Nonlinear superposition formulas based on imprimitive group action

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Abstract

Systems of nonlinear ordinary differential equations are constructed for which the general solution is expressed algebraically in terms of a finite number of particular solutions. The equations and the corresponding nonlinear superposition formula are based on a nonlinear action of the Lie Group $\mathrm{SL}(N, \mathbb{C})$ on a homogenous space M. The isotropy group of the origin of this space is a nonmaximal parabolic subgroup of $\mathrm{SL}(N, \mathbb{C})$. Such equations can occur as Bäcklund transformations for soliton equations on flag manifolds.

I. Introduction

Let us consider a system of n first order ordinary differential equations (ODEs)

$$\dot{y}^\mu = \eta^\mu(y,t) \qquad \mu = 1,\dots,n \tag{1.1}$$

where the dot denotes differentiation with respect to time t.

If the equations are linear we have a linear superposition formula: the general solution is a linear combination of n linearly independent particular solutions. More interestingly even if the system (??) is nonlinear, it may allow a nonlinear superposition formula

$$\vec{y}(t) = \vec{F}(\vec{y}_1,\dots,\vec{y}_m,c_1,\dots,c_n), \qquad \vec{y}\in\mathbb{C}^n \ (\text{or } \vec{y}\in\mathbb{R}^n), \tag{1.2}$$

where $\vec{y}_1,\dots,\vec{y}_m$ are particular solutions, $c_1,\dots,c_n$ are arbitrary constants and $\vec{y}(t)$ is the general solution.

S. Lie[??] established the conditions under which the system (??) allows a superposition formula (??), i. e. the general solution can be expressed as a function of a finite number m of particular solutions. Lie's result can be summed up as follows:

The system (??) allows a superposition formula (??) if and only if

1. It has the form

$$\dot{\vec{y}} = \sum_{k=1}^{r} Z_k(t)\vec{\xi}_k(\vec{y}). \tag{1.3}$$

2. The vector functions $\vec{\xi}_k(\vec{y})$ (independent of t) are such that the vector fields

$$\hat{X}_k = \sum_{\mu=1}^{n} \xi_k^\mu(\vec{y})\partial_{y^\mu} \tag{1.4}$$

generate a finite dimensional Lie algebra L:

$$[X_k,X_l] = \sum_{j=1}^{r} f_{klj}X_j. \tag{1.5}$$

The number of solutions m needed in the superposition formula (??) satisfies the relation

$$mn \geqq r, \tag{1.6}$$

where n is the number of equations and r is the dimension of the Lie algebra L.

Somewhat unexpectedly, it turned out that nonlinear ordinary differential equations with superposition formulas play an important role in soliton theory[??] where they occur as Bäcklund transformations.

In turn, Bäcklund transformations provide soliton superposition formulas, i. e. explicit formulas for multisoliton solutions that asymptotically correspond to a combination of independent solitons. Thus two very different types of nonlinear superposition formulas become linked via Bäcklund transformations for integrable nonlinear partial differential equations.

The classification and construction of all systems of n nonlinear ODEs with superposition formulas amounts to a classification of all finite dimensional subalgebras of diff(n), the infinite dimensional Lie algebra of vector fields in n dimensions. For $n=1$ such a classification is quite simple. Indeed the only finite-dimensional subalgebras of diff$(1,\mathbb{C})$ are sl$(2,\mathbb{C})$ and its subalgebras. For $n=2$ a complete classification exists and this is again due to S. Lie[??]. For $n \geqq 3$ the problem becomes intractable.

A more resctricted problem has however been solved[??], [??],[??] and that is the classificaton of indecomposable systems of ODEs with superposition formulas. These are systems satisfying eq. (??), (??) and (??) from which it is not possible to split off a subset of $l < n$ equations that also satisfy Lie's criteria and hence have a superposition formula of their own.

The classification of these indecomposable systems is best formulated in a geometric manner. Thus, let us view the variables $\{y_1, \ldots, y_n\}$ (for fixed t), as local coordinates on some manifold M. Let G be a Lie group acting transitively and effectively on M. We can then identify M with a quotient space $M \sim G/G_0$ where $G_0 \subset G$ is the isotropy group of the origin in M.

The system (??) is decomposable if local coordinates on M exist that can be divided into two subsets, $\{y_1, \ldots, y_n\} \sim \{x_1, \ldots, x_l, z_{l+1}, \ldots, z_n\}$ such that the vector fields (??) all simultaneously have the form

$$X_k = \sum_{a=1}^{l} f_k(x)\partial_{x_a} + \sum_{b=l+1}^{n} g_k(x,z)\partial_{z_b} \tag{1.7}$$

(i. e. the coefficients of ∂_{x_a} depend on the coordinates x only). Such coordinates exist if there exists a G-invariant foliation of the space M. The action of G on M is called primitive (in addition to being transitive and effective) if no such foliation exists. Locally, this can be expressed in terms of the Lie algebras L and L_0, corresponding to G and G_0. The system (??) is indecomposable if the pair of algebras (L_0, L) determines a transitive primitive Lie algebra. This means that L_0, the subalgebra of vector fields vanishing at the origin, must be a maximal subalgebra of L, and must not contain an ideal of L.

Transitive primitive Lie algebras have been classified [??], ..., [??]. In turn, this classification was used to classify indecomposable systems of ODEs with superposition formulas [??], [??], [??]. Several articles have been devoted to constructing systems of equations with superposition formulas, and to the superposition formulas themselves[??], ..., [??]. Supersymmetric versions of these equations have been constructed[??] as well as difference equations with superosition formulas[??], [??]. All cases considered so far correspond to transitive primitive Lie algebras.

The purpose of this article is to investigate the nonprimitive case and to show how the previously studied indecomposable systems serve as building blocks for decomposable ones.

We restrict ourselves to the Lie algebras sl$(N, \mathbb{C})$ (for arbitrary finite N) and make use of realizations of these algebras, constructed earlier[??].

II. Formulation of the problem and example of SL(2,$\mathbb{C}$)

II.1. General Formulation

Let us consider a system of ODEs as in eq. (??), allowing a superposition formula. In view of eq. (??) and (??) the right hand side of eq. (??) determines an element of the Lie algebra L for any fixed value of time t. As t varies, this element varies along some path in L. The general form of the solution is obtained by integrating the vector fields (??) and composing the results. Thus the solution will be given by the corresponding group action

$$\vec{y}(t) = g(t)\cdot\vec{u}, \tag{2.1}$$

where $g(t)$ is an element of the Lie group $G = \langle \exp L \rangle$ and $\vec{u}$ is a constant vector, related to the initial conditions for $\vec{y}(t)$. The group element $g(t)$ depends on $r = \dim L$ group parameters. These in turn depend on time t in such a way that $g(t)$ follows a path in the group G, corresponding to the path in L, determined by the equation, i. e. by the coefficients $Z_k(t)$, $k = 1, \ldots, r$.

The superposition formula (??) is obtained from the group action (??), once the time dependence in $g(t)$ is established. To do this, we assume that we know m solutions $\vec{y}_k(t)$, $k = 1, \ldots, m$ corresponding to the initial values u_k in (??). The m relations

$$\vec{y}_k(t) = g(t)\vec{u}_k \tag{2.2}$$

are then used to express all parameters in $g(t)$ in terms of the known solutions. Each solution provides n equations. The condition (??) simply means that we must have at least as many equations as unknowns. The actual number m of different solutions needed is obtained from the requirement that the only transformation $g(t_0)$ that simultaneously stabilizes all initial conditions u_k ($u_k = g(t_0)u_k$, $k = 1, \ldots, m$) is the identity transformation $g(t_0) = I$. This requirement determines the minimal number m and also the independence conditions on $u_1, \ldots, u_m$.

