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Contact geometry and linear differential equations

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Introduction

The aim of this paper is to present a method for studying differential equations based on contact geometry of the phase space. Let us clarify what we have in mind. The role played by symplectic geometry in the study of differential equations is well known. This refers first of all to the construction of (global) asymptotic expansions of solutions of equations with a small (large) parameter by the canonical operator method due to Maslov (see [1]–[4] and elsewhere). The same method can be used to obtain expansions of solutions of differential equations in smoothness, which, in turn, enables us to prove results on the solubility of equations in function spaces having to do with smoothness. From the point of view of geometry, this case is substantially different from the previous one: the principal object here is the *homogeneous* phase space, as are, actually, all the objects it contains: homogeneous Lagrangian manifolds, the "homogeneous" canonical Maslov operator and Fourier–Maslov integral operators (FMIO's), and so on. It is quite possible to construct a theory in this framework (see, for example, [5]–[10] and references contained therein), which, of course, is quite rigorous.

However, in reality a more appropriate geometrical apparatus is provided by the contact geometry(1) [11], [12] of a certain homogeneous space (that is, one obtained by factorizing a group action). In the study of differential equations one takes as the space to be factorized the cotangent bundle without the zero section, while the group is the group $\mathbb{R}_*$ of non-zero real numbers. The homogeneous space thus obtained is equipped with a natural contact structure, and this is the framework in which the theory is developed. Probably it is worth stressing at this point that factorization uses specifically the group $\mathbb{R}_*$ (and not the group $\mathbb{R}_+$ of positive real numbers), which leads to a classification of kernels of Fourier–Maslov integral operators that is more subtle than the standard classification obtained using the amplitude order (see, [5] for example). Let us explain this in more detail. Generically the kernel of a FMIO is locally a generalized function whose singularities lie on a certain submanifold S with singularities of codimension 1 (a so-called caustic). The singularities can be of two different types (for example, for distributions of order of homogeneity -1 it is $\delta(x)$ and v.p. $1/x$), where x is the coordinate transverse to S. Factorization of the phase space with respect to the group $\mathbb{R}_*$ leads (locally) to the construction of two different types of FMIO, which correspond to the usual homogeneity and to "sgn-homogeneity".(2) Here each type of homogeneity corresponds to a type of singularity of the kernel. When given as initial conditions, these singularities preserve their form up to the caustic and, which is the main point, can be represented as an image under the action of our FMIO.

On the other hand, on passing through a caustic the type of singularity can change, namely, one type of singularity can be transformed into another; this phenomenon is known under the name of discontinuity

(1) The naturalness of specifically contact geometry in such questions already follows from the definition of an operator of principal type (one of the main objects of study in this kind of question) as an operator, the principal symbol of which has no contact stationary points.

(2) We call a function $f(x)$ homogeneous if $f(\lambda x) = \lambda^s f(x)$ and sgn-homogeneous if $f(\lambda x) = \lambda^s \operatorname{sgn} \lambda f(x)$ $\forall \lambda \in \mathbb{R}_*$ and some $s \in \mathbb{C}$.

metamorphosis (see [1], [3]). And again the solution can be written down in terms of a FMIO (possibly of a different type). At the same time, the standard representation of a FMIO based on positive homogeneity (see, for example, [5]) does not capture these important effects, since in both cases the solution is expressed in terms of a Fourier integral operator of only one type. The situation in the study of lacunae of hyperbolic equations is similar.

Furthermore, the finer level of detail described above enables us to obtain quite elegant and transparent formulae for the Fourier transforms of homogeneous functions (compare with [14], [15]), and, incidentally, to axiomatize them (see §2).

After this paper was submitted, there appeared a paper of Palamodov [29], which also considers, in particular, questions of contact geometry in connection with the integral Fourier transform. Even though it has certain points of contact with the topic of the present article, [29] appeals, however, to the contact space obtained by factorizing the tangent bundle over the action of the group $\mathbb{R}_+$. In particular, that paper pays no heed at all to analysis on projective spaces, which plays an important role here.

Let us survey briefly the contents of our paper.

In §1 we present certain geometrical questions having to do with symplectic and contact geometry, and also establish the so-called classification lemma, which deals with the one-to-one correspondence between Lagrange (Legendre) manifolds and defining functions. In addition, we briefly review the main facts concerning the integration over real projective spaces of homogeneous forms. The formulae presented here form the technical apparatus needed for the subsequent analysis.

In §2 we develop the Fourier transform for homogeneous functions and present the main formulae in terms of integrals over projective spaces. The third section is an exposition of Fourier–Maslov integral operators in homogeneous (in the above sense) situations. This section contains the construction of FMIO's and the main composition and commutation theorems.

In §4 we apply this theory to problems of discontinuity propagation (metamorphosis) and to the study of lacunae of Green's function.

An important application of contact geometry is to be found in theorems dealing with the microlocal classification of Hamiltonians (pseudodifferential operators). These questions are considered in §5. The main points here are classification theorems for PDO's in a neighbourhood of a stationary point of the contact vector field and presentation of the corresponding normal forms of orbits under the action of the contact group.

Finally, in §6, the techniques of §3 and the results of §5 are used to study solubility and to prove finiteness theorems for operators of principal type, that is, operators whose principal symbol has no contact stationary points, and for operators of subprincipal type whose principal symbol can have contact stationary points.

We are grateful to V.P. Maslov for his constant interest and support.

§1. Technical preliminaries

1. Symplectic and contact geometry.

Let us first give a definition of a contact structure (compare with [11], [12]). Let S be a smooth odd-dimensional manifold, $\dim S = 2n-1$, and consider the fibre bundle

$$T_0^* S/\mathbb{R}_* \to S, \qquad (1)$$

where T_0^*S is the cotangent bundle T^*S to S without the zero section, and the action of the group $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$ on the fibre of T_0^*S is defined in the usual way. We note that since $T_0^*S \to T_0^*S/\mathbb{R}_*$ is a locally trivial bundle, each of its sections is locally covered by some differential 1-form.

Definition 1. A *contact structure* on a manifold S is a section α^* of the bundle (1) which is non-degenerate in the following sense: if α is a 1-form that covers α^*, then the form $d\alpha|_{\ker\alpha}$ is non-degenerate.

Let us now define a homogeneous symplectic structure. Let T be an even-dimensional smooth manifold, $\dim T = 2n$, on which the group $\mathbb{R}_*$ acts freely in such a way that the orbit space $S = T/\mathbb{R}_*$ admits the structure of a smooth manifold and the canonical projection is a smooth mapping. Let us denote the action of $\lambda \in \mathbb{R}_*$ on T by $\lambda \cdot t = F_t(\lambda) = F_\lambda(t)$; $F_t : \mathbb{R}_* \to T$, $F_\lambda : T \to T$.

Definition 2. A *homogeneous symplectic structure* on the manifold T is a non-degenerate 2-form ω on $t \in T$ that satisfies the condition

$$F_\lambda^*(\omega) = \lambda\omega.$$

Let $d/d\lambda$ be the radial vector field defined by the action of the group $\mathbb{R}_*$ on T. Let $s \in S$, t be a point of T that projects onto s, $Y \in T_s(S)$, and $Y' \in T_t(T)$, such that Y' projects onto Y. Let us define a form $\widetilde{\alpha}_t$ by

$$\tilde{\alpha}_t(Y) = \omega(d/d\lambda, Y').$$

It is not hard to see that the coset α^* of the form $\widetilde{\alpha}_t$ under the mapping (1) is independent of t and defines a contact structure on S. The converse is also true.

Proposition 1. Let (T_1, ω_1) *and* (T_2, ω_2) *be two homogeneous symplectic structures. If* $T_1/\mathbb{R}_* = T_2/\mathbb{R}_* = S$ *and the contact structures on* S *determined by the forms* ω_1 and ω_2 *coincide, then there exists a diffeomorphism from* T_1 *to* T_2 *that maps* ω_1 *into* ω_2 *and acts as a shift on* $\mathbb{R}_*$.

The direct product $T_1 \times T_2$ of two homogeneous symplectic manifolds is a homogeneous symplectic manifold with form $\omega_1 - \omega_2$. On the other hand, the product $S_1 \times S_2$ of two contact manifolds does not admit a contact structure (it is even-dimensional). We define the *contact product* $S_1 \underset{c}{\times} S_2$ by

$$S_1 \underset{c}{\times} S_2 = T_1 \times T_2/\mathbb{R}_*$$

(with the corresponding contact structure), where $T_1 \to S_1$, $T_2 \to S_2$ are symplectic coverings. We have the naturally defined mappings

$$\pi_i^* : S_1 \underset{c}{\times} S_2 \to S_i \quad (i = 1, 2); \qquad \Delta^* : S \to S_1 \underset{c}{\times} S_2$$

(projections and the diagonal mapping).

Let us discuss the concepts of symplectic and contact diffeomorphisms, vector fields, and the corresponding Hamiltonians.

Definition 3. A *homogeneous symplectic diffeomorphism* is a diffeomorphism $G : T_1 \to T_2$ such that $G^*(\omega_2) = \omega_1$ and $F_\lambda \circ G = G \circ F_\lambda$.

Definition 4. A *contact diffeomorphism* $g : S_1 \to S_2$ is a diffeomorphism for which $g^*(\alpha_2^*) = \alpha_1^*$.

We have the following statement.

***Proposition* 2.** *Any homogeneous symplectic diffeomorphism defines a contact diffeomorphism. Conversely, every contact diffeomorphism uniquely determines the homogeneous symplectic diffeomorphism covering it.*

If g is a contact diffeomorphism, G is a homogeneous symplectic diffeomorphism covering it, and $i_G : T \to T \times T$, $i_G(t) = (t, G(t))$, then the diagram

$$\begin{array}{ccc} T & \xrightarrow{i_G} & T \times T \\ \downarrow & & \downarrow \\ S & \xrightarrow{i_g^*} & S \underset{c}{\times} S \end{array} \tag{2}$$

uniquely determines the inclusion map i_g^*.

A vector field X' on a homogeneous symplectic space T is called *Hamiltonian* (more precisely, *homogeneous Hamiltonian*; homogeneity will not be mentioned explicitly in the following) if the local one-parameter group $\{G_\lambda\}$ consists of homogeneous symplectic diffeomorphisms. Similarly, a field X on a contact space S is called *a contact vector field* if the local one-parameter group of the field X consists of contact diffeomorphisms. Proposition 2 shows that the correspondence between homogeneous Hamiltonian fields on T and contact fields on the corresponding contact space S is a bijection.

Let us recall that a (local) *Hamiltonian* corresponding to a Hamiltonian field X' is a function H that satisfies the relation $dH = X' \rfloor \omega$. In the homogeneous case the Hamiltonian can be globally and uniquely determined by the formula $H = \omega(d/d\lambda, X')$. Then H is a homogeneous function of degree 1. The corresponding Hamiltonian vector field is denoted by $V(H)$, and the contact one by X_H.

Let us also discuss here the concepts of Lagrangian and Legendre manifolds. We recall that a manifold is called *Lagrangian* if $\omega|_L = 0$. A submanifold $l \subset S$ is called *Legendre* if $\alpha^*|_l = 0$.

***Proposition* 3.** *Any Lagrangian manifold $L \subset T$ that is invariant with respect to the action of $\mathbb{R}_*$ defines a Legendre manifold $l \subset S$ and vice versa.*

A typical example of a homogeneous symplectic manifold and of the corresponding contact manifold is the cotangent bundle T_0^*M to a manifold M, taken without the zero section, and the corresponding projective contact bundle $S^*M = T_0^*M/\mathbb{R}_*$. The latter can be represented as the manifold of contact elements of the manifold M, that is, as the manifold of pairs (x_0, L), where $L \subset T_{x_0}(M)$ is a hyperplane in the tangent space to M at the point x_0 (see Arnol'd [11]).

In conclusion, let us consider a method of setting up a Lagrangian (or the corresponding Legendre) manifold using a defining function.

Let $\Phi(x, \theta) = \Phi(x^1, ..., x^n; \theta_0, \theta_1, ..., \theta_m) \in O_0^1(U)$ be a C^∞-function defined in a neighbourhood U of a point (x_0, θ^0), $\theta^0 \neq 0$, that satisfies the following conditions:

1°. $\Phi(x, \lambda\theta) = \lambda\Phi(x, \theta)$, $\lambda \in \mathbb{R}_*$;

2°. $d\Phi(x, \theta) \neq 0$ on $\{(x, \theta) \mid \Phi(x, \theta) = 0\}$;

3°. $d\Phi_{\theta_0}, ..., d\Phi_{\theta_m}$ are linearly independent on $C_\Phi = \{(x, \theta) \mid \Phi_{\theta_0} = ... = \Phi_{\theta_m} = 0\}$.

Here subscripts indicate the corresponding partial derivatives, that is, $\Phi_{\theta_i} = \partial\Phi/\partial\theta_i$. Let us consider the mapping $\alpha : \mathbb{R}_x^n \times \mathbb{R}_\theta^m \to T^*(\mathbb{R}_x^n)$ defined by

$$\alpha(x, \theta) = (x, \Phi_x(x, \theta)). \tag{3}$$

***Lemma* 1.** *The mapping $\alpha|_{C_\Phi} : C_\Phi \to T^*(\mathbb{R}_x^n)$ is an immersion. The image $\alpha(C_\Phi)$ is a homogeneous Lagrangian manifold in $T^*(\mathbb{R}^n)$.*

We shall denote the Lagrangian manifold constructed from the function Φ by $L(\Phi)$, and call Φ the *defining* function of $L(\Phi)$. Let us note that (if necessary, renumbering the variables and making the neighbourhood U smaller) we can take $\theta_0 \neq 0$ in U; then the function $\Phi(x, \theta)$ is uniquely determined by the function $\Phi_1(x, \theta') = \Phi_1(x, \theta_1, ..., \theta_m) = \Phi(x, 1, \theta')$. We shall also denote the manifold $L(\Phi)$ by $L(\Phi_1)$.

Definition 5. Functions $\Phi_1'(x, \theta')$ and $\Phi_1''(x, \eta')$ will be called *directly equivalent* if they are connected by one of the three relations:

A. $\eta' = \theta'$ and there exists a function $\chi(x, \theta') \neq 0$ in U such that $\Phi_1'(x, \theta') = \chi(x, \theta')\Phi_1'(x, \theta')$;

B. there is a smooth change of variables $\eta' = \eta'(x, \theta')$ such that $\Phi_1'(x, \theta') = \Phi_1''(x, \eta'(x, \theta'))$;

C. $\theta' = (\theta_1, ..., \theta_m)$, $\eta' = \eta'(\theta', \theta_{m+1})$ and $\Phi_1''(x, \eta') = \Phi_1'(x, \theta') \pm \theta_{m+1}^2$.

Functions $\Phi_1'(x, \theta')$ and $\Phi_1''(x, \eta')$ will be called *equivalent* if they can be connected by a chain of directly equivalent functions.

The following lemma, which plays an important part in what follows, describes the role of Lagrangian manifolds in the classification of defining functions given in Definition 5.

