Manifolds and Transformation Groups

WS 05 — Technische Universität Darmstadt

Karl – Hermann Neeb

Inhalt

Manifolds and Transformation Groups

Introduction

In the basic calculus courses one mostly deals with (differentiable) functions on open subsets of $\mathbb{R}^n$, but as soon as one wants to solve equations of the form $f(x) = y$, where $f\colon U \to \mathbb{R}^m$ is a differentiable function and U is open in $\mathbb{R}^n$, one observes that the set $f^{-1}(y)$ of solutions behaves in a much more complicated manner than one is used to from Linear Algebra, where f is linear and $f^{-1}(y)$ is the intersection of U with an affine subspace. One way to approach differentiable manifolds is to think of them as the natural objects arising as solutions of non-linear equations as above (under some non-degeneracy condition on f, made precise by the Implicit Function Theorem). For submanifolds of $\mathbb{R}^n$, this is a quite natural approach, which immediately leads to the method of Lagrange multipliers to deal with extrema of differentiable functions under differentiable constraints. This is the external perspective on differentiable manifolds, which has the serious disadvantage that it depends very much on the surrounding space $\mathbb{R}^n$.

It is much more natural to adopt a more intrinsic perspective: an n-dimensional manifold is a topological space which locally looks like $\mathbb{R}^n$. More precisely, it arises by gluing open subsets of $\mathbb{R}^n$ in a smooth (differentiable) way. Below we shall make this more precise.

The theory of smooth manifolds has three levels:

(1) The **infinitesimal level**, where one deals with tangent spaces, tangent vectors and differentials of maps,

(2) the **local level**, which is analysis on open subsets of $\mathbb{R}^n$, and

(3) the **global level**, where one studies the global behavior of manifolds and other related structures.

These three levels are already visible in one-variable calculus: Suppose we are interested in the global maximum of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ which is a question about the global behavior of this function. The necessary condition $f'(x_0) = 0$ belongs to the infinitesimal level because it says something about the behavior of f infinitesimally close to the point x_0. The sufficient criterion for a local maximum: $f'(x_0) = 0$, $f''(x_0) < 0$ provides information on the local level. Everyone knows that this is far from being the whole story and that one really has to study global properties of f, such as $\lim_{x\to\pm\infty} f(x) = 0$, to guarantee the existence of global maxima.

I. Smooth manifolds

In this chapter we first recall the central definitions and results from calculus in several variables. Then we turn to the definition of a differentiable manifold and discuss several aspects of this concept.

I.1. Smooth maps in several variables

First we recall some facts and definitions from calculus in several variables, formulated in a way that will be convenient for us in the following.

Definition I.1.1. (Differentiable maps)

(a) Let $n, m \in \mathbb{N}$ and $U \subseteq \mathbb{R}^n$ be an open subset. A function $f\colon U \to \mathbb{R}^m$ is called *differentiable at* $x \in U$ if there exists a linear map $L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that for one (and hence for all norms on $\mathbb{R}^n$) we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L(h)}{\|h\|} = 0. \tag{1.1}$$

If f is differentiable in x, then for each $h \in \mathbb{R}^n$ we have

$$\lim_{t \to 0} \frac{1}{t}\big(f(x+th) - f(x)\big) = \lim_{t \to 0} \frac{1}{t} L(th) = L(h),$$

so that $L(h)$ is the directional derivative of f in x in the direction h. It follows in particular that condition (1.1) determines the linear map L uniquely. We therefore write

$$df(x)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x+th) - f(x)\big) = L(h)$$

and call the linear map $df(x)$ the *differential of* f *in* x.

(b) Let $e_1, \ldots, e_n$ denote the canonical basis vectors in $\mathbb{R}^n$. Then

$$\frac{\partial f}{\partial x_i}(x) := df(x)(e_i)$$

is called the *i-th partial derivative of* f *in* x. If f is differentiable in each $x \in U$, then the partial derivatives are functions

$$\frac{\partial f}{\partial x_i}\colon U \to \mathbb{R}^m,$$

and we say that f is *continuously differentiable*, or a C^1*-map*, if all its partial derivatives are continuous. For $k \geq 2$, the map f is said to be a C^k*-map* if it is C^1 and all its partial derivatives are C^{k-1}-maps. We say that f is *smooth* or a C^∞*-map* if it is C^k for each $k \in \mathbb{N}$.

(c) If $I \subseteq \mathbb{R}$ is an interval and $\gamma: I \to \mathbb{R}^n$ is a differentiable curve, we also write

$$\dot\gamma(t) = \gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

This is related to the notation from above by

$$\gamma'(t) = d\gamma(t)(e_1),$$

where $e_1 = 1 \in \mathbb{R}$ is the canonical basis vector. ■

Definition I.1.2. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. A map $f: U \to V$ is called C^k if it is C^k as a map $U \to \mathbb{R}^m$.

For $n \geq 1$ a C^k-map $f: U \to V$ is called a C^k*-diffeomorphism* if there exists a C^k-map $g: V \to U$ with

$$f \circ g = \mathrm{id}_V \quad \text{and} \quad g \circ f = \mathrm{id}_U.$$

Obviously, this is equivalent to f being bijective and f^{-1} being a C^k-map. Whenever such a diffeomorphism exists, we say that the domains U and V are C^k*-diffeomorphic.* For $k = 0$ we thus obtain the notion of a *homeomorphism.* ■

Theorem I.1.3. (Chain Rule) *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. Further let $f: U \to V$ be a C^k-map and $g: V \to \mathbb{R}^d$ a C^k-map. Then $g \circ f$ is a C^k-map, and for each $x \in U$ we have in* $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^d)$*:*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

■

The Chain Rule is an important tool which permits to "linearize" non-linear information. The following proposition is an example.

Proposition I.1.4. (Invariance of the dimension) *If the non-empty open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are C^1-diffeomorphic, then $m = n$.*

Proof. Let $f: U \to V$ be a C^1-diffeomorphism and $g: V \to U$ its inverse. Pick $x \in U$. Then the Chain Rule implies that

$$\mathrm{id}_{\mathbb{R}^n} = d(g \circ f)(x) = dg(f(x)) \circ df(x)$$

and

$$\mathrm{id}_{\mathbb{R}^m} = d(f \circ g)(f(x)) = df(x) \circ dg(f(x)),$$

so that $df(x): \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism. This implies that $m = n$. ■

Theorem I.1.5. (Inverse Function Theorem) *Let $U \subseteq \mathbb{R}^n$ be an open subset, $x_0 \in U$, $k \in \mathbb{N} \cup \{\infty\}$, and $f\colon U \to \mathbb{R}^n$ a C^k-map for which the linear map $df(x_0)$ is invertible. Then there exists an open neighborhood V of x_0 in U for which $f|_V\colon V \to f(V)$ is a C^k-diffeomorphism onto an open subset of $\mathbb{R}^n$.* ■

Corollary I.1.6. *Let $U \subseteq \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}^n$ be an injective C^k-map $(k \geq 1)$ for which $df(x)$ is invertible for each $x \in U$. Then $f(U)$ is open and $f\colon U \to f(U)$ is a C^k-diffeomorphism.*

Proof. First we use the Inverse Function Theorem to see that for each $x \in U$ the image $f(U)$ contains a neighborhood of $f(x)$, so that $f(U)$ is an open subset of $\mathbb{R}^n$. Since f is injective, the inverse function $g = f^{-1}\colon f(U) \to U$ exists. Now we apply the Inverse Function Theorem again to see that for each $x \in U$ there exists a neighborhood of $f(x)$ in $f(U)$ on which g is C^k. Therefore g is a C^k-map, and this means that f is a C^k-diffeomorphism. ■

Example I.1.7. That the injectivity assumption in Corollary I.1.6 is crucial is shown by the following example, which is a real description of the complex exponential function. We consider the smooth map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2).$$

Then the matrix of $df(x)$ with respect to the canonical basis is

$$[df(x)] = \begin{pmatrix} e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\ e^{x_1} \sin x_2 & e^{x_1} \cos x_2 \end{pmatrix}.$$

Its determinant is $e^{2x_1} \neq 0$, so that $df(x)$ is invertible for each $x \in \mathbb{R}^2$.

Polar coordinates immediately show that $f(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{(0,0)\}$, which is an open subset of $\mathbb{R}^2$, but the map f is not injective because it is 2π-periodic in x_2:

$$f(x_1, x_2 + 2\pi) = f(x_1, x_2).$$

Therefore the Inverse Function Theorem applies to each $x \in \mathbb{R}^2$, but f is not a global diffeomorphism. ■

Remark I.1.8. The best way to understand the Implicit Function Theorem is to consider the linear case first. Let $g\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. We are interested in conditions under which the equation $g(x, y) = 0$ can be solved for x, i.e., there is a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $g(x, y) = 0$ is equivalent to $x = f(y)$.

Since we are dealing with linear maps, there are matrices $A \in M_m(\mathbb{R})$ and $B \in M_{m,n}(\mathbb{R})$ with

$$g(x, y) = Ax + By \quad \text{for} \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

The unique solvability of the equation $g(x, y) = 0$ for x is equivalent to the unique solvability of the equation $Ax = -By$, which is equivalent to the invertibility of the matrix A. If $A \in \mathrm{GL}_m(\mathbb{R})$, we thus obtain the linear function

$$f\colon \mathbb{R}^n \to \mathbb{R}^m, \quad f(y) = -A^{-1}By$$

for which $x = f(y)$ is equivalent to $g(x, y) = 0$. ■

Theorem I.1.9. (Implicit Function Theorem) *Let $U \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be an open subset and $g: U \to \mathbb{R}^m$ be a C^k-function, $k \in \mathbb{N} \cup \{\infty\}$. Further let $(x_0, y_0) \in U$ with $g(x_0, y_0) = 0$ such that the linear map*

$$d_1 g(x_0, y_0): \mathbb{R}^n \to \mathbb{R}^n, \quad v \mapsto dg(x_0, y_0)(v, 0)$$

is invertible. Then there exist open neighborhoods V_1 of x_0 in $\mathbb{R}^m$ and V_2 of y_0 in $\mathbb{R}^n$ with $V_1 \times V_2 \subseteq U$, and a C^k-function $f: V_2 \to V_1$ with $f(y_0) = x_0$ such that

$$\{(x, y) \in V_1 \times V_2 : g(x, y) = 0\} = \{(f(y), y) : y \in V_2\}. \qquad \blacksquare$$

Definition I.1.10. (Higher derivatives) For $k \geq 2$, a C^k-map $f: U \to \mathbb{R}^m$ and $U \subseteq \mathbb{R}^n$ open, higher derivatives are defined inductively by

$$\begin{aligned} &d^k f(x)(h_1, \ldots, h_k) \\ :=& \lim_{t \to 0} \frac{1}{t}\big(d^{k-1} f(x + th_k)(h_1, \ldots, h_{k-1}) - d^{k-1} f(x)(h_1, \ldots, h_{k-1})\big). \end{aligned}$$

We thus obtain continuous maps

$$d^k f: U \times (\mathbb{R}^n)^k \to \mathbb{R}^m.$$

In terms of concrete coordinates and the canonical basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we then have

$$d^k f(x)(e_{i_1}, \ldots, e_{i_k}) = \frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}}(x). \qquad \blacksquare$$

Let V and W be vector spaces. We recall that a map $\beta: V^k \to W$ is called *k-linear* if all the maps

$$V \to W, \quad v \mapsto \beta(v_1, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_k)$$

are linear. It is said to be symmetric if

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \beta(v_1, \ldots, v_k)$$

holds for all permutations $\sigma \in S_k$.

Proposition I.1.11. *If $f \in C^k(U, \mathbb{R}^m)$ and $k \geq 2$, then the functions $(h_1, \ldots, h_k) \mapsto d^k f(x)(h_1, \ldots, h_k)$, $x \in U$, are symmetric k-linear maps.*

Proof. From the definition it follows inductively that $(d^k f)(x)$ is linear in each argument h_i, because if all other arguments are fixed, it is the differential of a C^1-function.

To verify the symmetry of $(d^k f)(x)$, we may also proceed by induction. Therefore it suffices to show that for $h_1, \ldots, h_{k-2}$ fixed, the map

$$\beta(v, w) := (d^k f(x))(h_1, \ldots, h_{k-2}, v, w)$$

is symmetric. This map is the second derivative $d^2F(x)$ of the function

$$F(x) := (d^{k-2}f)(x)(h_1, \ldots, h_{k-2}).$$

We may therefore assume that $k = 2$.

In view of the bilinearity, it suffices to observe that the Schwarz Lemma implies

$$(d^2F)(x)(e_j, e_i) = \Big(\frac{\partial^2}{\partial x_i x_j} F\Big)(x) = \Big(\frac{\partial^2}{\partial x_j x_i} F\Big)(x) = (d^2F)(x)(e_i, e_j). \qquad \blacksquare$$

Theorem I.1.12. (Taylor's Theorem) *Let $U \subseteq \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$ of class C^{k+1}. If $x + [0,1]h \subseteq U$, then we have the Taylor Formula*

$$\begin{aligned} f(x+h) &= f(x) + df(x)(h) + \ldots + \frac{1}{k!} d^k f(x)(h, \ldots, h) \\ &\quad + \frac{1}{k!} \int_0^1 (1-t)^k \big(d^{k+1} f(x+th)\big)(h, \ldots, h)\, dt. \end{aligned}$$

Proof. For each $i \in \{1, \ldots, m\}$ we consider the C^{k+1}-maps

$$F: [0,1] \to \mathbb{R}, \quad F(t) := f_i(x+th) \quad \text{with} \quad F^{(k)}(t) = d^k f_i(x+th)(h, \ldots, h)$$

and apply the Taylor Formula for functions $[0.1] \to \mathbb{R}$ to get

$$F(1) = F(0) + \ldots + \frac{F^{(k)}(0)}{k!} + \frac{1}{k!} \int_0^1 (1-t)^k F^{(k+1)}(t)\, dt. \qquad \blacksquare$$

I.2. The definition of a smooth manifold

Throughout this course we assume some familiarity with basic topological constructions and concepts, such as the quotient topology.

Before we turn to the concept of a smooth manifold, we recall the concept of a Hausdorff space. A topological space (X, τ) is called a *Hausdorff space* if for two different points $x, y \in X$ there exist disjoint open subsets O_x, O_y with $x \in O_x$ and $y \in O_y$. Recall that each metric space (X, d) is Hausdorff. In this case we may take $O_x := B_\varepsilon(x)$ and $O_y := B_\varepsilon(y)$ for any $\varepsilon < \frac{1}{2} d(x, y)$.

Definition I.2.1. Let M be a topological space.

(a) A pair (φ, U), consisting on an open subset $U \subseteq M$ and a homeomorphism $\varphi: U \to \varphi(U) \subseteq \mathbb{R}^n$ of U onto an open subset of $\mathbb{R}^n$ is called an *n-dimensional chart of M*.

(b) Two n-dimensional charts (φ, U) and (ψ, V) of M are said to be *C^k-compatible* ($k \in \mathbb{N} \cup \{\infty\}$) if $U \cap V = \emptyset$ or the map

$$\psi \circ \varphi^{-1}|_{\varphi(U \cap V)}: \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. Note that $\varphi: U \to \varphi(U)$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, so that $\varphi(U \cap V)$ is an open subset of $\varphi(U)$ and hence of $\mathbb{R}^n$.

(c) An *n-dimensional C^k-atlas of M* is a family $\mathcal{A} := (\varphi_i, U_i)_{i \in I}$ of n-dimensional charts of M with the following properties:

(A1) $\bigcup_{i \in I} U_i = M$, i.e., $(U_i)_{i \in I}$ is an open covering of M.

(A2) All charts (φ_i, U_i), $i \in I$, are pairwise C^k-compatible. For $U_{ij} := U_i \cap U_j$, this means that all maps

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})}: \varphi_i(U_{ij}) \to \varphi_j(U_{ij})$$

are C^k-maps because $\varphi_{ji}^{-1} = \varphi_{ij}$.

(d) A chart (φ, U) is called *compatible* with a C^k-atlas $(\varphi_i, U_i)_{i \in I}$ if it is C^k-compatible with all charts of the atlas $\mathcal{A}$. A C^k-atlas $\mathcal{A}$ is called *maximal* if it contains all charts compatible with it. A maximal C^k-atlas is also called a *C^k-differentiable structure on M*. For $k = \infty$ we call it a *smooth structure.* ■

Remark I.2.2. (a) In Definition I.2.1(b) we required that the map

$$\psi \circ \varphi^{-1}|_{\varphi(U \cap V)}: \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. Since φ and ψ are diffeomorphisms, this map always is a homeomorphism between open subsets of $\mathbb{R}^n$. The differentiability is an additional requirement.

(b) For $M = \mathbb{R}$ the maps (M, φ) and (M, ψ) with $\varphi(x) = x$ and $\psi(x) = x^3$ are 1-dimensional charts. These charts are not C^1-compatible: the map

$$\psi \circ \varphi^{-1}: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3$$

is smooth, but not a diffeomorphism, since its inverse $\varphi \circ \psi^{-1}$ is not differentiable.

(c) Any atlas $\mathcal{A}$ is contained in a unique maximal atlas: We simply add all charts compatible with $\mathcal{A}$, and thus obtain a maximal atlas. This atlas is unique (Exercise I.2). ■

Definition I.2.3. An *n-dimensional C^k-manifold* is a pair $(M, \mathcal{A})$ of a Hausdorff space M and a maximal n-dimensional C^k-atlas on M. For $k = \infty$ we call it a smooth manifold. ■

To specify a manifold structure, it suffices to specify a C^k-atlas $\mathcal{A}$ because this atlas is contained in a unique maximal one (Exercise I.2). In the following we shall never describe a maximal atlas. We shall always try to keep the number of charts as small as possible. For simplicity, we always assume in the following that $k = \infty$.

Example I.2.4. (Open subsets of $\mathbb{R}^n$) Let $U \subseteq \mathbb{R}^n$ be an open subset. Then U is a Hausdorff space with respect to the induced topology. The inclusion map $\varphi: U \to \mathbb{R}^n$ defines a chart (φ, U) which already defines a smooth atlas of U, turning U into an n-dimensional smooth manifold. ■

Example I.2.5. (The n-dimensional sphere) We consider the unit sphere

$$\mathbb{S}^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}$$

in $\mathbb{R}^n$, endowed with the subspace topology, turning it into a compact space.

(a) To specify a smooth manifold structure on $\mathbb{S}^n$, we consider the open subsets

$$U_i^\varepsilon := \{x \in \mathbb{S}^n : \varepsilon x_i > 0\}, \quad i = 0, \ldots, n, \quad \varepsilon \in \{\pm 1\}.$$

These $2(n+1)$ subsets form a covering of $\mathbb{S}^n$. We have homeomorphisms

$$\varphi_i^\varepsilon : U_i^\varepsilon \to B := \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$$

onto the open unit ball in $\mathbb{R}^n$, given by

$$\varphi_i^\varepsilon(x) = (x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

and with continuous inverse map

$$(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, \varepsilon\sqrt{1 - \|y\|_2^2}, y_{i+1}, \ldots, y_n).$$

This leads to charts $(\varphi_i^\varepsilon, U_i^\varepsilon)$ of $\mathbb{S}^n$.

It is easy to see that these charts are pairwise compatible. We have $\varphi_i^\varepsilon \circ (\varphi_i^{\varepsilon'})^{-1} = \mathrm{id}_B$, and for $i < j$, we have

$$\varphi_i^\varepsilon \circ (\varphi_j^{\varepsilon'})^{-1}(y) = (y_1, \ldots, y_i, y_{i+2}, \ldots, y_j, \varepsilon'\sqrt{1 - \|y\|_2^2}, y_{j+1}, \ldots, y_n),$$

which is a smooth map

$$\varphi_j^{\varepsilon'}(U_i^\varepsilon \cap U_j^{\varepsilon'}) \to \varphi_i^\varepsilon(U_i^\varepsilon \cap U_j^{\varepsilon'}).$$

(b) There is another atlas of $\mathbb{S}^n$ consisting only of two charts, where the maps are slightly more complicated.

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_+ : U_+ := \mathbb{S}^n \setminus \{e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 - y_0} y$$

and

$$\varphi_- : U_- := \mathbb{S}^n \setminus \{-e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 + y_0} y.$$

Both maps are bijective with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2}\Big)$$

(Exercise I.10). This implies that (φ_+, U_+) and (φ_-, U_-) are charts of $\mathbb{S}^n$. That both are smoothly compatible, hence a smooth atlas, follows from

$$(\varphi_+ \circ \varphi_-^{-1})(x) = (\varphi_- \circ \varphi_+^{-1})(x) = \frac{x}{\|x\|^2}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

which is the inversion in the unit sphere. ■

Example I.2.6. Let V be an n-dimensional real vector space. We know from Linear Algebra that V is isomorphic to $\mathbb{R}^n$, and that for each ordered basis $B := (b_1, \ldots, b_n)$ of V, the linear map

$$\varphi_B \colon \mathbb{R}^n \to V, \quad x = (x_1, \ldots, x_n) \mapsto \sum_{j=1}^n x_j b_j$$

is a linear isomorphism. Using such a linear isomorphism φ_B, we define a topology on V in such a way that φ_B is a homeomorphism, i.e., $O \subseteq V$ is open if and only if $\varphi_B^{-1}(O)$ is open in $\mathbb{R}^n$.

For any other choice of a basis $C = (c_1, \ldots, c_m)$ in V we recall from Linear Algebra that $m = n$ and that the map

$$\varphi_C^{-1} \circ \varphi_B \colon \mathbb{R}^n \to \mathbb{R}^n$$

is a linear isomorphism, hence a homeomorphism. This implies that for a subset $O \subseteq V$ the condition that $\varphi_B^{-1}(O)$ is open is equivalent to $\varphi_C^{-1}(O) = \varphi_C^{-1} \circ \varphi_B \circ \varphi_B^{-1}(O)$ being open. We conclude that the topology introduced on V by φ_B does not depend on the choice of a basis.

We thus obtain on V a natural topology for which it is homeomorphic to $\mathbb{R}^n$, hence in particular a Hausdorff space. From each coordinate map $\kappa_B := \varphi_B^{-1}$ we obtain a chart (κ_B, V) which already defines an atlas of V. We thus obtain on V the structure of an n-dimensional smooth manifold. That all these charts are compatible follows from the smoothness of the linear maps $\kappa_C \circ \kappa_B^{-1} = \varphi_C^{-1} \circ \varphi_B \colon \mathbb{R}^n \to \mathbb{R}^n$. ■

Example I.2.7. (Submanifolds of $\mathbb{R}^n$) A subset $M \subseteq \mathbb{R}^n$ is called a d-*dimensional submanifold* if for each $p \in M$ there exists an open neighborhood U of p in $\mathbb{R}^n$ and a diffeomorphism

$$\varphi \colon U \to \varphi(U) \subseteq \mathbb{R}^n$$

onto an open subset $\varphi(U)$ with

$$(SM) \qquad \varphi(U \cap M) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}).$$

Whenever this condition is satisfied, we call (φ, U) a *submanifold chart.*

A submanifold of codimension 1, i.e., $\dim M = n - 1$, is called a *smooth hyper-surface.*

We claim that M carries a natural d-dimensional manifold structure when endowed with the topology inherited from $\mathbb{R}^n$, which obviously turns it into a Hausdorff space.

In fact, for each submanifold chart (φ, U), we obtain a d-dimensional chart

$$(\varphi|_{U \cap M}, U \cap M).$$

For two such charts coming from (φ, U) and (ψ, V), we have

$$\psi \circ \varphi^{-1}|_{\varphi(U \cap V \cap M)} = (\psi|_{V \cap M}) \circ (\varphi|_{U \cap M})^{-1}|_{\varphi(U \cap V \cap M)},$$

which is a smooth map onto an open subset of $\mathbb{R}^d$. We thus obtain a smooth atlas of M. ■

The following proposition provides a particularly handy criterion to verify that the set of solutions of a non-linear equation is a submanifold. Let $f: U \to \mathbb{R}^m$ be a C^1-map. We call $y \in \mathbb{R}^m$ a *regular value of* f if for each $x \in U$ with $f(x) = y$ the differential $df(x)$ is surjective. Otherwise y is called a *singular value of* f.

Proposition I.2.8. *Let $U \subseteq \mathbb{R}^n$ be an open subset, $f: U \to \mathbb{R}^m$ a smooth map and $y \in f(U)$ such that $df(x)$ is surjective for each $x \in f^{-1}(y)$. Then $M := f^{-1}(y)$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$, hence in particular a smooth manifold.*

Proof. Let $d := n - m$ and observe that $d \geq 0$ because $df(x): \mathbb{R}^n \to \mathbb{R}^m$ is surjective for each $x \in M$. We have to show that for each $x_0 \in M$ there exists an open neighborhood V of x_0 in $\mathbb{R}^n$ and a diffeomorphism

$$\varphi: V \to \varphi(V) \subseteq \mathbb{R}^n$$

with

$$\varphi(V \cap M) = \varphi(V) \cap (\mathbb{R}^d \times \{0\}).$$

After a permutation of the coordinates, we may w.l.o.g. assume that the vectors

$$df(x_0)(e_{d+1}), \ldots, df(x_0)(e_n)$$

form a basis of $\mathbb{R}^m$. Then we consider the map

$$\varphi: U \to \mathbb{R}^n, \quad x = (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d, f_1(x) - y_1, \ldots, f_m(x) - y_m).$$

In view of

$$d\varphi(x_0)(e_j) = \begin{cases} e_j & \text{for } j \leq d \\ df(x_0)(e_j) & \text{for } j > d, \end{cases}$$

it follows that the linear map $d\varphi(x_0): \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Hence the Inverse Function Theorem implies the existence of an open neighborhood $V \subseteq U$ of x_0 for which $\varphi|_V: V \to \varphi(V)$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$.

Since

$$M = \{p \in U: \varphi(p) = (\varphi_1(p), \ldots, \varphi_d(p), 0, \ldots, 0)\} = \varphi^{-1}(\mathbb{R}^d \times \{0\}),$$

it follows that

$$\varphi(M \cap V) = \varphi(V) \cap (\mathbb{R}^d \times \{0\}). \qquad \blacksquare$$

Examples I.2.9. The preceding proposition is particularly easy to apply for hypersurfaces, i.e., to the case $m = 1$. Then $f: U \to \mathbb{R}$ is a smooth function and the condition that $df(x)$ is surjective simply means that $df(x) \neq 0$, i.e., that there exists some j with $\frac{\partial f}{\partial x_j}(x) \neq 0$.

(a) Let $A = A^T \in M_n(\mathbb{R})$ be a symmetric matrix and

$$f(x) := x^\top A x = \sum_{i,j=1}^n a_{ij} x_i x_j$$

the corresponding quadratic form.

We want to show that the corresponding *quadric*

$$Q := \{x \in \mathbb{R}^n : f(x) = 1\}$$

is a submanifold of $\mathbb{R}^n$. To verify the criterion from Proposition I.2.8, we assume that $f(x) = 1$ and note that

$$df(x)v = v^\top Ax + x^\top Av = 2v^\top Ax$$

(Exercise; use Exercise I.13). Therefore $df(x) = 0$ is equivalent to $Ax = 0$, which is never the case if $x^\top Ax = 1$. We conclude that all level surfaces of f are smooth hypersurfaces of $\mathbb{R}^n$.

For $A = E_n$ (the identity matrix), we obtain in particular the $(n-1)$-dimensional unit sphere $Q = \mathbb{S}^{n-1}$.

For $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and nonzero λ_i we obtain the hyperboloids

$$Q = \Big\{x \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i x_i^2 = 1\Big\}$$

which degenerate to hyperbolic cyclinders if some λ_i vanish.

(b) For singular values the level sets may or may not be submanifolds: For the quadratic form

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = x_1 x_2$$

the value 0 is singular because $f(0,0) = 0$ and $df(0,0) = 0$. The inverse image is

$$f^{-1}(0) = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}),$$

which is not a submanifold of $\mathbb{R}^2$ (Exercise).

For the quadratic form

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = x_1^2 + x_2^2$$

the value 0 is singular because $f(0,0) = 0$ and $df(0,0) = 0$. The inverse image is

$$f^{-1}(0) = \{(0,0)\},$$

which is a zero-dimensional submanifold of $\mathbb{R}^2$.

For the quadratic form

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$$

the value 0 is singular because $f(0) = 0$ and $f'(0) = 0$. The inverse image is

$$f^{-1}(0) = \{0\},$$

which is a submanifold of $\mathbb{R}$.

(c) On $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ we consider the quadratic function

$$f: M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}): A^\top = A\}, \quad X \mapsto XX^\top.$$

Then

$$f^{-1}(\mathbf{1}) = \mathrm{O}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}): g^\top = g^{-1}\}$$

is the *orthogonal group.*

To see that this is a submanifold of $M_n(\mathbb{R})$, we note that

$$df(X)(Y) = XY^\top + YX^\top$$

(Exercise I.13). If $f(X) = \mathbf{1}$, we have $X^\top = X^{-1}$, so that for any $Z \in \mathrm{Sym}_n(\mathbb{R})$ the matrix $Y := \frac{1}{2}ZX$ satisfies

$$XY^\top + YX^\top = \frac{1}{2}(XX^\top Z + ZXX^\top) = Z.$$

Therefore $df(X)$ is surjective in each orthogonal matrix X, and Proposition I.2.8 implies that $\mathrm{O}_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$ of dimension

$$d = n^2 - \dim\big(\mathrm{Sym}_n(\mathbb{R})\big) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}. \qquad \blacksquare$$

Examples I.2.10. (Graßmannians and Stiefel manifolds) We write $\mathrm{Gr}_k(\mathbb{R}^n) = \mathrm{Gr}_{k,n}$ for the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. We want to endow this set with a manifold structure.

First we consider the set

$$S_{k,n} := \{(v_1, \ldots, v_k) \in (\mathbb{R}^n)^k : v_1, \ldots, v_k \text{ lin. indep.}\}.$$

This is a subset of $(\mathbb{R}^n)^k \cong \mathbb{R}^{nk}$. The linear independence of a k-tuple of vectors $(v_1, \ldots, v_k)$ is equivalent to the matrix $V := (v_1, \ldots, v_k) \in M_{n,k}(\mathbb{R})$ having maximal rank k. This in turn means that there exists a k-element subset $I \subseteq \{1, \ldots, n\}$ such that the matrix

$$V_I = (v_{ij})_{i \in I, j=1,\ldots,k}, \quad v_j = \sum_i v_{ij} e_i$$

(obtained from erasing the rows of V corresponding to numbers $i \notin I$) is invertible. We conclude that

$$S_{k,n} = \bigcup_I \{V \in M_{n,k}(\mathbb{R}): \det(V_I) \neq 0\}$$

is a union of open subsets, hence an open subset of $M_{n,k}(\mathbb{R})$. Here we use that the map $V \mapsto \det(V_I)$ is continuous, which is clear from the fact that it is a polynomial in entries of V. As an open subset, $S_{k,n}$ carries a natural manifold structure of dimension kn. It is called the *Stiefel manifold of rank k in $\mathbb{R}^n$*.

We have a surjective map

$$S\colon S_{k,n} \to \mathrm{Gr}_{k,n}, \quad (v_1, \ldots, v_k) \mapsto \mathrm{span}(v_1, \ldots, v_k),$$

and two matrices $X, Y \in S_{k,n}$ satisfy $S(X) = S(Y)$ if and only if their columns span the same subspace of $\mathbb{R}^n$, which means that there exists some $g \in \mathrm{GL}_k(\mathbb{R})$ with $Y = Xg$ (Exercise I.19(a)). We now endow $\mathrm{Gr}_{k,n}$ with the quotient topology, turning S into a topological quotient map. This means that a subset $O \subseteq \mathrm{Gr}_{k,n}$ is open if and only if $S^{-1}(O)$ is open in $S_{k,n}$.

From this construction it is not so easy to see directly that the topology on $\mathrm{Gr}_{k,n}$ is Hausdorff. To see this, we recall the Gram–Schmidt process converting a linearly independent k-tuple $(v_1, \ldots, v_k) \in S_{k,n}$ into an orthonormal k-tuple $(w_1, \ldots, w_k)$. It is defined inductively by solving the equations

$$v_i = \sum_{j=1}^{i} \langle v_i, w_j \rangle w_j$$

inductively for $w_1, w_2, \ldots, w_k$. Let $T_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ denote the group of upper triangular matrices with positive diagonals and $F_{k,n} \subseteq S_{k,n}$ the subset of all orthonormal k-tuples (the set of *k-frames*). Then the Gram–Schmidt argument shows that the multiplication map

$$F_{k,n} \times T_n^+(\mathbb{R}) \to S_{k,n}, \quad (W, b) \mapsto Wb$$

is a homeomorphism. This implies that

$$\mathrm{Gr}_{k,n} = S_{k,n}/\sim\; \cong\; (F_{k,n} \times B)/\sim\; \cong\; F_{k,n}/\sim,$$

where $W \sim W'$ in $F_{k,n}$ holds if and only if there is an orthogonal matrix $g \in \mathrm{O}_k(\mathbb{R})$ with $W' = Wg$ (Exercise I.19(b)). Next we observe that the metric on $F_{k,n}$ inherited from the Euclidean metric on the row space $(\mathbb{R}^k)^n$ is invariant under right multiplication with orthogonal matrices. For two elements $W, W' \in F_{k,n}$ with $S(W) \neq S(W')$ we therefore have

$$\varepsilon := \mathrm{dist}(W \cdot \mathrm{O}_k(\mathbb{R}), W' \cdot \mathrm{O}_k(\mathbb{R})) = \inf\{d(Wg, W'g')\colon g, g' \in \mathrm{O}_k(\mathbb{R})\} > 0$$

(Exercise I.21) because the sets $W \cdot \mathrm{O}_k(\mathbb{R})$ and $W' \cdot \mathrm{O}_k(\mathbb{R})$ are compact and disjoint. Now the $\varepsilon/2$-neighborhoods of these sets are open disjoint $\sim$-saturated sets, so that their images in $\mathrm{Gr}_{k,n}$ are disjoint open subsets separating $S(W)$ and $S(W')$. This proves that $\mathrm{Gr}_{k,n}$ is a Hausdorff space.

For each k-element subset $I \subseteq \{1, \ldots, n\}$ we observe that

$$U_I := \{S(X) \in \mathrm{Gr}_{k,n}\colon \det(X_I) \neq 0\}$$

is open because $S^{-1}(U_I)$ is open in $S_{k,n}$. We write $I' := \{1, \ldots, n\} \setminus I$ and define a map

$$\varphi_I\colon U_I \to M_{n-k,k}(\mathbb{R}), \quad S(X) \mapsto X_{I'} X_I^{-1}.$$

Since $X_I \in \mathrm{GL}_k(\mathbb{R})$ is invertible and $X_{I'}$ can be any $k \times (n-k)$-matrix, the map φ_I is surjective. It is also injective because $X_{I'}X_I^{-1} = Y_{I'}Y_I^{-1}$ implies that $Y_{I'} = X_{I'}X_I^{-1}Y_I$, but we also have $Y_I = X_I X_I^{-1} Y_I$, which leads to $Y = XX_I^{-1}Y_I$, and hence to $S(X) = S(Y)$. This shows that φ_I is bijective. It is continuous because $\varphi_I \circ S$ is continuous (as a consequence of the definition of the quotient topology; Exercise I.22). We have

$$\varphi_I^{-1}(Z) = S(\widetilde{Z}),$$

where $\widetilde{Z} \in M_{n,k}(\mathbb{R})$ is the unique matrix with $\widetilde{Z}_I = E_k$ (the identity matrix) and $\widetilde{Z}_{I'} = Z$. Since S is continuous, it follows that φ_I^{-1} is also continuous.

For two k-element subsets I, J we have

$$\varphi_J \circ \varphi_I^{-1}(Z) = \varphi_J(S(\widetilde{Z})) = \widetilde{Z}_{J'}\widetilde{Z}_J^{-1},$$

which defines a smooth map on $\varphi_I(U_I \cap U_J)$. Therefore the charts (φ_I, U_I) form a smooth atlas on $\mathrm{Gr}_{k,n}$, turning it into a smooth manifold of dimension $k(n-k)$, called the *Graßmannian* or *Graßmann manifold of k-dimensional subspaces of* $\mathbb{R}^n$. ■

Examples I.2.11. (Projective space) As an important special case of the preceding example, we obtain the *real projective space*

$$\mathbb{P}(\mathbb{R}^n) := \mathrm{Gr}_1(\mathbb{R}^n)$$

of all 1-dimensional subspaces of $\mathbb{R}^n$. It is a smooth manifold of dimension $n-1$. For $n = 2$, this space is called the *projective line* and for $n = 3$ it is called the *projective plane* (it is a 2-dimensional manifold, thus also called a *surface*).

Let us write $[x] := [x_1, \ldots, x_n] := \mathbb{R}x$ for elements of $\mathbb{P}_n(\mathbb{R})$. We use the terminology from Example I.2.10. Then

$$U_i = \{[x] \colon x_i \neq 0\}$$

and the chart $\varphi_i \colon U_i \to M_{n-1,1} \cong \mathbb{R}^{n-1}$ is given by

$$\varphi_i([x]) = (x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i).$$

These charts are called *homogeneous coordinates.* They play a fundamental role in projective geometry.

In this case $S_{1,n} = \mathbb{R}^n \setminus \{0\}$ and $S(x) = \mathbb{R}x$. The set $F_{1,n}$ of orthonormal bases of 1-dimensional subspaces of $\mathbb{R}^n$ can be identified with $\mathbb{S}^n$, and we have $\mathrm{O}_1(\mathbb{R}) = \{\pm\mathbf{1}\}$. This leads to the quotient map

$$q \colon \mathbb{S}^n = F_{1,n} \to \mathrm{Gr}_{1,n} = \mathbb{P}(\mathbb{R}^n), \quad q(x) = \mathbb{R}x$$

with the fibers $\{\pm x\}$. Therefore $\mathbb{P}(\mathbb{R}^n)$ is obtained from the sphere by identifying antipodal points. ■

Remark I.2.12. (The gluing picture) Let M be an n-dimensional manifold with an atlas $\mathcal{A} = (\varphi_i, U_i)_{i\in A}$ and $V_i := \varphi_i(U_i)$ the corresponding open subsets of $\mathbb{R}^n$.

Note that we have used the topology of M to define the notion of a chart. We now explain how the topological space M can be reconstructed from the atlas $\mathcal{A}$. We first consider the set

$$S := \bigcup_{i\in I} \{i\} \times V_i,$$

which we consider as the disjoint union of the open subset $V_i \subseteq \mathbb{R}^n$. We endow S with the topology of the disjoint sum, i.e., a subset $O \subseteq S$ is open if and only if all its intersections with the subsets $\{i\} \times V_i \cong V_i$ are open.

We now consider the surjective map

$$\Phi: S \to M, \quad (i,x) \mapsto \varphi_i^{-1}(x).$$

On each subset $\{i\} \times V_i$ this map is a homeomorphism onto U_i. Hence Φ is continuous, surjective and open, which means that it is a quotient map, i.e., that the topology on M coincides with the quotient topology on $S/\sim$, where

$$(i,x) \sim (j,y) \quad \Longleftrightarrow \quad \varphi_i(\varphi_j^{-1}(y)) = x.$$

In this sense we can think of M as obtained by gluing of the patches $U_i \cong V_i$, where $x_i \in \varphi_i(U_{ij}) \subseteq V_i$ is identified with the point $x_j = \varphi_j(\varphi_i^{-1}(x_i)) \in V_j$. ■

Example I.2.13. (a) We discuss an example of a "non-Hausdorff manifold". We endow the set $S := (\{1\}\times\mathbb{R}) \cup (\{2\}\times\mathbb{R})$ with the disjoint sum topology and define an equivalence relation on S by

$$(1,x) \sim (2,y) \quad \Longleftrightarrow \quad x = y \neq 0,$$

so that all classes except $[1,0]$ and $[2,0]$ contain 2 points. The topological quotient space

$$M := S/\sim = \{[1,x]: x \in \mathbb{R}\} \cup \{[2,0]\} = \{[2,x]: x \in \mathbb{R}\} \cup \{[1,0]\}$$

is the union of a real line with an extra point, but the two points $[1,0]$ and $[2,0]$ have no disjoint open neighborhoods.

The subsets $U_j := \{[j,x]: x \in \mathbb{R}\}$, $j = 1,2$, of M are open, and the maps

$$\varphi_j: U_j \to \mathbb{R}, [j,x] \mapsto x,$$

are homeomorphism defining a smooth atlas on M (Exercise I.16).

(b) We discuss an example of a 1-dimensional smooth manifold whose underlying set is $\mathbb{R}^2$. Let $M := \mathbb{R}^2$ and for $y \in \mathbb{R}$ we put $U_y := \mathbb{R} \times \{y\}$ (the horizontal lines). Then M is a disjoint union of the subsets U_y, and we define a topology on M by declaring $O \subseteq M$ to be open if and only if all intersections $O \cap U_y$ are open, when considered as subsets of $\mathbb{R}$. Then the map $\varphi_y: U_y \to \mathbb{R}, (x,y) \mapsto x$ form a smooth 1-dimensional atlas of M.

Since all subsets U_y are open and connected, these sets form the connected components of M. As there are uncountably many, the topology of M does not have a countable basis. ■

Example I.2.14. We discuss an example of a manifold which is not separable. We define a new topology on $\mathbb{R}^2$ by defining $O \subseteq \mathbb{R}^2$ to be open if and only if for each $y \in \mathbb{R}$ the set

$$O^y := \{x \in \mathbb{R} \colon (x,y) \in O\}$$

is open in $\mathbb{R}$. This defines a topology on $\mathbb{R}^2$ for which all sets $U_y := \mathbb{R} \times \{y\}$ are open and the maps

$$\varphi_y \colon U_y \to \mathbb{R}, \quad (x,y) \mapsto x$$

are homeomorphisms. We this obtain a smooth 1-dimensional manifold structure on $\mathbb{R}^2$ for which is has unbountably many connected components, namely the subsets U_y, $y \in \mathbb{R}$. ■

Remark I.2.15. (Coordinates versus parameterizations)

(a) Let (φ, U) be an n-dimensional chart of the smooth manifold M. Then $\varphi \colon U \to \mathbb{R}^n$ has n components $\varphi_1, \ldots, \varphi_n$ which we consider as coordinate functions on U. Sometimes it is convenient to write $x_i := \varphi_i(p)$ for $p \in U$, so that $(x_1(p), \ldots, x_n(p))$ are the coordinates of $p \in U$ w.r.t. the chart (φ, U).

If we have another chart (ψ, V) of M with $U \cap V \neq \emptyset$, then any $p \in U \cap V$ has a second tuple of coordinates, $x_i'(p) := \psi_i(p)$, given by the components of ψ. Now the change of coordinates is given by

$$x'(x) = \psi(\varphi^{-1}(x)) \quad \text{and} \quad x(x') = \varphi(\psi^{-1}(x')).$$

In this sense the maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ describe how we translate between the x-coordinates and the x'-coordinates.

(b) Instead of putting the focus on coordinates, which are functions on open subsets of the manifold, one can also parameterize open subset of M. This is done by maps $\varphi \colon V \to M$, where V is an open subset of some $\mathbb{R}^n$ and $(\varphi^{-1}, \varphi(V))$ is a chart of M. Then the point $p \in M$ corresponding to the parameter values $(x_1, \ldots, x_n) \in V$ is $p = \varphi(x)$. In this picture the lines

$$t \mapsto \varphi(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$$

are curves on M, called the parameter lines. ■

New manifolds from old ones

Definition I.2.16. (Open subsets are manifolds) Let M be a smooth manifold and $N \subseteq M$ an open subset. Then N carries a natural smooth manifold structure.

Let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an atlas of M. Then $V_i := U_i \cap N$ and $\psi_i := \varphi_i|_{V_i}$ define a smooth atlas $\mathcal{B} := (\psi_i, V_i)_{i \in I}$ of N (Exercise). ■

Definition I.2.17. (Products of manifolds) Let M and N be smooth manifolds of dimensions d, resp., k and

$$M \times N = \{(m,n) : m \in M, n \in N\}$$

the product set, which we endow with the product topology.

We show that $M \times N$ carries a natural structure of a smooth $(d+k)$-dimensional manifold. Let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an atlas of M and $\mathcal{B} = (\psi_j, V_j)_{j \in J}$ an atlas of N. Then the product sets $W_{ij} := U_i \times V_j$ are open in $M \times N$ and the maps

$$\gamma_{ij} := \varphi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^d \times \mathbb{R}^k \cong \mathbb{R}^{d+k}, \quad (x,y) \mapsto (\varphi_i(x), \psi_j(y))$$

are homeomorphisms onto open subsets of $\mathbb{R}^{d+k}$. On $\gamma_{i'j'}(W_{ij} \cap W_{i'j'})$ we have

$$\gamma_{ij} \circ \gamma_{i'j'}^{-1} = (\varphi_i \circ \varphi_{i'}^{-1}) \times (\psi_j \circ \psi_{j'}^{-1}),$$

which is a smooth map. Therefore $(\varphi_{ij}, W_{ij})_{(i,j) \in I \times J}$ is a smooth atlas on $M \times N$. ∎

I.3. Smooth maps

Definition I.3.1. Let M and N be smooth manifolds.

(a) A continuous map $f : M \to N$ is called *smooth*, if for each chart (φ, U) of M and each chart (ψ, V) of N the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is smooth. Note that $\varphi(f^{-1}(V) \cap U)$ is open because f is continuous.

We write $C^\infty(M,N)$ for the set of smooth maps $M \to N$.

(b) A map $f : M \to N$ is called a *diffeomorphism*, or a *smooth isomorphism*, if there exists a smooth map $g : N \to M$ with

$$f \circ g = \mathrm{id}_N \quad \text{and} \quad g \circ f = \mathrm{id}_M .$$

This condition obviously is equivalent to f being bijective and its inverse f^{-1} being a smooth map.

We write $\mathrm{Diff}(M)$ for the set of all diffeomorphisms of M. ∎

Lemma I.3.2. *Compositions of smooth maps are smooth. In particular, the set* $\mathrm{Diff}(M)$ *is a group (with respect to composition) for each smooth manifold* M.*

* For each manifold M the identity $\mathrm{id}_M : M \to M$ is a smooth map, so that this lemma leads to the "category of smooth manifolds". The objects of this category are smooth manifolds and the morphisms are the smooth maps. In the following we shall use consistently category theoretical language, but we won't go into the formal details of category theory.

Proof. Let $f: M \to N$ and $g: N \to L$ be smooth maps. Pick charts (φ, U) of M and (γ, W) of L. To see that the map $\gamma \circ (g \circ f) \circ \varphi^{-1}$ is smooth on $\varphi((g \circ f)^{-1}(W))$, we have to show that each element $x = \varphi(p)$ in this set has a neighborhood on which it is smooth. Let $q := f(p)$ and note that $g(q) \in W$. We choose a chart (ψ, V) of N with $q \in V$. We then have

$$\gamma \circ (g \circ f) \circ \varphi^{-1} = (\gamma \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1})$$

on the open neighborhood $\varphi\big(f^{-1}(V) \cap (g \circ f)^{-1}(W)\big)$ of x. Since compositions of smooth maps on open domains in $\mathbb{R}^n$ are smooth by the Chain Rule (Theorem I.1.3), $\gamma \circ (g \circ f) \circ \varphi^{-1}$ is smooth on $\varphi((g \circ f)^{-1}(W))$. This proves that $g \circ f: M \to L$ is a smooth map. ■

Remark I.3.3. (a) If $I \subseteq \mathbb{R}$ is an open interval, then a smooth map $\gamma: I \to M$ is called a *smooth curve*.

For a not necessarily open interval $I \subseteq \mathbb{R}$, a map $\gamma: I \to \mathbb{R}^n$ is called smooth if all derivatives $\gamma^{(k)}$ exist in all points of I and define continuous functions $I \to \mathbb{R}^n$. Based on this generalization of smoothness for curves, a curve $\gamma: I \to M$ is said to be smooth, if for each chart (φ, U) of M the curves

$$\varphi \circ \gamma: \gamma^{-1}(U) \to \mathbb{R}^n$$

are smooth.

A curve $\gamma: [a, b] \to M$ is called *piecewise smooth* if γ is continuous and there exists a subdivision $x_0 = a < x_1 < \ldots, < x_N = b$ such that $\gamma|_{[x_i, x_{i+1}]}$ is smooth for $i = 0, \ldots, N-1$.

(b) Smoothness of maps $f: M \to \mathbb{R}^n$ can be checked more easily. Since the identity is a chart of $\mathbb{R}^n$, the smoothness condition simply means that for each chart (φ, U) of M the map

$$f \circ \varphi^{-1}: \varphi(f^{-1}(V) \cap U) \to \mathbb{R}^n$$

is smooth.

(c) If U is an open subset of $\mathbb{R}^n$, then a map $f: U \to M$ to a smooth m-dimensional manifold M is smooth if and only if for each chart (φ, V) of M the map

$$\varphi \circ f: f^{-1}(V) \to \mathbb{R}^n$$

is smooth.

(d) Any chart (φ, U) of a smooth n-dimensional manifold M defines a diffeomorphism $U \to \varphi(U) \subseteq \mathbb{R}^n$, when U is endowed with the canonical manifold structure as an open subset of M.

In fact, by definition, we may use (φ, U) as an atlas of U. Then the smoothness of φ is equivalent to the smoothness of the map $\varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$, which is trivial. Likewise, the smoothness of $\varphi^{-1}: \varphi(U) \to U$ is equivalent to the smoothness of $\varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$. ■

Lie groups

In the context of smooth manifolds, the natural class of groups are those endowed with a manifold structure compatible with the group structure.

Definition I.3.4. A *Lie group* is a group G, endowed with the structure of a smooth manifold, such that the group operations

$$m_G: G \times G \to G, \quad (x,y) \mapsto xy \quad \text{and} \quad \eta_G: G \to G, \quad x \mapsto x^{-1}$$

are smooth. ■

We shall see later that the requirement of η_G being smooth is redundant (Exercise II.12).

Examples I.3.5. (a) We consider the additive group $G := (\mathbb{R}^n, +)$, endowed with the natural n-dimensional manifold structure. A corresponding chart is given by $(\mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n)$, which shows that the corresponding product manifold structure on $\mathbb{R}^n \times \mathbb{R}^n$ is given by the chart $(\mathrm{id}_{\mathbb{R}^n} \times \mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n \times \mathbb{R}^n) = (\mathrm{id}_{\mathbb{R}^{2n}}, \mathbb{R}^{2n})$, hence coincides with the natural manifold structure on $\mathbb{R}^{2n}$. Therefore the smoothness of addition and inversion in G follows from the smoothness of the maps

$$\mathbb{R}^{2n} \to \mathbb{R}^n, \quad (x,y) \mapsto x+y \quad \text{and} \quad \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto -x.$$

(b) Next we consider the group $G := \mathrm{GL}_n(\mathbb{R})$ of invertible $(n\times n)$-matrices. If $\det: M_n(\mathbb{R}) \to \mathbb{R}$ denotes the determinant function

$$A = (a_{ij}) \mapsto \det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

then det is a polynomial, hence in particular continuous, and therefore $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R}^\times)$ is an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Hence G carries a natural manifold structure.

We claim that G is a Lie group. The smoothness of the multiplication map follows directly from the smoothness of the bilinear multiplication map

$$M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad (A,B) \mapsto \Big(\sum_{k=1}^n a_{ik} b_{kj}\Big)_{i,j=1,\ldots,n},$$

which is given in each component by a polynomial function in the $2n^2$ variables a_{ij} and b_{ij} (cf. Exercise I.13).

The smoothness of the inversion map follows from Cramer's Rule

$$g^{-1} = \frac{1}{\det g}(b_{ij}), \quad b_{ij} = (-1)^{i+j} \det(G_{ji}),$$

where $G_{ij} \in M_{n-1}(\mathbb{R})$ is the matrix obtained by erasing the i-th row and the j-th column in g.

(c) The circle group: We have already seen how to endow the circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

with a manifold structure (Example I.2.4). Identifying it with the unit circle

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

in $\mathbb{C} \cong \mathbb{R}^2$, it also inherits a group structure, given by

$$(x, y) \cdot (x', y') := (xx' - yy', xy' + x'y) \quad \text{and} \quad (x, y)^{-1} = (x, -y).$$

With these explicit formulas, it is easy to verify that $\mathbb{T}$ is a Lie group (Exercise I.6).

(d) (The n-dimensional torus) We have already seen how to endow spheres with a manifold structure. Therefore we do already have a natural manifold structure on the *n-dimensional torus*

$$\mathbb{T}^n := (\mathbb{S}^1)^n.$$

The corresponding direct product group structure

$$(t_1, \ldots, t_n)(s_1, \ldots, s_n) := (t_1 s_1, \ldots, t_n s_n)$$

turns $\mathbb{T}^n$ into a Lie group (Exercise I.9). ∎

Notes on Chapter I

The notion of a smooth manifold is more subtle than one may think on the surface. One of these subtleties arises from the fact that a topological space may carry different smooth manifold structures which are not diffeomorphic. Important examples of low dimension are the 7-sphere $\mathbb{S}^7$ and $\mathbb{R}^4$. Actually 4 is the only dimension n for which $\mathbb{R}^n$ carries two non-diffeomorphic smooth structures. At this point it is instructive to observe that two smooth structures might be diffeomorphic without having the same maximal atlas: The charts $(\varphi, \mathbb{R})$ and $(\psi, \mathbb{R})$ on $\mathbb{R}$ given by

$$\varphi(x) = x \quad \text{and} \quad \psi(x) = x^3$$

define two different smooth manifold structures $\mathbb{R}_\varphi$ and $\mathbb{R}_\psi$, but the map

$$\gamma : \mathbb{R}_\psi \to \mathbb{R}_\varphi, \quad x \mapsto x^3$$

is a diffeomorphism.

Later we shall see that there are also purely topological subtleties due to the fact that the topology might be "too large". The regularity assumption which is needed in many situations is the paracompactness of the underlying Hausdorff space.

Exercises for Chapter I

Exercise I.1. Let $M := \mathbb{R}$, endowed with its standard topology. Show that C^k-compatibility of 1-dimensional charts is not an equivalence relation. Hint: Consider the two charts from Remark I.2.2(b) and the chart (ζ, W) with $\zeta(x) = x$ and $W =]1, 2[$. ■

Exercise I.2. Show that each n-dimensional C^k-atlas is contained in a unique maximal one. Hint: Add all n-dimensional charts which are C^k-compatible with the atlas. ■

Exercise I.3. Let If M_i, $i = 1, \ldots, n$, be smooth manifolds of dimension d_i. Show that the product space $M := M_1 \times \ldots \times M_n$ carries the structure of a $(d_1 + \ldots + d_n)$-dimensional manifold. ■

Exercise I.4. (Relaxation of the smoothness definition) Let M and N be smooth manifolds. Show that a map $f\colon M \to N$ is smooth if and only if for each point $x \in M$ there exists a chart (φ, U) of M with $x \in U$ and a chart (ψ, V) of N with $f(x) \in V$ such that the map

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(f^{-1}(V)) \to \psi(V)$$

is smooth. ■

Exercise I.5. Show that the set $A := C^\infty(M, \mathbb{R})$ of smooth real-valued functions on M is a real algebra. If $g \in A$ is non-zero and $U := g^{-1}(\mathbb{R}^\times)$, then $\frac{1}{g} \in C^\infty(U, \mathbb{R})$. ■

Exercise I.6. Show that the natural group structure on $\mathbb{T} \cong \mathbb{S}^1$ turns it into a Lie group. ■

Exercise I.7. Let $f_1\colon M_1 \to N_1$ and $f_2\colon M_2 \to N_2$ be smooth maps. Show that the map

$$f_1 \times f_2\colon M_1 \times M_2 \to N_1 \times N_2, \quad (x, y) \mapsto (f_1(x), f_2(y))$$

is smooth. ■

Exercise I.8. Let $f_1\colon M \to N_1$ and $f_2\colon M \to N_2$ be smooth maps. Show that the map

$$(f_1, f_2)\colon M \to N_1 \times N_2, \quad x \mapsto (f_1(x), f_2(x))$$

is smooth. ■

Exercise I.9. Let $G_1, \ldots, G_n$ be Lie groups and

$$G := G_1 \times \ldots \times G_n,$$

endowed with the direct product group structure

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') := (g_1 g_1', \ldots, g_n g_n')$$

and the product manifold structure. Show that G is a Lie group. ■

Exercise I.10. (a) Verify the details in Example I.5, where we describe an atlas of $\mathbb{S}^n$ by stereographic projections.

(b) Show that the two atlasses of $\mathbb{S}^n$ constructed in Example I.5 and the atlas obtained from the realization of $\mathbb{S}^n$ as a quadric in $\mathbb{R}^{n+1}$ define the same differentiable structure. ■

Exercise I.11. Let N be an open subset of the smooth manifold M. Show that if $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ is a smooth atlas of M, $V_i := U_i \cap N$ and $\psi_i := \varphi_i|_{V_i}$, then $\mathcal{B} := (\psi_i, V_i)_{i \in I}$ is a smooth atlas of N. ■

Exercise I.12. Smoothness is a local property: Show that a map $f\colon M \to N$ between smooth manifolds is smooth if and only if for each $p \in M$ there is an open neighborhood U such that $f|_U$ is smooth. ■

Exercise I.13. Let $d_1, \ldots, d_k \in \mathbb{N}$ and

$$\beta\colon \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k} \to \mathbb{R}^d$$

be a k-linear map. Show that β is smooth with

$$d\beta(x_1, \ldots, x_k)(h_1, \ldots, h_k) = \sum_{j=1}^k \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_k). \qquad ■$$

Exercise I.14. Let $V_1, \ldots, V_k$ and V be finite-dimensional real vector space and

$$\beta\colon V_1 \times \ldots \times V_k \to V$$

be a k-linear map. Show that β is smooth with

$$d\beta(x_1, \ldots, x_k)(h_1, \ldots, h_k) = \sum_{j=1}^k \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_k). \qquad ■$$

Exercise I.15. Let V and W be finite-dimensional real vector spaces and $\beta\colon V \times V \to W$ a bilinear map. Show that $G := W \times V$ is a Lie group with respect to

$$(w, v)(w', v') := (w + w' + \beta(v, v'), v + v'). \qquad ■$$

Exercise I.16. Show that the space M defined in Example I.2.13 is not Hausdorff, but that the two maps $\varphi_j([j,x]) := x$, $j = 1,2$, define a smooth atlas of M. ■

Exercise I.17. A map $f\colon X \to Y$ between topological spaces is called a *quotient map* if a subset $O \subseteq Y$ is open if and only if $f^{-1}(O)$ is open. Show that:

(1) If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are open quotient maps, then the cartesian product

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2, \quad (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2)$$

is a quotient map.

(2) If $f\colon X \to Y$ is a quotient map and we define on X an equivalence relation by $x \sim y$ if $f(x) = f(y)$, then the map $\overline{f}\colon X/\sim \to Y$ is a homeomorphism if $X/\sim$ is endowed with the qotient topology.

