 & 0 & s & -s^2/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -s \\ 0 & -s & 1 & 0 & 0 & s^2/2 \\ 0 & 0 & 0 & 1 & -s & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Case 1 If $a_3 \neq 0$ by setting $s_4 = \tan^{-1}(a_1/a_3)$, $s_1 = -a_4/a_3$, $s_6 = -a_2/a_3$, $s_2 = 1$ and $s_5 = s_6 = 0$ coefficients a_1, a_2 and a_4 are vanished. By scaling if needed, we can suppose $a_3 = 1$ and so after adding vector X_7, X gets the form $\mathcal{A}_1^1$.

Case 2 If $a_3 = 0$ and $a_1 \neq 0$ by setting $s_1 = -a_2/a_1$, $s_2 = 1$, $s_3 = a_4/a_1$, $s_5 = 0$ and $s_6 = 0$ coefficients a_2 and a_4 are vanished. By scaling if needed, we can suppose $a_1 = 1$ and so after adding vector X_7, X gets the form $\mathcal{A}_1^2$.

Case 3 If $a_1 = a_3 = 0$ and $a_4 \neq 0$ by setting $s_1 = a_6/a_4$, $s_2 = 1$, $s_3 = s_4 =$

$s_5 = s_6 = 0$ the coefficient a_6 is vanished. By scaling if needed, we can suppose $a_4 = 1$ and so after adding vector X_7, X gets the form $\mathcal{A}_1^3$.

Case 4 If $a_1 = a_3 = a_4 = 0$ and $a_2 \neq 0$ by setting $s_1 = -a_5/a_2$, $s_2 = 1$, $s_3 = -a_6/a_2$ and $s_4 = s_5 = s_6 = 0$ the coefficients a_5 and a_6 are vanished. By scaling if needed, we can suppose $a_2 = 1$ and so after adding vector X_7, X gets the form $\mathcal{A}_1^4$.

Case 5 If $a_1 = a_3 = a_4 = a_2 = 0$ and $a_6 \neq 0$ by setting $s_1 = 0$, $s_2 = 1$, $s_3 = 0$ and $s_4 = \tan^{-1}(a_5/a_6)$ and $s_5 = s_6 = 0$ the coefficient a_5 is vanished. By scaling if needed, we can suppose $a_6 = 1$ and so after adding vector X_7, X gets the form $\mathcal{A}_1^5$.

Case 6 If $a_1 = a_2 = a_3 = a_4 = a_6 = 0$ by scaling if needed, we can suppose $a_5 = 1$ and so after adding vector X_7, X gets the form $\mathcal{A}_1^6$.

Because of X_7 is an one dimensional subalgebra of $\mathcal{L}^7$, independent from $\mathcal{L}^6$, so the case $\mathcal{A}_1^8$ must be regarded. □

Theorem. *Two dimensional subalgebras of $\mathcal{L}^7$ are:*

$$\mathcal{A}_2^1 : \langle X_3 + aX_5 + bX_6 + cX_7, X_1 - bX_5 + aX_6 \rangle, \quad \text{(abelian)} \ (\mathrm{a}^2 + \mathrm{b}^2 \neq 0)$$
$$\mathcal{A}_2^2 : \langle X_3 + cX_7, aX_1 + bX_2 \rangle), \quad \text{(non} - \text{abelian)}$$
$$\mathcal{A}_2^3 : \langle X_4 + aX_2 + bX_5 + cX_7, X_6 + \Big(\frac{1 + a^2}{b}\Big) X_2 + aX_5 \rangle, \quad \text{(abelian)} \ (b \neq 0)$$
$$\mathcal{A}_2^4 : \langle X_4 + aX_7, X_2 \rangle, \quad \text{(abelian)}$$
$$\mathcal{A}_2^5 : \langle X_2 + cX_7, aX_5 + bX_6 \rangle, \quad \text{(non} - \text{abelian)}$$
$$\mathcal{A}_2^6 : \langle X_3 + aX_5 + bX_6, X_7 \rangle, \quad \text{(abelian)}$$
$$\mathcal{A}_2^7 : \langle X_1 + aX_5 + bX_6, X_7 \rangle, \quad \text{(abelian)}$$
$$\mathcal{A}_2^8 : \langle X_4 + aX_2 + bX_5, X_7 \rangle, \quad \text{(abelian)}$$
$$\mathcal{A}_2^9 : \langle X_2, X_7 \rangle, \quad \text{(abelian)}$$
$$\mathcal{A}_2^{10} : \langle X_6, X_7 \rangle, \quad \text{(abelian)}$$
$$\mathcal{A}_2^{11} : \langle X_5, X_7 \rangle, \quad \text{(abelian)}$$