***Lemma* 2** (classification).[1] *The relation* $L(\Phi_1') = L(\Phi_1'')$ *holds if and only if there exist changes of variables*

$$\theta' = \theta'(x, \tau_1, \dots, \tau_k, \zeta_1', \dots, \zeta_l') \quad \text{and} \quad \eta' = \eta'(x, \tau_1, \dots, \tau_k, \zeta_1'', \dots, \zeta_s''),$$

and functions $\chi_1(x, \theta')$, $\chi_2(x, \eta') \neq 0$ *such that*

$$\chi_1(x, \theta')\Phi_1'(x, \theta')|_{\theta'=\theta'(x,\tau,\zeta')} = \Phi(x, \tau) \pm (\zeta_1')^2 \pm \cdots \pm (\zeta_l')^2, \tag{4}$$

$$\chi_2(x, \eta')\Phi_1''(x, \eta')|_{\eta'=\eta'(x,\tau,\zeta'')} = \Phi(x, \tau) \pm (\zeta_1'')^2 \pm \cdots \pm (\zeta_s'')^2, \tag{5}$$

and these changes of variables can be chosen in such a way that the numbers of squares with a negative coefficient in the right-hand sides of (4), (5) *are the same as the negative inertia indices of the matrices* $\partial^2\Phi'/\partial\theta'\partial\theta'$, $\partial^2\Phi''/\partial\eta'\partial\eta'$, *respectively.*

2. Integral calculus on $\mathbb{RP}^n$.

Much of the analysis below will make substantial use of integration in the projective spaces $\mathbb{RP}^n$. Since for even n these spaces are non-orientable, we shall need the concept of odd forms. In view of this, we present a brief review of the theory of odd forms (for more details see de Rham [17]).

An *odd form* $\omega \in \Lambda_1^k(M)$ of degree k is an object whose local expression in the local coordinates $(U, x^1, ..., x^n)$ on a manifold M is

$$\omega = \sum_{i_1,\dots,i_k} a_{i_1\dots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the coefficients $a_{i_1\dots i_k}(x) \in C^\infty(U)$ transform under changes of coordinates as follows: the coefficients $a_{i_1\dots i_k}(x)$ are related to the coefficients $b_{j_1\dots j_k}(y)$ in the local coordinates $(V, y^1, ..., y^n)$ on the intersection $U \cap V$ by

$$b_{j_1\dots j_k}(y) = \sum_{i_1,\dots,i_k} a_{i_1\dots i_k}(x(y)) \det \frac{\partial(x^{i_1}, \dots, x^{i_k})}{\partial(y^{j_1}, \dots, y^{j_k})} \operatorname{sgn} \det \frac{\partial x}{\partial y}. \tag{6}$$

The space of usual (even) forms we shall denote here by $\Lambda_0^k(M)$. Let us note that for odd forms we can define the concept of the exterior differential, which is also an odd form, and of the exterior product operation. Then the parity of the exterior product is expressed in terms of the parities of the components of the product by the usual rule of signs.

Let us indicate an important particular case. The space $\Lambda_0^1(M)$ is called the *space of pseudoscalars*. It is not hard to see that defining an orientation is equivalent to defining a pseudoscalar ε satisfying $\varepsilon^2 = 1$. Therefore an orientation ε on M induces an isomorphism between the spaces $\Lambda_1^k(M)$ and $\Lambda_0^k(M)$ of odd and even forms on M by the relation $\omega \to \varepsilon\omega$.

Now let $f : M \to N$ be a C^∞-mapping of manifolds. A mapping f is called *oriented* if for any pair $U \subset M$, $V \subset N$ of oriented open sets such that

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[1] The proof of this lemma can be found in Hörmander [19].

$f(U) \subset V$ there is a correspondence between the orientations ε_U and ε_V of these sets that is compatible with respect to intersections. If f is oriented, the formula

$$f^*(\omega) = \varepsilon_U f^*(\varepsilon_V \omega)$$

(ε_U and ε_V are associated under the orientation of f) defines the induced mapping $f^* : \Lambda_1^k(N) \to \Lambda_1^k(M)$.

The law of transformation (6) of coefficients of an odd form enables us to define an integral of an odd form of maximal degree with compact support over a manifold M.

To integrate forms of arbitrary degree on a manifold M, the concepts of odd and even chains on M of dimension k ($0 \leqslant k \leqslant n$) are introduced. Namely, an *odd singular simplex of dimension* k is a C^∞ mapping σ of an oriented standard simplex (Δ^k, ε) into the manifold M, $\sigma : (\Delta^k, \varepsilon) \to M$. An *even singular simplex of dimension* k is an oriented mapping σ of the standard simplex Δ^k into the manifold M, $\sigma : \Delta^k \to M$. An *odd* (*even*) *chain on* M *of dimension* k is a formal finite linear combination of odd (even) singular complexes with integer coefficients. Obviously, the set $C_\sigma^k(M)$, $\sigma = 0, 1$, of chains on M forms a group under addition.

Next, an integral of an even form ω of degree k over a singular simplex $\sigma : (\Delta^k, \varepsilon) \to M$ is defined by the formula

$$\int_\sigma \omega = \int_{\Delta^k} \varepsilon \sigma^*(\omega),$$

while an integral of an odd form ω over an even simplex $\sigma : \Delta^k \to M$ (the mapping σ is oriented!) is defined by the formula

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^*(\omega).$$

The concept of integral extends by linearity to the groups $C_\sigma^k(M)$. Note that only integrals of even forms over odd chains and of odd forms over even chains are defined!

We define the boundary operator $\partial : C_\sigma^k(M) \to C_\sigma^{k-1}(M)$ in the spaces $C_\sigma^k(M)$ in the usual way. Stokes' theorem is still valid:

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Now let $\mathbb{R}^{n+1}$ be the space with coordinates $x^0, \ldots, x^n$, $\mathbb{R}_*^{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$, and $\mathbb{RP}^n$ projective space of dimension n. Let $(x^0 : x^1 : \ldots : x^n)$ be the homogeneous coordinates in $\mathbb{RP}^n$, and $U_i = \{x^i \neq 0\} \subset \mathbb{RP}^n$ affine coordinate maps with coordinates $\widehat{x^i} = (x^0, \ldots, \widehat{x}^i, \ldots, x^n)$ (the hat here and below means that the object beneath is omitted). In the following we shall use the usual summation conventions.

We introduce forms on $\mathbb{R}^{n+1}$ (here $\rfloor$ is the inner product (substitution) operation; see Sternberg [18]) as follows:

$$
\begin{aligned}
dx &= dx^0 \wedge \cdots \wedge dx^n, \\
dx^{\hat{i}} &= \frac{\partial}{\partial x^i} \rfloor dx = (-1)^i dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \\
dx^{\widehat{ij}} &= \frac{\partial}{\partial x^j} \rfloor \left(\frac{\partial}{\partial x^i} \rfloor dx \right) \\
&= \begin{cases} (-1)^{i+j-1} dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n, & i < j, \\ 0, & i = j, \\ (-1)^{i+j} dx^0 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n, & i > j. \end{cases}
\end{aligned}
$$

Observe that in the forms $dx^{\hat{i}}$ and $dx^{\widehat{ij}}$ the indices are lower ones!

We shall denote by $d/d\lambda$ the radial vector field $x^i\partial/\partial x^i$ induced by the action of the group $\mathbb{R}_*$ on $\mathbb{R}_*^{n+1}$. Note that the projection π of the fibre bundle $\pi : \mathbb{R}_*^{n+1} \to \mathbb{RP}^n$ has the natural orientation determined by that of the fibre defined by the vector field $d/d\lambda$.

Let us introduce Leray forms in $\mathbb{R}_*^{n+1}$, which play an important part in analysis on $\mathbb{RP}^n$:

$$\omega = x^j dx^{\hat{j}} \in \Lambda_\sigma^n(\mathbb{R}_*^{n+1}), \tag{7}$$

$$\omega_j = x^k dx^{\widehat{jk}} \in \Lambda_\sigma^{n-1}(\mathbb{R}_*^{n+1}). \tag{8}$$

Since $\mathbb{R}_*^{n+1}$ has the standard orientation, Leray forms can be regarded as both odd and even forms. We shall consider the following two spaces of homogeneous functions ($k \in \mathbb{Z}$, $\sigma \in \{0, 1\}$):

the space $O_0^k(\mathbb{R}_*^{n+1})$ consisting of C^∞-functions on $\mathbb{R}_*^{n+1}$ that are homogeneous of degree k with respect to the action of the group $\mathbb{R}_*$ given by $\lambda(x^0, \ldots, x^n) = (\lambda x^0, \ldots, \lambda x^n)$:

$$f(\lambda x^0, \ldots, \lambda x^n) = \lambda^k f(x^0, \ldots, x^n)$$

for all $f \in O_0^k(\mathbb{R}_*^{n+1})$ and any number $\lambda \neq 0$;

the space $O_1^k(\mathbb{R}_*^{n+1})$ consisting of C^∞ functions on $\mathbb{R}_*^{n+1}$ that are anti-homogeneous of degree k, that is:

$$f(\lambda x^0, \ldots, \lambda x^n) = \operatorname{sgn} \lambda \cdot \lambda^k f(x^0, \ldots, x^n)$$

for all $f \in O_1^k(\mathbb{R}_*^{n+1})$ and any number $\lambda \neq 0$.

Theorem 1. *For odd n and $\sigma \in \{0, 1\}$ we have the relations*

$$\pi^*(\Lambda_\sigma^n(\mathbb{RP}^n)) = \{f\omega \mid f \in O_0^{-(n+1)}(\mathbb{R}_*^{n+1})\}, \tag{9}$$

$$\pi^*(\Lambda_\sigma^{n-1}(\mathbb{RP}^n)) = \{f^j\omega_j \mid f^j \in O_0^{-n}(\mathbb{R}_*^{n+1})\}. \tag{10}$$

For even n and $\sigma \in \{0, 1\}$ we have the relations

$$\pi^*(\Lambda^n_\sigma(\mathbb{RP}^n)) = \{f\cdot\omega \mid f \in O_\sigma^{-(n+1)}(\mathbb{R}^{n+1}_*)\}, \tag{11}$$

$$\pi^*(\Lambda^{n-1}_\sigma(\mathbb{RP}^n)) = \{f^j\omega_j \mid f^j \in O_\sigma^{-n}(\mathbb{R}^{n+1}_*)\}. \tag{12}$$

Therefore Theorem 1 shows that in the structure of the pre-images of forms there is a substantial difference between the orientable (n odd) and non-orientable (n even) cases.

Let $f \in O^k_0(\mathbb{R}^{n+1}_*)$, $g \in O_\delta^{-(n+k+1)}(\mathbb{R}^{n+1}_*)$, $\delta \equiv \sigma+n+1 (\mathrm{mod}\ 2)$. Then Theorem 1 shows that the pairing

$$O^k_\sigma(\mathbb{RP}^{n+1}_*) \times O_\delta^{-(n+k+1)}(\mathbb{R}^{n+1}_*) \to \mathbb{R}, \quad \langle f, g\rangle = \int_{\mathbb{RP}^n} fg\omega \tag{13}$$

is well defined.

***Proposition* 1.** *If f and g are such that the pairing* (13) *of f and $\partial g/\partial x^i$ is defined, then*

$$\langle f, \partial g/\partial x^i\rangle = -\langle \partial f/\partial x^i, g\rangle. \tag{14}$$

Let $A \in GL(n+1, \mathbb{R})$, $A : \mathbb{R}^{n+1}_* \to \mathbb{R}^{n+1}_*$, and A^* the induced mapping $\mathbb{RP}^n \to \mathbb{RP}^n$.

***Proposition* 2.** *We have the relation*

$$\langle A^*f, A^*g\rangle = |\det A|^{-1}\langle f, g\rangle. \tag{15}$$

Let us introduce the spaces $\tilde{O}^k_\sigma(\mathbb{R}^{n+1}_*)$:

$$\tilde{O}^k_\sigma(\mathbb{R}^{n+1}_*) = O^k_\sigma(\mathbb{R}^{n+1}_*), \qquad k > -n-1;$$

$$\tilde{O}^k_\sigma(\mathbb{R}^{n+1}_*) = \{f \in O^k_\sigma(\mathbb{R}^{n+1}_*) \mid f = \frac{\partial g^i}{\partial x^i}, g^i \in \tilde{O}^{k+1}_\sigma(\mathbb{R}^{n+1}_*)\}, \quad k \le -n-1$$

(these are spaces of functions that can be written in divergence form).

***Proposition* 3.** *For $k \leqslant -n-1$ we have the relations*

$$\tilde{O}^k_\sigma(\mathbb{R}^{n+1}_*) = \begin{cases} O^k_\sigma(\mathbb{R}^{n+1}_*), & n+\sigma \text{ even};\\ \{f \in O^k_\sigma(\mathbb{R}^{n+1}_*) \mid \langle x^\alpha, f\rangle = 0,\ |\alpha| = -(n+k+1)\}, & n+\sigma \text{ odd}.\end{cases}$$

Here $\alpha = (\alpha_0, \ldots, \alpha_n)$ is a multi-index.

3. Homogeneous and formally homogeneous generalized functions on $\mathbb{R}^{n+1}$. Let us define the following spaces of generalized functions:

$$\mathcal{D}'O^k_\sigma(\mathbb{R}^{n+1}_*) = \Big\{ f \in \mathcal{D}'(\mathbb{R}^{n+1}_*) \mid \Big(f, \varphi\Big(\frac{x}{\lambda}\Big)\Big) = \lambda^{n+k+1}(\operatorname{sgn}\lambda)^{n+\sigma+1} \times (f, \varphi), \lambda \in \mathbb{R}_*, \varphi \in C_0^\infty(\mathbb{R}^{n+1}_*)\Big\};$$

$$\mathcal{D}'O_\sigma^k(\mathbb{R}^{n+1}) = \Big\{ f \in \mathcal{D}'(\mathbb{R}^{n+1}) \mid \left(f, \varphi\left(\frac{x}{\lambda}\right)\right) = \lambda^{n+k+1}(\operatorname{sgn}\lambda)^{n+\sigma+1} \times (f,\varphi), \lambda \in \mathbb{R}_*, \varphi \in C_0^\infty(\mathbb{R}^{n+1}) \Big\};$$

$$O_\sigma^k(\mathbb{R}^{n+1}) = \{ f \in \mathcal{D}'O_\sigma^k(\mathbb{R}^{n+1}) | (f(x) \in C^\infty \text{ outside } \{0\}\}.$$

Let $\mathcal{D}'\Lambda_\sigma^k(M)$ be the space dual to $\Lambda_{1-\sigma}^{n-k}(M)$ on the n-dimensional manifold M (the space of flows of degree k and parity σ on M; see de Rham [17]).

Remark 1. Let us note that the pairing (13) can be extended to a bilinear mapping $O_\sigma^k(\mathbb{R}_*^{n+1}) \times \mathcal{D}'O_\delta^{-(n+k+1)}(\mathbb{R}_*^{n+1}) \to \mathbb{R}$ ($\sigma+\delta \equiv n+1 (\mathrm{mod}\ 2)$); with respect to this pairing the above spaces are dual to each other.

