INVARIANTS OF DIFFERENTIAL EQUATIONS DEFINED BY VECTOR FIELDS

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Abstract. We determine the most general group of equivalence transformations for a family of differential equations defined by an arbitrary vector field on a manifold. We also find all invariants and differential invariants for this group up to the second order. A result on the characterization of classes of these equations by the invariant functions is also given.

1. Introduction

In Lie theory, the invariance of functions and other objects under a transformation group $\mathcal{G}$ acting on an n-dimensional manifold V is usually characterized by the vanishing of functions under some vector fields generating the group action, and this vanishing is represented by a system of partial differential equations of the form

$$\sum_i A_i(X)\,\partial_{x^i} F(X) = 0 \tag{1.1}$$

where $X = (x^1, ..., x^n)$ is a local coordinates system on V and where F is the unknown function. The importance of linear partial differential equations of the form (1.1) usually referred to as determining equations for the invariant objects cannot be overstated. Indeed, they characterize invariant equations as well as their invariant solutions, and they have a similar importance in the study of Lie algebras and in representation theory. In physics, invariant operators of dynamical groups characterize specific properties of physical systems and provide mass formulaes and energy spectra [1, 2]. Invariants of physical symmetry groups also provide quantum numbers useful in the classification of elementary particles [3]. It would therefore be desirable to consider the group of equivalence transformations of equations of the form (1.1) and to determined all functions invariant under this group. Such functions are simply called invariants of the differential equation (1.1).

A method for the determination of invariants of linear and nonlinear equations build on an idea suggested by Lie himself [4] was developed in [5]. The method is based on the fact that the invariant functions for the infinite group of equivalence transformations of a given system of equations are precisely the invariants of the system of differential equations. That is, these functions are invariant under the group of transformations that confine the system of equations to a prescribed family of equations. The method also yields singular invariant equations, and has been used to complete the problem of determination of the Laplace invariants in [6], and in [7] to characterize linearizable second order ODE's.

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In the present paper we find the most general group G of equivalence transformations leaving unchanged, except for its coefficients A_i, an equation of the form

$$\sum_{i=1}^{n} A_i(X)\,\partial_{x^i} U = 0 \tag{1.2}$$

where $(x^1, \dots, x^n, U) \in \mathbb{R}^n \times \mathbb{R} = M$, and where we assume that none of the coefficients A_i for $i = 1, \dots, n$ vanishes identically. We then find the invariants and differential invariants up to the second order for this group and for an arbitrary number n of independent variables in the equation. We first treat with more details the cases $n = 2, 3$ before giving some generalizations of the results. Next, by investigating the regularity of the action of G on M, we show how the invariants found can be used to characterize families of equations of the form (1.2).

2. The group of equivalence transformations

Owing to the linearity of equation (1.2), any invertible change of the dependent variable U and the independent variables $(x^1, \dots, x^n) = X$ that preserves the form of the equation should be of the form

$$X = \psi(Y) \tag{2.1a}$$

$$U = H(Y)V(Y), \qquad H(Y) \neq 0 \tag{2.1b}$$

where $Y = (y^1, \dots, y^n)$ is the new set of independent variables, V is the new dependent variable and H is an arbitrary function.

Theorem 1. *The most general group G of equivalence transformations of equation* (1.2) *consists of the set of all invertible changes of variables of the form*

$$x^i = \psi^i(Y) \equiv \psi^i(y^i), \qquad \text{for } i = 1, \dots, n \tag{2.2a}$$

$$U = V. \tag{2.2b}$$

That is, each $\psi^i(Y)$ is a function a the single variable y^i, and G does not involve a change of the dependent variable.

Proof. Under the general change of variables (2.1a), and by setting $\phi = \psi^{-1}$, equation (1.2) takes the form

$$\sum_j B_j(Y)\partial_{y^j}(U) = 0 \tag{2.3a}$$

where

$$B_j(Y) = \sum_i^n A_i(\psi(Y))\frac{\partial \phi^j}{\partial x^i}(\psi(Y)) = \sum_i^n A_i(X)\frac{\partial \phi^j}{\partial x^i}(X) \tag{2.3b}$$

Equation (2.3a) together with the expression of U given by (2.1b) shows that none of the coefficients B_j should vanish identically. However, the expression of B_j in (2.3b) shows that if $\phi^j(X)$ depends on more than one of the variables x^i, for $i = 1, \dots, n$ it can be chosen as an invariant of an appropriate vector field, and so that $B_j = 0$. Hence $\phi^j(X) \equiv \phi^j(x^{qj})$, for some $qj \in \{1, \dots, n\}$ and because of the invertibility of ϕ, ϕ^j must be a nonconstant map and all the variables x^{qj} must be distinct for $j = 1, \dots, n$. This implies in particular that $x^i = \psi^i(y^{ki})$ must also be a nonconstant function of a single variable. If we let σ be the permutation that maps the ordered set $\{y^1, \dots, y^n\}$ onto the ordered set $\{y^{k1}, \dots, y^{kn}\}$, then the ith component of $\psi \circ \sigma^{-1}$ depends exactly on y^i alone. On account of the arbitrariness of ψ, we may replace ψ, by $\psi \circ \sigma^{-1}$, and thus that we may always assume that $x^i = \psi^i(y^i)$, and equivalently $y^i = \phi^i(x^i)$. This reduces the expression for $B_j(Y)$ in (2.3b) to the form

$$B_j = A_j(\psi(Y)) \frac{\partial \phi^j(x^j)}{\partial x^j} = \frac{A_j(\psi(Y))}{\psi^{j\,\prime}(y^j)} \neq 0, \tag{2.4}$$

where $\psi^{j\,\prime} = \partial \psi^j / \partial y^j$. Substituting (2.1b) into (2.3a) and expanding, equation (1.2) takes the form

$$\sum_j H\, B_j\, \partial_{y^j} V + V \left(\sum_j B_j\, \partial_{y^j} H \right) = 0. \tag{2.5}$$