(3) The map $q\colon \mathbb{R}^n \to \mathbb{T}^n, \quad x \mapsto (e^{2\pi i x_j})_{j=1,\ldots,n}$ is a quotient map.

(4) The map $\overline{q}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n, \quad [x] \mapsto (e^{2\pi i x_j})_{j=1,\ldots,n}$ is a homeomorphism. ■

Exercise I.18. Let M be a compact smooth manifold containing at least two points. Then each atlas of M contains at least two charts. In particular the atlas of $\mathbb{S}^n$ obtained from stereographic projections is minimal. ■

Exercise I.19. (a) Let $\mathbb{K}$ be a field and $V = (v_1, \ldots, v_k), W = (w_1, \ldots, w_k) \in (\mathbb{K}^n)^k \cong M_{n,k}(\mathbb{K})$ be two linearly independent k-tuples of elements of the vector space $\mathbb{K}^n$. Show that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(w_1, \ldots, w_k)$$

is equivalent to the existence of some $g \in \mathrm{GL}_k(\mathbb{K})$ with $W = Vg$ in $M_{n,k}(\mathbb{K})$.

(b) If $\mathbb{K} = \mathbb{R}$ and V and W are orthonormal k-tuples, then

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(w_1, \ldots, w_k)$$

is equivalent to the existence of some $g \in \mathrm{O}_k(\mathbb{R})$ with $W = Vg$ in $M_{n,k}(\mathbb{R})$. ■

Exercise I.20. (The Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$) Let

$$T_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$$

denote the subgroup of upper triangular matrices with positive diagonal entries. Show that the multiplication map

$$\mu\colon \mathrm{O}_n(\mathbb{R}) \times T_n^+(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (a,b) \mapsto ab$$

is a homeomorphism. Hint: Interprete invertible $(n \times n)$-matrices as bases of $\mathbb{R}^n$. Use the Gram–Schmidt algorithm to see that μ is surjective and that is has a continuous inverse map. ■

Exercise I.21. Let (M,d) be a metric space.
(a) For each non-empty subset $X \subseteq M$ the function

$$d_X: M \to \mathbb{R}, \quad z \mapsto \inf\{d(z,x): x \in X\}$$

is a continuous function satisfying $\overline{X} = \{z \in M: d_X(z) = 0\}$. The function d_X measures the distance from the subset X.
(b) For two subsets $X, Y \subseteq M$ we define their distance by

$$\mathrm{dist}(X,Y) := \inf\{d(x,y): x \in X, y \in Y\} = \inf\{d_X(y): y \in Y\}.$$

Show that if Y is compact and X closed with $X \cap Y = \emptyset$, then $\mathrm{dist}(X,Y) > 0$. ■

Exercise I.22. Let X and Y be topological spaces and $q: X \to Y$ a quotient map, i.e., q is surjective and $O \subseteq Y$ is open if and only if $q^{-1}(O)$ is open in X. Show that a map $f: Y \to Z$ (Z a topological space) is continuous if and only if the map $f \circ q: X \to Z$ is continuous. ■

Exercise I.23. Let M and B be smooth manifolds. A smooth map $\pi: M \to B$ is said to defined a *(locally trivial) fiber bundle with typical fiber F over the base manifold B* if each $b_0 \in B$ has an open neighborhood U for which there exists a diffeomorphism

$$\varphi: \pi^{-1}(U) \to U \times F,$$

satisfying $\mathrm{pr}_U \circ \varphi = \pi$, where $\mathrm{pr}_U: U \times F \to U, (u,f) \mapsto u$ is the projection onto the first factor. Then the pair (φ, U) is called a *bundle chart.*

Show that:
(1) If $(\varphi, U), (\psi, V)$ are bundle charts, then

$$\varphi \circ \psi^{-1}(b,f) = (b, g_{\varphi\psi}(b)(f))$$

holds for a function $g_{\varphi\psi}: U \cap V \to \mathrm{Diff}(F)$.
(2) If (γ, W) is another bundle chart, then

$$g_{\varphi\varphi} = \mathrm{id}_F \quad \text{and} \quad g_{\varphi\psi} g_{\psi\gamma} = g_{\varphi\gamma} \quad \text{on} \quad U \cap V \cap W.$$

Exercise I.24. Show that a function $f: \mathbb{R} \to \mathbb{R}$ is a diffeomorphism if and only if either
(1) $f' > 0$ and $\lim_{x\to\pm\infty} f(x) = \pm\infty$.
(2) $f' < 0$ and $\lim_{x\to\pm\infty} f(x) = \mp\infty$. ■

Exercise I.25. Let $B \in \mathrm{GL}_n(\mathbb{R})$ be an invertible matrix which is symmetric or skew-symmetric. Show that:
(1) $G := \{g \in \mathrm{GL}_n(\mathbb{R}): g^\top B g = B\}$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.
(2) If $B = B^\top$, then B is a regular value of the smooth function

$$f: M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R}), \quad x \mapsto x^\top B x.$$

Hint: $df(x)y = y^\top Bx + x^\top By = z$ can be solved with the Ansatz $y := \frac{1}{2}xB^{-1}z$.

(3) If $B = -B^\top$, then B is a regular value of the smooth function

$$f\colon M_n(\mathbb{R}) \to \mathrm{Skew}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R})\colon A^\top = -A\}, \quad x \mapsto x^\top Bx.$$

(4) G is a submanifold of $M_n(\mathbb{R})$.

(5) For $B = I_{p,q} := \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$ the *indefinite orthogonal group*

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R})\colon g^\top I_{p,q} g = I_{p,q}\}$$

is a submanifold on $M_n(\mathbb{R})$ of dimension $\frac{n(n-1)}{2}$.

(6) For $J = \begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix} \in M_2(M_n(\mathbb{R})) \cong M_{2n}(\mathbb{R})$ the *symplectic group*

$$\mathrm{Sp}_{2n}(\mathbb{R}) := \{g \in \mathrm{GL}_{2n}(\mathbb{R})\colon g^\top Jg = J\}$$

is a submanifold on $M_{2n}(\mathbb{R})$ of dimension $n(2n+1)$. ∎

II. Tangent vectors and tangent maps

The real strength of the theory of smooth manifolds is due to the fact that it permits to analyze differentiable structures in terms of their derivatives. To model these derivatives appropriately, we introduce the tangent bundle TM of a smooth manifold, tangent maps of smooth maps and smooth vector fields.

II.1. Tangent vectors and tangent bundle of a manifold

To understand the idea behind tangent vectors and the tangent bundle of a manifold, we first take a closer look at the special case of an open subset $U \subseteq \mathbb{R}^n$.

Remark II.1.1. (a) We think of a *tangent vector in* $p \in U$ as a vector $v \in \mathbb{R}^n$ attached to the point p. In particular, we distinguish tangent vectors attached to different points. A good way to visualize this is to think of v as an arrow starting in p. In this sense we write

$$T_p(U) := \{p\} \times \mathbb{R}^n$$

for the set of all tangent vectors in p, the *tangent space in* p. It carries a natural real vector space structure, given by

$$(p,v) + (p,w) := (p, v+w) \quad \text{and} \quad \lambda(p,v) := (p, \lambda v)$$

for $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

The collection of all tangent vectors, the *tangent bundle*, is denoted

$$T(U) := \bigcup_{p \in U} T_p(U) = \{(p,v) \colon p \in U, v \in \mathbb{R}^n\} \cong U \times \mathbb{R}^n.$$

(b) If $f \colon U \to V$ is a C^1-map between open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, then the directional derivatives permit us to "extend" f to tangent vectors by its *tangent map*

$$T(f) \colon T(U) \cong U \times \mathbb{R}^n \to TV \cong V \times \mathbb{R}^m, \quad (p,v) \mapsto (f(p), df(p)v).$$

For each $p \in U$ the map Tf restricts to a linear map

$$T_p(f) \colon T_p(U) \to T_{f(p)}(V), \quad (p,v) \mapsto (f(p), df(p)v). \tag{2.1.1}$$

The main difference to the map df is the book keeping; here we keep track of what happens to the point p and the tangent vector v.

If $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^d$ are open subsets and $f: U \to V$ and $g: V \to W$ are C^1-maps, then the Chain Rule implies that

$$\begin{aligned} T(g \circ f)(p, v) &= (g(f(p)), d(g \circ f)(p)v) = (g(f(p)), dg(f(p))df(p)v) \\ &= T_{f(p)}(g) \circ T_p(f)v, \end{aligned}$$

so that, in terms of tangent maps, the Chain Rule takes the simple form

$$T(g \circ f) = T(g) \circ T(f). \tag{2.1.2}$$

We clearly have $T(\mathrm{id}_U) = \mathrm{id}_{T(U)}$ and for a diffeomorphism f we thus obtain from the Chain Rule $T(f^{-1}) = T(f)^{-1}$.

(c) As the terminology suggests, tangent vectors arise as tangent vectors of curves. If $\gamma:]a, b[\to U$ is a C^1-curve, then its tangent map satisfies

$$T(\gamma)(t, v) = (\gamma(t), d\gamma(t)v) = (\gamma(t), v \cdot \gamma'(t)) \quad \text{on} \quad T(]a, b[) \cong]a, b[\times \mathbb{R},$$

and in particular

$$T(\gamma)(t, 1) = (\gamma(t), \gamma'(t))$$

is the *tangent vector in* $\gamma(t)$ to the curve γ. ∎

We now turn to the definition of a tangent vector of a smooth manifold. The subtle point of this definition is that tangent vectors can only be defined indirectly. The most direct way is to describe them in terms of tangent vector in local charts and then identify those tangent vectors in the charts which define the same tangent vector of M.

Definition II.1.2. Let M be an n-dimensional C^1-manifold.

For each chart (φ, U) of M, we write elements of the tangent bundle $T(\varphi(U))$ as pairs $(x, v)_{(\varphi, U)}$ to keep track of the chart, so that two tangent vectors corresponding to different charts can never be equal.

We now introduce an equivalence relation on the disjoint union of all tangent bundles corresponding to charts by

$$\begin{aligned} (x, v)_{(\varphi, U)} \sim (y, w)_{(\psi, V)} \quad &:\Leftrightarrow \quad y = \psi(\varphi^{-1}(x)), \quad w = d(\psi \circ \varphi^{-1})(x)v \\ &\Leftrightarrow \quad (y, w) = T(\psi \circ \varphi^{-1})(x, v). \end{aligned}$$

That this is indeed an equivalence relation follows from the Chain Rule for Tangent maps. In fact, the symmetry follows from

$$\begin{aligned} (x, v)_{(\varphi, U)} \sim (y, w)_{(\psi, V)} &\Leftrightarrow \quad (y, w) = T(\psi \circ \varphi^{-1})(x, v) \\ &\Leftrightarrow \quad (x, v) = T(\psi \circ \varphi^{-1})^{-1}(y, w) \\ &\Leftrightarrow \quad (x, v) = T(\varphi \circ \psi^{-1})(y, w) \\ &\Leftrightarrow \quad (y, w)_{(\psi, V)} \sim (x, v)_{(\varphi, U)} \end{aligned}$$

and the transitivity from

$$\begin{aligned}
& (x,v)_{(\varphi,U)} \sim (y,w)_{(\psi,V)}, \quad (y,w)_{(\psi,V)} \sim (z,u)_{(\gamma,W)} \\
\Rightarrow \quad & (y,w) = T(\psi\circ\varphi^{-1})(x,v), \quad (z,u) = T(\gamma\circ\psi^{-1})(y,w) \\
\Rightarrow \quad & (z,u) = T(\gamma\circ\psi^{-1})T(\psi\circ\varphi^{-1})(x,v) = T(\gamma\circ\psi^{-1}\circ\psi\circ\varphi^{-1})(x,v) \\
& \quad = T(\gamma\circ\varphi^{-1})(x,v) \\
\Rightarrow & (x,v)_{(\varphi,U)} \sim (z,u)_{(\gamma,W)}.
\end{aligned}$$

An equivalence class $[(x,v)_{(\varphi,U)}]$ is called a *tangent vector in* $p \in M$ if $\varphi(p) = x$. We write $T_p(M)$ for the set of all tangent vectors in p, called the *tangent space of* M *in* p.

Fix a point $p \in M$. To get a better picture of the tangent space in p, we fix a chart (φ, U) with $p \in U$ and put $x := \varphi(p)$. Then each element of $T_p(M)$ can be written in a unique fashion as $[(x,v)_{(\varphi,U)}]$ for some $v \in \mathbb{R}^n$.

For $v_1, v_2, w_1, w_2 \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and any other chart (ψ, V) of M with $\psi(p) = y$ we have

$$\begin{aligned}
& (x,v_1)_{(\varphi,U)} \sim (y,w_1)_{(\psi,V)}, \quad (x,v_2)_{(\varphi,U)} \sim (y,w_2)_{(\psi,V)} \\
\Rightarrow \quad & (x, v_1 + \lambda v_2)_{(\varphi,U)} \sim (y, w_1 + \lambda w_2)_{(\psi,V)}.
\end{aligned}$$

We thus obtain on $T_p(M)$ a well-defined vector space structure by

$$[(x,v_1)_{(\varphi,U)}] + [(x,v_2)_{(\varphi,U)}] := [(x, v_1 + v_2)_{(\varphi,U)}]$$

and

$$\lambda[(x,v)_{(\varphi,U)}] := [(x, \lambda v)_{(\varphi,U)}]$$

(Exercise II.4). It is immediately clear from this definition, that the map

$$\mathbb{R}^n \to T_p(M), \quad v \mapsto [(x,v)_{(\varphi,U)}]$$

is an isomorphism of real vector spaces.

The disjoint union $TM := \dot{\bigcup}_{p\in M} T_p(M)$ is called the *tangent bundle* of M and the map $\pi_M \colon TM \to M$ mapping each element of $T_p(M)$ to p is called the *canonical projection of the tangent bundle.* ■

Remark II.1.3. If U is an open subset of $\mathbb{R}^n$ and $\iota_U \colon U \to \mathbb{R}^n$ is the inclusion map, then we can use the chart (ι_U, U) to describe tangent vectors to U. The map

$$T(U) \to U \times \mathbb{R}^n, \quad [(x,v)_{(\iota_U,U)}] \to (x,v)$$

is bijective, and we recover the picture from Remark II.1.1. In the following we shall always identify TU with $U \times \mathbb{R}^n$, which means that we simply write (x,v) instead of $[(x,v)_{(\iota_U,U)}]$. ■

We shall see later how to introduce a manifold structure on $T(M)$. First we extend the concept of a tangent map to the manifold level.

Definition II.1.4. (Tangent maps) Let M and N be smooth manifolds and $f: M \to N$ a smooth map. We want to define a tangent map $Tf: TM \to TN$ in such a way that it restricts for each $p \in M$ to a linear map

$$T_p(f): T_p(M) \to T_{f(p)}(N)$$

which in the special case of open subsets of $\mathbb{R}^n$ coincides with the map in (2.1.1).

This is done as follows. For $p \in M$ we choose a chart (φ, U) of M with $p \in U$ and a chart (ψ, V) of N with $f(p) \in V$. For $[(x,v)_{(\varphi,U)}] = [(y,v')_{(\varphi',U')}] \in T_p(M)$ we then have

$$T(\psi \circ f \circ \varphi^{-1})(x,v) = T(\psi \circ f \circ (\varphi')^{-1})(y,v')$$

because

$$(y,v') = T(\varphi' \circ \varphi^{-1})(x,v).$$

We therefore obtain a well-defined linear map

$$T_p(f): T_p(M) \to T_{f(p)}(N), \quad [(x,v)_{(\varphi,U)}] \mapsto \big[\big(T(\psi \circ f \circ \varphi^{-1})(x,v)\big)_{(\psi,V)}\big].$$

This map is called the *tangent map* of f in p. We also combine all these tangent maps $T_p(f)$ to a map

$$T(f): T(M) \to T(N) \quad \text{with} \quad T(f)|_{T_p(M)} = T_p(f), \quad p \in M. \qquad \blacksquare$$

Remark II.1.5. (a) For the special case where $f = \varphi: U \to \varphi(U) \subseteq \mathbb{R}^n$ is a chart of M, the definition of the tangent map leads to

$$T_p(\varphi)[(x,v)_{(\varphi,U)}] = [(x,v)_{(\iota_{\varphi(U)},U)}] = (x,v),$$

where the last equality is our identification of $T(U)$ with $U \times \mathbb{R}^n$ (Remark II.1.3). In particular, we see that the maps

$$T_p(\varphi): T_p(M) \to \mathbb{R}^n, \quad p \in U,$$

are linear isomorphisms. From now on we shall never need the clumsy notation $[(x,v)_{(\varphi,U)}]$ for tangent vectors because we can use the maps $T_p(\varphi)$ instead.

We may use this observation to write for a smooth map $f: M \to N$ the tangent map $T_p(f): T_p(M) \to T_{f(p)}(M)$ in the form

$$T_p(f) = T_{f(p)}(\psi)^{-1} \circ T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) \circ T_p(\varphi), \tag{2.1.3}$$

which is a direct reformulation of the definition.

(b) Suppose that M and N are finite-dimensional real vector spaces of dimensions m and n, and $\kappa_B: M \to \mathbb{R}^n$, $\kappa_C: N \to \mathbb{R}^m$ are the corresponding coordinate maps obtained from choosing bases B in M and C in N. Then any linear map $f: M \to N$ is a smooth map, and

$$\kappa_C \circ f \circ \kappa_B^{-1}: \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto [f]^C_B \cdot x$$

is the linear map given by the matrix $[f]^C_B$ describing f with respect to the bases B and C. The formula

$$f = \kappa_C^{-1} \circ (\kappa_C \circ f \circ \kappa_B^{-1}) \circ \kappa_B$$

reconstructs the map f from the corresponding matrix (compare with (2.1.3). $\blacksquare$

Definition II.1.6. If V is a finite-dimensional vector space, then we identify its tangent bundle TV with the product set $V \times V$, where the canonical projection $\pi_V: TV \to V$ corresponds to the projection onto the first factor.* For each smooth V-valued map $f: M \to V$, the tangent map

$$Tf: TM \to TV \cong V \times V$$

now has two components.

For the special case where M is an open subset of $\mathbb{R}^n$, we have $Tf(x,v) = (f(x), df(x)v)$, which motivates the notation df for the second component of Tf. In this sense we have

$$Tf(v) = (f(x), df(x)v) \quad \text{for} \quad v \in T_x(M). \qquad \blacksquare$$

Lemma II.1.7. (Chain Rule for tangent maps) *For smooth maps $f: M \to N$ and $g: N \to L$ we have*

$$T(g \circ f) = T(g) \circ T(f).$$

Proof. We recall from Lemma I.3.2 that $g \circ f: M \to L$ is a smooth map, so that $T(g \circ f)$ is defined.

Let $p \in M$ and pick charts (φ, U) of M with $p \in U$, (ψ, V) of N with $f(p) \in V$ and (γ, W) of L with $g(f(p)) \in W$. We then obtain with Remarks II.1.1 and II.1.5

$$\begin{aligned} T_p(g \circ f) &= T_{g(f(p))}(\gamma)^{-1} T_{\varphi(p)}(\gamma \circ g \circ f \circ \varphi^{-1}) T_p(\varphi) \\ &\overset{(2.1.2)}{=} T_{g(f(p))}(\gamma)^{-1} T_{\psi(f(p))}(\gamma \circ g \circ \psi^{-1}) T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) T_p(\varphi) \\ &\overset{(2.1.3)}{=} T_{f(p)}(g) T_{f(p)}(\psi)^{-1} T_{f(p)}(\psi) T_p(f) T_p(\varphi)^{-1} T_p(\varphi) \\ &= T_{f(p)}(g) T_p(f). \end{aligned}$$

Since p was arbitrary, this implies the lemma. $\blacksquare$

So far we only considered the tangent bundle $T(M)$ of a smooth manifold M as a set, but this set also carries a natural topology and a smooth manifold structure.

Definition II.1.8. (Manifold structure on $T(M)$) Let M be a smooth manifold. First we introduce a topology on $T(M)$.

For each chart (φ, U) of M we have a tangent map

$$T(\varphi): T(U) \to T(\varphi(U)) \cong \varphi(U) \times \mathbb{R}^n,$$

where we consider $T(U) = \bigcup_{p \in U} T_p(M)$ as a subset of $T(M)$. We define a topology on $T(M)$ by declaring a subset $O \subseteq T(M)$ to be open if for each

* In fact, if $\varphi: V \to \mathbb{R}^n$ is a linear isomorphism, then its tangent map $T\varphi: TV \to T(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$ also is a linear isomorphism.

chart (φ, U) of M the set $T(\varphi)(O \cap T(U))$ is an open subset of $T(\varphi(U))$. It is easy to see that this defines indeed a Hausdorff topology on $T(M)$ for which all the subsets $T(U)$ are open and the maps $T(\varphi)$ are homeomorphisms onto open subsets of $\mathbb{R}^{2n}$ (Exercise II.1).

Since for two charts (φ, U), (ψ, V) of M the map

$$T(\varphi \circ \psi^{-1}) = T(\varphi) \circ T(\psi)^{-1} \colon T(\psi(V)) \to T(\varphi(U))$$

is smooth, for each atlas $\mathcal{A}$ of M the collection $(T(\varphi), T(U))_{(\varphi,U)\in\mathcal{A}}$ is a smooth atlas of $T(M)$ defining on $T(M)$ the structure of a smooth manifold. ■

Lemma II.1.9. *If $f \colon M \to N$ is a smooth map, then its tangent map $T(f)$ is smooth.*

Proof. Let $p \in M$ and choose charts (φ, U) and (ψ, V) of M, resp., N with $p \in U$ and $f(p) \in V$. Then the map

$$T(\psi) \circ T(f) \circ T(\varphi)^{-1} = T(\psi \circ f \circ \varphi^{-1}) \colon T(\varphi(f^{-1}(V) \cap U)) \to T(V)$$

is smooth, and this implies that $T(f)$ is a smooth map. ■

Remark II.1.10. For smooth manifolds $M_1, \ldots, M_n$, the projection maps

$$\pi_i \colon M_1 \times \cdots \times M_n \to M_i, \quad (p_1, \ldots, p_n) \mapsto p_i$$

induce a diffeomorphism

$$(T(\pi_1), \ldots, T(\pi_n)) \colon T(M_1 \times \cdots \times M_n) \to TM_1 \times \cdots \times TM_n$$

(Exercise II.5). ■

II.2. Submanifolds

Definition II.2.1. (Submanifolds) (a) Let M be a smooth n-dimensional manifold. A subset $S \subseteq M$ is called a *d-dimensional submanifold* if for each $p \in M$ there exists a chart (φ, U) of M with $p \in U$ such that

$$(SM) \qquad \varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}).$$

A submanifold of codimension 1, i.e., $\dim S = n - 1$, is called a *smooth hypersurface.*

(b) As we shall see in the following lemma, the concept of an initial submanifold is weaker: An *initial submanifold of M* is a subset $S \subseteq M$, endowed with a smooth manifold structure, such that the inclusion map $i_S \colon S \to M$ is smooth, and, moreover, for each other smooth manifold N a map $f \colon N \to S$ is smooth if and only if $i_S \circ f \colon N \to M$ is smooth. The latter condition means that a map into S is smooth if and only if it is smooth, when considered as a map into M. ■

Remark II.2.2. (a) Any discrete subset S of M is a 0-dimensional submanifold.

(b) If $n = \dim M$, any open subset $S \subseteq M$ is an n-dimensional submanifold. If, conversely, $S \subseteq M$ is an n-dimensional submanifold, then the definition immediately shows that S is open. ■

Lemma II.2.3. *Any submanifold S of a manifold M has a natural manifold structure, turning it into an initial submanifold.*

Proof. (a) We endow S with the subspace topology inherited from M, which turns it into a Hausdorff space. For each chart (φ, U) satisfying (SM), we obtain a d-dimensional chart

$$(\varphi|_{U\cap S}, U \cap S)$$

of S. For two such charts coming from (φ, U) and (ψ, V), we have

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap S)} = (\psi|_{V\cap S}) \circ (\varphi|_{U\cap S})^{-1}|_{\varphi(U\cap V\cap S)},$$

which is a smooth map onto an open subset of $\mathbb{R}^d$. We thus obtain a smooth atlas on S.

(b) To see that i_S is smooth, let $p \in S$ and (φ, U) a chart satisfying (SM). Then

$$\varphi \circ i_S \circ (\varphi|_{S\cap U})^{-1} \colon \varphi(U) \cap (\mathbb{R}^d \times \{0\}) \to \varphi(U) \subseteq \mathbb{R}^n$$

is the inclusion map, hence smooth. This implies that i_S is smooth.

(c) If $f \colon N \to S$ is smooth, then the composition $i_S \circ f$ is smooth (Lemma I.3.2). Suppose, conversely, that $i_S \circ f$ is smooth. Let $p \in N$ and choose a chart (φ, U) of M satisfying (SM) with $f(p) \in U$. Then the map

$$\varphi \circ i_S \circ f|_{f^{-1}(U)} \colon f^{-1}(U) \to \varphi(U) \subseteq \mathbb{R}^n$$

is smooth, but its values lie in

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}).$$

Therefore $\varphi \circ i_S \circ f|_{f^{-1}(U)}$ is also smooth as a map into $\mathbb{R}^d$, which means that

$$\varphi|_{U\cap S} \circ f|_{f^{-1}(U)} \colon f^{-1}(U) \to \varphi(U \cap S) \subseteq \mathbb{R}^d$$

is smooth, and hence that f is smooth as a map $N \to S$. ■

Remark II.2.4. (Tangent spaces of submanifolds) From the construction of the manifold structure on S, it follows that for each $p \in S$ and each chart (φ, U) satisfying (SM), we may identify the tangent space $T_p(S)$ with the subspace $T_p(\varphi)^{-1}(\mathbb{R}^d)$ mapped by $T_p(\varphi)$ onto the subspace $\mathbb{R}^d$ of $\mathbb{R}^n$. ■

Lemma II.2.5. *A subset S of a smooth manifold M carries at most one initial submanifold structure, i.e., for any two smooth manifolds structures S_1, S_2 on the same set S the map $\varphi: S_1 \to S_2$ is a diffeomorphism.*

Proof. Since $i_{S_2} \circ \varphi = i_{S_1}: S_1 \to M$ is smooth, the map φ is smooth. Likewise, we see that the inverse map $\varphi^{-1}: S_2 \to S_1$ is smooth, showing that φ is a diffeomorphism $S_1 \to S_2$. ∎

We are now ready to prove a manifold version of the fact that inverse images of regular values are submanifolds.

Theorem II.2.6. (Regular Value Theorem) *Let M and N be smooth manifolds of dimension n, resp., m, $f: M \to N$ a smooth map, and $y \in f(M)$ such that the linear map $T_x(f)$ is surjective for each $x \in f^{-1}(y)$. Then $S := f^{-1}(y)$ is an $(n-m)$-dimensional submanifold of M.*

In the preceding situation, y is called a *regular value* of the function f, otherwise it is called a *singular value.*

Proof. We note that $d := n - m > 0$ because $T_x(f): T_x(M) \cong \mathbb{R}^n \to \mathbb{R}^m \cong T_{f(x)}(N)$ is surjective for some $x \in M$.

Let $p \in S$ and choose charts (φ, U) of M with $p \in U$ and (ψ, V) of N with $f(p) \in V$. Then the map

$$F := \psi \circ f \circ \varphi^{-1}: \varphi(U \cap f^{-1}(V)) \to \mathbb{R}^m$$

is a smooth map, and for each $x \in F^{-1}(\psi(y)) = \varphi(S \cap U)$ the linear map

$$T_x(F) = T_{f(x)}(\psi) \circ T_x(f) \circ T_{\varphi(x)}(\varphi^{-1}): \mathbb{R}^n \to \mathbb{R}^m$$

is surjective. Therefore Proposition I.2.8 implies the existence of an open subset $U' \subseteq \varphi(U \cap f^{-1}(V))$ containing $\varphi(p)$ and a diffeomorphism $\gamma: U' \to \gamma(U')$ with

$$\gamma(U' \cap \varphi(U \cap S)) = (\mathbb{R}^d \times \{0\}) \cap \gamma(U').$$

Then $(\gamma \circ \varphi, \varphi^{-1}(U'))$ is a chart of M with

$$(\gamma \circ \varphi)(S \cap \varphi^{-1}(U')) = \gamma(\varphi(S \cap U) \cap U') = (\mathbb{R}^d \times \{0\}) \cap \gamma(U').$$

This shows that S is a d-dimensional submanifold of M. ∎

Remark II.2.7. If $S \subseteq M$ is a submanifold, then we may identify the tangent spaces $T_p(S)$ with the subspaces $\mathrm{im}(T_p(i_S))$ of $T_p(M)$, where $i_S: S \to M$ is the smooth inclusion map (cf. Remark II.2.4). If, in addition, $S = f^{-1}(y)$ for some regular value y of the smooth map $f: M \to N$, then we have

$$T_p(S) = \ker T_p(f) \quad \text{for} \quad p \in S.$$

To verify this relation, we recall that we know already that $\dim S = n - m = \dim T_p(S)$. On the other hand, $f \circ i_S = y: S \to N$ is the constant map, so that $T_p(f \circ i_S) = T_p(f) \circ T_p(i_S) = 0$, which leads to $T_p(S) \subseteq \ker T_p(f)$. Since $T_p(f)$ is surjective by assumption, equalitiy follows by comparing dimensions. ∎

II.3. Vector fields

Definition II.3.1. (a) A *(smooth) vector field* X on M is a smooth section of the tangent bundle $\pi_M: TM \to M$, i.e., a smooth map $X: M \to TM$ with $\pi_M \circ X = \mathrm{id}_M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M.

(b) If $f \in C^\infty(M, V)$ is a smooth function on M with values in some finite-dimensional vector space V and $X \in \mathcal{V}(M)$, then we obtain a smooth function on M via

$$X.f := df \circ X: M \to TM \to V.$$

(cf. Definition II.1.6). ■

Remark II.3.2. (a) If U is an open subset of $\mathbb{R}^n$, then $TU = U \times \mathbb{R}^n$ with the bundle projection

$$\pi_U: U \times \mathbb{R}^n \to U, \quad (x, v) \mapsto x.$$

Therefore each smooth vector field is of the form $X(x) = (x, \widetilde{X}(x))$ for some smooth function $\widetilde{X}: U \to \mathbb{R}^n$, and we may thus identify $\mathcal{V}(U)$ with the space $C^\infty(U, \mathbb{R}^n)$ of smooth $\mathbb{R}^n$-valued functions on U.

(b) The space $\mathcal{V}(M)$ carries a natural vector space structure given by

$$(X + Y)(p) := X(p) + Y(p), \quad (\lambda X)(p) := \lambda X(p)$$

(Exercise II.2).

More generally, we can multiply vector fields with smooth functions

$$(fX)(p) := f(p)X(p), \quad f \in C^\infty(M, \mathbb{R}), X \in \mathcal{V}(M).$$ ■

The Lie bracket of vector fields

Before we turn to the Lie bracket on the space $\mathcal{V}(M)$ of smooth vector fields on a manifold M, we take a closer look at the underlying concepts.

Definition II.3.3. Let L be a real vector space L. A map $[\cdot,\cdot]: L \times L \to L$ is called a *Lie bracket* if

(L1) $[\cdot,\cdot]$ is *bilinear*, i.e., linear in each argument separately.

(L2) $[x, x] = 0$ for $x \in L$ (the bracket is alternating).

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x, y, z \in L$ (Jacobi identity).

A vector space L, endowed with a Lie bracket $[\cdot,\cdot]$, is called a *Lie algebra*. ■

Lemma II.3.4. *Let $U \subseteq \mathbb{R}^n$ be an open subset. Then we obtain a Lie bracket on the space $C^\infty(U, \mathbb{R}^n)$ by*

$$[f, g](x) := dg(x)f(x) - df(x)g(x) \quad \textit{for} \quad x \in U.$$

Proof. (L1) and (L2) are obvious from the definition.

To verify the Jacobi identity, we first observe that for a function of the form $F(x) := df(x)g(x)$, we have

$$dF(x)v = (d^2 f)(x)(g(x), v) + df(x)dg(x)(v) \tag{2.2.1}$$

(cf. Definition I.1.10 for $d^2 f$). In fact, to derive (2.2.1), we write the map F as the composition $F = \beta \circ (df, g)$, where

$$\beta\colon \mathrm{End}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n, \quad (\varphi, v) \mapsto \varphi(v)$$

is the evaluation map which is bilinear, hence smooth with

$$d\beta(\varphi, v)(\psi, w) = \varphi(w) + \psi(v)$$

(Exercise I.14). We further have

$$(d(df))(x)w = (d^2 f)(x)(w, \cdot) = \frac{d}{dt}\Big|_{t=0} df(x + tw, \cdot),$$

so that the Chain Rule leads to

$$\begin{aligned} dF(x)v &= d\beta(df(x), g(x))(d(df)(x)v, dg(x)v) \\ &= \beta(df(x), dg(x)v) + \beta((d^2 f)(x)(v, \cdot), g(x)) \\ &= df(x)dg(x)v + (d^2 f)(x)(v, g(x)) = (d^2 f)(x)(g(x), v) + df(x)dg(x)v. \end{aligned}$$

We write (2.2.1) symbolically as

$$dF = (d^2 f)(g, \cdot) + df.dg.$$

We now calculate

$$\begin{aligned} [f, [g, h]] &= d[g, h].f - df.[g, h] \\ &= d(dh.g - dg.h).f - df.(dh.g - dg.h) \\ &= d^2h.(g, f) + dh.dg.f - d^2g.(h, f) - dg.dh.f \\ &\quad - df.dh.g + df.dg.h \\ &= d^2h.(g, f) - d^2g.(h, f) + dh.dg.f - dg.dh.f \\ &\quad - df.dh.g + df.dg.h. \end{aligned}$$

Summing all cyclic permutations and using the symmetry of second derivatives, which implies $d^2 f.(g, h) = d^2 f.(h, g)$, we see that all terms cancel. ∎

Remark II.3.5. For any open subset $U \subseteq \mathbb{R}^n$, the map

$$\mathcal{V}(U) \to C^\infty(U, \mathbb{R}^n), \quad X \to \widetilde{X}$$

with $X(x) = (x, \widetilde{X}(x))$ is a linear isomorphism. We use this map to transfer the Lie bracket on $C^\infty(U, \mathbb{R}^n)$, defined in Lemma II.3.4, to a Lie bracket on $\mathcal{V}(U)$, determined by

$$[X,Y]\widetilde{\ }(x) = [\widetilde{X}, \widetilde{Y}](x) = d\widetilde{Y}(x)\widetilde{X}(x) - d\widetilde{X}(x)\widetilde{Y}(x). \qquad \blacksquare$$

Our goal is to use the Lie brackets on the space $\mathcal{V}(U)$ and local charts to define a Lie bracket on $\mathcal{V}(M)$. The following lemma will be needed to ensure consistency in this process:

Lemma II.3.6. *Let $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ be open subsets. Further let $\varphi\colon M \to N$ be a smooth map. Suppose that the function $X_N, Y_N \in C^\infty(N, \mathbb{R}^m)$ and $X_M, Y_M \in C^\infty(M, \mathbb{R}^n)$ are φ-related in the sense that $X_N \circ \varphi = d\varphi.X_M$ and $Y_N \circ \varphi = d\varphi.Y_M$. Then the Lie brackets are also φ-related:*

$$[X_N, Y_N] \circ \varphi = d\varphi.[X_M, Y_M].$$

Proof. Taking derivatives of $Y_N \circ \varphi = d\varphi.Y_M$ in the direction of $X_M(p)$, we obtain with the Chain Rule and (2.2.1):

$$dY_N\big(\varphi(p)\big)d\varphi(p)X_M(p) = (d^2\varphi)(p)(Y_M(p), X_M(p)) + d\varphi(p)dY_M(p)X_M(p),$$

which leads to

$$\begin{aligned} dY_N\big(\varphi(p)\big)X_N\big(\varphi(p)\big) &= dY_N\big(\varphi(p)\big)d\varphi(p)X_M(p) \\ &= (d^2\varphi)(p)(Y_M(p), X_M(p)) + d\varphi(p)dY_M(p)X_M(p), \end{aligned}$$

and hence, with the symmetry of the second derivatives, to

$$\begin{aligned} [X_N, Y_N](\varphi(p)) &= dY_N\big(\varphi(p)\big)X_N\big(\varphi(p)\big) - dX_N\big(\varphi(p)\big)Y_N\big(\varphi(p)\big) \\ &= (d^2\varphi)(p)(Y_M(p), X_M(p)) - (d^2\varphi)(p)(X_M(p), Y_M(p)) \\ &\quad + d\varphi(p)dY_M(p)X_M(p) - d\varphi(p)dX_M(p)Y_M(p) \\ &= d\varphi(p)dY_M(p)X_M(p) - d\varphi(p)dX_M(p)Y_M(p) \\ &= d\varphi(p)[X_M, Y_M](p). \end{aligned}$$

$\blacksquare$

Proposition II.3.7. *For a vector field $X \in \mathcal{V}(M)$ and a chart (φ, U) of M, we write $X_\varphi := T\varphi \circ X \circ \varphi^{-1}$ for the corresponding vector field on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.*

For $X, Y \in \mathcal{V}(M)$, there exists a vector field $[X, Y] \in \mathcal{V}(M)$ which is uniquely determined by the property that for each chart (φ, U) of M we have

$$[X, Y]_\varphi = [X_\varphi, Y_\varphi]. \tag{2.2.2}$$

Proof. If (φ, U) and (ψ, V) are charts of M, the vector fields X_φ on $\varphi(U)$ and X_ψ on $\psi(V)$ are related by

$$T(\psi \circ \varphi^{-1}) \circ X_\varphi = X_\psi \circ \psi \circ \varphi^{-1},$$

which is equivalent to

$$d(\psi \circ \varphi^{-1}) \circ \widetilde{X}_\varphi = \widetilde{X}_\psi \circ \psi \circ \varphi^{-1},$$

for the corresponding $\mathbb{R}^n$-valued smooth functions. Therefore Lemma II.3.6 implies that for $X, Y \in \mathcal{V}(M)$ we have

$$T(\psi \circ \varphi^{-1}) \circ [X_\varphi, Y_\varphi] = [X_\psi, Y_\psi] \circ \psi \circ \varphi^{-1},$$

but this relation is equivalent to

$$T(\varphi)^{-1} \circ [X_\varphi, Y_\varphi] \circ \varphi = T(\psi)^{-1} \circ [X_\psi, Y_\psi] \circ \psi,$$

which is an identity of vector fields on the open subset $U \cap V$.

Hence there exists a unique vector field $[X, Y] \in \mathcal{V}(M)$, satisfying

$$[X, Y]|_U = T(\varphi)^{-1} \circ [X_\varphi, Y_\varphi] \circ \varphi$$

for each chart (φ, U), i.e., $[X, Y]_\varphi = [X_\varphi, Y_\varphi]$ on $\varphi(U)$. ∎

Proposition II.3.8. $(\mathcal{V}(M), [\cdot, \cdot])$ *is a Lie algebra.*

Proof. Clearly (L1) and (L2) are satisfied. To verify the Jacobi identity, let $X, Y, Z \in \mathcal{V}(M)$ and (φ, U) be a chart of M. For the vector field $J(X, Y, Z) := \sum_{\text{cyc.}} [X, [Y, Z]] \in \mathcal{V}(M)$ we then obtain from the definition of the bracket and Remark II.3.5:

$$J(X, Y, Z)_\varphi = J(X_\varphi, Y_\varphi, Z_\varphi) = 0$$

because $[\cdot, \cdot]$ is a Lie bracket on $\mathcal{V}(\varphi(U))$. This means that $J(X, Y, Z)$ vanishes on U, but since the chart (φ, U) was arbitrary, $J(X, Y, Z) = 0$. ∎

We shall see later in this course that the following lemma is an extremely important tool. If $f \colon M \to N$ is a smooth map, then we call two vector fields $X_M \in \mathcal{V}(M)$ and $X_N \in \mathcal{V}(N)$ *f-related* if

$$X_N \circ f = Tf \circ X_M \colon M \to TN. \tag{2.2.3}$$

Lemma II.3.9. (Related Vector Field Lemma) *Let M and N be smooth manifolds, $f \colon M \to N$ a smooth map, $X_N, Y_N \in \mathcal{V}(N)$ and $X_M, Y_M \in \mathcal{V}(M)$.*

If X_M is f-related to X_N and Y_M is f-related to Y_N, then the Lie bracket $[X_M, Y_M]$ is f-related to $[X_N, Y_N]$.

Proof. We have to show that for each $p \in M$ we have

$$[X_N, Y_N](f(p)) = T_p(f)[X_M, Y_M](p).$$

Let (φ, U) be a chart of M with $p \in U$ and (ψ, V) a chart of N with $f(p) \in V$. Then the vectors fields $(X_M)_\varphi$ and $(X_N)_\psi$ are $\psi \circ f \circ \varphi^{-1}$-related on $\varphi(f^{-1}(V) \cap U)$:

$$\begin{aligned}
T(\psi \circ f \circ \varphi^{-1})(X_M)_\varphi &= T(\psi \circ f \circ \varphi^{-1})T(\varphi) \circ X_M \circ \varphi^{-1} \\
&= T(\psi) \circ T(f) \circ X_M \circ \varphi^{-1} \\
&= T(\psi) \circ X_N \circ f \circ \varphi^{-1} = (X_N)_\psi \circ (\psi \circ f \circ \varphi^{-1}),
\end{aligned}$$

and the same holds for the vector fields $(Y_M)_\varphi$ and $(Y_N)_\psi$, hence for their Lie brackets.

Now the definition of the Lie bracket on $\mathcal{V}(N)$ and $\mathcal{V}(M)$ implies that

$$\begin{aligned}
T_{f(p)}(\psi)T_p(f)[X_M, Y_M](p) &= T_p(\psi \circ f)[X_M, Y_M](p) \\
&= T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})T_p(\varphi)[X_M, Y_M](p) \\
&= T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})[(X_M)_\varphi, (Y_M)_\varphi](\varphi(p)) \\
&\overset{II.3.6}{=} [(X_N)_\psi, (Y_N)_\psi](\psi(f(p))) \\
&= T_{f(p)}(\psi)[X_N, Y_N](f(p)),
\end{aligned}$$

and since the linear map $T_{f(p)}(\psi)$ is injective, the assertion follows. ■

Notes on Chapter II

Vector fields and their zeros play an important role in the topology of manifolds. To each manifold M we assoicate the maximal number $\alpha(M) = k$ for which there exists smooth vector fields $X_1, \ldots, X_k \in \mathcal{V}(M)$ which are linearly independent in each point of M. A manifold is called *parallelizable* if $\alpha(M) = \dim M$ (which is the maximal value). Clearly $\alpha(\mathbb{R}^n) = n$, so that $\mathbb{R}^n$ is parallelizable, but it is a deep theorem that the the n-sphere $\mathbb{S}^n$ is only parallelizable if $n = 0, 1, 3$ or 7. This in turn has important applications on the existence of real division algebras, namely that they only exist in dimensions 1, 2, 4 or 8 (This is the famous $1-2-4-8$ *Theorem*). Another important result in topology is $\alpha(\mathbb{S}^2) = 0$, i.e., each vector field on the 2-sphere has a zero (Hairy Ball Theorem).

There is another approach to the Lie bracket of vector fields, based on the identification of $\mathcal{V}(M)$ with the space of derivations of the algebra $C^\infty(M, \mathbb{R})$. This requires localization arguments which in turn rest on the assumption that the underlying manifold is paracompact. Althought it requires less work, once the localization machinery is available, we think that it is more natural to see that the Lie bracket can be obtained on this early stage of the theory. Another advantage of the direct approach is that it also works for very general infinite-dimensional manifolds without any change.

We shall see later that the Lie bracket on the space $\mathcal{V}(M)$ on vector fields is closely related to the commutator in the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. This fact is part of a more general correspondence in the theory of Lie groups which associates to each Lie group G a Lie algebra $\mathbf{L}(G)$ given by a suitable bracket on the tangent space $T_{\mathbf{1}}(G)$ which is defined in terms of the Lie bracket of vector fields.

Exercises for Chapter II

Exercise II.1. Let M be a smooth manifold. We call a subset $O \subseteq T(M)$ open if for each chart (φ, U) of M the set $T(\varphi)(O \cap T(U))$ is an open subset of $T(\varphi(U))$. Show that:
(1) This defines a topology on $T(M)$.
(2) All subsets $T(U)$ are open.
(3) The maps $T(\varphi)$ are homeomorphisms onto open subsets of $\mathbb{R}^{2n} \cong T(\mathbb{R}^n)$.
(4) The projection $\pi_M : T(M) \to M$ is continuous.
(5) $T(M)$ is Hausdorff. Hint: Use (4) to separate points in different tangent spaces by disjoint open sets. ■

Exercise II.2. Let M be a smooth manifold. Show that

$$(X+Y)(p) := X(p) + Y(p) \quad (\lambda X)(p) := \lambda X(p), \quad \lambda \in \mathbb{R},$$

defines on $\mathcal{V}(M)$ the structure of a real vector space.

Show also that the multiplication with smooth functions defined by

$$(fX)(p) := f(p)X(p)$$

satisfies for $X, Y \in \mathcal{V}(M)$ and $f, g \in C^\infty(M, \mathbb{R})$:
(1) $f(X+Y) = fX + fY$.
(2) $f(\lambda X) = \lambda \cdot f(X) = (\lambda f)X$ for $\lambda \in \mathbb{R}$.
(3) $(f+g)X = fX + gX$.
(4) $f(gX) = (fg)(X)$. ■

Exercise II.3. Let M be a smooth manifold, $X, Y \in \mathcal{V}(M)$ and $f, g \in C^\infty(M, \mathbb{R})$. Show that
(1) $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$, i.e., the map $f \mapsto X(f)$ is a derivation.
(2) $(fX)(g) = f \cdot X(g)$.
(3) $[X, Y](f) = X(Y(f)) - Y(X(f))$. ■

Exercise II.4. Let M be a smooth n-dimensional manifold and $p \in M$. Show that the operations

$$[(x, v_1)_{(\varphi,U)}] + [(x, v_2)_{(\varphi,U)}] := [(x, v_1 + v_2)_{(\varphi,U)}]$$

and

$$\lambda[(x, v)_{(\varphi,U)}] := [(x, \lambda v)_{(\varphi,U)}]$$

on $T_p(M)$ are well-defined and define a vector space structure for which $T_p(M) \cong \mathbb{R}^n$. ■

Exercise II.5. For smooth manifolds $M_1, \ldots, M_n$, the projection maps

$$\pi_i \colon M_1 \times \cdots \times M_n \to M_i, \quad (p_1, \ldots, p_n) \mapsto p_i$$

induce a diffeomorphism

$$(T(\pi_1), \ldots, T(\pi_n)) \colon T(M_1 \times \cdots \times M_n) \to TM_1 \times \cdots \times TM_n.$$ ■

Exercise II.6. Let N and $M_1, \ldots, M_n$ be a smooth manifolds. Show that a map

$$f \colon N \to M_1 \times \cdots \times M_n$$

is smooth if and only if all its component functions $f_i \colon N \to M_i$ are smooth. ■

Exercise II.7. Let $(A, \cdot)$ be an associative algebra. Show that the commutator bracket

$$[a, b] := ab - ba$$

is a Lie bracket on A. ■

Exercise II.8. Let $(A, \cdot)$ be a, not necessarily associative, algebra, i.e., A is a vector space endowed with a bilinear map $A \times A \to A, (a, b) \mapsto ab$. We call a map $D \in \mathrm{End}(A)$ a *derivation* if

$$D(ab) = D(a)b + aD(b) \quad \text{for all} \quad a, b \in A.$$

Show that the set $\mathrm{der}(A)$ of derivations of A is a Lie subalgebra of $\mathrm{End}(A)$, where the latter is endowed with the commutator bracket (cf. Exercise II.7). ■

Exercise II.9. Let $f \colon M \to N$ be a smooth map between manifolds, $\pi_{TM} \colon TM \to M$ the tangent bundle projection and $\sigma_M \colon M \to TM$ the zero section. Show that for each smooth map $f \colon M \to N$ we have

$$\pi_{TN} \circ Tf = f \circ \pi_{TM} \quad \text{and} \quad \sigma_N \circ f = Tf \circ \sigma_M.$$ ■

Exercise II.10. Let M be a smooth manifold. Show that:

(a) For each vector field, the map $C^\infty(M, \mathbb{K}) \to C^\infty(M, \mathbb{K}), f \mapsto \mathcal{L}_X f := X.f$ is a derivation.

(b) The map $\mathcal{V}(M) \to \mathrm{der}(C^\infty(M, \mathbb{K})), X \mapsto \mathcal{L}_X$ from (a) is a homomorphism of Lie algebras.

(c) If M is an open subset of some $\mathbb{R}^n$, then the map $X \mapsto \mathcal{L}_X$ is injective. ■

Exercise II.11. (Inverse Function Theorem for manifolds) Let $f\colon M \to N$ be a smooth map and $p \in M$ such that $T_p(f)\colon T_p(M) \to T_p(N)$ is a linear isomorphism. Show that there exists an open neighborhood U of p in M such that the restriction $f\,|_U\colon U \to f(U)$ is a diffeomorphism onto an open subset of N. ■

Exercise II.12. (Automatic smoothness of the inversion) Let G be an n-dimensional smooth manifold, endowed with a group structure for which the multiplication map m_G is smooth. Show that:

(1) $T_{(g,h)}(m_G) = T_g(\rho_h) + T_h(\lambda_g)$ for $\lambda_g(x) = gx$ and $\rho_h(x) = xh$.

(2) $T_{(\mathbf{1},\mathbf{1})}(m_G)(v,w) = v + w$.

(3) The inverse map $\eta_G\colon G \to G, g \mapsto g^{-1}$ is smooth if it is smooth in a neighborhood of $\mathbf{1}$.

(4) The inverse map η_G is smooth. Hint: Apply the Inverse Function Theorem to the map

$$\Phi\colon G \times G \to G \times G, \quad (x,y) \mapsto (x, xy).$$ ■

III. Some Multilinear Algebra

In this chapter we lay the algebraic foundation for differential forms on open subsets on $\mathbb{R}^n$ and on smooth manifolds.

Throughout this section, we write $\mathbb{K}$ for a field. All vector spaces are vector spaces over $\mathbb{K}$.

III.1. Alternating maps

Identifying the space $M_n(\mathbb{K})$ of $(n \times n)$-matrices with entries in $\mathbb{K}$ with the space $(\mathbb{K}^n)^n$ of n-tuples of (column) vectors, the determinant function $\det\colon M_n(\mathbb{K}) \to \mathbb{K}$ can also be viewed as a map

$$\det\colon (\mathbb{K}^n)^n \to \mathbb{K}.$$

This is the prototype of an alternating n-linear map. In this section we deal with more general alternating maps, but we shall also see how general alternating maps can be expressed in terms of determinants.

Definition III.1.1. Let V and W be $\mathbb{K}$-vector spaces. A map $\omega\colon V^k \to W$ is called *alternating* if

(A1) (Multilinearity) ω is linear in each argument

$$\begin{aligned}&\omega(v_1, \ldots, v_{i-1}, \lambda v_i + \mu v_i', v_{i+1}, \ldots, v_k)\\ &= \lambda\omega(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) + \mu\omega(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_k).\end{aligned}$$

(A2) $\omega(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for some pair of indices $i \neq j$.

We write $\mathrm{Alt}^k(V, W)$ for the set of alternating k-linear maps $V^k \to W$. Clearly, sums and scalar multiples of alternating maps are alternating, so that $\mathrm{Alt}^k(V, W)$ carries a natural vector space structure. For $k = 0$ we shall follow the convention that $\mathrm{Alt}^0(V, W) := W$ is the set of constant maps, which are considered to be 0-linear. ■

Example III.1.2. (a) From Linear Algebra we know the k-linear map

$$\omega\colon (\mathbb{K}^k)^k \to \mathbb{K}, \quad \omega(v_1, \ldots, v_k) = \det(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_{1,\sigma(1)} \cdots v_{k,\sigma(k)}.$$

(b) If L is a $\mathbb{K}$-vector space, then any Lie bracket $[\cdot, \cdot]$ on L is an alternating bilinear map.

(c) If $V = \mathbb{K}^{2n}$, then

$$\beta(x, y) := \sum_{i=1}^n x_i y_{i+n} - x_{i+n} y_i$$

is an alternating bilinear map $V \times V \to \mathbb{K}$. ■

Remark III.1.3. Any alternating bilinear map $\beta: V \times V \to W$ is *skew-symmetric*, i.e.,

$$\beta(v, w) = -\beta(w, v).$$

In fact,

$$\beta(v, w) + \beta(w, v) = \beta(v, v) + \beta(v, w) + \beta(w, v) + \beta(w, w) = \beta(v + w, v + w) = 0.$$

If, conversely, β is skew-symmetric, then we obtain $\beta(v, v) = -\beta(v, v)$, and therefore $2\beta(v, v) = 0$. If $\operatorname{char} \mathbb{K} \neq 2$, this implies that β is alternating, but for $\operatorname{char} \mathbb{K} = 2$ there are skew-symmetric (=symmetric) maps which are not alternating.

In fact, if $\mathbb{K} = \mathbb{F}_2 = \{0, 1\}$ is the two element field, then the multiplication map $\beta: \mathbb{K} \times \mathbb{K} \to \mathbb{K}, (x, y) \mapsto xy$ is (skew-)symmetric but not alternating. ∎

Below we shall see how general alternating maps can be expressed in terms of determinants.

Proposition III.1.4. *For any $\omega \in \operatorname{Alt}^k(V, W)$ we have:*

(1) $\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega(v_1, \ldots, v_k)$ *for each permutation $\sigma \in S_k$, and if $\operatorname{char} \mathbb{K} \neq 2$, then any k-linear map with this property is alternating.*

(2) *For $b_1, \ldots, b_k \in V$ and linear combinations $v_j = \sum_{i=1}^k a_{ij} b_i$ we have*

$$\omega(v_1, \ldots, v_k) = \det(A)\omega(b_1, \ldots, b_k),$$

where $A := (a_{ij}) \in M_k(\mathbb{K})$.

(3) $\omega(v_1, \ldots, v_k) = 0$ *if $v_1, \ldots, v_k$ are linearly dependent.*

(4) *For $b_1, \ldots, b_n \in V$ and linear combinations $v_j = \sum_{i=1}^n a_{ij} b_i$ we have*

$$\omega(v_1, \ldots, v_k) = \sum_I \det(A_I)\omega(b_{i_1}, \ldots, b_{i_k}),$$

where $A := (a_{ij}) \in M_{n,k}(\mathbb{K})$, $I = \{i_1, \ldots, i_k\}$ is a k-element subset of $\{1, \ldots, n\}$, $1 \leq i_1 < \ldots < i_k \leq n$, and $A_I := (a_{ij})_{i \in I, j=1,\ldots,k} \in M_k(\mathbb{K})$.

Proof. (1) Let $\sigma := \tau = (i\ j)$ be the transposition of i and j. Fix $v_1, \ldots, v_k \in V$. Then we obtain an alternating bilinear map

$$\omega'(a, b) := \omega(v_1, \ldots, v_{i-1}, a, v_{i+1}, \ldots, v_{j-1}, b, v_{j+1}, \ldots, v_k).$$

Then ω' is skew-symmetric (Remark III.1.3), and this implies that (1) holds for $\sigma = \tau$.

We know from Linear Algebra that any permutation $\sigma \in S_k$ is a product $\tau_1 \cdots \tau_k$ of transpositions, where $\operatorname{sgn}(\sigma) = (-1)^k$. We now argue by induction. We have already verified the case $k = 1$. We may thus assume that the assertion holds for $\sigma' := \tau_2 \cdots \tau_k$. We then obtain

$$\begin{aligned}\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) &= \omega(v_{\tau_1\sigma'(1)}, \ldots, v_{\tau_1\sigma'(k)}) = -\omega(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)}) \\ &= -\operatorname{sgn}(\sigma')\omega(v_1, \ldots, v_k) = \operatorname{sgn}(\sigma)\omega(v_1, \ldots, v_k).\end{aligned}$$

Suppose, conversely, that $\omega: V^k \to W$ is a k-linear map satisfying (1). If $v_i = v_j$ for $i < j$ and $\tau = (i\ j)$, then (1) leads to

$$\omega(v_1, \ldots, v_k) = -\omega(v_{\tau(1)}, \ldots, v_{\tau(k)}) = -\omega(v_1, \ldots, v_k)$$

and hence to $2\omega(v_1, \ldots, v_k) = 0$. If $2 \in \mathbb{K}^\times$, i.e., $\operatorname{char} \mathbb{K} \neq 2$, ω satisfies (A2).

(2) For the following calculation we note that if $\sigma: \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a map which is not bijective, then (A2) implies that $\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0$. We therefore get with (1)

$$\begin{aligned}
\omega(v_1, \ldots, v_k) &= \omega\Big(\sum_{i=1}^k a_{i1} b_i, \ldots, \sum_{i=1}^k a_{ik} b_i\Big) \\
&= \sum_{i_1, \ldots, i_k = 1}^k a_{i_1 1} \cdots a_{i_k k} \cdot \omega(b_{i_1}, \ldots, b_{i_k}) \\
&= \sum_{\sigma \in S_k} a_{\sigma(1)1} \cdots a_{\sigma(k)k} \cdot \omega(b_{\sigma(1)}, \ldots, b_{\sigma(k)}) \\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(k)k} \cdot \omega(b_1, \ldots, b_k) \\
&= \det(A) \cdot \omega(b_1, \ldots, b_k).
\end{aligned}$$

(3) follows immediately from (2) because the linear dependence of $v_1, \ldots, v_k$ implies that $\det A = 0$.

