Structure and Geometry of Lie Groups

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Preface

Nowadays there are plenty of textbooks on Lie groups to choose from, so we feel we should explain why we decided to add another one to the row. Most of the readily available books on Lie groups either aim at an elementary introduction mostly restricted to matrix groups or else they try to provide the background on semisimple Lie groups needed in harmonic analysis and unitary representation theory with as little general theory as possible. In [HN91] we tried to exhibit the basic principles of Lie theory rather than specific material stressing the exponential function as the means of translating problems and solutions between the global and the infinitesimal level. In that book, written in German for the German student population which typically did not know differential geometry but was well versed in advanced linear algebra, we avoided abstract differentiable manifolds by combining matrix groups with covering arguments. Having introduced the basic principles we demonstrated their power by proving a number of standard and not so standard results on the structure of Lie groups. The choice of results included owed a lot to Hochschild's book [Ho65] which even then was not so easy to come by.

This book builds on [HN91], but after twenty years of teaching and research in Lie theory we found it indispensable to also have the differential geometry of Lie groups available. Even though this is not apparent from the text, the reason for this is the large number of applications and further developments of Lie theory in which differential manifolds are essential. Moreover, we decided to include a number of structural results we found to be useful in the past but not readily available in the textbook literature. The basic line of thought now is:

- Simple examples: Matrix groups
- Tools from algebra: Lie algebras
- Tools from geometry: Smooth manifolds
- The basic principles: Lie groups, their Lie algebras and the exponential function
- Structure theory: General Lie groups and special classes
- Testing methods on examples: The topology of classical groups
- A slight extension: Several connected components

While this book offers plenty of tested material for different introductory courses such as *Matrix groups*, *Lie groups*, *Lie algebras*, or *Differentiable Manifolds*, it is not a textbook to follow from A to Z. Moreover, it contains advanced material one would not typically include in a first course. In fact, some of the advanced material has not appeared in any monograph before. This and the fact that we wanted the book to be self-contained, is the reason for its considerable length. In order to still keep it within reasonable limits, for some topics which are well-covered in the textbook literature, we decided to include only what was needed for the further developments in the book. This applies e.g. to the standard structure and classification theories of semisimple

Lie algebras. Thus we do *not* want to suggest that this book can replace previous textbooks. It is rather meant as a *true addition* to the existing textbook literature on Lie groups.

As was mentioned before, we are well aware of the fact that modern mathematics abounds with applications of Lie theory while this book hardly mentions any of them. The reason is that most applications require additional knowledge of the field in which these applications occur, so describing them would have meant either extensive story-telling or else a considerable expansion in length of this book. Neither option seemed attractive to us, so we leave it to future books to give detailed accounts of the beautiful ways in which Lie theory enters different fields of mathematics.

Even though there was a forerunner book and many lecture notes produced for various courses over the years, in compiling this text we produced many typos and made some mistakes. Many of them were shown to us by a small army of enthusiastic proof readers to whom we are extremely grateful: Hanno Becker, Jan Emonds, Hasan Gündogoan, Michael Klotz, Stéphane Merigon, Norman Metzner, Wolfgang Palzer, Matthias Peter, Henrik Seppänen, and Stefan Wagner read major parts of the manuscripts and there were others who looked at particular sections. Of course we know that also the final version will contain mistake and we assume full responsibility for those.

We also would like to thank Ilka Agricola and Thomas Friedrich for some background information on the early history of Lie theory.

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Introduction

To locate the theory of Lie groups within mathematics, one can say that Lie groups are groups with some additional structure that permits us to apply analytic techniques such as differentiation in a group theoretic context.

In the elementary courses on one variable calculus one studies functions on three levels:

(1) abstract functions between sets,
(2) continuous functions, and
(3) differentiable functions.

Going from level (1) to level (3), we refine the available tools at each step. At level (1) we have no structure at all to do anything, at level (2) we obtain results like the Intermediate Value Theorem or the Maximal Value Theorem saying that each function on a compact interval takes a maximal value. The latter result is a useful existence theorem, but it provides no help at all to calculate maximal values. For that we need refined tools such as the derivative of a function and a translation mechanism between properties of a function and its derivative. The situation is quite similar when we study groups. There is a level (1) consisting of abstract group theory which is particularly interesting for finite groups because the finiteness assumption is a powerful tool in the structure theory of finite groups. For infinite groups G it is good to have a topology on G which is compatible with the group structure in the sense that the group operations are continuous, so that we are at level (2), and G is called a *topological group*. If we want to apply calculus techniques to study a group, we need *Lie groups* [1], i.e. groups which at the same time are differentiable manifolds such that the group operations are smooth.

For Lie groups we also need a translation mechanism telling us how to pass from group theoretic properties of G to properties of its "derivative" $\mathbf{L}(G)$, which in technical terms is the tangent space $T_{\mathbf{1}}(G)$ of G at the identity. We think of $\mathbf{L}(G)$ as a "linear" object attached to the "nonlinear" object G, because $\mathbf{L}(G)$ is a vector space endowed with an additional algebraic structure $[\cdot,\cdot]$, the *Lie bracket*, turning it into a *Lie algebra*. This algebraic structure is a bilinear map $\mathbf{L}(G) \times \mathbf{L}(G) \to \mathbf{L}(G)$, called the Lie bracket, satisfying the axioms

$$[x,x] = 0 \quad \text{and} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{for} \quad x,y,z \in \mathbf{L}(G),$$

which can be considered as linearized versions of the group axioms. The connecting element between the group and its Lie algebra is the exponential function

[1] The Norwegian mathematician Marius Sophus Lie (1842–1899) was the first to study differentiability properties of groups in a systematic way. In the 1890s Sophus Lie developed his theory of differentiable groups (called continuous groups at a time when the concept of a topological space was not yet developed) to study symmetries of differential equations.

$$\exp_G\colon\ \mathbf{L}(G) \to G$$

for which we have the Product Formula

$$\exp_G(x+y) = \lim_{k\to\infty} \big(\exp_G(\tfrac{1}{k}x)\exp_G(\tfrac{1}{k}y)\big)^k$$

and the Commutator Formula

$$\exp_G([x,y]) = \lim_{k\to\infty} \Big(\exp_G\big(\tfrac{1}{k}x\big)\exp_G\big(\tfrac{1}{k}y\big)\exp_G\big(-\tfrac{1}{k}x\big)\exp_G\big(-\tfrac{1}{k}y\big)\Big)^{k^2}$$

connecting the algebraic operations (addition and Lie bracket on $\mathbf{L}(G)$) to the group operations (multiplication and commutator). For the important class of matrix groups $G \subseteq \mathrm{GL}_n(\mathbb{R})$, the Lie algebra $\mathbf{L}(G)$ is a set of matrices and the exponential function is simply given by the power series $\exp_G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

An important property of the Lie algebra $\mathbf{L}(G)$ is that we can extend $\mathbf{L}$ to smooth homomorphism $\varphi\colon G_1 \to G_2$ of Lie groups by putting $\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi)$ (the tangent map in $\mathbf{1}$) to obtain the so-called *Lie functor*, assigning to Lie groups Lie algebras and to group homomorphisms homomorphisms of Lie algebras. The compatibility of all that with the exponential function is encoded in the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(G_2) \\ \downarrow{\scriptstyle \exp_{G_1}} & & \downarrow{\scriptstyle \exp_{G_2}} \\ G_1 & \xrightarrow{\varphi} & G_2. \end{array}$$

The exponential function of a Lie group always maps sufficiently small 0-neighborhoods U in $\mathbf{L}(G)$ diffeomorphically to identity neighborhoods in G, so that the local structure of G is completely encoded in the multiplication

$$x * y := (\exp_G|_U)^{-1}(\exp_G x \exp_G y),$$

which turns out to be given by a universal power series,

$$x * y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}\big[x,[x,y]\big] + \frac{1}{12}\big[y,[y,x]\big] + \dots.$$

Its summands are obtained by iterated Lie brackets whose precise structure we know after fundamental work of H. F. Baker, J. E. Campbell, E. B. Dynkin and F. Hausdorff.

The basic philosophy of Lie theory now is that the local structure of the group G is determined by its Lie algebra $\mathbf{L}(G)$, and that the description of the global structure of a Lie group requires additional information that can be obtained in topological terms involving covering theory.

In Part I of this book, we approach the general concept of a Lie group by first discussing certain groups of matrices and groups arising in geometric contexts (Chapter 1). All these groups will later turn out to be Lie groups. In Chapter 2 we study the central tool in the theory of matrix groups that permits us to reverse the differentiation process from a Lie group G to its Lie algebra $\mathbf{L}(G)$: the exponential function $\exp_G\colon \mathbf{L}(G) \to G$, which is obtained by restriction from the matrix exponential function used in the theory of linear differential equations with constant coefficients. Chapter 3 treats Lie algebras of matrix groups and provides methods to calculate these Lie algebras effectively.

In Part II we study Lie algebras as independent algebraic structures. We start in Chapter 4 by working out the standard approach: What are the substructures? Under which conditions does a substructure lead to a quotient structure? What are the simple structures? Does one have composition series? This leads to concepts like Lie subalgebras and ideals, nilpotent, solvable, and semisimple Lie algebras. In Chapter 5 we introduce Cartan subalgebras and the associated root and weight decompositions as tools to study the structure of (semi)simple Lie algebras. Further, we define *abstract root systems* and associated Weyl groups. Even though representation theory is not in the focus of this book, we provide in Chapter 6 the basic theory as it repeatedly plays an important role in structural questions. In particular, we introduce the universal enveloping algebra and prove the Poincaré–Birkhoff–Witt (PBW) Theorem on the structure of the enveloping algebra which implies in particular that each finite dimensional Lie algebra sits in an associative algebra which has the same modules. From this we derive Serre's Theorem on the presentation of semisimple Lie algebras in terms of generators and relations, the Highest Weight Theorem on the classification of the simple finite dimensional modules, and Ado's Theorem on the existence of a faithful finite dimensional representation of a finite dimensional Lie algebra. Finally, we introduce basic cohomology theory for Lie algebras and describe extensions of Lie algebras.

In Part III we provide an introduction to Lie groups based on the theory of smooth manifolds. The basic concepts and results from differential geometry needed for this are introduced in Chapter 7. In particular, we discuss vector fields on smooth manifolds and their integration to local flows. Chapter 8 is devoted to the subject proper of this book: Lie groups, defined as smooth manifolds with group structure such that all structure maps are smooth. Here we introduce the key tools of Lie theory. The Lie functor which associates a Lie algebra with a Lie group and the exponential function from the Lie algebra to the Lie group. They provide the means to translate global problems into infinitesimal ones and to lift infinitesimal solutions to local and, with the help of some additional topology, global ones. As a first set of applications of these methods we identify the Lie group structures of closed subgroups of Lie groups and show how to construct Lie groups from local and infinitesimal data. Further we explain covering theory for Lie groups. Finally, we prove Yamabe's Theorem asserting that any arcwise connected subgroup of a Lie group carries a natural Lie group structure,, and this allows us to equip *any* subgroup of a Lie group with a canonical Lie group structure.

As we have explained before, a key method in Lie theory is to study the structure of Lie groups by translation group theoretic problems into linear algebra problems via the Lie functor, solving these problems and translating the solutions back using the exponential function. In Part IV we illustrate this general method by deriving a number of important structural results about Lie groups. Since in practice Lie groups often occur as symmetry groups which are not connected but have a finite number of connected components, we prove the results in this generality whenever it is possible without too much extra effort.

We start with quotient structures in Chapter 10, which also leads to homogeneous spaces, semidirect products, and eventually to a complete description of connected nilpotent and 1-connected solvable Lie groups.

In Chapter 11 we turn our attention to compact Lie groups and their covering groups. Again we first study the corresponding Lie algebras which are, by abuse of notation, called *compact*. Then we prove Weyl's Theorem saying that the simply connected covering of a semisimple compact Lie group is compact. Further, we prove the important fact that a compact connected Lie group is the union of its maximal tori and show that such a Lie group is the semidirect product of its (semisimple) commutator subgroup and a torus subgroup. We also show that each compact Lie group is linear, i.e., can be realized as a closed subgroup of some $\mathrm{GL}_n(\mathbb{R})$. It is possible to describe the fundamental group in terms of the Lie algebra and the exponential map. In this context, we introduce the analytic Weyl group and a number of relevant lattices (i.e., discrete additive subgroups of maximal rank) in the Lie algebra $\mathfrak{t}$ of a maximal torus T and its dual $\mathfrak{t}^*$. The techniques are finally extended a little to prove that fixed point sets of automorphisms of simply connected groups are connected, a fact that is very useful e.g. in the study of symmetric spaces.

Chapter 12 is devoted to the Cartan and the Iwasawa decomposition of noncompact semisimple Lie groups. These two decompositions are really only the starting point for a very rich structure theory which, in contrast to some other topics we present in this book, is very well covered in the existing literature (see e.g. [Wa88] and [Kn02]). Therefore, we decided to keep this chapter brief.

In Chapter 13 we return to the general structure theory and show that each Lie group with finitely many connected components admits a maximal compact subgroup which is unique up to conjugation. In fact, it turns out that the group is diffeomorphic to a product of the maximal compact subgroup and a finite dimensional vector space (the Manifold Splitting Theorem 13.3.11). In particular, the topology of a Lie group with finitely many connected components is completely determined by any of its maximal compact subgroups. Before we can prove that we have to characterize the center of a connected Lie group as a certain subset of the exponential image (Theorem 13.2.8). The techniques developed for the proof of the Manifold Splitting Theorem also allow us to prove Dixmier's Theorem which characterizes the 1-connected solvable Lie groups for which the exponential function is a diffeomorphism. Finally, we study in detail under which circumstances one finds integral subgroups which are *not* closed, i.e., proper dense subgroups of their closure. In particular, we give a series of verifiable sufficient conditions for an integral subgroup to be closed. These results build on the classification of finitely generated abelian groups for which we provide a proof in Appendix 13.6.

In Chapter 14 we explain how to complexify Lie groups. It turns out that each Lie group G has a universal complexification $G_{\mathbb{C}}$, but G does in general not embed into $G_{\mathbb{C}}$. If G is compact, however, it does embed into its universal complexification, and this gives rise to the class of *linearly complex reductive Lie groups*. They can be characterized by the existence of a holomorphic faithful representation and the fact that all holomorphic representations are completely reducible, hence the name (see Theorem 14.3.11). On the way to this characterization, we study *abelian* complex connected Lie groups in some detail and introduce the *linearizer* of a complex group, which measures how far the group is from being complex linear.

In the literature one finds a lot of different notions of *reductive groups*, for many of which one imposes extra linearity properties. This is why in Chapter 15 we take a closer look at the structural implications of the existence of a faithful continuous finite dimensional representation of a Lie group. In particular, we introduce a *real linearizer* and the notion of *linearly real reductive groups*. Combining these notions with suitable Levi complements we obtain a characterization of connected Lie groups which admit such faithful representations (see Theorems 15.2.7 and 15.2.9). The results of this chapter rely heavily on the results of Chapter 14. Conversely, we use the results of Chapter 15 to complete the discussion of the existence faithful of holomorphic representations in Section 15.3.

In Chapter 16 we apply the general results to compact and noncompact classical groups in order to provide explicit structural and topological information. In particular, we determine connected components and fundamental groups. Moreover, we include a rather detailed discussion of spin groups which builds on the material on Clifford algebras and related groups presented in Appendix B.3. Here we also explain a number of isomorphisms of low dimensional groups. This discussion, as well as the one on conformal groups in Section 16.4 exemplifies the way Lie theory can be used to study groups defined in geometric terms. For a more detailed information of this kind we refer to [GW09] for the classical and to [Ad96] for the exceptional Lie groups.

The examples from Chapter 16 show that many geometrically defined Lie groups have several connected components. While only the connected component of the identity is accessible to the methods built on the exponential function, there are still tools to analyze nonconnected Lie groups. In Chapter 17 we present some of these tools. The key notion is that of an extension of a discrete group by a (connected) Lie group. We explain how to classify such extensions in terms of group cohomology and apply this result to characterize those Lie groups with finite number of connected components which admit a simply connected covering group.

Fundamental notation

Throughout this book $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. All vector spaces will be $\mathbb{K}$-vector spaces if not otherwise specified. We write $M_n(\mathbb{K})$ for the ring of $(n \times n)$-matrices with entries in $\mathbb{K}$, $\mathbf{1}$ for the identity matrix and $\mathrm{GL}_n(\mathbb{K})$ for its group of units, the general linear group. Further, we write $\mathbb{N} := \{1, 2, \cdots\}$ for the set of natural number and denote (half-)open intervals as $]a, b] := \{x \in \mathbb{R} \colon a < x \leq b\}$, $[a, b[:= \{x \in \mathbb{R} \colon a \leq x < b\}$, and $]a, b[:= \{x \in \mathbb{R} \colon a < x < b\}$.

Part I

Matrix Groups

1

Concrete Matrix Groups

In this chapter we mainly study the general linear group $\mathrm{GL}_n(\mathbb{K})$ of invertible $n \times n$-matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and introduce some of its subgroups. In particular, we discuss some of the connections between matrix groups and certain symmetry groups of geometric structures like bilinear or sesquilinear forms. In Section 1.3 we introduce also groups of matrices with entries in the quaternions $\mathbb{H}$.

1.1 The General Linear Group

We start with some notation. We write $\mathrm{GL}_n(\mathbb{K})$ for the group of invertible matrices in $M_n(\mathbb{K})$ and note that

$$\mathrm{GL}_n(\mathbb{K}) = \{g \in M_n(\mathbb{K}) \colon (\exists h \in M_n(\mathbb{K}))\, hg = gh = \mathbf{1}\}.$$

Since the invertibility of a matrix can be tested with its determinant ([La93, Prop. XIII.4.6]),

$$\mathrm{GL}_n(\mathbb{K}) = \{g \in M_n(\mathbb{K}) \colon \det g \neq 0\}.$$

This group is called the *general linear group*.

On the vector space $\mathbb{K}^n$ we consider the *euclidian norm*

$$\|x\| := \sqrt{|x_1|^2 + \ldots + |x_n|^2}, \quad x \in \mathbb{K}^n,$$

and on $M_n(\mathbb{K})$ the corresponding *operator norm*

$$\|A\| := \sup\{\|Ax\| \colon x \in \mathbb{K}^n, \|x\| \leq 1\}$$

which turns $M_n(\mathbb{K})$ into a Banach space. On every subset $S \subseteq M_n(\mathbb{K})$ we shall always consider the subspace topology inherited from $M_n(\mathbb{K})$ (otherwise we shall say so). In this sense $\mathrm{GL}_n(\mathbb{K})$ and all its subgroups carry a natural topology.

Lemma 1.1.1. *The group* $\mathrm{GL}_n(\mathbb{K})$ *has the following properties:*

(i) $\mathrm{GL}_n(\mathbb{K})$ *is open in* $M_n(\mathbb{K})$.
(ii) *The multiplication map* $m\colon \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ *and the inversion map* $\eta\colon \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ *are smooth and in particular continuous.*

Proof. (i) Since the determinant function

$$\det\colon M_n(\mathbb{K}) \to \mathbb{K}, \quad \det(a_{ij}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

is continuous and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ is open in $\mathbb{K}$, the set $\mathrm{GL}_n(\mathbb{K}) = \det^{-1}(\mathbb{K}^\times)$ is open in $M_n(\mathbb{K})$.

(ii) For $g \in \mathrm{GL}_n(\mathbb{K})$ we define $b_{ij}(g) := \det(g_{mk})_{m\neq j, k\neq i}$. According to Cramer's Rule, the inverse of g is given by

$$(g^{-1})_{ij} = \frac{(-1)^{i+j}}{\det g} b_{ij}(g).$$

The smoothness of the inversion therefore follows from the smoothness of the determinant (which is a polynomial) and the polynomial functions b_{ij} defined on $M_n(\mathbb{K})$.

For the smoothness of the multiplication map, it suffices to observe that

$$(ab)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

is the (ik)-entry in the product matrix. Since all these entries are quadratic polynomials in the entries of a and b, the product is a smooth map. □

Definition 1.1.2. A *topological group* G is a Hausdorff space G, endowed with a group structure, such that the multiplication map $m_G\colon G \times G \to G$ and the inversion map $\eta\colon G \to G$ are continuous, when $G \times G$ is endowed with the product topology.

Lemma 1.1.1(ii) says in particular that $\mathrm{GL}_n(\mathbb{K})$ is a topological group. It is clear that the continuity of group multiplication and inversion is inherited by every subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$, so that every subgroup G of $\mathrm{GL}_n(\mathbb{K})$ also is a topological group.

We write a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ also as (a_{ij}) and define

$$A^\top := (a_{ji}), \quad \overline{A} := (\overline{a_{ij}}), \quad \text{and} \quad A^* := \overline{A}^\top = (\overline{a_{ji}}).$$

Note that $A^* = A^\top$ is equivalent to $\overline{A} = A$, which means that all entries of A are real. Now we can define the most important classes of matrix groups.

Definition 1.1.3. We introduce the following notation for some important subgroups of $\mathrm{GL}_n(\mathbb{K})$:

(1) The *special linear group* : $\mathrm{SL}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon \det g = 1\}$.
(2) The *orthogonal group* : $\mathrm{O}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^\top = g^{-1}\}$.
(3) The *special orthogonal group* : $\mathrm{SO}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K})$.
(4) The *unitary group* : $\mathrm{U}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^* = g^{-1}\}$. Note that $\mathrm{U}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$, but $\mathrm{O}_n(\mathbb{C}) \neq \mathrm{U}_n(\mathbb{C})$.
(5) The *special unitary group* : $\mathrm{SU}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{U}_n(\mathbb{K})$.

One easily verifies that these are indeed groups. One simply has to use that $(ab)^\top = b^\top a^\top$, $\overline{ab} = \overline{a}\overline{b}$ and that

$$\det\colon \mathrm{GL}_n(\mathbb{K}) \to (\mathbb{K}^\times, \cdot)$$

is a group homomorphism.
We write $\mathrm{Herm}_n(\mathbb{K}) := \{A \in M_n(\mathbb{K})\colon A^* = A\}$ for the set of *hermitian matrices*. For $\mathbb{K} = \mathbb{C}$ this is not a vector subspace of $M_n(\mathbb{K})$, but it is always a real subspace. A matrix $A \in \mathrm{Herm}_n(\mathbb{K})$ is called *positive definite* if for each $0 \neq z \in \mathbb{K}^n$ we have $\langle Az, z\rangle > 0$, where

$$\langle z, w\rangle := \sum_{j=1}^n z_j \overline{w_j}$$

is the natural scalar product on $\mathbb{K}^n$. We write $\mathrm{Pd}_n(\mathbb{K}) \subseteq \mathrm{Herm}_n(\mathbb{K})$ for the subset of positive definite matrices.

Lemma 1.1.4. *The groups*

$$\mathrm{U}_n(\mathbb{C}), \quad \mathrm{SU}_n(\mathbb{C}), \quad \mathrm{O}_n(\mathbb{R}) \quad \textit{and} \quad \mathrm{SO}_n(\mathbb{R})$$

are compact.

Proof. Since all these groups are subsets of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, we have to show that they are closed and bounded.

Bounded: In view of

$$\mathrm{SO}_n(\mathbb{R}) \subseteq \mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{U}_n(\mathbb{C}) \quad \text{and} \quad \mathrm{SU}_n(\mathbb{C}) \subseteq \mathrm{U}_n(\mathbb{C}),$$

it suffices to see that $\mathrm{U}_n(\mathbb{C})$ is bounded. Let $g_1, \ldots, g_n$ denote the rows of the matrix $g \in M_n(\mathbb{C})$. Then $g^* = g^{-1}$ is equivalent to $gg^* = \mathbf{1}$, which means that $g_1, \ldots, g_n$ form an orthonormal basis for $\mathbb{C}^n$ with respect to the scalar product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ which induces the norm $\|z\| = \sqrt{\langle z, z\rangle}$. Therefore $g \in \mathrm{U}_n(\mathbb{C})$ implies $\|g_j\| = 1$ for each j, so that $\mathrm{U}_n(\mathbb{C})$ is bounded.

Closed: The functions

$$f, h\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad f(A) := AA^* - \mathbf{1} \quad \text{and} \quad h(A) := AA^\top - \mathbf{1}$$

are continuous. Therefore the groups

$$\mathrm{U}_n(\mathbb{K}) := f^{-1}(\mathbf{0}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{K}) := h^{-1}(\mathbf{0})$$

are closed. Likewise $\mathrm{SL}_n(\mathbb{K})$ is closed, and therefore the groups $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{R})$ are also closed because they are intersections of closed subsets. □

1.1.1 The Polar Decomposition

Proposition 1.1.5 (Polar decomposition). *The multiplication map*

$$m\colon \mathrm{U}_n(\mathbb{K}) \times \mathrm{Pd}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K}), \quad (u,p) \mapsto up$$

is a homeomorphism. In particular, each invertible matrix g can be written in a unique way as a product $g = up$ of a unitary matrix u and a positive definite matrix p.

Proof. We know from linear algebra that for each hermitian matrix A there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ consisting of eigenvectors of A, and that all the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are real ([La93, Thm. XV.6.4]). From that it is obvious that A is positive definite if and only if $\lambda_j > 0$ holds for each j. For a positive definite matrix A, this has two important consequences:

(1) A is invertible, and A^{-1} satisfies $A^{-1}v_j = \lambda_j^{-1}v_j$.

(2) There exists a unique positive definite matrix B with $B^2 = A$ which will be denoted $\sqrt{A}$: We define B with respect to the basis $(v_1, \ldots, v_n)$ by $Bv_j = \sqrt{\lambda_j}v_j$. Then $B^2 = A$ is obvious and since all λ_j are real and the v_j are orthonormal, B is positive definite because

$$\Big\langle B\Big(\sum_i \mu_i v_i\Big), \sum_j \mu_j v_j\Big\rangle = \sum_{i,j} \mu_i \overline{\mu_j} \langle Bv_i, v_j\rangle = \sum_{j=1}^n |\mu_j|^2 \sqrt{\lambda_j} > 0$$

for $\sum_j \mu_j v_j \neq 0$. It remains to verify the uniqueness. So assume that C is positive definite with $C^2 = A$. Then $CA = C^3 = AC$ implies that C preserves all eigenspaces of A, so that we find an orthonormal basis $w_1, \ldots, w_n$ consisting of simultaneous eigenvectors of C and A (cf. Exercise 1.1.1). If $Cw_j = \alpha_j w_j$, we have $Aw_j = \alpha_j^2 w_j$, which implies that C acts on the λ-eigenspace of A by multiplication with $\sqrt{\lambda}$, which shows that $C = B$.

From (1) we derive that the image of the map m is contained in $\mathrm{GL}_n(\mathbb{K})$.
m **is surjective:** Let $g \in \mathrm{GL}_n(\mathbb{K})$. For $0 \neq v \in \mathbb{K}^n$ we then have

$$0 < \langle gv, gv\rangle = \langle g^*gv, v\rangle,$$

showing that g^*g is positive definite. Let $p := \sqrt{g^*g}$ and define $u := gp^{-1}$. Then

$$uu^* = gp^{-1}p^{-1}g^* = gp^{-2}g^* = g(g^*g)^{-1}g^* = gg^{-1}(g^*)^{-1}g^* = \mathbf{1}$$

implies that $u \in \mathrm{U}_n(\mathbb{K})$, and it is clear that $m(u,p) = g$.
m **is injective:** If $m(u,p) = m(w,q) = g$, then $g = up = wq$ implies that

$$p^2 = p^*p = (up)^*up = g^*g = (wq)^*wq = q^2,$$

so that p and q are positive definite square roots of the same positive definite matrix g^*g, hence coincide by (2) above. Now $p = q$, and therefore $u = gp^{-1} = gq^{-1} = w$.

It remains to show that m is a homeomorphism. Its continuity is obvious, so that it remains to prove the continuity of the inverse map m^{-1}. Let $g_j = u_j p_j \to g = up$. We have to show that $u_j \to u$ and $p_j \to p$. Since $\mathrm{U}_n(\mathbb{K})$ is compact, the sequence (u_j) has a subsequence (u_{j_k}) converging to some $w \in \mathrm{U}_n(\mathbb{K})$. Then $p_{j_k} = u_{j_k}^{-1} g_{j_k} \to w^{-1}g =: q \in \mathrm{Herm}_n(\mathbb{K})$ and $g = wq$. For each $v \in \mathbb{K}^n$ we then have

$$0 \leq \langle p_{j_k} v, v\rangle \to \langle qv, v\rangle,$$

showing that all eigenvalues of q are ≥ 0. Moreover, $q = w^{-1}g$ is invertible, and therefore q is positive definite. Now $m(u, p) = m(w, q)$ yields $u = w$ and $p = q$. Since each convergent subsequence of (u_j) converges to u, the sequence itself converges to u (Exercise 1.1.9), and therefore $p_j = u_j^{-1} g_j \to u^{-1}g = p$. □

We shall see later that the set $\mathrm{Pd}_n(\mathbb{K})$ is homeomorphic to a vector space (Proposition 2.3.5), so that, topologically, the group $\mathrm{GL}_n(\mathbb{K})$ is a product of the compact group $\mathrm{U}_n(\mathbb{K})$ and a vector space. Therefore the "interesting" part of the topology of $\mathrm{GL}_n(\mathbb{K})$ is contained in the compact group $\mathrm{U}_n(\mathbb{K})$.

Remark 1.1.6 (Normal forms of unitary and orthogonal matrices). We recall some facts from linear algebra:

(a) For each $u \in \mathrm{U}_n(\mathbb{C})$, there exists an orthonormal basis $v_1, \ldots, v_n$ consisting of eigenvectors of g ([La93, Thm. XV.6.7]). This means that the unitary matrix s whose columns are the vectors $v_1, \ldots, v_n$ satisfies

$$s^{-1}us = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

where $uv_j = \lambda_j v$ and $|\lambda_j| = 1$.

The proof of this normal form is based on the existence of an eigenvector v_1 of u which in turn follows from the existence of a zero of the characteristic polynomial. Since u is unitary, it preserves the hyperplane $v_1^{\perp}$ of dimension $n-1$. Now one uses induction to obtain an orthonormal basis $v_2, \ldots, v_n$ consisting of eigenvectors.

(b) For elements of $\mathrm{O}_n(\mathbb{R})$, the situation is more complicated because real matrices do not always have real eigenvectors.

Let $A \in M_n(\mathbb{R})$ and consider it as an element on $M_n(\mathbb{C})$. We assume that A does not have a real eigenvector. Then there exists an eigenvector $z \in \mathbb{C}^n$ corresponding to some eigenvalue $\lambda \in \mathbb{C}$. We write $z = x + iy$ and $\lambda = a + ib$. Then

$$Az = Ax + iAy = \lambda z = (ax - by) + i(ay + bx).$$

Comparing real and imaginary part yields

$$Ax = ax - by \quad \text{and} \quad Ay = ay + bx.$$

Therefore the two-dimensional subspace generated by x and y in $\mathbb{R}^n$ is invariant under A.

This can be applied to $g \in \mathrm{O}_n(\mathbb{R})$ as follows. The argument above implies that there exists an invariant subspace $W_1 \subseteq \mathbb{R}^n$ with $\dim W_1 \in \{1, 2\}$. Then

$$W_1^\perp := \{v \in \mathbb{R}^n : \langle v, W_1 \rangle = \{0\}\}$$

is a subspace of dimension $n - \dim W_1$ which is also invariant (Exercise 1.1.14), and we apply induction to see that $\mathbb{R}^n$ is a direct sum of g-invariant subspaces $W_1, \ldots, W_k$ of dimension ≤ 2. Therefore the matrix g is conjugate by an orthogonal matrix s to a block matrix of the form

$$d = \mathrm{diag}(d_1, \ldots, d_k),$$

where d_j is the matrix of the restriction of the linear map corresponding to g to W_j.

To understand the structure of the d_j, we have to take a closer look at the case $n \leq 2$. For $n = 1$ the group $\mathrm{O}_1(\mathbb{R}) = \{\pm 1\}$ consists of two elements, and for $n = 2$ an element $r \in \mathrm{O}_2(\mathbb{R})$ can be written as

$$r = \begin{pmatrix} a & \mp b \\ b & \pm a \end{pmatrix} \qquad \text{with} \qquad \det r = \pm(a^2 + b^2) = \pm 1,$$

because the second column contains a unit vector orthogonal to the first one. With $a = \cos\alpha$ and $b = \sin\alpha$ we get

$$r = \begin{pmatrix} \cos\alpha & \mp\sin\alpha \\ \sin\alpha & \pm\cos\alpha \end{pmatrix}.$$

For $\det r = -1$, we obtain

$$r^2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \mathbf{1},$$

but this implies that r is an orthogonal reflection with the two eigenvalues ± 1 (Exercise 1.1.13), hence has two orthogonal eigenvectors.

In view of the preceding discussion, we may therefore assume that the first m of the matrices d_j are of the rotation form

$$d_j = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix},$$

that $d_{m+1}, \ldots, d_\ell$ are -1, and that $d_{\ell+1}, \ldots, d_n$ are 1:

$$\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 &&&&&&&&& \\ \sin\alpha_1 & \cos\alpha_1 &&&&&&&&& \\ && \ddots &&&&&&&& \\ &&& \cos\alpha_m & -\sin\alpha_m &&&&&& \\ &&& \sin\alpha_m & \cos\alpha_m &&&&&& \\ &&&&& -1 &&&&& \\ &&&&&& \ddots &&&& \\ &&&&&&& -1 &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{pmatrix}.$$

For $n = 3$ we obtain in particular the normal form

$$d = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

From this normal form we immediately read off that $\det d = 1$ is equivalent to d describing a rotation around an axis consisting of fixed points (the axis is $\mathbb{R}e_3$ for the normal form matrix).

Proposition 1.1.7. (a) *The group* $\mathrm{U}_n(\mathbb{C})$ *is arcwise connected.*

(b) *The group* $\mathrm{O}_n(\mathbb{R})$ *has the two arc components*

$$\mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{R})_- := \{g \in \mathrm{O}_n(\mathbb{R}) \colon \det g = -1\}.$$

Proof. (a) First we consider $\mathrm{U}_n(\mathbb{C})$. To see that this group is arcwise connected, let $u \in \mathrm{U}_n(\mathbb{C})$. Then there exists an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors of u (Remark 1.1.6(a)). Let $\lambda_1, \ldots, \lambda_n$ denote the corresponding eigenvalues. Then the unitarity of u implies that $|\lambda_j| = 1$, and we therefore find $\theta_j \in \mathbb{R}$ with $\lambda_j = e^{\theta_j i}$. Now we define a continuous curve

$$\gamma \colon [0,1] \to \mathrm{U}_n(\mathbb{C}), \quad \gamma(t)v_j := e^{t\theta_j i} v_j, \ j = 1, \ldots, n.$$

We then have $\gamma(0) = \mathbf{1}$ and $\gamma(1) = u$. Moreover, each $\gamma(t)$ is unitary because the basis $(v_1, \ldots, v_n)$ is orthonormal.

(b) For $g \in \mathrm{O}_n(\mathbb{R})$ we have $gg^\top = \mathbf{1}$ and therefore $1 = \det(gg^\top) = (\det g)^2$. This shows that

$$\mathrm{O}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R}) \dot\cup \mathrm{O}_n(\mathbb{R})_-$$

and both sets are closed in $\mathrm{O}_n(\mathbb{R})$ because det is continuous. Therefore $\mathrm{O}_n(\mathbb{R})$ is not connected and hence not arcwise connected. If we show that $\mathrm{SO}_n(\mathbb{R})$ is arcwise connected and $x, y \in \mathrm{O}_n(\mathbb{R})_-$, then $\mathbf{1}, x^{-1}y \in \mathrm{SO}_n(\mathbb{R})$ can be connected by an arc $\gamma \colon [0,1] \to \mathrm{SO}_n(\mathbb{R})$, and then $t \mapsto x\gamma(t)$ defines an arc $[0,1] \to \mathrm{O}_n(\mathbb{R})_-$ connecting x to y. So it remains to show that $\mathrm{SO}_n(\mathbb{R})$ is connected.

Let $g \in \mathrm{SO}_n(\mathbb{R})$. In the normal form of g discussed in Remark 1.1.6, the determinant of each two-dimensional block is 1, so that the determinant is the product of all -1-eigenvalues. Hence their number is even, and we can write each consecutive pair as a block

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

This shows that with respect to some orthonormal basis for $\mathbb{R}^n$ the linear map defined by g has a matrix of the form

$$g = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & & & & & & \\ \sin\alpha_1 & \cos\alpha_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\alpha_m & -\sin\alpha_m & & & \\ & & & \sin\alpha_m & \cos\alpha_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

Now we obtain an arc $\gamma\colon [0,1] \to \mathrm{SO}_n(\mathbb{R})$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = g$ by

$$\gamma(t) := \begin{pmatrix} \cos t\alpha_1 & -\sin t\alpha_1 & & & & & & \\ \sin t\alpha_1 & \cos t\alpha_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos t\alpha_m & -\sin t\alpha_m & & & \\ & & & \sin t\alpha_m & \cos t\alpha_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

□

Corollary 1.1.8. *The group* $\mathrm{GL}_n(\mathbb{C})$ *is arcwise connected and the group* $\mathrm{GL}_n(\mathbb{R})$ *has two arc-components given by*

$$\mathrm{GL}_n(\mathbb{R})_\pm := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon \pm \det g > 0\}.$$

Proof. If $X = A \times B$ is a product space, then the arc-components of X are the sets of the form $C \times D$, where $C \subseteq A$ and $D \subseteq B$ are arc-components (easy Exercise!). The polar decomposition of $\mathrm{GL}_n(\mathbb{K})$ yields a homeomorphism

$$\mathrm{GL}_n(\mathbb{K}) \cong \mathrm{U}_n(\mathbb{K}) \times \mathrm{Pd}_n(\mathbb{K}).$$

Since $\mathrm{Pd}_n(\mathbb{K})$ is an open convex set, it is arcwise connected (Exercise 1.1.6). Therefore the arc-components of $\mathrm{GL}_n(\mathbb{K})$ are in one-to-one correspondence with those of $\mathrm{U}_n(\mathbb{K})$ which have been determined in Proposition 1.1.7. □

1.1.2 Normal Subgroups of $\mathrm{GL}_n(\mathbb{K})$

We shall frequently need some basic concepts from group theory which we recall in the following definition.

Definition 1.1.9. Let G be a group with identity element e.

(a) A subgroup $N \subseteq G$ is called *normal* if $gN = Ng$ holds for all $g \in G$. We write this as $N \trianglelefteq G$. The normality implies that the quotient set G/N (the set of all cosets of the subgroup N) inherits a natural group structure by

$$gN \cdot hN := ghN$$

for which eN is the identity element and the quotient map $q\colon G \to G/N$ is a surjective group homomorphism with kernel $N = \ker q = q^{-1}(eN)$.

On the other hand, all kernels of group homomorphisms are normal subgroups, so that the normal subgroups are precisely those which are kernels of group homomorphisms.

It is clear that G and $\{e\}$ are normal subgroups. We call G *simple* if $G \neq \{e\}$ and these are the only normal subgroups.

(b) The subgroup $Z(G) := \{g \in G\colon (\forall x \in G)gx = xg\}$ is called the *center of* G. It obviously is a normal subgroup of G. For $x \in G$ the subgroup

$$Z_G(x) := \{g \in G\colon gx = xg\}$$

is called its *centralizer*. Note that $Z(G) = \bigcap_{x \in G} Z_G(x)$.

(c) If $G_1, \ldots, G_n$ are groups, then the product set $G := G_1 \times \ldots \times G_n$ has a natural group structure given by

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') := (g_1 g_1', \ldots, g_n g_n').$$

The group G is called the *direct product* of the groups G_j, $j = 1, \ldots, n$. We identify G_j with a subgroup of G. Then all subgroups G_j are normal subgroups and $G = G_1 \cdots G_n$.

In the following we write $\mathbb{R}_+^\times :=]0, \infty[$.

Proposition 1.1.10. (a) $Z(\mathrm{GL}_n(\mathbb{K})) = \mathbb{K}^\times \mathbf{1}$.
(b) *The multiplication map*

$$\varphi\colon (\mathbb{R}_+^\times, \cdot) \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})_+, \quad (\lambda, g) \mapsto \lambda g$$

is a homeomorphism and a group isomorphism, i.e., an isomorphism of topological groups.

Proof. (a) It is clear that $\mathbb{K}^\times \mathbf{1}$ is contained in the center of $\mathrm{GL}_n(\mathbb{K})$. To see that each matrix $g \in Z(\mathrm{GL}_n(\mathbb{K}))$ is a multiple of $\mathbf{1}$, we consider the elementary matrix $E_{ij} := (\delta_{ij})$ with the only nonzero entry 1 in position (i, j). For $i \neq j$ we then have $E_{ij}^2 = 0$, so that $(\mathbf{1} + E_{ij})(\mathbf{1} - E_{ij}) = \mathbf{1}$ implies that $T_{ij} := \mathbf{1} + E_{ij} \in \mathrm{GL}_n(\mathbb{K})$. From the relation $gT_{ij} = T_{ij}g$ we immediately get $gE_{ij} = E_{ij}g$ for $i \neq j$, so that for $k, \ell \in \{1, \ldots, n\}$ we get

$$g_{ki}\delta_{j\ell} = (gE_{ij})_{k\ell} = (E_{ij}g)_{k\ell} = \delta_{ik}g_{j\ell}.$$

For $k = i$ and $\ell = j$ we obtain $g_{ii} = g_{jj}$ and for $k = j = \ell$, we get $g_{ji} = 0$. Therefore $g = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{K}$.

(b) It is obvious that φ is a group homomorphism and that φ is continuous. Moreover, the map

$$\psi\colon\ \mathrm{GL}_n(\mathbb{R})_+ \to \mathbb{R}_+^\times \times \mathrm{SL}_n(\mathbb{R}), \quad g \mapsto ((\det g)^{\frac{1}{n}}, (\det g)^{-\frac{1}{n}} g)$$

is continuous and satisfies $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$. Hence φ is a homeomorphism. ◻

Remark 1.1.11. The subgroups

$$Z(\mathrm{GL}_n(\mathbb{K})) \quad \text{and} \quad \mathrm{SL}_n(\mathbb{K})$$

are normal subgroups of $\mathrm{GL}_n(\mathbb{K})$. Moreover, for $\mathrm{GL}_n(\mathbb{R})$ the subgroup $\mathrm{GL}_n(\mathbb{R})_+$ is a proper normal subgroup and the same holds for $\mathbb{R}_+^\times\mathbf{1}$. One can show that these examples exhaust all normal arcwise connected subgroups of $\mathrm{GL}_n(\mathbb{K})$.

Exercises for Section 1.1

Exercise 1.1.1. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$. We write $V_\lambda(A) := \ker(A - \lambda\mathbf{1})$ for the *eigenspace of* A corresponding to the eigenvalue λ and $V^\lambda(A) := \bigcup_{n\in\mathbb{N}} \ker(A - \lambda\mathbf{1})^n$ for the *generalized eigenspace of* A corresponding to λ.

(a) If $A, B \in \mathrm{End}(V)$ commute, then

$$BV^\lambda(A) \subseteq V^\lambda(A) \quad \text{and} \quad BV_\lambda(A) \subseteq V_\lambda(A)$$

holds for each $\lambda \in \mathbb{K}$.

(b) If $A \in \mathrm{End}(V)$ is diagonalizable and $W \subseteq V$ is an A-invariant subspace, then $A|_W \in \mathrm{End}(W)$ is diagonalizable.

(c) If $A, B \in \mathrm{End}(V)$ commute and both are diagonalizable, then they are simultaneously diagonalizable, i.e., there exists a basis for V which consists of eigenvectors of A and B.

(d) If $\dim V < \infty$ and $\mathcal{A} \subseteq \mathrm{End}(V)$ is a commuting set of diagonalizable endomorphisms, then $\mathcal{A}$ can be simultaneously diagonalized, i.e., V is a direct sum of simultaneous eigenspaces of $\mathcal{A}$.

(e) For any function $\lambda\colon \mathcal{A} \to V$, we write $V_\lambda(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} V_{\lambda(a)}(a)$ for the corresponding simultaneous eigenspace. Show that the sum $\sum_\lambda V_\lambda(\mathcal{A})$ is direct.

(f) If $\mathcal{A} \subseteq \mathrm{End}(V)$ is a finite commuting set of diagonalizable endomorphisms, then $\mathcal{A}$ can be simultaneously diagonalized.

(g) Find a commuting set of diagonalizable endomorphisms of a vector space V which cannot be diagonalized simultaneously.

Exercise 1.1.2. Let G be a topological group. Let G_0 be the *identity component*, i.e., the the connected component of the identity in G. Show that G_0 is an open and closed normal subgroup of G.

Exercise 1.1.3. $\mathrm{SO}_n(\mathbb{K})$ is a closed normal subgroup of $\mathrm{O}_n(\mathbb{K})$ of index 2 and, for every $g \in \mathrm{O}_n(\mathbb{K})$ with $\det(g) = -1$,

$$\mathrm{O}_n(\mathbb{K}) = \mathrm{SO}_n(\mathbb{K}) \cup g\,\mathrm{SO}_n(\mathbb{K})$$

is a disjoint decomposition.

Exercise 1.1.4. For each subset $M \subseteq M_n(\mathbb{K})$ the *centralizer*

$$Z_{\mathrm{GL}_n(\mathbb{K})}(M) := \{g \in \mathrm{GL}_n(\mathbb{K}) : (\forall m \in M) gm = mg\}$$

is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$.

Exercise 1.1.5. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the map $z = x+iy \mapsto (x, y)$ and write $I(x, y) := (-y, x)$ for the real linear endomorphism of $\mathbb{R}^{2n}$ corresponding to multiplication with i. Then

$$\mathrm{GL}_n(\mathbb{C}) \cong Z_{\mathrm{GL}_{2n}(\mathbb{R})}(\{I\})$$

yields a realization of $\mathrm{GL}_n(\mathbb{C})$ as a closed subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$.

Exercise 1.1.6. A subset C of a real vector space V is called a *convex cone* if C is convex and $\lambda C \subseteq C$ for each $\lambda > 0$.

Show that $\mathrm{Pd}_n(\mathbb{K})$ is an open convex cone in $\mathrm{Herm}_n(\mathbb{K})$.

Exercise 1.1.7. Show that

$$\gamma : (\mathbb{R}, +) \to \mathrm{GL}_2(\mathbb{R}), \quad t \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

is a continuous group homomorphism with $\gamma(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\operatorname{im} \gamma = \mathrm{SO}_2(\mathbb{R})$.

Exercise 1.1.8. Show that the group $\mathrm{O}_n(\mathbb{C})$ is homeomorphic to the topological product of the subgroup

$$\mathrm{O}_n(\mathbb{R}) \cong \mathrm{U}_n(\mathbb{C}) \cap \mathrm{O}_n(\mathbb{C}) \quad \text{and the set} \quad \mathrm{Pd}_n(\mathbb{C}) \cap \mathrm{O}_n(\mathbb{C}).$$

Exercise 1.1.9. Let (X, d) be a compact metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Show that $\lim_{n\to\infty} x_n = x$ is equivalent to the condition that each convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to x.

Exercise 1.1.10. If $A \in \mathrm{Herm}_n(\mathbb{K})$ satisfies $\langle Av, v\rangle = 0$ for each $v \in \mathbb{K}^n$, then $A = 0$.

Exercise 1.1.11. Show that for a complex matrix $A \in M_n(\mathbb{C})$ the following are equivalent:
(1) $A^* = A$.
(2) $\langle Av, v\rangle \in \mathbb{R}$ for each $v \in \mathbb{C}^n$.

Exercise 1.1.12. (a) Show that a matrix $A \in \mathrm{Herm}_n(\mathbb{K})$ is hermitian if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ with $Av_j = \lambda_j v_j$.

(b) Show that a complex matrix $A \in M_n(\mathbb{C})$ is unitary if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ and $Av_j = \lambda_j v_j$.

(c) Show that a complex matrix $A \in M_n(\mathbb{C})$ is *normal*, i.e., satisfies $A^*A = AA^*$, if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and $\lambda_j \in \mathbb{C}$ with $Av_j = \lambda_j v_j$.

Exercise 1.1.13. (a) Let V be a vector space and $\mathbf{1} \neq A \in \mathrm{End}(V)$ with $A^2 = \mathbf{1}$ (A is called an *involution*). Show that

$$V = \ker(A - \mathbf{1}) \oplus \ker(A + \mathbf{1}).$$

(b) Let V be a vector space and $A \in \mathrm{End}(V)$ with $A^3 = A$. Show that

$$V = \ker(A - \mathbf{1}) \oplus \ker(A + \mathbf{1}) \oplus \ker A.$$

(c) Let V be a vector space and $A \in \mathrm{End}(V)$ an endomorphism for which there exists a polynomial p of degree n with n different zeros $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $p(A) = 0$. Show that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Exercise 1.1.14. Let $\beta\colon V \times V \to \mathbb{K}$ be a bilinear map and $g : V \to V$ with $\beta(gv, gw) = \beta(v, w)$ be a β-isometry. For a subspace $E \subseteq V$ we write

$$E^\perp := \{v \in V\colon (\forall w \in E)\ \beta(v, w) = 0\}$$

for its *orthogonal space*. Show that $g(E) = E$ implies that $g(E^\top) = E^\top$.

Exercise 1.1.15 (Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$). Let

$$T_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$$

denote the subgroup of upper triangular matrices with positive diagonal entries. Show that the multiplication map

$$\mu\colon\ \mathrm{O}_n(\mathbb{R}) \times T_n^+(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (a, b) \mapsto ab$$

is a homeomorphism.

Exercise 1.1.16. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$. Show that

$$Z(M_n(\mathbb{K})) := \{z \in M_n(\mathbb{K})\colon (\forall x \in M_n(\mathbb{K}))\, zx = xz\} = \mathbb{K}\mathbf{1}.$$

1.2 Groups and Geometry

In Definition 1.1.3 we have defined certain matrix groups by concrete conditions on the matrices. If we think of matrices as linear maps described with respect to a basis, we have to adopt a more abstract point of view. Similarly, one can study symmetry groups of bilinear forms on a vector space V without fixing a certain basis a priori. Actually it is much more convenient to choose a basis for which the structure of the bilinear form is as simple as possible.

1.2.1 Isometry Groups

Definition 1.2.1 (Groups and bilinear forms).

(a) (The abstract general linear group) Let V be a $\mathbb{K}$-vector space. We write $\mathrm{GL}(V)$ for the group of linear automorphisms of V. This is the group of invertible elements in the ring $\mathrm{End}(V)$ of all linear endomorphisms of V.

If V is an n-dimensional $\mathbb{K}$-vector space and $v_1, \ldots, v_n$ is a basis for V, then the map

$$\Phi\colon M_n(\mathbb{K}) \to \mathrm{End}(V), \quad \Phi(A)v_k := \sum_{j=1}^{n} a_{jk} v_j$$

is a linear isomorphism which describes the passage between linear maps and matrices. In view of $\Phi(\mathbf{1}) = \mathrm{id}_V$ and $\Phi(AB) = \Phi(A)\Phi(B)$, we obtain a group isomorphism

$$\Phi|_{\mathrm{GL}_n(\mathbb{K})}\colon\ \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}(V).$$

(b) Let V be an n-dimensional vector space with basis $v_1, \ldots, v_n$ and $\beta\colon V \times V \to \mathbb{K}$ a bilinear map. Then $B = (b_{jk}) := (\beta(v_j, v_k))_{j,k=1,\ldots,n}$ is an $(n \times n)$-matrix, but this matrix should NOT be interpreted as the matrix of a linear map. It is the matrix of a bilinear map to $\mathbb{K}$, which is something different. It describes β in the sense that

$$\beta\Big(\sum_j x_j v_j, \sum_k y_k v_k\Big) = \sum_{j,k=1}^{n} x_j b_{jk} y_k = x^\top B y,$$

where $x^\top B y$ with column vectors $x, y \in \mathbb{K}^n$ is viewed as a matrix product whose result is a (1×1)-matrix, i.e., an element of $\mathbb{K}$.

We write

$$\mathrm{Aut}(V, \beta) := \{g \in \mathrm{GL}(V)\colon (\forall v, w \in V)\, \beta(gv, gw) = \beta(v, w)\}$$

for the *isometry group of the bilinear form* β. Then it is easy to see that

$$\Phi^{-1}(\mathrm{Aut}(V, \beta)) = \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^\top B g = B\}.$$

If β is symmetric, we also write $\mathrm{O}(V, \beta) := \mathrm{Aut}(V, \beta)$ and if β is skew-symmetric, we write $\mathrm{Sp}(V, \beta) := \mathrm{Aut}(V, \beta)$.

If $v_1, \ldots, v_n$ is an orthonormal basis for β, i.e., $B = \mathbf{1}$, then

$$\Phi^{-1}(\mathrm{Aut}(V, \beta)) = \mathrm{O}_n(\mathbb{K})$$

is the orthogonal group defined in Section 1.1. Note that orthonormal bases can only exist for symmetric bilinear forms (Why?).

For $V = \mathbb{K}^{2n}$ and the block (2×2)-matrix

$$B := \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix}$$

we see that $B^\top = -B$, and the group

$$\mathrm{Sp}_{2n}(\mathbb{K}) := \{g \in \mathrm{GL}_{2n}(\mathbb{K}) \colon g^\top B g = B\}$$

is called the *symplectic group*. The corresponding skew-symmetric bilinear form on $\mathbb{K}^{2n}$ is given by

$$\beta(x, y) = x^\top B y = \sum_{i=1}^n x_i y_{n+i} - x_{n+i} y_i.$$

(c) A symmetric bilinear form β on V is called *nondegenerate* if $\beta(v, V) = \{0\}$ implies $v = 0$. For $\mathbb{K} = \mathbb{C}$ every nondegenerate symmetric bilinear form β possesses an orthonormal basis (this builds on the existence of square roots of nonzero complex numbers; see Exercise 1.2.1), so that for every such form β we get

$$\mathrm{O}(V, \beta) \cong \mathrm{O}_n(\mathbb{C}).$$

For $\mathbb{K} = \mathbb{R}$ the situation is more complicated, since negative real numbers do not have a square root in $\mathbb{R}$. There might not be an orthonormal basis, but if β is nondegenerate, there always exists an orthogonal basis $v_1, \ldots, v_n$ and $p \in \{1, \ldots, n\}$ such that $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, n$. Let $q := n - p$ and $I_{p,q}$ denote the corresponding matrix

$$I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_{p+q}(\mathbb{R}).$$

Then $\mathrm{O}(V, \beta)$ is isomorphic to the group

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon g^\top I_{p,q} g = I_{p,q}\},$$

where $\mathrm{O}_{n,0}(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$.