Proof: Like the previous theorem we determine two dimensional subalgebras of $\mathcal{L}^6$ at first. So get an optional vector $Y = b_1X_1 + \cdots + b_6X_6$ and a vector X of one dimensional subalgebras. Suppose $\mathfrak{h} = \text{Span}\{X, Y\}$ is a two dimensional subalgebra of $\mathcal{L}^6$ and $\mathfrak{g}$ be the corresponding subalgebra of $\mathcal{L}^7$; so we have:

Case 1 If $X = X_3 + aX_5 + bX_6$ then because $\mathfrak{h}$ must be closed under the Lie bracket, we set $[X, Y] = \alpha X + \beta Y$. Now two states may be happened:

a) If $a, b \neq 0$ or $a = 0, b \neq 0$ or $a \neq 0, b = 0$ then we have $\alpha = \beta = 0$ and $Y = b_1(X_1 - bX_5 + aX_6) + b_3X$. By choosing suitable basis we can get $Y = X_1 - bX_5 + aX_6$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^1$ and the algebra is abelian.

b) If $a = b = 0$ then we have $\alpha = -b_2, \beta = 0$ and $Y = b_1X_1 + b_2X_2 + b_3X_3$. By choosing suitable basis we can get $Y = b_1X_1 + b_2X_2$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^2$ and the algebra is not abelian.

Case 2 If $X = X_1 + aX_5 + bX_6$ then because $\mathfrak{h}$ must be closed under the Lie bracket, we set $[X, Y] = \alpha X + \beta Y$. Now two states may be happened:

a) If $a, b \neq 0$ or $a = 0, b \neq 0$ or $a \neq 0, b = 0$ then we have $\alpha = \beta = 0$ and $Y = b_1X + b_3(X_3 + bX_5 - aX_6)$. By choosing suitable basis we can get $Y = X_3 + bX_5 - aX_6$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^1$ and the algebra is abelian.

b) If $a = b = 0$ then we have $\alpha = -b_2, \beta = 0$ and $Y = b_1X_1 + b_2X_2 + b_3X_3$. By choosing suitable basis we can get $Y = b_2X_2 + b_3X_3$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^2$ and the algebra is not abelian.

Case 3 If $X = X_4 + aX_2 + bX_5$ then because $\mathfrak{h}$ must be closed under the Lie bracket, we set $[X, Y] = \alpha X + \beta Y$. Now two states may be happened:

a) If $b \neq 0$ then we have $\alpha = \beta = 0$ and $Y = b_4X + b_6((1+a^2)/b)X_2 + aX_5 + X_6)$. By choosing suitable basis we can get $Y = X_6 + ((1 + a^2)/b)X_2 + aX_5$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^3$ and the algebra is abelian.

b) If $b = 0$ then we have $\alpha = \beta = 0$ and $Y = b_2X_2 + b_4X_4$. By choosing suitable basis we can get $Y = X_2$ and $X = X_4$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^4$ and the algebra is abelian.

Case 4 If $X = X_2$ then because $\mathfrak{h}$ must be closed under the Lie bracket, we set $[X, Y] = \alpha X + \beta Y$. Now three states may be happened:

a) We have $\alpha = \beta = 0$ and $Y = b_2X_2 + b_4X_4$. By choosing suitable basis we can get $Y = X_4$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^4$ and the algebra is abelian.

b) We have $\alpha = -b_2$, $\beta = 1$ and $Y = b_1X_1 + b_2X_2 + b_3X_3$. By choosing suitable basis we can get $Y = b_1X_1 + b_3X_3$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^2$ and the algebra is not abelian.

c) We have $\alpha = b_2$, $\beta = -1$ and $Y = b_2X_2 + b_5X_5 + b_6X_6$. By choosing suitable basis we can get $Y = b_5X_5 + b_6X_6$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^5$ and the algebra is not abelian.

Case 5 If $X = X_6$ then because $\mathfrak{h}$ must be closed under the Lie bracket, we set $[X, Y] = \alpha X + \beta Y$. Now $\alpha = b_2$, $\beta = 0$ and $Y = b_2X_2 + b_5X_5 + b_6X_6$. By choosing suitable basis we can get $Y = b_2X_2 + b_5X_5$ and . After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^5$ and the algebra is not abelian.

Case 6 If $X = X_5$ then because $\mathfrak{h}$ must be closed under the Lie bracket, we set $[X, Y] = \alpha X + \beta Y$. Now $\alpha = b_2$, $\beta = 0$ and $Y = b_2X_2 + b_5X_5 + b_6X_6$. By choosing suitable basis we can get $Y = b_2X_2 + b_6X_6$. After adding X_7, $\mathfrak{g}$ forms as $\mathcal{A}_2^5$ and the algebra is not abelian.