The fact that the coefficient of V appearing in (2.5) must identically vanish and the arbitrariness of the n coefficients A_j in the expression of B_j in (2.4) show that $\partial_{y^j} H(Y) = 0$, for all $j = 1, \dots, n$. Thus $H(Y) \neq 0$ is a constant function and without loss of generality we may assume that $H = 1$. This last equality transforms equation (2.5) to the form

$$\sum_j B_j(Y) \partial_{y^j} V(Y) = 0, \tag{2.6}$$

which is of the prescribed form. This completes the proof of the theorem. □

Remark . It should also be noted that under the general change of variables (2.1), it is always possible, by the well-known result on the rectification of vector fields, to put (1.2) in the form

$$\partial_{y^1}(HV) = 0, \qquad \text{that is,} \qquad (\partial_{y^1} H)V + H(\partial_{y^1} V) = 0$$

Thus if we allow some of the coefficients A_i to vanish, then the only additional condition to be imposed on the change of variables (2.1) would be $\partial_{y^1} H = 0$, and all equations of the form (1.2) would be equivalent. There are clearly no invariant functions or invariant equations of any order in such case.

We now move on to determine the infinitesimal generators of the group G. As already noted, equation (2.2a) implies that $y^i = \phi^i(x^i)$, for $i = 1, \dots, n$, and this shows that the infinitesimal transformation of (2.2) has the form

$$y^i \approx x^i + \epsilon \xi^i(x^i), \qquad V \approx U, \tag{2.7}$$

where the functions ξ^i are also arbitrary, due to the arbitrariness of the functions ψ^i. The first prolongation of this transformation has the form

$$\partial_{y^i} V \approx \partial_{x^i} U + \epsilon(-\xi^{i\,\prime} \partial_{x^i} U), \tag{2.8}$$

which implies that

$$\partial_{x^i} U \approx \partial_{y^i} V + \epsilon(\xi^{i\,\prime} \partial_{y^i} V,) \tag{2.9}$$

where ∂_x is the differential operator $\partial / \partial x$, for any variable x. A substitution of equation (2.9) into the original equation (1.2) yields the infinitesimal transformation of that equation in the form

$$\sum_i^n (A_i + \epsilon A_i\, \xi^{i\,\prime}) \partial_{y^i} V = 0. \tag{2.10}$$

This shows that the infinitesimal transformation $\tilde{A}$ of the coefficient A_i is given by

$$\tilde{A}_i \approx A_i + \epsilon A_i {\xi^i}'.$$

The infinitesimal generators of the equivalence transformation G therefore has the form

$$\mathcal{V} = \sum_i^n \xi^i \partial_{x^i} + \sum_i^n A_i\, {\xi^i}' \partial_{A_i} \tag{2.11}$$

3. Zeroth-order invariants

We would like to first recall very briefly certain elementary facts about the invariant functions of a given transformation group. Suppose that the infinitesimal generators of an r-parameters group of transformations G acting on the $\mathcal{Q}$-dimensional manifold M are of the form

$$\mathcal{V}_k = \sum_j \xi^{kj} \partial_{x^j}, \qquad \text{for } k = 1, \dots, r. \tag{3.1}$$

The invariant functions and invariant equations of G are determined by

$$\mathcal{V}_k\,(F) = 0 \tag{3.2a}$$

$$\mathcal{V}_k\,(F)\Big|_{F=0} = 0 \tag{3.2b}$$

respectively, for $k = 1, \dots, r$. The number of fundamental invariants of G does not exceed $\mathcal{Q} - \tau$, where τ is the rank of the matrix $\left(\xi^{kj}\right)_{k,j}$ of coefficients of the r operators $\mathcal{V}_k$. Each of these functions naturally gives rise to an invariant equation. Invariant equations $F = 0$, where F is not an invariant function, and obtained by imposing the additional condition $\tau < \mathcal{Q}$ to the second equation of (3.2b) are often referred to as singular invariant equations. Using a Lie linearization test, such equations were recently shown [7] to characterize all linearizable second order ordinary differential equations.

When some of the independent variables x^j in the expression of the $\mathcal{V}_k$ can be taken as depend variables for other objects such as a differential equation, the generators $\mathcal{V}_k$ can be extended to involve higher order derivatives of the dependent variables. If $\mathcal{V}$ is a given infinitesimal generator of G, then we shall often use the same symbol $\mathcal{V}$ to represent both $\mathcal{V}$ and its m-th prolongation $\mathcal{V}^{(m)}$. Similarly, the mth jet space of M will often be denoted simply by M.

Since the general change of variables (2.2) is merely a change of the independent variables and does not involve the dependent variable U, this variable is trivially an invariant for G. We shall therefore ignore this variable in our search for the invariant functions of G whose general form for the zeroth-order operator (2.11) is $F(x^1, \dots, x^n, A_1, \dots, A_n)$.

Theorem 2. *The group of equivalence transformations G of* (1.2) *has neither invariant functions nor invariant equations.*

Proof. Rewriting the generic generator $\mathcal{V}$ in (2.11) as a linear combination of the arbitrary functions ξ^i and their derivatives gives

$$\mathcal{V} = \sum_i^n \xi^i (\partial_{x^i}) + \sum_i^n {\xi^i}' (A_i\, \partial_{A_i}),$$

and this proves the first part of the theorem at once, on account of the arbitrariness of the functions ξ^i. To show that G has no invariant equation, we use an elementary technique similar to that used in [7]. Suppose that $F(x^1, \dots, x^n, A_1, \dots, A_n) = 0$ is a nontrivial invariant equation for G, and so it explicitly involves at least one of the variables, say x^1, in the set $\left\{x^1, \dots, x^n, A_1, \dots, A_n\right\}$. Then solving the equation for x^1 reduces it to the equivalent form $x^1 = K(x^2, \dots, x^n, A_1, \dots, A_n)$. The arbitrariness of the functions ξ^i and their derivatives implies again that we must have in particular

$$\partial_{x^1}\left(x^1 - K\right)\Big|_{x^1 = K} = 0.$$

But this last condition cannot hold because $\partial_{x^1}\left(x^1 - K\right) = 1$, and this completes the proof of the theorem. □

