Differential Topology of Fiber Bundles

Karl-Hermann Neeb

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Contents

Chapter 1

Basic Concepts

1.1 The concept of a fiber bundle

Definition 1.1.1 A C^k-*fiber bundle* ($k \in \mathbb{N}_0 \cup \{\infty\}$) is a quadruple (E, M, F, q), consisting of C^k-manifolds E, M and F and a C^k-map $q \colon E \to B$ with the following property of local triviality: Each point $b \in B$ has an open neighborhood U for which there exists a C^k-diffeomorphism

$$\varphi_U \colon U \times F \to q^{-1}(U),$$

satisfying

$$q \circ \varphi_U = p_U \colon U \times F \to U, \quad (u, f) \mapsto u.$$

We use the following terminology:

- E is called the *total space*.
- B is called the *base space*.
- F is called the *fiber type*.
- q is called the *bundle projection*.
- The sets $E_b := q^{-1}(b)$ are called the *fibers of* q.
- φ_U is called a *bundle chart*.
- $E_U := q^{-1}(U)$ is called the *restriction of* E *to* U.
- (E, B, F, q) is called an F-*bundle* over M.

Remark 1.1.2 (Restriction of fiber bundles) If (E, M, F, q) is a C^k-F-bundle over M, $N \subseteq M$ is an open subset and $E_N := q^{-1}(N)$, then $(E_N, N, F, q|_{E_N})$ is a C^k-F-bundle over N.

Example 1.1.3 If M is a smooth n-dimensional manifold, then the tangent bundle $T(M)$ with the projection

$$q\colon T(M) \to M, \quad T_p(M) \ni v \mapsto p$$

yields a smooth fiber bundle $(T(M), M, \mathbb{R}^n, q)$.

Examples 1.1.4 (a) If $\sigma\colon G \times M \to M, (g, m) \mapsto g.m$ is a smooth action of the Lie group G on M which is free and proper, then the Quotient Theorem leads to a smooth fiber bundle $(M, M/G, G, q)$, where

$$M/G := \{G.m\colon m \in M\}$$

is the set of G-orbits in M and $q\colon M \to M/G$ is the canonical map. In the following we always write $[m] := q(m) = G.m$.

(b) The *Möbius strip* is an example for the situation under (b). Here

$$M := (\mathbb{R}\times]-1, 1[)/\mathbb{Z},$$

where the group $\mathbb{Z}$ acts freely and properly by

$$n.(x, y) := (x + n, (-1)^n y).$$

(c) The *Klein bottle* arises similarly as

$$K := \mathbb{R}^2/\Gamma,$$

where $\Gamma \subseteq \mathrm{Diff}(\mathbb{R}^2)$ is the subgroup generated by the two elements

$$\sigma_1(x, y) := (x + 1, -y) \quad \text{and} \quad \sigma_2(x, y) := (x, y + 1).$$

The relation $\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}$ implies that Γ is a non-abelian semidirect product group. The map

$$\varphi\colon \mathbb{Z} \rtimes_\alpha \mathbb{Z} \to \Gamma, \quad \varphi(m, n) := \sigma_2^m\sigma_1^n$$

is an isomorphism for $\alpha(n)(m) := (-1)^n m$.

Examples 1.1.5 (a) The Möbius strip carries a natural fiber bundle structure

$$q\colon M = (\mathbb{R}\times]-1,1[)/\mathbb{Z} \to \mathbb{S}^1 := \{z \in \mathbb{C}^\times : |z| = 1\}, \quad q([x,y]) := e^{2\pi i x}.$$

We thus obtain a smooth fiber bundle $(M, \mathbb{S}^1,]-1,1[, q)$ whose fiber is the open interval $]-1,1[$.

(b) The Klein bottle has a similar bundle structure, defined by

$$q\colon K = \mathbb{R}^2/\Gamma \to \mathbb{S}^1, \quad q([x,y]) := e^{2\pi i x},$$

which leads to the fiber bundle $(K, \mathbb{S}^1, \mathbb{S}^1, q)$. Hence K is an $\mathbb{S}^1$-bundle over $\mathbb{S}^1$.

Examples 1.1.6 Let G be a Lie group and $H \subseteq G$ be a closed subgroup. Then the right action of H on G, by

$$G \times H \to G, \quad (g,h) \mapsto gh$$

is proper, so that the Quotient Theorem yields a manifold structure on the space $G/H = \{gH \colon g \in G\}$ of left cosets of H in G.

If, in addition, H is a normal subgroup, then G/H carries a natural group structure defined by

$$gH \cdot g'H := gg'H.$$

Since the group operations on G/H are smooth, we thus obtain a Lie group structure on G/H.

Definition 1.1.7 (a) We call a sequence

$$\cdots \xrightarrow{\varphi_{i-1}} G_{i-1} \xrightarrow{\varphi_i} G_i \xrightarrow{\varphi_{i+1}} G_{i+1} \xrightarrow{\varphi_{i+2}} G_{i+2} \xrightarrow{\varphi_{i+3}} \cdots$$

of group homomorphisms *exact in* G_i if

$$\operatorname{im}(\varphi_i) = \ker(\varphi_{i+1}).$$

We call the sequence *exact* if it is exact in each G_i.

An exact sequence is said to be *short* if it is of the form

$$\mathbf{1} \to G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \to \mathbf{1}.$$

A short sequence is exact if and only if φ is injective, ψ is surjective and $\mathrm{im}(\varphi) = \ker(\psi)$.

(b) An *extension of Lie groups* is a short exact sequence of morphisms of Lie groups

$$\mathbf{1} \to N \xrightarrow{\iota} G \xrightarrow{q} Q \to \mathbf{1}.$$

Then N can be identified with the closed subgroup $\iota(N) = \ker q$ of G, so that $Q \cong G/\iota(N)$. As we have seen in Example 1.1.6, the right action of N on G by $g.n := g \cdot \iota(n)$ leads to an N-fiber bundle (G, Q, N, q).

In view of the analogy with group extensions, we may think of F-fiber bundles over the manifold M as "extensions" of M by F.

1.2 Coverings

Definition 1.2.1 A fiber bundle (E, B, F, q) is called a *covering* if $\dim F = 0$, i.e., F is a discrete space.

Example 1.2.2 (a) If Γ is a discrete group, then an action $\sigma \colon \Gamma \times M \to M$ on M is proper if and only if for each pair K and Q of compact subsets of M, the set

$$\{g \in \Gamma \colon g.K \cap Q \neq \emptyset\}$$

is finite (Exercise).

If the action is free and proper, then $q \colon M \to M/\Gamma$ is a covering (cf. Examples 1.1.4).

(b) If Γ is a discrete subgroup of the Lie group G, then the quotient map $q \colon G \to G/\Gamma$ is a covering (cf. Example 1.1.6).

Example 1.2.3 The action of the discrete group $\mathbb{Q}$ by translations on $\mathbb{R}$ is free but not proper. The quotient group $\mathbb{R}/\mathbb{Q}$ carries no natural manifold structure.

Examples 1.2.4 (a) $M = \mathbb{R}$, $\Gamma = \mathbb{Z}$ with $\sigma(x, y) := x + y$ leads to the quotient Lie group $\mathbb{T} := \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.

(b) $M = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$ with $\sigma(x, y) := x + y$ leads to $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

(c) $M = \mathbb{C}$, $\Gamma = 2\pi i\mathbb{Z}$ with $\sigma(x, y) := x + y$ leads to $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C}^\times$, where the quotient map can be realized by the exponential function $\exp \colon \mathbb{C} \to \mathbb{C}^\times$.

(d) $M = \mathbb{C}$, $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$ with $\sigma(x, y) := x + y$ leads to an *elliptic curve* $\mathbb{C}/\Gamma$. These are one-dimensional complex manifolds which are, as real smooth manifolds, diffeomorphic to $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, but they are not necessarily diffeomorphic to $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ as a complex manifold.

1.3 Morphisms of Bundles

Definition 1.3.1 Let (E_i, B_i, F_i, q_i), $i = 1, 2$, be fiber bundles.

(a) A smooth map $\varphi_E \colon E_1 \to E_2$ is called a *morphism of fiber bundles* if there exists a smooth map $\varphi_B \colon B_1 \to B_2$ with

$$q_2 \circ \varphi_E = \varphi_B \circ q_1.$$

Since q_1 is surjective, the map φ_B is uniquely determined by φ_E.

(b) A morphism $\varphi_E \colon E_1 \to E_2$ of fiber bundles is called an *isomorphism* if there exists a morphism $\psi_E \colon E_2 \to E_1$ satisfying

$$\varphi_E \circ \psi_E = \mathrm{id}_{E_2} \quad \text{and} \quad \psi_E \circ \varphi_E = \mathrm{id}_{E_1}.$$

This condition is equivalent to φ_E being a diffeomorphism mapping fibers of E_1 into fibers of E_2 (Exercise). If an isomorphism $\varphi_E \colon E_1 \to E_2$ exists, then the corresponding bundles are said to be *isomorphic*.

(c) If $B_1 = B_2 = B$ and $F_1 = F_2 = F$, then a morphism $\varphi_E \colon E_1 \to E_2$ of fiber bundles is called an *equivalence* if $\varphi_B = \mathrm{id}_B$. Accordingly, the corresponding bundles are said to be *equivalent*. It is easy to see that we thus obtain an equivalence relation on the class of all F-fiber bundles over B. The set of all equivalence classes is denoted $\mathrm{Bun}(B, F)$. It is a major problem in differential topology to calculate these sets for concrete manifolds M and F.

(d) A fiber bundle (E, B, F, q) is called *trivial* if it is equivalent to the bundle $(B \times F, B, F, p_B)$, where $p_B(b, f) = b$. It is easy to see that this requirement is equivalent to E being isomorphic to $(B \times F, B, F, p_B)$ (Exercise).

Remark 1.3.2 (a) A smooth map $\varphi_E \colon E_1 \to E_2$ is a morphism of fiber bundles if and only if it maps each fiber $E_{1,b}$, $b \in B_1$, into some fiber $E_{2,c}$ of E_2 for some $c \in B_2$. Then $\varphi_B \colon B_1 \to B_2$ is defined by $\varphi_B(b) = c$, and since q_1 is a submersion, the smoothness of φ_B follows from the smoothness of $\varphi_B \circ q_1 = q_2 \circ \varphi_E$.

Definition 1.3.3 (The group of bundle automorphisms) The set $\mathrm{Aut}(E)$ of automorphisms of the fiber bundle (E, B, F, q) is a group under composition of maps and

$$\Gamma\colon \mathrm{Aut}(E) \to \mathrm{Diff}(B), \quad r_B(\varphi_E) = \varphi_B$$

is a group homomorphism. Its kernel is the group

$$\mathrm{Gau}(E) := \{\varphi \in \mathrm{Aut}(E)\colon q \circ \varphi = q\}$$

of self-equivalences of E, resp., *gauge transformations*. We thus obtain an exact sequence of groups

$$\mathbf{1} \to \mathrm{Gau}(E) \to \mathrm{Aut}(E) \xrightarrow{\Gamma} \mathrm{Diff}(B).$$

The automorphism group of a trivial bundle

For two manifolds B and F we consider the trivial bundle $E := B \times F$, defined by the projection $p_B\colon B \times F \to B$. Each diffeomorphism $\varphi \in \mathrm{Diff}(B)$ lifts to a bundle automorphism by $\widetilde{\varphi}(b, f) := (\varphi(b), f)$, which leads to a group homomorphism

$$\sigma\colon \mathrm{Diff}(B) \to \mathrm{Aut}(B \times F), \quad \varphi \mapsto \widetilde{\varphi}$$

splitting Γ (Definition 1.3.3).

We call a map $\gamma\colon B \to \mathrm{Diff}(F)$ *smooth* if the corresponding map

$$\widetilde{\gamma}\colon B \times F \to F, \quad \widetilde{\gamma}(b, f) := \gamma(b)(f)$$

is smooth and write $C^\infty(B, \mathrm{Diff}(F))$ for the set of smooth maps in this sense. We would like to turn this set into a group with respect to the pointwise multiplication, but it is not obvious that it is closed under inversion. This is a consequence of the following lemma.

Lemma 1.3.4 *The map*

$$\Phi\colon C^\infty(B, \mathrm{Diff}(F)) \to \mathrm{Gau}(B \times F), \quad \gamma \mapsto \Phi_\gamma, \quad \Phi_\gamma(b, f) := (b, \gamma(b)(f))$$

is a bijection satisfying

$$\Phi_{\gamma_1} \circ \Phi_{\gamma_2} = \Phi_{\gamma_1 \cdot \gamma_2}.$$

In particular, $C^\infty(B, \mathrm{Diff}(F))$ is a group with respect to the pointwise product:

$$(\gamma_1 \cdot \gamma_2)(b) := \gamma_1(b) \circ \gamma_2(b)$$

and Φ is a group isomorphism.

Proposition 1.3.5 *By* $\alpha(\varphi)(\gamma) := \gamma \circ \varphi^{-1}$ *we obtain a group homomorphism*

$$\alpha\colon \operatorname{Diff}(B) \to \operatorname{Aut}(C^\infty(B, \operatorname{Diff}(F))),$$

and the map

$$C^\infty(B, \operatorname{Diff}(F)) \rtimes_\alpha \operatorname{Diff}(B) \to \operatorname{Aut}(B \times F), \quad (\gamma, \varphi) \mapsto \Phi_\gamma \circ \widetilde{\varphi}$$

is a group isomorphism.

Pullbacks

Proposition 1.3.6 *Let* $f\colon X \to Z$ *and* $g\colon Y \to Z$ *be smooth maps and assume that either* f *or* g *is a submersion. Then the* fiber product

$$X \times_Z Y := \{(x, y) \in X \times Y \colon f(x) = g(y)\}$$

is a closed submanifold of $X \times Y$*. If* f *is a submersion, then the same holds for the projection*

$$p_Y \colon X \times_Z Y \to Y.$$

Definition 1.3.7 (Pullbacks of fiber bundles) If (E, B, F, q) is a fiber bundle and $f\colon X \to B$ a smooth map, then

$$f^*E := \{(x, e) \in X \times E \colon f(x) = q(e)\}$$

carries the structure of an F-fiber bundle, defined by the projection $p_X \colon f^*E \to X$. This bundle is called the *pullback of* E *by* f.

Remark 1.3.8 (a) If E is trivial, then any pullback f^*E is also trivial.
(b) If f is constant, then f^*E is also trivial.

Proposition 1.3.9 *The image of* $\Gamma\colon \operatorname{Aut}(E) \to \operatorname{Diff}(B)$ *coincides with the subgroup*

$$\operatorname{Diff}(B)_E := \{\varphi \in \operatorname{Diff}(B) \colon \varphi^*E \sim E\}.$$

Example 1.3.10 The Hopf fibration

$$q\colon \mathbb{S}^3 \to \mathbb{S}^2 \cong \mathbb{P}_1(\mathbb{C}) \cong \mathbb{S}^3/\mathbb{T}, \quad (z_1, z_2) \mapsto \mathbb{T}.(z_1, z_2),$$

defines a fiber bundle $(E, B, F, q) = (\mathbb{S}^3, \mathbb{S}^2, \mathbb{S}^1, q)$ with the property that $\operatorname{Diff}(B)_E \neq \operatorname{Diff}(B)$. As we shall see later, the diffeomorphism

$$\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad \varphi(x) = -x$$

satisfies $\varphi^*E \not\sim E$, so that $\varphi \notin \operatorname{Diff}(B)_E$. To verify such an assertion, one needs to know that $\operatorname{Bun}(\mathbb{S}^2, \mathbb{S}^1) \cong \mathbb{Z}$ (this set even has a natural group structure!), where E corresponds to the element 1 and φ^*E to the element -1.

1.4 Principal Bundles

Definition 1.4.1 Let G be a Lie group and $k \in \mathbb{N} \cup \{\infty\}$. A C^k-*principal bundle* is a quintuple (P, M, G, q, σ), where $\sigma\colon P \times G \to P$ is a C^k-right action with the property of local triviality: Each point $m \in M$ has an open neighborhood U for which there exists a C^k-diffeomorphism

$$\varphi_U \colon U \times G \to q^{-1}(U),$$

satisfying $q \circ \varphi_U = p_U$ and the equivariance property

$$\varphi_U(u, gh) = \varphi_U(u, g).h \quad \text{for} \quad u \in U, g, h \in G.$$

Remark 1.4.2 (a) For each principal bundle (P, M, G, q, σ), the quadruple (P, M, G, q) is a fiber bundle.

(b) The right action of G on P is free and proper and the natural map $\overline{q}\colon P/G \to M, p.G \mapsto q(p)$, is a diffeomorphism.

(c) Conversely, in view of the Quotient Theorem, each free and proper right C^k-action $\sigma\colon M \times G \to M$ defines the principal bundle $(M, M/G, G, q, \sigma)$.

Example 1.4.3 (a) The Hopf fibration

$$q\colon \mathbb{S}^3 \to \mathbb{S}^3/\mathbb{T} \cong \mathbb{S}^2 \cong \mathbb{P}_1(\mathbb{C}), \quad (z_1, z_2) \mapsto \mathbb{T}.(z_1, z_2),$$

is a $\mathbb{T}$-principal bundle over $\mathbb{S}^2$.

(b) More generally, we may consider for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $m \geq n$ the action of the compact Lie group $\mathrm{U}_n(\mathbb{K})$ on the set

$$\begin{aligned} S &:= \{V \in M_{m,n}(\mathbb{K})\colon V^*V = \mathbf{1}_n\} \\ &= \{(v_1, \ldots, v_n) \in (\mathbb{K}^m)^n \cong M_{m,n}(\mathbb{K})\colon \langle v_i, v_j\rangle = \delta_{ij}\} \end{aligned}$$

of orthonormal n-frames in $\mathbb{K}^m$ by $V.g := Vg$ (matrix product). Identifying S with the set of isometric embeddings

$$\varphi_V \colon \mathbb{K}^n \to \mathbb{K}^m, \quad x \mapsto Vx,$$

we see that the map $\varphi \mapsto \mathrm{im}(\varphi)$ yields a bijection

$$S/\,\mathrm{U}_n(\mathbb{K}) \to \mathrm{Gr}_{n,m}(\mathbb{K})$$

of the orbit space onto the Graßmannian of n-dimensional subspaces in $\mathbb{K}^m$.

For $n = 1$ and $d := \dim \mathbb{K} \in \{1, 2, 4\}$, we have

$$U_1(\mathbb{K}) = \{\lambda \in \mathbb{K}\colon |\lambda| = 1\} \cong \mathbb{S}^{d-1}$$

and

$$\mathrm{Gr}_{1,n}(\mathbb{K}) = \mathbb{P}_{n-1}(\mathbb{K}) = \mathbb{P}(\mathbb{K}^n)$$

is the $(n-1)$-dimensional projective space over $\mathbb{K}$.

(c) There is also a non-compact picture of the Graßmannian. For that we consider the set

$$S := \{V \in M_{m,n}(\mathbb{K})\colon \operatorname{rank} V = n\}$$

and let $\mathrm{GL}_n(\mathbb{K})$ act by right multiplication. Identifying S with the set of linear embeddings

$$\varphi\colon \mathbb{K}^n \to \mathbb{K}^m,$$

we see that the map $\varphi \mapsto \mathrm{im}(\varphi)$ yields a bijection

$$S/\,\mathrm{GL}_n(\mathbb{K}) \to \mathrm{Gr}_{n,m}(\mathbb{K})$$

of the orbit space onto the Graßmannian of n-dimensional subspaces in $\mathbb{K}^m$.

Definition 1.4.4 Let (P_i, M_i, G, q_i), $i = 1, 2$, be principal bundles.

(a) A smooth map $\varphi_P\colon P_1 \to P_2$ is called a *morphism of principal bundles* if it is G-equivariant, i.e.,

$$\varphi_P(p.g) = \varphi_P(p).g \quad \text{for} \quad p \in P_1, g \in G.$$

Then it is in particular a morphism of fiber bundles and induces a smooth map $\varphi_M\colon M_1 \to M_2$.

(b) A morphism $\varphi_P\colon P_1 \to P_2$ of fiber bundles is called an *isomorphism* if there exists a morphism $\psi_P\colon P_2 \to P_1$ satisfying

$$\varphi_P \circ \psi_P = \mathrm{id}_{P_2} \quad \text{and} \quad \psi_P \circ \varphi_P = \mathrm{id}_{P_1}.$$

If an isomorphism $\varphi_P\colon P_1 \to P_2$ exists, then the corresponding principal bundles are said to be *isomorphic.*

(c) If $M_1 = M_2 = M$, then a morphism $\varphi_P\colon P_1 \to P_2$ of principal bundles is called an *equivalence* if $\varphi_M = \mathrm{id}_M$. Accordingly, the corresponding bundles are said to be *equivalent.* It is easy to see that we thus obtain an equivalence relation on the class of all principal G-bundles over M. The set of all equivalence classes is denoted $\mathrm{Bun}(M, G)$. For finite-dimensional Lie groups, the description of this set is considerably easier than the corresponding classification problem for fiber bundles.

(d) A principal bundle (P, M, G, q, σ) is called *trivial* if it is equivalent to the bundle $(M \times G, M, G, p_M, \sigma)$, where $p_M(m, g) = m$ and $\sigma((m, g), h) = (m, gh)$.

The following lemma is a convenient tool to detect isomorphisms of principal bundles.

Proposition 1.4.5 *A morphism $\varphi_P \colon P_1 \to P_2$ of G-principal bundles is an isomorphism if and only if the induced map $\varphi_M \colon M_1 \to M_2$ is a diffeomorphism.*

Proposition 1.4.6 *A principal G-bundle $q \colon P \to M$ is trivial if and only if it has a smooth section.*

Proof. If the bundle is trivial, then it clearly has a smooth section. If, conversely, $s \colon M \to P$ is a smooth section, then the map

$$\varphi \colon M \times G \to P, \quad (m, g) \mapsto s(m)g$$

is a smooth morphism of principal G-bundle with $\varphi_M = \mathrm{id}_M$, hence an equivalence by Proposition 1.4.5. ■

Definition 1.4.7 (The group of bundle automorphisms) For a principal bundle (P, M, G, q, σ), the *automorphism group* is

$$\mathrm{Aut}(P) = \{\varphi \in \mathrm{Diff}(P) \colon (\forall g \in G)\ \varphi \circ \sigma_g = \sigma_g \circ \varphi\} = \mathrm{Diff}(P)^G,$$

i.e., the group of all diffeomorphisms of P commuting with the G-action.

Proposition 1.4.8 (The automorphism group of a trivial principal bundle) *Consider the trivial G-bundle $(M \times G, M, G, p_M, \sigma)$ with $\sigma((m, g), h) := (m, g).h := (m, gh)$. Then the group $\mathrm{Diff}(M)$ acts by G-bundle isomorphisms on $M \times G$ via*

$$\varphi.(m, g) := \widetilde{\varphi}(m, g) := (\varphi(m), g)$$

and the group $C^\infty(M,G)$, on which we define the group structure by the pointwise product, acts by gauge transformations via

$$\gamma.(m,g) := \Phi_\gamma(m,g) := (m, \gamma(m)g).$$

Moreover, $\alpha(\varphi)(\gamma) := \gamma \circ \varphi^{-1}$ *defines is a group homomorphism*

$$\alpha\colon \operatorname{Diff}(M) \to \operatorname{Aut}(C^\infty(M,G)),$$

and the map

$$C^\infty(M,G) \rtimes_\alpha \operatorname{Diff}(M) \to \operatorname{Aut}(M \times G), \quad (\gamma,\varphi) \mapsto \Phi_\gamma \circ \widetilde{\varphi},$$

is an isomorphism of groups. In particular,

$$\operatorname{Gau}(M \times G) \cong C^\infty(M,G).$$

Definition 1.4.9 (Pullbacks of principal bundles) If (P,M,G,q,σ) is a principal bundle and $f\colon X \to M$ a smooth map, then the *pullback bundle of P by f.*

$$f^*P := \{(x,p) \in X \times P \colon f(x) = q(p)\}$$

carries the structure of a G-principal bundle, defined by the projection $p_X\colon f^*P \to X$ and the G-action $(x,p).g := (x, p.g)$.

Remark 1.4.10 (a) If P is trivial, then any pullback f^*P is also trivial.

(b) If f is constant, then f^*P is also trivial.

(c) For the projection map $q\colon P \to M$, the pullback q^*P is a trivial G-principal bundle over P. In fact, the map $s(p) := (p,p)$ defines a smooth section (cf. Proposition 1.4.6).

Proposition 1.4.11 *The image of* $\Gamma\colon \operatorname{Aut}(P) = \operatorname{Diff}(P)^G \to \operatorname{Diff}(M)$ *coincides with the subgroup*

$$\operatorname{Diff}(M)_P := \{\varphi \in \operatorname{Diff}(M) \colon \varphi^*P \sim P\}.$$

1.5 Vector Bundles

Definition 1.5.1 Let V be a vector space. A *vector bundle* is a fiber bundle $(\mathbb{V}, M, V, q)$ for which all fibers $\mathbb{V}_m$, $m \in M$, carry vector space structures, and each point $m \in M$ has an open neighborhood U for which there exists a C^k-diffeomorphism

$$\varphi_U : U \times V \to q^{-1}(U) = \mathbb{V}_U,$$

satisfying $q \circ \varphi_U = p_U$ and all maps

$$\varphi_{U,x} : V \to \mathbb{V}_x, \quad v \mapsto \varphi_U(x, v)$$

are linear isomorphisms.

Example 1.5.2 For each smooth n-dimensional manifold M, the tangent bundle TM is a vector bundle $(TM, M, \mathbb{R}^n, q)$.

Definition 1.5.3 If $q : \mathbb{V} \to M$ is a smooth vector bundle, then the space

$$\Gamma\mathbb{V} := \{s \in C^\infty(M, \mathbb{V}) : q \circ s = \mathrm{id}_M\}$$

of smooth sections carries a vector space structure defined by

$$(s_1 + s_2)(m) := s_1(m) + s_2(m) \quad \text{and} \quad (\lambda s)(m) := \lambda s(m).$$

Moreover, for each $f \in C^\infty(M) := C^\infty(M, \mathbb{R})$ and $s \in \Gamma\mathbb{V}$, the product

$$(fs)(m) := f(m)s(m)$$

is a smooth section of $\mathbb{V}$. We thus obtain on $\Gamma\mathbb{V}$ the structure of a $C^\infty(M)$-module, i.e., the following relations hold for $f, f_1, f_2 \in C^\infty(M)$ and $s, s_1, s_2 \in \Gamma\mathbb{V}$:

$$(f_1 + f_2)s = f_1 s + f_2, \quad (f_1 f_2)s = f_1(f_2 s), \quad f(s_1 + s_2) = fs_1 + fs_2.$$

Definition 1.5.4 A morphisms of vector bundles $f : \mathbb{V}_1 \to \mathbb{V}_2$ is a morphism of fiber bundles which is fiberwise linear. Accordingly we define isomorphisms, equivalences and triviality of a vector bundle.

In a similar way as for principal bundles, we obtain:

Proposition 1.5.5 (The automorphism group of a trivial vector bundle) *Consider the trivial vector bundle* $(M \times V, M, V, p_M)$*. Then the group* $\mathrm{Diff}(M)$ *acts by vector bundle isomorphisms on* $M \times V$ *via*

$$\varphi.(m,v) := \widetilde{\varphi}(m,v) := (\varphi(m), v)$$

and the group $C^\infty(M, \mathrm{GL}(V))$*, on which we define the group structure by the pointwise product, acts by gauge transformations via*

$$\gamma.(m,v) := \Phi_\gamma(m,v) := (m, \gamma(m)v).$$

Moreover, $\alpha(\varphi)(\gamma) := \gamma \circ \varphi^{-1}$ *defines is a group homomorphism*

$$\alpha\colon \mathrm{Diff}(M) \to \mathrm{Aut}(C^\infty(M, \mathrm{GL}(V))),$$

and the map

$$C^\infty(M, \mathrm{GL}(V)) \rtimes_\alpha \mathrm{Diff}(M) \to \mathrm{Aut}(M \times V), \quad (\gamma, \varphi) \mapsto \Phi_\gamma \circ \widetilde{\varphi},$$

is an isomorphism of groups. In particular,

$$\mathrm{Gau}(M \times V) \cong C^\infty(M, \mathrm{GL}(V)).$$

Remark 1.5.6 If $q\colon \mathbb{V} \to M$ is a trivial vector bundle with n-dimensional fiber, then there exist n sections $s_1, \ldots, s_n \in \Gamma\mathbb{V}$ which are linearly independent in each point of M. Then the map

$$s\colon M \times \mathbb{R}^n \to \mathbb{V}, \quad (m,x) \mapsto \sum_{i=1}^n x_i s_i(x)$$

defines an isomorphism of vector bundles.

Therefore, for a vector bundle $\mathbb{V}$, the maximal number k for which there exists a set $s_1, \ldots, s_k$ of sections which are everywhere linearly independent measures the "degree of triviality" of the bundle. We call this number $\alpha(\mathbb{V})$.

The famous 1-2-4-8-Theorem asserts that the tangent bundle $T(\mathbb{S}^n)$ is trivial if and only if $n = 0, 1, 3, 7$. For all other n-dimensional spheres we have $\alpha(T(\mathbb{S}^n)) < n$. For spheres these numbers can be computed with the theory of Clifford algebras. For $n = 2$, the Hairy Ball Theorem (Satz vom Igel) asserts that each vector field has a zero, i.e., that $\alpha(T(\mathbb{S}^2)) = 0$.

Proposition 1.5.7 *For a vector bundle* $(\mathbb{V}, M, V, q)$*, the assignment*

$$(\varphi.s)(m) := \varphi(s(\varphi_M^{-1}(m)))$$

defines a representation of the automorphism group $\mathrm{Aut}(\mathbb{V})$ *on* $\Gamma\mathbb{V}$*.*

1.6 Associated bundles and structure groups

Definition 1.6.1 Let (P, M, G, q, σ) be a principal bundle and

$$\tau\colon G \times F \to F, \quad (g, f) \mapsto g.f$$

be a smooth action of G on F. Then

$$\tau(P) := P \times_\tau F := P \times_G F := (P \times F)/G$$

is the set of G-orbits of G in $P \times F$ under the left action

$$g.(p, f) := (p.g^{-1}, g.f)$$

(which is free and proper). We write $[p, f] := G.(p, f)$ for the G-orbit of (p, f). Then

$$\widetilde{q}\colon P \times_G F \to M, \quad [p, f] \mapsto q(p)$$

defines an F-fiber bundle $(P \times_G F, M, F, \widetilde{q})$.

The so obtained bundle is called the *bundle associated to P by τ* and G is called the *structure group* of the associated bundle $P \times_G F$.

Lemma 1.6.2 *For an associated bundle $E = P \times_G F$, the map*

$$\Phi\colon \operatorname{Aut}(P) \to \operatorname{Aut}(E), \quad \Phi(\varphi)([p, f]) := [\varphi(p), f]$$

is a group homomorphism with $\Phi(\varphi)_M = \varphi_M$.

$$\begin{array}{ccc} \operatorname{Aut}(P) & \xrightarrow{\Phi} & \operatorname{Aut}(E) \\ \downarrow \Gamma_P & & \downarrow \Gamma_E \\ \operatorname{Diff}(M) & \xrightarrow{\text{id}} & \operatorname{Diff}(M) \end{array}$$

The following proposition provides a realization of section of associated bundles in terms of smooth functions on P. This is convenient in many situations.

Proposition 1.6.3 (Sections of associated bundles) *If (P, M, G, q, σ) is a principal bundle and $\tau\colon G \times F \to F$ a smooth action, then we write*

$$C^\infty(P, F)^G := \{\alpha \in C^\infty(P, F)\colon (\forall g \in G)(\forall p \in P)\ \alpha(p.g) = g^{-1}.\alpha(p)\}$$

for the space of equivariant smooth functions. Then the map

$$\Psi\colon C^\infty(P, F)^G \to \Gamma(P \times_G F),$$

defined by $\Psi(\alpha)(q(p)) := [p, \alpha(p)]$, is a bijection.

Example 1.6.4 If (P, M, G, q, σ) is a principal bundle and $\pi\colon G \to \mathrm{GL}(V)$ is a smooth representation of G on V, then the associated bundle

$$\mathbb{V} := P \times_\pi V := P \times_G V$$

is a vector bundle. The canonical homomorphism $\mathrm{Aut}(P) \to \mathrm{Aut}(\mathbb{V})$ defines a natural representation of the group $\mathrm{Aut}(P)$ on the vector space

$$\Gamma\mathbb{V} \cong C^\infty(P, V)^G \quad \text{by} \quad (\varphi.\alpha)(p) := \alpha(\varphi^{-1}(p))$$

(cf. Proposition 1.6.3).

Example 1.6.5 (Induced representations of Lie groups) Let G be a Lie group and $H \subseteq G$ a closed subgroup. Then $q\colon G \to G/H$ defines an H-principal bundle over the homogeneous space $M = G/H$ and the left action of G on itself defined by the group multiplication yields an action of G by H-bundle automorphism:

$$g.(g'.h) = gg'h = (gg').h.$$

The map

$$G \to \mathrm{Diff}(G)^H = \mathrm{Aut}_H(G), \quad g \mapsto \lambda_g$$

is the corresponding group homomorphism.

In, particular, we obtain for each smooth representation $\pi\colon H \to \mathrm{GL}(V)$ a representation of G on the space $\Gamma(\mathbb{V}_\pi)$ of smooth sections of the vector bundle $\mathbb{V}_\pi := G \times_\pi V$:

$$(\pi_G(g)s)(xH) = g.s(g^{-1}xH).$$

Identifying $\Gamma(\mathbb{V}_\pi)$ with $C^\infty(G, V)^H$, this representation corresponds to

$$\pi_G(g)\alpha = \alpha \circ \lambda_g^{-1}.$$

This representation is called an *induced representation*. More precisely, it is called the representation of G induced by the representation π of H.

The conjugation bundle

Definition 1.6.6 Each Lie group G acts on itself by conjugation

$$C\colon G \times G \to G, \quad (g,x) \mapsto c_g(x) := gxg^{-1}.$$

For each principal bundle (P, M, G, q, σ) we thus obtain an associated bundle

$$C(P) := P \times_G G = (P \times G)/G, \quad g.(p,h) = (p.g^{-1}, ghg^{-1}),$$

called the *conjugation bundle of* P.

Its space of smooth sections can be identified with

$$\begin{aligned}\Gamma(C(P)) &\cong C^\infty(P,G)^G \\ &= \{f \in C^\infty(P,G)\colon (\forall p \in P)(\forall g \in G)\ f(p.g) = g^{-1} f(p) g\}.\end{aligned}$$

We see in particular that the space of smooth sections carries a group structure, defined by pointwise multiplication.

The following proposition shows that this space of sections is isomorphic to the group of gauge transformations. We thus obtain a nice description of gauge transformations by smooth functions, even if the bundle is not trivial.

Proposition 1.6.7 *Each equivariant function $f \in C^\infty(P,G)^G$ defines a gauge transformation of P by*

$$\varphi_f(p) := p.f(p)$$

and the map

$$\Phi\colon \Gamma(C(P)) \cong C^\infty(P,G)^G \to \mathrm{Gau}(P), \quad f \mapsto \varphi_f$$

is an isomorphism of groups.

Chapter 2

Bundles and Cocycles

In this chapter we first explain how fiber bundles can be constructed from trivial bundles by a glueing process, defined by a collection of transition functions. This leads to a completely local description on the bundle in terms of bundle charts. This is applied in Section 2.2 to obtain a convenient local description of sections of associated bundle in terms of families of local sections of trivial bundles, satisfying certain transition relations. Finally, we use the local description of bundles to show that each vector bundle $\mathbb{V}$ with fiber V is associated in a natural way to a $\mathrm{GL}(V)$-principal bundle, called the frame bundle $\mathrm{Fr}(\mathbb{V})$ of $\mathbb{V}$.

2.1 Local description of bundles

Fiber bundles from transition functions

Let (E, B, F, q) be a fiber bundle and $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of B, for which the restrictions E_{U_i} are trivial, so that we have bundle charts

$$\varphi_i := \varphi_{U_i} \colon U_i \times F \to E_{U_i}.$$

For $i, j \in I$ we put $U_{ij} := U_i \cap U_j$. If this set is non-empty, then the map

$$\varphi_i^{-1} \circ \varphi_j \colon U_{ij} \times F \to U_{ij} \times F$$

is a self equivalence, i.e., a gauge transformation, of the trivial bundle $U_{ij} \times F$, hence of the form

$$\varphi_i^{-1} \circ \varphi_j(b, f) = (b, g_{ij}(b)(f)), \tag{2.1}$$

where $g_{ij} \in C^\infty(U_{ij}, \mathrm{Diff}(F))$, in the sense of Lemma 1.3.4. It is clear from the construction that the functions g_{ij} satisfy the relations

$$g_{ii} = \mathbf{1} \quad \text{and} \quad g_{ij}g_{jk} = g_{ik} \quad \text{on} \quad U_{ijk} := U_i \cap U_j \cap U_k. \tag{2.2}$$

Proposition 2.1.1 (a) *If* $(g_{ij})_{i,j\in I}$ *is a collection of functions* $g_{ij} \in C^\infty(U_{ij}, \mathrm{Diff}(F))$ *satisfying* (2.2)*, then there exists a bundle* (E, B, F, q) *and bundle charts*

$$\varphi_{U_i} : U_i \times F \to E_{U_i}$$

such that (2.1) *holds.*

(b) *Two bundles constructed as in* (a) *for families* (g_{ij}) *and* (g'_{ij}) *are equivalent if and only if there exist smooth functions* $h_i \in C^\infty(U_i, \mathrm{Diff}(F))$ *with*

$$g_{ij} = h_i \cdot g'_{ij} \cdot h_j^{-1} \quad on \quad U_{ij}.$$

Proof. (Sketch) (a) We consider the disjoint union

$$\widetilde{E} := \dot\cup_{i\in I}\{i\} \times U_i \times F.$$

Then

$$(i, x, f) \sim (j, x', f') \quad \Longleftrightarrow x' = x, f' = g_{ji}(x)(f)$$

defines an equivalence relation on $\widetilde{E}$. The quotient topology turns the set $E := \widetilde{E}/\sim$ of equivalence classes into a topological space and on which the projection map

$$q \colon E \to B, \quad q([(i, x, f)]) := x$$

is well-defined and continuous.

For each $i \in I$ we have a map

$$\varphi_i \colon U_i \times F \to E, \quad (x, f) \mapsto [(i, x, f)]$$

which is easily seen to be injective. These maps satisfy

$$\varphi_{ij}(x, f) := \varphi_i^{-1} \circ \varphi_j(x, f) = (x, g_{ij}(x)(f)), \quad x \in U_{ij}.$$

Since for each open subset $O \subseteq U_i \times F$, the subsets

$$\varphi_{ji}(O \cap (U_{ij} \times F)) \subseteq U_j \times F$$

are open, the image $\varphi_i(O)$ is open in E. Hence φ_i is an open embedding. In particular, all subsets $\varphi_i(U_i \times F)$ of E are Hausdorff.

Since the map q is continuous, two elements $p_1 \neq p_2 \in E$ with $q(p_1) \neq q(p_2)$ have disjoint open neighborhoods. If $q(p_1) = q(p_2)$, both points lie in some set $\varphi_i(U_i \times F)$, which is open and Hausdorff, so that p_1 and p_2 has disjoint open neighborhoods contained in this subset. Therefore E is Hausdorff.

Since the maps φ_{ij} are diffeomorphisms, there exists a unique smooth manifold structure on E for which all maps

$$\varphi_i \colon U_i \times F \to E, \quad (x, f) \mapsto [(i, x, f)]$$

are diffeomorphisms onto open subsets. We thus obtain on E the structure of a fiber bundle (E, B, F, q), for which the maps φ_i, $i \in I$, form a bundle atlas with transitions functions g_{ij}.