(d) Let V be an n-dimensional complex vector space and $\beta \colon V \times V \to \mathbb{C}$ a sesquilinear form, i.e., β is linear in the first and antilinear in the second argument. Then we also choose a basis $v_1, \ldots, v_n$ in V and define $B = (b_{jk}) := (\beta(v_j, v_k))_{j,k=1,\ldots,n}$, but now we obtain

$$\beta\Big(\sum_j x_j v_j, \sum_k y_k v_k\Big) = \sum_{j,k=1}^n x_j b_{jk} \overline{y_k} = x^\top B \overline{y}.$$

We write

$$\mathrm{U}(V,\beta) := \{g \in \mathrm{GL}(V) \colon (\forall v, w \in V)\, \beta(gv, gw) = \beta(v,w)\}$$

for the corresponding *unitary group* and find

$$\Phi^{-1}(\mathrm{U}(V,\beta)) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^\top B \overline{g} = B\}.$$

If $v_1, \ldots, v_n$ is an orthonormal basis for β, i.e., $B = \mathbf{1}$, then

$$\Phi^{-1}(\mathrm{U}(V,\beta)) = \mathrm{U}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* = g^{-1}\}$$

is the unitary group over $\mathbb{C}$. We call β *hermitian* if it is sesquilinear and satisfies $\beta(y,x) = \overline{\beta(x,y)}$. In this case one has to face the same problems as for symmetric forms on real vector spaces, but there always exists an orthogonal basis $v_1, \ldots, v_n$ and $p \in \{1, \ldots, n\}$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, n$. With $q := n - p$ and

$$I_{p,q} := \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_n(\mathbb{C})$$

we then define the *indefinite unitary groups* by

$$\mathrm{U}_{p,q}(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^\top I_{p,q} \overline{g} = I_{p,q}\}.$$

Since $I_{p,q}$ has real entries,

$$\mathrm{U}_{p,q}(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* I_{p,q} g = I_{p,q}\},$$

where $\mathrm{U}_{n,0}(\mathbb{C}) = \mathrm{U}_n(\mathbb{C})$.

Definition 1.2.2. (a) Let V be a vector space. We consider the *affine group* $\mathrm{Aff}(V)$ of all maps $V \to V$ of the form

$$\varphi_{v,g}(x) = gx + v, \quad g \in \mathrm{GL}(V), v \in V.$$

We write elements $\varphi_{v,g}$ of $\mathrm{Aff}(V)$ simply as pairs (v,g). Then the composition in $\mathrm{Aff}(V)$ is given by

$$(v,g)(w,h) = (v + gw, gh),$$

$(0, \mathbf{1})$ is the identity, and inversion is given by

$$(v,g)^{-1} = (-g^{-1}v, g^{-1}).$$

For $V = \mathbb{K}^n$ we put $\mathrm{Aff}_n(\mathbb{K}) := \mathrm{Aff}(\mathbb{K}^n)$. Then the map

$$\Phi\colon\ \mathrm{Aff}_n(\mathbb{K}) \to \mathrm{GL}_{n+1}(\mathbb{K}), \quad \Phi(v,g) = \begin{pmatrix} [g] & v \\ 0 & 1 \end{pmatrix}$$

is an injective group homomorphism, where $[g]$ denotes the matrix of the linear map with respect to the canonical basis for $\mathbb{K}^n$.

(b) (The euclidian isometry group) Let $V = \mathbb{R}^n$ and consider the euclidian metric $d(x, y) := \|x - y\|_2$ on $\mathbb{R}^n$. We define

$$\mathrm{Iso}_n(\mathbb{R}) := \{g \in \mathrm{Aff}(\mathbb{R}^n) \colon (\forall x, y \in V)\ d(gv, gw) = d(v, w)\}.$$

This is the group of *affine isometries* of the euclidian n-space. Actually one can show that every isometry of a normed space $(V, \|\cdot\|)$ is an affine map (Exercise 1.2.5). This implies that

$$\mathrm{Iso}_n(\mathbb{R}) = \{g \colon \mathbb{R}^n \to \mathbb{R}^n \colon (\forall x, y \in \mathbb{R}^n)\ d(gv, gw) = d(v, w)\}.$$

1.2.2 Semidirect Products

We have seen in Definition 1.1.9 how to form direct products of groups. If $G = G_1 \times G_2$ is a direct product of the groups G_1 and G_2, then we identify G_1 and G_2 with the corresponding subgroups of $G_1 \times G_2$, i.e., we identify $g_1 \in G_1$ with (g_1, e) and $g_2 \in G_2$ with (e, g_2). Then G_1 and G_2 are normal subgroups of G and the product map

$$m \colon G_1 \times G_2 \to G, \quad (g_1, g_2) \mapsto g_1 g_2 = (g_1, g_2)$$

is a group isomorphism, i.e., each element $g \in G$ has a unique decomposition $g = g_1 g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$.

The affine group $\mathrm{Aff}(V)$ has a structure which is similar. The translation group $V \cong \{(v, \mathbf{1}) \colon v \in V\}$ and the linear group $\mathrm{GL}(V) \cong \{0\} \times \mathrm{GL}(V)$ are subgroups, and each element (v, g) has a unique representation as a product $(v, \mathbf{1})(0, g)$, but in this case $\mathrm{GL}(V)$ is not a normal subgroup, whereas V is normal. The following lemma introduces a concept that is important to understand the structure of groups which have similar decompositions.

In the following we write $\mathrm{Aut}(G)$ for the set of automorphisms of the group G and note that this set is a group under composition of maps. In particular the inverse of a group automorphism is an automorphism.

Lemma 1.2.3. (a) *Let N and H be groups, write* $\mathrm{Aut}(N)$ *for the group of all automorphisms of N, and suppose that $\delta \colon H \to \mathrm{Aut}(N)$ is a group homomorphism. Then we define a multiplication on $N \times H$ by*

$$(n, h)(n', h') := (n\delta(h)(n'), hh'). \tag{1.1}$$

This multiplication turns $N \times H$ into a group denoted by $N \rtimes_\delta H$, where $N \cong N \times \{e\}$ is a normal subgroup, $H \cong \{e\} \times H$ is a subgroup, and each element $g \in N \rtimes_\delta H$ has a unique representation as $g = nh$, $n \in N$, $h \in H$.

(b) *If, conversely, G is a group, $N \trianglelefteq G$ a normal subgroup and $H \subseteq G$ a subgroup with the property that the multiplication map $m \colon N \times H \to G$ is bijective, i.e., $NH = G$ and $N \cap H = \{e\}$, then*

$$\delta\colon H \to \mathrm{Aut}(N), \quad \delta(h)(n) := hnh^{-1} \tag{1.2}$$

is a group homomorphism, and the map

$$m\colon N \rtimes_\delta H \to G, \quad (n,h) \mapsto nh$$

is a group isomorphism.

Proof. (a) We have to verify the associativity of the multiplication and the existence of an inverse. The associativity follows from

$$\begin{aligned}
&\big((n,h)(n',h')\big)(n'',h'')\\
&= (n\delta(h)(n'), hh')(n'',h'') = (n\delta(h)(n')\delta(hh')(n''), hh'h'')\\
&= (n\delta(h)(n')\delta(h)\big(\delta(h')(n'')\big), hh'h'') = (n\delta(h)\big(n'\delta(h')(n'')\big), hh'h'')\\
&= (n,h)(n'\delta(h')(n''), h'h'') = (n,h)\big((n',h')(n'',h'')\big).
\end{aligned}$$

With (1.1) we immediately get the formula for the inverse

$$(n,h)^{-1} = (\delta(h^{-1})(n^{-1}), h^{-1}). \tag{1.3}$$

(b) Since

$$\delta(h_1h_2)(n) = h_1h_2n(h_1h_2)^{-1} = h_1(h_2nh_2^{-1})h_1^{-1} = \delta(h_1)\delta(h_2)(n),$$

the map $\delta\colon H \to \mathrm{Aut}(N)$ is a group homomorphism. Moreover, the multiplication map m satisfies

$$m(n,h)m(n',h') = nhn'h' = (nhn'h^{-1})hh' = m((n,h)(n',h')),$$

hence is a group homomorphism. It is bijective by assumption. □

Definition 1.2.4. The group $N \rtimes_\delta H$ constructed in Lemma 1.2.3 from the data (N,H,δ) is called the *semidirect product* of N and H with respect to δ. If it is clear from the context what δ is, then we simply write $N \rtimes H$ instead of $N \rtimes_\delta H$.

If δ is trivial, i.e., $\delta(h) = \mathrm{id}_N$ for each $h \in H$, then $N \rtimes_\delta H \cong N \times H$ is a direct product. In this sense semidirect products generalize direct products. Below we shall see several concrete examples of groups which can most naturally be described as semidirect products of known groups.

One major point in studying semidirect products is that for any normal subgroup $N \trianglelefteq G$, we think of the groups N and G/N as building blocks of the group G. For each semidirect product $G = N \rtimes H$ we have $G/N \cong H$, so that the two building blocks N and $G/N \cong H$ are the same, although the groups might be quite different, f.i. $\mathrm{Aff}(V)$ and $V \times \mathrm{GL}(V)$ are very different groups: In the latter group $N = V \times \{\mathbf{1}\}$ is a central subgroup and in the first group it is not. On the other hand there are situations where G cannot be build from N and $H := G/N$ as a semidirect product. This works if and only if there exists a group homomorphism $\sigma\colon G/N \to G$ with $\sigma(gN) \in gN$ for each $g \in G$. An example where such a homomorphism does not exist is

$$G = C_4 := \{z \in \mathbb{C}^\times : z^4 = 1\} \quad \text{and} \quad N := C_2 := \{z \in \mathbb{C}^\times : z^2 = 1\} \trianglelefteq G.$$

In this case $G \ncong N \rtimes H$ for any group H because then $H \cong G/N \cong C_2$, so that the fact that G is abelian would lead to $G \cong C_2 \times C_2$, contradicting the existence of elements of order 4 in G.

Example 1.2.5. (a) We know already the following examples of semidirect products from Definition 1.2.2: The affine group $\mathrm{Aff}(V)$ of a vector space is isomorphic to the semidirect product

$$\mathrm{Aff}(V) \cong V \rtimes_\delta \mathrm{GL}(V), \quad \delta(g)(v) = gv.$$

Similarly, we have

$$\mathrm{Aff}_n(\mathbb{R}) \cong \mathbb{R}^n \rtimes_\delta \mathrm{GL}_n(\mathbb{R}), \quad \delta(g)(v) = gv.$$

We furthermore have the subgroup $\mathrm{Iso}_n(\mathbb{R})$, which, in view of

$$\mathrm{O}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall x \in \mathbb{R}^n)\|gx\| = \|x\|\}$$

(cf. Exercise 1.2.6) satisfies

$$\mathrm{Iso}_n(\mathbb{R}) \cong \mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R}).$$

The group of *euclidian motions of* $\mathbb{R}^n$ is the subgroup

$$\mathrm{Mot}_n(\mathbb{R}) := \mathbb{R}^n \rtimes \mathrm{SO}_n(\mathbb{R})$$

of those isometries preserving orientation.

(b) For each group G we can form the semidirect product group

$$G \rtimes_\delta \mathrm{Aut}(G), \quad \delta(\varphi)(g) = \varphi(g).$$

Example 1.2.6 (The concrete Galilei[1] group). We consider the vector space

$$M := \mathbb{R}^4 \cong \mathbb{R}^3 \times \mathbb{R}$$

as the space of pairs (q, t) describing *events* in a four-dimensional (nonrelativistic) *spacetime*. Here q stands for the spatial coordinate of the event and t for the (absolute) time of the event. The set M is called *Galilei spacetime*. There are three types of symmetries of this spacetime:

[1] Galileo Galilei (1564–1642), was an italian mathematician and philosopher. He held professorships in Pisa and Padua, later he worked at the court in Florence. The Galilei group is the symmetry group of nonrelativistic kinematics in three dimensions.

(1) The special Galilei transformations:

$$G_v \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (q+vt,t) = \begin{pmatrix} \mathbf{1} & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ t \end{pmatrix}$$

describing movements with constant velocity v.
(2) Rotations:

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (Aq,t), \quad A \in \mathrm{SO}_3(\mathbb{R}),$$

(3) Space translations

$$T_v \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (q+v,t),$$

and time translations

$$T_\beta \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}, \quad (q,t) \mapsto (q,t+\beta).$$

All these maps are affine maps on $\mathbb{R}^4$. The subgroup $\Gamma \subseteq \mathrm{Aff}_4(\mathbb{R})$ generated by the maps in (1), (2) and (3) is called the *proper (orthochrone) Galilei group*. The full *Galilei group* Γ_{ext} is obtained if we add the time reversion $T(q,t) := (q,-t)$ and the space reflection $S(q,t) := (-q,t)$. Both are not contained in Γ.

Roughly stated, *Galilei's relativity principle* states that *the basic physical laws of closed systems are invariant under transformations of the proper Galilei group* (see [Sch95], Sect. II.2, for more information on this perspective). It means that Γ is the natural symmetry group of nonrelativistic mechanics.

To describe the structure of the group Γ, we first observe that by (3) it contains the subgroup $\Gamma_t \cong (\mathbb{R}^4,+)$ of all spacetime translations. The maps under (1) and (2) are linear maps on $\mathbb{R}^4$. They generate the group

$$\Gamma_\ell := \{(v,A) \colon A \in \mathrm{SO}_3(\mathbb{R}), v \in \mathbb{R}^3\},$$

where we write (v,A) for the affine map given by $(q,t) \mapsto (Aq+vt,t)$. The composition of two such maps is given by

$$(v,A).\big((v',A').(q,t)\big) = (A(A'q+v't)+vt,t) = (AA'q+(Av'+v)t,t),$$

so that the product in Γ_ℓ is

$$(v,A)(v',A') = (v+Av',AA').$$

We conclude that

$$\Gamma_\ell \cong \mathbb{R}^3 \rtimes \mathrm{SO}_3(\mathbb{R})$$

is isomorphic to the group $\mathrm{Mot}_3(\mathbb{R})$ of motions of euclidian space. We thus obtain the description

$$\Gamma \cong \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \mathrm{SO}_3(\mathbb{R})) \cong \mathbb{R}^4 \rtimes \mathrm{Mot}_3(\mathbb{R}),$$

where $\mathrm{Mot}_3(\mathbb{R})$ acts on $\mathbb{R}^4$ by $(v, A).(q, t) := (Aq + vt, t)$, which corresponds to the natural embedding $\mathrm{Aff}_3(\mathbb{R}) \to \mathrm{GL}_4(\mathbb{R})$ discussed in Example 1.2.2.

For the extended Galilei group one easily obtains

$$\Gamma_{\mathrm{ext}} \cong \Gamma \rtimes \{S, T, ST, \mathbf{1}\} \cong \Gamma \rtimes (C_2 \times C_2),$$

because the group $\{S, T, ST, \mathbf{1}\}$ generated by S and T is a four element group intersecting the normal subgroup Γ trivially. Therefore the description as a semidirect product follows from the second part of Lemma 1.2.3.

Example 1.2.7 (The concrete Poincaré group). In the preceding example we have viewed four-dimensional spacetime as a product of space $\mathbb{R}^3$ with time $\mathbb{R}$. This picture changes if one wants to incorporate special relativity. Here the underlying spacetime is *Minkowski space*, which is $M = \mathbb{R}^4$, endowed with the *Lorentz form*

$$\beta(x, y) := x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$

The group

$$L := \mathrm{O}_{3,1}(\mathbb{R}) \cong \mathrm{O}(\mathbb{R}^4, \beta)$$

is called the *Lorentz group*. This is the symmetry group of relativistic (classical) mechanics.

The Lorentz group has several important subgroups:

$$L_+ := \mathrm{SO}_{3,1}(\mathbb{R}) := L \cap \mathrm{SL}_4(\mathbb{R}) \quad \text{and} \quad L^\uparrow := \{g \in L \colon g_{44} \geq 1\}.$$

The condition $g_{44} \geq 1$ comes from the observation that for $e_4 = (0, 0, 0, 1)^\top$ we have

$$-1 = \beta(e_4, e_4) = \beta(ge_4, ge_4) = g_{14}^2 + g_{24}^2 + g_{34}^2 - g_{44}^2,$$

so that $g_{44}^2 \geq 1$. Therefore either $g_{44} \geq 1$ or $g_{44} \leq -1$. To understand geometrically why $L^\uparrow$ is a subgroup, we consider the quadratic form

$$q(x) := \beta(x, x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$$

on $\mathbb{R}^4$. Since q is invariant under L, the action of the group L on $\mathbb{R}^4$ preserves the double cone

$$C := \{x \in \mathbb{R}^4 \colon q(x) \leq 0\} = \{x \in \mathbb{R}^4 \colon |x_4| \geq \|(x_1, x_2, x_3)\|\}.$$

Let

$$C_\pm := \{x \in C \colon \pm x_4 \geq 0\} = \{x \in \mathbb{R}^4 \colon \pm x_4 \geq \|(x_1, x_2, x_3)\|\}.$$

Then $C = C_+ \cup C_-$ with $C_+ \cap C_- = \{0\}$ and the sets $C_\pm$ are both convex cones, as follows easily from the convexity of the norm function on $\mathbb{R}^3$ (Exercise). Each element $g \in L$ preserves the set $C \setminus \{0\}$ which has the two arc-components

$C_\pm \setminus \{0\}$. The continuity of the map $g\colon C \setminus \{0\} \to C \setminus \{0\}$ now implies that we have two possibilities. Either $gC_+ = C_+$ or $gC_+ = C_-$. In the first case, $g_{44} \geq 1$ and in the latter case $g_{44} \leq -1$.

In the physical literature one sometimes finds $\mathrm{SO}_{3,1}(\mathbb{R})$ as the notation for $L_+^\uparrow := L_+ \cap L^\uparrow$, which is inconsistent with the standard notation for matrix groups.

The *(proper) Poincaré group* is the corresponding affine group

$$P := \mathbb{R}^4 \rtimes L_+^\uparrow.$$

This group is the identity component of the *inhomogeneous Lorentz group* $\mathbb{R}^4 \rtimes L$. Some people use the name Poincaré group only for the universal covering group $\widetilde{P}$ of P which is isomorphic to $\mathbb{R}^4 \rtimes \mathrm{SL}_2(\mathbb{C})$, as we shall see below in Example 8.5.16(3).

The topological structure of the Poincaré- and Lorentz group will become more transparent when we have refined information on the polar decomposition obtained from the exponential function (Example 3.3.4). Then we shall see that the Lorentz group L has four arc-components

$$L_+^\uparrow, \quad L_+^\downarrow, \quad L_-^\uparrow \quad \text{and} \quad L_-^\downarrow,$$

where

$$L_\pm := \{g \in L\colon \det g = \pm 1\}, \quad L^\downarrow := \{g \in L\colon g_{44} \leq -1\}$$

and

$$L_\pm^\uparrow := L_\pm \cap L^\uparrow, \quad L_\pm^\downarrow := L_\pm \cap L^\downarrow.$$

Exercises for Section 1.2

Exercise 1.2.1. (a) Let β be a symmetric bilinear form on a finite-dimensional complex vector space V. Show that there exists an orthogonal basis $v_1, \ldots, v_n$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = 0$ for $j > p$.

(b) Show that each invertible symmetric matrix $B \in \mathrm{GL}_n(\mathbb{C})$ can be written as $B = AA^\top$ for some $A \in \mathrm{GL}_n(\mathbb{C})$.

Exercise 1.2.2. Let β be a symmetric bilinear form on a finite-dimensional real vector space V. Show that there exists an orthogonal basis $v_1, \ldots, v_{p+q}$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$, $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, p+q$, and $\beta(v_j, v_j) = 0$ for $j > p+q$.

Exercise 1.2.3. Let β be a skew-symmetric bilinear form on a finite-dimensional vector space V which is nondegenerate in the sense that $\beta(v, V) = \{0\}$ implies $v = 0$. Show that there exists a basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ of V with

$$\beta(v_i, w_j) = \delta_{ij} \quad \text{and} \quad \beta(v_i, v_j) = \beta(w_i, w_j) = 0.$$

Exercise 1.2.4 (Metric characterization of midpoints). Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ distinct points. Let

$$M_0 := \{z \in X : \|z - x\| = \|z - y\| = \tfrac{1}{2}\|x - y\|\} \quad \text{and} \quad m := \frac{x+y}{2}.$$

For a subset $A \subseteq X$ we define its *diameter*

$$\delta(A) := \sup\{\|a - b\| : a, b \in A\}.$$

Show that:

(1) If X is a pre-Hilbert space (i.e., a vector space with a hermitian scalar product), then $M_0 = \{m\}$ is a one-element set.
(2) $\|z - m\| \leq \frac{1}{2}\delta(M_0) \leq \frac{1}{2}\|x - y\|$ for $z \in M_0$.
(3) For $n \in \mathbb{N}$ we define inductively:

$$M_n := \{p \in M_{n-1} : (\forall z \in M_{n-1})\ \|z - p\| \leq \tfrac{1}{2}\delta(M_{n-1})\}.$$

Then, for each $n \in \mathbb{N}$:
(a) M_n is a convex set.
(b) M_n is invariant under the point reflection $s_m(a) := 2m - a$ in m.
(c) $m \in M_n$.
(d) $\delta(M_n) \leq \frac{1}{2}\delta(M_{n-1})$.
(4) $\bigcap_{n \in \mathbb{N}} M_n = \{m\}$.

Exercise 1.2.5 (Isometries of normed spaces are affine maps). Let $(X, \|\cdot\|)$ be a normed space endowed with the metric $d(x, y) := \|x - y\|$. Show that each isometry $\varphi\colon (X, d) \to (X, d)$ is an affine map by using the following steps:

(1) It suffices to assume that $\varphi(0) = 0$ and to show that this implies that φ is a linear map.
(2) $\varphi(\frac{x+y}{2}) = \frac{1}{2}(\varphi(x) + \varphi(y))$ for $x, y \in X$.
(3) φ is continuous.
(4) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in 2^{\mathbb{Z}} \subseteq \mathbb{R}$.
(5) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for $x, y \in X$.
(6) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{R}$.

Exercise 1.2.6. Let $\beta\colon V \times V \to V$ be a symmetric bilinear form on the vector space V and

$$q\colon V \to V, \quad v \mapsto \beta(v, v)$$

the corresponding quadratic form. Then for $\varphi \in \operatorname{End}(V)$ the following are equivalent:

(1) $(\forall v \in V)\ q(\varphi(v)) = q(v)$.
(2) $(\forall v, w \in V)\ \beta(\varphi(v), \varphi(w)) = \beta(v, w)$.

Exercise 1.2.7. We consider $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where the elements of $\mathbb{R}^4$ are considered as space time events (q, t), $q \in \mathbb{R}^3$, $t \in \mathbb{R}$. On $\mathbb{R}^4$ we have the linear (time) functional

$$\Delta \colon \mathbb{R}^4 \to \mathbb{R}, (x, t) \mapsto t$$

and we endow $\ker \Delta \cong \mathbb{R}^3$ with the euclidian scalar product

$$\beta(x, y) := x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Show that

$$H := \{g \in \mathrm{GL}_4(\mathbb{R}) \colon g \ker \Delta \subseteq \ker \Delta, g|_{\ker \Delta} \in \mathrm{O}_3(\mathbb{R})\} \cong \mathbb{R}^3 \rtimes (\mathrm{O}_3(\mathbb{R}) \times \mathbb{R}^\times)$$

and

$$G := \{g \in H \colon \Delta \circ g = \Delta\} \cong \mathbb{R}^3 \rtimes \mathrm{O}_3(\mathbb{R}).$$

In this sense the linear part of the Galilei group (extended by the space reflection S) is isomorphic to the symmetry group of the triple $(\mathbb{R}^4, \beta, \Delta)$, where Δ represents a universal time function and β is the scalar product on $\ker \Delta$. In the relativistic picture (Example 1.2.7), the time function is combined with the scalar product in the Lorentz form.

Exercise 1.2.8. On the four-dimensional real vector space $V := \mathrm{Herm}_2(\mathbb{C})$ we consider the symmetric bilinear form β given by

$$\beta(A, B) := \mathrm{tr}(AB) - \mathrm{tr}\, A \,\mathrm{tr}\, B.$$

Show that:

(1) The corresponding quadratic form is given by $q(A) := \beta(A, A) = -2 \det A$.
(2) Show that $(V, \beta) \cong \mathbb{R}^{3,1}$ by finding a basis $E_1, \ldots, E_4$ of $\mathrm{Herm}_2(\mathbb{C})$ with

$$q(a_1 E_1 + \ldots + a_4 E_4) = a_1^2 + a_2^2 + a_3^2 - a_4^2.$$

(3) For $g \in \mathrm{GL}_2(\mathbb{C})$ and $A \in \mathrm{Herm}_2(\mathbb{C})$ the matrix gAg^* is hermitian and satisfies

$$q(gAg^*) = |\det(g)|^2 q(A).$$

(4) For $g \in \mathrm{SL}_2(\mathbb{C})$ we define a linear map $\rho(g) \in \mathrm{GL}(\mathrm{Herm}_2(\mathbb{C}))$ by $\rho(g)(A) := gAg^*$. Then we obtain a homomorphism

$$\rho \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R}).$$

(5) Show that $\ker \rho = \{\pm \mathbf{1}\}$.

Exercise 1.2.9. Let $\beta \colon V \times V \to \mathbb{K}$ be a bilinear form.
(1) Show that there exists a unique symmetric bilinear form β_+ and a unique skew-symmetric bilinear form β_- with $\beta = \beta_+ + \beta_-$.
(2) $\mathrm{Aut}(V, \beta) = \mathrm{O}(V, \beta_+) \cap \mathrm{Sp}(V, \beta_-)$.

Exercise 1.2.10. (a) Let G be a group, $N \subseteq G$ a normal subgroup and $q\colon G \to G/N, g \mapsto gN$ the quotient homomorphism. Show that:
(1) If $G \cong N \rtimes_\delta H$ for a subgroup H, then $H \cong G/N$.
(2) There exists a subgroup $H \subseteq G$ with $G \cong N \rtimes_\delta H$ if and only if there exists a group homomorphism $\sigma\colon G/N \to G$ with $q \circ \sigma = \mathrm{id}_{G/N}$.

(b) Show that

$$\mathrm{GL}_n(\mathbb{K}) \cong \mathrm{SL}_n(\mathbb{K}) \rtimes_\delta \mathbb{K}^\times$$

for a suitable homomorphism $\delta\colon \mathbb{K}^\times \to \mathrm{Aut}(\mathrm{SL}_n(\mathbb{K}))$.

Exercise 1.2.11. Show that $\mathrm{O}_{p,q}(\mathbb{C}) \cong \mathrm{O}_{p+q}(\mathbb{C})$ for $p, q \in \mathbb{N}_0$, $p + q > 0$.

Exercise 1.2.12. Let (V, β) be a euclidian vector space, i.e., a real vector space endowed with a positive definite symmetric bilinear form β. An element $\sigma \in \mathrm{O}(V, \beta)$ is called an *orthogonal reflection* if $\sigma^2 = \mathbf{1}$ and $\ker(\sigma - \mathbf{1})$ is a hyperplane. Show that for any finite-dimensional euclidian vector space (V, β), the orthogonal group $\mathrm{O}(V, \beta)$ is generated by reflections.

Exercise 1.2.13. (i) Show that, if n is odd, each $g \in \mathrm{SO}_n(\mathbb{R})$ has the eigenvalue 1.
(ii) Show that each $g \in \mathrm{O}_n(\mathbb{R})_-$ has the eigenvalue -1.

Exercise 1.2.14. Let V be a $\mathbb{K}$-vector space. An element $\varphi \in \mathrm{GL}(V)$ is called a *transvection* if $\dim_{\mathbb{K}}\big(\mathrm{im}\,(\varphi - \mathrm{id}_V)\big) = 1$ and $\mathrm{im}\,(\varphi - \mathrm{id}_V) \subseteq \ker(\varphi - \mathrm{id}_V)$. Show that:

(i) For each transvection φ, there exist a $v_\varphi \in V$ and a $\alpha_\varphi \in V^*$ such that $\varphi(v) = v - \alpha_\varphi(v)v_\varphi$ and $\alpha_\varphi(v_\varphi) = 0$.
(ii) For each transvection φ, there exist a $v_\varphi \in V$ and a $\alpha_\varphi \in V^*$ such that $\varphi(v) = v - \alpha_\varphi(v)v_\varphi$ and $\alpha_\varphi(v_\varphi) = 0$.
(ii) If $\dim V < \infty$, then $\det(\varphi) = 1$ for each transvection φ.
(iii) If $\psi \in \mathrm{GL}(V)$ commutes with all transvections, then every element of V is an eigenvector of ψ, so that $\psi \in \mathbb{K}^\times \mathrm{id}_V$.
(iv) $Z(\mathrm{GL}(V)) = \mathbb{K}^\times \mathbf{1}$.
(v) If $\dim V < \infty$, then $Z(\mathrm{SL}(V)) = \Gamma \mathbf{1}$, where $\Gamma := \{z \in \mathbb{K}^\times : z^n = 1\}$.

Exercise 1.2.15. Let V be a finite-dimensional $\mathbb{K}$-vector space for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and β be a skew symmetric bilinear form on V. Show that:

(i) A transvection $\varphi(v) = v - \alpha_\varphi(v)v_\varphi$ preserves β if and only if

$$(\forall v, w \in V): \quad \alpha_\varphi(v)\beta(v_\varphi, w) = \alpha_\varphi(w)\beta(v_\varphi, v).$$

If, in addition, β is nondegenerate, we call φ a *symplectic transvection*.
(ii) If β is nondegenerate and $\psi \in \mathrm{GL}(V)$ commutes with all symplectic transvections, then every vector in v is an eigenvector of ψ.

Exercise 1.2.16. Let V be a finite-dimensional $\mathbb{K}$-vector space for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and and β be a non-degenerate symmetric bilinear form on V. An involution $\varphi \in \mathrm{O}(V, \beta)$ is called an *orthogonal reflection* if $\dim_{\mathbb{K}} \big(\mathrm{im}\,(\varphi - \mathrm{id}_V)\big) = 1$. Show that:

(i) For each orthogonal reflection φ, there exists a non-isotropic $v_\varphi \in V$ such that $\varphi(v) = v - 2\frac{\beta(v, v_\alpha)}{\beta(v_\alpha, v_\alpha)}$.

(ii) If $\psi \in \mathrm{GL}(V)$ commutes with all orthogonal reflections, then every non-isotropic vector for β is an eigenvector of ψ, and this implies that $\psi \in \mathbb{K}^\times \mathrm{id}_V$.

(iv) $Z\big(\mathrm{O}(V, \beta)\big) = \{\pm \mathbf{1}\}$.

1.3 Quaternionic Matrix Groups

It is an important conceptual step to extend the real number field $\mathbb{R}$ to the field $\mathbb{C}$ of complex numbers. There are numerous motivations for this extension. The most obvious one is that not every algebraic equation with real coefficients has a solution in $\mathbb{R}$, and that $\mathbb{C}$ is *algebraically closed* in the sense that every nonconstant polynomial, even with complex coefficients, has zeros in $\mathbb{C}$. This is the celebrated Fundamental Theorem of Algebra. For analysis, the main point in passing from $\mathbb{R}$ to $\mathbb{C}$ is that the theory of holomorphic functions permits to understand many functions showing up in real analysis from a more natural viewpoint, which leads to a thorough understanding of singularities and of integrals which can be computed with the calculus of residues.

It therefore is a natural question whether there exists an extension $\mathbb{F}$ of the field $\mathbb{C}$ which would similarly enrich analysis and algebra if we pass from $\mathbb{C}$ to $\mathbb{F}$. It is an important algebraic result that there exists no finite-dimensional field extension of $\mathbb{R}$ other than $\mathbb{C}$ (cf. Exercise 1.3.4). This is most naturally obtained in Galois theory, i.e., the theory of extending fields by adding zeros of polynomials. It is closely related to the fact that every real polynomial is a product of linear factors and factors of degree 2. Fortunately this does not mean that one has to give up, but that one has to sacrifice one of the axioms of a field to obtain something new.

We call a unital (associative) algebra A a *skew field* or a *division algebra* if every nonzero element $a \in A^\times$ is invertible, i.e., $A = A^\times \cup \{0\}$. Now the question is: Are there any division algebras which are finite-dimensional real vector spaces, apart from $\mathbb{R}$ and $\mathbb{C}$. Here the answer is yes: there is the four-dimensional division algebra $\mathbb{H}$ of *quaternions*, and this is the only finite-dimensional real noncommutative division algebra.

The easiest way to define the quaternions is to take

$$\mathbb{H} := \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in M_2(\mathbb{C})\colon a, b \in \mathbb{C} \right\}.$$

Lemma 1.3.1. *$\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$ which is a division algebra.*

Proof. It is clear that $\mathbb{H}$ is a real vector subspace of $M_2(\mathbb{C})$. For the product of elements of $\mathbb{H}$ we obtain

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} c & -\overline{d} \\ d & \overline{c} \end{pmatrix} = \begin{pmatrix} ac - \overline{b}d & -a\overline{d} - \overline{b}\overline{c} \\ bc + \overline{a}d & -b\overline{d} + \overline{a}\overline{c} \end{pmatrix} \in \mathbb{H}.$$

This implies that $\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$.

We further have

$$\det \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = |a|^2 + |b|^2, \tag{1.4}$$

so that every nonzero element of $\mathbb{H}$ is invertible in $M_2(\mathbb{C})$, and its inverse

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}^{-1} = \frac{1}{|a|^2 + |b|^2} \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} \tag{1.5}$$

is again contained in $\mathbb{H}$. □

A convenient basis for $\mathbb{H}$ is given by

$$\mathbf{1}, \quad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K := IJ = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Then the multiplication in $\mathbb{H}$ is completely determined by the relations

$$I^2 = J^2 = K^2 = -\mathbf{1} \quad \text{and} \quad IJ = -JI = K.$$

Here $\mathbb{C} \cong \mathbb{R}\mathbf{1} + \mathbb{R}I$, but $\mathbb{H}$ is not a complex algebra because the multiplication in $\mathbb{H}$ is not a complex bilinear map.

Since $\mathbb{H}$ is a division algebra, its group of units is $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$, and (1.4) implies that

$$\mathbb{H}^\times = \mathbb{H} \cap \mathrm{GL}_2(\mathbb{C}).$$

On $\mathbb{H}$ we consider the euclidean norm given by

$$|x| := \sqrt{\det x}, \qquad \left| \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right| = \sqrt{|a|^2 + |b|^2}.$$

From the multiplicativity of the determinant we immediately derive that

$$|xy| = |x| \cdot |y| \quad \text{for} \quad x, y \in \mathbb{H}. \tag{1.6}$$

It follows in particular that $\mathbb{S} := \{x \in \mathbb{H} : |x| = 1\}$ is a subgroup of $\mathbb{H}$. In terms of complex matrices, we have $\mathbb{S} = \mathrm{SU}_2(\mathbb{C})$.

Many of the standard results from linear algebra generalize from vector spaces and matrices over fields to modules and matrices over division rings. If the division ring is noncommutative, however, one has to be careful on which side one wants to let the ring act. We want to recover the usual identification of linear maps with matrices acting from the left on column vectors such that the composition of maps corresponds with matrix multiplication. To this end one has to consider the column vectors with entries in $\mathbb{H}$ as a *right* $\mathbb{H}$-module via componentwise multiplication. See Exercises 1.3.1 and 1.3.2 for the basics of quaternionic linear algebra (a systematic treatment of linear algebra on division rings can be found in [Bou70], Chap. II).

In contrast to bases, linear maps and representing matrices, determinants do not have a straightforward generalization to linear algebra over division rings. Thus we cannot characterize the *quaternionic general linear group* $\mathrm{GL}_n(\mathbb{H})$ of invertible elements in the ring $M_n(\mathbb{H})$ of $n\times n$-matrices with entries in $\mathbb{H}$ via an $\mathbb{H}$-valued determinant.

Proposition 1.3.2. *View $M_n(\mathbb{H})$ as a real subalgebra of $M_{2n}(\mathbb{C})$ writing each entry of $A \in M_n(\mathbb{H})$ as a complex 2×2-matrix. Then*

$$\mathrm{GL}_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) : \det_{\mathbb{C}}(A) \neq 0\},$$

where $\det_{\mathbb{C}} \colon M_{2n}(\mathbb{C}) \to \mathbb{C}$ *is the ordinary determinant.*

Proof. It suffices to show that $M_n(\mathbb{H}) \cap \mathrm{GL}_{2n}(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{H})$. So pick $A \in M_n(\mathbb{H})$ which is invertible in $M_{2n}(\mathbb{C})$. Then the left multiplication λ_A by A on $M_n(\mathbb{H})$ is injective, hence bijective. Thus we have $A^{-1} = \lambda_A(\mathbf{1}) \in M_n(\mathbb{H})$. □

It follows from Proposition 1.3.2 that $\mathrm{GL}_n(\mathbb{H})$ is a (closed) subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$. Moreover, it allows us to define the *quaternionic special linear group*

$$\mathrm{SL}_n(\mathbb{H}) := \mathrm{GL}_n(\mathbb{H}) \cap \mathrm{SL}_{2n}(\mathbb{C}).$$

Observe that $\mathbb{H}$ as a subset of $M_2(\mathbb{C})$ can be characterized as

$$\mathbb{H} = \{A \in M_2(\mathbb{C}) : \overline{A}J = JA\},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the matrix used to build the symplectic group $\mathrm{Sp}_2(\mathbb{K})$ in Definition 1.2.1. Thus $\mathrm{GL}_n(\mathbb{H})$, viewed as a subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ is given by

$$\mathrm{GL}_n(\mathbb{H}) = \{A \in \mathrm{GL}_{2n}(\mathbb{C}) : \overline{A}J_n = J_nA\},$$

where J_n is the block diagonal matrix in $M_{2n}(\mathbb{C})$ having J as diagonal entries.

It turns out that inside $\mathrm{GL}_n(\mathbb{H})$ one can define analogs of unitary groups which are closely related to the symplectic groups. We note first that we can write the norm on $\mathbb{H}$ as

$$|x| = \sqrt{x^*x},$$

where $x^* = a\mathbf{1} - bI - cJ - dK$ for $x = a\mathbf{1} + bI + cJ + dK$. We extend this conjugation to matrices with entries in $\mathbb{H}$ setting

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{mn} \end{pmatrix}^* = \begin{pmatrix} x_{11}^* & x_{21}^* & \cdots & x_{m1}^* \\ x_{12}^* & x_{22}^* & \cdots & x_{m2}^* \\ \vdots & \ddots & \ddots & \vdots \\ x_{1n}^* & x_{2n}^* & \cdots & x_{nm}^* \end{pmatrix}.$$

Note that with respect to the embedding $M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ this involution agrees with the standard involution $A \mapsto A^* = \overline{A}^\top$ on $M_{2n}(\mathbb{C})$. Now

$$\mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}, \quad (v, w) \mapsto v^* w$$

defines a *quaternionic inner product* on $\mathbb{H}^n$ and $v \mapsto |v| := \sqrt{v^* v}$ is a euclidean norm on the real vector space $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

Definition 1.3.3. For $p + q = n \in \mathbb{N}$ view the matrix $I_{p,q}$ as an element of $M_n(\mathbb{H})$ and define *quaternionic unitary groups* via

$$\mathrm{U}_{p,q}(\mathbb{H}) := \{g \in \mathrm{GL}_n(\mathbb{H}) : g^* I_{p,q} g = I_{p,q}\}.$$

If p or q is zero, then we simply write $\mathrm{U}_n(\mathbb{H})$.

Proposition 1.3.4. *Viewed as a subset of* $\mathrm{GL}_{2n}(\mathbb{C})$*, the quaternionic unitary group* $\mathrm{U}_{p,q}(\mathbb{H})$*, is given by*

$$\mathrm{U}_{p,q}(\mathbb{H}) = \mathrm{U}_{2p,2q}(\mathbb{C}) \cap \mathrm{Sp}(\mathbb{C}^{2n}, \beta),$$

where $\beta\colon \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ *is the skew-symmetric bilinear form given by the matrix* $J_n^\top I_{2p,2q}$*. The group* $\mathrm{Sp}(\mathbb{C}^{2n}, \beta)$ *is conjugate to* $\mathrm{Sp}_{2n}(\mathbb{C})$ *in* $\mathrm{GL}_{2n}(\mathbb{C})$*. In particular,* $\mathrm{U}_n(\mathbb{H})$ *is isomorphic to a compact subgroup of* $\mathrm{Sp}_{2n}(\mathbb{C})$*.*

Proof. Let $g \in \mathrm{U}_{p,q}(\mathbb{H})$ be viewed as an element of $\mathrm{GL}_{2n}(\mathbb{C})$. Then we have $g^* I_{2p,2q} g = I_{2p,2q}$ and $\overline{g} J_n = J_n g$. Therefore $J_n^\top g^* = g^\top J_n^\top$ and

$$g^\top J_n^\top I_{2p,2q} g = J_n^\top I_{2p,2q}. \qquad \square$$

Exercises for Section 1.3

For the first two exercises recall that a right module M over a (noncommutative) ring R is an abelian group M together with a map $M \times R \to M, (m, r) \mapsto mr$ such that $r \mapsto (m \mapsto mr)$ defines a ring homomorphism $R \to \mathrm{End}(M)$.

Exercise 1.3.1. Let V be a right $\mathbb{H}$-module. Show that

(i) V is free, i.e. it admits an $\mathbb{H}$-basis.
(ii) If V is finitely generated as an $\mathbb{H}$-module, then it admits a finite $\mathbb{H}$-basis. In this case all $\mathbb{H}$-bases have the same number of elements. This number is called the *dimension* of V over $\mathbb{H}$ and denoted by $\dim_{\mathbb{H}}(V)$.

Exercise 1.3.2. Let V and W be two right $\mathbb{H}$-modules with $\mathbb{H}$-bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$. Given an $\mathbb{H}$-linear map $\varphi\colon V \to W$, write

$$\varphi(v_j) = \sum_{k=1}^{n} w_k a_{kj}$$

with $a_{kj} \in \mathbb{H}$. Show that

(i) If $\varphi(v) = w$ with $v = \sum_{r=1}^{m} v_r x_r$ and $W = \sum_{s=1}^{s} w_s y_s$, then

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ y_2 \\ \vdots \\ x_m \end{pmatrix}.$$

(ii) The map $\varphi \mapsto (a_{kj})$ is a bijection between the set of $\mathbb{H}$-linear maps $\varphi\colon V \to W$ and matrices $A \in M_n(\mathbb{H})$ intertwining the composition of maps with the ordinary matrix multiplication (whenever composition makes sense).

Exercise 1.3.3. Show that the group $\mathrm{U}_n(\mathbb{H})$ is compact and connected.

Exercise 1.3.4. Show that each finite-dimensional complex division algebra is one-dimensional.

Notes on Chapter 1

The material covered in this chapter is standard and only touches the surfaces of what is known about the structure of matrix groups. For much more detailed presentations see [GW09] or [Gr01].

2

The Matrix Exponential Function

In this chapter we study one of the central tools in Lie theory: the matrix exponential function. This function has various applications in the structure theory of matrix groups. First of all, it is naturally linked to the one-parameter subgroups, and it turns out that the local group structure of $\mathrm{GL}_n(\mathbb{K})$ in a neighborhood of the identity is determined by its one-parameter subgroups.

In the first section of this chapter we provide some tools to show that matrix valued functions defined by convergent power series are actually smooth. This is applied in the subsequent sections to the exponential and the logarithm functions. Then we discuss restrictions of the exponential function to certain subsets such as small 0-neighborhoods, the set of nilpotent matrices and the set of hermitian matrices. Finally, we derive the Baker–Campbell–Dynkin–Hausdorff formula expressing the product of two exponentials near the identity in terms of the Hausdorff series which involves only commutator brackets.

In the following chapter, we shall use the matrix exponential function to generalize the polar decomposition given in Proposition 1.1.5 to a larger class of groups. This will lead to topological information on various concrete matrix groups.

2.1 Smooth Functions Defined by Power Series

First we put the structure that we have on the space $M_n(\mathbb{K})$ of $(n\times n)$-matrices into a slightly more general context.

Definition 2.1.1. (a) A vector space A together with a bilinear map $A \times A \to A, (x,y) \mapsto x \cdot y$ (called multiplication) is called an (*associative*) *algebra* if the multiplication is associative in the sense that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{for} \quad x, y, z \in A.$$

We write $xy := x \cdot y$ for the product of x and y in A.

The algebra A is called *unital* if it contains an element $\mathbf{1}$ satisfying $\mathbf{1}a = a\mathbf{1} = a$ for each $a \in A$.

(b) A norm $\|\cdot\|$ on an algebra A is called *submultiplicative* if

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for all} \quad a, b \in A.$$

Then the pair $(A, \|\cdot\|)$ is called a *normed algebra*. If, in addition, A is a complete normed space, then it is said to be a *Banach algebra*.

Remark 2.1.2. Any finite-dimensional normed space is complete, so that each finite-dimensional normed algebra is a Banach algebra.

Example 2.1.3. Endowing $M_n(\mathbb{K})$ with the operator norm with respect to the euclidian norm on $\mathbb{K}^n$ defines on $M_n(\mathbb{K})$ the structure of a unital Banach algebra.

Lemma 2.1.4. *If A is a unital Banach algebra, then we endow the vector space $TA := A \oplus A$ with the norm $\|(a,b)\| := \|a\| + \|b\|$ and the multiplication*

$$(a,b)(a',b') := (aa', ab' + a'b).$$

Then TA is a unital Banach algebra with identity $(\mathbf{1}, 0)$.

We put $\varepsilon := (0,1)$. Then each element of TA can be written in a unique fashion as $(a,b) = a + b\varepsilon$ and the multiplication satisfies

$$(a + b\varepsilon)(a' + b'\varepsilon) = aa' + (ab' + a'b)\varepsilon.$$

In particular, $\varepsilon^2 = 0$.

Proof. That TA is a unital algebra is a trivial verification. That the norm is submultiplicative follows from

$$\begin{aligned}\|(a,b)(a',b')\| &= \|aa'\| + \|ab' + a'b\| \leq \|a\| \cdot \|a'\| + \|a\| \cdot \|b'\| + \|a'\| \cdot \|b\| \\ &\leq (\|a\| + \|b\|)(\|a'\| + \|b'\|) = \|(a,b)\| \cdot \|(a',b')\|.\end{aligned}$$

This proves that $(TA, \|\cdot\|)$ is a unital normed algebra, the unit being $\mathbf{1} = (\mathbf{1}, 0)$. The completeness of TA follows easily from the completeness of A (Exercise). □

Lemma 2.1.5. *Let $c_n \in \mathbb{K}$ and $r > 0$ with $\sum_{n=0}^{\infty} |c_n| r^n < \infty$. Further let A be a finite-dimensional unital Banach algebra. Then*

$$f\colon B_r(0) := \{x \in A\colon \|x\| < r\} \to A, \quad x \mapsto \sum_{n=0}^{\infty} c_n x^n$$

defines a smooth function. Its derivative is given by

$$\mathtt{d}f(x) = \sum_{n=0}^{\infty} c_n \mathtt{d}p_n(x),$$

where $p_n(x) = x^n$ is the n^{th} power map whose derivative is given by

$$\mathtt{d}p_n(x)y = x^{n-1}y + x^{n-2}yx + \ldots + xyx^{n-2} + yx^{n-1}.$$

For $\|x\| < r$ and $y \in A$ with $xy = yx$ we obtain in particular

$$\mathtt{d}p_n(x)y = nx^{n-1}y \quad \textit{and} \quad \mathtt{d}f(x)y = \sum_{n=1}^{\infty} c_n n x^{n-1} y.$$

Proof. First we observe that the series defining $f(x)$ converges for $\|x\| < r$ by the Comparison Test (for series in Banach spaces). We shall prove by induction over $k \in \mathbb{N}$ that all such functions f are C^k-functions.

Step 1: First we show that f is a C^1-function. We define $\alpha_n \colon A \to A$ by

$$\alpha_n(h) := x^{n-1}h + x^{n-2}hx + \ldots + xhx^{n-2} + hx^{n-1}.$$

Then α_n is a continuous linear map with $\|\alpha_n\| \leq n\|x\|^{n-1}$. Furthermore

$$p_n(x+h) = (x+h)^n = x^n + \alpha_n(h) + r_n(h),$$

where

$$\begin{aligned} \|r_n(h)\| &\leq \binom{n}{2}\|h\|^2\|x\|^{n-2} + \binom{n}{3}\|h\|^3\|x\|^{n-3} + \ldots + \|h\|^n \\ &= \sum_{k\geq 2}\binom{n}{k}\|h\|^k\|x\|^{n-k}. \end{aligned}$$

In particular $\lim_{h\to 0}\frac{\|r_n(h)\|}{\|h\|} = 0$, and therefore p_n is differentiable in x with $\mathtt{d}p_n(x) = \alpha_n$. The series

$$\beta(h) := \sum_{n=0}^{\infty} c_n \alpha_n(h)$$

converges absolutely in $\mathrm{End}(A)$ by the Ratio Test since $\|x\| < r$:

$$\sum_{n=0}^{\infty} |c_n|\|\alpha_n\| \leq \sum_{n=0}^{\infty} |c_n| \cdot n \cdot \|x\|^{n-1} < \infty.$$

We thus obtain a linear map $\beta(x) \in \mathrm{End}(A)$ for each x with $\|x\| < r$.

Now let h satisfy $\|x\| + \|h\| < r$, i.e., $\|h\| < r - \|x\|$. Then

$$f(x+h) = f(x) + \beta(x)(h) + r(h), \quad r(h) := \sum_{n=2}^{\infty} c_n r_n(h),$$

where

$$\begin{aligned}\|r(h)\| &\leq \sum_{n=2}^{\infty} |c_n| \|r_n(h)\| \leq \sum_{n=2}^{\infty} |c_n| \sum_{k=2}^{n} \binom{n}{k} \|h\|^k \|x\|^{n-k} \\ &\leq \sum_{k=2}^{\infty} \Big(\sum_{n=k}^{\infty} |c_n| \binom{n}{k} \|x\|^{n-k} \Big) \|h\|^k < \infty\end{aligned}$$

follows from $\|x\| + \|h\| < r$ because

$$\sum_{k} \sum_{n \geq k} |c_n| \binom{n}{k} \|x\|^{n-k} \|h\|^k = \sum_{n} |c_n| (\|x\| + \|h\|)^n \leq \sum_{n} |c_n| r^n < \infty.$$

Therefore the continuity of real-valued functions represented by a power series yields

$$\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = \sum_{k=2}^{\infty} \Big(\sum_{n=k}^{\infty} |c_n| \binom{n}{k} \|x\|^{n-k} \Big) 0^{k-1} = 0.$$

This proves that f is a C^1-function with the required derivative.
Step 2: To complete our proof by induction, we now show that if all functions f as above are C^k, then they are also C^{k+1}. In view of Step 1, this implies that they are smooth.

To set up the induction, we consider the Banach algebra TA from Lemma 2.1.4 and apply Step 1 to this algebra to obtain a smooth function

$$\begin{aligned}&F\colon \{x + \varepsilon h \in TA\colon \|x\| + \|h\| = \|x + \varepsilon h\| < r\} \to TA \\ &F(x + \varepsilon h) = \sum_{n=0}^{\infty} c_n \cdot (x + \varepsilon h)^n,\end{aligned}$$

We further note that $(x + \varepsilon h)^n = x^n + \mathtt{d}p_n(x)h \cdot \varepsilon$. This implies the formula

$$F(x + \varepsilon h) = f(x) + \varepsilon \mathtt{d}f(x)h,$$

i.e., that the extension F of f to TA describes the first order Taylor expansion of f in each point $x \in A$. Our induction hypothesis implies that F is a C^k-function.

Let $x_0 \in A$ with $\|x_0\| < r$ and pick a basis $h_1, \ldots, h_d$ for A with $\|h_i\| < r - \|x_0\|$. Then all functions $x \mapsto \mathtt{d}f(x)h_i$ are defined and C^k on a neighborhood of x_0, and this implies that the function

$$B_r(0) \to \mathrm{Hom}(A, A), \quad x \mapsto \mathtt{d}f(x)$$

is C^k. This in turn implies that f is C^{k+1}. □

The following proposition shows in particular that inserting elements of a Banach algebra in power series is compatible with composition.

Proposition 2.1.6. (a) *On the set P_R of power series of the form*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{K}$$

and converging on the open disc $B_R(0) := \{z \in \mathbb{K}: |z| < R\}$, we define for $r < R$:

$$\|f\|_r := \sum_{n=0}^{\infty} |a_n| r^n.$$

Then $\|\cdot\|_r$ is a norm with the following properties:

(1) *$\|\cdot\|_r$ is submultiplicative: $\|fg\|_r \leq \|f\|_r \|g\|_r$.*
(2) *The polynomials $f_N(z) := \sum_{n=0}^{N} a_n z^n$ satisfy $\|f - f_N\|_r \to 0$.*
(3) *If A is a finite-dimensional Banach algebra and $x \in A$ satisfies $\|x\| < R$, then $f(x) := \sum_{n=0}^{\infty} a_n x^n$ converges. Moreover,*

$$\|f(x)\| \leq \|f\|_r \quad \textit{for} \quad \|x\| \leq r < R,$$

and

$$(f \cdot g)(x) = f(x)g(x) \quad \textit{for} \quad f, g \in P_R.$$

(b) *If $g \in P_S$ with $\|g\|_s < R$ for all $s < S$ and $f \in P_R$, then $f \circ g \in P_S$ defines an analytic function on the open disc of radius S, and for $x \in A$ with $\|x\| < S$ we have $\|g(x)\| < R$ and the Composition Formula*

$$f(g(x)) = (f \circ g)(x). \tag{2.1}$$

Proof. (1) First we note that P_R is the set of all power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $\|f\|_r < \infty$ holds for all $r < R$. We leave the easy argument that $\|\cdot\|_r$ is a norm to the reader. If $\|f\|_r, \|g\|_r < \infty$ holds for $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the Cauchy Product Formula (Exercise 2.1.3) implies that

$$\|fg\|_r = \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} a_k b_{n-k} \Big| r^n \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k|\, |b_{n-k}| r^k r^{n-k} = \|f\|_r \|g\|_r.$$

(2) follows immediately from $\|f - f_N\|_r = \sum_{n>N} |a_n| r^n \to 0$.

(3) The relation $\|f(x)\| \leq \|f\|_r$ follows from $\|a_n x^n\| \leq |a_n| r^n$ and the Domination Test for absolutely converging series in a Banach space. The relation $(f \cdot g)(x) = f(x)g(x)$ follows directly from the Cauchy Product Formula because the series $f(x)$ and $g(x)$ converge absolutely (Exercise 2.1.3).