4. First-order differential invariants

The first prolongation $\mathcal{V}^{(1)}$ of the infinitesimal generator (2.11) of G has the form

$$\mathcal{V}^{(1)} = \mathcal{V} + \sum_i^n A_i \xi^{i\,\prime\prime} \frac{\partial}{\partial A_{ii}} + \sum_i^n \sum_{j \neq i} A_{ji} \left(\xi^{j\,\prime} - \xi^{i\,\prime}\right) \frac{\partial}{\partial A_{ji}} \tag{4.1a}$$

where we have used the notation

$$A_{ij} = \frac{\partial A_i}{\partial x^j}, \qquad \text{for } i, j \in \{1, \dots, n\}. \tag{4.1b}$$

In terms of the linear combination of the arbitrary functions ξ^i and their derivatives, this expression takes the form

$$\begin{aligned} \mathcal{V}^{(1)} &= \sum_i^n \xi^i \frac{\partial}{\partial x^i} + \sum_i^n \xi^{i\,\prime\prime} \frac{\partial}{\partial A_{ii}} \\ &+ \sum_i^n \xi^{i\,\prime} \left[\frac{\partial}{\partial A_i} + \sum_{j \neq i} \left(A_{ij} \frac{\partial}{\partial A_{ij}} - A_{ji} \frac{\partial}{\partial A_{ji}}\right)\right]. \end{aligned} \tag{4.2}$$

Equation (4.2) clearly shows that any first-order differential invariant of G should be independent of all the independent variables x^i, as well as the variables A_{ii}. This last condition reduces $\mathcal{V}^{(1)}$ to the form

$$\mathcal{V}^{(1)} = \sum_i^n \xi^{i\,\prime} \mathcal{V}_{\xi^{i\,\prime}} \tag{4.3a}$$

where

$$\mathcal{V}_{\xi^{i\,\prime}} = A_i \partial_{A_i} + \sum_{j \neq i} \left(A_{ij} \frac{\partial}{\partial A_{ij}} - A_{ji} \frac{\partial}{\partial A_{ji}}\right). \tag{4.3b}$$

As we are considering the arbitrary functions A_i in (1.2) in their most general form, we may assume that the functions $A_{ij} = \partial_{x^j} A_i$ do not vanish identically. It then follows from (4.3) that the generators of the first prolongation of G depend on $\mathcal{Q} = n^2$ independent variables. If however we assume that exactly p of the functions A_{ij} vanish identically, then this number p is invariant under the action of G and $\mathcal{Q} = n^2 - p$.

Theorem 3. *Consider the n operators $\mathcal{V}_{\xi^{i\,\prime}}$ of (4.3).*

(a) *The rank of the coefficients matrix $\mathcal{M}$ of the operator $\mathcal{V}_{\xi^i{}'}$ is n, which is maximal.*
(b) *The $\mathcal{V}_{\xi^i{}'}$ form an n-dimensional commutative Lie algebra.*
(c) *The number fundamental first-order differential invariants of the group G of equivalence transformations of equation (1.2) is $n(n-1)$.*

Proof. In any coordinate system of the form $\{A_1, \dots, A_n, \dots\}$ on the extended jet space on which the first prolongation of G operates, equation (4.3) shows that the first n columns of $\mathcal{M}$ is represented by the matrix diag $\{A_1, \dots, A_n\}$ which has rank n, owing to the fact that none of the coefficients A_i is zero, and this proves part (a) and shows that the n vectors $\mathcal{V}_{\xi^i{}'}$ are linearly independent. For part (b), if for any $k \in \{1, \dots, n\}$ we write

$$\mathcal{V}_{\xi^k{}'} = A_k \partial_{A_k} + \sum_{q \neq k} \left(A_{kq} \frac{\partial}{\partial A_{kq}} - A_{qk} \frac{\partial}{\partial A_{qk}} \right)$$

then we readily see that the commutator $\left[\mathcal{V}_{\xi^i{}'}, \mathcal{V}_{\xi^k{}'}\right]$ is a linear combination of the identically vanishing commutators

$$\begin{aligned} &\left[A_{ij}\partial_{A_{ij}} - A_{ji}\partial_{A_{ji}}, A_{kq}\partial_{A_{kq}} - A_{qk}\partial_{A_{qk}}\right], \qquad \left[A_i\partial_{A_i}, A_k\partial_{A_k}\right] \\ &\left[A_i\partial_{A_i}, A_{kq}\partial_{A_{kq}} - A_{qk}\partial_{A_{qk}}\right], \qquad \left[A_{ij}\partial_{A_{ij}} - A_{ji}\partial_{A_{ji}}, A_k\partial_{A_k}\right], \end{aligned}$$

This fact together with part (a) proves (b). Since the number of independent variables involved in the complete system of n operators $\mathcal{V}_{\xi^i{}'}$ is n^2, the number of their functionally independent invariants is precisely $n^2 - \text{rank}(\mathcal{M})$, which is $n(n-1)$. □

The most practical way to find the $n^2 - n$ first order differential invariants of G would be to compute these invariants for low dimensions of M, i.e. for $n = 2, 3$ and then make use of the symmetry inherent in equation (1.2) to find the invariants in the general case.