(b) If E' is the bundle constructed from the functions g'_{ij} are in (a), then any bundle equivalence $\varphi_E \colon E' \to E$ is given in the bundle charts over the open sets U_i by some $h_i \in C^\infty(U_i, \mathrm{Diff}(E))$ because

$$\varphi_i^{-1} \circ \varphi_E \circ \varphi'_i \in \mathrm{Gau}(U_i \times F), \quad (b, f) \mapsto (b, h_i(b)(f))$$

(cf. Lemma 1.3.4). Then we immediately get from

$$\varphi_j^{-1} \circ \varphi_i \circ \varphi_i^{-1} \circ \varphi_E \circ \varphi'_i \circ \varphi_i'^{-1} \circ \varphi'_j = \varphi_j^{-1} \circ \varphi_E \circ \varphi'_j$$

the relation

$$g_{ji} \cdot h_i \cdot g'_{ij} = h_j \quad \text{on} \quad U_{ij},$$

and hence

$$g_{ij} = h_i \cdot g'_{ij} \cdot h_j^{-1}. \tag{2.3}$$

If, conversely, (2.3) is satisfied by the family $(h_i)_{i \in I}$, then

$$\varphi_E(\varphi'(b, f)) := \varphi_i(b, h_i(b)(f))$$

yields a well-defined bundle equivalence $E' \to E$ because for $(b, f) \in U_{ij} \times F$ the relation

$$\varphi'_j(b, f) = \varphi'_i(b, g'_{ij}(b)(f))$$

implies that

$$\varphi_i(b, h_i(b) \circ g'_{ij}(b)(f)) = \varphi_i(b, g_{ij}(b) \circ h_j(b)(f)) = \varphi_j(b, h_j(b)(f)).$$

■

Principal bundles

Now let G be a Lie group and $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of the smooth manifold M. We define the set

$$\check{Z}^1(\mathcal{U}, G) := \{(g_{ij}) \in \prod_{ij} C^\infty(U_{ij}, G)\colon g_{ii} = \mathbf{1}, g_{ij}g_{jk} = g_{ik} \text{ on } U_{ijk}, i, j, k \in I\}$$

of *smooth Čech-*1*-cocycles with values in G with respect to* $\mathcal{U}$. Since the groups $C^\infty(U_i, G)$ are naturally isomorphic to the group $\mathrm{Gau}(U_i \times G)$ of gauge transformations of the trivial G-bundle (Proposition 1.4.8), it follows as in the preceding subsection, that each bundle atlas $(\varphi_i, P_{U_i})_{i\in I}$ of a G-principal bundle (P, M, G, q, σ) leads to a Čech-1-cocycle (g_{ij}), defined by

$$\varphi_i^{-1} \circ \varphi_j(x, g) = (x, g_{ij}(x)g), \tag{2.4}$$

Conversely, a slight variation of Proposition 2.1.1 implies that for each element $(g_{ij}) \in \check{Z}^1(\mathcal{U}, G)$ there exists a principal bundle (P, M, G, q, σ) which is trivial on each U_i and for which there exist bundle charts whose transition functions are given by the (g_{ij}). Moreover, the bundle constructed for two 1-cocycles (g_{ij}) and (g'_{ij}) are equivalent if and only if there exists a collection of smooth functions $h_i\colon U_i \to G$, satisfying

$$g_{ij} = h_i g'_{ij} h_j^{-1} \quad \text{on} \quad U_{ij}.$$

Definition 2.1.2 To get hold of the algebraic structure behing these constructions, we define the set

$$\check{C}^0(\mathcal{U}, G) := \prod_{i\in I} C^\infty(U_i, G),$$

of *Čech-*0*-cochains*, which carries a natural group structure, given by the pointwise product in each factor. This group acts naturally on the set $\check{Z}^1(\mathcal{U}, G)$ by

$$(h_i) * (g_{ij}) := (h_i g_{ij} h_j^{-1}).$$

The set of orbits for this group action is denoted

$$\check{H}^1(\mathcal{U}, G) := \check{Z}^1(\mathcal{U}, G)/\check{C}^0(\mathcal{U}, G)$$

and called the *first Čech cohomology set with respect to* $\mathcal{U}$ *with values in* G. This is a set with a base point $[\mathbf{1}]$, given by the orbit of the cocycle given by the constant functions $g_{ij} = \mathbf{1}$. If G is not abelian, this set carries no natural group structure.

Collecting the information obtained so far, we have:

Proposition 2.1.3 *The construction in Proposition 2.1.1 yields a bijection between the elements of* $\check{H}^1(\mathcal{U}, G)$ *and principal bundles* (P, M, G, q, σ) *for which all restrictions* P_{U_i} *are trivial.*

Since a general G-bundle over M need not be trivialized by a given open cover $\mathcal{U}$, one has to use refinements of open covers. We call an open cover $\mathcal{V} = (V_j)_{j\in J}$ a *refinement* of the open cover $\mathcal{U} = (U_i)_{i\in I}$ if there exists for each $j \in J$ an element $\alpha(j) \in I$ with $V_j \subseteq U_{\alpha(j)}$. Then we have a natural map

$$r_{\mathcal{U},\mathcal{V}} \colon \check{H}^1(\mathcal{U}, G) \to \check{H}^1(\mathcal{V}, G), \quad [(g_{ij})_{i,j\in I}] \mapsto [(g_{\alpha(i),\alpha(j)}|_{V_{ij}})_{i,j\in J}].$$

The corresponding direct limit set is denoted

$$\check{H}^1(M, G) := \varinjlim{}_{\mathcal{U}} \check{H}^1(\mathcal{U}, G).$$

Theorem 2.1.4 *The cohomology set* $\check{H}^1(M, G)$ *parameterizes the equivalence classes of principal* G*-bundles over* M*.*

Remark 2.1.5 If G is abelian, then the restriction maps $r_{\mathcal{U},\mathcal{V}}$ are group homomorphisms, so that the direct limit $\check{H}^1(M, G)$ inherits a natural group structure.

2.2 Sections of associated bundles

In this section (P, M, G, q, σ) denotes a principal bundle and $(\varphi_i, U_i)_{i\in I}$ a corresponding bundle atlas with transition functions $g_{ij} \in C^\infty(U_{ij}, G)$, defined by

$$\varphi_i(x, g) = \varphi_j(x, g_{ji}(x)g).$$

Let $\tau \colon G \times F \to F$ be a smooth action of G on the manifold F and form the associated bundle $P_\tau = P \times_\tau F$. Then P_τ is also trivialized over each U_i by the canonical bundle charts

$$\varphi^\tau(x, f) := [\varphi_i(x, \mathbf{1}), f],$$

so that each smooth section $s \colon M \to P_\tau$ leads on each U_i to a smooth function $s_i \colon U_i \to F$, defined by

$$s(x) = \varphi_i^\tau(x, s_i(x)), \quad x \in U_i.$$

Proposition 2.2.1 *A family $s_i \in C^\infty(U_i, F)$, $i \in I$, defines a section of P_τ if and only if the relation*

$$s_j(x) = g_{ji}(x).s_i(x) \quad \text{holds for} \quad x \in U_{ij}.$$

Remark 2.2.2 In Proposition 1.6.3 we have seen that the map

$$\Gamma(P_\tau) \to C^\infty(P, F)^G, \quad s \mapsto \alpha, \quad s(q(p)) = [p, \alpha(p)]$$

is a bijection.

The passage between the family $(s_i)_{i \in I}$ defining a section s and this picture is established by the formula

$$\alpha(\varphi_i(x, g)) = g^{-1}.s_i(x).$$

In fact, for $x \in U_i$ and $p = \varphi_i(x, \mathbf{1})$, we have

$$s(x) = \varphi_i^\tau(x, s_i(x)) = [\varphi_i(x, \mathbf{1}), s_i(x)].$$

Remark 2.2.3 For the bundle P itself, the trivializations φ_i define smooth local sections

$$\gamma_i \colon U_i \to P, \quad \gamma_i(x) = \varphi_i(x, \mathbf{1}),$$

satisfying

$$\gamma_i = \gamma_j \cdot g_{ji} \quad \text{on} \quad U_{ij}.$$

A global section $s \colon M \to P$ now corresponds to a family of smooth functions $s_i \colon U_i \to G$, satisfying

$$s(x) = \gamma_i(x) \cdot s_i(x), \quad x \in U_i, \quad s_j = g_{ji} s_i \quad \text{on} \quad U_{ij}.$$

Proposition 2.2.4 *For a principal bundle (P, M, G, q, σ), the following are equivalent:*

(a) *P has a global smooth section.*

(b) *P is trivial.*

(c) *There exists a smooth G-equivariant function $f \colon P \to G$.*

Proof. (a) $\Rightarrow$ (b): If $s \colon M \to P$ is a smooth section, then the map $\varphi \colon M \times G \to P, (m, g) \mapsto s(m)g$ is a G-bundle equivalence (cf. Proposition 1.4.5).
(b) $\Rightarrow$ (c): If $\psi \colon P \to M \times G$ is a G-bundle equivalence, then $\mathrm{pr}_G \circ \psi \colon P \to G$ is smooth and G-equivariant.
(c) $\Rightarrow$ (b): The map $\varphi := (q, f) \colon P \to M \times G$ is a G-bundle equivalence.
(b) $\Rightarrow$ (a) is trivial. ∎

Example 2.2.5 Let $C(P) = P \times_C G$ denote the conjugation bundle whose space of sections $\Gamma(C(P))$ is isomorphic to the group of gauge transformations. In view of Proposition 2.2.1, the sections of this bundle are given in terms of the local trivializations by a family of smooth functions $s_i \colon U_i \to G$, satisfying

$$s_j = g_{ji} \cdot s_i \cdot g_{ji}^{-1} = g_{ji} \cdot s_i \cdot g_{ij} \quad \text{on} \quad U_{ij}. \tag{2.5}$$

The corresponding local gauge transformations are given by

$$\psi_i \colon U_i \times G \to U_i \times G, \quad \psi_i(x, g) = (x, s_i(x)g).$$

We may also write the transformation law (2.5) as

$$g_{ji} = s_j g_{ji} s_i^{-1},$$

which identifies the tuple $(s_i)_{i \in I}$ as an element in the group $\check{C}^0(\mathcal{U}, G)$, stabilizing the cocycle (g_{ij}). This shows that

$$\mathrm{Gau}(P) \cong \check{C}^0(\mathcal{U}, G)_{(g_{ij})}.$$

2.3 Vector bundles as associated bundles

In Section 1.6 we have already seen how to obtain vector bundles as associated bundle of a principal bundle (P, M, G, q, σ) via representations (π, V) of the structure group G. The following theorem provides a converse:

Theorem 2.3.1 (a) *If $(P, M, \mathrm{GL}(V), q, \sigma)$ is a $\mathrm{GL}(V)$-principal bundle and (π, V) the identical representation of $\mathrm{GL}(V)$ on V, then $\mathbb{V} := P \times_\pi V$ is a vector bundle with fiber V.*

(b) *If, conversely, $(\mathbb{V}, M, V, q)$ is a vector bundle over M, then its frame bundle*

$$\mathrm{Fr}(\mathbb{V}) = \bigcup_{m \in M} \mathrm{Iso}(V, \mathbb{V}_m)$$

carries the structure of a $\mathrm{GL}(V)$-principal bundle with respect to the action $\varphi.g := \varphi \circ g$. The evaluation map

$$\mathrm{Fr}(\mathbb{V}) \times V \to \mathbb{V}, \quad (\varphi, v) \mapsto \varphi(v)$$

induces a bundle equivalence

$$\mathrm{Fr}(\mathbb{V}) \times_\pi V \to \mathbb{V}, \quad [(\varphi, v)] \mapsto \varphi(v).$$

Remark 2.3.2 The preceding theorem easily implies that for two vector bundles $\mathbb{V}_1$, $\mathbb{V}_2$ with fiber V over M we have

$$\mathbb{V}_1 \sim \mathbb{V}_2 \quad \Longleftrightarrow \quad \mathrm{Fr}(\mathbb{V}_1) \sim \mathrm{Fr}(\mathbb{V}_2).$$

This leads to a bijection of the set $\mathrm{Bun}(M, \mathrm{GL}(V))$ of equivalence classes of $\mathrm{GL}(V)$-bundles over M and the set $\mathrm{Vbun}(M, V)$ of equivalence classes of V-vector bundles over M.

Using the frame bundle, we can immediately attach to a vector bundle several new vector bundles in a functorial manner.

Definition 2.3.3 Let $(\mathbb{V}, M, V, q)$ be a vector bundle over M and $\mathrm{Fr}(\mathbb{V})$ its frame bundle.

(a) The dual bundle is defined as the vector bundle associated to $\mathrm{Fr}(\mathbb{V})$ by the dual representation

$$\pi^*\colon\ \mathrm{GL}(V) \to \mathrm{GL}(V^*), \quad \pi^*(g)(\alpha) := \alpha \circ g^{-1},$$

$$\mathbb{V}^* := \mathrm{Fr}(\mathbb{V}) \times_{\pi^*} V^*.$$

We then have $\mathbb{V}^*_m \cong (\mathbb{V}_m)^*$ in each $m \in M$.

(b) More generally, we can use the tensor representation

$$\pi_{r,s} := \pi^{\otimes r} \otimes (\pi^*)^{\otimes s}\colon\ \mathrm{GL}(V) \to \mathrm{GL}(V^{\otimes r} \otimes (V^*)^{\otimes s}),$$

to define the tensor bundle

$$T^{r,s}(\mathbb{V}) := \mathbb{V}^{\otimes r} \otimes (\mathbb{V}^*)^{\otimes s},$$

whose fibers are the spaces

$$T^{r,s}(\mathbb{V})_m := \mathbb{V}_m^{\otimes r} \otimes (\mathbb{V}^*_m)^{\otimes s}.$$

In particular

$$T^{(0,1)}(\mathbb{V}) \cong \mathbb{V}^* \quad \text{and} \quad T^{(1,1)}(\mathbb{V}) \cong \mathrm{End}(\mathbb{V}).$$

(c) Of particular importance are also symmetric and alternating tensor bundles. For example the representation

$$\mathrm{Sym}^2(\pi)\colon\ \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^2(V, \mathbb{R})), \quad (g.\beta)(v_1, v_2) := \beta(g^{-1}v_1, g^{-1}v_2)$$

on the space $\mathrm{Sym}^2(V,\mathbb{R})$ of real-valued symmetric bilinear forms on V leads to the bundle

$$\mathrm{Sym}^2(\mathbb{V}) := \mathrm{Fr}(\mathbb{V}) \times_{\mathrm{Sym}^2(\pi)} \mathrm{Sym}^2(V,\mathbb{R}),$$

whose fibers are the spaces

$$\mathrm{Sym}^2(\mathbb{V})_m \cong \mathrm{Sym}^2(\mathbb{V}_m,\mathbb{R})$$

of symmetric bilinear forms on $\mathbb{V}_m$.

A *Riemannian bundle metric on* $\mathbb{V}$ is a smooth section $g = (g_m)_{m\in M}$ of this bundle with the addition property that each g_m is positive definite. Using partitions of unity, it is easy to show that such bundle metrics always exist.

Definition 2.3.4 (a) The preceding constructions apply in particular to the tangent bundle $\mathbb{V} = TM$ of a smooth manifold M. In this case we also write

$$T^*(M) := T(M)^*$$

for the *cotangent bundle* and

$$T^{(r,s)}(M) := T^{(r,s)}(TM)$$

for the tensor bundles over M.

(b) Suppose that $n = \dim M$, so that TM is an $\mathbb{R}^n$-vector bundle with structure group $\mathrm{GL}_n(\mathbb{R})$. For any finite-dimensional vector space V, the representation

$$\mathrm{Alt}^k(\pi)\colon\ \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}(\mathrm{Alt}^k(\mathbb{R}^n,V)), \quad (g.\alpha)(v_1,\ldots,v_k) := \alpha(g^{-1}v_1,\ldots,g^{-1}v_k)$$

leads to the bundle $\mathrm{Alt}^k(TM,V)$ whose sections form the space

$$\Omega^k(M,V) := \Gamma\,\mathrm{Alt}^k(TM,V)$$

of *V-valued k-forms on M*. The fibers of $\mathrm{Alt}^k(TM,V)$ are the spaces

$$\mathrm{Alt}^k(TM,V)_m \cong \mathrm{Alt}^k(T_m(M),V)$$

of alternating k-linear maps $T_m(M)^k \to V$.

Chapter 3

Cohomology of Lie Algebras

This chapter is of a purely algebraic nature. Here we discuss the algebraic essentials behind the geometric theory of connections, their curvature and characteristic classes of principal bundles. The key point which makes this isolation of the algebraic content possible is that differential forms on a manifold can be viewed as Lie algebra cochains for the Lie algebra $\mathcal{V}(M)$ of smooth vector fields with values in the module $C^\infty(M)$ of smooth functions on M.

3.1 The Chevalley–Eilenberg Complex

Definition 3.1.1 Let V and W be vector spaces and $p \in \mathbb{N}$. A multilinear map $f\colon W^p \to V$ is called *alternating* if

$$f(w_{\sigma_1}, \ldots, w_{\sigma_p}) = \operatorname{sgn}(\sigma) f(w_1, \ldots, w_p)$$

for $w_i \in W$ and $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_p$. We write $\operatorname{Alt}^p(V, W)$ for the set of p-linear alternating maps and $\operatorname{Mult}^p(V, W)$ for the space of all p-linear maps $V^p \to W$. For $p = 0$ we put $\operatorname{Mult}^0(V, W) := \operatorname{Alt}^0(V, W) := W$.

Definition 3.1.2 Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module.
(a) For $p \in \mathbb{N}_0$, we write $C^p(\mathfrak{g}, V) := \operatorname{Alt}^p(\mathfrak{g}, V)$ for the space of alternating p-linear mappings $\mathfrak{g}^p \to V$ and call the elements of $C^p(\mathfrak{g}, V)$ *p-cochains*. We also define

$$C(\mathfrak{g}, V) := \bigoplus_{k=0}^{\infty} C^k(\mathfrak{g}, V).$$

On $C^p(\mathfrak{g}, V)$ we define the *(Chevalley–Eilenberg) differential* $\mathtt{d}$ by

$$\mathtt{d}\omega(x_0, \ldots, x_p) := \sum_{j=0}^{p} (-1)^j x_j \cdot \omega(x_0, \ldots, \widehat{x}_j, \ldots, x_p) + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p),$$

where $\widehat{x}_j$ means that x_j is omitted. Observe that the right hand side defines for each $\omega \in C^p(\mathfrak{g}, V)$ an element of $C^{p+1}(\mathfrak{g}, V)$ because it is alternating (Exercise!). Putting the differentials on all the spaces $C^p(\mathfrak{g}, V)$ together, we obtain a linear map $\mathtt{d} = \mathtt{d}_{\mathfrak{g}} \colon C(\mathfrak{g}, V) \to C(\mathfrak{g}, V)$.

The elements of the subspace

$$Z^k(\mathfrak{g}, V) := \ker(\mathtt{d}|_{C^k(\mathfrak{g},V)})$$

as called *k-cocycles*, and the elements of the spaces

$$B^k(\mathfrak{g}, V) := \mathtt{d}\big(C^{k-1}(\mathfrak{g}, V)\big) \quad \text{and} \quad B^0(\mathfrak{g}, V) := \{0\}$$

are called *k-coboundaries*. We will see below that $\mathtt{d}^2 = 0$, which implies that $B^k(\mathfrak{g}, V) \subseteq Z^k(\mathfrak{g}, V)$, so that it makes sense to define the k^{th} *cohomology space of* $\mathfrak{g}$ *with values in the module* V:

$$H^k(\mathfrak{g}, V) := Z^k(\mathfrak{g}, V)/B^k(\mathfrak{g}, V).$$

(b) We further define for each $x \in \mathfrak{g}$ and $p > 0$ the *insertion map* or *contraction*

$$i_x \colon C^p(\mathfrak{g}, V) \to C^{p-1}(\mathfrak{g}, V), \quad \big(i_x\omega\big)(x_1, \ldots, x_{p-1}) = \omega(x, x_1, \ldots, x_{p-1}).$$

We further define i_x to be 0 on $C^0(\mathfrak{g}, V)$.

Remark 3.1.3 For elements of low degree we have in particular:

$$\begin{aligned}
p = 0: &\quad \mathtt{d}\omega(x) &&= x \cdot \omega \\
p = 1: &\quad \mathtt{d}\omega(x, y) &&= x \cdot \omega(y) - y \cdot \omega(x) - \omega([x, y]) \\
p = 2: &\quad \mathtt{d}\omega(x, y, z) &&= x \cdot \omega(y, z) - y \cdot \omega(x, z) + z \cdot \omega(x, y) \\
& &&\quad -\omega([x, y], z) + \omega([x, z], y) - \omega([y, z], x) \\
& &&= x \cdot \omega(y, z) + y \cdot \omega(z, x) + z \cdot \omega(x, y) \\
& &&\quad -\omega([x, y], z) - \omega([y, z], x) - \omega([z, x], y).
\end{aligned}$$

Example 3.1.4 (a) This means that

$$Z^0(\mathfrak{g}, V) = V^{\mathfrak{g}} := \{v \in V : \mathfrak{g} \cdot v = \{0\}\}$$

is the maximal trivial submodule of V. Since $B^0(\mathfrak{g}, V)$ is trivial by definition, we obtain

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}.$$

(b) The elements $\alpha \in Z^1(\mathfrak{g}, V)$ are also called *crossed homomorphisms*. They are defined by the condition

$$\alpha([x, y]) = x \cdot \alpha(y) - y \cdot \alpha(x), \quad x, y \in \mathfrak{g}.$$

The elements $\alpha(x) \cdot v := x \cdot v$ of the subspace $B^1(\mathfrak{g}, V)$ are also called *principal crossed homomorphisms*. It follows immediately from the definition of a $\mathfrak{g}$-module that each principal crossed homomorphism is a crossed homomorphism.

If V is a trivial module, then it is not hard to compute the cohomology spaces in degree one. In view of $\{0\} = \mathtt{d}V = \mathtt{d}C^0(\mathfrak{g}, V) = B^1(\mathfrak{g}, V)$, we have $H^1(\mathfrak{g}, V) = Z^1(\mathfrak{g}, V)$, and the condition that $\alpha \colon \mathfrak{g} \to V$ is a crossed homomorphism reduces to $\alpha([x, y]) = \{0\}$ for $x, y \in \mathfrak{g}$. This leads to

$$H^1(\mathfrak{g}, V) \cong \mathrm{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], V) \cong \mathrm{Hom}_{\mathrm{Liealg}}(\mathfrak{g}, V).$$

Example 3.1.5 Let $\mathfrak{g}$ be an abelian Lie algebra and V a trivial $\mathfrak{g}$-module. Then $\mathtt{d} = 0$, so that $H^p(\mathfrak{g}, V) = C^p(\mathfrak{g}, V) = \mathrm{Alt}^p(\mathfrak{g}, V)$ holds for each $p \in \mathbb{N}_0$.

Our first goal will be to show that $\mathtt{d}^2 = 0$. This can be proved directly by an awkward computation. We will follow another way which is more conceptual and leads to additional insights and tools which are useful in other situations: Let (ρ_V, V) be a $\mathfrak{g}$-module. Then the representations ρ_j of $\mathfrak{g}$ on the space $\mathrm{Mult}^p(\mathfrak{g}, V)$ of p-linear V-valued maps on $\mathfrak{g}$, defined by

$$\big(\rho_j(x)\omega\big)(x_1, \ldots, x_p) := -\omega(x_1, \ldots, x_{j-1}, \mathrm{ad}\, x(x_j), x_{j+1}, \ldots, x_p)$$

do pairwise commute. Therefore the sum of these representations is again a representation, and since they also commute with composition with $\rho_V(x)$, we obtain:

Lemma 3.1.6 *We have a representation $\mathfrak{g}$ on $C(\mathfrak{g}, V)$, given on $\alpha \in C^p(\mathfrak{g}, V)$*

$$\mathcal{L}_x\alpha = \rho_V(x) \circ \alpha + \sum_{i=1}^{p} \rho_i(x)\alpha,$$

i.e.,

$$\begin{aligned}\mathcal{L}_x\omega(x_1, \ldots, x_p)\\ &= x \cdot \omega(x_1, \ldots, x_p) - \sum_{j=1}^{p} \omega(x_1, \ldots, x_{j-1}, [x, x_j], x_{j+1}, \ldots, x_p)\\ &= x \cdot \omega(x_1, \ldots, x_p) + \sum_{j=1}^{p} (-1)^j \omega([x, x_j], x_1, \ldots, \widehat{x}_j, \ldots, x_p).\end{aligned}$$

Note that $\rho_0 := \sum_{i=1}^{p} \rho_i$ is the representation on $C^p(\mathfrak{g}, V)$ corresponding to the trivial module structure on V.

Lemma 3.1.7 (Cartan Formula) *The representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}\big(C(\mathfrak{g}, V)\big)$ satisfies the Cartan formula*

$$\mathcal{L}_x = \mathtt{d} \circ i_x + i_x \circ \mathtt{d}. \tag{3.1}$$

Proof. Using the insertion map i_{x_0}, we can rewrite the formula for the differential as

$$\begin{aligned}&\big(i_{x_0}\mathtt{d}\omega\big)(x_1, \ldots, x_p)\\ &= x_0 \cdot \omega(x_1, \ldots, x_p) - \sum_{j=1}^{p} (-1)^{j-1} x_j \cdot \omega(x_0, \ldots, \widehat{x}_j, \ldots, x_p)\\ &+ \sum_{j=1}^{p} (-1)^j \omega([x_0, x_j], x_1, \ldots, \widehat{x}_j, \ldots, x_p)\\ &+ \sum_{1 \le i < j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p)\end{aligned}$$

$$
\begin{aligned}
&= x_0 \cdot \omega(x_1, \ldots, x_p) - \sum_{j=1}^{p} \omega(x_1, \ldots, x_{j-1}, [x_0, x_j], x_{j+1}, \ldots, x_p) \\
&\quad - \sum_{j=1}^{p} (-1)^{j-1} x_j \cdot \omega(x_0, \ldots, \widehat{x}_j, \ldots, x_p) \\
&\quad - \sum_{1 \leq i < j} (-1)^{i+j} \omega(x_0, [x_i, x_j], \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p) \\
&= (\mathcal{L}_{x_0}\omega)(x_1, \ldots, x_p) - \mathtt{d}\big(i_{x_0}\omega\big)(x_1, \ldots, x_p).
\end{aligned}
$$

This proves the Cartan formula. ■

Lemma 3.1.8 *For $x, y \in \mathfrak{g}$, we have $i_{[x,y]} = [i_x, \mathcal{L}_y]$.*

Proof. The explicit formula for $\mathcal{L}_y$ (Lemma 3.1.6) yields for $x = x_1$: $i_x\mathcal{L}_y = \mathcal{L}_y i_x - i_{[y,x]}$. ■

Lemma 3.1.9 *For each $x \in \mathfrak{g}$, we have $[\mathcal{L}_x, \mathtt{d}] = 0$.*

Proof. In view of Lemma 3.1.8, we obtain with the Cartan formula

$$
\begin{aligned}
[\mathcal{L}_x, \mathcal{L}_y] &= [\mathtt{d} \circ i_x, \mathcal{L}_y] + [i_x \circ \mathtt{d}, \mathcal{L}_y] \\
&= [\mathtt{d}, \mathcal{L}_y] \circ i_x + \mathtt{d} \circ i_{[x,y]} + i_{[x,y]} \circ \mathtt{d} + i_x \circ [\mathtt{d}, \mathcal{L}_y] \\
&= [\mathtt{d}, \mathcal{L}_y] \circ i_x + \mathcal{L}_{[x,y]} + i_x \circ [\mathtt{d}, \mathcal{L}_y],
\end{aligned}
$$

so that the fact that ρ is a representation leads to

$$[\mathtt{d}, \mathcal{L}_y] \circ i_x + i_x \circ [\mathtt{d}, \mathcal{L}_y] = 0. \tag{3.2}$$

We now prove by induction over k that $[\mathtt{d}, \mathcal{L}_y]$ vanishes on $C^k(\mathfrak{g}, V)$. For $\omega \in C^0(\mathfrak{g}, V) \cong V$, we have

$$
\begin{aligned}
&\big([\mathtt{d}, \mathcal{L}_y]\omega\big)(x) = \mathtt{d}(y \cdot \omega)(x) - \big(y \cdot (\mathtt{d}\omega)\big)(x) \\
&= x \cdot (y \cdot \omega) - \big(y \cdot (x \cdot \omega) - \mathtt{d}\omega([y, x])\big) = [x, y] \cdot \omega + [y, x] \cdot \omega = 0.
\end{aligned}
$$

Suppose that $[\mathtt{d}, \mathcal{L}_y]C^k(\mathfrak{g}, V) = \{0\}$. Then (3.2) implies that

$$i_x[\mathtt{d}, \mathcal{L}_y]C^{k+1}(\mathfrak{g}, V) = -[\mathtt{d}, \mathcal{L}_y]i_x C^{k+1}(\mathfrak{g}, V) \subseteq [\mathtt{d}, \mathcal{L}_y]C^k(\mathfrak{g}, V) = \{0\}$$

for each $x \in \mathfrak{g}$. Hence $[\mathtt{d}, \mathcal{L}_y]C^{k+1}(\mathfrak{g}, V) = \{0\}$. By induction, this leads to $[\mathtt{d}, \mathcal{L}_y] = 0$ for each $y \in \mathfrak{g}$. ■

Proposition 3.1.10 $\mathtt{d}^2 = 0$.

Proof. We put Lemma 3.1.9 into the Cartan Formula (3.1) and get

$$0 = [\mathtt{d}, \mathcal{L}_x] = \mathtt{d}^2 \circ i_x - i_x \circ \mathtt{d}^2. \tag{3.3}$$

We use this formula to show by induction over k that $\mathtt{d}^2$ vanishes on $C^k(\mathfrak{g}, V)$. For $\omega \in C^0(\mathfrak{g}, V) \cong V$, we have $\mathtt{d}\omega(x) = x \cdot \omega$ and

$$\mathtt{d}^2\omega(x,y) = x\cdot\mathtt{d}\omega(y) - y\cdot\mathtt{d}\omega(x) - (\mathtt{d}\omega)([x,y]) = x\cdot(y\cdot\omega) - y\cdot(x\cdot\omega) - [x,y]\cdot\omega = 0.$$

If $\mathtt{d}^2(C^k(\mathfrak{g}, V)) = \{0\}$, we use (3.3) to see that

$$i_x\mathtt{d}^2(C^{k+1}(\mathfrak{g}, V)) = \mathtt{d}^2 i_x C^{k+1}(\mathfrak{g}, V) \subseteq \mathtt{d}^2(C^k(\mathfrak{g}, V)) = \{0\}$$

for all $x \in \mathfrak{g}$, and hence that $\mathtt{d}^2(C^{k+1}(\mathfrak{g}, V)) = \{0\}$. By induction on k, this proves $\mathtt{d}^2 = 0$. ■

Since the differential commutes with the action of $\mathfrak{g}$ on the graded vector space $C(\mathfrak{g}, V)$ (Lemma 3.1.9), the space of k-cocycles and of k-coboundaries is $\mathfrak{g}$-invariant, so that we obtain a natural representation of $\mathfrak{g}$ on the quotient spaces $H^k(\mathfrak{g}, V)$.

Lemma 3.1.11 *The action of* $\mathfrak{g}$ *on* $H^k(\mathfrak{g}, V)$ *is trivial, i.e.,*

$$\mathcal{L}_{\mathfrak{g}}(Z^k(\mathfrak{g}, V)) \subseteq B^k(\mathfrak{g}, V).$$

Proof. In view of Lemma 3.1.7, we have for $\omega \in Z^k(\mathfrak{g}, V)$ the relation

$$\mathcal{L}_x\omega = i_x\mathtt{d}\omega + \mathtt{d}\big(i_x\omega\big) = \mathtt{d}\big(i_x\omega\big) \in B^k(\mathfrak{g}, V).$$

Hence the $\mathfrak{g}$-action induced on the cohomology space $H^k(\mathfrak{g}, V)$ is trivial. ■

Extensions and Cocycles

In this section we interprete the cohomology spaces in degree 2 in terms of extensions of Lie algebras.

Definition 3.1.12 (a) Let $\mathfrak{g}$ and $\mathfrak{n}$ be Lie algebras. A *short exact sequence* of Lie algebra homomorphisms

$$\mathbf{0} \to \mathfrak{n} \xrightarrow{\iota} \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to \mathbf{0}$$

(this means ι injective, q surjective, and $\operatorname{im} \iota = \ker q$) is called an *extension of* $\mathfrak{g}$ *by* $\mathfrak{n}$. If we identify $\mathfrak{n}$ with its image in $\widehat{\mathfrak{g}}$, this means that $\widehat{\mathfrak{g}}$ is a Lie algebra containing $\mathfrak{n}$ as an ideal satisfying $\widehat{\mathfrak{g}}/\mathfrak{n} \cong \mathfrak{g}$. If $\mathfrak{n}$ is abelian (central) in $\widehat{\mathfrak{g}}$, then the extension is called *abelian (central)*. Two extensions $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_1 \twoheadrightarrow \mathfrak{g}$ and $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_2 \twoheadrightarrow \mathfrak{g}$ are called *equivalent* if there exists a Lie algebra homomorphism $\varphi\colon \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the diagram

$$\begin{array}{ccccc} \mathfrak{n} & \xrightarrow{\iota_1} & \widehat{\mathfrak{g}}_1 & \xrightarrow{q_1} & \mathfrak{g} \\ \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{n}}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{g}}} \\ \mathfrak{n} & \xrightarrow{\iota_2} & \widehat{\mathfrak{g}}_2 & \xrightarrow{q_2} & \mathfrak{g} \end{array}$$

commutes. It is easy to see that this implies that φ is an isomorphism of Lie algebras (Exercise).

(b) We call an extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q = \mathfrak{n}$ *trivial*, or say that the extension *splits*, if there exists a Lie algebra homomorphism $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. In this case the map

$$\mathfrak{n} \rtimes_\delta \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (a, x) \mapsto a + \sigma(x)$$

is an isomorphism, where the semidirect sum is defined by the homomorphism

$$\delta\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n}), \quad \delta(x)(a) := [\sigma(x), a].$$

For a trivial central extension we have $\delta = 0$ and therefore $\widehat{\mathfrak{g}} \cong \mathfrak{n} \times \mathfrak{g}$.

(c) A particular important case arises if $\mathfrak{n}$ is abelian. Then each Lie algebra extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{n}$ leads to a $\mathfrak{g}$-module structure on $\mathfrak{n}$ defined by $q(x) \cdot n := [x, n]$, which is well defined because $[\mathfrak{n}, \mathfrak{n}] = \{0\}$. It is easy to see that equivalent extensions lead to the same module structure (Exercise). Therefore it makes sense to write $\mathrm{Ext}_\rho(\mathfrak{g}, \mathfrak{n})$ for the set of equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{n}$ corresponding to the module structure given by the representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n}) = \mathrm{der}(\mathfrak{n}).$$

For a $\mathfrak{g}$-module V, we also write $\mathrm{Ext}(\mathfrak{g}, V) := \mathrm{Ext}_{\rho_V}(\mathfrak{g}, V)$, where ρ_V is the representation of $\mathfrak{g}$ on V corresponding to the module structure.

Proposition 3.1.13 *For an element $\omega \in C^2(\mathfrak{g}, V)$, the formula*

$$[(v,x),(v',x')] = (x\cdot v' - x'\cdot v + \omega(x,x'), [x,x'])$$

defines a Lie bracket on $V \times \mathfrak{g}$ if and only if $\omega \in Z^2(\mathfrak{g}, V)$. For a cocycle $\omega \in Z^2(\mathfrak{g}, V)$ we write $\mathfrak{g}_\omega := V \oplus_\omega \mathfrak{g}$ for the corresponding Lie algebra. Then we obtain for each cocycle ω an extension of $\mathfrak{g}$ by the abelian ideal V:

$$\mathbf{0} \to V \hookrightarrow \mathfrak{g}_\omega \twoheadrightarrow \mathfrak{g} \to \mathbf{0}.$$

This extension splits if and only if ω is a coboundary.

The map $\widetilde{\Gamma}\colon Z^2(\mathfrak{g}, V) \to \mathrm{Ext}(\mathfrak{g}, V), \omega \mapsto [\mathfrak{g}_\omega]$, defined by assigning to ω the equivalence class of the extension $\mathfrak{g}_\omega$ induces a bijection

$$\Gamma\colon H^2(\mathfrak{g}, V) \to \mathrm{Ext}(\mathfrak{g}, V), \quad [\omega] \mapsto [\mathfrak{g}_\omega].$$

Therefore $H^2(\mathfrak{g}, V)$ classifies the abelian extensions of $\mathfrak{g}$ by V for which the corresponding representation of $\mathfrak{g}$ on V is given by the module structure on V.

Proof. An easy calculation shows that $\mathfrak{g}_\omega = V \oplus_\omega \mathfrak{g}$ is a Lie algebra if and only if ω is a 2-cocycle, i.e., an element of $Z^2(\mathfrak{g}, V)$.

To see that every abelian Lie algebra extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q = V$ (as a $\mathfrak{g}$-module) is equivalent to some $\mathfrak{g}_\omega$, let $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be a linear map with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then the map

$$V \times \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (v,x) \mapsto v + \sigma(x)$$

is a bijection, and it becomes an isomorphism of Lie algebras if we endow $V \times \mathfrak{g}$ with the bracket of $\mathfrak{g}_\omega$ for

$$\omega(x,y) := [\sigma(x), \sigma(y)] - \sigma([x,y]). \tag{3.4}$$

This implies that $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is equivalent to $\mathfrak{g}_\omega$, and therefore that $\widetilde{\Gamma}$ is surjective.

Two Lie algebras $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega'}$ are equivalent as V-extensions of $\mathfrak{g}$ if and only if there exists a linear map $\varphi\colon \mathfrak{g} \to V$ such that the map

$$\widetilde{\varphi}\colon \mathfrak{g}_\omega = V \times \mathfrak{g} \to \mathfrak{g}_{\omega'} = V \times \mathfrak{g}, \quad (a,x) \mapsto (a + \varphi(x), x)$$

is a Lie algebra homomorphism. This means that

$$\begin{aligned}\widetilde{\varphi}([(a,x),(a',x')]) &= \widetilde{\varphi}(x\cdot a' - x'\cdot a + \omega(x,x'), [x,x']) \\ &= \big(x\cdot a' - x'\cdot a + \omega(x,x') + \varphi([x,x']), [x,x']\big)\end{aligned}$$

equals

$$\begin{aligned}[\widetilde{\varphi}(a,x),\widetilde{\varphi}(a',x')] &= (x\cdot(a'+\varphi(x')) - x'\cdot(a+\varphi(x)) + \omega'(x,x'),[x,x']) \\ &= (x\cdot a' - x'\cdot a + x\cdot\varphi(x') - x'\cdot\varphi(x) + \omega'(x,x'),[x,x']),\end{aligned}$$

which is equivalent to

$$\omega'(x,x') - \omega(x,x') = \varphi([x,x']) - x\cdot\varphi(x') + x'\cdot\varphi(x) = -(\mathtt{d}\varphi)(x,x').$$

Therefore $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega'}$ are equivalent abelian extensions of $\mathfrak{g}$ if and only if $\omega'-\omega$ is a coboundary. Hence $\widetilde{\Gamma}$ induces a bijection $\Gamma\colon H^2(\mathfrak{g},V)\to \mathrm{Ext}(\mathfrak{g},V)$. ■

3.2 Differential forms as Lie algebra cochains

Let M be a smooth manifold, $\mathfrak{g} := \mathcal{V}(M)$ the Lie algebra of smooth vector fields on M and W a vector space. We consider the $\mathfrak{g}$-module $V := C^\infty(M,W)$ of smooth W-valued functions on M. We want to identify the space $\Omega^p(M,W)$ of W-valued p-forms with a subspace of the cochain space $C^p(\mathfrak{g},V)$. This is done as follows: To each $\omega\in\Omega^p(M,W)$ we associate the element $\widetilde{\omega}\in C^p(\mathfrak{g},V)$, defined by

$$\widetilde{\omega}(X_1,\ldots,X_p)(m) := \omega_m(X_1(p),\ldots,X_p(m))$$

and observe that $\widetilde{\omega}$ is $C^\infty(M)$-multilinear. We then have the following theorem:

Theorem 3.2.1 *The map*

$$\Phi\colon \Omega^p(M,W)\to \mathrm{Alt}^p_{C^\infty(M)}(\mathcal{V}(M),C^\infty(M,W)),\quad \omega\mapsto\widetilde{\omega}$$

is a bijection.

This will follow from three lemmas:

Lemma 3.2.2 *Let M be a smooth manifold, $m\in M$ and U an open neighborhood of m. Then there exists a smooth function $\chi\in C^\infty(M)$ with the following properties:*

(a) $\chi = 1$ *in a neighborhood of m.*

(b) $\mathrm{supp}(\chi) := \overline{\{m \in M \colon \chi(m) \neq 0\}}$ *is a compact subset of* U. *In particular,* $\chi = 0$ *on* $M \setminus U$.

Lemma 3.2.3 *For each* $v \in T_m(M)$, *there exists a smooth vector field* $X \in \mathcal{V}(M)$ *with* $X(m) = v$.

Proof. Let (φ, U) be a chart of M with $m \in U$ and χ as in Lemma 3.2.2. Then

$$X(q) := \begin{cases} \chi(q) T_q(\varphi)^{-1} T_m(\varphi) v & \text{for } q \in U \\ 0 & \text{for } q \notin \mathrm{supp}(\chi) \end{cases}$$

defines a smooth vector field on M with $X(m) = v$. ■

Lemma 3.2.4 *If* $\eta \in \mathrm{Alt}^p_{C^\infty(M)}(\mathcal{V}(M), C^\infty(M, W))$ *and* $X_{i_0}(m) = 0$ *for some* i, *then* $\eta(X_1, \ldots, X_p)(m) = 0$.

Proof. Since η is alternating, we may w.l.o.g. assume that $i_0 = 1$.
Case 1: Suppose first that X_1 vanishes in a neighborhood U of m. Then we pick χ as in Lemma 3.2.2. Now $X_1 = (1 - \chi) X_1$ implies that

$$\begin{aligned} \eta(X_1, \ldots, X_p)(m) &= \eta((1 - \chi) X_1, \ldots, X_p)(m) \\ &= (1 - \chi(m)) \eta(X_1, \ldots, X_p)(m) = 0 \end{aligned}$$

follows from $\chi(m) = 1$.
Case 2: Let (φ, U) be a chart with $m \in U$ and χ as in Lemma 3.2.2. Then

$$\begin{aligned} \eta(X_1, \ldots, X_p)(m) &= \chi(m)^2 \eta(X_1, \ldots, X_p)(m) = \chi(m)^2 \eta(X_1, \ldots, X_p)(m) \\ &= \eta(\chi^2 X_1, \ldots, X_p)(m). \end{aligned}$$

The smooth vector fields $Y_i := (\varphi^{-1})_* e_i \in \mathcal{V}(U)$ form in each $x \in U$ a basis of $T_x(M)$, so that

$$X_1|_U = \sum_{j=1}^n a_j Y_j \quad \text{with} \quad a_j \in C^\infty(U).$$

Then

$$\chi^2 X_1 = \sum_{j=1}^n (\chi \cdot a_j)(\chi \cdot Y_j),$$

where we consider χa_j and χY_j as globally defined on M by 0 outside $\mathrm{supp}(\chi)$. From $X_1(m) = 0$ we now derive $a_j(m) = 0$ for each j, and therefore

$$\eta(X_1, \ldots, X_p)(m) = \sum_{j=1}^{n} (\chi \cdot a_j)(m)\eta(\chi Y_j, X_2, \ldots, X_p)(m) = 0.$$

■

Proof. **(of Theorem 3.2.1)** To see that Φ is injective, suppose that $\Phi(\omega) = \widetilde{\omega} = 0$. For $m \in M$ and $v_1, \ldots, v_p \in T_m(M)$, we then use Lemma 3.2.3 to find smooth vector fields X_i with $X_i(m) = v_i$. Then

$$\omega_m(v_1, \ldots, v_p) = \widetilde{\omega}(X_1, \ldots, X_p)(m) = 0$$

leads to ω_m, and hence to $\omega = 0$ because m was arbitrary.