We obviously have mappings

$$\mu_{k,\sigma} : \mathcal{D}'O_\sigma^k(\mathbb{R}^{n+1}) \to \mathcal{D}'O_\sigma^k(\mathbb{R}_*^{n+1}), \tag{16}$$

$$\tilde{\mu}_{k,\sigma} : O_\sigma^k(\mathbb{R}^{n+1}) \to O_\sigma^k(\mathbb{R}_*^{n+1}). \tag{17}$$

In what follows we shall be mainly dealing with functions on $O_\sigma^k(\mathbb{R}^{n+1})$, $O_\sigma^k(\mathbb{R}_*^{n+1})$. Let us try to construct an inverse operator to the operator (17). For this purpose we introduce the operator

$$\operatorname{reg} : O_c^k(\mathbb{R}_*^{n+1}) \to \mathcal{D}'(\mathbb{R}^{n+1})$$

using the formula

$$(\operatorname{reg} f, \varphi) = \int_{|x|\le 1} f(x) \left[\varphi(x) - \sum_{|\alpha|=0}^{-(n+k+1)} \frac{1}{\alpha!} x^\alpha \varphi^{(\alpha)}(0) \right] dx + \int_{|x|\ge 1} f(x) \left[\varphi(x) - \sum_{|\alpha|=0}^{-(n+k+2)} \frac{1}{\alpha!} x^\alpha \varphi^{(\alpha)}(0) \right] dx \tag{18}$$

(in the case of a negative upper limit the sum is taken to be equal to zero).

We have the following assertion.

***Proposition* 4.** *For $k > -n-1$ or for even $n+\sigma$ the operator* (17) *is invertible, and its inverse is the operator* (18). *For $k \leqslant -n-1$ and odd $n+\sigma$, the kernel of the operator* (17) *is the set of linear combinations of derivatives of δ-functions of order $-(n+k+1)$, while the image is the space $O_\sigma^k(\mathbb{R}_*^{n+1})$. On this space the operator* (18) *is the inverse of* (17) *modulo* Ker $\tilde{\mu}_{k,\sigma}$.

Proposition 4 can easily be generalized to the operator (16).

Let us also state the extension of Theorem 1 to generalized functions.

***Theorem* 2.** *For odd n and $\sigma \in \{0, 1\}$ we have the relations*

$$\pi^*(\mathcal{D}'\Lambda^n_\sigma(\mathbb{RP}^n)) = \{f\omega \mid f \in \mathcal{D}'O_0^{-(n+1)}(\mathbb{R}^{n+1}_*)\}, \tag{19}$$

$$\pi^*(\mathcal{D}'\Lambda^{n-1}_\sigma(\mathbb{RP}^n)) == \{f^j\omega_j \mid f_j \in \mathcal{D}'O_0^{-n}(\mathbb{R}^{n+1}_*)\}. \tag{20}$$

For even n and $\sigma \in \{0, 1\}$ we have the relations

$$\pi^*(\mathcal{D}'\Lambda^n_\sigma(\mathbb{RP}^n)) = \{f\cdot\omega \mid f \in \mathcal{D}'O_\sigma^{-(n+1)}(\mathbb{R}^{n+1}_*)\}, \tag{21}$$

$$\pi^*(\mathcal{D}'\Lambda^{n-1}_\sigma(\mathbb{RP}^n)) = \{f^j\cdot\omega_j \mid f_j \in \mathcal{D}'O_\sigma^{-n}(\mathbb{R}^{n+1}_*)\}. \tag{22}$$

Let us note that the operations π^* are defined on generalized functions and flows since $\pi : \mathbb{R}^{n+1}_* \to \mathbb{RP}^n$ is a fibre bundle with the natural fibre orientation.

To conclude this section, let us introduce the following spaces, larger (in general) than $O^k_\sigma(\mathbb{R}^{n+1})$:

$$\overset{\circ}{O}{}^k_\sigma(\mathbb{R}^{n+1}) = \begin{cases} O^k_\sigma(\mathbb{R}^{n+1}), & n+\sigma \text{ even or } k > -n-1; \\ \{f \mid f = \operatorname{reg} g + \sum_{|\alpha|=-(n+k+1)} C_\alpha\delta^{(\alpha)}(x),\ g \in O^k_\sigma(\mathbb{R}^{n+1}_*)\}, & \\ \qquad n+\sigma \text{ odd and } k \leqslant -n-1. & \end{cases} \tag{23}$$

These spaces will play a principal part in the axiomatic description of the Fourier transform.

§2. Fourier transforms of homogeneous functions

1. Statement of the problem.

The Fourier transform of a generalized function has the following property: for any matrix $A \in GL(n+1, \mathbb{R})$ we have the relation

$$(\mathcal{F}f)(p) = |\det A|\mathcal{F}(A^*f)({}^tAp), \tag{1}$$

where $\mathcal{F}f$ is the Fourier transform of the function f, $A^*f(x) = f(Ax)$, and ${}^tA$ is the transpose of the matrix A. It turns out that the property (1) actually characterizes the transform $\mathcal{F}$: more precisely, if

$$F : \overset{\circ}{O}{}^k_\sigma(\mathbb{R}^{n+1}) \to \mathcal{D}'(\mathbb{R}_{n+1*}) \tag{2}$$

is a continuous mapping satisfying (1), then (up to a choice of constants) this mapping coincides with the projection of the Fourier transform onto the space $\mathcal{D}'(\mathbb{R}_{n+1*})$. Let us note that apart from the case $k \geqslant 0$, $\sigma = 0$, the space $\mathcal{D}'(\mathbb{R}_{n+1*})$ can be replaced by the space $\mathcal{D}'(\mathbb{R}_{n+1})$; in the case $k \geqslant 0$, $\sigma = 0$ we neglect the distributions concentrated at the origin.

In parallel with the proof of this fact, we shall obtain explicit formulae for the transform F in terms of the pairing (1.8); these will later be of importance in the construction of Fourier–Maslov integral operators.

In subsection 2 of this section we present a lemma which plays a basic role in the proof of the theorem on the Fourier transform; subsection 3 contains the statement of the main theorem of this section and a sketch of its proof.

Finally, in subsection 4 we show how the constants in the transformations (2) for different k, σ have to be coordinated in order to guarantee the usual commutation relations of the Fourier transform with the operators of differentiation and multiplication by an independent variable.

2. The main lemma.
In this subsection we formulate a lemma concerning the structure of homogeneous generalized functions satisfying the equation

$$(3)\quad Q(p, Ax) = Q({}^tAp, x)|\det A|^{\alpha}(\operatorname{sgn}\det A)^{\beta},\quad A \in GL(n+1, \mathbb{R}).$$

***Lemma* 2.1.** *Let $n > 1$ and let $Q(p, x) \in \mathcal{D}'(\mathbb{R}_{n+1*}\times\mathbb{R}_{n+1*})$ be a distribution satisfying the relation* (3) *and homogeneous in the variable x of degree k and type σ:*

$$(4)\qquad Q(p, \lambda x) = \lambda^k(\operatorname{sgn}\lambda)^{\sigma} Q(p, x).$$

Then $\alpha = \beta = 0$ and there exists a function $K(z) \in O^k_{\sigma}(\mathbb{R}^1)$ such that $Q(p, x) = K(p\cdot x)$.

The proof of this lemma is based on the following geometrical fact. If $\mu : GL(n+1, \mathbb{R}) \to \mathbb{C}_* = \mathbb{C}\setminus\{0\}$ is a continuous homomorphism of the group $GL(n+1, \mathbb{R})$ into the multiplicative group $\mathbb{C}_*$, then $\mu(A) = |\det A|^{\alpha}(\operatorname{sgn}\det A)^{\beta}$ for some $\alpha \in \mathbb{C}$, $\beta \in \{0, 1\}$.

3. Structure theorem for the Fourier transform of a homogeneous function.
***Theorem* 2.1.** *Let $F_{k,\sigma} : \overset{0}{O}{}^k_{\sigma}(\mathbb{R}^{n+1}) \to \mathcal{D}'(\mathbb{R}_{n+1*})$ be a continuous mapping satisfying the relation* (1). *Then it is given by the formulae*

$$(5)\quad (F_{k,\sigma}f)(p) = C_{k,\sigma}\int_{\mathbb{RP}^n} \delta^{(n+k)}(x\cdot p)f(x)\omega(x),\ k \geq -n,\ n+\sigma\ \textit{even};$$

$$(6)\quad (F_{k,\sigma}f)(p) = C_{k,\sigma}\int_{\mathbb{RP}^n} \text{v.p.}\frac{f(x)\omega(x)}{(x\cdot p)^{n+k+1}},\ k \geq -n,\ n+\sigma\ \textit{odd};$$

$$(7)\quad (F_{k,\sigma}f)(p) = C_{k,\sigma}\int_{\mathbb{RP}^n} \operatorname{sgn}(x\cdot p)(x\cdot p)^{-(n+k+1)}f(x)\omega(x),\ k \leq -n-1,\ n+\sigma\ \textit{even};$$

$$(8)\quad \begin{aligned}(F_{k,\sigma}f)(p) = {}& C_{k,\sigma}\int_{\mathbb{RP}^n} (x\cdot p)^{-(n+k+1)}\ln\frac{|p\cdot x|}{|x|}f(x)\omega(x)\\ &+ C^{(1)}_{k,\sigma}\int_{\mathbb{RP}^n} (x\cdot p)^{-(n+k+1)}f(x)\omega(x),\ k \leq -n-1,\ n+\sigma\ \textit{odd};\end{aligned}$$

here $f = \operatorname{reg} f|_{\mathbb{R}_*^{n+1}}$. *Also, if* $f \in O_\sigma^k(\mathbb{R}_*^{n+1})$ *and (in the case of* (8)) *satisfies the orthogonality conditions* $\langle x^\alpha, f\rangle = 0$, $|\alpha| = -(n+k+1)$, *then*

$$(F_{k,\sigma}f)(p) \in O_{(\sigma+n+1)\bmod 2}^{-(n+k+1)}(\mathbb{R}_*^{n+1}),$$

and in this case the last term in (8) *can be omitted.*

Remark. Let us observe at this point that formulae of the type (5) $-$(8) *for the Fourier transform* are well known (for example, in the paper [27] of Semyaninskii). The point of Theorem 2.1 is that *any* continuous mapping of $\overset{\circ}{O}{}_\sigma^k(\mathbb{R}^{n+1})$ into $\mathcal{D}'(\mathbb{R}_{n+1*})$ satisfying (1) is given by these formulae, that is, (1) characterizes the Fourier transform. In particular, we are *not using* the Fourier transform formula in the derivation of formulae (5)$-$(8).

Proof (*sketch*). First let $k \geqslant -n$ or $n+\sigma$ be even. Then by (1.18) $\overset{\circ}{O}{}_\sigma^k(\mathbb{R}^{n+1}) = O_\sigma^k(\mathbb{R}^{n+1})$. Using Remark 1.1, we see that in this case

$$(Ff)(p) = \int_{\mathbb{RP}^n} Q(p,x)f(x)\omega(x),$$

where $Q \in \mathcal{D}'(\mathbb{R}_{n+1} \times \mathbb{R}_*^{n+1})$ satisfies the conditions of Lemma 2.1 (this follows from the fact that F satisfies (1)). Applying this lemma, we see that

$$Ff(p) = \operatorname{reg} \int_{\mathbb{RP}^n} K(x,p)f(x)\omega(x) + T,$$

where $K(z) \in O_{(\sigma+n+1)\bmod 2}^{-(r+k+1)}(\mathbb{R}^1)$, and T is a functional concentrated at zero. Except when $k \geqslant 0$, $\sigma = 0$, it follows from homogeneity considerations that $T = 0$; in the case $k \geqslant 0$, $\sigma = 0$ the function T disappears under projection onto $\mathcal{D}'(\mathbb{R}_*^{n+1})$. It remains to note that by Theorem 1.2 we have the equality $\mathcal{D}'O_\sigma(\mathbb{R}^1) = O_k^\sigma(\mathbb{R}^1)$, and therefore $K(z) = \delta^{(n+k)}(z)$ for $k \geqslant -n$ and $n+\sigma$ even, $K(z) = \text{v.p. } z^{-(n+k+1)}$ for $k \geqslant -n$ and $n+\sigma$ odd, and $K(z) = z^{(n+k+1)} \operatorname{sgn} z$ for $k \leqslant -n-1$ and $n+\sigma$ odd. The remainder of the theorem is easily verified using (5)$-$(7).

Now let $k \leqslant -n-1$ and let $n+\sigma$ be odd. In this case for any $f \in \overset{\circ}{O}{}_\sigma^k(\mathbb{R}^{n+1})$ we have

$$f = \operatorname{reg} g + \sum_{|\alpha|=-(n+k+1)} C_\alpha \delta^{(\alpha)}(x),$$

so that

$$(9) \qquad (Ff)(p) = \int_{\mathbb{RP}^n} Q(p,x)g(x)\omega(x) + \sum_{|\alpha|=-(n+k+1)} C_\alpha \Psi_\alpha(p),$$

where $\Psi_\alpha(p) = F(\delta^{(\alpha)})(p)$, $Q \in \mathcal{D}'(\mathbb{R}_{n+1} \times \mathbb{R}_*^{n+1})$. Using the property (1), we can show that $\Psi_\alpha(p) = C_0 p^\alpha$, where C_0 is independent of α. Further, for

$C_\alpha = 0$ we have

$$(A^* f)(x) = A^*(\operatorname{reg} g)$$
$$= \operatorname{reg} A^* g + \frac{2(-1)^{n+k+1}}{|\det A|} \sum_{|\alpha|=-(n+k+1)} \frac{1}{\alpha!}\langle x^\alpha \ln \frac{|x|}{|A^{-1}x|}, g\rangle \delta^{(\alpha)}(A^{-1}x).$$

From the last equality, relation (9) and the property (1) we derive an equation for $Q(p, x)$:

$$(10)\quad Q({}^t Ap, A^{-1}x) + \frac{2C_0(-1)^{n+k+1}}{[-(n+k+1)]!}(p \cdot x)^{-n+k+1} \ln \frac{|x|}{|A^{-1}x|} = Q(p, x).$$

The general solution of this equation has the form

$$Q(p, x) = \frac{2C_0(-1)^{n+k+1}}{[-(n+k+1)]!}(p \cdot x)^{-(n+k+1)} \ln \frac{|p \cdot x|}{|x|} + C_1 (p \cdot x)^{-(n+k+1)},$$

which concludes the proof of (8). The statement on homogeneity is verified directly.

4. Commutation formulae and the choice of constants.

It is well known that the Fourier transform $\mathcal{F}$ satisfies the following commutation relations with the operators of differentiation and multiplication by an independent variable:

$$(11)\qquad \frac{\partial}{\partial p_j}(\mathcal{F} f)(p) = -i\mathcal{F}(x^j \varphi)(p), \quad j = 0, 1, \ldots, n.$$

If we demand that the transformation (2) satisfy (11), then much of the freedom in the choice of constants is lost. Let us present the results of computations for the constants $C_{k,\sigma}$.