For $n = 2$ and $n = 3$ we write equation (1.2) in the form

$$aU_x + bU_y = 0, \qquad \text{and} \qquad aU_x + bU_y + cU_z = 0, \tag{4.4}$$

respectively. In case $n = 2$, the operators $\mathcal{V}_{\xi^i{}'}$ are given by

$$\mathcal{V}_{\xi^1{}'} = a\partial_a + a_y\partial_{a_y} - b_x\partial_{b_x}, \qquad \mathcal{V}_{\xi^2{}'} = b\partial_b - a_y\partial_{a_y} + b_x\partial_{b_x}.$$

Solving the system of equations $\mathcal{V}_{\xi^i{}'}(F) = 0$, for $i = 1, 2$ by the method of characteristics shows that G has a fundamental system of invariants consisting of the two functions

$$T_{12} = \frac{a_y\, b}{a}, \qquad \text{and} \qquad T_{21} = \frac{b_x\, a}{b}.$$

In case $n = 3$, the three operators $\mathcal{V}_{\xi^i{}'}$ are given by

$$\begin{aligned} \mathcal{V}_{\xi^1{}'} &= a\partial_a + (a_y\partial_{a_y} - b_x\partial_{b_x}) + (a_z\partial_{a_z} - c_x\partial_{c_x}) \\ \mathcal{V}_{\xi^2{}'} &= b\partial_b + (b_x\partial_{b_x} - a_y\partial_{a_y}) + (b_z\partial_{b_z} - c_y\partial_{c_y}) \\ \mathcal{V}_{\xi^3{}'} &= c\partial_c + (c_x\partial_{c_x} - a_z\partial_{a_z}) + (c_y\partial_{c_y} - b_z\partial_{b_z}) \end{aligned}$$

and the corresponding set of six invariants is found to be

$$T_{12} = \frac{a_y\, b}{a}, \quad T_{21} = \frac{b_x\, a}{b}, \quad T_{13} = \frac{a_z\, c}{a}$$
$$T_{31} = \frac{c_x\, a}{c}, \quad T_{23} = \frac{b_z\, c}{b}, \quad T_{32} = \frac{c_y\, b}{c}.$$

The form of the invariants found for $n = 2, 3$ together with part (c) of Theorem 3 asserting that the number of invariants in the general case is $n(n-1)$, which is $2\binom{n}{2}$, suggest that all invariants can be found by associating with each subset of two elements of the set of n coefficients of the differential equation a pair of invariants according to a very simple rule.

Theorem 4. *The $n(n-1)$ fundamental invariants T_{ij} of the group G of equivalence transformations of equation* (1.2) *are given by*

$$T_{ij} = \frac{A_{ij} A_j}{A_i}, \qquad \text{for } i \neq j, \qquad \text{where} \quad A_{ij} = \frac{\partial A_i}{\partial x^j}, \tag{4.5}$$

and where A_i and A_j run over the set coefficients of the equation.

Proof. It is easily verified that the identity $\mathcal{V}_{\xi^i\,'}(T_{kq}) = 0$ holds for all $i = 1, \ldots, n$ and for all $k \neq q$. Next, the functions A_{ij} for $i, j \in \{1, \ldots, n\}$ are functionally independent by assumption, and each T_{ij} depends on exactly one of them. □

Note that if we restrict the action of G to a sub-family of equations of the form (1.2) for which exactly p of the function A_{ij} vanish identically, then the maximal number of functionally independent first order differential invariants is $n(n-1)-p$.

5. Second-order differential invariants

The second prolongation of the generator (2.11) of G has the form

$$\begin{aligned}
\mathcal{V}^{(2)} = \;& \mathcal{V} + \sum_j^n (A_j)\xi^j\,''\partial_{A_{jj}} + \left(A_j \xi^j\,'' + A_{jj}\xi^j\,'' - A_{jjj}\xi^j\,'\right)\partial_{A_{jjj}} \\
& + \sum_{i \neq j} A_{ji}(\xi^j\,' - \xi^i\,')\partial_{A_{ji}} + 2\left(A_{ji}\xi^j\,'' - A_{jji}\xi^i\,'\right)\partial_{A_{jji}} \\
& + \left[(\xi^j\,' - 2\xi^i\,')A_{jii} - A_{ji}\xi^i\,''\right]\partial_{A_{jii}} \\
& + 2\sum_{\substack{i,k \neq j \\ i<k}} (\xi^j\,' - \xi^i\,' - \xi^k\,')A_{jik}\,\partial_{A_{jik}},
\end{aligned} \tag{5.1a}$$

where as usual

$$A_{ji} = \frac{\partial A_j}{\partial x^i}, \qquad A_{jik} = \frac{\partial A_j}{\partial x^i \partial x^k}, \qquad \text{etc.} \tag{5.1b}$$

Rewriting this expression as a linear combination of the arbitrary functions ξ^i and their derivatives shows that any invariant function should be independent from the independent variables and from variables of the form A_{iii} for $i = 1, \ldots, n$. This reduces the expression of $\mathcal{V}^{(2)}$ to the form

$$\mathcal{V}^{(2)} = \sum_i^n \xi^i\,'\mathcal{V}_{\xi^i\,'} + \xi^i\,''\mathcal{V}_{\xi^i\,''} \tag{5.2a}$$

where

$$\mathcal{V}_{\xi^{i\,\prime}} = A_i \partial_{A_i} + \sum_{j\neq i} \left(A_{ij}\partial_{A_{ij}} - A_{ji}\partial_{A_{ji}} + A_{ijj}\partial_{A_{ijj}} - 2\sum_k^n A_{jik}\partial_{A_{jik}} \right) + 2\sum_{\substack{j,k\neq i\\ j<k}} A_{ijk}\partial_{A_{ijk}} \tag{5.2b}$$

$$\mathcal{V}_{\xi^{i\,\prime\prime}} = A_i\,\partial_{A_{ii}} + \sum_{j\neq i} \left(2A_{ij}\,\partial_{A_{iij}} - A_{ji}\,\partial_{A_{jii}}\right). \tag{5.2c}$$

It readily follows from equations (5.2) that the second order differential invariants of G depend in general on $n+n\binom{n+2}{2}-2n$ variables, that is on $n^2(3+n)/2$ variables. This is the dimension of the subspace of the extended jet space $M^{(2)}$ of M on which the second prologation of G acts.