(b) We may w.l.o.g. assume that $\mathbb{K} = \mathbb{C}$ because everything on the case $\mathbb{K} = \mathbb{R}$ can be obtained by restriction. Our assumption implies that $g(B_S(0)) \subseteq B_R(0)$, so that $f \circ g$ defines a holomorphic function on the open disc $B_S(0)$. For $s < S$ and $\|g\|_s < r < R$ we then derive

$$\sum_{n=0}^{\infty} \|a_n g^n\|_s \leq \sum_{n=0}^{\infty} |a_n| \|g\|_s^n \leq \|f\|_r.$$

Therefore the series $f \circ g = \sum_{n=0}^{\infty} a_n g^n$ converges absolutely in P_S with respect to $\| \cdot \|_s$, and we thus obtain the estimate

$$\|f \circ g\|_s = \lim_{N \to \infty} \| \sum_{n=0}^{N} a_n g^n \|_s \leq \sum_{n=0}^{\infty} |a_n| \|g\|_s^n \leq \|f\|_r.$$

For $s := \|x\|$ we obtain $\|g(x)\| \leq \|g\|_s < R$, so that $f(g(x))$ is defined. For $s < r < R$ we then have

$$\|f(g(x)) - f_N(g(x))\| \leq \|f - f_N\|_r \to 0.$$

Likewise

$$\|(f \circ g)(x) - (f_N \circ g)(x)\| \leq \|(f \circ g) - (f_N \circ g)\|_s \leq \|f - f_N\|_r \to 0,$$

and we get

$$(f \circ g)(x) = \lim_{N \to \infty} (f_N \circ g)(x) = \lim_{N \to \infty} f_N(g(x)) = f(g(x))$$

because the Composition Formula trivially holds if f is a polynomial. □

Exercises for Section 2.1

Exercise 2.1.1. Let $X_1, \ldots, X_n$ be finite-dimensional normed spaces and $\beta \colon X_1 \times \ldots \times X_n \to Y$ an n-linear map.

(a) Show that β is continuous.
(b) Show that there exists a constant $C \geq 0$ with

$$\|\beta(x_1, \ldots, x_n)\| \leq C \|x_1\| \cdots \|x_n\| \quad \text{for} \quad x_i \in X_i.$$

(c) Show that β is differentiable with

$$\mathtt{d}\beta(x_1, \ldots, x_n)(h_1, \ldots, h_n) = \sum_{j=1}^{n} \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_n).$$

Exercise 2.1.2. Let Y be a Banach space and $a_{n,m}$, $n, m \in \mathbb{N}$, elements in Y with

$$\sum_{n,m} \|a_{n,m}\| := \sup_{N \in \mathbb{N}} \sum_{n,m \leq N} \|a_{n,m}\| < \infty.$$

(a) Show that

$$A := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m}$$

and that both iterated sums exist.

(b) Show that for each sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets $S_n \subseteq \mathbb{N}\times\mathbb{N}$, $n \in \mathbb{N}$, with $S_n \subseteq S_{n+1}$ and $\bigcup_n S_n = \mathbb{N}\times\mathbb{N}$ we have

$$A = \lim_{n\in\mathbb{N}} \sum_{(j,k)\in S_n} a_{j,k}.$$

Exercise 2.1.3 (Cauchy Product Formula). Let X, Y, Z be Banach spaces and $\beta\colon X \times Y \to Z$ a continuous bilinear map. Suppose that $x := \sum_{n=0}^{\infty} x_n$ is absolutely convergent in X and that $y := \sum_{n=0}^{\infty} y_n$ is absolutely convergent in Y. Then

$$\beta(x, y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \beta(x_k, y_{n-k}).$$

2.2 Elementary Properties of the Exponential Function

After the preparations of the preceding section, it is now easy to see that the matrix exponential function defines a smooth map on $M_n(\mathbb{K})$. In this section we describe some elementary properties of this function. As group theoretic consequences for $\mathrm{GL}_n(\mathbb{K})$, we show that it has no small subgroups and that all one-parameter groups are smooth and given by the exponential function.

For $x \in M_n(\mathbb{K})$, we define

$$e^x :\ = \sum_{k=0}^{\infty} \frac{1}{k!} x^k. \tag{2.2}$$

The absolute convergence of the series on the right follows directly from the estimate

$$\sum_{k=0}^{\infty} \frac{1}{k!}\|x^k\| \le \sum_{k=0}^{\infty} \frac{1}{k!}\|x\|^k = e^{\|x\|}$$

and the Comparison Test for absolute convergence of a series in a Banach space. We define the *exponential function of* $M_n(\mathbb{K})$ by

$$\exp\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad \exp(x) := e^x.$$

Proposition 2.2.1. *The exponential function* $\exp\colon M_n(\mathbb{K}) \to M_n(\mathbb{K})$ *is smooth. For* $xy = yx$ *it satisfies*

$$\mathtt{d}\exp(x)y = \exp(x)y = y\exp(x) \tag{2.3}$$

and in particular

$$\mathtt{d}\exp(0) = \mathrm{id}_{M_n(\mathbb{K})}.$$

Proof. To verify the formula for the differential, we note that for $xy = yx$, Lemma 2.1.5 implies that

$$\mathtt{d}\exp(x)y = \sum_{k=1}^{\infty} \frac{1}{k!} k x^{k-1} y = \sum_{k=0}^{\infty} \frac{1}{k!} x^k y = \exp(x)y.$$

For $x = 0$, the relation $\exp(0) = \mathbf{1}$ now implies in particular that $\mathtt{d}\exp(0)y = y$. □

Lemma 2.2.2. *Let $x, y \in M_n(\mathbb{K})$.*

(i) *If $xy = yx$, then $\exp(x + y) = \exp x \exp y$.*
(ii) $\exp(M_n(\mathbb{K})) \subseteq \mathrm{GL}_n(\mathbb{K})$, $\exp(0) = \mathbf{1}$, *and* $(\exp x)^{-1} = \exp(-x)$.
(iii) *For $g \in \mathrm{GL}_n(\mathbb{K})$ the relation $ge^x g^{-1} = e^{gxg^{-1}}$ holds.*

Proof. (i) Using the general form of the Cauchy Product Formula (Exercise 2.1.3), we obtain

$$\begin{aligned}\exp(x + y) &= \sum_{k=0}^{\infty} \frac{(x + y)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} x^\ell y^{k-\ell} \\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{x^\ell}{\ell!} \frac{y^{k-\ell}}{(k-\ell)!} = \Big(\sum_{p=0}^{\infty} \frac{x^p}{p!}\Big)\Big(\sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!}\Big).\end{aligned}$$

(ii) From (i) we derive in particular $\exp x \exp(-x) = \exp 0 = \mathbf{1}$, which implies (ii).

(iii) is a consequence of $gx^n g^{-1} = (gxg^{-1})^n$ and the continuity of the conjugation map $c_g(x) := gxg^{-1}$ on $M_n(\mathbb{K})$. □

Remark 2.2.3. (a) For $n = 1$, the exponential function

$$\exp\colon M_1(\mathbb{R}) \cong \mathbb{R} \to \mathbb{R}^\times \cong \mathrm{GL}_1(\mathbb{R}), \quad x \mapsto e^x$$

is injective, but this is not the case for $n > 1$. In fact,

$$\exp \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix} = \mathbf{1}$$

follows from

$$\exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R}.$$

This example is nothing but the real picture of the relation $e^{2\pi i} = 1$.

Proposition 2.2.4. *For each sufficiently small open neighborhood U of 0 in $M_d(\mathbb{K})$, the map*

$$\exp|_U : U \to \mathrm{GL}_d(\mathbb{K})$$

is a diffeomorphism onto an open neighborhood of $\mathbf{1}$ in $\mathrm{GL}_d(\mathbb{K})$.

Proof. We have already seen that exp is a smooth map, and that $\mathtt{d}\exp(\mathbf{0}) = \mathrm{id}_{M_d(\mathbb{K})}$. Therefore the assertion follows from the Inverse Function Theorem. □

If U is as in Proposition 2.2.4 and $V = \exp(U)$, we define

$$\log_V \colon = (\exp|_U)^{-1} \colon V \to U \subseteq M_d(\mathbb{K}).$$

We shall see below why this function deserves to be called a *logarithm function*.

Theorem 2.2.5 (No Small Subgroup Theorem). *There exists an open neighborhood V of $\mathbf{1}$ in $\mathrm{GL}_d(\mathbb{K})$ such that $\{\mathbf{1}\}$ is the only subgroup of $\mathrm{GL}_d(\mathbb{K})$ contained in V.*

Proof. Let U be as in Proposition 2.2.4 and assume further that U is convex and bounded. We set $U_1 := \frac{1}{2}U$. Let $G \subseteq V := \exp U_1$ be a subgroup of $\mathrm{GL}_d(\mathbb{K})$ and $g \in G$. Then we write $g = \exp x$ with $x \in U_1$ and assume that $x \neq 0$. Let $k \in \mathbb{N}$ be maximal with $kx \in U_1$ (the existence of k follows from the boundedness of U). Then

$$g^{k+1} = \exp(k+1)x \in G \subseteq V$$

implies the existence of $y \in U_1$ with $\exp(k+1)x = \exp y$. Since $(k+1)x \in 2U_1 = U$ follows from $\frac{k+1}{2}x \in [0,k]x \subseteq U_1$, and $\exp|_U$ is injective, we obtain $(k+1)x = y \in U_1$, contradicting the maximality of k. Therefore $g = \mathbf{1}$. □

A *one-parameter (sub)group* of a group G is a group homomorphism $\gamma \colon (\mathbb{R}, +) \to G$. The following result describes the differentiable one-parameter subgroups of $\mathrm{GL}_n(\mathbb{K})$.

Theorem 2.2.6 (One-parameter Group Theorem). *For each $x \in M_n(\mathbb{K})$, the map*

$$\gamma \colon (\mathbb{R}, +) \to \mathrm{GL}_n(\mathbb{K}), \quad t \mapsto \exp(tx)$$

is a smooth group homomorphism solving the initial value problem

$$\gamma(0) = \mathbf{1} \quad \text{and} \quad \gamma'(t) = \gamma(t)x \quad \text{for } t \in \mathbb{R}.$$

Conversely, every continuous one-parameter group $\gamma \colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ is of this form.

Proof. In view of Lemma 2.2.2(i) and the differentiability of exp, we have

$$\lim_{h\to 0} \frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big) = \lim_{h\to 0} \frac{1}{h}\big(\gamma(t)\gamma(h) - \gamma(t)\big) = \gamma(t) \lim_{h\to 0} \frac{1}{h}\big(e^{hx} - \mathbf{1}\big) = \gamma(t)x.$$

Hence γ is differentiable with $\gamma'(t) = x\gamma(t) = \gamma(t)x$. From that it immediately follows that γ is smooth with $\gamma^{(n)}(t) = x^n\gamma(t)$ for each $n \in \mathbb{N}$.

Although we will not need it for the completeness of the proof, we first show that each one-parameter group $\gamma\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ which is differentiable in 0 has the required form. For $x := \gamma'(0)$, the calculation

$$\gamma'(t) = \lim_{s\to 0} \frac{\gamma(t+s) - \gamma(t)}{s} = \lim_{s\to 0} \gamma(t)\frac{\gamma(s) - \gamma(0)}{s} = \gamma(t)\gamma'(0) = \gamma(t)x$$

implies that γ is differentiable and solves the initial value problem

$$\gamma'(t) = \gamma(t)x, \qquad \gamma(0) = \mathbf{1}.$$

Therefore the Uniqueness Theorem for Linear Differential Equations implies that $\gamma(t) = \exp tx$ for all $t \in \mathbb{R}$.

It remains to show that each continuous one-parameter group γ of $\mathrm{GL}_d(\mathbb{K})$ is differentiable in 0. As in the proof of Theorem 2.2.5, let U be a convex symmetric (i.e., $U = -U$) 0-neighborhood in $M_d(\mathbb{K})$ satisfying the properties described in Proposition 2.2.4 and $U_1 := \frac{1}{2}U$. Since γ is continuous in 0, there exists an $\varepsilon > 0$ such that $\gamma([-\varepsilon, \varepsilon]) \subseteq \exp(U_1)$. Then $\alpha(t) := (\exp|_U)^{-1}(\gamma(t))$ defines a continuous curve $\alpha\colon [-\varepsilon, \varepsilon] \to U_1$ with $\exp(\alpha(t)) = \gamma(t)$ for $|t| \leq \varepsilon$. For any such t we then have

$$\exp\left(2\alpha(\tfrac{t}{2})\right) = \exp(\alpha(\tfrac{t}{2}))^2 = \gamma(\tfrac{t}{2})^2 = \gamma(t) = \exp(\alpha(t)),$$

so that the injectivity of exp on U yields

$$\alpha(\tfrac{t}{2}) = \tfrac{1}{2}\alpha(t) \quad \text{ for } \quad |t| \leq \varepsilon.$$

Inductively we thus obtain

$$\alpha(\tfrac{t}{2^k}) = \tfrac{1}{2^k}\alpha(t) \quad \text{ for } \quad |t| \leq \varepsilon, k \in \mathbb{N}. \tag{2.4}$$

In particular, we obtain

$$\alpha(t) \in \frac{1}{2^k}U_1 \quad \text{ for } \quad |t| \leq \frac{\varepsilon}{2^k}.$$

For $n \in \mathbb{Z}$ with $|n| \leq 2^k$ and $|t| \leq \frac{\varepsilon}{2^k}$ we now have $|nt| \leq \varepsilon$, $n\alpha(t) \in \frac{n}{2^k}U_1 \subseteq U_1$, and

$$\exp(n\alpha(t)) = \gamma(t)^n = \gamma(nt) = \exp(\alpha(nt)).$$

Therefore the injectivity of exp on U_1 yields

$$\alpha(nt) = n\alpha(t) \quad \text{ for } \quad n \leq 2^k, |t| \leq \frac{\varepsilon}{2^k}. \tag{2.5}$$

Combining (2.4) and (2.5), leads to

$$\alpha(\tfrac{n}{2^k}t) = \tfrac{n}{2^k}\alpha(t) \quad \text{ for } \quad |t| \leq \varepsilon, k \in \mathbb{N}, |n| \leq 2^k.$$

Since the set of all numbers $\frac{nt}{2^k}$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$, $|n| \leq 2^k$, is dense in the interval $[-t, t]$, the continuity of α implies that

$$\alpha(t) = \frac{t}{\varepsilon}\alpha(\varepsilon) \quad \text{for} \quad |t| \leq \varepsilon.$$

In particular, α is smooth and of the form $\alpha(t) = tx$ for some $x \in M_d(\mathbb{K})$. Hence $\gamma(t) = \exp(tx)$ for $|t| \leq \varepsilon$, but then $\gamma(nt) = \exp(ntx)$ for $n \in \mathbb{N}$ leads to $\gamma(t) = \exp(tx)$ for each $t \in \mathbb{R}$. □

Exercises for Section 2.2

Exercise 2.2.1. Let $D \in M_n(\mathbb{K})$ be a diagonal matrix. Calculate its operator norm with respect to the euclidean norm on $\mathbb{K}^n$.

Exercise 2.2.2. If $g \in M_n(\mathbb{K})$ satisfies $\|g - \mathbf{1}\| < 1$, then $g \in \mathrm{GL}_n(\mathbb{K})$.

Exercise 2.2.3. (a) Calculate e^{tN} for $t \in \mathbb{K}$ and the matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 1 \\ 0 & & \dots & & 0 \end{pmatrix} \in M_n(\mathbb{K}).$$

(b) If A is a block diagonal matrix $\mathrm{diag}(A_1, \ldots, A_k)$, then e^A is the block diagonal matrix $\mathrm{diag}(e^{A_1}, \ldots, e^{A_k})$.
(c) Calculate e^{tA} for a matrix $A \in M_n(\mathbb{C})$ given in Jordan Normal Form.

Exercise 2.2.4. Recall that a matrix x is said to be *nilpotent* if $x^d = 0$ for some $d \in \mathbb{N}$ and y is called *unipotent* if $y - \mathbf{1}$ is nilpotent.

Let $a, b \in M_n(\mathbb{K})$ be commuting matrices.

(a) If a and b are nilpotent, then $a + b$ is nilpotent.
(b) If a and b are unipotent, then ab is unipotent.

Exercise 2.2.5 (Jordan decomposition).
(a) (Additive Jordan decomposition) Show that each complex matrix $X \in M_n(\mathbb{C})$ can be written in a unique fashion as

$$X = X_s + X_n \quad \text{with} \quad [X_s, X_n] = 0,$$

where X_n is nilpotent and X_s diagonalizable.
(b) $A \in M_n(\mathbb{C})$ commutes with a diagonalizable matrix D if and only if A preserves all eigenspaces of D.
(c) $A \in M_n(\mathbb{C})$ commutes with X if and only if it commutes with X_s and X_n.

Exercise 2.2.6 (Multiplicative Jordan decomposition). (a) Show that each invertible complex matrix $g \in \mathrm{GL}_n(\mathbb{C})$ can be written in a unique fashion as

$$g = g_s g_u, \quad \text{with} \quad g_s g_u = g_u g_s,$$

where g_u is unipotent and g_s diagonalizable.
(b) If $X = X_s + X_n$ is the additive Jordan decomposition, then $e^X = e^{X_s} e^{X_n}$ is the multiplicative Jordan decomposition of e^X.

Exercise 2.2.7. Let $A \in M_n(\mathbb{C})$. Show that the set $e^{\mathbb{R}A} = \{e^{tA} : t \in \mathbb{R}\}$ is bounded in $M_n(\mathbb{C})$ if and only if A is diagonalizable with purely imaginary eigenvalues.

Exercise 2.2.8. Let $U \in M_n(\mathbb{C})$. Then the set $\{U^n : n \in \mathbb{Z}\}$ is bounded if and only if U is diagonalizable and $\mathrm{Spec}\,(U) \subseteq \{z \in \mathbb{C} : |z| = \mathbf{1}\}$.

Exercise 2.2.9. Show that:
(a) $\exp(M_n(\mathbb{R}))$ is contained in the identity component $\mathrm{GL}_n(\mathbb{R})_+$ of $\mathrm{GL}_n(\mathbb{R})$. In particular the exponential function of $\mathrm{GL}_n(\mathbb{R})$ is not surjective because this group is not connected.
(b) The exponential function $\exp : M_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})_+$ is not surjective.
(c) Give also a direct argument why g is not of the form e^X.

Exercise 2.2.10. Let $V \subseteq M_n(\mathbb{C})$ be a commutative subspace, i.e., an abelian Lie subalgebra. Then $A := e^V$ is an abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$ and

$$\exp : (V, +) \to (A, \cdot)$$

is a group homomorphism whose kernel consists of diagonalizable elements whose eigenvalues are contained in $2\pi i\mathbb{Z}$.

Exercise 2.2.11. For $X, Y \in M_n(\mathbb{C})$ the following are equivalent:

(1) $e^X = e^Y$.
(2) $X_n = Y_n$ (the nilpotent Jordan components) and $e^{X_s} = e^{Y_s}$.

Exercise 2.2.12. For $A \in M_n(\mathbb{C})$ the relation $e^A = \mathbf{1}$ holds if and only if A is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$.

2.3 The Logarithm Function

In this section we apply the tools from Section 2.1 to the logarithm series. Since its radius of convergence is 1, it defines a smooth function $\mathrm{GL}_n(\mathbb{K}) \supseteq B_1(\mathbf{1}) \to M_n(\mathbb{K})$, and we shall see that it thus provides a smooth inverse of the exponential function.

Lemma 2.3.1. *The series* $\log(\mathbf{1} + x) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$ *converges for* $x \in M_d(\mathbb{K})$ *with* $\|x\| < 1$ *and defines a smooth function*

$$\log \colon B_1(\mathbf{1}) \to M_d(\mathbb{K}).$$

For $\|x\| < 1$ *and* $y \in M_d(\mathbb{K})$ *with* $xy = yx$*,*

$$(\mathtt{d}\log)(\mathbf{1} + x)y = (\mathbf{1} + x)^{-1} y.$$

Proof. The convergence follows from

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{r^k}{k} = \log(1+r) < \infty$$

for $|r| < 1$, so that the smoothness follows from Lemma 2.1.5.

If x and y commute, then the formula for the derivative in Lemma 2.1.5 leads to

$$(\mathtt{d}\log)(\mathbf{1}+x)y = \sum_{k=1}^{\infty}(-1)^{k+1}x^{k-1}y = (\mathbf{1}+x)^{-1}y$$

(here we used the Neumann series; cf. Exercise 2.2.2). □

Proposition 2.3.2. (a) *For $x \in M_d(\mathbb{K})$ with $\|x\| < \log 2$,*

$$\log(\exp x) = x.$$

(b) *Every $g \in \mathrm{GL}_d(\mathbb{K})$ with $\|g - \mathbf{1}\| < 1$ satisfies* $\exp(\log g) = g$.

Proof. (a) We apply Proposition 2.1.6 with $g = \exp \in P_S$, $S = \log 2$, $R = e^{\log 2} = 2$ and $\|\exp\|_s \leq e^s \leq e^S = 2$ for $s < S$. We thus obtain $\log(\exp x) = x$ for $\|x\| < \log 2$.

(b) Next we apply Proposition 2.1.6 with $f = \exp$, $S = 1$ and $g(z) = \log(1+z)$ to obtain $\exp(\log g) = g$. □

2.3.1 The Exponential Function on Nilpotent Matrices

Proposition 2.3.3. *Let*

$$U := \{g \in \mathrm{GL}_d(\mathbb{K})\colon (g-\mathbf{1})^d = 0\}$$

be the set of unipotent matrices and

$$N := \{x \in M_d(\mathbb{K})\colon x^d = 0\} = U - \mathbf{1}$$

the set of nilpotent matrices. Then $\exp_U := \exp|_N\colon N \to U$ is a homeomorphism whose inverse is given by

$$\log_U\colon g \mapsto \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(g-\mathbf{1})^k}{k} = \sum_{k=1}^{d-1}(-1)^{k+1}\frac{(g-\mathbf{1})^k}{k}.$$

Proof. First we observe that for $x \in N$ we have

$$e^x - \mathbf{1} = xa \quad \text{with} \quad a := \sum_{n=1}^{d}\frac{1}{n!}x^{n-1}.$$

In view of $xa = ax$, this leads to $(e^x - \mathbf{1})^d = x^d a^d = 0$. Therefore $\exp_U(N) \subseteq U$. Similarly we obtain for $g \in U$ that

$$\log_U(g) = (g - \mathbf{1}) \sum_{k=1}^{d} (-1)^{k+1} \frac{(g-\mathbf{1})^{k-1}}{k} \in N.$$

For $x \in N$, the curve

$$F \colon \mathbb{R} \to M_d(\mathbb{K}), \quad t \mapsto \log_U \exp_U(tx)$$

is a polynomial function and Proposition 2.3.2 implies that $F(t) = tx$ for $\|tx\| < \log 2$. This implies that $F(t) = tx$ for each $t \in \mathbb{R}$ and hence that $\log_U \exp_U(x) = F(1) = x$.

Likewise we see that for $g = \mathbf{1} + x \in U$ the curve

$$G \colon \mathbb{R} \to M_d(\mathbb{K}), \quad t \mapsto \exp_U \log_U(\mathbf{1} + tx)$$

is polynomial with $G(t) = \mathbf{1} + tx$ for $\|tx\| < 1$. Therefore $\exp_U \log_U(g) = F(1) = \mathbf{1} + x = g$. This proves that the functions $\exp_U$ and $\log_U$ are inverse to each other. □

Corollary 2.3.4. *Let $X \in \mathrm{End}(V)$ be a nilpotent endomorphism of the $\mathbb{K}$-vector space V and $v \in V$. Then the following are equivalent:*

(1) $Xv = 0$.
(2) $e^X v = v$.

Proof. Clearly $Xv = 0$ implies $e^X v = \sum_{n=0}^{\infty} \frac{1}{n!} X^n v = v$. If, conversely, $e^X v = v$, then $Xv = \log(e^X)v = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(e^X - \mathbf{1})^k}{k} v = 0$. □

2.3.2 The Exponential Function on Hermitian Matrices

For the following proof, we recall that for a hermitian $d \times d$-matrix A we have

$$\|A\| = \max\{|\lambda| \colon \det(A - \lambda\mathbf{1}) = 0\}$$

(Exercise 2.2.1).

Proposition 2.3.5. *The restriction*

$$\exp_P := \exp|_{\mathrm{Herm}_d(\mathbb{K})} \colon \mathrm{Herm}_d(\mathbb{K}) \to \mathrm{Pd}_d(\mathbb{K})$$

is a diffeomorphism onto the open subset $\mathrm{Pd}_d(\mathbb{K})$ of $\mathrm{Herm}_d(\mathbb{K})$.

Proof. We have $(e^x)^* = e^{x^*}$, which implies that $\exp x$ is hermitian if x is hermitian. Moreover, if $\lambda_1, \ldots, \lambda_d$ are the real eigenvalues of x, then $e^{\lambda_1}, \ldots, e^{\lambda_d}$ are the eigenvalues of e^x. Therefore e^x is positive definite for each hermitian matrix x.

If, conversely, $g \in \mathrm{Pd}_d(\mathbb{K})$, then let $v_1, \ldots, v_d$ be an orthonormal basis of eigenvectors for g with $gv_j = \lambda_j v_j$. Then $\lambda_j > 0$ for each j, and we define

$\log_H(g) \in \mathrm{Herm}_d(\mathbb{K})$ by $\log_H(g)v_j := (\log \lambda_j)v_j$, $j = 1, \ldots, d$. From this construction of the logarithm function it is clear that

$$\log_H \circ \exp_P = \mathrm{id}_{\mathrm{Herm}_d(\mathbb{K})} \quad \text{and} \quad \exp_P \circ \log_H = \mathrm{id}_{\mathrm{Pd}_d(\mathbb{K})}.$$

For two real numbers $x, y > 0$, we have $\log(xy) = \log x + \log y$. From this we obtain for $\lambda > 0$ the relation

$$\log_H(\lambda g) = (\log \lambda)\,\mathbf{1} + \log_H(g) \tag{2.6}$$

by following what happens on each eigenspace of g.

The relation

$$\log(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-\mathbf{1})^k}{k}$$

for $x \in \mathbb{R}$ with $|x-1| < 1$ implies that for $\|g-\mathbf{1}\| < 1$ we have

$$\log_H(g) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(g-\mathbf{1})^k}{k}.$$

This proves that $\log_H$ is smooth in $B_1(\mathbf{1}) \cap \mathrm{Herm}_d(\mathbb{K})$, hence in a neighborhood of g_0 if $\|g_0 - \mathbf{1}\| < 1$ (Lemma 2.3.1). This condition means that for each eigenvalue μ of g_0 we have $|\mu - 1| < 1$ (Exercise 2.3.1). If it is not satisfied, then we choose $\lambda > 0$ such that $\|\lambda g\| < 2$. Then $\|\lambda g - \mathbf{1}\| < 1$, and we obtain with (2.6) the formula

$$\log_H(g) = -(\log \lambda)\mathbf{1} + \log_H(\lambda g) = -(\log \lambda)\mathbf{1} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\lambda g - \mathbf{1})^k}{k}.$$

Therefore $\log_H$ is smooth on the entire open cone $\mathrm{Pd}_d(\mathbb{K})$, so that $\log_H = \exp_P^{-1}$ implies that $\exp_P$ is a diffeomorphism. □

With Proposition 1.1.5, we thus obtain:

Corollary 2.3.6. *The group* $\mathrm{GL}_d(\mathbb{K})$ *is homeomorphic to*

$$\mathrm{U}_d(\mathbb{K}) \times \mathbb{R}^m \quad \textit{with} \quad m := \dim_{\mathbb{R}}(\mathrm{Herm}_d(\mathbb{K})) = \begin{cases} \frac{d(d+1)}{2} & \textit{for } \mathbb{K} = \mathbb{R} \\ d^2 & \textit{for } \mathbb{K} = \mathbb{C}. \end{cases}$$

Exercises for Section 2.3.

Exercise 2.3.1. Show that for a hermitian matrix $A \in \mathrm{Herm}_n(\mathbb{K})$ and the euclidian norm $\|\cdot\|$ on $\mathbb{K}^n$ we have

$$\|A\| := \sup\{\|Ax\| \colon \|x\| \leq 1\} = \max\{|\lambda| \colon \ker(A - \lambda\mathbf{1}) = 0\}.$$

Exercise 2.3.2. The exponential function $\exp : M_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is surjective.

2.4 The Baker–Campbell–Dynkin–Hausdorff Formula

In this section we derive a formula which expresses the product $\exp x \exp y$ of two sufficiently small elements as the exponential image $\exp(x * y)$ of an element $x * y$ which can be described in terms of iterated commutator brackets. This implies in particular that the group multiplication in a small $\mathbf{1}$-neighborhood of $\mathrm{GL}_n(\mathbb{K})$ is completely determined by the commutator bracket. To obtain these results, we express $\log(\exp x \exp y)$ as a power series $x * y$ in two variables. The (local) multiplication $*$ is called the *Baker–Campbell–Dynkin–Hausdorff Multiplication* [1] and the identity

$$\log(\exp x \exp y) = x * y$$

the *Baker–Campbell–Dynkin–Hausdorff Formula* (BCDH). The derivation of this formula requires some preparation. We start with the *adjoint representation of* $\mathrm{GL}_n(\mathbb{K})$. This is the group homomorphism

$$\mathrm{Ad}\colon \mathrm{GL}_n(\mathbb{K}) \to \mathrm{Aut}(M_n(\mathbb{K})), \quad \mathrm{Ad}(g)x = gxg^{-1},$$

where $\mathrm{Aut}(M_n(\mathbb{K}))$ stands for the group of algebra automorphisms of $M_n(\mathbb{K})$. For $x \in M_n(\mathbb{K})$, we further define a linear map

$$\mathrm{ad}(x)\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad \mathrm{ad}\, x(y) := [x, y].$$

Lemma 2.4.1. *For each* $x \in M_n(\mathbb{K})$,

$$\mathrm{Ad}(\exp x) = \exp(\mathrm{ad}\, x). \tag{2.7}$$

Proof. We define the linear maps

$$\lambda_x\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad y \mapsto xy, \quad \rho_x\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad y \mapsto yx.$$

Then $\lambda_x \rho_x = \rho_x \lambda_x$ and $\mathrm{ad}\, x = \lambda_x - \rho_x$, so that Lemma 2.2.2(ii) leads to

$$\mathrm{Ad}(\exp x)y = e^x y e^{-x} = e^{\lambda_x} e^{-\rho_x} y = e^{\lambda_x - \rho_x} y = e^{\mathrm{ad}\, x} y.$$

This proves (2.7). □

Proposition 2.4.2. *Let* $x \in M_n(\mathbb{K})$ *and* $\lambda_{\exp x}(y) := (\exp x)y$ *the left multiplication by* $\exp x$. *Then*

$$\mathtt{d}\exp(x) = \lambda_{\exp x} \circ \frac{\mathbf{1} - e^{-\mathrm{ad}\, x}}{\mathrm{ad}\, x} \colon M_n(\mathbb{K}) \to M_n(\mathbb{K}),$$

where the fraction on the right means $\Phi(\mathrm{ad}\, x)$ *for the entire function*

$$\Phi(z) := \frac{1 - e^{-z}}{z} = \sum_{k=1}^{\infty} \frac{(-z)^{k-1}}{k!}.$$

The series $\Phi(x)$ *converges for each* $x \in M_n(\mathbb{K})$.

[1] See [Bak01, Bak05, Cam97, Cam98, Dyn53, Hau06]

Proof. First let $\alpha\colon [0,1] \to M_n(\mathbb{K})$ be a smooth curve. Then

$$\gamma(t,s) := \exp(-s\alpha(t))\frac{d}{dt}\exp(s\alpha(t))$$

defines a map $[0,1]^2 \to M_n(\mathbb{K})$ which is C^1 in each argument and satisfies $\gamma(t,0) = 0$ for each t. We calculate

$$\begin{aligned}\frac{\partial\gamma}{\partial s}(t,s) &= \exp(-s\alpha(t))\cdot(-\alpha(t))\frac{d}{dt}\exp(s\alpha(t)) \\ &\quad + \exp(-s\alpha(t))\cdot\frac{d}{dt}\Big(\alpha(t)\exp(s\alpha(t))\Big) \\ &= \exp(-s\alpha(t))\cdot(-\alpha(t))\frac{d}{dt}\exp(s\alpha(t)) \\ &\quad + \exp(-s\alpha(t))\cdot\Big(\alpha'(t)\exp(s\alpha(t)) + \alpha(t)\frac{d}{dt}\exp(s\alpha(t))\Big) \\ &= \mathrm{Ad}(\exp(-s\alpha(t)))\alpha'(t) = e^{-s\,\mathrm{ad}\,\alpha(t)}\alpha'(t).\end{aligned}$$

Integration over $[0,1]$ with respect to s now leads to

$$\gamma(t,1) = \gamma(t,0) + \int_0^1 e^{-s\,\mathrm{ad}\,\alpha(t)}\alpha'(t)\,ds = \int_0^1 e^{-s\,\mathrm{ad}\,\alpha(t)}\,ds\cdot\alpha'(t).$$

Next we note that, for $x \in M_n(\mathbb{K})$,

$$\begin{aligned}\int_0^1 e^{-s\,\mathrm{ad}\,x}\,ds &= \int_0^1\sum_{k=0}^\infty \frac{(-\,\mathrm{ad}\,x)^k}{k!}s^k\,ds = \sum_{k=0}^\infty(-\,\mathrm{ad}\,x)^k\int_0^1\frac{s^k}{k!}\,ds \\ &= \sum_{k=0}^\infty\frac{(-\,\mathrm{ad}\,x)^k}{(k+1)!} = \Phi(\mathrm{ad}\,x).\end{aligned}$$

We thus obtain for $\alpha(t) = x + ty$ with $\alpha(0) = x$ and $\alpha'(0) = y$ the relation

$$\exp(-x)\mathtt{d}\exp(x)y = \gamma(0,1) = \int_0^1 e^{-s\,\mathrm{ad}\,x}y\,ds = \Phi(\mathrm{ad}\,x)y.$$

□

Lemma 2.4.3. *For*

$$\Phi(z) = \frac{1-e^{-z}}{z} := \sum_{k=1}^\infty(-1)^{k-1}\frac{z^{k-1}}{k!}, \qquad z \in \mathbb{C}$$

and

$$\Psi(z) = \frac{z\log z}{z-1} := z\sum_{k=0}^\infty\frac{(-1)^k}{k+1}(z-1)^k \quad \text{for } |z-1| < 1$$

we have

$$\Psi(e^z)\Phi(z) = 1 \quad \text{for} \quad z \in \mathbb{C}, |z| < \log 2.$$

Proof. If $|z| < \log 2$, then $|e^z - 1| < 1$ and we obtain from $\log(e^z) = z$:

$$\Psi(e^z)\Phi(z) = \frac{e^z z}{e^z - 1}\frac{1 - e^{-z}}{z} = 1.$$

□

In view of the Composition Formula (2.1) (Proposition 2.1.6), the same identity as in Lemma 2.4.3 holds if we insert matrices $L \in \mathrm{End}(\mathfrak{gl}_n(\mathbb{K}))$ with $\|L\| < \log 2$ into the power series Φ and Ψ:

$$\Psi(\exp L)\Phi(L) = (\Psi \circ \exp)(L)\Phi(L) = ((\Psi \circ \exp) \cdot \Phi)(L) = \mathrm{id}_{\mathfrak{gl}_n(\mathbb{K})}. \qquad (2.8)$$

Here we use that $\|L\| < \log 2$ implies that all expressions are defined and in particular that $\|\exp L - \mathbf{1}\| < 1$, as a consequence of the estimate

$$\|\exp L - \mathbf{1}\| \leq e^{\|L\|} - 1. \qquad (2.9)$$

The derivation of the BCDH formula follows a similar scheme as the proof of Proposition 2.4.2. Here we consider $x, y \in V_o := B(0, \log\sqrt{2})$. For $\|x\|, \|y\| < r$ the estimate (2.9) leads to

$$\begin{aligned}\|\exp x \exp y - \mathbf{1}\| &= \|(\exp x - \mathbf{1})(\exp y - \mathbf{1}) + (\exp y - \mathbf{1}) + (\exp x - \mathbf{1})\| \\ &\leq \|\exp x - \mathbf{1}\| \cdot \|\exp y - \mathbf{1}\| + \|\exp y - \mathbf{1}\| + \|\exp x - \mathbf{1}\| \\ &< (e^r - 1)^2 + 2(e^r - 1) = e^{2r} - 1.\end{aligned}$$

For $r < \log\sqrt{2} = \frac{1}{2}\log 2$ and $|t| \leq 1$ we obtain in particular

$$\|\exp x \exp ty - \mathbf{1}\| < e^{\log 2} - 1 = 1.$$

Therefore $\exp x \exp ty$ lies for $|t| \leq 1$ in the domain of the logarithm function (Lemma 2.3.1). We therefore define for $t \in [-1, 1]$:

$$F(t) = \log(\exp x \exp ty).$$

To estimate the norm of $F(t)$, we note that for $g := \exp x \exp ty$, $|t| \leq 1$, and $\|x\|, \|y\| < r$ we have

$$\begin{aligned}\|\log g\| &\leq \sum_{k=1}^{\infty} \frac{\|g - 1\|^k}{k} = -\log(1 - \|g - \mathbf{1}\|) \\ &< -\log(1 - (e^{2r} - 1)) = -\log(2 - e^{2r}).\end{aligned}$$

For $r := \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2}) < \frac{\log 2}{2} = \log\sqrt{2}$ and $\|x\|, \|y\| < r$ this leads to

$$\|F(t)\| < -\log(2 - e^{2r}) = \log(\tfrac{2}{\sqrt{2}}) = \log(\sqrt{2}). \qquad (2.10)$$

Next we calculate $F'(t)$ with the goal to obtain the BCDH formula as $F(1) = F(0) + \int_0^1 F'(t)\, dt$. For the derivative of the curve $t \mapsto \exp F(t)$ we get

$$\begin{aligned}(\mathtt{d}\exp)\big(F(t)\big)F'(t) &= \frac{d}{dt}\exp(F(t)) = \frac{d}{dt}\exp x \exp ty \\ &= (\exp x \exp ty)y = (\exp F(t))y.\end{aligned}$$

Using Proposition 2.4.2, we obtain

$$\begin{aligned}y &= \big(\exp F(t)\big)^{-1}(\mathtt{d}\exp)\big(F(t)\big)F'(t) \\ &= \frac{\mathbf{1} - e^{-\operatorname{ad} F(t)}}{\operatorname{ad} F(t)}F'(t) = \Phi(\operatorname{ad} F(t))F'(t). \qquad (2.11)\end{aligned}$$

We claim that $\|\operatorname{ad}(F(t))\| < \log 2$. From $\|ab - ba\| \leq 2\|a\|\,\|b\|$ we derive

$$\|\operatorname{ad} a\| \leq 2\|a\| \quad \text{for} \quad a \in \mathfrak{gl}_n(\mathbb{K}).$$

Therefore, by (2.10),

$$\|\operatorname{ad} F(t)\| \leq 2\|F(t)\| < 2\log(\sqrt{2}) = \log 2,$$

so that (2.11) and (2.8) lead to

$$F'(t) = \Psi\big(\exp(\operatorname{ad} F(t))\big)y. \qquad (2.12)$$

Proposition 2.4.4. *For* $\|x\|, \|y\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$ *we have*

$$\log(\exp x \exp y) = x + \int_0^1 \Psi\big(\exp(\operatorname{ad} x)\exp(t \operatorname{ad} y)\big)y\,dt \in \mathfrak{g},$$

with Ψ *as in* Lemma 2.4.3.

Proof. With (2.12), Lemma 2.4.1 and the preceding remarks we get

$$\begin{aligned}F'(t) &= \Psi\big(\exp(\operatorname{ad} F(t))\big)y \\ &= \Psi\big(\operatorname{Ad}(\exp F(t))\big)y = \Psi\big(\operatorname{Ad}(\exp x \exp ty)\big)y \\ &= \Psi\big(\operatorname{Ad}(\exp x)\operatorname{Ad}(\exp ty)\big)y = \Psi\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty)\big)y.\end{aligned}$$

Moreover, we have $F(0) = \log(\exp x) = x$. By integration we therefore obtain

$$\log(\exp x \exp y) = x + \int_0^1 \Psi\big(\exp(\operatorname{ad} x)\exp(t \operatorname{ad} y)\big)y\,dt.$$

□

Proposition 2.4.5. *For* $x, y \in \mathfrak{gl}_n(\mathbb{K})$ *and* $\|x\|, \|y\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$,

$$\begin{aligned}x * y &:= \log(\exp x \exp y) \\ &= x + \\ &\sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1} \ldots (\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m}{p_1! q_1! \ldots p_k! q_k! m!} y.\end{aligned}$$

Proof. We only have to rewrite the expression in Proposition 2.4.4:

$$\begin{aligned}
&\int_0^1 \Psi\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty)\big)y\,dt \\
&= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty) - \mathrm{id}\big)^k}{(k+1)}\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty)\big)y\,dt \\
&= \int_0^1 \sum_{\substack{k\geq 0\\ p_i+q_i>0}} \frac{(-1)^k}{(k+1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} ty)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} ty)^{q_k}}{p_1!q_1!\ldots p_k!q_k!}\exp(\operatorname{ad} x)y\,dt \\
&= \sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} \frac{(-1)^k}{(k+1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m}{p_1!q_1!\ldots p_k!q_k!m!}y\int_0^1 t^{q_1+\ldots+q_k}\,dt \\
&= \sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} \frac{(-1)^k(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m y}{(k+1)(q_1+\ldots+q_k+1)p_1!q_1!\ldots p_k!q_k!m!}.
\end{aligned}$$

□

The power series in Proposition 2.4.5 is called the *Hausdorff Series.* We observe that it does not depend on n. For practical purposes it often suffices to know the first terms of the Hausdorff Series:

Corollary 2.4.6. *For $x, y \in \mathfrak{gl}_n(\mathbb{K})$ and $\|x\|, \|y\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$,*

$$x * y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \ldots$$

Proof. One has to collect the summands in Proposition 2.4.5 corresponding to $p_1 + q_1 + \ldots + p_k + q_k + m \leq 2$. □

Product and Commutator Formula

We have seen in Proposition 2.2.1 that the exponential image of a sum $x + y$ can be computed easily if x and y commute. In this case we also have for the commutator $[x, y] := xy - yx = 0$ the formula $\exp[x, y] = \mathbf{1}$. The following proposition gives a formula for $\exp(x + y)$ and $\exp[x, y]$ in the general case.

If g, h are elements of a group G, then $(g, h) := ghg^{-1}h^{-1}$ is called their *commutator*. On the other hand, we call for two matrices $A, B \in M_n(\mathbb{K})$ the expression

$$[A, B] := AB - BA$$

their *commutator bracket.*

Proposition 2.4.7. *For $x, y \in M_d(\mathbb{K})$ the following assertions hold:*

(i) (Trotter Product Formula) $\lim_{k\to\infty}\left(e^{\frac{1}{k}x}e^{\frac{1}{k}y}\right)^k = e^{x+y}$.

(ii) (Commutator Formula) $\lim_{k\to\infty}\left(e^{\frac{1}{k}x}e^{\frac{1}{k}y}e^{-\frac{1}{k}x}e^{-\frac{1}{k}y}\right)^{k^2} = e^{xy-yx}$.

Proof. (i) From Corollary 2.4.6 we obtain that $\lim_{k\to\infty} k\cdot\left(\frac{x}{k} * \frac{y}{k}\right) = x+y$. Applying the exponential function, we obtain (i).

(ii) We consider the function

$$\gamma(t) := tx * ty * (-tx) * (-ty),$$

which is defined and smooth on some interval $[-\varepsilon,\varepsilon]\subseteq\mathbb{R}$, $\varepsilon>0$. In view of

$$\exp(x*y*(-x)) = \exp x\exp y\exp(-x) = \exp\big(\mathrm{Ad}(\exp x)y\big) = \exp(e^{\mathrm{ad}\,x}y)$$

for x, y small enough (Lemma 2.4.1), we have

$$x * y * (-x) = e^{\mathrm{ad}\,x}y, \tag{2.13}$$

and therefore Taylor expansion with respect to t yields

$$\begin{aligned}\gamma(t) &= tx * ty * (-tx) * (-ty) = e^{t\,\mathrm{ad}\,x}ty * (-ty)\\ &= (ty + t^2[x,y] + \frac{t^3}{2}[x,[x,y]] + \ldots) * (-ty)\\ &= ty + t^2[x,y] - ty + [ty,-ty] + t^2r(t) = t^2[x,y] + t^2r(t),\end{aligned}$$

where $\lim_{t\to0} r(t) = 0$. We now have

$$\gamma(0) = \gamma'(0) = 0 \quad \text{and} \quad \frac{\gamma''(0)}{2} = [x,y].$$

This leads to

$$\lim_{k\to\infty} k^2\cdot\big(\frac{1}{k}x\big) * \big(\frac{1}{k}y\big) * \big(-\frac{1}{k}x\big) * \big(-\frac{1}{k}y\big) = \frac{\gamma''(0)}{2} = [x,y]. \tag{2.14}$$

Applying exp leads to the Commutator Formula. □

Notes on Chapter 2

Many of the results discussed in this chapter are valid in much greater generality. The Baker–Campbell-Dynkin–Hausdorff formula, for example, holds for general Lie groups as we will see in Chapter 8. Other results which are obtained via converging power series can easily be generalized to subgroups of Banach algebras.

3

Linear Lie Groups

We call a closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ a *linear Lie group*. In this section we shall use the exponential function to assign to each linear Lie group G a vector space

$$\mathbf{L}(G) := \{x \in M_n(\mathbb{K}) \colon \exp(\mathbb{R}x) \subseteq G\},$$

called the *Lie algebra of* G. This subspace carries an additional algebraic structure because, for $x, y \in \mathbf{L}(G)$, the commutator $[x, y] = xy - yx$ is contained in $\mathbf{L}(G)$, so that $[\cdot, \cdot]$ defines a skew-symmetric bilinear operation on $\mathbf{L}(G)$. As a first step, we shall see how to calculate $\mathbf{L}(G)$ for concrete groups and to use it to generalize the polar decomposition to a large class of linear Lie groups.

3.1 The Lie Algebra of a Linear Lie Group

We start with the introduction of the concept of a Lie algebra.

Definition 3.1.1. (a) Let k be a field and L a k-vector space. A bilinear map $[\cdot, \cdot] \colon L \times L \to L$ is called a *Lie bracket* if

(L1) $[x, x] = 0$ for $x \in L$ and
(L2) $\big[x, [y, z]\big] = \big[[x, y], z\big] + \big[y, [x, z]\big]$ for $x, y, z \in L$ (Jacobi identity).[1]

A *Lie algebra* [2] (over k) is a k-vector space L, endowed with a Lie bracket. A subspace $E \subseteq L$ of a Lie algebra is called a *subalgebra* if $[E, E] \subseteq E$. A *homomorphism* $\varphi \colon L_1 \to L_2$ of Lie algebras is a linear map with $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for $x, y \in L_1$. A Lie algebra is said to be *abelian* if $[x, y] = 0$ holds for all $x, y \in L$.

[1] Carl Gustav Jacob Jacobi (1804–1851), mathematician in Berlin and Königsberg (Kaliningrad). He found his famous identity about 1830 in the context of Poisson brackets, which are related to Hamiltonian Mechanics and Symplectic Geometry.
[2] The notion of a Lie algebra was coined in the 1920s by Hermann Weyl.

The following lemma shows that each associative algebra also carries a natural Lie algebra structure.

Lemma 3.1.2. *Each associative algebra A is a Lie algebra A_L with respect to the commutator bracket*

$$[a, b] := ab - ba.$$

Proof. (L1) is obvious. For (L2) we calculate

$$[a, bc] = abc - bca = (ab - ba)c + b(ac - ca) = [a, b]c + b[a, c],$$

and this implies

$$[a, [b, c]] = [a, b]c + b[a, c] - [a, c]b - c[a, b] = [[a, b], c] + [b, [a, c]].$$

□

Definition 3.1.3. A closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ is called a *linear Lie group*. For each subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ we define the set

$$\mathbf{L}(G) := \{x \in M_n(\mathbb{K}) \colon \exp(\mathbb{R}x) \subseteq G\}$$

and observe that $\mathbb{R}\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$ follows immediately from the definition.

We could also define this notion in more abstract terms by considering a finite-dimensional $\mathbb{K}$-vector space V and call a closed subgroup $G \subseteq \mathrm{GL}(V)$ a linear Lie group. Then

$$\mathbf{L}(G) = \{x \in \mathrm{End}(V) \colon \exp(\mathbb{R}x) \subseteq G\}.$$

In the following we shall use both pictures.

From Lemma 3.1.2 we know that the associative algebra $M_n(\mathbb{K})$ is a Lie algebra with respect to the matrix commutator $[x, y] := xy - yx$. We denote this Lie algebra by $\mathfrak{gl}_n(\mathbb{K}) := M_n(\mathbb{K})_L$. We likewise write $\mathfrak{gl}(V) := \mathrm{End}(V)_L$ for a vector space V.

The next proposition assigns a Lie algebra to each linear Lie group.

Proposition 3.1.4. *If $G \subseteq \mathrm{GL}(V)$ is a closed subgroup, then $\mathbf{L}(G)$ is a real Lie subalgebra of $\mathfrak{gl}(V)$ and we obtain a map*

$$\exp_G \colon \mathbf{L}(G) \to G, \quad x \mapsto e^x.$$

We call $\mathbf{L}(G)$ the *Lie algebra of G*.

In particular,

$$\mathbf{L}(\mathrm{GL}(V)) = \mathfrak{gl}(V) \quad \text{and} \quad \mathbf{L}(\mathrm{GL}_n(\mathbb{K})) = \mathfrak{gl}_n(\mathbb{K}).$$

Proof. Let $x, y \in \mathbf{L}(G)$. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we then have $\exp \frac{t}{k}x, \exp \frac{t}{k}y \in G$ and with the Trotter Formula (Proposition 2.4.7), we get for all $t \in \mathbb{R}$:

$$\exp(t(x+y)) = \lim_{k\to\infty}\left(\exp\frac{tx}{k}\exp\frac{ty}{k}\right)^k \in G$$

because G is closed. Therefore $x + y \in \mathbf{L}(G)$.

Similarly we use the Commutator Formula to get

$$\exp t[x,y] = \lim_{k\to\infty}\left(\exp\frac{tx}{k}\exp\frac{y}{k}\exp -\frac{tx}{k}\exp -\frac{y}{k}\right)^{k^2} \in G,$$

hence $[x, y] \in \mathbf{L}(G)$. □

Lemma 3.1.5. *Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a subgroup. If $\mathrm{Hom}(\mathbb{R}, G)$ denotes the set of all continuous group homomorphisms $(\mathbb{R}, +) \to G$, then the map*

$$\Gamma\colon\ \mathbf{L}(G) \to \mathrm{Hom}(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) = \exp(tx)$$

is a bijection.

Proof. For each $x \in \mathbf{L}(G)$, the map γ_x is a continuous group homomorphism (Theorem 2.2.6), and since $x = \gamma_x'(0)$, the map Γ is injective. To see that it is surjective, let $\gamma\colon \mathbb{R} \to G$ be a continuous group homomorphism and $\iota\colon G \to \mathrm{GL}_n(\mathbb{K})$ the natural embedding. Then $\iota \circ \gamma\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ is a continuous group homomorphism, so that there exists an $x \in \mathfrak{gl}_n(\mathbb{K})$ with $\gamma(t) = \iota(\gamma(t)) = e^{tx}$ for all $t \in \mathbb{R}$ (Theorem 2.2.6). This implies that $x \in \mathbf{L}(G)$, and therefore that $\gamma_x = \gamma$. □

Remark 3.1.6. The preceding lemma implies in particular that for a linear Lie group the set $\mathbf{L}(G)$ can also be defined in terms of the topological group structure on G as $\mathcal{L}(G) := \mathrm{Hom}(\mathbb{R}, G)$, the set of continuous one-parameter groups. From the Trotter Formula and the Commutator Formula we also know that the Lie algebra structure on $\mathcal{L}(G)$ can be defined intrinsically by

$$(\lambda\gamma)(t) := \gamma(\lambda t),$$

$$(\gamma_1 + \gamma_2)(t) := \lim_{n\to\infty}\left(\gamma_1(\tfrac{t}{n})\gamma_2(\tfrac{t}{n})\right)^{\frac{1}{n}}$$

and

$$[\gamma_1, \gamma_2](t) := \lim_{n\to\infty}\left(\gamma_1(\tfrac{t}{n})\gamma_2(\tfrac{1}{n})\gamma_1(-\tfrac{t}{n})\gamma_2(-\tfrac{1}{n})\right)^{\frac{1}{n^2}}.$$

This shows that the Lie algebra $\mathbf{L}(G)$ does not depend on the special realization of G as a group of matrices.

Example 3.1.7. We consider the homomorphism

$$\Phi\colon \mathbb{K}^n \to \mathrm{GL}_{n+1}(\mathbb{K}), \quad x \mapsto \begin{pmatrix} \mathbf{1} & x \\ 0 & 1 \end{pmatrix}$$

and observe that Φ is an isomorphism of the topological group $(\mathbb{K}^n, +)$ onto a linear Lie group.

The continuous one-parameter groups $\gamma\colon \mathbb{R} \to \mathbb{K}^n$ are easily determined because $\gamma(nt) = n\gamma(t)$ for all $n \in \mathbb{Z}$, $t \in \mathbb{R}$, implies further $\gamma(q) = q\gamma(1)$ for all $q \in \mathbb{Q}$ and hence, by continuity, $\gamma(t) = t\gamma(1)$ for all $t \in \mathbb{R}$. Since $(\mathbb{K}^n, +)$ is abelian, the Lie bracket on the Lie algebra $\mathbf{L}(\mathbb{K}^n, +)$ vanishes, and we obtain

$$\mathbf{L}(\mathbb{K}^n, +) = (\mathbb{K}^n, 0) \cong \mathbf{L}(\Phi(\mathbb{K}^n)) = \left\{ \begin{pmatrix} \mathbf{0} & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{K}^n \right\}$$

(Exercise).

3.1.1 Functorial Properties of the Lie Algebra

So far we have assigned to each linear Lie group G its Lie algebra $\mathbf{L}(G)$. We shall also see that this assignment can be extended to continuous homomorphisms between linear Lie groups in the sense that we assign to each such homomorphism $\varphi\colon G_1 \to G_2$ a homomorphism $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ of Lie algebras, and this assignment satisfies

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)} \quad \text{and} \quad \mathbf{L}(\varphi_1 \circ \varphi_2) = \mathbf{L}(\varphi_1) \circ \mathbf{L}(\varphi_2)$$

for a composition $\varphi_1 \circ \varphi_2$ of two continuous homomorphisms $\varphi_1\colon G_2 \to G_1$ and $\varphi_2\colon G_3 \to G_2$. In the language of category theory, this means that $\mathbf{L}$ defines a functor from the category of linear Lie groups (where the morphisms are the continuous group homomorphisms) to the category of real Lie algebras.