***Theorem* 2.2.** *If, in addition to the conditions of Theorem* 2.1, *the mapping F also satisfies the commutation formulae* (11), *then the constants $C_{k,\sigma}$ in* (5)–(8) *are*

$$(12)\qquad C_{k,\sigma} = \lambda \frac{i^{n+k}}{(2\pi)^{n/2-1}}, \qquad k \geq -n, n+\sigma \text{ even};$$

$$(13)\qquad C_{k,\sigma} = \lambda \frac{i^{n+k}}{2(2\pi)^{n/2-1}[-(n+k+1)]!}, \qquad k \leq -n-1, n+\sigma \text{ even};$$

$$(14)\qquad C_{k,\sigma} = \mu \frac{2(-i)^{n+k+1}}{(2\pi)^{n/2}}(n+k)!, \qquad k \geq -n, n+\sigma \text{ odd};$$

$$(15)\qquad C_{k,\sigma} = \mu \frac{2i^{-(n+k+1)}}{(2\pi)^{n/2}} \cdot \frac{1}{[-(n+k+1)]!}, \qquad k \leq -n-1, n+\sigma \text{ odd},$$

where λ, μ are arbitrary constants.

Let us note that the constants λ, μ depend on the normalization of the Fourier transform. If this normalization is chosen so that the formulae for the direct and inverse Fourier transforms are symmetric, then $\lambda = \mu = 1$.

§3. Fourier–Maslov integral operators

1. The Maslov canonical operator.
As in §1.1, let $\Phi(x, \theta) = \Phi(x^1, ..., x^n;\ \theta_0, \theta_1, ..., \theta_m) \in O_0^1(U)$ be a C^∞-function defined in a neighbourhood U of a point (x_0, θ^0) and satisfying conditions 1–3 of §1.1. Here the spaces O_σ^k are constructed relative to the action of $\mathbb{R}_*$ used in condition 1. Let us note that in accordance with 1 the neighbourhood U can be taken to be invariant with respect to the action of the group $\mathbb{R}_*$.

Now let $a(x, \theta) \in O_\sigma^k(U)$. We set $s = k+m$ and define the local elements $F_{s,\sigma}[\Phi, a]$ by the formulae

$$(1)\quad F_{s,\sigma}[\Phi,a]=$$
$$=\begin{cases} \frac{(-1)^s\pi i}{\pi^{m/2}}\int_{\mathbb{RP}_m}\delta^{(s)}(\Phi(x,\theta))a(x,\theta)\omega(\theta), & s\geqslant 0,\ m+\sigma \text{ even};\\ \frac{s!}{\pi^{m/2}}\int_{\mathbb{RP}_m}\text{v.p.}\,\frac{a(x,\theta)\omega(\theta)}{[\Phi(x,\theta)]^{s+1}}, & s\geqslant 0,\ m+\sigma \text{ odd};\\ \frac{(-1)^s\pi i}{2\pi^{m/2}(-s-1)!}\int_{\mathbb{RP}_m}\operatorname{sgn}[\Phi(x,\theta)][\Phi(x,\theta)]^{-(s+1)}a(x,\theta)\omega(\theta), & m+\sigma \text{ even},\ s<0;\\ \frac{(-1)^s}{\pi^{m/2}(-s-1)!}\int_{\mathbb{RP}_m}[\Phi(x,\theta)]^{-(s+1)}\ln|\Phi(x,\theta)|a(x,\theta)\omega(\theta), & m+\sigma \text{ odd},\ s<0. \end{cases}$$

Let us note that the last integral is defined modulo C^∞-functions.

It can be shown that by condition 2 these formulae define a generalized function that belongs for all $\varepsilon > 0$ to the space

$$(2)\qquad F_s[\Phi,a]\in H^{-s-1/2-\varepsilon}(\mathbb{R}^n).$$

Let us now present two lemmas, which (together with the classification lemma, see §1.1) enable us to compare local elements (1) on common domains of definition. For that let us introduce in $O_0^0(U)$, a module of $O_\sigma^k(U)$, the submodule $J_\sigma^k(\Phi)$, defined by the ideal $J(\Phi) \subset O_0^0(U)$ generated by $\{\Phi_{\theta_0}, ..., \Phi_{\theta_m}\}$ (the gradient ideal).

***Lemma* 1** (on the gradient ideal). *If $a(x, \theta) \in J(\Phi)$, $a(x, \theta) = b_i(x, \theta)\Phi_{\theta_i}(x, \theta)$, then modulo C^∞ we have the congruence*

$$(3)\qquad F_{s,\sigma}[\Phi,a]\equiv F_{s-1,\sigma}[\Phi,b],\quad b=\partial b_i(x,\theta)/\partial\theta_i.$$

In addition to the inclusion (2) Lemma 1 shows, in particular, that $F_{s,\theta}[\Phi, a] \in C^\infty$ outside the projection of the set C_Φ on the space $\mathbb{R}^{n+1}$.

To state the following lemma, we note that by localization in (x, θ) the integrals in (1) can be rewritten in the affine coordinate map of the space $\mathbb{RP}^n$. Assuming for definiteness that $\theta_0 \neq 0$ in U, we have, for example, for the case $s \geqslant 0$, $m+\sigma$ even

$$(4)\qquad F_{s,\sigma}[\Phi,a]=\frac{(-1)^s\pi i}{\pi^{m/2}}\int_{\mathbb{R}_m}\delta^{(s)}(\Phi_1(x,\theta'))a_1(x,\theta')\,d\theta'.$$

In the other cases the corresponding formulae have a similar form. Here $\Phi_1(x, \theta') = \Phi(x, 1, \theta_1, ..., \theta_m)$, $a_1(x, \theta') = a(x, 1, \theta_1, ..., \theta_m)$. For the integrals (4) we shall use the notation $F_{s,\sigma}[\Phi, a] = F_{s,\sigma}[\Phi_1, a_1]$.

***Lemma* 2** (on stabilization). *We have the following congruences modulo C^∞:*

$$(5)\quad F_{s+1,\sigma}[\Phi_1(x,\theta')+\theta_{m+1}^2+\theta_{m+2}^2, a_1(x,\theta')\zeta(\theta_{m+1})\zeta(\theta_{m+2})] \equiv F_{s,\sigma}[\Phi_1(x,\theta'), a_1(x,\theta')];$$

$$(6)\quad F_{s+1,\sigma}[\Phi_1(x,\theta')+\theta_{m+1}^2-\theta_{m+2}^2, a_1(x,\theta')\zeta(\theta_{m+1})\zeta(\theta_{m+2})] \equiv iF_{s,1-\sigma}[\Phi_1(x,\theta'), a_1(x,\theta')];$$

$$(7)\quad F_{s+1,\sigma}[\Phi_1(x,\theta')-\theta_{m+1}^2-\theta_{m+2}^2, a_1(x,\theta')\zeta(\theta_{m+1})\zeta(\theta_{m+2})] \equiv -F_{s,\sigma}[\Phi_1(x,\theta'), a_1(x,\theta')].$$

Here $\zeta(\eta) \in C_0^\infty(\mathbb{R}^1)$, $\zeta(\eta) \equiv 1$ in a neighbourhood of $\eta = 0$.

Lemma 2 shows, in particular, the difference between the distributions $F_{s,\sigma}[\Phi_1, a_1]$ for $\sigma = 0$ and $\sigma = 1$. To demonstrate this difference, let us assume that $\mathrm{Hess}_{\theta'}\Phi_1(x, \theta')$ is positive definite, $k \geqslant 0$, and m is even. Then, using the Morse lemma and the stabilization lemma, we obtain

$$(8)\quad F_{s,0}[\Phi_1, a_1] = (-1)^k \pi i \delta(s(x))\varphi(x),$$

$$(9)\quad F_{s,1}[\Phi_1, a_1] = k! \cdot \mathrm{v.p.} \frac{\varphi(x)}{[s(x)]^{k+1}},$$

where $s(x) = \Phi_1|_{C_\Phi}$, $\varphi(x) = a_1|_{C_\Phi}$. The function (8) is concentrated on the manifold $s(x) = 0$, while the function (9) does not have that property. If on the other hand (under the same assumptions on Φ_1) the number m is odd, then (8), (9) take the form

$$(10)\quad F_{s,\sigma}[\Phi_1, a_1] = \Gamma(k+1/2)\chi((-1)^\sigma s(x))|s(x)|^{-(k+3/2)},$$

where $\chi(\eta)$ is the characteristic function of the ray $\mathbb{R}_+$. Formulae (8)–(10) show that the cases of $\sigma = 0$ and $\sigma = 1$, as well as the cases of even and odd m, are significantly different from the point of view of the functions defined by the integrals (1).

To compare the local elements, let us introduce the spaces ($q = \frac{1}{2}m - s$)

$$I_q(\Phi) = \{f(x) | f(x) = F_{s,\sigma}[\Phi, a] + \tilde{f}(x), \tilde{f}(x) \in C^\infty\}.$$

It is easy to see that $I_{q'}(\Phi) \subset I_q(\Phi)$ for $q' > q$, so the spaces $I_q(\Phi)$ form a descending filtration, which by (2) agrees with the scale H^t.

Let us note that in general the spaces $I_q(\Phi)$ depend on Φ, although the associated gradation $I_q(\Phi)/I_{q+1}(\Phi)$ depends only on the Lagrangian manifold $L(\Phi)$. This fact follows from the classification lemma and Lemmas 1, 2.

Now let L be a homogeneous Lagrangian manifold, and $U \subset L$ an open homogeneous set on which L is defined by the function $\Phi(x, \theta): L \cap U = L(\Phi)$.

Let α be the mapping defined in §1.1. Let us assume that there is on L a homogeneous n-form μ of degree r. Let us define the function

$$F[\Phi,\mu](x,\theta) = \frac{\alpha^*\mu \wedge d\Phi_{\theta_0} \wedge \cdots \wedge d\Phi_{\theta_m}}{dx \wedge d\theta_0 \wedge \cdots \wedge d\theta_m}. \tag{11}$$

It is assumed that $\alpha^*\mu$ is extended from C_Φ to its neighbourhoods in an arbitrary way, so that $F[\Phi, \mu]$ is defined modulo $J(\Phi)$.

We shall need to take the square root of the function (11). Observe that by disconnectedness of $\mathbb{R}_*$ the neighbourhood U decomposes into two disconnected components. There are four branches of $\sqrt{F[\Phi,\mu]}$; two of these belong to $O_0^{\frac{1}{2}(r-m-1)}$ and the other two to $O_1^{\frac{1}{2}(r-m-1)}$. We shall denote them by $\sqrt{F[\Phi,\mu]_0}$ and $\sqrt{F[\Phi,\mu]_1}$.

For each of $\sigma = 0, 1$ let us choose one of the roots and fix it.

We define a *local canonical operator* k_σ^U in the coordinate map (U, Φ) of type σ to be the mapping

$$k_\sigma^U : O_0^k(L) \to I_{-k-\frac{r-1}{2}}(\Phi), \tag{12}$$

given by

$$k_\sigma^U(\varphi) = 2^{-m/2} F_{s,\sigma}[\Phi, \varphi\{F[\Phi,\mu]\}_\sigma^{1/2}]. \tag{13}$$

We have the following assertion.

***Proposition* 1.** *Let (U_1, Φ_2), (U_2, Φ_2) be two coordinate maps on L, $U_1 \cap U_2 \neq \varnothing$. Then on $U_1 \cap U_2$ we have the congruence*

$$k_{\sigma_1}^{U_1}(\varphi) \equiv e^{i\pi d_{12}} k_{\sigma_2}^{U_2}(\varphi) \pmod{I_{-k-\frac{r-1}{2}+1}}, \tag{14}$$

where

$$\sigma_1 - \sigma_2 \equiv C_{12} = \operatorname{ind}_- \frac{\partial^2\Phi_1}{\partial\theta\,\partial\theta} - \operatorname{ind}_- \frac{\partial^2\Phi_2}{\partial\theta'\partial\theta'} \pmod 2, \tag{15}$$

$$d_{12} = \frac{1}{2\pi}[\arg F[\Phi_1,\mu] - \arg F[\Phi_2,\mu]] + \frac{1}{2}C_{12}. \tag{16}$$

Now let $\{U_\alpha, \Phi_\alpha\}$ be a locally finite covering of L by homogeneous coordinate maps with defining function Φ_α. Equations (15) and (16) define Čech cohomology classes c, $d \in H^1(L, \mathbb{Z}_2)$. It can be shown that for any manifold L the class c is trivial. We shall call a Lagrangian manifold L on which $d = 0$ *quantized*, and d will be called the *Maslov class index*. Thus, we have the following theorem.

***Theorem* 1** (on cocylicity). *On a quantized manifold there exists a collection of types $\{\sigma_\alpha\}$ and arguments* $\arg F[\Phi_\alpha, \mu]$ *such that the operators* (12) *coincide modulo $I_{-k-(r-1)/2+1}$ in the intersections of the corresponding coordinate maps.*

Thus, we can define the Maslov canonical operator

$$k_\sigma^{(L,\mu)} : O_0^k(L) \to I_{-k-\frac{r-1}{2}}(L)/I_{-k-\frac{r-1}{2}+1}(L),$$

determined in each coordinate map by the relation (13). Then, obviously, there are two ways (for connected L) to choose $\{\sigma_\alpha\}$, so there are two types of Maslov canonical operator on L.

2. Fourier–Maslov integral operators.
The analysis presented below repeats the well-known construction for defining Fourier integral operators (see [5], [8], [19], [20]) and differs from these references by using the $\mathbb{R}_*$-equivariant canonical operator introduced above (instead of the $\mathbb{R}_+$-equivariant one used in these papers). Therefore we restrict ourselves to a brief presentation, and do not stop to prove the theorems we shall formulate.

We define a *Fourier–Maslov integral operator* to be an integral operator with a canonically representable kernel, that is,

$$\Phi_\sigma^{(L,\varphi,\mu)}(f) = \frac{i^{n-1}}{(2\pi)^{n/2-1}} \int_{\mathbb{R}^n} k_\sigma^{(L,\mu)}(\varphi)(x,y)f(y)\,dy. \tag{17}$$

The set of Fourier–Maslov integral operators is too large (in particular, it includes boundary and coboundary operators), and we shall consider only some of its subsets. Namely, let $g : T^*(\mathbb{R}^n_x) \to T^*(\mathbb{R}^n_y)$ be a homogeneous canonical transformation. The projection $L \to T^*(\mathbb{R}^n_y)$ is a diffeomorphism, which enables us to regard a function $\varphi \in O_0^k(L)$ as a function on $T^*(\mathbb{R}^n_y)$ and to define a measure μ by the relation $\mu = g^*((dy \wedge dq)^n)$. The operator we obtain in this fashion will be denoted by $T_g^\sigma(\varphi)$, and the set of such operators by Op_k. The inclusion (2) gives us some a priori estimates for the operators in Op_k:

$$Tg(\varphi) : H^{k+n-1/2+\varepsilon}(\mathbb{R}^n_y) \to H^{-k-n-1/2-\varepsilon}(\mathbb{R}^n_x). \tag{18}$$

Let $T^*(\mathbb{R}^n_z) \xrightarrow{g_1} T^*(\mathbb{R}^n_y) \xrightarrow{g_2} T^*(\mathbb{R}^n_x)$ be homogeneous canonical transformations, $\varphi_1 \in O_0^{k_1}(T^*(\mathbb{R}^n_z))$, $\varphi_2 \in O_0^{k_2}(T^*(\mathbb{R}^n_y))$. We have the following assertion.