II.2. Example of the algebra sl(2,ℂ)

For $n = 1$ the situation is very simple. The only finite dimensional subalgebras of $\mathrm{diff}(1,\mathbb{C})$ are $\mathrm{sl}(2,\mathbb{C})$ and its one and two-dimensional subalgebras. Up to local diffeomorphisms there exists just one realization of $\mathrm{sl}(2,\mathbb{C})$ as a subalgebra of $\mathrm{diff}(1,\mathbb{C})$, namely

$$X_1 = \partial_y, \quad X_2 = y\partial_y, \quad X_3 = y^2\partial_y. \tag{2.3}$$

The equation (??) in this case is the Riccati equation

$$\dot{y} = Z_1(t) + Z_2(t)y + Z_3(t)y^2. \tag{2.4}$$

The $\mathrm{SL}(2,\mathbb{C})$ group transformations corresponding to the realization (??) are projective transformations of $\mathbb{C}^1$. Thus, eq. (??) in this case is

$$y(t) = \frac{g_{11}u + g_{21}}{g_{12}u + g_{22}}, \quad g_{11}g_{22} - g_{21}g_{12} = 1. \tag{2.5}$$

In this case we have $n = 1$ (one equation), $r = \dim \mathrm{sl}(2,\mathbb{C}) = 3$, hence the number of solutions m needed to reconstruct the group element $g(t) = \{g_{ik}(t)\}$ satisfies $m \geqq 3$. In fact we have $m = 3$. Indeed let $y_1(t)$, $y_2(t)$ and $y_3(t)$ be any 3 different solutions of eq. (??), corresponding e. g. to the choices $u_1 = 0$, $u_2 \to \infty$, $u_3 = 1$, respectively. Substituting into eq. (??) we express g_{12}, g_{21} and g_{22} in terms of $y_i(t)$ and g_{11} (which cancels out) and obtain the (well known) nonlinear superposition formula

$$y(t) = \frac{uy_2(y_1 - y_3) + y_1(y_3 - y_2)}{u(y_1 - y_3) + y_3 - y_2}. \tag{2.6}$$

Choosing $u = 0$ as the origin of the space M, we see that it is stabilized by the maximal parabolic subgroup of $\mathrm{SL}(2,\mathbb{C})$ and we obtain the identification

$$\begin{aligned} &M \sim \mathbb{C} \sim G/G_0, \quad G \sim \mathrm{SL}(2,\mathbb{C}), \\ &G_0 \sim \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{11}^{-1} \end{pmatrix}. \end{aligned} \tag{2.7}$$

Now let us turn to a case that has not been considered before, namely $n = 2$ when the realization of the algebra $\mathrm{sl}(2,\mathbb{C})$ is not primitive. The group $\mathrm{SL}(2,\mathbb{C})$ has two inequivalent one dimensional subgroups,

$$G_{0,1} \sim \begin{pmatrix} g_{11} & 0 \\ 0 & g_{11}^{-1} \end{pmatrix}, \quad G_{0,2} \sim \begin{pmatrix} 1 & g_{12} \\ 0 & 1 \end{pmatrix}, \tag{2.8}$$

i. e. the maximal torus $G_{0,1}$ and the unipotent group $G_{0,2}$. Hence we must obtain 2 inequivalent $n = 2$ realizations of $\mathrm{sl}(2,\mathbb{C})$. In appropriate coordinates (y_1, y_2) the coefficients of ∂_{y_1} in all vector fields will depend on y_1 only, those of y_2 can depend on both y_1 and y_2 (see eq. (??)). The coefficients of ∂_{y_1} will hence be as in eq. (??). The analysis is easy to perform and we simply present the result:

$$\begin{aligned} &X_1 = \partial_{y_1}, \quad X_2 = y_1\partial_{y_1} + y_2\partial_{y_2}, \quad X_3 = y_1^2\partial_{y_1} + (2y_1y_2 + ky_2^2)\partial_{y_2}, \\ &k = 0 \text{ or } 1. \end{aligned} \tag{2.9}$$

The 2 inequivalent realizations correspond to $k = 0$ and $k = 1$, respectively. S. Lie in his classification of the $n = 2$ case gave these two reaizations in a different, but equivalent form[??]. The form (??) was already used by Krause and Michel[??].

The system of ODEs (??) corresponding to (??) are

$$\begin{aligned} \dot{y}_1 &= Z_1(t) + Z_2(t)y_1 + Z_3(t)y_1^2 \\ \dot{y}_2 &= \qquad\qquad Z_2(t)y_2 + Z_3(t)(2y_1y_2 + ky_2^2) \end{aligned} \quad . \tag{2.10}$$

We now wish to obtain superposition formulas for the above system, separately for $k = 0$ and $k = 1$.

Integrating the vector fields (??) we obtain the group action

$$\begin{aligned} y_1 &= \frac{g_{11}u_1 + g_{21}}{g_{12}u_1 + g_{22}}, \\ y_2 &= \frac{u_2}{(g_{12}u_1 + g_{22})[g_{12}(u_1 + ku_2) + g_{22}]}. \end{aligned} \tag{2.11}$$

Let us choose the point $(u_1, u_2) = (0, 1)$ as the origin. For $k = 0$ we see that it is stabilized by the unipotent group $G_{0,2}$ of eq. (??). For $k = 1$ the stabilizer of the origin is the subgroup

$$\tilde{G}_{0,1} \sim \begin{pmatrix} g_{11} & g_{11} - \dfrac{1}{g_{11}} \\ 0 & \dfrac{1}{g_{11}} \end{pmatrix}. \tag{2.12}$$

This is conjugate to $G_{0,1}$ of eq. (??), however if we transform $\tilde{G}_{0,1}$ into $G_{0,1}$, the origin is shifted to $(u_1, u_2) = (0, \infty)$. We prefer to stay with the more convenient origin.

The relation $mn \geqq r$ for eq. (??) is $2m \geqq 3$, hence $m \geqq 2$, and it turns out that 2 solutions are indeed sufficient to reconstruct the group element $g(t)$.

Let us assume that (v_1, v_2) and (w_1, w_2) are solutions of eq. (??). The four components of $\vec{v}$ and $\vec{w}$ are then not independent, but satisfy the relation

$$\frac{(v_1 - w_1)[v_1 - w_1 + k(v_2 - w_2)]}{v_2 w_2} = R(k), \quad k = 0,\, 1, \tag{2.13}$$

where $R(k)$ is a constant. This relation is the analogue of the famous unharmonic ratio of 4 solutions of the Riccati equation:

$$\frac{(y_1 - y_2)}{(y_4 - y_3)} \frac{(y_1 - y_3)}{(y_4 - y_2)} = K = \text{const.} \tag{2.14}$$

We assume

$$v_1(0) \neq w_1(0), \; v_2(0)w_2(0) \neq 0, \tag{2.15}$$

and for $k = 1$

$$R(1) \neq 1.$$

With no loss of generality we can assume that the initial conditions for the two known solutions are

$$\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad ab \neq 0. \tag{2.16}$$

Substituting these two solutions and their initial conditions into eq. (??), we can solve for $g_{ik}(t)$. We mention that the reconstruction is more efficient in the imprimitive case than in the primitive one. Indeed for the Riccati equation we need to know 3 solutions, for the system (??) only one.

Let us consider the two cases separately.

1) $k = 0$:

We obtain

$$g_{11}=v_1\sqrt{\frac{a}{v_2}}-w_1\sqrt{\frac{b}{w_2}},\quad g_{12}=\sqrt{\frac{a}{v_2}}-\sqrt{\frac{b}{w_2}},\quad g_{21}=w_1\sqrt{\frac{b}{w_2}},\quad g_{22}=\sqrt{\frac{b}{w_2}}. \tag{2.17}$$

Furthermore, the invariant $R(0)$ of eq. (??) is equal to

$$R(0)=\frac{(v_1-w_1)^2}{v_2w_2}=\frac{1}{ab} \tag{2.18}$$

and hence

$$g_{11}g_{22}-g_{12}g_{21}=\frac{v_1-w_2}{\sqrt{v_2w_2}}\sqrt{ab}=1. \tag{2.19}$$

2) $k=1$:

We use the invariant (??)

$$R(1)=\frac{(v_1-w_1)(v_1-w_1+v_2-w_2)}{v_2w_2}=\frac{a-b+1}{ab} \tag{2.20}$$

to express v_2 in terms of v_1, w_1 and w_2 and obtain

$$\begin{aligned} g_{11}&=\frac{b[w_1(v_1-w_2)-w_1^2]+(b-1)v_1w_2}{[b(1-b)w_2(w_1-v_1)(w_1+w_2-v_1)]^{1/2}},\\ g_{22}&=\frac{b(w_1+w_2-v_1)}{[b(1-b)w_2(w_1-v_1)(w_1+w_2-v_1)]^{1/2}},\\ g_{21}&=g_{22}w_1,\quad g_{12}=\frac{1}{v_1}[g_{11}+g_{22}(v_1-u_1)]. \end{aligned} \tag{2.21}$$

The result of this section can be summed up as follows:

Theorem 1. Two inequivalent realizations of sl$(2,\mathbb{C})$ by vector fields in two dimensions exist, given by eq. (??) with $k=0$ or $k=1$ respectively. The corresponding group actions on the homogeneous space $\mathrm{SL}(2,\mathbb{C})/G_{0,k}$ is given in eq. (??).