To see that Φ is surjective, let $\eta \in \mathrm{Alt}^p_{C^\infty(M)}(\mathcal{V}(M), C^\infty(M, W))$. For $m \in M$, we then define $\omega_m \in \mathrm{Alt}^p(T_m(M), W)$ by

$$\omega_m(v_1, \ldots, v_p) := \eta(X_1, \ldots, X_p)(m), \tag{3.5}$$

where $X_i \in \mathcal{V}(M)$ satisfies $X_i(m) = v_i$. In view of Lemma 3.2.4, the right hand side of (3.5) does not depend on the choice of the vector fields X_i. To see that the $(\omega_m)_{m \in M}$ define a smooth differential form on M, we note that for any chart (φ, U) and the corresponding basic fields $Y_i := T(\varphi)^{-1}e_i$ we obtain global vector fields χY_i with χ as in Lemma 3.2.2. Now

$$\omega(Y_1, \ldots, Y_p)(u) = \omega(\chi Y_1, \ldots, \chi Y_p)(u) = \eta(\chi Y_1, \ldots, \chi Y_p)(u)$$

is smooth on the neighborhood $\chi^{-1}(1)$ of m. This proves that $\omega \in \Omega^p(M, W)$ and we clearly have $\widetilde{\omega} = \eta$. ■

In the following we will identify $\Omega^p(M, W)$ with the subspace of $C^\infty(M)$-multilinear elements in $C^p(\mathcal{V}(M), C^\infty(M, W))$.

The elements of the space $\Omega^p(M, W)$ are called *smooth W-valued p-forms on M*, and

$$\Omega(M, W) := \bigoplus_{p \in \mathbb{N}_0} \Omega^p(M, W)$$

is the space of *exterior W-valued forms on M*. The restriction of $\mathtt{d}$ to these spaces is called the *exterior differential.* The space $\Omega(M, W)$ is invariant under the differential $\mathtt{d}$ and the $\mathfrak{g}$-action given by the *Lie derivative* $\mathcal{L}_X\omega$. Together with the exterior derivative, the spaces $\Omega^p(M,W)$ now form the *de Rham complex*

$$\cdots \xrightarrow{\mathtt{d}} \Omega^2(M,W) \xrightarrow{\mathtt{d}} \Omega^1(M,W) \xrightarrow{\mathtt{d}} \Omega^0(M,W) = C^\infty(M,W).$$

The cohomology groups of this subcomplex are the *de Rham cohomology groups (with values in W)*

$$H^p_{\mathrm{dR}}(M,W) := \frac{\ker\left(\mathtt{d}|_{\Omega^p(M,W)}\right)}{\mathtt{d}(\Omega^{p-1}(M,W))}$$

of M. Here $Z^p_{\mathrm{dR}}(M,W) := \ker\left(\mathtt{d}|_{\Omega^p(M,W)}\right)$ is the space of *closed forms* and $B^p_{\mathrm{dR}}(M,W) := \mathtt{d}(\Omega^{p-1}(M,W))$ is the space of *exact forms.*

Remark 3.2.5 Since this is valid in the abstract context of Lie algebra cochains, it follows in particular that the Lie derivative $\mathcal{L}_X$, the insertion map i_X and the exterior differential $\mathtt{d}$ satisfy the relations:

$$\mathcal{L}_X = i_X \circ \mathtt{d} + \mathtt{d} \circ i_X, \quad [\mathcal{L}_X, i_Y] = i_{[X,Y]} \quad \text{and} \quad [\mathcal{L}_X, \mathtt{d}] = 0.$$

Here we use that the representation of $\mathcal{V}(M)$ on $\Omega^p(M,\mathbb{R})$ defined by the Lie derivative is given by the same formula as in Lemma 3.1.6.

3.3 Multiplication of Lie algebra cochains

Definition 3.3.1 Let $\mathfrak{g}$ and V_i, $i = 1,2,3$, be vector spaces and

$$m \colon V_1 \times V_2 \to V_3, \quad (v_1, v_2) \mapsto v_1 \cdot_m v_2$$

be a bilinear map. For $\alpha \in \mathrm{Alt}^p(\mathfrak{g}, V_1)$ and $\beta \in \mathrm{Alt}^q(\mathfrak{g}, V_2)$ we define

$$\alpha \wedge_m \beta \in \mathrm{Alt}^{p+q}(\mathfrak{g}, V_3)$$

by

$$\begin{aligned}&(\alpha \wedge_m \beta)(x_1, \ldots, x_{p+q}) \\ &:= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma) \alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) \cdot_m \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}).\end{aligned}$$

For $p = q = 1$ we have in particular

$$(\alpha \wedge_m \beta)(x, y) = \alpha(x) \cdot_m \beta(y) - \alpha(y) \cdot_m \beta(x).$$

For $p = 1$ and arbitrary q we get

$$(\alpha \wedge_m \beta)(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i \alpha(x_i) \cdot_m \beta(x_0, \ldots, \widehat{x_i}, \ldots, x_p). \tag{3.6}$$

In the following we write for a p-linear map $\alpha\colon \mathfrak{g}^p \to V$ and $\sigma \in S_p$:

$$\alpha^\sigma(x_1, \ldots, x_p) := \alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) \tag{3.7}$$

and

$$\mathrm{Alt}(\alpha) := \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\alpha^\sigma.$$

In this sense we have

$$\alpha \wedge_m \beta = \frac{1}{p!q!} \mathrm{Alt}(\alpha \cdot_m \beta),$$

where $(\alpha \cdot_m \beta)(x_1, \ldots, x_{p+q}) := \alpha(x_1, \ldots, x_p) \cdot_m \beta(x_{p+1}, \ldots, x_{p+q})$.

Lemma 3.3.2 *The contraction maps*

$$i_x\colon \mathrm{Alt}^n(\mathfrak{g}, V_i) \to \mathrm{Alt}^{n-1}(\mathfrak{g}, V_i), \quad i = 1, 2, 3; n \in \mathbb{N}_0,$$

satisfy

$$i_x(\alpha \wedge_m \beta) = (i_x\alpha) \wedge_m \beta + (-1)^p \alpha \wedge_m i_x\beta \tag{3.8}$$

for $\alpha \in \mathrm{Alt}^p(\mathfrak{g}, V_1), \beta \in \mathrm{Alt}^q(\mathfrak{g}, V_2)$.

Remark 3.3.3 (The action of $\mathrm{End}(\mathfrak{g})$) For $D \in \mathrm{End}(\mathfrak{g})$, $\alpha \in \mathrm{Mult}^p(\mathfrak{g}, V)$ and $\sigma \in S_p$ we put

$$(D.\alpha)(x_1, \ldots, x_p) := -\sum_{i=1}^{p} \alpha(x_1, \ldots, x_{i-1}, Dx_i, x_{i+1}, \ldots, x_p).$$

Then we have the following relations:

(a) $D.\alpha^\sigma = (D.\alpha)^\sigma$ (cf. (3.7)).

(b) $D.\operatorname{Alt}(\alpha) = \operatorname{Alt}(D.\alpha)$.

(c) $D.(\alpha \cdot_m \beta) = (D.\alpha) \cdot_m \beta + \alpha \cdot_m D.\beta$.

(d) $D.(\alpha \wedge_m \beta) = (D.\alpha) \wedge_m \beta + \alpha \wedge_m D.\beta$ for $\alpha \in \operatorname{Alt}^p(\mathfrak{g}, V_1)$, $\beta \in \operatorname{Alt}^q(\mathfrak{g}, V_2)$.

Remark 3.3.4 (Associativity properties) Suppose we have four bilinear maps

$$m_{12}\colon V_1 \times V_2 \to W, \quad m_3\colon W \times V_3 \to U$$

$$m_{23}\colon V_2 \times V_3 \to X, \quad m_1\colon V_1 \times X \to U,$$

satisfying the associativity relation

$$v_1 \cdot_{m_1} (v_2 \cdot_{m_{23}} v_3) = (v_1 \cdot_{m_{12}} v_2) \cdot_{m_3} v_3$$

for $v_i \in V_i$, $i = 1, 2, 3$. Then we obtain for $\alpha_i \in \operatorname{Alt}^{p_i}(\mathfrak{g}, V_i)$ the relation

$$(\alpha_1 \cdot_{m_{12}} \alpha_2) \cdot_{m_3} \alpha_3 = \alpha_1 \cdot_{m_1} (\alpha_2 \cdot_{m_{23}} \alpha_3)$$

which in turn leads to

$$(\alpha_1 \wedge_{m_{12}} \alpha_2) \wedge_{m_3} \alpha_3 = \alpha_1 \wedge_{m_1} (\alpha_2 \wedge_{m_{23}} \alpha_3).$$

Example 3.3.5 An important special case of the situation discussed in the previous remark is the following. Let V be a vector space and $\operatorname{End}(V)$ the algebra of its linear endomorphisms. Then we have two bilinear maps given by evaluation

$$\operatorname{ev}\colon \operatorname{End}(V) \times V \to V, \quad (\varphi, v) \mapsto \varphi(v)$$

and composition

$$C\colon \operatorname{End}(V) \times \operatorname{End}(V) \to \operatorname{End}(V), \quad (\varphi, \psi) \mapsto \varphi \circ \psi.$$

They satisfy the associativity relation

$$\operatorname{ev}(C(\varphi, \psi), v) = (\varphi \circ \psi)(v) = \varphi(\psi(v)) = \operatorname{ev}(\varphi, \operatorname{ev}(\psi, v)).$$

For $\alpha \in \operatorname{Alt}^p(\mathfrak{g}, \operatorname{End}(V))$, $\beta \in \operatorname{Alt}^q(\mathfrak{g}, \operatorname{End}(V))$ and $\gamma \in \operatorname{Alt}^r(\mathfrak{g}, V)$ this leads to the relation

$$\alpha \wedge_{\mathrm{ev}} (\beta \wedge_{\mathrm{ev}} \gamma) = (\alpha \wedge_C \beta) \wedge_{\mathrm{ev}} \gamma.$$

Remark 3.3.6 (Commutativity properties) Now we consider a bilinear map

$$m\colon V \times V \to V$$

and $\alpha \in \mathrm{Alt}^p(\mathfrak{g}, V)$, $\beta \in \mathrm{Alt}^q(\mathfrak{g}, V)$.

If m is symmetric, then we find for their wedge product:

$$\beta \wedge_m \alpha = (-1)^{pq} \alpha \wedge_m \beta \tag{3.9}$$

and if m is skew-symmetric, we have

$$\beta \wedge_m \alpha = (-1)^{pq+1} \alpha \wedge_m \beta. \tag{3.10}$$

For the proof we use that the permutation $\gamma \in S_{p+q}$ exchanging the first p elements with the last p ones, satisfies $\mathrm{sgn}(\gamma) = (-1)^{pq}$.

Proposition 3.3.7 *Let $\mathfrak{g}$ be a Lie algebra, V_i, $i = 1, 2, 3$, be $\mathfrak{g}$-modules and $m\colon V_1 \times V_2 \to V_3$ be a $\mathfrak{g}$-invariant bilinear map, i.e.,*

$$x.m(v_1, v_2) = m(x.v_1, v_2) + m(v_1, x.v_2), \quad x \in \mathfrak{g}, v_i \in V_i.$$

Then we have for $\alpha \in C^p(\mathfrak{g}, V_1)$ and $\beta \in C^q(\mathfrak{g}, V_2)$ the relations

$$\mathcal{L}_x(\alpha \wedge_m \beta) = \mathcal{L}_x\alpha \wedge_m \beta + \alpha \wedge_m \mathcal{L}_x\beta \tag{3.11}$$

and

$$\mathtt{d}_{\mathfrak{g}}(\alpha \wedge_m \beta) = \mathtt{d}_{\mathfrak{g}}\alpha \wedge_m \beta + (-1)^p \alpha \wedge_m \mathtt{d}_{\mathfrak{g}}\beta. \tag{3.12}$$

Proof. The relation (3.11) follows easily from Remark 3.3.3(d).

For (3.12), we argue by induction on p and q. For $p = 0$ we have

$$(\alpha \wedge_m \beta)(x_1, \ldots, x_q) = \alpha \cdot \beta(x_1, \ldots, x_q)$$

and

$$\begin{aligned}
&\mathtt{d}_{\mathfrak{g}}(\alpha \wedge_m \beta)(x_0, \ldots, x_q) \\
&= \sum_{i=0}^{q} (-1)^i x_i.(\alpha \cdot \beta)(x_0, \ldots, \widehat{x_i}, \ldots, x_q) \\
&\quad + \sum_{i<j} (-1)^{i+j} \alpha \cdot \beta([x_i, x_j], \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_q) \\
&= \sum_{i=0}^{q} (-1)^i (x_i.\alpha) \cdot \beta(x_0, \ldots, \widehat{x_i}, \ldots, x_q) + \alpha \cdot (\mathtt{d}_{\mathfrak{g}}\beta)(x_0, \ldots, x_q)
\end{aligned}$$

and

$$\begin{aligned}
(\mathtt{d}_{\mathfrak{g}}\alpha \wedge_m \beta)(x_0, \ldots, x_q) &= \frac{1}{q!} \sum_{\sigma \in S_{q+1}} \operatorname{sgn}(\sigma)(\mathtt{d}_{\mathfrak{g}}\alpha)(x_{\sigma(0)}) \cdot \beta(x_{\sigma(1)}, \ldots, x_{\sigma(q)}) \\
&= \frac{1}{q!} \sum_{i=0}^{q} \sum_{\sigma(0)=i} \operatorname{sgn}(\sigma)(x_i.\alpha) \cdot \beta(x_{\sigma(1)}, \ldots, x_{\sigma(q)}) \\
&= \frac{1}{q!} \sum_{i=0}^{q} (-1)^i (x_i.\alpha) \cdot \operatorname{Alt}(\beta)(x_0, \ldots, \widehat{x_i}, \ldots, x_q) \\
&= \sum_{i=0}^{q} (-1)^i (x_i.\alpha) \cdot \beta(x_0, \ldots, \widehat{x_i}, \ldots, x_q).
\end{aligned}$$

This proves the assertion for $p = 0$. A similar argument works for $q = 0$. We now assume that $p, q \geq 1$ and that the assertion holds for the pairs $(p-1, q)$ and $(p, q-1)$. Then we obtain with the Cartan formulas and Lemma 3.3.2 for $x \in \mathfrak{g}$:

$$\begin{aligned}
&i_x(\mathtt{d}_{\mathfrak{g}}\alpha \wedge_m \beta + (-1)^p \alpha \wedge_m \mathtt{d}_{\mathfrak{g}}\beta) \\
&= (i_x \mathtt{d}_{\mathfrak{g}}\alpha) \wedge_m \beta + (-1)^{p+1} \mathtt{d}_{\mathfrak{g}}\alpha \wedge_m i_x\beta + (-1)^p i_x\alpha \wedge_m \mathtt{d}_{\mathfrak{g}}\beta + \alpha \wedge_m i_x \mathtt{d}_{\mathfrak{g}}\beta \\
&= \mathcal{L}_x\alpha \wedge_m \beta - \mathtt{d}_{\mathfrak{g}}(i_x\alpha) \wedge_m \beta + (-1)^{p+1} \mathtt{d}_{\mathfrak{g}}\alpha \wedge_m i_x\beta \\
&\quad + (-1)^p i_x\alpha \wedge_m \mathtt{d}_{\mathfrak{g}}\beta + \alpha \wedge_m \mathcal{L}_x\beta - \alpha \wedge_m \mathtt{d}_{\mathfrak{g}}(i_x\beta) \\
&= \mathcal{L}_x(\alpha \wedge_m \beta) - \mathtt{d}_{\mathfrak{g}}(i_x\alpha \wedge_m \beta) + (-1)^{p+1} \mathtt{d}_{\mathfrak{g}}(\alpha \wedge_m i_x\beta) \\
&= \mathcal{L}_x(\alpha \wedge_m \beta) - \mathtt{d}_{\mathfrak{g}}(i_x(\alpha \wedge_m \beta)) = i_x(\mathtt{d}_{\mathfrak{g}}(\alpha \wedge_m \beta)).
\end{aligned}$$

Since x was arbitrary, the assertion follows. ■

Remark 3.3.8 The preceding lemma implies that products of two cocycles are cocycles and that the product of a cocycle with a coboundary is a coboundary, so that we obtain bilinear maps

$$H^p(\mathfrak{g}, V_1) \times H^q(\mathfrak{g}, V_2) \to H^{p+q}(\mathfrak{g}, V_3), \quad ([\alpha], [\beta]) \mapsto [\alpha \wedge_m \beta]$$

which can be combined to a product $H^\bullet(\mathfrak{g}, U) \times H^\bullet(\mathfrak{g}, V) \to H^\bullet(\mathfrak{g}, W)$.

Example 3.3.9 If M is a smooth manifold and $V_i = C^\infty(M, \mathbb{R})$, then we consider the space $\Omega^p(M, \mathbb{R})$ of real-valued p-forms on M as a subspace of $C^p(\mathcal{V}(M), C^\infty(M, \mathbb{R}))$. Then the product

$$m(f_1, f_2) := f_1 f_2$$

on $C^\infty(M,\mathbb{R})$ defines the usual wedge product of differential forms. From Proposition 3.3.7 we derive immediately the relations

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_x\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$$

and

$$\mathtt{d}(\alpha \wedge \beta) = \mathtt{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathtt{d}\beta$$

for $\alpha \in \Omega^p(M,\mathbb{R})$ and $\beta \in \Omega^q(M,\mathbb{R})$. This implies in particular that the product of two closed forms is closed and if $\alpha = \mathtt{d}\gamma$ is exact and β closed, then

$$\mathtt{d}(\gamma \wedge \beta) = \alpha \wedge \beta$$

is exact. We thus obtain a well-defined product

$$H^p_{\rm dR}(M,\mathbb{R}) \times H^q_{\rm dR}(M,\mathbb{R}) \to H^{p+q}_{\rm dR}(M,\mathbb{R}), \quad ([\alpha],[\beta]) \mapsto [\alpha] \wedge [\beta] := [\alpha \wedge \beta].$$

This leads to an associative algebra structure on the space $H_{\rm dR}(M,\mathbb{R}) := \bigoplus_{p\in\mathbb{N}_0} H^p_{\rm dR}(M,\mathbb{R})$, called the *cohomology algebra of M*. This algebra is *graded commutative*, i.e.,

$$[\alpha] \wedge [\beta] = (-1)^{pq}[\beta] \wedge [\alpha]$$

for $[\alpha] \in H^p_{\rm dR}(M,\mathbb{R})$, $[\beta] \in H^q_{\rm dR}(M,\mathbb{R})$ (cf. Remark 3.3.6).

Definition 3.3.10 A *Lie superalgebra* (over a field $\mathbb{K}$ with $2,3 \in \mathbb{K}^\times$) is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ with a bilinear map $[\cdot,\cdot]$ satisfying

(LS1) $[\alpha,\beta] = (-1)^{pq+1}[\beta,\alpha]$ for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$.

(LS2) $[\alpha,[\beta,\gamma]] = [[\alpha,\beta],\gamma] + (-1)^{pq}[\beta,[\alpha,\gamma]]$ for $\alpha \in \mathfrak{g}_p$, $\beta \in \mathfrak{g}_q$ and $\gamma \in \mathfrak{g}_r$.

Note that (LS1) implies that

$$[\alpha,\alpha] = 0 = [\beta,[\beta,\beta]] \quad \text{for} \quad \alpha \in \mathfrak{g}_{\overline{0}}, \beta \in \mathfrak{g}_{\overline{1}}. \tag{3.13}$$

Example 3.3.11 (a) Suppose that V is a Lie algebra, considered as a trivial $\mathfrak{g}$-module. The bilinear bracket on $C^\bullet(\mathfrak{g},V) := \bigoplus_{p\in\mathbb{N}_0} C^p(\mathfrak{g},V)$ defined by

$$C^p(\mathfrak{g},V) \times C^q(\mathfrak{g},V) \to C^{p+q}(\mathfrak{g},V), \quad (\alpha,\beta) \mapsto [\alpha,\beta] := \alpha \wedge_{[\cdot,\cdot]} \beta,$$

turns the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $C^\bullet(\mathfrak{g},V) = C^{\rm even}(\mathfrak{g},V) \oplus C^{\rm odd}(\mathfrak{g},V)$ into a Lie superalgebra.

In fact, (LS1) follows from Remark 3.3.6. The relation (LS2) for $\deg \alpha = p$, $\deg \beta = q$ and $\deg \gamma = r$ can be obtained from Remark 3.3.4 and the Jacobi identity as follows. Let $b_V : V \times V \to V$ denote the Lie bracket on V. Then

$$[\alpha, [\beta, \gamma]] = \frac{1}{p!q!r!} \operatorname{Alt}(\alpha \cdot_{b_V} (\beta \cdot_{b_V} \gamma)).$$

The Jacobi identity in V implies

$$\alpha \cdot_{b_V} (\beta \cdot_{b_V} \gamma) = (\alpha \cdot_{b_V} \beta) \cdot_{b_V} \gamma + \big(\beta \cdot_{b_V} (\alpha \cdot_{b_V} \gamma)\big)^\sigma$$

for a permutation $\sigma \in S_{p+q+r}$ exchanging

$$\{1, \ldots, p\} \quad \text{by} \quad \{q+1, \ldots, q+p\}$$

without changing the order in the subsets. Then $\operatorname{sgn}(\sigma) = (-1)^{pq}$ and applying the alternator Alt yields (LS2).

(b) If M is a smooth manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and

$$\Omega(M, \mathfrak{g}) := \bigoplus_{p=0}^{\infty} \Omega^p(M, \mathfrak{g}),$$

then $\Omega^p(M, \mathfrak{g}) \subseteq \operatorname{Alt}^p(\mathcal{V}(M), C^\infty(M, \mathfrak{g}))$ for each $p \in \mathbb{N}_0$, and on $C^\infty(M, \mathfrak{g})$ we have the Lie bracket, defined pointwise by:

$$[f, g](m) := [f(m), g(m)].$$

This Lie bracket is $\mathcal{V}(M)$-invariant in the sense that

$$X[f, g] = [Xf, g] + [f, Xg],$$

so that Proposition 3.3.7 applies to the corresponding wedge product:

$$\begin{aligned} &[\alpha, \beta](X_1, \ldots, X_{p+q}) \\ &:= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma)[\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)}), \beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})]. \end{aligned}$$

We thus obtain on $\Omega(M, \mathfrak{g})$ the structure of a Lie superalgebra. Its bracket is compatible with the exterior differential and the Lie derivative in the sense that

$$\mathcal{L}_X[\alpha, \beta] = [\mathcal{L}_X \alpha, \beta] + [\alpha, \mathcal{L}_X \beta]$$

and

$$\mathtt{d}[\alpha, \beta] = [\mathtt{d}\alpha, \beta] + (-1)^p [\alpha, \mathtt{d}\beta].$$

3.4 Covariant derivatives and curvature

Definition 3.4.1 Let $\mathfrak{g}$ be a Lie algebra and V a vector space, considered as a trivial $\mathfrak{g}$-module, so that we have the corresponding Chevalley–Eilenberg complex $(C(\mathfrak{g}, V), \mathtt{d}_{\mathfrak{g}})$.

We now twist the differential in this complex with a linear map

$$S\colon \mathfrak{g} \to \mathrm{End}(V),$$

i.e., an element $S \in C^1(\mathfrak{g}, \mathrm{End}(V))$. First we note that the bilinear evaluation map $\mathrm{ev}\colon \mathrm{End}(V) \times V \to V$ leads to a linear operator

$$S_\wedge\colon C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V), \quad \alpha \mapsto S \wedge_{\mathrm{ev}} \alpha.$$

The corresponding *covariant differential on* $C(\mathfrak{g}, V)$ is defined by

$$\mathtt{d}_S := S_\wedge + \mathtt{d}_{\mathfrak{g}}\colon C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V), \quad p \in \mathbb{N}_0.$$

In view of (3.6) this can also be written as

$$\begin{aligned}(\mathtt{d}_S\alpha)(x_0, \ldots, x_p) := &\sum_{j=0}^{p} (-1)^j S(x_j).\alpha(x_0, \ldots, \widehat{x}_j, \ldots, x_p) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p).\end{aligned}$$

Proposition 3.4.2 *Let* $R_S(x, y) = [S(x), S(y)] - S([x, y])$ *for* $x, y \in \mathfrak{g}$. *Then*

$$R_S := \mathtt{d}_{\mathfrak{g}} S + \tfrac{1}{2}[S, S] \in C^2(\mathfrak{g}, \mathrm{End}(V)),$$

and for $\alpha \in C^p(\mathfrak{g}, V)$ *we have*

$$\mathtt{d}_S^2 \alpha = R_S \wedge_{\mathrm{ev}} \alpha. \tag{3.14}$$

In particular $\mathtt{d}_S^2 = 0$ *if and only if* S *is a homomorphism of Lie algebras, i.e.,* $R_S = 0$.

Proof. For $\alpha \in C^p(\mathfrak{g}, V)$ we get

$$\begin{aligned}\mathtt{d}_S^2 \alpha &= \mathtt{d}_S(S \wedge_{\mathrm{ev}} \alpha + \mathtt{d}_{\mathfrak{g}}\alpha) \\ &= (S \wedge_{\mathrm{ev}} (S \wedge_{\mathrm{ev}} \alpha)) + S \wedge_{\mathrm{ev}} \mathtt{d}_{\mathfrak{g}}\alpha + \mathtt{d}_{\mathfrak{g}}(S \wedge_{\mathrm{ev}} \alpha) + \mathtt{d}_{\mathfrak{g}}^2 \alpha \\ &= (S \wedge_C S) \wedge_{\mathrm{ev}} \alpha + S \wedge_{\mathrm{ev}} \mathtt{d}_{\mathfrak{g}}\alpha + (\mathtt{d}_{\mathfrak{g}} S \wedge_{\mathrm{ev}} \alpha - S \wedge_{\mathrm{ev}} \mathtt{d}_{\mathfrak{g}}\alpha) \\ &= (S \wedge_C S) \wedge_{\mathrm{ev}} \alpha + \mathtt{d}_{\mathfrak{g}} S \wedge_{\mathrm{ev}} \alpha = (S \wedge_C S + \mathtt{d}_{\mathfrak{g}} S) \wedge_{\mathrm{ev}} \alpha.\end{aligned}$$

Now

$$(S \wedge_C S)(x,y) = S(x)S(y) - S(y)S(x) = [S(x), S(y)] = \tfrac{1}{2}[S,S](x,y),$$

proves (3.14).

For $v \in V \cong C^0(\mathfrak{g}, V)$ we obtain in particular $(\mathtt{d}_S^2 v)(x,y) = R_S(x,y)v$, showing that $\mathtt{d}_S^2 = 0$ on $C^\bullet(\mathfrak{g}, V)$ is equivalent to $R_S = 0$, which means that $S \colon \mathfrak{g} \to (\mathrm{End}(V), [\cdot,\cdot])$ is a homomorphism of Lie algebras. ■

The following proposition provides an abstract algebraic version of identities originating from the context of differential forms.

Proposition 3.4.3 *Suppose that V is a Lie algebra, considered as a trivial $\mathfrak{g}$-module. Let $\sigma \in C^1(\mathfrak{g}, V)$ and put $S = \mathrm{ad} \circ \sigma$ and*

$$R_\sigma := \mathtt{d}_{\mathfrak{g}}\sigma + \tfrac{1}{2}[\sigma,\sigma] \in C^2(\mathfrak{g}, V) \qquad \text{i.e.,} \qquad R_\sigma(x,y) = [\sigma(x), \sigma(y)] - \sigma([x,y]).$$

Then the following assertions hold:

(1) $\mathtt{d}_S^2\alpha = [R_\sigma, \alpha]$ *for* $\alpha \in C^p(\mathfrak{g}, V)$.

(2) R_σ *satisfies the abstract Bianchi identity* $\mathtt{d}_S R_\sigma = 0$, *i.e.,*

$$\sum_{\text{cyc.}} [\sigma(x), R(y,z)] - R([x,y],z) = 0.$$

(3) *For* $\gamma \in C^1(\mathfrak{g}, V)$ *we have*

$$R_{\sigma+\gamma} = R_\sigma + R_\gamma + [\sigma,\gamma] = R_\sigma + \mathtt{d}_S\gamma + \tfrac{1}{2}[\gamma,\gamma].$$

Proof. (1) Since $\mathrm{ad} \colon V \to \mathrm{End}(V)$ is a homomorphism of Lie algebras, the definition of R_σ and Proposition 3.4.2 immediately lead for $\alpha \in C^p(\mathfrak{g}, V)$ to

$$\mathtt{d}_S^2\alpha = R_S \wedge_{\mathrm{ev}} \alpha = (\mathrm{ad} \circ R_\sigma) \wedge_{\mathrm{ev}} \alpha = [R_\sigma, \alpha]$$

(Example 3.3.11).

(2) From (3.10) and Proposition 3.3.7, we further get

$$\mathtt{d}_{\mathfrak{g}}[\sigma,\sigma] = [\mathtt{d}_{\mathfrak{g}}\sigma, \sigma] - [\sigma, \mathtt{d}_{\mathfrak{g}}\sigma] = [\mathtt{d}_{\mathfrak{g}}\sigma, \sigma] + [\mathtt{d}_{\mathfrak{g}}\sigma, \sigma] = 2[\mathtt{d}_{\mathfrak{g}}\sigma, \sigma].$$

Now the abstract Bianchi identity follows with Example 3.3.11 from

$$\begin{aligned}\mathtt{d}_S R_\sigma &= (\mathtt{d}_{\mathfrak{g}} + S_\wedge)R_\sigma = \mathtt{d}_{\mathfrak{g}}^2\sigma + \tfrac{1}{2}\mathtt{d}_{\mathfrak{g}}[\sigma,\sigma] + S \wedge R_\sigma = [\mathtt{d}_{\mathfrak{g}}\sigma,\sigma] + [\sigma, R_\sigma]\\ &= [\mathtt{d}_{\mathfrak{g}}\sigma,\sigma] - [R_\sigma,\sigma] = -\tfrac{1}{2}[[\sigma,\sigma],\sigma] = 0.\end{aligned}$$

(3) This relation follows from

$$\begin{aligned}R_{\sigma+\gamma} &= \mathtt{d}_{\mathfrak{g}}\sigma + \tfrac{1}{2}[\sigma,\sigma] + \mathtt{d}_{\mathfrak{g}}\gamma + \tfrac{1}{2}([\sigma,\gamma]+[\gamma,\sigma]+[\gamma,\gamma]) = R_\sigma + R_\gamma + [\sigma,\gamma]\\ &= R_\sigma + \mathtt{d}_{\mathfrak{g}}\gamma + \tfrac{1}{2}[\gamma,\gamma] + S\wedge\gamma = R_\sigma + \mathtt{d}_S\gamma + \tfrac{1}{2}[\gamma,\gamma].\end{aligned}$$

■

Lemma 3.4.4 *Suppose that $m\colon V\times V \to V$ is a bilinear map and that $S\colon \mathfrak{g}\to \operatorname{der}(V,m)$ is linear. Then we have for $\alpha\in C^p(\mathfrak{g},V)$ and $\beta\in C^q(\mathfrak{g},V)$ the relation*

$$\mathtt{d}_S(\alpha\wedge_m\beta) = \mathtt{d}_S\alpha\wedge_m\beta + (-1)^p\alpha\wedge_m \mathtt{d}_S\beta. \tag{3.15}$$

Proof. We have $\mathtt{d}_S = \mathtt{d}_{\mathfrak{g}} + S_\wedge$ and Proposition 3.3.7 implies the assertion for $S=0$. It therefore remains to show that

$$S\wedge(\alpha\wedge_m\beta) = (S\wedge\alpha)\wedge_m\beta + (-1)^p\alpha\wedge_m(S\wedge\beta).$$

We recall that

$$S\wedge(\alpha\wedge_m\beta) = \frac{1}{p!q!}\operatorname{Alt}(S\cdot(\alpha\cdot_m\beta))$$

and note that $S(\mathfrak{g})\subseteq\operatorname{der}(V,m)$ implies that

$$S\cdot(\alpha\cdot_m\beta)) = (S\cdot\alpha)\cdot_m\beta + \big(\alpha\cdot_m(S\cdot\beta)\big)^\sigma,$$

where $\sigma = (1\ 2\ \ldots p+1)\in S_{p+q+1}$ is a cycle of length $p+1$. Now the lemma follows from $\operatorname{sgn}(\sigma) = (-1)^p$. ■

3.5 Lecomte's generalization of the Chern–Weil map

To define Lecomte's characteristic map, we first have to explain how to multiply k-tuples of Lie algebra cochains with k-linear maps.

Definition 3.5.1 For a k-linear map $f \colon \mathfrak{n}^k \to V$ and $\varphi_1, \ldots, \varphi_k \in C^2(\mathfrak{g}, \mathfrak{n})$, we define

$$f_{\varphi_1,\ldots,\varphi_k} \in C^{2k}(\mathfrak{g}, V)$$

by

$$f_{\varphi_1,\ldots,\varphi_k}(x_1, \ldots, x_{2k}) ; = \sum_{\sigma_{2i-1} < \sigma_{2i}} \operatorname{sgn}(\sigma) f(\varphi_1(x_{\sigma_1}, x_{\sigma_2}), \ldots, \varphi_k(x_{\sigma_{2k-1}}, x_{\sigma_{2k}})).$$

This means that

$$f_{\varphi_1,\ldots,\varphi_k} = \frac{1}{2^k} \operatorname{Alt}(f \circ (\varphi_1, \varphi_2, \ldots, \varphi_k)),$$

so that we may consider $f_{\varphi_1,\ldots,\varphi_k}$ as a k-fold wedge product of the φ_i, defined by f.

The following remark explains why it suffices to work with symmetric maps.

Remark 3.5.2 (a) If $f \in \operatorname{Mult}^k(\mathfrak{n}, V)$ is not symmetric and $f_s = \frac{1}{k!} \sum_{\sigma \in S_k} f^\sigma$ is its symmetrization, then we have $f^\sigma = (f_s)^\sigma$ because for any permutation $\sigma \in S_k$, the relation

$$f_{\varphi_{\sigma(1)},\ldots,\varphi_{\sigma(k)}} = f_{\varphi_1,\ldots,\varphi_k}$$

follows from the fact that

$$f \circ (\varphi_{\sigma(1)}, \ldots, \varphi_{\sigma(k)}) = (f \circ (\varphi_1, \ldots, \varphi_k))^{\widetilde{\sigma}}$$

holds for an even permutation $\widetilde{\sigma} \in S_{2k}$ (Here we use the notation from (3.7)).

(b) We may also consider $f \in \operatorname{Sym}^k(\mathfrak{n}, V)$ as a linear map

$$\widetilde{f} \colon \mathfrak{n}^{\otimes k} \to V \quad \text{with} \quad \widetilde{f}(x_1 \otimes \cdots \otimes x_k) = f(x_1, \ldots, x_k).$$

Writing $\wedge_\otimes$ for the wedge products

$$C^p(\mathfrak{g}, \mathfrak{n}^{\otimes k}) \times C^q(\mathfrak{g}, \mathfrak{n}^{\otimes m}) \to C^{p+q}(\mathfrak{g}, \mathfrak{n}^{\otimes(k+m)})$$

defined by the canonical multiplication $\mathfrak{n}^{\otimes k} \times \mathfrak{n}^{\otimes m} \to \mathfrak{n}^{\otimes(k+m)}$, $(x, y) \mapsto x \otimes y$ (cf. Definition 3.3.1), we find the formula

$$f_{\varphi_1,\ldots,\varphi_k} = \widetilde{f} \circ (\varphi_1 \wedge_\otimes \cdots \wedge_\otimes \varphi_k), \tag{3.16}$$

expressing $f_{\varphi_1,\ldots,\varphi_k}$ simply as a composition of a linear map with an iterated wedge product (cf. Remark 3.3.4).

Definition 3.5.3 If $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a homomorphism of Lie algebras, then we define the *pullback map*

$$q^*\colon C^p(\mathfrak{g}, V) \to C^p(\widehat{\mathfrak{g}}, V),$$

by

$$(q^*\omega)(x_1, \ldots, x_p) := \omega(q(x_1), \ldots, q(x_p))$$

and observe that

$$\mathtt{d}_{\widehat{\mathfrak{g}}} \circ q^* = q^* \circ \mathtt{d}_{\mathfrak{g}}.$$

In particular, q^* induces a well-defined map

$$q^*\colon H^\bullet(\mathfrak{g}, V) \to H^\bullet(\widehat{\mathfrak{g}}, V), \quad [\omega] \mapsto [q^*\omega].$$

Now we turn to the subject proper of this section, a general algebraic construction of P. Lecomte assigning certain cohomology classes to Lie algebra extension.

Let

$$\mathbf{0} \to \mathfrak{n} \to \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to \mathbf{0}$$

be an extension of Lie algebras and V be a $\mathfrak{g}$-module which we also consider as a $\widehat{\mathfrak{g}}$-module with respect to the action $x.v := q(x).v$ for $x \in \widehat{\mathfrak{g}}$. Further, let $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be a linear section and define $R_\sigma \in C^2(\mathfrak{g}, \mathfrak{n})$ by

$$R_\sigma(x, y) = [\sigma(x), \sigma(y)] - \sigma([x, y]).$$

For $f \in \mathrm{Sym}^k(\mathfrak{n}, V)$, we define

$$f_\sigma := f_{R_\sigma, \ldots, R_\sigma} \in C^{2k}(\mathfrak{g}, V) \tag{3.17}$$

in the sense of Definition 3.5.1.

Lemma 3.5.4 *If f is symmetric and $\widehat{\mathfrak{g}}$-equivariant, then $\mathtt{d}_{\mathfrak{g}} f_\sigma = 0$ and its cohomology class $[f_\sigma] \in H^{2k}(\mathfrak{g}, V)$ does not depend on the choice of σ.*

Proof. Let $\widetilde{f}\colon \mathfrak{n}^{\otimes k} \to V$ be the linear map defined by f and recall from (3.16) that

$$f_{\varphi_1, \ldots, \varphi_k} = \widetilde{f} \circ (\varphi_1 \wedge_\otimes \varphi_2 \wedge_\otimes \cdots \wedge_\otimes \varphi_k).$$

We put $S(x) := \mathrm{ad}(\sigma(x))$. From

$$x.f(y_1, \ldots, y_k) = \sum_{i=1}^k f(y_1, \ldots, S(x)y_i, \ldots, y_k)$$

we now derive with Remark 3.5.2(b), Lemma 3.4.4 and the Bianchi identity $\mathtt{d}_S R_\sigma = 0$ the relation

$$\begin{aligned}\mathtt{d}_{\mathfrak{g}} f_\sigma &= \mathtt{d}_{\mathfrak{g}}\big(f_{R_\sigma,\ldots,R_\sigma}\big) = \mathtt{d}_{\mathfrak{g}}\big(\widetilde{f} \circ (R_\sigma \wedge_\otimes \cdots \wedge_\otimes R_\sigma)\big) \\ &= \widetilde{f} \circ \mathtt{d}_S(R_\sigma \wedge_\otimes \cdots \wedge_\otimes R_\sigma) \\ &= \widetilde{f} \circ \big(\mathtt{d}_S R_\sigma \wedge_\otimes \cdots \wedge_\otimes R_\sigma) + R_\sigma \wedge_\otimes \mathtt{d}_S R_\sigma \wedge \otimes \cdots + \ldots\big) = 0.\end{aligned}$$

Next we show that $[f_\sigma]$ does not depend on σ. Let $\sigma' \in C^1(\mathfrak{g}, \widehat{\mathfrak{g}})$ be another section, so that $D := \sigma' - \sigma \in C^1(\mathfrak{g}, \mathfrak{n})$. For $t \in \mathbb{R}$ we consider the section $\sigma_t := \sigma + tD = (1-t)\sigma + t\sigma'$ and put $S_t(x) := \mathrm{ad}(\sigma_t(x))$. Then Proposition 3.4.3(3) implies that

$$R_{\sigma_{t+h}} = R_{\sigma_t} + h\mathtt{d}_{S_t} D + \frac{h^2}{2}[D, D]$$

and therefore

$$\frac{d}{dt} R_{\sigma_t} = \mathtt{d}_{S_t} D.$$

From the symmetry of f we now derive with the Product Rule (cf. Remark 3.5.2(b) and Lemma 3.4.4)

$$\frac{d}{dt} f_{\sigma_t} = \frac{d}{dt} f_{R_{\sigma_t},\ldots,R_{\sigma_t}} = k f_{\mathtt{d}_{S_t} D, R_{\sigma_t},\ldots,R_{\sigma_t}}.$$

Again, we use Remark 3.5.2(b), Lemma 3.4.4 and the Bianchi identity $\mathtt{d}_{S_t} R_{\sigma_t} = 0$ to obtain

$$\begin{aligned}&\mathtt{d}_{\mathfrak{g}}\big(f_{D,R_{\sigma_t},\ldots,R_{\sigma_t}}\big) = \mathtt{d}_{\mathfrak{g}}\big(\widetilde{f} \circ (D \wedge_\otimes R_{\sigma_t} \wedge_\otimes \cdots \wedge_\otimes R_{\sigma_t})\big) \\ &= \widetilde{f} \circ \mathtt{d}_{S_t}(D \wedge_\otimes R_{\sigma_t} \wedge_\otimes \cdots \wedge_\otimes R_{\sigma_t}) \\ &= \widetilde{f} \circ \big(\mathtt{d}_{S_t} D \wedge_\otimes R_{\sigma_t} \wedge_\otimes \cdots \wedge_\otimes R_{\sigma_t}) - D \wedge_\otimes \mathtt{d}_{S_t} R_{\sigma_t} \wedge_\otimes \cdots + \ldots\big) \\ &= \widetilde{f} \circ (\mathtt{d}_{S_t} D \wedge_\otimes R_{\sigma_t} \wedge_\otimes \cdots \wedge_\otimes R_{\sigma_t}) \\ &= f_{\mathtt{d}_{S_t} D, R_{\sigma_t},\ldots,R_{\sigma_t}}.\end{aligned}$$

We combine all this to

$$\frac{d}{dt} f_{\sigma_t} = k\mathtt{d}_{\mathfrak{g}} f_{D,R_{\sigma_t},\ldots,R_{\sigma_t}},$$

and find that

$$f_{\sigma'} - f_\sigma = \int_0^1 \frac{d}{dt} f_{\sigma_t}\, dt = \mathtt{d}_{\mathfrak{g}} \int_0^1 k f_{D,R_{\sigma_t},\ldots,R_{\sigma_t}}\, dt$$

is a coboundary. ■

Remark 3.5.5 (Polynomial functions) A map $p\colon \mathfrak{n} \to V$ is called a *V-valued polynomial of degree k* if there exists a symmetric k-linear map $f\colon \mathfrak{n}^k \to V$ with

$$p(x) = \frac{1}{k!}\widetilde{p}(x, x, \ldots, x), \quad x \in \mathfrak{n}.$$

For each polynomial of degree k, we have

$$p(tx) = t^k p(x), \quad x \in \mathfrak{n}, t \in \mathbb{K}.$$

Moreover, for $v_1, \ldots, v_k \in \mathfrak{n}$, the map

$$\mathbb{K}^k \to V, \quad (t_1, \ldots, t_k) \mapsto p(t_1 v_1 + t_2 v_2 + \ldots + t_k v_k)$$

is a V-valued polynomial function in k-variables and we recover the k-linear map $\widetilde{p}$ via

$$\widetilde{p}(v_1, \ldots, v_k) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_i=0} p(t_1 v_1 + t_2 v_2 + \ldots + t_k v_k),$$

where the partial derivatives are to be understood in the formal sense, i.e., $\widetilde{p}(v_1, \ldots, v_k)$ is the "coefficient" of the monomial $t_1 t_2 \cdots t_k$ in the polynomial $p(t_1 v_1 + \ldots + t_k v_k)$.