***Theorem* 2** (on commutation). *We have the congruence*

$$T_{g_2}^{\sigma_2}(\varphi_2) \circ T_{g_1}^{\sigma_1}(\varphi_1) \equiv \pm T_{g_2 \circ g_1}^{\sigma_2+\sigma_1}(\varphi_1 \cdot g_1^*(\varphi_2)) \bmod \mathrm{Op}_{k_1+k_2-1}, \tag{19}$$

where the sign depends on the choice of the arguments of measures in all the operators involved.

Next, we define a *pseudodifferential operator* to be a Fourier–Maslov integral operator with $g = \mathrm{id}$. Let us observe that in this case there exists a global defining function $\Phi = p(x-y)$ of the corresponding Lagrangian manifold. The function $\varphi(x, p)$, the amplitude of the operator $T_{\mathrm{id}}^\sigma(\sigma)$, is called in this case the symbol of the corresponding pseudodifferential operator, and the corresponding operator is denoted by $\varphi_\sigma(x, -i\partial/\partial x)$. By computing the "radial part" of the integral, which defines a classical pseudodifferential operator, it is easy to show that the class of operators introduced is included in the class of classical pseudodifferential operators (see [9]).

Theorem 2 has the following corollaries.

***Corollary* 1.** *The following congruence holds:*

$$P_{\sigma_1}(x, -i\frac{\partial}{\partial x}) \circ T_g^{\sigma_2}(\varphi) \equiv T_g^{\sigma_1+\sigma_2}(P(x,p)\cdot\varphi)\,(\operatorname{mod} \mathrm{Op}_{k_1+k_2-1}).$$

Here k_1 and k_2 are the orders of the operators $P_{\sigma_1}(x, -i\partial/\partial x)$ and $T_g^{\sigma_2}(\varphi)$, respectively.

***Corollary* 2.** *The operator*

$$T_{g^{-1}}^{\sigma_1}(1) \circ P_{\sigma_2}\left(x, -\frac{\partial}{\partial x}\right) \circ T_g^{\sigma_1}(1)$$

modulo operators of order $k-1$ is a pseudodifferential operator with principal symbol $g^*(P)$.

Let us now study the action of Fourier–Maslov integral operators on canonical distributions.

***Proposition* 2.** *We have*

$$T_g^{\sigma_1}(\varphi)\{k_{\sigma_2}^{(L,\mu)}(\psi)\} \equiv k_{\sigma_1+\sigma_2}^{(g(L),(g^{-1})^*\mu)}(\varphi|_L\cdot\psi), (\operatorname{mod} I_{-k-\frac{r-1}{2}-s}), \tag{20}$$

where s is the order of the operator $T_g^{\sigma_1}(\varphi)$.

For a pseudodifferential operator (20) becomes

$$P_{\sigma_1}(x, -i\partial/\partial x)k_\sigma^{(L,\mu)}(\varphi) \equiv k_{\sigma_1+\sigma_2}^{(L,\mu)}(P|_L\cdot\varphi). \tag{21}$$

Formulae (20) and (21) enable us to claim that the following assertions hold.

***Corollary* 3.** *A pseudodifferential operator of order r has order r in the scale $I_q(L)$ for any Lagrangian manifold L.*

***Corollary* 4.** *If the principal symbol $P(x, p)$ of a pseudodifferential operator $P_\sigma(x, -i\partial/\partial x)$ of order r vanishes on L, then this operator has order $r-1$ in the scale $I_q(L)$.*

In the situation of the last corollary, (21) can be refined. Namely, we have the following result.

***Theorem* 3** (on commutation). *If the principal symbol $H(x, p)$ of $H_\sigma(x, -i\partial/\partial x)$ vanishes on L, and μ is invariant with respect to the Hamiltonian vector field $V(H)$ defined by the symbol H, then we have the commutation formula*

$$H_{\sigma_1}(x, -i\partial/\partial x)k_{\sigma_2}^{(L,\mu)}(\varphi) \equiv k_{\sigma_1+\sigma_2}^{(L,\mu)}(\mathcal{P}\varphi), (\operatorname{mod} I_{-k-\frac{r-1}{2}-s-1}), \tag{22}$$

where

$$\mathcal{P}\varphi = V(H)\varphi - \frac{1}{2}H_{px}\varphi$$

is the transport operator.

Let us note that (22) admits a generalization to a congruence modulo arbitrarily smooth functions.

To conclude this section, let us observe that the usual estimates of Fourier–Maslov integral operators can be obtained either by the method of the auxiliary Cauchy problem (see, for example, [5]), or by computing A^*A (see [20]). Namely, we have the following theorem.

***Theorem* 4.** *Let the operator* $T_g^\sigma(\varphi) \in \mathrm{Op}_k$. *Then this operator acts on the spaces*

$$T_g^\sigma(\varphi) : H^s(\mathbb{R}_y^n) \to H^{s-k}(\mathbb{R}_x^n).$$

Let us note that the proof of Theorem 4 (by any of the methods described above) is based on the composition theorem and on the a priori estimates (18) for the operators $T_g^\sigma(\varphi)$.

§4. Examples and applications

In this section we shall consider applications of the theory of the canonical operator to three classical problems: the discontinuity propagation problem, the discontinuity metamorphosis problem, and the investigation of Green's function for the Cauchy problem.

1. Preliminary remarks.

First of all, we shall consider the relatively simple example of the Cauchy problem for hyperbolic equations:

$$\left\{\begin{array}{l} \frac{\partial^2 u}{\partial t^2} = \Delta u, \\ u|_{t=0} = \delta(x), \\ u_t|_{t=0} = 0. \end{array}\right. \tag{1}$$

Let us compare solutions obtained by $\mathbb{R}_+$-equivariant theory (as is usually done) with those obtained by $\mathbb{R}_*$-equivariant theory from the point of view of their naturalness. First let us note that the initial datum of problem (1) can be given the canonical representation

$$\delta(x) = \left(\frac{1}{2\pi}\right)^n \int e^{ixp}\, dp = k^{(L,\mu)}(\varphi),$$

where $L = \{(x, p) \mid x = 0\}$, $\mu = dp$, $\varphi = (1/2\pi)^n$. Therefore it is natural to seek solutions $u(x, t)$ in the form of an image of the canonical operator

$$u(x,t) = k^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi})$$

for some Lagrangian manifold $\mathcal{L}$ in the phase space $\mathbb{R}^{n+1}_{x,t} \times \mathbb{R}_{n+1,p,E}$. From Theorem 2 of §3 it can be seen that the manifold $\mathcal{L}$ must lie in the zero level set of the Hamiltonian $E^2 - p^2$ (and therefore $\mathcal{L}$ must be invariant with respect to the vector field $V(E^2 - p^2) = 2E\frac{\partial}{\partial t} - 2p\frac{\partial}{\partial x}$). Hence it follows that the manifold $\mathcal{L}$ can be constructed in the following way. First we lift L into the space $\mathbb{R}^{n+1}_{x,t} \times \mathbb{R}_{n+1,p,E}$ so that this lift is contained in $\{t = 0\} \cap \{E^2 - p^2 = 0\}$. As a result we obtain an initial manifold for the Hamiltonian system. Next, the manifold $\mathcal{L}$ is constructed as the phase flow of the manifold $\mathcal{L}_0$ along the trajectories of the Hamiltonian system corresponding to the Hamiltonian $E^2 - p^2$:

$$\dot{E} = 0,\ \dot{p} = 0\ ,\dot{t} = 2E,\ \dot{x} = -2p.$$

Since $\mathcal{L}_0$ can be represented as the union of two connected components

$$\mathcal{L}_0^{\pm}\{x,t,p,E \mid x = 0,\ t = 0,\ E = \pm|p|\},$$

it follows that the manifold $\mathcal{L}$ can also be represented as $\mathcal{L}^+ \cup \mathcal{L}^-$. From the point of view of $\mathbb{R}_+$-equivariant theory it is natural to represent the solution $u(x,t) = k^{(\mathcal{L},\bar{\mu})}(\widetilde{\varphi})$ as a sum

$$u(x,t) = u^+(x,t) + u^-(x,t),\quad u^{\pm}(x,t) = k^{(\mathcal{L}^{\pm},\bar{\mu})}(\tilde{\varphi}),$$

of terms corresponding to the connected components of $\mathcal{L}$.

Since equation (1) is quite simple, we can write down explicit expressions for the elements of the canonical operators appearing in the above formula. Namely,

$$\mathcal{L}^{\pm} = \{(x,t,p,E) \mid E = \pm|p|,\ x = \mp tp/|p|\}$$

are the equations of the manifolds $\mathcal{L}^{\pm}$; the corresponding actions are given by

$$S^{\pm}(p,t) = \pm t|p|.$$

The invariant measure μ has density $\pm\frac{1}{2}|p|$, while the amplitude functions depend only on the coordinate p. Therefore

$$u^{\pm}(x,t) = \int e^{\pm it|p| + ipx} \varphi_{\pm}(p)\, dp$$

with some functions $\varphi_{\pm}(p)$. These functions can easily be determined from the initial data of problem (1):

$$\varphi_{\pm}(p) = \frac{1}{2(2\pi)^n}.$$

Therefore the asymptotics of the solution of problem (1), obtained by using the $\mathbb{R}_+$-equivariant theory of the Maslov canonical operator, is given by the formula

$$u(x,t) = \frac{1}{2(2\pi)^n} \int e^{it|p| + ipx}\, dp + \frac{1}{2(2\pi)^n} \int e^{-it|p| + ipx}\, dp. \tag{2}$$

Since (1) is an equation with constant coefficients, the solution obtained is not only an asymptotic one, but also an exact solution of the problem. Formula (2) shows that the solution $u(x,\ t)$ decomposes into a sum of two functions, which correspond to the connected components of $\mathcal{L}$.

To see whether this kind of decomposition is natural, let us compute the solution (2) for $n = 1$. We have

$$u^+(x,t) = \frac{1}{2}\{\delta_-(x-t) + \delta_+(x+t)\},$$
(3)
$$u^-(x,t) = \frac{1}{2}\{\delta_-(x+t) + \delta_+(x-t)\},$$

where

$$\delta_+(x) = \left(\frac{1}{2\pi}\right)^n \int_0^\infty e^{ixp}\,dp, \quad \delta_-(x) = \left(\frac{1}{2\pi}\right)^n \int_{-\infty}^0 e^{ixp}\,dp,$$

and

(4) $$u(x,t) = \frac{1}{2}\big(\delta(x+t) + \delta(x-t)\big).$$

It can be seen that the solution (4) is concentrated on the "light cone" $x = \pm t$, although each of its components $u^\pm(x, t)$ also has support outside this cone. A similar situation also obtains for any odd number of spatial variables. Hence it follows that the decomposition of the solution (4) as the sum of functions (3) is unnatural. This is hardly surprising, since the decomposition $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$ is itself unnatural. Indeed, the principal symbol $E^2 - p^2$ is a homogeneous function with respect to the group $\mathbb{R}_*$, and the connected components $\mathcal{L}_0^\pm$ of the initial manifold $\mathcal{L}_0$ are not invariant with respect to the action of this group.

As we saw above, the $\mathbb{R}_*$-equivariant theory of the Maslov canonical operator leads to a different decomposition of the function $k^{(L,\mu)}(\varphi)$ (where L is an $\mathbb{R}_*$-equivariant manifold). This decomposition is induced by the decomposition of the amplitude φ (which can only be $\mathbb{R}_+$-homogeneous) into the sum $\varphi_o + \varphi_e$ of its odd and even components.

The decomposition $k^{(L,\mu)}(\varphi) = k^{(L,\mu)}(\varphi_o) + k^{(L,\mu)}(\varphi_e)$ is in this case a decomposition into a sum of two distributions of different types (for example, solutions of the Cauchy problem (1) with different initial data can be distributions of the form $\delta(S(x, t))$ or v.p. $\dfrac{1}{S(x,t)}$). These considerations enable us, in particular, to study the problem of discontinuity metamorphosis of the solution (on passage through a focal (caustic) point), as well as the problem of finding the lacunae of Green's function of the Cauchy problem for hyperbolic equations.

To conclude, let us observe that Gårding [28] noticed the usefulness of combining the contributions of points in $\mathcal{L}^+$ and $\mathcal{L}^-$ that correspond to antipodal points. However, in that paper he presents only a local analysis of the result of such a combination. On the other hand, using the techniques of the $\mathbb{R}_*$-equivariant version of the canonical operator method introduced above, we can also consider global questions on the nature of the singularities of the fundamental solutions of strictly hyperbolic operators (see subsection 3 below).

In particular, our constructions immediately imply the stability of type of such singularities under small deformations of the Lagrangian manifold. We will also obtain topological conditions for lacunarity of the fundamental solution (connected with the concept of sharpness of the wave front, which was introduced in [28]), which are different from the well-known Petrovskii conditions.

2. The discontinuity propagation problem.
We shall consider this problem using as an example the Cauchy problem

$$\text{(5)} \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} = A\left(x, -\frac{\partial}{\partial x}\right) u, \\ u|_{t=0} = f(x),\ u_t|_{t=0} = g(x), \end{cases}$$

where $A(x, -\partial/\partial x)$ is a second-order elliptic differential operator:

$$\text{(6)} \qquad A\left(x, -\frac{\partial}{\partial x}\right) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j}, \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2.$$

As the Cauchy data we take canonically representable functions

$$f(x) = k_\sigma^{(L,\mu)}(\varphi), \quad g(x) = k_\sigma^{(L,\mu)}(\psi),$$

where $L \subset T^*(\mathbb{R}^n)$ is a homogeneous Lagrangian manifold, μ is some homogeneous measure on it, φ, ψ are homogeneous functions on L, and $\sigma = 0, 1$.

We shall seek a solution of the problem (5) (a formally asymptotic one) in the form $u = k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi})$, where $\mathcal{L}$ is a homogeneous Lagrangian manifold in $T^*(\mathbb{R}^{n+1}_{x,t})$, $\tilde{\mu}$ is some homogeneous measure on $\mathcal{L}$, and $\tilde{\varphi}$ is a homogeneous function on $\mathcal{L}$. By Theorem 3.3, for the equation in (5) to hold we must require that

$$\text{(7)} \qquad \mathcal{L} \subset \operatorname{char} \mathcal{H} = \{(x,t,p,\tau) | \mathcal{H}(x,t,p,\tau) = \tau^2 - \sum a_{ij}p_ip_j = 0\},$$
$$\widehat{P}\tilde{\varphi} = 0,$$

where $\widehat{P}$ is the transport operator on $\mathcal{L}$. In particular, $\mathcal{L}$ is invariant with respect to the field $V(\mathcal{H})$. We shall obtain initial conditions for $\mathcal{L}, \tilde{\varphi}$ from the initial conditions in (5).