(4) First we expand

$$\begin{aligned}
\omega(v_1, \ldots, v_k) &= \omega\Big(\sum_{i=1}^n a_{i1} b_i, \ldots, \sum_{i=1}^n a_{ik} b_i\Big) \\
&= \sum_{i_1, \ldots, i_k = 1}^n a_{i_1 1} \cdots a_{i_k k} \cdot \omega(b_{i_1}, \ldots, b_{i_k}).
\end{aligned}$$

If $|\{i_1, \ldots, i_k\}| < k$, then (A2) implies that $\omega(b_{i_1}, \ldots, b_{i_k}) = 0$ because two entries coincide. If $|\{i_1, \ldots, i_k\}| = k$, there exists a permutation $\sigma \in S_k$ with $i_{\sigma(1)} < \ldots < i_{\sigma(k)}$. We therefore get

$$\begin{aligned}
\omega(v_1, \ldots, v_k) &= \sum_{1 \le i_1 < \ldots < i_k \le n} \ \sum_{\sigma \in S_k} a_{i_{\sigma(1)} 1} \cdots a_{i_{\sigma(k)} k} \cdot \omega(b_{i_{\sigma(1)}}, \ldots, b_{i_{\sigma(k)}}) \\
&\overset{(1)}{=} \sum_{1 \le i_1 < \ldots < i_k \le n} \ \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a_{i_{\sigma(1)} 1} \cdots a_{i_{\sigma(k)} k} \cdot \omega(b_{i_1}, \ldots, b_{i_k}) \\
&= \sum_I \det(A_I) \omega(b_{i_1}, \ldots, b_{i_k}),
\end{aligned}$$

where the sum is to be extended over all k-element subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, where $i_1 < \ldots < i_k$. ■

Corollary III.1.5. (1) *If* $\dim V < k$, *then* $\mathrm{Alt}^k(V,W) = \{0\}$.
(2) *Let* $\dim V = n$ *and* $b_1, \ldots, b_n$ *be a basis of* V. *Then the map*

$$\Phi\colon \mathrm{Alt}^k(V,W) \to W^{\binom{n}{k}}, \quad \Phi(\omega) = (\omega(b_{i_1}, \ldots, b_{i_k}))_{i_1 < \ldots < i_k}$$

is a linear isomorphism. We obtain in particular $\dim\big(\mathrm{Alt}^k(V,\mathbb{K})\big) = \binom{n}{k}$.
(3) *If* $\dim V = k$ *and* $b_1, \ldots, b_k$ *is a basis of* V, *then the map*

$$\Phi\colon \mathrm{Alt}^k(V,W) \to W, \quad \Phi(\omega) = \omega(b_1, \ldots, b_k)$$

is a linear isomorphism.

Proof. (1) In Proposition III.1.4(2) we may choose $b_k = 0$.

(2) First we show that Φ is injective. So let $\omega \in \mathrm{Alt}^k(V,W)$ with $\Phi(\omega) = 0$. We now write any k elements $v_1, \ldots, v_k \in V$ with respect to the basis elements as $v_j = \sum_{i=1}^n a_{ij} b_i$ and obtain with Proposition III.1.4(4):

$$\omega(v_1, \ldots, v_k) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \det(A_I)\omega(b_{i_1}, \ldots, b_{i_k}) = 0.$$

To see that Φ is surjective, we pick for each k-element subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $1 \leq i_1 < \ldots < i_k \leq n$ an element $w_I \in W$. Then the tuple (w_I) is a typical element of $W^{\binom{n}{k}}$.

Expressing k elements $v_1, \ldots, v_k$ in terms of the basis elements $b_1, \ldots, b_n$ via $v_j = \sum_{i=1}^n a_{ij} b_i$, we obtain an $(n \times k)$-matrix A. We now define an alternating k-linear map $\omega \in \mathrm{Alt}^k(V,W)$ by

$$\omega(v_1, \ldots, v_k) := \sum_I \det(A_I) w_I.$$

The k-linearity of ω follows directly from the k-linearity of the maps

$$(v_1, \ldots, v_k) \mapsto \det(A_I).$$

For $i_1 < \ldots < i_k$ we further have $\omega(b_{i_1}, \ldots, b_{i_k}) = w_I$ because in this case $A_I \in M_k(\mathbb{K})$ is the identity matrix and all other matrices $A_{I'}$ have some vanishing columns. This implies that $\Phi(\omega) = (w_I)$, and hence that Φ is surjective.

(3) is a special case of (2). ■

III.2. The exterior product

In this subsection we assume that $\mathrm{char}(\mathbb{K}) = 0$.

Definition III.2.1. (The alternator) Let V and W be vector spaces. For a k-linear map $\omega\colon V^k \to W$ we define a new k-linear map by

$$\mathrm{Alt}(\omega)(v_1,\ldots,v_k) := \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\omega(v_{\sigma(1)},\ldots,v_{\sigma(k)}).$$

Writing

$$\omega^\sigma(v_1,\ldots,v_k) := \omega(v_{\sigma(1)},\ldots,v_{\sigma(k)}),$$

we then have

$$\mathrm{Alt}(\omega) = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\omega^\sigma.$$

The map $\mathrm{Alt}(\cdot)$ is called the *alternator*. We claim that it turns any k-linear map into an alternating k-linear map. To see this, we first note that for $\sigma,\pi\in S_k$ we have

$$\begin{aligned}(\omega^\sigma)^\pi(v_1,\ldots,v_k) &= (\omega^\sigma)(v_{\pi(1)},\ldots,v_{\pi(k)}) \\ &= \omega(v_{\pi\sigma(1)},\ldots,v_{\pi\sigma(k)}) = \omega^{\pi\sigma}(v_1,\ldots,v_k).\end{aligned} \tag{3.2.1}$$

This implies that

$$\begin{aligned}\mathrm{Alt}(\omega)^\pi &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)(\omega^\sigma)^\pi = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\omega^{\pi\sigma} = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\pi^{-1}\sigma)\omega^\sigma \\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma)\omega^\sigma = \mathrm{sgn}(\pi)\,\mathrm{Alt}(\omega) \\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma\pi^{-1})\omega^\sigma = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\omega^{\sigma\pi} \\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)(\omega^\pi)^\sigma = \mathrm{Alt}(\omega^\pi).\end{aligned}$$

Applying this argument to a transposition τ, it follows that $\mathrm{Alt}(\omega)$ is skew-symmetric in any pair of arguments (v_i,v_j), and since $\mathrm{char}\,\mathbb{K}\neq 2$, this implies that $\mathrm{Alt}(\omega)$ is alternating (Proposition III.1.4(i)). ■

Remark III.2.2. (a) We observe that if ω is alternating, then $\omega^\sigma = \mathrm{sgn}(\sigma)\omega$ for each permutation σ (Proposition III.1.4), and therefore

$$\mathrm{Alt}(\omega) = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma)\omega = \frac{1}{k!}\sum_{\sigma\in S_k}\omega = \omega.$$

(b) For $k=2$ we have

$$\mathrm{Alt}(\omega)(v_1,v_2) = \frac{1}{2}(\omega(v_1,v_2)-\omega(v_2,v_1))$$

and for $k=3$:

$$\begin{aligned}\mathrm{Alt}(\omega)(v_1,v_2,v_3) = \frac{1}{6}\Big(&\omega(v_1,v_2,v_3)-\omega(v_2,v_1,v_3)+\omega(v_2,v_3,v_1) \\ &-\omega(v_3,v_2,v_1)+\omega(v_3,v_1,v_2)-\omega(v_1,v_3,v_2)\Big).\end{aligned}$$

■

Definition III.2.3. Let $p, q \in \mathbb{N}_0$. For two multilinear maps

$$\omega_1 : V_1 \times \ldots \times V_p \to \mathbb{K}, \quad \omega_2 : V_{p+1} \times \ldots \times V_{p+q} \to \mathbb{K}$$

we define the *tensor product*

$$\omega_1 \otimes \omega_2 : V_1 \times \cdots \times V_{p+q} \to \mathbb{K}$$

by

$$(\omega_1 \otimes \omega_2)(v_1, \ldots, v_{p+q}) := \omega_1(v_1, \ldots, v_p)\omega_2(v_{p+1}, \ldots, v_{p+q}).$$

It is clear that $\omega_1 \otimes \omega_2$ is a $(p+q)$-linear map.

For $\lambda \in \mathbb{K}$ (the set of 0-linear maps) and a p-linear map ω as above, we obtain in particular

$$\lambda \otimes \omega := \omega \otimes \lambda := \lambda\omega.$$

For two alternating maps $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$ and $\beta \in \mathrm{Alt}^q(V, \mathbb{K})$ we define their *exterior product*:

$$\alpha \wedge \beta := \frac{(p+q)!}{p!q!} \mathrm{Alt}(\alpha \otimes \beta) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)(\alpha \otimes \beta)^\sigma.$$

We have seen in Definition III.2.1 that $\alpha \wedge \beta$ is alternating, so that we obtain a bilinear map

$$\wedge : \mathrm{Alt}^p(V, \mathbb{K}) \times \mathrm{Alt}^q(V, \mathbb{K}) \to \mathrm{Alt}^{p+q}(V, \mathbb{K}), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta.$$

On the direct sum

$$\mathrm{Alt}(V, \mathbb{K}) := \bigoplus_{p \in \mathbb{N}_0} \mathrm{Alt}^p(V, \mathbb{K})$$

we now obtain a bilinear product by putting

$$\Big(\sum_p \alpha_p\Big) \wedge \Big(\sum_q \beta_q\Big) := \sum_{p,q} \alpha_p \wedge \beta_q.$$

As before, we identify $\mathrm{Alt}^0(V, \mathbb{K})$ with $\mathbb{K}$ and obtain

$$\lambda\alpha = \lambda \wedge \alpha = \alpha \wedge \lambda$$

for $\lambda \in \mathrm{Alt}^0(V, \mathbb{K}) = \mathbb{K}$ and $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$.

The so obtained algebra $(\mathrm{Alt}(V, \mathbb{K}), \wedge)$ is called the *exterior algebra* or the *Graßmann algebra of the vector space* V. ■

We now take a closer look at the structure of the exterior algebra.

Lemma III.2.4. *The exterior algebra is associative, i.e., for* $\alpha \in \mathrm{Alt}^p(V,\mathbb{K})$, $\beta \in \mathrm{Alt}^q(V,\mathbb{K})$ *and* $\gamma \in \mathrm{Alt}^r(V,\mathbb{K})$ *we have*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

Proof. First we recall from Definition III.2.1 that for any n-linear map $\omega\colon V^n \to W$ and $\pi \in S_n$ we have

$$\mathrm{Alt}(\omega^\pi) = \mathrm{sgn}(\pi)\,\mathrm{Alt}(\omega).$$

We identify S_{p+q} in the natural way with the subgroup of S_{p+q+r} fixing the numbers $p+q+1,\ldots,p+q+r$. We thus obtain

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \frac{(p+q+r)!}{(p+q)!r!}\,\mathrm{Alt}((\alpha\wedge\beta)\otimes\gamma)\\
&= \frac{(p+q+r)!}{p!q!(p+q)!r!}\sum_{\sigma\in S_{p+q}}\mathrm{sgn}(\sigma)\,\mathrm{Alt}((\alpha\otimes\beta)^\sigma\otimes\gamma)\\
&= \frac{(p+q+r)!}{p!q!(p+q)!r!}\sum_{\sigma\in S_{p+q}}\mathrm{sgn}(\sigma)\,\mathrm{Alt}((\alpha\otimes\beta\otimes\gamma)^\sigma)\\
&\overset{III.2.1}{=} \frac{(p+q+r)!}{p!q!(p+q)!r!}\sum_{\sigma\in S_{p+q}}\mathrm{Alt}(\alpha\otimes\beta\otimes\gamma)\\
&= \frac{(p+q+r)!}{p!q!r!}\,\mathrm{Alt}(\alpha\otimes\beta\otimes\gamma) = \frac{(p+q+r)!}{p!q!r!}\,\mathrm{Alt}(\alpha\otimes(\beta\otimes\gamma))\\
&= \ldots = \frac{(p+q+r)!}{p!(q+r)!}\,\mathrm{Alt}(\alpha\otimes(\beta\wedge\gamma)) = \alpha\wedge(\beta\wedge\gamma).
\end{aligned}$$

■

From the associativity asserted in the preceding lemma, it follows that the multiplication in $\mathrm{Alt}(V,\mathbb{K})$ is associative, We may therefore suppress brackets and define

$$\omega_1 \wedge \ldots \wedge \omega_n := (\ldots((\omega_1\wedge\omega_2)\wedge\omega_3)\cdots\wedge\omega_n).$$

Remark III.2.5. (a) From the calculation in the preceding proof we know that for three elements $\alpha_i \in \mathrm{Alt}^{p_i}(V,\mathbb{K})$ the triple product in the associative algebra $\mathrm{Alt}(V,\mathbb{K})$ satisfies

$$\alpha_1\wedge\alpha_2\wedge\alpha_3 = \frac{(p_1+p_2+p_3)!}{p_1!p_2!p_3!}\,\mathrm{Alt}(\alpha_1\otimes\alpha_2\otimes\alpha_3).$$

Inductively this leads for n elements $\alpha_i \in \mathrm{Alt}^{p_i}(V,\mathbb{K})$ to

$$\alpha_1\wedge\ldots\wedge\alpha_n = \frac{(p_1+\ldots+p_n)!}{p_1!\cdots p_n!}\,\mathrm{Alt}(\alpha_1\otimes\cdots\otimes\alpha_n)$$

(Exercise III.2).

(b) For $\alpha_i \in \mathrm{Alt}^1(V,\mathbb{K}) \cong V^*$, we obtain in particular

$$\begin{aligned}
(\alpha_1\wedge\ldots\wedge\alpha_n)(v_1,\ldots,v_n) &= n!\,\mathrm{Alt}(\alpha_1\otimes\cdots\otimes\alpha_n)(v_1,\ldots,v_n)\\
&= \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\alpha_1(v_{\sigma(1)})\cdots\alpha_n(v_{\sigma(n)}) = \det(\alpha_i(v_j)).
\end{aligned}$$

■

Proposition III.2.6. *The exterior algebra is graded commutative, i.e., for* $\alpha \in \mathrm{Alt}^p(V,\mathbb{K})$ *and* $\beta \in \mathrm{Alt}^q(V,\mathbb{K})$ *we have*

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

Proof. Let $\sigma \in S_{p+q}$ denote the permutation defined by

$$\sigma(i) := \begin{cases} i+p & \text{for } 1 \leq i \leq q \\ i-q & \text{for } q+1 \leq i \leq p+q \end{cases}$$

which moves the first q elements to the last q positions. Then we have

$$\begin{aligned} &(\beta \otimes \alpha)^\sigma(v_1, \ldots, v_{p+q}) = (\beta \otimes \alpha)(v_{\sigma(1)}, \ldots, v_{\sigma(p+q)}) \\ &= \beta(v_{p+1}, \ldots, v_{p+q})\alpha(v_1, \ldots, v_p) = (\alpha \otimes \beta)(v_1, \ldots, v_{p+q}). \end{aligned}$$

This leads to

$$\begin{aligned} \alpha \wedge \beta &= \frac{(p+q)!}{p!q!} \mathrm{Alt}(\alpha \otimes \beta) = \frac{(p+q)!}{p!q!} \mathrm{Alt}((\beta \otimes \alpha)^\sigma) \\ &= \mathrm{sgn}(\sigma) \frac{(p+q)!}{p!q!} \mathrm{Alt}(\beta \otimes \alpha) = \mathrm{sgn}(\sigma)(\beta \wedge \alpha). \end{aligned}$$

On the other hand $\mathrm{sgn}(\sigma) = (-1)^F$, where

$$\begin{aligned} F &:= |\{(i,j) \in \{1, \ldots, p+q\} \colon i < j, \sigma(j) < \sigma(i)\}| \\ &= |\{(i,j) \in \{1, \ldots, p+q\} \colon i \leq q, j > q\}| = pq \end{aligned}$$

is the number of inversions of σ. Putting everything together, the lemma follows. ∎

Corollary III.2.7. *If* $\alpha \in \mathrm{Alt}^p(V,\mathbb{K})$ *and* p *is odd, then*

$$\alpha \wedge \alpha = 0.$$

Proof. In view of Proposition III.2.6, we have

$$\alpha \wedge \alpha = (-1)^{p^2} \alpha \wedge \alpha = -\alpha \wedge \alpha,$$

which leads to $\alpha \wedge \alpha = 0$ because $\mathrm{char}\, \mathbb{K} \neq 2$. ∎

Corollary III.2.8. *If* $\alpha_1, \ldots, \alpha_k \in V^* = \mathrm{Alt}^1(V,\mathbb{K})$ *and* $\beta_j = \sum_{i=1}^k a_{ij}\alpha_i$, *then*

$$\beta_1 \wedge \ldots \wedge \beta_k = \det(A) \cdot \alpha_1 \wedge \ldots \wedge \alpha_k \quad \textit{for} \quad A = (a_{ij}) \in M_k(\mathbb{K}).$$

Proof. We consider the k-fold multiplication map

$$\Phi \colon (V^*)^k \to \mathrm{Alt}^k(V,\mathbb{K}), \quad (\gamma_1, \ldots, \gamma_k) \mapsto \gamma_1 \wedge \ldots \wedge \gamma_k.$$

Since the exterior product is bilinear, this map is k-linear. It is also alternating, because repeated application of Proposition III.2.6 and Corollary III.2.7 leads for $\gamma_i = \gamma_j$ to

$$\begin{aligned} &\gamma_1 \wedge \ldots \wedge \gamma_i \wedge \ldots \wedge \gamma_j \wedge \ldots \wedge \gamma_k \\ &= (-1)^{j-i-1} \gamma_1 \wedge \ldots \wedge \gamma_{i-1} \wedge \gamma_{i+1} \ldots \wedge \gamma_{j-1} \wedge (\gamma_i \wedge \gamma_j) \wedge \ldots \wedge \gamma_k \\ &= (-1)^{j-i-1} \gamma_1 \wedge \ldots \wedge \gamma_{i-1} \wedge \gamma_{i+1} \ldots \wedge \gamma_{j-1} \wedge \underbrace{\gamma_i \wedge \gamma_i}_{=0} \wedge \ldots \wedge \gamma_k = 0. \end{aligned}$$

Since Φ is alternating, the assertion follows from Proposition III.1.4. ∎

Corollary III.2.9. *If* $\dim V = n$, $b_1, \ldots, b_n$ *is a basis of* V, *and* $b_1^*, \ldots, b_n^*$ *the dual basis of* V^*, *then the products*

$$b_I^* := b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*, \quad I = (i_1, \ldots, i_k), \quad 1 \leq i_1 < \ldots < i_k \leq n,$$

form a basis of $\mathrm{Alt}^k(V, \mathbb{K})$.

Proof. For $J = (j_1, \ldots, j_k)$ with $j_1 < \ldots < j_k$ we then get with Remark III.2.5(b)

$$b_I^*(b_{j_1}, \ldots, b_{j_k}) = \det(b_{i_l}^*(b_{j_m})_{l,m=1,\ldots,k}) = \begin{cases} 1 & \text{for } I = J \\ 0 & \text{for } I \neq J. \end{cases}$$

If follows in particular that the elements b_I are linearly independent, and since $\dim \mathrm{Alt}^k(V, \mathbb{K}) = \binom{n}{k}$ (Corollary III.1.5), the assertion follows. ∎

Remark III.2.10. (a) From Corollary III.1.5 it follows in particular that

$$\dim \mathrm{Alt}(V, \mathbb{K}) = \sum_{k=0}^{\dim V} \binom{\dim V}{k} = 2^{\dim V}$$

if V is finite-dimensional.

(b) If V is infinite-dimensional, then it has an infinite basis $(b_i)_{i \in I}$ (this requires Zorn's Lemma). In addition, the set I carries a linear order $\leq$ (this requires the Well Ordering Theorem), and for each k-element subset $J = \{j_1, \ldots, j_k\} \subseteq J$ with $j_1 < \ldots < j_k$ we thus obtain an element

$$b_J^* := b_{j_1}^* \wedge \ldots \wedge b_{j_k}^*.$$

Applying the b_J^* to k-tuples of basis elements shows that they are linearly independent, so that for each $k > 0$ the space $\mathrm{Alt}^k(V, \mathbb{K})$ is infinite-dimensional. ∎

Definition III.2.11. Let $\varphi \colon V_1 \to V_2$ be a linear map and W a vector space. For each p-linear map $\alpha \colon V_2^p \to W$ we define its *pull-back by* φ:

$$(\varphi^* \alpha)(v_1, \ldots, v_p) := \alpha(\varphi(v_1), \ldots, \varphi(v_p))$$

for $v_1, \ldots, v_p \in V_1$.

It is clear that $\varphi^* \alpha$ is a p-linear map $V_1^p \to W$ and that $\varphi^* \alpha$ is alternating if α has this property. ∎

Remark III.2.12. If $\varphi \colon V_1 \to V_2$ and $\psi \colon V_2 \to V_3$ are linear maps and $\alpha \colon V_3^p \to W$ is p-linear, then

$$(\psi \circ \varphi)^* \alpha = \varphi^*(\psi^* \alpha),$$

so that we formally have

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$

(Exercise). ∎

Proposition III.2.13. *Let $\varphi: V_1 \to V_2$ be a linear map. Then the pull-back map*

$$\varphi^*: \mathrm{Alt}(V_2, \mathbb{K}) \to \mathrm{Alt}(V_1, \mathbb{K})$$

is a homomorphism of algebras with unit.

Proof. For $\alpha \in \mathrm{Alt}^p(V_2, \mathbb{K})$ and $\beta \in \mathrm{Alt}^q(V_2, \mathbb{K})$ we have

$$\begin{aligned}\varphi^*(\alpha \wedge \beta) &= \frac{(p+q)!}{p!q!}\varphi^*(\mathrm{Alt}(\alpha \otimes \beta)) = \frac{(p+q)!}{p!q!}\mathrm{Alt}(\varphi^*(\alpha \otimes \beta)) \\ &= \frac{(p+q)!}{p!q!}\mathrm{Alt}(\varphi^*\alpha \otimes \varphi^*\beta) = \varphi^*\alpha \wedge \varphi^*\beta.\end{aligned}$$

■

Remark III.2.14. The results in this section remain valid for alternating forms with values in any commutative algebra A. Then $\mathrm{Alt}(V, A) = \bigoplus_{p \in \mathbb{N}_0} \mathrm{Alt}^p(V, A)$ also carries an associative, graded commutative algebra structure defined by

$$\alpha \wedge \beta := \frac{(p+q)!}{p!q!}\mathrm{Alt}(\alpha \otimes \beta),$$

where

$$(\alpha \otimes \beta)(v_1, \ldots, v_{p+q}) := \alpha(v_1, \ldots, v_p) \cdot \beta(v_{p+1}, \ldots, v_{p+q})$$

for $\alpha \in \mathrm{Alt}^p(V, A)$, $\beta \in \mathrm{Alt}^q(V, A)$.

This applies in particular to the 2-dimensional real algebra $A = \mathbb{C}$. ■

III.3. Orientations on vector spaces

Throughout this section, all vector spaces are real and finite-dimensional.

Definition III.3.1. (a) Let V be an n-dimensional real vector space. Then space $\mathrm{Alt}^n(V, \mathbb{R})$ is one-dimensional. Any non-zero element μ of this space is called a *volume form* on V.

(b) We define an equivalence relation on the set $\mathrm{Alt}^n(V, \mathbb{R}) \setminus \{0\}$ of volume forms by $\mu_1 \sim \mu_2$ if there exists a $\lambda > 0$ with $\mu_2 = \lambda\mu_1$ and write $[\mu]$ for the equivalence class of μ. These equivalence classes are called *orientations of* V. If $O = [\mu]$ is an orientation, then we write $-O := [-\mu]$ for the *opposite orientation.*

An *oriented vector space* is a pair (V, O), where V is a finite-dimensional real vector space and $O = [\mu]$ an orientation on V.

(c) An ordered basis $(b_1, \ldots, b_n)$ of $(V, [\mu])$ is said to be *positively oriented* if $\mu(b_1, \ldots, b_n) > 0$, and *negatively oriented* otherwise.

(d) An invertible linear map $\varphi: (V, [\mu_V]) \to (W, [\mu_W])$ between oriented vector spaces is called *orientation preserving* if $[\varphi^*\mu_W] = [\mu_V]$. Otherwise φ is called *orientation reversing.*

(e) We endow $\mathbb{R}^n$ with the canonical orientation defined by the determinant form

$$\mu(x_1, \ldots, x_n) := \det(x_1, \ldots, x_n) = \det(x_{ij})_{i,j=1,\ldots,n}.$$

■

Remark III.3.2. (a) If $B := (b_1, \ldots, b_n)$ is a basis of V, then Corollary III.2.9 implies that we obtain a volume form by

$$\mu_B := b_1^* \wedge \ldots \wedge b_n^*,$$

and since $\mu_B(b_1, \ldots, b_n) = \det(b_i^*(b_j)) = \det(\mathbf{1}) = 1$, the basis B is positively oriented with respect to the orientation $[\mu_B]$. We call $[\mu_B]$ the *orientation defined by the basis* B.

(b) The terminology "volume form" corresponds to the interpretation of $\mu(v_1, \ldots, v_n)$ as an "oriented" volume of the flat

$$[0,1]v_1 + \ldots + [0,1]v_n$$

generated by the n-tuple $(v_1, \ldots, v_n)$. Note that $\mu_B(v_1, \ldots, v_n) = \det(b_i^*(v_j))$. ■

Lemma III.3.3. *If μ_V is a volume form on V and $\varphi \in \mathrm{End}(V)$, then*

$$\varphi^*\mu_V = \det(\varphi)\mu_V.$$

In particular, φ is orientation preserving if and only if $\det(\varphi) > 0$.

Proof. Let $B = (b_1, \ldots, b_n)$ be a positively oriented basis of V and $A = [\varphi]_B$ the matrix of φ with respect to B, i.e., $\varphi(b_j) = \sum_i a_{ij} b_i$. Then

$$\begin{aligned}(\varphi^*\mu_V)(b_1, \ldots, b_n) &= \mu_V(\varphi(b_1), \ldots, \varphi(b_n)) = \det(A)\mu_V(b_1, \ldots, b_n) \\ &= \det(\varphi)\mu_V(b_1, \ldots, b_n)\end{aligned}$$

follows from Proposition III.1.4(2), and this implies the assertion. ■

Example III.3.4. (a) If $V = \mathbb{R}^2$ and $\varphi \in \mathrm{GL}(V)$ is the reflection in a line, then $\det(\varphi) < 0$ implies that φ is orientation reversing. The same holds for the reflection in a hyperplane in $\mathbb{R}^n$.

(b) Rotations of $\mathbb{R}^3$ around an axis are orientation preserving.

(c) In $V = \mathbb{C}$, considered as a real vector space, we have the natural basis $B = (1, i)$. A corresponding volume form is given by

$$\mu(z, w) := \mathrm{Im}(\overline{z}w) = \mathrm{Re}\, z \,\mathrm{Im}\, w - \mathrm{Im}\, z \,\mathrm{Re}\, w$$

because $\mu(1, i) = \mathrm{Im}(i) = 1 > 0$.

Each complex linear map $\varphi \colon \mathbb{C} \to \mathbb{C}$ is given by multiplication with some complex number $x + iy$, and the corresponding matrix with respect to B is

$$[\varphi]_B = \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

so that $\det(\varphi) = x^2 + y^2 > 0$ whenever $\varphi \neq 0$. We conclude that each non-zero complex linear map $V \to V$ is orientation preserving. ■

Proposition III.3.5. *Let V be a complex vector space, viewed as a real one, and $\varphi\colon V \to V$ a complex linear map. Then $\det_{\mathbb{R}}(\varphi) = |\det_{\mathbb{C}}(\varphi)|^2$. In particular, each invertible complex linear map is orientation preserving.*

Proof. Let $B_{\mathbb{C}} = (b_1, \ldots, b_n)$ be a complex basis of V, so that $B = (b_1, \ldots, b_n, ib_1, \ldots, ib_n)$ is a real basis of V. Further let $b_j^* \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, $j = 1, \ldots, n$, denote the complex dual basis. In $\mathrm{Alt}^{2n}(V, \mathbb{C}) \cong \mathbb{C}$ we then consider the element

$$\mu := b_1^* \wedge \ldots \wedge b_n^* \wedge \overline{b_1^*} \wedge \ldots \wedge \overline{b_n^*}.$$

That μ is non-zero follows from

$$\mu(b_1, \ldots, b_n, ib_1, \ldots, ib_n) = \det\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix} = \det\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^n$$
$$= (-i - i)^n = (-2i)^n \neq 0.$$

If $A = [\varphi]_{B_{\mathbb{C}}} \in M_n(\mathbb{C})$ is the matrix of φ with respect to $B_{\mathbb{C}}$, then we have

$$\varphi^* b_j^* = \sum_{k=1}^n a_{jk} b_k^* \quad \text{and} \quad \varphi^* \overline{b_j^*} = \sum_{k=1}^n \overline{a_{jk}} \overline{b_k^*}.$$

As in the proof of Lemma III.3.3, we now see that

$$\varphi^*(b_1^* \wedge \ldots \wedge b_n^*) = \det_{\mathbb{C}}(A) \cdot b_1^* \wedge \ldots \wedge b_n^*$$

and

$$\varphi^*(\overline{b_1^*} \wedge \ldots \wedge \overline{b_n^*}) = \overline{\det_{\mathbb{C}}(A)} \cdot \overline{b_1^*} \wedge \ldots \wedge \overline{b_n^*},$$

which leads with Proposition III.2.13 and Lemma III.3.3 to

$$\det_{\mathbb{R}}(\varphi)\mu = \varphi^*\mu = \det_{\mathbb{C}}(A)\overline{\det_{\mathbb{C}}(A)}\mu = |\det_{\mathbb{C}}(A)|^2\mu = |\det_{\mathbb{C}}(\varphi)|^2\mu. \qquad \blacksquare$$

Exercises for Chapter III

Exercise III.1. Fix $n \in \mathbb{N}$. Show that:

(1) For each matrix $A \in M_n(\mathbb{K})$ we obtain a bilinear map

$$\beta_A\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}, \quad \beta_A(x, y) := \sum_{i,j=1}^n a_{ij} x_i y_j.$$

(2) A can be recovered from β_A via $a_{ij} = \beta_A(e_i, e_j)$.

(3) Each bilinear map $\beta\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is of the form $\beta = \beta_A$ for a unique matrix $A \in M_n(\mathbb{R})$.

(4) $\beta_{A^\top}(x, y) = \beta_A(y, x)$.

(5) β_A is skew-symmetric if and only if A is so.

(6) β_A is alternating if and only if A is skew-symmetric and all its diagonal entries a_{ii} vanish. For $\mathrm{char}\,\mathbb{K} \neq 2$ the second condition is redundant. $\blacksquare$

Exercise III.2. Show that for $\alpha_1, \ldots, \alpha_n \in \mathrm{Alt}^{p_i}(V, \mathbb{K})*$ the exterior product satisfies

$$\alpha_1 \wedge \ldots \wedge \alpha_n = \frac{(p_1 + \ldots + p_n)!}{p_1! \cdots p_n!} \mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n) \qquad \blacksquare$$

IV. Differential forms on open subsets of $\mathbb{R}^n$

In this chapter we introduce differential form on open subsets $U \subseteq \mathbb{R}^n$. In the next chapter we shall extend this theory to differential forms on smooth manifolds.

IV.1. Basic definitions

Definition IV.1.1. Let $p \in \mathbb{N}_0$, $U \subseteq \mathbb{R}^n$ an open subset and E a finite-dimensional real vector space. An *E-valued p-form*, or *differential form of degree p*, on U is a function

$$\omega: U \to \mathrm{Alt}^p(\mathbb{R}^n, E).$$

This means that we assign to each $x \in U$ an alternating p-linear map $\omega_x := \omega(x): (\mathbb{R}^n)^p \to E$. A p-form ω is said to be *smooth* if the map ω is smooth. Recall that $\dim \mathrm{Alt}^p(\mathbb{R}^n, E) = \dim E \cdot \binom{n}{p}$ is finite, so that smoothness is well-defined in this context.

We write $\Omega^p(U, E)$ for the vector space of smooth E-valued p-forms on U. Here the vector space structure is the obvious one given by

$$(\omega + \eta)_x := \omega_x + \eta_x \quad \text{and} \quad (\lambda\omega)_x := \lambda\omega_x \quad \text{for} \quad x \in U.$$

We also form the direct sum of all these spaces

$$\Omega(U, E) := \bigoplus_{p=0}^{\infty} \Omega^p(U, E). \qquad \blacksquare$$

Example IV.1.2. (a) For $p = 0$ we have $\mathrm{Alt}^0(\mathbb{R}^n, E) = E$, so that smooth E-valued 0-forms are simply smooth functions $f: U \to E$. In this sense we identify

$$C^\infty(U, E) \cong \Omega^0(U, E).$$

(b) For $p = 1$ we obtain the so-called *Pfaffian forms*, which are smooth maps

$$\omega: U \to \mathrm{Hom}(\mathbb{R}^n, E).$$

For each smooth function $f: U \to E$ its differential df is a smooth E-valued 1-form:

$$(df)_x(v) := df(x)(v).$$

(c) Constant p-forms correspond to elements of $\mathrm{Alt}^p(\mathbb{R}^n, E)$, so that we obtain a natural embedding

$$\mathrm{Alt}^p(\mathbb{R}^n, E) \hookrightarrow \Omega^p(U, E)$$

as the subspace of constant forms. $\blacksquare$

Now we turn to algebraic operations on differential forms. We have already observed the vector space structure of $\Omega^p(U, \mathbb{R})$.

Definition IV.1.3. (a) For $f \in C^\infty(U,\mathbb{R})$ and $\omega \in \Omega^p(U,E)$ we define the product $f\omega \in \Omega^p(U,E)$ by

$$(f\omega)_x := f(x)\omega(x).$$

Since the scalar multiplication map $S\colon \mathbb{R} \times \mathrm{Alt}^p(\mathbb{R}^n,E) \to \mathrm{Alt}^p(\mathbb{R}^n,E)$ is bilinear, hence smooth, the p-form $f\omega = S \circ (f,\omega)$ is smooth. We thus obtain a multiplication map

$$C^\infty(U,\mathbb{R}) \times \Omega^p(U,E) \to \Omega^p(U,E), \quad (f,\omega) \mapsto f\omega.$$

This map turns $\Omega^p(U,E)$ into a *module* of the algebra $C^\infty(U,\mathbb{R})$, i.e., it is bilinear and associative in the sense that $f(g\omega) = (fg)\omega$ for $f,g \in C^\infty(U,\mathbb{R})$, $\omega \in \Omega^p(U,E)$ (Exercise).

(b) We further have the *exterior product*

$$\Omega^p(U,\mathbb{R}) \times \Omega^q(U,\mathbb{R}) \to \Omega^{p+q}(U,\mathbb{R}), \quad (\omega,\eta) \mapsto \omega \wedge \eta, \quad (\omega \wedge \eta)_x := \omega_x \wedge \eta_x$$

(Definition III.2.3). Here we also use the smoothness of the bilinear map

$$m : \mathrm{Alt}^p(\mathbb{R}^n,\mathbb{R}) \times \mathrm{Alt}^q(\mathbb{R}^n,\mathbb{R}) \to \mathrm{Alt}^{p+q}(\mathbb{R}^n,\mathbb{R})$$

(Exercise I.14) to see that $\omega \wedge \eta = m \circ (\omega,\eta)$ is a smooth map.

For $p = 0$ we have $\Omega^0(U,\mathbb{R}) \cong C^\infty(U,\mathbb{R})$, which leads to

$$f \wedge \eta = f\eta, \quad f \in C^\infty(U,\mathbb{R}), \eta \in \Omega^q(U,\mathbb{R}). \qquad \blacksquare$$

From Lemma III.2.4 and Proposition III.2.6 we immediately obtain:

Proposition IV.1.4. *The space $\Omega(U,\mathbb{R})$ is a graded commutative associative algebra with respect to the exterior product.*

Proof. The associativity follows immediately from the associativity of the exterior algebra $\mathrm{Alt}(\mathbb{R}^n,\mathbb{R})$ because for each $x \in U$ we have

$$\big((\alpha \wedge \beta) \wedge \gamma\big)_x = (\alpha_x \wedge \beta_x) \wedge \gamma_x = \alpha_x \wedge (\beta_x \wedge \gamma_x) = \big(\alpha \wedge (\beta \wedge \gamma)\big)_x.$$

We likewise get for $\alpha \in \Omega^p(U,\mathbb{R})$ and $\beta \in \Omega^q(U,\mathbb{R})$ with Prop. III.2.6:

$$(\alpha \wedge \beta)_x = \alpha_x \wedge \beta_x = (-1)^{pq}\beta_x \wedge \alpha_x = (-1)^{pq}(\beta \wedge \alpha)_x. \qquad \blacksquare$$

Definition IV.1.5. (Basic forms) For each $j \in \{1,\ldots,n\}$ we define the *basic differential* 1*-forms* $dx_j \in \Omega^1(U,\mathbb{R})$ by

$$(dx_j)_x(v) = v_j, \quad \text{resp.,} \quad (dx_j)_x = e_j^*,$$

where $e_1^*,\ldots,e_n^*$ is the dual basis of the standard basis $e_1,\ldots,e_n$. Then each dx_j is a constant 1-form, hence in particular smooth.

We have seen in Corollary III.2.9 that the products

$$e_I^* := e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*, \quad I = \{i_1, \ldots, i_k\}, \quad 1 \leq i_1 < \ldots < i_k \leq n,$$

form a basis of $\mathrm{Alt}^k(\mathbb{R}^n, \mathbb{R})$. We define the k-form $dx_I \in \Omega^k(U, \mathbb{R})$ by

$$(dx_I)_x := (dx_{i_1} \wedge \ldots \wedge dx_{i_k})_x = e_I^* \quad \text{for} \quad x \in U.$$

If $\omega \in \Omega^k(U, \mathbb{R})$ is an arbitrary smooth k-form and I as above, then

$$\omega_I(x) := \omega_x(e_{i_1}, \ldots, e_{i_k})$$

defines a smooth function on U (Exercise), and for each $x \in U$ we have

$$\omega_x = \sum_I \omega_x(e_{i_1}, \ldots, e_{i_k}) e_I^* = \sum_I \omega_I(x)(dx_I)_x.$$

In this sense we have

$$\omega = \sum_I \omega_I \cdot dx_I \tag{4.1}$$

in $\Omega^k(U, \mathbb{R})$. This is called the *basic representation of* ω, and the forms $dx_I \in \Omega^p(U, \mathbb{R})$ are called *basic forms.* ■

Example IV.1.6. (a) For $p = 1$ we have the basic 1-forms $dx_1, \ldots, dx_n$, and each smooth 1-form $\omega \in \Omega^1(U, \mathbb{R})$ can be written in a unique fashion as

$$\omega = \sum_{i=1}^n \omega_i dx_i, \quad \omega_i \in C^\infty(U, \mathbb{R}).$$

For each smooth function $f : U \to \mathbb{R}$, the differential df has the basic representation

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

because

$$(df)_x(e_i) = df(x)(e_i) = \frac{\partial f}{\partial x_i}(x).$$

(b) For $p = n$ we have only one basic n-form $dx_1 \wedge \ldots \wedge dx_n$, and each smooth n-form $\omega \in \Omega^n(U, \mathbb{R})$ can be written in a unique fashion as

$$\omega = f \cdot dx_1 \wedge \ldots \wedge dx_n \quad \text{with} \quad f \in C^\infty(U, \mathbb{R}).$$

In particular, we have

$$\Omega^n(U, \mathbb{R}) \cong C^\infty(U, \mathbb{R}).$$ ■

IV.2. Pullbacks of differential forms

Definition IV.2.1. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^k$ be open and $\varphi: V \to U$ a smooth map. For $\omega \in \Omega^p(U, E)$ we then obtain a smooth p-form

$$\varphi^*\omega \in \Omega^p(V, E), \quad (\varphi^*\omega)_x := (d\varphi(x))^*\omega_{\varphi(x)},$$

called the *pullback of* ω *by* φ. This means that

$$(\varphi^*\omega)_x(v_1, \ldots, v_p) = \omega_{\varphi(x)}(d\varphi(x)v_1, \ldots, d\varphi(x)v_p).$$

From this formula it is clear that $\varphi^*\omega$ is smooth if ω is smooth. ∎

Proposition IV.2.2. *The pullback of differential forms has the following properties:*

(1) *The map* $\varphi^*: \Omega^p(U, E) \to \Omega^p(V, E)$ *is linear.*

(2) $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$ *for* $\alpha \in \Omega^p(U, \mathbb{R}), \beta \in \Omega^q(U, \mathbb{R})$.

(3) $\varphi^* f = f \circ \varphi$ *and* $\varphi^*(df) = d(\varphi^* f)$ *for* $f \in C^\infty(U, E) = \Omega^0(U, E)$.

(4) *If* $\psi: U \to W$ *is another smooth map and* $W \subseteq \mathbb{R}^d$ *is open, then*

$$(\psi \circ \varphi)^*\omega = \varphi^*\psi^*\omega \quad \textit{for} \quad \omega \in \Omega^p(W, E).$$

Proof. (1) is a trivial consequence of the definitions.

(2) For each $x \in V$ we have

$$\begin{aligned}(\varphi^*(\alpha \wedge \beta))_x &= (d\varphi(x))^*(\alpha \wedge \beta)_{\varphi(x)} = (d\varphi(x))^*(\alpha_{\varphi(x)} \wedge \beta_{\varphi(x)}) \\ &\overset{III.2.13}{=} (d\varphi(x))^*\alpha_{\varphi(x)} \wedge (d\varphi(x))^*\beta_{\varphi(x)} = (\varphi^*\alpha)_x \wedge (\varphi^*\beta)_x \\ &= \big(\varphi^*\alpha \wedge \varphi^*\beta\big)_x.\end{aligned}$$

(3) The relation $\varphi^* f = f \circ \varphi$ holds by definition. Now the Chain Rule leads to

$$d(\varphi^* f)(x) = d(f \circ \varphi)(x) = df(\varphi(x))d\varphi(x) = (d\varphi(x))^*(df)_{\varphi(x)} = (\varphi^* df)_x.$$

(4) In view of Remark III.2.13 and the Chain Rule, we have for each $x \in V$:

$$\begin{aligned}((\psi \circ \varphi)^*\omega)_x &= (d(\psi \circ \varphi)(x))^*\omega_{\psi(\varphi(x))} = (d\psi(\varphi(x))d\varphi(x))^*\omega_{\psi(\varphi(x))} \\ &= d\varphi(x)^* d\psi(\varphi(x))^*\omega_{\psi(\varphi(x))} = d\varphi(x)^*(\psi^*\omega)_{\varphi(x)} = (\varphi^*(\psi^*\omega))_x.\end{aligned}$$

∎

Remark IV.2.3. It is instructive to take a closer look at the basic representation of pullbacks. Let $\varphi\colon V \to U$ be a smooth map ($V \subseteq \mathbb{R}^m$ and $U \subseteq \mathbb{R}^n$ open) and

$$\omega = \sum_I \omega_I dx_I = \sum_{i_1<\ldots<i_k} \omega_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \in \Omega^k(U,\mathbb{R}), \quad \omega_I \in C^\infty(U,\mathbb{R}).$$

Then Proposition IV.2.2(1/2) implies that

$$\begin{aligned}\varphi^*\omega &= \sum_I (\varphi^*\omega_I)\varphi^*(dx_I) = \sum_{i_1<\ldots<i_k} (\omega_{i_1,\ldots,i_k}\circ\varphi)\varphi^*(dx_{i_1}) \wedge \ldots \wedge \varphi^*(dx_{i_k}) \\ &= \sum_{i_1<\ldots<i_k} (\omega_{i_1,\ldots,i_k}\circ\varphi) d(\varphi^* x_{i_1}) \wedge \ldots \wedge d(\varphi^* x_{i_k}) \\ &= \sum_{i_1<\ldots<i_k} (\omega_{i_1,\ldots,i_k}\circ\varphi) d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_k}.\end{aligned}$$

In view of Example IV.1.6, we have

$$d\varphi_j = \sum_{i=1}^m \frac{\partial \varphi_j}{\partial x_i} dx_i. \tag{4.2}$$

For $I = \{i_1,\ldots,i_k\}$ with $1 \leq i_1 < \ldots < i_k \leq n$ we write $\varphi_I\colon V \to \mathbb{R}^k$ for the function whose components are given by $\varphi_{i_1},\ldots,\varphi_{i_k}$. For a matrix $A \in M_{n,m}(\mathbb{R})$ and $J = \{j_1,\ldots,j_k\}$ with $1 \leq j_1 < \ldots < j_k \leq m$ we write

$$A_I^J = (a_{ij})_{i\in I, j\in J} \in M_k(\mathbb{R})$$

for the submatrix of size $k \times k$, defined by erasing all rows not corresponding to elements in I and all columns not corresponding to elements of J. Then Proposition III.1.4(4) and (4.2) lead to

$$d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_k} = \sum_J \det([d\varphi_I]^J) dx_{j_1} \wedge \ldots \wedge dx_{j_k} = \sum_J \det([d\varphi]_I^J) dx_J.$$

Combining all this, we obtain

$$\varphi^*\omega = \sum_I (\omega_I\circ\varphi) \sum_J \det([d\varphi]_I^J) dx_J = \sum_J \Big(\sum_I (\omega_I\circ\varphi)\det([d\varphi]_I^J)\Big) dx_J. \tag{4.3}$$

■

Examples IV.2.4. (1) Pullbacks of 1-forms: For any smooth 1-form $\omega = \sum_i \omega_i dx_i$ on the open subset $U \subseteq \mathbb{R}^n$ and any smooth map $\varphi\colon V \to U$, $V \subseteq \mathbb{R}^m$ open, we have

$$\varphi^*\omega = \sum_{j=1}^m \Big(\sum_{i=1}^n (\omega_i\circ\varphi)\frac{\partial\varphi_i}{\partial x_j}\Big) dx_j.$$

(2) Pullbacks of 2-forms: For any smooth 2-form

$$\omega = \sum_{i_1<i_2} \omega_{i_1,i_2} dx_{i_1} \wedge dx_{i_2}$$

we have

$$\varphi^*\omega = \sum_{j_1<j_2} \Big(\sum_{i_1<i_2} (\omega_{i_1,i_2} \circ \varphi) \begin{vmatrix} \frac{\partial \varphi_{i_1}}{\partial x_{j_1}} & \frac{\partial \varphi_{i_1}}{\partial x_{j_2}} \\ \frac{\partial \varphi_{i_2}}{\partial x_{j_1}} & \frac{\partial \varphi_{i_2}}{\partial x_{j_2}} \end{vmatrix} \Big) dx_{j_1} \wedge dx_{j_2}.$$

(3) For the special case $k = m$ (4.3) reduces to

$$\varphi^*\omega = \Big(\sum_I (\omega_I \circ \varphi) \det([d\varphi]_I) \Big) \ dx_1 \wedge \ldots \wedge dx_m. \tag{4.4}$$

If $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear map with $\varphi(V) \subseteq U$ and $A \in M_{n,m}(\mathbb{R})$ the corresponding matrix, then $\varphi(x) = Ax$ and $[d\varphi(x)] = A$, so that we obtain for each $\omega \in \Omega^m(U, \mathbb{R})$:

$$\varphi^*\omega = \Big(\sum_I (\omega_I \circ \varphi) \det(A_I) \Big) dx_1 \wedge \ldots \wedge dx_m.$$

(4) For the special case $k = m = n$ and $\omega = f dx_1 \wedge \ldots \wedge dx_n$, (4.4) further simplifies to

$$\varphi^*\omega = (f \circ \varphi) \cdot \det([d\varphi]) \ dx_1 \wedge \ldots \wedge dx_n. \tag{4.5}$$

■

Example IV.2.5. We consider the smooth map

$$\varphi\colon V :=]0, \infty[\times]0, 2\pi[\to \mathbb{R}^3, \quad \varphi(r, \theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ r \end{pmatrix}.$$

Then V is open in $\mathbb{R}^2$, so that $m = 2$ and $n = 3$. Direct caculation yields the Jacobi matrix

$$[d\varphi(r, \theta)] = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \\ 1 & 0 \end{pmatrix}.$$

(a) Let $\omega := -x_2 dx_1 + x_1 dx_2 \in \Omega^1(\mathbb{R}^3, \mathbb{R})$, i.e., $\omega = \omega_1 dx_1 + \omega_2 dx_2$ with $\omega_1(x) = -x_2$ and $\omega_2(x) = x_1$. Then

$$\begin{aligned} (\varphi^*\omega)_{(r,\theta)} &= -\varphi_2 d\varphi_1 + \varphi_1 d\varphi_2 = -\varphi_2 \big(\frac{\partial \varphi_1}{\partial r} dr + \frac{\partial \varphi_1}{\partial \theta} d\theta \big) + \varphi_1 \big(\frac{\partial \varphi_2}{\partial r} dr + \frac{\partial \varphi_2}{\partial \theta} d\theta \big) \\ &= (-\sin\theta(-\sin\theta) + \cos\theta\cos\theta) d\theta = d\theta. \end{aligned}$$

(b) For $\omega = dx_3$ we have

$$\varphi^*\omega = d\varphi_3 = dr.$$

■

k-dimensional volumes in $\mathbb{R}^n$

In this subsection we discuss some interesting applications of the calculus of differential forms to k-dimensional volumina of subsets of $\mathbb{R}^n$. We start with a generalization of the Product Theorem for determinants of square matrices to rectangular ones.

Proposition IV.2.6. (Cauchy–Binet Formula) *For matrices $A \in M_{k,n}(\mathbb{R})$ and $B \in M_{n,k}(\mathbb{R})$, $k \leq n$, we have*

$$\det(AB) = \sum_J \det(A^J)\det(B_J),$$

where the sum is extended over all k-element subsets $J = \{j_1, \ldots, j_k\}$ of $\{1, \ldots, n\}$, A^J is the matrix consisting of the columns in positions $j_1, \ldots, j_k$, and B_J is the matrix consisting of the rows in positions $j_1, \ldots, j_k$.

Proof. We consider the linear maps

$$\varphi_A: \mathbb{R}^n \to \mathbb{R}^k, \quad x \mapsto Ax \quad \text{and} \quad \varphi_B: \mathbb{R}^k \to \mathbb{R}^n, \quad y \mapsto By.$$

Then $\varphi_{AB} \in \mathrm{End}(\mathbb{R}^k)$ satisfies

$$\varphi_{AB}^*(dx_1 \wedge \ldots \wedge dx_k) = \det(AB) \cdot dx_1 \wedge \ldots \wedge dx_k$$

(Example IV.2.4(4)). On the other hand,

$$\varphi_{AB}^*(dx_1 \wedge \ldots \wedge dx_k) = \varphi_B^*\varphi_A^*(dx_1 \wedge \ldots \wedge dx_k)$$

and (4.3) imply that

$$\varphi_A^*(dx_1 \wedge \ldots \wedge dx_k) = \sum_J \det([d\varphi_A]^J)\Big)dx_J = \sum_J \det(A^J)dx_J.$$

We further obtain with (4.4)

$$\varphi_B^* dx_J = \det([d\varphi_B]_J)dx_1 \wedge \ldots \wedge dx_k = \det(B_J)dx_1 \wedge \ldots \wedge dx_k.$$

Combining these formulas, we get

$$\begin{aligned}\varphi_B^*\varphi_A^*(dx_1 \wedge \ldots \wedge dx_k) &= \sum_J \det(A^J)\varphi_B^* dx_J \\ &= \Big(\sum_J \det(A^J)\det(B_J)\Big)dx_1 \wedge \ldots \wedge dx_k.\end{aligned}$$

This implies the lemma. ∎

Corollary IV.2.7. *For two matrices $A, B \in M_{n,k}(\mathbb{R})$ we have*

$$\det(A^\top B) = \sum_J \det(A_J)\det(B_J),$$

and in particular

$$\det(A^\top A) = \sum_J \det(A_J)^2.$$

Proof. This follows from Proposition IV.2.6 because

$$\det\big((A^\top)^J\big) = \det\big((A_J)^\top\big) = \det(A_J). \qquad \blacksquare$$

Remark IV.2.8. For k vectors $v_1, \ldots, v_k$, forming the columns of the matrix $A \in M_{n,k}(\mathbb{R})$, we define the k-dimensional volume of the flat

$$S := \sum_{j=1}^k [0,1]v_j \quad \text{by} \quad \mathrm{vol}_k(S) := \sqrt{\det(A^\top A)}.$$

(a) Here are some justifying arguments for this interpretation. The first requirement is that if $v_1, \ldots, v_k \in \mathbb{R}^k$, considered as the subspace $\mathbb{R}^k \times \{0\}$ of $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$, then we should have

$$\mathrm{vol}_k\Big(\sum_{j=1}^k [0,1]v_j\Big) = \mathrm{vol}\Big(\sum_{j=1}^k [0,1]v_j\Big) = |\det(\widetilde{A})|,$$

where $\widetilde{A}$ is the matrix obtained from A by erasing the rows $k+1, \ldots, n$. In fact, if all these rows vanish, we have $A^\top A = \widetilde{A}^\top \widetilde{A}$, which leads to

$$\sqrt{\det(A^\top A)} = \sqrt{\det(\widetilde{A}^\top \widetilde{A})} = \sqrt{\det(\widetilde{A})^2} = |\det(\widetilde{A})|.$$

The next requirement is that the k-dimensional volume of S should not change under isometries of $\mathbb{R}^n$. This means in particular that for each orthogonal matrix $Q \in \mathrm{O}_n(\mathbb{R})$ we should have

$$\mathrm{vol}_k(S) = \mathrm{vol}_k(QS).$$

For any k-tuple of vectors $v_1, \ldots, v_k$ there exists an orthogonal matrix $Q \in \mathrm{O}_n(\mathbb{R})$ with $Qv_j \in \mathbb{R}^k$ for each j (Exercise), i.e., the rows $k+1, \ldots, n$ of $QA \in M_{n,k}(\mathbb{R})$ vanish. With the preceding arguments, we thus arrive at

$$\begin{aligned}\mathrm{vol}_k(S) &= \mathrm{vol}_k(QS) = |\det(\widetilde{(QA)})| = \sqrt{\det((QA)^\top QA)} \\ &= \sqrt{\det((A^\top Q^\top QA)} = \sqrt{\det(A^\top A)}.\end{aligned}$$

(b) For any k-element subset $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$, let

$$p_J \colon \mathbb{R}^n \to \mathbb{R}^k, \quad x \mapsto (x_{j_1}, \ldots, x_{j_k})$$

denote the corresponding projection onto $\mathbb{R}^k$. Then each set $p_J(S)$ is a flat spanned by the vectors $p_J(v_1), \ldots, p_J(v_k)$ which are the columns of the matrix A_J. Therefore Corollary IV.2.7 implies that

$$\mathrm{vol}_k(S)^2 = \det(A^\top A) = \sum_J \det(A_J)^2 = \sum_J \mathrm{vol}_k(p_J(S))^2.$$

For $k = 1$ and $S = [0,1]v$ we have $A = (v) \in M_{n,1}(\mathbb{R})$, so that

$$\mathrm{vol}_1(S)^2 = \det(A^\top A) = \|v\|^2 = \sum_{j=1}^n v_j^2$$

(Pythagoras' Theorem).

For $k = 2$, $n = 3$ and the parallelogram $S = [0,1]v + [0,1]w$ in $\mathbb{R}^3$ we then have

$$\mathrm{vol}_2(S)^2 = \sum_{i<j} \mathrm{vol}_2(p_{ij}(S))^2, \quad \text{where} \quad p_{ij} \colon \mathbb{R}^3 \to \mathbb{R}^2, x \mapsto (x_i, x_j).$$

With

$$A = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix},$$

this leads to

$$\mathrm{vol}_2(S)^2 = \sum_{i<j} \det(A_{ij})^2 = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}^2 + \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix}^2 + \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix}^2 = \|v \times w\|^2,$$

where

$$v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

is the vector product in $\mathbb{R}^3$. ■

Definition IV.2.9. If $V \subseteq \mathbb{R}^k$ is open and $\varphi \colon V \to \mathbb{R}^n$ is an injective C^1-map, then we define the *k-dimensional volume of* $\varphi(K)$, $K \subseteq V$ a compact subset, by

$$\mathrm{vol}_k(\varphi(K)) := \int_K \sqrt{g(x)}\, dx,$$

where $g(x) := \det([d\varphi(x)]^\top [d\varphi(x)])$. ■

Example IV.2.10. If $V = \mathbb{R}^k$ and $\varphi(x) = \sum_{j=1}^k x_j v_j$, then $A := [d\varphi(x)] \in M_{n,k}(\mathbb{R})$ is the matrix whose columns are $v_1, \ldots, v_k$. If $C \subseteq \mathbb{R}^k$ is the closed unit cube, then $S := \varphi(C) = \sum_{j=1}^k [0,1]v_j$ is a k-dimensional flat, and we get

$$\mathrm{vol}_k(S) = \int_C \sqrt{\det(A^\top A)}\, dx = \sqrt{\det(A^\top A)} \int_C dx = \sqrt{\det(A^\top A)}.$$

In this sense Definition IV.2.9 is consistent with Remark IV.2.8. ■

IV.3. The exterior differential

In this section we discuss the exterior differential for forms on open subsets of $\mathbb{R}^n$. For each p the exterior differential maps p-forms to $(p+1)$-forms and generalizes the map $f \mapsto df$, assigning to each smooth function a 1-form. We shall also see that on domains in $\mathbb{R}^3$, the exterior differential provides a unified treatment of the operators grad, rot and div from 3-dimensional vector analysis.

Definition IV.3.1. (a) For $p \in \mathbb{N}_0$, $U \subseteq \mathbb{R}^n$ open, and $\omega \in \Omega^p(U, E)$ we defined the *exterior differential* $d\omega \in \Omega^{p+1}(U, E)$ as follows. In terms of the basic representation

$$\omega = \sum_I \omega_I dx_I,$$

we define

$$d\omega := \sum_I d\omega_I \wedge dx_I = \sum_I \sum_{i=1}^n \frac{\partial \omega_I}{\partial x_i} dx_i \wedge dx_I.$$

Note that for $I = \{i_1, \ldots, i_p\}$ with $i_1 < \ldots < i_p$ we have

$$dx_i \wedge dx_I = \begin{cases} 0 & \text{if } i \in I \\ (-1)^k dx_{I \cup \{i\}} & \text{if } i_k < i < i_{k+1} \end{cases}$$

because

$$dx_i \wedge dx_I = (-1)^k dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_i \wedge dx_{i_{k+1}} \wedge \ldots \wedge dx_{i_p} = (-1)^k dx_{I \cup \{i\}}.$$

We define $\mathrm{sgn}(I, i) := (-1)^k$ if $i_k < i < i_{k+1}$. Then we may rewrite the formula for $d\omega$ as

$$d\omega = \sum_I \sum_{i \notin I} \mathrm{sgn}(I, i) \frac{\partial \omega_I}{\partial x_i} dx_{I \cup \{i\}}.$$

For $J = \{j_0, \ldots, j_p\}$ with $1 \leq j_0 < \ldots < j_p \leq n$, this means that $d\omega = \sum_J (d\omega)_J dx_J$ with

$$(d\omega)_J = \sum_{i=0}^p (-1)^i \frac{\partial \omega_{J \setminus \{j_i\}}}{\partial x_{j_i}}. \tag{4.6}$$

(b) We call ω *closed* if $d\omega = 0$ and *exact* if there exists some $\eta \in \Omega^{p+1}(U, E)$ with $d\eta = \omega$. ■

To get a better feeling for the meaning of the exterior differential, we take a closer look at some examples.

Examples IV.3.2. (1) $p=0$: For $f \in C^\infty(U,R) = \Omega^0(U,E)$ we have

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i,$$

which is consistent with the previous definition of the differential df, interpreted as a 1-form.

