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Differential Invariants of SL(2) and SL(3)-actions on $\mathbf{R}^2$

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Abstract

The main purpose of this paper is calculation of differential invariants which arise from prolonged actions of two Lie groups SL(2) and SL(3) on the nth jet space of $\mathbf{R}^2$. It is necessary to calculate nth prolonged infinitesimal generators of the action.

Keywords: Differential invariant, infinitesimal generator, generic orbit, prolongation.

Introduction

Differential invariants theory is one of the most important concept in differential equations theory and differential geometry. The study of this theory will help us to analyze applications of geometry in differential equations. In this paper we will study some properties of two Lie group actions SL(2) and SL(3). After finding the Lie algebras of each groups, we will calculate all infinitesimal generators and their nth prolongations. Next we are going to find the number of functionally independent differential invariants up to order n (i_n), the number of strictly functionally independent differential invariants of order n (j_n), generic orbit dimensions of nth prolonged action (s_n) and isotropic

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group dimensions of prolonged action (h_n). It is necessary to realize that the symbol $u_{(k)}$ means kth derivative of u with respect to x.

1 Jet and Prolongation

This section is devoted to study of proper geometric context for "jet spaces" or "jet bundles", well known to 19th century practitioners, but first formally defined by Ehresmann. Prolongation will defined after study of jet to find the differential invariants.

1.1 Introduction

In this part we are going to define some fundamental concepts of differential equations theory. The two most important are Jet and Prolongation but it needs to analyze some definitions of differential equations.

Definition 1.1 *Suppose M and N are p-dimensional and q-dimensional Euclidean manifolds respectively, i.e. $M \simeq \mathbf{R}^p$ and $N \simeq \mathbf{R}^q$. The total space will be the Euclidean space $E = M \times N \simeq \mathbf{R}^{p+q}$ coordinatized by the independent and dependent variables $(x^1, ..., x^p, u^1, ..., u^q)$.*

Let f is a scalar-valued function with p independent and q dependent variables then it has $p_k = \binom{p+k-1}{k}$ different kth order partial derivatives and suppose the function characterized by $u = f(x)$.

Definition 1.2 *Let $N^{(n)}$ is an appropriate space for representation of the nth Tailor expansion of f, thus $N^{(n)}$ has the following decomposition*

$$N^{(n)} \simeq \mathbf{R}^q \times \mathbf{R}^{p_1 q} \times \mathbf{R}^{p_2 q} \times \cdots \times \mathbf{R}^{p_n q},$$

then $N^{(n)}$ is a $q\binom{p+n}{n}$ dimensional manifold. Define $J^{(n)} = J^{(n)}E = M \times N^{(n)}$, then $J^{(n)}$ is called the nth jet space of the total space E, which is a $p + q\binom{p+n}{n}$ dimensional Euclidean manifold with vector bundle structure whose fibers have $q\binom{p+n}{n}$ dimension.

According to the last definition, the nth jet space of $\mathbf{R}^2$ has dimension $n+2$ since we can write it in the form of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, ($M \simeq \mathbf{R}$ and $N \simeq \mathbf{R}$).

Theorem 1.3 *Suppose G be a Lie group acting on E and $\mathbf{v}$ is a member of Lie algebra. Let $\Phi_x : G \to E$ be a smooth map defined by $\Phi_x(g) = g.x$, then the infinitesimal generator $\widetilde{v}$ corresponding v is given by $\Phi_{x*}(v\mid_g) = \widetilde{v}\mid_{g.x}$, where Φ_{x*} is the push forward map of Φ_x.*

Proof *See [2] for a proof.*

Let G be a Lie group acting on the total space E, then all of infinitesimal generators of the group action have the following form $v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$.

Definition 1.4 *Nth prolonged of v denoted by $v^{(n)}$ is a vector field on the $J^{(n)}E$, which is an infinitesimal generator of nth prolonged action of G on E.*

Nth prolonged action of f represented by $u^{(n)} = f^{(n)}(x)$, is a function from M to $N^{(n)}$ (also known as the $n-jet$ and denoted by $j_n f$), defined by evaluating all the partial derivatives of f up to order n.