Proposition 3.1.8. *Let $\varphi\colon G_1 \to G_2$ be a continuous group homomorphism of linear Lie groups. Then the derivative*

$$\mathbf{L}(\varphi)(x) := \frac{d}{dt}\Big|_{t=0} \varphi(\exp_{G_1}(tx))$$

exists for each $x \in \mathbf{L}(G_1)$ and defines a homomorphism of Lie algebras $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}, \tag{3.1}$$

i.e., the following diagram commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \uparrow{\scriptstyle \exp_{G_1}} & & \uparrow{\scriptstyle \exp_{G_2}} \\ \mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(G_2). \end{array}$$

Then $\mathbf{L}(\varphi)$ is the uniquely determined linear map satisfying (3.1).

Proof. For $x \in \mathbf{L}(G_1)$ we consider the homomorphism $\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1)$ given by $\gamma_x(t) = e^{tx}$. According to Lemma 3.1.5, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for some $y \in \mathbf{L}(G_2)$, because $\varphi \circ \gamma_x \colon \mathbb{R} \to G_2$ is a continuous group homomorphism. Then clearly $y = (\varphi \circ \gamma_x)'(0) = \mathbf{L}(\varphi)x$. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x),$$

which is (3.1).

Conversely, every linear map $\psi \colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ \psi = \varphi \circ \exp_{G_1}$$

satisfies

$$\varphi \circ \exp_{G_1}(tx) = \exp_{G_2}(\psi(tx)) = \exp_{G_2}(t\psi(x)),$$

and therefore

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} \exp_{G_2}(t\psi(x)) = \psi(x).$$

Next we show that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. From the definition of $\mathbf{L}(\varphi)$ we immediately get for $x \in \mathbf{L}(G_1)$:

$$\exp_{G_2}(s\,\mathbf{L}(\varphi)(tx)) = \varphi(\exp_{G_1}(stx)) = \exp_{G_2}(ts\,\mathbf{L}(\varphi)(x)), \quad s, t \in \mathbb{R},$$

which leads to $\mathbf{L}(\varphi)(tx) = t\,\mathbf{L}(\varphi)(x)$.

Since φ is continuous, the Trotter Formula implies that

$$\begin{aligned}
&\exp_{G_2}(\mathbf{L}(\varphi)(x+y)) = \varphi\big(\exp_{G_1}(x+y)\big) \\
&= \lim_{k\to\infty} \varphi\Big(\exp_{G_1}\frac{1}{k}x \exp_{G_1}\frac{1}{k}y\Big)^k = \lim_{k\to\infty} \Big(\varphi\Big(\exp_{G_1}\frac{1}{k}x\Big)\varphi\Big(\exp_{G_1}\frac{1}{k}y\Big)\Big)^k \\
&= \lim_{k\to\infty} \Big(\exp_{G_2}\frac{1}{k}\,\mathbf{L}(\varphi)(x) \exp_{G_2}\frac{1}{k}\,\mathbf{L}(\varphi)(y)\Big)^k \\
&= \exp_{G_2}\big(\mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)\big)
\end{aligned}$$

for all $x, y \in \mathbf{L}(G_1)$. Therefore $\mathbf{L}(\varphi)(x+y) = \mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)$ because the same formula holds with tx and ty instead of x and y. Hence $\mathbf{L}(\varphi)$ is additive and therefore linear.

We likewise obtain with the Commutator Formula

$$\varphi(\exp[x,y]) = \exp[\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$$

and thus $\mathbf{L}(\varphi)([x,y]) = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$. □

Corollary 3.1.9. *If $\varphi_1 : G_1 \to G_2$ and $\varphi_2 \colon G_2 \to G_3$ are continuous homomorphisms of linear Lie groups, then*

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Moreover, $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$.

Proof. We have the relations

$$\varphi_1 \circ \exp_{G_1} = \exp_{G_2} \circ \mathbf{L}(\varphi_1) \quad \text{and} \quad \varphi_2 \circ \exp_{G_2} = \exp_{G_3} \circ \mathbf{L}(\varphi_2),$$

which immediately lead to

$$(\varphi_2 \circ \varphi_1) \circ \exp_{G_1} = \varphi_2 \circ \exp_{G_2} \circ \mathbf{L}(\varphi_1) = \exp_{G_3} \circ (\mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)),$$

and the uniqueness assertion of Proposition 3.1.8 implies that

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Clearly $\mathrm{id}_{\mathbf{L}(G)}$ is a linear map satisfying $\exp_G \circ \mathrm{id}_{\mathbf{L}(G)} = \mathrm{id}_G \circ \exp_G$, so that the uniqueness assertion of Proposition 3.1.8 implies $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$. □

Corollary 3.1.10. *If $\varphi\colon G_1 \to G_2$ is an isomorphism of linear Lie groups, then $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras.*

Proof. Since φ is an isomorphism of linear Lie groups, it is bijective and $\psi := \varphi^{-1}$ also is a continuous homomorphism. We then obtain with Corollary 3.1.9 the relations $\mathrm{id}_{\mathbf{L}(G_2)} = \mathbf{L}(\mathrm{id}_{G_2}) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$ and likewise

$$\mathrm{id}_{\mathbf{L}(G_1)} = \mathbf{L}(\psi) \circ \mathbf{L}(\varphi).$$

Hence $\mathbf{L}(\varphi)$ is an isomorphism with $\mathbf{L}(\varphi)^{-1} = \mathbf{L}(\psi)$. □

Definition 3.1.11. If V is a vector space and G a group, then a homomorphism $\varphi\colon G \to \mathrm{GL}(V)$ is called a *representation of* G *on* V. If $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation of* $\mathfrak{g}$ *on* V.

As a consequence of Proposition 3.1.8, we obtain

Corollary 3.1.12. *If $\varphi\colon G \to \mathrm{GL}(V)$ is a continuous representation of the linear Lie group G, then $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathbf{L}(G)$.*

The representation $\mathbf{L}(\varphi)$ obtained in Corollary 3.1.12 from the group representation φ is called the *derived representation*. This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)x} = \frac{d}{dt}\Big|_{t=0} \varphi(\exp tx).$$

3.1.2 The Adjoint Representation

Let $G \subseteq \mathrm{GL}(V)$ be a linear Lie group and $\mathbf{L}(G) \subseteq \mathfrak{gl}(V)$ the corresponding Lie algebra. For $g \in G$ we define the conjugation automorphism $c_g \in \mathrm{Aut}(G)$ by $c_g(x) := gxg^{-1}$. Then

$$\begin{aligned}\mathbf{L}(c_g)(x) &= \frac{d}{dt}\Big|_{t=0} c_g(\exp tx) = \frac{d}{dt}\Big|_{t=0} g(\exp tx)g^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} \exp(tgxg^{-1}) = gxg^{-1}\end{aligned}$$

(Proposition 2.2.1), and therefore $\mathbf{L}(c_g) = c_g|_{\mathbf{L}(G)}$. We define the *adjoint representation of G on $\mathbf{L}(G)$* by

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G)), \quad \mathrm{Ad}(g)(x) := \mathbf{L}(c_g)x = gxg^{-1}.$$

(That this is a representation follows immediately from the explicit formula).

For each $x \in \mathbf{L}(G)$, the map $G \to \mathbf{L}(G), g \mapsto \mathrm{Ad}(g)(x) = gxg^{-1}$ is continuous and each $\mathrm{Ad}(g)$ is an automorphism of the Lie algebra $\mathbf{L}(G)$. Therefore Ad is a continuous homomorphism from the linear Lie group G to the linear Lie group $\mathrm{Aut}(\mathbf{L}(G)) \subseteq \mathrm{GL}(\mathbf{L}(G))$. The derived representation

$$\mathbf{L}(\mathrm{Ad})\colon \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation. First we define for $x \in \mathbf{L}(G)$:

$$\mathrm{ad}(x)\colon \mathbf{L}(G) \to \mathbf{L}(G), \quad \mathrm{ad}\,x(y) := [x, y] = xy - yx.$$

Lemma 3.1.13. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$.

Proof. In view of Proposition 3.1.8 this is an immediate consequence of the relation $\mathrm{Ad}(\exp x) = e^{\mathrm{ad}\,x}$ (Lemma 2.4.1). □

Exercises for Section 3.1

Exercise 3.1.1. (a) If $(G_j)_{j\in J}$ is a family of linear Lie groups in $\mathrm{GL}_n(\mathbb{K})$, then their intersection $G := \bigcap_{j\in J} G_j$ also is a linear Lie group.

(b) If $(G_j)_{j\in J}$ is a family of subgroups of $\mathrm{GL}_n(\mathbb{K})$, then

$$\mathbf{L}(\bigcap_{j\in J} G_j) = \bigcap_{j\in J} \mathbf{L}(G_j).$$

Exercise 3.1.2. Let $G := \mathrm{GL}_n(\mathbb{K})$ and $V := P_k(\mathbb{K}^n)$ the space of homogeneous polynomials of degree k in $x_1, \ldots, x_n$, considered as functions $\mathbb{K}^n \to \mathbb{K}$. Show that:

(1) $\dim V = \binom{k+n-1}{n-1}$.
(2) We obtain a continuous representation $\rho\colon G \to \mathrm{GL}(V)$ of G on V by $(\rho(g)f)(x) := f(g^{-1}x)$.
(3) The elementary matrix E_{ij} with $E_{ij}e_k = \delta_{jk}e_i$ satisfies

$$\mathbf{L}(\rho)(E_{ij}) = -x_j \frac{\partial}{\partial x_i}.$$

Exercise 3.1.3. If $X \in \mathrm{End}(V)$ is nilpotent, then $\mathrm{ad}\, X \in \mathrm{End}(\mathrm{End}(V))$ is also nilpotent.

Exercise 3.1.4. If $X, Y \in M_n(\mathbb{K})$ are nilpotent, then the following are equivalent:

(1) $\exp X \exp Y = \exp Y \exp X$.
(2) $[X, Y] = 0$.

Exercise 3.1.5. If $(V, \cdot)$ is an associative algebra, then $\mathrm{Aut}(V, \cdot) \subseteq \mathrm{Aut}(V, [\cdot, \cdot])$.

Exercise 3.1.6. (a) $\mathrm{Ad} : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{Aut}\,\big(\mathfrak{gl}_n(\mathbb{K})\big)$ is a group homomorphism.

(b) For each Lie algebra $\mathfrak{g}$, the operators $\mathrm{ad}\, x(y) := [x, y]$ are derivations and the map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras.

Exercise 3.1.7. Let V be a finite-dimensional vector space, $F \subseteq V$ a subspace and $\gamma : [0, T] \to V$ a continuous curve with $\gamma([0, T]) \subseteq F$. Then for all $t \in [0, T]$:

$$I_t := \int_0^t \gamma(\tau)\, d\tau \in F.$$

Exercise 3.1.8. On each finite-dimensional Lie algebra $\mathfrak{g}$ there exists a norm with

$$\|[x, y]\| \leq \|x\|\|y\| \qquad \forall x, y \in \mathfrak{g},$$

i.e., $\|\,\mathrm{ad}\, x\| \leq \|x\|$.

Exercise 3.1.9. Let $\mathfrak{g}$ be a Lie algebra with a norm as in Exercise 3.1.8. Then for $\|x\| + \|y\| < \ln 2$ the Hausdorff series

$$x * y = x + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\mathrm{ad}\, x)^{p_1} (\mathrm{ad}\, y)^{q_1} \ldots (\mathrm{ad}\, x)^{p_k} (\mathrm{ad}\, y)^{q_k} (\mathrm{ad}\, x)^m}{p_1! q_1! \ldots p_k! q_k! m!} y$$

converges absolutely.

Exercise 3.1.10. Let V and W be vector spaces and $q : V \times V \to W$ a skew-symmetric bilinear map. Then

$$[(v, w), (v', w')] := \big(0, q(v, v')\big)$$

is a Lie bracket on $\mathfrak{g} := V \times W$. For $x, y, z \in \mathfrak{g}$ we have $\big[x, [y, z]\big] = 0$.

Exercise 3.1.11. Let $\mathfrak{g}$ be a Lie algebra with $\big[x, [y, z]\big] = 0$ for $x, y, z \in \mathfrak{g}$. Then

$$x * y := x + y + \frac{1}{2}[x, y]$$

defines a group structure on $\mathfrak{g}$. An example for such a Lie algebra is the three-dimensional *Heisenberg algebra*

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{K} \right\}.$$

3.2 Calculating Lie Algebras of Linear Lie Groups

In this section we shall see various techniques to determine the Lie algebra of a linear Lie group.

Example 3.2.1. The group $G := \mathrm{SL}_n(\mathbb{K}) = \det^{-1}(1) = \ker\det$ is a linear Lie group. To determine its Lie algebra, we first claim that

$$\det(e^x) = e^{\operatorname{tr} x} \tag{3.2}$$

holds for $x \in M_n(\mathbb{K})$. To verify this claim, we consider

$$\det\colon M_n(\mathbb{K}) \cong (\mathbb{K}^n)^n \to \mathbb{K}$$

as a multilinear map, where each matrix x is considered as an n-tuple of its column vectors $x_1, \ldots, x_n$. Then Exercise 2.1.1(c) implies that

$$\begin{aligned}&(\mathtt{d}\det)(\mathbf{1})(x) = (\mathtt{d}\det)(e_1, \ldots, e_n)(x_1, \ldots, x_n)\\ &= \det(x_1, e_2, \ldots, e_n) + \ldots + \det(e_1, \ldots, e_{n-1}, x_n) = x_{11} + \ldots + x_{nn} = \operatorname{tr} x.\end{aligned}$$

Now we consider the curve $\gamma\colon \mathbb{R} \to \mathbb{K}^\times \cong \mathrm{GL}_1(\mathbb{K}), t \mapsto \det(e^{tx})$. Then γ is a continuous group homomorphism, hence of the form $\gamma(t) = e^{at}$ for $a = \gamma'(0)$ (Theorem 2.2.6). On the other hand the Chain Rule implies

$$a = \gamma'(0) = \mathtt{d}\det(\mathbf{1})\big(\mathtt{d}\exp(\mathbf{0})(x)\big) = \operatorname{tr}(x),$$

and this implies (3.2). We conclude that

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{K}) &:= \mathbf{L}(\mathrm{SL}_n(\mathbb{K})) = \{x \in M_n(\mathbb{K})\colon (\forall t \in \mathbb{R})\, 1 = \det(e^{tx}) = e^{t \operatorname{tr} x}\}\\ &= \{x \in M_n(\mathbb{K})\colon \operatorname{tr} x = 0\}.\end{aligned}$$

Lemma 3.2.2. *Let V and W be finite-dimensional vector spaces and $\beta\colon V \times V \to W$ a bilinear map. For $(x, y) \in \mathrm{End}(V) \times \mathrm{End}(W)$ the following are equivalent:*

(1) $e^{ty}\beta(v, v') = \beta(e^{tx}v, e^{tx}v')$ *for all* $t \in \mathbb{R}$ *and all* $v, v' \in V$.
(2) $y\beta(v, v') = \beta(xv, v') + \beta(v, xv')$ *for all* $v, v' \in V$.

Proof. (1) $\Rightarrow$ (2): Taking the derivative in $t = 0$, the relation (1) leads to

$$y\beta(v, v') = \beta(xv, v') + \beta(v, xv'),$$

where we use the Product and the Chain Rule (Exercise 2.1.1(c)).

(2) $\Rightarrow$ (1): If (2) holds, then we obtain inductively

$$y^n.\beta(v, v') = \sum_{k=0}^{n} \binom{n}{k} \beta(x^k v, x^{n-k} v').$$

For the exponential series this leads with the general Cauchy Product Formula (Exercise 2.1.3) to

$$\begin{aligned} e^y\beta(v,v') &= \sum_{n=0}^{\infty}\frac{1}{n!}y^n.\beta(v,v') = \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\sum_{k=0}^{n}\binom{n}{k}\beta(x^k v, x^{n-k}v)\Big) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\beta\left(\frac{1}{k!}x^k v, \frac{1}{(n-k)!}x^{n-k}v'\right) \\ &= \beta\left(\sum_{k=0}^{\infty}\frac{1}{k!}x^k v, \sum_{m=0}^{\infty}\frac{1}{m!}x^m v'\right) = \beta\left(e^x v, e^x v'\right). \end{aligned}$$

Since (2) also holds for the pair (tx, ty) for all $t \in \mathbb{R}$, this completes the proof. □

Proposition 3.2.3. *Let V and W be finite-dimensional vector spaces and $\beta\colon V \times V \to W$ a bilinear map. For the group*

$$\mathrm{Aut}(V,\beta) = \{g \in \mathrm{GL}(V)\colon (\forall v, v' \in V)\ \beta(gv, gv') = \beta(v, v')\},$$

we then have

$$\mathfrak{aut}(V,\beta) := \mathbf{L}(\mathrm{Aut}(V,\beta)) = \{x \in \mathfrak{gl}(V)\colon (\forall v, v' \in V)\ \beta(xv, v') + \beta(v, xv') = 0\}.$$

Proof. We only have to observe that $X \in \mathbf{L}(\mathrm{Aut}(V,\beta))$ is equivalent to the pair $(X, 0)$ satisfying condition (1) in Lemma 3.2.2. □

Example 3.2.4. (a) Let $B \in M_n(\mathbb{K})$, $\beta(v,w) = v^\top Bw$, and

$$G := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^\top Bg = B\} \cong \mathrm{Aut}(\mathbb{K}^n, \beta).$$

Then Proposition 3.2.3 implies that

$$\begin{aligned} \mathbf{L}(G) &= \{x \in \mathfrak{gl}_n(\mathbb{K})\colon (\forall v, v' \in V)\ \beta(xv, v') + \beta(v, xv') = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{K})\colon (\forall v, v' \in V)\ v^\top x^\top Bv' + v^\top Bxv' = 0\} \\ &= \{x \in \mathfrak{gl}_n(\mathbb{K})\colon x^\top B + Bx = 0\}. \end{aligned}$$

In particular, we obtain

$$\mathfrak{o}_n(\mathbb{K}) := \mathbf{L}(\mathrm{O}_n(\mathbb{K})) = \{x \in \mathfrak{gl}_n(\mathbb{K})\colon x^\top = -x\} =: \mathrm{Skew}_n(\mathbb{K}),$$

$$\mathfrak{o}_{p,q}(\mathbb{K}) := \mathbf{L}(\mathrm{O}_{p,q}(\mathbb{K})) = \{x \in \mathfrak{gl}_{p+q}(\mathbb{K})\colon x^\top I_{p,q} + I_{p,q}x = 0\},$$

and

$$\mathfrak{sp}_n(\mathbb{K}) := \mathbf{L}(\mathrm{Sp}_{2n}(\mathbb{K})) := \{x \in \mathfrak{gl}_{2n}(\mathbb{K})\colon x^\top B + Bx = 0\},$$

where $B = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$.

(b) Applying Proposition 3.2.3 with $V = \mathbb{C}^n$ and $W = \mathbb{C}$, considered as real vector spaces, we also obtain for a hermitian form $\beta\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $(z, w) \mapsto w^* I_{p,q} z$:

$$\begin{aligned}\mathfrak{u}_{p,q}(\mathbb{C}) &:= \mathbf{L}(\mathrm{U}_{p,q}(\mathbb{C}))\\ &= \{x \in \mathfrak{gl}_n(\mathbb{C})\colon (\forall z, w \in \mathbb{C}^n)\, w^* I_{p,q} xz + w^* x^* I_{p,q} z = 0\}\\ &= \{x \in \mathfrak{gl}_n(\mathbb{C})\colon I_{p,q} x + x^* I_{p,q} = 0\}.\end{aligned}$$

In particular, we get

$$\mathfrak{u}_n(\mathbb{C}) := \mathbf{L}(\mathrm{U}_n(\mathbb{C})) = \{x \in \mathfrak{gl}_n(\mathbb{C})\colon x^* = -x\} =: \mathrm{Aherm}_n(\mathbb{C}).$$

Example 3.2.5. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{K}$-Lie algebra and

$$\mathrm{Aut}(\mathfrak{g}) := \{g \in \mathrm{GL}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\ g[x, y] = [gx, gy]\}.$$

To calculate the Lie algebra of G, we use Lemma 3.2.2 with $V = W = \mathfrak{g}$ and $\beta(x, y) = [x, y]$. Then we see that $D \in \mathfrak{aut}(\mathfrak{g}) := \mathbf{L}(\mathrm{Aut}(\mathfrak{g}))$ is equivalent to (D, D) satisfying the conditions in Lemma 3.2.2, and this leads to

$$\mathfrak{aut}(\mathfrak{g}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{g})) = \{D \in \mathfrak{gl}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\, D.[x, y] = [D.x, y] + [x, D.y]\}$$

The elements of this Lie algebra are called *derivations of* $\mathfrak{g}$, and $\mathfrak{aut}(\mathfrak{g})$ is also denoted $\mathrm{der}(\mathfrak{g})$. Note that the condition on an endomorphism of $\mathfrak{g}$ to be a derivation resembles the Leibniz Rule (Product Rule).

Remark 3.2.6. We call a linear Lie group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ a *complex linear Lie group* if $\mathbf{L}(G) \subseteq \mathfrak{gl}_n(\mathbb{C})$ is a complex subspace, i.e., $i\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$. Since Proposition 3.1.4 only ensures that $\mathbf{L}(G)$ is a real subspace, this definition makes sense.

For example $\mathrm{U}_n(\mathbb{C})$ is not a complex linear Lie group because

$$i\mathfrak{u}_n(\mathbb{C}) = \mathrm{Herm}_n(\mathbb{C}) \not\subseteq \mathfrak{u}_n(\mathbb{C}).$$

On the other hand $\mathrm{O}_n(\mathbb{C})$ is a complex linear Lie group because

$$\mathfrak{o}_n(\mathbb{C}) = \mathrm{Skew}_n(\mathbb{C})$$

is a complex subspace of $\mathfrak{gl}_n(\mathbb{C})$.

Exercises for Section 3.2

Exercise 3.2.1. Show that the following groups are linear Lie groups and determine their Lie algebras.

(1) $N := \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i > j)\ g_{ij} = 0, g_{ii} = 1\}$.
(2) $B := \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i > j)\ g_{ij} = 0\}$.

(3) $D := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall i \neq j)\ g_{ij} = 0\}$.
Note that $B \cong N \rtimes D$ is a semidirect product.
(4) A a finite-dimensional associative algebra and

$$G := \mathrm{Aut}(A) := \{g \in \mathrm{GL}(A) \colon (\forall a, b \in A)\ g(ab) = g(a)g(b)\}.$$

Exercise 3.2.2. Realize the two groups $\mathrm{Mot}_n(\mathbb{R})$ and $\mathrm{Aff}_n(\mathbb{R})$ as linear Lie groups in $\mathrm{GL}_{n+1}(\mathbb{R})$.

(1) Determine their Lie algebras $\mathfrak{mot}_n(\mathbb{R})$ and $\mathfrak{aff}_n(\mathbb{R})$.
(2) Calculate the exponential function $\exp \colon \mathfrak{aff}_n(\mathbb{R}) \to \mathrm{Aff}_n(\mathbb{R})$ in terms of the exponential function of $M_n(\mathbb{R})$.

Exercise 3.2.3. Let V be a finite-dimensional $\mathbb{K}$-vector space and $W \subseteq V$ a subspace. Show that

$$\mathrm{GL}(V)_W := \{g \in \mathrm{GL}(V) \colon gW = W\}$$

is a closed subgroup of $\mathrm{GL}(V)$ with

$$\mathbf{L}(\mathrm{GL}(V)_W) = \mathfrak{gl}(V)_W := \{X \in \mathfrak{gl}(V) \colon X.W \subseteq W\}.$$

Exercise 3.2.4. Show that for $n = p + q$ we have

$$\mathrm{O}_{p,q}(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K}) \cong \mathrm{O}_p(\mathbb{K}) \times \mathrm{O}_q(\mathbb{K}).$$

3.3 Polar Decomposition of Certain Algebraic Lie Groups

In this subsection we show that the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ can be used to obtain polar decompositions of many subgroups.

Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a linear Lie group. If $g = ue^x \in G$ (u unitary and x hermitian) implies that $u \in G$ and $e^x \in G$, then $g^* = e^x u^{-1} \in G$. Therefore a necessary condition for G to be adapted to the polar decomposition of $\mathrm{GL}_n(\mathbb{K})$ is that G is invariant under the map $g \mapsto g^*$. So we assume that this condition is satisfied. For $x \in \mathbf{L}(G)$, we then obtain from $(e^{tx})^* = e^{tx^*}$ that $x^* \in \mathbf{L}(G)$. Hence each element $x \in \mathbf{L}(G)$ can be written as

$$x = \frac{1}{2}(x - x^*) + \frac{1}{2}(x + x^*),$$

where both summands are in $\mathbf{L}(G)$. This implies that

$$\mathbf{L}(G) = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where} \quad \mathfrak{k} := \mathbf{L}(G) \cap \mathfrak{u}_n(\mathfrak{k}), \quad \mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Herm}_n(\mathbb{K}).$$

We also need a condition which ensures that $e^x \in G$, $x \in \mathrm{Herm}_n(\mathbb{K})$, implies $x \in \mathbf{L}(G)$.

Definition 3.3.1. We call a subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$ *algebraic* if there exists a family $(p_j)_{j\in J}$ of real polynomials

$$p_j(x) = p_j(x_{11}, x_{12}, \ldots, x_{nn}) \in \mathbb{R}[x_{11}, \ldots, x_{nn}]$$

in the entries of the matrix $x \in M_n(\mathbb{R})$ such that

$$G = \{x \in \mathrm{GL}_n(\mathbb{R})\colon (\forall j \in J)\ p_j(x) = 0\}.$$

Lemma 3.3.2. *Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be an algebraic subgroup, $y \in M_n(\mathbb{R})$ diagonalizable and $e^y \in G$. Then $y \in \mathbf{L}(G)$, i.e., $e^{\mathbb{R}y} \subseteq G$.*

Proof. Suppose that $A \in \mathrm{GL}_n(\mathbb{R})$ is such that AyA^{-1} is a diagonal matrix. Then $\widetilde{p}_j(x) = p_j(A^{-1}xA)$, $j \in J$, is also a set of polynomials in the entries of x and $e^y \in G$ is equivalent to

$$e^{AyA^{-1}} = Ae^yA^{-1} \in \widetilde{G} := AGA^{-1} = \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall j)\, \widetilde{p}_j(g) = 0\}.$$

Therefore we may assume that $y = \mathrm{diag}(y_1, \ldots, y_n)$ is a diagonal matrix. Now the polynomial $q_j(t) := p_j(e^{ty})$ has the form

$$\begin{aligned} q_j(t) &= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}_0^n} a_{k_1,\ldots,k_n}(e^{ty_1})^{k_1}\cdots(e^{ty_n})^{k_n} \\ &= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}_0^n} a_{k_1,\ldots,k_n} e^{t(k_1y_1+\ldots+k_ny_n)} \end{aligned}$$

(a finite sum). Therefore it can be written as $q_j(t) = \sum_{k=1}^m \lambda_k e^{tb_k}$, with $b_1 > \ldots > b_m$, where each b_k is a sum of the entries y_l of y. If q_j does not vanish identically on $\mathbb{R}$, then we may assume that $\lambda_1 \neq 0$. This leads to

$$\lim_{t\to\infty} e^{-tb_1} q_j(t) = \lambda_1 \neq 0,$$

which contradicts $q_j(\mathbb{Z}) = \{0\}$, which in turn follows from $e^{\mathbb{Z}y} \subseteq G$. Therefore each polynomial q_j vanishes identically, and hence $e^{\mathbb{R}y} \subseteq G$. □

Proposition 3.3.3 (Polar decomposition for real algebraic groups). *Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be an algebraic subgroup with $G = G^\top$. We define $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$. Then the map*

$$\varphi\colon K \times \mathfrak{p} \to G, \quad (k, x) \mapsto ke^x$$

is a homeomorphism.

Proof. Let $g \in G$ and write it as $g = ue^x$ with $u \in \mathrm{O}_n(\mathbb{R})$ and $x \in \mathrm{Sym}_n(\mathbb{R})$ (Proposition 2.3.5 and the polar decomposition). Then

$$e^{2x} = g^\top g \in G,$$

where $x \in \mathrm{Sym}_n(\mathbb{R})$ is diagonalizable. Therefore Lemma 3.3.2 implies that $e^{\mathbb{R}x} \subseteq G$, so that $x \in \mathfrak{p}$. Hence $u = ge^{-x} \in G \cap \mathrm{O}_n(\mathbb{R}) = K$. We conclude that φ is a surjective map. Furthermore Proposition 1.1.5 on the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ implies that φ is injective, hence bijective. The continuity of φ^{-1} also follows from the continuity of the inversion in $\mathrm{GL}_n(\mathbb{R})$ (cf. Proposition 1.1.5). □

Example 3.3.4. Proposition 3.3.3 applies to the following groups:

(a) $G = \mathrm{SL}_n(\mathbb{R})$ is $p^{-1}(0)$ for the polynomial $p(x) = \det x - 1$, and we obtain

$$\mathrm{SL}_n(\mathbb{R}) = K \exp \mathfrak{p} \cong K \times \mathfrak{p}$$

with

$$K = \mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathfrak{p} = \{x \in \mathrm{Sym}_n(\mathbb{R})\colon\ \mathrm{tr}\, x = 0\}.$$

For $\mathrm{SL}_2(\mathbb{R})$, we obtain in particular a homeomorphism

$$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^2 \cong \mathbb{S}^1 \times \mathbb{R}^2.$$

(b) $G = \mathrm{O}_{p,q} := \mathrm{O}_{p,q}(\mathbb{R})$ is defined by the condition $g^\top I_{p,q} g = I_{p,q}$. These are n^2 polynomial equations, one for each entry of the matrix. Moreover, $g \in \mathrm{O}_{p,q}$ implies

$$I_{p,q} = I_{p,q}^{-1} = (g^\top I_{p,q} g)^{-1} = g^{-1} I_{p,q} (g^\top)^{-1}$$

and hence $g I_{p,q} g^\top = I_{p,q}$, i.e., $g^\top \in \mathrm{O}_{p,q}$. Therefore $\mathrm{O}_{p,q}^\top = \mathrm{O}_{p,q}$, and all the assumptions of Proposition 3.3.3 are satisfied. In this case,

$$K = \mathrm{O}_{p,q} \cap \mathrm{O}_n \cong \mathrm{O}_p \times \mathrm{O}_q,$$

(Exercise 3.2.4) and topologically we obtain

$$\mathrm{O}_{p,q} \cong \mathrm{O}_p \times \mathrm{O}_q \times (\mathfrak{o}_{p,q} \cap \mathrm{Sym}_n(\mathbb{R})).$$

In particular, we see that for $p, q > 0$ the group $\mathrm{O}_{p,q}$ has four arc-components because O_p and O_q have two arc-components (Proposition 1.1.7).

For the subgroup $\mathrm{SO}_{p,q}$ we have one additional polynomial equation, so that it is also algebraic. Here we have

$$\begin{aligned} K_S := K \cap \mathrm{SO}_{p,q} &\cong \{(a,b) \in \mathrm{O}_p \times \mathrm{O}_q\colon\ \det(a)\det(b) = 1\} \\ &\cong \big(\mathrm{SO}_p \times \mathrm{SO}_q\big) \dot\cup \big(\mathrm{O}_{p,-} \times \mathrm{O}_{q,-}\big), \end{aligned}$$

so that $\mathrm{SO}_{p,q}$ has two arc-components if $p, q > 0$ (cf. the discussion of the Lorentz group in Example 1.2.7).

(c) We can also apply Proposition 3.3.3 to the subgroup $\mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}(\mathbb{R})$ which is defined by the condition $gI = Ig$, where $I\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to the componentwise multiplication with i on $\mathbb{C}^n$. These are

$4n^2 = (2n)^2$ polynomial equations defining $\mathrm{GL}_n(\mathbb{C})$. In this case we obtain a new proof of the polar decomposition of $\mathrm{GL}_n(\mathbb{C})$ because

$$K = \mathrm{GL}_n(\mathbb{C}) \cap \mathrm{O}_{2n}(\mathbb{R}) = \mathrm{U}_n(\mathbb{C})$$

and

$$\mathfrak{p} = \mathfrak{gl}_n(\mathbb{C}) \cap \mathrm{Sym}_{2n}(\mathbb{R}) = \mathrm{Herm}_n(\mathbb{C}).$$

Example 3.3.5. Let $X \in \mathrm{Sym}_n(\mathbb{R})$ be a nonzero symmetric matrix and consider the subgroup $G := \exp(\mathbb{Z}X) \subseteq \mathrm{GL}_n(\mathbb{R})$. Since $\exp X$ is symmetric, we then have $G^\top = G$. Moreover, if $\lambda_1 \leq \ldots \leq \lambda_k$ are the eigenvalues of X, then

$$\|\exp(nX) - \mathbf{1}\| = \max(|e^{n\lambda_k} - 1|, |e^{n\lambda_1} - 1|) \geq \max(|e^{\lambda_k} - 1|, |e^{\lambda_1} - 1|)$$

implies that G is a discrete subset of $\mathrm{GL}_n(\mathbb{R})$, hence a closed subgroup, and therefore a linear Lie group. On the other hand, the fact that G is discrete implies that $\mathbf{L}(G) = \{0\}$. This example show that the assumption that G is algebraic is indispensable for Proposition 3.3.3 because

$$G \cap O_n(\mathbb{R}) = \{\mathbf{1}\} \quad \text{and} \quad \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R}) = \{0\}.$$

Exercises for Section 3.3

Exercise 3.3.1. Show that the groups $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ have polar decompositions and describe their intersections with $\mathrm{O}_{2n}(\mathbb{R})$.

Exercise 3.3.2. Let $B \in \mathrm{Herm}_n(\mathbb{K})$ with $B^2 = \mathbf{1}$ and consider the automorphism $\tau(g) = Bg^{-\top}B^{-1}$ of $\mathrm{GL}_n(\mathbb{K})$. Show that:
(1) $\mathrm{Aut}(\mathbb{C}^n, B) = \{g \in \mathrm{GL}_n(\mathbb{K}) \colon \tau(g) = g\}$.
(2) $\mathrm{Aut}(\mathbb{C}^n, B)$ is adapted to the polar decomposition by showing that if $g = ue^x$ is the polar decomposition of g, then $\tau(g) = g$ is equivalent to $\tau(u) = u$ and $\tau(x) = x$.
(3) $\mathrm{Aut}(\mathbb{C}^n, B)$ is adapted to the polar decomposition by using that it is an algebraic group.

Notes on Chapter 3

As for Chapter 1 it has to be noted that the results presented are only the tip of the iceberg. The guiding principle for our choice of material was to present facts which either generalize to general classes of Lie groups or else serve as tools in the study of such classes of Lie groups.

The BCDH formula was first studied by J. E. Campbell in [Cam97, Cam98], and in [Hau06], F. Hausdorff studied the Hausdorff series on a formal level, showing that the formal expansion of $\log(e^x e^y)$ can be expressed in terms of Lie polynomials. Part of his results had been obtained earlier by H. F. Baker ([Bak01, Bak05]). In [Dyn47, Dyn53], E. B. Dynkin determined the summands in the Hausdorff series explicitly.

Part II

Lie Algebras

4

Elementary Structure Theory of Lie Algebras

Lie algebras form the infinitesimal counterparts of Lie groups. We have already seen in the preceding chapters how matrix groups give rise to Lie algebras of matrices, and we shall also see in Chapter 8 below how to associate a Lie algebra to any Lie group. This correspondence is the guiding motivation behind the theory of finite-dimensional Lie algebras to which we now turn in some depth.

In this part we study Lie algebras as independent objects and thus provide tools to solve the linear algebra problems which we will encounter when translating Lie group questions into Lie algebra questions in Part IV. We start in the present chapter by working out the standard analysis of an algebraic structure for Lie algebras: What are the substructures? Under which condition does a substructure lead to a quotient structure? What are the simple structures? Does one have composition series? This leads to concepts like Lie subalgebras and ideals, nilpotent, solvable, and semisimple Lie algebras. Key results in this context are Engel's Theorems on nilpotent Lie algebras, Lie's Theorem for solvable Lie algebras and Cartan's criteria for solvability and semisimplicity. The latter are first instances in which one recognizes the usefulness of the Cartan–Killing form, which is a specific structural element for Lie algebras.

4.1 Basic Concepts

In this section we provide the basic definitions and concepts concerning Lie algebras. In particular, we discuss ideals, quotients, homomorphisms and the elementary connections between these concepts.

4.1.1 Definitions and Examples

We start by recalling the definition of a Lie algebra from Definition 3.1.1.

Definition 4.1.1. Let $\mathfrak{g}$ be a vector space. A *Lie bracket* on $\mathfrak{g}$ is a bilinear map $[\cdot\,,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying

(L1) $[x,y] = -[y,x]$ for $x,y\in\mathfrak{g}$,
(L2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ for $x,y,z\in\mathfrak{g}$ (*Jacobi identity*).

For any Lie bracket on $\mathfrak{g}$, the pair $(\mathfrak{g},[\cdot\,,\cdot])$ is called a *Lie algebra*.

Example 4.1.2. (cf. Def. 2.1.1) A vector space $\mathcal{A}$ together with a bilinear map $\cdot\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is called an *(associative) algebra*, if

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c \quad \text{for} \quad a,b,c\in\mathcal{A}.$$

Then the commutator

$$[a,b] := a\cdot b - b\cdot a$$

defines a Lie bracket on $\mathcal{A}$ (Exercise). We write $\mathcal{A}_L := (\mathcal{A},[\cdot,\cdot])$ for this Lie algebra.

Example 4.1.3. (a) Let V be a vector space and $\mathrm{End}(V)$ be the set of linear endomorphisms of V. Then $\mathrm{End}(V)$ is an associative algebra and we write $\mathfrak{gl}(V) := \mathrm{End}(V)_L$ for the corresponding Lie algebra.

(b) The space $M_n(\mathbb{K})$ of $(n\times n)$-matrices with entries in $\mathbb{K}$ is an associative algebra with respect to matrix multiplication. We write $\mathfrak{gl}_n(\mathbb{K}) := M_n(\mathbb{K})_L$ for the corresponding Lie algebra.

Definition 4.1.4. (a) Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A linear map $\alpha\colon\mathfrak{g}\to\mathfrak{h}$ is called a *homomorphism* if

$$\alpha([x,y]) = [\alpha(x),\alpha(y)] \quad \text{for} \quad x,y\in\mathfrak{g}.$$

An *isomorphism* of Lie algebras is a homomorphism α for which there exists a homomorphism $\beta\colon\mathfrak{h}\to\mathfrak{g}$ with $\alpha\circ\beta = \mathrm{id}_{\mathfrak{h}}$ and $\beta\circ\alpha = \mathrm{id}_{\mathfrak{g}}$. It is easy to see that this condition is equivalent to α being bijective (Exercise).

A *representation of a Lie algebra* $\mathfrak{g}$ *on the vector space* V is a homomorphism $\alpha\colon\mathfrak{g}\to\mathfrak{gl}(V)$. We also write (α,V) for a representation α of $\mathfrak{g}$ on V.

(b) Let $\mathfrak{g}$ be a Lie algebra and U,V be subsets of $\mathfrak{g}$. We write

$$[U,V] := \mathrm{span}\{[u,v]\colon u\in U, v\in V\}$$

for the smallest subspace containing all brackets $[u,v]$ with $u\in U$ and $v\in V$.

(c) A linear subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Lie subalgebra* if $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$. We then write $\mathfrak{h}<\mathfrak{g}$. If we even have $[\mathfrak{g},\mathfrak{h}]\subseteq\mathfrak{h}$, we call $\mathfrak{h}$ an *ideal* of $\mathfrak{g}$ and write $\mathfrak{h}\trianglelefteq\mathfrak{g}$.

(d) The Lie algebra $\mathfrak{g}$ is called *abelian* if $[\mathfrak{g},\mathfrak{g}] = \{0\}$, which means that all brackets vanish.

Remark 4.1.5. From the definitions it is immediately clear that the image of a homomorphism $\alpha\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras is a subalgebra of $\mathfrak{g}_2$. Moreover, $\alpha^{-1}(\mathfrak{h})$ is an ideal in $\mathfrak{g}_1$ if $\mathfrak{h} \trianglelefteq \mathfrak{g}_2$, and $\alpha^{-1}(\mathfrak{h})$ is a subalgebra if $\mathfrak{h} < \mathfrak{g}_2$. In particular, the *kernel* $\ker\alpha$ of a Lie algebra homomorphism α is always an ideal.

Examples 4.1.6. (i) Let $\mathfrak{g}$ be Lie algebra. Then the *center*

$$\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid (\forall y \in \mathfrak{g})\, [x, y] = 0\}$$

of $\mathfrak{g}$ is an ideal in $\mathfrak{g}$.

(ii) For each Lie algebra $\mathfrak{g}$, the subspace $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, called the *commutator algebra* of $\mathfrak{g}$.

(iii) Every one-dimensional subspace of a Lie algebra is a subalgebra since the Lie bracket is skew-symmetric.

(iv) The set

$$\mathfrak{o}_n(\mathbb{K}) := \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid x = -x^\top\}$$

is a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$, the *orthogonal Lie algebra*.

(v) The set

$$\mathfrak{u}_n(\mathbb{C}) := \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid x = -x^*\}$$

is a real subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, the *unitary Lie algebra*.

(vi) The set

$$\mathfrak{sl}_n(\mathbb{K}) := \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid \mathrm{tr}(x) = 0\}$$

is an ideal in $\mathfrak{gl}_n(\mathbb{K})$, where $\mathrm{tr}(x)$ denotes the *trace* of X. It is called the *special linear Lie algebra*.

(vii) Let $J_n := \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{K})$ and note that $J_n^\top = -J_n$. Then the set

$$\mathfrak{sp}_n(\mathbb{K}) := \{x \in \mathfrak{gl}_{2n}(\mathbb{K})\colon x^\top J_n + J_n x = 0\}$$

is a Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{K})$, called the *symplectic Lie algebra*. Writing elements of $\mathfrak{gl}_{2n}(\mathbb{K})$ as (2×2)-block matrices, one easily verifies that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}_n(\mathbb{K}) \quad \iff \quad B = B^\top, C = C^\top, A^\top = -D.$$

(viii) The sets

$$\mathfrak{n} = \{x = (x_{ij}) \in \mathfrak{gl}_n(\mathbb{K}) \mid (\forall i \geq j)\, x_{ij} = 0\}$$

and

$$\mathfrak{b} = \{x = (x_{ij}) \in \mathfrak{gl}_n(\mathbb{K}) \mid (\forall i > j)\, x_{ij} = 0\}$$

are subalgebras of $\mathfrak{gl}_n(\mathbb{K})$.

(ix) Let V be a subspace of a Lie algebra $\mathfrak{g}$. The *normalizer*

$$\mathfrak{n}_{\mathfrak{g}}(V) = \{x \in \mathfrak{g} \mid [x, V] \subseteq V\}$$

of V in $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$.

Example 4.1.7. Let V be a vector space. A tuple $\mathcal{F} = (V_0, \ldots, V_n)$ of subspaces with

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

is called a *flag* in V. Then

$$\mathfrak{g}(\mathcal{F}) := \{x \in \mathfrak{gl}(V)\colon (\forall j)\ xV_j \subseteq V_j\}$$

is a Lie subalgebra of $\mathfrak{gl}(V) = \mathrm{End}(V)_L$.

To visualize this Lie algebra, we shall describe linear maps by suitable block matrices. If V is a vector space which is a direct sum $V = W_1 \oplus \ldots \oplus W_n$ of subspaces W_j, $j = 1, \ldots, n$, then we write an endomorphism $A \in \mathrm{End}(V)$ as an $(n \times n)$-block matrix

$$A = (A_{jk})_{j,k=1,\ldots,n} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix},$$

where $A_{jk} \in \mathrm{Hom}(W_k, W_j)$ is uniquely determined by the requirement that the image of $v = (v_1, \ldots, v_n) \in V$ is

$$Av = \Big(\sum_{k=1}^{n} A_{jk} v_k\Big)_{j=1,\ldots,n}.$$

Applying this kind of visualization to the Lie algebra $\mathfrak{g}(\mathcal{F})$, we choose in V_j a subspace W_j with $V_j \cong V_{j-1} \oplus W_j$. For each j we then have $V_j \cong W_1 \oplus \ldots \oplus W_j$, and in particular $V \cong W_1 \oplus \ldots \oplus W_n$. Now the elements of $\mathfrak{g}(\mathcal{F})$ are those endomorphisms of V corresponding to upper triangular matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix}.$$

Definition 4.1.8. Let $\mathfrak{g}$ be a Lie algebra. A linear map $\delta\colon \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* if

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)] \quad \text{for} \quad x, y \in \mathfrak{g}.$$

The set of all derivations is denoted by $\mathrm{der}(\mathfrak{g})$.

Definition 4.1.9. Let $\mathfrak{g}$ be a Lie algebra and $X \in \mathfrak{g}$. Then the Jacobi identity implies that the linear map

$$\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x, y]$$

is a derivation. Derivations of this form are called *inner derivations*. The map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is called the *adjoint representation*. That it is a representation follows directly from the Jacobi identity, which implies that

$$\mathrm{ad}[x, y] = [\mathrm{ad}\, x, \mathrm{ad}\, y].$$

Proposition 4.1.10. *For any Lie algebra* $\mathfrak{g}$*,*

(i) $\mathrm{der}(\mathfrak{g}) < \mathfrak{gl}(\mathfrak{g})$ *and* $\mathrm{ad}(\mathfrak{g}) \trianglelefteq \mathrm{der}(\mathfrak{g})$ *is an ideal. In particular*

$$[D, \mathrm{ad}\, x] = \mathrm{ad}(Dx) \quad \textit{for} \quad D \in \mathrm{der}(\mathfrak{g}), x \in \mathfrak{g}. \tag{4.1}$$

(ii) $\ker(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g})$

Proof. (i) The first part is a special case of Exercise 4.1.4 and for the second one verifies (4.1) by direct calculation.

(ii) is trivial. □

4.1.2 Representations and Modules

In this short subsection we introduce some terminology concerning representations of Lie algebras and the corresponding concept of a Lie algebra module.

Definition 4.1.11. Let $\mathfrak{g}$ be a Lie algebra, and V a vector space. Suppose that

$$\mathfrak{g} \times V \to V, \quad (x, v) \mapsto x \cdot v$$

is a bilinear map. If

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \text{for} \quad x, y \in \mathfrak{g}, v \in V,$$

then V is called a $\mathfrak{g}$*-module*.

Definition 4.1.12. (a) Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. A subspace $W \subseteq V$ is called a $\mathfrak{g}$*-submodule* if $\mathfrak{g} \cdot W \subseteq W$.

(b) A $\mathfrak{g}$-module V is called *simple* if it is nonzero and there are no submodules except $\{0\}$ and V. It is called *semisimple*, if V is the direct sum of simple submodules.

(c) If V and W are $\mathfrak{g}$-modules, then a linear map $\varphi\colon V \to W$ is called *homomorphism of* $\mathfrak{g}$*-modules* if for all $x \in \mathfrak{g}$ and all $v \in V$,

$$\varphi(x \cdot v) = x \cdot \varphi(v).$$

We write $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ for the vector space of all $\mathfrak{g}$-module homomorphisms from V to W and note that the set $\mathrm{End}_{\mathfrak{g}}(V) := \mathrm{Hom}_{\mathfrak{g}}(V, V)$ of module endomorphisms of V is an associative subalgebra of $\mathrm{End}(V)$.

If $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$ is bijective, then the inverse map $\psi\colon W \to V$ is also a homomorphism of $\mathfrak{g}$-modules (Exercise) satisfying $\varphi \circ \psi = \mathrm{id}_W$ and $\psi \circ \varphi = \mathrm{id}_V$. Therefore φ is called an *isomorphism of* $\mathfrak{g}$*-modules*. The set of isomorphisms $V \to V$ is the group $\mathrm{Aut}_{\mathfrak{g}}(V) := \mathrm{End}_{\mathfrak{g}}(V)^\times$ of units in the algebra $\mathrm{End}_{\mathfrak{g}}(V)$.

Example 4.1.13. (a) Any Lie algebra $\mathfrak{g}$ carries a natural $\mathfrak{g}$-module structure defined by the adjoint representation $x \cdot y := [x, y]$. The $\mathfrak{g}$-submodules of $\mathfrak{g}$ are precisely the ideals (cf. Definition 4.1.9).

(b) If $\mathfrak{g} = \mathbb{K}$ is the one-dimensional Lie algebra and V a $\mathbb{K}$-vector space, then any endomorphism $D \in \operatorname{End}(V)$ determines a $\mathfrak{g}$-module structure on V defined by $t \cdot v := tD(v)$. Clearly, each $\mathfrak{g}$-module structure of V is of this form for $D(v) = 1 \cdot v$.

Remark 4.1.14. If $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then a $\mathfrak{g}$-module structure on V is defined by $x \cdot v = \pi(x)v$. Conversely, for every $\mathfrak{g}$-module V, the map $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ defined by $\pi(x)v = x \cdot v$ is a representation. Thus representations of $\mathfrak{g}$ and $\mathfrak{g}$-modules are equivalent concepts.

Definition 4.1.15. A representation (π, V) of a Lie algebra $\mathfrak{g}$ is called *irreducible* if V is a simple $\mathfrak{g}$-module. It is called *completely reducible* if V is a semisimple $\mathfrak{g}$-module.

4.1.3 Quotients and Semidirect Sums

We have already seen that the kernel of a homomorphism of Lie algebras is an ideal. The following proposition implies in particular that each ideal is the kernel of a surjective homomorphism of Lie algebras.

Proposition 4.1.16. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ be an ideal in $\mathfrak{g}$. Then the quotient space $\mathfrak{g}/\mathfrak{n} = \{x + \mathfrak{n} \colon x \in \mathfrak{g}\}$ is a Lie algebra with respect to the bracket*

$$[x + \mathfrak{n}, y + \mathfrak{n}] := [x, y] + \mathfrak{n}.$$

The quotient map $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ is a surjective homomorphism of Lie algebras with kernel $\mathfrak{n}$.

Proof. The decisive step in the proof is to show that the Lie bracket is well defined. But this immediately follows from the definition of an ideal. All other properties one can easily derive from the respective properties of the Lie bracket on $\mathfrak{g}$ (Exercise). □

Theorem 4.1.17. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras.*

(i) *If $\alpha\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism, then*

$$\alpha(\mathfrak{g}) \cong \mathfrak{g}/\ker\alpha.$$

For any ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$ with $\mathfrak{i} \subseteq \ker\alpha$, there is exactly one homomorphism $\beta\colon \mathfrak{g}/\mathfrak{i} \to \mathfrak{h}$ with $\beta \circ \pi = \alpha$, where $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ is the quotient map.

(ii) *If $\mathfrak{i}, \mathfrak{j} \trianglelefteq \mathfrak{g}$ are ideals with $\mathfrak{i} \subseteq \mathfrak{j}$, then $\mathfrak{j}/\mathfrak{i} \trianglelefteq \mathfrak{g}/\mathfrak{i}$, and $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$.*

(iii) *If $\mathfrak{i}, \mathfrak{j} \trianglelefteq \mathfrak{g}$ are two ideals, then $\mathfrak{i} + \mathfrak{j}$ and $\mathfrak{i} \cap \mathfrak{j}$ are ideals of $\mathfrak{g}$, and*

$$\mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j}) \cong (\mathfrak{i} + \mathfrak{j})/\mathfrak{j}.$$

Proof. Exercise. □

We have already seen that we obtain with each ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a quotient algebra $\mathfrak{g}/\mathfrak{n}$, so that we may consider the two Lie algebras $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$ as two pieces into which $\mathfrak{g}$ is decomposed. It is therefore a natural question how we may build a Lie algebra $\mathfrak{g}$ from two Lie algebras $\mathfrak{n}$ and $\mathfrak{h}$ in such a way that $\mathfrak{n} \trianglelefteq \mathfrak{g}$ and $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{h}$. The following definition describes one such construction (cf. Section 6.6 for more information).

Definition 4.1.18. Let $\mathfrak{n}$ and $\mathfrak{h}$ be Lie algebras and $\alpha\colon \mathfrak{h} \to \operatorname{der}(\mathfrak{n})$ be a homomorphism. Then the direct sum $\mathfrak{n} \oplus \mathfrak{h}$ of the vector spaces $\mathfrak{n}$ and $\mathfrak{h}$ is a Lie algebra with respect to the bracket

$$[(x, y), (x', y')] := (\alpha(y)x' - \alpha(y')x + [x, x'], [y, y'])$$

for $x, x' \in \mathfrak{n}, y, y' \in \mathfrak{h}$. This Lie algebra is called the *semidirect sum* with respect to α of $\mathfrak{n}$ and $\mathfrak{h}$. It is denoted by $\mathfrak{n} \rtimes_\alpha \mathfrak{h}$. If $\alpha = 0$, then $\mathfrak{n} \rtimes_\alpha \mathfrak{h}$ is called the *direct sum* of $\mathfrak{n}$ and $\mathfrak{h}$, and it is denoted by $\mathfrak{n} \oplus \mathfrak{h}$.

The subspace $\{(x, 0) \in \mathfrak{n} \rtimes_\alpha \mathfrak{h}\}$ is an ideal in $\mathfrak{n} \rtimes_\alpha \mathfrak{h}$ isomorphic to $\mathfrak{n}$ and $\{(0, y) \in \mathfrak{n} \rtimes_\alpha \mathfrak{h}\}$ is a subalgebra of $\mathfrak{n} \rtimes_\alpha \mathfrak{h}$ isomorphic to $\mathfrak{h}$ (cf. Exercise 4.1.2).

For a derivation $D \in \operatorname{der} \mathfrak{n}$ we simply write $\mathfrak{n} \rtimes_D \mathbb{K}$ for the semidirect sum defined by $\alpha(t) := tD$.

Example 4.1.19. Let $\mathfrak{h}_3(\mathbb{R})$ be the 3-dimensional vector space with the basis p, q, z equipped with the skew-symmetric bracket determined by

$$[p, q] = z, \quad [p, z] = [q, z] = 0.$$

Then $\mathfrak{h}_3(\mathbb{R})$ is a Lie algebras called the three-dimensional *Heisenberg algebra*. It is isomorphic to the algebra $\mathfrak{n}$ in Example 4.1.6(viii) for $n = 3$. The linear endomorphism of $\mathfrak{h}_3(\mathbb{R})$ defined by

$$Dz = 0, \quad Dp = q \quad \text{and} \quad Dq = -p$$

then is a derivation of $\mathfrak{h}_3(\mathbb{R})$, so that we obtain a Lie algebra $\mathfrak{g} := \mathfrak{h}_3(\mathbb{R}) \rtimes_D \mathbb{R}$, called the *oscillator algebra*. Writing $h := (0, 1)$ for the additional basis element in $\mathfrak{g}$, the nonzero brackets of basis elements are

$$[p, q] = z, \quad [h, p] = q \quad \text{and} \quad [h, q] = -p.$$

Example 4.1.20. If $\mathcal{F} = (V_0, \ldots, V_n)$ is a flag in the vector space V (Example 4.1.7), then we know already the associated Lie algebra

$$\mathfrak{g}(\mathcal{F}) = \{x \in \mathfrak{gl}(V) \colon (\forall j) x V_j \subseteq V_j\}.$$

It is easy to see that

$$\mathfrak{g}_n(\mathcal{F}) := \{x \in \mathfrak{gl}(V) \colon (\forall j > 0)\ xV_j \subseteq V_{j-1}\}.$$

is an ideal of $\mathfrak{g}(\mathcal{F})$.