Theorem 2. The nonlinear ODEs (??) for $k=0$ and $k=1$ have superposition formulas given by the imprimitive group action (??). The group elements $g_{ik}(t)$, $i,k\in\{1,2\}$ are reconstructed from any two solutions $\vec{v}=(v_1,v_2)$ and $\vec{w}=(w_1,w_2)$ with the initial conditions satisfying eq. (??). The explicit reconstruction formulas are given in eq. (??) for $k=0$ and (??) for $k=1$ respectively.

III. Induced representations of SL(N,ℂ) and parabolic subgroups

III.1. General theory

Let us consider the Lie algebra sl$(N,\mathbb{C})$ realized by matrices $X\in\mathbb{C}^{N\times N}$, $\mathrm{Tr}\,X=0$. We shall make use of several subalgebras of sl$(N,\mathbb{C})$. The **Borel** subalgebra is the maximal solvable subalgebra and it can be realized by the set of all traceless upper triangular matrices. A **parabolic** subalgebra of sl$(N,\mathbb{C})$ is any subalgebra containing the Borel subalgebra. A **maximal** parabolic subalgebra is one that is not contained in any proper subalgebra of sl$(N,\mathbb{C})$. All parabolic subalgebras can be realized by block triangular matrices. Maximal parabolic subalgebras correspond to case of precisely two blocks on the diagonal.

A classification of all maximal parabolic subalgebras of sl$(N,\mathbb{C})$ is obtained by taking all decompositions of N into $N=r_1+r_2$, $\min(r_1,r_2)\geqq 1$ and writing all sets of matrices of the form

$$p(r_1,r_2)=\left\{X\in\mathbb{C}^{N\times N}\,\middle|\,\mathrm{Tr}\,X=0,\ X=\begin{pmatrix}A_{11}&A_{12}\\0&A_{22}\end{pmatrix}\right\},$$
$$N=r_1+r_2, \tag{3.1}$$
$$A_{11}\in\mathbb{C}^{r_1\times r_1},\quad A_{22}\in\mathbb{C}^{r_2\times r_2},\quad A_{12}\in\mathbb{C}^{r_1\times r_2}.$$

Similarly, a maximal parabolic subgroup of the group corresponds to the same partition of N and satisfies

$$P(r_1,r_2)=\left\{G\in\mathbb{C}^{N\times N}\,\middle|\,G=\begin{pmatrix}G_{11}&G_{12}\\0&G_{22}\end{pmatrix},\ \det G_{11}\det G_{22}=1\right\}. \tag{3.2}$$

The homogeneous space $M\sim\mathrm{SL}(N,\mathbb{C})/P(r_1,r_2)$ was constructed in an earlier article[??] (as a Grassmanian). Local coordinates on this space were introduced as matrix elements of a matrix $W\in\mathbb{C}^{r_1\times r_2}$. The group action on this space is represented by matrix fractional linear transformations

$$\tilde W=(G_{11}W+G_{12})(G_{21}W+G_{22})^{-1}. \tag{3.3}$$

The corresponding Lie algebra $\mathrm{sl}(N,\mathbb{C})$ is represented by vector fields with (specific) quadratic nonlinearities. The group action is primitive, the corresponding nonlinear ordinary differential equations with superposition formulas are matrix Riccati equations:

$$\dot W=A+BW+WC+WDW, \tag{3.4}$$
$$W,\,A\in\mathbb{C}^{r_1\times r_2},\quad B\in\mathbb{C}^{r_1\times r_1},\quad C\in\mathbb{C}^{r_2\times r_2},\quad D\in\mathbb{C}^{r_2\times r_1}.$$

In particular for $r_2=1$ we obtain projective Riccati equations corresponding to the projective action of $\mathrm{sl}(N,\mathbb{C})$ on $\mathbb{C}^{N-1}$. The corresponding superposition formula is given by eq. (??) in which $W=\mathrm{const}$ represents the initial data, $\tilde W=\tilde W(t)$ the general solution and the matrices $G_{ik}(t)$ can be reconstructed from $N+1$ known solutions[??], [??].

Series of nonprimitive realizations of the Lie algebra $\mathrm{sl}(N,\mathbb{C})$ for $N\geqq 3$ can be obtained using the theory of induced representations[??], [??]. The homogeneous spaces that we construct are $M\sim G/G_0$ where G is $\mathrm{SL}(N,\mathbb{C})$ and G_0 is a (nonmaximal) parabolic subgroup of $\mathrm{SL}(N,\mathbb{C})$. More specifically, in this article $P(N)$ is group of matrices

$$G_0\sim g=\begin{pmatrix}g_{11}&g_{12}&\cdots&\cdots&g_{1N}\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ g_{N-2,1}&\cdots&g_{N-2,N-2}&g_{N-2,N-1}&g_{N-2,N}\\ 0&\cdots&0&g_{N-1,N-1}&g_{N-1,N}\\ 0&\cdots&0&0&g_{NN}\end{pmatrix}. \tag{3.5}$$

We use the Borel subgroup $B\subset\mathrm{SL}(N,\mathbb{C})$ to induce representations of $\mathrm{SL}(N,\mathbb{C})$ on spaces of functions $f(Z)$ where Z is a point on the space $\mathrm{SL}(N,\mathbb{C})/B$.

To obtain the group action explicitly, we use the defining representation of $\mathrm{SL}(N,\mathbb{C})$ by $N\times N$ matrices and write the Gauss decomposition

$$g=\xi DZ,\quad \xi=\begin{pmatrix}1&\xi_{12}&\xi_{13}&\cdots&\xi_{1N}\\0&1&\xi_{23}&\cdots&\xi_{2N}\\ \\0&0&\ddots&\cdots&\cdots\\0&\cdots&0&1&\xi_{N-1,N}\\0&\cdots&&0&1\end{pmatrix},\quad Z=\begin{pmatrix}1&0&&\cdots&0\\z_{21}&1&0&\cdots&0\\ \\z_{31}&z_{32}&\ddots&\cdots&\cdots\\&\cdots&&1&0\\z_{N1}&\cdots&&z_{N,N-1}&1\end{pmatrix},$$
$$D=\mathrm{diag}(d_{11},\ldots,d_{NN}). \tag{3.6}$$

The matrix elements z_{ij}, $1\leqq j<i\leqq N$ are local coordinates on $\tilde M\sim\mathrm{SL}(N,\mathbb{C})/B$ (the Borel subgroup is represented by the matrices ξD). The action of the group $\mathrm{SL}(N,\mathbb{C})$ on $\tilde M$ is given in local coordinates as

$$Zg=\xi D\tilde Z. \tag{3.7}$$

Explicitly we obtain the group action

$$\tilde{Z} = F(g, Z) \tag{3.8}$$

by eliminating d_{ii} and ξ_{ik} from eq. (??) using a subset of these N^2 equations and substituting into the remaining $\frac{N(N-1)}{2}$ equations.

III.2. Example of the group SL(3,ℂ)

Let us illustrate the procedure for the simplest nontrivial case, namely SL$(3,\mathbb{C})$. Eq. (??) in this case is

$$\begin{pmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & \xi_{12} & \xi_{13} \\ 0 & 1 & \xi_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \tilde{z}_{21} & 1 & 0 \\ \tilde{z}_{31} & \tilde{z}_{32} & 1 \end{pmatrix}. \tag{3.9}$$

We rewrite eq. (??) symbolically as

$$R_{ik} = 0.$$

Eq. $R_{i3} = 0$ gives us d_{33}, ξ_{23} and ξ_{13} in terms of z_{ik} and g_{ik}. Eq. $R_{3k} = 0$ for $k = 1$ and 2 give us $\tilde{z}_{32}$ and $\tilde{z}_{31}$ in terms of z_{ik} and g_{ik}, eq. $R_{22} = 0$ and $R_{21} = 0$ give us d_{22} and finally $\tilde{z}_{21}$.