(2) $p=1$: For $\omega = \sum_i \omega_i dx_i$ we obtain with (4.6):

$$d\omega = \sum_{i<j} \Big(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\Big)\, dx_i \wedge dx_j.$$

If $\omega = df$, then $\omega_i = \frac{\partial f}{\partial x_i}$, so that the Schwarz Lemma implies that

$$\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} = \frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j} = 0.$$

Therefore $d(df) = 0$.

(3) $p = n-1$, $\omega \in \Omega^{n-1}(U,\mathbb{R})$: Then we write

$$\omega = \sum_{i=1}^n (-1)^{i-1} F_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n,$$

where $\widehat{dx_i}$ indicates that the factor dx_i is omitted. In terms of the basic representation, this means that

$$F_i = (-1)^{i-1} \omega_{\{1,\ldots,i-1,i+1,\ldots,n\}}.$$

We then have

$$\begin{aligned} d\omega &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial F_i}{\partial x_i} dx_i \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\ &= \Big(\sum_{i=1}^n \frac{\partial F_i}{\partial x_i}\Big) dx_1 \wedge \ldots \wedge dx_n. \end{aligned}$$

Interpreting the functions F_i as the components of a smooth vector field $F\colon U \to \mathbb{R}^n$, this means that

$$d\omega = (\operatorname{div} F) \cdot dx_1 \wedge \ldots \wedge dx_n,$$

where

$$\operatorname{div} F := \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$$

is the *divergence of* F.

(4) For $k = n$ we have $\omega = f dx_1 \wedge \ldots \wedge dx_n$, and therefore $d\omega = 0$. Note that this follows already from $\Omega^{n+1}(U,E) = \{0\}$. ∎

Remark IV.3.3. (Vector analysis in $\mathbb{R}^3$) Let $U \subseteq \mathbb{R}^3$ be an open subset.

We associate to a smooth vector field $F \in C^\infty(U, \mathbb{R}^3)$ the 1-form

$$\omega_F := \sum_i F_i dx_i$$

and the 2-form

$$\eta_F := F_1 dx_2 \wedge dx_3 - F_2 dx_1 \wedge dx_3 + F_3 dx_1 \wedge dx_2.$$

We have already seen that this identification implies that

$$d\eta_F = (\operatorname{div} F) dx_1 \wedge dx_2 \wedge dx_3.$$

It is also clear that for each smooth function $f\colon U \to \mathbb{R}$

$$df = \omega_{\operatorname{grad} f}$$

is the 1-form associated to the gradient of f.

Moreover, we have

$$d\omega_F = \eta_{\operatorname{rot} F},$$

where the *rotation* of F is defined by

$$\operatorname{rot} F := \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}.$$

In this sense we obtain uniform interpretations of the three operations grad, rot and div from vector analysis in $\mathbb{R}^3$ in terms of the exterior differential. These are the historical roots of the concept of a differential form.

In the physics literature one also finds the following interpretations. The $\mathbb{R}^3$-valued 1-form

$$d\vec{s} := \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \in \Omega^1(U, \mathbb{R}^3),$$

called the *vectorial line element*, the $\mathbb{R}^3$-valued 2-form

$$d\vec{F} := \begin{pmatrix} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{pmatrix} \in \Omega^2(U, \mathbb{R}^3),$$

called the *vectorial surface element*, and

$$dV := dx_1 \wedge dx_2 \wedge dx_3 \in \Omega^3(U, \mathbb{R})$$

is called the *volume element*.

We then have

$$df = \operatorname{grad} f \cdot d\vec{s}$$

(where $\cdot$ stands for the scalar product on $\mathbb{R}^3$),

$$d(F \cdot d\vec{s}) = \operatorname{rot}(F) \cdot d\vec{F},$$

and

$$d(G \cdot d\vec{F}) = \operatorname{div}(G) \cdot dV.$$

We shall see later how these interpretations can be nicely justified in the context of integration of p-forms. ■

We now turn to further properties of the exterior differential:

Proposition IV.3.4. *The exterior differential*

$$d\colon \Omega^p(U,E) \to \Omega^{p+1}(U,E)$$

has the following properties:
(a) *It is a linear map.*
(b) *For* $\alpha \in \Omega^p(U,\mathbb{R})$, $\beta \in \Omega^q(U,\mathbb{R})$ *we have* $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.
(c) $d(d\omega) = 0$ *for any* $\omega \in \Omega^p(U,\mathbb{R})$.
(d) *If* $\varphi\colon V \to U$ *is smooth, where* $V \subseteq \mathbb{R}^k$ *is open, then*

$$d(\varphi^*\omega) = \varphi^*(d\omega), \quad \omega \in \Omega^p(U,E).$$

Proof. (a) is a trivial consequence of the definition.

(b) We write α and β in terms of their basic representation

$$\alpha = \sum_I \alpha_I dx_I \quad \text{and} \quad \beta = \sum_J \beta_J dx_J.$$

We then have

$$\alpha \wedge \beta = \sum_{I,J} \alpha_I \beta_J dx_I \wedge dx_J = \sum_{I \cap J = \emptyset} \alpha_I \beta_J dx_I \wedge dx_J,$$

and if I and J are disjoint, we have

$$dx_I \wedge dx_J = \varepsilon_{I,J} dx_{I \cup J},$$

where $\varepsilon_{I,J} \in \{\pm 1\}$. Hence (a) implies that

$$\begin{aligned}
d(\alpha \wedge \beta) &= \sum_{I,J} d(\alpha_I \beta_J) \wedge dx_I \wedge dx_J \\
&= \sum_{I,J} d\alpha_I \wedge \beta_J \wedge dx_I \wedge dx_J + \alpha_I d\beta_J \wedge dx_I \wedge dx_J \\
&= \sum_{I,J} (d\alpha_I \wedge dx_I) \wedge (\beta_J dx_J) + (-1)^p \alpha_I dx_I \wedge (d\beta_J \wedge dx_J) \\
&= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.
\end{aligned}$$

Here we have used that $dx_I \wedge d\beta_J = (-1)^p d\beta_J \wedge dx_I$ (Proposition IV.1.4).

(c) In view of (a), we may assume that $\omega = f dx_I$ for some p-element subset I and a smooth function $f\colon U \to E$. For $p = 0$ we have already seen in Example IV.2.3 that $d(d\omega) = d(df) = 0$.

We may therefore assume that $p > 0$. Then $d\omega = df \wedge dx_I$, and (b) implies that

$$d(d\omega) = d(df) \wedge dx_I - df \wedge d(dx_I) = 0,$$

since $d(dx_I) = 0$ by definition.

(d) In view of (a) and the linearity of the pullback map φ^*, we may assume that $\omega = f dx_I$, as in (c). Then $d\omega = df \wedge dx_I$ leads to

$$\begin{aligned}\varphi^*(d\omega) &\overset{IV.2.2.}{=} \varphi^*(df) \wedge \varphi^*(dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \\ &\overset{IV.2.2.}{=} d(\varphi^* f) \wedge \varphi^*(dx_{i_1}) \wedge \ldots \wedge \varphi^*(dx_{i_p}) \\ &\overset{IV.2.2.}{=} d(\varphi^* f) \wedge d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_p}.\end{aligned}$$

In view of

$$\varphi^*\omega = \varphi^* f \cdot \varphi^* dx_I = \varphi^* f \cdot d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_p}$$

and (b), we have

$$d(\varphi^*\omega) = d(\varphi^* f) \wedge d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_p} + \varphi^* f \cdot d(d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_p}),$$

but (c) and iterated application of (b) lead to

$$d(d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_p}) = 0.$$

This proves (d). ■

Definition IV.3.5. (de Rham cohomology) Let $U \subseteq \mathbb{R}^n$ be an open subset and E a finite-dimensional vector space. We write

$$Z^p_{\mathrm{dR}}(U, E) := \{\omega \in \Omega^p(U, E) \colon d\omega = 0\}$$

for the space of closed forms and

$$B^p_{\mathrm{dR}}(U, E) := \{\omega \in \Omega^p(U, E) \colon (\exists \eta \in \Omega^{p-1}(U, E))\ d\eta = \omega\}$$

for the space of exact forms. In view of $d \circ d = 0$, all exact forms are closed, i.e.,

$$B^p_{\mathrm{dR}}(U, E) \subseteq Z^p_{\mathrm{dR}}(U, E),$$

and we can form the quotient vector space

$$H^p_{\mathrm{dR}}(U, E) := Z^p_{\mathrm{dR}}(U, E)/B^p_{\mathrm{dR}}(U, E).$$

It is called the *p-th E-valued de Rham cohomology space of U*. ■

The important point in the definition of the de Rham cohomology spaces is that they are topological invariants, i.e., if two domains U and U' are homeomorphic, then their cohomology spaces are isomorphic. One can show that for each starlike domain U and, more generally, for each contractible domain, all cohomology spaces $H^p_{\mathrm{dR}}(U, E)$ vanish (Poincaré Lemma). This means that each closed p-form is exact. We shall come back to this point later. It is the first step to the application of differential forms in algebraic topology.

IV.4. Fiber integration

In this section we describe the technique of fiber integration. It leads in particular to a proof of the Poincaré Lemma, which ensures that for smoothly contractible domains $U \subseteq \mathbb{R}^n$, the de Rham cohomology groups $H^p_{\rm dR}(U,E)$ vanish for $p > 0$.

First we slightly generalize the domain of smooth functions, resp., differential forms on subsets of $\mathbb{R}^n$.

Definition IV.4.1. Let $U \subseteq \mathbb{R}^n$ be a subset with dense interior U^0 and E a finite-dimensional vector space.

(a) A function $f\colon U \to E$ is called a *C^k-function*, $k \in \mathbb{N} \cup \{\infty\}$, if $f \mid_{U^0}$ is a C^k-function and all partial derivatives of order $\leq k$ extend to continuous functions on all of U. We write $C^k(U,E)$ for the space of C^k-functions $f\colon U \to E$.

(b) An *E-valued p-form on U of class C^k* is a C^k-map

$$\omega\colon U \to \mathrm{Alt}^p(\mathbb{R}^n, E).$$

This means that ω has a basic representation of the form

$$\omega = \sum_I \omega_I dx_I \quad \text{with} \quad \omega_I \in C^k(U,E). \qquad \blacksquare$$

Definition IV.4.2. (Fiber integration) Let $U \subseteq \mathbb{R}^n$ be open and $I := [0,1]$ the unit interval. We write elements of $I \times U$ as pairs (t,x) and, accordingly,

$$dt = dx_0, \quad dx_1, \ldots, dx_n$$

for the basic 1-forms on $\mathbb{R} \times \mathbb{R}^n \supseteq I \times U$. We further write $(e_0, \ldots, e_n)$ for the canonical basis of $\mathbb{R} \times \mathbb{R}^n$. Note that $I \times U$ has dense interior $]0,1[\times U$, so that the notion of a smooth function and a smooth differential form on $I \times U$ are defined (Definition IV.4.1).

For $\omega \in \Omega^{p+1}(I \times U, E)$ we define $\mathcal{F}(\omega) \in \Omega^p(U,E)$ by

$$\mathcal{F}(\omega)_x(v_1, \ldots, v_p) = \int_0^1 \omega_{(t,x)}(e_0, v_1, \ldots, v_p)\, dt.$$

Since integrals of smooth functions of $(n+1)$-variable are smooth functions of n-variables (differentiation under the integral), $\mathcal{F}(\omega)$ defines indeed a smooth E-valued p-form on U. For smooth functions $f \in C^\infty(I \times U, E)$ we put $\mathcal{F}(f) = 0$. We thus obtain a series of linear maps

$$\mathcal{F}\colon \Omega^{p+1}(I \times U, E) \to \Omega^p(U,E),$$

called *fiber integrals*.

Note that if ω is a $(p+1)$-form of class C^k, then $\mathcal{F}(\omega)$ is also of class C^k. $\blacksquare$

Remark IV.4.3. Let us evaluate the fiber integrals of some basic forms.

For $\omega = f dt \wedge dx_J$ with a p-element subset $J \subseteq \{1, \ldots, n\}$ we have

$$(dt \wedge dx_J)(e_0, v_1, \ldots, v_p) = dx_J(v_1, \ldots, v_p)$$

for $v_i \in \mathbb{R}^n$ because $dt(v_i) = 0$ for $i = 1, \ldots, p$. Hence

$$\begin{aligned}\mathcal{F}(\omega)_x(v_1, \ldots, v_p) &= \int_0^1 f(t,x)(dt \wedge dx_J)(e_0, v_1, \ldots, v_p)\, dt \\ &= \Big(\int_0^1 f(t,x)\, dt\Big) \cdot dx_J(v_1, \ldots, v_p),\end{aligned}$$

so that

$$\mathcal{F}(\omega)_x = \Big(\int_0^1 f(t,x)\, dt\Big) \cdot dx_J.$$

For $\omega = f dx_J$ with a $(p+1)$-element subset $J \subseteq \{1, \ldots, n\}$ we have $\omega_x(e_0, \ldots) = 0$, and therefore $\mathcal{F}(\omega) = 0$. ■

Theorem IV.4.4. (The Homotopy Formula) *For each $\omega \in \Omega^p(I \times U, E)$ we have*

$$\mathcal{F}(d\omega) + d\mathcal{F}(\omega) = j_1^*\omega - j_0^*\omega,$$

where $j_t \colon U \to I \times U, x \mapsto (t,x)$.

Proof. Since fiber integration and the exterior differential are linear, it suffices to verify the formula for differential forms of the type $\omega = f dx_J$, $f \in C^\infty(I \times U, E)$.

Case 1: $0 \in J$, i.e., $\omega = f dt \wedge dx_K$, $K \subseteq \{1, \ldots, n\}$ a $(p-1)$-element subset. Then

$$d\omega = df \wedge dt \wedge dx_K = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i \wedge dt \wedge dx_K = -\sum_{i=1}^n \frac{\partial f}{\partial x_i} dt \wedge dx_i \wedge dx_K.$$

Therefore Remark IV.4.3 implies that

$$\mathcal{F}(d\omega) = -\sum_{i=1}^n \Big(\int_0^1 \frac{\partial f(t,x)}{\partial x_i}\, dt\Big) dx_i \wedge dx_K.$$

On the other hand,

$$\mathcal{F}(\omega) = \Big(\int_0^1 f(t,x)\, dt\Big) dx_K,$$

so that differentiation under the integral sign leads to

$$d\mathcal{F}(\omega) = \sum_{i=1}^n \Big(\int_0^1 \frac{\partial f(t,x)}{\partial x_i}\, dt\Big) dx_i \wedge dx_K = -\mathcal{F}(d\omega).$$

Finally, we observe that $j_t^*(dt) = 0$ for each t because the t-component of the maps $j_t: U \to I \times U$ is constant (Exercise IV.2). Therefore

$$\mathcal{F}(d\omega) + d\mathcal{F}(\omega) = 0 = j_1^*\omega - j_0^*\omega$$

in this case.

Case 2: $0 \notin J$, i.e., $\omega = f dx_J$, $J \subseteq \{1, \ldots, n\}$ a p-element subset. Then

$$d\omega = df \wedge dx_J = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_J = \frac{\partial f}{\partial t} dt \wedge dx_J + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_J,$$

and therefore

$$\mathcal{F}(d\omega)_x = \Big(\int_0^1 \frac{\partial f(t,x)}{\partial t}\, dt\Big) \cdot dx_J = f(1,x) dx_J - f(0,x) dx_J = (j_1^*\omega)_x - (j_0^*\omega)_x.$$

Since $\mathcal{F}(\omega) = 0$ (Remark IV.4.3), the assertion also follows in this case. ■

The Homotopy Formula has many applications. Below we only discuss some.

Corollary IV.4.5. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open and $\omega \in \Omega^p(V, E)$ a closed p-form. Further let $\varphi: I \times U \to V$ be a smooth map and $\varphi_t(x) := \varphi(t,x)$. Then the p-forms*

$$\varphi_1^*\omega - \varphi_0^*\omega \in \Omega^p(U, E)$$

are exact, so that the induced maps

$$\varphi_1^*: H^p_{\mathrm{dR}}(V, E) \to H^p_{\mathrm{dR}}(U, E), \quad [\omega] \mapsto [\varphi_1^*\omega]$$

and

$$\varphi_0^*: H^p_{\mathrm{dR}}(V, E) \to H^p_{\mathrm{dR}}(U, E), \quad [\omega] \mapsto [\varphi_0^*\omega]$$

coincide.

Two such maps $\varphi_0, \varphi_1: U \to V$ are called *(smoothly homotopic)*.

Proof. In view of $\varphi_t = \varphi \circ j_t$, $j_t(x) = (t,x)$, we have

$$\begin{aligned}\varphi_1^*\omega - \varphi_0^*\omega &= j_1^*\varphi^*\omega - j_0^*\varphi^*\omega = \mathcal{F}(d(\varphi^*\omega)) + d\mathcal{F}(\varphi^*\omega) \\ &= \mathcal{F}(\varphi^* d\omega) + d\mathcal{F}(\varphi^*\omega) = d\mathcal{F}(\varphi^*\omega).\end{aligned}$$

■

Definition IV.4.6. An open subset $U \subseteq \mathbb{R}^n$ is called *smoothly contractible* if there exists a smooth map $\varphi: I \times U \to U$ with $\varphi_1 = \mathrm{id}_U$ and φ_0 constant. ■

Examples IV.4.7. (a) If $U \subseteq \mathbb{R}^n$ is starlike with respect to x_0, then U is smoothly contractible. We simply take

$$\varphi(t,x) := tx + (1-t)x_0.$$

(b) If $U \subseteq \mathbb{R}^n$ is smoothly contractible and $f\colon U \to V \subseteq \mathbb{R}^n$ a diffeomorphism, then the domain $f(U)$ is also smoothly contractible. In fact, if $\varphi\colon I \times U \to U$ as in Definition IV.4.6, then

$$\psi(t,x) := f\big(\varphi(t, f^{-1}(x))\big)$$

defines a smooth map $I \times V \to V$ with

$$\psi_1 = f \circ \varphi_1 \circ f^{-1} = \mathrm{id}_V \quad \text{ and } \quad \psi_0 = f \circ \varphi_0 \circ f^{-1}$$

is constant. ■

Theorem IV.4.8. (Poincaré Lemma) *Let $U \subseteq \mathbb{R}^n$ be smoothly contractible. Then*

$$H^0_{\mathrm{dR}}(U,E) \cong E$$

(the space of constant functions), and

$$H^p_{\mathrm{dR}}(U,E) = 0 \quad \textit{for} \quad p > 0.$$

In other words, each locally constant function on U is constant, and each closed p-form, $p > 0$, on U is exact.

Proof. Let $\varphi\colon I \times U \to U$ be a smooth map with $\varphi_1 = \mathrm{id}_U$ and $\varphi_0 = x_0$ constant.

For the case $p = 0$ we observe thay the curve $I \to U, t \mapsto \varphi(t,x)$ links x to x_0, so that U is arcwise connected. Hence each closed 0-form, i.e., each locally constant function, is constant.

Now we assume that $p > 0$. Then Corollary IV.4.5 implies for each closed p-form ω on U, that $\varphi_1^*\omega - \varphi_0^*\omega$ is exact. As $\varphi_1 = \mathrm{id}_U$, we have $\varphi_1^*\omega = \omega$, and since φ_0 is constant and ω is of positive degree, $\varphi_0^*\omega = 0$ (Exercise IV.2). Hence each closed form ω on U is exact with $\omega = d\mathcal{F}(\varphi^*\omega)$. ■

Corollary IV.4.9. $H^0_{\mathrm{dR}}(\mathbb{R}^n, E) \cong E$ *and* $H^p_{\mathrm{dR}}(\mathbb{R}^n, E) = 0$ *for* $p > 0$. ■

Corollary IV.4.10. *If $U \subseteq \mathbb{R}^n$ is open, $x \in U$ and $\omega \in \Omega^p(U,E)$ a closed p-form, $p > 0$, then there exists a neighborhood V of x such that $\omega|_V$ is exact.*

Proof. Choose a convex neighborhood V of x and apply the Poincaré Lemma. ■

IV.5. Integration of differential forms on $\mathbb{R}^n$

In this short section we define integrals of n-form on open subsets of $\mathbb{R}^n$ and prove the corresponding transformation formula by reducing it to the well-known formula for Lebesgue, resp., Riemann integrals.

Definition IV.5.1. Let $U \subseteq \mathbb{R}^n$ be open.

(a) For any continuous n-form $\omega = f dx_1 \wedge \ldots \wedge dx_n$, $f \in C(U, \mathbb{R})$, and each compact subset $K \subseteq U$, we define

$$\int_K \omega := \int_K f dx_1 \wedge \ldots \wedge dx_n := \int_K f(x)\, dx,$$

where the integral on the right is interpreted as a Lebesgue integral. At this point one can also work with the n-dimensional Riemann integral, but then one has to assume that K is a Riemann measurable subset, i.e., its boundary has measure zero.

(b) For a function $f: U \to \mathbb{R}$ we define its *support* by

$$\mathrm{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}},$$

which is the smallest closed subset for which f vanishes on its complement.

For any compactly supported continuous function $f: U \to \mathbb{R}$ we define

$$\int_U f dx_1 \wedge \ldots \wedge dx_n := \int_{\mathrm{supp}(f)} f(x)\, dx.$$

Note that the compact subset $\mathrm{supp}(f) \subseteq U$ can be covered by finitely many non-overlapping small cubes of the form $C_x := \prod_{i=1}^n [x_i, x_i + \varepsilon[$, so that the integral on the right makes always sense as a Riemann integral. ■

Definition IV.5.2. Let $U, V \subseteq \mathbb{R}^n$ be open subsets. A C^1-map $\varphi: U \to V$ is said to be *orientation preserving* if

$$\det(d\varphi(x)) > 0 \quad \text{for all} \quad x \in U$$

and *orientation resersing* if

$$\det(d\varphi(x)) < 0 \quad \text{for all} \quad x \in U.$$

Note that, if U is connected and $d\varphi(x)$ is invertible for each $x \in U$, then φ is either orientation preserving or reversing (Exercise IV.4).

Proposition IV.5.3. (Oriented Transformation Formula) *Let $U, V \subseteq \mathbb{R}^n$ be open subsets and $\varphi: U \to V$ a C^1-diffeomorphism. Futher let ω be a continuous n-form on V and $A \subseteq U$ compact. Then*

$$\int_{\varphi(A)} \omega = \varepsilon \int_A \varphi^*\omega,$$

where $\varepsilon = 1$ if φ is orientation preserving and $\varepsilon = -1$ if φ is orientation reversing.

Proof. We write $\omega = f dx_1 \wedge \ldots \wedge dx_n$ for some $f \in C(V, \mathbb{R})$. We then have

$$\varphi^*\omega = (f \circ \varphi) d\varphi_1 \wedge \ldots \wedge d\varphi_n = (f \circ \varphi) \det(d\varphi(x)) dx_1 \wedge \ldots \wedge dx_n$$

(Example IV.2.4(4)), which leads to

$$\int_A \varphi^*\omega = \int_A (f \circ \varphi)(x) \det(d\varphi(x)) dx.$$

On the other hand, the Transformation Formula for Lebesgue (Riemann) integrals leads to

$$\begin{aligned} \int_{\varphi(A)} \omega = \int_{\varphi(A)} f(x) dx &= \int_A f(\varphi(x)) |\det(d\varphi(x))|\, dx \\ &= \varepsilon \int_A f(\varphi(x)) \cdot \det(d\varphi(x))\, dx. \end{aligned}$$

■

The following definition shows how the path integrals showing up in Analysis II or Complex Analysis can be viewed as integrals of 1-forms.

Definition IV.5.4. Let $U \subseteq \mathbb{R}^n$ be an open subset and ω a continuous E-valued 1-form on U. Then for each smooth path $\gamma: [a, b] \to U$ we define

$$\int_\gamma \omega := \int_{[a,b]} \gamma^*\omega.$$

We have $(\gamma^*\omega)_t = f(t)\, dt$ with

$$f(t) = (d\gamma(t)^*\omega_{\gamma(t)})(1) = \omega_{\gamma(t)}(d\gamma(t)(1)) = \omega_{\gamma(t)}(\gamma'(t)),$$

which leads to

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt.$$

More explicitly, we obtain for $\omega = \sum_i \omega_i dx_i$ the relation

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt = \sum_{i=1}^n \int_a^b \omega_i(\gamma(t))\gamma_i'(t)\, dt.$$

If $\gamma\colon [a,b] \to U$ is piecewise smooth in the sense that γ is continuous and there exist

$$t_0 = a < t_1 < \ldots < t_k = b$$

such that the restrictions $\gamma_j := \gamma|_{[t_j, t_{j+1}]}$ are smooth, then we put

$$\int_\gamma \omega := \sum_{j=0}^{k-1} \int_{\gamma_j} \omega$$

and observe that the right hand side does not depend on the subdivision of the interval $[a,b]$. ■

Lemma IV.5.5. *If $f \in C^1(U,\mathbb{R})$ and $\gamma\colon [a,b] \to U$ a piecewise smooth path, then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

In particular, the integral vanishes if γ is closed, i.e., $\gamma(a) = \gamma(b)$.

Proof. The function $f \circ \gamma\colon [a,b] \to \mathbb{R}$ is continuous and piecewise C^1 with $(f\circ\gamma)'(t) = df(\gamma(t))(\gamma'(t))$. Hence the Fundamental Theorem of Calculus implies that

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b (f \circ \gamma)'(t)\, dt = \int_\gamma df. \qquad ■$$

Remark IV.5.6. If $U \subseteq \mathbb{C} \cong \mathbb{R}^n$ is open and $f\colon U \to \mathbb{C}$ is a holomorphic function, then we associate to f the *holomorphic* 1*-form*

$$\omega = f dz \in \Omega^1(U,\mathbb{C}).$$

Here $dz \in \Omega^1(U,\mathbb{C})$ is the 1-form with the basic representation $dz = dx + idy$, where we write $z = x + iy$ for elements of U. For $f = u + iv$ this means that

$$f dz = (u + iv)(dx + idy) = (udx - vdy) + i(vdx + udy).$$

For any piecewise smooth path $\gamma = \gamma_1 + i\gamma_2\colon [a,b] \to U$ we then have

$$(dz)_{\gamma(t)}(\gamma'(t)) = \gamma_1'(t) + i\gamma_2'(t) = \gamma'(t),$$

so that

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt,$$

which is the formula usually used for path integrals in Complex Analysis. ■

Exercises for Chapter IV

Exercise IV.1. Let $\omega \in \Omega^p(U,E)$, where U is an open subset of $\mathbb{R}^n$ and E a finite-dimensional vector space. Show that for any p-tuple $(v_1,\ldots,v_p) \in (\mathbb{R}^n)^p$ the function

$$x \mapsto \omega_x(v_1,\ldots,v_p), \quad U \to E$$

is smooth. Hint: If $f\colon U \to V$ is smooth and $\alpha\colon V \to W$ a linear map, then $\alpha \circ f\colon U \to W$ is smooth. ■

Exercise IV.2. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. Show that if $\varphi\colon U \to V$ is a constant map and $\omega \in \Omega^p(V,E)$ with $p > 0$, then $\varphi^*\omega = 0$. ■

Exercise IV.3. (Explicit Poincaré Lemma) Let $U \subseteq \mathbb{R}^n$ be star-like with respect to x_0 and $\omega \in \Omega^p(U,E)$ a closed p-form, $p > 0$. Show that:

(1) $\omega = d\eta$ holds for the $(p-1)$-form η defined by

$$\eta_x(v_1,\ldots,v_{p-1}) := \int_0^1 t^{p-1}\omega_{tx+(1-t)x_0}(x - x_0, v_1,\ldots,v_{p-1})\,dt.$$

Hint: Have a look at the proof of the Poincaré Lemma to see that $\eta = \mathcal{F}(\varphi^*\omega)$ for a suitable map $\varphi\colon I \times U \to U$.

(2) For $j \not\in I$ and $|I| = p-1$, we have

$$(dx_{i_1} \wedge \ldots dx_{i_k} \wedge dx_j \wedge dx_{i_{k+1}} \wedge \ldots \wedge dx_{p-1})(x, e_{i_1},\ldots,e_{i_{p-1}}) = \operatorname{sgn}(I,j)x_j,$$

where $\operatorname{sgn}(I,j) := (-1)^k$ if $i_k < j < i_{k+1}$. Hint: Remark III.2.5(b).

(3) Suppose that $\omega = \sum_J \omega_J dx_J$ is the basic representation of ω. Assume $x_0 = 0$ and show that the basic representation $\eta = \sum_I \eta_I dx_I$ satisfies

$$\eta_I(x) = \int_0^1 (-1)^{p-1}\Big(\sum_{j\not\in I} \operatorname{sgn}(I,j)\omega_{I\cup\{j\}}(tx)x_j\Big)\,dt.$$

(4) Specialize in (3) to the case $p = 1$ and compare with the formula you know from Analysis II. ■

Exercise IV.4. Let $U,V \subseteq \mathbb{R}^n$ be open subsets and $\varphi\colon U \to V$ a C^1-map with $d\varphi(x)$ invertible for each $x \in U$. Show that if U is connected, then then φ is either orientation preserving or reversing. ■

Exercise IV.5. (Lie brackets of vector fields; revisited) Let $U \subseteq \mathbb{R}^n$ be open.

(a) Let $X, Y \in \mathcal{V}(U)$ be smooth vector fields on U and $f, g \in C^\infty(U, \mathbb{R})$. Show that

$$[fX, gY] = f(X.g)Y - g(Y.f)X + fg[X, Y].$$

(b) Let $\vec{e}_i(x) = (x, e_i)$ denote the basic vector field. Show that each vector field $X \in \mathcal{V}(U)$ has a unique representation of the form $X = \sum_{i=1}^n f_i \vec{e}_i, \quad f_i \in C^\infty(U, \mathbb{R})$.

(c) Show that

$$\Big[\sum_i f_i \vec{e}_i, \sum_j g_j \vec{e}_j\Big] = \sum_{j=1}^n \Big(\sum_{i=1}^n f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i}\Big) \vec{e}_j. \qquad \blacksquare$$

Exercise IV.6. (Vector fields and differential forms) For $\omega \in \Omega^p(U, \mathbb{R})$ and $X_1, \ldots, X_p \in \mathcal{V}(U)$ we define a smooth function $\omega(X_1, \ldots, X_p)$ by

$$\omega(X_1, \ldots, X_p)(x) := \omega_x(X_1(x), \ldots, X_p(x)), x \in U.$$

Show that for $p = 1$ we have

$$d\omega(X_1, X_2) = X_1.\omega(X_2) - X_2.\omega(X_1) - \omega([X_1, X_2]).$$

Proceed along the following steps:

(1) Both sides are $C^\infty(U, \mathbb{R})$-linear in X_1 and X_2. Hint: Ex. IV.5.

(2) Verify the formula for the basic vector fields $X_1 = \vec{e}_i$ and $X_2 = \vec{e}_j$.

V. Differential forms on manifolds

In this chapter we eventually turn to differential forms on smooth manifolds and how they can be integrated. This requires several new concepts which are developed along the way. One is the concept of a smooth partition of unity, and another one of central importance is the notion of an oriented manifold.

V.1. Basic operations on differential forms

Throughout this section M denotes a smooth n-dimensional manifold.

Definition V.1.1. (a) Let $k \in \mathbb{N}_0$ and E a finite-dimensional real vector space. An *E-valued k-form*, or *differential form of degree k*, on M is a function

$$\omega: M \to \dot{\bigcup}_{p\in M} \mathrm{Alt}^k(T_p(M), E) \quad \text{with} \quad \omega(p) \in \mathrm{Alt}^k(T_p(M), E), p \in M.$$

This means that we assign to each $p \in M$ an alternating k-linear map $\omega_p := \omega(p): T_p(M)^k \to E$.

(b) If $f: M \to N$ is a smooth map and ω an E-valued k-form on N, then

$$(f^*\omega)_p := T_p(f)^*\omega_{f(p)}$$

defines an E-valued k-form on M.

If $g: L \to M$ is another smooth map, then we have

$$(f \circ g)^*\omega = g^*(f^*\omega) \tag{5.1.1}$$

because the Chain Rule and Remark III.2.12 imply that

$$\begin{aligned}\big((f \circ g)^*\omega\big)_p &= T_p(f \circ g)^*\omega_{f(g(p))} = \big(T_{g(p)}(f) \circ T_p(g)\big)^*\omega_{f(g(p))} \\ &= T_p(g)^*T_{g(p)}(f)^*\omega_{f(g(p))} = T_p(g)^*(f^*\omega)_{g(p)} = (g^*(f^*\omega))_p.\end{aligned}$$

(c) We call a k-form ω on M *smooth* if for all charts (φ, U) the pullbacks $(\varphi^{-1})^*\omega$ are smooth k-forms on U. It actually suffices that for each point $p \in M$ there exists a chart (φ, U) with $p \in M$ and such that $(\varphi^{-1})^*\omega$ is smooth (Exercise V.1).

We write $\Omega^k(M, E)$ for the set of smooth E-valued k-forms on M and put

$$\Omega(M, E) := \bigoplus_{k\in\mathbb{N}_0} \Omega^k(M, E). \qquad \blacksquare$$

Example V.1.2. (a) For $k = 0$ we have $\mathrm{Alt}^0(T_p(M), E) = E$, so that smooth E-valued 0-forms are simply smooth functions $f\colon M \to E$. In this sense we identify

$$C^\infty(M, E) \cong \Omega^0(M, E).$$

(b) For $p = 1$ we obtain the so-called *Pfaffian forms*, or 1-forms. Each smooth 1-form $\omega \in \Omega^1(M, E)$ can be viewed as a function

$$\omega\colon T(M) \to E, \quad v \in T_p(M) \mapsto \omega_p(v)$$

which is linear on each space $T_p(M)$.

For each diffeomorphism $\psi\colon V \subseteq \mathbb{R}^n \to \psi(V) \subseteq M$ the map $T(\psi)\colon T(V) \cong V \times \mathbb{R}^n \to T(\psi(V)) \subseteq T(M)$ is a diffeomorphism which is linear on each space $T_x(V)$. The function $T(V) \to E$ associated to the pullback $\psi^*\omega$ is then given by

$$(x, v) \mapsto (\psi^*\omega)_x(v) = \omega_{\psi(x)}(T_x(\psi)v) = \big(\omega \circ T(\psi)\big)(x, v).$$

Therefore ω is a smooth 1-form if and only if the corresponding function $T(M) \to E$ is smooth.

For each smooth function $f\colon M \to E$ its differential df is a smooth 1-form:

$$(df)_x(v) := df(x)(v).$$

The corresponding smooth function $df\colon T(M) \to E$ is simply the second component of the smooth function $T(f)\colon T(M) \to T(E) \cong E \times E$ (cf. Definition II.1.6). ■

Now we turn to algebraic operations on differential forms on manifolds.

Definition V.1.3. Let E be a finite-dimensional vector space and M a smooth n-dimensional manifold.

(a) For each $k \in \mathbb{N}_0$ the set $\Omega^k(M, E)$ carries a natural vector space structure defined by

$$(\omega + \eta)_p := \omega_p + \eta_p, \quad (\lambda\omega)_p := \lambda\omega_p$$

for $\omega, \eta \in \Omega^k(M, E), \lambda \in \mathbb{R}$, $p \in M$.

(b) For $f \in C^\infty(M, \mathbb{R})$ and $\omega \in \Omega^k(M, E)$ we define the product $f\omega \in \Omega^k(M, E)$ by

$$(f\omega)_p := f(p)\omega_p.$$

For each diffeomorphism $\psi\colon V \to \psi(V) \subseteq M$ we then have

$$(\psi^*(f\omega))_x = T_x(\psi)^*((f\omega)_{\psi(x)}) = f(\psi(x))T_x(\psi)^*\omega_{\psi(x)} = \big((f \circ \psi) \cdot (\psi^*\omega)\big)_x,$$

i.e., $\psi^*(f\omega) = (f \circ \psi) \cdot \psi^*\omega$. Therefore Definition IV.1.3 implies that $f\omega$ is a smooth k-form on M. We thus obtain on $\Omega^k(M, E)$ the structure of a *module* of the algebra $C^\infty(M, \mathbb{R})$ (Exercise).

(c) More generally, we have the *exterior product*

$$\Omega^p(M,\mathbb{R}) \times \Omega^q(M,\mathbb{R}) \to \Omega^{p+q}(M,\mathbb{R}), \quad (\omega,\eta) \mapsto \omega \wedge \eta, \quad (\omega \wedge \eta)_x := \omega_x \wedge \eta_x$$

(Definition III.2.3).

To verify that the exterior product of smooth forms is smooth, we observe that for each diffeomorphism $\psi: V \to \psi(V) \subseteq M$ we have

$$\begin{aligned} &(\psi^*(\omega \wedge \eta))_x = T_x(\psi)^*(\omega \wedge \eta)_{\psi(x)} = T_x(\psi)^*\big(\omega_{\psi(x)} \wedge \eta_{\psi(x)}\big) \\ &= (T_x(\psi)^*\omega_{\psi(x)}) \wedge (T_x(\psi)^*\eta_{\psi(x)}) = (\psi^*\omega)_x \wedge (\psi^*\eta)_x = (\psi^*\omega \wedge \psi^*\eta)_x. \end{aligned}$$

Since exterior products of smooth forms on V are smooth, we obtain the corresponding assertion for smooth forms on M.

For $p = 0$ we have $\Omega^0(M,\mathbb{R}) \cong C^\infty(M,\mathbb{R})$, which leads to

$$f \wedge \omega = f\omega, \quad f \in C^\infty(M,\mathbb{R}), \omega \in \Omega^p(M,\mathbb{R}). \qquad \blacksquare$$

From Lemma III.2.4 and Proposition III.2.6 we obtain as in the proof of Proposition IV.1.4:

Proposition V.1.4. *The space $\Omega(M,\mathbb{R})$ is a graded commutative associative algebra with respect to the exterior product.* ■

Proposition V.1.5. *Let $f: M \to N$ be a smooth map between manifolds and E a finite-dimensional real vector space. Then the pullback operation has the following properties:*

(1) *$f^*\Omega^k(N,E) \subseteq \Omega^k(M,E)$, i.e., the pullback of a smooth form is smooth. Moreover, the map $f^*: \Omega^k(N,E) \to \Omega^k(N,E)$ is linear.*

(2) *$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$ for $\alpha \in \Omega^p(M,\mathbb{R}), \beta \in \Omega^q(M,\mathbb{R})$.*

(3) *$f^*g = g \circ f$ and $f^*(dg) = d(f^*g)$ for $g \in C^\infty(N,E) = \Omega^0(N,E)$.*

Proof. (1) The main point is the smoothness of $f^*\omega$. The linearity of f^* is clear.

(a) First we consider the special case where M is an open subset of some $\mathbb{R}^n$. If (φ, U) is a chart of N, then we have on $f^{-1}(U)$ the identity

$$f^*\omega = f^*(\varphi^{-1} \circ \varphi)^*\omega = f^*\varphi^*\big((\varphi^{-1})^*\omega\big) = (\varphi \circ f)^*\big((\varphi^{-1})^*\omega\big),$$

so that the smoothness of $f^*\omega$ follows from Definition IV.2.1, the smoothness of $\varphi \circ f$ and the smoothness of the k-form $(\varphi^{-1})^*\omega$ on $\varphi(U)$.

(b) Now we consider the general case. Let (φ, U) be an n-dimensional chart of M. Then we have on $\varphi(U) \subseteq \mathbb{R}^n$:

$$(\varphi^{-1})^*(f^*\omega) = (f \circ \varphi^{-1})^*\omega,$$

and since $f \circ \varphi^{-1}: \varphi(U) \to N$ is smooth, the assertion follows from (a).

(2) For each $x \in M$ we have

$$\begin{aligned}(f^*(\alpha \wedge \beta))_x &= T_x(f)^*(\alpha \wedge \beta)_{f(x)} = T_x(f)^*(\alpha_{f(x)} \wedge \beta_{f(x)}) \\ &\overset{III.2.13}{=} T_x(f)^*\alpha_{f(x)} \wedge T_x(f)^*\beta_{f(x)} = (f^*\alpha)_x \wedge (f^*\beta)_x \\ &= \big(f^*\alpha \wedge f^*\beta\big)_x.\end{aligned}$$

(3) The relation $f^*g = g \circ f$ holds by definition. In view of

$$T(g \circ f) = T(g) \circ T(f),$$

the second component $d(g \circ f)$ of $T(g \circ f)$ coincides with $dg \circ T(f) = f^*dg$. ∎

Definition V.1.6. Let M be a smooth manifold, E a finite-dimensional vector space and $\omega \in \Omega^k(M, E)$ a smooth E-valued k-form.

Let $p \in M$ and $(\varphi, U), (\psi, V)$ be two n-dimensional charts of M with $p \in U \cap V$. On the open subset $\varphi(U \cap V) \subseteq \mathbb{R}^n$ we then have

$$(\psi \circ \varphi^{-1})^*((\psi^{-1})^*\omega) = (\varphi^{-1})^*\psi^*(\psi^{-1})^*\omega = (\varphi^{-1})^*\omega,$$

so that Proposition IV.3.4 implies that

$$\begin{aligned}d\big((\varphi^{-1})^*\omega\big) &= d\big((\psi \circ \varphi^{-1})^*(\psi^{-1})^*\omega\big) = (\psi \circ \varphi^{-1})^*d\big((\psi^{-1})^*\omega\big) \\ &= (\varphi^{-1})^*\psi^*d\big((\psi^{-1})^*\omega\big).\end{aligned}$$

We conclude that

$$\varphi^*d\big((\varphi^{-1})^*\omega\big) = \psi^*d\big((\psi^{-1})^*\omega\big)$$

holds on $U \cap V$. We may therefore define a smooth $(k+1)$-form $d\omega \in \Omega^{k+1}(M, E)$ by

$$d\omega|_U := \varphi^*d\big((\varphi^{-1})^*\omega\big)$$

for any chart (φ, U) of M. The preceding arguments imply that we thus obtain a well-defined E-valued differential form on M. ∎

We now turn to further properties of the exterior differential:

Proposition V.1.7. *The exterior differential*

$$d\colon \Omega^p(M, E) \to \Omega^{p+1}(M, E)$$

has the following properties:

(a) *It is a linear map.*

(b) *For $\alpha \in \Omega^p(M, \mathbb{R})$, $\beta \in \Omega^q(M, \mathbb{R})$ we have $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.*

(c) *$d(d\omega) = 0$ for any $\omega \in \Omega^p(M, E)$.*

(d) *If $f\colon M \to N$ is smooth and $\omega \in \Omega^p(N, E)$, then $d(f^*\omega) = f^*(d\omega)$.*

Proof. (a) is a trivial consequence of the definition.

(b) Let (φ, U) be a chart of M. In view of Definition V.1.6, Proposition V.1.5, and Proposition IV.3.4, we then have

$$\begin{aligned} d(\alpha \wedge \beta) &= \varphi^* d\big((\varphi^{-1})^*(\alpha \wedge \beta)\big) = \varphi^* d\big((\varphi^{-1})^*\alpha \wedge (\varphi^{-1})^*\beta\big) \\ &= \varphi^* \Big(d\big((\varphi^{-1})^*\alpha\big) \wedge (\varphi^{-1})^*\beta + (-1)^p (\varphi^{-1})^*\alpha \wedge d\big((\varphi^{-1})^*\beta\big)\Big) \\ &= \varphi^* d\big((\varphi^{-1})^*\alpha\big) \wedge \beta + (-1)^p \alpha \wedge \varphi^* d\big((\varphi^{-1})^*\beta\big) \\ &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta. \end{aligned}$$

(c) For any chart (φ, U) of M we have on U:

$$d(d\omega) = \varphi^*\big(d((\varphi^{-1})^*(d\omega))\big) = \varphi^*\big(d(d[(\varphi^{-1})^*\omega])\big) = 0$$

because d^2 vanishes on $\Omega^k(\varphi(U), E)$ (Proposition IV.3.4).

(d) Let $p \in M$, (φ, U) be a chart of M with $p \in U$ and (ψ, V) a chart of N with $f(p) \in V$. Then $f^{-1}(V) \cap U$ is an open neighborhood of p, and it suffices to verify the identity $d(f^*\omega) = f^*(d\omega)$ on this open set.

With Proposition IV.3.4 we obtain

$$\begin{aligned} d(f^*\omega) &= \varphi^*(d[(\varphi^{-1})^* f^*\omega]) = \varphi^*(d[(\varphi^{-1})^* f^*\psi^*(\psi^{-1})^*\omega]) \\ &= \varphi^*(d[(\psi \circ f \circ \varphi^{-1})^*(\psi^{-1})^*\omega]) = \varphi^*(\psi \circ f \circ \varphi^{-1})^* d[(\psi^{-1})^*\omega] \\ &= (\psi \circ f)^*(\psi^{-1})^* d\omega = f^*\psi^*(\psi^{-1})^* d\omega = f^* d\omega. \end{aligned}$$

This proves (d). ■

Definition V.1.8. (de Rham cohomology) Let M be a smooth n-dimensional manifold and E a finite-dimensional vector space. We write

$$Z^p_{\mathrm{dR}}(M, E) := \{\omega \in \Omega^p(M, E) \colon d\omega = 0\}$$

for the space of *closed p-forms* and

$$B^p_{\mathrm{dR}}(M, E) := \{\omega \in \Omega^p(M, E) \colon (\exists \eta \in \Omega^{p-1}(M, E))\ d\eta = \omega\}$$

for the space of *exact p-forms*. In view of $d \circ d = 0$, all exact forms are closed, i.e.,

$$B^p_{\mathrm{dR}}(M, E) \subseteq Z^p_{\mathrm{dR}}(M, E),$$

so that we can form the quotient vector space

$$H^p_{\mathrm{dR}}(M, E) := Z^p_{\mathrm{dR}}(M, E) / B^p_{\mathrm{dR}}(M, E).$$

It is called the *p-th E-valued de Rham cohomology space of M*. Its elements $[\alpha] := \alpha + B^p_{\mathrm{dR}}(M, E)$ are called *cohomology classes.* ■

Lemma V.1.9. *The space* $H_{\mathrm{dR}}(M,\mathbb{R}) := \bigoplus_{p=0}^{\infty} H^p_{\mathrm{dR}}(M,\mathbb{R})$ *carries the structure of an associative, graded commutative algebra, defined by*

$$[\alpha]\wedge[\beta] := [\alpha\wedge\beta], \quad \alpha\in Z^p_{\mathrm{dR}}(M,\mathbb{R}), \beta\in Z^q_{\mathrm{dR}}(M,\mathbb{R}).$$

Proof. If α is a closed p-form and β is a closed q-form, then

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta = 0$$

shows that $\alpha\wedge\beta$ is closed. Therefore the subspace

$$Z_{\mathrm{dR}}(M,\mathbb{R}) := \bigoplus_{p=0}^{\infty} Z^p_{\mathrm{dR}}(M,\mathbb{R}) \subseteq \Omega(M,\mathbb{R})$$

of closed forms is a subalgebra with respect to the exterior multiplication.

Moreover, the subspace

$$B_{\mathrm{dR}}(M,\mathbb{R}) := \bigoplus_{p=0}^{\infty} B^p_{\mathrm{dR}}(M,\mathbb{R}) \subseteq Z_{\mathrm{dR}}(M,\mathbb{R})$$

is an ideal because if α is a closed p-form and $\beta = d\gamma$ is an exact q-form, then

$$\alpha\wedge\beta = \alpha\wedge d\gamma = (-1)^p d(\alpha\wedge\gamma)$$

is exact. We conclude that

$$[\alpha]\wedge[\beta] := [\alpha\wedge\beta]$$

yields a well-defined multiplication on $H_{\mathrm{dR}}(M,\mathbb{R})$. From the subalgebra $Z_{\mathrm{dR}}(M,\mathbb{R})$ it inherits the associativity and the graded commutativity. ■

An important point of the de Rham cohomology spaces is that they are topological invariants, i.e., if two smooth manifolds M and N are homeomorphic, then their cohomology spaces $H^p_{\mathrm{dR}}(M,E)$ and $H^p_{\mathrm{dR}}(N,E)$ are isomorphic for each finite-dimensional vector space E. The following statement is weaker in the sense that φ and its inverse are assumed to be smooth.

Proposition V.1.10. *If* $\varphi\colon M\to N$ *is a smooth map, then the pullback defines an algebra homomorphism*

$$\varphi^*\colon H_{\mathrm{dR}}(N,\mathbb{R}) \to H_{\mathrm{dR}}(M,\mathbb{R}).$$

If, in addition, φ *is a diffeomorphism, then* φ^* *induces an isomorphism of algebras.*

Proof. It clearly suffices to verify the first part, because the second statement then follows by applying the first one to φ and φ^{-1}.

Proposition V.1.5 implies that φ^* preserves the exterior product and that it commutes with the exterior differential, hence maps closed forms to closed forms and exact forms to exact forms. In particular, we obtain well-defined maps

$$\varphi^*\colon H^p_{\mathrm{dR}}(N,\mathbb{R}) \to H^p_{\mathrm{dR}}(M,\mathbb{R}), \quad [\alpha]\mapsto[\varphi^*\alpha],$$

and combining them to a linear map $\varphi^*\colon H_{\mathrm{dR}}(N,\mathbb{R}) \to H_{\mathrm{dR}}(M,\mathbb{R})$, we obtain an algebra homomorphism because

$$\begin{aligned}\varphi^*([\alpha]\wedge[\beta]) &= \varphi^*([\alpha\wedge\beta]) = [\varphi^*(\alpha\wedge\beta)] = [\varphi^*\alpha\wedge\varphi^*\beta]\\ &= [\varphi^*\alpha]\wedge[\varphi^*\beta] = \varphi^*[\alpha]\wedge\varphi^*[\beta].\end{aligned}$$

■

Using a manifold version of fiber integration, one can show that for each (smoothly) contractible smooth manifold M, all cohomology spaces $H^p_{\rm dR}(M,E)$ vanish (Poincaré Lemma). This means that each closed p-form is exact. We shall come back to this point later. It is the first step to the application of differential forms in algebraic topology.

V.2. Partitions of unity

In this section we shall introduce a central tool for the analysis on manifolds: smooth partitions of unity. They are used in various situations to localize problems, i.e., to turn them into problems on open subsets of $\mathbb{R}^n$, which are usually easier to solve.

To obtain sufficiently fine smooth partitions of unity on a manifold, we have to impose a condition on the underlying topological space.

Definition V.2.1. (a) A topological space X is said to be *σ-compact* if there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} X_n$.

(b) If $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ are open covers of the topological space X, then we call $(V_j)_{j\in J}$ a *refinement* of $(U_i)_{i\in I}$ if for each $j\in J$ there exists some $i_j\in I$ with $V_j\subseteq U_{i_j}$.

A family $(S_i)_{i\in I}$ of subsets of X is called *locally finite* if each point $p\in X$ has a neighborhood V intersecting only finitely many of the sets S_i.

A topological space X is said to be *paracompact* if each open cover has a locally finite refinement. ■

Lemma V.2.2. *If X is a σ-compact locally compact topological space, then there exists an exhaustion of X by compact subsets, i.e., a sequence $(K_n)_{n\in\mathbb{N}}$ with $\bigcup_{n\in\mathbb{N}} K_n = X$ and $K_n\subseteq K^0_{n+1}$ for each $n\in\mathbb{N}$.*

Proof. Since X is σ-compact, there exists a sequence of compact subsets $(Q_n)_{n\in\mathbb{N}}$ with $\bigcup_n Q_n = X$. We shall construct the new sequence K_n inductively. We put $K_1 := Q_1$ and assume that $K_1,\ldots,K_n$ are already constructed such that $K_i\subseteq K^0_{i+1}$ for $i=1,\ldots,n-1$ and $Q_1\cup\ldots\cup Q_n\subseteq K_n$.

Put $K_n' := K_n\cup\bigcup_{j=1}^{n+1} Q_j$. For each point $x\in K_n'$ we pick a compact neighborhood U_x of x. Then the sets U_x^0 form an open cover of the compact set K_n'. Hence there exist finitely many points $x_1,\ldots,x_m\in K_n'$ with

$$K_n'\subseteq U^0_{x_1}\cup\ldots\cup U^0_{x_m}.$$

Then $K_{n+1} := U_{x_1}\cup\ldots\cup U_{x_m}$ is a compact subset of X with $K_n\subseteq K_n'\subseteq K^0_{n+1}$ and $Q_j\subseteq K_{n+1}$ for $j\leq n+1$. We thus obtain a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X with $\bigcup_n K_n\supseteq\bigcup_n Q_n = X$ and $K_n\subseteq K^0_{n+1}$ for each $n\in\mathbb{N}$. ■

Theorem V.2.3. *For a finite-dimensional (topological) manifold M, the following are equivalent:*

(1) *M is paracompact.*

(2) *Each connected component of M is σ-compact.*

Proof. (1) $\Rightarrow$ (2): Since M is a manifold, any point $p \in M$ has a compact neighborhood V_p, and the open sets $(V_p^0)_{p\in M}$ form an open cover of M. As M is paracompact, this cover has a locally finite refinement $(U_j)_{j\in J}$. Since the sets V_p are compact, the U_j have compact closures.

Let $x \in M$. We show that the connected component of M containing x is σ-compact. Put $K_1 := \overline{U_{j_x}}$, where $j_x \in J$ is chosen such that $x \in U_{j_x}$. Next we use Exercise V.9 to see that only finitely many U_j intersect K_1. We define K_2 as the union of the closures of all the U_j intersecting K_1. We then proceed inductively and define K_{n+1} as the union of the closures of the U_j intersecting K_n. We thus obtain a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of M. As the U_j intersecting K_n cover this set, we further have

$$K_n \subseteq \bigcup_{U_j\cap K_n\neq\emptyset} U_j \subseteq K_{n+1}^0,$$

and

$$K := \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} K_n^0$$

is an open subset of M containing x. Its complement is the union of all U_j not intersecting any K_n, hence an open subset of M. Therefore K is closed and open, which implies that it contains the connected component M_x of x, and we conclude that

$$M_x = \bigcup_{n\in\mathbb{N}} (M_x \cap K_n)$$

is σ-compact.

(2) $\Rightarrow$ (1): Let $\mathcal{U} := (U_i)_{i\in I}$ be an open cover of M. Since M is a manifold, hence locally connected, its connected components are open subsets. Therefore it suffices to construct for each connected component a refinement of the open cover induced on it by $\mathcal{U}$. Therefore we may assume that M is connected, and, in view of (2), that M is σ-compact.

As M is locally compact, we find with Lemma V.2.3 a sequence of compact subsets of M with $K_n \subseteq K_{n+1}^0$ and $\bigcup_n K_n = M$. We put $K_{-1} := K_0 := \emptyset$. For each $p \in K_n \setminus K_{n-1}^0$ we choose some open neighborhood V_p contained in some covering set U_i and in $K_{n+1}^0 \setminus K_{n-2}$. Then finitely many

$$V_{p_1}^n, \ldots, V_{p_{m_n}}^n$$

of these sets cover the compact set $K_n \setminus K_{n-1}^0$. Hence the collection of all the sets $(V_{p_j}^n)_{n\in\mathbb{N}, j\leq m_n}$ is an open cover of M which is locally finite. In fact, K_n is not intersected by any set $V_{p_j}^m$ with $m > n+1$. We have thus found a locally finite refinement $\mathcal{V}$ of the open cover $\mathcal{U}$. ∎

Definition V.2.4. A *smooth partition of unity* on a smooth manifold M is a family $(\psi_i)_{i\in I}$ of smooth functions $\psi_i \in C^\infty(M,\mathbb{R})$ such that

(P1) $0 \leq \psi_i$ for each $i \in I$.

(P2) Local finiteness: each point $p \in M$ has a neighborhood U such that $\{i \in I : \psi_i|_U \neq 0\}$ is finite.

(P3) $\sum_i \psi_i = 1$.

Note that (P2) implies that in each $p \in M$

$$\sum_{i\in I} \psi_i(p) = \sum_{\psi_i(p)\neq 0} \psi_i(p)$$

is a finite-sum, so that it is well-defined, even if I is an infinite set.

If $\mathcal{U} = (U_j)_{j\in J}$ is an open cover, then a partition of unity $(\psi_j)_{j\in J}$ is said to be *associated to* $\mathcal{U}$ *if* $\mathrm{supp}(\psi_j) \subseteq U_j$ *holds for each* $j \in J$. ■

Lemma V.2.5. (Existence of bump functions) *Let M be a smooth manifold, $p \in M$ and $U \subseteq M$ an open neighborhood of p. Then there exists a smooth function $f: M \to \mathbb{R}$ with*

(1) $0 \leq f \leq 1$.

(2) $f(p) = 1$.

(3) $\mathrm{supp}(f) \subseteq U$.

Proof. Let (φ, V) be an n-dimensional chart of M with $p \in V \subseteq U$. Then $\varphi(V) \subseteq \mathbb{R}^n$ is an open subset and $x := \varphi(p) \in \varphi(V)$. Let $r > 0$ be such that the closed ball $B_r(x) := \{y \in \mathbb{R}^n : \|x - y\|_2 \leq r\}$ is contained in $\varphi(V)$.

From Exercise V.6(c) we obtain a smooth function $h: \mathbb{R} \to \mathbb{R}$ with $h(0) = 1$, $0 \leq h \leq 1$ and $\mathrm{supp}(h) =]-\infty, r]$. Then the function

$$g: \mathbb{R}^n \to \mathbb{R}, \quad y \mapsto h(\|x - y\|_2^2)$$

is smooth with $g(x) = 1$, $0 \leq g \leq 1$ and $g(y) = 0$ for $\|x - y\| \geq r$.

We define

$$f: M \to \mathbb{R}, \quad x \mapsto \begin{cases} g(\varphi(x)) & \text{for } x \in V \\ 0 & \text{for } x \notin V. \end{cases}$$

Then $f = g \circ \varphi$ is smooth on the open set V, and $f = 0$ is smooth on $M \setminus \varphi^{-1}(B_r(x))$, which is also open. Hence f is smooth, $f(p) = 1$, $0 \leq f \leq 1$ and $\mathrm{supp}(f) = \varphi^{-1}(B_r(x)) \subseteq V$. ■

The main result of this section is the following theorem:

Theorem V.2.6. *If M is paracompact and $(U_j)_{j\in J}$ is a locally finite open cover of M, then there exists an associated smooth partition of unity on M.*

Proof. Since M is a manifold, hence locally connected, its connected components are open subsets. Therefore it suffices to construct on each connected component a smooth partition of unity associated to the corresponding open cover. Hence we may w.l.o.g. assume that M is connected and σ-compact.