Now it's time to give a formula to calculate nth prolongations. The following theorem gives an explicit formula for the prolonged vector field.

First of all we need two important definition.

Definition 1.5 *The characteristic of a vector field v on E is a q-tuple of functions $Q(x,u^{(1)})$, depending on x and u and first order derivatives of u, defined by*

$$Q^\alpha(x,u^{(1)}) = \varphi^\alpha(x,u^{(1)}) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1,...q.$$

Definition 1.6 *Let $F(x,u^{(n)})$ be a differential function of order n. (A smooth, real valued function $F : J^{(n)} \to \mathbf{R}$, defined on an open subset of the nth jet space is called a $differential$ $function$ of order n.) The total derivative F with respect to x^i is the (n+1)st order differential function D_iF satisfying*

$$D_i[F(x,f^{(n+1)}(x))] = \frac{\partial}{\partial x^i}F(x,f^{(n)}(x)).$$

Theorem 1.7 *Let v be an infinitesimal generator on E, and let $Q = (Q^1, ..., Q^q)$ be its characteristic. The nth prolongation of v is given explicitly by*

$$v^{(n)} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{\sharp J=j=0}^{n} \varphi_\alpha^J(x,u^{(j)})\frac{\partial}{\partial u_J^\alpha} ,$$

with coefficients $\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$, where $J = (j_1, ..., j_k)$ is a multi-indices where $1 \le j_k \le p$ and $\sharp J = k$.

Proof *See [3; Theorem 2.36] for a proof.*

Theorem 1.8 *(Infinitesimal Method, [2])A function $I : J^{(n)} \rightarrow \mathbf{R}$ is a differential invariant for a connected transformation group G if and only if it is annihilated by all prolonged infinitesimal generators:*

$$v^{(n)}(I(x,u^{(n)})) = 0 , \qquad v \in Lie(G) .$$

Here $x = (x^1, ..., x^p)$ is independent variable and $u^{(n)}$ is a coordinate chart on $N^{(n)}$.

Definition 1.9 *The collection $\{I_1, ..., I_k\}$ of arbitrary differential invariants is called functionally independent if $dI_1 \wedge ... \wedge dI_k \neq 0$, and strictly functionally independent if $dx \wedge du^{(n-1)} \wedge dI_1 \wedge ... \wedge dI_k \neq 0$.*

Now we are ready to give some formulas for calculating i_n, j_n, s_n and h_n.

According to the prolonged action of G on $J^{(n)}$ define $V^{(n)}$ denoted a subset of $J^{(n)}$ which consists of all points contained in the orbit of maximal dimension (generic orbit). Then on $V^{(n)}$

$$i_n = \dim J^{(n)} - s_n = \dim J^{(n)} - \dim G + h_n ,$$

thus the number of independent differential invariants of order less or equal to n, forms a nondecreasing sequence $i_0 \le i_1 \le i_2 \le \cdots$.

The difference $j_n = i_n - i_{n-1}$, is the number of strictly functionally independent differential invariants of order n.

Note that j_n cannot exceed the number of independent derivative coordinate of order n, so if

$$q_n = \dim J^{(n)} - \dim J^{(n-1)} = q\binom{p+n}{n} - q\binom{p-n-1}{n-1} = q\binom{p+n-1}{n} ,$$

is the number of derivative coordinate of order n, so $j_n \leq q_n$, which implies that the elementary inequalities $i_{n-1} \leq i_n \leq i_{n-1} + q_n$.

The maximal orbit dimension s_n is also a nondecreasing function of n, bounded by r, the dimension of G itself: $s_0 \leq s_1 \leq s_2 \leq \cdots \leq r$.