In this sense polynomials of degree k and symmetric k-linear maps can be considered as the same mathematical structures.

Proposition 3.5.6 *Let V be an associative algebra. Then we define on* $\mathrm{Sym}(\mathfrak{n}, V)$ *a multiplication by*

$$\alpha \vee \beta := \frac{1}{k!m!} \sum_{\sigma \in S_{k+m}} (\alpha \cdot \beta)^\sigma, \quad \alpha \in \mathrm{Sym}^k(\mathfrak{n}, V), \beta \in \mathrm{Sym}^m(\mathfrak{n}, V),$$

where

$$(\alpha \cdot \beta)(x_1, \ldots, x_{k+m}) := \alpha(x_1, \ldots, x_k)\beta(x_{k+1}, \ldots, x_{k+m}).$$

Then the map

$$\Phi\colon \mathrm{Sym}^\bullet(\mathfrak{n}, V) \to \mathrm{Pol}^\bullet(\mathfrak{n}, V), \quad \Phi(f)(x) := \frac{1}{k!} f(x, x, \ldots, x), \quad f \in \mathrm{Sym}^k(\mathfrak{n}, V),$$

defines an isomorphism of associative algebras.

Proof. For that we only have to observe that for $\alpha \in \mathrm{Sym}^k(\mathfrak{n}, V)$ and $\beta \in \mathrm{Sym}^m(\mathfrak{n}, V)$ we have

$$\frac{1}{(k+m)!}(\alpha \vee \beta)(x, \ldots, x) = \frac{1}{k!m!}\alpha(x, \ldots, x)\beta(x, \ldots, x).$$

■

Theorem 3.5.7 (Lecomte) (a) *For each $k \in \mathbb{N}_0$, we have a natural map*

$$C_k \colon \mathrm{Sym}^k(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}} \to H^{2k}(\mathfrak{g}, V), \quad f \mapsto \frac{1}{k!}[f_\sigma].$$

(b) *Suppose, in addition, that $m_V \colon V \times V \to V$ is an associative multiplication and that $\mathfrak{g}$ acts on V by derivations, i.e., m_V is $\mathfrak{g}$-invariant. Then $(C^\bullet(\mathfrak{g}, V), \wedge_{m_V})$ is an associative algebra, inducing an algebra structure on $H^\bullet(\mathfrak{g}, V)$. Further, $(\mathrm{Sym}^\bullet(\mathfrak{n}, V), \vee_{m_V})$ is an associative algebra, and the maps $(C_k)_{k \in \mathbb{N}_0}$ combine to an algebra homomorphism*

$$C \colon \mathrm{Sym}^\bullet(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}} \to H^{2\bullet}(\mathfrak{g}, V).$$

Proof. (a) follows from Lemma 3.5.4.

(b) With the notation from (3.7), we have for $f_i \in \mathrm{Sym}^{k_i}(\mathfrak{n}, V)$:

$$f_1 \vee_{m_V} f_2 = \frac{1}{k_1!k_2!} \sum_{\sigma \in S_{k_1+k_2}} (f_1 \cdot_{m_V} f_2)^\sigma.$$

We therefore have

$$\begin{aligned}
(f_1 \vee_{m_V} f_2)_{\varphi_1, \ldots, \varphi_{k_1+k_2}} &= \frac{1}{k_1!k_2!} \sum_{\sigma \in S_{k_1+k_2}} (f_1 \cdot f_2)^\sigma_{\varphi_1, \ldots, \varphi_{k_1+k_2}} \\
&= \frac{(k_1+k_2)!}{k_1!k_2!}(f_1 \cdot f_2)_{\varphi_1, \ldots, \varphi_{k_1+k_2}} \\
&= \frac{(k_1+k_2)!}{k_1!k_2!}(f_1)_{\varphi_1, \ldots, \varphi_{k_1}}(f_2)_{\varphi_{k_1+1}, \ldots, \varphi_{k_1+k_2}}.
\end{aligned}$$

This implies that for each σ, the maps

$$C_k \colon \mathrm{Sym}^k(\mathfrak{n}, V) \to C^{2k}(\mathfrak{g}, V), \quad f \mapsto \frac{1}{k!}[f_\sigma]$$

define an algebra homomorphism

$$C\colon \operatorname{Sym}^\bullet(\mathfrak{n}, V) \to C^{2\bullet}(\mathfrak{g}, V),$$

satisfying $C\big(\operatorname{Sym}^\bullet(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}}\big) \subseteq Z^{2\bullet}(\mathfrak{g}, V)$, hence inducing a homomorphism

$$C\colon \operatorname{Sym}^\bullet(\mathfrak{n}, V)^{\widehat{\mathfrak{g}}} \to H^{2\bullet}(\mathfrak{g}, V).$$

In view of Lemma 3.5.4, this map does not depend on the choice of σ. ■

Example 3.5.8 If $\mathfrak{n}$ is abelian, then we take $V := \mathfrak{n}$ and note that id_V is equivariant. Then (3.4) in the proof of Proposition 3.1.13 implies that $C(\mathrm{id}_V) = [R_\sigma] \in H^2(\mathfrak{g}, \mathfrak{n})$ is the characteristic class of the abelian Lie algebra extension $\widehat{\mathfrak{g}}$. As Proposition 3.1.13 asserts, this class determines the extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{n}$ uniquely in the sense that two extensions with the same class are equivalent.

Chapter 4

Smooth Functions with Values in Lie Groups

In this chapter we provide some basic results on Lie groups and their Lie algebras which are relevant for the understanding of the basic differential theory of fiber bundles.

4.1 Lie Groups and Their Lie Algebras

Let G be a Lie group. We recall that we define the *Lie algebra* $\mathfrak{g} = \mathbf{L}(G)$ of G, as the vector space $T_{\mathbf{1}}(G)$, endowed with the Lie bracket, obtained by identifying this space with the space $\mathcal{V}(G)_l$ of left invariant vector fields on G, i.e.,

$$\mathrm{ev}\colon \mathcal{V}(G)_l \to \mathbf{L}(G), \quad X \mapsto X(\mathbf{1})$$

is an isomorphism of Lie algebras.

Examples 4.1.1 (a) $G = \mathrm{GL}(V)$ for a finite-dimensional vector space. Since $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$, we have a natural trivialization of the tangent bundle. A vector field X on G is left invariant if and only if it is of the form $X(v) = Av$ for $A \in \mathrm{End}(V)$. This leads to

$$\mathbf{L}(\mathrm{GL}(V)) := \mathfrak{gl}(V) := (\mathrm{End}(V), [\cdot,\cdot]), \quad [A,B] := AB - BA.$$

(b) For $G = \mathbb{R}^n$, the left invariant vector fields are the constant ones. They form an abelian Lie algebra, so that $\mathbf{L}(\mathbb{R}^n) = \mathbb{R}^n$, with the trivial Lie bracket.

Proposition 4.1.2 *The map*

$$\Phi\colon G \times \mathfrak{g} \to TG, \quad (g, x) \mapsto g.x := T_{\mathbf{1}}(\lambda_g)x$$

is a bundle equivalence. In particular, the tangent bundle of each Lie group is trivial.

Definition 4.1.3 We define the *(left) Maurer–Cartan form* $\kappa_G \in \Omega^1(G, \mathfrak{g})$ by

$$\kappa_G := \mathrm{pr}_{\mathfrak{g}} \circ \Phi^{-1}\colon TG \to \mathfrak{g},$$

i.e.,

$$\kappa_G(g.x) = x \quad \text{for } g \in G, x \in \mathfrak{g}.$$

In the literature one also finds the notation $\kappa_G = g^{-1}\mathtt{d}g$ which is slightly ambiguous. We will not use it here.

The Exponential Function of a Lie Group

Lemma 4.1.4 *On a Lie group G, each left invariant vector field X is complete, i.e., has a global flow $\Phi^X\colon \mathbb{R} \times G \to G$.*

Definition 4.1.5 The exponential function of a Lie group G is defined by

$$\exp_G\colon \mathfrak{g} \to G, \quad \exp_G(x) = \Phi_1^{x_l}(\mathbf{1}),$$

where $x_l \in \mathcal{V}(G)$ denotes the left invariant vector field with $x_l(\mathbf{1}) = x$.

Remark 4.1.6 The curves $\gamma_x\colon \mathbb{R} \to G, \gamma_x(t) := \exp_G(tx)$ are smooth group homomorphisms with $\gamma_x'(0) = x$. In particular

$$T_0(\exp_G) = \mathrm{id}_{\mathfrak{g}},$$

so that $\exp_G$ is a local diffeomorphism in 0.

Derived Actions

Proposition 4.1.7 (The derived action/representation) *Let G be a Lie group and M be a smooth manifold.*

(1) *If* $\sigma\colon M \times G \to M$, $(m,g) \mapsto m.g =: \sigma^m(g)$ *is a smooth right action of* G *on* M, *then*

$$\dot\sigma\colon \mathbf{L}(G) \to \mathcal{V}(M), \qquad \dot\sigma(x)(m) := T_{(m,\mathbf{1})}(\sigma)(0,x) = T_{\mathbf{1}}(\sigma^m)(x)$$

defines a homomorphism of Lie algebras.

(ii) *If* $\sigma\colon G \times M \to M$ *is a smooth left action of* G *on* M, *then*

$$\dot\sigma\colon \mathbf{L}(G) \to \mathcal{V}(M), \qquad \dot\sigma(x)(m) := -T_{(\mathbf{1},m)}(\sigma)(x,0)$$

defines a homomorphism of Lie algebras.

(iii) *If* $\pi\colon G \to \mathrm{GL}(V)$ *is a smooth representation of* G *on* V, *i.e.*, $\sigma(g,v) := \pi(g)v$ *defines a smooth action of* G *on* V, *then*

$$\mathbf{L}(\pi)(x)v := T_{(\mathbf{1},v)}(\sigma)(x,0) = -\dot\sigma(x)(v)$$

defines a homomorphism of Lie algebras $\mathbf{L}(\pi)\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$.

Proof. (i) We pick $m \in M$ and write $\sigma^m\colon G \to M, g \mapsto m.g := \sigma(m,g)$ for the orbit map of m. Then $\sigma^m \circ \lambda_g = \sigma^{m.g}$ leads to

$$T(\sigma^m)(x_l(g)) = T_{\mathbf{1}}(\sigma^{m.g})x = \dot\sigma(x)_{m.g} = (\dot\sigma(x) \circ \sigma^m)(g),$$

which means that the vector fields x_l and $\dot\sigma(x)$ are σ^m-related. We conclude that for $x, y \in \mathbf{L}(G)$ the vector fields $[x_l, y_l] = [x,y]_l$ and $[\dot\sigma(x), \dot\sigma(y)]$ are also σ^m-related, which leads to

$$[\dot\sigma(x), \dot\sigma(y)](m) = T_{\mathbf{1}}(\sigma^m)[x,y]_l(\mathbf{1}) = T_{(m,\mathbf{1})}(\sigma)(0,[x,y]) = \dot\sigma([x,y])(m).$$

(ii) If σ is a left action, then $\check\sigma(m,g) := \sigma(g^{-1},m)$ defines a right action, and $T_{\mathbf{1}}(\eta_G)x = -x$ (Exercise 4.1.12) implies that

$$\dot\sigma(x) = -T_{(\mathbf{1},m)}(\sigma)(x,0) = T_{(m,\mathbf{1})}(\check\sigma)(0,x),$$

so that the assertion follows from (i).

(iii) For linear vector fields $X_A(x) = Ax$, $X_B(x) = Bx$, $A, B \in \mathfrak{gl}(V)$, we have

$$[X_A, X_B](x) = \mathtt{d}X_B(x)X_A(x) - \mathtt{d}X_A(x)X_B(x) = BAx - ABx = -[A,B]x,$$

so that the corresponding map $\mathfrak{gl}(V) \to \mathcal{V}(V), A \mapsto -X_A$ is a homomorphism of Lie algebras. Therefore (iii) follows from (ii). ∎

Remark 4.1.8 (a) For $G = \mathbb{R}$, smooth actions on manifolds are the same as global flows $\Phi\colon \mathbb{R} \times M \to M$. Thinking of a flow as a smooth right action, its infinitesimal generator is the vector field

$$X = \dot{\Phi}(1) \in \mathcal{V}(M).$$

It satisfies $\Phi = \Phi^X$.

(b) For a general Lie group G, the homomorphism $\dot{\sigma}$ can be calculated with the one-parameter groups $\gamma_x(t) = \exp_G(tx)$ via

$$\dot{\sigma}(x)(m) = \frac{d}{dt}\Big|_{t=0} m.\exp_G(tx).$$

Proposition 4.1.9 *If $\varphi\colon G \to H$ is a morphism of Lie groups, then*

$$\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$$

is a homomorphism of Lie algebras.

This completes the definition of the *Lie functor* $\mathbf{L}$, which assigns to a Lie group G its Lie algebra its Lie algebra $\mathbf{L}(G)$ and to each morphism φ of Lie groups a morphism of Lie algebras $\mathbf{L}(\varphi)$. It follows immediately from the Chain Rule that

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)} \quad \text{and} \quad \mathbf{L}(\varphi_1 \circ \varphi_2) = \mathbf{L}(\varphi_1) \circ \mathbf{L}(\varphi_2),$$

so that $\mathbf{L}$ is indeed a (covariant) functor, i.e., compatible with composition of morphisms.

Remark 4.1.10 If $\varphi \in \mathrm{Aut}(G)$ is an automorphism of the Lie group G, then $\mathbf{L}(\varphi) \in \mathrm{Aut}(\mathbf{L}(G))$.

Example 4.1.11 [The adjoint representation] For each Lie group G, the conjugation action $C(g,x) = gxg^{-1}$ defines a homomorphism

$$C\colon G \to \mathrm{Aut}(G), g \mapsto c_g, \quad c_g(x) = gxg^{-1},$$

so that

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}), \quad \mathrm{Ad}(g) := \mathbf{L}(c_g)$$

also is a group homomorphism. This defines a smooth representation of G on its Lie algebra $\mathfrak{g}$, called the *adjoint representation*.

Writing $g.v := T(\lambda_g)v$ and $v.g := T(\rho_g)v$ for the canonical left, resp., right action of G on its tangent bundle TG, the adjoint action can also be written as

$$\mathrm{Ad}(g)x = g.(x.g^{-1}) = (g.x).g^{-1} \in \mathfrak{g} = T_{\mathbf{1}}(G) \subseteq T(G).$$

The corresponding derived representation is given by

$$\mathbf{L}(\mathrm{Ad})(x)(y) = [x, y] = \mathrm{ad}\, x(y).$$

Exercises for Section 4.1

Exercise 4.1.12 Let G be a Lie group with multiplication $m_G \colon G \times G \to G$ and inversion $\eta_G \colon G \to G$. Show that:

(1) For $g, h \in G$ and $v \in T_g(G)$ and $w \in T_h(G)$, we have

$$T_{(g,h)}(m_G)(v, w) = T_g(\rho_h)v + T_h(\lambda_g)w.$$

(2) $T_{\mathbf{1}}(\eta_G) = -\,\mathrm{id}_{\mathfrak{g}}$.

Exercise 4.1.13 Let G be a Lie group and $x_l \in \mathcal{V}(G)$ the left invariant vector field with $x_l(\mathbf{1}) = x$. Show that:

(a) $\Phi^X_{st} = \Phi^{sX}_t$ for any $X \in \mathcal{V}(M)$.

(b) $\gamma_x \colon \mathbb{R} \to G, \gamma_x(t) := \exp_G(tx)$, is a group homomorphism.

(c) The local flow maps $\Phi^{x_l}_t$ commute with all left translations λ_g.

(d) $\Phi^{x_l}_t(g) = g \exp_G(tx)$ is the flow of x_l.

4.2 The Local Fundamental Theorem

In this section we introduce the logarithmic derivative $\delta(f) \in \Omega^1(M, \mathfrak{g})$ of a Lie group-valued smooth function $f \colon M \to G$. We shall see that any logarithmic derivative satisfies the Maurer–Cartan equation and that, conversely, this equation implies local integrability of an element of $\Omega^1(M, \mathfrak{g})$ (the Local Fundamental Theorem).

Definition 4.2.1 For a smooth function $f\colon M \to G$ with values in the Lie group G, the $\mathfrak{g}$-valued 1-form

$$\delta(f) := f^*\kappa_G \colon TM \to \mathfrak{g}$$

in $\Omega^1(M, \mathfrak{g})$ is called its *logarithmic derivative*. In $m \in M$ we have

$$\delta(f)_m = f(m)^{-1} \cdot T_m(f) = f(m)^{-1}(\mathtt{d}f)_m \colon T_m(M) \to \mathfrak{g},$$

which justifies the common notation $\delta(f) = f^{-1}\mathtt{d}f$. In this sense, the relation $\delta(\mathrm{id}_G) = \kappa_G$ justifies the notation $g^{-1}\mathtt{d}g$ for κ_G if one denotes the identity of G simply by g.

Lemma 4.2.2 *For each $f \in C^\infty(M, G)$, the 1-form $\alpha := \delta(f)$ satisfies the Maurer–Cartan equation*

$$\mathtt{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

Proof. First we show that κ_G satisfies the Maurer–Cartan equation. It suffices to evaluate $\mathtt{d}\kappa_G$ on left invariant vector fields x_l, y_l, where $x, y \in \mathfrak{g}$. Since $\kappa_G(x_l) = x$ is constant, we have

$$\begin{aligned} \mathtt{d}\kappa_G(x_l, y_l) &= x_l.\kappa_G(y_l) - y_l.\kappa_G(x_l) - \kappa_G([x_l, y_l]) = -\kappa_G([x, y]_l) = -[x, y] \\ &= -\frac{1}{2}[\kappa_G, \kappa_G](x_l, y_l). \end{aligned}$$

Since κ_G satisfies the MC equation, $\alpha = f^*\kappa_G$ satisfies

$$\mathtt{d}\alpha = f^*\mathtt{d}\kappa_G = -\frac{1}{2}f^*[\kappa_G, \kappa_G] = -\frac{1}{2}[f^*\kappa_G, f^*\kappa_G] = -\frac{1}{2}[\alpha, \alpha],$$

which is the Maurer–Cartan equation. ■

Example 4.2.3 For $G = \mathbb{R}^n$ we identify $TG = T(\mathbb{R}^n)$ with $\mathbb{R}^n \times \mathbb{R}^n$ and obtain $\kappa_G(x, v) = v$. Therefore we have for $f \in C^\infty(M, \mathbb{R}^n)$

$$\delta(f) = \mathtt{d}f.$$

Since $\mathfrak{g} = \mathbf{L}(G)$ is abelian, for $\alpha \in \Omega^1(M, \mathfrak{g})$, the MC equation simple is $\mathtt{d}\alpha = 0$, which means that α is a closed 1-form. Closedness is a necessary condition for α to be exact, i.e., of the form $\mathtt{d}f$ for a smooth function $f\colon M \to \mathbb{R}^n$.

Remark 4.2.4 (a) Let $I \subseteq \mathbb{R}$ be an open interval. Then $\Omega^2(I, \mathfrak{g}) = \{0\}$, so that each $\alpha \in \Omega^1(I, \mathfrak{g})$ trivially satisfies the MC equation. Moreover, the map

$$C^\infty(I, \mathfrak{g}) \to \Omega^1(I, \mathfrak{g}), \quad f \mapsto \mathtt{d}x \cdot f$$

is a linear isomorphism. We may therefore identify 1-forms on I with smooth curves in $\mathfrak{g}$.

For a smooth curve $\gamma \colon I \to G$ this identification leads to

$$\delta(\gamma)_t = \gamma(t)^{-1}\gamma'(t) = T_{\gamma(t)}(\lambda_{\gamma(t)^{-1}})\gamma'(t).$$

(b) If $U \subseteq \mathbb{R}^2$ is an open subset, then

$$\mathcal{V}(U) \cong C^\infty(U) \cdot \frac{\partial}{\partial x} \oplus C^\infty(U) \cdot \frac{\partial}{\partial y}.$$

Therefore each $\alpha \in \Omega^1(U, \mathfrak{g})$ is given by the two smooth functions

$$f := \alpha\Big(\frac{\partial}{\partial x}\Big) \quad \text{and} \quad g := \alpha\Big(\frac{\partial}{\partial y}\Big).$$

In these terms, the MC equation is equivalent to the PDE

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = [g, f].$$

If $f \colon M \to G$ is a smooth function and $\alpha \in \Omega^1(M, \mathfrak{g})$, then we use the short hand notation

$$\big(\mathrm{Ad}(f).\alpha\big)_m := \mathrm{Ad}(f(m)) \circ \alpha_m$$

and note that $\mathrm{Ad}(f)\alpha \in \Omega^1(M, \mathfrak{g})$.

Lemma 4.2.5 *For $f, g \in C^\infty(M, G)$, the following assertions hold:*

(1) *The map $f^{-1} \colon M \to G, m \mapsto f(m)^{-1}$ is smooth with*

$$\delta(f^{-1}) \;=\; -\mathrm{Ad}(f).\delta(f).$$

(2) *We have the following product and quotient rules:*

$$\delta(fg) = \delta(g) + \mathrm{Ad}(g^{-1}).\delta(f)$$

and

$$\delta(fg^{-1}) = \mathrm{Ad}(g).(\delta(f) - \delta(g)).$$

(3) (Uniqueness Lemma) *If M is connected, then the relation $\delta(f_1) = \delta(f_2)$ is equivalent to the existence of a $g \in G$ with $f_2 = \lambda_g \circ f_1$. In particular*

$$\delta(f_1) = \delta(f_2) \quad \textit{and} \quad f_1(m_0) = f_2(m_0) \quad \textit{for some} \quad m_0 \in M$$

imply $f_1 = f_2$.

Proof. (1), (2): Writing $fg = m_G \circ (f, g)$, we obtain from

$$T_{(g,h)}(m_G)(v, w) = T_g(\rho_h)v + T_h(\lambda_g)w = v.h + g.w$$

(Exercise 4.1.12) the relation

$$T(fg) = T(m_G) \circ (T(f), T(g)) = T(f).g + f.T(g) \colon M \to T(G),$$

which immediately leads to

$$\delta(fg) = (fg)^{-1}.(T(f).g + f.T(g)) = g^{-1}.\delta(f).g + \delta(g) = \mathrm{Ad}(g)^{-1}.\delta(f) + \delta(g).$$

From the Product Rule and $\delta(ff^{-1}) = 0$, the formula for $\delta(f^{-1})$ follows, and by combining this with the Product Rule, we get the Quotient Rule.

(3) If $\delta(f_1) = \delta(f_2)$, then the Quotient Rule implies that $\delta(f_1 f_2^{-1}) = 0$, so that $f_1 f_2^{-1}$ is locally constant. This implies (3). ■

Lie group homomorphisms and Maurer–Cartan forms

Lemma 4.2.6 *Let $\varphi \colon G \to H$ be a morphism of Lie groups. Then*

$$\delta(\varphi) = \varphi^*\kappa_H = \mathbf{L}(\varphi) \circ \kappa_G,$$

and for each smooth map $f \colon M \to G$ we have

$$\delta(\varphi \circ f) = \mathbf{L}(\varphi) \circ \delta(f).$$

Proof. For $g \in G$ we have $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$, which implies that for $g \in G$ and $x \in \mathbf{L}(G)$ we have

$$\delta(\varphi)(g.x) = \varphi(g)^{-1}.T_g(\varphi)(g.x) = T_{\mathbf{1}}(\varphi)x = \mathbf{L}(\varphi)x,$$

which immediately shows that $\delta(\varphi) = \mathbf{L}(\varphi) \circ \kappa_G$.

The second assertion follows from

$$(\varphi \circ f)^*\kappa_G = f^*\varphi^*\kappa_G = f^*(\mathbf{L}(\varphi) \circ \kappa_G) = \mathbf{L}(\varphi) \circ (f^*\kappa_G) = \mathbf{L}(\varphi) \circ \delta(f).$$

■

Proposition 4.2.7 *If G is connected and $\varphi_1, \varphi_2 : G \to H$ are morphism of Lie groups with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$, then $\varphi_1 = \varphi_2$.*

Proof. In view of Lemma 4.2.6, $\delta(\varphi_1) = \mathbf{L}(\varphi_1) \circ \kappa_G = \mathbf{L}(\varphi_2) \circ \kappa_G = \delta(\varphi_2)$, so that the uniqueness assertion follows from Lemma 4.2.5(3). ■

The preceding observation has some interesting consequences:

Corollary 4.2.8 *If G is a connected Lie group, then*

$$\ker \mathrm{Ad} = Z(G) := \{g \in G \colon (\forall h \in G)\ gh = hg\}$$

is the center of G.

Proof. Let $c_g(x) = gxg^{-1}$. In view of Proposition 4.2.7, for $g \in G$ the conditions $c_g = \mathrm{id}_G$ and $\mathbf{L}(c_g) = \mathrm{Ad}(g) = \mathrm{id}_{\mathbf{L}(G)}$ are equivalent. This implies the assertion. ■

The Local Fundamental Theorem

Now we turn to the local version of the Fundamental Theorem of Calculus for Lie group-valued smooth functions.

Definition 4.2.9 Let G be a Lie group, $I = [0,1]$ and $\xi \in C^\infty(I, \mathfrak{g})$ be a smooth curve in its Lie algebra. Then the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \delta(\gamma) = \gamma^{-1} \cdot \gamma' = \xi \tag{4.1}$$

has a unique solution γ_ξ. We write

$$\mathrm{evol}_G \colon C^\infty(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

for the corresponding evolution map.

Definition 4.2.10 Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$. We call a $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$ *integrable* if there exists a smooth function $f \colon M \to G$ with $\delta(f) = \alpha$. The 1-form α is said to be *locally integrable* if each point $m \in M$ has an open neighborhood U such that $\alpha|_U$ is integrable.

We know already from Lemma 4.2.2 that any locally integrable element $\alpha \in \Omega^1(M, \mathfrak{g})$ satisfies the MC equation and we want to show that the converse is also true. We start with the first crucial case where the MC equation is non-trivial, namely smooth 1-forms on the square.

Remark 4.2.11 Let $I = [0,1]$ be the unit interval. A smooth $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(I^2, \mathfrak{g})$ can be written as

$$\alpha = \mathtt{d}x \cdot v + \mathtt{d}y \cdot w \quad \text{with} \quad v, w \in C^\infty(I^2, \mathfrak{g}),$$

and we have already seen in Remark 4.2.4 that it satisfies the MC equation if and only if

$$\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x} = [v, w]. \tag{4.2}$$

Suppose that the two smooth functions $v, w \colon I^2 \to \mathfrak{g}$ satisfy (4.2). We define a function $f \colon I^2 \to G$ by

$$f(x, 0) := \gamma_{v(\cdot,0)}(x) \quad \text{and} \quad f(x, y) := f(x, 0) \cdot \gamma_{w(x,\cdot)}(y)$$

(here we use the notation from Definition 4.2.10.

The smoothness of f follows from the smooth dependence of the solutions of ODEs from parameters. Now

$$\delta f = \mathtt{d}x \cdot \widehat{v} + \mathtt{d}y \cdot w \quad \text{with} \quad \widehat{v}(x, 0) = v(x, 0) \quad \text{for} \quad x \in I.$$

We now show that $\widehat{v} = v$, so that $\delta(f) = \alpha$. The Maurer–Cartan equation for δf implies

$$\frac{\partial \widehat{v}}{\partial y} - \frac{\partial w}{\partial x} = [\widehat{v}, w],$$

so that subtraction of this equation from (4.2) leads to

$$\frac{\partial (v - \widehat{v})}{\partial y} = [v - \widehat{v}, w].$$

As $(v - \widehat{v})(x, 0) = 0$, the uniqueness of solutions of linear ODEs for a given initial value, applied to $\eta(t) := (v - \widehat{v})(x, t)$, x fixed, implies that $(v - \widehat{v})(x, y) = 0$ for all $x, y \in I$, hence that $v = \widehat{v}$. This means that $\delta(f) = \mathtt{d}x \cdot v + \mathtt{d}y \cdot w = \alpha$.

Theorem 4.2.12 (Local Fundamental Theorem) *Let U be an open convex subset of $\mathbb{R}^n$, G a Lie group with Lie algebra $\mathfrak{g}$ and $\alpha \in \Omega^1(U, \mathfrak{g})$ satisfying the Maurer–Cartan equation. Then α is integrable.*

Proof. We may w.l.o.g. assume that $x_0 = 0 \in U$. For $x \in U$ we then consider the smooth curve

$$\xi_x \colon I \to \mathfrak{g}, \quad t \mapsto \alpha_{tx}(x).$$

Since ξ depends smoothly on x, the function

$$f \colon U \to G, \quad x \mapsto \mathrm{evol}_G(\xi_x)$$

is smooth.

First we show that $f(sx) = \gamma_{\xi_x}(s)$ holds for each $s \in I$. For $s \in [0,1]$, the curve $\eta(t) := \gamma_{\xi_x}(st)$ satisfies

$$\delta(\eta)_t = s\xi_x(st) = s\alpha_{stx}(x) = \alpha_{stx}(sx) = \xi_{sx}(t),$$

so that

$$f(sx) = \mathrm{evol}(\xi_{sx}) = \eta(1) = \gamma_{\xi_x}(s).$$

For $x, x+h \in U$, we now consider the smooth map

$$\beta \colon I \times I \to U, \quad (s,t) \mapsto t(x+sh)$$

and the smooth function $F := f \circ \beta$. Then the preceding considerations imply $F(s,0) = f(0) = \mathbf{1}$ and

$$\begin{aligned} \frac{\partial F}{\partial t}(s,t) &= \frac{d}{dt} f(t(x+sh)) = \frac{d}{dt}\gamma_{\xi_{x+sh}}(t) = F(s,t).\xi_{x+sh}(t) \\ &= F(s,t).\alpha_{t(x+sh)}(x+sh) = F(s,t).(\beta^*\alpha)_{(s,t)}\Big(\frac{\partial}{\partial t}\Big). \end{aligned}$$

As we have seen in Remark 4.2.11, these two relations already imply that

$$\delta(F) = \beta^*\alpha \quad \text{on} \quad I \times I.$$

We therefore obtain

$$\frac{\partial}{\partial s} f(x+sh) = \frac{\partial}{\partial s} F(s,1) = F(s,1).\alpha_{x+sh}(h) = f(x+sh).\alpha_{x+sh}(h),$$

and for $s = 0$ this leads to $T_x(f)(h) = f(x).\alpha_x(h)$, which means that $\delta(f) = \alpha$. ■

Exercises for Section 4.2

Exercise 4.2.13 Let M be a smooth manifold and G a Lie group. Show that for each smooth function $f\colon M \to G$, each $\alpha \in \Omega^1(M, \mathfrak{g})$ and each smooth map $\varphi\colon N \to M$, we have

$$\varphi^*(\mathrm{Ad}(f).\alpha) = \mathrm{Ad}(\varphi^* f).\varphi^*\alpha.$$

4.3 Some Covering Theory

In this section we recall some of the main results on coverings of topological spaces needed to develop coverings of Lie groups and manifolds. In particular, this material is needed to extend the local Fundamental Theorem to a global one. The proofs not written in detail can all be found in the chapter on covering theory in Bredon's book [Br93].

The Fundamental Group

To define the notion of a simply connected space, we first have to define its fundamental group. The elements of this group are homotopy classes of loops. The present section develops this concept and provides some of its basic properties.

Definition 4.3.1 Let X be a topological space, $I := [0,1]$, and $x_0 \in X$. We write

$$P(X, x_0) := \{\gamma \in C(I, X)\colon \gamma(0) = x_0\}$$

and

$$P(X, x_0, x_1) := \{\gamma \in P(X, x_0)\colon \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X, x_0, x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H\colon I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

(for $H_t(s) := H(t, s)$) and

$$(\forall t \in I) \quad H(t, 0) = x_0,\ H(t, 1) = x_1.$$

It is easy to show that $\sim$ is an equivalence relation (Exercise 4.3.17), called *homotopy*. The homotopy class of α is denoted $[\alpha]$.

We write $\Omega(X, x_0) := P(X, x_0, x_0)$ for the set of loops based at x_0. For $\alpha \in P(X, x_0, x_1)$ and $\beta \in P(X, x_1, x_2)$, we define a product $\alpha * \beta$ in $P(X, x_0, x_2)$ by

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Lemma 4.3.2 *If $\varphi\colon [0,1] \to [0,1]$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then for each $\alpha \in P(X, x_0, x_1)$, we have $\alpha \sim \alpha \circ \varphi$.*

Proof. Use $H(t, s) := \alpha(ts + (1-t)\varphi(s))$. ∎

Proposition 4.3.3 *The following assertions hold:*

(1) *$\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ implies $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$, so that we obtain a well-defined product*

$$[\alpha] * [\beta] := [\alpha * \beta]$$

of homotopy classes.

(2) *If x_0 also denotes the constant map $I \to \{x_0\} \subseteq X$, then*

$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \textit{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) *(Associativity) $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ for $\alpha \in P(X, x_0, x_1)$, $\beta \in P(X, x_1, x_2)$ and $\gamma \in P(X, x_2, x_3)$.*

(4) *(Inverse) For $\alpha \in P(X, x_0, x_1)$ and $\overline{\alpha}(t) := \alpha(1-t)$ we have*

$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) *(Functoriality) For any continuous map $\varphi\colon X \to Y$ with $\varphi(x_0) = y_0$ we have*

$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta)$$

and $\alpha \sim \beta$ implies $\varphi \circ \alpha \sim \varphi \circ \beta$.

Proof. (1) If H^α is a homotopy from α_1 to α_2 and H^β a homotopy from β_1 to β_2, then we put

$$H(t, s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

(cf. Exercise 4.3.16).
(2) For the first assertion we use Lemma 4.3.2 and

$$x_0 * \alpha = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For the second, we have

$$\alpha * x_1 = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{4} \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(4)

$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \leq \frac{1-t}{2} \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \overline{\alpha}(2s-1) & \text{for } s \geq \frac{1+t}{2}. \end{cases}$$

(5) is trivial. ■

Definition 4.3.4 From the preceding definition, we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in x_0 carries a natural group structure. This group is called the *fundamental group of X with respect to x_0*.

A pathwise connected space X is called *simply connected* (or 1-*connected*) if $\pi_1(X, x_0)$ vanishes for some $x_0 \in X$ (which implies that is trivial for each $x_0 \in X$; Exercise 4.3.19).

Lemma 4.3.5 (Functoriality of the fundamental group) *If $f\colon X \to Y$ is a continuous map with $f(x_0) = y_0$, then*

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$

is a group homomorphism. Moreover, we have

$$\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)} \quad \textit{and} \quad \pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g).$$

Proof. This follows directly from Proposition 4.3.3(5). ■

Theorem 4.3.6 (The Path Lifting Property) *Let $q\colon X \to Y$ be a covering map and $\gamma\colon [0,1] \to Y$ a path. Let $x_0 \in X$ be such that $q(x_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma}\colon [0,1] \to X$ such that*

$$q \circ \widetilde{\gamma} = \gamma \quad \text{and} \quad \widetilde{\gamma}(0) = x_0.$$

Theorem 4.3.7 (The Covering Homotopy Theorem) *Let $I := [0,1]$, and $q\colon X \to Y$ be a covering map and $H\colon I^2 \to Y$ be a homotopy with fixed endpoints of the paths $\gamma := H_0$ and $\eta := H_1$. For any lift $\widetilde{\gamma}$ of γ there exists a unique lift $G\colon I^2 \to X$ of H with $G_0 = \widetilde{\gamma}$. Then $\widetilde{\eta} := G_1$ is the unique lift of η starting in the same point as $\widetilde{\gamma}$ and G is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\eta}$. In particular, lifts of homotopic curves in Y starting in the same point are homotopic in X.*

Corollary 4.3.8 *If $q\colon X \to Y$ is a covering with $q(x_0) = y_0$, then the corresponding homomorphism*

$$\pi_1(q)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [q \circ \gamma]$$

is injective.

Proof. If γ, η are loops in x_0 with $[q \circ \gamma] = [q \circ \eta]$, then the Covering Homotopy Theorem 4.3.7 implies that γ and η are homotopic. Therefore $[\gamma] = [\eta]$ shows that $\pi_1(q)$ is injective. ■

Corollary 4.3.9 *If Y is simply connected and X is arcwise connected, then each covering map $q\colon X \to Y$ is a homeomorphism.*

Proof. Since q is an open continuous map, it remains to show that q is injective. So pick $x_0 \in X$ and $y_0 \in Y$ with $q(x_0) = y_0$. If $x \in X$ also satisfies $q(x) = y_0$, then there exists a path $\alpha \in P(X, x_0, x)$ from x_0 to x. Now $q \circ \alpha$ is a loop in Y, hence contractible because Y is simply connected. Now the Covering Homotopy Theorem implies that the unique lift α of $q \circ \alpha$ starting in x_0 is a loop, and therefore that $x_0 = x$. This proves that q is injective. ■

The following theorem provides a more powerful tool, from which the preceding corollary easily follows. We recall that a topological space X is called *locally arcwise connected* if each point $x \in X$ possesses an arcwise connected neighborhood. All manifolds have this property because each point of a manifold M has an open neighborhood homeomorphic to an open convex set in $\mathbb{R}^n$.

Theorem 4.3.10 (The Lifting Theorem) *Assume that $q\colon X \to Y$ is a covering map with $q(x_0) = y_0$, that W is arcwise connected and locally arcwise connected, and that $f\colon W \to Y$ is a given map with $f(w_0) = y_0$. Then a continuous map $g\colon W \to X$ with*

$$g(w_0) = x_0 \quad \text{and} \quad q \circ g = f$$

exists if and only if

$$\pi_1(f)(\pi_1(W, w_0)) \subseteq \pi_1(q)(\pi_1(X, x_0)), \text{ i.e. } \operatorname{im}(\pi_1(f)) \subseteq \operatorname{im}(\pi_1(q)). \tag{4.3}$$

If g exists, then it is uniquely determined. Condition (4.3) *is in particular satisfied if W is simply connected.*

Corollary 4.3.11 (Uniqueness of simply connected coverings) *Suppose that Y is locally arcwise connected. If $q_1 : X_1 \to Y$ and $q_2 : X_2 \to Y$ are two simply connected arcwise connected coverings, then there exists a homeomorphism $\varphi\colon X_1 \to X_2$ with $q_2 \circ \varphi = q_1$.*

Proof. Since Y is locally arcwise connected, both covering spaces X_1 and X_2 also have this property. Pick points $x_1 \in X_1$, $x_2 \in X_2$ with $y := q_1(x_1) = q_2(x_2)$. According to the Lifting Theorem 4.3.10, there exists a unique lift $\varphi\colon X_1 \to X_2$ of q_1 with $\varphi(x_1) = x_2$. We likewise obtain a unique lift $\psi\colon X_2 \to X_1$ of q_2 with $\psi(x_2) = x_1$. Then $\varphi \circ \psi\colon X_1 \to X_1$ is a lift of id_Y fixing x_1, so that the uniqueness of lifts implies that $\varphi \circ \psi = \mathrm{id}_{X_1}$. The same argument yields $\psi \circ \varphi = \mathrm{id}_{X_2}$, so that φ is a homeomorphism with the required properties. ■

Theorem 4.3.12 *Each manifold M has a simply connected covering $q_M : \widetilde{M} \to M$.*

Definition 4.3.13 Let $q\colon X \to Y$ be a covering. A homeomorphism $\varphi\colon X \to X$ is called a *deck transformation* of the covering if $q \circ \varphi = \mathrm{id}_Y$. This means that φ permutes the elements in the fibers of q. We write $\mathrm{Deck}(X, q)$ for the group of deck transformations.