Let $(K_i)_{i\in\mathbb{N}}$ be a sequence of compact subsets of M with $\bigcup_{n\in\mathbb{N}} K_n = M$ and put $K_0 := \emptyset$. For $p \in M$ let i_p be the largest integer with $p \in M \setminus K_{i_p}$. We then have $p \in K_{i_p+1} \subseteq K^0_{i_p+2}$. Choose a $j_p \in J$ with $p \in U_{j_p}$ and let $\psi_p \in C^\infty(M,\mathbb{R})$ as in Lemma V.2.5 with

$$\operatorname{supp}(\psi_p) \subseteq U_{j_p} \cap (K^0_{i_p+2} \setminus K_{i_p}).$$

Then $W_p := \psi_p^{-1}(]0,\infty[)$ is an open neighborhood of p. For each $i \geq 1$, choose a finite set of points p in M whose corresponding neighborhoods W_p cover the compact set $K_i \setminus K^0_{i-1}$. We order the corresponding functions ψ_p in a sequence $(\psi_i)_{i\in\mathbb{N}}$. Their supports form a locally finite family of subsets of M because for only finitely many of them, the supports intersect a given set K_i. Moreover, the sets $\psi_i^{-1}(]0,\infty[)$ cover M. Therefore

$$\psi := \sum_j \psi_j$$

is a smooth function which is everywhere positive (Exercise V.5). Therefore we obtain smooth functions

$$\varphi_i := \frac{\psi_i}{\psi}, \quad i \in \mathbb{N}.$$

Then the functions φ_i form a smooth partition of unity on M.

We now define another partition of unity, associated to the open cover $(U_j)_{j\in J}$ as follows. For each $i \in \mathbb{N}$ we pick a $j_i \in J$ with $\operatorname{supp}(\varphi_i) \subseteq U_{j_i}$ and define

$$\alpha_j := \sum_{j_i = j} \varphi_i.$$

As the sum on the right hand side is locally finite, the functions α_j are smooth and

$$\operatorname{supp}(\alpha_j) \subseteq \bigcup_{j_i=j} \operatorname{supp}(\varphi_i) \subseteq U_j$$

(Exercise V.8). We further observe that only countably many of the α_j are non-zero, that $0 \leq \alpha_j$, $\sum_j \alpha_j = 1$, and that the supports form a locally finite family because the cover $(U_j)_{j\in J}$ is locally finite. ■

Corollary V.2.7. *Let M be a paracompact smooth manifold, $K \subseteq M$ a closed subset and $U \subseteq M$ an open neighborhood of K. Then there exists a smooth function $f\colon M \to \mathbb{R}$ with*

(1) $0 \leq f \leq 1$.

(2) $f|_K = 1$.

(3) $\operatorname{supp}(f) \subseteq U$.

Proof. In view of Theorem V.2.6, there exists a smooth partition of unity associated to the open cover $\{U, M \setminus K\}$. This is a pair of smooth functions (f,g) with $\operatorname{supp}(f) \subseteq U$, $\operatorname{supp}(g) \subseteq M \setminus K$, $0 \leq f, g$, and $f + g = 1$. We thus have (1) and (3), and (2) follows from $g|_K = 0$. ■

V.3. Direct limit spaces and the smooth long line

In this section we discuss an example of a strange one-dimensional manifold, the *smooth long line*. To construct this long line, we first have to take a closer look at direct limit constructions.

Direct limit spaces

Definition V.3.1. (a) (Direct limit sets) Let $(I, \leq)$ be a *directed set*, i.e., $\leq$ is a partial order on I (i.e., a reflexive, transitive and antisymmetric relation) and for $i, j \in I$ there exists a $k \in I$ with $i, j \leq k$.

Now suppose we are given sets $(X_i)_{i \in I}$ and maps

$$\varphi_{ij} \colon X_j \to X_i, \quad j \leq i,$$

satisfying

$$\varphi_{ii} = \mathrm{id}_{X_i}, \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}, \quad k \leq j \leq i. \tag{5.3.1}$$

We then call $(X_i, \varphi_{ij})_{j \leq i \in I}$ a *directed system of maps* and the maps φ_{ij} the connecting maps.

Then we obtain on the disjoint union $\widetilde{X} := \dot{\bigcup}_{i \in I}(\{i\} \times X_i)$ an equivalence relation by

$$(i, x_i) \sim (j, x_j) \quad \Longleftrightarrow \quad (\exists k \geq i, j)\ \varphi_{ki}(x_i) = \varphi_{kj}(x_j).$$

Reflexivity and symmetry of this relation are obvious. To see that this relation is transitive, suppose that $(i, x_i) \sim (j, x_j) \sim (r, x_r)$, and that $i, j \leq k$, $j, r \leq \ell$ with

$$\varphi_{ki}(x_i) = \varphi_{kj}(x_j), \quad \varphi_{\ell j}(x_j) = \varphi_{\ell r}(x_r).$$

For $s \geq k, \ell$, we then obtain

$$\varphi_{si}(x_i) = \varphi_{sk}\varphi_{ki}(x_i) = \varphi_{sk}\varphi_{kj}(x_j) = \varphi_{s\ell}\varphi_{\ell j}(x_j) = \varphi_{s\ell}\varphi_{\ell r}(x_r) = \varphi_{sr}(x_r),$$

which implies the transitivity of $\sim$.

We write $X := \{[x] \colon x \in \widetilde{X}\}$ for the set of equivalence classes in $\widetilde{X}$. We then obtain maps

$$\varphi_i \colon X_i \to X, \quad x \mapsto [(i, x)]$$

satisfying

$$\varphi_j \circ \varphi_{ji} = \varphi_i \quad \text{for} \quad i \leq j. \tag{5.3.2}$$

The set X, together with the maps φ_i, $i \in I$, is called the *direct limit of the family* (X_i, φ_{ij}) and it is denoted by

$$X = \varinjlim X_i.$$

This set is the directed union of the subsets $\varphi_i(X_i)$, $i \in I$. It has the universal property that for each collection of maps $f_i: X_i \to Y$ satisfying

$$f_j \circ \varphi_{ji} = f_i \quad \text{for} \quad i \leq j \tag{5.3.3}$$

there exists a unique map $f: X \to Y$ with $f \circ \varphi_j = f_j$ for each $j \in J$ (Exercise V.12).

(b) Now assume, in addition, that each X_i is a topological space and that all maps $\varphi_{ji}: X_i \to X_j$ are continuous. We then define a topology on X by defining $O \subseteq X$ to be open if and only if for each i the set $\varphi_i^{-1}(O)$ is open in X_i. It is easy to see that we thus obtain a topology on X (Exercise V.10). We call X the topological direct limit space of the family (X_i, φ_{ij}).

It has the universal property that for each collection of continuous maps $f_i: X_i \to Y$ into some topological space Y, satisfying (5.3.3), there exists a unique continuous map $f: X \to Y$ with $f \circ \varphi_j = f_j$ for each $j \in J$ (Exercises V.10/12). ■

Lemma V.3.2. (a) *If all the maps $\varphi_{ji}: X_i \to X_j$ are injective, then all the maps $\varphi_i: X_i \to X$ are injective.*

(b) *If all the maps $\varphi_{ji}: X_i \to X_j$ are open embeddings, then all the maps $\varphi_i: X_i \to X$ are open embeddings. If, in addition, all spaces X_i are Hausdorff, then X is Hausdorff.*

(c) *If all space X_i are n-dimensional smooth manifolds and the maps $\varphi_{ji}: X_i \to X_j$ are diffeomorphic embeddings, then X carries the structure of a smooth n-dimensional manifold for which all the maps $\varphi_i: X_i \to X$ are diffeomorphic embeddings.*

Proof. (a) Suppose that $\varphi_i(x) = \varphi_i(y)$ for $x, y \in X_i$. Then $[(i,x)] = [(i,y)]$ implies the existence of some $j > i$ with $\varphi_{ji}(x) = \varphi_{ji}(y)$, and this leads to $x = y$ since φ_{ij} is injective.

(b) From (a) it follows that each φ_j is injective. Let $U \subseteq X_i$ be an open subset. We have to show that $\varphi_i(U)$ is an open subset of X, which in turn means that for each $j \in J$ the set

$$\begin{aligned}\varphi_j^{-1}(\varphi_i(U)) &= \{x_j \in X_j : [(j, x_j)] \in \varphi_i(U)\} \\ &= \{x_j \in X_j : (\exists k \geq i, j)\varphi_{kj}(x_j) \in \varphi_{ki}(U)\} = \bigcup_{k \geq i,j} \varphi_{kj}^{-1}(\varphi_{ki}(U))\end{aligned}$$

is open. This is the case because the sets $\varphi_{ki}(U)$ are open and the maps φ_{kj} are continuous.

If, in addition, all X_i are Hausdorff, then X is also Hausdorff. In fact, take $x, y \in X$. Then there exists some $\varphi(X_i)$ containing both x and y, and in $\varphi(X_i) \cong X_i$ we find disjoint neighborhoods of x and y, but these are also disjoint neighborhoods in X.

(c) Now we assume that all X_j are n-dimensional smooth manifolds. From (b) we now that X is a Hausdorff space and that each φ_j is an open embedding. Therefore each chart (ψ_j, V_j) of X_j yields a chart $(\psi_j \circ \varphi_j^{-1}, \varphi_j(V_j))$ of X.

Two such charts $(\psi_i \circ \varphi_i^{-1}, \varphi_i(V_i))$ and $(\psi_j \circ \varphi_j^{-1}, \varphi_j(V_j))$ of X are smoothly compatible because we may choose $k \geq i, j$ to see that

$$\varphi_i \circ \psi_i^{-1} = \varphi_k \circ \varphi_{ki} \circ \psi_i^{-1}$$

leads to

$$\begin{aligned}(\psi_i \circ \varphi_i^{-1}) \circ (\psi_j \circ \varphi_j^{-1})^{-1} &= \psi_i \circ \varphi_{ki}^{-1} \circ \varphi_k^{-1} \circ \varphi_k \circ \varphi_{kj} \circ \psi_j^{-1} \\ &= \psi_i \circ (\varphi_{ki}^{-1} \circ \varphi_{kj}) \circ \psi_j^{-1}\end{aligned}$$

which is smooth because it is a composition of smooth maps.

Therefore the family of all charts (ψ_j, V_j) yields a smooth n-dimensional atlas of X. ■

The long line

Definition V.3.3. A linearly ordered set $(X, \leq)$ is called *well-ordered* if each non-empty subset $S \subseteq X$ has a minimal element.

An isomorphy class of well-ordered sets is called an *ordinal number*. ■

Examples V.3.4. The set $\omega := (\mathbb{N}, \leq)$ is well-ordered, but there are more complicated well-ordered sets:

$$\omega + 1: \quad 1, 2, \ldots, \omega,$$

$$\omega + 2: \quad 1, 2, \ldots, \omega, \omega + 1,$$

$$2\omega: \quad 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots$$

$$3\omega + 1: \quad 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, 2\omega, 2\omega + 1, 2\omega + 2, \ldots, 3\omega$$

etc.

Another example of a countable well-ordered set is $\omega^2 := (\mathbb{N} \times \mathbb{N}, \leq)$, endowed with the lexicographic ordering:

$$(n, m) \leq (n', m') \quad \Longleftrightarrow \quad n < n' \quad \text{or} \quad n = n', m \leq m'. \qquad ■$$

In the following construction we shall need:

Lemma V.3.5. *Let $c < a < b \leq \infty$ in $\mathbb{R}$ and $f\colon]-\infty, a[\to \mathbb{R}$ a smooth function with $f' > 0$. Then there exists a smooth function $g\colon]-\infty, b[\to \mathbb{R}$ with $g' > 0$ and $g = f$ on $]-\infty, c]$.*

Proof. Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $0 \leq \alpha \leq 1$, $\alpha(x) = 1$ for $x \leq c$ and $\alpha(x) = 0$ for $x > \frac{a+c}{2}$ (Exercise V.6). The function

$$\beta\colon]-\infty, b[\to \mathbb{R}, \quad \beta(x) := \alpha(x)f'(x) + (1 - \alpha(x))$$

is smooth because $\alpha(x) = 0$ for $x > \frac{a+c}{2}$. It is everywhere positive and β coincides with f' for $x \leq c$ and satisfies $\beta = 1$ on $x > \frac{a+c}{2}$. Therefore

$$g(x) := f(c) + \int_c^x \beta(t)\, dt$$

is a smooth function on $]-\infty, b[$ which coincides with f on $]-\infty, c]$ and satisfies $g'(x) = \beta(x) > 0$ for all $x < b$. ■

Lemma V.3.6. *There exists an uncountable well-ordered set $(\Omega, \leq)$ with the property that for each $\beta \in \omega$ the set $\downarrow\beta := \{\alpha \in \Omega\colon \alpha \leq \beta\}$ is countable. Any such well-ordered set $(\Omega, \leq)$ has the property that each countable subset of Ω is bounded.*

Proof. First we use the Well-Ordering Theorem, which ensures the existence of a well-ordering $\leq$ on $\mathbb{R}$. If the orderet set $(\mathbb{R}, \leq)$ does already have the property required above, we are done. Otherwise there exists an $\alpha \in \mathbb{R}$ for which $\downarrow\alpha$ is uncountable. Since $\leq$ is a well-ordering, there is a minimal such α, and then we may put $\Omega := \{\beta \in \mathbb{R}\colon \beta < \alpha\}$. By construction, this set is uncountable and has all required properties.

Now let $(\Omega, \leq)$ be an uncountable well-ordered set in which all set $\downarrow\beta$ are countable. To see that each countable subset $M \subseteq \Omega$ bounded, let us assume that M is countable and unbounded. Then there exists for each $\beta \in \Omega$ an element $\gamma \in M$ with $\beta \in \downarrow\gamma$. This means that $\Omega = \bigcup_{\gamma \in M} \downarrow\gamma$, a countable union of countable sets, contradicting the uncountability of Ω. ■

We shall use Ω to construct the *long line* as a direct limit manifold of a system $(X_\alpha, \varphi_{\beta\alpha})_{\alpha \leq \beta \in \Omega}$, with $X_\alpha = \mathbb{R}$ for each α (Lemma V.3.2(c)).

Theorem V.3.7. *There exists a directed family $(\mathbb{R}_\alpha, \varphi_{\alpha',\alpha})_{\alpha \leq \alpha' \in \Omega}$ of smooth maps $\varphi_{\alpha',\alpha}\colon \mathbb{R} \to \mathbb{R}$ such that for $\alpha < \alpha'$ we have*

(1) $\varphi'_{\alpha',\alpha} > 0$.

(2) $\varphi_{\alpha',\alpha}(\mathbb{R}) =]-\infty, x[$ *for some* $x \in \mathbb{R}$.

Proof. Our first major step of the proof is to show that for each $\beta \in \Omega$ there exists such a directed system over the well-ordered index set $(\downarrow\beta, \leq)$.

Step 1: If β is the minimal element, then we simply put $\varphi_{\beta,\beta} := \mathrm{id}_\mathbb{R}$, and we are done.

Step 2: Suppose that β is the successor of some $\gamma \in \Omega$, and that a directed system with the required properties exists on $(\downarrow\gamma, \leq)$. Let $f(x) := -e^{-x}$ and note that this function defines a diffeomorphism $f\colon \mathbb{R} \to]-\infty, 0[$. For $\alpha < \beta$ we also have $\alpha \leq \gamma$, so that we can define

$$\varphi_{\beta,\beta} := \mathrm{id}_{\mathbb{R}} \quad \text{and} \quad \varphi_{\beta,\alpha} := \begin{cases} f & \text{for } \alpha = \gamma \\ f \circ \varphi_{\gamma,\alpha} & \text{for } \alpha < \gamma. \end{cases}$$

Now each map $\varphi_{\alpha',\alpha}$ with $\alpha \leq \alpha' \leq \beta$ is smooth with $\varphi'_{\alpha',\alpha} > 0$, hence defines a smooth embedding $\mathbb{R} \to \mathbb{R}$ as an open interval bounded from above, but not from below. Moreover, for $\alpha < \alpha' < \beta$ we have $\alpha' \leq \gamma$, and therefore

$$\varphi_{\beta,\alpha'} \circ \varphi_{\alpha',\alpha} = f \circ \varphi_{\gamma,\alpha'} \circ \varphi_{\alpha',\alpha} = f \circ \varphi_{\gamma,\alpha} = \varphi_{\beta,\alpha}.$$

Step 3: Suppose that β is not the successor of any $\gamma \in \Omega$, and that a directed system with the required properties exists on the countable well-ordered set $I_\beta := \{\alpha \in \Omega \colon \alpha < \beta\}$.

Let

$$X := \varinjlim\, (\mathbb{R}_\alpha, \varphi_{\alpha',\alpha})_{\alpha \leq \alpha' < \beta}$$

be the corresponding direct limit space. In view of Lemma V.3.2(c), it carries the structure of a smooth 1-dimensional manifold. We want to construct maps $\varphi_{\beta,\alpha}$ for $\alpha < \beta$ by putting $\varphi_{\beta,\alpha} := f \circ \varphi_\alpha$ for a smooth function $f\colon X \to \mathbb{R}$ satisfying $(f \circ \varphi_\alpha)' > 0$ for each $\alpha < \beta$.

To find the function f, we first recall that I_β is countable, so that the elements of this set can be written as a sequence $(\alpha_n)_{n\in\mathbb{N}}$. We then define a new sequence

$$\gamma_n := \max\{\alpha_1, \ldots, \alpha_n\}$$

and observe that this sequence is increasing with $\beta = \sup\{\gamma_n \colon n \in \mathbb{N}\}$. Passing to a subsequence, if necessary, we may w.l.o.g. assume that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is strictly increasing.

Since $X = \bigcup_{\alpha<\beta} X_\alpha$, where $X_\alpha := \varphi_\alpha(\mathbb{R})$ is an open subset diffeomorphic to $\mathbb{R}$ and $X_\alpha \subseteq X_\beta$ for $\alpha \leq \beta$, the subsets $X_n := X_{\gamma_n} \cong \mathbb{R}$ satisfy $\overline{X_n} \subseteq X_{n+1}$ with $\bigcup_n X_n = X$.

On $X_2 = \mathrm{im}(\varphi_{\gamma_2})$ we define

$$f_2 \colon X_2 \to \mathbb{R}, \quad x \mapsto -e^{-\varphi_{\gamma_2}^{-1}(x)}$$

and observe that $(f_2 \circ \varphi_{\gamma_2})' > 0$ and $\mathrm{im}(f_2) =]-\infty, 0[$. On $X_3 = \mathrm{im}(\varphi_{\gamma_3})$ we have

$$\varphi_{\gamma_3}^{-1}(X_2) = \varphi_{\gamma_3}^{-1}(\varphi_{\gamma_2}(\mathbb{R})) = \varphi_{\gamma_3,\gamma_2}(\mathbb{R}) =]-\infty, x_2[$$

and $\varphi_{\gamma_3}^{-1}(X_1)$ is a proper subinterval, unbounded from below. We now use Lemma V.3.5 to find a smooth function $f_3 \colon X_3 \to \mathbb{R}$ with $(f_3 \circ \varphi_{\gamma_3})' > 0$ and $f_3 = f_2$ on X_1. We proceed inductively to obtain $f_n \colon X_n \to \mathbb{R}$ with $(f_n \circ \varphi_{\gamma_n})' > 0$ and $f_n = f_{n-1}$ on X_{n-2} for $n \geq 3$.

We now put

$$f(x) := f_n(x) \quad \text{for} \quad x \in X_{n-2}$$

and note that this does not depend on the choice of n, so that we obtain a smooth function $f: X \to \mathbb{R}$ with $f = f_{n+2}$ on X_n, satisfying $(f \circ \varphi_{\gamma_n})' > 0$ for each n. For other elements $\alpha < \beta$ there exists an n with $\gamma_n \geq \alpha$, which leads to

$$(f \circ \varphi_\alpha)' = (f \circ \varphi_{\gamma_n} \circ \varphi_{\gamma_n,\alpha})' = \Big((f \circ \varphi_{\gamma_n})' \circ (\varphi_{\gamma_n,\alpha})\Big) \cdot \varphi'_{\gamma_n,\alpha} > 0.$$

We now put

$$\varphi_{\beta,\alpha} := f \circ \varphi_\alpha \quad \text{for} \quad \alpha < \beta$$

and obtain, as in Step 2, a directed system on $\downarrow\!\beta$ with the required properties.

Step 4: We now consider the collection $\mathcal{P}$ of all directed systems $P = (\mathbb{R}_\alpha, \varphi_{\alpha'\alpha})_{\alpha \leq \alpha' \in \Omega_P}$, where $\Omega_P \subseteq \Omega$ is a subset with $\downarrow\!\Omega_P = \Omega_P$, i.e., $\beta \leq \beta'$, $\beta' \in \Omega_P$ implies $\beta \in \Omega_P$. We define $P_1 \leq P_2$ if $\Omega_{P_1} \subseteq \Omega_{P_2}$ and P_1 is a subsystem of P_2. Then the union of any chain in $(\mathcal{P}, \leq)$ yields another element of $\mathcal{P}$, i.e., $(\mathcal{P}, \leq)$ is inductively ordered. Now Zorn's Lemma implies the existence of a maximal element $P = P_{\max}$. We claim that $\Omega_P = \Omega$, and this completes our proof. If this is not the case, then $\Omega_P \neq \Omega$, hence $\Omega_P = \downarrow\!\Omega_P$ implies that Ω_P is bounded, and since Ω is well-ordered, it follows that it either is of the form $\downarrow\!\gamma$ or I_γ for $\gamma = \sup \Omega_P$. In the first case $\Omega_P = \downarrow\!\gamma$, we apply Step 2 with the successor β of γ to obtain a contradiction ot the maximality of P, and in the second case we apply Step 2/3 with $\beta = \gamma$ to obtain a contradiction to the maximality of P. Hence $\Omega_P \neq \Omega$ leads to a contradiction, and the proof is complete. ■

Definition V.3.8. Using a system of smooth maps as in Theorem V.3.7, we define the *long line* L as the direct limit of the system $((\mathbb{R})_{\alpha\in\Omega}, \varphi_{\beta\alpha})$, which is a smooth one-dimensional manifold (Lemma V.3.2(c)). ■

Proposition V.3.9. *The long line* L *has the following properties:*

(1) L *is arcwise connected.*

(2) L *is not* σ*-compact, hence not paracompact.*

Proof. (1) Let $x, y \in L$. Since L is the union of the subsets $L_\alpha := \varphi_\alpha(\mathbb{R}) \cong \mathbb{R}$ and $L_\alpha \subseteq L_\beta$ for $\alpha \leq \beta$, we find an $\alpha \in \Omega$ with $x, y \in L_\alpha$. Since $L_\alpha \cong \mathbb{R}$ is arcwise connected, the two points x and y lie in the same arc-component of L. Hence L is arcwise connected.

(2) Let $K \subseteq L$ be a compact subset. Then the sets $(L_\alpha)_{\alpha\in\Omega}$ form an open cover of K for which there exists a finite subcover. As this cover is totally ordered, K is contained in some L_α.

If $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact subsets of L, there exists for each n an element $\alpha_n \in \Omega$ with $K_n \subseteq L_{\alpha_n}$. Then the subset $S := \{\alpha_n : n \in \mathbb{N}\}$ is bounded in Ω (Lemma V.3.6). Hence $\gamma := \sup(S)$ exists, and we obtain $\bigcup_n K_n \subseteq L_\gamma \neq L$, which implies that L is not σ-compact. ■

V.4. Oriented manifolds

As we shall see below, integration of differential forms over a manifold requires an orientation. In Section III.3 we have already seen the concept of an orientation of a vector space and how such orientations can be specified. The idea behind an orientation of a smooth manifold is that we want each tangent space to be endowed with an orientation which does not change locally.

Definition V.4.1. An *oriented manifold* is a pair (M, O), where M is a smooth n-dimensional manifold and O is a collection of orientations O_p of the tangent spaces $T_p(M)$ such that there exists an atlas $\mathcal{A} = (\varphi_i, U_i)_{i\in I}$, for which all maps $T_p(\varphi_i): T_p(M) \to \mathbb{R}^n, p \in U_i$, are orientation preserving.

If O is an orientation on M, we write $-O$ for the *opposite orientation*, i.e., a basis in $T_p(M)$ is positively oriented for $(-O)_p$ if and only if it is negatively oriented for O_p. ■

Examples V.4.2. Each open subset $U \subseteq \mathbb{R}^n$ is orientable. We simply endow each tangent space $T_x(U) \cong \mathbb{R}^n$ with the canonical orientation of $\mathbb{R}^n$. ■

We are now looking for a criterion to decide for a given manifold whether it possesses some orientation. We call such manifolds *orientable*.

Definition V.4.3. Let (M, O^M) and (N, O^N) be oriented smooth manifolds of the same dimension. A smooth map $\varphi: M \to N$ is called *orientation preserving* if all the maps $T_x(\varphi): T_x(M) \to T_{\varphi(x)}(N)$ are orientation preserving. ■

Remark V.4.4. Let $U, V \subseteq \mathbb{R}^n$ be open subsets. A diffeomorphism $\varphi: U \to V$ is orientation preserving if and only if all the tangent maps

$$T_x(\varphi) = d\varphi(x) \in \mathrm{GL}_n(\mathbb{R})$$

are orientation preserving, i.e.,

$$\det\big(d\varphi(x)\big) > 0 \quad \text{for all} \quad x \in U$$

(cf. Lemma III.3.5). ■

Proposition V.4.5. *A smooth manifold M is orientable if and only if it possesses an atlas $\mathcal{A} = (\varphi_i, U_i)_{i\in I}$ for which all transition maps*

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are orientation preserving.

Proof. (1) We first assume that M is orientable and that O is an orientation on M. Let $\mathcal{A} = (\varphi_i, U_i)_{i\in I}$ be a smooth atlas for M for which all maps

$T_p(\varphi_i): T_p(M) \to \mathbb{R}^n$ are orientation preserving. Then for $p \in U_i \cap U_j$ the linear maps

$$T_p(\varphi_i): T_p(M) \to \mathbb{R}^n, \quad T_p(\varphi_j): T_p(M) \to \mathbb{R}^n$$

are both orientation preserving, which implies that

$$T_{\varphi_j(p)}(\varphi_{ij}) = T_{\varphi_j(p)}(\varphi_i \circ \varphi_j^{-1}) = T_p(\varphi_i) \circ T_p(\varphi_j)^{-1} \in \mathrm{GL}_n(\mathbb{R})$$

is orientation preserving.

(2) Conversely, let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an atlas with orientation preserving transition maps. Then for each $p \in U_i \cap U_j$ the map

$$T_p(\varphi_i)^{-1} \circ T_p(\varphi_j) \in \mathrm{GL}(T_p(M))$$

has positive determinant because it is conjugate to

$$T_p(\varphi_j) \circ T_p(\varphi_i)^{-1} \in \mathrm{GL}_n(\mathbb{R}).$$

We may therefore define consistently an orientation O_p on $T_p(M)$ for which all maps $T_p(\varphi_i): T_p(M) \to \mathbb{R}^n$, $p \in U_i$, are orientation preserving. We thus obtain an orientation on M. ∎

Lemma V.4.6. *Let $f : (M, O^M) \to (N, O^N)$ be a smooth map between oriented manifolds of the same dimension such that all tangent maps $T_x(f): T_x(M) \to T_{f(x)}(N)$ are invertible. If $T_{x_0}(f)$ is orientation preserving for some $x_0 \in M$, then the same holds for each x in the connected component of M containing x_0.*

Proof. Let (φ, U) be an orientation preserving chart for O^M with $x_0 \in U$ and likewise (ψ, V) with $f(x_0) \in V$ for O^N. We consider the connected component W of $f^{-1}(V) \cap U$ containing x_0. Then $\varphi(W)$ is an open connected subset of $\mathbb{R}^n$ and

$$F := \psi \circ f \circ \varphi^{-1}: \varphi(W) \to \psi(f(W)) \subseteq \mathbb{R}^n$$

is a smooth map whose differentials are invertible. In $\varphi(x_0)$ our assumption implies that

$$\det(dF(\varphi(x_0))) > 0$$

because $T_{x_0}(\varphi)$, $T_{x_0}(f)$, and $T_{f(x_0)}(\psi)$ are orientation preserving. As W is connected, F is orientation preserving (Exercise IV.4), which implies that f is orientation preserving on the open neighborhood W of x_0.

We conclude that the set $M_1 \subseteq M$ of those points $x \in M$ for which $T_x(f)$ is orientation preserving is an open subset of M. It likewise follows that M_2, the set of all points x for which $T_x(f)$ is orientation reversing, resp., orientation preserving w.r.t. $-O^N$, is open. We now have $x_0 \in M_1$ and $M = M_1 \dot\cup M_2$, so that the connected component of x_0 lies in M_1. ∎

Proposition V.4.7. *If M is connected and orientable, then it carries exactly two orientations.*

Proof. If O and O' are orientations on M, then we apply the preceding Lemma to $f = \mathrm{id}_M: (M, O) \to (M, O')$ to see that f either is orientation preserving or reversing on all of M. Hence $O' = O$ or $O' = -O$. ∎

Example V.4.8. (The Möbius strip) We consider the open rectangle

$$X := \{(x,y) \in \mathbb{R}^2 \colon -1 < y < 1, 0 < x < 3\}.$$

We define an equivalence relation on X by the equivalence classes

$$[(x,y)] := \begin{cases} \{(x,y)\} \colon 1 \leq x \leq 2\} \\ \{(x,y), (2+x, -y) \colon 0 < x < 1\}. \end{cases}$$

We endow the set $M := X/\sim$ of equivalence classes with the quotient topology. The equivalence relation models a certain gluing of the subset $]2,3[\times]-1,1[$ with the subset $]0,1[\times]-1,1[$. The space M is called the *open Möbius strip*.

We claim that M is a Hausdorff space and that it carries the structure of a smooth 2-dimensional manifold which is not orientable.

To verify the Hausdorff property, we note that the map

$$\widetilde{F} \colon X \to \mathbb{R}^3, \quad F([(x,y)]) := \begin{pmatrix} \cos \pi x \\ \sin \pi x \\ 0 \end{pmatrix} + y \begin{pmatrix} \cos \pi x \sin(\frac{\pi}{2}x) \\ \sin \pi x \sin(\frac{\pi}{2}x) \\ \cos(\frac{\pi}{2}x) \end{pmatrix}$$

factors through a smooth *injective* map $F \colon M \to \mathbb{R}^3$. It follows in particular that M is Hausdorff.

The smooth manifold structure on M can be obtained by observing that the two subsets

$$X_1 :=]0,2[\times]-1,1[\quad \text{and} \quad X_2 :=]1,3[\times]-1,1[$$

cover X, and that the quotient map $q \colon X \to M$ restricts to diffeomorphisms on X_1 and X_2.

To see that M is not orientable, we argue by contradiction. If this were the case, then the map $q \colon X \to M$ satisfies the assumption of Lemma V.4.6, which then implies that $q \colon X \to M$ is either orientation preserving or reversing. For $x < 1$ we have $q(x,y) = q(x+2,-y)$, which equals $q(\tau(x,y))$ for the orientation reversing map $\tau(x,y) = (x+2,-y)$ (a glide reflection). As τ is orientation reversing, the differentials $dq(x,y)$ and $dq(\tau(x,y))$ cannot be both orientation preserving (Exercise V.15). ■

Definition V.4.9. Let M be an n-dimensional smooth manifold. Each element $\mu \in \Omega^n(M,\mathbb{R})$ satisfying $\mu_p \neq 0$ for each $p \in M$ is called a *volume form*. ■

Lemma V.4.10. *If there exists a volume form on M, then M is orientable.*

Proof. Let μ be a volume form on M. Then each μ_p is a volume form on $T_p(M)$, hence defines an orientation O_p on this space (Definition III.3.1).

To see that this defines an orientation on M, we have to find for each $p \in M$ a chart (φ, U) with $p \in U$ such that for each $q \in U$ the map $T_q(\varphi) \colon T_q(M) \to \mathbb{R}^n$ is orientation preserving.

Let (φ, U) be any chart of M with $p \in U$ for which U is connected (this can always be achieved by shrinking the chart domain). Then $\varphi(U)$ is an open connected subset of $\mathbb{R}^n$ and $\mu_\varphi := (\varphi^{-1})^*\mu$ is a volume form on $\varphi(U)$, hence of the form

$$\mu_\varphi = f \cdot dx_1 \wedge \ldots \wedge dx_n,$$

where $f\colon \varphi(U) \to \mathbb{R}$ is a nowhere vanishing smooth function. Since $\varphi(U)$ is connected, we either have $f > 0$ or $f < 0$.

If $f > 0$, then all maps $T_q(\varphi)$ are orientation preserving and we are done. If $f < 0$, they are orientation reversing. In the latter case we define a new chart $(\widetilde{\varphi}, U)$ of M by

$$\widetilde{\varphi} := (-\varphi_1, \varphi_2, \ldots, \varphi_n).$$

Then $\widetilde{\varphi} \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}^n$ is the linear map whose matrix is $\mathrm{diag}(-1, 1, \ldots, 1)$. Then $\widetilde{\varphi} \circ \varphi^{-1}$ is orientation reversing, and therefore all maps $T_q(\widetilde{\varphi})\colon T_p(M) \to \mathbb{R}^n$ are orientation preserving. ■

Example V.4.11. Each Lie group G is orientable: In fact, let $\mu_{\mathbf{1}}$ be a volume form on the tangent space $T_{\mathbf{1}}(G)$. We now define a smooth n-form μ on G by

$$\mu_g(T_{\mathbf{1}}(\lambda_g).x_1, \ldots, T_{\mathbf{1}}(\lambda_g).x_n) := \mu_{\mathbf{1}}(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in T_{\mathbf{1}}(G)$$

The smoothness of this n-form follows from the smoothness of the left invariant vector fields $x_l(g) := T_{\mathbf{1}}(\lambda_g)x$, $x \in T_{\mathbf{1}}(G)$ (cf. Exercise V.13). ■

Proposition V.4.12. *A smooth paracompact manifold M is orientable if and only if it possesses a volume form.*

Proof. In view of the preceding lemma, we only have to show that the orientability of M implies the existence of a volume form. This requires smooth partitions of unity.

Let $\mathcal{A} = \{(\varphi_i, U_i)\colon i \in I\}$ be a locally finite smooth atlas of M for which all transition maps

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are orientation preserving (Proposition V.4.5). Further let $(\psi_i)_{i \in I}$ be a associated smooth partition of unity.

Let $\eta_i := dx_1 \wedge \ldots \wedge dx_n$ denote the canonical volume form on $\varphi_i(U_i)$. Then $\omega_i := \psi_i \varphi_i^* \eta_i$ is a smooth n-form on U_i whose support is a closed subset of M contained in U_i. We may therefore consider ω_i as a smooth n-form on M.

Now we put $\omega := \sum_i \omega_i$. Since this sum is locally finite in the sense that each point $p \in M$ has a neighborhood intersecting the support of only finitely many ω_i, ω is a smooth n-form on M.

For each $p \in M$ there exists an i with $\psi_i(p) > 0$. Then $p \in U_i$, and we have $\omega_i(p) \neq 0$. If we also have $\omega_j(p) \neq 0$, then

$$\varphi_i^* \eta_i = (\varphi_{ij} \circ \varphi_j)^* \eta_i = \varphi_j^* \varphi_{ij}^* \eta_i = \varphi_j^*(\det(d\varphi_{ij})\eta_j) = \varphi_j^*(\det(d\varphi_{ij})) \cdot \varphi_j^* \eta_j$$

implies that $\omega_j(p) \in \mathbb{R}^+ \omega_i(p)$, and therefore $\omega(p) \neq 0$. This implies that ω is a volume form on M. ■

Exercises for Section V

Exercise V.1. Show that an E-valued k-form ω on the smooth manifold M is smooth if and only if for each point $p \in M$ there exists a chart (φ, U) with $p \in M$ such that $(\varphi^{-1})^*\omega$ is smooth (cf. Definition V.1.1.). ■

Exercise V.2. Let M be a smooth manifold and E a finite-dimensional vector space. Show that the multiplication map

$$C^\infty(M,\mathbb{R}) \times \Omega^p(M,E), \quad (f,\omega) \mapsto f\omega, \quad (f\omega)_p := f(p)\omega_p$$

defines on $\Omega^k(M,E)$ the structure of a $C^\infty(M,\mathbb{R})$-module. ■

Exercise V.3. Let $r > 0$. Show that there exists a diffeomorphism $\alpha\colon \mathbb{R} \to]-r,r[$ satisying $\alpha(x) = x$ for all x in some 0-neighborhood. ■

Exercise V.4. Show that for each open ball $B \subseteq \mathbb{R}^n$ there exists a diffeomorphism $\varphi\colon \mathbb{R}^n \to B$. Hint: Let $r > 0$ be the radius of B and consider a function of the form $\varphi(x) = \alpha(\|x\|_2)\frac{x}{\|x\|_2}$ with α as in Exercise VI.1. ■

Exercise V.5. Let E be a finite-dimensional vector space. A family $(f_j)_{j\in J}$ of smooth E-valued functions on M is called *locally finite* if each point $p \in M$ has a neighborhood U for which the set $\{j \in J\colon f_j\,|_U \neq 0\}$ is finite. Show that this implies that $f := \sum_{j\in J} f_j$ defines a smooth E-valued function on M. ■

Exercise V.6. (a) The function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-\frac{1}{x}} & \text{for } x > 0 \end{cases}$$

is smooth and strictly increasing on $[0,\infty[$. Hint: Show that for each $n \in \mathbb{N}$ we have for $x > 0$: $f^{(n)}(x) = P_n(\frac{1}{x})f(x)$ for a polynomial P_n.

(b) For $a < b$ the function $g(x) := f(x-a)f(b-x)$ is a non-negative smooth function with $\mathrm{supp}(g) = [a,b]$ and a unique maximum in $x = \frac{a+b}{2}$.

(c) The function

$$h_{a,b}(x) := \frac{\int_x^b g(t)\,dt}{\int_a^b g(t)\,dt}$$

is smooth with

$$h_{a,b}(x) = \begin{cases} 1 & \text{for } x < a \\ 0 & \text{for } x > b \\ \in]0,1[& \text{for } a < x < b. \end{cases}$$

■

Exercise V.7. Let X be a topological space and $(K_n)_{n\in\mathbb{N}}$ a sequence of compact subsets of X with $\bigcup_{n\in\mathbb{N}} K_n = X$ and $K_n \subseteq K_{n+1}^0$ for each $n \in \mathbb{N}$. Show that for each compact subset $C \subseteq X$ there exists an $n \in \mathbb{N}$ with $C \subseteq K_n$.■

Exercise V.8. A family $(S_i)_{i\in I}$ of subsets of a topological space X is said to be *locally finite* if each point $p \in X$ has a neighborhood intersecting only finitely many S_i. Show that if $(S_i)_{i\in I}$ is a locally finite family of closed subsets of X, then $\bigcup_{i\in I} S_i$ is closed. ■

Exercise V.9. Let $(S_i)_{i\in I}$ be a locally finite family of subsets of the topological space X. Show that each compact subset $K \subseteq X$ intersects only finitely many of the sets S_i. ■

Exercise V.10. (Final topologies) Let X be a set and $f_i: X_i \to X$ be maps, where each X_i is a topological space. The *final topology on* X *with respect to the maps* $(f_i)_{i\in I}$ is the finest topology for which all maps f_i are continuous. Show that:

(1) $\tau := \{O \subseteq X : (\forall i \in I) f_i^{-1}(O) \text{ open}\}$ defines a topology on X.

(2) Show that τ is the finest topology on X for which all maps f_i are continuous.

(3) A map $g: X \to Y$, Y a topological space, is continuous if and only if all maps $g \circ f_i$ are continuous. ■

Exercise V.11. (Initial topologies) Let X be a set and $f_i: X \to X_i$ be maps, where each X_i is a topological space. The *initial topology on* X *with respect to the maps* $(f_i)_{i\in I}$ is the coarsest topology for which all maps f_i are continuous. Show that:

(1) The topology on X generated by the sets of the form $f_i^{-1}(O)$, $O \subseteq X_i$ open, has the property required above.

(2) A map $g: Y \to X$ is continuous if and only if all maps $f_i \circ g$ are continuous.■

Exercise V.12. Suppose that the set X is the direct limit of the system $(X_i, \varphi_{ji})_{i\leq j\in I}$. Show that if we are given maps $f_i: X_i \to Y$ satisfying

$$f_j \circ \varphi_{ji} = f_i \quad \text{for} \quad i \leq j,$$

then there exists a unique map $f: X \to Y$ with $f \circ \varphi_j = f_j$ for each $j \in J$. ■

Exercise V.13. (Vector fields on a Lie group) Let G be a Lie group with neutral element $\mathbf{1}$, multiplication map m_G, and $T(G)$ its tangent bundle. We write $\lambda_x(y) := xy$ and $\rho_y(x) := xy$ for the left and right multiplications on G. Show that:

(1) We obtain for each $x \in T_{\mathbf{1}}(G)$ a smooth vector field on G by

$$x_l(g) := T_{\mathbf{1}}(\lambda_g)(x).$$

Hint: (Exercise IV.5 shows that $x_l(g) = T_{(g,\mathbf{1})}(m_G)(0, x)$).

(2) Each Lie group is parallelizable. ■

Exercise V.14. Let G be a Lie group. Show that:

(1) All connected components of G are diffeomorphic. Hint: The maps $\lambda_g: G \to G, x \mapsto gx$ are diffeomorphisms.

(2) If $U \subseteq G$ is an open set and $V \subseteq G$ arbitrary, then

$$V \cdot U := \{vu: v \in V, u \in U\}$$

is open in G.

(3) If $U, V \subseteq G$ are compact, then $U \cdot V$ is compact.

(4) If $H \subseteq G$ is an open subgroup, then it is also closed. Hint: Consider the cosets of H in G.

(5) If $\mathbf{1} \in U = U^{-1} \subseteq G$ is an open subset, then each set $U^n := U^{n-1} \cdot U$ defines an ascending sequence of open subsets of G and $\bigcup_{n \in \mathbb{N}} U^n$ is an open closed subgroup.

(6) If $\mathbf{1} \in U = U^{-1} \subseteq G$ is a compact identity neighborhood, then each set U^n is compact and $\bigcup_n U^n$ contains the identity component of G.

(7) Each finite-dimensional Lie group G is paracompact. ■

Exercise V.15. Let (M, O) be an oriented manifold, $U \subseteq \mathbb{R}^n$ open and $f: U \to M$ a smooth orientation preserving map. Show that $f(x) = f(y)$ implies that

$$\det(T_x(f)^{-1} \circ T_y(f)) > 0.$$

■

Exercise V.16. Let M be a smooth manifold. A subgroup $\Gamma \subseteq \mathrm{Diff}(M)$ is called *admissible* if for each $p \in M$ there exists a neighborhood U such that the sets $\gamma.U$, $\gamma \in \Gamma$, do not overlap. Show that:

(a) For any admissible group Γ, we endow the set $X := M/\Gamma$ of Γ-orbits in M with the quotient topology. Show that the canonical map $q: M \to X$ is a local homeomorphism, i.e., each point $p \in M$ has a neighborhood U for which $q|_U: U \to q(U)$ is a diffeomorphism.

(b) The space X carries a unique structure of a smooth manifold for which q is a local diffeomorphism. ■

Exercise V.17. Show that the following subgroups Γ of the group $\mathrm{Mot}_2(\mathbb{R}) \subseteq \mathrm{Diff}(\mathbb{R}^2)$ of motions of the euclidean plane are admissible in the sense of Exercise V.16 and verify the following statements:

(1) $\Gamma = \langle \tau_1, \tau_2 \rangle$, where $\tau_1(x, y) = (x+1, y)$ and $\tau_2(x, y) = (x, y+1)$. In this case $\Gamma \cong \mathbb{Z}^2$ and $\mathbb{R}^2/\Gamma \cong \mathbb{T}^2$.

(2) $\Gamma = \langle \tau \rangle$, where $\tau(x, y) = (x+1, y)$. In this case $\Gamma \cong \mathbb{Z}$ and $\mathbb{R}^2/\Gamma$ is diffeomorphic to the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$.

(3) $\Gamma = \langle \tau \rangle$, where $\tau(x, y) = (x+1, -y)$. In this case $\Gamma \cong \mathbb{Z}$ and $\mathbb{R}^2/\Gamma$ is diffeomorphic to the Möbius strip.

(4) $\Gamma = \langle \tau_1, \tau_2 \rangle$, where $\tau_1(x, y) = (x+1, y)$ and $\tau_2(x, y) = (-x, y+1)$. In this case Γ is not abelian, $\mathbb{R}^2/\Gamma$ is compact and called the *Klein bottle*. ■

V.5. Manifolds with boundary

We have defined differentiable manifolds by charts consisting of homeomorphisms onto an open subset of $\mathbb{R}^n$. Manifolds with boundary are more general objects defined by charts which are homeomorphisms onto an open subset of closed half spaces of $\mathbb{R}^n$.

Definition V.5.1. Let M be a Hausdorff space and

$$H_n := \{x \in \mathbb{R}^n : x_1 \leq 0\}$$

the *closed left half space* in $\mathbb{R}^n$. Then

$$\partial H_n = \{x \in \mathbb{R}^n : x_1 = 0\}$$

is a hyperplane.

Note that the interior of H_n is dense and that this property is inherited by each open subset $V \subseteq H_n$ (Exercise V.18).

(a) A pair (φ, U), consisting of an open subset $U \subseteq M$ and a homeomorphism $\varphi: U \to \varphi(U) \subseteq H_n$ of U onto an open subset of H_n is called an H_n*-chart of* M.

(b) Two H_n-charts (φ, U) and (ψ, V) of M are said to be C^k*-compatible* ($k \in \mathbb{N} \cup \{\infty\}$) if $U \cap V = \emptyset$ or the map

$$\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism, i.e., $\psi \circ \varphi^{-1}$ and its inverse are smooth, as maps on subsets of $\mathbb{R}^n$ with dense interior (Definition IV.4.1).

(c) An H_n*-atlas of* M *of order* C^k is a family $\mathcal{A} := (\varphi_i, U_i)_{i \in I}$ of H_n-charts of M with $\bigcup_{i \in I} U_i = M$ and all charts in $\mathcal{A}$ are pairwise C^k-compatible.

(d) An H_n-atlas of order C^k is called *maximal* if it contains all charts which are C^k-compatible with it.

(e) An n*-dimensional* C^k*-manifold with boundary* is a pair $(M, \mathcal{A})$ of a Hausdorff space M and a maximal H_n-atlas of order C^k on M. For $k = \infty$ we call it a *smooth manifold with boundary.* ■

Lemma V.5.2. *Let* $U \subseteq H_n$ *be open,* E *a finite-dimensional vector space, and* $\psi: U \to E$ *a* C^k*-map. Then the restriction*

$$\psi_\partial := \psi|_{U \cap \partial H_n} : U \cap \partial H_n \to E$$

is a C^k*-map.*

Proof. By definition, ψ_∂ is continuous. Pick $p \in U \cap \partial H_n$ and let $U_1 \subseteq U$ be an open convex neighborhood of p in H_n. For each $q \in U_1 \cap H_n^0$ we then have for $\varepsilon \in]0,1[$:

$$\psi(q) = \psi(p + (q-p)) = \psi(p + \varepsilon(q-p)) + \int_\varepsilon^1 d\psi(p + t(q-p))(q-p)\,dt$$
$$\to \psi(p) + \int_0^1 d\psi(p + t(q-p))(q-p)\,dt$$

for $\varepsilon \to 0$. By continuity of both sides, we get

$$\psi(q) = \psi(p) + \int_0^1 d\psi(p + t(q-p))(q-p)\,dt$$

for each $q \in U_1 \cap \partial H_n$. This implies that for each $v \in \{0\} \times \mathbb{R}^{n-1} = T_p(\partial H_n)$, we have

$$\lim_{h\to 0} h^{-1}(\psi(p+hv) - \psi(p)) = \lim_{h \to 0} \int_0^1 d\psi(p + thv)(v)\,dt = d\psi(p)(v).$$

We conclude that ψ_∂ is a C^1-map with $d\psi_\partial(q)(v) = d\psi(q)(v)$. Iterating this argument in the sense that we apply it again to the partial derivatives of ψ, implies that ψ_∂ is C^k and that all its partial derivatives of order $\leq k$ coincide with the continuous extensions of the partial derivatives of ψ to $U \cap \partial H_n$. ■

Lemma V.5.3. *Let $U, V \subseteq H_n$ be open subsets and $\psi: U \to V$ a C^k-diffeomorphism. Then the following assertions hold:*

(1) *$\psi(U^0) = V^0$ and $\psi(\partial H_n \cap U) = \partial H_n \cap V$.*

(2) *The restriction $\psi_\partial := \psi|_{U \cap \partial H_n}$ is a C^k-diffeomorphism onto $V \cap \partial H_n$.*

(3) *If ψ is orientation preserving, then the same holds for ψ_∂.*

Proof. (1) The existence of a smooth map $\varphi: V \to U$ with $\varphi \circ \psi = \mathrm{id}_U$ and $\psi \circ \varphi = \mathrm{id}_V$ implies that $d\psi(x)$ is invertible for each $x \in U$. Hence the Inverse Function Theorem implies that $\psi(U^0)$ is an open subset of V, hence contained in V^0. As the same argument applies to φ, we also have $\varphi(V^0) \subseteq U^0$, which leads to $V^0 = \psi(\varphi(V^0)) \subseteq \psi(U^0)$. This proves that $\psi(U^0) = V^0$, and now

$$\psi(U \cap \partial H_n) = \psi(U \setminus U^0) = V \setminus V^0 = V \cap \partial H_n.$$

(2) In view of Lemma V.5.2, ψ_∂ is smooth. Since the same argument applies to φ_∂, ψ_∂ is a diffeomorphism onto its image.

(3) Assume that ψ is orientation preserving. The Jacobi matrix of ψ in $x \in \partial H_n$ is of the form

$$[d\psi(x)] = \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1}(x) & \mathbf{0} \\ * & [d\psi_\partial(x)] \end{pmatrix}.$$

As ψ maps U into H_n, we have $\frac{\partial \psi_1}{\partial x_1}(x) > 0$ for each $x \in \partial H_n \cap U$ (Exercise V.19). Hence

$$0 < \det(d\psi(x)) = \frac{\partial \psi_1}{\partial x_1}(x) \cdot \det\big(d\psi_\partial(x)\big)$$

implies $\det\big(d\psi_\partial(x)\big) > 0$, so that ψ_∂ is orientation preserving. ■

Definition V.5.4. (Boundary points) Let M be an n-dimensional manifold with boundary. The preceding lemma shows that if (φ, U) is a H_n-chart of M and $p \in M$ with $\varphi(p) \in \partial H_n$, then $\psi(p) \in \partial H_n$ holds for all H_n-charts (ψ, V) of M. Points with this property are called *boundary points of* M. We write ∂M for the set of all boundary points and $M^0 := M \setminus \partial M$ for the *interior of* M. ■

Lemma V.5.5. *The boundary ∂M of a smooth n-dimensional manifold M with boundary is a smooth $(n-1)$-dimensional manifold and M^0 is a smooth n-dimensional manifold without boundary.*

Proof. As a subset of the Hausdorff space M, ∂M is a Hausdorff space. For each H_n-chart (φ, U) of M, the restriction $\varphi' := \varphi|_{U \cap \partial M}$ is a homeomorphism $U \cap \partial M \to \varphi(U) \cap \partial H_n$, hence an $(n-1)$-dimensional chart of ∂M. Further Lemma V.5.3(2) shows that all these charts are smoothly compatible, hence define a smooth $(n-1)$-dimensional manifold structure on ∂M.

That M^0 is a smooth n-dimensional manifold without boundary follows easily from the definition. To obtain charts of M^0, we pick for each point $p \in M^0$ an H_n-chart (φ, U) with $p \in U$. Then $\varphi(p) \in H_n^0$, so that $U' := \varphi^{-1}(H_n^0)$ also is an open neighborhood of p, and $(\varphi|_{U'}, U')$ is a chart of M^0. The compatibility of all these charts follows from the compatibility of the chart of M. ■

Examples V.5.6. (a) H_n is an n-dimensional manifold with boundary and $\partial H_n = \{0\} \times \mathbb{R}^{n-1}$ is its topological boundary.

More generally, for each non-zero linear functional $f: V \to \mathbb{R}$ (V a finite-dimensional vector space) and $c \in \mathbb{R}$ the set $M := \{x \in V: f(x) \leq c\}$ is a smooth manifold with boundary. In fact, let $\psi_1 := f$ and extend to a basis $\psi_1, \ldots, \psi_n$ of V^*. Then

$$\varphi := (f - c, \psi_2, \ldots, \psi_n): V \to \mathbb{R}^n$$

is an affine isomorphism mapping M to H_n.

(b) Let M be a smooth n-dimensional manifold, $f: M \to \mathbb{R}$ a smooth function and $c \in \mathbb{R}$ a regular value of f, i.e., $df(x) \neq 0$ for each $x \in S := f^{-1}(c)$. Then

$$N := \{x \in M: f(x) \leq c\}$$

carries the structure of a smooth n-dimensional manifold with boundary $\partial N = S$ (Exercise).

We know from the Regular Value Theorem II.2.6 that S is an $(n-1)$-dimensional smooth submanifold of M. In view of the proof of Proposition I.2.8, there exists for each $p \in S$ a chart (φ, U) of M with $p \in U$ such that

$$\varphi(U \cap S) = \varphi(U) \cap (\{0\} \times \mathbb{R}^{n-1}) \quad \text{and} \quad \varphi_1 = f - c.$$

Then

$$\varphi(U \cap N) = \varphi(U) \cap H_n.$$

(c) $\mathbb{B}^n := \{x \in \mathbb{R}^n: \|x\|_2 \leq 1\}$ is a smooth submanifold with boundary. We simply apply (b) to the function $f(x) := \|x\|_2^2$.

(d) If M is an n-dimensional smooth manifold without boundary and $I := [0,1]$, then $N := I \times M$ is a smooth manifold with boundary $\partial N = (\{0\} \times M) \cup (\{1\} \times M)$ (Exercise V.23). ■

Definition V.5.7. (Tangent vectors) We define the tangent bundle of H_n by $T(H_n) := H_n \times \mathbb{R}^n$, considered as a subset of $T(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$. Accordingly, we put $T_x(H_n) = \mathbb{R}^n$ for each $x \in H_n$. For a boundary point $x \in \partial H_n$ a vector $v \in T_x(H_n)$ is called *inward (pointing)* if $v_1 < 0$ and *outward (pointing)* if $v_1 > 0$.

For a smooth manifold M with boundary, we now define tangent vectors and the tangent bundle $T(M) = \bigcup_{p\in M} T_p(M)$ in the same way as for manifolds without boundary in such a way that for each H_n-chart (φ, U) the tangent map $T(\varphi)\colon T(U) \to T(\varphi(U)) = \varphi(U) \times \mathbb{R}^n$ is a diffeomorphism. We then have $T_p(M) \cong \mathbb{R}^n$ for each $p \in M$, and in particular for all boundary points. This means that for $p \in \partial M$ the subspace $T_p(\partial M)$ is a hyperplane in $T_p(M)$. If $p \in \partial M$, then a tangent vector $v \in T_p(M)$ is said to be *inward pointing* if for all charts (φ, U) with $p \in U$ the vector $T_p(\varphi)v \in T_{\varphi(p)}(H_n)$ is inward pointing. Likewise we call v *outward pointing* if for all charts (φ, U) with $p \in U$ the vector $T_p(\varphi)v \in T_{\varphi(p)}(H_n)$ is outward pointing. ■

Definition V.5.8. (Differential forms) Let M be an n-dimensional smooth manifold with boundary, $k \in \mathbb{N}_0$ and E a finite-dimensional real vector space. An *E-valued k-form* on M is a function

$$\omega\colon M \to \dot{\bigcup}_{p\in M} \mathrm{Alt}^k(T_p(M), E) \quad \text{with} \quad \omega(p) \in \mathrm{Alt}^k(T_p(M), E), p \in M.$$

It is said to be *smooth* if for each H_n-chart (φ, U) the differential form $(\varphi^{-1})^*\omega$ is smooth on $\varphi(U)$, in the sense of Definition IV.4.1. We write $\Omega^k(M, E)$ for the set of smooth E-valued k-forms on M. ■

Definition V.5.9. An *oriented manifold with boundary* is a pair (M, O), where M is a smooth n-dimensional manifold and O is a collection of orientations O_p of the tangent spaces $T_p(M)$ such that there exists an atlas $\mathcal{A} = (\varphi_i, U_i)_{i\in I}$ consisting of H_n-charts and $(-H_n)$-charts, for which all maps $T_p(\varphi_i)\colon T_p(M) \to \mathbb{R}^n, p \in U_i$, are orientation preserving. ■

Remark V.5.10. If $\dim M > 1$, then we can avoid $(-H_n)$-charts because if (φ, U) is an orientation preserving $(-H_n)$-chart of M, then

$$\widetilde{\varphi} := (-\varphi_1, -\varphi_2, \varphi_3, \ldots, \varphi_n)$$

defines an orientation preserving H_n-chart $(\widetilde{\varphi}, U)$.

For $\dim M = 1$ we might not have enough H_n-charts to provide an atlas. The simplest example is $-H_1 = [0, \infty[$, endowed with the induced orientation from $\mathbb{R}$. Then there exists no orientation preserving H_1-chart (φ, U) with $0 \in U$. In fact, that φ is orientation preserving means that $\varphi' > 0$. ■

Definition V.5.11. We define an *orientation of a 0-dimensional manifold* M as a function assigning to each $p \in M$ a sign $O_p \in \{\pm 1\}$. ■

Proposition V.5.12. *If (M, O) is an oriented manifold of dimension $n > 1$ with boundary, then ∂M carries a unique orientation O_∂ determined by the property that for each $p \in \partial M$ and each outward pointing tangent vector $v_1 \in T_p(M)$ a basis $(v_2, \ldots, v_n)$ of $T_p(\partial M)$ is positively oriented if and only if the basis $(v_1, \ldots, v_n)$ of $T_p(M)$ is positively oriented. If $\dim M = 1$, then we define an orientation of ∂M by $O_p = 1$ if there exists an orientation preserving H_1-chart (φ, U) with $p \in U$ and $O_p = -1$ otherwise.*

Proof. First we observe that if $v_1' \in T_p(M)$ is another outward pointing tangent vector, then $v_1' \in \lambda v_1 + \operatorname{span}\{v_2, \ldots, v_n\}$ for some $\lambda > 0$, so that $(v_1', v_2, \ldots, v_n)$ is positively oriented if and only if $(v_1, v_2, \ldots, v_n)$ is. We thus obtain an orientation on each tangent space of ∂M.

If (φ, U) is an orientation preserving chart of (M, O), then the corresponding chart $(\varphi_\partial, U \cap \partial M)$ is orientation preserving with respect to the orientation of $\partial H_n \cong \mathbb{R}^{n-1}$ for which $(w_2, \ldots, w_n) \in \partial H_n$ is positively oriented if and only if $(e_1, w_2, \ldots, w_n)$ is positively oriented in $\mathbb{R}^n$. This implies that the orientation O_∂ on ∂M is compatible with the atlas of ∂M obtained from the orientation preserving charts of (M, O) (cf. Lemma V.5.5). ■

V.6. Integration of differential forms

In this section we shall see how to integrate smooth n-forms over oriented manifolds (with boundary) and prove Stokes' Integral Theorem.