To find a subalgebra complementary to this ideal, we choose subspaces $W_0, \ldots, W_{n-1}$ of V with $V_{j+1} \cong V_j \oplus W_j$ for $j = 0, \ldots, n-1$. Then

$$\mathfrak{g}_s(\mathcal{F}) := \{X \in \mathfrak{gl}(V) \colon (\forall j)\ XW_j \subseteq W_j\} \subseteq \mathfrak{g}(\mathcal{F})$$

is a subalgebra with

$$\mathfrak{g}(\mathcal{F}) \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \mathfrak{g}_s(\mathcal{F}) \quad \text{and} \quad \mathfrak{g}_s(\mathcal{F}) \cong \bigoplus_{j=1}^{n} \mathfrak{gl}(W_j).$$

Describing the elements of $\mathfrak{g}(\mathcal{F})$ as in Example 4.1.7 by block matrices, the semidirect decomposition of the Lie algebra $\mathfrak{g}(\mathcal{F})$ corresponds to the decomposition of an upper triangular matrix as a sum of a strictly upper triangular matrix and a diagonal matrix. For $n = 3$ we have in particular:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}}_{\in \mathfrak{g}_s(\mathcal{F})} + \underbrace{\begin{pmatrix} 0 & A_{12} & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix}}_{\in \mathfrak{g}_n(\mathcal{F})}.$$

4.1.4 Complexification and Real Forms

Up to now, the base field did not really play a role in our considerations. Nevertheless, it is important (just think about basic linear algebra) to note that, for instance, homomorphisms of complex Lie algebras are supposed to be complex linear. Therefore, we also consider the *complexification* of a real Lie algebra. For this, we briefly recall how to calculate with the complexification of a vector space.

Let V be an $\mathbb{R}$-vector space. The *complexification* $V_{\mathbb{C}}$ of V is the tensor product $\mathbb{C} \otimes_{\mathbb{R}} V$. This is a complex vector space with respect to the scalar multiplication $\lambda \cdot (z \otimes v) := \lambda z \otimes v$. Identifying V with the subspace $1 \otimes V$ of $V_{\mathbb{C}}$, each element of $V_{\mathbb{C}}$ can be written in a unique fashion as $z = x + iy$ with $x, y \in V$. If $\{v_1, \ldots, v_n\}$ is a real basis for V, then $\{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ is a complex basis for $V_{\mathbb{C}}$.

Proposition 4.1.21. *Let $\mathfrak{g}$ be a real Lie algebra.*

(i) $\mathfrak{g}_{\mathbb{C}}$ *is a complex Lie algebra with respect to the complex bilinear Lie bracket, defined by*

$$[x + iy, x' + iy'] := ([x, x'] - [y, y']) + i([x, y'] + [y, x']),$$

and satisfying

$$[z \otimes v, z' \otimes v'] = zz' \otimes [v, v'].$$

(ii) $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \cong [\mathfrak{g}, \mathfrak{g}]_{\mathbb{C}}$ *as complex Lie algebras.*

Proof. Elementary calculations. □

Definition 4.1.22. Let $\mathfrak{g}$ be a complex Lie algebra. A real Lie algebra $\mathfrak{h}$ with $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{g}$ is called a *real form* of $\mathfrak{g}$.

We have seen that for every real Lie algebra, we can assign a complexification in a unique way. However, nonisomorphic real algebras can have isomorphic complexifications, resp., complex Lie algebras can have nonisomorphic real forms.

Example 4.1.23. Let $\mathfrak{so}_3(\mathbb{R}) = \mathfrak{o}_3(\mathbb{R}) \cap \mathfrak{sl}_3(\mathbb{R}) = \mathfrak{o}_3(\mathbb{R})$. Then the complexifications of $\mathfrak{so}_3(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{R})$ are both isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. To see this, we consider the bases

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{R})$, and

$$x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

of $\mathfrak{so}_3(\mathbb{R})$. Then

$$[h, u] = 2t, \quad [h, t] = 2u, \quad [u, t] = 2h$$

and

$$[x, y] = z, \quad [z, x] = y, \quad [y, z] = x.$$

Let

$$\mathfrak{h} = \mathbb{R}ih + \mathbb{R}u + \mathbb{R}it.$$

Then $\mathfrak{h}$ is a Lie algebra with $\mathfrak{h}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ which is isomorphic to $\mathfrak{so}_3(\mathbb{R})$ via

$$ih \mapsto 2x, \; u \mapsto 2y, \; it \mapsto 2z.$$

We note that $\mathfrak{h}$ coincides with $\mathfrak{su}_2(\mathbb{C})$. Since, obviously, $\mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, it only remains to show that $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$ are not isomorphic. For this, it suffices to check that $\mathbb{R}h + \mathbb{R}(u + t)$ is a two-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{R})$, while $\mathfrak{so}_3(\mathbb{R})$ has no two-dimensional subalgebra. Namely, the latter is isomorphic to $\mathbb{R}^3$ with the vector product (Exercise 4.1.6), and since the vector product of two vectors is orthogonal to these vectors, a plane cannot be a subalgebra.

Example 4.1.24 (A complex Lie algebra with no real form). On the abelian Lie algebra $V := \mathbb{C}^2$ we consider the linear operator D, defined by $De_1 = 2e_1$ and $De_2 = ie_2$ with respect to the canonical basis. Then we form the three-dimensional complex Lie algebra $\mathfrak{g} := V \rtimes_D \mathbb{C}$ and note that $V = [\mathfrak{g}, \mathfrak{g}]$ is a 2-dimensional ideal of $\mathfrak{g}$.

Suppose that $\mathfrak{g}$ has a real form. Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be the corresponding complex conjugation, which is an involutive automorphism of $\mathfrak{g}$. Then

$$\sigma(V) = \sigma([\mathfrak{g}, \mathfrak{g}]) = [\sigma(\mathfrak{g}), \sigma(\mathfrak{g})] = [\mathfrak{g}, \mathfrak{g}] = V,$$

so that σ induces an antilinear involution σ_V on V. Let $\sigma(0, 1) = (v_0, \lambda)$ and note that $\sigma(V) = V$ implies that $\lambda \neq 0$. Applying σ again, we see that

$$(0, 1) = \sigma^2(0, 1) = \sigma_V(v_0) + \overline{\lambda}\sigma(0, 1) = (\sigma_V(v_0) + \overline{\lambda}v_0, \overline{\lambda} \cdot \lambda).$$

We conclude that $|\lambda| = 1$. Further, $\sigma \circ \mathrm{ad}(0, 1) \circ \sigma = \mathrm{ad}(\sigma(0, 1)) = \mathrm{ad}(v_0, \lambda)$ implies by restricting to V that

$$\sigma_V \circ D \circ \sigma_V = \lambda D.$$

If $v \in V$ is a D-eigenvector with $Dv = \alpha v$, then

$$D(\sigma_V v) = \sigma_V(\lambda D v) = \overline{\lambda}\overline{\alpha}\sigma_V(v).$$

This means that $\sigma_V(V_\alpha(D)) = V_{\overline{\lambda}\overline{\alpha}}(D)$. In particular, σ_V permutes the D-eigenspaces. Now $|\lambda| = 1$ and $|i| \neq 2$ show that σ_V preserves both eigenspaces. For $\alpha = 2$, this leads to $\overline{\lambda} = 1$, so that $\lambda = 1$. For $\alpha = i$ we now arrive at the contradiction $-i = \overline{\lambda}\overline{\alpha} = \alpha = i$.

This example is minimal because each complex Lie algebra of dimension 2 has a real form (cf. Example 4.4.2).

Exercises for Section 4.1

Exercise 4.1.1. Let A be an associative algebra and A_L be the associated Lie algebra (cf. Example 4.1.2).

(i) A *derivation* of A is a linear map $\delta\colon A \to A$ such that

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \forall a, b \in A.$$

Then $\mathrm{der}(A) \subseteq \mathrm{der}(A_L)$, i.e., every derivation of the associative algebra A is a derivation of the Lie algebra A_L, too.

(ii) $[a, bc] = [a, b]c + b[a, c]$ for $a, b, c \in A$.

(iii) In general $\mathrm{der}(A) \neq \mathrm{der}(A_L)$.

(iv) If A is commutative, then $A \cdot \mathrm{der}(A) \subseteq \mathrm{der}(A)$.

Exercise 4.1.2. Let U be an open subset of $\mathbb{R}^{2n}$ and $\mathfrak{g} = C^\infty(U, \mathbb{R})$ be the set of smooth functions on U and write $q_1, \ldots, q_m, p_1, \ldots, p_m$ for the coordinates with respect to a basis. Then $\mathfrak{g}$ is a Lie algebra with respect to the *Poisson bracket*

$$\{f, g\} := \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}.$$

Exercise 4.1.3. Let U be an open subset of $\mathbb{R}^n$, $A = C^\infty(U, \mathbb{R})$, and $\mathfrak{g} = C^\infty(U, \mathbb{R}^n)$. For $f \in A$ and $X \in \mathfrak{g}$, we define

$$\mathcal{L}_X f := Xf := \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i}.$$

(i) The maps $\mathcal{L}_X$ are derivations of the algebra A.
(ii) If $\mathcal{L}_X = 0$, then $X = 0$.
(iii) The commutator of two such operators has the form $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, where the bracket on $\mathfrak{g}$ is defined by (cf. Definition 7.1.1)

$$[X, Y](p) := \mathtt{d}Y(p)X(p) - \mathtt{d}X(p)Y(p),$$

resp.,

$$[X, Y]_i = \sum_{j=1}^n X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j}.$$

(iv) $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra.
(v) To each $A \in \mathfrak{gl}_n(\mathbb{R})$, we associate the linear vector field $X_A(x) := Ax$. Show that, for $A, B \in M_n(\mathbb{R})$, we have $X_{[A,B]} = -[X_A, X_B]$.

Exercise 4.1.4. Let A be a vector space and $m \colon A \times A \to A$ a bilinear map. Then the space

$$\operatorname{der}(A, m) := \{D \in \mathfrak{gl}(A) \colon (\forall a, b \in A)\ Dm(a, b) = m(Da, b) + m(a, Db)\}$$

of derivations is a Lie subalgebra of $\mathfrak{gl}(A)$.

Exercise 4.1.5. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{n} \trianglelefteq \mathfrak{g}$ an ideal and $\mathfrak{h} < \mathfrak{g}$ a Lie subalgebra with $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and $\mathfrak{n} \cap \mathfrak{h} = \{0\}$. Then

$$\delta \colon \mathfrak{h} \to \operatorname{der} \mathfrak{n}, \quad \delta(x) := \operatorname{ad} x|_{\mathfrak{n}}$$

defines a homomorphism of Lie algebras and the map

$$\Phi \colon \mathfrak{n} \rtimes_\delta \mathfrak{h} \to \mathfrak{g}, \quad (x, y) \mapsto x + y$$

is an isomorphism of Lie algebras.

Exercise 4.1.6. On $\mathbb{R}^3$ we define the vector product by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} := \begin{pmatrix} v_2w_3 - v_3y_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}.$$

Show that $(\mathbb{R}^3, \times)$ is a Lie algebra and that the map

$$\Phi\colon \mathbb{R}^3 \to \mathfrak{so}_3(\mathbb{R}), \quad \Phi(v) := v_1x + v_2y + v_3z$$

(in the notation of Example 4.1.23) is an isomorphism of Lie algebras.

Exercise 4.1.7. Show that $[\mathfrak{gl}_n(\mathbb{K}), \mathfrak{gl}_n(\mathbb{K})] = \mathfrak{sl}_n(\mathbb{K})$ and $\mathfrak{z}(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}\mathbf{1}$.

Exercise 4.1.8. On the algebra $A := C^\infty(\mathbb{R}, \mathbb{R})$, consider the operators

$$Pf(x) := f'(x), \quad Qf(x) := xf(x) \quad \text{and} \quad Zf(x) = f(x).$$

Then the Lie subalgebra of $\mathfrak{gl}(A)$ generated by P, Q and Z is isomorphic to the Heisenberg algebra $\mathfrak{h}_3(\mathbb{R})$, i.e.,

$$[P, Q] = Z \quad \text{and} \quad [P, Z] = [Q, Z] = 0.$$

Adding also the operator

$$Hf(x) := \frac{1}{2}\Big(\frac{d^2 f}{dx^2}(x) + x^2 f(x)\Big), \quad H = \frac{1}{2}(P^2 + Q^2),$$

we obtain a four-dimensional subalgebra, isomorphic to the oscillator algebra (Example 4.1.19).

Exercise 4.1.9. Show that for two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the Lie algebra $\mathfrak{g}$, the subspace $[\mathfrak{a}, \mathfrak{b}]$ also is an ideal.

Exercise 4.1.10. Let (π, V) be a representation of the Lie algebra $\mathfrak{g}$ on V and $W \subseteq V$ a $\mathfrak{g}$-invariant subspace, i.e., $\pi(\mathfrak{g})W \subseteq W$. Then

$$\overline{\pi}\colon \mathfrak{g} \to \mathfrak{gl}(V/W), \quad \overline{\pi}(x)(v + W) := \pi(x)v + W$$

defines a representation of $\mathfrak{g}$ on the quotient space V/W.

Exercise 4.1.11. For the following Lie algebras, find a *faithful*, i.e., injective, finite-dimensional representation: $\mathfrak{sl}_2(\mathbb{K})$, the Heisenberg algebra, the oscillator algebra, and the abelian Lie algebra $\mathbb{R}^n$.

4.2 Nilpotent Lie Algebras

In the following, we shall encounter several important classes of Lie algebras that play a central role in the structure theory of finite-dimensional Lie algebras. The first of these two classes, nilpotent Lie algebras, are those for which iterated brackets $[x_1, [x_2, [x_3, [x_4, \cdots]]]]$ of sufficiently large order vanish. The most important result on nilpotent Lie algebras is Engel's Theorem which translates nilpotency of a Lie algebra into a pointwise condition. Typical examples of nilpotent Lie algebras are Lie algebras of strictly upper triangular (block) matrices.

Definition 4.2.1. Let $\mathfrak{g}$ be a Lie algebra. We define its *descending (lower) central series* inductively by

$$C^1(\mathfrak{g}) := \mathfrak{g} \quad \text{and} \quad C^{n+1}(\mathfrak{g}) := [\mathfrak{g}, C^n(\mathfrak{g})].$$

In particular, $C^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is the commutator algebra. The Lie algebra $\mathfrak{g}$ is called *nilpotent*, if there is an $n \in \mathbb{N}$ with $C^n(\mathfrak{g}) = \{0\}$. By induction, one immediately sees that each $C^n(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, so that $C^{n+1}(\mathfrak{g}) \subseteq C^n(\mathfrak{g})$. Hence, for finite-dimensional Lie algebra, the nilpotency of $\mathfrak{g}$ is equivalent to the vanishing of the ideal $C^\infty(\mathfrak{g}) := \bigcap_{n\in\mathbb{N}} C^n(\mathfrak{g})$.

Example 4.2.2. (i) The Heisenberg algebra $\mathfrak{h}_3(\mathbb{R})$ is nilpotent.
(ii) Every abelian Lie algebra is nilpotent.
(iii) If $\mathcal{F} = (V_0, \ldots, V_n)$ is a flag in the vector space V and we put $V_i := \{0\}$ for $i < 0$, then $\mathfrak{g}_n(\mathcal{F})$ is a nilpotent Lie algebra. In fact, an easy induction leads to

$$C^m\big(\mathfrak{g}_n(\mathcal{F})\big)V_n \subseteq V_{n-m}$$

and therefore to $C^n\big(\mathfrak{g}_n(\mathcal{F})\big) = \{0\}$.

Proposition 4.2.3. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) *If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are nilpotent.*
(ii) *If $\mathfrak{a} < \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}/\mathfrak{a}$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*
(iii) *If $\mathfrak{g} \neq \{0\}$ is nilpotent, then $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$.*
(iv) *If $\mathfrak{g}$ is nilpotent, then there is an $n \in \mathbb{N}$ with $\mathrm{ad}(x)^n = 0$ for all $x \in \mathfrak{g}$, i.e., the $\mathrm{ad}(x)$ are nilpotent as linear maps.*
(v) *If $\mathfrak{i} \trianglelefteq \mathfrak{g}$, then all the spaces $C^n(\mathfrak{i})$ are ideals of $\mathfrak{g}$.*

Proof. (i) If $\mathfrak{h} < \mathfrak{g}$, then $[\mathfrak{h}, \mathfrak{h}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $C^n(\mathfrak{h}) \subseteq C^n(\mathfrak{g})$ follows by induction. Therefore each subalgebra of a nilpotent Lie algebra is nilpotent.

For a homomorphism $\alpha\colon \mathfrak{g} \to \mathfrak{h}$, we obtain inductively

$$C^n(\alpha(\mathfrak{g})) = \alpha(C^n(\mathfrak{g})) \quad \text{for each } n \in \mathbb{N}. \tag{4.2}$$

Thus, if $C^n(\mathfrak{g}) = \{0\}$, then $C^n(\mathrm{im}\ \alpha) = \{0\}$.

(ii) If $\mathfrak{g}/\mathfrak{a}$ is nilpotent, then there is an $n \in \mathbb{N}$ with $C^n(\mathfrak{g}/\mathfrak{a}) = \{0\}$, so that (4.2), applied to the quotient homomorphism $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$, leads to $C^n(\mathfrak{g}) \subseteq \mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$ and thus to $C^{n+1}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{z}(\mathfrak{g})] = \{0\}$.
(iii) If $\mathfrak{g} \neq \{0\}$ is nilpotent, for some $n \in \mathbb{N}$, we have $C^n(\mathfrak{g}) = \{0\}$ and $C^{n-1}(\mathfrak{g}) \neq \{0\}$. Then $[\mathfrak{g}, C^{n-1}(\mathfrak{g})] = \{0\}$ implies that the nontrivial ideal $C^{n-1}(\mathfrak{g})$ is contained in the center.
(iv) If $C^{n+1}(\mathfrak{g}) = \{0\}$, then $(\operatorname{ad} x)^n \mathfrak{g} \subseteq C^{n+1}(\mathfrak{g}) = \{0\}$.
(v) In view of Exercise 4.1.9, this follows by induction. □

In Proposition 4.2.3, we have seen that for every nilpotent Lie algebra, all the endomorphisms $\operatorname{ad}(x)$, $x \in \mathfrak{g}$, are nilpotent. Now our aim is to show that a finite-dimensional Lie algebra, for which every $\operatorname{ad} x$ is nilpotent, is nilpotent itself. We start with a simple lemma, the proof of which we leave to the reader as an exercise (cf. Exercises 3.1.3 and 4.1.10).

Lemma 4.2.4. (i) *Let V be a finite-dimensional vector space, $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a subalgebra and $x \in \mathfrak{g}$. If $x \in \mathfrak{gl}(V)$ is nilpotent, then $\operatorname{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ is also nilpotent.*
(ii) *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} < \mathfrak{g}$. Then*

$$\operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}), \quad \operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}(x)(y + \mathfrak{h}) := [x, y] + \mathfrak{h}$$

defines a representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$.

Theorem 4.2.5 (Engel's Theorem on Linear Lie Algebras). *Let $V \neq \{0\}$ be a finite-dimensional vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra. If all $x \in \mathfrak{g}$ are nilpotent, i.e., x^n for some $n \in \mathbb{N}$, then there exists a nonzero $v_o \in V$ with $\mathfrak{g}(v_o) = \{0\}$.*

Proof. We proceed by induction on $\dim \mathfrak{g}$. For $\dim \mathfrak{g} = 0$ the assertion holds trivially for each nonzero $v_o \in V$.

Next we assume that $\dim \mathfrak{g} > 0$ and pick a proper subalgebra $\mathfrak{h} < \mathfrak{g}$ of maximal dimension. According to Lemma 4.2.4, for each $x \in \mathfrak{h}$ the operators $\operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}(x)$ are nilpotent. Now our induction hypothesis implies the existence of some $x_o \in \mathfrak{g} \backslash \mathfrak{h}$ with $\operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}(\mathfrak{h})(x_o) = \{0\}$, i.e., $[\mathfrak{h}, x_o] \subseteq \mathfrak{h}$. This implies that $\mathbb{K}x_o + \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and, again by maximality of $\mathfrak{h}$, it follows that $\mathbb{K}x_o + \mathfrak{h} = \mathfrak{g}$. The induction hypothesis also implies that the space $V_o := \{v \in V\colon \mathfrak{h}(v) = \{0\}\}$ is nonzero. Moreover,

$$yx(w) = xy(w) - [x, y](w) \in x\mathfrak{h}(w) + \mathfrak{h}(w) = \{0\}$$

for $x \in \mathfrak{g}, y \in \mathfrak{h}, w \in V_o$, implies that $\mathfrak{g}(V_o) \subseteq V_o$. Since $x_o|_{V_o}$ is also nilpotent, there exists a nonzero $v_o \in V_o$ with $x_o(v_o) = 0$. Putting all this together, we arrive at $\mathfrak{g}(v_o) = \mathfrak{h}(v_o) + \mathbb{K}x_o(v_o) = \{0\}$. □

Exercise 4.2.8 discusses an interesting Lie algebra of nilpotent endomorphisms of an infinite-dimensional space, showing in particular that Engel's Theorem does not generalize to infinite-dimensional spaces.

From Theorem 4.2.5, one also gets the induction step by which one shows that every subalgebra of $\mathfrak{gl}(V)$ which consists of nilpotent elements can be written as a Lie algebra of triangular matrices.

Definition 4.2.6. Let V be an n-dimensional vector space. A *complete flag* in V is a flag $(V_0, \ldots, V_n)$ with with $\dim V_k = k$ for each k.

Corollary 4.2.7. *Let V be a finite-dimensional vector space and $\mathfrak{g} < \mathfrak{gl}(V)$ such that all elements of $\mathfrak{g}$ are nilpotent. Then there exists a complete flag $\mathcal{F}$ in V with $\mathfrak{g} \subseteq \mathfrak{g}_n(\mathcal{F})$. In particular, there is a basis for V with respect to which the elements of $\mathfrak{g}$ correspond to strictly upper triangular matrices. In particular, $\mathfrak{g}$ is nilpotent.*

Proof. In view of Theorem 4.2.5, there exists a nonzero $v_1 \in V$ with $\mathfrak{g}(v_1) = \{0\}$. We set $V_1 := \mathbb{K}v_1$. Then

$$\alpha\colon \mathfrak{g} \to \mathfrak{gl}(V/V_1), \quad \alpha(x)(v + V_1) := x(v) + V_1$$

is a representation of $\mathfrak{g}$ on V/V_1 (Exercise 4.1.10), and $\alpha(\mathfrak{g})$ consists of nilpotent endomorphisms. We now proceed by induction on $\dim V$, so that the induction hypothesis implies that V/V_1 possesses a complete flag $\mathcal{F}_1 = (W_1, \ldots, W_k)$ with $\alpha(\mathfrak{g}) \subseteq \mathfrak{g}_n(\mathcal{F}_1)$. Then $\{0\}$, together with the preimage of the flag $\mathcal{F}_1$ in V is a complete flag $\mathcal{F}$ in V with $\mathfrak{g} \subseteq \mathfrak{g}_n(\mathcal{F})$. Since $\mathfrak{g}_n(\mathcal{F})$ is nilpotent (Example 4.2.2(iii)), the subalgebra $\mathfrak{g}$ is also nilpotent. □

Now we are able to prove the announced criterion for the nilpotency of a Lie algebra.

Theorem 4.2.8 (Engel's Characterization Theorem for Nilpotent Lie Algebras). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if for each $x \in \mathfrak{g}$ the operator $\operatorname{ad} x$ is nilpotent.*

Proof. We have already seen in Proposition 4.2.3 that for each $x \in \mathfrak{g}$ the operator $\operatorname{ad} x$ is nilpotent. It remains to show the converse.

If $\operatorname{ad} x$ is nilpotent for each $x \in \mathfrak{g}$, then Corollary 4.2.7 implies that the Lie algebra $\operatorname{ad}(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent. Now Proposition 4.2.3(ii) shows that $\mathfrak{g}$ is also nilpotent. □

Exercises for Section 4.2

Exercise 4.2.1. Let V be a finite-dimensional complex vector space and $x \in \operatorname{End}(V)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of $\operatorname{ad} x$ are all numbers of the form

$$\lambda_i - \lambda_j, \quad i, j = 1, \ldots, n.$$

Exercise 4.2.2. Let $\mathfrak{g}$ be Lie algebra, $\mathfrak{h}$ a subalgebra and $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \setminus \mathfrak{h}$. Then $\mathfrak{h} + \mathbb{R}x \cong \mathfrak{h} \rtimes_\alpha \mathbb{R}x$ for $\alpha(tx) = \operatorname{ad}(tx)|_{\mathfrak{h}}$.

Exercise 4.2.3. Let $\mathfrak{g}$ be the Heisenberg algebra. Determine a basis for $\mathfrak{g}$ with respect to which $\operatorname{ad}\mathfrak{g}$ consists of upper triangular matrices.

Exercise 4.2.4. Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ be a nonzero ideal in $\mathfrak{g}$. Show that the intersection of $\mathfrak{h}$ with the center of $\mathfrak{g}$ is not trivial.

Exercise 4.2.5. Show: If $\mathfrak{a}$, $\mathfrak{b}$ are nilpotent ideals of the Lie algebra $\mathfrak{g}$, then $\mathfrak{a}+\mathfrak{b}$ is also nilpotent.

Exercise 4.2.6. Give an example of a Lie algebra $\mathfrak{g}$ which contains a nilpotent ideal $\mathfrak{n}$ for which $\mathfrak{g}/\mathfrak{n}$ is nilpotent and $\mathfrak{g}$ is not nilpotent.

Exercise 4.2.7. For each Lie algebra $\mathfrak{g}$, we have

$$[C^n(\mathfrak{g}), C^m(\mathfrak{g})] \subseteq C^{n+m}(\mathfrak{g}) \quad \text{for} \quad n, m \in \mathbb{N}.$$

Exercise 4.2.8. This exercise shows why Engel's Theorem does not generalize to infinite-dimensional spaces. We consider the vector space $V = \mathbb{K}^{(\mathbb{N})}$ with the basis $\{e_i : i \in \mathbb{N}\}$. In terms of the rank-one-operators $E_{ij} \in \operatorname{End}(V)$, defined by $E_{ij}e_k = \delta_{jk}e_i$, we consider the Lie algebra

$$\mathfrak{g} := \operatorname{span}\{E_{ij} : i > j\}$$

(strictly lower triangular matrices). Show that:

(a) $C^n(\mathfrak{g}) = \operatorname{span}\{E_{ij} : i \geq j+n\}$, $n \in \mathbb{N}$. In particular, we have $C^\infty(\mathfrak{g}) = \bigcap_{n\in\mathbb{N}} C^n(\mathfrak{g}) = \{0\}$, i.e., $\mathfrak{g}$ is *residually nilpotent*.
(b) $\mathfrak{g}$ consists of nilpotent endomorphisms of finite rank.
(c) $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
(d) $V^{\mathfrak{g}} = \{v \in V : \mathfrak{g}\cdot v = \{0\}\} = \{0\}$ (compare with Engel's Theorem).

4.3 The Jordan Decomposition

In this section we develop a tool that will be of crucial importance throughout the structure theory of Lie algebras: the Jordan decomposition of an endomorphism of a finite-dimensional vector space. Although the existence of the Jordan decomposition can be derived from the Jordan normal form, the proof of the Jordan decomposition is less involved because it does not specify the structure of the nilpotent component. Since we need various properties of the Jordan decomposition, we give a direct self-contained proof which does not require more than some elementary properties of polynomials.

Definition 4.3.1. Let V be a vector space and $M \in \operatorname{End}(V)$.
(a) For $\lambda \in \mathbb{K}$, we define the *eigenspace* with respect to λ as

$$V_\lambda(M) := \ker(M - \lambda\mathbf{1})$$

and the *generalized eigenspace* as

$$V^\lambda(M) := \bigcup_{n\in\mathbb{N}} \ker(M - \lambda\mathbf{1})^n.$$

Note that the ascending sequence $\ker(M - \lambda\mathbf{1})^n$ is eventually constant if V is finite-dimensional. We call λ an *eigenvalue* if $V_\lambda(M) \neq \{0\}$.
(b) We call M *diagonalizable* if $V = \bigoplus_{\lambda\in\mathbb{K}} V_\lambda(M)$, i.e., V is a direct sum of the eigenspaces of M.
(c) We call M *nilpotent* if there exists an $n \in \mathbb{N}$ with $M^n = 0$. If M is nilpotent, then $V = V^0(M)$.
(d) We call M *split* if there is a nonzero polynomial $f \in \mathbb{K}[X]$ with $f(M) = 0$ which decomposes as a product of linear factors. This is always the case for $\mathbb{K} = \mathbb{C}$.
(e) For $\mathbb{K} = \mathbb{R}$, we call M *semisimple* if the endomorphism $M_\mathbb{C}$ of $V_\mathbb{C}$, defined by $M_\mathbb{C}(z \otimes v) = z \otimes Mv$ is diagonalizable (cf. Exercise 4.3.5).

Lemma 4.3.2. *For two commuting endomorphisms M, N of the finite-dimensional vector space V, the following assertions hold:*

(a) *If M and N are diagonalizable, then $M + N$ is diagonalizable.*
(b) *If M and N are nilpotent, then $M + N$ is nilpotent.*

Proof. (a) Exercise 1.1.1(a)-(c).
(b) Suppose that $M^m = N^n = 0$. Then $[M, N] = 0$ implies that

$$(M + N)^k = \sum_{i+j=k} \binom{k}{i} M^i N^j.$$

If $k \geq n + m - 1$, then either $i \geq m$ or $j \geq n$, so that all summands vanish. Hence $(M + N)^k = 0$. □

Theorem 4.3.3 (Jordan Decomposition Theorem). *Let V be a finite-dimensional vector space and $M \in \mathrm{End}(V)$ a split endomorphism. Then there exists a diagonalizable endomorphism M_s and a nilpotent endomorphism M_n such that*

(i) $M = M_s + M_n$.
(ii) $V^\lambda(M_s) = V_\lambda(M_s) = V^\lambda(M)$ *for each* $\lambda \in \mathbb{K}$.
(iii) *There exist polynomials $P, Q \in \mathbb{K}[X]$ with $P(0) = Q(0) = 0$ such that $M_s = P(M)$ and $M_n = Q(M)$.*
(iv) *If $L \in \mathrm{End}(V)$ commutes with M, then it also commutes with M_s and M_n.*
(v) (Uniqueness of the Jordan decomposition) *If $S, N \in \mathrm{End}(V)$ commute, S is diagonalizable and N nilpotent with $M = S + N$, then $S = M_s$ and $N = M_n$.*

Proof. Let $f \in \mathbb{K}[X]$ be the minimal polynomial of M, i.e., a generator of the ideal $I_M := \{f \in \mathbb{K}[X]\colon f(M) = 0\}$ with leading coefficient 1. By assumption, I_M contains a nonzero polynomial which is a product of linear factors, so that Exercise 4.3.6 implies that f also has this property. Hence there exist pairwise different $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ and $k_i \in \mathbb{N}$ such that f can be written as

$$f = (X - \lambda_1)^{k_1}(X - \lambda_2)^{k_2} \cdots (X - \lambda_m)^{k_m}.$$

Put $f_i := f/(X - \lambda_i)^{k_i}$. Then the ideal

$$I := (f_1) + \ldots + (f_m) \subseteq \mathbb{K}[X]$$

is generated by some element g ($\mathbb{K}[X]$ is a principal ideal domain, a simple consequence of Euclid's Algorithm) which is the greatest common divisor of the polynomials f_i. The fact that the $f_1, \ldots, f_m$ have no nontrivial common divisor (cf. Exercise 4.3.6) implies that g is constant, so that $I = \mathbb{K}[X]$. Hence $1 \in I$, so that there exist polynomials $r_1, \ldots, r_m \in \mathbb{K}[X]$ with

$$1 = r_1 f_1 + \ldots + r_m f_m.$$

Put $E_i := (r_i f_i)(M) \in \mathrm{End}(V)$ and note that $\sum_i E_i = \mathrm{id}_V$. If $i \neq j$, then f divides $r_i f_i r_j f_j$, so that $f(M) = 0$ leads to $E_i E_j = 0$, and thus $E_i^2 = E_i\big(\sum_{j=1}^m E_j\big) = E_i$. Therefore the E_i are pairwise commuting projections onto subspaces V_i with $V = \bigoplus_{i=1}^m V_i$ (since $\sum_{i=1}^m E_i = \mathbf{1}$). Now $M_s := \sum_{i=1}^m \lambda_i E_i$ is diagonalizable with $V_i = V_{\lambda_i}(M_s)$.

Since M commutes with each E_i, it preserves the subspaces V_i, and therefore $f_i(M)$ preserves V_i. The relation

$$\mathrm{id}_{V_i} = E_i|_{V_i} = r_i(M) f_i(M)|_{V_i}$$

shows that the restriction of $f_i(M)$ to V_i is invertible. Therefore $f(M) = 0$ leads to

$$(M - \lambda_i \mathbf{1})^{k_i}(V_i) = (M - \lambda_i \mathbf{1})^{k_i} f_i(M)(V_i) = f(M)(V_i) = \{0\},$$

i.e., $V_i \subseteq V^{\lambda_i}(M)$.

With $M_n := M - M_s$ and $k_0 := \max\{k_i\colon i = 1, \ldots, m\}$ we finally get $M_n^{k_0} = 0$, proving (i).

(ii) We have to show that $V_i = V^{\lambda_i}(M)$. We know already that $V_i \subseteq V^{\lambda_i}(M)$. So let $v \in V^{\lambda_i}(M)$ and write v as $v = \sum_{j=1}^m v_j$ with $v_j \in V_j$. Then the invariance of V_j under M implies that $v_j \in V^{\lambda_i}(M)$. If $v_j \neq 0$, then there exists a nonzero eigenvector $v_j' \in V_{\lambda_i}(M) \cap V_j$ (put $v_j' = (M - \lambda_i \mathbf{1})^k v_j$, where k is maximal with the property that this vector is nonzero). Then $(M - \lambda_j \mathbf{1})^{n_j} v_j' = (\lambda_i - \lambda_j)^{n_j} v_j' = 0$, hence $\lambda_j = \lambda_i$, i.e., $j = i$. This implies that $v = v_i \in V_i$ and therefore $V^{\lambda_i}(M) = V_i$.

(iii) By construction, $M_s = P_1(M)$ and $M_n = Q_1(M)$ for $P_1 = \sum_i \lambda_i r_i f_i$ and $Q_1 = X - P_1$. It remains to be seen that these polynomials can be chosen with trivial constant term. If one eigenvalue λ_j vanishes, then $\{0\} \neq V_0 := \ker M \subseteq V_j$ and $M_s|_{V_0} = 0$ implies that P_1 has no constant term. Then $Q_1 = X - P_1$ likewise has no constant term and (iii) holds with $P := P_1$ and $Q := Q_1$.

If all eigenvalues λ_j are nonzero, then $f(0) \neq 0$ and (iii) holds with

$$P := P_1 - \frac{P_1(0)}{f(0)} f \quad \text{and} \quad Q := Q_1 - \frac{Q_1(0)}{f(0)} f.$$

(iv) is a direct consequence of (iii).
(v) Since N and S commute with $M = N + S$, (iii) shows that they both commute with M_s and M_n. Then Lemma 4.3.2 shows that

$$S - M_s = M_n - N$$

is nilpotent as well as diagonalizable, which leads to $0 = S - M_s = M_n - N$. □

Definition 4.3.4. The decomposition $M = M_s + M_n$ is called the *Jordan decomposition* of M, M_s is called the *semisimple Jordan component* and M_n the *nilpotent Jordan component* of M.

Example 4.3.5. If M is a Jordan block $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then the Jordan decomposition is

$$M = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{M_s} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{M_n}.$$

The matrix $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is diagonalizable and therefore $M = M_s$. In this case

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

is not the Jordan decomposition, but the first summand is diagonalizable and the second summand is nilpotent. These summands do not commute.

The preceding theorem does not apply to all endomorphisms of real vector spaces. We now explain how this problem can be overcome, so that we also obtain a Jordan decomposition for endomorphisms of real vector spaces.

Definition 4.3.6 (Jordan decomposition in the real case). If V is a real finite-dimensional vector space and $M \in \mathrm{End}(V)$, then $M_{\mathbb{C}} \in \mathrm{End}(V_{\mathbb{C}})$, defined by $M_{\mathbb{C}}(z \otimes v) := z \otimes Mv$ has a Jordan decomposition

$$M_{\mathbb{C}} = M_{\mathbb{C},s} + M_{\mathbb{C},n}.$$

Let $\sigma\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ be the antilinear map defined by $\sigma(z \otimes v) := \overline{z} \otimes v$ and define for any complex linear $A \in \mathrm{End}(V_{\mathbb{C}})$ the complex linear endomorphism $\overline{A} := \sigma \circ A \circ \sigma \in \mathrm{End}(V_{\mathbb{C}})$. Then $\overline{M_{\mathbb{C}}} = M_{\mathbb{C}}$ leads to

$$M_{\mathbb{C}} = \overline{M_{\mathbb{C}}} = \overline{M_{\mathbb{C},s}} + \overline{M_{\mathbb{C},n}},$$

where the summands on the right commute, the first is diagonalizable and the second is nilpotent (Exercise). Hence the uniqueness of the Jordan decomposition yields

$$\overline{M_{\mathbb{C},s}} = M_{\mathbb{C},s} \quad \text{and} \quad \overline{M_{\mathbb{C},n}} = M_{\mathbb{C},n}.$$

In view of Exercise 4.3.1, this implies the existence of $M_s \in \mathrm{End}(V)$ and $M_n \in \mathrm{End}(V)$, with

$$(M_s)_{\mathbb{C}} = M_{\mathbb{C},s} \quad \text{and} \quad (M_n)_{\mathbb{C}} = M_{\mathbb{C},n}.$$

Then $M = M_s + M_n$, and this is called the *Jordan decomposition of* M. It is uniquely characterized by the properties that $[M_s, M_n] = 0$, M_s is semisimple and M_n is nilpotent (Exercise).

Proposition 4.3.7 (Properties of the Jordan decomposition). *Let V be a finite-dimensional vector space and $M \in \mathrm{End}(V)$.*
(i) *If $M' \in \mathrm{End}(V')$ and $f\colon V \to V'$ satisfy $f \circ M = M' \circ f$, then*

$$f \circ M_s = M'_s \circ f \quad \textit{and} \quad f \circ M_n = M'_n \circ f.$$

(ii) *If $W \subseteq V$ is an M-invariant subspace, then*

$$(M|_W)_s = M_s|_W \quad \textit{and} \quad (M|_W)_n = M_n|_W.$$

In particular, W is invariant under M_s and M_n. If $\overline{M}$ denotes the induced endomorphism of V/W, then

$$(\overline{M})_s = \overline{M_s} \quad \textit{and} \quad (\overline{M})_n = \overline{M_n}.$$

(iii) *If $U \subseteq W$ are subspaces of V with $MW \subseteq U$, then $M_sW \subseteq U$ and $M_nW \subseteq U$.*

Proof. (i) Let $W := V \oplus V'$, $L := M \oplus M'$, and consider the linear map $\varphi\colon W \to W$ defined by $\varphi(v, v') = \big(0, f(v)\big)$. Then $\varphi \circ L = L \circ \varphi$ and thus $\varphi L_s = L_s \varphi$ and $\varphi L_n = L_n \varphi$. Further, $L_s = M_s \oplus M'_s$ and $L_n = M_n \oplus M'_n$ follows from the uniqueness of the Jordan decomposition (Theorem 4.3.3(v)) and the semisimplicity of $M_s \oplus M'_s$. This shows that

$$M'_s \circ f = f \circ M_s \quad \text{and} \quad M'_n \circ f = f \circ M_n.$$

(ii) Apply (i) to the inclusion $j\colon W \to V$ and the quotient map $p\colon V \to V/W$.
(iii) For $\mathbb{K} = \mathbb{C}$, this follows from Theorem 4.3.3(iii) and the real case is obtained by complexification. □

Proposition 4.3.8. *Let V be finite-dimensional and $x \in \mathfrak{gl}(V)$. If x is nilpotent (diagonalizable), then so is* $\mathrm{ad}\, x$.

This proposition can be obtained by combining Lemma 4.2.4(i) with Exercise 4.2.1. We give an alternative proof using the Jordan decomposition.

Proof. Put $L_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V), y \mapsto xy$ and $R_x : \mathfrak{gl}(V) \to \mathfrak{gl}(V), y \mapsto yx$. Then $\mathrm{ad}\, x = L_x - R_x$ and $[L_x, R_x] = 0$. In view of Lemma 4.3.2, it suffices to see that L_x and R_x are nilpotent, resp., diagonalizable whenever x has this property.

If $x^n = 0$, then $L_x^n = L_{x^n} = 0 = R_x^n$. If x is diagonalizable and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x, we consider the decomposition $V = \bigoplus_{j=1}^n V_{\lambda_j}(x)$. We write any $y \in \mathrm{End}(V)$ as $y = \sum_{j,k=1}^n y_{jk}$ with $y_{jk} V_{\lambda_k}(x) \subseteq V_{\lambda_j}(x)$. Then $L_x y_{jk} = \lambda_j y_{jk}$ and $R_x y_{jk} = \lambda_k y_{jk}$ imply that L_x and R_x are diagonalizable on $\mathfrak{gl}(V) = \mathrm{End}(V)$. □

Corollary 4.3.9. *For each endomorphism $x \in \mathfrak{gl}(V)$ of the finite-dimensional vector space V, the Jordan decomposition of $\mathrm{ad}\, x$ is given by*

$$\mathrm{ad}\, x = \mathrm{ad}(x_s) + \mathrm{ad}(x_n).$$

Proof. Proposition 4.3.8 implies for $\mathbb{K} = \mathbb{C}$ that $\mathrm{ad}(x_s)$ is diagonalizable and for $\mathbb{K} = \mathbb{R}$ that $(\mathrm{ad}\, x_s)_{\mathbb{C}} = \mathrm{ad}((x_s)_{\mathbb{C}})$ is diagonalizable. Further $\mathrm{ad}(x_n)$ is nilpotent, and $[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}[x_s, x_n] = 0$, so that the assertion follows from the uniqueness of the Jordan decomposition of $\mathrm{ad}\, x$. □

Proposition 4.3.10. *If A is a finite-dimensional algebra and $D \in \mathrm{der}(A)$, then the Jordan components D_s and D_n are also derivations of A.*

Proof. **First proof:** Let $m : A \otimes A \to A$ denote the linear map defined by the algebra multiplication. Then $D \in \mathrm{der}(A)$ is equivalent to the relation

$$D \circ m = m \circ (D \otimes \mathrm{id}_A + \mathrm{id}_A \otimes D).$$

Next we observe that

$$D \otimes \mathrm{id}_A + \mathrm{id}_A \otimes D = (D_s \otimes \mathrm{id}_A + \mathrm{id}_A \otimes D_s) + (D_n \otimes \mathrm{id}_A + \mathrm{id}_A \otimes D_n)$$

is the Jordan decomposition (Exercise!), so that Proposition 4.3.7 implies that

$$D_s \circ m = m \circ (D_s \otimes \mathrm{id}_A + \mathrm{id}_A \otimes D_s),$$

which means that $D_s \in \mathrm{der}(A)$, and hence that $D_n = D - D_s \in \mathrm{der}(A)$ because $\mathrm{der}(A)$ is a linear space.

Second proof: Since $\mathrm{der}(A)$ is a vector space, it suffices to show that $D_s \in \mathrm{der}(A)$. Furthermore, $D \in \mathrm{der}(A)$ is equivalent to $D_{\mathbb{C}} \in \mathrm{der}(A_{\mathbb{C}})$, so that we may assume that $\mathbb{K} = \mathbb{C}$.

For $a, b \in A$ and $\lambda, \mu \in \mathbb{K}$ we have for all $n \in \mathbb{N}$ the formula

$$\big(D - (\lambda + \mu)\mathbf{1}\big)^n (ab) = \sum_{k=0}^{n} \binom{n}{k} (D - \lambda\mathbf{1})^k (a) \cdot (D - \mu\mathbf{1})^{n-k} (b)$$

(Exercise 4.3.7). It follows that for $a \in A_\lambda(D_s) = A^\lambda(D)$ and $b \in A_\mu(D_s) = A^\mu(D)$, we have $ab \in A^{\lambda+\mu}(D) = A_{\lambda+\mu}(D_s)$. Furthermore

$$D_s(a)b + aD_s(b) = \lambda ab + \mu ab = (\lambda + \mu)ab = D_s(ab).$$

Since $A = \sum_{\lambda \in \mathbb{K}} A_\lambda(D_s)$, it follows that $D_s \in \operatorname{der}(A)$. □

Lemma 4.3.11 (Fitting decomposition). *Let V be a finite-dimensional vector space and $T \in \operatorname{End}(V)$. If $V^+(T) := \bigcap_{n \in \mathbb{N}} T^n(V)$, then*

$$V = V^0(T) \oplus V^+(T).$$

Proof. The sequence $T^n(V)$ is decreasing and $\dim V < \infty$ implies that there exists some n with $T^{n+1}(V) = T^n(V)$. Then

$$T|_{T^n(V)} \colon T^n(V) \to T^n(V)$$

is surjective, hence bijective, and we see that $V^+(T) = V^n(T)$. On the intersection $V^0(T) \cap V^+(T)$, the restriction of T is nilpotent and bijective at the same time. This is only possible if the intersection is $\{0\}$. Therefore it remains to see that $V = V^0(T) + V^+(T)$.

First we assume that T is split. We consider the generalized eigenspace decomposition

$$V = V^0(T) \oplus \bigoplus_{\lambda \neq 0} V^\lambda(T)$$

and note that each $V^\lambda(T)$ is T-invariant. For $\lambda \neq 0$, we have $V^\lambda(T) \cap \ker T = \{0\}$, so that $T|_{V^\lambda(T)}$ is injective, hence surjective. This implies that

$$\bigoplus_{\lambda \neq 0} V^\lambda(T) \subseteq V^+(T)$$

and thus $V = V^0(T) + V^+(T)$.

If T is not split, then $\mathbb{K} = \mathbb{R}$ and we consider the complexification $T_\mathbb{C} \in \operatorname{End}(V_\mathbb{C})$. Then the preceding argument implies that

$$V_\mathbb{C} = V^0(T_\mathbb{C}) \oplus V^+(T_\mathbb{C}) = V^0(T)_\mathbb{C} \oplus V^+(T)_\mathbb{C},$$

where we use $T^n(V)_\mathbb{C} = T_\mathbb{C}^n(V_\mathbb{C})$ for $n \in \mathbb{N}$ in the second equality. We conclude that $(V^0(T) + V^+(T))_\mathbb{C} = V_\mathbb{C}$, and this entails that $V^0(T) + V^+(T) = V$. □

The space $V^+(T)$ is called the *Fitting one component*. In this context the generalized eigenspace $V^0(T)$ is called the *Fitting null component*.

Exercises for Section 4.3

Exercise 4.3.1. Let V be a real vector space and

$$V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V = (1 \otimes V) \oplus (i \otimes V)$$

its complexification. We identify V with the real subspace $1 \otimes V$, so that

$$V_{\mathbb{C}} \cong V \oplus iV.$$

Show that:

(i) $\sigma(z \otimes v) := \overline{z} \otimes v$ defines an antilinear involution of $V_{\mathbb{C}}$ whose fixed point space is V.
(ii) A complex subspace $E \subseteq V_{\mathbb{C}}$ is of the form $W_{\mathbb{C}}$ for some real subspace of V if and only if $\sigma(E) = E$.
(iii) For each $M \in \mathrm{End}(V)$, the complexification $M_{\mathbb{C}} \in \mathrm{End}(V_{\mathbb{C}})$, defined by $M_{\mathbb{C}}(z \otimes v) := z \otimes Mv$ commutes with σ.
(iv) For $A \in \mathrm{End}(V_{\mathbb{C}})$ the following are equivalent
   (a) A commutes with σ.
   (b) A preserves the real subspace V.
   (c) $A = M_{\mathbb{C}}$ for some $M \in \mathrm{End}(V)$.

Exercise 4.3.2. Let V be a complex vector space and $M \in \mathrm{End}(V)$. Show that M is diagonalizable if and only if each M-invariant subspace $W \subseteq V$ possesses an M-invariant complement.

Exercise 4.3.3. Let V be a real vector space, $A \in \mathrm{End}(V)$ and $z \in V_{\mathbb{C}}$ an eigenvector of $A_{\mathbb{C}}$ with respect to the eigenvalue λ. Show that if $z = x + iy$ with $x, y \in V$ and $\lambda = a + ib$, then

$$Ax = ax - by \quad \text{and} \quad Ay = ay + bx.$$

In particular, the 2-dimensional subspace $E := \mathrm{span}\{x, y\} \subseteq V$ is invariant under A.

Exercise 4.3.4. Let $A \in M_2(\mathbb{R})$ with no real eigenvalue. Then there exists a basis $x, y \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$ with

$$Ax = ax - by \quad \text{and} \quad Ay = ay + bx.$$

Exercise 4.3.5. Let V be a real vector space and $M \in \mathrm{End}(V)$. Show that M is semisimple if and only if each M-invariant subspace $W \subseteq V$ possesses an M-invariant complement.

Exercise 4.3.6. Let $f \in \mathbb{K}[X]$ be a polynomial of the form

$$f = (X - \lambda_1)^{k_1}(X - \lambda_2)^{k_2} \cdots (X - \lambda_m)^{k_m}$$

and $g \in \mathbb{K}[X]$ a divisor of f with leading coefficient 1. Show that there exist $\ell_i \leq k_i$ with

$$g = (X - \lambda_1)^{\ell_1}(X - \lambda_2)^{\ell_2} \cdots (X - \lambda_m)^{\ell_m}.$$

Exercise 4.3.7. Show that for each algebra A, a derivation $D \in \operatorname{der}(A)$ and $\lambda, \mu \in \mathbb{K}$, we have for $a, b \in A$:

$$\big(D - (\lambda + \mu)\mathbf{1}\big)^n(ab) = \sum_{k=0}^{n} \binom{n}{k} (D - \lambda\mathbf{1})^k(a) \cdot (D - \mu\mathbf{1})^{n-k}(b).$$

Exercise 4.3.8. Let V be a finite-dimensional vector space over $\mathbb{K}$ and $A \in \operatorname{End}(V)$. Then the multiplicity of the root 0 of its characteristic polynomial $\det(A - X\mathbf{1})$ coincides with $\dim V^0(A)$.

4.4 Solvable Lie Algebras

In this section we turn to the class of solvable Lie algebras. They are defined in a similar fashion as nilpotent ones and each nilpotent Lie algebra is solvable. The central results on solvable Lie algebras are Lie's Theorem on representations of solvable Lie algebras and Cartan's Solvability Criterion in terms of vanishing of

$$\operatorname{tr}(\operatorname{ad}[x, y] \operatorname{ad} z) \quad \text{for} \quad x, y, z \in \mathfrak{g}.$$

As we shall see later on, similar techniques apply to semisimple Lie algebras.

Definition 4.4.1. (a) Let $\mathfrak{g}$ be a Lie algebra. The *derived series* of $\mathfrak{g}$ is defined by

$$D^0(\mathfrak{g}) := \mathfrak{g} \quad \text{and} \quad D^n(\mathfrak{g}) := [D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})]$$

for $n \in \mathbb{N}$.

From $D^1(\mathfrak{g}) \subseteq \mathfrak{g}$ we inductively see that $D^n(\mathfrak{g}) \subseteq D^{n-1}(\mathfrak{g})$. Further, an easy induction shows that all $D^n(\mathfrak{g})$ are ideals of $\mathfrak{g}$ (Exercise 4.1.9). The derived series is a descending series of ideals.
(b) The Lie algebra $\mathfrak{g}$ is said to be *solvable*, if there exists an $n \in \mathbb{N}$ with $D^n(\mathfrak{g}) = \{0\}$.

Example 4.4.2. (i) The oscillator algebra is solvable, but not nilpotent.
(ii) Every nilpotent Lie algebra is solvable because $D^n(\mathfrak{g}) \subseteq C^{n+1}(\mathfrak{g})$ follows easily by induction.
(iii) Consider $\mathbb{R}$ and $\mathbb{C}$ as abelian real Lie algebras and write $I \in \operatorname{End}_{\mathbb{R}}(\mathbb{C})$ for the multiplication with i. Then the Lie algebra $\mathbb{C} \rtimes_I \mathbb{R}$ is solvable, but not nilpotent. It is isomorphic to $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, where $\mathfrak{g}$ is the oscillator algebra.
(iv) Let $\mathfrak{g}$ be a 2-dimensional nonabelian Lie algebra with basis x, y. Then $0 \neq [x, y] = ax + by$ for $(a, b) \neq (0, 0)$. Assume w.l.o.g. that $b \neq 0$ and put $v := b^{-1}x$ and $w := ax + by$. Then $[v, w] = w$. This implies in particular that all nonabelian two-dimensional Lie algebras are isomorphic because they have a basis (v, w), with $[v, w] = w$. Then $D^1(\mathfrak{g}) = \mathbb{K}w$ and $D^2(\mathfrak{g}) = \{0\}$, so that $\mathfrak{g}$ is solvable. On the other hand $C^n(\mathfrak{g}) = \mathbb{K}w$ for each $n > 1$, so that $\mathfrak{g}$ is not nilpotent.

A natural matrix realization of this Lie algebra is

$$\mathfrak{aff}_1(\mathbb{K}) = \begin{pmatrix} \mathbb{K} & \mathbb{K} \\ 0 & 0 \end{pmatrix} \quad \text{with the basis} \quad v := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad w := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Proposition 4.4.3. *For a Lie algebra $\mathfrak{g}$, the following assertions hold:*

(i) *If $\mathfrak{g}$ is solvable, then all subalgebras and homomorphic images of $\mathfrak{g}$ are solvable.*
(ii) *Solvability is an extension property: If $\mathfrak{i}$ is a solvable ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{i}$ is solvable, then $\mathfrak{g}$ is solvable.*
(iii) *If $\mathfrak{i}$ and $\mathfrak{j}$ are solvable ideals of $\mathfrak{g}$, then the ideal $\mathfrak{i}+\mathfrak{j}$ is solvable.*
(iv) *If $\mathfrak{i} \trianglelefteq \mathfrak{g}$ is an ideal, then the $D^n(\mathfrak{i})$ are ideals in $\mathfrak{g}$.*

Proof. (i) If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, then $D^n(\mathfrak{h}) \subseteq D^n(\mathfrak{g})$ follows by induction, and if $\alpha\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then we obtain

$$D^n(\alpha(\mathfrak{g})) = \alpha(D^n(\mathfrak{g})) \tag{4.3}$$

by induction. This implies (i).