On the other hand, since the orbit cannot increase in dimension any more than the increase in dimension of the jet spaces themselves, we have the elementary inequalities $s_{n-1} \leq s_n \leq s_{n-1} + q_n$, governing the orbit dimension.

2 ACTION OF LIE GROUP SL(2) AND SL(3)

In this section we are going to define an action of each Lie group on $\mathbf{R}^2$, then we will give the infinitesimal generators arise from Lie algebra and their prolongations, next we will calculate $i_n, j_n, \cdots$.

For more details see [2], [3] and [6].

2.1 Action of SL(2)

Define an action of SL(2) on $\mathbf{R}^2$,

$$(x,u) \longmapsto \Big(\frac{ax+b}{cx+d}, u\Big) \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2) \text{ and } (x,u) \in \mathbf{R}^2 .$$

The Lie algebra of SL(2) generates by the following matrices:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the infinitesimal generators corresponding to the above matrices are:

$$\mathbf{v}_1 = \frac{\partial}{\partial x} \qquad \mathbf{v}_2 = x\frac{\partial}{\partial x} \qquad \mathbf{v}_3 = x^2\frac{\partial}{\partial x}\ .$$

According to the theorem 1.7, the three following vector fields are the corresponding prolonged infinitesimal generators:

$$\begin{aligned}
\mathbf{v}_1^{(n)} &= \frac{\partial}{\partial x} \\
\mathbf{v}_2^{(n)} &= x\frac{\partial}{\partial x} - \cdots - nu_{(n)}\frac{\partial}{\partial u_{(n)}} \\
\mathbf{v}_3^{(n)} &= x^2\frac{\partial}{\partial x} - \cdots - \Big[n(n-1)u_{(n-1)} + 2nxu_{(n)}\Big]\frac{\partial}{\partial u_{(n)}}\ .
\end{aligned}$$

According to the Theorem 1.8, the first three differential invariants of order 0,1,2 are functions of u, thus $i_0 = i_1 = i_2 = 1$, moreover the third one depends both on u and

$$\frac{2u_{(1)}u_{(3)} - 3u_{(1)}^2}{2u_{(1)}^4}\ ,$$

thus $i_3 = 2$, so the calculations show that $i_4 = 3, \cdots, i_n = n-1$ and also $s_0 = 1, s_1 = 2$ and $s_2 = s_3 = \cdots = s_n = 3$, but $j_1 = j_2 = 0$ and $j_0 = j_3 = \cdots = j_n = 1$, so $h_0 = 2, h_1 = 1$ and $h_2 = \cdots = h_n = 0$.

2.2 Action of SL(3)

Similarly, we define an action of SL(3) on $\mathbf{R}^2$,

$$(x,u) \longmapsto \Big(\frac{ax+bu+c}{hx+ju+k}, \frac{dx+eu+f}{hx+ju+k}\Big)$$

where $\begin{pmatrix} a & b & c \\ d & e & f \\ h & j & k \end{pmatrix} \in \mathrm{SL}(3)$ and $(x,u) \in \mathbf{R}^2$. The Lie algebra of SL(3) generates by the following matrices:

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

therefore the infinitesimal generators are:

$$\mathbf{v}_1 = \frac{\partial}{\partial x} \qquad \mathbf{v}_2 = \frac{\partial}{\partial u} \qquad \mathbf{v}_3 = x\frac{\partial}{\partial x} \qquad \mathbf{v}_4 = u\frac{\partial}{\partial u} \qquad \mathbf{v}_5 = x\frac{\partial}{\partial u} \qquad \mathbf{v}_6 = u\frac{\partial}{\partial x}$$
$$\mathbf{v}_7 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u} \qquad \mathbf{v}_8 = xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u},$$