Cases $\mathcal{A}_2^6, \cdots, \mathcal{A}_2^{11}$ are concluded by adding X_7 to each one dimensional subalgebras. □

Theorem *Three dimensional subalgebras of $\mathcal{L}^7$ are:*

$$\begin{aligned}
\mathcal{A}_3^1 &: \langle X_3, aX_1 + bX_2, aX_4 - bX_6 + cX_7 \rangle, \quad (a \neq 0, b \neq 0)\\
\mathcal{A}_3^2 &: \langle X_3, X_2, X_6 + aX_7 \rangle,\\
\mathcal{A}_3^3 &: \langle X_3, X_1, aX_2 + bX_4 + cX_7 \rangle,\\
\mathcal{A}_3^4 &: \langle X_2, aX_5 + bX_6, aX_1 + bX_3 + cX_7 \rangle, \quad (a \neq 0, b \neq 0)\\
\mathcal{A}_3^5 &: \langle X_2, X_5, X_6 + aX_7 \rangle,\\
\mathcal{A}_3^6 &: \langle X_2, X_5, X_1 + aX_7 \rangle,\\
\mathcal{A}_3^7 &: \langle X_3 + aX_5 + bX_6, X_1 - bX_5 + aX_6, X_7 \rangle,\\
\mathcal{A}_3^8 &: \langle X_3, aX_1 + bX_2, X_7 \rangle),
\end{aligned}$$

$\mathcal{A}_3^9$: $\langle X_4 + aX_2 + bX_5, X_6 + ((1 + a^2)/b)X_2 + aX_5, X_7\rangle$,
$\mathcal{A}_3^{10}$: $\langle X_4, X_2, X_7\rangle$,
$\mathcal{A}_3^{11}$: $\langle X_2, aX_5 + bX_6, X_7\rangle$.

Proof: Like the previous theorem we determine three dimensional subalgebras of $\mathcal{L}^6$ at first. So get an optional vector $Z = b_1X_1 + \cdots + b_6X_6$ and vectors X, Y of two dimensional subalgebras. Suppose $\mathfrak{h} = \mathrm{Span}\{X, Y, Z\}$ is a three dimensional subalgebra of $\mathcal{L}^6$ and $\mathfrak{g}$ be the corresponding subalgebra of $\mathcal{L}^7$; so we have:

Case 1 If $X = X_3 + aX_5 + bX_6$ and $Y = X_1 - bX_5 + aX_6$ (not a=b=0) then we have:

a) If $a \neq 0$ and $b \neq 0$ then $Z = b_1(X_1 - bX_5 + aX_6) + b_3(X_3 + aX_5 + bX_6)$, so in this case we receive a two dimensional algebra.

b) If $a = 0$ and $b \neq 0$ then $X = X_3 + bX_6$, $Y = X_1 - bX_5$ and $Z = b_1(X_1 - bX_5) + b_3(X_3 + bX_6)$, so in this case we receive a two dimensional algebra.

c) If $a \neq 0$ and $b = 0$ then $X = X_3 + aX_5$, $Y = X_1 + aX_6$ and $Z = b_1(X_1 + aX_6) + b_3(X_3 + aX_5)$, so in this case we receive a two dimensional algebra.

Case 2 If $X = X_3$ and $Y = aX_1 + bX_2$ then we have several states:

a) If $a \neq 0$ and $b \neq 0$ then $Z = b_1(X_1 + (b/a)X_2) + b_3X_3 + aX_4 - bX_6$ and by choosing a suitable basis we have $Z = aX_4 - bX_6$, so we have the case $\mathcal{A}_3^1$ and other states concluding to two dimensional algebras.

b) If $a = 0$ and $b \neq 0$ then $X = X_3$, $Y = X_2$ and we have two states:

b-1) $Z = b_2X_2 + b_3X_3 + b_6X_6$ and we can get $Z = X_6$, after adding X_7 this is the case $\mathcal{A}_3^2$.

b-2) $Z = b_1X_1 + b_2X_2 + b_3X_3$ and we can get $Z = X_1$, after adding X_7 this is the case $\mathcal{A}_3^3$.

c) If $a \neq 0$ and $b = 0$ then $X = X_3$, $Y = X_1$ and $Z = b_2X_2 + b_4X_4$, so after adding X_7 this is the case $\mathcal{A}_3^3$.