Example 4.3.14 (a) For the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, the deck transformations have the form

$$\varphi(z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$

(b) For the covering map $q\colon \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, the deck transformations have the form

$$\varphi(z) = z + n, \quad n \in \mathbb{Z}.$$

Proposition 4.3.15 *Let $q\colon \widetilde{X} \to X$ be a simply connected covering of X with base point $\widetilde{x}_0$. For each $[\gamma] \in \pi_1(X, x_0)$ we write $\varphi_{[\gamma]} \in \mathrm{Deck}(\widetilde{X}, q)$ for the unique lift of id_X mapping $\widetilde{x}_0$ to the endpoint $\widetilde{\gamma}(1)$ of the canonical lift $\widetilde{\gamma}$ of γ starting in $\widetilde{x}_0$. Then the map*

$$\Phi\colon \pi_1(X, x_0) \to \mathrm{Deck}(\widetilde{X}, q), \quad \Phi([\gamma]) = \varphi_{[\gamma]}$$

is an isomorphism of groups.

Proof. The composition $\varphi_{[\gamma]} \circ \varphi_{[\eta]}$ is a deck transformation mapping $\widetilde{x}_0$ to the endpoint of $\varphi_{[\gamma]} \circ \widetilde{\eta}$ which coincides with the endpoint of the lift of η starting in $\widetilde{\gamma}(1)$. Hence it also is the endpoint of the lift of the loop $\gamma * \eta$. Therefore Φ is a group homomorphism.

To see that Φ is injective, we note that $\varphi_\gamma = \mathrm{id}_{\widetilde{X}}$ implies that $\widetilde{\gamma}(1) = \widetilde{x}_0$, so that $\widetilde{\gamma}$ is a loop, and hence that $[\gamma] = [x_0]$.

For the surjectivity, let φ be a deck transformation and $y := \varphi(\widetilde{x}_0)$. If α is a path from $\widetilde{x}_0$ to y, then $\gamma := q \circ \alpha$ is a loop in x_0 with $\alpha = \widetilde{\gamma}$, so that $\varphi_{[\gamma]}(\widetilde{x}_0) = y$, and the uniqueness of lifts implies that $\varphi = \varphi_{[\gamma]}$. ■

Exercises for Section 4.3

Exercise 4.3.16 If $f\colon X \to Y$ is a map between topological spaces and $X = X_1 \cup \ldots \cup X_n$ holds with closed subsets $X_1, \ldots, X_n$, then f is continuous if and only if all restrictions $f|_{X_i}$ are continuous.

Exercise 4.3.17 Show that the homotopy relation on $P(X, x_0, x_1)$ is an equivalence relation. Hint: Exercise 4.3.16 helps to glue homotopies.

Exercise 4.3.18 Show that for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected. For the proof, proceed along the following steps:
(a) Let $\gamma : [0,1] \to \mathbb{S}^n$ be continuous. Then there exists an $m > 0$ such that $\|\gamma(t) - \gamma(t')\| < \frac{1}{2}$ for $|t - t'| < \frac{1}{m}$.
(b) Define $\widetilde{\alpha} : [0,1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0, \ldots, m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0,1] \to \mathbb{S}^n$.

(c) $\alpha \sim \gamma$. Hint: Consider $H(t,s) := \frac{(1-s)\gamma(t)+s\alpha(t)}{\|(1-s)\gamma(t)+s\alpha(t)\|}$.
(d) α is not surjective. The image of α is the central projection of a polygonal arc on the sphere.
(e) If $\beta \in \Omega(\mathbb{S}^1, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map). Hint: Let $p \in \mathbb{S}^n \setminus \operatorname{im}\beta$. Using stereographic projection, where p corresponds to the point at infinity, show that $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible.
(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$.

Exercise 4.3.19 Let X be a topological space x_0, x_1 and $\alpha \in P(X, x_0, x_1)$ a path from x_0 to x_1. Show that the map

$$C\colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense the fundamental group does not depend on the base point if X is arcwise connected.

4.4 The Fundamental Theorem for Lie group-valued functions

Proposition 4.4.1 *Let M be a connected manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and $\alpha \in \Omega^1(M, \mathfrak{g})$ a 1-form. If α is locally integrable, then there exists a connected covering $q\colon \widehat{M} \to M$ such that $q^*\alpha$ is integrable. If, in addition, M is simply connected, then α is integrable.*

Proof. We consider the product set $P := M \times G$ with the two projection maps $F\colon P \to G$ and $q\colon P \to M$. We define a topology on P as follows. For each pair (U, f), consisting of an open subset $U \subseteq M$ and a smooth function $f\colon U \to G$ with $\delta(f) = \alpha|_U$, the graph $\Gamma(f, U) := \{(x, f(x))\colon x \in U\}$ is a subset of P.

Step 1: We claim that the sets $\Gamma(f, U)$ form a basis for a topology τ on P. In fact, let $p = (m, g) \in \Gamma(f_1, U_1) \cap \Gamma(f_2, U_2)$. We choose a connected open neighborhood U of m, contained in $U_1 \cap U_2$. Then $f_1(m) = f_2(m) = g$ and $\delta(f_1|_U) = \delta(f_2|_U) = \alpha|_U$ imply that $f_1 = f_2$ on U (Lemma 4.2.5(c)), so that $p \in \Gamma(f_1|_U, U) \subseteq \Gamma(f_1, U_1) \cap \Gamma(f_2, U_2)$. This proves our claim. We endow P with the topology τ.

Step 2: τ is Hausdorff: It is clear that the projection $q\colon P \to M$ is continuous with respect to τ. It therefore suffices to show that two different points $(m, g_1), (m, g_2) \in P$ can be separated by open subsets. If U is a connected open neighborhood of m on which α is integrable, we find a smooth function $f_1 \colon U \to G$ with $f_1(m) = g_1$ and $\delta(f_1) = \alpha$. Now $f_2 := g_2^{-1} g_1 f_1$ satisfies $f_2(m) = g_2$ and $\delta(f_2) = \alpha$. Hence $\Gamma(f_1, U)$ is an open neighborhood of (m, g_1) and $\Gamma(f_2, U)$ an open neighborhood of (m, g_2). If both intersect, there exists a point $x \in U$ with $f_1(x) = f_2(x)$, so that the connectedness of U implies $f_1 = f_2$, contradicting $g_1 = g_2$ (Lemma 4.2.5(c)). Hence P is Hausdorff.

Step 3: The restrictions $q|_{\Gamma(f,U)}$ are homeomorphisms. We know already that q is continuous. Let $\sigma := (\mathrm{id}_U, f) \colon U \to \Gamma(f, U) \subseteq P$. Then, for each basic neighborhood $\Gamma(g, V)$, we have

$$\alpha^{-1}(\Gamma(g, V)) = \{x \in U \cap V \colon f(x) = g(x)\}.$$

In view of the uniqueness assertion of Lemma 4.2.5(c), the latter set is open, so that σ is continuous. Now $q \circ \sigma = \mathrm{id}_U$ implies that $q|_{\Gamma(f,U)}$ is a homeomorphism onto U.

Step 4: The mapping $q \colon P \to M$ is a covering. Let $x \in M$. Since α is integrable, there exists a connected open neighborhood U_x of x such that $\alpha|_{U_x}$ is locally integrable. Then there exists for each $g \in G$ a smooth function $f_g \colon U_x \to G$ with $f_g(x) = g$ and $\delta(f_g) = \alpha|_U$. Now $q^{-1}(U) = U \times G = \bigcup_{g \in G} \Gamma(f_g, U)$ is a disjoint union of open subsets of P, as we have seen in Step 1, and in Step 3 we have seen that it restricts to homeomorphisms on each of these sets. Hence q is a covering.

Step 5: We conclude that P carries a natural manifold structure for which q is a local diffeomorphism (Exercise!). For this manifold structure, the function $F \colon P \to G$ is smooth, because for each basic open set $\Gamma(f, U)$, U connected, the inverse of the corresponding restriction of U is given by

$$\sigma_{(f,U)} := (q|_{\Gamma(f,U)})^{-1} \colon U \to \Gamma(f, U), \quad x \mapsto (x, f(x)),$$

and $F \circ \sigma_{(f,U)} = f$ is smooth. Moreover, $f \circ q = F|_{\Gamma(f,U)}$ leads to

$$\delta(F) = q^*\delta(f) = q^*\alpha$$

on each set $\Gamma(f, U)$, hence on all of P.

Fix a point $m_0 \in M$. Then the connected component $\widehat{M}$ of $(m_0, \mathbf{1})$ in P is a connected covering manifold of M with the required properties.

If, in addition, M is simply connected, then q is a trivial covering, hence a diffeomorphism (Corollary 4.3.9), and therefore α is integrable. ■

Let $\alpha \in \Omega^1(M, \mathfrak{g})$. If $\gamma \colon I = [0,1] \to M$ is a piecewise smooth loop, then $\gamma^*\alpha \in \Omega^1(I, \mathfrak{g}) \cong C^\infty(I, \mathfrak{g})$ and we get the element $\mathrm{evol}_G(\gamma^*\alpha) \in G$.

Lemma 4.4.2 *Suppose that M is connected and $\alpha \in \Omega^1(M, \mathfrak{g})$ is locally integrable. Pick $m_0 \in M$. Then the element $\mathrm{evol}_G(\gamma^*\alpha)$ does not change under homotopies with fixed endpoints and*

$$\mathrm{per}_\alpha^{m_0} \colon \pi_1(M, m_0) \to G, \quad [\gamma] \mapsto \mathrm{evol}_G(\gamma^*\alpha)$$

is a group homomorphism.

Proof. Let $q_M \colon \widetilde{M} \to M$ denote a simply connected covering manifold of M (Theorem 4.3.12) and choose a base point $\widetilde{m}_0 \in \widetilde{M}$ with $q_M(\widetilde{m}_0) = m_0$. Then the $\mathfrak{g}$-valued 1-form $q_M^*\alpha$ on $\widetilde{M}$ also satisfies the Maurer–Cartan equation, so that Proposition 4.4.1 implies the existence of a unique smooth function $\widetilde{f} \colon \widetilde{M} \to G$ with $\delta(\widetilde{f}) = q_M^*\alpha$ and $\widetilde{f}(\widetilde{m}_0) = \mathbf{1}$.

We write

$$\sigma \colon \pi_1(M, m_0) \times \widetilde{M} \to \widetilde{M}, \quad (d, m) \mapsto d.m = \sigma_d(m)$$

for the left action of the fundamental group $\pi_1(M, m_0)$ on $\widetilde{M}$ by deck transformations (cf. Proposition 4.3.15). Then

$$\delta(\widetilde{f} \circ \sigma_d) = \sigma_d^* q_M^*\alpha = q_M^*\alpha = \delta(\widetilde{f})$$

for each $d \in \pi_1(M, m_0)$ implies the existence of a function

$$\chi \colon \pi_1(M, m_0) \to G \quad \text{with} \quad \widetilde{f} \circ \sigma_d = \chi(d) \cdot \widetilde{f}, \quad d \in \pi_1(M, m_0).$$

For $d_1, d_2 \in \pi_1(M, m_0)$, we then have

$$\widetilde{f} \circ \sigma_{d_1 d_2} = \widetilde{f} \circ \sigma_{d_1} \circ \sigma_{d_2} = (\chi(d_1) \cdot \widetilde{f}) \circ \sigma_{d_2} = \chi(d_1) \cdot (\widetilde{f} \circ \sigma_{d_2}) = \chi(d_1)\chi(d_2) \cdot \widetilde{f},$$

hence χ is a group homomorphism.

We now pick a piecewise smooth curve $\widetilde{\gamma} \colon I \to \widetilde{M}$ with $q_M \circ \widetilde{\gamma} = \gamma$ (Theorem 4.3.6) and observe that

$$\delta(\widetilde{f} \circ \widetilde{\gamma}) = \widetilde{\gamma}^* q_M^*\alpha = \gamma^*\alpha,$$

so that

$$\chi([\gamma]) = \widetilde{f}([\gamma].\widetilde{m}_0) = \widetilde{f}(\widetilde{\gamma}(1)) = \mathrm{evol}_G(\gamma^*\alpha).$$

This completes the proof. ∎

Definition 4.4.3 For each locally integrable $\alpha \in \Omega^1(M, \mathfrak{g})$, the homomorphism

$$\mathrm{per}_\alpha := \mathrm{per}_\alpha^{m_0} \colon \pi_1(M, m_0) \to G \quad \text{with} \quad \mathrm{per}_\alpha^{m_0}([\gamma]) = \mathrm{evol}_G(\gamma^*\alpha),$$

for each piecewise smooth loop $\gamma \colon I \to M$ in m_0, is called the *period homomorphism of α with respect to m_0*.

The following theorem is a global version of the Fundamental Theorem of calculus for functions with values in Lie groups.

Theorem 4.4.4 (Fundamental Theorem for Lie group-valued functions) *Let M be a smooth manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and $\alpha \in \Omega^1(M, \mathfrak{g})$. Then the following assertions hold:*

(1) *α is locally integrable if and only if it satisfies the Maurer–Cartan equation $\mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0$.*

(2) *If M is 1-connected and α is locally integrable, then it is integrable.*

(3) *If M is connected, α is locally integrable, and $m_0 \in M$, then $\mathrm{per}_\alpha^{m_0}$ vanishes if and only if α is integrable.*

Proof. (1) If α is locally integrable, then Lemma 4.2.2 implies that it satisfies the Maurer–Cartan equation. If, conversely, it satisfies the MC equation, then Lemma 4.2.2 implies its local integrability.

(2) Proposition 4.4.1.

(3) The function $\widetilde{f}$ constructed in the proof of Lemma 4.4.2 above factors through a smooth function on M if and only if the period homomorphism is trivial. This implies (3). ■

Corollary 4.4.5 *Let M be a smooth connected manifold, $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $\alpha \in \Omega^1(M, \mathbb{R}^n)$. Then the following assertions hold:*

(1) *α is locally integrable if and only if α is closed.*

(2) *α is integrable to a smooth G-valued function if and only if all integrals of α over piecewise smooth loops are contained in $\mathbb{Z}^n$.*

Corollary 4.4.6 *Let M be a connected smooth connected manifold and $m_0 \in M$. Then the map*

$$P\colon H^1_{\rm dR}(M,\mathbb{R}) \to \mathrm{Hom}(\pi_1(M,m_0),\mathbb{R}), \quad [\alpha] \mapsto \mathrm{per}_\alpha^{m_0}$$

is injective.

Proof. To see that P is well-defined, we note that $\mathrm{per}_{\mathtt{d}f} = 0$ for any smooth function $f\colon M \to \mathbb{R}$. If, conversely, $\mathrm{per}_\alpha^{m_0}\,\alpha = 0$ holds for a closed 1-form $\alpha \in \Omega^1(M,\mathbb{R})$, then the Fundamental Theorem implies that $\alpha = \mathtt{d}f$ for some smooth function $f\colon M \to \mathbb{R}$, so that $[\alpha] = 0$. ■

Remark 4.4.7 Let M be a connected smooth manifold, $m_0 \in M$, and G a Lie group with Lie algebra $\mathfrak{g}$. We write

$$\mathrm{MC}(M,\mathfrak{g}) := \Big\{\alpha \in \Omega^1(M,\mathfrak{g})\colon \mathtt{d}\alpha + \frac{1}{2}[\alpha,\alpha] = 0\Big\}$$

for the set of solutions of the MC equation. If $\mathfrak{g}$ is abelian, then $\mathrm{MC}(M,\mathfrak{g})$ is the space of closed $\mathfrak{g}$-valued 1-forms.

From the Fundamental Theorem, we obtain a sequence of maps

$$G \hookrightarrow C^\infty(M,G) \xrightarrow{\delta} \mathrm{MC}(M,\mathfrak{g}) \xrightarrow{\mathrm{per}^{m_0}} \mathrm{Hom}(\pi_1(M,m_0),G),$$

which is exact as a sequence of pointed sets, i.e., in each place, the inverse image of the base point is the range of the preceding map.

Applications of the Fundamental Theorem

Proposition 4.4.8 *Let G and H be Lie groups. Assume that G is connected and that $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a morphism of Lie algebras. If $\varphi\colon G \to H$ is a smooth map with*

$$\varphi(\mathbf{1}_G) = \mathbf{1}_H \quad \textit{and} \quad \varphi^*\kappa_H = \psi \circ \kappa_G,$$

then φ is a homomorphism of Lie groups.

Proof. Let $x \in G$. Then we obtain a smooth map

$$f = \lambda_{\varphi(x)^{-1}} \circ \varphi \circ \lambda_x \colon G \to H, \quad y \mapsto \varphi(x)^{-1}\varphi(xy)$$

with $f(\mathbf{1}_G) = \mathbf{1}_H$. Further, we obtain with Lemma 4.2.6

$$\begin{aligned}\delta(f) &= \delta(\lambda_{\varphi(x)^{-1}} \circ \varphi \circ \lambda_x) = \delta(\varphi \circ \lambda_x) = \lambda_x^* \varphi^* \kappa_H \\ &= \lambda_x^*(\psi \circ \kappa_G) = \psi \circ (\lambda_x^* \kappa_G) = \psi \circ \kappa_G = \delta(\varphi).\end{aligned}$$

Since G is connected and $f(\mathbf{1}_G) = \varphi(\mathbf{1}_G)$, we obtain $f = \varphi$ from the Uniqueness Lemma. This means that $\varphi(xy) = \varphi(x)\varphi(y)$ for $x, y \in G$. ∎

Theorem 4.4.9 *If H is a Lie group, G is a 1-connected Lie group, and $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a continuous homomorphism of Lie algebras, then there exists a unique Lie group homomorphism $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. The uniqueness assertion follows from Proposition 4.2.7. On G we consider the smooth $\mathbf{L}(H)$-valued 1-form given by $\alpha := \psi \circ \kappa_G$. That it satisfies the Maurer–Cartan equation follows from

$$\mathtt{d}\alpha = \psi \circ \mathtt{d}\kappa_G = -\frac{1}{2}\psi \circ [\kappa_G, \kappa_G] = -\frac{1}{2}[\psi \circ \kappa_G, \psi \circ \kappa_G] = -\frac{1}{2}[\alpha, \alpha].$$

Therefore the Fundamental Theorem implies the existence of a unique smooth function $\varphi\colon G \to H$ with $\delta(\varphi) = \alpha$ and $\varphi(\mathbf{1}_G) = \mathbf{1}_H$. In view of Proposition 4.4.8, the function φ is a morphism of Lie groups, and we clearly have $\mathbf{L}(\varphi) = \alpha(\mathbf{1}) = \psi$. ∎

Corollary 4.4.10 *If G_1 and G_2 are simply connected Lie groups with isomorphic Lie algebras, then G_1 and G_2 are isomorphic.*

Proof. Let $\psi\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ be an isomorphism of Lie algebras. Then there exists a unique morphism of Lie groups $\varphi\colon G_1 \to G_2$ with $\mathbf{L}(\varphi) = \psi$, and likewise there exists a morphism $\widehat{\varphi}\colon G_2 \to G_1$ with $\mathbf{L}(\widehat{\varphi}) = \psi^{-1}$. Then $\mathbf{L}(\varphi \circ \widehat{\varphi}) = \mathrm{id}_{\mathbf{L}(G_2)}$ and $\mathbf{L}(\widehat{\varphi} \circ \varphi) = \mathrm{id}_{\mathbf{L}(G_1)}$, together with the uniqueness, leads to $\widehat{\varphi} \circ \varphi = \mathrm{id}_{G_1}$ and $\varphi \circ \widehat{\varphi} = \mathrm{id}_{G_2}$. Thus φ is an isomorphism of Lie groups. ∎

Chapter 5

Connections on Principal Bundles

In this chapter we introduce connections on principal bundles. They are the key tool to the differential theory of fiber bundles. In particular, they are crucial for the calculation of characteristic classes.

Throughout this chapter, (P, M, G, q, σ) denotes a principal bundle, $\mathfrak{g} = \mathbf{L}(G)$. We recall the notation $\sigma_g(p) := p.g$ and $\sigma^p(g) = p.g$.

5.1 The Lie Algebra of Infinitesimal Bundle Automorphisms

Definition 5.1.1 Motivated by the relation

$$\mathrm{Aut}(P) := \{\varphi \in \mathrm{Diff}(P) \colon (\forall g \in G)\, \varphi \circ \sigma_g = \sigma_g \circ \varphi\},$$

we consider the Lie algebra

$$\mathfrak{aut}(P) := \{X \in \mathcal{V}(P) \colon (\forall g \in G)\, (\sigma_g)_* X = X\}$$

of G-invariant vector fields on P.

Lemma 5.1.2 *The following assertions hold for* $\mathfrak{aut}(P)$*:*

(a) $\mathfrak{aut}(P)$ *is a Lie subalgebra of* $\mathcal{V}(P)$*.*

(b) *We have a well-defined map*

$$q_*\colon \mathfrak{aut}(P) \to \mathcal{V}(M), \quad (q_*X)(q(p)) := T_p(q)X(p)$$

which is a homomorphism of Lie algebras.

(c) $\mathfrak{aut}(P)$ *is a* $C^\infty(M)$*-module w.r.t.*

$$(fX)(p) := f(q(p))X(p)$$

and q_* *is* $C^\infty(M)$*-linear.*

(d) $[\dot\sigma(x), Y] = 0$ *for* $x \in \mathfrak{g}$ *and* $Y \in \mathfrak{aut}(P)$.

Proof. (a) That $X \in \mathfrak{aut}(P)$ is equivalent to X being σ_g-related to itself for each $g \in G$. Therefore the Related Vector Field Lemma implies that $\mathfrak{aut}(P)$ is a Lie subalgebra.
(b) The relation $(\sigma_g)_*X = X$ means that for each $p \in P$ we have

$$T(\sigma_g)X(p) = X(p.g).$$

Hence

$$T(q)X(p.g) = T(q)T(\sigma_g)X(p) = T(q \circ \sigma_g)X(p) = T(q)X(p)$$

implies that q_*X is well-defined. Since $(q_*X) \circ q = T(q) \circ X$ is smooth and $q\colon P \to M$ is a submersion, the vector field q_*X is smooth.

For $X, Y \in \mathfrak{aut}(P)$ we use that the vector fields X and q_*X are q-related to see that $[X, Y]$ and $[q_*X, q_*Y]$ are q-related. This implies that $q_*[X, Y] = [q_*X, q_*Y]$.
(c) For $g \in G$ we have $(\sigma_g)_*(fX) = f \cdot (\sigma_g)_*X = f \cdot X$.
(d) follows from

$$[\dot\sigma(x), Y] = \mathcal{L}_{\dot\sigma(x)}Y = \frac{d}{dt}\Big|_{t=0} (\sigma_{\exp_G(-tx)})_*Y = \frac{d}{dt}\Big|_{t=0} Y = 0.$$

■

Definition 5.1.3 The ideal

$$\mathfrak{gau}(P) := \ker(q_*) \trianglelefteq \mathfrak{aut}(P)$$

describes those vector fields which are tangent to the G-orbits in P.

Recall the derived action

$$\dot{\sigma} \colon \mathfrak{g} \to \mathcal{V}(P), \quad \dot{\sigma}(x)_p = \frac{d}{dt}\Big|_{t=0} p.\exp(tx).$$

We know from Proposition 4.1.7 that $\dot{\sigma}$ is a homomorphism of Lie algebras.

Lemma 5.1.4 (a) *For each $p \in P$, the map*

$$\dot{\sigma}_p \colon \mathfrak{g} \to \ker(T_p(q)) \subseteq T_p(P), \quad x \mapsto \dot{\sigma}(x)(p) = T_{\mathbf{1}}(\sigma^p)(x)$$

is a linear isomorphism.

(b) *If $f \colon N \to G$ and $s \colon N \to P$ are smooth functions, then the function $s.f \colon N \to P, x \mapsto s(x).f(x)$, satisfies*

$$T_x(s.f) = T(\sigma_{f(x)}) \circ T_x(s) + \dot{\sigma}_{s(x).f(x)} \circ \delta(f)_x. \tag{5.1}$$

Proof. (a) Since this is a local assertion, we may assume that $P = M \times G$ is the trivial bundle. Then

$$\dot{\sigma}(x)(m,g) = (0, g.x) \in T_{(m,g)}(M \times G) \cong T_m(M) \times T_g(G),$$

so that the assertion follows from Proposition 4.1.2.
(b) We simply calculate with the Chain Rule

$$\begin{aligned} T_x(s.f)v &= T_x(\sigma \circ (s,f))v \\ &= T_{s(x),f(x)}(\sigma)(T_x(s)v, 0) + T_{(s(x),f(x))}(\sigma)(0, T_x(f)v) \\ &= T(\sigma_{f(x)})T_x(s)v + T_{(s(x),f(x))}(\sigma)(0, f(x).\delta(f)_x v) \\ &= T(\sigma_{f(x)})T_x(s)v + T(\sigma^{s(x).f(x)})(\delta(f)_x v) \\ &= T(\sigma_{f(x)})T_x(s)v + \dot{\sigma}(\delta(f)_x v). \end{aligned}$$

■

Definition 5.1.5 For each $p \in P$, we define the *vertical subspace* of $T_p(P)$ by

$$V_p(P) := \dot{\sigma}_p(\mathfrak{g}) = \ker(T_p(q))$$

and call its elements the *vertical* tangent vectors.

Proposition 5.1.6 (The Lie algebra $\mathfrak{gau}(P)$) *Let* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ *denote the adjoint action of* G *on its Lie algebra* $\mathfrak{g}$ *and*

$$\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{g}$$

the corresponding associated bundle. Then the following assertions hold:

(a) *The map*

$$\Psi\colon C^\infty(P,\mathfrak{g})^G \cong \Gamma\,\mathrm{Ad}(P) \to \mathfrak{gau}(P), \quad \Psi(\xi)(p) := -\dot{\sigma}_p(\xi(p))$$

is an isomorphism of Lie algebras, where $C^\infty(P,\mathfrak{g})^G$ *is endowed with the pointwise Lie bracket*

$$[\xi,\eta](p) := [\xi(p),\eta(p)].$$

(b) *The map*

$$\exp_*\colon C^\infty(P,\mathfrak{g})^G \to C^\infty(P,G)^G \cong \mathrm{Gau}(P), \quad \xi \mapsto \exp_G \circ \xi$$

associates to each $\xi \in C^\infty(P,\mathfrak{g})^G$ *the smooth flow*

$$\Phi^\xi_t(p) := p.\exp_G(t\xi(p))$$

by bundle automorphisms. Its infinitesimal generator is $-\Psi(\xi)$.

(c) (The adjoint action of $\mathrm{Gau}(P)$) *For* $h \in C^\infty(P,G)^G$ *and* $\varphi_h(p) = p.h(p)$, *we have*

$$(\varphi_h)_*\Psi(f) = \Psi(\mathrm{Ad}(h).f).$$

Proof. The first part is an immediate consequence of Proposition 1.6.3 which asserts that

$$\Gamma(\mathrm{Ad}(P)) \cong C^\infty(P,\mathfrak{g})^G.$$

We first show (b) and (c), then (a).
(b) Since the exponential function $\exp_G\colon \mathfrak{g} \to G$ is equivariant:

$$\exp_G(\mathrm{Ad}(g)x) = \exp_G(\mathbf{L}(c_g)x) = c_g(\exp_G(x)) = g\exp_G(x)g^{-1},$$

it follows that

$$(\exp_G)_*(C^\infty(P,\mathfrak{g})^G) \subseteq C^\infty(P,G)^G.$$

For each $\xi \in C^\infty(P, \mathfrak{g})$,

$$\Phi^\xi(t,p) := p.\exp_G(t\xi(p))$$

defines a smooth flow on P whose infinitesimal generator is the vector field $-\Psi(\xi)$. From $\varphi_{\exp_G \circ \xi} \in \mathrm{Gau}(P)$ we thus derive

$$(\sigma_g)_*\Psi(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_g \circ \Phi^\xi_{-t} \circ \sigma_g^{-1} = \Psi(\xi) \quad \text{for} \quad g \in G.$$

Since $\Psi(\xi)$ is clearly vertical, we find that $\Psi(\xi) \in \mathfrak{gau}(P)$.
(c) Since $\varphi := \varphi_h \in \mathrm{Gau}(P) \subseteq \mathrm{Aut}(P)$, we have $\varphi \circ \sigma_g = \sigma_g \circ \varphi$, which leads to

$$T_p(\varphi)\dot\sigma_p(x) = \dot\sigma_{\varphi(p)}(x),$$

i.e.,

$$\varphi_*\dot\sigma(x) = \dot\sigma(x).$$

This leads to

$$\begin{aligned}\big(\varphi_*\Psi(\xi)\big)(p) &= T(\varphi)\Psi(\xi)(p.h(p)^{-1}) = -T(\varphi)\dot\sigma_{p.h(p)^{-1}}(\xi(p.h(p)^{-1}))\\ &= -\dot\sigma_p(\xi(p.h(p)^{-1})) = -\dot\sigma_p(\mathrm{Ad}(h(p))\xi(p)) = \Psi(\mathrm{Ad}(h)\xi)(p).\end{aligned}$$

(a) For $\varphi(p) = p.\exp_G(-t\xi(p)) = \Phi^{\Psi(\xi)}_t(p)$, we derive

$$\begin{aligned}[\Psi(\xi), \Psi(\eta)] &= \mathcal{L}_{\Psi(\xi)}\Psi(\eta) = \frac{d}{dt}\Big|_{t=0} (\Phi^{\Psi(\xi)}_{-t})_*\Psi(\eta)\\ &= \frac{d}{dt}\Big|_{t=0} \Psi(\mathrm{Ad}(\exp_G(t\xi)).\eta) = \frac{d}{dt}\Big|_{t=0} \Psi(e^{t\,\mathrm{ad}\,\xi}.\eta) = \Psi([\xi,\eta]).\end{aligned}$$

This proves that Ψ is a homomorphism of Lie algebras. Since all the maps $\dot\sigma_p$ are injective, Ψ is injective. To see that it is surjective, let $X \in \mathfrak{gau}(P)$ and define

$$\xi(p) := -\dot\sigma_p^{-1}(X(p)).$$

Then $\xi\colon P \to \mathfrak{g}$ is a function with $\Psi(\xi) = X$. Further

$$\begin{aligned}X(p.g) = T(\sigma_g)X(p) &= -T(\sigma_g)\dot\sigma(\xi(p)) = -T(\sigma_g)T(\sigma^p)\xi(p)\\ &= -T(\sigma^{p.g})T(c_g^{-1})\xi(p) = -T(\sigma^{p.g})\,\mathrm{Ad}(g^{-1})\xi(p)\end{aligned}$$

implies that

$$\xi(p.g) = \mathrm{Ad}(g^{-1})\xi(p).$$

The smoothness of ξ can be verified locally, so that we may assume that the bundle is trivial. Then $X(m,g) = (0, \widetilde{X}(m,g))$ for a smooth function $\widetilde{X} \colon M \times G \to TG$ with $\widetilde{X}(m,g) \in T_g(G)$. Now $\dot\sigma_{(m,g)}(x) = (0, g.x)$ implies that

$$\xi(m,g) = -g^{-1}.\widetilde{X}(m,g) = -\kappa_G(\widetilde{X}(m,g))$$

is a smooth function. This proves that Ψ is bijective. ∎

We collect some useful formulas for the calculation with the group of gauge transformations and its Lie algebra.

Lemma 5.1.7 *Using the map $f \mapsto \varphi_f$, we identify $C^\infty(P,G)^G$ with* $\mathrm{Gau}(P)$ *(Proposition 1.6.7). We then have*

(a) $\varphi \circ \varphi_f \circ \varphi^{-1} = \varphi_{f \circ \varphi^{-1}}$ *for* $\varphi \in \mathrm{Aut}(P)$.

(b) $\varphi_* \Psi(\xi) = \Psi(\xi \circ \varphi^{-1})$ *for* $\xi \in C^\infty(P, \mathfrak{g})^G$ *and* $\varphi \in \mathrm{Aut}(P)$.

(c) $[X, \Psi(\xi)] = \Psi(X.\xi)$ *for* $X \in \mathfrak{aut}(P)$ *and* $\xi \in C^\infty(P,\mathfrak{g})^G$.

Proof. (a) For $p \in P$ we have $\varphi \circ \varphi_f \circ \varphi^{-1}(p) = \varphi(\varphi^{-1}(p).f(\varphi^{-1}(p))) = p.f(\varphi^{-1}(p))$.
(b) Applying (a) to $f(p) = \exp_G(t\xi(p))$ for $\xi \in C^\infty(P,\mathfrak{g})^G$ and taking derivatives in $t = 0$, we obtain (b).
(c) For $X \in \mathfrak{aut}(P)$ we get with (b):

$$[X, \Psi(\xi)] = \mathcal{L}_X \Psi(\xi) = \frac{d}{dt}\Big|_{t=0} (\Phi^X_{-t})_* \Psi(\xi) = \frac{d}{dt}\Big|_{t=0} \Psi(\xi \circ \Phi^X_t) = \Psi(X.\xi).$$

∎

Remark 5.1.8 For the trivial bundle $P = M \times G$ with $\sigma_g(m,h) = (m, hg)$, we have

$$\mathfrak{aut}(P) \cong \mathfrak{gau}(P) \rtimes \mathcal{V}(M) \cong C^\infty(P,\mathfrak{g})^G \rtimes \mathcal{V}(M) \cong C^\infty(M,\mathfrak{g}) \rtimes \mathcal{V}(M),$$

where $\mathcal{V}(M)$ acts on $C^\infty(M,\mathfrak{g})$ by $(X.\xi)(m) := \mathtt{d}\xi(m)X(m)$.

5.2 Paracompactness and Partitions of Unity

In this section we recall a central tool for the analysis on manifolds: smooth partitions of unity. They are used in various situations to localize problems, i.e., to turn them into problems on open subsets of $\mathbb{R}^n$, which are usually easier to solve.

To obtain sufficiently fine smooth partitions of unity on a manifold, we have to impose a condition on the underlying topological space.

Definition 5.2.1 (a) A topological space X is said to be *σ-compact* if there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} X_n$.

(b) If $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ are open covers of the topological space X, then we call $(V_j)_{j\in J}$ a *refinement* of $(U_i)_{i\in I}$ if for each $j \in J$ there exists some $i_j \in I$ with $V_j \subseteq U_{i_j}$.

A family $(S_i)_{i\in I}$ of subsets of X is called *locally finite* if each point $p \in X$ has a neighborhood V intersecting only finitely many of the sets S_i.

A topological space X is said to be *paracompact* if each open cover has a locally finite refinement.

Theorem 5.2.2 *For a finite-dimensional (topological) manifold M, the following are equivalent:*

(1) *M is paracompact.*

(2) *Each connected component of M is σ-compact.*

Definition 5.2.3 A *smooth partition of unity* on a smooth manifold M is a family $(\psi_i)_{i\in I}$ of smooth functions $\psi_i \in C^\infty(M,\mathbb{R})$ such that

(P1) $0 \leq \psi_i$ for each $i \in I$.

(P2) Local finiteness: each point $p \in M$ has a neighborhood U such that $\{i \in I : \psi_i|_U \neq 0\}$ is finite.

(P3) $\sum_i \psi_i = 1$.

Note that (P2) implies that in each $p \in M$

$$\sum_{i\in I} \psi_i(p) = \sum_{\psi_i(p)\neq 0} \psi_i(p)$$

is a finite-sum, so that it is well-defined, even if I is an infinite set.

If $\mathcal{U} = (U_j)_{j\in J}$ is an open cover, then a partition of unity $(\psi_j)_{j\in J}$ is said to be *associated to* $\mathcal{U}$ if $\operatorname{supp}(\psi_j) \subseteq U_j$ holds for each $j \in J$.

Theorem 5.2.4 *If M is paracompact and $(U_j)_{j\in J}$ is a locally finite open cover of M, then there exists an associated smooth partition of unity on M.*

Corollary 5.2.5 *Let M be a paracompact smooth manifold, $F \subseteq M$ a closed subset and $U \subseteq M$ an open neighborhood of F. Then there exists a smooth function $f\colon M \to \mathbb{R}$ with*

(1) $0 \leq f \leq 1$.

(2) $f|_F = 1$.

(3) $\mathrm{supp}(f) \subseteq U$.

Proof. In view of Theorem 5.2.4, there exists a smooth partition of unity associated to the open cover $\{U, M \setminus F\}$. This is a pair of smooth functions (f, g) with $\mathrm{supp}(f) \subseteq U$, $\mathrm{supp}(g) \subseteq M \setminus F$, $0 \leq f, g$, and $f + g = 1$. We thus have (1) and (3), and (2) follows from $g|_F = 0$. ■

Applications to affine bundles

Let $G = (V, +)$ be the additive group of a vector space. Then a principal G-bundle is called an *affine bundle*. Although the fibers of an affine bundle (P, M, V, q, σ) are isomorphic to V as an affine space, they do not carry a natural vector space structure; affine bundle are NOT vector bundles.

As an immediate consequence of the existence of smooth partitions of unity, we obtain:

Theorem 5.2.6 *Each affine bundle over a paracompact manifold M is trivial.*

Proof. The main point of the proof is that convex combinations make sense in affine bundles. Let (P, M, V, q, σ) be an affine bundle over M, then we define for $x_1, \ldots, x_p \in P_m$ and $\lambda_i \in \mathbb{R}$ with $\sum_i \lambda_i = 1$ the convex combination $\sum_i \lambda_i x_i$ as follows.

If $\varphi_U\colon U \times V \to P_U$ is a bundle chart and $x_i = \varphi_U(m, v_i)$, we put

$$\sum_i \lambda_i x_i := \varphi_U(m, \sum_i \lambda_i v_i).$$

If $\varphi_W\colon W \times V \to P_W$ is another chart with $m \in W$, then $\varphi_W(m,v) = \varphi_U(m, g(m)+v)$ for a smooth function $g\colon U \cap W \to V$ and

$$x_i = \varphi_W(m, v_i - g(m)),$$

so that

$$\begin{aligned}\varphi_W\Big(m, \sum_i \lambda_i(v_i - g(m))\Big) &= \varphi_W\Big(m, \sum_i \lambda_i v_i - \sum_i \lambda_i g(m)\Big) \\ &= \varphi_W\Big(m, (\sum_i \lambda_i v_i) - g(m)\Big) = \varphi_U\Big(m, \sum_i \lambda_i v_i\Big).\end{aligned}$$

Therefore the convex combination $\sum_i \lambda_i x_i$ is well-defined.

Now let $(\varphi_i, U_i)_{i\in I}$ be a bundle atlas. In view of the paracompactness of M, we may w.l.o.g. assume that the open cover $\mathcal{U} = (U_i)_{i\in I}$ is locally finite, and hence that there exists a subordinated smooth partition of unity $(\chi_i)_{i\in I}$. The functions $s_i\colon U_i \to P_{U_i}$, $s_i(m) := \varphi_i(m,0)$ define smooth sections of P_{U_i} and therefore

$$s(m) := \sum_{\lambda_i(m)\neq 0} \chi_i(m) s_i(m)$$

is in each point $m \in M$ a convex combination of elements of P_m. Moreover, the local finiteness of $\mathcal{U}$ implies that s is a smooth section of P, so that P is trivial (Proposition 2.2.4). ■

5.3 Bundle-valued Differential Forms

We have already seen that for a vector space V, the space $\Omega^k(M,V)$ can be identified in a natural way with the space $\mathrm{Alt}^k_{C^\infty(M)}(\mathcal{V}(M), C^\infty(M,V))$ of $C^\infty(M)$-multilinear k-cochains of the Lie algebra $\mathcal{V}(M)$ with values in the $C^\infty(M)$-module $C^\infty(M,V)$ (Theorem 3.2.1). This identification suggests the following definition of differential forms with values in vector bundles.

Definition 5.3.1 Let $(\mathbb{V}, M, V, q)$ be a vector bundle. Then we put

$$\Omega^k(M,\mathbb{V}) := \mathrm{Alt}^k_{C^\infty(M)}(\mathcal{V}(M), \Gamma\mathbb{V}),$$

and call the elements of this space $\mathbb{V}$-valued k-forms.

Remark 5.3.2 (Local description of bundle-valued forms in trivializations) Let $(\varphi_i, U_i)_{i\in I}$ be a vector bundle atlas of $\mathbb{V}$ with transition functions $g_{ij} \in C^\infty(U_{ij}, \mathrm{GL}(V))$. Each $\mathbb{V}$-valued differential form $\omega \in \Omega^k(M, \mathbb{V})$ yields a family of differential forms

$$\omega_i \in \Omega^k(U_i, \mathbb{V}_{U_i}) \cong \Omega^k(U_i, V)$$

(Exercise 5.3.7) satisfying the transformation rule

$$\omega_j = g_{ji} \cdot \omega_i \quad \text{on} \quad U_{ij}.$$

Conversely, each family $(\omega_i)_{i\in I}$ satisfying the above transformation rule yields a $\mathbb{V}$-valued k-form ω by

$$\omega(X_1, \ldots, X_k)(m) := \varphi_i(m, \omega_i(X_1(m), \ldots, X_k(m))) \quad \text{for} \quad m \in U_i. \quad (5.2)$$

Definition 5.3.3 We call a differential form $\omega \in \Omega^k(P, V)$ *horizontal* if $i_{\dot\sigma(x)}\omega = 0$ holds for each $x \in \mathfrak{g}$. We write $\Omega^k(P, V)_{\mathrm{hor}}$ for the subspace of horizontal V-valued k-forms on P.

If $\pi\colon G \to \mathrm{GL}(V)$ is a representation, then $\omega \in \Omega^k(P, V)$ is called *equivariant* if

$$\sigma_g^*\omega = \pi(g)^{-1} \circ \omega$$

holds for each $g \in G$. We write $\Omega^k(P, V)^G$ for the space G-equivariant k-forms with values in V. Both carry natural $C^\infty(M)$-module structures.