Definition V.6.1. Let (M, O) be an n-dimensional oriented paracompact smooth manifold (with boundary), $\omega \in \Omega^n(M, E)$ a continuous E-valued differential form on M and $A \subseteq M$ a compact subset. We want to define the integral $\int_{(A,O)} \omega$ of ω over A with respect to the orientation O.

First we consider the case $n = 0$ (cf. Definition V.5.10). Then $A \subseteq M$ is a finite subset and $\omega = f\colon M \to E$ is a function. We then define

$$\int_{(A,O)} f := \sum_{p \in A} O_p \cdot f(p).$$

Now we turn to the case $n > 0$. We distinguish two cases.

Case 1: There exists a positively oriented chart (φ, U) with $A \subseteq U$. Then we define

$$\int_{(A,O)} \omega := \int_{\varphi(A)} (\varphi^{-1})^* \omega,$$

where the right hand side is an integral of the continuous n-form $(\varphi^{-1})^*\omega$ on $\varphi(U)$ over the compact subset $\varphi(A) \subseteq \varphi(U)$ (cf. Definition IV.5.1).

To see that this integral is well-defined, assume that (ψ, V) is another positively oriented chart with $A \subseteq V$. Then $A \subseteq U \cap V$, and the map

$$\eta := \varphi \circ \psi^{-1}\colon \psi(U \cap V) \to \varphi(U \cap V)$$

is an orientation preserving diffeomorphism. In view of the Oriented Transformation Formula (Prop. IV.5.3), we obtain

$$\int_{\varphi(A)} (\varphi^{-1})^*\omega = \int_{\varphi(A)} ((\eta \circ \psi)^{-1})^*\omega = \int_{\eta(\psi(A))} (\eta^{-1})^*(\psi^{-1})^*\omega = \int_{\psi(A)} (\psi^{-1})^*\omega.$$

Case 2: (General case) Let $(\varphi_i, U_i)_{i\in I}$ be a positively oriented atlas of M. Since M is paracompact, the open cover $(U_i)_{i\in I}$ has a locally finite refinement, and Theorem V.2.6 guarantees the existence of an associated partition of unity $(\psi_j)_{j\in J}$. Then for each j the n-form $\psi_j\omega$ has compact support contained in some U_{i_j}, so that $\int_{(A\cap \mathrm{supp}(\psi_j), O)} \psi_j\omega$ is defined as in Case 1. The local finiteness implies that $A \cap \mathrm{supp}(\psi_j) \neq \emptyset$ for only finitely many j, so that we may define

$$\int_{(A,O)} \omega := \sum_{j\in J} \int_{(A\cap \mathrm{supp}(\psi_j), O)} \psi_j\omega.$$

Again, we have to verify that the right hand side does not depend on the choice of the partition of unity. So let $(\eta_k)_{k\in K}$ be another partition of unity for which $\mathrm{supp}(\eta_k)$ is contained in a chart domain of some positively oriented chart of M. We then have

$$\begin{aligned}
&\sum_{j\in J} \int_{A\cap \mathrm{supp}(\psi_j)} \psi_j\omega = \sum_{j\in J} \int_{A\cap \mathrm{supp}(\psi_j)} \sum_{k\in K} \eta_k\psi_j\omega \\
&= \sum_{j\in J}\sum_{k\in K} \int_{A\cap \mathrm{supp}(\psi_j)} \eta_k\psi_j\omega = \sum_{j\in J}\sum_{k\in K} \int_{A\cap \mathrm{supp}(\eta_k)} \eta_k\psi_j\omega \\
&= \sum_{k\in K} \int_{A\cap \mathrm{supp}(\eta_k)} \sum_{j\in J} \eta_k\psi_j\omega = \sum_{k\in K} \int_{A\cap \mathrm{supp}(\eta_k)} \eta_k\omega.
\end{aligned}$$

■

Remark V.6.2. Let (M, O) be an oriented n-dimensional manifold. If $-O$ is the opposite orientation on M, then

$$\int_{(A,-O)} \omega = -\int_{(A,O)} \omega.$$

If $\dim M = 0$, this is immediate from the definition. To verify this, we may w.l.o.g. assume that $\dim M > 0$ and $A \subseteq U$ for some positively oriented chart (φ, U) of (M, O). Let $\tau \in \mathrm{End}(\mathbb{R}^n)$ be defined by

$$\tau(x) = (-x_1, x_2, \ldots, x_n).$$

Then $(\tau \circ \varphi, U)$ is positively oriented for $(M, -O)$, and we have

$$\begin{aligned}
\int_{(A,-O)} \omega &= \int_{\tau(\varphi(A))} ((\tau\circ\varphi)^{-1})^*\omega = \int_{\tau(\varphi(A))} \tau^*(\varphi^{-1})^*\omega \\
&= -\int_{\varphi(A)} (\varphi^{-1})^*\omega = -\int_{(A,O)} \omega
\end{aligned}$$

by the Oriented Transformation Formula (Prop. IV.5.3). ■

Theorem V.6.3. (Stokes' Theorem) *Let (M, O) be an oriented n-dimensional manifold with boundary, $n > 0$, and endow ∂M with the induced orientation O_∂. For each compactly supported $(n-1)$-form ω of class C^1, we then have*

$$(SF) \qquad \int_{(M,O)} d\omega = \int_{(\partial M, O_\partial)} \omega.$$

Proof. Let $(\varphi_i, U_i)_{i\in I}$ be a positively oriented atlas of M. Since M is paracompact, the open cover $(U_i)_{i\in I}$ has a locally finite refinement, and Theorem V.2.6 guarantees the existence of an associated partition of unity $(\psi_j)_{j\in J}$. Then for each j the n-form $\omega_j := \psi_j\omega$ has compact support contained in some U_{i_j}. Since both sides of (SF) are linear in ω, it suffices to assume that $\mathrm{supp}(\omega) \subseteq U_i$ for some i.

If $\dim M > 1$, then we may assume that all charts of M are H_n-charts, but for $n = 1$ we also have to admit $-H_1$-charts (cf. Proposition V.5.12).

By definition, we then have for any H_n-chart (φ_i, U_i):

$$\int_{(M,O)} d\omega = \int_{H_n} (\varphi_i^{-1})^*(d\omega) = \int_{H_n} d((\varphi_i^{-1})^*\omega)$$

and

$$\int_{(\partial M, O_\partial)} \omega = \int_{\partial H_n} (\varphi_i^{-1})^*\omega,$$

which reduces the assertion to the case $M = H_n$, resp., to H_1 or $-H_1$ for $n = 1$.

As in Example IV.3.2, we write the $(n-1)$-form ω as

$$\omega = \sum_{i=1}^n (-1)^{i-1} f_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

and recall that

$$d\omega = \Big(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\Big) dx_1 \wedge \ldots \wedge dx_n.$$

Next we observe that the restriction of $dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$ to ∂H_n vanishes for $i > 1$ because dx_1 vanishes on each tangent vector of ∂H_n. As the support of ω is compact,

$$\mathrm{supp}(\omega) \subseteq [-R, 0] \times [-R, R]^{n-1}$$

for some $R > 0$. We thus obtain

$$\begin{aligned}\int_{\partial H_n} \omega &= \int_{\partial H_n} f_1(0, x_2, \ldots, x_n) dx_2 \wedge \ldots \wedge dx_n \\ &= \int_{[-R,R]^{n-1}} f_1(0, x_2, \ldots, x_n) dx_2 \ldots dx_n.\end{aligned}$$

On the other hand, we have

$$\int_{H_n} d\omega = \int_{[-R,0]\times[-R,R]^{n-1}} \Big(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\Big) dx_1 \ldots dx_n.$$

For $i > 1$ we have

$$\int_{-R}^{R} \frac{\partial f_i}{\partial x_i}(x_1^0, x_2^0, \ldots, x_i, \ldots, x_n^0) dx_i$$
$$= f_i(x_1^0, x_2^0, \ldots, R, \ldots, x_n^0) - f_i(x_1^0, x_2^0, \ldots, -R, \ldots, x_n^0) = 0,$$

so that Fubini's Theorem implies that

$$\int_{[-R,0]\times[-R,R]^{n-1}} \frac{\partial f_i}{\partial x_i} dx_1 \ldots dx_n = 0.$$

This leaves only the first summand:

$$\begin{aligned}\int_{H_n} d\omega &= \int_{[-R,0]\times[-R,R]^{n-1}} \frac{\partial f_1}{\partial x_1} dx_1 \ldots dx_n \\ &= \int_{[-R,R]^{n-1}} f_1(0, x_2, \ldots, x_n) - f_1(-R, x_2, \ldots, x_n)\, dx_2 \ldots dx_n \\ &= \int_{[-R,R]^{n-1}} f_1(0, x_2, \ldots, x_n)\, dx_2 \ldots dx_n.\end{aligned}$$

Since this coincides with $\int_{\partial H_n} \omega$, the proof is complete. ∎

Corollary V.6.4. *If (M, O) is a compact oriented smooth n-dimensional manifold without boundary and ω an $(n-1)$-form on M, then*

$$\int_M d\omega = 0.$$ ∎

Remark V.6.5. Let (M, O) be a compact oriented manifold. The preceding corollary shows that the integral of each exact n-form ω over M vanishes.

If μ is a volume form on M compatible with the orientation (Proposition V.4.12), then we have for each orientation preserving chart (φ, U) that $(\varphi^{-1})^*\mu = f\, dx_1 \wedge \ldots \wedge dx_n$ for some positive function f. This implies that $\int_A \omega > 0$ for each compact subset $A \subseteq M$ with non-empty interior. In particular, we obtain $\int_M \mu > 0$. Therefore $[\mu] \in H^n_{\rm dR}(M, \mathbb{R})$ is non-zero, and the map

$$I\colon H^n_{\rm dR}(M, \mathbb{R}) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

is surjective. As a particular consequence of de Rham's Theorem (which we won't prove here), this map is also injective: an n-form $\omega \in \Omega^n(M, \mathbb{R})$ is exact if and only if $\int_M \omega = 0$. It follows in particular that

$$H^n_{\rm dR}(M, \mathbb{R}) \cong \mathbb{R}.$$ ∎

Example V.6.6. (The 1-dim. case) For $a < b \in \mathbb{R}$ the interval $M := [a, b]$ is a 1-dimensional compact manifold with boundary. For any smooth function f on M, considered as a 0-form, we therefore obtain

$$\int_a^b f'(t)\,dt = \int_{(M,O)} df = \int_{(\partial M, O_\partial)} f = f(b) - f(a)$$

because the orientation on $\partial[a, b]$ induces by the standard orientation of $\mathbb{R}$ assigns 1 to b and -1 to a (Proposition V.5.12). ■

Example V.6.7. Let $M \subseteq \mathbb{R}^n$ be a compact n-dimensional submanifold with smooth boundary and

$$\omega := \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n.$$

Then $d\omega = n dx_1 \wedge \ldots \wedge dx_n$, so that Stokes' Theorem leads to

$$n \operatorname{vol}_n(M) = \int_M d\omega = \int_{(\partial M, O_\partial)} \omega.$$

We can thus express the volume of M by a surface integral:

$$\operatorname{vol}_n(M) = \frac{1}{n} \int_{(\partial M, O_\partial)} \omega.$$

For $n = 2$ we obtain in particular

$$\operatorname{vol}_2(M) = \frac{1}{2} \int_{(\partial M, O_\partial)} x_1 dx_2 - x_2 dx_1. \qquad ■$$

Gauß' Theorem

Theorem V.6.8. *Let $M \subseteq \mathbb{R}^n$ be a compact subset with smooth boundary and $F \colon M \to \mathbb{R}^n$ a C^1-function (a vector field). Then*

$$\int_M \operatorname{div}(F) dx_1 \cdots dx_n = \int_{\partial M} \langle F, \nu \rangle (x) dS(x), \tag{6.2}$$

where dS stands for the surface measure of ∂M (cf. Definition V.2.9) *and $\nu(x) \in T_x(\partial M)^\perp$ is the exterior unit normal vector.* ■

We now explain how Gauß' Theorem can be derived from Stokes' Theorem. For the proof we shall need the following lemma.

Lemma V.6.9. *If* $B \in M_{n,n-1}(\mathbb{R})$ *has rank* $n-1$ *and* $B_i \in M_{n-1}(\mathbb{R})$ *is obtained by erasing the* i*-th row of* B*, then*

$$v := \frac{1}{\sqrt{\det(B^\top B)}}((-1)^{i+1}\det B_i)_{i=1,\ldots,n}$$

is a unit vector in $\mathrm{im}(B)^\perp$ *with* $\det(v, Be_1, \ldots, Be_{n-1}) > 0$.

Proof. For each i we put $b_i := Be_i$. Then the matrix $(b_i, B) \in M_n(\mathbb{R})$ is singular. Expanding the determinant $\det(b_i, B) = 0$ with respect to the first column, it follows that $v \perp b_i$, hence that $v \perp \mathrm{im}(B)$.

We further have

$$\|v\|^2 = \frac{1}{\sqrt{\det(B^\top B)}}\sum_{i=1}^n (\det B_i)^2 = 1$$

(Corollary IV.2.7) and

$$\det(v, Be_1, \ldots, Be_{n-1}) = \det(v|B) = \sum_{i=1}^n (-1)^{i+1} v_i \det(B_i)$$
$$= \frac{1}{\sqrt{\det(B^\top B)}}\sum_{i=1}^n \det(B_i)^2 > 0.$$

This completes the proof. ■

Proof. (Gauß' Theorem) To see how (6.2) follows from Stokes' Theorem, we consider the $(n-1)$-form

$$\omega := \sum_{i=1}^n (-1)^{i-1} F_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n,$$

so that $d\omega = (\mathrm{div}\, F) dx_1 \wedge \ldots \wedge dx_n$ implies that

$$\int_M \mathrm{div}(F)\, dx_1 \cdots dx_n = \int_M d\omega = \int_{(\partial M, O_\partial)} \omega.$$

If $U \subseteq \mathbb{R}^{n-1}$ is an open subset and $\psi\colon U \to \partial M$ is a parametrization, i.e., ψ is a C^1-map with $\mathrm{rk}(d\psi(x)) = n-1$ for each $x \in U$, then

$$dS(x) = \sqrt{g(x)}\, dx,$$

where $g(x) := \det([d\psi(x)]^\top [d\psi(x)])$.

For any parametrization $\psi\colon U \to \partial M$ the preceding lemma implies that the exterior unit normal vector $\nu(\psi(x))$ is given by

$$\nu(\psi(x)) = \frac{1}{\sqrt{g(x)}}((-1)^{i+1}\det[d\psi(x)]_i)_{i=1,\ldots,n},$$

which leads to

$$\langle F,\nu\rangle(\psi(x))\sqrt{g(x)} = \sum_{i=1}^n (-1)^{i+1} F_i(\psi(x)) \det[d\psi(x)]_i.$$

On the other hand

$$\begin{aligned}(\psi^*\omega)(x_2,\ldots,x_n) &= \sum_{i=1}^n (-1)^{i+1} F_i(\psi(x)) d\psi_1 \wedge \ldots \wedge \widehat{d}\psi_i \wedge \ldots \wedge d\varphi_n \\ &= \sum_{i=1}^n (-1)^{i+1} F_i(\psi(x)) \det([d\psi(x)]_i) dx_2 \wedge \ldots \wedge dx_n.\end{aligned}$$

We conclude that for each compact subset $K \subseteq U$ we have

$$\int_{\psi(K)} \omega = \int_K \psi^*\omega = \int_K \langle F,\nu\rangle(\psi(x))\sqrt{g(x)}\, dx_2 \cdots dx_n = \int_K \langle F,\nu\rangle(\psi(x)) dS(x).$$

■

Remark V.6.10. Originally, Stokes' Theorem was concerned with integrals over regions on surfaces in $\mathbb{R}^3$, surrounded by (piecewise) smooth curves: Let $M \subseteq \mathbb{R}^3$ be an oriented hypersurface and $A \subseteq M$ a compact subset with smooth boundary. Let $F\colon A \to \mathbb{R}^3$ be a C^1-vector field and $\nu\colon M \to \mathbb{R}^3$ a unit normal vector field defined by the property that for each positively oriented basis (b_1, b_2) of $T_p(M)$, the basis $(\nu(p), b_1, b_2)$ of $\mathbb{R}^3$ is positively oriented. Then we have

$$\int_A \langle \mathrm{rot}(F),\nu\rangle\, dS = \int_{(\partial A, O_\partial)} \sum_{i=1}^3 F_i dx_i.$$

To see how this assertion follows from Stokes' Theorem, observe that $\omega := \sum_{i=1}^3 F_i dx_i$ defines a C^1-1-Form on the 2-dimensional oriented manifold A with boundary. Therefore Stokes' Theorem asserts that

$$\begin{aligned}&\int_{(\partial A, O_\partial)} \sum_{i=1}^3 F_i dx_i = \int_{(\partial A, O_\partial)} \omega \overset{(SF)}{=} \int_{(A, O_A)} d\omega \\ &= \int_{(A,O_A)} (\mathrm{rot}\, F)_1 dx_2 \wedge dx_3 - (\mathrm{rot}\, F)_2 dx_1 \wedge dx_3 + (\mathrm{rot}\, F)_3 dx_1 \wedge dx_2\end{aligned}$$

(Remark IV.3.3). Now we argue as in the proof of Gauß' Theorem to see that this equals the surface integral $\int_{(A,O_A)} \langle \mathrm{rot}\, F, \nu\rangle\, dS$. ■

Remark V.6.11. We briefly discuss an application to electrodynamics. Let $U \subseteq \mathbb{R}^3$ be open and $I \subseteq \mathbb{R}$ an interval. In electrodynamics the *magnetic* and the *electrical field* are modeled by time-dependent vector fields, hence functions

$$E, B\colon U \times I \to \mathbb{R}^3.$$

One of Maxwell's equations asserts that

$$\mathrm{rot}\, E = -\frac{\partial B}{\partial t}. \tag{ME}$$

Let $M \subseteq \mathbb{R}^3$ be a compact oriented surface with boundary. Then the integral

$$\Phi(t) := \int_{(M,O)} \langle B(x,t), \nu(x)\rangle\, dS(x)$$

is the *magnetic flux through* M *at time* t. Differentiation under the integral sign leads to

$$\begin{aligned}\Phi'(t) &= \int_{(M,O)} \langle \frac{\partial B(x,t)}{\partial t}, \nu(x)\rangle\, dS(x) \overset{(ME)}{=} -\int_{(M,O)} \langle \mathrm{rot}\, E(x,t), \nu(x)\rangle\, dS(x)\\ &\overset{(SF)}{=} -\int_{(\partial M, O_\partial)} \sum_{i=1}^3 E_i dx_i = -\oint_{\partial M} E ds.\end{aligned}$$

The physical interpretation of this equation is that any change of the magnetic flux through a surface M bounded by a wire (an electric conductor) induces an electric current through the boundary wire. ■

More Exercises for Chapter V

Exercise V.18. Show that each subset $U \subseteq \mathbb{R}^n$, which is an open subset of a subset V with dense interior, has dense interior. ■

Exercise V.19. Let $U \subseteq H_n$ be an open subset and $\psi\colon U \to H_n$ be a smooth map. Show that for each $x \in U \cap \partial H_n$ with $\psi(x) \in \partial H_n$ we have

$$\frac{\partial \psi_1}{\partial x_1} \geq 0.$$

If, in addition, ψ is a diffeomorphism onto an open subset of H_n, then

$$\frac{\partial \psi_1}{\partial x_1} > 0$$

for each $x \in U \cap \partial H_n$. ■

Exercise V.20. Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ be the n-dimensional sphere.

(1) Show that $\mathbb{S}^n$ is orientable. Hint: It is the boundary of an orientable manifold with boundary.

(2) The map $\tau\colon \mathbb{S}^n \to \mathbb{S}^n, x \mapsto -x$ is orientation preserving if and only if n is odd. ■

Exercise V.21. Let $\mathbb{T}^n = (\mathbb{S}^1)^n$ be the n-dimensional torus, realized as $\mathbb{R}^n/\mathbb{Z}^n$. Show that:

(1) $\mathbb{T}^n$ is orientable.

(2) Each $g \in \mathrm{GL}_n(\mathbb{Z})$ induces a diffeomorphism φ_g of $\mathbb{T}^n$ by $[x] \mapsto [gx]$.

(3) φ_g is orientation preserving if and only if $\det(g) = 1$. ■

Exercise V.22. Show that the real projective space $\mathbb{P}_n(\mathbb{R}) := \mathbb{P}(\mathbb{R}^{n+1})$ is not orientable if n is odd. Hint: Consider the quotient map $q\colon \mathbb{S}^n \to \mathbb{P}_n(\mathbb{R})$ whose fibers are the sets $\{\pm x\}$, $x \in \mathbb{S}^n$, and use Exercise V.20. ■

Exercise V.23. Show that:

(1) If M is an n-dimensional smooth manifold without boundary and M' is a smooth k-dimensional manifold with boundary, then $N := M \times M'$ is a smooth $(n+k)$-dimensional manifold with boundary $\partial N = M \times \partial M'$.

(2) If M is an n-dimensional smooth manifold without boundary and $I = [0,1]$, then $N := I \times M$ is a smooth manifold with boundary $\partial N = \{0\} \times M \cup \{1\} \times M$. ■

Exercise V.23. Let $V = V_1 \times V_2$ be a direct product of two finite-dimensional vector spaces. Show that:

(1) (Product orientation) If O_{V_1}, resp., O_{V_2} are orientations of V_1, resp., V_2, then we obtain a unique orientation O_V on V for which any basis of the form $(b_1, \ldots, b_n, c_1, \ldots, c_m)$, where $(b_1, \ldots, b_n)$ is a positively oriented basis of (V_1, O_{V_1}) and $(c_1, \ldots, c_m)$ is a positively oriented basis of (V_2, O_{V_2}), is positively oriented.

(2) (Induced orientation) If O_{V_1}, resp., O_V are orientations of V_1, resp., V, then we obtain a unique orientation O_{V_2} on V_2 for which a basis of the form $(c_1, \ldots, c_m)$ is positively oriented if for each positively oriented basis $(b_1, \ldots, b_n)$ of (V_1, O_{V_1}) the basis $(b_1, \ldots, b_n, c_1, \ldots, c_m)$ is positively oriented in (V, O_V). ■

Exercise V.24. Let $M \subseteq \mathbb{R}^2$ be a compact subset with smooth boundary.

(a) Let E a finite-dimensional vector space and $f, g \in C^1(M, E)$. Prove Green's Integral Theorem:

$$\int_{(\partial M, O_\partial)} f dx + g dy = \int_M \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big) dx \wedge dy.$$

(b) Show that if $f\colon M \to \mathbb{C}$ is holomorphic on M (in the sense of differentiable functions on sets with dense interior), then

$$\int_{(\partial M, O_\partial)} f dz = 0.$$ ■

Exercise V.25. We realize $\mathbb{S}^1 = \mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$ and write $\omega \in \Omega^1(\mathbb{T}, \mathbb{R})$ as $\omega = f(t)dt$ for a smooth 1-periodic function f. Show that

(1) ω is exact if and only if $\int_0^1 f(t)\, dt = \int_{\mathbb{T}} \omega = 0$.

(2) Derive that $\dim H^1_{\mathrm{dR}}(\mathbb{S}^1, \mathbb{R}) = 1$. ■

Exercise V.26. We realize $\mathbb{T}^n$ as $\mathbb{R}^n/\mathbb{Z}^n$ and write $\omega \in \Omega^n(\mathbb{T},\mathbb{R})$ as $\omega = f(t_1,\ldots,f_n)dt_1 \wedge \ldots \wedge dt_n$ for a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, periodic in each variable.

(1) Expand the function f in a Fourier series

$$f(t) = \sum_{m\in\mathbb{Z}^n} a_m e_m(t), \quad e_m(t) = e^{2\pi i\langle m,t\rangle} = e^{2\pi i(m_1t_1+\ldots+m_nt_n)}$$

and show that $f = \frac{\partial g}{\partial t_j}$ for a periodic function g if and only if a_m vanishes for all m with $m_j = 0$.

(2) ω is exact if and only if $\int_{[0,1]^n} f(t)\,dt = \int_{\mathbb{T}^n} \omega = 0$. Hint: The exactness of ω means that $f = \operatorname{div} G$ for a periodic vector field G.

(3) Derive that $\dim H^n_{\rm dR}(\mathbb{T}^n,\mathbb{R}) = 1$. ■

VI. Vector fields and local flows

In this section we turn to the geometric nature of vector fields as infinitesimal generators of local flows on manifolds. This provides a natural perspective on (autonomous) ordinary differential equations.

VI.1. Integral curves of vector fields

Throughout this section M denotes an n-dimensional smooth manifold.

Definition VI.1.1. Let $X \in \mathcal{V}(M)$ and $I \subseteq \mathbb{R}$ an open interval containing 0. A differentiable map $\gamma: I \to M$ is called an *integral curve* of X if

$$\gamma'(t) = X(\gamma(t)) \qquad \text{for each} \quad t \in I.$$

Note that the preceding equation implies that γ' is continuous and further that if γ is C^k, then γ' is also C^k. Therefore integral curves are automatically smooth.

If $J \supseteq I$ is an interval containing I, then an integral curve $\eta: J \to M$ is called an *extension* of γ if $\eta|_I = \gamma$. An integral curve γ is said to be *maximal* if it has no proper extension. ■

Remark VI.1.2. (a) If $U \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, then we write a vector field $X \in \mathcal{V}(U)$ as $X(x) = (x, F(x))$, where $F: M \to \mathbb{R}^n$ is a smooth function. A curve $\gamma: I \to U$ is an integral curve of X if and only if it satisfies the ordinary differential equation

$$\gamma'(t) = F(\gamma(t)) \quad \text{for all} \quad t \in I.$$

(b) If (φ, U) is a chart of the manifold M and $X \in \mathcal{V}(M)$, then a curve $\gamma: I \to M$ is an integral curve of X if and only if the curve $\eta := \varphi \circ \gamma$ is an integral curve of the vector field $X_\varphi := T(\varphi) \circ X \circ \varphi^{-1} \in \mathcal{V}(\varphi(U))$ because

$$X_\varphi(\eta(t)) = T_{\gamma(t)}(\varphi)X(\gamma(t)) \quad \text{and} \quad \eta'(t) = T_{\gamma(t)}(\varphi)\gamma'(t). \qquad ■$$

Remark VI.1.3. A curve $\gamma: I \to M$ is an integral curve of X if and only if $\widetilde{\gamma}(t) := \gamma(-t)$ is an integral curve of the vector field $-X$.

More generally, for $a, b \in \mathbb{R}$, the curve $\eta(t) := \gamma(at+b)$ is an integral curve of the vector field aX. ■

Definition VI.1.4. Let $a < b \in [-\infty, \infty]$. For a continuous curve $\gamma:]a, b[\to M$ we say that

$$\lim_{t \to b} \gamma(t) = \infty$$

if for each compact subset $K \subseteq M$ there exists a $c < b$ with $\gamma(t) \notin K$ for $t > c$. We likewise define

$$\lim_{t \to a} \gamma(t) = \infty. \qquad ■$$

Theorem VI.1.5. (Existence and Uniqueness Theorem for integral curves) *Let $X \in \mathcal{V}(M)$ and $p \in M$. Then there exists a unique maximal integral curve $\gamma_p\colon I_p \to M$ with $\gamma_p(0) = p$. If $a := \inf I > -\infty$, then $\lim_{t\to a} \gamma_p(t) = \infty$ and if $b := \sup I < \infty$, then $\lim_{t\to b} \gamma_p(t) = \infty$.*

Proof. We have seen in Remark VI.1.2 that in local charts, integral curves are solutions of an ordinary differential equation with a smooth right hand side. We now reduce the proof to the Local Existence- and Uniqueness Theorem for ODE's.

Uniqueness: Let $\gamma, \eta\colon I \to M$ be two integral curves of X with $\gamma(0) = \eta(0) = p$. The continuity of the curves implies that that $J := \{t \in I\colon \gamma(t) = \eta(t)\}$ is a closed subset of I, and it is clear that $0 \in J$. In view of the Local Uniqueness Theorem for ODE's, there exists for each $t_0 \in J$ an $\varepsilon > 0$ with $[t_0, t_0 + \varepsilon] \subseteq J$, and likewise $[t_0 - \varepsilon, t_0] \subseteq J$ (Remark VI.1.3). Therefore J is also open. Now the connectedness of I implies $I = J$, so that $\gamma = \eta$.

Existence: The Local Existence Theorem implies the existence of some integral curve $\gamma\colon I \to M$ on some open interval containing 0. For any other integral curve $\eta\colon J \to M$, the intersection $I \cap J$ is an interval containing 0, so that the Uniqueness assertion implies that $\eta = \gamma$ on $I \cap J$.

Let $I \subseteq \mathbb{R}$ be the union of all open intervals I_j on which there exists an integral curve $\gamma_j\colon I_j \to M$ of X with $\gamma_j(0) = p$. Then the preceding argument shows that

$$\gamma(t) := \gamma_j(t) \quad \text{for} \quad t \in I_j$$

defines an integral curve of X on I, which is maximal by definition. The uniqueness of the maximal integral curve also follows from its definition.

Limit condition: Suppose that $b := \sup I < \infty$. If $\lim_{t\to b} \gamma(t) = \infty$ does not hold, then there exists a compact subset $K \subseteq M$ and a sequence $t_m \in I$ with $t_m \to b$ and $\gamma(t_m) \in K$. As K can be covered with finitely many closed subsets homeomorphic to a closed subsets of a ball in $\mathbb{R}^n$, after passing to a suitable subsequence, we may w.l.o.g. assume that K itself is homeomorphic to a compact subset of $\mathbb{R}^n$. Then a subsequence of $(\gamma(t_m))_{m\in\mathbb{N}}$ converges, and we may replace the original sequence by this subsequence, hence assume that $q := \lim_{m\to\infty} \gamma(t_m)$ exists.

The Local Existence Theorem for ODE's implies the existence of a compact neighborhood $V \subseteq M$ of q and $\varepsilon > 0$ such that the initial value problem

$$\eta(0) = x, \quad \eta' = X \circ \eta$$

has a solution on $[-\varepsilon, \varepsilon]$ for each $x \in V$. Pick $m \in \mathbb{N}$ with $t_m > b - \varepsilon$ and $\gamma(t_m) \in V$. Further let $\eta\colon [-\varepsilon, \varepsilon] \to M$ be an integral curve with $\eta(0) = \gamma(t_m)$. Then

$$\gamma(t) := \eta(t - t_m) \quad \text{for} \quad t \in [t_m - \varepsilon, t_m + \varepsilon],$$

defines an extension of γ to the interval $I \cup]t_m, t_m + \varepsilon[$ strictly containing $]a, b[$, hence contradicting the maximality of I. This proves that $\lim_{t\to b} \gamma(t) = \infty$. Replacing X by $-X$, we also obtain $\lim_{t\to a} \gamma(t) = \infty$. ■

If $q = \gamma_p(t)$ is a point on the unique integral curve of X through $p \in M$, then $I_q = I_p - t$ and

$$\gamma_q(s) := \gamma_p(t+s)$$

is the unique maximal integral curve through q.

Example VI.1.6. (a) On $M = \mathbb{R}$ we consider the vector field X given by the function $F(s) = (1+s^2)$, i.e., $X(s) = (s, 1+s^2)$. The corresponding ODE is

$$\gamma'(s) = X(\gamma(s)) = 1 + \gamma(s)^2.$$

For $\gamma(0) = 0$ the function $\gamma(s) := \tan(s)$ on $I :=]-\frac{\pi}{2}, \frac{\pi}{2}[$ is the unique maximal solution because

$$\lim_{t\to\frac{\pi}{2}} \tan(t) = \infty \quad \text{and} \quad \lim_{t\to-\frac{\pi}{2}} \tan(t) = -\infty.$$

(b) Let $M :=]-1, 1[$ and $X(s) = (s, 1)$, so that the corresponding ODE is $\gamma'(s) = 1$. Then the unique maximal solution is

$$\gamma(s) = s, \quad I =]-1, 1[.$$

Note that we also have in this case

$$\lim_{s\to\pm 1} \gamma(s) = \infty$$

if we consider γ as a curve in the non-compact manifold M.

For $M = \mathbb{R}$ the same vector field has the maximal integral curve

$$\gamma(s) = s, \quad I = \mathbb{R}.$$

(c) For $M = \mathbb{R}$ and $X(s) = (s, -s)$, the differential equation is $\gamma'(t) = -\gamma(t)$, so that we obtain the maximal integral curves $\gamma(t) = \gamma_0 e^{-t}$. For $\gamma_0 = 0$ this curve is constant, and for $\gamma_0 \neq 0$ we have $\lim_{t\to\infty} \gamma(t) = 0$, hence $\lim_{t\to\infty} \gamma(t) \neq \infty$. This shows that maximal integral curves do not always leave every compact subset of M if they are define on an interval unbounded from above. ■

The preceding example shows in particular that the global existence of integral curves can also be destroyed by deleting parts of the manifold M, i.e., by considering $M' := M \setminus K$ for some closed subset $K \subseteq M$.

Definition VI.1.7. A vector field $X \in \mathcal{V}(M)$ is said to be *complete* if all its maximal integral curves are defined on all of $\mathbb{R}$. ■

Corollary VI.1.8. *All vector fields on a compact manifold M are complete.* ■

VI.2. Local flows

Definition VI.2.1. Let M be a smooth manifold. A *local flow on* M is a smooth map

$$\Phi: U \to M,$$

where $U \subseteq \mathbb{R} \times M$ is an open subset containing $\{0\} \times M$, such that for each $x \in M$ the intersection $I_x := U \cap (\mathbb{R} \times \{x\})$ is an interval and

$$\Phi(0, x) = x \quad \text{and} \quad \Phi\big(t, \Phi(s, x)\big) = \Phi(t + s, x)$$

hold for all t, s, x for which both sides are defined. The maps

$$\alpha_x: I_x \to M, \qquad t \mapsto \Phi(t, x)$$

are called the *flow lines*. The flow Φ is said to be *global* if $U = \mathbb{R} \times M$. ■

Lemma VI.2.2. *If $\Phi: U \to M$ is a local flow, then*

$$X^\Phi(x) := \frac{d}{dt}\Big|_{t=0} \Phi(t, x) = \alpha_x'(0)$$

defines a smooth vector field.

It is called the *velocity field* or the *infinitesimal generator* of the local flow Φ. ■

Lemma VI.2.3.. *If $\Phi: U \to M$ is a local flow on M, then the flow lines are integral curves of the vector field X^Φ. In particular, the local flow Φ is uniquely determined by the vector field X^Φ.*

Proof. Let $\alpha_x: I_x \to M$ be a flow line and $s \in I_x$. For sufficiently small $t \in \mathbb{R}$ we then have

$$\alpha_x(s + t) = \Phi(s + t, x) = \Phi\big(t, \Phi(s, x)\big) = \Phi\big(t, \alpha_x(s)\big),$$

so that taking derivatives in $t = 0$ leads to $\alpha_x'(s) = X^\Phi(\alpha_x(s))$.

That Φ is uniquely determined by the vector field X^Φ follows from the uniqueness of integral curves (Theorem VI.1.5). ■

Theorem VI.2.4. *Each smooth vector field X is the velocity field of a unique maximal local flow defined by*

$$\mathcal{D}_X := \bigcup_{x \in M} I_x \times \{x\} \quad \textit{and} \quad \Phi(t, x) := \gamma_x(t) \quad \textit{for} \quad (t, x) \in \mathcal{D}_X,$$

where $\gamma_x: I_x \to M$ is the unique maximal integral curve through $x \in M$.

Proof. If $(s,x), \big(t,\Phi(s,x)\big)$ and $(s+t,x) \in \mathcal{D}_X$, the relation

$$\Phi(s+t,x) = \Phi\big(t,\Phi(s,x)\big) \quad \text{and} \quad I_{\Phi(s,x)} = I_{\gamma_x(s)} = I_x - s$$

follow from the fact that both curves

$$t \mapsto \Phi(t+s,x) = \gamma_x(t+s) \quad \text{and} \quad t \mapsto \Phi\big(t,\Phi(s,x)\big) = \gamma_{\Phi(s,x)}(t)$$

are integral curves of X with the initial value $\Phi(s,x)$, hence coincide. That the flow Φ is maximal is a direct consequence of the maximality of its integral curves.

We claim that all maps $\Phi_t \colon M_t := \{x \in M \colon (t,x) \in \mathcal{D}_X\} \to M, x \mapsto \Phi(t,x)$ are injective. In fact, if $p := \Phi_t(x) = \Phi_t(y)$, then $\gamma_x(t) = \gamma_y(t)$, and on $[0,t]$ the curves $s \mapsto \gamma_x(t-s), \gamma_y(t-s)$ are integral curves of $-X$, starting in p. Hence the Uniqueness Theorem VI.1.5 implies that they coincide in $s = t$, which mans that $x = \gamma_x(0) = \gamma_y(0) = y$. From this argument it further follows that $\Phi_t(M_t) = M_{-t}$ and $\Phi_t^{-1} = \Phi_{-t}$.

It remains to show that $\mathcal{D}_X$ is open and Φ smooth. The local Existence Theorem provides for each $x \in M$ an open neighborhood U_x diffeomorphic to a cube and some $\varepsilon_x > 0$, as well as a smooth map

$$\varphi_x \colon]-\varepsilon_x, \varepsilon_x[\times U_x \to M, \quad \varphi_x(t,y) = \gamma_y(t) = \Phi(t,y).$$

Hence $]-\varepsilon_x, \varepsilon_x[\times U_x \subseteq \mathcal{D}_X$, and the restriction of Φ to this set is smooth. Therefore Φ is smooth on a neighborhood of $\{0\} \times M$ in $\mathcal{D}_X$.

Now let J_x be the set of all $t \in [0,\infty[$, for which $\mathcal{D}_X$ contains a neighborhood of $[0,t] \times \{x\}$ on which Φ is smooth. The interval J_x is open in $\mathbb{R}^+ := [0,\infty[$ by definition. We claim that $J_x = I_x \cap \mathbb{R}^+$. This entails that $\mathcal{D}_X$ is open because the same argument applies to $I_x \cap]-\infty, 0]$.

We assume the contrary and find a minimal $\tau \in I_x \cap \mathbb{R}^+ \setminus J_x$, because this interval is closed. Put $p := \Phi(\tau,x)$ and pick a product set $I \times W \subseteq \mathcal{D}_X$, where W is a neighborhood of p and $I =]-2\varepsilon, 2\varepsilon[$ a 0-neighborhood, such that $2\varepsilon < \tau$ and $\Phi : I \times W \to M$ is smooth. By assumption, there exists a neighborhood V of x such that Φ is smooth on $[0,\tau-\varepsilon] \times V \subseteq \mathcal{D}_X$. Then $\Phi_{\tau-\varepsilon}$ is smooth on V. We now define

$$V' := \Phi_{\tau-\varepsilon}^{-1}\big(\Phi_\varepsilon^{-1}(W)\big) \cap V = \Phi_\tau^{-1}(W) \cap V.$$

Then $[0,\tau+\varepsilon] \times V'$ is a neighborhood of $[0,\tau+\varepsilon] \times \{x\}$ in $\mathcal{D}_X$ on which Φ is smooth, because it is a composition of smooth maps:

$$]\tau - 2\varepsilon, \tau + 2\varepsilon[\times V' \to M, \quad (t,y) \mapsto \Phi\big(t-\tau, \Phi(\varepsilon, \Phi(\tau-\varepsilon, y))\big).$$

We thus arrive at the contradiction $\tau \in J_x$.

This completes the proof of the openness of $\mathcal{D}_X$ and the smoothness of Φ. The uniqueness of the flow follows from the uniqueness of the integral curves. ■

Remark VI.2.5. Let $X \in \mathcal{V}(M)$ be a complete vector field and $\Phi: \mathbb{R} \times M \to M$ the corresponding global flow. The maps $\Phi_t: x \mapsto \Phi(t, x)$ then satisfy
(A1) $\Phi_0 = \mathrm{id}_M$.
(A2) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for $t, s \in \mathbb{R}$.

It follows in particular that $\Phi_t \in \mathrm{Diff}(M)$ with $\Phi_t^{-1} = \Phi_{-t}$, so that we obtain a group homomorphism

$$\gamma_X: \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \Phi_t.$$

With respect to the terminology introduced below, (A1) and (A2) mean that Φ defines a *smooth action of* $\mathbb{R}$ *on* M. As Φ is determined by the vector field X, we call it the infinitesimal generator of this action. In this sense the smooth $\mathbb{R}$-actions on a manifold M are in one-to-one correspondence with the complete vector fields on M. ∎

Remark VI.2.6. Let $\Phi^X: \mathcal{D}_X \to M$ be the maximal local flow of a vector field X on M. Let $M_t := \{x \in M: (t, x) \in \mathcal{D}_X\}$, and observe that this is an open subset of M. We have already seen in the proof of Theorem VI.2.2 above, that all the smooth maps $\Phi_t^X: M_t \to M$ are injective with $\Phi_t^X(M_t) = M_{-t}$ and $(\Phi_t^X)^{-1} = \Phi_{-t}^X$ on the image. It follows in particular, that $\Phi_t^X(M_t) = M_{-t}$ is open, and that

$$\Phi_t^X: M_t \to M_{-t}$$

is a diffeomorphism whose inverse is Φ_{-t}^X. ∎

The Lie derivative of vector fields

Before we turn to actions of higher dimensional groups, we take a closer look at the interaction of local flows and vector fields.

Let $X \in \mathcal{V}(M)$ and $\Phi^X: \mathcal{D}_X \to M$ its maximal local flow. For a second vector field $Y \in \mathcal{V}(M)$, we define a smooth vector field on the open subset $M_{-t} \subseteq M$ by

$$(\Phi_t^X)_* Y := T(\Phi_t^X) \circ Y \circ \Phi_{-t}^X = T(\Phi_t^X) \circ Y \circ (\Phi_t^X)^{-1}$$

(cf. Remark VI.2.6) and define the *Lie derivative* by

$$\mathcal{L}_X Y := \lim_{t \to 0} \frac{1}{t}((\Phi_{-t}^X)_* Y - Y) = \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^X)_* Y,$$

which is defined on all of M since for each $p \in M$ the vector $((\Phi_t^X)_* Y)(p)$ is defined for sufficiently small t and depends smoothly on t.

Theorem VI.2.7. *$\mathcal{L}_X Y = [X, Y]$ for $X, Y \in \mathcal{V}(M)$.*

Proof. Fix $p \in M$. It suffices to show that $\mathcal{L}_X Y$ and $[X, Y]$ coincide in p. We may therefore work in a local chart, hence assume that $M = U$ is an open subset of $\mathbb{R}^n$.

Identifying vector fields with smooth $\mathbb{R}^n$-valued functions, we then have

$$[X, Y](x) = dY(x)X(x) - dX(x)Y(x), \quad x \in U.$$

On the other hand,

$$\begin{aligned}((\Phi^X_{-t})_* Y)(x) &= T(\Phi^X_{-t}) \circ Y \circ \Phi^X_t(x) \\ &= d(\Phi^X_{-t})(\Phi^X_t(x))Y(\Phi^X_t(x)) = \big(d(\Phi^X_t)(x)\big)^{-1} Y(\Phi^X_t(x)).\end{aligned}$$

To calculate the derivative of this expression with respect to t, we first observe that it does not matter if we first take derivatives with respect to t and then with respect to x or vice versa. This leads to

$$\frac{d}{dt}\Big|_{t=0} d(\Phi^X_t)(x) = d\Big(\frac{d}{dt}\Big|_{t=0} \Phi^X_t\Big)(x) = dX(x).$$

Next we note that for any smooth curve $\alpha\colon [-\varepsilon, \varepsilon] \to \mathrm{GL}_n(\mathbb{R})$ with $\alpha(0) = \mathbf{1}$ we have

$$(\alpha^{-1})'(t) = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1},$$

and in particular $(\alpha^{-1})'(0) = -\alpha'(0)$. Combining all this, we obtain with the Product Rule

$$\mathcal{L}_X(Y)(x) = -dX(x)Y(x) + dY(x)X(x) = [X, Y](x). \qquad \blacksquare$$

Corollary VI.2.8. *If $X, Y \in \mathcal{V}(M)$ are complete vector fields, then their global flows $\Phi^X, \Phi^Y \colon \mathbb{R} \to \mathrm{Diff}(M)$ commute if and only if X and Y commute, i.e., $[X, Y] = 0$.*

Proof. (1) Suppose first that Φ^X and Φ^Y commute, i.e.,

$$\Phi^X(t) \circ \Phi^Y(s) = \Phi^Y(s) \circ \Phi^X(t) \quad \text{for } t, s \in \mathbb{R}.$$

Let $p \in M$ and $\gamma_p(s) := \Phi^Y_s(p)$ the Y-integral curve through p. We then have

$$\gamma_p(s) = \Phi^Y_s(p) = \Phi^X_t \circ \Phi^Y_s \circ \Phi^X_{-t}(p),$$

and passing to the derivative in $s = 0$ yields

$$Y(p) = \gamma_p'(0) = T(\Phi^X_t)Y(\Phi^X_{-t}(p)) = \big((\Phi^X_t)_* Y\big)(p).$$

Passing now to the derivative in $t = 0$, we arrive at $[X, Y] = \mathcal{L}_X(Y) = 0$.

(2) Now we assume $[X,Y]=0$. First we show that $(\Phi_t^X)_*Y=Y$ holds for all $t\in\mathbb{R}$. For $t,s\in\mathbb{R}$ we have

$$(\Phi_{t+s}^X)_*Y=(\Phi_t^X)_*(\Phi_s^X)_*Y,$$

so that

$$\frac{d}{dt}(\Phi_t^X)_*Y=-(\Phi_t^X)_*\mathcal{L}_X(Y)=0$$

for each $t\in\mathbb{R}$. Since for each $p\in M$ the curve

$$\mathbb{R}\to T_p(M),\quad t\mapsto\big((\Phi_t^X)_*Y\big)(p)$$

is smooth, and its derivative vanishes, it is constant $Y(p)$. This shows that $(\Phi_t^X)_*Y=Y$ for each $t\in\mathbb{R}$.

For $\gamma(s):=\Phi_t^X\Phi_s^Y(p)$ we now have $\gamma(0)=\Phi_t^X(p)$ and

$$\gamma'(s)=T(\Phi_t^X)\circ Y(\Phi_s^Y(p))=Y(\Phi_t^X\Phi_s^Y(p))=Y(\gamma(s)),$$

so that γ is an integral curve of Y. We conclude that $\gamma(s)=\Phi_s^Y(\Phi_t^X(p))$, and this means that the flows of X and Y commute. ■

Remark VI.2.9. Let $X,Y\in\mathcal{V}(M)$ be two complete vector fields and Φ^X, resp., Φ^Y their local flows. We then consider the commutator map

$$F\colon\mathbb{R}^2\to\mathrm{Diff}(M),\quad(t,s)\mapsto\Phi_t^X\circ\Phi_s^Y\circ\Phi_{-t}^X\circ\Phi_{-s}^Y.$$

We know from Corollary VI.2.8 that it vanishes if and only if $[X,Y]=0$, but there is also a more direct way from F to the Lie bracket.

In fact, we first observe that

$$\frac{\partial F}{\partial s}(t,0)=(\Phi_t^X)_*Y-Y,$$

and hence that

$$\frac{\partial^2F}{\partial t\partial s}(0,0)=[Y,X].$$

Here we use that if $I\subseteq\mathbb{R}$ is an interval and $\alpha\colon I\to\mathrm{Diff}(M)$ and $\beta\colon I\to\mathrm{Diff}(M)$ are maps for which

$$\widehat{\alpha}\colon I\times M\to M,\quad(t,x)\mapsto\alpha(t)(x)\quad\text{ and }\quad\widehat{\beta}\colon I\times M\to M,\quad(t,x)\mapsto\beta(t)(x)$$

are smooth, then the curve $\gamma(t):=\alpha(t)\circ\beta(t)$ also has this property (by the Chain Rule), and if $\alpha(0)=\beta(0)=\mathrm{id}_M$, then γ satisfies

$$\gamma'(0)=\alpha'(0)\circ\beta(0)+T(\alpha(0))\circ\beta'(0)=\alpha'(0)+\beta'(0).$$ ■

Exercises for Chapter VI

Exercise VI.1. Let $M := \mathbb{R}^n$. For a matrix $A \in M_n(\mathbb{R})$ we consider the linear vector field

$$X_A(x) := Ax.$$

Calculate the maximal flow $\Phi^X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ of this vector field. ■

Exercise VI.2. Let $M := \mathbb{R}^n$. For a matrix $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ we consider the affine vector field

$$X_{A,b}(x) := Ax + b.$$

(1) Calculate the maximal flow $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ of this vector field. Hint: For each $t \in \mathbb{R}$ the map Φ_t is affine and the translation part is $\frac{e^{tA} - \mathbf{1}}{A} b$.

(2) Let

$$\mathfrak{aff}_n(\mathbb{R}) := \begin{pmatrix} M_n(\mathbb{R}) & \mathbb{R}^n \\ 0 & 0 \end{pmatrix} \subseteq M_{n+1}(\mathbb{R})$$

be the affine Lie algebra on $\mathbb{R}^n$, realized as a Lie subalgebra of $M_{n+1}(\mathbb{R})$, endowed with the commutator bracket. Show that the map

$$\varphi : \mathfrak{aff}_n(\mathbb{R}) \to \mathcal{V}(\mathbb{R}^n), \qquad \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mapsto -X_{A,b}$$

is a homomorphism of Lie algebras. ■

VII. Lie group actions on manifolds

In the preceding chapter we discussed local flows on manifolds and how they are related to smooth vector fields. In particular, we have seen that global flows are in one-to-one correspondence with complete vector fields. Global flows can also be viewed as smooth actions of the one-dimensional Lie group $(\mathbb{R},+)$ on a manifold.

In this chapter we now turn to actions of more general Lie groups on manifolds. This requires some preparation. First we define the Lie algebra $\mathbf{L}(G)$ of a Lie group G, which is obtained as the space of left invariant vector fields on G. From the global flows of these vector fields, we obtain the exponential function

$$\exp_G: \mathbf{L}(G) \to G.$$

It translates between the group G and its Lie algebra $\mathbf{L}(G)$, which can be viewed as a "first order approximation" of the group. We then turn to actions of Lie groups on manifolds and discuss how they are related to finite-dimensional Lie algebras of vector fields.

VII.1. Lie groups and their Lie algebras

Throughout this section, G denotes a Lie group with multiplication map $m_G: G \times G \to G, (x,y) \mapsto xy$, inversion map $\eta_G: G \to G, x \mapsto x^{-1}$, and neutral element $\mathbf{1}$. For $g \in G$ we write $\lambda_g: G \to G, x \mapsto gx$ for the left multiplication map, $\rho_g: G \to G, x \mapsto xg$ for the right multiplication map, and $c_g: G \to G, x \mapsto gxg^{-1}$ for the conjugation with g.

A *morphism of Lie groups* is a smooth homomorphism of Lie groups $\varphi: G_1 \to G_2$.

Remark VII.1.1. All maps λ_g, ρ_g and c_g are smooth. Moreover, they are bijective with $\lambda_{g^{-1}} = \lambda_g^{-1}$, $\rho_{g^{-1}} = \rho_g^{-1}$ and $c_{g^{-1}} = c_g^{-1}$, so that they are diffeomorphisms of G onto itself.

In addition, the maps c_g are automorphisms of G, so that we obtain a group homomorphism

$$C: G \to \mathrm{Aut}(G), \quad g \mapsto c_g,$$

where $\mathrm{Aut}(G)$ stands for the *group of automorphisms of the Lie group* G, i.e., the group automorphisms which are diffeomorphisms. ■

Lemma VII.1.2. (a) *As usual, we identify $T(G \times G)$ with $T(G) \times T(G)$. Then the tangent map*

$$T(m_G): T(G \times G) \cong T(G) \times T(G) \to T(G), \quad (v,w) \mapsto v \cdot w := Tm_G(v,w)$$

defines a Lie group structure on $T(G)$ *with identity element* $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$ *and inversion* $T(\eta_G)$. *The canonical projection* $\pi_{T(G)}: T(G) \to G$ *is a morphism of Lie groups with kernel* $(T_{\mathbf{1}}(G), +)$ *and the zero section* $\sigma: G \to T(G), g \mapsto 0_g \in T_g(G)$ *is a homomorphism of Lie groups with* $\pi_{T(G)} \circ \sigma = \mathrm{id}_G$.

(b) *The map*

$$\Phi: G \times T_{\mathbf{1}}(G) \to T(G), \quad (g, x) \mapsto g.x := 0_g.x = Tm_G(0_g, x) = T(\lambda_g)x$$

is a diffeomorphism.

Proof. (a) Since the multiplication map $m_G: G \times G \to G$ is smooth, the same holds for its tangent map

$$Tm_G: T(G \times G) \cong T(G) \times T(G) \to T(G).$$

Let $\varepsilon_G: G \to G, g \mapsto \mathbf{1}$ be the constant homomorphism. Then the group axioms for G are encoded in the relations

(1) $m_G \circ (m_G \times \mathrm{id}_G) = m_G \circ (\mathrm{id}_G \times m_G)$ (associativity),

(2) $m_G \circ (\eta_G, \mathrm{id}_G) = m_G \circ (\mathrm{id}_G, \eta_G) = \varepsilon_G$ (inversion), and

(3) $m_G \circ (\varepsilon_G, \mathrm{id}_G) = m_G \circ (\mathrm{id}_G, \varepsilon_G) = \mathrm{id}_G$ (unit element).

Using the functoriality of T and its compatibility with products, we see that these properties carry over to the corresponding maps on $T(G)$:

(1)
$$\begin{aligned} T(m_G) \circ T(m_G \times \mathrm{id}_G) &= T(m_G) \circ (T(m_G) \times \mathrm{id}_{T(G)}) \\ &= T(m_G) \circ (\mathrm{id}_{T(G)} \times T(m_G)) \end{aligned}$$

(associativity),

(2) $T(m_G) \circ (T(\eta_G), \mathrm{id}_{T(G)}) = T(m_G) \circ (\mathrm{id}_{T(G)}, T(\eta_G)) = T(\varepsilon_G)$ (inversion), and

(3) $T(m_G) \circ (T(\varepsilon_G), \mathrm{id}_{T(G)}) = T(m_G) \circ (\mathrm{id}_{T(G)}, T(\varepsilon_G)) = \mathrm{id}_{T(G)}$ (unit element).

Here we only have to observe that the tangent map $T(\varepsilon_G)$ maps each $v \in T(G)$ to $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$, which is the neutral element of $T(G)$. We conclude that $T(G)$ is a Lie group with multiplication $T(m_G)$, inversion $T(\eta_G)$, and unit element $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$.

The definition of the tangent map implies that the zero section $\sigma: G \to T(G), g \mapsto 0_g$ satisfies

$$Tm_G \circ (\sigma \times \sigma) = \sigma \circ m_G, \quad Tm_G(0_g, 0_h) = 0_{m_G(g,h)} = 0_{gh},$$

which means that it is a morphism of Lie groups. That $\pi_{T(G)}$ also is a morphism of Lie groups follows likewise from the relation

$$\pi_{T(G)} \circ Tm_G = m_G \circ (\pi_{T(G)} \times \pi_{T(G)}),$$

which also is an immediate consequence of the definition of the tangent map: it maps $T_g(G) \times T_h(G)$ into $T_{gh}(G)$.

For $v \in T_g(G)$ and $w \in T_h(G)$ the linearity of $T_{(g,h)}(m_G)$ implies that

$$\begin{aligned} Tm_G(v,w) &= T_{(g,h)}(m_G)(v,w) = T_{(g,h)}(m_G)(v,0) + T_{(g,h)}(m_G)(0,w) \\ &= T_g(\rho_h)v + T_h(\lambda_g)w, \end{aligned}$$

and in particular $T_{(\mathbf{1},\mathbf{1})}(m_G)(v,w) = v + w$, so that the multiplication on the normal subgroup $\ker \pi_{T(G)} = T_{\mathbf{1}}(G)$ is simply given by addition.

(b) The smoothness of Φ follows from the smoothness of the multiplication of $T(G)$ and the smoothness of the zero section $\sigma\colon G \to T(G), g \mapsto 0_g$. That Φ is a diffeomorphism follows from the following explicit formula for its inverse: $\Phi^{-1}(v) = (\pi_{T(G)}(v), \pi_{T(G)}(v)^{-1}.v)$, so that its smoothness follows from the smoothness of $\pi_{T(G)}$ (its first component), and the smoothness of the multiplication on $T(G)$. ■

Definition VII.1.3. (The Lie algebra of G) A vector field $X \in \mathcal{V}(G)$ is called *left invariant* if

$$X \circ \lambda_g = T(\lambda_g) \circ X$$

holds for each $g \in G$, i.e., $(\lambda_g)_* X = X$. We write $\mathcal{V}(G)^l$ for the set of left invariant vector fields in $\mathcal{V}(G)$. Clearly $\mathcal{V}(G)^l$ is a linear subspace of $\mathcal{V}(G)$.

Writing the left invariance as $X = T(\lambda_g) \circ X \circ \lambda_g^{-1}$, we see that it means that X is λ_g-related to itself. Therefore Lemma II.3.9 implies that if X and Y are left-invariant, their Lie bracket $[X,Y]$ is also λ_g-related to itself for each $g \in G$, hence left invariant. We conclude that the vector space $\mathcal{V}(G)^l$ is a Lie subalgebra of $(\mathcal{V}(G), [\cdot,\cdot])$.

Next we observe that the left invariance of a vector field X implies that for each $g \in G$ we have $X(g) = g.X(\mathbf{1})$ (cf. Lemma VII.1.2(b)), so that X is completely determined by its value $X(\mathbf{1}) \in T_{\mathbf{1}}(G)$. Conversely, for each $x \in T_{\mathbf{1}}(G)$, we obtain a left invariant vector field $x_l \in \mathcal{V}(G)^l$ with $x_l(\mathbf{1}) = x$ by $x_l(g) := g.x$. That this vector field is indeed left invariant follows from

$$x_l \circ \lambda_h(g) = x_l(hg) = (hg).x = h.(g.x) = T(\lambda_h)x_l(g)$$

for all $h, g \in G$. Hence

$$T_{\mathbf{1}}(G) \to \mathcal{V}(G)^l, \quad x \mapsto x_l$$

is a linear bijection. We thus obtain a Lie bracket $[\cdot,\cdot]$ on $T_{\mathbf{1}}(G)$ satisfying

$$[x,y]_l = [x_l, y_l] \quad \text{for all} \quad x, y \in T_{\mathbf{1}}(G). \tag{7.1.1}$$

The Lie algebra

$$\mathbf{L}(G) := (T_{\mathbf{1}}(G), [\cdot,\cdot]) \cong \mathcal{V}(G)^l$$

is called *the Lie algebra of G*. ■

Proposition VII.1.4. (Functoriality of the Lie algebra) *If* $\varphi: G \to H$ *is a homomorphism of Lie groups, then the tangent map*

$$\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi): \mathbf{L}(G) \to \mathbf{L}(H)$$

is a homomorphism of Lie algebras.