Theorem 5. *The set of operators* $\left\{\mathcal{V}_{\xi^{i\,\prime}}\right\}_{i=1}^n$ *and* $\left\{\mathcal{V}_{\xi^{i\,\prime\prime}}\right\}_{i=1}^n$ *given in* (5.2) *each generate an* n *dimensional commutative Lie algebra.*

Proof. Thanks to the term $A_i\partial_{A_i}$ appearing in the expression of each generator $\mathcal{V}_{\xi^{i\,\prime}}$ as the only term involving ∂_{A_i}, the coefficients matrix of these operators admits a submatrix of the form $\operatorname{diag}\{A_1,\dots,A_n\}$, which is clearly of rank n, showing that the $\mathcal{V}_{\xi^{i\,\prime}}$ generate an n-dimensional space. Similarly, as the term $A_i\partial_{A_{ii}}$ appears in the same manner in the expression of each generator $\mathcal{V}_{\xi^{i\,\prime\prime}}$, the set $\left\{\mathcal{V}_{\xi^{i\,\prime\prime}}\right\}_{i=1}^n$ also generate an n dimensional space. For each pair $\{i,k\}$, it is easy to see as in the proof of Theorem 3 that each of the commutators $[\mathcal{V}_{\xi^{i\,\prime}},\mathcal{V}_{\xi^{k\,\prime}}]$ and $[\mathcal{V}_{\xi^{i\,\prime\prime}},\mathcal{V}_{\xi^{k\,\prime\prime}}]$ is a linear combination of identically vanishing commutators. This completes the proof of the theorem. □

If we denote by $\xi^{i\,(j)}$ the jth derivative of ξ^i, then Theorem 5 asserts that for j fixed, the $\mathcal{V}_{\xi^{i(j)}}$'s form a commutative Lie algebra for $j=1,2$. However, the set of all operator $\mathcal{V}_{\xi^{i(j)}}$ for $i=1,\dots,n$ and $j=1,2$ that determine the second order differential invariants of G do not form a Lie algebra in general when they are considered together, as this easily appears from the low dimensional cases.

Indeed, if for $n=2,3$ we rewrite equation (1.2) as in (4.4), then for $n=2$, we have

$$\begin{aligned}
\mathcal{V}_{\xi^{1\,\prime}} &= a\frac{\partial}{\partial a} + a_y\frac{\partial}{\partial a_y} + a_{yy}\frac{\partial}{\partial a_{yy}} - b_x\frac{\partial}{\partial b_x} - 2b_{xx}\frac{\partial}{\partial b_{xx}} - 2b_{xy}\frac{\partial}{\partial b_{xy}}\\
\mathcal{V}_{\xi^{2\,\prime}} &= b\frac{\partial}{\partial b} - a_y\frac{\partial}{\partial a_y} - 2a_{xy}\frac{\partial}{\partial a_{xy}} - 2a_{yy}\partial_{a_{yy}} + b_x\frac{\partial}{\partial b_x} + b_{xx}\frac{\partial}{\partial b_{xx}}\\
\mathcal{V}_{\xi^{1\,\prime\prime}} &= a\frac{\partial}{\partial a_x} + 2a_y\frac{\partial}{\partial a_{xy}} - b_x\frac{\partial}{\partial b_{xx}}\\
\mathcal{V}_{\xi^{2\,\prime\prime}} &= -a_y\frac{\partial}{\partial a_{yy}} + b\frac{\partial}{\partial b_y} + 2b_x\frac{\partial}{\partial b_{xy}}.
\end{aligned}$$

In this case we have

$$[\mathcal{V}_{\xi^{1\,\prime}},\mathcal{V}_{\xi^{1\,\prime\prime}}] = \mathcal{V}_{\xi^{1\,\prime\prime}}, \qquad \text{and} \qquad [\mathcal{V}_{\xi^{2\,\prime}},\mathcal{V}_{\xi^{2\,\prime\prime}}] = \mathcal{V}_{\xi^{2\,\prime\prime}}$$

However, the span of $\left\{V_{\xi^{1\,\prime}}, V_{\xi^{2\,\prime}}, \mathcal{V}_{\xi^{1\,\prime\prime}}, \mathcal{V}_{\xi^{2\,\prime\prime}}\right\}$ does not contain the commutator $[\mathcal{V}_{\xi^{i\,(j)}},\mathcal{V}_{\xi^{k\,(p)}}]$ for any sets $\{i,k\}$ and $\{j,p\}$ of distinct elements. For instance, we have

$$[\mathcal{V}_{\xi^1\,(2)}, \mathcal{V}_{\xi^2\,(1)}] = -2a_y \frac{\partial}{\partial\, a_{xy}}.$$

We have a similar situation in the case of three independent variables. The operators $\mathcal{V}_{\xi^i\,(j)}$ are given in this case by