Proposition 5.3.4 *If $\mathbb{V} = P \times_\pi V$ is associated to P by the representation (π, V) of G, then $\omega \mapsto \widetilde{\omega}$, defined by*

$$\omega_{q(p)}(T(q)v_1, \ldots, T(q)v_k) = [p, \widetilde{\omega}_p(v_1, \ldots, v_k)]$$

yields a well-defined $C^\infty(M)$-linear bijection

$$\Omega^k(M, \mathbb{V}) \to \Omega^k(P, V)^G_{\mathrm{hor}}.$$

Proof. (a) From the local description in (5.2) in Remark 5.3.2, it follows that the forms $\widetilde{\omega}_p \in \mathrm{Alt}^k(T_p(M), V)$ are well-defined and combine to an element of $\Omega^k(P, V)$ with $i_v\widetilde{\omega}_p = 0$ for each vertical vector v. From the relation

$$\begin{aligned}
[p, \widetilde{\omega}_p(v_1, \ldots, v_k)] &= \omega_{q(p)}(T(q)v_1, \ldots, T(q)v_k) \\
&= [p.g, \widetilde{\omega}_{p.g}(T(\sigma_g)v_1, \ldots, T(\sigma_g)v_k)] \\
&= [p, \pi(g)\widetilde{\omega}_{p.g}(T(\sigma_g)v_1, \ldots, T(\sigma_g)v_k)]
\end{aligned}$$

we further derive that

$$\sigma_g^*\widetilde{\omega} = \pi(g)^{-1} \circ \widetilde{\omega}.$$

Therefore $\widetilde{\omega} \in \Omega^k(P, V)^G_{\rm hor}$.

(b) Suppose, conversely, that $\eta \in \Omega^k(P, V)^G_{\rm hor}$. We choose a bundle atlas (φ_i, U_i) which leads to the local sections $s_i(m) := \varphi_i(m, \mathbf{1})$. We thus obtain a family of differential forms

$$\omega_i := s_i^*\eta \in \Omega^k(U_i, V).$$

We recall from Remark 2.2.3 the relation $s_i = s_j g_{ji}$ and note that for $m \in U_{ij}$ and $v \in T_m(M)$ the difference

$$T(s_i)v - T(\sigma_{g_{ji}(m)})T(s_j)v = \dot{\sigma}_{s_i(m)}(\delta(g_{ji})_m v)$$

(cf. Lemma 5.1.4(b)) is a vertical vector. We thus obtain

$$\begin{aligned}
&\omega_i(m)(v_1, \ldots, v_k) = \eta_{s_i(m)}(T(s_i)v_1, \ldots, T(s_i)v_k) \\
&= \eta_{s_j(m)g_{ji}(m)}(T(\sigma_{g_{ji}(m)})T(s_j)v_1, \ldots, T(\sigma_{g_{ji}(m)})T(s_j)v_k) \\
&= \pi(g_{ji}(m))^{-1}.\eta_{s_j(m)}(T(s_j)v_1, \ldots, T(s_j)v_k) = \pi(g_{ji}(m))^{-1}.\omega_j(v_1, \ldots, v_k).
\end{aligned}$$

This means that $\omega_j = \pi(g_{ji}) \cdot \omega_i$, so that Remark 5.3.2 implies that there exists an element $\omega \in \Omega^k(M, \mathbb{V})$ with

$$\begin{aligned}
\omega(X_1, \ldots, X_k)(m) &= [\varphi_i(m, \mathbf{1}), \omega_i(X_1(m), \ldots, X_k(m))] \\
&= [s_i(m), \omega_i(X_1(m), \ldots, X_k(m))] \\
&= [s_i(m), \eta_{s_i(m)}(T(s_i)X_1(m), \ldots, T(s_i)X_k(m))].
\end{aligned}$$

Since this expression also equals

$$[s_i(m), \widetilde{\omega}_{s_i(m)}(T(s_i)X_1(m), \ldots, T(s_i)X_k(m))],$$

it follows that $\widetilde{\omega} = \eta$. ■

With the preceding proposition, we now easily see that the full automorphism group $\mathrm{Aut}(P)$ acts linearly on the space of bundle-valued differential forms:

Proposition 5.3.5 *If $\mathbb{V} = P \times_\pi V$ is associated to P by the representation (π, V) of G, then for $\varphi \in \mathrm{Aut}(P)$ and $\alpha \in \Omega^k(P,V)^G_{\mathrm{bas}} \cong \Omega^k(M,\mathbb{V})$, the pullback $\varphi^*\alpha$ is also contained in $\Omega^k(P,V)^G_{\mathrm{bas}}$, so that*

$$\varphi.\alpha := (\varphi^{-1})^*\alpha$$

defines a representation of $\mathrm{Aut}(P)$ on $\Omega^k(P,V)^G_{\mathrm{bas}}$. For a gauge transformation φ_f, $f \in C^\infty(P,G)^G$, we simply have

$$(\varphi_f^*\alpha)_p = \pi(f(p))^{-1} \circ \alpha_p.$$

Proof. That $\varphi^*\alpha$ is also equivariant follows from

$$\sigma_g^*(\varphi^*\alpha) = \sigma_g^*\varphi^*\alpha = \varphi^*\sigma_g^*\alpha = \varphi^*(\pi(g)^{-1} \circ \alpha) = \pi(g)^{-1} \circ (\varphi^*\alpha).$$

Further, $\varphi_*\dot\sigma(x) = \dot\sigma(x)$ for $x \in \mathfrak{g}$ leads to

$$i_{\dot\sigma(x)}(\varphi^*\alpha) = \varphi^*(i_{\dot\sigma(x)}\alpha) = 0.$$

Therefore the space of horizontal equivariant differential forms is $\mathrm{Aut}(P)$-invariant and we thus obtain a representation by $\varphi.\alpha := (\varphi^{-1})^*\alpha$.

For $f \in C^\infty(P,G)^G$ and the corresponding gauge transformation $\varphi_f(p) = p.f(p)$, we obtain with Lemma 5.1.4(b)

$$T_p(\varphi_f)v = T_p(\sigma_{f(p)})v + \dot\sigma_{p.f(p)}(\delta(f)_p v).$$

Since the second summand is a vertical vector, we obtain for any $\alpha \in \Omega^k(P,V)^G_{\mathrm{hor}}$ the relation

$$\begin{aligned}(\varphi_f^*\alpha)_p(v_1,\ldots,v_k) &= \alpha_{p.f(p)}(T(\sigma_{f(p)})v_1,\ldots,T(\sigma_{f(p)})v_k) \\ &= (\sigma_{f(p)}^*\alpha)_p(v_1,\ldots,v_k) = \pi(f(p))^{-1}\alpha_p(v_1,\ldots,v_k).\end{aligned}$$

■

We also take a brief look at the Lie algebra level of this representation.

Remark 5.3.6 Let $\pi\colon G \to \mathrm{GL}(V)$ be a representation of V.

The subspace $\Omega^k(P,V)_{\mathrm{hor}}$ of horizontal V-valued k-forms is invariant under the action of $\mathfrak{aut}(P) \subseteq \mathcal{V}(P)$ by the Lie derivative. In fact, for $x \in \mathfrak{g}$ and $Y \in \mathfrak{aut}(P)$ we have

$$[Y, \dot\sigma(x)] = -\mathcal{L}_{\dot\sigma(x)}Y = 0$$

(Lemma 5.1.2(d)) because $(\sigma_g)_*Y = Y$ for each $g \in G$. This implies that

$$[\mathcal{L}_Y, i_{\dot\sigma(x)}] = i_{[Y,\dot\sigma(x)]} = 0,$$

so that we obtain for $\omega \in \Omega^k(M,V)_{\rm hor}$:

$$i_{\dot\sigma(x)}(\mathcal{L}_Y\omega) = \mathcal{L}_Y(i_{\dot\sigma(x)}\omega) = 0.$$

From $(\sigma_g)_*Y = Y$ for each $Y \in \mathfrak{aut}(P)$ we also derive that $\Omega^k(P,V)^G$ is invariant under $\mathfrak{aut}(P)$. In fact, for $\omega \in \Omega^k(P,V)^G$, $g \in G$ and $Y \in \mathfrak{aut}(P)$, we have

$$(\sigma_g)^*(\mathcal{L}_Y\omega) = \mathcal{L}_Y(\sigma_g^*\omega) = \mathcal{L}_Y(\pi(g)^{-1}\circ\omega) = \pi(g)^{-1}\circ(\mathcal{L}_Y\omega).$$

Combining the preceding two observations, we see that $\mathfrak{aut}(P)$ also acts on their intersection $\Omega^k(P,V)^G_{\rm hor} \cong \Omega^k(M,\mathbb{V})$ by the Lie derivative.

Exercises for Section 5.3

Exercise 5.3.7 Let $(\mathbb{V},M,V,q)$ be a vector bundle and $\omega \in \Omega^k(M,\mathbb{V}) = \mathrm{Alt}^k_{C^\infty(M)}(\mathcal{V}(M),\Gamma\mathbb{V})$ be a $\mathbb{V}$-valued k-form. Further, let $U \subseteq M$ be an open subset. We want to show that ω defines a *restriction* $\omega_U \in \Omega^k(M,\mathbb{V}_U)$. For $X_1,\ldots,X_k \in \mathcal{V}(U)$ and $m \in M$ we pick a smooth function $\chi : U \to M$ with compact support and $\chi(m) = 1$ in a neighborhood of m. Then we may consider χX_i as an element of $\mathcal{V}(M)$ and put

$$\omega_U(X_1,\ldots,X_k)(m) := \omega(\chi X_1,\ldots,\chi X_k)(m).$$

Show that this definition does not depend on the choice of the function χ.

Exercise 5.3.8 Let $\sigma\colon P\times G \to P$ be a smooth right action and $f\colon P \to G$ an equivariant smooth function, i.e., $f(p.g) = f(p)g$ for $p \in P$ and $g \in G$. Show that $\delta(f) \in \Omega^1(P,\mathfrak{g})$ is an equivariant 1-form, i.e.,

$$\sigma_g^*\delta(f) = \mathrm{Ad}(g)^{-1}\circ\delta(f).$$

Exercise 5.3.9 Let $\mathbb{V} = P\times_\pi V$ be a vector bundle over M associated to P via the representation (π,V) of G. Recall that the G-bundle q^*P over P has a canonical section defined by $s(p) := (p,p)$. Show that:

(1) The map

$$\varphi\colon P \times V \to q^*\mathbb{V}, \quad (p, v) \mapsto (p, [p, v])$$

is a global bundle map. In particular, $q^*\mathbb{V}$ is trivial.

(2) $\Gamma(q^*\mathbb{V}) \cong C^\infty(P, V)$, where a section s corresponds to the function $f\colon P \to V$, defined by $s(p) = (p, [p, f(p)])$.

(3) If $\omega \in \Omega^k(M, \mathbb{V})$ is a bundle-valued k-form, then we obtain a $q^*\mathbb{V}$-valued k-form by

$$(q^*\omega)_p(v_1, \ldots, v_k) := \omega_{q(p)}(T(q)v_1, \ldots, T(q)v_k) \in \mathbb{V}_{q(p)} \cong (q^*\mathbb{V})_p.$$

(4) With respect to the trivialization of $q^*\mathbb{V}$ from (1), the form $q^*\omega \in \Omega^k(P, q^*\mathbb{V})$ corresponds to the lift $\widetilde{\omega} \in \Omega^k(P, V)^G_{\text{hor}}$, defined in Proposition 5.3.4.

5.4 Connection 1-Forms

Connections on principal bundles are the fundamental tool in the theory of differentiable fiber bundles. They are most conveniently defined in terms of Lie algebra-valued 1-forms.

Definition 5.4.1 A 1-form $\theta \in \Omega^1(P, \mathfrak{g})$ is called a *connection* 1*-form* if

(C1) $\theta(\dot{\sigma}(x)) = x$ for each $x \in \mathfrak{g}$.

(C2) $\sigma_g^*\theta = \mathrm{Ad}(g)^{-1} \circ \theta$ for each $g \in G$, i.e., $\theta \in \Omega^1(P, \mathfrak{g})^G$ (cf. Definition 5.3.3).

We write $\mathcal{C}(P)$ for the space of connection 1-forms. This is obviously an affine subspace of $\Omega^1(P, \mathfrak{g})$ whose translation vector space is

$$\Omega^1(P, \mathfrak{g})^G_{\text{bas}} \cong \Omega^1(M, \mathrm{Ad}(P)).$$

Definition 5.4.2 For each $p \in P$, we recall the vertical subspace of $T_p(P)$, defined by

$$V_p(P) := \dot{\sigma}_p(\mathfrak{g}) = \ker(T_p(q)).$$

If, in addition, $\theta \in \mathcal{C}(P)$ is a connection 1-form, then

$$H_p(P) := \ker \theta_p$$

is a vector space complement to $V_p(P)$, called the *horizontal subspace*. Vectors in $H_p(P)$ are called *horizontal*, those in $V_p(P)$ are called *vertical*. We then have

$$T_p(P) = V_p(P) \oplus H_p(P)$$

and

$$T(\sigma_g)H_p(P) = H_{p.g}(P), \quad T(\sigma_g)V_p(P) = V_{p.g}(P) \quad \text{for} \quad g \in G,$$

i.e., the decomposition into vertical and horizontal subspace is G-invariant.

Since a horizontal k-form $\omega \in \Omega^k(P,V)_{\mathrm{hor}}$ vanishes on all tuples containing a vertical vector, they are completely determined by their values on the horizontal subspace $H_p(P)$. This justifies the terminology.

Remark 5.4.3 Note that each connection 1-form $\theta \in \mathcal{C}(P)$ can be reconstructed from the family of horizontal subspaces because, for each $p \in P$, we have

$$T_p(P) = \dot\sigma_p(\mathfrak{g}) \oplus H_p(P),$$

and on $v = \dot\sigma_p(x)+w$, $w \in H_p(P)$, we have $\theta_p(v) = x$. Therefore the subspace $H_p(P)$ determines the linear map $\theta_p \colon T_p(P) \to \mathfrak{g}$ completely.

Proposition 5.4.4 *The group* $\mathrm{Aut}(P)$ *acts on the affine space* $\mathcal{C}(P)$ *of connection* 1*-forms from the right via* $\theta.\varphi := \varphi^*\theta$ *by affine maps.*

For each smooth function $f \colon P \to G$ *and the corresponding map* $\varphi_f \colon P \to P, p \mapsto p.f(p)$*, we have*

$$\varphi_f^*\theta = \delta(f) + \mathrm{Ad}(f)^{-1}.\theta. \tag{5.3}$$

Proof. Let $\theta \in \mathcal{C}(P)$ be a connection 1-form.
(a) Since each $\varphi \in \mathrm{Aut}(P)$ commutes with each σ_g, the pullback $\varphi^*\theta$ is also equivariant: [1]

$$\sigma_g^*(\theta.\varphi) = \sigma_g^*\varphi^*\theta = \varphi^*\sigma_g^*\theta = \varphi^*(\mathrm{Ad}(g)^{-1} \circ \theta) = \mathrm{Ad}(g)^{-1} \circ (\varphi^*\theta).$$

Next we note that $\varphi_*\dot\sigma(x) = \dot\sigma(x)$ leads to

$$(\varphi^*\theta)(\dot\sigma(x)) = \varphi^*(\theta(\dot\sigma(x))) = \varphi^*x = x.$$

[1] In general we have $\varphi^*(\theta(X)) = \varphi^*\theta((\varphi^{-1})_*X)$.

(b) First we use Lemma 5.1.4(b)) to get the formula

$$T_p(\varphi_f)v = T_p(\sigma_{f(p)})v + \dot\sigma_{p.f(p)}(\delta(f)_p v).$$

We thus obtain

$$\begin{aligned}(\varphi_f^*\theta)v &= \theta_{p.f(p)}T_p(\varphi_f)v = \theta_{p.f(p)}\dot\sigma(\delta(f)_p v) + \theta_{p.f(p)}T(\sigma_{f(p)})v \\ &= \theta_{p.f(p)}\dot\sigma(\delta(f)_p v) + (\sigma_{f(p)}^*\theta)v = \delta(f)_p v + \mathrm{Ad}(f(p))^{-1}\theta(v).\end{aligned}$$

This means that $\varphi_f^*\theta = \delta(f) + \mathrm{Ad}(f)^{-1}.\theta$. ■

Remark 5.4.5 (a) If $P = M\times G$ is the trivial principal bundle with $\sigma_g(x,h) = (x,hg)$, then

$$\Omega^1(M,\mathrm{Ad}(P)) \cong \Omega^1(P,\mathrm{Ad}(P))^G_{\mathrm{hor}} \cong \Omega^1(M,\mathfrak{g})$$

parameterizes the space of connection 1-forms.

The Maurer–Cartan form $\kappa_G \in \Omega^1(G,\mathfrak{g})$ is the unique connection 1-form for the trivial G-bundle $\{*\}\times G \cong G$ because in this case $\dot\sigma(x)_g = g.x = x_l(g)$ is a left invariant vector field, so that $\kappa_G(x_l) = x$. We further obtain with the Product Rule for logarithmic derivatives

$$\rho_g^*\kappa_G = \delta(\rho_g) = \delta(\mathrm{id}_G\cdot g) = \mathrm{Ad}(g)^{-1}\circ\kappa_G. \tag{5.4}$$

If $p_G\colon M\times G \to G$ is the G-projection and $p_M\colon M\times G\to M$ is the M-projection, this implies that each connection 1-form $\theta \in \mathcal{C}(M\times G)$ is of the form

$$\theta = p_G^*\kappa_G + (\mathrm{Ad}\circ p_G)^{-1}.p_M^*A, \quad A\in\Omega^1(M,\mathfrak{g}).$$

It is uniquely determined by the property that, for the canonical section $s(x) := (x,\mathbf{1})$, we have $A = s^*\theta$. For $A = 0$ we obtain the *canonical connection 1-form* of the trivial bundle

$$\theta_{\mathrm{can}} := p_G^*\kappa_G = \delta(p_G),$$

i.e., the logarithmic derivative of p_G.

(b) Now let $(\varphi_i,U_i)_{i\in I}$ be a bundle atlas of P and $s_i(x) := \varphi_i(x,\mathbf{1})$ be the corresponding local sections. Then we obtain a collection of 1-forms

$$A_i := s_i^*\theta \in \Omega^1(U_i,\mathfrak{g}),$$

called the *local gauge potentials.*

We rewrite the relation $s_i = s_j g_{ji}$ with $g_{ji} \in C^\infty(U_{ij}, G)$ as

$$s_i = \varphi_{q^*g_{ji}} \circ s_j,$$

where $\varphi_{q^*g_{ji}}(x,h) = (x, hg_{ji}(x))$ (Proposition 5.4.4). Then we use formula (5.3) in Proposition 5.4.4 (for the restriction $P_{U_{ij}}$) to obtain with $q \circ s_i = \mathrm{id}_{U_i}$:

$$\begin{aligned} A_i = s_i^*\theta = s_j^*\varphi_{q^*g_{ji}}^*\theta &= s_j^*\big(\delta(q^*g_{ji}) + \mathrm{Ad}(q^*g_{ji})^{-1}.\theta\big) \\ &= s_j^*q^*\delta(g_{ji}) + s_j^*\big((q^*\,\mathrm{Ad}(g_{ji})^{-1}).\theta\big) = \delta(g_{ji}) + \mathrm{Ad}(g_{ji})^{-1}.s_j^*\theta \\ &= \delta(g_{ji}) + \mathrm{Ad}(g_{ji})^{-1}.A_j. \end{aligned}$$

We thus obtain the relation

$$A_i = \delta(g_{ji}) + \mathrm{Ad}(g_{ji})^{-1}.A_j = \delta(g_{ji}) + \mathrm{Ad}(g_{ij}).A_j. \tag{5.5}$$

Using smooth partitions of unity, we obtain:

Proposition 5.4.6 *If M is paracompact, then each principal bundle (P, M, G, q, σ) possesses a connection 1-form.*

Proof. Let $(\varphi_i)_{i\in I}$ be a bundle atlas with the bundle charts

$$\varphi_i \colon U_i \times G \to P_{U_i}.$$

Since M is paracompact, we may w.l.o.g. assume that $\mathcal{U} = (U_i)_{i\in I}$ is locally finite. Then there exists a subordinate smooth partition of unity $(\chi_i)_{i\in I}$.

Let $\theta_i \in \mathcal{C}(P_{U_i})$ be any connection 1-form of the trivial bundle (Remark 5.4.5. Then

$$\theta := \sum_{i\in I}(q^*\chi_i)\theta_i$$

defines an element of $\mathcal{C}(P)$. Indeed, since the support of χ is a closed subset of M, contained in U_i, we may consider the forms $(q^*\chi_i)\theta_i$, extended by 0 outside U_i, as smooth $\mathfrak{g}$-valued 1-forms on P. Since the covering is locally finite, it follows that $\theta \in \Omega^1(P, \mathfrak{g})$ is a smooth 1-form.

Now (C1) follows from $\sum_{i\in I}\chi_i(m) = 1$ for each $m \in M$, and the equivariance of θ follows from the equivariance of each θ_i. ∎

Definition 5.4.7 (Horizontal lifts) Let $\theta \in \mathcal{C}(P)$ be a connection 1-form and $H_p(P) = \ker \theta_p$. For $X \in \mathcal{V}(M)$, we define a vector field $\widetilde{X} \in \mathcal{V}(P)$ by

$$\widetilde{X}(p) := (T_p(q)|_{H_p(P)})^{-1} X(q(p)).$$

The vector field $\widetilde{X}$ clearly satisfies

$$q_* \widetilde{X} = X \quad \text{and} \quad \theta(\widetilde{X}) = 0.$$

That $\widetilde{X}$ is smooth can be verified locally, so that we may assume that $P = M \times G$ is trivial. Then θ is of the form $\theta = p_G^* \kappa_G + (\mathrm{Ad} \circ p_G)^{-1}.p_M^* A$ for some $A \in \Omega^1(M, \mathfrak{g})$. Now $\theta_{(m,g)}(v, w) = 0$ is equivalent to

$$g^{-1}.w + \mathrm{Ad}(g)^{-1} A_x(v) = 0,$$

which is equivalent to $w = -A_x(w).g$. Hence the horizontal lift $\widetilde{X}$ of $X \in \mathcal{V}(M)$ is given by

$$\widetilde{X}(m, g) = (X(m), -A_m(X(m)).g),$$

which is smooth.

Proposition 5.4.8 *For any $\theta \in \mathcal{C}(P)$, the horizontal lift defines a $C^\infty(M)$-linear map*

$$\tau_\theta \colon \mathcal{V}(M) \to \mathfrak{aut}(P), \quad X \mapsto \widetilde{X} \quad \textit{with} \quad q_* \circ \tau_\theta = \mathrm{id}_{\mathcal{V}(M)}.$$

In particular, $q_* \colon \mathfrak{aut}(P) \to \mathcal{V}(M)$ is surjective. For each $C^\infty(M)$-linear section τ of q_*, there exists a unique $\theta \in \mathcal{C}(P)$ with $\tau = \tau_\theta$.

Proof. (1) First we show that $\widetilde{X} \in \mathfrak{aut}(P)$:

$$\begin{aligned}\widetilde{X}(p.g) &= (T_{p.g}(q)|_{H_{p.g}(P)})^{-1} X(q(p)) = T(\sigma_g)(T_p(q)|_{H_p(P)})^{-1} X(q(p)) \\ &= T(\sigma_g)\widetilde{X}(p).\end{aligned}$$

(2) $(f.X)\widetilde{\ }(p) = f(q(p))\widetilde{X}(p) = (f.\widetilde{X})(p)$.
(3) Let $\tau \colon \mathcal{V}(M) \to \mathfrak{aut}(P)$ be a $C^\infty(M)$-linear cross section of q_*. Since τ is $C^\infty(M)$-linear, it defines over each open subset $U \subseteq M$ a $C^\infty(U)$-linear cross section of q_*. If P_U is trivial, then we thus obtain a $C^\infty(M)$-linear map

$$\tau_U \colon \mathcal{V}(U) \to \mathfrak{aut}(P_U) \cong C^\infty(U, \mathfrak{g}) \rtimes \mathcal{V}(U).$$

This map is of the form

$$\tau_U(X) = (-A(X), X)$$

with a $C^\infty(U)$-linear map $A\colon \mathcal{V}(U) \to C^\infty(U, \mathfrak{g})$, and this means that $A \in \Omega^1(U, \mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form. In terms of vector fields on $U \times G$, we then have

$$\tau_U(X)(m, g) = (X(m), -A(X)(m)g)$$

(cf. Definition 5.4.7). We conclude that τ_U coincides with the horizontal section with respect to the connection 1-form

$$\theta_U = p_G^*\kappa_G + (\mathrm{Ad} \circ p_G)^{-1}.p_U^* A.$$

Now $\theta_U \in \mathcal{C}(P_U)$ is a connection 1-form for which τ_U is the corresponding horizontal lift. From the uniqueness of θ_U we derive that for each open subset $V \subseteq U$ we have $\theta_V = \theta_U|_{P_V}$, so that the collection of the θ_U's defines an element $\theta \in \mathcal{C}(P)$ with $\tau_\theta = \tau$. ∎

Corollary 5.4.9 *If M is paracompact, then the homomorphism $q_*\colon \mathfrak{aut}(P) \to \mathcal{V}(M)$ is surjective and we obtain a short exact sequence of Lie algebras*

$$\mathbf{0} \to \mathfrak{gau}(P) \to \mathfrak{aut}(P) \xrightarrow{q_*} \mathcal{V}(M) \to \mathbf{0}.$$

Induced connections

Definition 5.4.10 Let $\varphi\colon G \to H$ be a morphism of Lie groups and (P, M, G, q, σ) a principal bundle. Then φ defines a smooth action of G on H by $g.h := \varphi(g)h$, so that we obtain the associated bundle

$$P_\varphi := P \times_\varphi H = (P \times H)/G,$$

where G acts on $P \times H$ by $g.(p, h) = (p.g, \varphi(g)^{-1}h)$. Since the G-action on $P \times H$ commutes with the H-action by right multiplications $\widetilde{\sigma}((p, h), h') := (p, hh')$, there exists a unique smooth action

$$\widetilde{\sigma}\colon P_\varphi \times H \to P_\varphi, \quad [p, h].h' := [p, hh'].$$

For the smoothness of this action we simply note that the map

$$P \times H \times H \to P_\varphi, \quad (p, h, h') \mapsto [p, hh']$$

is smooth and the projection $P \times H \to P_\varphi$ is a submersion.

If (φ, U) is a bundle chart of P on U, then the associated chart

$$\widetilde{\varphi}_U \colon U \times H \to (P_\varphi)_U, \quad (x, h) \mapsto [\varphi(x, \mathbf{1}), h]$$

is a bundle chart and it is obvious that it is H-equivariant. Therefore P_φ is an H-principal bundle. It is called the *H-principal bundle induced from P via φ.*

Proposition 5.4.11 *Let $\varphi \in \mathrm{Hom}(G, H)$ and*

$$\psi \colon P \to P_\varphi, \quad p \mapsto [p, \mathbf{1}]$$

the canonical bundle map. Then, for each connection 1-form $\theta \in \mathcal{C}(P)$, there exists a unique connection 1-form $\theta' \in \mathcal{C}(P_\varphi)$ with

$$\psi^*\theta' = \mathbf{L}(\varphi) \circ \theta.$$

Then θ' is called the *connection on P_φ induced by θ.*

Proof. Let $(\varphi_i, U_i)_{i \in I}$ be a bundle atlas of P and $s_i(x) := \varphi_i(x, \mathbf{1})$ be the corresponding local sections, so that θ is represented by the gauge potentials

$$A_i := s_i^*\theta \in \Omega^1(U_i, \mathfrak{g}),$$

satisfying

$$A_i = \delta(g_{ji}) + \mathrm{Ad}(g_{ji})^{-1}.A_j.$$

Then $B_i := \mathbf{L}(\varphi) \circ A_i \in \Omega^1(U_i, \mathfrak{h})$ is a collection of 1-forms satisfying

$$B_i = \mathbf{L}(\varphi) \circ \delta(g_{ji}) + \mathbf{L}(\varphi) \circ \mathrm{Ad}(g_{ji})^{-1}.A_j = \delta(\varphi \circ g_{ji}) + \mathrm{Ad}(\varphi \circ g_{ji})^{-1}.B_j.$$

Since the functions $\varphi \circ g_{ji}$ are the transition functions for the associated bundle P_φ, we see that there exists a unique connection 1-form $\theta' \in \mathcal{C}(P_\varphi)$ with $\widetilde{s}_i^*\theta' = B_i$ for the local sections

$$\widetilde{s}_i(x) = \psi(s_i(x)) = [\varphi_i(x, \mathbf{1}), \mathbf{1}].$$

Then

$$s_i^*\psi^*\theta' = \widetilde{s}_i^*\theta' = B_i = \mathbf{L}(\varphi) \circ A_i = \mathbf{L}(\varphi) \circ s_i^*\theta = s_i^*\big(\mathbf{L}(\varphi) \circ \theta\big)$$

for each i implies that $\psi^*\theta' = \mathbf{L}(\varphi) \circ \theta$.

To see that θ' is uniquely determined by this property, we observe that it implies

$$\widetilde{s}_i^*\theta' = s_i^*\psi^*\theta' = s_i^*(\mathbf{L}(\varphi) \circ \theta) = \mathbf{L}(\varphi) \circ s_i^*\theta = \mathbf{L}(\varphi) \circ A_i = B_i.$$

∎

Exercises for Section 5.4

Exercise 5.4.12 If $\varphi\colon P_1 \to P_2$ is an isomorphism of G-principal bundles and $\theta \in \mathcal{C}(P_2)$, then $\varphi^*\theta \in \mathcal{C}(P_1)$ also is a connection 1-form.

Chapter 6

Curvature

In this last chapter we discuss curvature issues of principal bundles. First we define the curvature of a connection 1-form θ and find that it is a 2-form with values in the bundle $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$. Then we explain in Section 6.2 how the theory of characteristic classes that we encountered in the algebraic context of Lie algebra extensions can be adapted to obtain characteristic classes of principal bundles derived from G-invariant polynomials on the Lie algebra $\mathfrak{g}$. If a principal bundle P is flat in the sense that it carries a connection with vanishing curvature, then all its characteristic classes are trivial. In Section 6.3 we take a closer look at flat bundles over a connected manifold M and explain how they can be classified by the cohomology set $H^1(\pi_1(M), G)$, where the corresponding notion of equivalence also takes the connection into account, hencs is much finer than bundle equivalence. We conclude this section with a brief discussion of bundles with abelian structure group.

Throughout this chapter, (P, M, G, q, σ) denotes a principal bundle and $\mathfrak{g} = \mathbf{L}(G)$. We recall the notation $\sigma_g(p) := p.g$ and $\sigma^p(g) = p.g$.

6.1 The Curvature of a Connection 1-Form

Definition 6.1.1 If $\theta \in \mathcal{C}(P)$ is a connection 1-form, then

$$F(\theta) := \mathtt{d}\theta + \tfrac{1}{2}[\theta, \theta] \in \Omega^2(P, \mathfrak{g})$$

is called the *curvature of* θ.

We call the connection *flat* if $F(\theta) = 0$, i.e., if θ satisfies the Maurer–Cartan equation. The bundle P is called *flat* if it permits a flat connection 1-form θ.

Lemma 6.1.2 *For each connection* 1*-form* θ *we have*

$$F(\theta) \in \Omega^2(P, \mathfrak{g})^G_{\rm hor} \cong \Omega^2(M, \mathrm{Ad}(P)).$$

Proof. Since θ is equivariant and $\mathtt{d}$ commutes with pullbacks, we obtain for each $g \in G$:

$$\begin{aligned} \sigma_g^* F(\theta) &= \sigma_g^* \mathtt{d}\theta + \tfrac{1}{2}\sigma_g^*[\theta,\theta] = \mathtt{d}(\sigma_g^*\theta) + \tfrac{1}{2}[\sigma_g^*\theta, \sigma_g^*\theta] \\ &= \mathtt{d}(\mathrm{Ad}(g)^{-1} \circ \theta) + \tfrac{1}{2}[\mathrm{Ad}(g)^{-1} \circ \theta, \mathrm{Ad}(g)^{-1} \circ \theta] \\ &= \mathtt{d}(\mathrm{Ad}(g)^{-1} \circ \theta) + \tfrac{1}{2}\,\mathrm{Ad}(g)^{-1} \circ [\theta, \theta] \\ &= \mathrm{Ad}(g)^{-1} \circ F(\theta). \end{aligned}$$

To see that $F(\theta)$ is horizontal, we show that $i_v F(\theta)_p = 0$ for each vertical vector $v \in V_p(P)$. If $w \in V_p(P)$, then there exist $x, y \in \mathfrak{g}$ with $v = \dot\sigma_p(x)$ and $w = \dot\sigma_p(y)$. Now

$$F(\theta)_p(v,w) = F(\theta)(\dot\sigma(x), \dot\sigma(y))_p,$$

and this is the value in p of the function

$$\begin{aligned} & \dot\sigma(x).\theta(\dot\sigma(y)) - \dot\sigma(y).\theta(\dot\sigma(x)) - \theta([\dot\sigma(x), \dot\sigma(y)]) + \tfrac{1}{2}[\theta,\theta](\dot\sigma(x), \dot\sigma(y)) \\ = \; & -[x,y] + [x,y] = 0 \end{aligned}$$

because the functions $\theta(\dot\sigma(x)) = x$ and $\theta(\dot\sigma(y)) = y$ are constant.

For $Y \in \mathcal{V}(M)$ and its horizontal lift $\widetilde{Y}$, we further have

$$F(\theta)(\dot\sigma(x), \widetilde{Y}) = \dot\sigma(x).\theta(\widetilde{Y}) - \widetilde{Y}.\theta(\dot\sigma(x)) - \theta([\dot\sigma(x), \widetilde{Y}]) + \tfrac{1}{2}[\theta,\theta](\dot\sigma(x), \widetilde{Y}) = 0$$

because $\theta(\widetilde{Y}) = 0$ and the G-invariance of the vector field $\widetilde{Y}$ implies $[\dot\sigma(x), \widetilde{Y}] = 0$. This proves that $F(\theta) \in \Omega^2(P, \mathfrak{g})^G_{\rm hor}$. ■

Remark 6.1.3 (Local description of the curvature) Let $(\varphi_i, U_i)_{i \in I}$ be a bundle atlas for (P, M, G, q, σ) and $s_i \colon U_i \to P, x \mapsto \varphi_i(x, \mathbf{1})$, the corresponding local sections. Then any connection 1-form $\theta \in \mathcal{C}(P)$ is determined by the gauge potentials $A_i := s_i^*\theta \in \Omega^1(U_i, \mathfrak{g})$ (cf. Remark 5.4.5). For the curvature we obtain accordingly

$$s_i^* F(\theta) = F(A_i) = \mathtt{d}A_i + \tfrac{1}{2}[A_i, A_i] \in \Omega^2(U_i, \mathfrak{g}).$$

Remark 6.1.4 Let $X, Y \in \mathcal{V}(M)$ and $\widetilde{X}, \widetilde{Y}$ the horizontal lifts. The curvature 2-form $F(\theta) \in \Omega^2(P, \mathfrak{g})$ satisfies

$$F(\theta)(\widetilde{X}, \widetilde{Y}) = \mathtt{d}\theta(\widetilde{X}, \widetilde{Y}) + [\theta(\widetilde{X}), \theta(\widetilde{Y})] = -\theta([\widetilde{X}, \widetilde{Y}]) = \theta([X,Y]^{\sim} - [\widetilde{X}, \widetilde{Y}]).$$

Restricting θ to $\mathfrak{aut}(P) \subseteq \mathcal{V}(P)$, we also obtain a map

$$\theta\colon \mathfrak{aut}(P) \to C^\infty(P, \mathfrak{g})^G, \quad X \mapsto \theta(X)$$

because $(\sigma_g)_* X = X$ implies that

$$\sigma_g^*(\theta(X)) = (\sigma_g^*\theta)(X) = \mathrm{Ad}(g)^{-1} \circ \theta(X).$$

On the ideal $\mathfrak{gau}(P) \trianglelefteq \mathfrak{aut}(P)$, we then obtain a bijection

$$\theta\colon \mathfrak{gau}(P) \to C^\infty(P, \mathfrak{g})^G,$$

inverting the isomorphism of Lie algebras (Proposition 5.1.6)

$$\Psi\colon C^\infty(P, \mathfrak{g})^G \to \mathfrak{gau}(P), \quad \Psi(\xi)(p) = -\dot{\sigma}_p(\xi(p))$$

in the sense that

$$\theta(\Psi(\xi)) = -\xi \qquad \text{and} \qquad \Psi(\theta(X)) = -X \quad \text{for} \quad X \in \mathfrak{gau}(P).$$

For the $C^\infty(M)$-linear section

$$\tau = \tau_\theta\colon \mathcal{V}(M) \to \mathfrak{aut}(P), \quad X \mapsto \widetilde{X}$$

defined by the horizontal lift, we have

$$R_\tau(X, Y) := [\tau(X), \tau(Y)] - \tau([X, Y]) = [\widetilde{X}, \widetilde{Y}] - [X, Y]^{\sim} \in \mathfrak{gau}(P),$$

and the calculation from above now shows that R_τ and $F(\theta)$ are related by

$$\Psi\big(F(\theta)(\widetilde{X}, \widetilde{Y})\big) = R_\tau(X, Y). \tag{6.1}$$

We conclude in particular that R_τ is $C^\infty(M)$-bilinear.

Exercises for Section 6.1

Exercise 6.1.5 Show that for a principal bundle (P, M, G, q, σ), the logarithmic derivative defines a map

$$\delta\colon \operatorname{Gau}(P) \cong C^\infty(P, G)^G \to \Omega^1(P, \mathfrak{g})^G.$$

Exercise 6.1.6 Let $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ be a surjective homomorphism of Lie algebras with kernel $\mathfrak{n}$. Then $\mathfrak{n}$ is a $\widehat{\mathfrak{g}}$-module with respect to the action $x.y := [x, y]$. Let $-\theta\colon \widehat{\mathfrak{g}} \to \mathfrak{n}$ be a linear projection, i.e., $\theta(x) = -x$ for $x \in \mathfrak{n}$. Show that:

(i) $\widehat{\mathfrak{g}} \cong \mathfrak{n} \oplus \ker\theta$ and there exists a unique linear map $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$ and $\sigma(\mathfrak{g}) = \ker\theta$.

(ii) The element $F(\theta) := \mathsf{d}_{\widehat{\mathfrak{g}}}\theta + \frac{1}{2}[\theta, \theta] \in C^2(\widehat{\mathfrak{g}}, \mathfrak{n})$ satisfies

  (a) $F(\theta)$ is horizontal, i.e., $i_x F(\theta) = 0$ for $x \in \mathfrak{n}$.

  (b) $F(\theta)(\sigma(x), \sigma(y)) = [\sigma(x), \sigma(y)] - \sigma([x, y]) =: R_\sigma(x, y)$ for $x, y \in \mathfrak{g}$.

  (c) $F(\theta) = q^* R_\sigma$.

(iii) The following are equivalent

  (a) $F(\theta) = 0$, i.e., θ satisfies the MC equation.

  (b) σ is a homomorphism of Lie algebras.

  (c) $\ker\theta$ is a subalgebra of $\widehat{\mathfrak{g}}$.

6.2 Characteristic Classes

Chern–Weil Theory

We want to assign to a principal bundle (P, M, G, q, σ) characteristic cohomology classes in the de Rham cohomology $H_{\mathrm{dR}}(M, \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Here the term characteristic means that equivalent bundles should lead to the same characteristic classes.

The idea is to apply Lecomte's abstract construction which assigns to a Lie algebra extension

$$\mathbf{0} \to \mathfrak{n} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to \mathbf{0}$$

and a symmetric k-linear $\widehat{\mathfrak{g}}$-invariant map $\widetilde{f}\colon \mathfrak{n}^k \to V$ with values in a $\mathfrak{g}$-module V a cohomology class $[f_\sigma] \in H^{2k}(\mathfrak{g}, V)$.

To make this fit to the context of principal bundles, we consider the short exact sequence of Lie algebras

$$\mathbf{0} \to \mathfrak{gau}(P) \to \mathfrak{aut}(P) \xrightarrow{q_*} \mathcal{V}(M) \to \mathbf{0} \tag{6.2}$$

which is an extension of the Lie algebra $\mathcal{V}(M)$ by the Lie algebra $\mathfrak{gau}(P) \cong C^\infty(P, \mathfrak{g})^G \cong \Gamma \operatorname{Ad}(P)$.

Let $p\colon \mathfrak{g} \to \mathbb{K}$ be a homogeneous polynomial function of degree k which is $\operatorname{Ad}(G)$-invariant. Then the corresponding symmetric k-linear map

$$\widetilde{p}\colon \mathfrak{g}^k \to \mathbb{K} \quad \text{with} \quad p(x) = \frac{1}{k!}\widetilde{p}(x, x, \ldots, x)$$

is also $\operatorname{Ad}(G)$-invariant:

$$\widetilde{p}(\operatorname{Ad}(g)x_1, \ldots, \operatorname{Ad}(g)x_k) = \widetilde{p}(x_1, \ldots, x_k) \quad \text{for} \quad x_i \in \mathfrak{g}, g \in G.$$

Taking derivatives in $g = \mathbf{1}$ at $x \in \mathfrak{g}$, this leads with the Product Rule to

$$\sum_{i=1}^k \widetilde{p}(x_1, \ldots, [x, x_i], \ldots, x_k) = 0,$$

i.e., that $\widetilde{p} \in \operatorname{Sym}^k(\mathfrak{g}, \mathbb{K})^{\mathfrak{g}}$.