Defining $z_{31} = x_1$, $z_{32} = x_2$ and $z_{21} = x_3$ and similarly for $\tilde{z}_{31}$, $\tilde{z}_{32}$ and $\tilde{z}_{21}$ we obtain the group transformations, namely

$$\begin{aligned} \tilde{x}_1 &= \frac{g_{11}x_1 + g_{21}x_2 + g_{31}}{g_{13}x_1 + g_{23}x_2 + g_{33}}, \\ \tilde{x}_2 &= \frac{g_{12}x_1 + g_{22}x_2 + g_{32}}{g_{13}x_1 + g_{23}x_2 + g_{33}}, \\ \tilde{x}_3 &= \frac{(-x_1 + x_2x_3)A_{1123} + x_3A_{1133} + A_{2133}}{(-x_1 + x_2x_3)A_{1223} + x_3A_{1233} + A_{2233}}, \end{aligned} \tag{3.10}$$

where

$$A_{ijkl} = \begin{vmatrix} g_{ij} & g_{il} \\ g_{kj} & g_{kl} \end{vmatrix} = g_{ij}g_{kl} - g_{il}g_{kj}. \tag{3.11}$$

We see that the elements of the last row in Z, namely $z_{31} = x_1$, $z_{32} = x_2$ transform independently of the second row ($z_{21} = x_3$). Thus, we have a realization of SL$(3,\mathbb{C})$ on a flag manifold. Indeed $\{x_1, x_2\}$ are local coordinates on the space SL$(3,\mathbb{C})/P(2,1)$ and we have a primitive action on this subspace (the projective action). Further $\{x_1, x_2, x_3\}$ are local coordinates on SL$(3,\mathbb{C})/B$ and on this space the action is imprimitive (we have $P(3) \sim B$).

The linear representations of SL$(3,\mathbb{C})$ on functions $f(Z)$ is given as

$$T_g f(x_1, x_2, x_3) = f(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3), \tag{3.12}$$

with $\tilde{x}_i$ as in eq. (??). Calculating the infinitesimal operators in this representation we find

$$\begin{aligned} &\hat{E}_{31} = \partial_1,\ \hat{E}_{32} = \partial_2,\ \hat{E}_{21} = x_2\partial_1 + \partial_3, \\ &\hat{E}_{11} = x_1\partial_1 + x_3\partial_3,\ \hat{E}_{22} = x_2\partial_2 - x_3\partial_3,\ \hat{E}_{12} = x_1\partial_2 - x_3^2\partial_3, \\ &\hat{E}_{13} = x_1(x_1\partial_1 + x_2\partial_2) - x_3(x_3x_2 - x_1)\partial_3, \\ &\hat{E}_{23} = x_2(x_1\partial_1 + x_2\partial_2) - (x_3x_2 - x_1)\partial_3, \\ &\partial_i \equiv \partial_{x_i}. \end{aligned} \tag{3.13}$$

The nonlinear ordinary differential equations with a superposition formula corresponding to the action of SL$(3,\mathbb{C})$ on SL$(3,\mathbb{C})/B$ can be read off from (??) using Lie's result (??), (??). They are

$$\begin{aligned} \dot{x}_1 &= Z_{31} + Z_{11}x_1 + Z_{21}x_2 + Z_{13}x_1^2 + Z_{23}x_1x_2, \\ \dot{x}_2 &= Z_{32} + Z_{12}x_1 + Z_{22}x_2 + Z_{13}x_1x_2 + Z_{23}x_2^2, \\ \dot{x}_3 &= Z_{21} + (Z_{11} - Z_{22})x_3 - Z_{12}x_3^2 - Z_{13}x_3(x_3x_2 - x_1) - Z_{23}(x_3x_2 - x_1), \end{aligned} \tag{3.14}$$

where $Z_{ik} = Z_{ik}(t)$ are arbitrary functions of time t.

We mention that the origin of the coordinate system in $\mathrm{SL}(3,\mathbb{C})/P(2,1)$ is the point $(x_1, x_2) = (0,0)$ and in $\mathrm{SL}(3,\mathbb{C})/P(3)$ it is $(x_1, x_2, x_3) = (0,0,0)$.

Notice that the first two equations in eq. (??) involve quadratic nonlinearities, while the third equation involves cubic ones. However, if x_1 and x_2 are known (from the first two equations), then the equation for $\dot{x}_3$ reduces to a Riccati equation.

III.3. The group SL(4,$\mathbb{C}$)

For $\mathrm{SL}(4,\mathbb{C})$ we no longer have $P(4) \sim B$ and the flag manifold consists of 3 spaces

$$\mathrm{SL}(4,\mathbb{C})/P(3,1) \subset \mathrm{SL}(4,\mathbb{C})/P(4) \subset \mathrm{SL}(4,\mathbb{C})/B. \tag{3.15}$$

We restrict ourselves to the space $\mathrm{SL}(4,\mathbb{C})/P(4)$, put $x_1 = z_{41}$, $x_2 = z_{42}$, $x_3 = z_{43}$, $x_4 = z_{31}$ and $x_5 = z_{32}$ and use eq. (??) to obtain

$$\begin{aligned}
\tilde{x}_1 &= \frac{x_1 g_{11} + x_2 g_{21} + x_3 g_{31} + g_{41}}{x_1 g_{14} + x_2 g_{24} + x_3 g_{34} + g_{44}}, \\
\tilde{x}_2 &= \frac{x_1 g_{12} + x_2 g_{22} + x_3 g_{32} + g_{42}}{x_1 g_{14} + x_2 g_{24} + x_3 g_{34} + g_{44}}, \\
\tilde{x}_3 &= \frac{x_1 g_{13} + x_2 g_{23} + x_3 g_{33} + g_{43}}{x_1 g_{14} + x_2 g_{24} + x_3 g_{34} + g_{44}}, \\
\tilde{x}_4 &= \frac{(x_1x_5 - x_2x_4)A_{1124} + (x_1 - x_3x_4)A_{1134} + (x_2 - x_3x_5)A_{2134} - x_4A_{1144} - x_5A_{2144} - A_{3144}}{(x_1x_5 - x_2x_4)A_{1324} + (x_1 - x_3x_4)A_{1334} + (x_2 - x_3x_5)A_{2334} - x_4A_{1344} - x_5A_{2344} - A_{3344}}, \\
\tilde{x}_5 &= \frac{(x_1x_5 - x_2x_4)A_{1224} + (x_1 - x_3x_4)A_{1234} + (x_2 - x_3x_5)A_{2234} - x_4A_{1244} - x_5A_{2244} - A_{3244}}{(x_1x_5 - x_2x_4)A_{1324} + (x_1 - x_3x_4)A_{1334} + (x_2 - x_3x_5)A_{2334} - x_4A_{1344} - x_5A_{2344} - A_{3344}}.
\end{aligned} \tag{3.16}$$

The infinitesimal operators can again be obtained as vector fields, using the representation $T_g f(Z) = f(\tilde{Z})$ with Z and $\tilde{Z}$ related as in eq. (??). Instead of writing them out we give the corresponding nonlinear ordinary differential equations, namely

$$\begin{aligned}
\dot{x}_1 &= Z_{41} + Z_{11}x_1 + Z_{21}x_2 + Z_{31}x_3 + x_1(Z_{14}x_1 + Z_{24}x_2 + Z_{34}x_3), \\
\dot{x}_2 &= Z_{42} + Z_{12}x_1 + Z_{22}x_2 + Z_{32}x_3 + x_2(Z_{14}x_1 + Z_{24}x_2 + Z_{34}x_3), \\
\dot{x}_3 &= Z_{43} + Z_{13}x_1 + Z_{23}x_2 + Z_{33}x_3 + x_3(Z_{14}x_1 + Z_{24}x_2 + Z_{34}x_3), \\
\dot{x}_4 &= Z_{31} + (Z_{11} - Z_{33})x_4 + Z_{21}x_5 - x_4(Z_{13}x_4 + Z_{23}x_5) + (x_1 - x_3x_4)(Z_{14}x_4 + Z_{24}x_5 + Z_{34}), \\
\dot{x}_5 &= Z_{32} + Z_{12}x_4 + (Z_{22} - Z_{33})x_5 - x_5(Z_{13}x_4 + Z_{23}x_5) + (x_2 - x_3x_5)(Z_{14}x_4 + Z_{24}x_5 + Z_{34}),
\end{aligned} \tag{3.17}$$

where Z_{ik} are arbitrary functions of t. Notice that $\{x_1, x_2, x_3\}$ satisfy projective Riccati equations. The last two equations involve cubic nonlinearities. However, similarly as in the $\mathrm{sl}(3,\mathbb{C})$ case the equations "decompose". If $\{x_1, x_2, x_3\}$ are known, then the equations for $\{x_4, x_5\}$ satisfy projective Riccati equations, based on algebra $\mathrm{sl}(3,\mathbb{C})$.