Let us consider the projection

$$\pi : T^*(\mathbb{R}^{n+1}_{x,t})|_{t=0} \to T^*(\mathbb{R}^n_x),$$
$$(x,p,\tau) \mapsto (x,p)$$

and denote by $\mathcal{L}_0$ the intersection $\pi^{-1}(L) \cap \operatorname{char} \mathcal{H}$. By condition (6) $\mathcal{L}_0$ is a smooth manifold. It is obviously isotropic in $T^*(\mathbb{R}^{n+1}_{x,t})$ and gives a two-sheeted covering of L under the projection π. Let $\mathcal{L}$ be the phase flow of $\mathcal{L}_0$ with respect to the vector field $V(\mathcal{H})$. By the construction of $\mathcal{L}$ it is clear that the first of conditions (7) holds. We shall choose the measure $\tilde{\mu}$ on $\mathcal{L}$ to be invariant with respect to $V(\mathcal{H})$ and to coincide for $t = 0$ with the measure

$\pi^*(\mu) \wedge dt$. By definition of $\mathcal{L}, \tilde{\mu}$, and the two-sheeted covering $\mathcal{L} \cap \{t = 0\} \to L$, we have

$$k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi})|_{t=0} = k_\sigma^{(L,\mu)}(\varphi_1),$$

where φ_1 is the sum of the values of $\tilde{\varphi}$ on the two sheets of the covering $\mathcal{L} \cap \{t = 0\} \to L$.

Now set $\tilde{\varphi}_1|_{t=0} = \frac{1}{2}\pi^*\varphi$, where $\tilde{\varphi}_1$ satisfies the transport equation. Then

$$\left[\frac{\partial^2}{\partial t^2} - A\left(x, -\frac{\partial}{\partial x}\right)\right] k_0^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}_1) \equiv 0,$$

and the initial conditions

$$k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}_1)|_{t=0} = \frac{1}{2}k_\sigma^{(L,\mu)}(2\varphi) = k_\sigma^{(L,\mu)}(\varphi) = f,$$
$$\frac{\partial}{\partial t}k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}_1)|_{t=0} = \frac{1}{2}k_\sigma^{(L,\mu)}(\tau_1\varphi + \tau_2\varphi) = 0$$

hold, where τ_1, τ_2 are the values of τ on the two sheets of the covering $\mathcal{L} \cap \{t = 0\} \to L$, which obviously have different signs. Therefore the function $u_1 = k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}_1)$ satisfies (5) with $f(x) = k_\sigma^{(\mathcal{L},\mu)}(\varphi)$, $g(x) = 0$.

Similarly, let us set $\tilde{\varphi}_2|_{t=0} = 1/(2\tau)\pi^*(\psi)$, and demand that $\tilde{\varphi}_2$ satisfy the transport equation. Then equation (5) is satisfied in the corresponding modulus, while the initial conditions are

$$k_\sigma^{(\mathcal{L},\tilde{\mu})}(\varphi_2)|_{t=0} = \frac{1}{2}k_\sigma^{(L,\mu)}\left(\frac{1}{\tau_1}\psi + \frac{1}{\tau_2}\psi\right) = 0,$$
$$\frac{\partial}{\partial t}k_\sigma^{(\mathcal{L},\tilde{\mu})}(\varphi_2)|_{t=0} = \frac{1}{2}k_\sigma^{(L,\mu)}(2\psi) = k_\sigma^{(L,\mu)}(\psi) = g,$$

and therefore the function $u_2 = k_\sigma^{(\mathcal{L},\widetilde{\mu})}(\tilde{\varphi}_2)$ satisfies (5) with $f(x) = 0$, $g(x) = k_\sigma^{(\mathcal{L},\mu)}(\psi)$.

So finally we have the solution of (5) in the form

$$u(x,t) = k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}_1) + k_\sigma^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}_2),$$

that is, the solution of (5) is defined by a Maslov canonical operator of the same type as the initial conditions. If the manifold L is non-singular, then for $|t| \leqslant \varepsilon(x)$ the manifold $\mathcal{L}$ will also be non-singular ($\varepsilon(x)$ is a sufficiently small positive function of x).

3. The discontinuity metamorphosis problem.

Let us continue the study of the problem from the previous subsection, taking (for definiteness) initial conditions of the form

$$f(x) = \varphi_0(x)\delta(S(x)), \quad g(x) = 0,$$

where $S(x)$ is a function for which $dS(x) \neq 0$ when $S(x) = 0$. The discontinuities of f are concentrated on the manifold $X = \{S(x) = 0\}$

and here

$$f(x) = k_0^{(L,\mu)}(\varphi_0),$$

where $L = N^*(X)$ for some choice of the measure μ. We shall assume that the manifold $\mathcal{L}$ constructed in the previous subsection is a manifold in general position and denote by $\widetilde{X}$ the projection of this manifold onto the space $[x, t]$, and $\Sigma\widetilde{X}$ will stand for the projection of the cycle of its singularities onto $[x, t]$, $\Sigma\widetilde{X} \subset \widetilde{X}$. It is clear that on the part of $\widetilde{X}\backslash\Sigma\widetilde{X}$ that has points in common with $t = 0$, in view of subsection 2 the solution has the form

$$u(x,t) = k_0^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}) = \tilde{\varphi}(x,t)\delta(S(x,t)). \tag{8}$$

But if (x_0, t_0) lies in a different component of $\widetilde{X}\backslash\Sigma\widetilde{X}$, then the type σ of the local canonical operator can change. By the cocyclicity theorem, the type σ can be determined in a neighbourhood of (x_0, t_0) in the following way. Let l be a curve on $\mathcal{L}$, the initial point of which has $t = 0$, while the terminal point projects onto (x_0, t_0), and l crosses the singularity cycle transversely. Let us compute the number

$$\operatorname{ind} l \equiv \sum_{i=1}^{k} C_{\alpha\beta}^{(i)} \pmod 2, \tag{9}$$

where the sum is determined as follows. Let $\{U_{\alpha_0}, \ldots, U_{\alpha_k}\}$ be a sequence of coordinate maps covering l, such that U_{α_0} and U_{α_k} are non-singular. Then $C_{\alpha\beta}^{(i)}$ are the values of the cochain C, computed from (3.15) for the intersection $U_{\alpha_{i-1}} \cap U_{\alpha_i}$. We shall call the number (9) the index of the curve l. If it is even, then the type σ in a neighbourhood of (x_0, t_0) is 0, but if it is odd, then $\sigma = 1$. Thus, when ind l is even, the asymptotic behaviour of $u(x, t)$ in a neighbourhood of (x_0, t_0) is given by (8), while for odd ind l it is given by the formula

$$u(x,t) = k_1^{(\mathcal{L},\tilde{\mu})}(\tilde{\varphi}) = \tilde{\varphi}(x,t)\,\mathrm{v.p.}\frac{1}{S(x,t)}.$$

Taking into account that the functions $\delta(z)$ and v.p. $\frac{1}{z}$ are related by v.p. $\frac{1}{z} = \mathcal{R}(\delta(z))$, where $\mathcal{R}$ is the Riemann–Hilbert operator, and that $\mathcal{R}$ is idempotent, we obtain the formula

$$u(x,t) = \tilde{\varphi}(x,t)\left(\mathcal{R}^{\operatorname{ind} l}\delta\right)(S(x,t)),$$

which can be applied to any non-singular point (x_0, t_0).[1]

Let us now consider more general initial conditions for the problem (5):

$$f(x) = \varphi_0(x)k(S(x)), \quad g(x) = 0, \tag{10}$$

[1] More precisely, in the right-hand side of the formula we should have a sum over all the coordinate maps of $\mathcal{L}$ that project onto a neighbourhood of the point (x_0, t_0).

where the requirements on $S(x)$ are the same as before, and $k(z)$ is a generalized function that belongs to C^∞ outside $z = 0$. Let us write the initial condition (10) in the form

$$f(x) = \int \varphi_0(x)\delta(S(x) - z)k(z)\,dz$$

(the integral is defined as a generalized function of x, since the supports of the singularities $z = S(x)$ and $z = 0$ of the factors intersect transversely). By the above argument, the solution of the Cauchy problem with initial data

$$f(x) = \varphi_0(x)\delta(S(x) - z), \quad g(x) = 0$$

(z is a parameter) has the form

$$U(x, t, z) = \tilde{\varphi}(x, t)(\mathcal{R}^{\mathrm{ind}\, l}\delta)(S(x, t) - z),$$

where $S(x, t) - z$ is a non-singular action on the manifold $\mathcal{L}_z$ constructed as in subsection 2 from the manifold $L = N^*(\{S(x) - z = 0\})$.

For (x_0, t_0), which is non-singular with respect to $\mathcal{L}_0$, this point is also non-singular for $|z| < \varepsilon$ and the index ind l does not change with z (homotopic stability). Hence it can be seen that

$$\begin{aligned} u(x,t) &= \int U(t, x, z)k(z)\,dz = \int \tilde{\varphi}(x,t)(\mathcal{R}^{\mathrm{ind}\, l}\delta)(S(x,t) - z)k(z)\,dz \\ &= \tilde{\varphi}(x,t)(\mathcal{R}^{\mathrm{ind}\, l}k)(S(x,t)). \end{aligned}$$

This also gives the formula for the metamorphosis of the discontinuity in the general case.

4. Investigation of Green's function of the Cauchy problem.
In this subsection we shall study Green's function for the Cauchy problem (5), that is, the solution of this problem with the initial data

$$f(x) = 0, \quad g(x) = \delta(x - y),$$

where $y = (y^1, \ldots, y^n)$ is an n-dimensional parameter. First of all it is necessary to represent $\delta(x-y)$ as a canonical operator. Since $\delta(x-y)$ is the kernel of a unitary PDO, for odd n we have

$$\delta(x - y) = \frac{(-1)^{n-1}}{(2\pi i)^{n-1}} \int_{[p]^*} \delta^{(n-1)}(p \cdot (x - y))\omega(p), \tag{11}$$

while for even n

$$\delta(x - y) = \frac{1}{2(2\pi i)^n} \int_{[p]^*} \mathrm{v.p.}\frac{1}{[p \cdot (x - y)]^n}\omega(p). \tag{12}$$

It is not hard to see that the integrals (11) and (12) correspond to a Lagrangian manifold $L \subset T^*(\mathbb{R}^n_x \times \mathbb{R}^n_y)$, which is the graph of the identity canonical transformation. Hence

$$\delta(x - y) = k_0^{(L,\mu)}(1),$$

where μ is the measure $(dp \wedge dx)^n$ and L has been described above. Let us consider a sufficiently small t. Then

$$S(x,y,p,t) = p \cdot (x-y) + \tau|_L t + o(t).$$

Since $\tau|_L = \pm\sqrt{\sum_{i,j=1}^n a_{ij}(x)p_i p_j}$, for small non-zero t the matrix $\|\partial^2 S/\partial p_i \partial p_j\|$, $i, j = 2, \ldots, n$, has for $p_1 = 1$ non-zero determinant and zero negative inertial index. Therefore for such t the manifold $\mathcal{L}$ is non-singular. Since the initial datum for the transport equation is $1/\tau$ and therefore the order of homogeneity of the amplitude of the canonical operator is -1, for odd n and small t we have by Lemma 3.2

$$G(x,y,t) = \tilde{\varphi}(x,t)\delta^{(\frac{n-3}{2})}(S(x,t)). \tag{13}$$

For even n the number of "removable squares" in formulae of the form (3.5)−(3.7) will be odd, and therefore these formulae are inapplicable. In this case it can be shown that

$$G(x,y,t) = \tilde{\varphi}(x,t)(S(x,t))_+^{(n-1)/2}, \tag{14}$$

where the function z_+^α equals z^α for $z > 0$ and 0 for $z < 0$. The difference between the functions (13) and (14) is that the function (13) is concentrated on the light cone, while the function (14) is not. However, (13), (14) are asymptotic, not exact, equalities.

Definition 1. We shall say that a function $f(x)$ is *concentrated on a set X modulo the space $H^s(\mathbb{R}^{n+1})$* if in the equivalence class $\{f(x) \bmod H^s(\mathbb{R}^{n+1})\}$ there exists a function with support in X.

Thus the function (13) is concentrated on the light cone (mod H^s) for $s = -(n-3)/2 + 1/2$, while the function (14) is not (for any s larger than $-n/2$).

Let us note that as a caustic is traversed, the situation changes. Namely, if ind l changes by an odd number, then passing through a caustic the solution for even n will be concentrated on the light cone, while for odd n it will not.

Let us note that similar results in terms of sharpness of the front were obtained in the work of Tvorogov [30] for quasi-hyperbolic equations. This result was obtained by using the method of paired Fourier integral operators introduced by Gårding [28].

§5. Microlocal classification of pseudodifferential operators

1. Microlocal equivalence.

Definition 1. We shall say that pseudodifferential operators $\widehat{H}_1 = H_1(x, -i\partial/\partial x)$ and $\widehat{H}_2 = H_2(x, -i\partial/\partial x)$ are *microlocally equivalent* at a point $(x_0, p_0) \in T_0^*M$ if there exists an elliptic Fourier integral operator $\widehat{\Phi}$, associated with some canonical transformation, such that in a neighbourhood of (x_0, p_0) the full symbols of the operators $\widehat{H}_2$ and $\widehat{\Phi}^{-1}\widehat{Q}\widehat{H}_1\widehat{\Phi}$ coincide; here $\widehat{Q}$ is some elliptic PDO.

Certain particular cases of the problem of microlocal classification of PDO's have been considered in a number of papers (see [23], [24], [25]). A complete solution of the problem is contained in [21] and [22].

Let us state the main results of [21], [22].

2. Elliptic operators and operators of principal type.

An elliptic operator $\widehat{H}$ is an operator whose Hamiltonian is non-zero at a point $(x_0, p_0) \in T_0^*M$:

$$H(x_0, p_0) \neq 0.$$

It is clear that any two elliptic operators are equivalent. Therefore, all elliptic operators form an orbit, as a representative of which we can take, for example, the Laplacian.

Now let $H(x_0, p_0) = 0$. We call such a point a characteristic point. In this case the main role in the classification is played by the contact vector field X_H of the Hamiltonian H. In this subsection we shall assume that at a (characteristic) point of the Hamiltonian H the contact vector field does not vanish: $X_H(x_0, p_0) \neq 0$. Such an operator is called an operator of (microlocally) principal type. It turns out ([20], [26]) that in this case also there exists precisely one orbit of the group: any two operators of principal type are microlocally equivalent. A representative of this orbit is, for example, the operator $-i\partial/\partial x^1$.