The space $V := C^\infty(M, \mathbb{K})$ carries a natural $\mathcal{V}(M)$-module structure defined by $X.f := \mathtt{d}f \circ X$ and each invariant polynomial p defines on $\mathfrak{gau}(P)^k$ a $C^\infty(M)$-k-linear map

$$p^M\colon \big(C^\infty(P, \mathfrak{g})^G\big)^k \to V, \quad p^M(\xi_1, \ldots, \xi_k)(q(p)) := \widetilde{p}(\xi_1(p), \ldots, \xi_k(p)).$$

That this is well-defined follows from the fact that the equivariance of the $\xi_i \in C^\infty(P, \mathfrak{g})^G$ and the invariance of $\widetilde{p}$ implies that the right hand side is constant on the G-orbits in P, so that it factors through a smooth function on M.

Next we recall that the action of $X \in \mathfrak{aut}(P)$ on $\Psi(\xi) \in \mathfrak{gau}(P)$ is given by $[X, \Psi(\xi)] = \Psi(X.\xi)$ (Lemma 5.1.7), to see that

$$q^*\big((q_*X).p^M(\xi_1, \ldots, \xi_k)\big) = X.(q^*p^M(\xi_1, \ldots, \xi_k)) = X.\widetilde{p}(\xi_1, \ldots, \xi_k)$$
$$= \sum_{i=1}^k \widetilde{p}^M(\xi_1, \ldots, X.\xi_i, \ldots, \xi_k) = \sum_{i=1}^k q^*p^M(\xi_1, \ldots, X.\xi_i, \ldots, \xi_k),$$

which means that

$$p^M \in \mathrm{Sym}^k(C^\infty(P,\mathfrak{g})^G, V)^{\mathfrak{aut}(P)}.$$

Therefore we can apply Lecomtes construction to p^M to obtain characteristic classes

$$[p^M_\tau] \in H^{2k}(\mathcal{V}(M), V)$$

for each linear section $\tau\colon \mathcal{V}(M) \to \mathfrak{aut}(P)$. Actually, we can do better. We are not interested in general linear sections without geometric meaning. In our context it is much more natural to consider $C^\infty(M)$-linear sections, and we have seen in Proposition 5.4.8 that each section with this property is the horizontal lift $X \mapsto \widetilde{X} = \tau(X)$ for some connection 1-form $\theta \in \mathcal{C}(P)$. If τ is $C^\infty(M)$-linear, the same holds for

$$R_\tau\colon \mathcal{V}(M) \times \mathcal{V}(M) \to \mathfrak{gau}(P), \quad R_\tau(X,Y) = [\tau(X),\tau(Y)] - \tau([X,Y]),$$

resp., the corresponding map

$$F(\theta) \in \mathrm{Alt}^2_{C^\infty(M)}(\mathcal{V}(M), C^\infty(P,\mathfrak{g})^G) \cong \Omega^2(P,\mathfrak{g})^G_{\mathrm{hor}}$$

(Remark 6.1.4). This implies that

$$p^M_\theta := p^M_{F(\theta),\ldots,F(\theta)} \in C^{2k}(\mathcal{V}(M), C^\infty(M,\mathbb{K}))$$

is $C^\infty(M)$-linear, hence a $\mathbb{K}$-valued differential form (Theorem 3.2.1).

Proposition 6.2.1 *The form p^M_θ is closed and its cohomology class $[p^M_\theta] \in H^{2k}_{\mathrm{dR}}(M,\mathbb{K})$ does not depend on the connection θ.*

Proof. Since p^M_θ defines a Lie algebra cocycle (Lemma 3.5.4), it is a closed differential form. From the proof of Lemma 3.5.4, we also recall that, if θ' is another connection 1-form and

$$D := \theta' - \theta,$$

which is an element of $\Omega^1(P,\mathfrak{g})^G_{\mathrm{hor}}$, then

$$p^M_{\theta'} - p^M_\theta = \mathtt{d}_{\mathcal{V}(M)} \int_0^1 k p^M_{D,F(\theta_t),\ldots,F(\theta_t)}\, dt$$

hold for $\theta_t := (1-t)\theta + t\theta' = \theta + tD$. Since, for each $t \in [0,1]$,

$$p^M_{D,F(\theta_t),\ldots,F(\theta_t)} \in \Omega^{2k-1}(P,\mathbb{K})^G_{\mathrm{hor}} \cong \Omega^{2k-1}(M,\mathbb{K})$$

is a differential form in $\Omega^{2k-1}(M,\mathbb{K})$ depending polynomially on t, it follows that $p^M_{\theta'} - p^M_\theta$ is exact. ∎

Definition 6.2.2 For a Lie group G we write

$$I(G,\mathbb{K}) := \mathrm{Pol}(\mathfrak{g},\mathbb{K})^G$$

for the space of $\mathbb{K}$-valued polynomials invariant under the adjoint action of G on $\mathfrak{g}$. We also put

$$I(G) := I(G,\mathbb{R}).$$

Clearly, this is an algebra, and for each G-invariant polynomial all its homogeneous components are also G-invariant, so that

$$I(G,\mathbb{K}) = \bigoplus_{d=0}^{\infty} \mathrm{Pol}^d(\mathfrak{g},\mathbb{K})^G.$$

Remark 6.2.3 If G is connected, then the invariance of a polynomial p of degree d on $\mathfrak{g}$ is equivalent to the invariance of the corresponding d-linear symmetric map $\widetilde{p}\colon \mathfrak{g}^d \to \mathbb{K}$ under the Lie algebra, which means that

$$\sum_{i=1}^{d} \widetilde{p}(x_1,\ldots,[x,x_i],\ldots,x_d) = 0 \quad \text{for} \quad x,x_1,\ldots,x_d \in \mathfrak{g}.$$

If G is not connected, then the algebra $I(G,\mathbb{K})$ may be strictly smaller than $I(G_0,\mathbb{K})$. As we shall seen in Example 6.2.10, this happens for $G = \mathrm{O}_{2m}(\mathbb{R})$.

Theorem 6.2.4 (Abstract Chern–Weil Homomorphism) *Let G be a Lie group and (P,M,G,q,σ) a principal bundle. Then we have an algebra homomorphism*

$$W_P\colon I(G,\mathbb{K}) \to H^{2\bullet}_{\mathrm{dR}}(M,\mathbb{K}), \quad W_P(p) = \frac{1}{k!}[p^M_\theta].$$

This homomorphism does not depend on the connection $\theta \in \mathcal{C}(P)$.

For $\mathbb{K} = \mathbb{R}$ the homomorphism if often simply called the *Weil homomorphism*, whereas it is called the *Chern–Weil homomorphism* for $\mathbb{K} = \mathbb{C}$.

Proof. This follows immediately by combining Theorem 3.5.7 with Proposition 6.2.1. ■

Remark 6.2.5 Let $\varphi\colon P_1 \to P_2$ be an equivalence of G-principal bundles over M. We claim that the corresponding Weil homomorphisms

$$W_i\colon I(G) \to H^{2\bullet}_{\mathrm{dR}}(M, \mathbb{K}), \quad i = 1, 2,$$

coincide. In fact, let $\theta_2 \in \mathcal{C}(P_2)$ be a connection 1-form and note that $\theta_1 := \varphi^*\theta_2$ is a connection 1-form on P_1. Then $\varphi^*F(\theta_2) = F(\theta_1)$. For $X, Y \in \mathcal{V}(M)$ we then have

$$\big(\varphi^*F(\theta_2)\big)(\tau_2(X), \tau_2(Y)) = F(\theta_1)(\tau_1(X), \tau_1(Y))$$

because the respective horizontal lifts $\tau_i(X) \in \mathfrak{aut}(P)$ of $X \in \mathcal{V}(M)$ satisfy $\varphi_*\tau_1(X) = \tau_2(X)$. This implies that $\varphi^* q_2^* p^M_{\theta_2} = q_1^* p^M_{\theta_1}$, so that $q_2 \circ \varphi = q_1$ leads to $p^M_{\theta_2} = p^M_{\theta_1}$ and therefore to $W_1 = W_2$. This means that the homomorphism W leads indeed to characteristic classes of the bundle P.

Proposition 6.2.6 *If $(P_i, M, G, q_i, \sigma_i)$, $i = 1, 2$, are principal bundles whose Weil homomorphisms*

$$W_{P_i}\colon I(G) \to H^{2\bullet}_{\mathrm{dR}}(M, \mathbb{K})$$

do not coincide, then $P_1 \not\sim P_2$.

Proposition 6.2.7 *If P is a flat G-bundle, then $W_P = 0$.*

Since W_P vanishes for a flat bundle P, we have:

Corollary 6.2.8 *If $W_P \neq 0$, then the bundle P does not possess any flat connection.*

Invariant polynomials for classical groups

Of particular importance in the theory of fiber bundles are the bundles with compact structure groups. This is due to the fact that each finite-dimensional Lie group G with finitely many components has a maximal compact subgroup K, and for this subgroup the natural map

$$\mathrm{Bun}(M, K) \to \mathrm{Bun}(M, G), \quad [P] \mapsto [P \times_K G]$$

defined by induction (cf. Definition 5.4.10) is a bijection (cf. [Hu94]).

For $\mathrm{GL}_n(\mathbb{C})$ the maximal compact subgroup is $\mathrm{U}_n(\mathbb{C})$ and for $\mathrm{GL}_n(\mathbb{R})$ the maximal compact subgroup is $\mathrm{O}_n(\mathbb{R})$. The set $\mathrm{Bun}(M, \mathrm{GL}_n(\mathbb{C}))$ classifies n-dimensional complex vector bundles over M and $\mathrm{Bun}(M, \mathrm{GL}_n(\mathbb{R}))$ classifies n-dimensional real vector bundles. Therefore we take a closer look at the two compact groups $\mathrm{U}_n(\mathbb{C})$ and $\mathrm{O}_n(\mathbb{R})$.

Example 6.2.9 For $G = \mathrm{GL}_n(\mathbb{R})$ and $k \in \mathbb{N}_0$, we write P_k for the homogeneous polynomial on $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ of degree k defined by

$$\det\Big(t\mathbf{1} - \frac{A}{2\pi}\Big) = \sum_{k=0}^{n} P_k(A)t^{n-k}, \quad A \in \mathfrak{gl}_n(\mathbb{R}).$$

In particular, we have

$$P_0(A) = 1, \quad P_1(A) = -\frac{\operatorname{tr} A}{2\pi} \quad \text{and} \quad P_n(A) = (-1)^n \frac{\det(A)}{(2\pi)^n}.$$

The invariance of the polynomials P_k follows immediately from the invariance of the characteristic polynomial of A under conjugation. It is called the *k-th Pontrjagin polynomial*, and one can show that they are algebraically independent and generate $I(\mathrm{GL}_n(\mathbb{R}))$, so that

$$I(\mathrm{GL}_n(\mathbb{R})) = \mathbb{R}[p_1, \ldots, p_n]$$

(cf. [Bou90, Ch. VIII, §13, no. 1]). The key ingredient of the argument is to reduce the study of the algebra $I(\mathrm{GL}_n(\mathbb{R}))$, by restriction to diagonal matrices, to the isomorphic algebra $\mathbb{R}[x_1, \ldots, x_n]^{S_n}$ of symmetric polynomials (Chevalley's Restriction Theorem).

The elementary symmetric polynomials are defined by

$$\prod_{i=1}^{n}(t - a_i) = \sum_{k=0}^{n}(-1)^k \sigma_k(a) t^{n-k}, \quad a \in \mathbb{R}^n.$$

In particular,

$$\sigma_1(a) = \sum_{i=1}^{n} a_i, \quad \sigma_2(a) = \sum_{i<j} a_i a_j \quad \text{and} \quad \sigma_n(a) = a_1 \cdots a_n.$$

On diagonal matrices we clearly have

$$P_k(\operatorname{diag}(a_1, \ldots, a_n)) = \frac{(-1)^k}{(2\pi)^k} \sigma_k(a_1, \ldots, a_n).$$

If $(P, M, \mathrm{GL}_n(\mathbb{R}), q, \sigma)$ is a principal bundle, then the image $W_P(P_k) \in H^{2k}(M, \mathbb{R})$ under the Weil homomorphism is called the k-th *Pontrjagin class* of P. Accordingly, we define for a real vector bundle $\mathbb{V}$ with n-dimensional fiber $V \cong \mathbb{R}^n$ the Pontrjagin classes by $W_{\mathrm{Fr}(\mathbb{V})}(P_k)$ and the Pontrjagin classes of a manifold M are defined as the Pontrjagin classes of its tangent bundle TM. [1]

Example 6.2.10 The Lie algebra of $G = \mathrm{O}_n(\mathbb{R})$ is

$$\mathfrak{g} = \mathfrak{o}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \colon x^\top = -x\},$$

which implies that

$$\det\left(t\mathbf{1} - \frac{A}{2\pi}\right) = \det\left(t\mathbf{1} + \frac{A}{2\pi}\right)$$

for $A \in \mathfrak{o}_n(\mathbb{R})$. Hence the restriction of P_k to $\mathfrak{o}_n(\mathbb{R})$ is an even polynomial which implies that it vanishes if k is odd. As we have already mentioned above, each $\mathrm{GL}_n(\mathbb{R})$-bundle is induced from an $\mathrm{O}_n(\mathbb{R})$-bundle, which implies that its Pontrjagin classes $W_P(P_k)$ vanish if k is odd. Only the polynomials

$$P_2, P_4, \ldots, P_{2[\frac{n}{2}]}$$

may lead to non-trivial characteristic classes of $\mathrm{O}_n(\mathbb{R})$, resp., $\mathrm{GL}_n(\mathbb{R})$-bundles.

For the subgroup $\mathrm{SO}_n(\mathbb{R})$, the identity component of $\mathrm{O}_n(\mathbb{R})$, the picture only changes a little. Its Lie algebra $\mathfrak{so}_n(\mathbb{R}) = \mathfrak{o}_n(\mathbb{R})$ is the same, but since $\mathrm{SO}_n(\mathbb{R})$ is smaller, the space of invariant polynomials can be larger. If $n = 2m+1$ is odd, then

$$I(\mathrm{O}_n(\mathbb{R})) = I(\mathrm{SO}_n(\mathbb{R})) = \mathbb{R}[P_2, \ldots, P_{2m}]$$

and the p_i are algebraically independent (cf. [Bou90, Ch. VIII, §13, no. 2]), but if $n = 2m$ is even, then

$$I(\mathrm{O}_n(\mathbb{R})) = \mathbb{R}[P_2, \ldots, P_{2m}]$$

[1] The purpose of the 2π-factor in the definition of the Pontrjagin classes ensures that these classes are *integral*, i.e., lie in the image of the $\mathbb{Z}$-valued singular cohomology. Although this may seem artificial at first sight, it is crucial to compare the characteristic classes in de Rham cohomology with those obtained in the purely topological theory of fiber bundles.

with algebraically independent generators, whereas

$$I(\mathrm{SO}_n(\mathbb{R})) = \mathbb{R}[P_2, \ldots, P_{2m-2}, \widetilde{\mathrm{Pf}}]$$

also is also generated by m algebraically independent generators, where $\widetilde{\mathrm{Pf}}^2 = P_{2m}$ (cf. (cf. [Bou90, Ch. VIII, §13, no. 4]). We now take a closer look at this remarkable polynomial $\widetilde{\mathrm{Pf}}$ showing up for $n = 2m$.

We associate to the skew-symmetric matrix A the alternating bilinear form

$$\omega_A(x, y) := x^\top A y, \quad x, y \in \mathbb{R}^{2m}$$

with

$$\omega_A(e_i, e_j) = a_{ij} \quad \text{for} \quad i < j,$$

so that

$$\omega_A = \sum_{i<j} a_{ij} e_i^* \wedge e_j^*.$$

We now define the *Pfaffian of A* by the relation

$$\frac{1}{m!}\omega_A^m = \mathrm{Pf}(A) e_1^* \wedge \ldots \wedge e_{2m}^*$$

in the algebra $\mathrm{Alt}(\mathbb{R}^{2m}, \mathbb{R})$.

For $g \in \mathrm{O}_{2m}(\mathbb{R})$ we have $\mathrm{Ad}(g)A = gAg^{-1}$ and

$$\omega_{\mathrm{Ad}(g)A}(x, y) = \omega_A(g^{-1}x, g^{-1}y) = (g.\omega_A)(x, y).$$

Therefore

$$\omega_{\mathrm{Ad}(g)A}^m = m!\,\mathrm{Pf}(A) g.(e_1^* \wedge \ldots \wedge e_{2m}^*) = m!\,\mathrm{Pf}(A)\det(g)(e_1^* \wedge \ldots \wedge e_{2m}^*)$$

implies that

$$\mathrm{Pf}(\mathrm{Ad}(g)A) = \det(g)\,\mathrm{Pf}(A).$$

We conclude that Pf is invariant under $\mathrm{SO}_{2m}(\mathbb{R})$, but not under $\mathrm{O}_{2m}(\mathbb{R})$.

If $A = A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is a block diagonal matrix and, accordingly, $m = m_1 + m_2$, then $\omega_A = \omega_{A_1} + \omega_{A_2}$ is an orthogonal direct sum and

$$\omega_A^{m_1+m_2} = (\omega_{A_1} + \omega_{A_2})^{m_1+m_2} = \binom{m_1 + m_2}{m_1} \omega_{A_1}^{m_1} \wedge \omega_{A_2}^{m_2}$$

follows from $\omega_{A_i}^k = 0$ for $k > m_i$. We thus obtain

$$\mathrm{Pf}(A_1 \oplus A_2) = \mathrm{Pf}(A_1)\,\mathrm{Pf}(A_2). \tag{6.3}$$

For $m = 1$ and $A = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$ we have $\omega_A = x e_1^* \wedge e_2^*$, so that $\mathrm{Pf}(A) = x$ and $\det(A) = x^2$. From the normal form of skew-symmetric matrices, which is a block diagonal matrix with (2×2)-diagonal blocks, we thus obtain with (6.3)

$$\det(A) = \mathrm{Pf}(A)^2.$$

This shows that $\widetilde{\mathrm{Pf}} := \frac{1}{(2\pi)^m}\,\mathrm{Pf}$ satisfies $\widetilde{\mathrm{Pf}}^2 = P_{2m}$.

For an $\mathrm{SO}_{2m}(\mathbb{R})$-bundle P, the class

$$W_P(\widetilde{\mathrm{Pf}}) \in H^{2m}(M, \mathbb{R})$$

is called the *Euler class*. This class shows up in a remarkable generalization of the Gauß–Bonnet Theorem from Riemannian Geometry, namely the Chern–Gauß–Bonnet Theorem, asserting that for an oriented Riemannian manifold and the $\mathrm{SO}_{2m}(\mathbb{R})$-bundle P of positively oriented orthonormal $(2m)$-frames, we have

$$\int_M W_P(\widetilde{\mathrm{Pf}}) = \chi(M),$$

where $\chi(M)$ denotes the *Euler characteristic of M* (cf. [Du78, p.112]).

Example 6.2.11 For the complex Lie group $G = \mathrm{GL}_n(\mathbb{C})$, we are only interested in the space

$$I(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}[z_{11}, \ldots, z_{nn}]^{\mathrm{GL}_n(\mathbb{C})}$$

of invariant holomorphic polynomials on its Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. Here we consider the complex-valued invariant polynomials C_k which are the coefficients of t^{n-k} in the polynomial

$$\det\left(t\mathbf{1} - \frac{A}{2\pi i}\right) = \sum_{k=0}^{n} C_k(A) t^{n-k}, \quad A \in M_n(\mathbb{C}).$$

Their invariance follows immediately from the invariance of the characteristic polynomial of A under conjugation and, with the same argument as in the real case,

$$I(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{C}[C_1, \ldots, C_n],$$

where the Chern polynomials $C_1, \ldots, C_n$ are algebraically independent.

The image $W_P(C_k)$ of C_k under the Chern–Weil map is called the *k-th Chern class* of the bundle P. Note that the restriction of C_k to $\mathfrak{gl}_n(\mathbb{R})$ satisfies

$$i^k C_k(A) = P_k(A),$$

so that the m-th Pontrjagin class of an n-dimensional real vector bundle $\mathbb{V}$ is $(-1)^m$ times the $2m$-th Chern class of its complexification $\mathbb{V}_{\mathbb{C}}$. For a complex manifold M, the Chern classes of M are defined as the Chern classes of its tangent bundle TM.

Example 6.2.12 For the subgroup $\mathrm{U}_n(\mathbb{C})$, the Lie algebra is

$$\mathfrak{u}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \colon x^* = -x\}$$

which is a totally real subspace of $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$, i.e.,

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) \oplus i\mathfrak{u}_n(\mathbb{C}).$$

Therefore each polynomial function $p \colon \mathfrak{u}_n(\mathbb{C}) \to \mathbb{C}$ has a unique extension to a holomorphic polynomial $p_{\mathbb{C}} \colon \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$. This extension is invariant under $\mathrm{GL}_n(\mathbb{C})$ if and only if p is invariant under the connected Lie group $\mathrm{U}_n(\mathbb{C})$:

$$I_{\mathbb{C}}(\mathrm{GL}_n(\mathbb{C})) \cong I(\mathrm{U}_n(\mathbb{C}), \mathbb{C}).$$

For $A \in \mathfrak{u}_n(\mathbb{C})$, the relation $A = -A^* = -\overline{A}^\top$ implies that

$$C_k(A) = \overline{C_k(A)},$$

so that the Chern polynomials are real-valued on $\mathfrak{u}_n(\mathbb{R})$. We thus obtain

$$I(\mathrm{U}_n(\mathbb{C}), \mathbb{R}) \cong \mathbb{R}[C_1, \ldots, C_n].$$

In particular, the Chern classes of $\mathrm{U}_n(\mathbb{C})$-bundles take values in the real-valued cohomology $H_{\mathrm{dR}}(M, \mathbb{R})$.

6.3 Flat Bundles

A principal bundle (P, M, G, q, σ) is called *flat* if it has a connection 1-form θ with vanishing curvature $F(\theta) = 0$. Since all characteristic classes of a flat bundle vanish, the Lie algebraic methods provide no means to distinguish flat bundles. Actually it turns out that the theory of flat bundles essentially is a theory of bundles with discrete structure groups. In this section we show that for a connected manifold M, each flat bundle is associated to the universal covering $q_M\colon \widetilde{M} \to M$ with respect to a homomorphism $\chi\colon \pi_1(M) \to G$. In this sense flat bundles can be parameterized by the set $\mathrm{Hom}(\pi_1(M), G)$. In particular, flat bundles over simply connected manifolds are trivial. We shall also see that the equivalence classes of flat bundles, as bundles with connection, correspond to the quotient set

$$H^1(\pi_1(M), G) = \mathrm{Hom}(\pi_1(M), G)/G, \quad g.\chi := c_g \circ \chi \quad \text{for } g \in G,$$

of $\mathrm{Hom}(\pi_1(M), G)$ by the conjugation action of G. A crucial tool in our discussion is the Fundamental Theorem on Lie group valued functions, which is applied to a flat connection 1-form.

Constructing flat bundles

Let M be a connected manifold and $m_0 \in M$ a base point. We write $q_M\colon \widetilde{M} \to M$ for a simply connected covering manifold of M and

$$\pi_1(M) := \mathrm{Deck}(\widetilde{M}, q_M) = \{\varphi \in \mathrm{Diff}(\widetilde{M})\colon q_M \circ \varphi = q_M\}$$

for its group of deck transformation which is isomorphic to $\pi_1(M, m_0)$ (Proposition 4.3.15).

The trivial bundle $M \times G$ has a flat connection 1-form, given by $\theta = \delta(p_G)$, where $p_G\colon M \times G \to G$ is the projection (Remark 5.4.5).

A more interesting class of flat bundle is obtained as follows:

Definition 6.3.1 Let G be a Lie group and $\chi\colon \pi_1(M) \to G$ a group homomorphism. Then

$$P_\chi := (\widetilde{M} \times G)/\pi_1(M) = \widetilde{M} \times_\chi G,$$

where $\pi_1(M)$ acts on $\widetilde{M} \times G$ by

$$\varphi.(\widetilde{m}, g) = (\varphi(\widetilde{m}), \chi(\varphi)g)$$

is a G-principal bundle. It is the bundle associated to the $\pi_1(M)$-principal bundle $(\widetilde{M}, M, \pi_1(M), q_M, \sigma)$ with $\sigma(\widetilde{m}, \varphi) := \varphi^{-1}(m)$ with respect to the action of $\pi_1(M)$ on G by left multiplications $\varphi.g := \chi(\varphi)g$ (cf. Definition 5.4.10).

Lemma 6.3.2 *The G-bundle P_χ carries a unique connection 1-form $\theta_\chi \in \mathcal{C}(P_\chi)$ for which the map $s\colon \widetilde{M} \to P_\chi, s(\widetilde{m}) = [\widetilde{m}, \mathbf{1}]$ satisfies $s^*\theta_\chi = 0$. The connection θ_χ is flat and we also have*

$$q \circ s = q_M \quad \text{and} \quad s \circ d = s.\chi(d)^{-1} \quad \text{for} \quad d \in \pi_1(M).$$

Proof. Let $\theta \in \mathcal{C}(P_\chi)$ be the unique connection on P_χ with $s^*\theta = 0$. Its existence follows from Proposition 5.4.11 because 0 is the unique connection 1-form on the $\pi_1(M)$-bundle $\widetilde{M}$.

To see that θ is flat, we note that we obtain local sections $\widetilde{s}_i\colon U_i \to P_\chi$ from local sections $s_i\colon U_i \to \widetilde{M}$ by $\widetilde{s}_i = s \circ s_i$. Then the local gauge potentials

$$A_i = \widetilde{s}_i^*\theta = s_i^* s^*\theta = 0$$

vanish, and therefore θ is flat (Remark 6.1.3). ∎

Lemma 6.3.3 *If $\theta \in \mathcal{C}(P)$ and $s_1, s_2\colon \widetilde{M} \to P$ are two smooth maps with*

$$s_i^*\theta = 0 \quad \text{and} \quad q \circ s_i = q_M,$$

then there exists a unique $g \in G$ with $s_2 = s_1.g$.

Proof. There exists a smooth function $f\colon \widetilde{M} \to G$ with $s_2(x) = s_1(x).f(x)$ for each $x \in \widetilde{M}$. Then

$$T_x(s_2) = T(\sigma_{f(x)}) \circ T_x(s_1) + \dot{\sigma}_{s_2(x)} \circ \delta(f)_x$$

(Lemma 5.1.4(b)) implies that

$$0 = s_2^*\theta = \mathrm{Ad}(f)^{-1}.s_1^*\theta + \delta(f) = \delta(f),$$

so that f is constant, i.e., $s_2 = s_1.g$ for some $g \in G$. ∎

Definition 6.3.4 Let $(P_i, M, G, q_i, \sigma_i)$, $i = 1, 2$, be principal bundles and $\theta_i \in \mathcal{C}(P_i)$. Then a bundle morphism $\varphi\colon P_1 \to P_2$ is called a *morphism of bundles with connection* if, in addition, $\varphi^*\theta_2 = \theta_1$. Similarly, we define an equivalence of bundles with connection and write $\mathrm{CBun}(M, G)$ for the set of equivalence classes of G-bundles with connection over M.

Remark 6.3.5 (a) If M is paracompact, then each G-bundle over M has a connection (Proposition 5.4.6), so that the natural map

$$\Gamma\colon \mathrm{CBun}(M,G) \to \mathrm{Bun}(M,G), \quad [(P,\theta)] \mapsto [P]$$

is surjective. The fiber of this map over the class $[P]$ can be identified with the set of equivalence classes $[(P,\theta)]$ in $\mathrm{CBun}(M,G)$, which can be identified with the set

$$\mathcal{C}(P)/\operatorname{Gau}(P)$$

of orbits of the group $\operatorname{Gau}(P)$ of gauge transformations on the set $\mathcal{C}(P)$ of connections on P.

(b) It is an important and difficult question which bundles P admit a flat connection. For the case where M is a compact orientable surface and $G = \mathrm{GL}_2(\mathbb{R})_0$, the G-bundles with a flat connection have been classified by Milnor in [Mi58]. We have

$$\mathrm{Bun}(M,G) \cong \mathrm{Bun}(M,\mathrm{SO}_2(\mathbb{R})) \cong \mathrm{Bun}(M,\mathbb{T}) \cong \check{H}^1(M,\mathbb{T}) \cong H^2(M,\mathbb{Z}) \cong \mathbb{Z}$$

(where $H^2(M,\mathbb{Z})$ denotes the second singular cohomology group) and $n \in \mathbb{Z}$ corresponds to a flat bundle if and only if $|n| < g$, where g is the genus of M.

Lemma 6.3.6 *Let (P,θ) is a G-bundle with connection and $s\colon \widetilde{M} \to P$ a smooth map with $s^*\theta = 0$ and $q \circ s = q_M$. Then there exists a homomorphism $\chi_s\colon \pi_1(M) \to G$ with*

$$s \circ d = s.\chi(d)^{-1} \quad \textit{for} \quad d \in \pi_1(M)$$

and the map

$$\widetilde{M} \times G \to P, \quad (x,g) \mapsto s(x)g \tag{6.4}$$

induces an equivalence $(P_{\chi_s},\theta_{\chi_s}) \to (P,\theta)$ of bundles with connection.

Proof. For each $d \in \pi_1(M)$, the function $s \circ d\colon \widetilde{M} \to P_\chi$ also satisfies $(s \circ d)^*\theta = 0$, so that Lemma 6.3.3 implies that $s \circ d = s.\chi_s(d)^{-1}$ for some $\chi_s(d) \in G$. Then

$$\begin{aligned} s \circ (d_1 d_2) &= (s.\chi_s(d_1)^{-1}) \circ d_2 = (s \circ d_2).\chi_s(d_1)^{-1} = s \circ \chi_s(d_2)^{-1}\chi_s(d_1)^{-1} \\ &= s \circ (\chi_s(d_1)\chi_s(d_2))^{-1} \end{aligned}$$

implies that $\chi_s\colon \pi_1(M) \to G$ is a group homomorphism.

Now the map in (6.4) factors through a bundle equivalence

$$\varphi\colon P_{\chi_s} \to P, \quad [x,g] \mapsto s(x)g.$$

Further $\varphi^*\theta$ is the unique connection 1-form on P_{χ_s} vanishing on $[\widetilde{M} \times \{\mathbf{1}\}]$, so that φ is an equivalence of bundles with connection. ■

Proposition 6.3.7 *Let $\chi_1, \chi_2 \in \mathrm{Hom}(\pi_1(M), G))$ and $\theta_i = \theta_{\chi_i}$ be the canonical flat connection on P_{χ_i}. Then (P_{χ_1}, θ_1) and (P_{χ_2}, θ_2) are equivalent as bundles with connection if and only if there exists a $g \in G$ with*

$$\chi_2 = c_g \circ \chi_1.$$

In particular, the set

$$H^1(\pi_1(M), G) := \mathrm{Hom}(\pi_1(M), G)/G, \quad g.\chi := c_g \circ \chi \quad \text{for} \quad g \in G,$$

classifies the set of equivalence classes of flat G-bundles with connection over M.

Proof. We write $s_i\colon \widetilde{M} \to P_{\chi_i}$ for the canonical maps with $s_i^*\theta_i = 0$, $q \circ s_i = q_M$ and consider the corresponding homomorphisms $\chi_i := \chi_{s_i}$.

Suppose first that $\varphi\colon P_1 \to P_2$ is a bundle equivalence with $\varphi^*\theta_2 = \theta_1$. Then $\varphi \circ s_1\colon \widetilde{M} \to P_2$ satisfies

$$(\varphi \circ s_1)^*\theta_2 = s_1^*\varphi^*\theta_2 = s_1^*\theta_1 = 0,$$

so that Lemma 6.3.3 implies that $\varphi \circ s_1 = s_2.g$ for some $g \in G$. Comparing

$$\varphi \circ s_1 \circ d = (\varphi \circ s_1).\chi_1(d)^{-1}$$

with

$$(s_2.g) \circ d = s_2.\chi_2(d)g = (s_2.g).(g^{-1}\chi_2(d)g),$$

it follows that $\chi_2 = c_g \circ \chi_1$.

If, conversely, $\chi_2 = c_g \circ \chi_1$, then the map $s_2' := s_2.g\colon \widetilde{M} \to P_{\chi_2}$ satisfies

$$s_2' \circ d = (s_2 \circ d).g = s_2.\chi_2(d)^{-1}g = s_2.g\chi_1(d)^{-1} = s_2'.\chi_1(d)^{-1},$$

so that $\chi_{s_2'} = \chi_1$ and Lemma 6.3.6 implies that $(P_{\chi_1}, \theta_1) \sim (P_{\chi_2}, \theta_2)$ as bundles with connection. ■

Proposition 6.3.8 *For a homomorphism $\chi\colon \pi_1(M) \to G$, the following are equivalent:*

(i) *The bundle P_χ is trivial.*

(ii) *There exists a smooth function $f\colon \widetilde{M} \to G$ with $f(d.x) = \chi(d)f(x)$ for $d \in \pi_1(M)$, $x \in \widetilde{M}$.*

(iii) *There exists some $\alpha \in \mathrm{MC}(M, \mathfrak{g})$ and $m_0 \in M$ with $\mathrm{per}_\alpha^{m_0} = \chi$.*

Proof. (i) $\Leftrightarrow$ (ii): The bundle P_χ is trivial if and only if it has a smooth section $s\colon M \to P_\chi$. Since P_χ is a bundle associated to the $\pi_1(M)$-bundle $\widetilde{M}$, smooth sections of P_χ correspond to smooth functions $f\colon \widetilde{M} \to G$ which are equivariant in the sense that

$$f(x.d) = f(d^{-1}(x)) = \chi(d)^{-1}f(x), \quad d \in \pi_1(M), x \in \widetilde{M}$$

(Proposition 1.6.3).

(ii) $\Rightarrow$ (iii): Pick $\widetilde{m}_0 \in \widetilde{M}$ and put $m_0 := q_M(\widetilde{m}_0)$. Replacing f by $f \cdot f(\widetilde{m}_0)^{-1}$, we may assume that $f(\widetilde{m}_0) = \mathbf{1}$. Consider $\widetilde{\alpha} := \delta(f) \in \Omega^1(\widetilde{M}, \mathfrak{g})$. Then (ii) implies that for each $d \in \pi_1(M)$ we have

$$d^*\widetilde{\alpha} = d^*\delta(f) = \delta(d^*f) = \delta(f \circ d) = \delta(\chi(d) \cdot f) = \delta(f) = \widetilde{\alpha},$$

so that there exists a unique 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$ with $q_M^*\alpha = \widetilde{\alpha}$. Since $\widetilde{\alpha} = \delta(f)$ satisfies the Maurer–Cartan equation, the same holds for α.

Let $[\gamma] \in \pi_1(M, m_0)$ and $\widetilde{\gamma}\colon [0,1] \to \widetilde{M}$ be the lift of γ starting in $\widetilde{m}_0$. From Proposition 4.3.15 we recall the group isomorphism

$$\Phi\colon \pi_1(M, m_0) \to \pi_1(M), \quad \Phi([\gamma]) = \varphi_{[\gamma]}, \quad \varphi_{[\gamma]}(\widetilde{m}_0) = \widetilde{\gamma}(1).$$

Then $\widetilde{\gamma}^*\widetilde{\alpha} = \widetilde{\gamma}^*q_M^*\alpha = \gamma^*\alpha$ implies that

$$\begin{aligned}\mathrm{per}_\alpha^{m_0}([\gamma]) &= \mathrm{evol}_G(\gamma^*\alpha) = \mathrm{evol}_G(\widetilde{\gamma}^*\widetilde{\alpha}) = \mathrm{evol}_G(\widetilde{\gamma}^*\delta(f)) = \widetilde{f}(\widetilde{\gamma}(1)) \\ &= \widetilde{f}(\varphi_{[\gamma]}(\widetilde{m}_0)) = \chi(\varphi_{[\gamma]})\widetilde{f}(\widetilde{m}_0) = \chi(\varphi_{[\gamma]}).\end{aligned}$$

(iii) $\Rightarrow$ (ii): Since $\widetilde{M}$ is simply connected, there exists a smooth function $f\colon \widetilde{M} \to G$ with $\delta(f) = q_M^*\alpha$ and $f(\widetilde{m}_0) = \mathbf{1}$. From the calculation in the proof of Lemma 4.4.2 we know that f satisfies (ii) with $\chi = \mathrm{per}_\alpha^{m_0}$. ■

Remark 6.3.9 With the information of the preceding proposition, we can continue the exact sequence

$$G \hookrightarrow C^\infty(M,G) \xrightarrow{\delta} \mathrm{MC}(M,\mathfrak{g}) \xrightarrow{\mathrm{per}^{m_0}} \mathrm{Hom}(\pi_1(M,m_0),G)$$

from Remark 4.4.7 by

$$\mathrm{MC}(M,\mathfrak{g}) \xrightarrow{\mathrm{per}^{m_0}} \mathrm{Hom}(\pi_1(M,m_0),G) \xrightarrow{B} \mathrm{Bun}(M,G),$$

where $B(\chi) := [P_\chi]$. We know already that the image of the map B is contained in the set $\mathrm{Bun}(M,G)_{\mathrm{flat}}$ of classes of flat G-bundles. Below we shall see that this is precisely the image.

From flat bundles to associated bundles

Lemma 6.3.10 *If θ is a flat connection 1-form on P, $p_0 \in P$ a base point, $m_0 := q(p_0)$, and G is connected, then the corresponding period homomorphism*

$$\mathrm{per}_\theta^{p_0} \colon \pi_1(P,p_0) \to G$$

factors through the surjective homomorphism $\pi_1(q)\colon \pi_1(P,p_0) \to \pi_1(M,m_0)$ to a group homomorphism

$$\mathrm{per}_\theta^M \colon \pi_1(M,m_0) \to G.$$

Proof. If G is connected, then $\pi_0(G)$ is trivial, so that $\pi_1(q)$ is surjective by Theorem 6.3.17 in the appendix below. From the same theorem get $\ker \pi_1(q) = \mathrm{im}(\pi_1(\sigma^{p_0}))$. If $\eta\colon I = [0,1] \to G$ is a piecewise smooth loop in G, then

$$\begin{aligned}\mathrm{per}_\theta^{p_0}([\sigma^{p_0}\circ\eta]) &= \mathrm{evol}_G((\sigma^{p_0}\circ\eta)^*\theta) = \mathrm{evol}_G(\eta^*(\sigma^{p_0})^*\theta) = \mathrm{evol}_G(\eta^*\kappa_G)\\ &= \mathrm{evol}_G(\delta(\eta)) = \eta(0)^{-1}\eta(1) = \mathbf{1}.\end{aligned}$$

This proves the lemma. ∎

Proposition 6.3.11 *Let G be a connected Lie group and $q\colon P \to M$ a principal G-bundle over the connected manifold M.*

(1) *A smooth function $f\colon P \to G$ is G-equivariant with respect to the canonical right action of G on itself if and only if $\delta(f) \in \Omega^1(P,\mathfrak{g})$ is a connection 1-form.*

(2) *The 1-form $\theta \in \mathcal{C}(P) \subseteq \Omega^1(P,\mathfrak{g})$ is integrable if and only if θ is flat and the period homomorphism*

$$\mathrm{per}_\theta^M \colon \pi_1(M) \to G$$

vanishes.

(3) *The bundle P is trivial if and only if it admits a flat connection with vanishing periods.*

Proof. (1) If $f\colon P \to G$ is smooth and equivariant, then $\delta(f) \in \Omega^1(P,\mathfrak{g})$ and the equivariance of f implies that

$$\sigma_g^*\delta(f) = \delta(\sigma_g \circ f) = \delta(f \cdot g) = \mathrm{Ad}(g)^{-1} \circ \delta(f) \quad \text{and} \quad \delta(f)(\dot\sigma(x)) = x$$

for each $g \in G$ and $x \in \mathfrak{g}$. Hence $\delta(f)$ is a connection 1-form.

If, conversely, $f\colon P \to G$ is a smooth function for which $\theta := \delta(f)$ is a connection 1-form, then

$$\delta(f \circ \sigma_g) = \sigma_g^*\delta(f) = \mathrm{Ad}(g)^{-1} \circ \delta(f) = \delta(f \cdot g)$$

holds for each $g \in G$.

Fix $p \in P$. Then

$$\kappa_G = (\sigma^p)^*\theta = (\sigma^p)^*\delta(f) = \delta(f \circ \sigma^p)$$

and the connectedness of G imply that $f(p.g) = f(p)g$ for each $g \in G$. Hence $f \circ \sigma_g$ coincides with $f \cdot g$ in p, and now the connectedness of P (which follows from the connectedness of M and G), together with $\delta(f \circ \sigma_g) = \delta(f \cdot g)$ implies that $f \circ \sigma_g = f \cdot g$.

(2) This follows by combining Lemma 6.3.10 with the Fundamental Theorem 4.4.4 and the observation that θ is flat if and only if it satisfies the MC equation.

(3) This follows by combining (1) and (2) with the triviality criterion in Proposition 2.2.4(c), which states that P is trivial if and only if there exists an equivariant function $P \to G$. ∎

Theorem 6.3.12 *Each flat G-bundle (P,θ) is equivalent, as a bundle with connection, to some P_χ, $\chi \in \mathrm{Hom}(\pi_1(M), G)$. In particular, it is trivial if M is simply connected.*

Proof. **Step 1:** First we consider the case where M is simply connected. We show that (P,θ) has a horizontal section $s\colon M \to P$. We reduce the assertion to the case where G is connected. Let $G_0 \subseteq G$ be the identity component. Then the quotient manifold $\widehat{M} := P/G_0$ is a principal bundle with the discrete structure group $\pi_0(G) = G/G_0$ over M, so that the natural map $\widehat{q}\colon \widehat{M} \to M$ is a covering map. Since M is simply connected, each connected component of $\widehat{M}$ is mapped diffeomorphically onto M (Corollary 4.3.9). Since the connected components of $\widehat{M}$ are the image of the connected components of P under the quotient map (the fibers of this map are homeomorphic to G_0, hence connected), it follows that each connected component P_1 of P is a G_0-principal bundle over M.