III.4. Finite transformations and vector fields for general N

For general $N \geqq 3$ the formulas are quite similar to the above cases $N = 3, 4$ only somewhat more cumbersome to write. Dropping the calculations, we just present $\mathrm{SL}(N,\mathbb{C})$ group action on the space $\mathrm{SL}(N,\mathbb{C})/P(N)$:

$$\tilde{x}_i = \frac{\left(\sum_{j=1}^{N-1} g_{ji}x_j\right) + g_{Ni}}{\left(\sum_{j=1}^{N-1} g_{jN}x_j\right) + g_{NN}}, \quad i = 1, \dots, N-1, \tag{3.18}$$

$$\tilde{x}_{i+N-1} = F(i)/F(N-1), \quad i = 1, \dots, N-2,$$

where

$$F(i) = \left(\sum_{k=1}^{N-3} \sum_{l=1}^{N-2-k} (x_k x_{l+N+k-1} - x_{k+l} x_{k+N-1}) A_{ki(k+l)N} \right) +$$
$$+ \left(\sum_{k=1}^{N-2} (x_k - x_{N-1} x_{k+N-1}) A_{ki(N-1)N} \right) + \left(\sum_{k=1}^{N-2} (-x_{k+N-1}) A_{kiNN} \right) - A_{(N-1)iNN}.$$

We see that $\{x_1, \ldots, x_{N-1}\}$ transform according to the projective realization of $\mathrm{SL}(N,\mathbb{C})$. The remaining $N-2$ variables $\{x_N, \ldots, x_{2N-3}\}$ transform in a manner involving quadratic polynomials in the denominator and numerator of a fraction.

The nonlinear ODEs with superposition formulas can be read off from the vector fields representing the Lie algebra $\mathrm{sl}(N,\mathbb{C})$ in this realization, namely

$$\begin{aligned}
&\hat{E}_{Nj} = \partial_j, \quad 1 \leqq j \leqq N-1, \\
&\hat{E}_{ij} = x_i \partial_j + x_{N-1+i} \partial_{N-1+j}, \quad 1 \leqq i \leqq N-2,\, 1 \leqq j \leqq N-2, \\
&\hat{E}_{N-1\,j} = x_{N-1} \partial_j + \partial_{N+j-1}, \quad 1 \leqq j \leqq N-2, \\
&\hat{E}_{N-1\,N-1} = x_{N-1} \partial_{N-1} - \sum_{m=N}^{2N-3} x_m \partial_m, \\
&\hat{E}_{i\,N-1} = x_i \partial_{N-1} - x_{N-1+i} \sum_{m=N}^{2N-3} x_m \partial_m, \quad 1 \leqq i \leqq N, \\
&\hat{E}_{iN} = x_i \sum_{j=1}^{N-1} x_j \partial_j - x_{N-1+i} \sum_{m=N}^{2N-3} (x_m x_{N-1} - x_{m-N+1}) \partial_m, \quad 1 \leqq i \leqq N-2, \\
&\hat{E}_{N-1\,N} = x_{N-1} \sum_{j=1}^{N-1} x_j \partial_j - \sum_{m=N}^{2N-3} (x_m x_{N-1} - x_{m-N+1}) \partial_m.
\end{aligned} \tag{3.19}$$

The $\mathrm{sl}(N,\mathbb{C})$ equations can now be written in a quite compact form, namely

$$\dot{x}_j = Z_{Nj} + \sum_{i=1}^{N-1} Z_{ij} x_i + x_j \sum_{i=1}^{N-1} Z_{iN} x_i, \quad 1 \leqq j \leqq N-1, \tag{3.20}$$

$$\begin{aligned}
\dot{x}_{N-1+j} = {} & Z_{N-1\,j} + \sum_{i=1}^{N-2} Z_{ij} x_{N-1+i} - Z_{N-1\,N-1} x_{N-1+j} - x_{N-1+j} \sum_{i=1}^{N-2} Z_{i\,N-1} x_{N-1+i} + \\
& + (x_j - x_{N-1+j} x_{N-1}) \sum_{i=1}^{N-2} (Z_{iN} x_{N-1+i} + Z_{N-1\,N}), \quad 1 \leqq j \leqq N-2.
\end{aligned} \tag{3.21}$$

In (??) and (??) Z_{ik} are arbitrary functions of t. For $N = 3$ and $N = 4$ these formulas coincide with (??) and (??) respectively. The same comments pertain, namely eq. (??) are projective Riccati equations based on $\mathrm{sl}(N,\mathbb{C})$. Eq. (??) are projective Riccati equations based on $\mathrm{sl}(N-1,\mathbb{C})$ if $x_1, \ldots, x_{N-1}$ are known.

IV. Superposition formulas

IV.1. General comments

The superposition formula for the imprimitive $\mathrm{SL}(N,\mathbb{C})$ equations is given by the group action formula (??) in which x_i, $i=1,\ldots,2N-3$ is a constant vector, related to the initial conditions. The matrix elements $g_{ik}(t)$ must be expressed in terms of m particular solutions. The number m satisfies eq. (??). In our case that means that we must have

$$m(2N-3) \geqq N^2-1. \tag{4.1}$$

The actual number m, as well us conditions that must be imposed on the known solutions, is obtained from the requirement that m be the smallest number of solutions such that their joint isotropy group consists only of the identity transformation. We shall call such set a "fundamental set of solutions".

IV.2. Example of SL(3,ℂ)

Eq. (??) in this case is $3m \geqq 8$ and hence $m \geqq 3$.

The equations under consideration are given in eq. (??), the superposition formula has the form (??).

The fundamental set of solutions. Let us assume that we know 3 solutions of eq. (??), $\vec{x}(t)$, $\vec{y}(t)$ and $\vec{z}(t)$. Each solution is a 3 component vector. The group $\mathrm{SL}(3,\mathbb{C})$ acts on the space $M\times M\times M$ of these 3 solutions. Since we have $\dim \mathrm{SL}(3,\mathbb{C})=8$ the group can sweep out at most an 8 dimensional orbit, so there must exist at least one $\mathrm{SL}(3,\mathbb{C})$ invariant in this 9 dimensional space. We denote this invariant

$$R(x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3)=K, \tag{4.2}$$

where $\vec{x}$, $\vec{y}$ and $\vec{z}$ are 3 solutions of eq. (??). We calculate $\dot{R}=\frac{\mathrm{d}R}{\mathrm{d}t}$, replace $\dot{\vec{x}}$, $\dot{\vec{y}}$, $\dot{\vec{z}}$ using eq. (??) and require $\dot{R}=0$ for all functions $Z_{ik}(t)$. This provides us with 8 linear first order partial differential equations for R. These equations can be solved and we obtain a single elementary invariant, namely

$$R=\frac{(XZ)_{321}(ZY)_{321}(YX)_{321}}{(XY)_{321}(YZ)_{321}(ZX)_{321}}, \tag{4.3}$$

where we have defined

$$(XY)_{321}=x_3(x_2-y_2)-x_1+y_1, \tag{4.4}$$

etc. The quantity $(XY)_{321}$ and similarly defined $(XZ)_{321}$, $(YX)_{321}$, $(YZ)_{321}$, $(ZX)_{321}$ and $(ZY)_{321}$ play a crucial role in the reconstruction of the group elements $g_{ik}(t)$. For $\mathrm{SL}(3,\mathbb{C})$ the notation is somewhat redundant. However for $\mathrm{SL}(N,\mathbb{C})$, $N\geqq 4$ we will have quantities like $(XY)_{431}$, $(XY)_{532}$, etc. so for uniformity we keep the subscripts for $N=3$ as well.