$$\begin{aligned}
\mathcal{V}_{\xi^1\,\prime} =\;& a\frac{\partial}{\partial\, a} + a_y\frac{\partial}{\partial\, a_y} + a_z\frac{\partial}{\partial\, a_z} + a_{yy}\frac{\partial}{\partial\, a_{yy}} + 2\, a_{yz}\frac{\partial}{\partial\, a_{yz}} + a_{zz}\frac{\partial}{\partial\, a_{zz}}\\
&- b_x\frac{\partial}{\partial\, b_x} - 2b_{xx}\frac{\partial}{\partial\, b_{xx}} - 2b_{xy}\frac{\partial}{\partial\, b_{xy}} - 2b_{xz}\frac{\partial}{\partial\, b_{xz}} - c_x\frac{\partial}{\partial\, c_x}\\
&- 2c_{xx}\frac{\partial}{\partial\, c_{xx}} - 2c_{xy}\frac{\partial}{\partial\, c_{xy}} - 2c_{xz}\frac{\partial}{\partial\, c_{xz}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{\xi^2\,\prime} =\;& b\frac{\partial}{\partial\, b} - a_y\frac{\partial}{\partial\, a_y} - 2a_{xy}\frac{\partial}{\partial\, a_{xy}} - 2a_{yy}\frac{\partial}{\partial\, a_{yy}} - 2a_{yz}\frac{\partial}{\partial\, a_{yz}} + b_x\frac{\partial}{\partial\, b_x}\\
&+ b_z\frac{\partial}{\partial\, b_z} + b_{xx}\frac{\partial}{\partial\, b_{xx}} + 2b_{xz}\frac{\partial}{\partial\, b_{xz}} + b_{zz}\frac{\partial}{\partial\, b_{zz}} - c_y\frac{\partial}{\partial\, c_y}\\
&- 2c_{xy}\frac{\partial}{\partial\, c_{xy}} - 2c_{yy}\frac{\partial}{\partial\, c_{yy}} - 2c_{yz}\frac{\partial}{\partial\, c_{yz}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{\xi^3\,\prime} =\;& c\frac{\partial}{\partial\, c} - a_z\frac{\partial}{\partial\, a_z} - 2a_{xz}\frac{\partial}{\partial\, a_{xz}} - 2a_{yz}\frac{\partial}{\partial\, a_{yz}} - 2a_{zz}\frac{\partial}{\partial\, a_{zz}} - b_z\frac{\partial}{\partial\, b_z}\\
&- 2b_{xz}\frac{\partial}{\partial\, b_{xz}} - 2b_{yz}\frac{\partial}{\partial\, b_{yz}} - 2b_{zz}\frac{\partial}{\partial\, b_{zz}} + c_x\frac{\partial}{\partial\, c_x} + c_y\frac{\partial}{\partial\, c_y}\\
&+ c_{xx}\frac{\partial}{\partial\, c_{xx}} + 2c_{xy}\frac{\partial}{\partial\, c_{xy}} + c_{yy}\frac{\partial}{\partial\, c_{yy}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{\xi^1\,\prime\prime} &= a\frac{\partial}{\partial\, a_x} + 2a_y\frac{\partial}{\partial\, a_{xy}} + 2a_z\frac{\partial}{\partial\, a_{xz}} - b_x\frac{\partial}{\partial\, b_{xx}} - c_x\frac{\partial}{\partial\, c_{xx}}\\
\mathcal{V}_{\xi^2\,\prime\prime} &= -a_y\frac{\partial}{\partial\, a_{yy}} + b\frac{\partial}{\partial\, b_y} + 2b_x\frac{\partial}{\partial\, b_{xy}} + 2b_z\frac{\partial}{\partial\, b_{yz}} - c_y\frac{\partial}{\partial\, c_{yy}}\\
\mathcal{V}_{\xi^3\,\prime\prime} &= -a_z\frac{\partial}{\partial\, a_{zz}} - b_z\frac{\partial}{\partial\, b_{zz}} + c\frac{\partial}{\partial\, c_z} + 2c_x\frac{\partial}{\partial\, c_{xz}} + 2c_y\frac{\partial}{\partial\, c_{yz}}.
\end{aligned}$$

As in case $n = 2$, we have $[\mathcal{V}_{\xi^i\,\prime}, \mathcal{V}_{\xi^i\,\prime\prime}] = \mathcal{V}_{\xi^i\,\prime\prime}$. That is, $\left\{\mathcal{V}_{\xi^i\,\prime}, \mathcal{V}_{\xi^i\,\prime\prime}\right\}$ spans a solvable Lie algebra with nilradical $\left\{\mathcal{V}_{\xi^i\,\prime\prime}\right\}$, for $i = 1, 2, 3$. However, here again the span of $\left\{\mathcal{V}_{\xi^i\,(j)}\right\}_{i,j}$ does not contain the commutator $[\mathcal{V}_{\xi^i\,(j)}, \mathcal{V}_{\xi^k\,(p)}]$ for any sets $\{i, k\}$ and $\{j, p\}$ of distinct elements. Indeed, we have for instance

$$[\mathcal{V}_{\xi^1\,(2)}, \mathcal{V}_{\xi^3\,(1)}] = 2c_x \frac{\partial}{\partial\, c_{xz}}.$$

There is no guarantee in this case that the number of invariant attains its maximum which is $\mathcal{Q} - \tau$, with the usual notation. More over, they are much more difficult to find using the method of characteristic. we shall therefore attempt to determine the invariants of the second prolongation of G using the so-called method of total derivatives [8, 9].

Suppose that we are given a system of equations of the form (3.2a) where the $\mathcal{V}_k$'s are arbitrary linear differential operators given as in (3.1) and depend on a total of $\mathcal{Q}$ variables. Denote again by τ the rank of the coefficients matrix $\left(\xi^{kj}\right)_{k,j}$, and set $p = \mathcal{Q} - \tau$. Thus we can solve (3.2a) for τ of the variables $\partial_{x_t} F$ in terms of the remaining p others, and this gives rise to the Jacobian system

$$\Delta_t F \equiv \frac{\partial F}{\partial x_t} + \sum_{s=1}^{p} U_{s,t} \frac{\partial F}{\partial u_s} = 0, \qquad \text{for } t = 1, \dots, \tau, \tag{5.3}$$

where we have renamed the remaining p variables $x_{\tau+j}$ as u_j, for $j = 1, \dots, p$ and where the $U_{s,t}$'s are functions depending in general on the $\mathcal{Q}$ vriables $x_1, \dots, x_\tau$ and $u_1, \dots, u_p$. In this case the equivalent adjoint system of total differential equations takes the form

$$du_s = \sum_{t=1}^{\tau} U_{s,t} dx_t, \qquad \text{for } s = 1, \dots, p. \tag{5.4}$$

The equations (3.2a) and (5.4) are equivalent in the sense that they have the same integrals [8]. We denote by $\mathcal{M}_u$ the coefficients matrix $\{U_{s,t}\}$ that completely determines the adjoint system (5.4).