Case 3 If $X = X_4 + aX_2 + bX_5$ and $Y = X_6 + ((1 + a^2)/b)X_2 + aX_5$ then we have two states:

a) If $a \neq 0$ then so we have $Z = b_6(X_6 + ((1 + a^2)/b)X_2 + aX_5) + b_4(X_4 + aX_2 + bX_5)$, so we receive a two dimensional algebra.

b) If $a = 0$ then $X = X_4 + bX_5$, $Y = X_6 + (1/b)X_2$ and $Z = b_2(X_2 + bX_6) + b_4(X_4 + bX_5)$ and we receive a two dimensional algebra.

Case 4 If $X = X_4$ and $Y = X_2$ then we have $Z = b_2X_2 + b_4X_4$ and so we receive a two dimensional algebra.

Case 5 If $X = X_2$ and $Y = aX_5 + bX_6$ then we have several states:

a) If $a \neq 0$ and $b \neq 0$ then we have:

a-1) $Z = b_2X_2 + b_5(X_5 + (b/a)X_6)$ and we receive a two dimensional algebra.

a-2) $Z = b_1(X_1 + (b/a)X_3) + b_2X_2 + b_5(X_5 + (b/a)X_6)$ and by choosing a suitable basis we have $Z = aX_1 + bX_3$, so after adding X_7 we have the case $\mathcal{A}_3^4$.

a-3) $Z = b_2X_2 + b_3X_3 + b_6X_6$ and we receive a two dimensional algebra.

b) If $a \neq 0$ and $b = 0$ then $X = X_2$ and $Y = X_5$ then we have:

b-1) $Z = b_2X_2 + b_5X_5$ and we receive a two dimensional algebra.

b-2) $Z = b_2X_2 + b_5X_5 + b_6X_6$, so by choosing suitable basis $Z = X_6$ and after adding X_7 we receive case $\mathcal{A}_3^5$.

b-3) $Z = b_1X_1 + b_2X_2 + b_5X_5$, so by choosing suitable basis $Z = X_1$ and after adding X_7 we receive case $\mathcal{A}_3^6$.

c) If $a = 0$ and $b \neq 0$ then $X = X_2$ and $Y = X_6$ then we have

c-1) $Z = b_2X_2 + b_5X_5 + b_6X_6$ so by choosing a suitable basis we have $Z = X_5$, so we receive case $\mathcal{A}_3^6$.

c-2) $Z = b_2X_2 + b_3X_3 + b_6X_6$ so by choosing a suitable basis we have $Z = X_3$, so we receive case $\mathcal{A}_3^2$.

Cases $\mathcal{A}_3^7, \cdots, \mathcal{A}_3^{11}$ are concluded by adding X_7 to each two dimensional subalgebras of $\mathcal{L}^6$. □

Theorem *Four dimensional subalgebras of $\mathcal{L}^7$ are:*

$\mathcal{A}_4^1$: $\langle X_1, X_2, X_3, X_4 + aX_7 \rangle$,
$\mathcal{A}_4^2$: $\langle X_2, X_4, X_5, X_6 + aX_7 \rangle$,
$\mathcal{A}_4^3$: $\langle X_3, aX_1 + bX_2, aX_4 - bX_6, X_7 \rangle$, $(a \neq 0, b \neq 0)$,
$\mathcal{A}_4^4$: $\langle X_3, X_2, X_6, X_7 \rangle$,
$\mathcal{A}_4^5$: $\langle X_3, X_1, aX_2 + bX_4, X_7 \rangle$,
$\mathcal{A}_4^6$: $\langle X_2, aX_5 + bX_6, aX_1 + bX_3, X_7 \rangle$, $(a \neq 0, b \neq 0)$
$\mathcal{A}_4^7$: $\langle X_2, X_5, X_6, X_7 \rangle$,
$\mathcal{A}_4^8$: $\langle X_2, X_5, X_1, X_7 \rangle$,

Proof: We determine four dimensional subalgebras of $\mathcal{L}^6$ at first. So get an optional vector $W = b_1 X_1 + \cdots + b_6 X_6$ and vectors X, Y, Z of three dimensional subalgebras. Suppose $\mathfrak{h} = \text{Span}\{X, Y, Z, W\}$ is a four dimensional subalgebra of $\mathcal{L}^6$ and $\mathfrak{g}$ be the corresponding subalgebra of $\mathcal{L}^7$; so we have:

Case 1. If $X = X_3$, $Y = aX_1 + bX_2$ and $Z = aX_4 - bX_6$ $(a \neq 0, b \neq 0)$ then we conclude $W = b_1(X_1 + (b/a)X_2) + b_3 X_3 + b_1(X_4 - (b/a)X_6) + aX_4 - bX_6$ and so $W = (b_1/a)Y + b_3 X + (b_1/a)(Z + 1)$. Consequently we have a three dimensional algebra.