The initial condition for 3 solutions are $\vec{x}(0)$, $\vec{y}(0)$ and $\vec{z}(0)$. We shall use the transformation (??) (with constant coefficients) to standardize the initial conditions. Let us do this transformation in two steps:

$$g=g_2g_1,\quad g_1=\begin{pmatrix} g_{11} & g_{12} & 0\\ g_{21} & g_{22} & 0\\ g_{31} & g_{32} & g_{33}\end{pmatrix},\ g_2=\begin{pmatrix} a_{11} & 0 & a_{11}-a_{33}\\ 0 & a_{22} & a_{22}-a_{33}\\ 0 & 0 & a_{33}\end{pmatrix}. \tag{4.5}$$

Let us now assume that the first two components of $\vec{x}(0)$, $\vec{y}(0)$ and $\vec{z}(0)$ satisfy

$$\triangle=\begin{vmatrix} x_1(0)-z_1(0) & y_1(0)-z_1(0)\\ x_2(0)-z_2(0) & y_2(0)-z_2(0)\end{vmatrix}\neq 0. \tag{4.6}$$

This condition can be rewritten as

$$\triangle=x_1(0)(y_2(0)-z_2(0))+y_1(0)(z_2(0)-x_2(0))+z_1(0)(x_1(0)-y_2(0))\neq 0,$$

so that it is actually symmetric in the three two dimensional vectors

$$x_{\rm T}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}, \; y_{\rm T}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}, \; z_{\rm T}(0) = \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}. \tag{4.7}$$

If Eq. (??) is satisfied we can use the transformation (??) with the matrix g_1 to transform the 3 initial vectors into a more convenient form

$$g_1 : \{\vec{x}(0), \vec{y}(0), \vec{z}(0)\} \rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ \tilde{x}_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \tilde{y}_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \tilde{x}_3 \end{pmatrix} \right\} \tag{4.8}$$

with

$$\tilde{x}_3 = \frac{(XY)_{321} - (XZ)_{321}}{(XZ)_{321}}, \; \tilde{y}_3 = -\frac{(YZ)_{321}}{(YZ)_{321} - (YX)_{321}}, \; \tilde{z}_3 = -\frac{(ZY)_{321}}{(ZX)_{321}}$$

(all components evaluated at $t = 0$).

The initial conditions (??) imply that $(XY)_{321}$ and $(XZ)_{321}$ cannot vanish simultaneously (and similarly for $(YZ)_{321}$ and $(YX)_{321}$, or $(ZX)_{321}$ and $(ZY)_{321}$). To proceed further we need a stronger condition, namely that at least one of the following relations holds:

$$(XY)_{321}(XZ)_{321} \neq 0, \; (YZ)_{321}(YX)_{321} \neq 0, \; (ZX)_{321}(ZY)_{321} \neq 0 \tag{4.9}.$$

With no loss of generality we assume that the first of the above relations holds. We then transform further using the matrix g_2 and obtain

$$\begin{aligned} &g_2 g_1 : \{\vec{x}(0), \vec{y}(0), \vec{z}(0)\} \rightarrow \{\vec{x}_{\rm S}(0), \vec{y}_{\rm S}(0), \vec{z}_{\rm S}(0)\}, \\ &\vec{x}_{\rm S}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{y}_{\rm S}(0) = \begin{pmatrix} 0 \\ 1 \\ \alpha \end{pmatrix}, \vec{z}_{\rm S}(0) = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \end{aligned} \tag{4.10}$$

where

$$\alpha = \left[-1 + \frac{a_{33}}{a_{11}} \frac{(YX)_{321}}{(YZ)_{321}} \right]^{-1}, \quad \beta = -\frac{a_{11}}{a_{22}} \frac{(ZY)_{321}}{(ZX)_{321}}. \tag{4.11}$$

To standardize $\vec{x}_{\rm S}(0)$ we have already fixed the ratio a_{33}/a_{22} hence only one of the ratios a_{33}/a_{11} and a_{11}/a_{22} can be chosen freely. >From eq. (??) we immediately see 4 special cases:

$$\begin{aligned} &(YX)_{321} = 0 \Rightarrow \alpha = -1, \\ &(YZ)_{321} = 0 \Rightarrow \alpha = 0, \\ &(ZY)_{321} = 0 \Rightarrow \beta = 0, \\ &(ZX)_{321} = 0 \Rightarrow \beta \rightarrow \infty. \end{aligned} \tag{4.12}$$

Thus, if $(YX)_{321}(YZ)_{321} \neq 0$ we can standardize $\alpha \rightarrow 1$. Alternatively, if $(ZY)_{321}(ZX)_{321} \neq 0$ we can standardize $\beta \rightarrow 1$. The invariant R of eq. (??) for the standardized initial conditions (??) is

$$R = -\frac{(\alpha + 1)}{\alpha} \beta. \tag{4.13}$$

We recall that R is time independent and cannot be changed by group transformation of the initial conditions.

To see whether a reconstruction is possible from three solutions with initial conditions (??), let us calculate the stabilizer of these "standard" initial conditions. The stabilizer of the vector $\vec{x}_{\rm S}(0)$ and of the first two components of the other two vectors is

$$g = \begin{pmatrix} a_{11} & 0 & a_{11} - a_{22} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{22} \end{pmatrix}. \tag{4.14}$$

The remaining conditions for $\vec{y}_{\rm S}(0)$ and $\vec{z}_{\rm S}(0)$ to be stabilized are

$$\begin{aligned} \alpha(\alpha+1)(g_{11}-g_{22}) &= 0 \\ \beta g_{11} &= \beta g_{22} \end{aligned}. \tag{4.15}$$

Thus, the stabilizer is $g \sim I$ if we have at least one of the following conditions:

$$\alpha(\alpha+1) \neq 0, \quad 0 < |\beta| < \infty. \tag{4.16}$$

All other cases must be excluded.

Thus, if $\triangle = 0$, the stabilizer is always too large. The same is true if all the products in eq. (??) vanish (at $t = 0$).

Thus, a reconstruction is possible if and only if:

1) $\triangle \neq 0$,

2) At least 2 of the products in eq. (??) are nonzero.

Reconstruction of the group element. Let us reconstruct the $\mathrm{SL}(3,\mathbb{C})$ group element from the fundamental set of solutions, corresponding to the initial data (??) satisfying one of the conditions (??). We first make use of the first 2 components of the 3 solutions $\vec{x}(t)$, $\vec{y}(t)$, $\vec{z}(t)$. The first two equations in (??) then imply

$$\begin{aligned} x_1 &= \frac{g_{11}+g_{31}}{g_{13}+g_{33}}, \quad y_1 = \frac{g_{21}+g_{31}}{g_{23}+g_{33}}, \quad z_1 = \frac{g_{31}}{g_{33}}, \\ x_2 &= \frac{g_{12}+g_{32}}{g_{13}+g_{33}}, \quad y_2 = \frac{g_{22}+g_{32}}{g_{23}+g_{33}}, \quad z_2 = \frac{g_{32}}{g_{33}}. \end{aligned} \tag{4.17}$$

This allows us to express all off-diagonal elements $g_{ik}(t)$, $i \neq k$ in terms of the diagonal ones and the first two components of the three known solutions:

$$\begin{aligned} g_{12} &= \frac{1}{x_1}\big[g_{11}x_2 + g_{33}(z_1x_2 - x_1z_2)\big], \quad g_{13} = \frac{1}{x_1}\big[g_{11} + g_{33}(z_1 - x_1)\big], \\ g_{21} &= \frac{1}{y_2}\big[g_{22}y_1 + g_{33}(y_1z_2 - z_1y_2)\big], \quad g_{23} = \frac{1}{y_2}\big[g_{22} + g_{33}(z_2 - y_2)\big], \\ g_{31} &= g_{33}z_1, \quad g_{32} = g_{33}z_2. \end{aligned} \tag{4.18}$$

Next, we substitute the known solutions and initial conditions (??) into the third eq. in (??). We replace the off-diagonal elements using eq. (??) and obtain 3 linear equations for the diagonal elements g_{11}, g_{22} and g_{33}:

$$\begin{pmatrix} 0 & -(XY)_{321} & y_2(XZ)_{321} - z_2(XY)_{321} \\ -\alpha(YX)_{321} & 0 & (1+\alpha)x_1(YZ)_{321} - \alpha z_1(YX)_{321} \\ \beta y_2(ZX)_{321} & x_1(ZY)_{321} & \beta z_1y_2(ZX)_{321} + x_1z_2(ZY)_{321} \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{22} \\ g_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.19}$$

The rank r_M of the above matrix is $r_M \leqq 2$. A reconstruction is possible if $r_M = 2$; then we can express g_{11} and g_{22} linearly in terms of g_{33}. The rank is $r_M = 1$ if and only if both conditions (??) are violated (i. e. $\alpha = 0$ or $\alpha = -1$ and simultaneously $\beta = 0$ or $\beta \to \infty$).