(ii) Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ be the quotient map. We have already seen in (i) that $\pi(D^n(\mathfrak{g})) = D^n(\pi(\mathfrak{g}))$ for each n. If $\mathfrak{g}/\mathfrak{i}$ is solvable, then $\pi(D^n(\mathfrak{g}))$ vanishes for some $n \in \mathbb{N}$. Now $D^n(\mathfrak{g}) \subseteq \ker\pi = \mathfrak{i}$, so that $D^{n+k}(\mathfrak{g}) \subseteq D^k(\mathfrak{i})$ for each $k \in \mathbb{N}$. If $\mathfrak{i}$ is also solvable, we immediately derive that $\mathfrak{g}$ is solvable.

(iii) The ideal $\mathfrak{j}$ of $\mathfrak{i}+\mathfrak{j}$ is solvable and $(\mathfrak{i}+\mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/(\mathfrak{i}\cap\mathfrak{j})$ (Theorem 4.1.17(iii)) is solvable by (i). Hence (ii) implies that $\mathfrak{i}+\mathfrak{j}$ is solvable.

(iv) We only have to observe that for each ideal $\mathfrak{i}$, its commutator algebra $[\mathfrak{i},\mathfrak{i}]$ also is an ideal (Exercise 4.1.9). Then (iv) follows by induction. □

Example 4.4.4. If $\mathcal{F} = (V_0, \ldots, V_n)$ is a complete flag in the n-dimensional vector space V, then $\mathfrak{g}(\mathcal{F})$ is a solvable Lie algebra. In fact,

$$\mathfrak{g}(\mathcal{F}) \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \mathfrak{gl}_1(\mathbb{K})^n \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \mathbb{K}^n$$

(Example 4.1.20).

Since $\mathbb{K}^n \cong \mathfrak{g}(\mathcal{F})/\mathfrak{g}_n(\mathcal{F})$ is abelian and $\mathfrak{g}_n(\mathcal{F})$ nilpotent (Example 4.2.2(iii)), the solvability of $\mathfrak{g}(\mathcal{F})$ follows from Proposition 4.4.3(ii). Below we shall see that Lie's Theorem provides a converse for solvable subalgebras of $\mathfrak{gl}(V)$ and $\mathbb{K} = \mathbb{C}$; they are always contained in some $\mathfrak{g}(\mathcal{F})$ for a complete flag $\mathcal{F}$.

Definition 4.4.5. Proposition 4.4.3(iii) shows that in every finite-dimensional Lie algebra $\mathfrak{g}$, there is a largest solvable ideal. This ideal is called the *radical* of $\mathfrak{g}$, and is denoted by $\mathrm{rad}(\mathfrak{g})$.

Remark 4.4.6. The analog of Proposition 4.4.3(iii) for nilpotent Lie algebras is also valid (Exercise 4.2.5). Note that nilpotency is not an extension property, i.e., the analog of Proposition 4.4.3(ii) is false for nilpotent Lie algebras (cf. Example 4.4.2(iv)).

4.4.1 Lie's Theorem

Now we turn to solvable Lie subalgebras $\mathfrak{g}$ of $\mathfrak{gl}(V)$. In this context we do not want to make any assumption on the elements of $\mathfrak{g}$, as in Corollary 4.2.7.

Theorem 4.4.7. *Let V be a nonzero finite-dimensional complex vector space and $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a nonzero common eigenvector v for $\mathfrak{g}$, i.e., $\mathfrak{g}(v) \subseteq \mathbb{C}v$.*

Proof. We may w.l.o.g. assume that $\mathfrak{g} \neq \{0\}$. We proceed by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g} = \mathbb{C}x$, then every eigenvector of x (and such an eigenvector always exists) satisfies the requirement of the theorem. So let $\dim_{\mathbb{C}} \mathfrak{g} > 1$ and $\mathfrak{h}$ be a complex hyperplane in $\mathfrak{g}$ which contains $[\mathfrak{g}, \mathfrak{g}] = D^1(\mathfrak{g})$. Here we use that $D^1(\mathfrak{g})$ is a proper subspace because $\mathfrak{g}$ is solvable. In view of $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$, the subspace $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Now the induction hypothesis provides a nonzero common eigenvector v for $\mathfrak{h}$. If $x(v) = \lambda(x)v$ for $x \in \mathfrak{h}$, then $\lambda\colon \mathfrak{h} \to \mathbb{C}$ is a linear map and

$$v \in V_\lambda(\mathfrak{h}) := \{w \in V \mid (\forall x \in \mathfrak{h})\, x(w) = \lambda(x)w\}.$$

Suppose that $V_\lambda(\mathfrak{h})$ is $\mathfrak{g}$-invariant and pick $y \in \mathfrak{g}\backslash\mathfrak{h}$. Then there exists a nonzero eigenvector $v_o \in V_\lambda(\mathfrak{h})$ for y. Then v_o is a common eigenvector for $\mathfrak{g} = \mathfrak{h} + \mathbb{C}y$ and the proof is complete.

It remains to show that $V_\lambda(\mathfrak{h})$ is $\mathfrak{g}$-invariant. For this, we calculate as in the proof of Theorem 4.2.5:

$$yx(w) = xy(w) - [x, y](w) = \lambda(y)x(w) - \lambda([x, y])(w)$$

for $w \in V_\lambda(\mathfrak{h})$, $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$. Hence it suffices to show that $[\mathfrak{g}, \mathfrak{h}] \subseteq \ker \lambda$. For fixed $w \in V_\lambda(\mathfrak{h})$, $x \in \mathfrak{g}$ and $k \in \mathbb{N}$, we consider the space

$$W^k = \mathbb{C}w + \mathbb{C}x(w) + \ldots + \mathbb{C}x^k(w).$$

Since

$$yx^k(w) = xy(x^{k-1}w) - [x, y](x^{k-1}w) \tag{4.4}$$

and $y(w) = \lambda(y)w$ for $y \in \mathfrak{h}$, we see by induction on k that $\mathfrak{h}(W^k) \subseteq W^k$ for each $k \in \mathbb{N}$.

Now we choose $k_o \in \mathbb{N}$ maximal with respect to the property that

$$\{w, x(w), \ldots, x^{k_o}(w)\}$$

is a basis for W^{k_o}. Then $W^{k_o+m} = W^{k_o}$ for all $m \in \mathbb{N}$, and

$$\mathcal{F} = (\{0\}, W^1, \ldots, W^{k_o})$$

is a complete flag in W^{k_o} which is invariant under $\mathfrak{h}$. Thus, every $y \in \mathfrak{h}$ corresponds to an upper triangular matrix (y_{ij}) with respect to the above basis for W^{k_o}. The diagonal entries y_{ii} of this matrix are all equal to $\lambda(y)$ since $y(w) = \lambda(y)w$ and (4.4) imply by induction that

$$yx^k(w) \in \lambda(y)x^k(w) + W^{k-1}.$$

Since x and y leave the space W^{k_o} invariant, we have

$$[x,y]|_{W^{k_o}} = [x|_{W^{k_o}}, y|_{W^{k_o}}].$$

In particular, $[x,y]|_{W^{k_o}}$ is a commutator of two endomorphisms so that its trace vanishes. Finally $[x,y] \in \mathfrak{h}$ leads to

$$0 = \mathrm{tr}([x,y]|_{W^{k_o}}) = (k_o + 1)\lambda([x,y]),$$

so that $\lambda([x,y]) = 0$. □

Theorem 4.4.8 (Lie's Theorem). *Let V be a finite-dimensional complex vector space and $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a complete $\mathfrak{g}$-invariant flag in V.*

Proof. We may assume that V is nonzero. By Theorem 4.4.7, there is a nonzero common $\mathfrak{g}$-eigenvector $v_1 \in V$. Put $V_1 := \mathbb{C}v_1$. Then

$$\alpha\colon \mathfrak{g} \to \mathfrak{gl}(V/V_1), \quad \alpha(x)(v + V_1) := x(v) + V_1$$

defines a representation of $\mathfrak{g}$ on the quotient space V/V_1 (Exercise 4.1.10) and $\alpha(\mathfrak{g})$ is solvable. Proceeding by induction on $\dim V$, we may assume that there exists an $\alpha(\mathfrak{g})$-invariant complete flag in V/V_1, and the preimage in V, together with $\{0\}$, is a complete $\mathfrak{g}$-invariant flag in V. □

Remark 4.4.9. If we apply Lie's Theorem 4.4.8 to $V = \mathfrak{g}$ and $\mathrm{ad}(\mathfrak{g})$, where $\mathfrak{g}$ is solvable, we get a complete flag of ideals

$$\{0\} = \mathfrak{g}_0 < \mathfrak{g}_1 < \ldots < \mathfrak{g}_n = \mathfrak{g}$$

of $\mathfrak{g}$ with $\dim \mathfrak{g}_k = k$. Such a chain is called a *Hölder series* for $\mathfrak{g}$.

Definition 4.4.10. We call a representation (π, V) of the Lie algebra *nilpotent* if there exists an $n \in \mathbb{N}$ with $\rho(\mathfrak{g})^n = \{0\}$.

Corollary 4.4.11. *Let $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of the solvable Lie algebra $\mathfrak{g}$. Then the restriction to $[\mathfrak{g},\mathfrak{g}]$ is a nilpotent representation.*

Proof. (a) First we assume that $\mathbb{K} = \mathbb{C}$. Applying Lie's Theorem to the solvable subalgebra $\pi(\mathfrak{g})$ of $\mathfrak{gl}(V)$, we obtain a complete flag $\mathcal{F}$ with $\pi(\mathfrak{g}) \subseteq \mathfrak{g}(\mathcal{F})$. Then

$$\pi([\mathfrak{g},\mathfrak{g}]) \subseteq [\mathfrak{g}(\mathcal{F}), \mathfrak{g}(\mathcal{F})] \subseteq \mathfrak{g}_n(\mathcal{F})$$

implies the assertion.
(b) If $\mathbb{K} = \mathbb{R}$, then

$$\pi_{\mathbb{C}} \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}}) \cong \mathbb{C} \otimes \mathfrak{gl}(V), \quad \pi(z \otimes x) = z \otimes \pi(x)$$

defines a representation of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on the complex vector space $V_{\mathbb{C}}$. Using Proposition 4.1.21, we see by induction that $D^k(\mathfrak{g})_{\mathbb{C}} \cong D^k(\mathfrak{g}_{\mathbb{C}})$ for each $k \in \mathbb{N}$, so that $\mathfrak{g}_{\mathbb{C}}$ is also solvable. Now (a) applies and we obtain a complete flag $\mathcal{F}$ in $V_{\mathbb{C}}$ with $\pi_{\mathbb{C}}([\mathfrak{g}, \mathfrak{g}]) \subseteq \mathfrak{g}_n(\mathcal{F})$. In particular, each endomorphism $\pi(x)$, $x \in [\mathfrak{g}, \mathfrak{g}]$, is nilpotent. Finally Corollary 4.2.7 provides a complete flag $\mathcal{F}'$ in V with $\pi([\mathfrak{g}, \mathfrak{g}]) \subseteq \mathfrak{g}_n(\mathcal{F}')$, and the assertion follows. □

Corollary 4.4.12. *A Lie algebra $\mathfrak{g}$ is solvable if and only if its commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then $\mathfrak{g}$ is solvable because $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian and solvability is an extension property (Proposition 4.4.3(ii)).

If, conversely, $\mathfrak{g}$ is solvable, then Corollary 4.4.11 implies that the adjoint representation of $[\mathfrak{g}, \mathfrak{g}]$ on $\mathfrak{g}$, and hence on $[\mathfrak{g}, \mathfrak{g}]$, is nilpotent. From that we derive in particular that $C^\infty([\mathfrak{g}, \mathfrak{g}]) = \{0\}$, so that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

4.4.2 The Ideal $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$

Lemma 4.4.13. *Let $\mathfrak{g}$ be a Lie algebra and (ρ, V) a representation of $\mathfrak{g}$. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be a subspace for which there exists an $n \in \mathbb{N}$ with $\rho(\mathfrak{a})^n = \{0\}$ and $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ such that $\rho(x)$ is nilpotent. Then there exists an $N \in \mathbb{N}$ with $\rho(\mathfrak{a} + \mathbb{K}x)^N = \{0\}$.*

Proof. Replacing $\mathfrak{g}$ by $\rho(\mathfrak{g})$, we may w.l.o.g. assume that $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Let $m \in \mathbb{N}$ with $x^m = 0$. We claim that $(\mathbb{K}x + \mathfrak{a})^{nm} = \{0\}$.

Let $u = u_1 \cdots u_{nm}$ be a product of elements of $\{x\} \cup \mathfrak{a}$. We have to show that all such products vanish. For $a \in \mathfrak{a}$ we have

$$ax = xa + [a, x] \in xa + \mathfrak{a}.$$

This leads to

$$u_1 \cdots u_{nm} \in \sum_{r=0}^{nm} x^r \mathfrak{a}^t,$$

where t is the number of indices j with $u_j \in \mathfrak{a}$. Hence this product vanishes for $t \geq n$. If $t < n$, then there exists a j with $u_{j+1} \cdots u_{j+m} = x^m = 0$ because in this case $nm - t > n(m-1)$ factors are not contained in $\mathfrak{a}$, so that we always find a consecutive product of m such elements. We therefore have in all cases $u_1 \cdots u_{nm} = 0$. □

Proposition 4.4.14. *For any finite-dimensional representation (ρ, V) of the Lie algebra $\mathfrak{g}$, the restriction to the ideal $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ is nilpotent, i.e., there exists an $m \in \mathbb{N}$ with $\rho([\mathfrak{g}, \mathrm{rad}(\mathfrak{g})])^m = \{0\}$.*

Proof. Let $\mathfrak{r} := \mathrm{rad}(\mathfrak{g})$ and $\mathfrak{a} := [\mathfrak{g}, \mathfrak{r}]$. According to Corollary 4.4.11, the representation is nilpotent on the ideal $[\mathfrak{r}, \mathfrak{r}]$. Now let $\mathfrak{t} \subseteq [\mathfrak{g}, \mathfrak{r}]$ be a subspace containing $[\mathfrak{r}, \mathfrak{r}]$, which is maximal with respect to the property that the representation on V is nilpotent on $\mathfrak{t}$. Note that $\mathfrak{t}$ always is an ideal of $\mathfrak{r}$, hence in particular a subalgebra, because it contains the commutator algebra.

Assume that $\mathfrak{t} \neq [\mathfrak{g}, \mathfrak{r}]$. Then there exists an $x \in \mathfrak{g}$ and $y \in \mathfrak{r}$ with $[x, y] \notin \mathfrak{t}$. The subspace $\mathfrak{b} := \mathfrak{r} + \mathbb{K}x$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{r}$ is a solvable ideal of $\mathfrak{b}$, and $\mathfrak{b}/\mathfrak{r} \cong \mathbb{K}$ is abelian. Therefore $\mathfrak{b}$ is solvable (Proposition 4.4.3).

Again, we use Corollary 4.4.11 to see that the representation is nilpotent on $[\mathfrak{b}, \mathfrak{b}]$ and hence that $\rho([x, y])$ is nilpotent. Since $\mathfrak{t} \subseteq \mathfrak{r}$ and $[x, y] \in [\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{r}$, we have $\big[[x, y], \mathfrak{t}\big] \subseteq [\mathfrak{r}, \mathfrak{t}] \subseteq \mathfrak{t}$. Finally, the preceding Lemma 4.4.13 shows that the representation is nilpotent on the subspace $\mathbb{K}[x, y] + \mathfrak{t}$. This contradicts the maximality of $\mathfrak{t}$. We conclude that $\mathfrak{t} = [\mathfrak{g}, \mathfrak{r}]$, so that the representation is nilpotent on $[\mathfrak{g}, \mathfrak{r}]$. □

Applying the preceding proposition to the adjoint representation, using Engel's Theorem 4.2.8 we get:

Corollary 4.4.15. *The ideal* $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ *is nilpotent. In particular,* $\mathrm{ad}\, x$ *is nilpotent on* $\mathfrak{g}$ *for each* $x \in [\mathfrak{g}, \mathrm{rad}\, \mathfrak{g}]$.

4.4.3 Cartan's Solvability Criterion

This subsection is devoted to a characterization of solvable Lie algebras by properties of its elements. The result will be that $\mathfrak{g}$ is solvable if and only if $\mathrm{tr}(\mathrm{ad}\, x \,\mathrm{ad}\, y) = 0$ for $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$ (Cartan's criterion). Thus, we have to study the linear maps $\mathrm{ad}(x) \colon \mathfrak{g} \to \mathfrak{g}$.

Lemma 4.4.16. *Let V be a finite-dimensional vector space and $E \subseteq F$ be subspaces of $\mathfrak{gl}(V)$. Further, let*

$$x \in M := \{y \in \mathfrak{gl}(V) \mid [y, F] \subseteq E\}.$$

If $\mathrm{tr}(xy) = 0$ *for all* $y \in M$*, then* x *is nilpotent.*

Proof. Complexifying all vector spaces involved, we may assume without loss of generality that $\mathbb{K} = \mathbb{C}$. In particular, we know that x has a Jordan decomposition $x = x_s + x_n$. Now, let $\{v_1, \ldots, v_n\}$ be a basis for V consisting of eigenvectors of x_s. We denote the corresponding eigenvalues by λ_j for $j = 1, \ldots, n$. Let Q be the $\mathbb{Q}$-vector space in $\mathbb{C}$ which is spanned by the λ_j. We have to show that $Q = \{0\}$. To do this, we consider an $f \in Q^* := \mathrm{Hom}_{\mathbb{Q}}(Q, \mathbb{Q})$, the dual space (over $\mathbb{Q}$) of Q. The matrix

$$\begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix},$$

represents an element $y \in \mathfrak{gl}(V)$ with respect to the basis $\{v_1, \ldots, v_n\}$. As in the proof of Proposition 4.3.8, if we choose a basis $\{x^{ij}\}$ for $\mathfrak{gl}(V)$ with $x^{ij}(v_k) = \delta_{jk} v_i$, we get

$$\mathrm{ad}(x_s)x^{ij} = (\lambda_i - \lambda_j)x^{ij} \quad \text{and} \quad \mathrm{ad}(y)x^{ij} = f(\lambda_i - \lambda_j)x^{ij}.$$

Now, choose a polynomial $P \in \mathbb{C}[t]$ with

$$P(0) = 0 \quad \text{and} \quad P(\lambda_i - \lambda_j) = f(\lambda_i - \lambda_j)$$

for all pairs (i,j) (Exercise 4.4.6). Then

$$P\big(\mathrm{ad}(x_s)\big)x^{ij} = f(\lambda_i - \lambda_j)x^{ij} = \mathrm{ad}(y)x^{ij},$$

i.e., $P\big(\mathrm{ad}(x_s)\big) = \mathrm{ad}(y)$. Since $\mathrm{ad}(x_s)$ is the semisimple part of $\mathrm{ad}(x)$ by Corollary 4.3.9, it follows from $x \in M$ and Proposition 4.3.7(iii) that $\mathrm{ad}(x_s)F \subseteq E$. But then $P(0) = 0$ implies $\mathrm{ad}(y)F \subseteq E$, i.e., $y \in M$. Since $xy.v_i = \lambda_i f(\lambda_i)v_i$ for each i, our assumption and $y \in M$ leads to

$$\sum_{k=1}^{n} \lambda_k f(\lambda_k) = \mathrm{tr}(xy) = 0,$$

and therefore

$$\sum_{k=1}^{n} f(\lambda_k)^2 = f\Big(\sum_{k=1}^{n} \lambda_k f(\lambda_k)\Big) = 0.$$

Hence $f(\lambda_k) = 0$ for all λ_k which yields $f = 0$. But since $f \in Q^*$ was arbitrary, we see that $Q = \{0\}$. □

Theorem 4.4.17 (Cartan's Solvability Criterion). *Let V be a finite-dimensional vector space and $\mathfrak{g} < \mathfrak{gl}(V)$. Then the following are equivalent*

(i) $\mathfrak{g}$ *is solvable.*
(ii) $\mathrm{tr}(xy) = 0$ *for all* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and* $y \in \mathfrak{g}$.

Proof. $(ii) \Rightarrow (i)$: By Corollary 4.4.12, it suffices to show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. But to show that, by Corollary 4.2.7, we only have to prove that every element of $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. We want to apply Lemma 4.4.16 with $E = [\mathfrak{g}, \mathfrak{g}]$ and $F = \mathfrak{g}$, i.e., we set

$$M := \{y \in \mathfrak{gl}(V) \mid [y, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]\}.$$

Since the trace is linear, it is enough to show that $\mathrm{tr}([x, x']y) = 0$ for $x, x' \in \mathfrak{g}$ and $y \in M$. But this follows from $[x', y] \subseteq [\mathfrak{g}, \mathfrak{g}]$ and (ii):

$$\mathrm{tr}([x, x']y) = \mathrm{tr}(x[x', y]) = 0$$

(cf. Exercise 4.4.5).

(i) $\Rightarrow$ *(ii)*: Since $\operatorname{tr}(x_{\mathbb{C}}) = \operatorname{tr} x$ for $x \in \operatorname{End}(V)$, we may assume that $\mathbb{K} = \mathbb{C}$ (cf. also Exercise 4.4.2). Then by Lie's Theorem 4.4.8, there is a basis for V with respect to which all $x \in \mathfrak{g}$ are upper triangular matrices. In particular, all elements of $[\mathfrak{g}, \mathfrak{g}]$ are given by strictly upper triangular matrices. But multiplying an upper triangular matrix with a strictly upper triangular matrix yields a strictly upper triangular matrix which has zero trace. □

Corollary 4.4.18. *Let $\mathfrak{g}$ be a Lie algebra. Then the following statements are equivalent*

(i) $\mathfrak{g}$ *is solvable.*
(ii) $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ *for all* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and all* $y \in \mathfrak{g}$.

Proof. *(ii)* $\Rightarrow$ *(i)*: By the Cartan Criterion (Theorem 4.4.17), $\operatorname{ad}(\mathfrak{g})$ is solvable. Then it follows from $\operatorname{ad}(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and Proposition 4.4.3(ii) that $\mathfrak{g}$ is solvable.

(i) $\Rightarrow$ *(ii)*: Proposition 4.4.3(i) shows that $\operatorname{ad}(\mathfrak{g})$ is solvable, so that (ii) is an immediate consequence of Theorem 4.4.17. □

Exercises for Section 4.4

Exercise 4.4.1. Let $\mathfrak{g}$ be a Lie algebra and $\alpha\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V. Then $V \rtimes_\alpha \mathfrak{g}$ is a Lie algebra which contains V as an abelian ideal.

Exercise 4.4.2. (i) For a real Lie algebra $\mathfrak{g}$, we have

$$C^n(\mathfrak{g}_{\mathbb{C}}) = C^n(\mathfrak{g})_{\mathbb{C}}, \quad \text{and} \quad D^n(\mathfrak{g}_{\mathbb{C}}) = D^n(\mathfrak{g})_{\mathbb{C}}.$$

(ii) A finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent (solvable) if and only if $\mathfrak{g}_{\mathbb{C}}$ is nilpotent (solvable).

Exercise 4.4.3. Show that for the Heisenberg algebra $\mathfrak{h}_3$, the derivation algebra is isomorphic to $\mathbb{K}^2 \rtimes \mathfrak{gl}_2(\mathbb{K})$, where $\operatorname{ad}(\mathfrak{h}_3) \cong \mathbb{K}^2$ is an abelian ideal. Show that this Lie algebra is neither nilpotent nor solvable.

Exercise 4.4.4. Show that a representation (π, V) of a Lie algebra $\mathfrak{g}$ on a vector space V is nilpotent if and only if there is a flag $\mathcal{F}$ in V with $\pi(V) \subseteq \mathfrak{g}_n(\mathcal{F})$.

Exercise 4.4.5. A symmetric bilinear form $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ on a Lie algebra $\mathfrak{g}$ is called *invariant* if

$$\kappa([x,y],z) = \kappa(x,[y,z]) \quad \text{for} \quad x,y,z \in \mathfrak{g}.$$

Show that:

(i) The form $\kappa(x,y) := \operatorname{tr}(xy)$ on $\mathfrak{gl}(V)$ is invariant for each finite-dimensional vector space V.
(ii) For each representation (π, V) of the Lie algebra $\mathfrak{g}$, the form $\kappa_\pi(x,y) := \operatorname{tr}(\pi(x)\pi(y))$ is invariant.

(iii) For each Lie algebra $\mathfrak{g}$, the Cartan–Killing form $\kappa_{\mathfrak{g}}(x,y) := \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$ is invariant.
(iv) For each invariant symmetric bilinear form κ on $\mathfrak{g}$, its radical $\operatorname{rad}(\kappa) = \{x \in \mathfrak{g} \colon \kappa(x,\mathfrak{g}) = \{0\}\}$ is an ideal.
(v) For any invariant symmetric bilinear form κ on $\mathfrak{g}$, the trilinear map $\Gamma(\kappa)(x,y,z) := \kappa([x,y],z)$ is alternating, i.e.,

$$\Gamma(\kappa)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \operatorname{sgn}(\sigma)\Gamma(\kappa)(x_1,x_2,x_3)$$

for $\sigma \in S_3$ and $x_1, x_2, x_3 \in \mathfrak{g}$.
(vi) $\mathfrak{g}$ is solvable if and only if $\Gamma(\kappa_{\mathfrak{g}}) = 0$.

Exercise 4.4.6 (Interpolation polynomials). Let $\mathbb{K}$ be a field, $x_1, \ldots, x_n \in \mathbb{K}$ pairwise different, and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Then there exists a polynomial $f \in \mathbb{K}[t]$ with $f(x_i) = \lambda_i$ for $i = 1, \ldots, n$. Hint: Consider the polynomials $f_i(t) := \prod_{j \neq i} \frac{t - x_j}{x_i - x_j}$ of degree $n-1$.

Exercise 4.4.7. This exercise shows why Lie's Theorem does not generalize to infinite-dimensional spaces. We consider the vector space $V = \mathbb{K}^{(\mathbb{N})}$ with the basis $\{e_i \colon i \in \mathbb{N}\}$. In terms of the rank-one-operators $E_{ij} \in \operatorname{End}(V)$, defined by $E_{ij}e_k = \delta_{jk}e_i$, we consider the Lie algebra

$$\mathfrak{g} := \operatorname{span}\{E_{ij} \colon i \geq j\}$$

(lower triangular matrices). Show that:

(a) $D^n(\mathfrak{g}) = \operatorname{span}\{E_{ij} \colon i \geq j + 2^{n-1}\}$, $n \in \mathbb{N}$. In particular, we have $D^\infty(\mathfrak{g}) := \bigcap_{n\in\mathbb{N}} D^n(\mathfrak{g}) = \{0\}$, i.e., $\mathfrak{g}$ is *residually solvable*.
(b) $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ for an increasing sequence of finite-dimensional solvable subalgebras $\mathfrak{g}_n$ ($\mathfrak{g}$ is locally solvable).
(c) $\mathfrak{g}$ has no common eigenvector in V (compare with Lie's Theorem).

Exercise 4.4.8. Show that:

(a) A finite-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if there exists a sequence

$$\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \ldots \subseteq \mathfrak{g}_n = \mathfrak{g}$$

of subalgebras with $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$ for $i = 1, \ldots, n$.
(b) If $\mathfrak{g}$ is solvable, then there exists a sequence as in (a), satisfying, in addition, $\dim \mathfrak{g}_i = i$. Conclude that

$$\mathfrak{g}_{i+1} \cong \mathfrak{g}_i \rtimes_{D_i} \mathbb{K} \quad \text{for some} \quad D_i \in \operatorname{der}(\mathfrak{g}_i), \quad i = 1, \ldots, n-1.$$

This means that

$$\mathfrak{g} \cong \Big(\cdots ((\mathbb{K} \rtimes_{D_1} \mathbb{K}) \rtimes_{D_2} \mathbb{K}) \cdots \rtimes_{D_{n-1}} \mathbb{K} \Big).$$

Exercise 4.4.9. (Is there a Cartan Criterion for nilpotent Lie algebras?)

(a) If $\mathfrak{g}$ is nilpotent, then $\kappa_{\mathfrak{g}} = 0$.
(b) Consider the Lie algebra $\mathfrak{g} = \mathbb{C}^2 \rtimes_D \mathbb{C}$, where $\mathbb{C}^2$ is considered as an abelian Lie algebra and $D = \mathrm{diag}(1, i)$. This Lie algebra is not nilpotent, but $\kappa_{\mathfrak{g}} = 0$. If we consider $\mathfrak{g}$ as a 6-dimensional real Lie algebra, then its Cartan–Killing form also vanishes. Conclude that it is NOT true that a Lie algebra is nilpotent if and only if its Cartan–Killing form vanishes.

4.5 Semisimple Lie Algebras

In this section we encounter a third class of Lie algebras. Semisimple Lie algebras are a counterpart to the solvable and nilpotent Lie algebras because their ideal structure is quite simple. They can be decomposed as a direct sum of simple ideals. On the other hand, they have a rich geometric structure which even makes a complete classification of finite-dimensional semisimple Lie algebras possible. One can even show that every finite-dimensional Lie algebra is a semidirect sum of its radical and a semisimple subalgebra (cf. Levi's Theorem 4.6.6).

Definition 4.5.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is called *semisimple* if radical is trivial, i.e., $\mathrm{rad}(\mathfrak{g}) = \{0\}$. The Lie algebra $\mathfrak{g}$ is called *simple* if it is not abelian and it contains no ideals other than $\mathfrak{g}$ and $\{0\}$.

Lemma 4.5.2. *Each simple Lie algebra is semisimple.*

Proof. Let $\mathfrak{g}$ be a simple Lie algebra. Since the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is a nonzero ideal of $\mathfrak{g}$, it coincides with $\mathfrak{g}$. Hence $\mathfrak{g}$ is not solvable, so that $\mathrm{rad}(\mathfrak{g}) = \{0\}$. □

We shall see in Proposition 4.5.11 that a Lie algebra is semisimple if and only if it is a direct sum of simple ideals.

4.5.1 Cartan's Semisimplicity Criterion

In this subsection we prove the characterization of semisimple Lie algebras in terms of the Cartan–Killing form which can be defined for any Lie algebra.

Definition 4.5.3. In connection with the Cartan criterion for solvable Lie algebras, we have seen that the bilinear form

$$\kappa_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}, \quad \kappa_{\mathfrak{g}}(x, y) := \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$$

on a finite-dimensional Lie algebra is of interest. It is called the *Cartan–Killing form* of $\mathfrak{g}$. Its compatibility with the Lie algebra structure is expressed by its invariance

$$\kappa_{\mathfrak{g}}([x,y],z) = \kappa_{\mathfrak{g}}(x,[y,z]) \quad \text{for} \quad x,y,z,\in \mathfrak{g}$$

(Exercise 4.4.5). If $\mathfrak{g}$ is clear from the context, we sometimes write κ instead of $\kappa_{\mathfrak{g}}$.

Example 4.5.4. (i) With respect to the basis (h,u,t) for $\mathfrak{sl}_2(\mathbb{K})$ given in Example 4.1.23, the Cartan–Killing form has the matrix

$$\kappa = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

(ii) With respect to the basis (x,y,z) for $\mathfrak{so}_3(\mathbb{R})$, given in Example 4.1.23, the Cartan–Killing form has the matrix

$$\kappa = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(iii) With respect to the basis (h,p,q,z) for the oscillator algebra given in Example 4.1.19, the Cartan–Killing form has the matrix

$$\kappa = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In general, the Cartan–Killing form of a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ cannot be calculated in terms of the Cartan–Killing form of $\mathfrak{g}$, but for ideals we have:

Lemma 4.5.5. *For any ideal* $\mathfrak{i} \trianglelefteq \mathfrak{g}$, $\kappa_{\mathfrak{i}} = \kappa_{\mathfrak{g}}|_{\mathfrak{i}\times\mathfrak{i}}$.

Proof. If the image of $A \in \mathrm{End}(\mathfrak{g})$ is contained in $\mathfrak{i}$, then we pick a basis for $\mathfrak{g}$ which starts with a basis for $\mathfrak{i}$. With respect to this basis, we can write A as a block matrix

$$A = \begin{pmatrix} A|_{\mathfrak{i}} & * \\ 0 & 0 \end{pmatrix},$$

and this shows $\mathrm{tr}(A) = \mathrm{tr}(A|_{\mathfrak{i}})$. We apply this to $A = \mathrm{ad}(x)\,\mathrm{ad}(y)$ for $x,y \in \mathfrak{i}$ to obtain $\mathrm{tr}\left(\mathrm{ad}(x)\,\mathrm{ad}(y)\right) = \mathrm{tr}\left(\mathrm{ad}(x)|_{\mathfrak{i}}\,\mathrm{ad}(y)|_{\mathfrak{i}}\right) = \kappa_{\mathfrak{i}}(x,y)$. □

Remark 4.5.6. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{R}$-Lie algebra. Since a basis for $\mathfrak{g}$ is also a (complex) basis for $\mathfrak{g}_{\mathbb{C}}$, one immediately sees (cf. Exercise 4.5.3) that

$$\kappa_{\mathfrak{g}} = \kappa_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{g}\times\mathfrak{g}}.$$

Let V be a vector space and $\beta\colon V\times V \to \mathbb{K}$ be a symmetric bilinear form. Then we denote the orthogonal set

$$\{v \in V \mid (\forall w \in W)\, \beta(v,w) = 0\}$$

of a subspace W with respect to β by $W^{\perp,\beta}$. If β is the Cartan–Killing form of a Lie algebra, we simply write $\perp$ instead of $\perp,\beta$. The set $\mathrm{rad}(\beta) := V^{\perp,\beta}$ is called the *radical* of β. The form is called *degenerate* if $\mathrm{rad}(\beta) \neq \{0\}$. Using this notation, we can reformulate the Cartan Criterion 4.4.17 as follows:

Remark 4.5.7. In terms of the Cartan–Killing form, Cartan's Solvability Criterion states that $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g},\mathfrak{g}] \subseteq \mathrm{rad}(\kappa_{\mathfrak{g}})$ (cf. Exercise 4.5.7 for a description of the radical of $\kappa_{\mathfrak{g}}$ for a general Lie algebra).

Lemma 4.5.8. *For any ideal* $\mathfrak{j}$ *of a Lie algebra* $\mathfrak{g}$*, the following assertions hold:*

(i) *Its orthogonal space* $\mathfrak{j}^\perp$ *with respect to* $\kappa_{\mathfrak{g}}$ *also is an ideal.*
(ii) $\mathfrak{j} \cap \mathfrak{j}^\perp$ *is a solvable ideal.*
(iii) *If* $\mathfrak{j}$ *is semisimple, then* $\mathfrak{g}$ *decomposes as a direct sum* $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{j}^\perp$ *of Lie algebras.*

Proof. (i) For $x \in \mathfrak{j}^\perp$, $y \in \mathfrak{g}$ and $z \in \mathfrak{j}$, we find $\kappa_{\mathfrak{g}}([x,y],z) = \kappa_{\mathfrak{g}}(x,[y,z]) = 0$, so that $\mathfrak{j}^\perp$ is an ideal of $\mathfrak{g}$.

(ii) For $\mathfrak{i} := \mathfrak{j} \cap \mathfrak{j}^\perp$, the Cartan–Killing form $\kappa_{\mathfrak{g}}$ vanishes on $\mathfrak{i} \times \mathfrak{i}$. Hence $\mathrm{rad}(\kappa_{\mathfrak{i}}) = \mathfrak{i}$ by Lemma 4.5.5. In particular, $\mathfrak{i}$ is solvable by Remark 4.5.7.

(iii) If $\mathfrak{j}$ is semisimple, then (ii) implies that $\mathfrak{j} \cap \mathfrak{j}^\perp \subseteq \mathrm{rad}(\mathfrak{j}) = \{0\}$. Since $\mathfrak{j}^\perp$ is the kernel of the linear map $\mathfrak{g} \to \mathfrak{j}^*, x \mapsto \kappa_{\mathfrak{g}}(x,\cdot)$, we have $\dim \mathfrak{j}^\perp \geq \dim \mathfrak{g} - \dim \mathfrak{j}$, which implies $\mathfrak{j} + \mathfrak{j}^\perp = \mathfrak{g}$, so that $\mathfrak{g}$ is a direct sum of the vector subspaces $\mathfrak{j}$ and $\mathfrak{j}^\perp$. As both are ideals, $[\mathfrak{j},\mathfrak{j}^\perp] \subseteq \mathfrak{j} \cap \mathfrak{j}^\perp = \{0\}$, and we obtain a direct sum of Lie algebras. □

We can also characterize semisimplicity in terms of the Cartan–Killing form.

Theorem 4.5.9 (Cartan's Semisimplicity Criterion). *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\kappa_{\mathfrak{g}}$ *is nondegenerate, i.e.,* $\mathrm{rad}(\kappa_{\mathfrak{g}}) = \{0\}$*.*

Proof. With Lemma 4.5.8(ii), we see that $\mathfrak{g} \cap \mathfrak{g}^\perp = \mathrm{rad}(\kappa_{\mathfrak{g}})$ is a solvable ideal, so that $\mathrm{rad}(\kappa_{\mathfrak{g}}) \subseteq \mathrm{rad}(\mathfrak{g})$. In particular, $\kappa_{\mathfrak{g}}$ is nondegenerate if $\mathfrak{g}$ is semisimple.

Suppose, conversely, that $\mathfrak{g}$ is not semisimple and put $\mathfrak{r} := \mathrm{rad}(\mathfrak{g}) \neq \{0\}$. Let $n \in \mathbb{N}_0$ be maximal with $\mathfrak{h} := D^n(\mathfrak{r}) \neq \{0\}$. Then $\mathfrak{h}$ is an abelian ideal of $\mathfrak{g}$. For $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$, we then have $(\mathrm{ad}\, x\, \mathrm{ad}\, y)\mathfrak{g} \subseteq \mathfrak{h}$ and therefore $(\mathrm{ad}\, x\, \mathrm{ad}\, y)^2 = 0$. This implies that $\kappa_{\mathfrak{g}}(x,y) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$. Since $y \in \mathfrak{g}$ was arbitrary, this means that $x \in \mathrm{rad}(\kappa_{\mathfrak{g}})$, i.e., $\kappa_{\mathfrak{g}}$ is degenerate. □

Remark 4.5.10. In view of $\mathrm{rad}(\kappa_{\mathfrak{g}})_{\mathbb{C}} = \mathrm{rad}(\kappa_{\mathfrak{g}_{\mathbb{C}}})$ (cf. Exercise 4.5.2), Theorem 4.5.9 shows that a real Lie algebra $\mathfrak{g}$ is semisimple if and only if its complexification $\mathfrak{g}_{\mathbb{C}}$ is semisimple.

Proposition 4.5.11. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then there are simple ideals* $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ *of* $\mathfrak{g}$ *with*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k.$$

Every ideal $\mathfrak{i} \trianglelefteq \mathfrak{g}$ *is semisimple and a direct sum* $\mathfrak{i} = \bigoplus_{j \in I} \mathfrak{g}_j$ *for some subset* $I \subseteq \{1, \ldots, k\}$. *Conversely, each direct sum of simple Lie algebras is semisimple.*

Proof. Let $\mathfrak{j} \trianglelefteq \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, Lemma 4.5.8(ii) shows that $\mathfrak{j} \cap \mathfrak{j}^\perp = \{0\}$, so that $\kappa_\mathfrak{j} = \kappa_\mathfrak{g}|_{\mathfrak{j} \times \mathfrak{j}}$ is nondegenerate. Hence $\mathfrak{j}$ is semisimple. According to Lemma 4.5.8(iii), $\mathfrak{j}^\perp$ also is an ideal of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{j}^\perp$ (direct sum of Lie algebras). As $\mathfrak{j}$ and $\mathfrak{j}^\perp$ are semisimple, an induction on $\dim \mathfrak{g}$ now implies that $\mathfrak{g}$ decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

of simple ideals.

Finally, let $\mathfrak{i} \neq \{0\}$ be an ideal of $\mathfrak{g}$. Let $\pi_j \colon \mathfrak{g} \to \mathfrak{g}_j$ be the projections. Then we have $\pi_j(\mathfrak{i}) \neq \{0\}$ for at least one j. But since π_j is surjective $\pi_j(\mathfrak{i})$ is an ideal of $\mathfrak{g}_j$ and therefore equal to $\mathfrak{g}_j$. Thus

$$\mathfrak{g}_j = [\mathfrak{g}_j, \mathfrak{g}_j] = [\mathfrak{g}_j, \pi_j(\mathfrak{i})] = [\mathfrak{g}_j, \mathfrak{i}] \subseteq \mathfrak{i}$$

because $[\mathfrak{g}_j, \pi_l(\mathfrak{i})] = \{0\}$ for $l \neq j$. The argument shows that every $\mathfrak{g}_j$ with $\pi_j(\mathfrak{i}) \neq \{0\}$ is contained in $\mathfrak{i}$. But then $\mathfrak{i}$ is the direct sum of these $\mathfrak{g}_j$.

The preceding argument shows in particular that no nonzero ideal of a direct sum of simple Lie algebras is solvable, hence that any such direct sum is semisimple. □

Example 4.5.12. The Lie algebras $\mathfrak{sl}_2(\mathbb{K})$, $\mathfrak{so}_3(\mathbb{K})$ and $\mathfrak{su}_2(\mathbb{C})$ are simple. We have seen in Example 4.5.4 that the Cartan–Killing forms of $\mathfrak{sl}_2(\mathbb{K})$ and $\mathfrak{so}_3(\mathbb{K})$ are nondegenerate, so that they are semisimple. Further, $\mathfrak{su}_2(\mathbb{C})$ is a real form of $\mathfrak{sl}_2(\mathbb{C})$, hence semisimple by Remark 4.5.10. Now we apply Exercise 4.5.1.

Corollary 4.5.13. *For a semisimple Lie algebra* $\mathfrak{g}$*, the following assertions hold:*

(i) $\mathfrak{g}$ *is perfect, i.e.,* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.
(ii) *All homomorphic images of* $\mathfrak{g}$ *are semisimple.*
(iii) *Each ideal* $\mathfrak{n} \trianglelefteq \mathfrak{g}$ *is semisimple and there exists another ideal* $\mathfrak{c}$ *with* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{c}$.

Proof. (i) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}$ from Proposition 4.5.11. Since $\mathfrak{g}_j$ is not abelian, we have $[\mathfrak{g}_j, \mathfrak{g}_j] = \mathfrak{g}_j$, and therefore

$$[\mathfrak{g}, \mathfrak{g}] = \sum_{j=1}^{k} [\mathfrak{g}_j, \mathfrak{g}_j] = \sum_{j=1}^{k} \mathfrak{g}_j = \mathfrak{g}.$$

(ii) This follows by combining Proposition 4.5.11 with the Isomorphism Theorem 4.1.17(i).
(iii) This follows immediately from Proposition 4.5.11. □

In Example 4.1.9, we have seen that the adjoint representation gives derivations on the Lie algebra. In the case of semisimple Lie algebras, this representation in fact gives *all* derivations.

Theorem 4.5.14. *For a semisimple Lie algebra* $\mathfrak{g}$ *all derivations are inner, i.e.,*

$$\mathrm{ad}(\mathfrak{g}) = \mathrm{der}(\mathfrak{g}).$$

Proof. By Proposition 4.1.10(i), $\mathrm{ad}\,\mathfrak{g} \trianglelefteq \mathrm{der}(\mathfrak{g})$ is an ideal, and since $\mathfrak{z}(\mathfrak{g}) = \{0\}$, the ideal $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$ is semisimple. Therefore $\mathrm{der}\,\mathfrak{g}$ decomposes as a direct sum $\mathrm{ad}\,\mathfrak{g} \oplus \mathfrak{j}$ for the orthogonal complement $\mathfrak{j}$ of $\mathrm{ad}(\mathfrak{g})$ with respect to the Cartan–Killing form of $\mathrm{der}(\mathfrak{g})$ (Lemma 4.5.8(iii)). For $\delta \in \mathfrak{j}$ and $x \in \mathfrak{g}$ we then have

$$0 = [\delta, \mathrm{ad}\,x](y) = \delta([x, y]) - [x, \delta(y)] = [\delta(x), y] = \mathrm{ad}\,\big(\delta(x)\big).$$

This means that $\delta(x) \in \mathfrak{z}(\mathfrak{g}) = \{0\}$, i.e., $\delta = 0$. We conclude that $\mathfrak{j} = \{0\}$, so that $\mathrm{der}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g}$. □

4.5.2 Weyl's Theorem on Complete Reducibility

We have already seen how Engel's Theorem and Lie's Theorem provide important information on the structure of representations of nilpotent, resp., solvable Lie algebras. For semisimple Lie algebras, Weyl's Theorem, which asserts that each representation of a semisimple Lie algebra is completely reducible, plays a similar role. The crucial tool needed for the proof of Weyl's Theorem is the Casimir element.

Definition 4.5.15. Let β be a nondegenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$, $x_1, \ldots, x_k$ a basis for $\mathfrak{g}$ and $x^1, \ldots, x^k$ the dual basis with respect to β, i.e., $\beta(x_i, x^j) = \delta_{ij}$ (Kronecker delta). For any Lie algebra homomorphism $\rho\colon \mathfrak{g} \to A_L$, A an associative algebra, we define the *Casimir element*

$$\Omega(\beta, \rho) := \sum_{i=1}^{k} \rho(x_i)\rho(x^i).$$

The same argument that shows the independence of the trace of an operator (defined as the sum of its diagonal elements) of the choice of the basis, shows that $\Omega(\beta, \rho)$ does not depend on the choice of the basis $x_1, \ldots, x_k$ (cf. Exercise 4.5.9).

The Casimir element is a useful tool for the study of representation since it commutes with $\rho(\mathfrak{g})$:

Lemma 4.5.16. *For each nondegenerate invariant symmetric bilinear form* β *on* $\mathfrak{g}$ *and each homomorphism* $\rho\colon \mathfrak{g} \to A$*, the Casimir element* $\Omega(\beta, \rho) \in A$ *commutes with* $\rho(\mathfrak{g})$*.*

Proof. Let $z \in \mathfrak{g}$. Then we have

$$\operatorname{ad} z(x_j) = \sum_{k=1}^n a_{kj} x_k \quad \text{and} \quad \operatorname{ad} z(x^j) = \sum_{k=1}^n a^{kj} x^k$$

with two matrices (a_{ij}) and (a^{ij}) in $M_k(\mathbb{K})$. Then

$$a_{kj} = \beta([z, x_j], x^k) = -\beta(x_j, [z, x^k]) = -a^{jk},$$

and with this relation we obtain

$$\begin{aligned}
\Omega\rho(z) - \rho(z)\Omega &= \sum_{j=1}^n \rho(x_j)\rho(x^j)\rho(z) - \rho(z)\rho(x_j)\rho(x^j) \\
&= \sum_{j=1}^n \rho(x_j)(\rho(x^j)\rho(z) - \rho(z)\rho(x^j)) - (\rho(z)\rho(x_j) - \rho(x_j)\rho(z))\rho(x^j) \\
&= \sum_{j=1}^n \rho(x_j)\rho([x^j, z]) - \rho([z, x_j])\rho(x^j) \\
&= \sum_{j,k=1}^n (-a^{kj})\rho(x_j)\rho(x^k) - a_{kj}\rho(x_k)\rho(x^j) \\
&= \sum_{j,k=1}^n a_{jk}\rho(x_j)\rho(x^k) - a_{kj}\rho(x_k)\rho(x^j) \\
&= \sum_{j,k=1}^n a_{jk}\rho(x_j)\rho(x^k) - a_{jk}\rho(x_j)\rho(x^k) = 0.
\end{aligned}$$

□

Proposition 4.5.17. *Let $\mathfrak{g}$ be a semisimple Lie algebra and (ρ, V) a finite-dimensional representation. Then V is the direct sum of the $\mathfrak{g}$-modules*

$$V^{\mathfrak{g}} := \bigcap_{x \in \mathfrak{g}} \ker \rho(x) \quad \textit{and} \quad V_{\text{eff}} := \sum_{x \in \mathfrak{g}} \rho(x)(V).$$

Proof. Note that $\rho(\mathfrak{g})(V^{\mathfrak{g}}) = \{0\}$ and $\rho(\mathfrak{g})(V_{\text{eff}}) \subseteq V_{\text{eff}}$, so that $V^{\mathfrak{g}}$ and V_{eff} are indeed $\mathfrak{g}$-invariant. We argue by induction on $\dim V$. The case $\dim V = \{0\}$ is trivial. Since the statement of the proposition is obvious for $\rho = 0$, we may assume that $\rho \neq 0$.

Step 1: Let $\beta_\rho(x, y) = \operatorname{tr}\big(\rho(x)\rho(y)\big)$ denote the trace form on $\mathfrak{g}$ and $\mathfrak{a} := \operatorname{rad}(\beta_\rho)$ denote the radical of β_ρ, which is an ideal because β_ρ is invariant (Exercise 4.4.5). Let $\mathfrak{b} \trianglelefteq \mathfrak{g}$ be a complementary ideal, so that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of Lie algebras (Proposition 4.5.11). In view of Cartan's Solvability Criterion (Theorem 4.4.17), the Lie algebra $\rho(\mathfrak{a})$ is a solvable ideal of $\rho(\mathfrak{g})$

because the trace form vanishes on this Lie algebra. Since $\rho(\mathfrak{g})$ is semisimple, $\rho(\mathfrak{a}) \subseteq \mathrm{rad}(\rho(\mathfrak{g})) = \{0\}$, so that $\mathfrak{a} \subseteq \ker \rho$. Conversely, the ideal $\ker \rho$ is contained in $\mathrm{rad}(\beta_\rho)$, which leads to $\mathfrak{a} = \ker \rho$. It follows in particular that $\beta := \beta_\rho|_{\mathfrak{b} \times \mathfrak{b}}$ is nondegenerate on the semisimple Lie algebra $\mathfrak{b}$.
Step 2: Let

$$\Omega := \Omega(\beta, \rho|_{\mathfrak{b}}) := \sum_j \rho(x_j)\rho(x^j) \in \mathrm{End}(V)$$

be the associated Casimir element (Definition 4.5.15). Then Lemma 4.5.16 implies that

$$\Omega \in \mathrm{End}_{\mathfrak{b}}(V) := \{A \in \mathrm{End}(V) \colon (\forall x \in \mathfrak{b})\ A\rho(x) = \rho(x)A\}.$$

Since $\mathfrak{a} = \ker \rho$ this implies

$$\Omega \in \mathrm{End}_{\mathfrak{g}}(V) := \{A \in \mathrm{End}(V) \colon (\forall x \in \mathfrak{g})\ A\rho(x) = \rho(x)A\}.$$

Finally we note that

$$\mathrm{tr}\, \Omega = \sum_j \mathrm{tr}(\rho(x_j)\rho(x^i)) = \sum_j \beta(x_j, x^j) = \dim \mathfrak{b}.$$

Step 3: If V is the direct sum of two nonzero $\mathfrak{g}$-invariant subspaces, then $V^{\mathfrak{g}}$ and V_{eff} decompose accordingly, and we can use our induction hypothesis. Let $V = V^0(\Omega) \oplus V^+(\Omega)$ be the Fitting decomposition of V with respect to Ω. Since Ω commutes with $\mathfrak{g}$, both summands are $\mathfrak{g}$-invariant, so that we may assume that one of these summands is trivial.

Since we assume that $\mathfrak{b} \cong \rho(\mathfrak{g}) \neq \{0\}$, we have $\mathrm{tr}\, \Omega > 0$, so that Ω is not nilpotent and thus $V^+(\Omega)$ is nonzero. Hence $V^0(\Omega) = \{0\}$ and, consequently, $V = V^+(\Omega)$. Then Ω is invertible, so that $V = V^+(\Omega) \subseteq V_{\mathrm{eff}}$ and $V^{\mathfrak{g}} \subseteq V^0(\Omega) = \{0\}$. This completes the proof. □

Proposition 4.5.18. *For a finite-dimensional representation (ρ, V) of the Lie algebra $\mathfrak{g}$, the following are equivalent:*

(i) *Each $\mathfrak{g}$-invariant subspace of V possesses a $\mathfrak{g}$-invariant complement.*
(ii) *(ρ, V) is completely reducible.*
(iii) *V is a sum of $\mathfrak{g}$-invariant subspaces on which the representation is irreducible.*

Proof. (i) ⇒ (ii): For-dimensional reasons, each V contains a minimal nonzero $\mathfrak{g}$-invariant subspace V_1. Then there exists a $\mathfrak{g}$-invariant complement W, so that $V = V_1 \oplus W$. We now argue by induction on $\dim V$ and apply the induction hypothesis to the representation of $\mathfrak{g}$ on W.
(ii) ⇒ (iii) is trivial.
(iii) ⇒ (ii): Let $V_1, \ldots, V_n$ be a maximal set of $\mathfrak{g}$-invariant subspaces on which the representation of $\mathfrak{g}$ is irreducible and whose sum $W := \sum_{i=1}^n V_i$ is direct. We claim that $W = V$, which implies (ii). If W is a proper subspace, then

(iii) implies the existence of a minimal nonzero $\mathfrak{g}$-invariant subspace U not contained in W. Then $W \cap U = \{0\}$ follows from the minimality of U, so that the sum $U + \sum_i V_i$ is direct, contradicting the maximality of the set $\{V_1, \ldots, V_n\}$. This proves $W = V$.
(ii) $\Rightarrow$ (i): Let $V = \bigoplus_{i=1}^n V_i$ be a direct sum of minimal nonzero $\mathfrak{g}$-invariant subspaces and $W \subseteq V$ a $\mathfrak{g}$-invariant subspace. Further, let $J \subseteq \{1, \ldots, n\}$ be maximal with $W \cap (\sum_{i \in J} V_i) = \{0\}$. Then $W' := \sum_{i \in J} V_i$ satisfies $W \cap W' = \{0\}$ and it remains to see that $W + W' = V$.