and by calculating the prolonged infinitesimal generators, we have,

$$\begin{aligned}
\mathbf{v}_1^{(n)} &= \frac{\partial}{\partial x} \\
\mathbf{v}_2^{(n)} &= \frac{\partial}{\partial u} \\
\mathbf{v}_3^{(n)} &= x\frac{\partial}{\partial x} - \cdots - nu\frac{\partial}{\partial u_{(n)}} \\
\mathbf{v}_4^{(n)} &= u\frac{\partial}{\partial u} + \cdots + u_{(n)}\frac{\partial}{\partial u_{(n)}} \\
\mathbf{v}_5^{(n)} &= x\frac{\partial}{\partial u} + \frac{\partial}{\partial u_{(1)}} \\
\mathbf{v}_6^{(1)} &= u\frac{\partial}{\partial x} - u_{(1)}^2\frac{\partial}{\partial u_{(1)}} \\
\mathbf{v}_6^{(n)} &= u\frac{\partial}{\partial x} - \cdots - \Big\{ \sum_{i=2}^{(n-1)/2} \binom{n+1}{i} u_{(i)}u_{(n-i+1)} \\
&\quad + \frac{1}{2}\binom{n+1}{(n+1)/2}\Big[u_{((n+1)/2)}\Big]^2 + (n+1)u_{(n)}\frac{\partial}{\partial u_{(1)}}\Big\}\frac{\partial}{\partial u_{(n)}} \quad (n, \text{odd}) \\
\mathbf{v}_6^{(n)} &= u\frac{\partial}{\partial x} - \cdots - \Big[\sum_{i=2}^{n/2} \binom{n+1}{i} u_{(i)}u_{(n+i-1)} + (n+1)u_{(n)}u_{(1)}\Big]\frac{\partial}{\partial u_{(n)}} \qquad (n, \text{even}) \\
\mathbf{v}_7^{(n)} &= x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u} - \cdots - \Big[n(n-2) + (2n-1)xu_{(n)}\Big]\frac{\partial}{\partial u_{(n)}}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{v}_8^{(1)} &= xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} - (xu_{(1)} - u)u_{(1)}\frac{\partial}{\partial u_{(1)}} \\
\mathbf{v}_8^{(2)} &= xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial x} - (xu_{(1)} - u)u_{(1)}\frac{\partial}{\partial u_{(1)}} - 3xu_{(1)}u_{(2)}\frac{\partial}{\partial u_{(2)}} \\
\mathbf{v}_8^{(n)} &= xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} - \cdots - \left\{\binom{n}{(n+1)/2}\left[(n-2)u_{((n-1)/2)} + u_{((n+1)/2)}\right]u_{((n+1)/2)}\right. \\
&\quad \left. + \sum_{i=1+(n+1)/2}^{n}\left[\binom{n}{j-1}(n-2)u_{(j-1)} + \binom{n+1}{j}xu_{(j)}\right]u_{(i)}\right\} \\
&\quad (j = 1, ..., \frac{n-1}{2} \text{ and } n, \text{odd}) \\
\mathbf{v}_8^{(n)} &= xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} - \cdots - \left\{(\frac{n}{2})\binom{n}{n/2}[u_{(n/2)}]^2\right. \\
&\quad \left. + \sum_{i=1+n/2}^{n}\left[\binom{n+1}{j}xu_{(j)} + \binom{n}{j-1}(n-2)u_{(j-1)}\right]u_{(i)}\right\}\frac{\partial}{\partial u_{(n)}} \\
&\quad (j = 1, ..., \frac{n}{2} \text{ and } n, \text{even}) \ ,
\end{aligned}
$$

corresponding to minimum value of i in $\mathrm{v}_8{}^{(n)}$, j takes its maximum value, i.e. when i increases j decreases.