3. Operators of subprincipal type.

A substantially different picture occurs in the case when at a given (characteristic) point the contact vector field vanishes: $H(x_0, p_0) = X_H(x_0, p_0) = 0$. Now it is the next term of the Taylor expansion that starts to play a crucial role, the linear part of the contact vector field computed at the point (x_0, p_0), or more precisely its spectrum

$$\lambda_0, \lambda_1, \ldots, \lambda_{2n-1}.$$

These eigenvalues have a number of remarkable properties. First of all, one of the eigenvalues, which we shall denote by λ_0, is, up to sign, the proportionality constant between the Hamiltonian vector field[(1)] $V(H)$ and the "radial" one $\partial/\partial p$:

$$V(H)(x_0, p_0) = -\lambda_0 \frac{\partial}{\partial p}.$$

Furthermore, it can be shown that the eigenvalues can be renumbered so that we have the relation

$$\lambda_j + \lambda_{n+j-1} = \lambda_0, \quad j = 1, 2, \ldots, n-1.$$

[(1)] We take this to be non-zero at the point under consideration.

a) We shall assume initially that the following non-degeneracy (non-resonance) condition is satisfied:

$$\sum_j m_j \lambda_j \neq \lambda_0 \quad \left(m_j \geq 0, \sum_j m_j \geq 3 \right).$$

In this case two operators are equivalent if and only if the spectra of their linear parts are the same, so that an orbit is in bijective correspondence with the numbers $\lambda_0, \ldots, \lambda_n$. Moreover, the corresponding normal forms, the simplest representatives of the orbits, are exhibited [21], [22]. For example, if all the numbers λ_j, $j = 1, \ldots, n$, are real and distinct, then the operator $H(x, \widehat{p})$ can be reduced to the form

$$H(x, \widehat{p}) = -i \sum_{j=1}^{n} \lambda_{j-1} x^j \frac{\partial}{\partial x^j}.$$

In the case of complex eigenvalues λ_0, $\lambda_{1,2} = \sigma \pm i\tau$, the operator $H(x, \widehat{p})$ reduces to

$$H(x, \widehat{p}) = -i \left[\lambda_0 x^1 \frac{\partial}{\partial x^1} + \left(\sigma x^2 - \tau x^3 \right) \frac{\partial}{\partial x^2} + \left(\sigma x^3 + \tau x^2 \right) \frac{\partial}{\partial x^3} \right].$$

b) Assume now that the condition of a) does not hold, but that instead we have the weaker condition of absence of higher order resonances,

$$\sum_{j=0}^{2n-2} m_j \lambda_j \neq \lambda_0 \quad \left(m_j \geq 0 \quad \sum_j m_j \geq k+1 \right)$$

for some $k \geqslant 2$. The number k is then called the *multiplicity* of the point (x_0, p_0). In this situation the question of equivalence of operators is solved at the level of truncations of order k of the Taylor series of their Hamiltonians (jets of the field X_H of order $k-1$). In this case also the corresponding normal forms are exhibited [22].

§6. Equations of principal and subprincipal types

1. Equations of principal type.
Let $\widehat{H}$ be a pseudodifferential operator with a real symbol. In this section we shall study questions of solubility of the equation

$$\widehat{H} u = f \tag{1}$$

in the Sobolev spaces $H^s(\mathbb{R}^n)$.[(1)] Let us note that we can always take the order of the operator to be one; otherwise equation (1) can be multiplied by a suitable invertible elliptic pseudodifferential operator.

[(1)] For simplicity, we consider the equation on $\mathbb{R}^n$; extending the results to the case of equations on smooth manifolds does not present any difficulties.

Let us recall the definition of equations of principal type; here we shall distinguish the microlocal (in a neighbourhood of a point of the phase space), local (in a neighbourhood of a point on $\mathbb{R}^n_x$), and global (on an arbitrary compact set in $\mathbb{R}^n_x$) situations.

Definition 1. Equation (1) is called

1) *(microlocally) an equation of principal type in a neighbourhood of a point* $\alpha \in S^*\mathbb{R}^n$ if the contact vector field X_n corresponding to the Hamiltonian $H(x, p)$ of (1) does not vanish at α;

2) *(locally) an equation of principal type at a point* $x_0 \in \mathbb{R}^n$ if all the trajectories of the contact vector field X_n contained in the set

$$\text{char}\, H = \{\alpha \in S^*\mathbb{R}^n \,|\, H(\alpha) = 0\} \tag{2}$$

leave the fibre of the bundle $S^*\mathbb{R}^n$ lying over the point x_0 in finite time;

3) *(globally) an equation of principal type* if for any two compact sets $K_1, K_2 \subset \mathbb{R}^n$ there exists a number $T(K_1, K_2)$ such that the trajectories of the field X_n lying in char H that start above K_1 for $t = 0$ lie above the complement of the compact set K_2 for $|t| \geqslant T(K_1, K_2)$.

To each of these cases there corresponds its own concept of solubility of equation (1).

Definition 2. Equation (1) is called

1) *microlocally soluble at a point* $\alpha \in S^*\mathbb{R}^n$ if for every element $f \in H^s(\mathbb{R}^n)$ there exists a function $u \in H^s(\mathbb{R}^n)$ such that the wave front[1] $\mathrm{WF}(\widehat{H}u - f)$ of the difference $\widehat{H}u - f$ does not intersect a neighbourhood of the point α, and furthermore $\|u\|_s \leqslant C\|f\|_s$;

2) *locally soluble at a point* $x_0 \in \mathbb{R}^n$ if for every right-hand side $f \in H^s(\mathbb{R}^n)$ there exists a function $u \in H^s(\mathbb{R}^n)$ such that $\widehat{H}u - f = 0$ in a neighbourhood of the point x_0, and furthermore $\|u\|_s \leqslant C\|f\|_s$;

3) *globally soluble* if for any compact set K the kernel $N(K)$ of the operator $\widehat{H}^*$ adjoint to the operator $\widehat{H}: H^s_{\text{loc}}(K) \to H^s_{\text{loc}}(K)$ is finite-dimensional, independent of s, and for any function $f \in H^s_{\text{loc}}(K)$ orthogonal to $N(K)$ there exists a function $u \in H^s_{\text{loc}}(K)$ such that $\widehat{N}u = f$ in a neighbourhood of the compact set K; here also $\|u\|_s \leqslant C\|f\|_s$.

Here K is the closure of some domain $\overset{\circ}{K}$ and $H^s_{\text{loc}}(K)$ is the space of functions defined in some neighbourhood (depending on the function) of the compact set K and belonging to the space H^s_{loc} in that neighbourhood. Functions that are the same in $\overset{\circ}{K}$ are deemed to be the same element.

***Theorem* 1.** *If equation* (1) *is microlocally* (*respectively, locally, globally*) *of principal type, then it is microlocally* (*respectively, locally, globally*) *soluble.*

The result of Theorem 1 is well known (see, for example, the work of Egorov [10], or the book of Trèves [9]). However, we present here a proof

[1] Concerning the definition of a wave front, see [8], [9], and elsewhere.

based on $\mathbb{R}_*$-equivariant techniques of Fourier integral operators to illustrate the operators connected with a Lagrangian manifold with a boundary that arise in the process, as well as $\mathbb{R}_*$-equivariant geometric objects that arise in the construction of regularizers. We also present the scheme for constructing a *global* regularizer following Sternin [16].

2. Proof of Theorem 1.

We present here a sketch of the proof of Theorem 1. This proof relies on constructing a right regularizer for the operator $\widehat{H}$ and is based on a certain modification of Fourier–Maslov integral operators (for details see [16]).

In the bundle $S^*(\mathbb{R}^n_x \times \mathbb{R}^n_y)$ let us consider the set

$$l_0 = \Delta^* \cap \operatorname{char} \tilde{H}. \tag{3}$$

Here Δ^* has been defined in §1 as the image of the corresponding diagonal mapping, while $\widetilde{H}$ is the lift of the symbol H of the operator $\widehat{H}$ to $T^*_0\mathbb{R}^n_x \times T^*_0\mathbb{R}^n_y$ using the projection onto the first component. Let us denote by l the phase flow of the manifold (3) along trajectories of the contact vector field $X_{\tilde{H}}$. Let $\alpha \in \operatorname{char} H$ be a point on $S^*\mathbb{R}^n_x$, $\alpha = \pi_1\widetilde{\alpha}$, $\widetilde{\alpha} \in l_0$. For an equation of principal type we obviously have $X_H(\alpha) \neq 0$. Then it is not hard to see that l_0 is a submanifold in $S^*(\mathbb{R}^n \times \mathbb{R}^n)$ which is transversal to $X_{\tilde{H}}$ in a neighbourhood of the point $\widetilde{\alpha}$.

Without loss of generality we can assume that at the point $\widetilde{\alpha}$ (and therefore in some neighbourhood of that point) $p_1 > 0$. Then as coordinates on $S^*\mathbb{R}^n$ we can take the collection $(x^1, ..., x^n, p_2^*, ..., p_n^*) = (x,p)$, where $p_i^* = p_i/p_1$ (here (x, p) are the standard coordinates on $T^*\mathbb{R}^n_x$). In these coordinates, the field X_H is

$$X_H = -\left(\sum_{i=2}^{n} p_i^* H_{p_i^*}(x,1,p^*) - H(x,1,p^*)\right)\frac{\partial}{\partial x^1} + \sum_{i=2}^{n} H_{p_i^*}(x,1,p^*)\frac{\partial}{\partial x^i}$$
$$- \sum_{i=2}^{n} \left(H_{x^i}(x,1,p^*) - p_i H_{x^1}(x,1,p^*)\right)\frac{\partial}{\partial p_i^*}.$$

Since $X_H \neq 0$, we either have $H_{p^*_{i_0}}(x,1,p^*) \neq 0$ for some $i_0 \in \{2, ..., n\}$ or $(H_{x^{i_0}} - p^*_{i_0}H_{x^1})(x,1,p^*) \neq 0$, $i_0 \in \{2,..., n\}$ (the case $H_{p^*_{i_0}} = 0$, $\sum_{i=2}^n p_i^* H_{p_i^*} - H \neq 0$ is not possible since at that point $H = 0$). The first case is called non-singular and the second one singular.

On $S^*(\mathbb{R}^n_x \times \mathbb{R}^n_y)$ we shall use the system of coordinates

$$(x^1, \dots, x^n, p_1^*, \dots, p_n^*; y^1, \dots, y^n, q_2^*, \dots, q_n^*),$$

where (x, p) are the coordinates on $T^*_0\mathbb{R}^n_x$, (y, q) are the coordinates on $T^*_0\mathbb{R}^n_y$, $p_i^* = p_i/q_1$, $i = 1, ..., n$, and $q_i^* = q_i/q_1$, $i = 2, ..., n$.

In the non-singular case a system of coordinates on l in a neighbourhood of the point $\widetilde{\alpha}$ is

$$(x^1, \dots, x^n, y^{i_0}, q_2^*, \dots, \widehat{q}_{i_0}, \dots, q_n^*)$$

($q^*_{i_0}$ has been omitted). The defining function for l in this case is

$$(4) \qquad \Phi^{i_0}(x, y, q_2^*, \ldots, \widehat{q}_{i_0}, \ldots, q_n^*) = \left[y_1 + \sum_{\substack{i=2 \\ i \neq i_0}}^{n} y^i q_i^* \right] |_l - \left[y^1 + \sum_{\substack{i=2 \\ i \neq i_0}}^{n} y^i q_i^* \right].$$

In the singular case coordinates on l in a neighbourhood of the point $\widetilde{\alpha}$ are

$$(p_2^*, \ldots, p_n^*, u, q_{i_0}^*, y^2, \ldots, \widehat{y}^{i_0}, \ldots, y^n),$$

where $u = y^1 + q^*_{i_0} y^{i_0}$, while the defining function is given by the equation

(5)

$$\Phi_{i_0}(x, y, p_2^*, \ldots, p_n^*, u, q_{i_0}^*) = -\left[x^1 + \sum_{i=2}^{n} p_i^* x^i \right] |_l + (u - y^1 - q_{i_0}^* y^{i_0}) + \left(x^1 + \sum_{i=2}^{n} p_i^* x^i \right).$$

The manifold l is divided by the manifold l_0 into two parts, l_+ and l_-, according to the sign of the parameter t in the shift along trajectories of the field $X_{\tilde{H}}$. For definiteness, we shall assume that on l_+ we have the inequality $x^{i_0} > y^{i_0}$ in the non-singular case and $p^*_{i_0} > q^*_{i_0}$ in the singular case. Let us introduce operators $\widehat{\Phi}^{i_0}_{(l_+, \mu, \varphi)}$ and $\widehat{\Phi}_{i_0(l_+, \mu, \varphi)}$ as integral operators with kernels

$$(6) \quad K^{i_0}(x, y) = \begin{cases} \frac{i^n (-1)^n}{(2\pi)^n} \theta(x^{i_0} - y^{i_0}) \int \delta^{(n-2)}[\Phi^{i_0}(x, y, q_2^*, \ldots, \widehat{q}^*_{i_0}, \ldots, q_n^*)] \\ \quad \times \varphi(x, y^{i_0}, q_2^*, \ldots, \widehat{q}^*_{i_0}, \ldots, q_n^*) \sqrt{F(\Phi^{i_0}, \mu)}\, dq_2^* \ldots \widehat{dq^*_{i_0}} \ldots dq_n^* \\ \quad \text{for even } n, \\ \frac{i^n (n-2)!}{(2\pi)^n} \theta(x^{i_0} - y^{i_0}) \int \frac{\varphi(x, y^{i_0}, q_2^*, \ldots, \widehat{q}^*_{i_0}, \ldots, q_n^*)}{[\Phi^{i_0}(x, y, q_2^*, \ldots, \widehat{q}^*_{i_0}, \ldots, q_n^*)]^{n-1}} \\ \quad \times \sqrt{F(\Phi^{i_0}, \mu)}\, dq_2^* \ldots \widehat{dq^*_{i_0}} \ldots dq_n^* \\ \quad \text{for odd } n, \end{cases}$$

in the non-singular case and

$$(7) \quad K_{i_0}(x, y) = $$

$$= \begin{cases} \frac{i^n (-1)^n}{(2\pi)^n} \int \theta(p_{i_0}^* - q_{i_0}^*) \delta^{(n)}[\Phi_{i_0}(x, y, p_2^*, \ldots, p_n^*, u, q_{i_0}^*)] \\ \quad \times \varphi(p_2^*, \ldots, p_n^*, u, q_{i_0}^*, y^2, \ldots, \widehat{y}^{i_0}, \ldots, y^n) \sqrt{F(\Phi_{i_0}, \mu)}\, dp_2^* \ldots dp_n^*\, du\, dq_{i_0}^* \\ \quad \text{for even } n, \\ \frac{i^n \cdot n!}{(2\pi)^n} \int \theta(p_{i_0}^* - q_{i_0}^*) \frac{\varphi(p_2^*, \ldots, p_n^*, u, q_{i_0}^*, y^2, \ldots, \widehat{y}^{i_0}, \ldots, y^n)}{[\Phi_{i_0}(x, y, p_2^*, \ldots, p_n^*), u, q_{i_0}^*]^{n+1}} \\ \quad \times \sqrt{F(\Phi_{i_0}, \mu)}\, dp_2^* \ldots dp_n^*\, du\, dq_{i_0}^* \\ \quad \text{for odd } n, \end{cases}$$

in the singular case. Formulae (6), (7) define the operators $\widehat{\Phi}^{i_0}_{(l_+,\mu,\varphi)}$ and $\widehat{\Phi}_{i_0(l_+,\mu,\varphi)}$ only for functions φ concentrated in some neighbourhood $U(\widetilde{\alpha})$ of the point $\widetilde{\alpha}$. However, using the methods of §3, these operators can be defined for functions φ with support in the phase flow of the manifold $l_0 \cap U(\widetilde{\alpha})$ along the vector field $X_{\widetilde{H}}$.