Pick $i \in I$. If $i \in J$, then $V_i \subseteq W' \subseteq W + W'$. If $i \notin J$, then the maximality of J implies that $(W' + V_i) \cap W \neq \{0\}$ and hence $(W + W') \cap V_i \neq \{0\}$. Hence the minimality of V_i implies that $V_i \subseteq W + W'$, and this proves $V = W + W'$. □

Lemma 4.5.19. *If $\mathfrak{g}$ is a real Lie algebra and V a finite-dimensional $\mathfrak{g}$-module, then the following are equivalent:*

(i) *V is semisimple.*
(ii) *$V_{\mathbb{C}}$ is a semisimple complex $\mathfrak{g}$-module.*

Proof. (i) $\Rightarrow$ (ii): If V is semisimple, then V is a direct sum of simple submodules V_i, then $V_{\mathbb{C}}$ is the direct sum of the submodules $(V_i)_{\mathbb{C}}$. Hence it suffices to show that the complexification $W_{\mathbb{C}}$ of a simple real $\mathfrak{g}$-module W is semisimple. In fact, if $W_{\mathbb{C}}$ is not simple, then let $U \subseteq W_{\mathbb{C}}$ be a nonzero minimal complex submodule. This implies in particular that U is simple. Let $\sigma \colon W_{\mathbb{C}} \to W_{\mathbb{C}}$ be the conjugation involution defined by $\sigma(z \otimes w) = \overline{z} \otimes w$. Then σ commutes with the action of $\mathfrak{g}$ on $W_{\mathbb{C}}$, and therefore $\sigma(U)$ also is a simple complex submodule. Now $U + \sigma(U)$ is a complex σ-invariant submodule of $W_{\mathbb{C}}$, hence of the form $X_{\mathbb{C}}$ for $X := W \cap (U + \sigma(U))$ (Exercise 4.3.1(ii)). Then X is a nonzero $\mathfrak{g}$-submodule of W, so that the simplicity of W yields $X = W$ and thus $U + \sigma(U) = W_{\mathbb{C}}$. Now Proposition 4.5.18 shows that $W_{\mathbb{C}}$ is semisimple because it is the sum of two simple submodules.

(ii) $\Rightarrow$ (i): Let $W \subseteq V$ be a submodule. We have to show that there exists a module complement U (Proposition 4.5.18). Since $V_{\mathbb{C}}$ is semisimple, there exists a module complement X of $W_{\mathbb{C}}$ in $V_{\mathbb{C}}$, i.e., a complex linear projection $p \colon V_{\mathbb{C}} \to W_{\mathbb{C}}$ commuting with $\mathfrak{g}$. Let $q \colon \mathbb{C} \to \mathbb{R}, q(z) := \operatorname{Re} z$, denote the canonical projection onto $\mathbb{R}$. Then $q \otimes \mathrm{id}_V : V_{\mathbb{C}} \to V$ is a real linear projection commuting with $\mathfrak{g}$. Hence

$$P := (q \otimes \mathrm{id}_V) \circ p|_V : V \to W$$

is a $\mathfrak{g}$-equivariant real linear map with $P(V) = W$ and $P|_W = \mathrm{id}_W$. Hence P is a projection onto W and $\ker P$ is a complementary submodule. □

Lemma 4.5.20. *Let V be a finite-dimensional vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a commutative subalgebra consisting of semisimple elements. Then V is a semisimple $\mathfrak{g}$-module.*

Proof. In view of Lemma 4.5.19, it suffices to show that $V_{\mathbb{C}}$ is a semisimple complex $\mathfrak{g}$-module, resp., $\mathfrak{g}_{\mathbb{C}}$. On $V_{\mathbb{C}}$, each $x \in \mathfrak{g}$ is diagonalizable, and since $\mathfrak{g}$ is abelian, $\mathfrak{g}$ is simultaneously diagonalizable (Exercise 1.1.1(d)), so that $V_{\mathbb{C}}$ is a direct sum of one-dimensional submodules, hence semisimple. □

Theorem 4.5.21 (Weyl's Theorem on Complete Reducibility). *Each finite-dimensional representation of a semisimple Lie algebra is completely reducible.*

Proof. We argue by induction on the dimension of the representation (ρ, V) of the semisimple Lie algebra $\mathfrak{g}$. In view of Proposition 4.5.18, it suffices to show that each $\mathfrak{g}$-invariant subspace $W \subseteq V$ possesses a $\mathfrak{g}$-invariant complement U.
Step 1: Let $W \subseteq V$ be a $\mathfrak{g}$-invariant subspace of codimension 1. Then the representation $(\overline{\rho}, V/W)$, defined by $\overline{\rho}(x)(v+W) := \rho(x)v+W$ is one-dimensional. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is perfect and $\mathfrak{gl}_1(\mathbb{K}) \cong \mathbb{K}$ is abelian, $\overline{\rho} = 0$, i.e., $\rho(\mathfrak{g})V \subseteq W$. In view of Proposition 4.5.17, $V = V^{\mathfrak{g}} + V_{\text{eff}}$, and since V_{eff} is contained in W, there exists some $v_o \in V^{\mathfrak{g}} \setminus W$. Then $\mathbb{K}v_o$ is a $\mathfrak{g}$-invariant complement of W.
Step 2: Now let $W \subseteq V$ be an arbitrary $\mathfrak{g}$-invariant subspace. We define a representation of $\mathfrak{g}$ on $\operatorname{Hom}(V, W)$ by

$$\pi(x)\varphi := \rho(x)|_W \circ \varphi - \varphi \circ \rho(x)$$

(Exercise). Then the subspace

$$U := \{\varphi \in \operatorname{Hom}(V, W) \colon \varphi|_W \in \mathbb{K}\,\mathrm{id}_W\}$$

is $\mathfrak{g}$-invariant because we have for $\varphi \in U$ the relation $(\pi(x)\varphi)(W) = \{0\}$: For $\varphi|_W = \lambda\,\mathrm{id}_W$ and $w \in W$ we have

$$(\pi(x)\varphi)(w) = \rho(x)\varphi(w) - \varphi(\rho(x)w) = \rho(x)(\lambda w) - \lambda\rho(x)w = 0.$$

Therefore

$$U_0 := \{\varphi \in U \colon \varphi(W) = \{0\}\}$$

is a $\mathfrak{g}$-invariant subspace of U of codimension 1. Step 1 now implies the existence of a $\mathfrak{g}$-invariant $\varphi_0 \in U \setminus U_0$. The $\mathfrak{g}$-invariance of φ_0 means that $\varphi_0 \in \operatorname{Hom}_{\mathfrak{g}}(V, W)$ and since $\varphi_0|_W \in \mathbb{K}^\times \mathrm{id}_W$ is invertible, $\ker \varphi_0$ is a $\mathfrak{g}$-invariant subspace complementing W. □

Exercises for Section 4.5

Exercise 4.5.1. Show that the dimension of a simple Lie algebra is at least 3. Conclude that every semisimple Lie algebra of dimension ≤ 5 is simple.

Exercise 4.5.2. For a real Lie algebra $\mathfrak{g}$, we have:

(i) $\operatorname{rad}(\mathfrak{g}_{\mathbb{C}}) = \operatorname{rad}(\mathfrak{g})_{\mathbb{C}}$.
(ii) $\operatorname{rad}(\kappa_{\mathfrak{g}})_{\mathbb{C}} = \operatorname{rad}(\kappa_{\mathfrak{g}_{\mathbb{C}}})$.

(iii) $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{C}}$ is semisimple.

Exercise 4.5.3. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Show that the Cartan–Killing forms of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are related by

$$\kappa_{\mathfrak{g}}(x,y) = \kappa_{\mathfrak{g}_{\mathbb{C}}}(x,y) \quad \text{for} \quad x,y \in \mathfrak{g}.$$

Exercise 4.5.4. Verify the computations of the Cartan–Killing forms of $\mathfrak{sl}_2(\mathbb{K})$, $\mathfrak{so}_3(\mathbb{R})$ and of the oscillator algebra in Example 4.5.4.

Exercise 4.5.5. (i) Let $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of the Lie algebra $\mathfrak{g}$ on V and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be an ideal. Then the space

$$V_0(\mathfrak{n}) := \{v \in V \mid (\forall x \in \mathfrak{n})\ \alpha(x)v = 0\}$$

is $\mathfrak{g}$-invariant.

(ii) Let

$$\mathfrak{a}_0 = \{0\} \subseteq \mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_n = \mathfrak{g}$$

be a maximal chain of ideals of $\mathfrak{g}$ and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a nilpotent ideal. Then $[\mathfrak{n}, \mathfrak{a}_j] \subseteq \mathfrak{a}_{j-1}$ for $j > 0$.

Exercise 4.5.6. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Every nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ is orthogonal to $\mathfrak{g}$ with respect to the Cartan–Killing form.

Exercise 4.5.7. Show that $[\mathfrak{g},\mathfrak{g}]^{\perp} = \mathrm{rad}(\mathfrak{g})$ for every finite-dimensional Lie algebra. Here $\perp$ refers to the Cartan–Killing form.

Exercise 4.5.8. Each one-dimensional representation (π, V) of a perfect Lie algebra is trivial.

Exercise 4.5.9. Let V be a finite-dimensional vector space and V^* its dual space. Show that:

(a) The map $\gamma\colon V \otimes V^* \to \mathrm{End}(V)$, $\gamma(v,\alpha)(w) := \alpha(w)v$, is a linear isomorphism.

(b) If $v_1, \ldots, v_n$ is a basis for V and $v_1^*, \ldots, v_n^*$ the dual basis for V^*, defined by $v_j^*(v_i) = \delta_{ij}$, then $\gamma(\sum_{i=1}^n v_i \otimes v_i^*) = \mathrm{id}_V$.

(c) If $\beta\colon V \times V \to \mathbb{K}$ is a nondegenerate symmetric bilinear form, then

$$\widetilde{\gamma}\colon V \otimes V \to \mathrm{End}(V), \quad \widetilde{\gamma}(v \otimes w)(x) := \beta(x,w)v$$

is a linear isomorphism. If $v_1, \ldots, v_n$ is a basis for V and $v^1, \ldots, v^n \in V$ with $\beta(v^i, \cdot) = v_i^*$, $i = 1, \ldots, n$, then $\widetilde{\gamma}(\sum_{i=1}^n v_i \otimes v^i) = \mathrm{id}_V$.

Exercise 4.5.10. (cf. Exercise 12.1.4)

(i) Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Show that each invariant bilinear form κ on $\mathfrak{g}$ is a scalar multiple of the Cartan–Killing form $\kappa_{\mathfrak{g}}$.

(ii) Show that the result from (i) is false for simple algebras over $\mathbb{R}$.

4.6 The Theorems of Levi and Malcev

In the preceding sections we dealt in particular with solvable and semisimple Lie algebras separately. Now we shall address the question how a finite-dimensional Lie algebra $\mathfrak{g}$ decomposes into its maximal solvable ideal $\mathrm{rad}(\mathfrak{g})$ and the semisimple quotient $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$. The theorems of Levi and Malcev are fundamental for the structure theory of finite-dimensional Lie algebras. Levi's Theorem asserts the existence of a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ complementing the radical $\mathrm{rad}(\mathfrak{g})$, also called a Levi complement. As a consequence, $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \rtimes \mathfrak{s}$ is a semidirect sum. Malcev's Theorem asserts that all Levi complements are conjugate under the group of inner automorphisms of $\mathfrak{g}$, which is a uniqueness result.

4.6.1 Levi's Theorem

Lemma 4.6.1. *The quotient Lie algebra* $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple.*

Proof. Let $q\colon \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ be the quotient homomorphism and $\mathfrak{a} \trianglelefteq \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ a solvable ideal. Then $\mathfrak{b} := q^{-1}(\mathfrak{a}) \trianglelefteq \mathfrak{g}$ is an ideal containing $\mathrm{rad}(\mathfrak{g})$, for which $\mathfrak{a} \cong \mathfrak{b}/\mathrm{rad}(\mathfrak{g})$ is solvable. Since solvability is an extension property, $\mathfrak{b}$ is solvable, hence $\mathfrak{b} \subseteq \mathrm{rad}(\mathfrak{g})$, and thus $\mathfrak{a} = \{0\}$. This proves that $\mathrm{rad}(\mathfrak{g}/\mathrm{rad}(\mathfrak{g})) = \{0\}$, i.e., $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. □

Lemma 4.6.2. *If* $\alpha\colon \mathfrak{g} \to \mathfrak{h}$ *is a surjective homomorphism of Lie algebras, then* $\alpha(\mathrm{rad}\,\mathfrak{g}) = \mathrm{rad}\,\mathfrak{h}$.

Proof. Let $\mathfrak{r} := \mathrm{rad}\,\mathfrak{g}$. First we note that $\alpha(\mathfrak{r})$ is a solvable ideal of $\mathfrak{h}$, hence contained in $\mathrm{rad}(\mathfrak{h})$. Here we use that images of ideals under surjective homomorphisms are ideals: $[\mathfrak{h}, \alpha(\mathfrak{r})] = [\alpha(\mathfrak{g}), \alpha(\mathfrak{r})] = \alpha([\mathfrak{g}, \mathfrak{r}]) \subseteq \alpha(\mathfrak{r})$.

Let $\pi\colon \mathfrak{h} \to \mathfrak{h}/\alpha(\mathfrak{r})$ be the quotient homomorphism. We consider the homomorphism $\beta := \pi \circ \alpha\colon \mathfrak{g} \to \mathfrak{h}/\alpha(\mathfrak{r})$. In view of $\mathfrak{r} \subseteq \ker\beta$, β factors through a surjective homomorphism $\widetilde{\beta}\colon \mathfrak{g}/\mathfrak{r} \to \mathfrak{h}/\alpha(\mathfrak{r})$. Since $\mathfrak{g}/\mathfrak{r}$ is semisimple (Lemma 4.6.1), the homomorphic image $\mathfrak{h}/\alpha(\mathfrak{r})$ is also semisimple. Consequently $\pi(\mathrm{rad}\,\mathfrak{h}) \subseteq \mathrm{rad}\big(\mathfrak{h}/\alpha(\mathfrak{r})\big) = \{0\}$, i.e., $\mathrm{rad}\,\mathfrak{h} \subseteq \alpha(\mathfrak{r})$. We thus obtain $\mathrm{rad}\,\mathfrak{h} = \alpha(\mathfrak{r})$. □

Definition 4.6.3. An ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$ is called *characteristic*, if it is invariant under all derivations of $\mathfrak{g}$.

Lemma 4.6.4. *For the radical of the Lie algebra* $\mathfrak{g}$*, the following assertions hold:*

(i) $\mathrm{rad}(\mathfrak{g})$ *is a characteristic ideal.*
(ii) *If* $\mathfrak{a} \subseteq \mathfrak{g}$ *is an ideal, then* $\mathrm{rad}(\mathfrak{a}) = \mathrm{rad}(\mathfrak{g}) \cap \mathfrak{a}$.

Proof. (i) First we note that $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal of $\mathfrak{g}$ because for each derivation $D \in \operatorname{der} \mathfrak{g}$ and $x, y \in \mathfrak{g}$ we have $D([x,y]) = [Dx, y] + [x, Dy] \in [\mathfrak{g}, \mathfrak{g}]$. Next we note that the Cartan–Killing from is invariant (cf. Exercise 4.4.5) under $\operatorname{der}(\mathfrak{g})$:

$$\begin{aligned} \kappa_{\mathfrak{g}}(Dx, y) &= \operatorname{tr}\big(\operatorname{ad}(Dx)\operatorname{ad} y\big) = \operatorname{tr}\big([D, \operatorname{ad} x]\operatorname{ad} y\big) \\ &= -\operatorname{tr}\big(\operatorname{ad} x[D, \operatorname{ad} y]\big) = -\operatorname{tr}\big(\operatorname{ad} x \operatorname{ad}(Dy)\big) = -\kappa_{\mathfrak{g}}(x, Dy). \end{aligned}$$

Therefore $\operatorname{rad}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^{\perp, \kappa_{\mathfrak{g}}}$ (Exercise 4.5.7) is also invariant under $\operatorname{der}(\mathfrak{g})$.
(ii) Clearly, $\operatorname{rad}(\mathfrak{g}) \cap \mathfrak{a}$ is a solvable ideal of $\mathfrak{a}$, hence contained in $\operatorname{rad}(\mathfrak{a})$. Since $\operatorname{rad}(\mathfrak{a})$ is a characteristic ideal of $\mathfrak{a}$, it is invariant under the adjoint representation of $\mathfrak{g}$ on $\mathfrak{a}$, hence a solvable ideal of $\mathfrak{g}$. This proves that $\operatorname{rad}(\mathfrak{a}) \subseteq \operatorname{rad}(\mathfrak{g})$. □

We will need the following technical lemma.

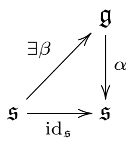
Lemma 4.6.5. *Let (ρ, V) be a representation of $\mathfrak{g}$ and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ an ideal. For $v \in V$ let $\mathfrak{z}_{\mathfrak{g}}(v) := \{x \in \mathfrak{g} \colon \rho(x)v = 0\}$. If $v \in V$ satisfies*

$$\rho(\mathfrak{g})v = \rho(\mathfrak{n})v \quad \text{and} \quad \mathfrak{z}_{\mathfrak{g}}(v) \cap \mathfrak{n} = \{0\},$$

then $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathfrak{z}_{\mathfrak{g}}(v)$.

Proof. Our assumption implies that $\mathfrak{z}_{\mathfrak{g}}(v) \cap \mathfrak{n} = \mathfrak{z}_{\mathfrak{n}}(v) = \{0\}$. The linear map $\varphi \colon \mathfrak{g} \to V, x \mapsto \rho(x)v$ satisfies $\varphi(\mathfrak{g}) = \varphi(\mathfrak{n})$, hence $\mathfrak{g} = \mathfrak{n} + \ker \varphi = \mathfrak{n} + \mathfrak{z}_{\mathfrak{g}}(v)$. Since $\mathfrak{z}_{\mathfrak{g}}(v)$ is a subalgebra, the assertion follows. □

Theorem 4.6.6 (Levi's Theorem). *If $\alpha \colon \mathfrak{g} \to \mathfrak{s}$ is a surjective homomorphism of Lie algebras and $\mathfrak{s}$ is semisimple, then there exists a homomorphism $\beta \colon \mathfrak{s} \to \mathfrak{g}$ with $\alpha \circ \beta = \operatorname{id}_{\mathfrak{s}}$.*



Proof. We argue by induction on the dimension of $\mathfrak{n} := \ker \alpha$. For $\mathfrak{n} = \{0\}$, there is nothing to show. So we assume that $\mathfrak{n} \neq \{0\}$.

Case 1: The ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is not minimal, i.e., there exists a nonzero ideal $\mathfrak{n}_1 \subseteq \mathfrak{n}$, different from $\mathfrak{n}$. Now α factors through a surjective homomorphism $\alpha_1 \colon \mathfrak{g}/\mathfrak{n}_1 \to \mathfrak{s}$ with

$$\dim(\ker \alpha_1) = \dim \mathfrak{n} - \dim \mathfrak{n}_1 < \dim \mathfrak{n}.$$

Therefore our induction hypothesis implies the existence of a homomorphism $\beta_1 \colon \mathfrak{s} \to \mathfrak{g}/\mathfrak{n}_1$ with $\alpha_1 \circ \beta_1 = \operatorname{id}_{\mathfrak{s}}$. Let $q \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}_1$ be the quotient map and $\mathfrak{b} := q^{-1}\big(\beta_1(\mathfrak{s})\big)$. Then $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ and the homomorphism

$$\alpha_2 := q|_{\mathfrak{b}} \colon \mathfrak{b} \to \beta_1(\mathfrak{s}) \cong \mathfrak{s}, \quad x \mapsto x + \mathfrak{n}_1$$

is surjective. In view of $\dim(\ker \alpha_2) = \dim \mathfrak{n}_1 < \dim \mathfrak{n}$, the induction hypothesis implies the existence of a homomorphism $\beta_2 \colon \beta_1(\mathfrak{s}) \to \mathfrak{b}$ with $\alpha_2 \circ \beta_2 = \mathrm{id}_{\beta_1(\mathfrak{s})}$. Now $\beta := \beta_2 \circ \beta_1 \colon \mathfrak{s} \to \mathfrak{g}$ is a homomorphism satisfying

$$\alpha \circ \beta = \alpha_1 \circ \alpha_2 \circ \widetilde{\beta_2} \circ \beta_1 = \alpha_1 \circ \beta_1 = \mathrm{id}_{\mathfrak{s}}\,.$$

Case 2: The ideal $\mathfrak{n}$ is minimal. Since $\mathfrak{s}$ is semisimple, the radical $\mathfrak{r} := \mathrm{rad}(\mathfrak{g})$ of $\mathfrak{g}$ is contained in $\mathfrak{n}$ (Lemma 4.6.2). If $\mathfrak{r} = \{0\}$, then $\mathfrak{g}$ is semisimple, and the assertion follows from Proposition 4.5.11 because $\mathfrak{g}$ contains an ideal complementing $\mathfrak{n}$. So let us assume that $\mathfrak{r} \neq \{0\}$. Then the minimality of $\mathfrak{n}$ shows that $\mathfrak{n} = \mathfrak{r}$ is abelian.

The representation $\rho \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n}), x \mapsto \mathrm{ad}\, x|_{\mathfrak{n}}$ satisfies $\mathfrak{n} \subseteq \ker \rho$ ($\mathfrak{n}$ is abelian), hence factors through a representation $\overline{\rho}$ of $\mathfrak{s}$ on $\mathfrak{n}$, determined by $\overline{\rho} \circ \alpha = \rho$. Since $\mathfrak{n}$ is a minimal ideal of $\mathfrak{g}$, we thus obtain on $\mathfrak{n}$ an irreducible representation of $\mathfrak{s}$. If $\overline{\rho} = 0$, then $\mathfrak{n}$ is central in $\mathfrak{g}$, and the adjoint representation $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{der}(\mathfrak{g})$ factors through a representation of $\mathfrak{s}$ on $\mathfrak{g}$. According to Weyl's Theorem, there exists an ideal of $\mathfrak{g}$ complementing $\mathfrak{n}$ (Proposition 4.5.18) and the proof is complete. We may therefore assume that $\overline{\rho}$ is nonzero.

We are now at the point where we can use Lemma 4.6.5. On $V := \mathrm{End}(\mathfrak{g})$, we consider the representation

$$\pi(x)\varphi := \mathrm{ad}\, x \circ \varphi - \varphi \circ \mathrm{ad}\, x = [\mathrm{ad}\, x, \varphi].$$

We consider the following three subspaces of V:

$$\begin{aligned} P := \mathrm{ad}\, \mathfrak{n} \subseteq Q &:= \{\varphi \in V \colon \varphi(\mathfrak{g}) \subseteq \mathfrak{n}, \varphi(\mathfrak{n}) = \{0\}\} \\ \subseteq R &:= \{\varphi \in V \colon \varphi(\mathfrak{g}) \subseteq \mathfrak{n}, \varphi|_{\mathfrak{n}} \in \mathbb{K}\, \mathrm{id}_{\mathfrak{n}}\}. \end{aligned}$$

Since $Q \subseteq R$ is the kernel of the linear map $\chi \colon R \to \mathbb{K}$, defined by $\varphi|_{\mathfrak{n}} = \chi(\varphi)\, \mathrm{id}_{\mathfrak{n}}$, we see that $\dim(R/Q) = 1$.

We claim that P, Q and R are $\mathfrak{g}$-invariant. To this end, let $y \in \mathfrak{g}$. For $x \in \mathfrak{n}$ we then have $[\mathrm{ad}\, y, \mathrm{ad}\, x] = \mathrm{ad}[y, x] \in P$, so that P is $\mathfrak{g}$-invariant. To see that R and Q are $\mathfrak{g}$-invariant, we show that $\pi(\mathfrak{g})R \subseteq Q$. So let $x \in \mathfrak{g}$, $\varphi \in R$ and $\varphi|_{\mathfrak{n}} = \lambda\, \mathrm{id}_{\mathfrak{n}}$. For $a \in \mathfrak{n}$ we then have

$$(\pi(x)\varphi)(a) = [x, \varphi(a)] - \varphi([x, a]) = [x, \lambda a] - \lambda[x, a] = 0,$$

hence $x.\varphi \in Q$. For $y \in \mathfrak{n}$ we get

$$[\mathrm{ad}\, y, \varphi] = \mathrm{ad}\, y \circ \varphi - \varphi \circ \mathrm{ad}\, y = -\lambda\, \mathrm{ad}\, y \in P.$$

This proves that $\pi(\mathfrak{n})R \subseteq P$. The ideal $\mathfrak{n}$ acts trivially on the quotient space R/P, which therefore inherits a representation of $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{n}$.

According to Weyl's Theorem 4.5.21, there exists an $\mathfrak{s}$-invariant subspace of R/P complementing Q/P. Clearly, this complement is one-dimensional, hence generated by the image $\overline{v}$ of one element $v \in R \setminus Q$, of which we may assume that $v|_{\mathfrak{n}} = \mathrm{id}_{\mathfrak{n}}$. As the one-dimensional representation of $\mathfrak{s}$ on $\mathbb{R}\overline{v}$ is trivial because $\mathfrak{s}$ is perfect (Exercise 4.5.8), we see that $\pi(\mathfrak{g})v \subseteq P$. Next we verify the assumptions of Lemma 4.6.5.

For $x \in \mathfrak{n}$ we have already seen that $\pi(x)v = [\mathrm{ad}\, x, v] = -\mathrm{ad}\, x$. If $\pi(x)v = 0$, then $\mathrm{ad}\, x = 0$, i.e., $x \in \mathfrak{z}(\mathfrak{g})$. Since $\mathfrak{n}$ is a minimal ideal of $\mathfrak{g}$ which is not central, we derive that $x = 0$. This leads to $\mathfrak{z}_{\mathfrak{n}}(v) = \{0\}$ and $\pi(\mathfrak{n})v = \mathrm{ad}\,\mathfrak{n} = P = \pi(\mathfrak{g})v$. Finally, we apply Lemma 4.6.5 to complete the proof. □

Definition 4.6.7. If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathrm{rad}(\mathfrak{g}) \trianglelefteq \mathfrak{g}$ its solvable radical, then we call a subalgebra $\mathfrak{s} \leq \mathfrak{g}$ complementing the radical $\mathrm{rad}(\mathfrak{g})$ a *Levi complement*. Note that $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \rtimes \mathfrak{s}$ holds for any Levi complement.

Corollary 4.6.8. *Each finite-dimensional Lie algebra* $\mathfrak{g}$ *contains a semisimple Levi complement.*

Proof. Let $\mathfrak{s} := \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ and $\alpha\colon \mathfrak{g} \to \mathfrak{s}$ the quotient map. According to Lemma 4.6.1, $\mathfrak{s}$ is semisimple. Hence Theorem 4.6.6 provides a homomorphism $\beta\colon \mathfrak{s} \to \mathfrak{g}$ with $\alpha \circ \beta = \mathrm{id}_{\mathfrak{s}}$. Then β is injective, so that $\beta(\mathfrak{s}) \cap \mathrm{rad}(\mathfrak{g}) = \{0\}$ as well as $\beta(\mathfrak{s}) + \mathrm{rad}(\mathfrak{g}) = \mathfrak{g}$. Thus $\beta(\mathfrak{s})$ is a semisimple Levi complement. □

Corollary 4.6.9. *If* $\mathfrak{s}$ *is a Levi complement in* $\mathfrak{g}$*, then*

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \rtimes \mathfrak{s}.$$

If $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$*, then* $[\mathfrak{g}, \mathfrak{g}]$ *is a Levi complement.*

Proof. The second assertion immediately follows from the first and the fact that $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ is equivalent to $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = \{0\}$.

For the first assertion we note that $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ leads to

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] + [\mathfrak{g}, \mathfrak{s}] = [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] + [\mathrm{rad}(\mathfrak{g}), \mathfrak{s}] + [\mathfrak{s}, \mathfrak{s}] = [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] + \mathfrak{s}.$$

□

Corollary 4.6.10. *If* $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ *is a surjective homomorphism of finite-dimensional Lie algebras,* $\mathfrak{s}$ *is semisimple and* $\alpha\colon \mathfrak{s} \to \mathfrak{g}$ *is a homomorphism, then there exists a homomorphism* $\widehat{\alpha}\colon \mathfrak{s} \to \widehat{\mathfrak{g}}$ *with* $q \circ \widehat{\alpha} = \alpha$*.*

Proof. Apply Levi's Theorem to the surjective homomorphism $q\colon q^{-1}(\alpha(\mathfrak{s})) \to \alpha(\mathfrak{s})$ and note that the homomorphic image $\alpha(\mathfrak{s})$ of $\mathfrak{s}$ is semisimple. □

Example 4.6.11. (a) Let V be a finite-dimensional vector space and $\mathfrak{g} = \mathfrak{gl}(V)$. Then

$$\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = \mathbb{K}\mathbf{1}$$

and $\mathfrak{sl}(V)$ is a Levi complement in $\mathfrak{g}$ (Exercise).

(b) Let V be a finite-dimensional vector space and $\mathcal{F} = (V_0, \ldots, V_n)$ a flag in V. Then

$$\mathrm{rad}(\mathfrak{g}(\mathcal{F})) = \{\varphi \in \mathfrak{g}(\mathcal{F}) \colon (\forall i)(\exists \lambda_i \in \mathbb{K})\ (\varphi - \lambda_i \mathbf{1})(V_i) \subseteq V_{i-1}\}$$

(Exercise). Note in particular that

$$\mathrm{rad}(\mathfrak{g}(\mathcal{F})) \supseteq \mathfrak{g}_n(\mathcal{F}) = \{\varphi \in \mathfrak{g}(\mathcal{F}) \colon (\forall i)\ \varphi(V_i) \subseteq V_{i-1}\}.$$

Choosing subspaces $W_1, \ldots, W_n \subseteq V$ with $V_i = W_1 \oplus \ldots \oplus W_i$, we find

$$\mathfrak{g}(\mathcal{F}) \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \bigoplus_{i=1}^n \mathfrak{gl}(W_i),$$

$$\mathrm{rad}(\mathfrak{g}(\mathcal{F})) \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \mathbb{K}^n \quad \text{and} \quad \mathfrak{g}(\mathcal{F})/\,\mathrm{rad}(\mathfrak{g}(\mathcal{F})) \cong \bigoplus_{i=1}^n \mathfrak{sl}(W_i).$$

Remark 4.6.12. If $\mathfrak{g}$ is a solvable Lie algebra, then $\mathfrak{g}$ is isomorphic to a nested semidirect sum

$$\big(\ldots\big((\mathfrak{g}_1 \rtimes_{\alpha_1} \mathfrak{g}_2) \rtimes_{\alpha_2} \mathfrak{g}_3\big) \ldots \rtimes_{\alpha_{n-1}} \mathfrak{g}_n\big)$$

in which every $\mathfrak{g}_j$ is isomorphic to $\mathbb{K}$.

Composing this with Levi's Theorem and using Proposition 4.5.11, we obtain a similar factorization for arbitrary finite-dimensional Lie algebras $\mathfrak{g}$, the only difference is that the $\mathfrak{g}_j$ are either one-dimensional or simple. In particular, we get a sequence

$$\mathfrak{a}_0 = \{0\} \subseteq \mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_k = \mathfrak{g}$$

of subalgebras of $\mathfrak{g}$ for which $\mathfrak{a}_{j-1}$ is an ideal in $\mathfrak{a}_j$ and the quotient $\mathfrak{a}_j/\mathfrak{a}_{j-1}$ is either one-dimensional or simple. Such a series is called *Jordan–Hölder series* of $\mathfrak{g}$.

4.6.2 Malcev's Theorem

Theorem 4.6.13 (Malcev's Theorem). *For two Levi complements $\mathfrak{s}$ and $\mathfrak{s}'$ in $\mathfrak{g}$, there exists some $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ with $e^{\mathrm{ad}\, x}\mathfrak{s}' = \mathfrak{s}$.*

Proof. Let $\mathfrak{r} := \mathrm{rad}(\mathfrak{g})$. We first consider some special cases.
(a) If $[\mathfrak{g}, \mathfrak{r}] = \{0\}$, then $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ is a direct sum of Lie algebras and $\mathfrak{r} = \mathfrak{z}(\mathfrak{g})$ is abelian. Therefore $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{s}', \mathfrak{s}'] = \mathfrak{s}'$, and there is nothing to show.
(b) If $[\mathfrak{g}, \mathfrak{r}] \neq \{0\}$ and $\mathfrak{r}$ is a minimal nonzero ideal of $\mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, $[\mathfrak{r}, \mathfrak{r}] = \{0\}$ (since $D^1(\mathfrak{r}) \neq \mathfrak{r}$), and $\mathfrak{z}(\mathfrak{g}) = \{0\}$ (because $\mathfrak{r} \not\subseteq \mathfrak{z}(\mathfrak{g})$). We define a map $h \colon \mathfrak{s}' \to \mathfrak{r}$ by $x + h(x) \in \mathfrak{s}$ for $x \in \mathfrak{s}'$, i.e., $-h$ is the projection of $\mathfrak{s}'$ to $\mathfrak{s}$ along $\mathfrak{r}$. Since $\mathfrak{s}$ is a subalgebra and $\mathfrak{r}$ is abelian, we have

$$[x + h(x), y + h(y)] = [x, y] + [x, h(y)] + [h(x), y] \in \mathfrak{s}.$$

Therefore

$$h([x, y]) = [x, h(y)] + [h(x), y].$$

This implies that

$$\pi(x)(r, t) := ([x, r] + th(x), 0)$$

defines a representation of $\mathfrak{s}'$ on $\mathfrak{r} \times \mathbb{K}$. The subspace $\mathfrak{r} \cong \mathfrak{r} \times \{0\}$ is $\mathfrak{s}'$-invariant. According to Weyl's Theorem, there exists an $\mathfrak{s}'$-invariant complement $\mathbb{K}(v, 1)$ of $\mathfrak{r}$ in $\mathfrak{r} \oplus \mathbb{K}$. As $\mathfrak{s}'$ is semisimple, $\pi(\mathfrak{s}')(v, 1) = \{0\}$, and hence $h(x) + [x, v] = 0$ for $x \in \mathfrak{s}'$. Now we have

$$e^{\mathrm{ad}\, v} x = x + [v, x] = x + h(x) \in \mathfrak{s}$$

for $x \in \mathfrak{s}'$ and thus $e^{\mathrm{ad}\, v}(\mathfrak{s}') \subseteq \mathfrak{s}$. Equality follows from $\dim \mathfrak{s} = \dim \mathfrak{g}/\mathfrak{r} = \dim \mathfrak{s}'$. This proves the theorem if $[\mathfrak{g}, \mathfrak{r}]$ is a nonzero minimal ideal.

(c) Finally, we turn to the general case. We argue by induction on $n := \dim \mathfrak{r}$. The case $n = 0$ is trivial, so that we assume that $n > 0$ and that the assertion holds for all Lie algebras $\mathfrak{h}$ with $\dim \mathrm{rad}(\mathfrak{h}) < n$. In view of (a), we may assume that $[\mathfrak{g}, \mathfrak{r}] \neq \{0\}$. As the ideal $[\mathfrak{g}, \mathfrak{r}]$ is nilpotent (Corollary 4.4.15), its center $\mathfrak{c} := \mathfrak{z}([\mathfrak{g}, \mathfrak{r}])$ is nonzero. Let $\mathfrak{m} \neq \{0\}$ be a minimal ideal of $\mathfrak{g}$ contained in $\mathfrak{c}$. If $\mathfrak{m} = \mathfrak{r}$, then we are in the situation of (b). We therefore assume $\mathfrak{m} \neq \mathfrak{r}$. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}_1 := \mathfrak{g}/\mathfrak{m}$ be the quotient map. Then $\mathfrak{r}_1 := \pi(\mathfrak{r})$ is the radical of $\mathfrak{g}_1$ (Lemma 4.6.2), and $\pi(\mathfrak{s})$ and $\pi(\mathfrak{s}')$ are Levi complements in $\mathfrak{g}/\mathfrak{m}$ because both are semisimple (Corollary 4.5.13) and complementing $\pi(\mathfrak{r})$. Now our induction hypothesis provides an $x_1 \in [\mathfrak{g}_1, \mathfrak{r}_1]$ with $e^{\mathrm{ad}\, x_1}\pi(\mathfrak{s}') = \pi(\mathfrak{s})$. Using $\pi([\mathfrak{g}, \mathfrak{r}]) = [\mathfrak{g}_1, \mathfrak{r}_1]$, we find an $x \in [\mathfrak{g}, \mathfrak{r}]$ with $\pi(x) = x_1$. Then $e^{\mathrm{ad}\, x_1}\pi(\mathfrak{s}') = \pi(e^{\mathrm{ad}\, x}\mathfrak{s}') \subseteq \pi(\mathfrak{s})$, i.e.,

$$e^{\mathrm{ad}\, x}\mathfrak{s}' \subseteq \mathfrak{h} := \mathfrak{s} + \mathfrak{m}.$$

Now $e^{\mathrm{ad}\, x}\mathfrak{s}'$ and $\mathfrak{s}$ are two Levi complements in the Lie algebra $\mathfrak{h}$ with $\dim \mathrm{rad}(\mathfrak{h}) = \dim \mathfrak{m} < n = \dim \mathfrak{r}$. Hence the induction hypothesis provides a $y \in \mathfrak{m}$ with $e^{\mathrm{ad}\, y} e^{\mathrm{ad}\, x}\mathfrak{s}' \subseteq \mathfrak{s}$. Since $\mathfrak{m}$ is central in $[\mathfrak{g}, \mathfrak{r}]$, we have $[x, y] = 0$ and therefore $e^{\mathrm{ad}\, y} e^{\mathrm{ad}\, x}\mathfrak{s}' = e^{\mathrm{ad}(x+y)}\mathfrak{s}' \subseteq \mathfrak{s}$. □

Malcev's Theorem has interesting consequences:

Corollary 4.6.14. *Each semisimple subalgebra of* $\mathfrak{g}$ *is contained in a Levi complement. In particular, the Levi complements are precisely the maximal semisimple subalgebras of* $\mathfrak{g}$.

Proof. Let $\mathfrak{r} := \mathrm{rad}(\mathfrak{g})$ be the radical of $\mathfrak{g}$, $\mathfrak{h} \subseteq \mathfrak{g}$ a semisimple subalgebra, and $\mathfrak{a} := \mathfrak{r} + \mathfrak{h}$. Then $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{r}$ is a solvable ideal of $\mathfrak{a}$. Since the solvable ideal $\mathrm{rad}(\mathfrak{a}) \cap \mathfrak{h}$ of the semisimple Lie algebra $\mathfrak{h}$ is trivial, we see that $\mathfrak{r} = \mathrm{rad}(\mathfrak{a})$. The ideal $\mathfrak{h} \cap \mathfrak{r}$ of $\mathfrak{h}$ is solvable and semisimple, hence trivial. This proves that $\mathfrak{h}$ is a Levi complement in $\mathfrak{a}$.

Let $\mathfrak{s}$ be a Levi complement in $\mathfrak{g}$. Then $\mathfrak{a} = \mathfrak{r} + (\mathfrak{a} \cap \mathfrak{s})$ is a semidirect sum and since $\mathfrak{a} \cap \mathfrak{s} \cong \mathfrak{a}/\mathfrak{r} \cong \mathfrak{h}$ is semisimple, $\mathfrak{a} \cap \mathfrak{s}$ is a Levi complement in $\mathfrak{a}$. According to Malcev's Theorem 4.6.13, there exists an $x \in [\mathfrak{a}, \mathfrak{r}]$ with $e^{\operatorname{ad} x}(\mathfrak{a} \cap \mathfrak{s}) = \mathfrak{h}$, i.e., $\mathfrak{h}$ is a contained in the Levi complement $e^{\operatorname{ad} x}(\mathfrak{s})$ of $\mathfrak{g}$. □

Corollary 4.6.15. *If* $\mathfrak{n} \trianglelefteq \mathfrak{g}$ *is an ideal of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ *a Levi decomposition, i.e.,* $\mathfrak{s}$ *is a Levi complement, then* $\mathfrak{n} = (\mathfrak{n} \cap \mathfrak{r}) \rtimes (\mathfrak{n} \cap \mathfrak{s})$ *is a Levi decomposition of* $\mathfrak{n}$.

Proof. We have already seen in Lemma 4.6.4 that $\mathfrak{n} \cap \mathfrak{r} = \operatorname{rad}(\mathfrak{n})$. If $\mathfrak{s}_{\mathfrak{n}}$ is a Levi complement in $\mathfrak{n}$, then Corollary 4.6.14 implies that the semisimple Lie algebra $\mathfrak{s}_{\mathfrak{n}}$ is contained in a Levi complement $\mathfrak{s}'$ of $\mathfrak{g}$. For $x \in [\mathfrak{g}, \mathfrak{r}]$ with $e^{\operatorname{ad} x}\mathfrak{s}' = \mathfrak{s}$ we now see that $e^{\operatorname{ad} x}\mathfrak{s}_{\mathfrak{n}} \subseteq \mathfrak{n} \cap \mathfrak{s}$, because

$$e^{\operatorname{ad} x}\mathfrak{n} \subseteq \mathfrak{n} + [x, \mathfrak{n}] \subseteq \mathfrak{n}.$$

Since the ideal $\mathfrak{n} \cap \mathfrak{s}$ of $\mathfrak{s}$ is semisimple (Corollary 4.5.13) and $\mathfrak{s}_{\mathfrak{n}}$ is maximal semisimple in $\mathfrak{n}$, we obtain $e^{\operatorname{ad} x}\mathfrak{s}_{\mathfrak{n}} = \mathfrak{n} \cap \mathfrak{s}$. This shows that $\mathfrak{n} \cap \mathfrak{s}$ is a Levi complement in $\mathfrak{n}$. □

4.7 Reductive Lie Algebras

We conclude this chapter with a brief discussion of reductive Lie algebras. This class of Lie algebras is only slightly larger than the class of semisimple Lie algebras, but they occur naturally. In particular, one often finds them as stabilizer subalgebras inside semisimple Lie algebras. Thus they appear frequently in proofs by induction on the dimension.

Definition 4.7.1. We call a finite-dimensional Lie algebra $\mathfrak{g}$ *reductive* if $\mathfrak{g}$ is a semisimple module with respect to the adjoint representation, i.e., for each ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$, there exists an ideal $\mathfrak{b} \trianglelefteq \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

Lemma 4.7.2. *For a reductive Lie algebra* $\mathfrak{g}$*, the following assertions hold:*

(i) *If* $\mathfrak{n} \trianglelefteq \mathfrak{g}$ *is an ideal, then* $\mathfrak{n}$ *and* $\mathfrak{g}/\mathfrak{n}$ *are reductive.*
(ii) $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ *and* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple.*
(iii) $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Proof. (i) Since $\mathfrak{g}$ is reductive, there exists an ideal $\mathfrak{b} \trianglelefteq \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{b}$. Then $[\mathfrak{b}, \mathfrak{n}] = \{0\}$, so that $\mathfrak{g}$ is a direct sum of Lie algebras. As submodules of the semisimple $\mathfrak{g}$-module $\mathfrak{g}$, the ideals $\mathfrak{n}$ and $\mathfrak{b}$ are semisimple $\mathfrak{g}$-modules, and since the complementary ideals do not act on each other, it follows that $\mathfrak{n}$ and $\mathfrak{b} \cong \mathfrak{g}/\mathfrak{n}$ are reductive Lie algebras.
(ii) Let $\mathfrak{a} \subseteq \mathfrak{g}$ be an ideal complement of $[\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g} = \mathfrak{a} \oplus [\mathfrak{g}, \mathfrak{g}]$, and $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$ implies that $\mathfrak{a}$ is central. Further, (i) implies that $[\mathfrak{g}, \mathfrak{g}]$ is reductive. To see that $\mathfrak{z}(\mathfrak{g})$ is not larger than $\mathfrak{a}$, we choose an ideal $\mathfrak{b}$ of $[\mathfrak{g}, \mathfrak{g}]$

complementing $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. Then $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{b}$ yields $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$, and hence $\mathfrak{z}(\mathfrak{g}) = \mathfrak{a}$.

Since $[\mathfrak{g}, \mathfrak{g}]$ is reductive, it is a direct sum of simple modules $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ for the adjoint representation. The preceding argument implies that none of these ideals is abelian, hence they are simple Lie algebras and thus $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

(iii) If $\mathfrak{z}(\mathfrak{g}) = \{0\}$, then (ii) implies that $\mathfrak{g}$ is semisimple. If, conversely, $\mathfrak{g}$ is semisimple, then $\mathfrak{z}(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g}) = \{0\}$. □

Proposition 4.7.3. *For a finite-dimensional Lie algebra $\mathfrak{g}$, the following are equivalent:*

(i) $\mathfrak{g}$ *is reductive.*
(ii) $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple.*
(iii) $\mathrm{rad}(\mathfrak{g})$ *is central in* $\mathfrak{g}$.

Proof. (i) ⇒ (ii) follows from Lemma 4.7.2.
(ii) ⇒ (iii): Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$ with $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$. Then $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{r}] \rtimes \mathfrak{s}$, so that the semisimplicity of $[\mathfrak{g}, \mathfrak{g}]$ implies that $[\mathfrak{g}, \mathfrak{r}] = \{0\}$.
(iii) ⇒ (i): If $\mathfrak{r}$ is central in $\mathfrak{g}$, then any Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a direct sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is a central ideal. Since $\mathfrak{z}(\mathfrak{s}) = \{0\}$, we immediately get $\mathfrak{z}(\mathfrak{g}) = \mathfrak{r}$, so that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}$. Thus $\mathfrak{g}$ is a direct sum of simple submodules with respect to the adjoint representation, hence a semisimple module and this means that $\mathfrak{g}$ is reductive. □

Exercises for Section 4.6

Exercise 4.7.1. Let $\mathcal{F} = (V_0, \ldots, V_n)$ be a flag in the finite-dimensional vector space V. Determine a Levi decomposition of the Lie algebra $\mathfrak{g}(\mathcal{F})$ (Example 4.6.11).

Exercise 4.7.2. Show that for every Jordan–Hölder series

$$\mathfrak{a}_0 = \{0\} \subseteq \mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_k = \mathfrak{g}$$

of subalgebras of $\mathfrak{g}$ (i.e., $\mathfrak{a}_{j-1}$ is an ideal in $\mathfrak{a}_j$ and the quotient $\mathfrak{a}_j/\mathfrak{a}_{j-1}$ is either one-dimensional or simple), the set of quotients $\{\mathfrak{a}_j/\mathfrak{a}_{j-1} \colon j = 1, \ldots, k\}$ does not depend on the Jordan–Hölder series (cf. Remark 4.6.12).

Notes on Chapter 4

The Jacobi identity was discovered around 1830 by Carl Gustav Jacob Jacobi (1804–1851) as an identity for the Poisson bracket $\{\cdot, \cdot\}$ on smooth functions on $\mathbb{R}^{2n}$ (Exercise 4.1.2).

The term Lie algebra was introduced in the 1920s by Hermann Weyl, following a suggestion of N. Jacobson. Lie himself was dealing mainly with

Lie algebras of vector fields (Exercise 4.1.3), which he called (infinitesimal) transformation groups. The term Lie group was introduced by É. Cartan.

The Jordan decompositions and the Jordan normal form are due to Camille Jordan (1838–1922). He wrote in the 1870s a text book on Galois theory of polynomial equations, thus making Galois' ideas available to the mathematical world. This promoted group theoretical ideas considerably. In particular, it inspired Sophus Lie to work on a "Galois theory" for differential equations, using symmetries of differential equations to understand the structure of their solutions.

In the original proof of his theorem, Weyl used the famous "unitary trick". For $\mathbb{K} = \mathbb{C}$ one can derive Weyl's theorem on complete reducibility from the representation theory of compact groups (cf. Exercise 12.2.5). For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ this works roughly as follows. One shows that the complex representations of $\mathfrak{g}$ are in one-to-one correspondence with the complex representations of $\mathfrak{sl}_n(\mathbb{C})$, resp., its real form $\mathfrak{su}_n(\mathbb{C})$, hence further with unitary representations of the compact simply connected Lie group $\mathrm{SU}_n(\mathbb{C})$. For unitary representations complete irreducibility is trivial. A purely algebraic proof was found later in the 1935 by H. B. G. Casimir and B. L. van der Waerden [CW35], after H. B. G. Casimir had dealt with the case $\mathfrak{sl}_2(\mathbb{C})$ using the operator named after him. Another algebraic proof was found in 1935 by R. Brauer [Br36]. A completely different approach based on Lie algebra cohomology has been developed by J. H. C. Whitehead (see the first Whitehead Lemma 6.5.26).

The original proof of Levi's Theorem for complex Lie algebras [Le05] was based on the classification of simple Lie algebras. The classification free proof for real Lie algebras given here goes back to J. H. C. Whitehead [Wh36]. The conjuagacy of the Levi complements was shown by A. I. Malcev in [Ma42].

For a detailed account of the early history of Lie theory up to 1926 we refer to the book [Haw00] of Th. Hawkins.

5

Root Decomposition

Since a simple Lie algebra $\mathfrak{g}$ has no other ideals than $\mathfrak{g}$ and $\{0\}$, we cannot analyze its structure by breaking it up into an ideal $\mathfrak{n}$ and the corresponding quotient algebra $\mathfrak{g}/\mathfrak{n}$. We therefore need refined tools to look inside simple Lie algebras. It turns out that Cartan subalgebras and the corresponding root decompositions provide such a tool.

Roots and root spaces have remarkable properties some of which one turns into a system of axioms for *abstract root systems*. We derive a number of additional properties from these axioms. Moreover, we define certain objects associated with abstract roots systems like Weyl groups and Weyl chambers. Using these structural elements one could proceed rather easily to a complete classification of complex simple Lie algebras, but we refrain from doing this since our emphasis is on structure rather than classification.

In the present chapter, we first develop the concept of a Cartan subalgebra and root decompositions for general Lie algebras. Then we turn to semisimple Lie algebras and we finally discuss the geometry of the root system.

5.1 Cartan Subalgebras

For a better understanding of the structure of a Lie algebra, one decomposes it into simultaneous eigenspaces of linear maps of the form $\operatorname{ad} x$. Cartan subalgebras $\mathfrak{h} \leq \mathfrak{g}$ provide maximal subsets of $\mathfrak{g}$, for which there exist simultaneous generalized eigenspaces for all operators $\operatorname{ad} x$, $x \in \mathfrak{h}$, whenever $\mathfrak{g}$ is a complex Lie algebra.

5.1.1 Weight and Root Decompositions

Root decompositions are the simultaneous eigenspace decompositions of the type mentioned above. They are special cases of weight decompositions.

Definition 5.1.1. Let (π, V) be a representation of the Lie algebra $\mathfrak{h}$. For a function $\lambda\colon \mathfrak{h} \to \mathbb{K}$, we define the corresponding *weight space* and the corresponding *generalized weight space* by

$$V_\lambda(\mathfrak{h}) := \bigcap_{x\in\mathfrak{h}} V_{\lambda(x)}(\pi(x)) \quad \text{and} \quad V^\lambda(\mathfrak{h}) := \bigcap_{x\in\mathfrak{h}} V^{\lambda(x)}(\pi(x)).$$

Any function $\lambda\colon \mathfrak{h} \to \mathbb{K}$ for which $V^\lambda(\mathfrak{h}) \neq \{0\}$ is called a *weight of the representation* (π, V). We write $\mathcal{P}_\mathfrak{h}(V)$ for the set of weights of (π, V).

Remark 5.1.2. Suppose that we have a subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}(V)$ for which

$$V = \bigoplus_\lambda V^\lambda(\mathfrak{h})$$

is a direct sum of generalized weight spaces which are $\mathfrak{h}$-invariant. We claim that $\mathfrak{h}$ is nilpotent. For each $x \in \mathfrak{h}$, let $x = x_s + x_n$ denote the Jordan decomposition, $\mathfrak{h}_s := \{x_s : x \in \mathfrak{h}\}$ and $\mathfrak{h}_n := \{x_n : x \in \mathfrak{h}\}$. For each weight λ we have

$$V^\lambda(\mathfrak{h}) = \bigcap_{x\in\mathfrak{h}} V_{\lambda(x)}(x_s),$$

so that the weight space decomposition yields a simultaneous diagonalization of the set $\mathfrak{h}_s$. Moreover, for each $x \in \mathfrak{h}$, all eigenspaces of x_s are $\mathfrak{h}$-invariant because they are a sum of certain weight spaces, and this implies that $[\mathfrak{h}, x_s] = 0$.

Let $\pi_\lambda\colon \mathfrak{h} \to \mathfrak{gl}(V^\lambda(\mathfrak{h}))$ denote the representation of $\mathfrak{h}$ on the weight space $V^\lambda(\mathfrak{h})$. For any $x \in \mathfrak{h}$, we then have $\pi_\lambda(x)_s = \lambda(x)\mathbf{1}$. With $n := \dim V^\lambda(\mathfrak{h})$ we now see that

$$\lambda(x) = \frac{1}{n}\operatorname{tr}(\pi_\lambda(x))$$

is a linear functional on $\mathfrak{h}$ vanishing on all brackets. In particular $\ker\lambda$ is an ideal of $\mathfrak{h}$. For $x, y \in \mathfrak{h}$, we further derive that

$$[\pi_\lambda(x) - \lambda(x)\mathbf{1}, \pi_\lambda(y) - \lambda(y)\mathbf{1}] = [\pi_\lambda(x), \pi_\lambda(y)] = \pi_\lambda([x, y]) \in \pi_\lambda(\ker\lambda),$$

so that $\mathfrak{a} := \{\pi_\lambda(x) - \lambda(x)\mathbf{1} : x \in \mathfrak{h}\}$ is a Lie subalgebra of $\mathfrak{gl}(V^\lambda(\mathfrak{h}))$ consisting of nilpotent elements, hence nilpotent by Corollary 4.2.7. Now

$$\pi_\lambda(\mathfrak{h}) \subseteq \mathfrak{a} + \mathbb{K}\mathbf{1} \cong \mathfrak{a} \oplus \mathbb{K}$$

is a subalgebra of a nilpotent Lie algebra, hence nilpotent.

Finally, we observe that we have an inclusion $\mathfrak{h} \hookrightarrow \bigoplus_\lambda \pi_\lambda(\mathfrak{h})$ of $\mathfrak{h}$ into a nilpotent Lie algebra, so that $\mathfrak{h}$ is nilpotent.

In the preceding remark we have seen that we can only expect that a representation decomposes into invariant generalized weight spaces if $\mathfrak{h}$ is nilpotent.

Lemma 5.1.3. *Let (π, V) be a finite-dimensional representation of the nilpotent Lie algebra $\mathfrak{h}$ such that every $\pi(x)$, $x \in \mathfrak{h}$, is split. Then each weight is linear and V decomposes as*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda(\mathfrak{h}).$$

Moreover, each generalized weight space $V^\lambda(\mathfrak{h})$ is $\mathfrak{h}$-invariant.

Proof. Since the assertion of the lemma only refers to the Lie algebra $\pi(\mathfrak{h})$, we may replace $\mathfrak{h}$ by $\pi(\mathfrak{h})$ and assume that $\mathfrak{h} \subseteq \mathfrak{gl}(V)$ is a nilpotent subalgebra consisting of split endomorphisms.