According to the Theorem 1.8 with a tedious calculation we have, $i_0 = i_1 = i_2 = i_3 = i_4 = i_5 = i_6 = 0$ and $i_7 = 1, i_8 = 2, \cdots, i_n = n - 6$, consequently $j_0 = j_1 = j_2 = j_3 = j_4 = j_5 = j_6 = 0$ and $j_7 = \cdots j_n = 1$, but $s_0 = 2, s_1 = 3, s_2 = 4, s_3 = 5, s_4 = 6, s_5 = 7$, thus $h_0 = 6, h_1 = 5, h_2 = 4, h_3 = 3, h_4 = 2, h_5 = 1$, and $h_6 = \cdots h_n = 0$. The differential invariants have so complicated forms in this action. For example the first nonconstant differential invariant (the invariant of 7th prolonged action) is a function of

$$
\begin{aligned}
&\frac{1}{18(40u_{(3)}^2 + 9u_{(2)}^2 u_{(5)} - 45u_{(2)}u_{(3)}u_{(4)})}\Big(11200u_{(3)}^8 - 33600u_{(2)}u_{(3)}^6 u_{(4)} + 6720u_{(2)}^2 u_{(3)}^5 u_{(5)} \\
&\quad +31500u_{(2)}^2 u_{(3)}^4 u_{(4)}^2 - 12600u_{(2)}^3 u_{(3)}^3 u_{(4)} u_{(5)} + 720u_{(2)}^4 u_{(3)}^3 u_{(7)} - 756u_{(2)}^4 u_{(3)}^2 u_{(5)}^2 + \cdots \\
&\quad -2835u_{(2)}^5 u_{(4)} u_{(5)}^2 - 189u_{(2)}^6 u_{(6)}^2\Big) \ ,
\end{aligned}
$$

and the second differential invariant arise from 8th prolonged action depends on the above phrase and the following one

$$\frac{1}{207360000}\Big((20412u_{(6)}^3+6561u_{(5)}^2u_{(8)}-26244u_{(5)}u_{(6)}u_{(7)})u_{(2)}^9$$
$$+(((131220u_{(6)}u_{(7)}-65610u_{(5)}u_{(8)})u_{(4)}+104976u_{(5)}^2u_{(7)}$$
$$\vdots$$

more than 10 lines calculation

$$\vdots$$
$$-201600000u_{(2)}u_{(3)}^{10}u_{(4)}+4480000u_{(3)}^{12})/(u_{(3)}^3$$
$$\frac{9}{40}u_{(2)}^2u_{(5)}-\frac{9}{8}u_{(2)}u_{(3)}u_{(4)})^4\Big)\ .$$

3 MORE DETAILS

In this section we will give some details where in special cases we can classify the differential invariants.

3.1 Differential Operators

Theorem 3.1 *([2, Proposition 5.15])Suppose $E=\mathbf{R}\times N$ (N is a q-dimensional manifold) be a total space. Let $I(x,u^{(n)})$ and $J(x,u^{(n)})$ be functionally independent differential invariants, at least one of which has order exactly n. Then the derivative $\frac{dJ}{dI}=\frac{D_xJ}{D_xI}$ is an (n+1)st order differential invariant.*

Definition 3.2 *If $I=I(x,u^{(n)})$ is any given differential invariant with one independent variable, then*

$$\mathcal{D}=\frac{d}{dI}=(D_xI)^{-1}D_x\ ,$$

is called an invariant differential operator for the prolonged group action.

Since if J is any differential invariant, so is $\mathcal{D}J$. Therefore, we can iterate $\mathcal{D}$, producing a sequence

$$\mathcal{D}^kJ=\frac{d^kJ}{dI^k}\qquad k=0,1,\ldots\ ,$$

of higher order differential invariants. The last theorem classifies differential operator on $\mathbf{R}^2$.

Theorem 3.3 *([2, Proposition 5.16])Suppose G is a group of point transformations acting on the jet space corresponding to $E = \mathbf{R} \times \mathbf{R}$. Then, for some $n \geq 0$, there are precisely two functionally independent differential invariants I and J of order n (or less). Furthermore for any $k \geq 0$, a complete system of functionally independent differential invariants of order $n+k$ is provided by $I, J, \mathcal{D}J, ..., \mathcal{D}^k J$, where $\mathcal{D}$ is the associated invariant differential operator.*

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