Let us now formulate an assertion concerning composition of the operators constructed above with the operator $\widehat{H}$.

***Proposition* 1.** *We have the following formulae:*

$$H\left(x,-\frac{\partial}{\partial x}\right)\circ\widehat{\Phi}^{i_0}_{(l_+,\mu,\varphi)}=\widehat{\Phi}^{i_0}_{(l_+,\mu,\mathcal{P}\varphi)}+S^{i_0}\left(x,-\frac{\partial}{\partial x}\right), \tag{8}$$

$$H\left(x,-\frac{\partial}{\partial x}\right)\circ\widehat{\Phi}_{i_0(l_+,\mu,\varphi)}=\widehat{\Phi}_{i_0(l_+,\mu,\mathcal{P}\varphi)}+S_{i_0}\left(x,-\frac{\partial}{\partial x}\right). \tag{9}$$

Here $\mathcal{P}$ is the transport operator (see Theorem 3.3), while S^{i_0}, S_{i_0} are pseudodifferential operators with symbols

$$S^{i_0}(x,p)=\Delta_{p_{i_0}}H(x,p)\cdot[\sqrt{F}\cdot\varphi]|_{l_0}, \tag{10}$$

$$S_{i_0}(x,p)=[\Delta_{x^{i_0}}H(x,p)-p_{i_0}\Delta_{x^1}H(x,p)][\sqrt{F}\cdot\varphi]|_{l_0}. \tag{11}$$

We shall denote by $\Delta_{p_{i_0}}$, Δ_{x^0}, Δ_{x^1} difference derivatives on char H:

$$\Delta_{p_{i_0}}H(x,p)=\frac{H(x,p)}{p_{i_0}-p_{i_0}|_{\operatorname{char}H}};\quad \Delta_{x^i}H(x,p)=\frac{H(x,p)}{x^i-x^i|_{\operatorname{char}H}},\ i=1,i_0.$$

Using equations (8), (9) of Proposition 1 we construct the microlocal regularizer $\widehat{R}$ of equation (1), that is, an operator such that $H(x,\,-\partial/\partial x)\circ\widehat{R}$ is a pseudodifferential operator whose symbol is equal to one in a neighbourhood of a point up to operators of arbitrarily low order. The operator $\widehat{R}$ has the form $\widehat{\Phi}^{i_0}_{(l_0,\mu,\varphi)}+T(x,\,-\partial/\partial x)$ in the non-singular case, and $\widehat{\Phi}_{i_0(l_+,\mu,\varphi)}+T(x,\,-\partial/\partial x)$ in the singular case ($T(x,\,-\partial/\partial x)$ is some pseudodifferential operator). Putting $u=\widehat{R}f$, we obtain the microlocal assertion of the theorem.

To prove the local statement we construct a local regularizer $\widehat{R}$, which is the sum of $\widehat{\Phi}^{i_0}_{(l_+,\mu,\varphi)}$ and $\widehat{\Phi}_{i_0(l_+,\mu,\varphi)}$ over different coordinate maps of the manifold and a pseudodifferential operator. Here the operator $\widehat{R}^*$ is a left regularizer for $\widehat{H}^*$. Therefore if v is a solution of the equation

$$\widehat{H}^*v=0, \tag{12}$$

with support in a small neighbourhood $U(x_0)$ of the point x_0, then

$$0=\widehat{R}^*H^*v=v+\widehat{Q}v, \tag{13}$$

where $\widehat{Q}$ is an arbitrarily smoothing operator. It follows from (13) that $v\in C^\infty(\mathbb{R}^n)$ and that the space $N(U(x_0))$ of solutions of equation (12) in the neighbourhood $U(x_0)$ is finite-dimensional. Since $N(U'(x_0))\subset N(U(x_0))$ for

$U'(x_0) \subset U(x_0)$, it follows that $N(U(x_0)) = 0$ for a sufficiently small neighbourhood $U(x_0)$ (there is no C^∞-function whose support is a single point). The remainder of the proof uses methods of functional analysis (see [16], [20]).

The construction of the global regularizer is similar to that of the local one. Since the set l is an everywhere regular immersion, we can construct a global regularizer $\widehat{R} : H^s_{\text{comp}}(\mathbb{R}^n) \to H^s_{\text{loc}}(\mathbb{R}^n)$ that satisfies

$$\widehat{H} \circ \widehat{R} = 1 + \widehat{Q}, \tag{14}$$

where the operator $\widehat{Q} : H^s_{\text{comp}}(\mathbb{R}^n) \to H^{s}_{\text{loc}}(\mathbb{R}^n)$ is continuous for every $s' > s$. In this case, however, for any compact set $K \subset M$ the space $N(K)$, even though it remains finite-dimensional, is not zero in general. Therefore for the solubility of equation (1) in $\overset{\circ}{K}$ it is necessary and sufficient for the right-hand side of the equation to satisfy finitely many orthogonality conditions.

3. Equations of subprincipal type.

In this subsection we shall consider the question of solubility of equation (1) under the assumption that the contact vector field X_H defined by the Hamiltonian H has finitely many stationary points (we are still assuming that the order of inhomogeneity of the function $H(x, p)$ is equal to one). As we shall show, in this case equation (1) is already insoluble in the whole scale of spaces H^s. Indeed, even the simplest equation $xu' = f(x)$ does not have any (even locally) smooth solution even for an infinitely smooth right-hand side. Furthermore, the kernel of the adjoint operator in this case does not consist of smooth functions alone.

We assume that the following conditions hold.

a) $dH \neq 0$ on the set char H (note that by homogeneity in the variables p the requirement $dH \neq 0$ is well defined, that is, it depends only on the point $\alpha \in S^*\mathbb{R}^n$).

b) There exist only finitely many points $\alpha^j \in S^*\mathbb{R}^n$, $j = 1, 2, \ldots, N$, where the field X_H vanishes: $X_H(\alpha^j) = 0$ (in what follows these points will be called singular).

c) The eigenvalues $\lambda_1^j, \ldots, \lambda_{2n-1}^j$ of the operator of the linear part of the contact vector field X_H in a neighbourhood of a point α^j satisfy the non-resonance condition [12], [21], [22]:

$$\sum_{i=1}^{2n-1} m_i \lambda_i^j \neq \lambda_1^j,$$

where the m_i are natural numbers with $\sum m_i \geqslant 3$. (Note that on a ray in $T_0^*\mathbb{R}^n$ corresponding to the point α^j the Hamiltonian vector field and the radial vector field $p\dfrac{\partial}{\partial p}$ are collinear. It can be shown that the coefficient of proportionality between these two vector fields is an eigenvalue of the above operator. This eigenvalue will always be denoted by λ_1^j in what follows.)

d) The numbers $\lambda_1, \ldots, \lambda_{2n-1}$ are real and pairwise distinct.

e) The field X_H has no finite motions. This means that, firstly, for any compact set $K \subset S^*\mathbb{R}^n$ each trajectory of the field X_H either leaves the compact set K or goes to a singular point, and that, secondly, there is no closed system of trajectories of X_H joining singular points.

It can be shown that by condition d) for any singular point α^j all the eigenvalues $\lambda_1^j, \ldots, \lambda_{2n-1}^j$ have the same sign. Let us set

$$s^j = \frac{|\lambda^j|}{2\lambda^j_{\min}}; \quad \sigma^j = \begin{cases} \frac{|\lambda^j|}{2\lambda^j_{\max}}, & \lambda^j_k < 0, \\ +\infty & \lambda^j_k > 0, \end{cases} \tag{15}$$

where

$$\lambda^j_{\min} = \min_{1\le k\le 2n-1} |\lambda^j_k|, \lambda^j_{\max} = \max_{1\le k\le 2n-1} |\lambda^j_k|, |\lambda^j| = |\lambda^j_1| + \cdots + |\lambda^j_{2n-1}|.$$

Furthermore, it is shown in [16] that under the above assumptions in a neighbourhood of every point α^j there are Fourier integral operators U^j of order k^j that microlocally reduce the operator $\widehat{H}$ to the form

$$\widehat{H}^j = i\sum_{k=1}^{n} \lambda^j_k x^k\, \partial/\partial x^k. \tag{16}$$

The numbers k^j are determined by the principal and subprincipal symbols of the operator $\widehat{H}$ at the point α^j. Let us define the numbers

$$s_{\min} = \max_j(s^j + k^j),$$
$$\sigma_{\max} = \min_j(\sigma^j + k^j),$$

where s^j, σ^j are defined by (15). For $s > s_{\min}$, $\sigma < \sigma_{\max}$ we shall define the operator

$$\widehat{H}_{s,\sigma} : H^{\sigma}_{\mathrm{loc}}(K) \to H^{s}_{\mathrm{loc}}(K). \tag{17}$$

As before, K is the closure of some domain $\overset{\circ}{K} \subset \mathbb{R}^n$, $K \Subset \mathbb{R}^n$. The domain of definition of the operator (17) consists of distributions $u \in H^{\sigma}_{\mathrm{loc}}(K)$ for which $\widehat{H}u \in H^{s}_{\mathrm{loc}}(K)$ ($\widehat{H}u$ is understood here in the sense of distributions). The operator (17) is closed and densely defined.

Let $N_{s,\sigma}(K)$ be the kernel of the adjoint operator

$$\widehat{H}^*_{s,\sigma} : H^{-s}_{\mathrm{comp}}(K) \to H^{-\sigma}_{\mathrm{comp}}(K). \tag{18}$$

Let us state the main theorem of this subsection.

***Theorem* 2.** *Under conditions* a)–e) *above for any compact set K and any numbers $s > s_{\min}$, $\sigma < \sigma_{\max}$,*

a) *the space $N_{s,\sigma}(K)$ is finite-dimensional, and*

b) *for any element $f \in H^{s}_{\mathrm{loc}}(K)$ that is orthogonal to $N_{s,\sigma}(K)$ there is a* *function* $u \in H^{\sigma}_{\mathrm{loc}}(K)$ *such that* $\widehat{H}_{s,\sigma}u = f$ *in* $\overset{\circ}{K}$ *and we have the inequality*

$$\|\varphi u\|_{\sigma} \leq C_{s,\sigma}\|\psi \widehat{H}_{s,\sigma}u\|_{s} \tag{19}$$

with a constant $C_{s,\sigma}$ *independent of* u.

The proof of Theorem 2 is based on the construction of a regularizer, that is, a continuous operator

$$\widehat{R}_{s,\sigma} : H^{s}_{\mathrm{comp}}(K) \to H^{\sigma}_{\mathrm{comp}}(K)$$

such that

$$\widehat{H}_{s,\sigma} \circ \widehat{R}_{s,\sigma} = \widehat{1} + \widehat{Q},$$

where $\widehat{1}$ is the identity operator, and $\widehat{Q} : H^{s}_{\mathrm{comp}}(K) \to C^{\infty}(K)$ is a continuous operator for $s > s_{\min}$. The full proof can be found in [16]; here we just indicate the main steps.

The point of departure in the construction of the regularizer is the classification of pseudodifferential operators in a neighbourhood of a stationary point of the contact vector field X_H. We shall call pseudodifferential operators $\widehat{H}_1 = H_1(x, -i\partial/\partial x)$ and $\widehat{H}_2 = H_2(x, -i\partial/\partial x)$ microlocally equivalent in a neighbourhood of a point α if there exists an elliptic Fourier integral operator $\widehat{U}$ associated with some homogeneous canonical transformation g such that in a neighbourhood of α the full symbols of the operators $\widehat{H}_2$ and $\widehat{U}^{-1}\widehat{H}_1\widehat{U}$ coincide. Here $\widehat{U}^{-1}$ is a Fourier integral operator which is the inverse of $\widehat{U}$ modulo infinitely smoothing operators.

It is known (see, for example, [22]) that any two operators of principal type are microlocally equivalent in a neighbourhood of any point. For operators with contact stationary points this is no longer so. We have the following theorem.

***Theorem* 3.** *Let* $\widehat{H}_1$ *and* $\widehat{H}_2$ *be two pseudodifferential operators with contact stationary points* α_1 *and* α_2, *satisfying the non-resonance condition* c). *Then* $\widehat{H}_1$ *is equivalent to* $\widehat{H}_2$ *if and only if they have the same spectrum of the operator of the linear part of the contact vector field.*

The reader can find the proof of this theorem in [22], which also contains a list of normal forms in equivalence classes. Under condition d) the corresponding normal form is (16).

The construction of the regularizer in a neighbourhood of a singular point employs one of the two regularizers of the operator (16):

$$\widehat{R}_0[f](x) = i\int_0^1 [f(t^{\lambda}x) - f(0)]\,\frac{dt}{t}, \tag{20}$$

$$\widehat{R}_{\infty}[f](x) = i\int_{+\infty}^1 f(t^{\lambda}x)\,\frac{dt}{t}, \tag{21}$$

where $t^\lambda x = (t^{\lambda_1}x^1, \ldots, t^{\lambda_n}x^n)$. Let us note that if $s > s_{\min}$ the operator (20) is well defined, since $s_{\min} \geqslant \pi/2$, and therefore the function $f \in H^s(\mathbb{R}^n)$ has trace $f(0)$ at the origin. It can be shown that the operators (20), (21) are bounded in the spaces

$$\widehat{R}_0 : H^s_{\mathrm{loc}}(\mathbb{R}^n) \to H^s_{\mathrm{loc}}(\mathbb{R}^n), \qquad s > |\lambda|/2\lambda_{\min};$$
$$\widehat{R}_\infty : H^s_{\mathrm{comp}}(\mathbb{R}^n) \to H^s_{\mathrm{comp}}(\mathbb{R}^n), s < |\lambda|/2\lambda_{\max}$$

(see [16]). For each singular point let us set $\alpha^j\widehat{\Phi}^j = \widehat{R}_0$ if $\lambda^j_k > 0$ and $\widehat{\Phi}^j = \widehat{R}_\infty$ if $\lambda^j_k < 0$. The expression for the "singular part" of the regularizer $\widehat{R}_{\mathrm{sing}}$ has the form

$$\widehat{R}_{\mathrm{sing}} = \sum_{j=1}^{N} (\widehat{U}^j)^{-1} \cdot \widehat{\Phi}^j \cdot U^j,$$

where the operators $\widehat{U}^j$ are as defined above. The construction of the non-singular part of the regularizer (similar to the constructions in subsection 2 of this section) and the "gluing together" of the global regularizer are described in detail in the article [16] cited above.

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