Proof. Let $x, y \in \mathbf{L}(G)$, and x_l, y_l be the corresponding left invariant vector fields. Then $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ for each $g \in G$ implies that

$$T(\varphi) \circ T(\lambda_g) = T(\lambda_{\varphi(g)}) \circ T(\varphi),$$

and applying this relation to $x, y \in T_{\mathbf{1}}(G)$, we get

$$T\varphi \circ x_l = \big(\mathbf{L}(\varphi)x\big)_l \circ \varphi \quad \text{and} \quad T\varphi \circ y_l = \big(\mathbf{L}(\varphi)y\big)_l \circ \varphi, \tag{7.1.2}$$

i.e., x_l is φ-related to $\big(\mathbf{L}(\varphi)x\big)_l$ and y_l is φ-related to $\big(\mathbf{L}(\varphi)y\big)_l$. Therefore Lemma II.3.8 implies that

$$T\varphi \circ [x_l, y_l] = [\big(\mathbf{L}(\varphi)x\big)_l, \big(\mathbf{L}(\varphi)y\big)_l] \circ \varphi.$$

Evaluating at $\mathbf{1}$, we obtain $\mathbf{L}(\varphi)[x, y] = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$, showing that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. ■

Remark VII.1.5. We obviously have $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$, and for two morphisms $\varphi_1: G_1 \to G_2$ and $\varphi_2: G_2 \to G_3$ of Lie groups, we have

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1),$$

as a consequence of the Chain Rule:

$$T_{\mathbf{1}}(\varphi_2 \circ \varphi_1) = T_{\varphi_1(\mathbf{1})}(\varphi_2) \circ T_{\mathbf{1}}(\varphi_1) = T_{\mathbf{1}}(\varphi_2) \circ T_{\mathbf{1}}(\varphi_1).$$

In this sense the preceding lemma implies that the assignments $G \mapsto \mathbf{L}(G)$ and $\varphi \mapsto \mathbf{L}(\varphi)$ define a functor

$$\mathbf{L}: \underline{\mathit{LieGrp}} \to \underline{\mathit{LieAlg}}$$

from the category $\underline{\mathit{LieGrp}}$ of Lie groups to the category $\underline{\mathit{LieAlg}}$ of (finite-dimensional) Lie algebras. ■

Corollary VII.1.6. *For each isomorphism of Lie groups* $\varphi: G \to H$ *the map* $\mathbf{L}(\varphi)$ *is an isomorphism of Lie algebras, and for each* $x \in \mathbf{L}(G)$ *we have*

$$\varphi_* x_l := T(\varphi) \circ x_l \circ \varphi^{-1} = \big(\mathbf{L}(\varphi)x\big)_l. \tag{7.1.3}$$

Proof. Let $\psi: H \to G$ be the inverse of φ. Then $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$ leads to $\mathbf{L}(\varphi) \circ \mathbf{L}(\psi) = \mathrm{id}_{\mathbf{L}(H)}$ and $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi) = \mathrm{id}_{\mathbf{L}(G)}$ (Remark VII.1.5).

The relation (7.1.3) follows from (7.1.2) in the proof of Proposition VII.1.4. ■

The exponential function of a Lie group

Proposition VII.1.7. *Each left invariant vector field X on G is complete.*

Proof. Let $g \in G$ and $\gamma: I \to G$ be the unique maximal integral curve of $X \in \mathcal{V}(G)^l$ with $\gamma(0) = g$.

For each $h \in G$ we have $(\lambda_h)_* X = X$, which implies that $\eta := \lambda_h \circ \gamma$ also is an integral curve of X (see the argument in Remark VI.1.2(b)). Put $h = \gamma(s)g^{-1}$ for some $s > 0$. Then

$$\eta(0) = (\lambda_h \circ \gamma)(0) = h\gamma(0) = hg = \gamma(s),$$

and the uniqueness of integral curves implies that $\gamma(t+s) = \eta(t)$ for all t in the interval $I \cap (I - s)$. In view of the maximality of I, it now follows that $I - s \subseteq I$, and hence that $I - ns \subseteq I$ for each $n \in \mathbb{N}$, so that the interval I is unbounded from below. Applying the same argument to some $s < 0$, we see that I is also unbounded from above. Hence $I = \mathbb{R}$, which means that X is complete. ∎

Definition VII.1.8. We now define the *exponential function*

$$\exp_G: \mathbf{L}(G) \to G, \quad \exp_G(x) := \gamma_x(1),$$

where $\gamma_x: \mathbb{R} \to G$ is the unique maximal integral curve of the left invariant vector field x_l, satisfying $\gamma_x(0) = \mathbf{1}$. This means that γ_x is the unique solution of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = x_l(\gamma(t)) = \gamma(t).x \quad \text{for all} \quad t \in \mathbb{R}.$$ ∎

Examples VII.1.9. (a) Let $G := (V, +)$ be the additive group of a finite-dimensional vector space. The left invariant vector fields on V are given by

$$x_l(w) := \frac{d}{dt}\Big|_{t=0} w + tx = x,$$

so that they are simply the constant vector fields. Hence

$$[x_l, y_l](0) = dx_l(y_l(0)) - dx_l(y_l(0)) = dx_l(y) - dy_l(x) = 0.$$

Therefore $\mathbf{L}(V)$ is an abelian Lie algebra.

For each $x \in V$ the flow of x_l is given by $\Phi^{x_l}(t, v) = v + tx$, so that

$$\exp_V(x) = \Phi^{x_l}(1, 0) = x, \quad \text{i.e.,} \quad \exp_V = \mathrm{id}_V.$$

(b) Now let $G := \mathrm{GL}_n(\mathbb{R})$ be the Lie group of invertible $(n \times n)$-matrices, which inherits its manifold structure from the embedding as an open subset of the vector space $M_n(\mathbb{R})$.

The left invariant vector field A_l corresponding to a matrix A is given by

$$A_l(g) = T_{\mathbf{1}}(\lambda_g)A = gA$$

because $\lambda_g(h) = gh$ extends to a linear endomorphism of $M_n(\mathbb{R})$. Therefore the unique solution

$$\gamma_A\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$$

of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = A_l(\gamma(t)) = \gamma(t)A$$

is nothing but the curve describing the fundamental system of the linear differential equation defined by the matrix A:

$$\gamma_A(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k.$$

It follows that

$$\exp_G(A) = e^A$$

is the matrix exponential function.

The Lie algebra $\mathbf{L}(G)$ of G is determined from

$$\begin{aligned}[A, B] &= [A_l, B_l](\mathbf{1}) = dB_l(\mathbf{1})A_l(\mathbf{1}) - dA_l(\mathbf{1})B_l(\mathbf{1}) \\ &= dB_l(\mathbf{1})A - dA_l(\mathbf{1})B = AB - BA.\end{aligned}$$

Therefore the Lie bracket on $\mathbf{L}(G) = T_{\mathbf{1}}(G) \cong M_n(\mathbb{R})$ is given by the commutator bracket. This Lie algebra is denoted $\mathfrak{gl}_n(\mathbb{R})$, to express that it is the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$.

(c) If V is a finite-dimensional real vector space, then $V \cong \mathbb{R}^n$, so that we can immediately use (b) to see that $\mathrm{GL}(V)$ is a Lie group with Lie algebra $\mathfrak{gl}(V) := (\mathrm{End}(V), [\cdot,\cdot])$ and exponential function

$$\exp_{\mathrm{GL}(V)}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \qquad \blacksquare$$

Lemma VII.1.10. (a) *For each $x \in \mathbf{L}(G)$ the curve*

$$\gamma_x\colon \mathbb{R} \to G$$

is a smooth homomorphism of Lie groups with $\gamma_x'(0) = x$.

(b) *The global flow of the left invariant vector field x_l is given by*

$$\Phi(t, g) = g\gamma_x(t) = g \exp_G(tx).$$

(c) *If $\gamma\colon \mathbb{R} \to G$ is a smooth homomorphism of Lie groups and $x := \gamma'(0)$, then $\gamma = \gamma_x$. In particular, the map*

$$\mathrm{Hom}(\mathbb{R}, G) \to \mathbf{L}(G), \quad \gamma \mapsto \gamma'(0)$$

is a bijection, where $\mathrm{Hom}(\mathbb{R}, G)$ stands for the set of morphisms, i.e., smooth homomorphisms, of Lie groups $\mathbb{R} \to G$.

Proof. (a), (b) Since γ_x is an integral curve of the smooth vector field x_l, it is a smooth curve. Hence the smoothness of the multiplication in G implies that $\Phi(t,g) := g\gamma_x(t)$ defines a smooth map $\mathbb{R} \times G \to G$. In view of the left invariance of x_l, we have for each $g \in G$ and $\Phi^g(t) := \Phi(t,g)$ the relation

$$(\Phi^g)'(t) = T(\lambda_g)\gamma_x'(t) = T(\lambda_g)x_l(\gamma_x(t)) = x_l(g\gamma_x(t)) = x_l(\Phi^g(t)).$$

Therefore Φ^g is an integral curve of x_l with $\Phi^g(0) = g$, and this proves that Φ is the unique maximal flows of the complete vector field x_l.

In particular, we obtain for $t, s \in \mathbb{R}$:

$$\gamma_x(t+s) = \Phi(t+s, \mathbf{1}) = \Phi(t, \Phi(s, \mathbf{1})) = \Phi(s, \mathbf{1})\gamma_x(t) = \gamma_x(s)\gamma_x(t).$$

Hence γ_x is a group homomorphism $(\mathbb{R}, +) \to G$.

(c) If $\gamma\colon (\mathbb{R}, +) \to G$ is a smooth group homomorphism, then

$$\Phi(t,g) := g\gamma(t)$$

defines a global flow on G whose infinitesimal generator is the vector field given by

$$X(g) = \frac{d}{dt}\Big|_{t=0} \Phi(t,g) = T(\lambda_g)\gamma'(0).$$

We conclude that $X = x_l$ for $x = \gamma'(0)$, so that X is a left invariant vector field. Since γ is its unique integral curve through 0, it follows that $\gamma = \gamma_x$. In view of (a), this proves (c). ■

Lemma VII.1.11. *If $x, y \in \mathbf{L}(G)$ commute, i.e., $[x,y] = 0$, then*

$$\exp_G(x+y) = \exp_G(x)\exp_G(y).$$

Proof. If x and y commute, then the corresponding left invariant vector fields commute, and Corollary VI.2.8 implies that their flows commute. We conclude that for all $t, s \in \mathbb{R}$ we have

$$\exp_G(tx)\exp_G(sy) = \exp_G(sy)\exp_G(tx). \tag{7.1.4}$$

Therefore

$$\gamma(t) := \exp_G(tx)\exp_G(ty)$$

is a smooth map $\mathbb{R} \to G$, and (7.1.4) implies that it is a group homomorphism. In view of

$$\gamma'(0) = dm_G(\mathbf{1}, \mathbf{1})(x,y) = x + y$$

(Lemma VII.1.2), Lemma VII.1.10(c) leads to $\gamma(t) = \exp_G(t(x+y))$, and for $t = 1$ we obtain the lemma. ■

Proposition VII.1.12. (Smooth dependence on parameters and initial conditions) *Let M be a smooth manifold, V a finite-dimensional vector space, $V_1 \subseteq V$ an open subset, and $\Psi\colon V_1 \to \mathcal{V}(M)$ a map for which the map*

$$\widehat{\Psi}\colon V_1 \times M \to T(M), \quad (v,p) \mapsto \Psi(v)(p)$$

is smooth (the vector field $\Psi(v)$ depends smoothly on v). Then there exists for each $(p_0, v_0) \in M \times V_1$ an open neighborhood U of p_0 in M, an open interval $I \subseteq \mathbb{R}$ containing 0, an open neighborhood W of v_0 in V_1, and a smooth map

$$\Phi\colon I \times U \times W \to M$$

such that for each $(p,v) \in U \times W$ the curve

$$\Phi_p^v\colon I \to M, \quad t \mapsto \Phi(t,p,v)$$

is an integral curve of the vector field $\Psi(v)$ with $\Phi_p^v(0) = p$.

Proof. The parameters do not cause any additional problems, which can be seen by the following trick: On the product manifold $N := V_1 \times M$ we consider the smooth vector field Y, given by

$$Y(v,p) := (0, \Psi(v)(p)).$$

Then the integral curves of Y are of the form

$$\gamma(t) = (v, \gamma_v(t)),$$

where γ_v is an integral curve of the smooth vector field $\Psi(v)$ on M. Therefore the assertion is an immediate consequence on the smoothness of the local flow of Y on $V_1 \times M$ (Theorem VI.2.4). ■

Proposition VII.1.13. *The exponential function*

$$\exp_G\colon \mathbf{L}(G) \to G$$

is smooth and satisfies

$$T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}.$$

In particular, $\exp_G$ is a local diffeomorphism in 0 in the sense that it maps some 0-neighborhood in $\mathbf{L}(G)$ diffeomorphically onto some $\mathbf{1}$-neighborhood in G.

Proof. Let $n \in \mathbb{N}$. In view of Lemma VII.1.11, we have

$$\exp_G(nx) = \gamma_x(n) = \gamma_x(1)^n = \exp_G(x)^n \tag{7.1.5}$$

for each $x \in \mathbf{L}(G)$. Since the n-fold multiplication map

$$G^n \to G, \quad (g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$$

is smooth, the n-th power map $G \to G, g \mapsto g^n$ is smooth. Therefore it suffices to verify the smoothness of $\exp_G$ in some 0-neighborhood W. Then (7.1.5) immediately implies smoothness in nW for each n, and hence on all of $\mathbf{L}(G)$.

The map $\Psi\colon \mathbf{L}(G) \to \mathcal{V}(G), x \mapsto x_l$ satisfies the assumptions of Proposition VII.1.12 because the map

$$\mathbf{L}(G) \times G \to T(G), \quad (x, g) \mapsto x_l(g) = g.x$$

is smooth (Lemma VII.1.2). In the terminology of Proposition VII.1.12, it now follows that the map

$$\Phi\colon \mathbb{R} \times \mathbf{L}(G) \times G \to G, \quad (t, x, g) \mapsto g\gamma_x(t) = g\exp_G(tx)$$

is smooth on a neighborhood of $(0, 0, \mathbf{1})$. In particular, for some $t > 0$, the map

$$x \mapsto \exp_G(tx)$$

is smooth on a 0-neighborhood of $\mathbf{L}(G)$, and this proves that $\exp_G$ is smooth in some 0-neighborhood.

Finally, we observe that

$$T_0(\exp_G)(x) = \frac{d}{dt}\Big|_{t=0} \exp_G(tx) = \gamma_x'(0) = x,$$

so that $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$. ■

Lemma VII.1.14. *The subgroup $\langle \exp_G(\mathbf{L}(G))\rangle$ of G generated by $\exp_G(\mathbf{L}(G))$ coincides with the connected component G_0 of $\mathbf{1}$.*

Proof. Since $\exp_G$ is a local diffeomorphism in 0 (Proposition VII.1.13), the Inverse Function Theorem implies that $\exp_G(\mathbf{L}(G))$ is a neighborhood of $\mathbf{1}$. We conclude that the subgroup $H := \langle \exp_G(\mathbf{L}(G))\rangle$ generated by the exponential image is a $\mathbf{1}$-neighborhood.

Then H^0 is a non-empty open subset of H, satisfying

$$H = H^0 H = \bigcup_{h \in H} H^0 h = \bigcup_{h \in H} \rho_h(H^0).$$

Since all sets $\rho_h(H^0)$ are open subsets of G, their union is open, and we conclude that H is an open subgroup of H. This implies that all cosets of H are open, and hence that

$$H = G \setminus \bigcup_{g \notin H} gH$$

is closed. Now H is open and closed, hence contains G_0.

On the other hand, $\exp_G$ is continuous, so that it maps the connected space $\mathbf{L}(G)$ into the identity component G_0 of G, which leads to $H \subseteq G_0$, and hence to equality. ■

Proposition VII.1.15. *Let $\varphi\colon G_1 \to G_2$ be a morphism of Lie groups and $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ its differential in $\mathbf{1}$. Then*

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1},$$

i.e., the following diagram commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{\quad\varphi\quad} & G_2 \\ \Big\uparrow{\scriptstyle \exp_{G_1}} & & \Big\uparrow{\scriptstyle \exp_{G_2}} \\ \mathbf{L}(G_1) & \xrightarrow{\quad\mathbf{L}(\varphi)\quad} & \mathbf{L}(G_2). \end{array}$$

Proof. For $x \in \mathbf{L}(G_1)$ we consider the smooth homomorphism

$$\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1), \quad \gamma_x(t) = \exp_{G_1}(tx).$$

According to Lemma VII.1.10, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for $y = (\varphi \circ \gamma_x)'(0) = \mathbf{L}(\varphi)x$, because $\varphi \circ \gamma_x\colon \mathbb{R} \to G_2$ is a smooth group homomorphism. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x)),$$

which we had to show. ■

Corollary VII.1.16. *If $\varphi_1, \varphi_2\colon G_1 \to G_2$ are morphisms of Lie groups with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$, then φ_1 and φ_2 coincide on the identity component of G_1.*

Proof. In view of Proposition VII.1.15, we have for $x \in \mathbf{L}(G_1)$:

$$\varphi_1(\exp_{G_1}(x)) = \exp_{G_2}(\mathbf{L}(\varphi_1)x) = \exp_{G_2}(\mathbf{L}(\varphi_2)x) = \varphi_2(\exp_{G_1}(x)),$$

so that φ_1 and φ_2 coincide on the image of $\exp_{G_1}$, hence on the subgroup generated by this set. Now the assertion follows from Lemma VII.1.14. ■

Proposition VII.1.17. *Let G be a Lie group with Lie algebra $\mathbf{L}(G)$. Then we have for $x, y \in \mathbf{L}(G)$ the following formulas:*

(1) (Product Formula) $\exp_G(x + y) = \lim_{k\to\infty} \big(\exp_G(\frac{1}{k}x)\exp_G(\frac{1}{k}y)\big)^k$.

(2) (Commutator Formula)

$$\exp_G([x, y]) = \lim_{k\to\infty} \Big(\exp_G(\frac{1}{k}x)\exp_G(\frac{1}{k}y)\exp_G(-\frac{1}{k}x)\exp_G(-\frac{1}{k}y)\Big)^{k^2}.$$

Proof. Let $U \subseteq \mathbf{L}(G)$ be an open 0-neighborhood for which

$$\exp_U := \exp_G|_U\colon U \to \exp_G(U)$$

is a diffeomorphisms onto an open subset of G. Put

$$U^1 := \{(x,y) \in U \times U : \exp_G(x)\exp_G(y) \in \exp_G(U)\}$$

and observe that this is an open subset of $U \times U$ containing $(0,0)$ because $\exp_G(U)$ is open and $\exp_G$ is continuous.

For $(x,y) \in U_1$ we then define

$$x * y := \exp_U^{-1}(\exp_G(x)\exp_G(y))$$

and thus obtain a smooth map

$$m : U^1 \to \mathbf{L}(G), \quad (x,y) \mapsto x * y.$$

(1) In view of $m(0,x) = m(x,0) = x$, we have

$$dm(0,0)(x,y) = dm(0,0)(x,0) + dm(0,0)(0,y) = x + y.$$

This implies that

$$\lim_{k\to\infty} k \cdot (\frac{1}{k}x * \frac{1}{k}y) = \lim_{k\to\infty} k \cdot \Big(m(\frac{1}{k}x, \frac{1}{k}y) - m(0,0)\Big) = dm(0,0)(x,y) = x + y.$$

Applying $\exp_G$, it follows that

$$\begin{aligned}\exp_G(x+y) &= \lim_{k\to\infty} \exp_G\Big(k \cdot (\frac{1}{k}x * \frac{1}{k}y)\Big) = \lim_{k\to\infty} \exp_G\Big(\frac{1}{k}x * \frac{1}{k}y\Big)^k \\ &= \lim_{k\to\infty} \Big(\exp_G(\frac{1}{k}x)\exp_G(\frac{1}{k}y)\Big)^k.\end{aligned}$$

(2) Now let $x_l^* := T(\exp_U)^{-1} \circ x_l \circ \exp_U$ be the smooth vector field on U corresponding to the left invariant vector field x_l on $\exp_G(U)$. Then x_l^* and x_l are $\exp_U$-related, so that Lemma II.3.8 on related vector fields leads to

$$[x_l^*, y_l^*](0) = T_0(\exp_U)[x_l^*, y_l^*](0) = [x_l, y_l](\exp_G(0)) = [x_l, y_l](\mathbf{1}) = [x,y].$$

The local flow of x_l through a point $\exp_U(y)$ is given by the curve $t \mapsto \exp_G(y)\exp_G(tx)$, which implies that the integral curve of x_l^* through $y \in U$ is given for t close to 0 by $\Phi_t^{x_l^*}(y) = y*tx$. We therefore obtain with Remark VI.2.9:

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t}\Big|_{t=s=0} tx * sy * (-tx) * (-sy) &= \frac{\partial^2}{\partial s \partial t}\Big|_{t=s=0} \Phi_{-s}^{y_l^*} \circ \Phi_{-t}^{x_l^*} \circ \Phi_s^{y_l^*} \circ \Phi_t^{x_l^*}(0) \\ &= [-x_l^*, -y_l^*](0) = [x_l^*, y_l^*](0) = [x,y].\end{aligned}$$

Note that $F(t,s) := tx * sy * (-tx) * (-sy)$ vanishes for $t = 0$ and $s = 0$.

For $f(t) := F(t,t)$ we have

$$f'(t) = \frac{\partial F}{\partial t}(t,t) + \frac{\partial F}{\partial s}(t,t)$$

and

$$f''(t) = \frac{\partial^2 F}{\partial t^2}(t,t) + 2\frac{\partial^2 F}{\partial t \partial s}(t,t) + \frac{\partial^2 F}{\partial s^2}(t,t),$$

and

$$\frac{\partial^2 F}{\partial t^2}(0,0) = 0 = \frac{\partial^2 F}{\partial s^2}(0,0),$$

so that

$$\frac{1}{2}f''(0) = \frac{\partial^2 F}{\partial t \partial s}(0,0) = [x,y],$$

which leads to

$$\lim_{k\to\infty} k^2\Big(\frac{1}{k}x * \frac{1}{k}y * (-\frac{1}{k}x) * (-\frac{1}{k}y)\Big) = \lim_{k\to\infty} k^2 f\Big(\frac{1}{k}\Big) = \frac{1}{2}f''(0) = [x,y].$$

Applying the exponential function, this leads to the Commutator Formula. ■

Exercises for Chapter VII

Exercise VII.1. Let $f_1, f_2 \colon G \to H$ be two group homomorphisms. Show that the pointwise product

$$f_1 f_2 \colon G \to H, \quad g \mapsto f_1(g) f_2(g)$$

is a homomorphism if $f_1(G)$ commutes with $f_2(G)$. ■

Exercise VII.2. Let M be a manifold and and V a finite-dimensional vector space with a basis $(b_1, \ldots, b_n)$. Let $f \colon M \to \mathrm{GL}(V)$ be a map. Show that the following are equivalent:
(1) f is smooth.
(2) For each $v \in V$ the map $f_v \colon M \to V, m \mapsto f(m)v$ is smooth.
(3) For each i the map $f \colon M \to V, m \mapsto f(m)b_i$ is smooth. ■

Exercise VII.3. A vector field X on a Lie group G is called *right invariant* if for each $g \in G$ the vector field $(\rho_g)_* X = T(\rho_g) \circ X \circ \rho_g^{-1}$ coincides with X. We write $\mathcal{V}(G)^r$ for the set of right invariant vector fields on G. Show that:
(1) The evaluation map $\mathrm{ev}_{\mathbf{1}} \colon \mathcal{V}(G)^r \to T_{\mathbf{1}}(G)$ is a linear isomorphism.
(2) If X is right invariant, then there exists a unique $x \in T_{\mathbf{1}}(G)$ such that $X(g) = x_r(g) := T_{\mathbf{1}}(\rho_g)x = x \cdot 0_g$ (w.r.t. multiplication in $T(G)$).
(3) If X is right invariant, then $\widetilde{X} := (\eta_G)_* X := T(\eta_G) \circ X \circ \eta_G^{-1}$ is left invariant and vice versa.
(4) Show that $(\eta_G)_* x_r = -x_l$ and $[x_r, y_r] = -[x,y]_r$ for $x, y \in T_{\mathbf{1}}(G)$.
(5) Show that each right invariant vector field is complete and determine its flow. ■

Exercise VII.4. Let M be a smooth manifold, $\varphi \in \mathrm{Diff}(M)$ and $X \in \mathcal{V}(M)$. Show that the following are equivalent:

(1) φ commutes with the flow maps $\Phi_t^X: M_t \to M$ of X, i.e., each set M_t is φ-invariant and $\Phi_t^X \circ \varphi = \varphi \circ \Phi_t^X$ holds on M_t.

(2) For each integral curve $\gamma: I \to M$ of X the curve $\varphi \circ \gamma$ also is an integral curve of X.

(3) $X = \varphi_* X = T(\varphi) \circ X \circ \varphi^{-1}$, i.e., X is φ-invariant. ∎

Exercise VII.5. Let G be a Lie group. Show that any map $\varphi: G \to G$ commuting with all left multiplications λ_g, $g \in G$, is a right multiplication. ∎

Exercise VII.6. Let $X, Y \in \mathcal{V}(M)$ be two commuting complete vector fields, i.e., $[X, Y] = 0$. Show that the vector field $X + Y$ is complete and that its flow is given by

$$\Phi_t^{X+Y} = \Phi_t^X \circ \Phi_t^Y \quad \text{for all} \quad t \in \mathbb{R}.$$ ∎

Exercise VII.7. Let V be a finite-dimensional vector space and $\mu_t(v) := tv$ for $t \in \mathbb{R}^\times$. Show that:

(1) A vector field $X \in \mathcal{V}(V)$ is linear if and only if $(\mu_t)_* X = X$ holds for all $t \in \mathbb{R}^\times$.

(2) A diffeomorphism $\varphi \in \mathrm{Diff}(V)$ is linear if and only if it commutes with all the maps μ_t, $t \in \mathbb{R}^\times$. ∎

VII.2. Closed subgroups of Lie groups and their Lie algebras

In this section we shall see that closed subgroups of Lie groups are always Lie groups and that for a closed subgroup H of G its Lie algebra can be computed as

$$\mathbf{L}(H) = \{x \in \mathbf{L}(G): \exp_G(\mathbb{R}x) \subseteq H\}.$$

This makes it particularly easy to verify that concrete groups of matrices are Lie groups and to determine their algebras.

Definition VII.2.1. Let G be a Lie group and $H \leq G$ a subgroup. We define the set

$$\mathbf{L}(H) := \{x \in \mathbf{L}(G): \exp_G(\mathbb{R}x) \subseteq H\}$$

and observe that $\mathbb{R}\,\mathbf{L}(H) \subseteq \mathbf{L}(H)$ follows immediately from the definition. ∎

Note that for each $x \in \mathbf{L}(G)$ the set

$$\{t \in \mathbb{R}: \exp_G(tx) \in H\} = \gamma_x^{-1}(H)$$

is a closed subgroup of $\mathbb{R}$, hence either discrete cyclic or equal to $\mathbb{R}$.

Example VII.2.2. We consider the Lie group $G := \mathbb{R} \times \mathbb{T}$ (the cylinder) with Lie algebra $\mathbf{L}(G) \cong \mathbb{R}^2$ and the exponential function

$$\exp_G(x, y) = (x, e^{2\pi i y}).$$

For the closed subgroup $H := \mathbb{R} \times \{\mathbf{1}\}$ we then see that $(x, y) \in \mathbf{L}(H)$ is equivalent to $y = 0$, but

$$\exp_G^{-1}(H) = \mathbb{R} \times \mathbb{Z}.$$

■

Proposition VII.2.3. *If $H \leq G$ is a closed subgroup, then $\mathbf{L}(H)$ is a real Lie subalgebra of $\mathbf{L}(G)$.*

Proof. Let $x, y \in \mathbf{L}(H)$. For $k \in \mathbb{N}$ we then have $\exp_G \frac{1}{k}x, \exp_G \frac{1}{k}y \in H$, and with the Product Formula (Proposition VII.1.17), we get

$$\exp_G(x + y) = \lim_{k\to\infty} \Big(\exp_G \frac{x}{k} \exp_G \frac{y}{k}\Big)^k \in H$$

because H is closed. Therefore $\exp_G(x + y) \in H$, and $\mathbb{R}\,\mathbf{L}(H) = \mathbf{L}(H)$ now implies $\exp_G(\mathbb{R}(x + y)) \subseteq H$, hence $x + y \in \mathbf{L}(H)$.

Similarly, we use the Commutator Formula to get

$$\exp_G[x, y] = \lim_{k\to\infty} \Big(\exp_G \frac{x}{k} \exp_G \frac{y}{k} \exp_G -\frac{x}{k} \exp_G -\frac{y}{k}\Big)^{k^2} \in H,$$

hence $\exp_G([x, y]) \in H$, and $\mathbb{R}\,\mathbf{L}(H) = \mathbf{L}(H)$ yields $[x, y] \in \mathbf{L}(H)$. ■

One can show that $\mathbf{L}(H)$ is a Lie algebra for any subgroup, but this requires Yamabe's Theorem on analytic subgroups of matrix groups, which we won't prove here.

THE IDENTITY NEIGHBORHOOD THEOREM

Theorem VII.2.4. *Let H be a closed subgroup of the Lie group G. Then each 0-neighborhood in $\mathbf{L}(H)$ contains an open 0-neighborhood V such that $\exp_G|_V : V \to \exp_G(V) \cap H$ is a homeomorphism onto an open subset of H.*

Proof. First we use Proposition VII.1.13 to find an open 0-neighborhood $V_o \subseteq \mathbf{L}(G)$ such that

$$\exp_{V_o} := \exp_G|_{V_o} : V_o \to W_o := \exp_G(V_o)$$

is a diffeomorphism between open sets. In the following we write $\log_{W_o} := (\exp_{V_o})^{-1}$ for the inverse function. Then the following assertions hold:

- $V_o \cap \mathbf{L}(H)$ is a 0-neighborhood in $\mathbf{L}(H)$.
- $W_o \cap H$ is a $\mathbf{1}$-neighborhood in H.
- $\exp_G(V_o \cap \mathbf{L}(H)) \subseteq W_o \cap H$
- $\exp|_{V_o \cap \mathbf{L}(H)}$ is injective.

If H is not closed, then it need not be true that

$$\exp(V_o \cap \mathbf{L}(H)) = W_o \cap H$$

because it might be the case that $W_o \cap H$ is much larger than $\exp(V_o \cap \mathbf{L}(H))$ (cf. "the dense wind" in the 2-torus). We do not even know whether $\exp(V_o \cap \mathbf{L}(H))$ is open in H. Before we can complete the proof, we need three lemmas.

Lemma VII.2.5. *Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $H \cap W_o$ with $g_k \neq \mathbf{1}$ for all $k \in \mathbb{N}$ and $g_k \to \mathbf{1}$. We put $y_k := \log_{W_o} g_k$ and fix a norm on $\mathbf{L}(G)$. Then every cluster point of the sequence $\left\{\frac{y_k}{\|y_k\|} : k \in \mathbb{N}\right\}$ is contained in $\mathbf{L}(H)$.*

Proof. Let x be such a cluster point. By replacing the original sequence by a subsequence, the Bolzano-Weierstraß Theorem implies that we may assume that

$$x_k := \frac{y_k}{\|y_k\|} \to x \in \mathbf{L}(G).$$

Note that this implies $\|x\| = 1$. Let $t \in \mathbb{R}$ and put $p_k := \frac{t}{\|y_k\|}$. Then $tx_k = p_k y_k$, $y_k \to \log_{W_o} \mathbf{1} = 0$ by assumption,

$$\exp_G(tx) = \lim_{k\to\infty} \exp_G(tx_k) = \lim_{k\to\infty} \exp_G(p_k y_k),$$

and

$$\exp_G(p_k y_k) = \exp_G(y_k)^{[p_k]} \exp_G\big((p_k - [p_k])y_k\big),$$

where $[p_k] = \max\{l \in \mathbb{Z} : l \leq p_k\}$ is the *Gauß function.* We therefore have

$$\|(p_k - [p_k])y_k\| \leq \|y_k\| \to 0$$

and eventually

$$\exp_G(tx) = \lim_{k\to\infty} (\exp_G y_k)^{[p_k]} = \lim_{k\to\infty} g_k^{[p_k]} \in H,$$

because H is closed. This implies $x \in \mathbf{L}(H)$. ■

Lemma VII.2.6. *Let $E \subseteq \mathbf{L}(G)$ be a vector subspace complementing the $\mathbf{L}(H)$. Then there exists a 0-neighborhood $U_E \subseteq E$ with*

$$H \cap \exp_G(U_E) = \{\mathbf{1}\}.$$

Proof. We argue by contradiction. If a neighborhood U_E with the required properties does not exist, then for each compact convex 0-neighborhood $V_E \subseteq E$ we have for each $k \in \mathbb{N}$:

$$(\exp_G \tfrac{1}{k} V_E) \cap H \neq \{\mathbf{1}\}.$$

For each $k \in \mathbb{N}$ we therefore find $y_k \in V_E$ with $\mathbf{1} \neq g_k := \exp_G(\frac{y_k}{k}) \in H$. Now the compactness of V_E implies that the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded, so that $\frac{y_k}{k} \to 0$, which implies $g_k \to \mathbf{1}$. Now let $x \in E$ be a cluster point of the sequence $\frac{y_k}{\|y_k\|}$ which lies in the compact set $S_E := \{z \in E : \|z\| = 1\}$, so that at least one cluster point exists. According to Lemma VII.2.5, we have $x \in \mathbf{L}(H) \cap E = \{0\}$ because $g_k \in H \cap W_o$ for k sufficiently large, so that Lemma VII.2.5 applies. We arrive at a contradiction to $\|x\| = 1$. This proves the lemma. ■

Lemma VII.2.7. *Let $E, F \subseteq \mathbf{L}(G)$ be vector subspaces with $E \oplus F = \mathbf{L}(G)$. Then the map*

$$\Phi: E \times F \to G, \quad (x, y) \mapsto \exp_G(x)\exp_G(y),$$

restricts to a diffeomorphism of a neighborhood of $(0,0)$ to an open $\mathbf{1}$-neighborhood in G.

Proof. The Chain Rule implies that

$$\begin{aligned} T_{(0,0)}(\Phi)(x,y) &= T_{(\mathbf{1},\mathbf{1})}(m_G) \circ (T_0(\exp_G)|_E, T_0(\exp_G)|_F)(x,y) \\ &= T_{(\mathbf{1},\mathbf{1})}(m_G)(x,y) = x + y, \end{aligned}$$

Since the addition map $E \times F \to \mathbf{L}(G) \cong T_{\mathbf{1}}(G)$ is bijective, the Inverse Function Theorem implies that Φ restricts to a diffeomorphism of an open neighborhood of $(0,0)$ in $E \times F$ onto an open neighborhood of $\mathbf{1}$ in G. ■

Now we are ready to complete the proof of Theorem VII.2.4. We choose E as above, a vector space complement to $\mathbf{L}(H)$, and define

$$\Phi: E \times \mathbf{L}(H) \to G, \quad (x, y) \mapsto \exp_G x \exp_G y.$$

According to Lemma VII.2.7, there exist open 0-neighborhoods $U_E \subseteq E$ and $U_H \subseteq \mathbf{L}(H)$ such that

$$\Phi|_{U_E \times U_H}: U_E \times U_H \to \exp_G(U_E)\exp_G(U_H)$$

is a diffeomorphism onto an open $\mathbf{1}$-neighborhood in G. Moreover, in view of Lemma VII.2.6, we may choose U_E so small that $\exp_G(U_E) \cap H = \{\mathbf{1}\}$.

Since $\exp_G(U_H) \subseteq H$, the condition

$$g = \exp_G x \exp_G y \in H \cap (\exp_G(U_E)\exp_G(U_H))$$

implies $\exp_G x = g(\exp_G y)^{-1} \in H \cap \exp_G U_E = \{\mathbf{1}\}$. Therefore

$$H \supseteq \exp_G(U_H) = H \cap (\exp_G(U_E)\exp_G(U_H))$$

is an open $\mathbf{1}$-neighborhood in H. This completes the proof of Theorem VII.2.4.■

The Identity Neighborhood Theorem has important consequences for the structure of closed subgroups.

Proposition VII.2.8. *Every closed subgroup H of a Lie group G is a submanifold, and $m_H := m_G|_{H \times H}$ induces a Lie group structure on H such that the inclusion map $\iota_H: H \to G$ is a morphism of Lie groups for which $\mathbf{L}(\iota_H): \mathbf{L}(H) \to \mathbf{L}(G)$ is the inclusion of $\mathbf{L}(H)$ in $\mathbf{L}(G)$.*

Proof. We recall the diffeomorphism

$$\Phi: U_E \times U_H \to \exp_G(U_E)\exp(U_H)$$

from the proof of Theorem VII.2.4, where $U_H \subseteq \mathbf{L}(H)$ and $U_E \subseteq E$ are open 0-neighborhoods and $\mathbf{L}(G) = E \oplus \mathbf{L}(H)$. We also recall that

$$\Phi(U_E \times U_H) \cap H = \exp_G(U_H) = \Phi(\{0\} \times U_H).$$

For $h \in H$, the set $U_h := \lambda_h(\mathrm{im}(\Phi)) = h\,\mathrm{im}(\Phi)$ is an open neighborhood of h in G. Moreover, the map

$$\varphi_h \colon U_h \to E \oplus \mathbf{L}(H) = \mathbf{L}(G), \quad x \mapsto \Phi^{-1}(h^{-1}x)$$

is a diffeomorphism onto the open subset $U_E \times U_H$ of $\mathbf{L}(G)$, and we have

$$\begin{aligned}\varphi_h(U_h \cap H) &= \varphi_h(h\,\mathrm{im}(\Phi) \cap H) = \varphi_h(h(\mathrm{im}(\Phi) \cap H)) \\ &= \varphi_h(h\exp_G(U_H)) = \{0\} \times U_H = (U_E \times U_H) \cap (\{0\} \times \mathbf{L}(H)).\end{aligned}$$

Therefore the family $(\varphi_h, U_h)_{h\in H}$ provides a submanifold atlas for H in G. This defines a manifold structure on H for which $\exp_G|_{U_H}$ is a local chart (Lemma II.2.3).

The map $m_H \colon H \times H \to H$ is a restriction of the multiplication map m_G of G, hence smooth as a map $H \times H \to G$, and since H is an initial submanifold of G, Lemma II.2.3 implies that m_H is smooth. With a similar argument we see that the inversion η_H of H is smooth. Therefore H is a Lie group and the inclusion map $\iota_H \colon H \to G$ a smooth homomorphism. ■

Examples VII.2.9. We take a closer look at closed subgroups of the Lie group $(V, +)$, where V is a finite-dimensional vector space. From Example VII.1.9 we know that $\exp_V = \mathrm{id}_V$. Let $H \subseteq V$ be a closed subgroup. Then

$$\mathbf{L}(H) = \{x \in V \colon \mathbb{R}x \subseteq H\} \subseteq H$$

is the largest vector subspace contained in H. Let $E \subseteq V$ be a vector space complement for $\mathbf{L}(H)$. Then $V \cong \mathbf{L}(H) \times E$, and we derive from $\mathbf{L}(H) \subseteq H$ that

$$H \cong \mathbf{L}(H) \times (E \cap H).$$

Lemma VII.2.6 implies the existence of some 0-neighborhood $U_E \subseteq E$ with $U_E \cap H = \{0\}$, and this implies that $H \cap E$ is discrete because 0 is an isolated point of $H \cap E$. Now Exercise VII.18 implies the existence of linearly independent elements $f_1, \ldots, f_k \in E$ with

$$E \cap H = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_k.$$

We conclude that

$$H \cong \mathbf{L}(H) \times \mathbb{Z}^k \cong \mathbb{R}^d \times \mathbb{Z}^k \quad \text{for} \quad d = \dim \mathbf{L}(H).$$

Note that $\mathbf{L}(H)$ coincides with the connected component H_0 of 0 in H. ■

Calculating Lie algebras of subgroups

Lemma VII.2.10. *Let V and W be finite-dimensional vector spaces and $\beta: V \times V \to W$ a bilinear map. For $(x, y) \in \mathrm{End}(V) \times \mathrm{End}(W)$ the following are equivalent:*
(1) $e^{ty}.\beta(v, v') = \beta(e^{tx}.v, e^{tx}.v')$ *for all* $t \in \mathbb{R}$ *and all* $v, v' \in V$.
(2) $y.\beta(v, v') = \beta(x.v, v') + \beta(v, x.v')$ *for all* $v, v' \in V$.

Proof. (1) $\Rightarrow$ (2): Taking the derivative in $t = 0$, (1) leads to

$$y.\beta(v, v') = \beta(x.v, v') + \beta(v, x.v'),$$

where we use the Product and the Chain Rule.

(2) $\Rightarrow$ (1): If (2) holds, then the smooth curve $\alpha(t) := e^{-ty}\beta(e^{tx}.v, e^{tx}v')$ satisfies $\alpha(0) = \beta(v, v')$ and

$$\alpha'(t) = e^{-ty}\big(- y.\beta(e^{tx}v, e^{tx}v') + \beta(xe^{tx}v, e^{tx}v') + \beta(e^{tx}v, xe^{tx}v')\big) = 0.$$

Hence α is constant, so that

$$\beta(v, v') = \alpha(0) = \alpha(t) = e^{-ty}\beta(e^{tx}v, e^{tx}v')$$

holds for $v, v' \in V$ and $t \in \mathbb{R}$. ∎

Proposition VII.2.11. *Let V and W be finite-dimensional vector spaces and $\beta: V \times V \to W$ a bilinear map. For the closed subgroup*

$$\mathrm{O}(V, \beta) := \{g \in \mathrm{GL}(V) : (\forall v, v' \in V)\ \beta(g.v, g.v') = \beta(v, v')\}$$

we then have

$$\mathfrak{o}(V, \beta) := \mathbf{L}(\mathrm{O}(V, \beta)) = \{x \in \mathfrak{gl}(V) : (\forall v, v' \in V)\ \beta(x.v, v') + \beta(v, x.v') = 0\}.$$

Proof. We only have to observe that $x \in \mathbf{L}(\mathrm{O}(V, \beta))$ is equivalent to the pair $(x, 0)$ satisfying condition (1) in Lemma VII.2.10. ∎

Example VII.2.12. Let $B \in M_n(\mathbb{R})$, $\beta(v, w) = v^\top Bw$, and

$$G := \mathrm{O}(\mathbb{R}^n, \beta) := \{g \in \mathrm{GL}_n(\mathbb{R}) : g^\top Bg = B\}.$$

Then Proposition VII.2.11 implies that

$$\begin{aligned}\mathbf{L}(G) &= \{x \in \mathfrak{gl}_n(\mathbb{R}) : (\forall v, v' \in V)\ \beta(x.v, v') + \beta(v, x.v') = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{R}) : (\forall v, v' \in V)\ v^\top x^\top Bv' + v^\top Bxv' = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{R}) : x^\top B + Bx = 0\}.\end{aligned}$$

For $B = \mathbf{1}$ we obtain the orthogonal group

$$\mathrm{O}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \colon g^\top = g^{-1}\},$$

so that its Lie algebra is the set

$$\mathfrak{o}_n(\mathbb{R}) := \mathbf{L}(\mathrm{O}_n(\mathbb{R})) = \{x \in \mathfrak{gl}_n(\mathbb{R}) \colon x^\top = -x\}$$

of skew symmetric matrices. We further get for $\mathrm{Sp}_{2n}(\mathbb{R}) = \mathrm{O}(\mathbb{R}^{2n}, B)$, $B = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$,

$$\mathfrak{sp}_{2n}(\mathbb{R}) := \mathbf{L}(\mathrm{Sp}_{2n}(\mathbb{R})) := \{x \in \mathfrak{gl}_{2n}(\mathbb{R}) \colon x^\top B + Bx = 0\},$$

(cf. Exercise I.25). ■

Example VII.2.13. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and

$$\mathrm{Aut}(\mathfrak{g}) := \{g \in \mathrm{GL}(\mathfrak{g}) \colon (\forall x, y \in \mathfrak{g})\ g.[x,y] = [g.x, g.y]\},$$

which is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ (Exercise!). To calculate the Lie algebra of G, put $V = W = \mathfrak{g}$ and $\beta(x,y) = [x,y]$. Then we see that $D \in \mathfrak{aut}(\mathfrak{g}) := \mathbf{L}(\mathrm{Aut}(\mathfrak{g}))$ is equivalent to (D, D) satisfying the conditions in Lemma VII.2.10, and this leads to

$$\mathfrak{aut}(\mathfrak{g}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{g})) = \{D \in \mathfrak{gl}(\mathfrak{g}) \colon (\forall x, y \in \mathfrak{g})\, D.[x,y] = [D.x, y] + [x, D.y]\}\,.$$

The elements of this Lie algebra are called *derivations of* $\mathfrak{g}$, and $\mathfrak{aut}(\mathfrak{g})$ is also denoted $\mathrm{der}(\mathfrak{g})$. Note that the condition on an endomorphism of $\mathfrak{g}$ to be a derivation resembles the Leibniz Rule (Product Rule). ■

VII.3. Smooth actions of Lie groups

In Chapter VI we already encountered smooth actions of the one-dimensional Lie group $(\mathbb{R}, +)$ on manifolds, and we have seen that these actions are in one-to-one correspondence to complete vector fields, which is the corresponding Lie algebra picture. Now we turn to smooth actions of general Lie groups.

Definition VII.3.1. Let M be a smooth manifold and G a Lie group. A *(smooth) (left) action of* G *on* M is a smooth map

$$\sigma \colon G \times M \to M$$

with the following properties:

(A1) $\sigma(\mathbf{1}, m) = m$ for all $m \in M$.

(A2) $\sigma(g_1, \sigma(g_2, m)) = \sigma(g_1 g_2, m)$ for $g_1, g_2 \in G$ and $m \in M$.
We also write

$$g.m := \sigma(g,m), \quad \sigma_g(m) := \sigma(g,m), \quad \sigma^m(g) := \sigma(g,m) = g.m. \qquad \blacksquare$$

Note that for each smooth action σ the map

$$\widehat{\sigma}: G \to \mathrm{Diff}(M), \quad g \mapsto \sigma_g$$

is a homomorphism, and that any homomorphism $\gamma: G \to \mathrm{Diff}(M)$ for which the map

$$\sigma_\gamma: G \times M \to M, \quad (g,m) \mapsto \gamma(g)(m)$$

is smooth defines a smooth action of G on M.

Remark VII.3.2. What we call an action is sometimes called a *left action*. Likewise one defines a *right action* as a smooth map $\sigma_R: M \times G \to M$ with

$$\sigma_R(m, \mathbf{1}) = m, \quad \sigma_R(\sigma_R(m, g_1), g_2) = \sigma_R(m, g_1 g_2).$$

For $m.g := \sigma_R(m, g)$, this takes the form

$$m.(g_1 g_2) = (m.g_1).g_2$$

of an associativity condition.

If σ_R is a smooth right action of G on M, then

$$\sigma_L(g,m) := \sigma_R(m, g^{-1})$$

defines a smooth left action of G on M. Conversely, if σ_L is a smooth left action, then

$$\sigma_R(m,g) := \sigma_L(g^{-1}, m)$$

defines a smooth left action. This translation is one-to-one, so that we may freely pass from one type of action to the other. $\blacksquare$

Example VII.3.3. (a) If $X \in \mathcal{V}(M)$ is a complete vector field and $\Phi: \mathbb{R} \times M \to M$ its global flow, then Φ defines a smooth action of $G = (\mathbb{R}, +)$ on M.

(b) If G is a Lie group, then the multiplication map $\sigma := m_G: G \times G \to G$ defines a smooth left action of G on itself. In this case $(m_G)_g = \lambda_g$ are the left multiplications.

The multiplication map also defines a defines a smooth right action on it self. The corresponding left action is

$$\sigma: G \times G \to G, \quad (g,h) \mapsto hg^{-1} \quad \text{with} \quad \sigma_g = \rho_g^{-1}.$$

We have a third action of G on itself: the *conjugation action*

$$\sigma: G \times G \to G, \quad (g,h) \mapsto ghg^{-1} \quad \text{with} \quad \sigma_g = c_g.$$

(c) We have a natural smooth action of the Lie group $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$:

$$\sigma\colon \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma(g,x) := gx.$$

We further have an action of $\mathrm{GL}_n(\mathbb{R})$ on $M_n(\mathbb{R})$:

$$\sigma\colon \mathrm{GL}_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad \sigma(g,A) = gAg^{-1}.$$

(d) On the set $M_{p,q}(\mathbb{R})$ of $(p\times q)$-matrices we have an action of the direct product Lie group $G := \mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R})$ by

$$\sigma((g,h),A) := gAh^{-1}.$$ ■

Definition VII.3.4. Let $\sigma\colon G \times M \to M, (g,m) \mapsto g.m$ be a group action. For $m \in M$ the set

$$\mathcal{O}_m := G.m := \{g.m\colon g \in G\} = \{\sigma(g,m)\colon g \in G\}$$

is called the *orbit of* m.

The action is said to be *transitive* if there exists only one orbit, i.e., for $x, y \in M$ there exists a $g \in G$ with $y = g.x$.

We write $M/G := \{\mathcal{O}_m\colon m \in M\}$ for the set of G-orbits on M. ■

Remark VII.3.5. If $\sigma\colon G \times X \to X$ is an action of G on X, then the orbits forma a partition of X (Exercise).

A subset $R \subseteq X$ is called a set of *representatives for the action* if each G-orbit in X meets R exactly once:

$$(\forall x \in X) \quad |R \cap \mathcal{O}_x| = 1.$$ ■

Example VII.3.6. (1) We consider the action of the circle group

$$\mathbb{T} = \{z \in \mathbb{C}^\times\colon |z| = 1\}$$

on $\mathbb{C}$ by

$$\sigma\colon \mathbb{T} \times \mathbb{C} \to \mathbb{C}, \quad t.z = tz.$$

The orbits of this action are concentric circles:

$$\mathcal{O}_z = \{tz\colon t \in \mathbb{T}\} = \{w \in \mathbb{C}\colon |w| = |z|\}.$$

A set of representatives is given by

$$R := [0,\infty[= \{r \in \mathbb{R}\colon r \geq 0\}.$$

(2) For $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and the action

$$\sigma\colon \mathrm{GL}_n(\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^n, \quad (g,x) \mapsto gx$$

there are only two orbits:

$$\mathcal{O}_0 = \{0\} \quad \text{and} \quad \mathcal{O}_x = \mathbb{K}^n \setminus \{0\} \quad \text{for} \quad x \neq 0.$$

Each non-zero vector $x \in \mathbb{K}^n$ can be complemented to a basis of $\mathbb{K}^n$, hence arises as a first column of an invertible matrix g. Then $ge_1 = x$ implies that $\mathcal{O}_x = \mathcal{O}_{e_1}$. We conclude that $\mathbb{K}^n \setminus \{0\} = \mathcal{O}_{e_1}$.

(3) For the conjugation action

$$\mathrm{GL}_n(\mathbb{K}) \times M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad (g,A) \mapsto gAg^{-1}$$

the orbits are the similarity classes of matrices $\mathcal{O}_A = \{gAg^{-1}\colon g \in \mathrm{GL}(\mathbb{K})\}$. ■

The derived action

The following proposition generalizes the passage from flows to vector fields to actions of general Lie groups.

Proposition VII.3.7. *Let G be a Lie group and $\sigma: G \times M \to M, (g,m) \mapsto g.m$ a smooth action of G on M. Then the assignment*

$$\dot\sigma: \mathbf{L}(G) \to \mathcal{V}(M), \qquad \text{with} \quad \dot\sigma(x)(m) := -T_{\mathbf{1}}(\sigma^m)(x) = -(T_{(\mathbf{1},m)}\sigma)(x,0)$$

is a homomorphism of Lie algebras.

Proof. First we observe that for each $x \in \mathbf{L}(G)$ the map $\dot\sigma(x)$ defines a smooth map $M \to T(M)$, and since $\dot\sigma(x)(m) \in T_{\sigma(\mathbf{1},m)}(M) = T_m(M)$, it is a smooth vector field on M.

To see that $\dot\sigma$ is a homomorphism of Lie algebras, we pick $m \in M$ and write

$$\varphi^m := \sigma^m \circ \eta_G: G \to M, \quad g \mapsto g^{-1}.m$$

for the reversed orbit map. Then

$$\varphi^m(gh) = (gh)^{-1}.m = h^{-1}.(g^{-1}.m) = \varphi^{g^{-1}.m}(h),$$

which can be written as

$$\varphi^m \circ \lambda_g = \varphi^{g^{-1}.m}.$$

Taking the differential in $\mathbf{1} \in G$, we obtain for each $x \in \mathbf{L}(G) = T_{\mathbf{1}}(G)$:

$$\begin{aligned} T_g(\varphi^m)x_l(g) &= T_g(\varphi^m)T_{\mathbf{1}}(\lambda_g)x = T_{\mathbf{1}}(\varphi^m \circ \lambda_g)x = T_{\mathbf{1}}(\varphi^{g^{-1}.m})x \\ &= T_{\mathbf{1}}(\sigma^{g^{-1}.m})T_{\mathbf{1}}(\eta_G)x = -T_{\mathbf{1}}(\sigma^{\varphi^m(g)})x = \dot\sigma(x)(\varphi^m(g)). \end{aligned}$$

This means that the left invariant vector field x_l on G is φ^m-related to the vector field $\dot\sigma(x)$ on M. Therefore Lemma II.3.9 implies that for $x, y \in \mathbf{L}(G)$ the vector field $[x_l, y_l]$ is φ^m-related to $[\dot\sigma(x), \dot\sigma(y)]$, which leads for each $m \in M$ to

$$\begin{aligned} \dot\sigma([x,y])(m) &= T_{\mathbf{1}}(\varphi^m)[x,y]_l(\mathbf{1}) = T_{\mathbf{1}}(\varphi^m)[x_l, y_l](\mathbf{1}) \\ &= [\dot\sigma(x), \dot\sigma(y)](\varphi^m(\mathbf{1})) = [\dot\sigma(x), \dot\sigma(y)](m). \end{aligned}$$

■

Lemma VII.3.8. *If $\sigma: G \times M \to M$ is a smooth action and $x \in \mathbf{L}(G)$, then the global flow of the vector field $\dot\sigma(x)$ is given by $\Phi^x(t,m) = \exp_G(-tx).m$. In particular,*

$$\dot\sigma(x)(m) = \frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m.$$

Proof. In the proof of the preceding proposition, we have seen that

$$T_g(\varphi^m)x_l(g) = \dot{\sigma}(x)(\varphi^m(g))$$

holds for the map $\varphi^m(g) = g^{-1}.m$. Therefore

$$\frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m = T_{\mathbf{1}}(\varphi^m)T_0(\exp_G)x = T_{\mathbf{1}}(\varphi^m)x = \dot{\sigma}(x)(m).$$

This proves the lemma. ∎

Representations of Lie groups

Definition VII.3.9. (a) We have already seen that for each finite-dimensional vector space V, the group $\mathrm{GL}(V)$ carries a natural Lie group structure. Let G be a Lie group. A smooth homomorphism

$$\pi: G \to \mathrm{GL}(V)$$

is a called a *representation of* G *on* V (cf. Exercise VII.2).

Any representations defines a smooth action of G on V via

$$\sigma(g, v) := \pi(g)(v).$$

In this sense representations are nothing but linear actions on vector spaces.

(b) If $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation of* $\mathfrak{g}$ *on* V. ∎

As a consequence of Proposition VII.1.4, we obtain

Proposition VII.3.10. *If* $\varphi: G \to \mathrm{GL}(V)$ *is a smooth representation of* G*, then* $\mathbf{L}(\varphi): \mathbf{L}(G) \to \mathfrak{gl}(V)$ *is a representation of its Lie algebra* $\mathbf{L}(G)$. ∎

Remark VII.3.11. If $\sigma: G \times V \to V$ is a linear action and $\pi: G \to \mathrm{GL}(V)$ the corresponding representation, then

$$\dot{\sigma}(x)(v) = -\mathbf{L}(\pi)(x)(v)$$

is a linear vector field. ∎

The representation $\mathbf{L}(\varphi)$ obtained in Proposition VII.2.3 from the group representation φ is called the *derived representation.* This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)(x) = \frac{d}{dt}\Big|_{t=0} \varphi(\exp_G tx) = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)(x)}.$$

The adjoint representation

Let G be a Lie group and $\mathbf{L}(G)$ its Lie algebra. For $g \in G$ we recall the conjugation automorphism $c_g \in \mathrm{Aut}(G), c_g(x) = gxg^{-1}$ and define

$$\mathrm{Ad}(g) := \mathbf{L}(c_g) \in \mathrm{Aut}(\mathbf{L}(G)).$$

Then

$$\mathrm{Ad}(g_1g_2) = \mathbf{L}(c_{g_1g_2}) = \mathbf{L}(c_{g_1})\,\mathbf{L}(c_{g_2}) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2)$$

shows that $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G))$ is a group homomorphism. To see that it is smooth, we observe that for each $x \in \mathbf{L}(G)$ we have

$$\mathrm{Ad}(g)x = \mathbf{L}(c_g)x = T_{\mathbf{1}}(c_g)x = T_{\mathbf{1}}(\lambda_g \circ \rho_{g^{-1}})x = T_{g^{-1}}(\lambda_g)T_{\mathbf{1}}(\rho_{g^{-1}})x = 0_g \cdot x \cdot 0_{g^{-1}}$$

in the Lie group $T(G)$ (Lemma VII.1.2). Since the multiplication in $T(G)$ is smooth, the representation Ad of G on $\mathbf{L}(G)$ is smooth (Exercise VII.2).

We know already that the derived representation

$$\mathbf{L}(\mathrm{Ad})\colon \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation.

For $x \in \mathbf{L}(G)$ we define

$$\mathrm{ad}(x)\colon \mathbf{L}(G) \to \mathbf{L}(G), \quad \mathrm{ad}\,x(y) := [x,y].$$

Lemma VII.3.12. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$.