For $n = 2$, by permuting the coordinate system on M so as to have in diag $\{a, b, a, b\}$ as the submatrix corresponding to the first 4 columns of the coefficients matrix for the system $\mathcal{S}_{22} = \left\{V_{\xi^1{}'}, V_{\xi^2{}'}, \mathcal{V}_{\xi^1{}''}, \mathcal{V}_{\xi^2{}''}\right\}$, we obtain the transposed matrix $\mathcal{M}_u^T$ of $\mathcal{M}_u$ in the form

$$\mathcal{M}_u^T = \begin{pmatrix} 0 & \frac{a_{yy}}{a} & -\frac{b_x}{a} & \frac{a_y}{a} & -\frac{2b_{xx}}{a} & -\frac{2b_{xy}}{a} \\ -\frac{2a_{xy}}{b} & -\frac{2a_{yy}}{b} & \frac{b_x}{b} & -\frac{a_y}{b} & \frac{b_{xx}}{b} & 0 \\ \frac{2a_y}{a} & 0 & 0 & 0 & -\frac{b_x}{a} & 0 \\ 0 & -\frac{a_y}{b} & 0 & 0 & 0 & \frac{2b_x}{b} \end{pmatrix}.$$

The corresponding system (5.4) of total differential equations can be solved using methods described in [8]. We get stuck with a problem of finding some integrating factors while trying to solve by the method of characteristics the equivalent system (3.2a) of linear partial differential equations for the system of operators $\mathcal{S}_{22}$. However, we readily get the following set of six functions by solving the corresponding adjoint system (5.4).

$$\begin{aligned} T_{12} &= \frac{a_y\, b}{a}, \quad K_{12} = \frac{a_{yy}\, b}{a_y} + b_y, \quad J_{12} = \frac{a_{xy}\, ab}{a_y} - 2a_x \\ T_{21} &= \frac{b_x\, a}{b}, \quad K_{21} = \frac{b_{xx}\, a}{b_x} + a_x, \quad J_{21} = \frac{b_{xy}\, ab}{b_x} - 2b_y. \end{aligned} \tag{5.5}$$

Although their number corresponds to the maximal number of functionally independent invariants in this case, not all of them are actually invariants because the system $\mathcal{S}_{22}$ is not complete. More precisely, only T_{ij} and K_{ij}, for $i, j = 1, 2$ are invariants, and not only the J_{ij}'s are not invariants, but also the equations $J_{i,j} = 0$ are not invariant equations.

Similarly for $n = 3$, the total number $\mathcal{Q}$ of variables defining the invariants is 27, and we have $\tau = 6$. By permuting again the coordinate system on M, so as to have diag $\{a, b, c, a, b, c\}$ as the first six columns of the coefficients matrix for the system of perators $\mathcal{S}_{32} = \left\{V_{\xi^1{}'}, V_{\xi^2{}'}, V_{\xi^3{}'}, \mathcal{V}_{\xi^1{}''}, \mathcal{V}_{\xi^2{}''}, \mathcal{V}_{\xi^3{}''}\right\}$, we obtain a more convenient representation of the 21×6 matrix $\mathcal{M}_u$. The transpose $\mathcal{M}_u^T$ of $\mathcal{M}_u$ in which only its first six columns are represented has the form

$$\mathcal{M}_u^T = \begin{pmatrix} 0 & 0 & \frac{a_{yy}}{a} & \frac{2a_{yz}}{a} & \frac{a_{zz}}{a} & -\frac{b_x}{a} & \dots \\ -\frac{2a_{xy}}{b} & 0 & -\frac{2a_{yy}}{b} & -\frac{2a_{yz}}{b} & 0 & \frac{b_x}{b} & \dots \\ 0 & -\frac{2a_{xz}}{c} & 0 & -\frac{2a_{yz}}{c} & -\frac{2a_{zz}}{c} & 0 & \dots \\ \frac{2a_y}{a} & \frac{2a_z}{a} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{a_y}{b} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -\frac{a_z}{c} & 0 & \dots \end{pmatrix},$$

where the dots represent the remaining 15 matrix columns. Solving the corresponding system (5.4) yields the expected maximal number of 21 functionally independent functions. But since here again the corresponding system of operators $\mathcal{S}_{32}$ is not complete, only 15 of them are actually invariants of G. Reverting back to the original notation $A_1 = a, A_2 = b$ and $A_3 = c$, and using (5.1b), these 15 invariants can be written in the form

$$T_{ij} = \frac{A_{ij}\, A_j}{A_i}, \quad K_{ij} = \frac{A_{ijj}\, A_j}{A_{ij}} + A_{jj}, \quad L_{ijk} = A_{ijk} \left(\frac{A_j A_k}{A_i} \right), \tag{5.6}$$

where $i, j \in \{1,2,3\}$, with $i \neq j$, and where $\{j,k\} = \{1,2,3\} \setminus \{i\}$ for $i = 1,2,3$. We have thus obtained the following result.

Theorem 6. *Let $\mathcal{N}$ be the maximal number of functionally independent invariants of the second prolongation of the group of equivalence transformations of* (1.2) *in n independent variables.*

(a) *For $n = 2$, $\mathcal{N} = 4$, and the invariants are the function T_{ij} and K_{ij} of* (5.5).
(b) *For $n = 3$, $\mathcal{N} = 15$, and the invariants are the functions T_{ij}, K_{ij} and L_{ijk} given by* (5.6).