If $\theta \in \mathcal{C}(P)$ is a flat connection 1-form, then $\theta|_{P_1} \in \mathcal{C}(P_1)$ also is a flat connection 1-form. Since $\pi_1(M)$ is trivial, Proposition 6.3.11(1),(2) imply the existence of a smooth G_0-equivariant function $f\colon P_1 \to G_0$ with $\delta(f) = \theta|_{P_1}$. Now Proposition 2.2.4(c) shows that P_1 has a smooth section $s\colon M \to P_1$ with $s(M) \subseteq f^{-1}(\mathbf{1})$. Hence $s^*\theta = s^*\delta(f) = \delta(f \circ s) = 0$, and s also is a section of P.

Step 2: Now we turn to the general case. Let $q_M\colon \widetilde{M} \to M$ be the universal covering map. Then q_M^*P is a G-bundle over $\widetilde{M}$ and $p_P^*\theta$ is a flat connection 1-form. In view of the preceding corollary, q_M^*P has a flat section $f\colon \widetilde{M} \to q_M^*P$, which means that projecting to P yields a smooth map $s := p_P \circ f\colon \widetilde{M} \to P$ with $q \circ s = q_M$ and $s^*\theta = f^*p_P^*\theta = 0$. Now Lemma 6.3.6 implies that $(P,\theta) \sim (P_{\chi_s}, \theta_{\chi_s})$ as bundles with connection. ■

Example 6.3.13 If $\dim M = 1$, then all 2-forms on M are trivial, which implies in particular that all G-bundles over M are flat.

For $M = \mathbb{R}$ this implies in particular that all G-bundles are trivial because $\mathbb{R}$ is simply connected.

For $M = \mathbb{S}^1$ we have $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, so that G-bundles with connection over $\mathbb{S}^1$ are classified by the set $H^1(\mathbb{Z}, G)$ of conjugacy classes in the group G.

Example 6.3.14 Assume that M is connected and G is discrete. Then $\mathcal{C}(P) = \{0\}$ and each G-bundle over M is flat, hence equivalent to a bundle of the form P_χ, $\chi \in \mathrm{Hom}(\pi_1(M), G)$ (Theorem 6.3.12). This result can be obtained quited directly from covering theory which implies the existence of a continuous map $s\colon \widetilde{M} \to P$ with $q \circ s = q_M$ (cf. the Lifting Theorem 4.3.10). Since q is a local diffeomorphism, the map s is actually smooth, and now Lemma 6.3.6 applies.

Since each equivalence of G-bundles automatically is an equivalence of G-bundles with connection, Proposition 6.3.7 implies that

$$\mathrm{Bun}(G, M) \cong H^1(\pi_1(M), G)$$

is the set of G-conjugacy class in $\mathrm{Hom}(\pi_1(M), G)$.

Gauge transformations correspond to equivariant functions $f\colon \widetilde{M} \to G$ with respect to the action of $\pi_1(M)$ on G, defined by $d.g := \chi(d)g\chi(d)^{-1}$ (Proposition 1.6.3). Since M is connected, all these functions are constant, and we find that

$$\mathrm{Gau}(P) \cong C_G(\mathrm{im}(\chi)) = \{g \in G\colon (\forall d \in \pi_1(M))\ \chi(d)g = g\chi(d)\}.$$

Remark 6.3.15 If G is a Lie group, then we write G_d for G, considered as a discrete group. We then have a natural map

$$H^1(\pi_1(M), G) \cong \mathrm{Bun}(M, G_d) \cong \check{H}^1(M, G_d) \to \mathrm{Bun}(M, G) \cong \check{H}^1(M, G)$$

(Theorem 2.1.4). As we have just seen, a G-bundle is flat if and only if it is associated to a G_d-bundle with respect to the natural map $G_d \to G$.

Proposition 6.3.16 *A principal bundle (P, θ) with connection is flat if and only if there exists a bundle atlas $(\varphi_i, U_i)_{i\in I}$ for which all local gauge potentials A_i vanish.*

Proof. If all A_i vanish, then Remark 6.1.3 implies that $F(\theta) = 0$.

If, conversely, $F(\theta) = 0$, we choose a bundle atlas $(\varphi_i, U_i)_{i\in I}$ for which all sets U_i are simply connected. Then $A_i := s_i^*\theta \in \Omega^1(U_i, G)$ satisfies the Maurer–Cartan equation, so that the Fundamental Theorem implies the existence of some $f_i\colon U_i \to G$ with $\delta(f_i) = A_i$. For the new section $\widetilde{s}_i := s_i.f_i^{-1}$, we then have

$$\widetilde{s}_i^*\theta = \mathrm{Ad}(f_i(x)) \circ s_i^*\theta + \delta(f_i^{-1}) = \mathrm{Ad}(f_i(x)) \circ \delta(f_i) + \delta(f_i^{-1}) = 0$$

(Lemma 4.2.5), where we have used that

$$T_x(\widetilde{s}_i) = T(\sigma_{f_i(x)}^{-1}) \circ T_x(s_i) + \dot{\sigma}_{\widetilde{s}(x)} \circ \delta(f_i^{-1})_x.$$

(Lemma 5.1.4(b)). Therefore the local gauge potentials vanish for the new bundle charts defined by $\widetilde{\varphi}(x, g) := \varphi_i(x, f_i(x)^{-1}g)$. ■

Appendix: The Exact Homotopy Sequence

Theorem 6.3.17 (Exact homotopy sequence of a principal bundle) *Let (P, M, G, q, σ) be a principal bundle, $p_0 \in P$, and $m_0 := q(p_0)$. Then there exists a group homomorphism*

$$\delta_1 \colon \pi_1(M, m_0) \to \pi_0(G)$$

such that the sequence

$$\pi_1(G) \xrightarrow{\pi_1(\sigma^{p_0})} \pi_1(P, p_0) \xrightarrow{\pi_1(q)} \pi_1(M, m_0) \xrightarrow{\delta_1} \pi_0(G) \to \pi_0(P, p_0) \to \pi_0(M, m_0)$$

is exact as a sequence of groups resp., pointed spaces. Here $\pi_0(M, m_0)$ denotes the set or arc-components of M with the component $[m_0]$ containing m_0 as a base point. The homomorphism δ_1 is defined by $\delta_1([\gamma]) = [g] = gG_0$ if

$$\widetilde{\gamma}(1) = \widetilde{\gamma}(0).g^{-1}$$

holds for a continuous lift $\widetilde{\gamma} \colon [0,1] \to P$ of the loop γ in P with $\widetilde{\gamma}(0) = p_0$.

Proof. (cf. [Br93, Thm. VII.6.7]) We give a direct argument for the exactness in $\pi_1(M, m_0)$. If $\gamma \colon [0,1] \to M$ is a loop in m_0, then there exists a lift $\widetilde{\gamma} \colon [0,1] \to P$ with $\widetilde{\gamma}(0) = p_0$. Then $\widetilde{\gamma}(1) \in P_{m_0} = p_0.G$. If $\delta_1([\gamma]) = \mathbf{1}$, then $\widetilde{\gamma}(1) \in p_0.G_0$. Let $\alpha \colon [0,1] \to G$ be a continuous path with $\alpha(0) = \mathbf{1}$ and $\widetilde{\gamma}(1).\alpha(1) = p_0$. Then $\widehat{\gamma}(t) := \widetilde{\gamma}(t).\alpha(t)$ is a closed lift of γ. This implies that $[\gamma] = \pi_1(q)[\widehat{\gamma}]$, so that $[\gamma] \in \operatorname{im}(\pi_1(q))$.

That $\delta_1 \circ \pi_1(q) = \mathbf{1}$ follows from the fact that for each loop $\widetilde{\gamma}$ in p_0, $\widetilde{\gamma}$ is a closed lift of $\gamma := q \circ \widetilde{\gamma}$. ■

Remark 6.3.18 Assume that M is connected but that P is not. Let $P_1 \subseteq P$ be a connected component. Using local trivializations, it is easy to see that $q(P_1)$ is open and closed in M, so that the connectedness of M implies that $q(P_1) = M$. The subgroup

$$G_1 := \{g \in G \colon P_1.g = P_1\}$$

of G is open because it contains G_0. Since the diffeomorphisms σ_g, $g \in G$, permute the connected components of P, the relation $\sigma_g(P_1) \cap P_1 \neq \emptyset$ implies $g \in G_1$, and from that one readily verifies with bundle charts of P that P_1 is a G_1-principal bundle over M.

One can show that the image of G_1 in $\pi_0(G) = G/G_0$ coincides with the range of the connecting homomorphism

$$\delta_1 \colon \pi_1(M, m_0) \to \pi_0(G)$$

of the exact homotopy sequence of P. In particular, the connectedness of P is equivalent to the surjectivity of δ_1.

Remark 6.3.19 If P is connected, then $P/G_0 \to M$ is a connected covering of M which is a $\pi_0(G)$-principal bundle, called the *associated squeezed bundle.* If M is simply connected, then this covering must be trivial which implies that G is connected. Note that this also follows from the exactness of the exact homotopy sequence in $\pi_0(G)$ (Theorem 6.3.17).

We claim that P/G_0 is the bundle associated to the connecting homomorphism $\delta_1 \colon \pi_1(M) \to \pi_0(G)$. Indeed, let $\widehat{M} := \widetilde{M} \times_{\delta_1} \pi_0(G)$ denote the covering of M associated to δ_1. If P is connected, then δ_1 is surjective (Remark 6.3.18) and therefore $\widehat{M}$ is connected.

Pick a base point $p_0 \in P$ and put $m_0 := q(p_0)$. Then the exactness of the long exact homotopy sequence of the bundle P implies that

$$\pi_1(q)(\pi_1(P, p_0)) = \ker \delta_1.$$

Therefore $q \colon P \to M$ lifts to a unique smooth map $\widehat{q} \colon P \to \widehat{M}$ with $\widehat{q}(p_0) = \widehat{m}_0$, where $\widehat{m}_0 = [\widetilde{m}_0, \mathbf{1}]$ is a base point of $\widehat{M}$ over m_0 (Lifting Theorem 4.3.10). The uniqueness of lifts now easily implies that $\widehat{q}$ is G-equivariant, and this entails that $\widehat{M} \cong P/G_0$.

6.4 Abelian Bundles

In this section we consider principal bundles with abelian structure groups. Let G be an abelian Lie group, $\pi_0(G) := G/G_0$ its group of connected component and $\pi_1(G)$ its fundamental group. Since the exponential function

$$\exp_G \colon \mathfrak{g} \to G$$

is a covering morphism of Lie groups, we have

$$G_0 \cong \mathfrak{g}/\ker \exp_G, \qquad \text{where} \qquad \ker \exp_G \cong \pi_1(G)$$

is a discrete subgroup of the vector space $\mathfrak{g}$. Further, since G_0 is a divisible abelian group, the short exact sequence

$$\mathbf{0} \to G_0 \to G \to \pi_0(G) \to \mathbf{0}$$

of abelian groups split, which implies that

$$G \cong G_0 \times \pi_0(G),$$

where $\pi_0(G)$ is considered as a 0-dimensional abelian Lie group. From that we immediately derive that

$$\mathrm{Bun}(M, G) \cong \check{H}^1(M, G) \cong H^1(M, G_0) \times \check{H}^1(M, \pi_0(G)),$$

and we have already seen that

$$\check{H}^1(M, \pi_0(G)) \cong \mathrm{Bun}(M, \pi_0(G)) \cong \mathrm{Hom}(\pi_1(M), \pi_0(G))$$

(Example 6.3.14). Therefore we may concentrate in the following on the case of connected groups.

From now on $G \cong \mathfrak{g}/\Gamma$, where $\Gamma \cong \pi_1(G)$ is a discrete subgroup of $\mathfrak{g}$. A case of particular importance is the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write $q_G = \exp_G \colon \mathfrak{g} \to G$ for the quotient map.

Flat abelian bundles

We continue the discussion from Remark 6.3.9 for the special case of abelian groups $G = \mathfrak{g}/\Gamma$. Since each $(\mathfrak{g}, +)$-bundle is trivial (Theorem 5.2.6), the map

$$\mathrm{MC}(M, \mathfrak{g}) = Z^1_{\mathrm{dR}}(M, \mathfrak{g}) \xrightarrow{\mathrm{per}^{m_0}} \mathrm{Hom}(\pi_1(M, m_0), \mathfrak{g})$$

is surjective, so that Corollary 4.4.6 shows that

$$P \colon H^1_{\mathrm{dR}}(M, \mathfrak{g}) \to \mathrm{Hom}(\pi_1(M, m_0), \mathfrak{g}), \quad [\alpha] \mapsto \mathrm{per}^{m_0}_\alpha \tag{6.5}$$

is a linear isomorphism.

Let us write

$$H_1(M) := \pi_1(M, m_0)/D^1(\pi_1(M, m_0))$$

for the quotient of $\pi_1(M, m_0)$ by its commutator group (its first derived group). This is the maximal abelian quotient group of $\pi_1(M, m_0)$, so that we have for each abelian group A a natural isomorphism:

$$\mathrm{Hom}(\pi_1(M, m_0), A) \cong \mathrm{Hom}(H_1(M), A).$$

This leads in particular to

$$H^1_{\mathrm{dR}}(M, \mathfrak{g}) \cong \mathrm{Hom}(\pi_1(M, m_0), \mathfrak{g}) \cong \mathrm{Hom}(H_1(M), \mathfrak{g}).$$

Next we note that q_G induces a homomorphism of abelian groups

$$(q_G)_* \colon \mathrm{Hom}(H_1(M), \mathfrak{g}) \to \mathrm{Hom}(H_1(M), G)$$

and the G-valued period map $\mathrm{per}_G^{m_0}$ can be factored as

$$\mathrm{per}_G^{m_0} = (q_G)_* \circ \mathrm{per}_{\mathfrak{g}}^{m_0}.$$

This implies that for the map

$$B \colon \mathrm{Hom}(\pi_1(M, m_0), G) \cong \mathrm{Hom}(H_1(M), G) \to \mathrm{Bun}(M, G),$$

we have

$$\ker B = \mathrm{im}(\mathrm{per}_G^{m_0}) = \mathrm{im}((q_G)_*).$$

This means that a homomorphism $\chi \colon \pi_1(M, m_0) \to G$ defines a trivial bundle P_χ if and only if χ lifts to a homomorphism $\widetilde{\chi} \colon \pi_1(M, m_0) \to \mathfrak{g}$. Therefore we may rewrite the information in a short exact sequence

$$\mathbf{0} \to \mathrm{Hom}(H_1(M), \mathfrak{g}) \to \mathrm{Hom}(H_1(M), G) \to \mathrm{Bun}(M, G).$$

If A and B are abelian group, then we write $\mathrm{Ext}(A, B)$ for the set of equivalence classes of abelian extensions of A by B. From abelian group theory, we know that $\mathrm{Ext}(A, B)$ carries a natural abelian group structure and that the short exact sequence

$$\mathbf{0} \to \Gamma \to \mathfrak{g} \to G \to \mathbf{0}$$

of abelian groups induces a long exact sequence

$$\begin{aligned} \mathbf{0} \to \mathrm{Hom}(H_1(M), \Gamma) &\to \mathrm{Hom}(H_1(M), \mathfrak{g}) \to \mathrm{Hom}(H_1(M), G) \\ &\to \mathrm{Ext}(H_1(M), \Gamma) \to \mathbf{0} \end{aligned}$$

because the divisibility of $\mathfrak{g}$ implies that $\operatorname{Ext}(H_1(M), \mathfrak{g}) = \mathbf{0}$. This implies that we may identify the set $\operatorname{Bun}(M, G)_{\text{flat}}$ of equivalence classes of flat G-bundles with

$$\operatorname{Bun}(M, G)_{\text{flat}} \cong \operatorname{Ext}(H_1(M), \Gamma). \tag{6.6}$$

As a discrete subgroup of $\mathfrak{g}$, the group Γ is isomorphic to $\mathbb{Z}^k$ for some k. Hence

$$\operatorname{Ext}(H_1(M), \Gamma) \cong \operatorname{Ext}(H_1(M), \mathbb{Z})^k.$$

If $H_1(M)$ is a finitely generated (which is the case if M is compact), then it is a direct sum of cyclic groups. This reduces the computation of the group of flat bundles to the determination of the groups $\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ for $m \in \mathbb{Z}$. For $m = 0$ the freeness of the group $\mathbb{Z}$ implies that

$$\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0.$$

If $m \neq 0$, then one can show that

$$\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \cong \operatorname{Hom}(m\mathbb{Z}, \mathbb{Z}) / \operatorname{Hom}(\mathbb{Z}, \mathbb{Z})|_{m\mathbb{Z}} \cong \mathbb{Z}/m\mathbb{Z}.$$

Curvature of abelian bundles

If G is abelian, then

$$\operatorname{Gau}(P) \cong C^\infty(P, G)^G \cong C^\infty(M, G),$$

where $f \in C^\infty(M, G)$ acts on P by $\varphi_{q^*f}(p) = p.f(q(p))$.

The bundle $\operatorname{Ad}(P)$ is trivial, so that the translation space of $\mathcal{C}(P)$ is $\Omega^1(M, \operatorname{Ad}(P)) \cong \Omega^1(M, \mathfrak{g})$ and $F(\theta) = \mathtt{d}\theta$ vanishes if and only if θ is closed.

Lemma 6.4.1 *For each connection 1-form $\theta \in \mathcal{C}(P)$, there exists a unique 2-form $\omega \in \Omega^2(M, \mathfrak{g})$ with*

$$q^*\omega = \mathtt{d}\theta = F(\theta).$$

This 2-form is closed.

Proof. Since $F(\theta)$ is a horizontal equivariant 2-form, the assumption that G is abelian implies that it is actually invariant. Therefore

$$\omega_{q(p)}(T(q)v_1, T(q)v_2) := F(\theta)_p(v_1, v_2)$$

is a well-defined 2-form on M with $F(\theta) = q^*\omega$. From

$$q^*\mathtt{d}\omega = \mathtt{d}q^*\omega = \mathtt{d}F(\theta) = \mathtt{d}^2\theta = 0$$

we derive that ω is closed because q is a submersion. ■

Definition 6.4.2 The 2-form $\omega \in \Omega^2(M, \mathfrak{g})$ with $q^*\omega = \mathtt{d}\theta$ is called the *curvature of* (P, θ).

Its cohomology class $[\omega] \in H^2_{\rm dR}(M, \mathfrak{g})$ is called the *Chern class of* P. It can be obtained by applying the Chern–Weil construction to the invariant linear map $\mathrm{id}_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{g}$, which leads to the 2-cohomology class

$$W_P(\mathrm{id}_{\mathfrak{g}}) = [p_\theta^M] = [\omega] \in H^2_{\rm dR}(M, \mathfrak{g}).$$

Remark 6.4.3 The Chern class does not depend on the connection θ. This does also follow directly from the fact that each other connection is of the form $\theta' = \theta + q^*\alpha$ for some $\alpha \in \Omega^1(M, \mathfrak{g})$, which leads to $F(\theta') = F(\theta) + q^*\mathtt{d}\alpha$, and hence to $\omega' = \omega + \mathtt{d}\alpha$.

Remark 6.4.4 Let M be a connected manifold and (P, M, G, q, σ) a principal bundle.

We take a closer look at the action of the group $\mathrm{Gau}(P) \cong C^\infty(M, G)$ of gauge transformations on the affine space $\mathcal{C}(P)$. According to Proposition 5.4.4, it is given by

$$\varphi_{q^*f}^*\theta = \delta(q^*f) + \theta = q^*\delta(f) + \theta.$$

In particular, φ_{q^*f} fixes θ if and only if q^*f is locally constant, which is equivalent to f being locally constant. If M is connected, this proves that

$$\mathrm{Gau}(P)_\theta \cong G,$$

is the subgroup of constant functions in $C^\infty(M, G)$.

Identifying $\Omega^1(P, \mathfrak{g})^G = q^*\Omega^1(M, \mathfrak{g})$ with $\Omega^1(M, \mathfrak{g})$, we have

$$\varphi_f^*\theta = \delta(f) + \theta,$$

which implies that the orbits of $\mathrm{Gau}(P)$ correspond to the cosets of the subgroup $\delta(C^\infty(M, G))$ in the space $\Omega^1(M, \mathfrak{g})$. In particular, the quotient group

$$\Omega^1(M, \mathfrak{g})/\delta(C^\infty(M, G))$$

acts simply transitively on the quotient space $\mathcal{C}(P)/\,\mathrm{Gau}(P)$.

Since the forms $\delta(f)$ are closed, the curvature induces a map

$$F \colon \mathcal{C}(P)/\,\mathrm{Gau}(P) \to \Omega^2(M, \mathfrak{g}), \quad \theta \mapsto F(\theta).$$

Its fibers are acted upon simply transitively by the group

$$H^1_{\rm dR}(M,\mathfrak{g},\Gamma) := Z^1_{\rm dR}(M,\mathfrak{g})/\delta(C^\infty(M,G)),$$

so that we obtain an exact sequence

$$\mathbf{0} \to H^1_{\rm dR}(M,\mathfrak{g},\Gamma) \to \Omega^1(M,\mathfrak{g})/\delta(C^\infty(M,G)) \xrightarrow{\mathtt{d}} \Omega^2(M,\mathfrak{g}). \qquad (6.7)$$

From the Fundamental Theorem 4.4.4 we obtain the exact sequence of abelian groups

$$C^\infty(M,G) \xrightarrow{\delta} Z^1_{\rm dR}(M,\mathfrak{g}) \xrightarrow{\mathrm{per}^{m_0}} \mathrm{Hom}(\pi_1(M,m_0),G),$$

showing that per^{m_0} induces an embedding

$$H^1_{\rm dR}(M,\mathfrak{g},\Gamma) \hookrightarrow \mathrm{Hom}(\pi_1(M,m_0),G), \quad [\alpha] \mapsto \mathrm{per}^{m_0}_\alpha$$

of abelian groups. In view of the isomorphism

$$P\colon H^1_{\rm dR}(M,\mathfrak{g}) \xrightarrow{\mathrm{per}^{m_0}} \mathrm{Hom}(\pi_1(M,m_0),\mathfrak{g})$$

from (6.5), we obtain

$$H^1_{\rm dR}(M,\mathfrak{g},\Gamma) \cong (q_G)_*\big(\mathrm{Hom}(\pi_1(M,m_0),\mathfrak{g})\big) \subseteq \mathrm{Hom}(\pi_1(M,m_0),G).$$

We know already that the set $\mathrm{Bun}(M,G) \cong \check{H}^1(M,G)$ carries a natural group structure given by multiplication of the corresponding Čech cocycles (Remark 2.1.5). It is therefore of some interest to see how the curvature can be calculated from the corresponding Čech cocycles.

Remark 6.4.5 If θ is a connection 1-form on a G-bundle P and $(U_i,\varphi_i)_{i\in I}$ a bundle atlas, then the corresponding local gauge potentials $A_i \in \Omega^1(U_i,\mathfrak{g})$ satisfy

$$A_i = \delta(g_{ji}) + A_j$$

(Remark 5.4.5) and the restriction of the curvature 2-form $\omega_i \in \Omega^2(U_i,\mathfrak{g})$ satisfies

$$\omega_i = F(A_i) = \mathtt{d}A_i.$$

Proposition 6.4.6 *Let $\theta \in \mathcal{C}(P)$ with curvature $\omega \in \Omega^2(M, \mathfrak{g})$. Then the subgroup $G = \mathrm{Gau}(P)_\theta \subseteq C^\infty(M, G) \cong \mathrm{Gau}(P)$ is central. The group*

$$\mathrm{Aut}(P, \theta) := \{\varphi \in \mathrm{Aut}(P) \colon \varphi^*\theta = \theta\}$$

is mapped by the natural map $\Gamma\colon \mathrm{Aut}(P) \to \mathrm{Diff}(M)$ into the symplectomorphism group

$$\mathrm{Sp}(M, \omega) := \{\varphi_M \in \mathrm{Diff}(M) \colon \varphi^*\omega = \omega\},$$

which leads to an exact sequence

$$\mathbf{1} \to G \to \mathrm{Aut}(P, \theta) \to \mathrm{Sp}(M, \omega).$$

Proof. If $\varphi^*\theta = \theta$ holds for $\theta \in \mathrm{Aut}(P)$, then the induced diffeomorphism $\varphi_M = \Gamma(\varphi)$ of M satisfies

$$q^*\varphi_M^*\omega = (\varphi_M \circ q)^*\omega = (q \circ \varphi)^*\omega = \varphi^* q^*\omega = \varphi^*\mathtt{d}\theta = \mathtt{d}(\varphi^*\theta) = \mathtt{d}\theta = q^*\omega,$$

which implies that $\varphi_M^*\omega = \omega$. ■

6.5 Appendices

Transformation behavior of the curvature

Lemma 6.5.1 *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $f \in C^\infty(M, G)$ and $\alpha \in \Omega^1(M, \mathfrak{g})$, we have*

$$\mathtt{d}(\mathrm{Ad}(f).\alpha) = \mathrm{Ad}(f).\big(\mathtt{d}\alpha + [\delta(f), \alpha]\big).$$

Proof. We consider $\mathrm{Ad}(f)$ as a smooth function $M \to \mathrm{End}(\mathfrak{g})$. Then Example 3.3.9 implies that

$$\mathtt{d}(\mathrm{Ad}(f).\alpha) = \mathtt{d}(\mathrm{Ad}(f)) \wedge \alpha + \mathrm{Ad}(f).\mathtt{d}\alpha,$$

where $\wedge$ is defined in terms of the evaluation map $\mathrm{End}(\mathfrak{g}) \times \mathfrak{g} \to \mathfrak{g}$. Now the definition of the left logarithmic derivative, Lemma 4.2.6 and $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$ (Example 4.1.11) lead to

$$\mathtt{d}(\mathrm{Ad}(f)) = \mathrm{Ad}(f) \cdot \delta(\mathrm{Ad}(f)) = \mathrm{Ad}(f) \cdot \mathrm{ad}(\delta(f)),$$

where $\cdot$ denotes the product in $\mathrm{End}(\mathfrak{g})$. We thus obtain

$$\mathtt{d}(\mathrm{Ad}(f).\alpha) \quad = \quad \mathrm{Ad}(f).(\mathtt{d}\alpha + \mathrm{ad}(\delta(f)) \wedge \alpha) = \mathrm{Ad}(f).(\mathtt{d}\alpha + [\delta(f), \alpha]).$$

■

Lemma 6.5.2 *Let M be a smooth manifold and G a Lie group with Lie algebra $\mathfrak{g}$. Then the prescription*

$$\alpha * f := \delta(f) + \mathrm{Ad}(f)^{-1}.\alpha$$

defines an affine right action of the group $C^\infty(M,G)$ on $\Omega^1(M,\mathfrak{g})$ and for

$$F(\alpha) := \mathtt{d}\alpha + \tfrac{1}{2}[\alpha,\alpha] \quad \textit{we have} \quad F(\alpha * f) = \mathrm{Ad}(f)^{-1}.F(\alpha).$$

In particular, the subset $\mathrm{MC}(M,\mathfrak{g})$ of solutions of the MC equation is invariant under this action.

Proof. That $\alpha * f$ defines an action of the group $C^\infty(M,G)$ follows with the Product Rule (Lemma 4.2.5) from

$$\begin{aligned}(\alpha * f_1) * f_2 &= \delta(f_2) + \mathrm{Ad}(f_2)^{-1}.(\delta(f_1) + \mathrm{Ad}(f_1)^{-1}.\alpha)\\ &= \delta(f_1 f_2) + \mathrm{Ad}(f_1 f_2)^{-1}.\alpha = \alpha * (f_1 f_2).\end{aligned}$$

Since $\delta(f)$ satisfies the MC equation, Lemma 6.5.1 implies that

$$\begin{aligned}&F(\alpha * f) = \mathtt{d}(\alpha * f) + \frac{1}{2}[\alpha * f, \alpha * f]\\ =\ & \mathtt{d}(\delta(f)) + \mathrm{Ad}(f)^{-1}.\big(\mathtt{d}\alpha + [\delta(f^{-1}),\alpha]\big) + \frac{1}{2}[\delta(f),\delta(f)]\\ &\quad +[\delta(f), \mathrm{Ad}(f)^{-1}.\alpha] + \frac{1}{2}[\mathrm{Ad}(f)^{-1}.\alpha, \mathrm{Ad}(f)^{-1}.\alpha]\\ =\ & \mathrm{Ad}(f)^{-1}.\big(\mathtt{d}\alpha + \frac{1}{2}[\alpha,\alpha]\big) + [\mathrm{Ad}(f^{-1}).\delta(f^{-1}), \mathrm{Ad}(f^{-1}).\alpha]\big)\\ &\quad +[\delta(f), \mathrm{Ad}(f)^{-1}.\alpha]\\ =\ & \mathrm{Ad}(f)^{-1}.F(\alpha),\end{aligned}$$

where we have used that $\delta(f^{-1}) = -\mathrm{Ad}(f).\delta(f)$. ∎

Example 6.5.3 We have seen in Proposition 5.4.4 that

$$\mathrm{Gau}(P) = \{\varphi_f \colon f \in C^\infty(P,G)^G\}$$

acts on $\mathcal{C}(P)$ by

$$\varphi_f^*\theta = \delta(f) + \mathrm{Ad}(f)^{-1}.\theta = \theta * f.$$

Therefore the preceding lemma implies that the curvature transforms under gauge transformations according to

$$F(\varphi_f^*\theta) = \mathrm{Ad}(f).F(\theta).$$

Chapter 7

Perspectives

To develop the theory of fiber bundles further from where we stand, one needs various tools from algebraic topology, differential geometry (integration of differential forms) and sheaf theory.

7.1 Abelian Bundles and More Cohomology

First of all one needs the singular cohomology of a topological space X.

Definition 7.1.1 (Singular homology and cohomology) (a) Let X be a topological space and

$$\Delta_p := \{x \in \mathbb{R}^{p+1} \colon (\forall i) x_i \geq 0, \sum_i x_i = 1\}$$

be the p-dimensional standard simplex. The group $C_p(X)$ is the space of singular p-chains is the free group over the set $C(\Delta_p, X)$ of continuous maps $\Delta_p \to X$. It consists of all formal linear combinations

$$\sum_j a_j f_j, \quad f_j \in C(\Delta_p, X) \quad \text{with } a_j \in \mathbb{Z}.$$

We now form the abelian group

$$C(X) := \bigoplus_{p=0}^{\infty} C_p(X)$$

of *singular chains.*

There is a natural boundary operator

$$\partial\colon C_p(X) \to C_{p-1}(X),$$

defined on $f \in C(\Delta_p, X)$ by

$$\partial f = \sum_{i=0} (-1)^i \partial_i f,$$

where

$$\partial_i f\colon \Delta_{p-1} \to X, \quad x \mapsto f(x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_p).$$

For $p = 0$ we put $\delta(f) := 0$. One easily verifies that the corresponding endomorphism

$$\partial\colon C(X) \to C(X)$$

satisfies $\partial^2 = 0$. The elements of the group

$$Z_p(X) := \ker(\delta|_{C_p(X)})$$

are called *p-cycles*, and the elements of the group

$$B_p(X) := \partial(C_{p+1}(X)) \subseteq Z_p(X)$$

are called *p-boundaries.* The quotient

$$H_p(X) := Z_p(X)/B_p(X)$$

is called the p-th *singular homology group of* X.

(b) Now let A be any abelian group. The elements of the group

$$C^p(X, A) := \mathrm{Hom}(C_p(X), A)$$

are called *singular p-cochains with values in* A. On the space

$$C(X, A) := \bigoplus_{p=0}^{\infty} C^p(X, A)$$

we now define a coboundary operator by

$$\mathtt{d}\colon C^p(X, A) \to C^{p+1}(X, A), \quad (\mathtt{d}f)(\sigma) := f(\partial\sigma)$$

and find that $\mathtt{d}^2 = 0$ because $\partial^2 = 0$. The elements of the group

$$Z^p(X, A) := \ker(\mathtt{d}|_{C^p(X,A)})$$

are called *p-cocycles*, and the elements of the group

$$B^p(X, A) := \mathtt{d}(C^{p-1}(X, A)) \subseteq Z^p(X, A)$$

are called *p-boundaries*. The quotient

$$H^p_{\mathrm{sing}}(X, A) := H^p(X, A) := Z^p(X, A)/B^p(X, A)$$

is called the p-th *singular cohomology group of X with values in A*.

(c) If X is a smooth manifold, it makes sense to speak of smooth maps $\sigma\colon \Delta_p \to X$. These are maps which are smooth on the relative interior of Δ_p in its affine span, and for which all partial derivatives extend continuously to the boundary. We use these cochains we obtain the smooth singular cohomology groups

$$H^p_{\mathrm{diff}}(X, A), \quad p \in \mathbb{N}_0.$$

We now have natural homomorphisms

$$\varphi_p\colon H^p_{\mathrm{diff}}(X, A) \to H^p_{\mathrm{sing}}(X, A)$$

and it is part of an important Theorem of de Rham that each φ_p is an isomorphism for each abelian group A.

If, in addition, $A = V$ is a vector space, then each differential form $\omega \in \Omega^p(X, V)$ defines an element $\widetilde{\omega}$ of $C^p_{\mathrm{diff}}(X, V)$ by

$$\widetilde{\omega}(\sigma) := \int_\sigma \omega := \int_{\Delta_p} \sigma^*\omega.$$

In view of Stoke's Theorem, which holds in this context:

$$\int_{\partial\sigma} \omega = \int_\sigma \mathtt{d}\omega,$$

so that we obtain a linear map

$$\psi\colon H^p_{\mathrm{dR}}(M, V) \to H^p_{\mathrm{diff}}(M, V), \quad [\omega] \mapsto [\widetilde{\omega}].$$

Again, de Rham's Theorem (cf. [Wa83]) asserts that this map is an isomorphism.

Definition 7.1.2 As we have already seen in Chapter 2, there is another type of cohomology that shows up naturally in bundle theory, namely Čech cohomology. We know already how to define $\check{H}^1(M,G)$ for a non-abelian Lie group G, so let us assume here that G is abelian.

We start with an open cover $\mathcal{U} = (U_i)_{i\in I}$ of the smooth manifold M. For $\alpha = (i_1,\ldots,i_p) \in I^p$, we write

$$U_\alpha := U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_p}$$

and write $I_p := \{\alpha \in I^p \colon U_\alpha \neq \emptyset\}$. Then we consider the group

$$\check{C}^p(\mathcal{U},G) := \prod_{\alpha\in I_{p+1}} C^\infty(U_\alpha,G)$$

of smooth p-cochains. We then have a coboundary map

$$\delta\colon \check{C}^p(\mathcal{U},G) \to \check{C}^{p+1}(\mathcal{U},G),$$

defined by

$$\delta(f)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1}(-1)^j f_{i_0,\ldots,i_{j-1},i_{j+1},\ldots,i_p}.$$

Again, one can show that $\delta^2 = 0$, which leads to cohomology groups

$$\check{H}^p(\mathcal{U},G) = \ker(\delta|_{\check{C}^p(\mathcal{U},G)})/\delta(\check{C}^{p-1}(\mathcal{U},G)).$$

If the open cover $\mathcal{V} = (V_j)_{j\in J}$ is a refinement of $\mathcal{U}$, then we have a natural homomorphism

$$r_{\mathcal{U},\mathcal{V}}\colon \check{H}^p(\mathcal{U},G) \to \check{H}^p(\mathcal{V},G)$$

and the corresponding direct limit group is denoted

$$\check{H}^p(M,G) := \varinjlim \check{H}^p(\mathcal{U},G).$$

Again, as a consequence of de Rham's Theorem, we obtain for a discrete abelian group G an isomorphism

$$\check{H}^p(M,G) = H^p_{\rm sing}(M,G) \cong H^p_{\rm diff}(M,G).$$

To analyze abelian bundles further, one needs results on these cohomology groups. We have already seen that for vector groups $G = (V,+)$, each G-bundles is trivial, which means that

$$\check{H}^1(M,V) = \mathbf{0}.$$

This generalizes to
$$\check{H}^p(M, V) = \mathbf{0} \quad \text{for} \quad p > 0.$$
If $G = \mathfrak{g}/\Gamma$ is a connected abelian Lie group, then the short exact sequence

$$\mathbf{0} \to \Gamma \to \mathfrak{g} \to G \to \mathbf{0}$$

induces a long exact cohomology sequence

$$\ldots \to \check{H}^p(M, \Gamma) \to \check{H}^p(M, \mathfrak{g}) \to \check{H}^p(M, G) \to \check{H}^{p+1}(M, \Gamma) \to$$

so that the vanishing of the groups $\check{H}^p(M, \mathfrak{g})$ for $p = 1, 2$ yields an isomorphism
$$\delta\colon \check{H}^1(M, G) \to \check{H}^2(M, \Gamma) \cong H^2_{\text{sing}}(M, \Gamma).$$
This result can be viewed as a classificaton of principal G-bundles in terms on singular homology:

$$\text{Bun}(M, G) \cong H^2_{\text{sing}}(M, \Gamma).$$

For $G = \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, we obtain in particular

$$\text{Bun}(M, \mathbb{T}) \cong H^2_{\text{sing}}(M, \mathbb{Z}).$$

For each connection on a G-bundle we have defined the curvature of a G-bundle as an element of $H^2_{\text{dR}}(M, \mathfrak{g})$. This corresponds to the natural homomorphism

$$R\colon H^2_{\text{sing}}(M, \Gamma) \to H^2_{\text{diff}}(M, \mathfrak{g}) \to H^2_{\text{dR}}(M, \mathfrak{g}).$$

Its image is called the Γ-*integral* de Rham cohomology of M and denoted by $H^2_{\text{dR}}(M, \mathfrak{g}, \Gamma)$. It consists of those cohomology classes $[\omega]$ for which the corresponding singular cocycle

$$\sigma \mapsto \int_\sigma \omega$$

defined by integration has values in the discrete subgroup Γ of $\mathfrak{g}$. The map R has a certain kernel corresponding to the flat G-bundles. We know already that
$$\text{Bun}(M, G)_{\text{flat}} \cong \text{Ext}(H_1(M), \Gamma).$$

In terms of homological algebra, this corresponds to the Universal Coefficient Theorem (cf. [Br93]), which provides for each abelian group A a short exact sequence

$$\mathbf{0} \to \operatorname{Ext}(H_1(M), A) \to H^2(M, A) \to \operatorname{Hom}(H_2(M), A) \to \mathbf{0}.$$

Applying this to $A = \mathfrak{g}$, we first find with $\operatorname{Ext}(\cdot, \mathfrak{g}) = \mathbf{0}$ ($\mathfrak{g}$ is divisible) that

$$H^2(M, \mathfrak{g}) \cong \operatorname{Hom}(H_2(M), \mathfrak{g})$$

and by applying it to $A = \Gamma$, this leads to the exact sequence

$$\mathbf{0} \to \operatorname{Ext}(H_1(M), \Gamma) \to H^2(M, \Gamma) \to \operatorname{Hom}(H_2(M), \Gamma) \cong H^2_{\mathrm{dR}}(M, \mathfrak{g}, \Gamma) \to 0.$$

7.2 Homotopy Theory of Bundles

To obtain a classification of G-bundles in more concrete terms than the non-abelian cohomology group $\check{H}^1(M, G)$, one translates the classification of bundles into a homotopy theoretic problem.

To do that, we first move from the smooth category to the topological category. Every Lie group G also is a topological group and we may thus consider locally trivial topological G-bundles (P, M, G, q, σ), where σ is simply a continuous action. That this change of perspective does not lead to a loss of information is contained in the theorem that the corresponding natural map

$$\operatorname{Bun}(M, G) \to \operatorname{Bun}(M, G)_{\mathrm{top}}$$

assigning to the smooth bundle the class of the corresponding topological bundle actually is a bijection.

The next major step is to show that for each topological group G there exists a universal bundle (EG, BG, G, q, σ) with the property that for any other G-bundle there exists a continuous map $f\colon M \to BG$ with

$$P \cong f^*EG$$

and that two bundles f_1^*EG and f_2^*EG are equivalent if and only if the two maps $f_i\colon M \to BG$ are homotopic. We thus obtain a bijection

$$[M, BG] \to \operatorname{Bun}(M, G)_{\mathrm{top}}, \quad [f] \mapsto [f^*EG]$$

which leads to a classification of G-bundles in terms of homotopy classes.

For simple manifolds such as spheres, the basic tools of homotopy theory now lead to

$$\mathrm{Bun}(\mathbb{S}^n, G) \cong [\mathbb{S}^n, BG] \cong \pi_{n-1}(G), \quad n > 1,$$

whereas we find for $n = 1$ that $\mathrm{Bun}(\mathbb{S}^1, G)$ can be identified with the set $H^1(\mathbb{Z}, \pi_0(G))$ of conjugacy classes in $\pi_0(G)$ (Example 6.3.13). Here we see quite explicitly how the classification of flat bundles (as bundles with connection) and the classifications of bundles differ. Each bundle over $\mathbb{S}^1$ is flat, so that

$$\mathrm{Bun}(\mathbb{S}^1, G) \cong \mathrm{Bun}(\mathbb{S}^1, G)_{\mathrm{flat}} \cong H^1(\mathbb{Z}, \pi_0(G))$$

(Example 6.3.13). but the equivalence classes of flat bundles with connection are parameterizes by the set $H^1(\mathbb{Z}, G)$. There is a natural surjective map

$$H^1(\mathbb{Z}, G) \to H^1(\mathbb{Z}, \pi_0(G)),$$

but it is far from being injective.

On the topological level the characteristic classes of a G-bundle $P \cong f^*EG$ are obtained from the natural homomorphism

$$f^* \colon H_{\mathrm{sing}}(BG, \mathbb{K}) \to H(M, \mathbb{K}) \cong H_{\mathrm{dR}}(M, \mathbb{K}).$$

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