The above results are particularly obvious at $t = 0$ when eq. (??) reduce to

$$\begin{pmatrix} 0 & 1 & -1 \\ -\alpha(\alpha+1) & 0 & \alpha(\alpha+1) \\ \beta & -\beta & 0 \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{22} \\ g_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.20}$$

Let us now sum up the results of this section as follows.

Theorem 3. 1. The general solution of the system of nonlinear ordinary differential equations (??) associate with the imprimitive action of $\mathrm{SL}(3,\mathbb{C})$ on the space $\mathrm{SL}(3,\mathbb{C})$ is given by the formula

$$\begin{aligned} v_1(t) &= \frac{g_{11}u_1 + g_{21}u_2 + g_{31}}{g_{13}u_1 + g_{23}u_2 + g_{33}}, \\ v_2(t) &= \frac{g_{12}u_1 + g_{22}u_2 + g_{32}}{g_{13}u_1 + g_{23}u_2 + g_{33}}, \end{aligned} \tag{4.21}$$

$$v_3(t) = \frac{(u_2u_3 - u_1)A_{11,23} + u_3A_{11,33} + A_{21,33}}{(u_2u_3 - u_1)A_{12,23} + u_3A_{12,33} + A_{22,33}},$$

$$A_{ij,kl}(t) = \begin{vmatrix} g_{ij} & g_{il} \\ g_{kj} & g_{kl} \end{vmatrix} = g_{ij}g_{kl} - g_{il}g_{kj}.$$

The 3 constants u_1, u_2 and u_3 are related to the initial conditions for the solution $\vec{v}(t)$.

2. The group elements $g_{ik}(t)$ (and hence also the quantities A_{ijkl}) are reconstructed from a set of 3 solutions $\vec{x}(t)$, $\vec{y}(t)$ and $\vec{z}(t)$. This fundamental set of solutions is quite generic and is subject to just 2 conditions:

(i) $\triangle \neq 0$ with $\triangle$ defined in eq. (??),

(ii) At least two of the inequalities (??) hold with $(XY)_{321}$ defined in eq. (??).

3. The off-diagonal elements $g_{ik}(t)$, $i \neq k$ are given in eq. (??). The diagonal ones are obtained by solving eq. (??).

Comments. 1. The reconstruction is linear in sense we only need to solve linear algebraic relations (once 3 solutions are known). All elements g_{ik} are proportional to $g_{33}(t)$ which cancels out from eq. (??).

2. A fundamental set of solutions for projective Riccati equations based on the primitive action of $\mathrm{SL}(3,\mathbb{C})$ consists of 4 solutions. In the imprimitive case we only need 3 solutions.

IV.3. The group SL(4,ℂ)

The $\mathrm{sl}(4,\mathbb{C})$ ODEs are given in eq. (??), the general form of the superposition formula in eq. (??). The number of equations is $n = 5$, the dimension of the Lie algebra is $r = 15$, hence $nm \geqq r$ implies $m \geqq 3$.

We shall show that we do actually need precisely $m = 3$ generically chosen particular solutions and give an explicit reconstruction of the group element. For the group $\mathrm{SL}(4,\mathbb{C})$ we have $nm = r$ (for $m = 3$) and no $\mathrm{SL}(4,\mathbb{C})$ invariant can be formed out of 3 solutions.

Let us assume that we know three solutions $\vec{x}$, $\vec{y}$, $\vec{z}$, each of them a 5-component vector.

Let us assume that the first 3 components of these vectors satisfy an independence condition for $t = 0$, namely

$$\operatorname{rank}\begin{pmatrix} x_1(0)-z_1(0) & y_1(0)-z_1(0) \\ x_2(0)-z_2(0) & y_2(0)-z_2(0) \\ x_3(0)-z_3(0) & y_3(0)-z_3(0) \end{pmatrix} = 2. \tag{4.22}$$

We can then use a constant coefficient $\mathrm{SL}(4,\mathbb{C})$ transformation to take the initial conditions into

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ x_4(0) \\ x_5(0) \end{pmatrix}, \ \vec{y}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ y_4(0) \\ y_5(0) \end{pmatrix}, \ \vec{z}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.23}$$

Let us make a further assumption namely that transformed initial conditions satisfy

$$x_5(0)y_4(0) \neq 0, \quad y_4(0) + y_5(0) \neq x_4(0) + x_5(0). \tag{4.24}$$

We can then standardize the initial conditions further, namely take them into

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \vec{y}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix}, \ \vec{z}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.25}$$

The stabilizer of these 3 vectors in $\mathrm{SL}(4,\mathbb{C})$ consists of matrices satisfying

$$g(0)=\begin{pmatrix} g_{11} & 0 & 0 & g_{11}-g_{44}\\ 0 & g_{22} & 0 & g_{22}-g_{44}\\ 0 & 0 & g_{22} & g_{44}-g_{22}\\ 0 & 0 & 0 & g_{44}\end{pmatrix}, \quad \alpha(g_{11}-g_{22})=0,\ (\alpha-1)(g_{11}-g_{44})=0. \qquad (4.26).$$

Thus, for $\alpha \neq 0$, $\alpha \neq 1$ we have $g \sim I$ and a unique reconstruction of the group elements $g_{ik}(t)$ is possible. We mention that even though α has no invariant meaning, the values $\alpha = 0$ and $\alpha = 1$ correspond to degenerate orbits of triplets of vectors. These values of α must be excluded from further considerations. Indeed, the stabilizer (??) for $\alpha = 0$ or $\alpha = 1$ is larger, since g_{11} and either g_{22} or g_{44} remain free.

Let us now perform a reconstruction, using 3 solutions satisfying eq. (??) with $\alpha(1-\alpha) \neq 0$. Substituting the components x_i, y_i and z_i, $i = 1, 2, 3$ into eq. (??) we obtain:

$$\begin{aligned}
&g_{41}=g_{44}z_1,\quad g_{42}=g_{44}z_2,\quad g_{43}=g_{44}z_3,\\
&g_{12}=\frac{1}{x_1}[g_{11}x_2+g_{44}(x_2z_1-x_1z_2)],\ g_{13}=\frac{1}{x_1}[g_{11}x_3+g_{44}(x_3z_1-x_1z_3)], \qquad (4.27)\\
&g_{14}=\frac{1}{x_1}[g_{11}+g_{44}(z_1-x_1)],\ g_{21}=\frac{1}{y_2}[g_{22}y_1+g_{44}(y_1z_2-y_2z_1)],\\
&g_{23}=\frac{1}{y_2}[g_{22}y_3+g_{44}(y_3z_2-y_2z_3)],\ g_{24}=\frac{1}{y_2}[g_{22}+g_{44}(z_2-y_2)].
\end{aligned}$$

Using the relations (??) w_4, w_5 and x_4, we obtain the remaining off-diagonal elements in terms of the diagonal ones (and known solutions):

$$\begin{aligned}
&g_{31}+(z_3z_4-z_1)g_{34}=g_{33}w_4,\\
&g_{32}+(z_3z_5-z_2)g_{34}=g_{33}w_5, \qquad (4.28)\\
&g_{34}=\frac{-(a_{22}+z_2a_{44})(XY)_{431}+a_{33}y_2(x_4-z_4)+a_{44}y_2(XZ)_{431}}{y_2[x_3x_4-x_1-z_3z_4+z_1]}.
\end{aligned}$$

The expressions $(XY)_{431}$ and $(XZ)_{431}$ are defined as in eq. (??).