Proof. Let $x, y \in \mathbf{L}(G)$ and x_l, y_l the corresponding left invariant vector fields. Corollary VII.1.6 implies for $g \in G$ the relation

$$(c_g)_* y_l = \big(\mathbf{L}(c_g)y\big)_l = \big(\mathrm{Ad}(g)y\big)_l.$$

On the other hand, the left invariance of y_l leads to

$$\begin{aligned}(c_g)_* y_l &= (\rho_g^{-1} \circ \lambda_g)_* y_l \\ &= (\rho_g^{-1})_*(\lambda_g)_* y_l = (\rho_g^{-1})_* y_l.\end{aligned}$$

Next we observe that $\Phi_t^{x_l} = \rho_{\exp_G(tx)}$ is the flow of the vector field x_l (Lemma VII.1.10), so that Theorem VI.2.7 implies that

$$\begin{aligned}[x_l, y_l] = \mathcal{L}_{x_l} y_l &= \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^{x_l})_* y_l = \frac{d}{dt}\Big|_{t=0} (c_{\exp_G(tx)})_* y_l \\ &= \frac{d}{dt}\Big|_{t=0} \big(\mathrm{Ad}(\exp_G(tx))y\big)_l.\end{aligned}$$

Evaluating in $\mathbf{1}$, we get

$$[x,y] = [x_l, y_l](\mathbf{1}) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp_G(tx))y = \mathbf{L}(\mathrm{Ad})(x)(y). \qquad \blacksquare$$

Combining Proposition VII.1.15 with Lemma VII.3.12, we obtain the important formula

$$\mathrm{Ad} \circ \exp_G = \exp_{\mathrm{Aut}(\mathbf{L}(G))} \circ \mathrm{ad},$$

i.e.,

$$\mathrm{Ad}(\exp_G(x)) = e^{\mathrm{ad}\, x} \quad \text{for} \quad x \in \mathbf{L}(G). \tag{7.3.1}$$

Stabilizers and orbits

Definition VII.3.13. Let $\sigma: G \times M \to M, (g,m) \mapsto g.m$ be an action of the group G on M. For $m \in M$ the subset

$$G_m := \{g \in G : g.m = m\}$$

is called the *stabilizer of* m.

For $g \in G$ we write

$$\mathrm{Fix}(g) := M^g := \{m \in M : g.m = m\}$$

for the set of fixed points of g in M. We then have

$$m \in M^g \quad \Longleftrightarrow \quad g \in G_m.$$

For a subset $S \subseteq M$ we write

$$G_S := \bigcap_{m \in S} G_m = \{g \in G : (\forall m \in S)\ g.m = m\}$$

and for $H \subseteq G$ we write

$$M^H := \{m \in M : (\forall h \in H)\ h.m = m\}$$

for the set of points in M fixed by H. $\blacksquare$

Lemma VII.3.14. *For each smooth action $\sigma: G \times M \to M$ the following assertions hold:*

(1) *For each $m \in M$ the stabilizer is a closed subgroup of G.*

(2) *For $m \in M$ and $g \in G$ we have*

$$G_{g.m} = gG_m g^{-1}.$$

(3) *If $S \subseteq M$ is a G-invariant subset, then $G_S \trianglelefteq G$ is a normal subgroup.*

Proof. (1) That G_m is a subgroup is a trivial consequence of the action axioms. Its closedness follows from the continuity of the orbit map $\sigma^m: G \to M, g \mapsto g.m$ and the closedness of the points of M.

(2) If $h \in G_m$, then

$$ghg^{-1}.(g.m) = ghg^{-1}g.m = gh.m = g.m,$$

hence $ghg^{-1} \in G_{g.m}$ and thus $gG_m g^{-1} \subseteq G_{g.m}$. Similarly we get $g^{-1}G_{g.m}g \subseteq G_{g^{-1}.(g.m)} = G_m$ and therefore $gG_m g^{-1} = G_{g.m}$.

(3) follows directly from (2). ■

The normal subgroup G_M consisting of all elements of G which do not move any element of M is called the *effectivity kernel* of the action. It is the kernel of the corresponding homomorphism $G \to \mathrm{Diff}(M)$.

Proposition VII.3.15. *Let $\sigma: G \times M \to M$ be a smooth action of G on M. Then the following assertions hold:*

(1) *$m \in M^G \Rightarrow \dot\sigma(x)(m) = 0$ for each $x \in \mathbf{L}(G)$. The converse holds if G is connected.*

(2) *If $\dot\sigma(\mathbf{L}(G))(m) = T_m(M)$, then the orbit $\mathcal{O}_m$ of m is open.*

Proof. (1) Suppose first that m is a fixed point and let $x \in \mathbf{L}(G)$. Then

$$\dot\sigma(x)(m) = \frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m = \frac{d}{dt}\Big|_{t=0} m = 0$$

(Lemma VII.3.8).

If, conversely, all vector fields $\dot\sigma(x)$ vanish in m, then m is a fixed point for all flows generated by these vector fields, which leads to $\exp_G(x).m = m$ for each $x \in \mathbf{L}(G)$. This means that $G_m \supseteq \langle \exp_G \mathbf{L}(G)\rangle$, which in turn is the identity component of G (Lemma VII.1.14). If G is connected, we get $G = G_m$.

(2) By assumption, the linear map $\varphi: \mathbf{L}(G) \to T_m(M), x \mapsto \dot\sigma(x)(m)$ is surjective. Let $E \subseteq \mathbf{L}(G)$ be a subspace for which $\varphi|_E: E \to T_m(M)$ is bijective. For the smooth map

$$\Phi: E \to M, \quad \Phi(x) := \exp_G(x).m$$

we then have

$$T_0(\Phi)(x) = T_{(\mathbf{1},m)}(\sigma)(x, 0) = -\dot\sigma(x)(m) = -\varphi(x).$$

Therefore the Inverse Function Theorem implies that there exists a 0-neighborhood $U_E \subseteq E$ for which $\Phi|_{U_E}$ is a diffeomorphism onto an open subset of M. It follows in particular that $G.m$ is a neighborhood of m.

Since all maps σ_g are diffeomorphisms of M, it follows that $\sigma_g(G.m) = gG.m = G.m$ also is a neighborhood of $g.m$, hence that $G.m$ is open. ■

Corollary VII.3.16. *For each $m \in M$ we have*

$$\mathbf{L}(G_m) = \{x \in \mathbf{L}(G) : \dot{\sigma}(x)(m) = 0\}. \qquad \blacksquare$$

The preceding proposition shows in particular that the orbit $\mathcal{O}_m$ is a submanifold if m is fixed point (zero-dimensional case) and if $\mathcal{O}_m$ is open. Our next goal is to show that orbits of smooth actions always carry a natural manifold structure. This leads us to the geometry of homogeneous spaces in the next section.

More exercises for Section VII.

Exercise VII.8. If $(H_j)_{j\in J}$ is a family of subgroups of the Lie group G, then $\mathbf{L}(\bigcap_{j\in J} H_j) = \bigcap_{j\in J} \mathbf{L}(H_j)$. $\blacksquare$

Exercise VII.9. Let $G := \mathrm{GL}_n(\mathbb{R})$ and $V := P_k(\mathbb{R}^n)$ the space of homogeneous polynomials of degree k in $x_1, \ldots, x_n$, considered as functions $\mathbb{R}^n \to \mathbb{R}$. Show that:

(1) $\dim V = \binom{k+n-1}{n-1}$.

(2) We obtain a smooth representation $\rho: G \to \mathrm{GL}(V)$ of G on V by

$$(\rho(g).f)(x) := f(g^{-1}x).$$

(3) For the elementary matrix $E_{ij} = (\delta_{ij})$ we have $\mathbf{L}(\rho)(E_{ij}) = -x_j \frac{\partial}{\partial x_i}$. Hint: $(\mathbf{1} + tE_{ij})^{-1} = \mathbf{1} - tE_{ij}$. $\blacksquare$

Exercise VII.10. If $X \in \mathfrak{gl}(V)$ is a nilpotent linear map, then $\operatorname{ad} X \in \operatorname{End}(\mathfrak{gl}(V))$ is also nilpotent. Hint: $\operatorname{ad} X = L_X - R_X$ holds for $L_X(Y) = XY$ and $R_X(Y) = YX$, and both summands commute. $\blacksquare$

Exercise VII.11. Let $\varphi: G \to H$ be a morphism of Lie groups. Show that

$$\mathbf{L}(\ker \varphi) = \ker \mathbf{L}(\varphi). \qquad \blacksquare$$

Exercise VII.12. We consider the homomorphism

$$\det: \mathrm{GL}_n(\mathbb{R}) \to (\mathbb{R}^\times, \cdot).$$

Show that:

(1) $T_{\mathbf{1}}(\det) = d(\det)(\mathbf{1}) = \operatorname{tr}$. Hint: Product rule for n-linear maps.

(2) Show that the Lie algebra of the *special linear group*

$$\mathrm{SL}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : \det(g) = 1\}$$

coincides with

$$\mathfrak{sl}_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) : \operatorname{tr} x = 0\}.$$

Hint: Exercise VII.11. $\blacksquare$

Exercise VII.13. Show that the orbits of a group action $\sigma: G \times M \to M$ form a partition of M. ■

Exercise VII.14. Show that the following maps define group actions and determine their orbits by naming a representative for each orbit ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).

(a) $\mathrm{GL}_n(\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^n, \quad (g, v) \mapsto gv$.
(b) $\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \quad (g, v) \mapsto gv$.
(c) $\mathrm{GL}_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad (g, x) \mapsto gxg^{-1}$.
(d) $\mathrm{U}_n(\mathbb{K}) \times \mathrm{Herm}_n(\mathbb{K}) \to \mathrm{Herm}_n(\mathbb{K}), \quad (g, x) \mapsto gxg^{-1}$.
(e) $\mathrm{GL}_n(\mathbb{K}) \times \mathrm{Herm}_n(\mathbb{K}) \to \mathrm{Herm}_n(\mathbb{K}), \quad (g, x) \mapsto gxg^*$.
(f) $\mathrm{O}_n(\mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n, \quad (g, (x, y)) \mapsto (gx, gy)$. ■

Exercise VII.15. For a complex number $\lambda \in \mathbb{C}$ consider the smooth action

$$\sigma: \mathbb{R} \times \mathbb{C} \to \mathbb{C}, \quad \sigma(t, z) := e^{t\lambda} z.$$

(1) Sketch the orbits of this action in dependence of λ.
(2) Under which conditions are there compact orbits?
(3) Describe the corresponding vector field. ■

Exercise VII.16. For complex numbers $\lambda_1, \lambda_2 \in \mathbb{C}$, consider the smooth action

$$\sigma: \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2, \quad \sigma(t, (z_1, z_2)) := (e^{t\lambda_1} z_1, e^{t\lambda_2} z_2).$$

(1) For which pairs (λ_1, λ_2) are there bounded orbits?
(2) For which pairs (λ_1, λ_2) are there compact orbits?
(3) Describe a situation where the closure of some orbit is compact, but the orbit itself is not. ■

Exercise VII.17. (a) Show that each submanifold S of a manifold M is locally closed, i.e., for each point $s \in S$ there exists an open neighborhood U of s in M such that $U \cap S$ is closed.

(b) Show that any locally closed subgroup H of a Lie group G is closed. Hint: Let $g \in \overline{H}$ and U an open $\mathbf{1}$-neighborhood in G for which $U \cap H$ is closed. Show that:

(1) $g \in HU^{-1}$, i.e., $g = hu^{-1}$ with $h \in H$, $u \in U$.
(2) $\overline{H}$ is a subgroup of G.
(3) $u \in \overline{H} \cap U = \overline{H \cap U} = H \cap U$.
(4) $g \in H$. ■

Exercise VII.18. Let $D \subseteq \mathbb{R}^n$ be a discrete subgroup. Then there exist linearly independent elements $v_1, \ldots, v_k \in \mathbb{R}^n$ with $D = \sum_{i=1}^k \mathbb{Z} v_i$. Hint: Use induction on $\dim \operatorname{span} D$.

(1) Show that D is closed.
(2) Reduce to $\operatorname{span} D = \mathbb{R}^n$.

(3) Every compact subset $C \subseteq \mathbb{R}^n$ intersects D in a finite subset.

(4) Assume that $\operatorname{span} D = \mathbb{R}^n$ and assume that there exists a basis $f_1, \ldots, f_n$ of $\mathbb{R}^n$, contained in D, such that the hyper-plane $F := \operatorname{span}\{f_1, \ldots, f_{n-1}\}$ satisfies $F \cap D = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_{n-1}$. Show that

$$\delta := \inf\{\lambda_n > 0 \colon (\exists \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}) \sum_{i=1}^n \lambda_i f_i \in D\} > 0.$$

Hint: It suffices to assume $0 \leq \lambda_i \leq 1$ for $i = 1, \ldots, n$ and to observe (4).

(5) Apply induction on n to find $f_1, \ldots, f_n$ as in (4) and pick $f_n' := \sum_{i=1}^n \lambda_i f_i \in D$ with $\lambda_n = \delta$. Show that $D = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_{n-1} + \mathbb{Z}f_n'$. ∎

Exercise VII.19. (The structure of connected abelian Lie groups) Let A be a connected abelian Lie group. Show that

(1) $\exp_A \colon (\mathbf{L}(A), +) \to A$ is a morphism of Lie groups.

(2) $\exp_A$ is surjective.

(3) $\Gamma_A := \ker \exp_A$ is a discrete subgroup of $(\mathbf{L}(A), +)$.

(4) $\mathbf{L}(A)/\Gamma_A \cong \mathbb{R}^k \times \mathbb{T}^m$ for some $k, m \geq 0$.In particular, it is a Lie group and the quotient map $q_A \colon \mathbf{L}(A) \to \mathbf{L}(A)/\Gamma_A$ is a smooth map (cf. Exercise IX.5).

(5) $\exp_A$ factors through a diffeomorphism $\varphi \colon \mathbf{L}(A)/\Gamma_A \to A$.

(6) $A \cong \mathbb{R}^k \times \mathbb{T}^m$ as Lie groups. ∎

Exercise VII.20. (Divisible groups) An abelian group D is called *divisible* if for each $d \in D$ and $n \in \mathbb{N}$ there exists an $a \in D$ with $a^n = d$. Show that:

(1)* If G is an abelian group, H a subgroup and $f \colon H \to D$ a homomorphism into a divisible group D, then there exists an extension of f to a homomorphism $\widetilde{f} \colon G \to D$. Hint: Use Zorn's Lemma to reduce the situation to the case where G is generated by H and one additional element.

(2) If G is an abelian group and D a divisible subgroup, then $G \cong D \times H$ for some subgroup H of G. Hint: Extend $\mathrm{id}_D \colon D \to D$ to a homomorphism $f \colon G \to D$ and define $H := \ker f$. ∎

Exercise VII.21. (General abelian Lie groups) Let A be an abelian Lie group. Show that:

(1) The identity component of A_0 is isomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ for some $k, m \in \mathbb{N}_0$.

(2) A_0 is divisible.

(3) $A \cong A_0 \times \pi_0(A)$, where $\pi_0(A) := A/A_0$.

(4) There exists a discrete abelian group D with $A \cong \mathbb{R}^k \times \mathbb{T}^m \times D$. ∎

VII.4. Transitive actions and homogeneous spaces

The main result of this section is that for any smooth action of a Lie group G on a smooth manifold M, all orbits carry a natural manifold structure. First we take a closer look at transitive actions, i.e., actions with a single orbit.

Definition VII.4.1. (a) Let G be a group and H a subgroup. We write

$$G/H := \{gH : g \in G\}$$

for the set of *left cosets of H in G*. Then

$$\sigma: G \times G/H \to G/H, \quad (g, xH) \mapsto gxH$$

defines a transitive action of G on the set G/H (easy exercise).

(b) Let G be a group and $\sigma_1: G \times M_1 \to M_1$ and $\sigma_2: G \times M_2 \to M_2$ two actions of the group G on sets. A map $f: M_1 \to M_2$ is called *G-equivariant* if

$$f(g.m) = g.f(m) \quad \text{holds for all} \quad g \in G, m \in M_1. \qquad \blacksquare$$

Remark VII.4.2. Let $\sigma: G \times M \to M$ be an action of the group G on the set M. Fix $m \in M$. Then the *orbit map*

$$\sigma^m: G \to \mathcal{O}_m \subseteq M, \quad g \mapsto g.m$$

factors through a bijective map

$$\overline{\sigma}^m: G/G_m \to \mathcal{O}_m, \quad gG_m \mapsto g.m$$

which is equivariant with respect to the G-actions on G/G_m and M (Exercise). $\blacksquare$

The preceding remark shows that if we want to obtain a manifold structure on orbits of smooth actions, it is natural to try to define a manifold structure on the coset spaces G/H for closed subgroups H of a Lie group G. To understand this manifold structure, we need the concept of a submersion of manifolds.

Submersions

Definition VII.4.3. Let $f: M_1 \to M_2$ be a smooth map and $m \in M_1$. The map f is called *submersive in m* if the differential $T_m(f)$ is surjective. Otherwise m is called a *critical point of f*.

The map f is said to be a *submersion* if $T_m(f)$ is surjective for each $m \in M_1$. $\blacksquare$

Lemma VII.4.4. *If $f\colon M_1 \to M_2$ is a surjective submersion, then there exists for each point $p \in M_2$ an open neighborhood $U \subseteq M_2$ and a smooth map $\sigma\colon U \to M_1$ with $f \circ \sigma = \mathrm{id}_U$.*

Proof. Our assumption implies in particular that p is a regular value of f, so that $S := f^{-1}(p)$ is a smooth submanifold of M_1 of dimension $k := d_1 - d_2$, where $d_j := \dim M_j$ (Theorem II.2.6). Let $s \in S$ and (φ, V) be a submanifold chart of M_1 with

$$\varphi(V \cap S) = (\mathbb{R}^k \times \{0\}) \cap \varphi(V) \quad \text{and} \quad \varphi(s) = (0,0).$$

Consider the open 0-neighborhood $W := \{x \in \mathbb{R}^{d_2}\colon (0,x) \in \varphi(V)\}$. Then the map

$$\alpha\colon W \to M_2, \quad x \mapsto f(\varphi^{-1}(0,x))$$

is smooth with

$$T_0(\alpha) = T_s(f)T_s(\varphi)^{-1}|_{\{0\}\times\mathbb{R}^{d_2}}.$$

As $T_s(\varphi)^{-1}$ maps $\{0\}\times\mathbb{R}^{d_2}$ to a subspace of $T_s(M_1)$ complementary to $T_s(S) = \ker T_s(f) \cong \mathbb{R}^k$, it follows that $T_0(\alpha)$ is bijective. Hence the Inverse Function Theorem implies the existence of an open 0-neighborhood $W_1 \subseteq W$ such that $\alpha|_{W_1}\colon W_1 \to U := \alpha(W_1)$ is a diffeomorphism onto an open subset of M_2. Now

$$\sigma := \varphi^{-1} \circ (\alpha|_{W_1})^{-1} : U \to M_1,$$

is smooth, and we have for each $x \in U$:

$$f(\sigma(x)) = f(\varphi^{-1}\alpha^{-1}(x)) = \alpha \circ \alpha^{-1}(x) = x. \qquad \blacksquare$$

Proposition VII.4.5. (Universal property of submersions) *Let $f\colon M_1 \to M_2$ be a surjective submersion and $g\colon M_1 \to N$ a smooth map which is constant on all fibers of f. Then there exists a unique map $\overline{g}\colon M_2 \to N$ with $\overline{g} \circ f = g$, and this map is smooth.*

Proof. Let $p \in M_2$, $U \subseteq M_2$ open containing p, and $\sigma\colon U \to M_1$ a smooth map with $f \circ \sigma = \mathrm{id}_U$ (Lemma VII.4.4). Then we have $\overline{g}\,|_U = g \circ \sigma$, which is smooth. Hence $\overline{g}$ is smooth on a neighborhood of p, and since $p \in M_2$ was arbitrary, the assertion follows. $\blacksquare$

Corollary VII.4.6. *If $f\colon M_1 \to M_2$ is a bijective submersion, then f is a diffeomorphism.*

Proof. Apply the preceding proposition with $N := M_1$ and $g = \mathrm{id}_{M_1}$. $\blacksquare$

Remark VII.4.7. The smooth map $f\colon M_1 := \mathbb{R} \to M_2 := \mathbb{R}, x \mapsto x^3$ is submersive in all points $x \neq 0$. The map $g = \mathrm{id}_{\mathbb{R}}\colon M_1 = \mathbb{R} \to N := \mathbb{R}$ is smooth and bijective, hence constant on the fibers of f, but the map $\overline{g}\colon M_2 \to N, x \mapsto x^{\frac{1}{3}}$ is not smooth in 0. This shows that the assumption in Proposition VII.4.5 that f is submersive is really needed. $\blacksquare$

The following theorem, which we cite here without proof, implies in particular the existence of submersive points for surjective smooth maps.

Theorem VII.4.8. (Sard) *Let M_1 be a smooth σ-compact manifold, $f: M_1 \to M_2$ a smooth map and M_1^c the set of critical points of f. Then $f(M_1^c)$ is a set of measure zero in M_2, i.e., for each chart (φ, U) of M_2 the set $\varphi(U \cap f(M_1^c))$ is of Lebesgue measure zero.* ■

Homogeneous spaces

Lemma VII.4.9. *Let $H \leq G$ be a closed subgroup of the Lie group G and $E \subseteq \mathbf{L}(G)$ be a vector space complement of $\mathbf{L}(H)$. Then there exists an open 0-neighborhood $V_E \subseteq E$ such that*

$$\varphi: V_E \times H \to \exp_G(V_E)H, \quad (x,h) \mapsto \exp_G(x)h$$

is a diffeomorphism onto an open subset of G.

Proof. We recall that H is a submanifold of G and a Lie group with respect to this manifold structure (Proposition VII.2.8). Consider the map

$$\Phi: E \times H \to G, \quad (x,h) \mapsto \exp_G(x)h.$$

We then have

$$T_{(0,\mathbf{1})}(\Phi)(v,w) = T_{(\mathbf{1},\mathbf{1})}(m_G)(T_0(\exp_G)v, w) = v + w,$$

so that $T_{(0,\mathbf{1})}(\Phi)$ is a bijective linear map. Hence there exists a 0-neighborhood $U_E \subseteq E$ and a $\mathbf{1}$-neighborhood $U_H \subseteq H$ such that

$$\Phi_1 := \Phi|_{U_E \times U_H}: U_E \times U_H \to \exp_G(U_E)U_H$$

is a diffeomorphism onto an open subset of G. We further recall from Lemma VII.2.6, that we may assume, in addition, that

$$\exp_G(U_E) \cap H = \{\mathbf{1}\}. \tag{7.4.1}$$

We now pick a small symmetric 0-neighborhood $V_E = -V_E \subseteq U_E$ such that $\exp_G(V_E)\exp_G(V_E) \subseteq \exp_G(U_E)U_H$. Its existence follows from the continuity of the multiplication in G. We claim that the map

$$\varphi: V_E \times H \to \exp_G(V_E)H$$

is a diffeomorphism onto an open subset of G. To this end, we first observe that

$$\varphi \circ (\mathrm{id}_{V_E} \times \rho_h) = \rho_h \circ \varphi \quad \text{for each} \quad h \in H,$$

i.e., $\varphi(x, h'h) = \varphi(x,h')h$, so that

$$T_{(x,h)}(\varphi) \circ (\mathrm{id}_E \times T_{\mathbf{1}}(\rho_h)) = T_{\varphi(x,\mathbf{1})}(\rho_h) \circ T_{(x,\mathbf{1})}(\varphi).$$

Since $T_{(x,\mathbf{1})}(\varphi) = T_{(x,\mathbf{1})}(\Phi)$ is invertible for each $x \in V_E$, $T_{(x,h)}(\varphi)$ is invertible for each $(x,h) \in V_E \times H$. This implies that φ is a local diffeomorphism in each point (x,h). To see that φ is injective, we observe that

$$\exp_G(x)h = \varphi(x,h) = \varphi(x',h') = \exp_G(x')h'$$

implies that

$$\exp_G(x)^{-1}\exp_G(x') = h(h')^{-1} \in \exp_G(V_E)^2 \cap H \subseteq (\exp_G(U_E)U_H) \cap H = U_H,$$

where we have used (7.4.1). We thus obtain

$$\exp_G(x') \in \exp_G(x)U_H,$$

so that the injectivity of Φ_1 yields $x = x'$, which in turn leads to $h = h'$. This prove that φ is injective and a local diffeomorphism, hence a diffeomorphism. ■

Theorem VII.4.10. *Let G be a Lie group and $H \leq G$ a closed subgroup. Then the coset space G/H, endowed with the quotient topology, carries a natural manifold structure for which the quotient map $q: G \to G/H, g \mapsto gH$ is a submersion.*

Moreover, $\sigma: G \times G/H \to G/H, (g, xH) \mapsto gxH$ defines a smooth action of G on G/H.

Proof. Let $E \subseteq \mathbf{L}(G)$ be a vector space complement of the subspace $\mathbf{L}(H)$ and V_E be as in Lemma VII.4.9.

Step 1 (The topology on G/H): We endow $M := G/H$ with the quotient topology. Since for each open subset $O \subseteq G$ the product OH is open (Exercise VII.22), the openness of $OH = q^{-1}(q(O))$ shows that q is an open map, i.e., maps open subsets to open subsets.

To see that G/H is a Hausdorff space, let $g_1, g_2 \in G$ with $g_1H \neq g_2H$, i.e., $g_1 \notin g_2H$. Since H is closed, there exists a $\mathbf{1}$-neighborhood U_1 in G with $U_1g_1 \cap g_2H = \emptyset$, and further a symmetric $\mathbf{1}$-neighborhood U_2 with $U_2^{-1}U_2 \subseteq U_1$. Then U_2g_1H and U_2g_2H are disjoint q-saturated open subset of G, so that

$$q(U_2g_1H) = q(U_2g_1) \quad \text{and} \quad q(U_2g_2H) = q(U_2g_2)$$

are disjoint open subsets of G/H, separating g_1H and g_2H. This shows that G/H is a Hausdorff space.

We also observe that the action map σ is continuous because $\mathrm{id}_G \times q: G \times G \to G \times G/H$ is a quotient map since q is open (cf. Exercise I.17) and

$$\sigma \circ (\mathrm{id}_G \times q) = q \circ m_G: G \times G \to G/H, \quad (g,x) \mapsto gxH$$

is continuous.

Step 2 (The atlas of G/H): Let $W := q(\exp_G(V_E))$ with V_E as in Lemma VII.4.9 and define a smooth map

$$p_E: q^{-1}(W) = \exp_G(V_E)H \to V_E \quad \text{by} \quad \varphi^{-1}(w) = (p_E(w), *)$$

Since $q^{-1}(W)$ is open in G, W is open in G/H. Moreover, a subset $O \subseteq W$ is open if and only if $q^{-1}(O) \subseteq q^{-1}(W)$ is open. Since $q^{-1}(O) = \exp_G(p_E(q^{-1}(O))H$ and $q^{-1}(W) \cong V_E \times H$, this is equivalent to $p_E(q^{-1}(O))$ being open in V_E. Therefore the map $\psi: W \to V_E, q(g) \mapsto p_E(g)$ is a homeomorphism and (ψ, W) is a chart of G/H.

For $g \in G$ we put $W_g := g.W$ and $\psi_g(x) = \psi(g^{-1}.x)$. Since all maps $\sigma_g: G/H \to G/H$ are homeomorphisms (by Step 1), we thus get charts $(\psi_g, W_g)_{g\in G}$, and it is clear that $\bigcup_{g\in G} W_g = G/H$.

We claim that this collection of homeomorphisms is a smooth atlas of G/H. Let $g_1, g_2 \in G$ and assume that $W_{g_1} \cap W_{g_2} \neq \emptyset$. We then have

$$\begin{aligned}\psi_{g_1} \circ \psi_{g_2}^{-1}(x) &= \psi({g_1}^{-1} g_2.\psi^{-1}(x)) = \psi({g_1}^{-1} g_2.q(\exp_G(x)) \\ &= \psi(q({g_1}^{-1} g_2 \exp_G(x)) = p_E({g_1}^{-1} g_2 \exp_G(x)).\end{aligned}$$

Since p_E is smooth, this map is smooth on its open domain, which shows that $(\psi_g, W_g)_{g\in G}$ is a smooth atlas of G/H.

Step 3 (Smoothness of the maps σ_g): For $g_1, g_2 \in G$ we have $\sigma_{g_1}(W_{g_2}) = W_{g_1 g_2}$ and $\psi_{g_1 g_2} \circ \sigma_{g_1} = \psi_{g_2}$, which immediately implies that $\sigma_{g_1}|_{W_{g_2}}: W_{g_2} \to W_{g_1 g_2}$ is smooth. Since g_2 was arbitrary, all maps σ_g, $g \in G$, are smooth. From $\sigma_g \circ \sigma_{g^{-1}} = \mathrm{id}_M$ we further derive that they are diffeomorphisms.

Step 4 (q is a submersion): The smoothness of q on $q^{-1}(W)$ follows from $\psi(q(x)) = p_E(x)$ and the smoothness of p_E on $q^{-1}(W)$. Moreover, $T_{\mathbf{1}H}(\psi)T_{\mathbf{1}}(q) = T_{\mathbf{1}}(p_E): \mathbf{L}(G) \to E$ is the linear projection onto E with kernel $\mathbf{L}(H)$, hence surjective. This proves that $T_{\mathbf{1}}(q)$ is surjective.

For each $g \in G$ we have $q \circ \lambda_g = \sigma_g \circ q$, so that Step 3 implies that q is smooth on all of G. Taking derivatives, we obtain

$$T_g(q) \circ T_{\mathbf{1}}(\lambda_g) = T_{\mathbf{1}H}(\sigma_g) \circ T_{\mathbf{1}}(q),$$

and since all σ_g are diffeomorphisms, this implies that all differentials $T_g(q)$ are surjective, hence that q is a submersion.

Step 5 (Smoothness of the action of G/H): Since q is a submersion, the product map $\mathrm{id}_G \times q: G \times G \to G \times G/H$ also is a submersion. In view of Proposition VII.4.5, it therefore suffices to show that

$$\sigma \circ (\mathrm{id}_G \times q): G \times G \to G/H$$

is a smooth map, which is follows from $\sigma \circ (\mathrm{id}_G \times q) = q \circ m_G$. ∎

The following corollary shows that for each smooth group action, all orbits carry natural manifold structures. Not all these manifold structures turn these orbits into submanifolds, as the dense wind (discussed below) shows.

Corollary VII.4.11. *Let $\sigma\colon G\times M\to M$ be a smooth action of the Lie group G on M. Then for each $m\in M$ the orbit map $\sigma^m\colon G\to M, g\mapsto g.m$ factors through a smooth bijective equivariant map*

$$\eta_m\colon G/G_m\to M,\quad gG_m\mapsto g.m,$$

whose image is the set $\mathcal{O}_m$.

Proof. The existence of the map η_m is clear (Remark VII.4.2). Since the quotient map $q\colon G\to G/G_m$ is a submersion, the smoothness of η_m follows from the smoothness of the map $\eta_m\circ q=\sigma^m$ (Proposition VII.4.5). ■

The preceding corollary provides on each orbit $\mathcal{O}_m$ of a smooth group action the structure of a smooth manifold. Its dimension is given by

$$\dim(G/G_m)=\dim G-\dim G_m=\dim\mathbf{L}(G)-\dim\mathbf{L}(G_m)=\dim\dot\sigma(\mathbf{L}(G))(m),$$

because $\mathbf{L}(G_m)$ is the kernel of the linear map

$$\mathbf{L}(G)\to T_m(M),\quad x\mapsto\dot\sigma(x)(m)$$

(Corollary VII.3.16). In this sense we may identify the subspace $\dot\sigma(\mathbf{L}(G))(m)\subseteq T_m(M)$ with the tangent space of the orbit $\mathcal{O}_m$.

In some case the orbit $\mathcal{O}_m$ may already have another manifold structure, f.i., if it is a submanifold of M. In this case the following proposition says that this manifold structure coincides with the one induced by identifying it with G/G_m.

Proposition VII.4.12. *If $\mathcal{O}_m$ is a submanifold of M, then the map*

$$\eta_m\colon G/G_m\to\mathcal{O}_m$$

is a diffeomorphism.

Proof. We recall from Lemma II.2.3 that $\mathcal{O}_m$ is an initial submanifold of M (Lemma II.2.3), so that the map η_m is also smooth as a map $\eta_m\colon G/G_m\to\mathcal{O}_m$. The equivariance of this map means that

$$\eta_m\circ\mu_g=\sigma_g\circ\eta_m\quad\text{ for }\quad\mu_g(xG_m)=gxG_m,\quad\sigma_g(y)=g.y.$$

For the differential in the base point $G_m\in G/G_m$, this implies that

$$T_{gG_m}(\eta_m)\circ T_{G_m}(\mu_g)=T_m(\sigma_g)\circ T_{G_m}(\eta_m).$$

Since the maps μ_g and σ_g are diffeomorphisms, it follows that the rank of all tangent maps $T_{gG_m}(\eta_m)$ is the same. As η_m is surjective, Sard's Theorem implies the existence of some submersive point, but since all ranks are equal, it follows that $\eta_m\colon G/G_m\to\mathcal{O}_m$ is a bijective submersion, hence a diffeomorphism (Corollary VII.4.6). ■

Corollary VII.4.13. *If $\sigma: G\times M \to M$ is a transitive smooth action of the Lie group G on the manifold M and $m \in M$, then the orbit map $\eta_m: G/G_m \to M$ is a G-equivariant diffeomorphism.* ■

Definition VII.4.14. The manifolds of the form $M = G/H$, where H is a closed subgroup of a Lie group G, are called *homogeneous spaces*. We know already that the canonical action of G on G/H is smooth and transitive, and Corollary VII.4.13 shows the converse, i.e., that each transitive action is equivalent to the action on some G/H because there exists an equivariant diffeomorphism. ■

Example VII.4.15. (Graßmannians) Let $M := \mathrm{Gr}_k(\mathbb{R}^n)$ denote the set of all k-dimensional subspaces of $\mathbb{R}^n$, the *Graßmann manifold of degree k*. We know from linear algebra that the natural action

$$\sigma: \mathrm{GL}_n(\mathbb{R}) \times \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n), \quad (g, F) \mapsto g(F)$$

is transitive (Exercise). Let $F := \mathrm{span}\{e_1, \ldots, e_k\}$. Writing elements of $M_n(\mathbb{R})$ as 2×2-block matrices, according to

$$M_n(\mathbb{R}) = \begin{pmatrix} M_k(\mathbb{R}) & M_{k,n-k}(\mathbb{R}) \\ M_{n-k,k}(\mathbb{R}) & M_{n-k}(\mathbb{R}) \end{pmatrix},$$

the stabilizer of F in $\mathrm{GL}_n(\mathbb{R})$ is

$$\mathrm{GL}(n, \mathbb{R})_F := \Big\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \mathrm{GL}_k(\mathbb{R}), b \in M_{k,n-k}(\mathbb{R}), d \in \mathrm{GL}_{n-k}(\mathbb{R}) \Big\},$$

which is a closed subgroup. Then the homogeneous space $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{R})_F$ carries a natural manifold structure, and since the orbit map of F induces a bijection

$$\eta_F: \mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{R})_F \to \mathrm{Gr}_k(\mathbb{R}^n), \quad g\,\mathrm{GL}_n(\mathbb{R})_F \mapsto g(F),$$

we obtain a manifold structure on $\mathrm{Gr}_k(\mathbb{R}^n)$ for which the natural action of $\mathrm{GL}_n(\mathbb{R})$ is smooth.

The dimension of $\mathrm{Gr}_k(\mathbb{R}^n)$ is given by

$$\dim \mathrm{GL}_n(\mathbb{R}) - \dim \mathrm{GL}_n(\mathbb{R})_F = n^2 - (k^2 + (n-k)^2 + k(n-k)) = k(n-k).$$

(Exercise: Show that this manifold structure coincides with the one from Example I.2.10).

Note that for $k = 1$ we obtain the manifold structure on the projective space $\mathbb{P}(\mathbb{R}^n)$. ■

Example VII.4.15. (Flag manifolds) A *flag* in $\mathbb{R}^n$ is a tuple

$$\mathcal{F} = (F_1, \ldots, F_m)$$

of subspaces of $\mathbb{R}^n$ with

$$F_1 \subseteq F_2 \subseteq \ldots \subseteq F_m.$$

Let $k_i := \dim F_i$ and call $(k_1, \ldots, k_m)$ the *signature of the flag*. We write $\mathrm{Fl}(k_1, \ldots, k_m)$ for the set of all flags of signature $(k_1, \ldots, k_m)$ in $\mathbb{R}^n$. Clearly,

$$\mathrm{Fl}(k_1, \ldots, k_m) \subseteq \mathrm{Gr}_{k_1}(\mathbb{R}^n) \times \ldots \times \mathrm{Gr}_{k_m}(\mathbb{R}^n).$$

We also have a natural action of $\mathrm{GL}_n(\mathbb{R})$ on the product of the Graßmann manifolds by

$$g.(F_1, \ldots, F_m) := (g(F_1), \ldots, g(F_m)).$$

To describe a base point, let

$$F_i^0 := \mathrm{span}\{e_1, \ldots, e_{k_i}\}$$

and note that

$$\mathcal{F}^0 := (F_1^0, \ldots, F_m^0) \in \mathrm{Fl}(k_1, k_1, \ldots, k_m).$$

From basic Linear Algebra, it follows that the action of $\mathrm{GL}_n(\mathbb{R})$ on the subset $\mathrm{Fl}(k_1, \ldots, k_m)$ is transitive, which is shown by choosing for each flag $\mathcal{F}$ of the given signature a basis $(b_i)_{1 \leq i \leq n}$ of $\mathbb{R}^n$ such that

$$F_i := \mathrm{span}\{b_1, \ldots, b_{k_i}\} \quad \text{for} \quad i = 1, \ldots, m.$$

Writing elements of $M_n(\mathbb{R})$ as $(m \times m)$-block matrices according to the partition

$$n = k_1 + (k_2 - k_1) + (k_3 - k_2) + \ldots + (k_m - k_{m-1}) + (n - k_m),$$

the stabilizer of $\mathcal{F}^0$ is given by

$$\mathrm{GL}(n, \mathbb{R})_{\mathcal{F}^0} = \{(g_{ij})_{i,j=1,\ldots,m} \colon (i > j \Rightarrow g_{ij} = 0); g_{ii} \in \mathrm{GL}_{k_i - k_{i-1}}(\mathbb{R})\},$$

which is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. We now proceed as above to get a manifold structure on the set $\mathrm{Fl}(k_1, \ldots, k_m)$, turning it into a homogeneous space, called a *flag manifold* (Exercise: Calculate the dimension of $\mathrm{Fl}(1, 2, 3, 4)(\mathbb{R}^6)$.) ■

Example VII.4.16. The orthogonal group $\mathrm{O}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \colon g^\top = g^{-1}\}$ acts smoothly on $\mathbb{R}^n$, and its orbits are the spheres

$$S(r) := \{x \in \mathbb{R}^n \colon \|x\| = r\}, \quad r \geq 0.$$

We know already that all these spheres carry natural manifold structures. Therefore Corollary VII.4.14 implies that for each $r > 0$ we have

$$S(r) \cong \mathbb{S}^{n-1} \cong \mathrm{O}_n(\mathbb{R}) / \mathrm{O}_n(\mathbb{R})_{e_1},$$

where

$$\mathrm{O}_n(\mathbb{R})_{e_1} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \colon a \in \{\pm 1\}, d \in O_{n-1}(\mathbb{R}) \right\} \cong (\mathbb{Z}/2\mathbb{Z}) \times \mathrm{O}_{n-1}(\mathbb{R}). \qquad ■$$

The dense wind in the 2-torus

Example VII.4.17. (Closed subgroups of $\mathbb{T}$) Let $H \subseteq \mathbb{T} \subseteq (\mathbb{C}^\times, \cdot)$ be a closed proper (=different from $\mathbb{T}$) subgroup. Since $\exp_{\mathbb{T}}$ is surjective and $\dim \mathbb{T} = 1$, it follows that $\mathbf{L}(H) = \{0\}$, so that the Identity Neighborhood Theorem implies that H is discrete, hence finite because $\mathbb{T}$ is compact.

On the other hand, $q^{-1}(H)$ is a closed proper subgroup of $\mathbb{R}$, hence cyclic (this is a very simple case of Exercise VII.18), which implies that $H = q(q^{-1}(H))$ is also cyclic. Therefore H is one of the groups

$$C_n := \{z \in \mathbb{T} : z^n = 1\}$$

of n-th roots of unity. ∎

Example VII.4.18. (Subgroups of $\mathbb{T}^2$) (a) Let $H \subseteq \mathbb{T}^2$ be a closed proper subgroup. Then $\mathbf{L}(H) \neq \mathbf{L}(\mathbb{T}^2)$ implies $\dim H < \dim \mathbb{T}^2 = 2$. Further, H is compact, so that the group $\pi_0(H)$ of connected components of H is finite.

If $\dim H = 0$, then H is finite, and for $n := |H|$ it is contained in a subgroup of the form $C_n \times C_n$, where $C_n \subseteq \mathbb{T}$ is the subgroup of n-th roots of unity (cf. Example VII.4.17).

If $\dim H = 1$, then H_0 is a compact connected 1-dimensional Lie group, hence isomorphic to $\mathbb{T}$ (Exercise VII.19). Therefore $H_0 = \exp_{\mathbb{T}^2}(\mathbb{R}x)$ for some $x \in \mathbf{L}(H)$ with $\exp_{\mathbb{T}^2}(x) = (e^{2\pi i x_1}, e^{2\pi i x_2}) = (1,1)$, which is equivalent to $x \in \mathbb{Z}^2$. We conclude that the Lie algebras of the closed subgroups are of the form $\mathbf{L}(H) = \mathbb{R}x$ for some $x \in \mathbb{Z}^2$.

(b) For each $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the image of the 1-parameter group

$$\gamma : \mathbb{R} \to \mathbb{T}^2, \quad t \mapsto (e^{i\theta t}, e^{it})$$

is not closed because γ is injective. Hence the closure of $\gamma(\mathbb{R})$ is a closed subgroup of dimension at least 2, which shows that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$. ∎

VII.5. Quotients of Lie groups

Theorem VII.5.1. *If N is a closed normal subgroup of the Lie group G, then the quotient group G/N carries a unique Lie group structure for which the quotient homomorphism $q : G \to G/N, g \mapsto gN$ is a submersion.*

Proof. Theorem VII.4.7 provides the manifold structure on G/N and that q is a submersion. Let $m_{G/N}$ denote the multiplication map on G/N. Since $q \times q : G \times G \to G/N \times G/N$ also is a submersion, the smoothness of $m_{G/N}$ follows from the smoothness of

$$m_{G/N} \circ (q \times q) = q \circ m_G : G \times G \to G/N$$

(Proposition VII.4.5). Likewise, the smoothness of the inversion $\eta_{G/N}$ follows from the smoothness of $\eta_{G/N} \circ q = q \circ \eta_G$. ∎

Corollary VII.5.2. *Let $\varphi: G \to H$ be a morphism of Lie groups and endow $G/\ker\varphi$ with its natural Lie group structure. Then φ factors through a smooth injective morphism of Lie groups $\overline{\varphi}: G/\ker\varphi \to H, g\ker\varphi \mapsto \varphi(g)$.*

If, in addition, $\mathbf{L}(\varphi)$ is surjective, then $\overline{\varphi}$ is a diffeomorphism onto an open subgroup of H.

Proof. The existence of the map $\overline{\varphi}$ is clear. It is also easy to see that it is a group homomorphism. Since the quotient map $q: G \to G/\ker\varphi$ is a submersion, the smoothness of $\overline{\varphi}$ follows from the smoothness of $\overline{\varphi} \circ q = \varphi$.

If, in addition, $\mathbf{L}(\varphi)$ is surjective, then $\overline{\varphi}$ is a morphism of Lie groups whose differential is an isomorphism. Since it is also injective, the Inverse Function Theorem implies that it is a diffeomorphism onto an open subgroup of H. ■

Even more exercises on Section VII.

Exercise VII.22. Let G be a topological group (multiplication and inversion are continuous). Then for each open subset $O \subseteq G$ and for each subset $S \subseteq G$ to product sets

$$OS = \{gh: g \in O, h \in S\} \quad \text{and} \quad SO = \{hg: g \in O, h \in S\}$$

are open (Hint: proof of Lemma VII.1.14). ■

VIII. From infinitesimal to global structures

In this brief concluding chapter we formulate the assumptions required to integrate homomorphisms of Lie algebras to group homomorphisms and likewise, to integrate homomorphisms $\mathbf{L}(G) \to \mathcal{V}(M)$ to smooth group actions. The condition showing up in this context is a requirement on the topology of the group G: If it is simply connected, then everything works, otherwise there are obstructions which vanish sometimes but not always.

The fundamental group

Definition VIII.1. Let X be a topological space, $I := [0,1]$, and $x_0 \in X$. We write

$$P(X,x_0) := \{\gamma \in C(I,X) \colon \gamma(0) = x_0\}$$

and

$$P(X,x_0,x_1) := \{\gamma \in P(X,x_0) \colon \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X,x_0,x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H \colon I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

and

$$(\forall t \in I) \quad H(t,0) = x_0, \; H(t,1) = x_1.$$

One easily verifies that we thus obtain an equivalence relation whose classes, the homotopy classes, are denoted $[\alpha]$.

We write $\Omega(X,x_0) := P(X,x_0,x_0)$ for the set of loops in x_0. For $\alpha \in P(X,x_0,x_1)$ and $\beta \in P(X,x_1,x_2)$ we define a product $\alpha * \beta \in P(X,x_0)$ by

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \qquad \blacksquare$$

Lemma VIII.2. *If $\varphi \colon [0,1] \to [0,1]$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then for each $\alpha \in P(X,x_0,x_1)$ we have $\alpha \sim \alpha \circ \varphi$.*

Proof. $H(t,s) := \alpha(t + (1-t)\varphi(s))$. $\blacksquare$

Proposition VIII.3. *The following assertions hold:*

(1) *$\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ implies $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$, so that we obtain a well-defined product*

$$[\alpha] * [\beta] := [\alpha * \beta]$$

of homotopy classes.

(2) *If* $x_0 \in \Omega(X, x_0)$ *denotes the constant map* $I \to X, t \mapsto x_0$, *then*

$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \textit{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) *(Associativity)* $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ *for* $\alpha \in P(X, x_0, x_1), \beta \in P(X, x_1, x_2)$, $\gamma \in P(X, x_2, x_3)$.

(4) *(Inverse) For* $\alpha \in P(X, x_0)$ *and* $\overline{\alpha}(t) := \alpha(1-t)$ *we have*

$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) *(Functoriality) For any continuous map* $\varphi: X \to Y$ *with* $\varphi(x_0) = y_0$ *we have*

$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta).$$

Proof. (1) If H^α is a homotopy from α_1 to α_2 and H^β a homotopy from β_1 to β_2, then we put

$$H(t,s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

(2) For the first assertion we use Lemma VIII.2 and

$$x_0 * \alpha = \alpha \circ \varphi$$

for

$$\varphi(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For the second we have $\alpha * x_0 = \alpha \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{4} \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(4)

$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \leq \frac{1-t}{2} \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \overline{\alpha}(2s-1) & \text{for } s \geq \frac{1+t}{2}. \end{cases}$$

(5) is trivial. ∎

Definition VIII.4. From the preceding definition we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in x_0 carries a natural group structure. This group is called the *fundamental group of X with respect to x_0*.

A pathwise connected space X is called *simply connected* if $\pi_1(X, x_0)$ vanishes for some $x_0 \in X$ (which implies that it vanishes for each $x_0 \in X$; Exercise). ■

Remark VIII.5. The map

$$\sigma: \pi_1(X, x_0) \times P(X, x_0)/\sim \to P(X, x_0)/\sim, \quad ([\alpha], [\beta]) \mapsto [\alpha * \beta] = [\alpha] * [\beta]$$

defines an action of the group $\pi_1(X, x_0)$ on the set $P(X, x_0)/\sim$ of homotopy classes of paths starting in x_0 (Lemma VIII.3). ■

At this point we can formulate the "integration part" of the Lie functor:

Theorem VIII.6. *Let G be a connected, simply connected Lie group. Then the following assertions hold:*
(1) *If H is a Lie group and $\psi: \mathbf{L}(G) \to \mathbf{L}(H)$ is a homomorphism of Lie algebras, then there exists a unique homomorphism of Lie groups with $\mathbf{L}(\varphi) = \psi$.*
(2) *If M is a smooth manifold and $\psi: \mathbf{L}(G) \to \mathcal{V}(M)$ is a homomorphism of Lie algebras, then there exists a unique smooth action $\sigma: G \times M \to M$ with $\dot{\sigma} = \psi$.* ■

The usefulness of the preceding theorem in concrete situations depends on whether one can check that a given Lie group G is simply connected.

Here are some tools:

Lemma VIII.7. (a) *If X is contractible, then $\pi_1(X, x_0) = \{[x_0]\}$ is trivial.*
(b) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. ■

Examples VIII.8. Here is a list of fundamental groups of concrete groups:
(1) $\pi_1(\mathbb{R}^n) = \{0\}$ because $\mathbb{R}^n$ is contractible.
(2) $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$ because $\pi_1(\mathbb{T}^n) \cong \pi_1(\mathbb{T})^n$ and $\pi_1(\mathbb{T}) \cong \mathbb{Z}$.
(3) $\pi_1(\mathrm{SL}_n(\mathbb{C})) = \pi_1(\mathrm{SU}_n(\mathbb{C})) = \{\mathbf{1}\}$ because

$$\mathrm{SL}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C}) \times \{x \in \mathrm{Herm}_n(\mathbb{C}): \operatorname{tr} x = 0\}$$

(Polar decomposition).
(4) $\pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \pi_1(\mathrm{U}_n(\mathbb{C})) \cong \mathbb{Z}$ because

$$\mathrm{GL}_n(\mathbb{C}) \cong \mathrm{U}_n(\mathbb{C}) \times \mathrm{Herm}_n(\mathbb{C}), \quad \mathrm{GL}_n(\mathbb{C}) \cong \mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times$$

(topologically).
(5)

$$\pi_1(\mathrm{SL}_n(\mathbb{R})) = \pi_1(\mathrm{GL}_n(\mathbb{R})) = \pi_1(\mathrm{O}_n(\mathbb{R})) = \pi_1(\mathrm{SO}_n(\mathbb{R})) \cong \begin{cases} \mathbb{Z} & \text{for } n = 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n > 2 \end{cases}.$$

■

The following lemma implies in particular, that fundamental groups of topological groups are always abelian.

Lemma VIII.8. *Let G be a topological group and consider the identity element $\mathbf{1}$ as a base point. Then the set $P(G,\mathbf{1})$ also carries a natural group structure given by the pointwise product $(\alpha\cdot\beta)(t) := \alpha(t)\beta(t)$ and we have*

(1) *$\alpha\sim\alpha'$, $\beta\sim\beta'$ implies $\alpha\cdot\beta\sim\alpha'\cdot\beta'$, so that we obtain another well-defined product*
$$[\alpha]\cdot[\beta] := [\alpha\cdot\beta]$$
of homotopy classes, defining a group structure on $P(G,\mathbf{1})/\sim$.

(2) *$\alpha\sim\beta \Longleftrightarrow \alpha\beta^{-1}\sim\mathbf{1}$, the constant map.*

(3) *(Commutativity) $[\alpha]\cdot[\beta]=[\beta]\cdot[\alpha]$ for $\alpha,\beta\in\Omega(G,\mathbf{1})$.*

(4) *(Consistency) $[\alpha]\cdot[\beta]=[\alpha]*[\beta]$ for $\alpha\in\Omega(G,\mathbf{1})$, $\beta\in P(G,\mathbf{1})$.*

Proof. (1) follows by composing homotopies with the product map m_G.

(2) follows from (1).

(3)
$$[\alpha][\beta]=[\alpha*\mathbf{1}][\mathbf{1}*\beta]=[(\alpha*\mathbf{1})(\mathbf{1}*\beta)]=[(\mathbf{1}*\beta)(\alpha*\mathbf{1})]=[\mathbf{1}*\beta][\alpha*\mathbf{1}]=[\beta][\alpha].$$

(4) $[\alpha][\beta]=[(\alpha*\mathbf{1})(\mathbf{1}*\beta)]=[\alpha*\beta]=[\alpha]*[\beta]$. ∎

As a consequence of (4), we can calculate the product of homotopy classes as a pointwise product of representatives.

Definition VIII.9. Let X and Y be topological spaces. A continuous map $q\colon X\to Y$ is called a *covering* if each $y\in Y$ has an open neighborhood U such that $q^{-1}(U)$ is a non-empty disjoint union of subsets $(V_i)_{i\in I}$, such that for each $i\in I$ the restriction $q|_{V_i}\colon V_i\to U$ is a homeomorphism.

Note that this condition implies in particular that q is surjective. ∎

Proposition VIII.10. *If $q\colon X\to Y$ is a covering and $x_0\in X$, $y_0\in Y$ satisfy $q(y_0)=x_0$, then the map*
$$q_*\colon P(X,x_0)\to P(Y,y_0),\quad \gamma\mapsto q\circ\gamma$$
is bijective (Path Lifting Property) and induces a bijection
$$\overline{q}_*\colon P(X,x_0)/\sim\to P(Y,y_0)/\sim,\quad [\gamma]\mapsto[q\circ\gamma]$$
(Homotopy Lifting Property). ∎

Corollary VIII.11. *If Y is arcwise connected and simply connected, then each covering map $q\colon X\to Y$, where X is arcwise connected, is a homeomorphism.*

Proof. By assumption, the evaluation maps
$$\mathrm{ev}_X\colon P(X,x_0)/\sim\to X,\quad [\alpha]\mapsto\alpha(1)$$

and

$$\mathrm{ev}_Y\colon P(Y, y_0)/\sim \to Y, \quad [\alpha] \mapsto \alpha(1)$$

are surjective. It is clear that they are well-defined, because homotopic paths have the same endpoints.

On the other hand, the simple connectedness of Y implies that the map ev_Y is also injective because $\alpha(1) = \beta(1)$ implies that $\alpha \sim \beta$. In fact, the path $\alpha * \overline{\beta} \in \Omega(Y, y_0)$ is a loop in y_0, hence contractible. Therefore

$$\beta \sim (\alpha * \overline{\beta}) * \beta \sim \alpha * (\overline{\beta} * \beta) \sim \alpha * \beta(1) = \alpha * \alpha(1) \sim \alpha$$

(Lemma VIII.3).

Now $q \circ \mathrm{ev}_X = \mathrm{ev}_Y \circ q_*$ implies that q is injective, hence a homeomorphism because it is surjective, continuous and open (Exercise). ∎

The preceding proof even suggests how to find for a given space X a simply connected covering space $\widetilde{X}$. To formulate its assumption, we need:

Definition VIII.12. A topological space X is called *semilocally simply connected* if each point $x_0 \in X$ has a neighborhood U such that each loop $\alpha \in P(U, x_0)$ is homotopic to $[x_0]$ in X, i.e., the homomorphism

$$\pi_1(U, x_0) \to \pi_1(X, x_0)$$

is trivial. ∎

Theorem VIII.13. *If X is arcwise connected, locally arcwise connected and semilocally simply connected, then the set $\widetilde{X} := P(X, x_0)/\sim$ carries a topology for which the evaluation map*

$$\mathrm{ev}_X\colon P(X, x_0)/\sim \to X, \quad [\gamma] \mapsto \gamma(1)$$

is an arcwise connected covering by a simply connected space. ∎

Conclusion

The problem to integrate homomorphisms of Lie algebras to homomorphisms of Lie groups and Lie algebras of vector fields to smooth group actions leads us to the question whether a given Lie group is simply connected. This question can naturally be dealt with in the context of covering theory, a branch of (algebraic) topology. The results mentioned above barely scratch the surface of this rich and interesting theory.

One of the outcomes of this theory is that for each conncted Lie group G there exists a simply connected Lie group $\widetilde{G}$ and a morphism $q_G\colon \widetilde{G} \to G$ of Lie groups which is a covering map with $\ker q_G \cong \pi_1(G)$. Moreover, $\mathbf{L}(G) = \mathbf{L}(\widetilde{G})$

and all Lie groups with the same Lie algebra are of the form $\widetilde{G}/D$ for some discrete central subgroup of $\widetilde{G}$.

For any Lie algebra homomorphism $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ one now has a Lie group homomorphism $\widetilde{\varphi}\colon \widetilde{G} \to H$ with $\mathbf{L}(\widetilde{\varphi}) = \psi$ and $\widetilde{\varphi}$ factors through a homomorphism $\varphi\colon G \to H$ if and only if $\ker q_G \subseteq \widetilde{\varphi}$. This is how one deals with the integration problem in practical situations.

The End

Literature on “Manifolds and Transformation Groups”

[Ar78] Arnold, V. I., “Mathematical Methods of Classical Mechanics,” Springer-Verlag, 1978.

(Contains a very nice introduction to differential forms and their applications in Hamiltonian mechanics.)

[Be00] Bertram, W., “The Geometry of Jordan and Lie Structures,” Lecture Notes in Math. **1754**, Springer-Verlag, Berlin, 2000.

(This book exploints the connection between differential geometric structures on symmetric spaces, Lie algebras and Jordan algebras.)

[Bo75] Boothby, W. M., “Introduction to differentiable manifolds and Riemannian geometry, 1975.

(A very good source for the material covered by this lecture.)

[Bre93] Bredon, G. E., “Topology and Geometry,” Graduate Texts in Mathematics **139**, Springer-Verlag, Berlin, 1993.

(An excellent reference for everything related to topology.)

[DK00] Duistermaat, J. J., and J. A. C. Kolk, “Lie groups,” Springer-Verlag, Universitext, 2000.

(The emphasis of this book lies on differential geometric aspects of Lie groups, such as actions on manifolds and compact groups.)

[GS84] Guillemin, V., and S. Sternberg, “Symplectic Techniques in Physics,” Cambridge University Press, 1984.

(The mathematical side of classical mechanics is symplectic geometry. This book develops the basic techniques of symplectic geometry and Lie group actions on symplectic manifolds.)

[Hel78] Helgason, S., “Differential Geometry, Lie Groups, and Symmetric Spaces,” Acad. Press, London, 1978.

(The all-time-classic on Riemannian symmetric spaces.)

[La99] Lang, S., “Fundamentals of Differential Geometry,” Graduate Texts in Math. **191**, Springer-Verlag, 1999.

(An approach to differential and Riemannian geometry where the manifolds are assumed to be modeled over Banach spaces.)

[La99] Lang, S., “Math Talks for Undergraduates,” Springer-Verlag, 1999.

(This nice booklet contains among many other things an elementary approach to the Riemmanian symmetric space of positive definite matrices.)

[Sch95] Schottenloher, M., "Geometrie und Symmetrie in der Physik," Vieweg Lehrbuch Mathematische Physik, Vieweg Verlag, 1995.

(This book deals with many aspects under which Lie groups arise as symmetry groups in physics. In particular it contains a nice introduction to principal bundles, connections etc., and how all this is linked to physics. The book is written in such a way that the first sections in each chapter can be read with minimal prerequisites and the later sections require more background knowledge.)

[Wa83] Warner, F. W., "Foundations of Differentiable Manifolds and Lie Groups," Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1983.

(A good source for de Rham's theorem which links de Rham cohomology to singular cohomology and other approaches of a more topological flavour.)