Case 2. If $X = X_3$, $Y = X_2$ and $Z = X_6$ then we have $W = b_2 Y + b_3 X + b_6 Z$ and so the generated algebra is three dimensional.

Case 3. If $X = X_3$, $Y = X_1$ and $Z = aX_2 + bX_4$ then we have several states:

a) If $a \neq 0$ and $b \neq 0$ then we have:

a-1) $W = b_1 Y + b_3 X + Z$ where $a = b_2$ and $b = b_4$. So the generated algebra is three dimensional.

a-2) $W = b_1 Y + (b_2/a) Z + b_3 X$ and so we have a three dimensional algebra.

b) If $a = 0$ and $b \neq 0$ then $X = X_3$, $Y = X_1$ and $Z = X_4$, so we have:

b-1) $W = b_1 Y + b_2 X + b_4 Z$, so the generated algebra is three dimensional.

b-2) $W = b_1Y + b_2X_2 + b_3X + b_4Z$ and by choosing suitable basis $W = X_2$ and after adding X_7 we receive case $\mathcal{A}_4^1$.

c) If $b = 0$ and $a \neq 0$ then $X = X_3$, $Y = X_1$ and $Z = X_2$, so we have:

c-1) $W = b_1Y + b_2Z + b_3X$, so the generated algebra is three dimensional.

c-2) $W = b_1Y + b_2Z + b_3X + b_4X_4$ and by choosing suitable basis $W = X_4$ and after adding X_7 we receive case $\mathcal{A}_4^1$.

Case 4. If $X = X_2$, $Y = aX_5 + bX_6$ and $Z = aX_1 + bX_3$, $(a \neq 0, b \neq 0)$, then we conclude $W = (b_1/a)Z + (b_5/a)Y + (1 + b_5/a)X$. Consequently we have a three dimensional algebra.

Case 5. If $X = X_2$, $Y = X_5$ and $Z = X_6$ then we have two states:

a) $W = b_2X + b_5Y + b_6Z$ and so we have a three dimensional algebra.

b) $W = b_2X + b_4X_4 + b_5Y + b_6Z$ and by choosing suitable basis $W = X_4$ and after adding X_7 we receive case $\mathcal{A}_4^2$.

Case 6. If $X = X_2$, $Y = X_5$ and $Z = X_1$ then $W = b_1Z + b_2X + b_5Y$. So the generated algebra is three dimensional.

Cases $\mathcal{A}_4^3, \cdots, \mathcal{A}_4^8$ are concluded by adding X_7 to each three dimensional subalgebras of $\mathcal{L}^6$. □

Theorem *Five dimensional subalgebras of $\mathcal{L}^7$ are:*

$$\mathcal{A}_5^1 \,:\, \langle X_1, X_2, X_3, X_4, X_7 \rangle, \qquad \mathcal{A}_5^2 \,:\, \langle X_2, X_4, X_5, X_6, X_7 \rangle.$$

Proof: We determine five dimensional subalgebras of $\mathcal{L}^6$ at first. So get an optional vector $T = b_1X_1 + \cdots + b_6X_6$ and vectors X, Y, Z, W of four dimensional subalgebras. Suppose $\mathfrak{h} = \mathrm{Span}\{X, Y, Z, W, T\}$ is a five dimensional subalgebra of $\mathcal{L}^6$ and $\mathfrak{g}$ be the corresponding subalgebra of $\mathcal{L}^7$; So, we have:

Case 1 If $X = X_1$, $Y = X_2$, $Z = X_3$ and $W = X_4$ then we have $T = b_1X + b_2Y + b_3Z + b_4W$, so the generated algebra is four dimensional.

Case 2 If $X = X_2$, $Y = X_4$, $Z = X_5$ and $W = X_6$ then we have $T = b_2X + b_4Y + b_5Z + b_6W$, so the generated algebra is four dimensional.

Thus five dimensional subalgebras of $\mathcal{L}^7$ are generated by adding X_7 to each four dimensional subalgebras. So we obtain cases $\mathcal{A}_5^1, \mathcal{A}_5^2$ □

Corollary *The only six dimensional subalgebra of $\mathcal{L}^7$ is $\mathcal{L}^6$.*

Proof According to the last theorem $\mathcal{L}^6$ doesn't have any five dimensional subalgebra to be added by X_7. So the only six dimensional subalgebra of $\mathcal{L}^7$, is $\mathcal{L}^6$ itself! □

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