Finally using the expressions for $x_5(t)$, $y_4(t)$ and $y_5(t)$ we obtain 3 linear relations for the diagonal elements g_{ii}, making it possible to express g_{11}, g_{22} and g_{33} linearly in terms of g_{44}:

$$\begin{aligned}
&a_{22}\Big\{(x_3x_5-x_2-z_3z_5+z_2)(XY)_{431}-(x_3x_4-x_1-z_3z_4+z_1)(XY)_{532}\Big\}+\\
&+a_{33}y_2\Big\{(x_3x_5-x_2-z_3z_5+z_2)(-x_4+z_4)+(x_3x_4-x_1-z_3z_4+z_1)(x_5-z_5)\Big\}+\\
&+a_{44}\Big\{(x_3x_5-x_2-z_3z_5+z_2)[-y_2(XZ)_{431}+z_2(XY)_{431}]+\\
&+(x_3x_4-x_1-z_3z_4+z_1)[y_2(XZ)_{532}-z_2(XY)_{532}]\Big\}=0, \qquad (4.29)\\
&-a_{11}\alpha y_2(x_3x_4-x_1-z_3z_4+z_1)(YX)_{431}+a_{22}x_1(y_3y_4-y_1-z_3z_4+z_1)(XY)_{431}+\\
&+a_{33}x_1y_2[(y_4-z_4)(x_3x_4-x_1-z_3z_4+z_1)-(x_4-z_4)(y_3y_4-y_1-z_3z_4+z_1)]+\\
&+a_{44}\Big\{\alpha y_2(x_3x_4-x_1-z_3z_4+z_1)[-z_1(YX)_{431}+x_1(YZ)_{431}]-\\
&-x_1(y_3y_4-y_1-z_3z_4+z_1)[y_2(XZ)_{431}-z_2(XY)_{431}]\Big\}=0, \qquad (4.30)\\
&-a_{11}\alpha y_2(x_3x_4-x_1-z_3z_4+z_1)(YX)_{532}+a_{22}x_1(y_3y_5-y_2-z_3z_5+z_2)(XY)_{431}+\\
&+a_{33}x_1y_2[(x_3x_4-x_1-z_3z_4+z_1)(y_5-z_5)-(y_3y_5-y_2-z_3z_5+z_2)(x_4-z_4)]+\\
&+a_{44}\Big\{\alpha y_2(x_3x_4-x_1-z_3z_4+z_1)[-z_1(YX)_{532}+x_1(YZ)_{532}]-\\
&-x_1(y_3y_5-y_2-z_3z_5+z_2)[y_2(XZ)_{431}-z_2(XY)_{431}]\Big\}=0. \qquad (4.31)
\end{aligned}$$

Again, we sum up the results as a theorem.

Theorem 4. The general solution of the equations (??) based on the imprimitive action of SL(4, ℂ) can be expressed in terms of 3 generically chosen particular solutions $\vec{x}(t)$, $\vec{y}(t)$, $\vec{z}(t)$. The general solution $u_a(t)$, $a = 1, \ldots, 5$ is given by eq. (??), where $z_1, \ldots, z_5$ are constants representing the initial conditions for $\vec{u}_a(t)$. The matrix elements $g_{ik}(t)$ are expressed in eq. (??), ..., (??) in terms of 3 solutions with initial conditions (??), satisfying $\alpha(\alpha-1) \neq 0$, α finite. □

Comments. 1. As in the case of SL(3, ℂ) the reconstruction of $g_{ik}(t)$ is linear.

2. In the primitive case we need 5 solutions for SL(4, ℂ), in the imprimitive case only 3.

IV.4. The group SL(N, ℂ) for $N \geqq 2$

Let us sum up without proof the main results valid for all N.

Theorem 5. 1. The nonlinear ODEs with superposition formulas, based on the action of SL(N, ℂ) on the space SL(N, ℂ)/$P(N)$ are given for all $N \geqq 3$ in eq. (??) and (??). The general form of the solution is given by eq. (??).

2. The number of equations for SL(N, ℂ) is $n = 2N - 3$. The group elements $g_{ij}(t)$ can be reconstructed from m particular generically chosen solutions with

$$m = \left\langle \begin{matrix} k+2 & \text{for } N = 2k+1 \\ k+1 & \text{for } N = 2k \end{matrix} \right. . \tag{4.32}$$

3. Such a fundamental set of m solutions τ satisfies constraints with

$$\tau = mn - N^2 + 1 = \left\langle \begin{matrix} 3k-2 & \text{for } N = 2k+1 \\ k-2 & \text{for } N = 2k \end{matrix} \right. . \tag{4.33}$$

4. The reconstruction of the group action is linear in the sense that it requires the solution of $2N - 3$ linear algebraic equations.

V. Conclusions

We mentioned in the Introduction that nonlinear ordinary equations with superposition formulas occur in soliton theory as Bäcklund transformations. For equations in the AKNS family[??] the underlying Lie algebra is sl(2, ℝ) and hence the Bäcklund transformations are essentially Riccati equations.

As an example consider the sine-Gordon equation and the Bäcklund transformation, relating two solutions, z_1 and z_2:

$$z_{i,xy} = \sin z_i, \quad i = 1, 2 \tag{5.1}$$

$$\begin{aligned} z_{1x} - z_{2x} &= 2a \sin \frac{z_1 + z_2}{2} \\ z_{1y} + z_{2y} &= \frac{2}{a} \sin \frac{z_1 - z_2}{2} \end{aligned} \tag{5.2}$$

The point transformation

$$u_i = \tan \frac{z_i}{4}, \quad i = 1, 2 \tag{5.3}$$

takes the above system into:

$$u_{i,xy} = \frac{u_i}{1 + u_i^2}(2u_{i,x}^2 - 1 + u_i^2) \tag{5.4}$$

$$\begin{aligned} u_{1,x} &= \frac{1}{1+u_2^2}[au_2 + u_{2,x} + a(1-u_2^2)u_1 + (u_{2,x} - au_2)u_1^2] \\ u_{1,y} &= \frac{1}{1+u_2^2}\Big[-\frac{u_2}{a} - u_{2,y} + \frac{1}{a}(1-u_2^2)u_1 + \Big(\frac{u_2}{a} - u_{2,y}\Big)u_1^2\Big] \end{aligned} \tag{5.5}$$

Eq. (??) are Riccati equations for u_1, once u_2 is given.

Wahlquist and Estabrook have proposed[??] a method for finding Bäcklund transformations for a given nonlinear partial differential equation. They introduced a prolongation structure, essentially an additional system of matrix equations. The compatibility conditions are solved by requiring that the right-hand sides of these equations lie in a finite-dimensional Lie algebra. This is the same condition that is required in Lie's theorem on ODEs with superposition formulas.

For integrable multifield equations the Bäcklund transformations are based on other Lie algebras and groups. Thus, for n-dimensional generalizations of the sine-Gordon equation and also the wave equation[??],[??] the Bäcklund transformations are matrix Riccati equations. Similarly, for Toda field theories (two-dimensional generalized Toda lattices) Bäcklund transformations are given[??],[??] that can be transformed into projective Riccati equations.[??] The Backlund transformations for nonlinear σ-models are again various types of matrix Riccati equations.[??],[??],[??]

All the above examples, and all other Bäcklund transformations for soliton equations that we are aware of, share a common feature. Namely, they have the form of nonlinear ODEs with superposition formulas based on transitive and primitive group actions.

The equations presented in this article correspond to imprimitive actions. We have a group G, in this article $\mathrm{SL}(N,\mathbb{C})$. We have a chain of subgroups $G_1 \subset G_2 \subset G_3 \subset \cdots \subset G_{n-1} \subset G_n \equiv G$, where G_{n-1} is maximal in G. Correspondingly, we have a flag of subspaces $G/G_n \subset G/G_{n-1} \subset \cdots \subset G/G_1$. The action of G on G/G_{n-1} is primitive on the other spaces not.

The integrable systems discussed above "live" either on Lie groups, like the σ-model, or on the Grassmannians, on which the group acts primitively. Now, there also exist integrable systems on flag manifolds.[??],[??] While the manifolds, in particular Grassmannians involved are a priori, infinite-dimensional, various reduction schemes lead to finite-dimensional ones.

It is in this direction that we hope that the equations obtained in this article will appear as Bäcklund transformations.

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