Contrary to the case of the first prolongation of G, a determination of all invariants of the second prolongation for larger values of n using only invariants of a lower order of n and a symmetry argument does not seem to be obvious. Indeed, the equations (5.5) and (5.6) show that for $n = 3$, the invariants of type T_{ij} and K_{ij} can be simply derived by symmetry from those for $n = 2$ without any further calculations. However, the invariants of type L_{ijk} in (5.6) cannot be obtained from (5.5) using only a symmetry argument. This makes it more difficult to find all the invariants for the second prolongation of G when $n \geq 4$. Nevertheless, we do have the following result which is solely based on a symmetry argument.

Theorem 7. *For $n \geq 3$, a fundamental set of invariants of the second prolongation of G includes all the invariants of type T_{ij}, K_{ij} and L_{ijk} of* (5.6), *whose total number is $n(n^2 + n - 2)/2$.*

Indeed, this result clearly follows from (5.6) and the symmetry inherent in (1.2), by noting that the total number of the T_{ij}, K_{ij} and L_{ijk} for $n \geq 3$ is

$$2\binom{n}{2} + 2\binom{n}{2} + 3\binom{n}{3} = \frac{1}{2}\, n(n^2 + n - 2). \tag{5.7}$$

Although we may not find all the invariants of the second prolongation of G for larger values of $n > 3$ using only symmetry arguments, it should be possible to predict their number. Denote by $M_k^{n,j}$ the number of fundamental invariants of the kth prolongation of equation (1.2) (with n independent variables) involving terms of the form A_I, where I is an index of the form $i_1 i_2 \ldots i_j$ with distinct $i_k \in \{1, \ldots, n\}$ for $k = 1, \ldots, j$. Note that the corresponding type of functions appears for the first time as invariants of (1.2) when the number of independent variables is j, where $2 \leq j \leq n$. If we also denote by M_k^n the number of invariants of the kth prolongation for n variables, then a closer look at equation (5.6) suggests that $A_2^{n,j} = j\binom{n}{j}$ and $M_2^n = M_1^n + W_n$, where

$$W_n = A_2^{n,2} + A_2^{n,3} + \cdots + A_2^{n,n} = \sum_{j=2}^{n} j \binom{n}{j}.$$

Using the properties of binomial coefficients, it can be shown that $\sum_2^n j\binom{n}{j}$ equals $n(2^{n-1} - 1)$. Since by Theorem 3 we have $M_1^n = n(n-1)$, our conjecture follows.

Conjecture. *For any value n of independent variables in equation* (1.2)*, the number M_2^n of functionally independent invariants of the second prolongation of G is $n(2^{n-1} + n - 2)$.*

This conjecture says that $M_2^4 = 40$, and $M_2^5 = 95$. By a result of Lie (see [6]), it is possible to find differential invariants of G of higher order than 2 using invariant differentiation, but we will not discuss that here.

6. Properties of the invariants

It follows from Theorem 1 that every element of the equivalence transformations group G of equation (1.2) can be represented by an invertible map ϕ of the form

$$\phi \colon \mathbb{R}^n \to \mathbb{R}^n : X = (x^1, \ldots, x^n) \mapsto Y = \phi(X) \equiv (\phi^1(x^1), \ldots, \phi^n(x^n)).$$

That is, the ith component of ϕ depends only on the single variable x^i. In a given coordinate system $X = (x^1, \ldots, x^n)$, each equation of the form (1.2) can be represented by the n-tuple $(A_i(X))_{i=1}^n$ or just $(A_i(X))$. It follows from equation (2.4) that under the action of $\phi \in G$, the differential equation $(A_i(X))$ is mapped to the differential equation $(B_i(Y)) = \phi \cdot (A_i(X))$, where

$$B_i(Y) = A_i(\phi^{-1}(Y)) {\phi^i}'(\psi^i(Y)) = \frac{A_i(\psi(Y))}{{\psi^i}'(y^i)},$$

and where $\psi = \phi^{-1}$. If θ is any other element of G, and Id is the identity transformation, it is easy to see that

$$\mathrm{Id} \cdot (A_i(X)) = (A_i(X)), \quad \text{and} \quad \theta \cdot (\phi \cdot (A_i(X))) = \theta \circ \phi \cdot (A_i(X)). \tag{6.1}$$

Thus if we denote by E_n the variety of all differential equations of the form (1.2), then (6.1) shows that the action of G on M induces another group action of G on E_n. This yields a partition of E_n into orbits which can be described by the original action of G on M. For a given element $(A_i(X))$ in E_n, we set

$$T_{ij}^A = A_{ij} A_j / A_i.$$

Theorem 8. *Suppose that the coefficients A_i, for $n = 1, \ldots, n$ in equation* (1.2) *are non vanishing and that exactly p of the functions A_{ij} in the expression of the generators $\mathcal{V}_{\xi^i{}'}$ of the first prolongation of G in* (4.3) *vanish identically. Then two differential equations $(A_i(X))$ and $(B_i(X))$ are equivalent, i.e. they belong to the same orbit of the first prolongation of G if and only if*

$$T_{ij}^A(X) = T_{ij}^B(X)$$

for all of the $n(n-1)-p$ *nonzero such functions* T_{ij}^A *and* T_{ij}^B.

Proof. If the coefficients A_i of (1.2) are non vanishing, then $\operatorname{diag}\{A_1,\dots,A_n\}$ has constant rank n, and by the expression of the generators in (4.3), G acts semi-regularly. More over, the expression of the corresponding invariant functions in (4.5) shows that this action is regular. Since the invariants of G are actually the invariants of the induced group action of G on E_n, the result follows from a theorem of [10, Theorem 2.34] stating that for regular group actions, two points lie in the same orbit if and only if they take on the same values under all invariant functions. □

7. Concluding Remarks

It follows from Theorem 8 above that all differential equations $(A_i(X))$ where $A_i = A_i(x^i)$ depends on x^i alone are equivalent. The expression of the generators in (4.3) shows that the action of G restricted to this family of equations has no invariant of any order.

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