For each $x \in \mathfrak{h}$, we have $\operatorname{ad} x(\mathfrak{h}) \subseteq \mathfrak{h}$, so that we also get $(\operatorname{ad} x)_s(\mathfrak{h}) \subseteq \mathfrak{h}$. Since $\operatorname{ad} x|_\mathfrak{h}$ is nilpotent, Corollary 4.3.9 shows that $0 = (\operatorname{ad} x)_s|_\mathfrak{h} = \operatorname{ad} x_s|_\mathfrak{h}$. It follows that $[x_s, \mathfrak{h}] = \{0\}$. In view of Theorem 4.3.3(iv), we further have $[x_s, y_s] = 0$ for $x, y \in \mathfrak{h}$, so that $\mathfrak{h}_s := \{x_s : x \in \mathfrak{h}\}$ is a commutative set of diagonalizable endomorphisms, hence simultaneously diagonalizable (Exercise 1.1.1(d)). Let $\widetilde{\lambda} : \mathfrak{h}_s \to \mathbb{K}$ be a map and $V_{\widetilde{\lambda}}(\mathfrak{h}_s) = \bigcap_{x \in \mathfrak{h}} V_{\widetilde{\lambda}(x_s)}(x_s)$ be the corresponding simultaneous eigenspace, so that

$$V = \bigoplus_{\widetilde{\lambda}} V_{\widetilde{\lambda}}(\mathfrak{h}_s)$$

(cf. Exercise 1.1.1(e)).

In view of $[\mathfrak{h}, \mathfrak{h}_s] = \{0\}$ and Exercise 1.1.1(a), $V_{\widetilde{\lambda}}(\mathfrak{h}_s)$ is $\mathfrak{h}$-invariant. Let $\pi_{\widetilde{\lambda}}$ denote the representation of $\mathfrak{h}$ on this subspace. For any $x \in \mathfrak{h}$, we then have $\pi_{\widetilde{\lambda}}(x)_s = \widetilde{\lambda}(x_s)\mathbf{1}$. For $n := \dim V_{\widetilde{\lambda}}(\mathfrak{h}_s)$ we now see that

$$\lambda(x) = \widetilde{\lambda}(x_s) = \frac{1}{n} \operatorname{tr}(\pi_{\widetilde{\lambda}}(x))$$

defines a linear functional on $\mathfrak{h}$, satisfying

$$V_{\widetilde{\lambda}}(\mathfrak{h}_s) = \bigcap_{x \in \mathfrak{h}} V_{\lambda(x)}(x_s) = \bigcap_{x \in \mathfrak{h}} V^{\lambda(x)}(x) = V^\lambda(\mathfrak{h}).$$

This completes the proof. ◻

Definition 5.1.4. If $\mathfrak{h}$ is a nilpotent subalgebra of the Lie algebra $\mathfrak{g}$, then the weights of the representation $\pi = \operatorname{ad}|_\mathfrak{h}$ which are different from zero are called *roots* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of all roots is denoted $\Delta(\mathfrak{g}, \mathfrak{h})$. The (generalized) weight spaces $\mathfrak{g}^\lambda(\mathfrak{h})$ are called *root spaces*. Sometimes we write $\mathfrak{g}^\lambda$ instead of $\mathfrak{g}^\lambda(\mathfrak{h})$. If $0 \neq \mu \in \mathfrak{h}^*$ is not a root, we put $\mathfrak{g}^\mu := \{0\}$.

Proposition 5.1.5. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$.*

(i) $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subseteq \mathfrak{g}^{\lambda+\mu}$ *for all* $\lambda, \mu \in \mathfrak{h}^*$.

(ii) $\mathfrak{g}^0$ *is a subalgebra of* $\mathfrak{g}$.

Proof. (i) For $x \in \mathfrak{g}^\lambda$, $y \in \mathfrak{g}^\mu$ and $h \in \mathfrak{h}$, we have

$$\begin{aligned}&\big(\operatorname{ad}(h) - \lambda(h) - \mu(h)\big)^n([x,y]) \\ &= \sum_{k=0}^{n} \binom{n}{k} [\big(\operatorname{ad}(h) - \lambda(h)\big)^k x, \big(\operatorname{ad}(h) - \mu(h)\big)^{n-k} y]\end{aligned}$$

(Exercise 4.3.7). If n is sufficiently large, then for every summand either the left factor or the right factor in the bracket vanishes, so that the whole sum vanishes. This proves that $[x,y] \in \mathfrak{g}^{\lambda+\mu}$.

(ii) is a direct consequence of (i). □

5.1.2 Cartan Subalgebras

In the following we want to decompose a Lie algebra into root spaces with respect to a nilpotent subalgebra $\mathfrak{h}$. Since we want such decompositions to be as fine as possible, $\mathfrak{g}^0(\mathfrak{h})$ should be as small as possible, hence equal to $\mathfrak{h}$. The following result prepares the definition of a Cartan subalgebra.

Proposition 5.1.6. *For a subalgebra* $\mathfrak{h}$ *of a finite-dimensional Lie algebra* $\mathfrak{g}$*, the following are equivalent:*

(C1) $\mathfrak{h}$ *is nilpotent and self-normalizing, i.e.,* $\mathfrak{h} = \mathfrak{n}_\mathfrak{g}(\mathfrak{h})$.
(C2) $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$.

If these conditions are satisfied, then $\mathfrak{h}$ *is a maximal nilpotent subalgebra of* $\mathfrak{g}$.

Proof. (C1) $\Rightarrow$ (C2): Since $\mathfrak{h}$ is nilpotent, we have $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h})$. Assume that $\mathfrak{g}^0(\mathfrak{h})$ strictly contains $\mathfrak{h}$ and consider the representation

$$\pi\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}^0(\mathfrak{h})/\mathfrak{h}), \quad \pi(h)(x + \mathfrak{h}) := [h,x] + \mathfrak{h}$$

whose image consists of nilpotent endomorphisms. By Theorem 4.2.5, there exists some $x \in \mathfrak{g}^0(\mathfrak{h}) \backslash \mathfrak{h}$ with $\operatorname{ad}(h)x \in \mathfrak{h}$ for all $h \in \mathfrak{h}$. But this contradicts $\mathfrak{n}_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$.

(C2) $\Rightarrow$ (C1): From $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h})$ we derive with Engel's Theorem that $\mathfrak{h}$ is nilpotent. To see that $\mathfrak{h}$ is self-normalizing, let $x \in \mathfrak{n}_\mathfrak{g}(\mathfrak{h})$. Then $\operatorname{ad}(h)x \in \mathfrak{h}$ for all $h \in \mathfrak{h}$, and therefore $\operatorname{ad}(h)^n x = 0$ for sufficiently large $n \in \mathbb{N}$. Hence $x \in \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$.

If (C1) and (C2) are satisfied and $\mathfrak{n} \supseteq \mathfrak{h}$ is a nilpotent subalgebra, then $\mathfrak{n} \subseteq \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$. Therefore $\mathfrak{h}$ is maximally nilpotent. □

Definition 5.1.7. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A nilpotent subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* of $\mathfrak{g}$ if it is self-normalizing, i.e., $\mathfrak{n}_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$.

Remark 5.1.8. (a) If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is the only Cartan subalgebra of $\mathfrak{g}$ because any Cartan subalgebra is maximally nilpotent (Proposition 5.1.6).

(b) If $\mathfrak{g}_j$ are Lie algebras with Cartan subalgebras $\mathfrak{h}_j$, then $\bigoplus_j \mathfrak{h}_j$ is a Cartan subalgebra of $\bigoplus_j \mathfrak{g}_j$.

Example 5.1.9. (i) Let $\mathfrak{g} = \mathbb{R}h + \mathbb{R}p + \mathbb{R}q + \mathbb{R}z$ be the oscillator algebra (cf. Example 4.1.19). Then $\mathfrak{h} = \mathbb{R}h + \mathbb{R}z$ is a Cartan subalgebra of $\mathfrak{g}$.

(ii) In $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{K})$, the Lie subalgebra $\mathfrak{h}$ of the diagonal matrices is a Cartan subalgebra (cf. Exercise 5.1.1).

(iii) Every one-dimensional subspace of $\mathfrak{g} = \mathfrak{so}_3(\mathbb{R})$ is a Cartan subalgebra.

(iv) Every Cartan subalgebra is maximally nilpotent (Proposition 5.1.6), but not every maximally nilpotent subalgebra is a Cartan subalgebra. The subalgebra

$$\mathfrak{n} := \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} < \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$$

is maximally nilpotent and not self-normalizing. Its normalizer is the only proper subalgebra

$$\mathfrak{b} := \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

containing $\mathfrak{n}$. It is solvable but not nilpotent.

Definition 5.1.10. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h} < \mathfrak{g}$ be a Cartan subalgebra. We call $\mathfrak{h}$ a *splitting Cartan subalgebra* if $\operatorname{ad} x$ is split for each $x \in \mathfrak{h}$. By Lemma 5.1.3 and Proposition 5.1.6, we then have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}^{\lambda}. \tag{5.1}$$

with respect to the Cartan subalgebra $\mathfrak{h}$.

If, in addition, each $\operatorname{ad} x$, $x \in \mathfrak{h}$, diagonalizable, we call $\mathfrak{h}$ a *toral Cartan subalgebra.* In this case $\mathfrak{g}^{\lambda} = \mathfrak{g}_{\lambda}$ for each $\lambda \in \mathfrak{h}^*$ and in particular $\mathfrak{h} = \mathfrak{g}_0$ implies that $\mathfrak{h}$ is abelian.

Proposition 5.1.11. (i) *A subalgebra $\mathfrak{h}$ of the real Lie algebra $\mathfrak{g}$ is a Cartan subalgebra if and only if $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$.*

(ii) *Let $\varphi\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ be a surjective homomorphism and $\mathfrak{h} < \mathfrak{g}$ a Cartan subalgebra. Then $\varphi(\mathfrak{h})$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}$.*

(iii) *If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in the subalgebra $\mathfrak{k}$, then $\mathfrak{h}$ also is a Cartan subalgebra of $\mathfrak{k}$.*

Proof. (i) This follows immediately from $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})_{\mathbb{C}} = \mathfrak{n}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{h}_{\mathbb{C}})$ and Exercise 4.4.2.
(ii) By (i), we may assume that $\mathbb{K} = \mathbb{C}$. Proposition 4.2.3 shows that $\widetilde{\mathfrak{h}} := \varphi(\mathfrak{h})$ is nilpotent. By Proposition 5.1.6, it suffices to show $\widetilde{\mathfrak{g}}^0(\widetilde{\mathfrak{h}}) = \widetilde{\mathfrak{h}}$. Clearly, $\varphi(\mathfrak{g}^0(\mathfrak{h})) = \varphi(\mathfrak{h}) = \widetilde{\mathfrak{h}}$ (Proposition 5.1.6). Moreover, Lemma 5.1.3 yields decompositions

$$\mathfrak{g} = \mathfrak{g}^0(\mathfrak{h}) \oplus \bigoplus_{0 \neq \lambda \in \mathfrak{h}^*} \mathfrak{g}^\lambda(\mathfrak{h}) \quad \text{and} \quad \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^0(\widetilde{\mathfrak{h}}) \oplus \bigoplus_{\widetilde{\lambda} \in \widetilde{\mathfrak{h}}^*} \widetilde{\mathfrak{g}}^{\widetilde{\lambda}}(\widetilde{\mathfrak{h}}). \tag{5.2}$$

If $x \in \mathfrak{g}^\lambda(\mathfrak{h})$, $\lambda \neq 0$ and $h \in \mathfrak{h}$, there is an $m \in \mathbb{N}$ with $(\operatorname{ad} h - \lambda(h))^m x = 0$, and since φ is a homomorphism, $(\operatorname{ad} \varphi(h) - \lambda(h))^m \varphi(x) = 0$. If $\varphi(x) \neq 0$ and $\varphi(h) = 0$, then $\lambda(h) = 0$, so that we can define $\widetilde{\lambda} \in \widetilde{\mathfrak{h}}^*$ via $\widetilde{\lambda}(\varphi(h)) := \lambda(h)$. We now have

$$\varphi(\mathfrak{g}^\lambda(\mathfrak{h})) \subseteq \begin{cases} \{0\} & \text{if } \varphi(\mathfrak{g}^\lambda(\mathfrak{h})) = \{0\}, \\ \widetilde{\mathfrak{g}}^{\widetilde{\lambda}}(\widetilde{\mathfrak{h}}) & \text{if } \varphi(\mathfrak{g}^\lambda(\mathfrak{h})) \neq \{0\}. \end{cases}$$

In particular, for each root λ, the image of $\mathfrak{g}^\lambda(\mathfrak{h})$ is contained in $\sum_{0 \neq \widetilde{\lambda}} \widetilde{\mathfrak{g}}^{\widetilde{\lambda}}(\widetilde{\mathfrak{h}})$. Since φ is surjective, it follows that $\widetilde{\mathfrak{h}} = \varphi(\mathfrak{h}) = \widetilde{\mathfrak{g}}^0(\widetilde{\mathfrak{h}})$.

(iii) The subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ is nilpotent and self-normalizing, hence a Cartan subalgebra. □

Proposition 5.1.12. *Let $\mathfrak{a} \subseteq \mathfrak{g}$ be an abelian subalgebra for which all operators $\operatorname{ad} x$, $x \in \mathfrak{a}$, are semisimple. Then the Cartan subalgebras of the centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ are precisely the Cartan subalgebras of $\mathfrak{g}$ containing $\mathfrak{a}$. In particular, such Cartan subalgebras exist.*

Proof. Let $\mathfrak{c} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{c}$. Since $\mathfrak{a}$ is central in $\mathfrak{c}$, we find $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{c}) \subseteq \mathfrak{n}_{\mathfrak{c}}(\mathfrak{h}) = \mathfrak{h}$. For the normalizer $\mathfrak{n} := \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$, we have

$$[\mathfrak{a}, \mathfrak{n}] \subseteq [\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{h}.$$

Since each $\operatorname{ad}_{\mathfrak{g}} x$, $x \in \mathfrak{a}$, is semisimple, Lemma 4.5.20 shows that $\mathfrak{g}$ is a semisimple $\mathfrak{a}$-module, so that there exists an $\mathfrak{a}$-invariant subspace $\mathfrak{d} \subseteq \mathfrak{n}$ with $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{d}$. Then $[\mathfrak{a}, \mathfrak{d}] \subseteq \mathfrak{d} \cap \mathfrak{h} = \{0\}$ implies $\mathfrak{d} \subseteq \mathfrak{c}$. Therefore $\mathfrak{n}$ is contained in $\mathfrak{c}$, and since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{c}$, we get $\mathfrak{n} = \mathfrak{h}$, showing that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

If, conversely, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, then $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h}) \subseteq \mathfrak{g}^0(\mathfrak{a}) = \mathfrak{g}_0(\mathfrak{a}) = \mathfrak{c}$. Since $\mathfrak{h}$ is self-normalizing in $\mathfrak{g}$, it is also self-normalizing in $\mathfrak{c}$, hence a Cartan subalgebra of $\mathfrak{c}$. □

Lemma 5.1.13. *For a nilpotent Lie algebra $\mathfrak{h} \subseteq \mathfrak{gl}(V)$, the following assertions hold:*

(i) *The set $\mathfrak{h}_s := \{x_s : x \in \mathfrak{h}\}$, consisting of all semisimple Jordan components of elements in $\mathfrak{h}$, is an abelian Lie algebra commuting with $\mathfrak{h}$.*
(ii) *The set $\mathfrak{h}_n := \{x_n : x \in \mathfrak{h}\}$ of all nilpotent Jordan components of elements in $\mathfrak{h}$ is a nilpotent Lie subalgebra of $\mathfrak{gl}(V)$.*
(iii) *$\widetilde{\mathfrak{h}} := \mathfrak{h}_n + \mathfrak{h}_s$ is a direct sum of Lie algebras.*

Proof. (i) For each $x \in \mathfrak{h}$, we have $\operatorname{ad} x(\mathfrak{h}) \subseteq \mathfrak{h}$, so that we get $(\operatorname{ad} x)_s(\mathfrak{h}) \subseteq \mathfrak{h}$. Since $\operatorname{ad} x|_{\mathfrak{h}}$ is nilpotent, and Proposition 4.3.7(ii)) and Corollary 4.3.9 show that

$$0 = (\operatorname{ad} x|_{\mathfrak{h}})_s = (\operatorname{ad} x)_s|_{\mathfrak{h}} = \operatorname{ad}(x_s)|_{\mathfrak{h}},$$

it follows that $[x_s, \mathfrak{h}] = \{0\}$. In view of Theorem 4.3.3(iv), $[x_s, y_s] = 0$ for $x, y \in \mathfrak{h}$, so that $\mathfrak{h}_s$ is a commutative subset of $\mathfrak{gl}(V)$.

To see that $\mathfrak{h}_s$ actually is a linear subspace, we may w.l.o.g. assume that $\mathbb{K} = \mathbb{C}$ (otherwise we consider the complexification $V_{\mathbb{C}}$). We recall from Lemma 5.1.3 the generalized weight space decomposition

$$V = \bigoplus_{\lambda \in \mathcal{P}(V)} V^\lambda(\mathfrak{h}), \qquad \text{where} \qquad \mathcal{P}(V) = \{\lambda \in \mathfrak{h}^* \colon V^\lambda(\mathfrak{h}) \neq \{0\}\}.$$

With respect to this decomposition, x_s corresponds to the diagonal operator acting on $V^\lambda(\mathfrak{h})$ by $\lambda(x)$. We may therefore identify $\mathfrak{h}_s$ with the image of $\mathfrak{h}$ under the linear map

$$\mathfrak{h} \to \mathbb{K}^{\mathcal{P}(V)}, \quad h \mapsto (\lambda(h))_\lambda,$$

which is a linear subspace. In particular, we see that $(x + y)_s = x_s + y_s$ for $x, y \in \mathfrak{h}$.

(ii) As before, we may assume that $\mathbb{K} = \mathbb{C}$. In view of (i), $\widetilde{\mathfrak{h}} := \mathfrak{h} + \mathfrak{h}_s$ is a Lie subalgebra of $\mathfrak{gl}(V)$ and all weights $\lambda \colon \mathfrak{h} \to \mathbb{C}$ extend to linear functionals $\widetilde{\lambda} \colon \widetilde{\mathfrak{h}} \to \mathbb{C}$, so that we arrive at the same generalized weight spaces $V^\lambda(\mathfrak{h}) = V^{\widetilde{\lambda}}(\widetilde{\mathfrak{h}})$. In view of $x_n = x - x_s$, $\widetilde{\mathfrak{h}}$ also contains $\mathfrak{h}_n$. Moreover, each element of $\widetilde{\mathfrak{h}}$ can be written as a sum $x + y_s$ for $x, y \in \mathfrak{h}$. Since $\operatorname{ad} x(y_s) = 0$, the same holds for $\operatorname{ad} x_n(y_s) = (\operatorname{ad} x)_n(y_s)$, so that $x + y_s = x_n + (x_s + y_s)$ is the Jordan decomposition of $x + y_s$. We further know from (i) that $(x + y)_s = x_s + y_s$, so that $\widetilde{\mathfrak{h}} = \mathfrak{h}_n + \mathfrak{h}_s$.

An element of $\widetilde{\mathfrak{h}}$ is of the form x_n for some $x \in \mathfrak{h}$ if and only if its semisimple component vanishes, i.e., $x \in \bigcap \ker \widetilde{\lambda}$. Since the functionals $\widetilde{\lambda} \colon \widetilde{\mathfrak{h}} \to \mathbb{C}$ vanish on the commutator algebras (Corollary 4.4.11), the intersection of their kernels is an ideal in $\widetilde{\mathfrak{h}}$. Since $\mathfrak{h}_s$ is central in $\widetilde{\mathfrak{h}}$ and $\mathfrak{h}_n \cap \mathfrak{h}_s = \{0\}$, $\widetilde{\mathfrak{h}} = \mathfrak{h}_n \oplus \mathfrak{h}_s$ is a direct sum of Lie algebras. □

5.1.3 Cartan Subalgebras and Regular Elements

Next we want to ensure that Cartan subalgebras actually exist. To this end, for every $x \in \mathfrak{g}$, we consider the generalized eigenspace $\mathfrak{g}^0(\operatorname{ad} x)$. This space is always different from zero since it obviously contains x.

Definition 5.1.14. (a) The number

$$\operatorname{rank}(\mathfrak{g}) := \min\{\dim \mathfrak{g}^0(\operatorname{ad} x) \mid x \in \mathfrak{g}\}$$

is called the *rank* of $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is called *regular* if $\dim \mathfrak{g}^0(\operatorname{ad} x) = \operatorname{rank}(\mathfrak{g})$. We write $\operatorname{reg}(\mathfrak{g})$ for the set of regular elements in $\mathfrak{g}$.

(b) Since $\dim \mathfrak{g}^0(\operatorname{ad} x)$ is the multiplicity of 0 as a root of the characteristic polynomial

$$\det(\operatorname{ad} x - t \operatorname{id}_{\mathfrak{g}}) = \sum_{k=0}^{n} p_k(x) t^k \tag{5.3}$$

of $\operatorname{ad} x$, we have

$$\operatorname{rank}(\mathfrak{g}) = \min\{k \in \mathbb{N} \mid p_k \not\equiv 0\}.$$

As the determinant is a polynomial function on $\operatorname{End}(\mathfrak{g})$, all the functions $p_k \colon \mathfrak{g} \to \mathbb{K}$ in (5.3) are polynomials (cf. Exercise 4.3.8).

Lemma 5.1.15. *The set* $\operatorname{reg}(\mathfrak{g})$ *of regular elements has the following properties:*

(i) $\mathfrak{g} \setminus \operatorname{reg}(\mathfrak{g})$ *is the zero set of a nonconstant polynomial.*
(ii) $\operatorname{reg}(\mathfrak{g})$ *is open and dense. For* $\mathbb{K} = \mathbb{C}$ *it is connected.*
(iii) $\operatorname{reg}(\mathfrak{g})$ *is invariant under the automorphism group* $\operatorname{Aut}(\mathfrak{g})$ *of* $\mathfrak{g}$.
(iv) *If* $\mathbb{K} = \mathbb{R}$, *then* $\operatorname{reg}(\mathfrak{g}) = \mathfrak{g} \cap \operatorname{reg}(\mathfrak{g}_{\mathbb{C}})$ *and* $\operatorname{rank}_{\mathbb{R}}(\mathfrak{g}) = \operatorname{rank}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$.

Proof. (i) Let $r := \operatorname{rank}(\mathfrak{g})$. Then (i) follows from the fact that the functions $p_k \colon \mathfrak{g} \to \mathbb{K}$ occurring in (5.3) are polynomials and

$$\operatorname{reg}(\mathfrak{g}) = \{x \in \mathfrak{g} \colon p_r(x) \neq 0\}.$$

(ii) Clearly $\operatorname{reg}(\mathfrak{g}) = p_r^{-1}(\mathbb{K}^\times)$ is open because p_r is continuous. Further, the zero set of the nonconstant polynomial p_r contains no open subset (Exercise 5.1.2), so that $\operatorname{reg}(\mathfrak{g})$ is also dense in $\mathfrak{g}$. If $\mathfrak{g}$ is a complex Lie algebra, then Exercise 5.1.3 further implies that $\operatorname{reg}(\mathfrak{g})$ is connected.

(iii) For $\gamma \in \operatorname{Aut}(\mathfrak{g})$ we have $\operatorname{ad} \gamma(x) = \gamma \circ \operatorname{ad} x \circ \gamma^{-1}$. Therefore $\operatorname{ad} \gamma(x)$ and $\operatorname{ad} x$ have the same characteristic polynomial. We conclude that all polynomials p_k are invariant under $\operatorname{Aut}(\mathfrak{g})$. In particular, $\operatorname{reg}(\mathfrak{g}) = p_r^{-1}(\mathbb{K}^\times)$ is invariant.

(iv) Since $\det(M_{\mathbb{C}}) = \det(M)$ for each $M \in \operatorname{End}(\mathfrak{g})$, the polynomials $p_k^{\mathbb{C}}$ on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ satisfy $p_k^{\mathbb{C}}|_{\mathfrak{g}} = p_k$ for each k. Hence Exercise 5.1.4 shows that p_k vanishes if and only $p_k^{\mathbb{C}}$ does. In particular, $\operatorname{rank}_{\mathbb{R}}(\mathfrak{g}) = \operatorname{rank}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$, and (iv) follows. □

For the following lemma, we recall from Example 3.2.5 that for each inner derivation $\operatorname{ad} x$, $e^{\operatorname{ad} x}$ is an automorphism of $\mathfrak{g}$. The elements of the group

$$\operatorname{Inn}(\mathfrak{g}) := \langle e^{\operatorname{ad} x} \colon x \in \mathfrak{g} \rangle$$

generated by these automorphisms are called *inner automorphisms*.

Lemma 5.1.16. *Let* $\mathfrak{h} \subseteq \mathfrak{g}$ *be a splitting Cartan subalgebra and* $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$. *Then the following assertions hold:*

(i) $\operatorname{reg}(\mathfrak{g}) \cap \mathfrak{h} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta} \ker \alpha$.
(ii) $\operatorname{rank}(\mathfrak{g}) = \dim \mathfrak{h}$.
(iii) $\operatorname{Inn}(\mathfrak{g})(\mathfrak{h} \cap \operatorname{reg}(\mathfrak{g}))$ *is an open subset of* $\mathfrak{g}$.

Proof. (i), (ii) For each $h \in \mathfrak{h}$ we have $\mathfrak{g}^0(\operatorname{ad} h) = \mathfrak{h} + \sum_{\alpha(h)=0} \mathfrak{g}^\alpha$ and this space is minimal if and only if no root vanishes on h. Since Δ is finite, such elements exist, and for any such element $\mathfrak{g}^0(\operatorname{ad} h) = \mathfrak{h}$.

It remains to show that $r := \operatorname{rank}(\mathfrak{g})$ is not strictly smaller than $\dim \mathfrak{h}$. To this end, we consider the map

$$\Phi\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}, \quad (a, b) \mapsto e^{\operatorname{ad} a} b.$$

Then

$$\mathtt{d}\Phi(0, b)(v, w) = [v, b] + w$$

for $v \in \mathfrak{g}$, $w \in \mathfrak{h}$. We therefore have

$$\operatorname{Im}\big(\mathtt{d}\Phi(0, b)\big) = [b, \mathfrak{g}] + \mathfrak{h} \supseteq \mathfrak{h} + \sum_{\alpha(b) \neq 0} \mathfrak{g}^\alpha$$

because $\alpha(b) \neq 0$ implies $\mathfrak{g}^\alpha \subseteq [b, \mathfrak{g}]$ since $\operatorname{ad} b|_{\mathfrak{g}^\alpha}$ is invertible. If no root vanishes on b, then $\mathtt{d}\Phi(0, b)$ is surjective, and the Implicit Function Theorem implies that the image of Φ is a neighborhood of $b = \Phi(0, b)$. Since $\operatorname{reg}(\mathfrak{g})$ is a dense subset of $\mathfrak{g}$ (Lemma 5.1.15), the image of Φ contains a regular element x which we write as $x = \Phi(a, h) = e^{\operatorname{ad} a} h$ for some $h \in \mathfrak{h}$ and $a \in \mathfrak{g}$. Then $h = e^{-\operatorname{ad} a} x$ is also regular (Lemma 5.1.15) and contained in $\mathfrak{h}$. Thus $\mathfrak{h}$ contains regular elements and the discussion above yields $r = \dim \mathfrak{h}$ and (i).

(iii) The argument above also shows that $\mathtt{d}\Phi(0, h)$ is surjective for each regular element $h \in \mathfrak{h}$. Since $\operatorname{reg}(\mathfrak{g}) \cap \mathfrak{h}$ is open, we see with the Implicit Function Theorem that there exist neighborhoods U_h of h in $\mathfrak{h} \cap \operatorname{reg}(\mathfrak{g})$ and V of 0 in $\mathfrak{g}$ such that $\Phi(V \times U_h) = e^{\operatorname{ad} V} U_h$ is an open subset of $\mathfrak{g}$. Since $e^{\operatorname{ad} V} U_h$ consists of regular elements, h is an interior point of

$$\operatorname{Inn}(\mathfrak{g})(\operatorname{reg}(\mathfrak{g}) \cap \mathfrak{h}).$$

As $\operatorname{Inn}(\mathfrak{g})$ acts by homeomorphisms on $\mathfrak{g}$, this shows that each point in $\operatorname{Inn}(\mathfrak{g})(\operatorname{reg}(\mathfrak{g}) \cap \mathfrak{h})$ is inner, hence that $\operatorname{Inn}(\mathfrak{g})(\operatorname{reg}(\mathfrak{g}) \cap \mathfrak{h})$ is open in $\mathfrak{g}$. □

Lemma 5.1.17. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h} < \mathfrak{g}$. If $x \in \mathfrak{h}$ is regular in $\mathfrak{g}$, then x is also regular in $\mathfrak{h}$.*

Proof. For $h \in \mathfrak{h}$ and $V := \mathfrak{g}/\mathfrak{h}$, consider the linear maps

$$A(h)\colon V \to V, \quad y + \mathfrak{h} \mapsto [h, y] + \mathfrak{h}$$

and $B(h) = \operatorname{ad}(h)|_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{h}$. We set

$$\det(A(h) - t\ \mathrm{id}) = \sum_{k=0}^{m} a_k(h) t^k \quad \text{and} \quad \det(B(h) - t\ \mathrm{id}) = \sum_{j=0}^{n} b_j(h) t^j.$$

We further put

$$d_A(h) := \dim V^0\big(A(h)\big), \quad d_B(h) := \dim \mathfrak{h}^0\big(B(h)\big),$$

and

$$r_A := \min_{h\in\mathfrak{h}} d_A(h), \quad r_B := \min_{h\in\mathfrak{h}} d_B(h).$$

Then $r_A = d_A(h)$ if and only if $a_{r_A}(h) \neq 0$, and similarly $r_B = d_B(h)$ if and only if $b_{r_B}(h) \neq 0$. We consider the set

$$S := \{h \in \mathfrak{h} \mid a_{r_A}(h) \cdot b_{r_B}(h) \neq 0\} = \{h \in \mathfrak{h} \mid r_A = d_A(h), r_B = d_B(h)\}$$

which is nonempty because b_{r_B} does not vanish on the open complement of the zero-set of a_{r_A} (Exercise 5.1.2).

Every element of S is clearly regular in $\mathfrak{h}$, so that it suffices to show that $x \in S$. If we identify $V = \mathfrak{g}/\mathfrak{h}$ with a vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$, we can write $\mathrm{ad}\, h$ in block form as

$$\mathrm{ad}\, h = \begin{pmatrix} B(h) & * \\ 0 & A(h) \end{pmatrix}$$

(cf. Example 4.1.7). This leads for $h \in \mathfrak{h}$ to the factorization

$$\det(\mathrm{ad}\, h - t\,\mathrm{id}) = \det(A(h) - t\,\mathrm{id})\det(B(h) - t\,\mathrm{id}),$$

and hence to $r := \mathrm{rank}\,(\mathfrak{g}) \leq r_A + r_B$ since $S \neq \emptyset$. On the other hand, the fact that $\mathfrak{h}$ contains the regular element x leads to $r = r_A + r_B$, so that $a_{r_A}(h)b_{r_B}(h) = p_r(h)$. We conclude that $x \in S \subseteq \mathrm{reg}(\mathfrak{h})$. □

The following theorem clarifies the connection between Cartan subalgebras and regular elements.

Theorem 5.1.18. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra.*

(i) *For any regular element $x \in \mathfrak{g}$, $\mathfrak{g}^0(\mathrm{ad}\, x)$ is a Cartan subalgebra of $\mathfrak{g}$.*
(ii) *Every Cartan subalgebra $\mathfrak{h}$ contains regular elements and if $x \in \mathfrak{h}$ is regular, then $\mathfrak{h} = \mathfrak{g}^0(\mathrm{ad}\, x)$.*
(iii) *All Cartan subalgebras have the same dimension* $\mathrm{rank}\,(\mathfrak{g})$.
(iv) *For any Cartan subalgebra $\mathfrak{h}$, the set* $\mathrm{Inn}(\mathfrak{g})(\mathfrak{h} \cap \mathrm{reg}(\mathfrak{g}))$ *is open in $\mathfrak{g}$.*

Proof. (i) Clearly, $x \in \mathfrak{h} := \mathfrak{g}^0(\mathrm{ad}\, x)$, so by Lemma 5.1.17, x is also regular in $\mathfrak{h}$. Since $\mathfrak{h} = \mathfrak{h}^0(\mathrm{ad}\, x)$, we have

$$\mathrm{rank}(\mathfrak{h}) = \dim(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g}).$$

Hence $\mathfrak{h}^0(\mathrm{ad}\, y) = \mathfrak{h}$ for all $y \in \mathfrak{h}$, i.e., each $\mathrm{ad}(y)|_{\mathfrak{h}}$ is nilpotent. Therefore $\mathfrak{h}$ is nilpotent by Corollary 4.2.8. We thus arrive at

$$\mathfrak{h} \subseteq \mathfrak{g}^0(\mathrm{ad}\, \mathfrak{h}) \subseteq \mathfrak{g}^0(\mathrm{ad}\, x) = \mathfrak{h},$$

and Proposition 5.1.6 shows that $\mathfrak{h}$ is a Cartan subalgebra.

(ii) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. If $x \in \mathfrak{h} \cap \mathrm{reg}(\mathfrak{g})$, then (i) implies that $\mathfrak{g}^0(\mathrm{ad}\, x)$ is a Cartan subalgebra containing $\mathfrak{h}$, so that the fact that Cartan subalgebras as maximally nilpotent yields $\mathfrak{h} = \mathfrak{g}^0(\mathrm{ad}\, x)$.

If $\mathbb{K} = \mathbb{C}$, then $\mathfrak{h}$ is splitting and Lemma 5.1.16 implies that $\mathfrak{h}$ contains regular elements. For $\mathbb{K} = \mathbb{R}$, we note that $\mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ is a Cartan subalgebra (Proposition 5.1.11), hence contains regular elements. This means that for $r := \mathrm{rank}\,(\mathfrak{g})$, the polynomial $p_r \colon \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ does not vanish on $\mathfrak{h}_{\mathbb{C}}$. Hence it does not vanish on $\mathfrak{h}$ (Exercise 5.1.4), and this means that $\mathfrak{h}$ contains regular elements of $\mathfrak{g}$.

(iii) follows immediately from (ii) and $\dim \mathfrak{g}^0(\mathrm{ad}\, x) = \mathrm{rank}\,(\mathfrak{g})$ for regular elements.

(iv) We know already that this is the case for $\mathbb{K} = \mathbb{C}$. In the real case, we also consider the map

$$\Phi \colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}, \quad (a, b) \mapsto e^{\mathrm{ad}\, a} b$$

with

$$\mathtt{d}\Phi(0, x)(v, w) = [v, x] + w.$$

If $x \in \mathfrak{h}$ is regular, then $\mathfrak{h} = \mathfrak{g}^0(\mathrm{ad}\, x)$, so that the induced map $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(y + \mathfrak{h}) := [x, y] + \mathfrak{h}$ is invertible because its kernel is trivial. This implies that $\mathfrak{g} \subseteq \mathfrak{h} + [x, \mathfrak{g}]$, which means that $\mathtt{d}\Phi(0, x)$ is surjective, so that the Implicit Function Theorem implies that $\Phi(\mathfrak{g} \times (\mathrm{reg}(\mathfrak{g}) \cap \mathfrak{h}))$ is a neighborhood of $x = \Phi(0, x)$. Now the same argument as in the complex case proves (iv) (Lemma 5.1.16). □

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Define an equivalence relation R on the set $\mathrm{reg}(\mathfrak{g})$ of regular elements via

$$x \sim y \quad :\Longleftrightarrow \quad \exists g \in \mathrm{Inn}(\mathfrak{g}) \text{ with } g\big(\mathfrak{g}^0(\mathrm{ad}\, x)\big) = \mathfrak{g}^0(\mathrm{ad}\, y).$$

Lemma 5.1.19. *The equivalence classes of $\sim$ are open subsets of* $\mathrm{reg}(\mathfrak{g})$.

Proof. Fix $x \in \mathrm{reg}(\mathfrak{g})$ and set $\mathfrak{h} := \mathfrak{g}^0(\mathrm{ad}\, x)$. In view of Theorem 5.1.18, the set $\mathrm{Inn}(\mathfrak{g})(\mathfrak{h} \cap \mathrm{reg}(\mathfrak{g}))$ is an open neighborhood of x. Each element in this set is of the form $g(y)$ for $y \in \mathrm{reg}(\mathfrak{g}) \cap \mathfrak{h}$, so that

$$\mathfrak{g}^0(\mathrm{ad}(g(y))) = g\big(\mathfrak{g}^0(\mathrm{ad}\, y)\big) = g(\mathfrak{h}) = g(\mathfrak{g}^0(\mathrm{ad}\, x))$$

implies that $g(y) \sim x$. This proves that all equivalence classes of $\sim$ are open. □

Theorem 5.1.20. *If $\mathfrak{g}$ is a finite-dimensional complex Lie algebra, then the group* $\mathrm{Inn}(\mathfrak{g})$ *acts transitively on the set of Cartan subalgebras of $\mathfrak{g}$.*

Proof. According to Lemma 5.1.15, $\mathrm{reg}(\mathfrak{g})$ is connected. On the other hand it is the disjoint union of the open equivalence classes of the relation $\sim$ (cf. Lemma 5.1.19). Hence only one such class exists. Since every Cartan subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{g}^0(\mathrm{ad}\, x)$ by Theorem 5.1.18, the assertion follows. □

Exercises for Section 5.1

Exercise 5.1.1. The diagonal matrices form a Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.

Exercise 5.1.2. Let V be a finite-dimensional vector space and $p\colon V \to \mathbb{K}$ a polynomial function vanishing on an open subset $U \subseteq V$. Show that $p = 0$.

Exercise 5.1.3. Let V be a complex finite-dimensional vector space and $p\colon V \to \mathbb{K}$ a nonzero polynomial function. Then $p^{-1}(\mathbb{C}^\times)$ is connected.

Exercise 5.1.4. Let V be a finite-dimensional real vector space and $V_{\mathbb{C}}$ its complexification. Show that a polynomial function $p\colon V_{\mathbb{C}} \to \mathbb{C}$ vanishes if and only if $p|_V$ vanishes.

5.2 The Classification of Simple $\mathfrak{sl}_2(\mathbb{K})$-Modules

As we shall see in Section 5.3 below, the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ is of particular importance because semisimple Lie algebras with splitting Cartan subalgebras contain many subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{K})$ and the collection of these subalgebras essentially determines the structure of the whole Lie algebra. Therefore the representation theory of $\mathfrak{sl}_2(\mathbb{K})$ plays a key role in the structure theory of these Lie algebras.

In the following, we shall use the following basis for $\mathfrak{sl}_2(\mathbb{K})$:

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.4}$$

satisfying

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h. \tag{5.5}$$

We start with a discussion of a concrete family of representations of $\mathfrak{sl}_2(\mathbb{K})$. It will turn out later that the study of this family already provides all irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{K})$.

Example 5.2.1. Let $V := \mathbb{K}[Z, Z^{-1}]$ be the algebra of Laurent polynomials in Z. For any $f \in V$ the operator $D := f\frac{d}{dZ}$ is a derivation of V (Product Rule) and any derivation of the algebra V is of this kind (Exercise 5.2.2). The Lie bracket on $\mathrm{der}(V)$ satisfies

$$\Big[f\frac{d}{dZ}, g\frac{d}{dZ}\Big] = (fg' - f'g)\frac{d}{dZ}. \tag{5.6}$$

For $\lambda \in \mathbb{K}$, we consider the operators

$$E := \frac{d}{dZ}, \quad F := -Z^2\frac{d}{dZ} + \lambda Z\mathbf{1}, \quad H := -2Z\frac{d}{dZ} + \lambda\mathbf{1}.$$

With (5.6) we obtain

$$\Big[Z^n \frac{d}{dZ}, Z^m \frac{d}{dZ}\Big] = (m-n)Z^{n+m-1}\frac{d}{dZ}$$

and hence the commutator relations

$$\begin{array}{rcl} [H,E] = & -2\big[Z\frac{d}{dZ}, \frac{d}{dZ}\big] = 2\frac{d}{dZ} & = \quad 2E, \\ [H,F] = & -2\big[Z\frac{d}{dZ}, -Z^2\frac{d}{dZ} + \lambda Z\mathbf{1}\big] = 2Z^2\frac{d}{dZ} - 2\lambda Z\mathbf{1} & = -2F, \\ [E,F] = & \big[\frac{d}{dZ}, -Z^2\frac{d}{dZ} + \lambda Z\mathbf{1}\big] = -2Z\frac{d}{dZ} + \lambda\mathbf{1} & = \quad H. \end{array}$$

Since these are precisely the commutator relations of $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{K})$, we obtain by

$$e \mapsto E, \quad f \mapsto F, \quad h \mapsto H$$

a representation $\rho_\lambda \colon \mathfrak{sl}_2(\mathbb{K}) \to \mathrm{End}(V)$, resp., an $\mathfrak{sl}_2(\mathbb{K})$-module structure on V.

To understand the structure of this module, we consider the action of the operators H, E and F on the canonical basis:

$$H \cdot Z^n = (\lambda - 2n)Z^n, \quad E \cdot Z^n = nZ^{n-1}, \quad F \cdot Z^n = (\lambda - n)Z^{n+1}. \quad (5.7)$$

In particular, we see that H is diagonalizable with one-dimensional eigenspaces. With this information, it is easy to determine all submodules. Any submodule is adapted to the eigenspace decomposition of H (Exercise 1.1.1(b)). Hence each submodule is of the form

$$V_J := \mathrm{span}\{Z^n \colon n \in J\}$$

for a subset $J \subseteq \mathbb{Z}$. From the formulas above we see that V_J is a submodule if and only if J satisfies the following conditions:
(i) If $n \in J$ and $n \neq 0$, then $n-1 \in J$.
(ii) If $n \in J$ and $\lambda \neq n$, then $n+1 \in J$.

If $\lambda \notin \mathbb{Z}$, then $\mathbb{K}[Z] = V_{\mathbb{N}_0}$ is the only nontrivial submodule of V. If $\lambda \in \mathbb{Z}$, then there are two possibilities. For $\lambda < 0$, the only proper subsets of $\mathbb{Z}$ satisfying (i) and (ii) are

$$\{\ldots, \lambda-1, \lambda\}, \quad \mathbb{N}_0, \quad \text{and} \quad \{\ldots, \lambda-1, \lambda\} \cup \mathbb{N}_0.$$

$$\circ \quad \circ \quad \circ \quad] \quad \circ \quad \circ \quad \cdots \quad \circ \quad [\quad \circ \quad \circ \quad \cdots .$$

For $\lambda \geq 0$, then the subsets of $\mathbb{Z}$ defining submodules are

$$\mathbb{N}_0, \quad \{\ldots, \lambda-1, \lambda\}, \quad \{0, 1, \ldots, \lambda-1, \lambda\}.$$

$$\circ \quad \circ \quad \circ \quad [\quad \circ \quad \circ \quad \cdots \quad \circ \quad] \quad \circ \quad \circ \quad \cdots .$$

In this case we obtain in particular a finite-dimensional submodule

$$L(\lambda) := \mathrm{span}\{1, Z, \ldots, Z^\lambda\}.$$

Since $L(\lambda)$ contains no nontrivial proper submodule, it is simple.

We have seen in the preceding example that for each $\lambda \in \mathbb{N}_0$, there exists a simple $\mathfrak{sl}_2(\mathbb{K})$-module of dimension $\lambda + 1$. Our next goal is to show that all simple finite-dimensional modules are isomorphic to some $L(\lambda)$.

The following lemma specializes for $\mu = 1$ to an assertion on the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$.

Lemma 5.2.2. *Let (e, h, f) be a triple of elements of an associative algebra A, satisfying the commutator relations*

$$[h, e] = 2e, \quad [h, f] = -2f \quad \textit{and} \quad [e, f] = h.$$

Then the following assertions hold:

(i) $[h, e^n] = 2ne^n$ *and* $[h, f^n] = -2nf^n$ *for* $n \in \mathbb{N}_0$.
(ii) *For* $n > 0$,

$$[f, e^n] = -ne^{n-1}\big(h + (n-1)\big) = -n\big(h - (n-1)\big)e^{n-1}$$

and

$$[e, f^n] = nf^{n-1}\big(h - (n-1)\big) = n\big(h + (n-1)\big)f^{n-1}.$$

Proof. (i) Since $\operatorname{ad} h(a) := ha - ah$ is a derivation of A and $[h, e] = 2e$ commutes with e, we obtain inductively $[h, e^n] = n[h, e]e^{n-1} = 2ne^n$. The second part of (i) is obtained similarly.
(ii) We calculate

$$\begin{aligned}
[f, e^n] &= \sum_{j=0}^{n-1} e^j [f, e] e^{n-j-1} = \sum_{j=0}^{n-1} e^j (-h) e^{n-j-1} \\
&= -\sum_{j=0}^{n-1} e^j [h, e^{n-j-1}] - \sum_{j=0}^{n-1} e^{n-1} h = -\Big(\sum_{j=0}^{n-1} 2(n-j-1)e^{n-1}\Big) - ne^{n-1}h \\
&= -\Big(\sum_{j=0}^{n-1} 2je^{n-1}\Big) - ne^{n-1}h = -n(n-1)e^{n-1} - ne^{n-1}h.
\end{aligned}$$

In view of (i), this equals

$$-n(n-1)e^{n-1} - nhe^{n-1} + n[h, e^{n-1}] = n(n-1)e^{n-1} - nhe^{n-1}.$$

This is the first part of (ii). The second part is reduced to the first one by considering the triple $(f, -h, e)$, satisfying the same commutation relations as (e, h, f). $\square$

Lemma 5.2.3. *Let V be a $\mathfrak{g}$-module and $\mathfrak{h} \subseteq \mathfrak{g}$ a subalgebra. Then, for any $\alpha, \beta \in \mathfrak{h}^*$,*

$$\mathfrak{g}_\alpha(\mathfrak{h}) \cdot V_\beta(\mathfrak{h}) \subseteq V_{\alpha+\beta}(\mathfrak{h}).$$

Proof. For $v_\beta \in V_\beta(\mathfrak{h})$, $x \in \mathfrak{h}$ and $y \in \mathfrak{g}_\alpha(\mathfrak{h})$ we have

$$x \cdot (y \cdot v_\beta) = [x, y] \cdot v_\beta + y \cdot (x \cdot v_\beta) = \alpha(x) y \cdot v_\beta + \beta(x) y \cdot v_\beta = (\alpha + \beta)(x) y \cdot v_\beta.$$

□

Proposition 5.2.4. *Let V be a finite-dimensional $\mathfrak{sl}_2(\mathbb{K})$-module and $v_0 \in V$ an element with $e \cdot v_0 = 0$ and $h \cdot v_0 = \lambda v_0$. Then*

(i) $\lambda \in \mathbb{N}_0$.
(ii) *v_0 generates a submodule isomorphic to $L(\lambda)$.*

Proof. (i) Let $V_\alpha := V_\alpha(h)$ be the h-eigenspace corresponding to the eigenvalue α on V, which is a weight space for the representation of the subalgebra $\mathfrak{h} = \mathbb{K}h$. ¿From $v_0 \in V_\lambda$ and $[h, f] = -2f$, we obtain with Lemma 5.2.3 the relation $h \cdot (f^n \cdot v_0) = (\lambda - 2n)(f^n \cdot v_0)$.

We further obtain with Lemma 5.2.3:

$$e \cdot (f^n \cdot v_0) = [e, f^n] \cdot v_0 + f^n \cdot \underbrace{(e \cdot v_0)}_{=0} = n f^{n-1}(h - n + 1) \cdot v_0 = n(\lambda - n + 1) f^{n-1} \cdot v_0.$$

This shows that the submodule W generated by v_0 is

$$W = \operatorname{span}\{f^n \cdot v_0 \colon n \in \mathbb{N}_0\}.$$

Since V is finite-dimensional, h has only finitely many eigenvalues on V. Hence there is a minimal $N \in \mathbb{N}_0$ with $f^{N+1} \cdot v_0 = 0$. ¿From $e \cdot (f^{N+1} \cdot v_0) = 0$ we derive that $\lambda = N \in \mathbb{N}_0$.
(ii) To see that $W \cong L(\lambda)$, we consider the following basis

$$v_k := \frac{f^k \cdot v_0}{\lambda(\lambda - 1) \cdots (\lambda - k + 1)}, \quad k = 0, \ldots, \lambda,$$

for W (note that the denominator never vanishes.). For this basis, we have

$$h \cdot v_k = (\lambda - 2k) v_k, \quad f \cdot v_k = (\lambda - k) v_{k+1},$$

$e \cdot v_0 = 0$ and, for $k > 0$,

$$\begin{aligned} e \cdot v_k &= \frac{k(\lambda - k + 1)}{\lambda(\lambda - 1) \cdots (\lambda - k + 1)} f^{k-1} \cdot v_0 \\ &= \frac{k}{\lambda(\lambda - 1) \cdots (\lambda - k + 2)} f^{k-1} \cdot v_0 = k v_{k-1}. \end{aligned}$$

With respect to this basis, e, f and h are represented by the same matrices as on $L(\lambda)$, and this shows that $W \cong L(\lambda)$. □

Lemma 5.2.5. *If V is a finite-dimensional real vector space and $a, b \in \mathfrak{gl}(V)$ with $[a, b] = b$, then b is nilpotent.*

Proof. Apply Proposition 4.4.14 to the solvable subalgebra $\mathbb{K}a + \mathbb{K}b \subseteq \mathfrak{gl}(V)$. Then $[a, b] = b$ implies that b is nilpotent. □

Theorem 5.2.6 (Classification of Simple $\mathfrak{sl}_2(\mathbb{K})$-Modules). *Each finite-dimensional simple $\mathfrak{sl}_2(\mathbb{K})$-module is isomorphic to some $L(\lambda)$, $\lambda \in \mathbb{N}_0$. For each $n \in \mathbb{N}$, there exists a simple $\mathfrak{sl}_2(\mathbb{K})$-module of dimension n which is unique up to isomorphism.*

Proof. Let (ρ, V) be a simple $\mathfrak{sl}_2(\mathbb{K})$-module. We consider the solvable subalgebra $\mathfrak{b} := \mathrm{span}\{e, h\}$. We apply Lemma 5.2.5 to $a := \frac{1}{2}\rho(h)$ and $b := \rho(e)$, to see that $\rho(e)$ is nilpotent. Let $d \in \mathbb{N}$ be minimal with $\rho(e)^d = 0$. Then Lemma 5.2.2 yields

$$0 = [\rho(f), \rho(e)^d] = -d(\rho(h) - (d-1)\mathbf{1})\rho(e)^{d-1},$$

so that each nonzero $v_0 \in \rho(e)^{d-1}(V)$ is an eigenvector of $\rho(h)$. In view of the simplicity of the module V, it is generated by v_0, and Proposition 5.2.4 shows that $V \cong L(\lambda)$. The remaining assertions are immediate from Example 5.2.1. □

Remark 5.2.7. A particular interesting representation of $\mathfrak{sl}_2(\mathbb{K})$ is the *oscillator representation*. Here we consider the space

$$\mathcal{P} = \mathbb{C}[x_1, \ldots, x_n]$$

of complex-valued polynomials on $\mathbb{R}^n$. Let $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$ be the *Laplacian*. We put $e := \frac{1}{2}\Delta$ and $f = \frac{1}{2}m_{r^2}$ (multiplication operator with $r^2 := \sum_j x_j^2$), and $h := E + \frac{n}{2}\mathbf{1}$, where $E = \sum_j x_j \frac{\partial}{\partial x_j}$ is the *Euler operator*, for which a homogeneous polynomial of degree d is an eigenvector of degree d.

It is easily verified that $(h, e, f) \in \mathrm{End}(\mathcal{P})$ satisfies the commutation relations of $\mathfrak{sl}_2(\mathbb{R})$, so that $\mathcal{P}$ is an $\mathfrak{sl}_2(\mathbb{K})$-module (Exercise 5.2.3). This module plays an important role in quantum mechanics of systems on $\mathbb{R}^n$ with full rotational symmetry. An important example is the spherical harmonic oscillator on $\mathbb{R}^3$, corresponding to the hydrogen atom.

Proposition 5.2.8. *For a finite-dimensional $\mathfrak{sl}_2(\mathbb{K})$-representation (ρ, V), the following assertions hold:*

(i) *$\rho(h)$ is diagonalizable and the set $\mathcal{P}_V$ of all eigenvalues is contained in $\mathbb{Z}$.*
(ii) *$\mathcal{P}_V = -\mathcal{P}_V$.*
(iii) *(String property) If $\alpha, \alpha + 2k \in \mathcal{P}_V$ for some $k \in \mathbb{N}_0$, then $\alpha + 2j \in \mathcal{P}_V$ for $j = 0, 1, \ldots, k$.*

Proof. In view of Weyl's Theorem 4.5.21, V is a direct sum of simple submodules $V_1, \ldots, V_m$, and Theorem 5.2.6 implies that $V_i \cong \mathbf{L}(\lambda_i)$ for some $\lambda_i \in \mathbb{N}_0$.

(i) and (ii) now follow immediately from the corresponding property of the modules $L(\lambda)$ (Example 5.2.1).

(iii) In view of (ii), we may w.l.o.g. assume that $\beta := \alpha + 2k$ satisfies $|\beta| \geq |\alpha|$. Then we pick some simple submodule $V_i \cong L(\lambda_i)$ of V such that β is an eigenvalue $\rho(h)|_{V_i}$. Then $\lambda_i - \beta \in 2\mathbb{N}_0$ and all integers in

$$[-|\beta|, |\beta|] \cap (\beta + 2\mathbb{Z})$$

are eigenvalues of $\rho(h)|_{V_i}$. This contains in particular the set of all integers of the form $\alpha + 2j$, $j = 0, 1, \ldots, k$, between α and β. □

We consider the element

$$\theta := e^{\operatorname{ad} e} e^{-\operatorname{ad} f} e^{\operatorname{ad} e} \in \operatorname{Aut}(\mathfrak{sl}_2(\mathbb{K}))$$

(Example 3.2.5) and

$$\sigma := \exp(e)\exp(-f)\exp(e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \operatorname{SL}_2(\mathbb{K}).$$

Then Lemma 2.4.1 implies for $z \in \mathfrak{sl}_2(\mathbb{K})$ the relation $\theta(z) = \sigma z \sigma^{-1}$, hence in particular

$$\theta(h) = -h, \quad \theta(e) = -f \quad \text{and} \quad \theta(f) = -e.$$

Lemma 5.2.9. *Let (ρ, V) be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{K})$ and $\sigma_V := e^{\rho(e)} e^{-\rho(f)} e^{\rho(e)} \in \operatorname{GL}(V)$. Then*

$$\sigma_V \rho(z) \sigma_V^{-1} = \rho(\sigma z \sigma^{-1})$$

for $z \in \mathfrak{sl}_2(\mathbb{K})$,

$$\sigma_V(V_\alpha(\rho(h))) = V_{-\alpha}(\rho(h))$$

for the eigenspaces of $\rho(h)$, and in particular

$$\dim V_\alpha(\rho(h)) = \dim V_{-\alpha}(\rho(h)).$$

Proof. For $z \in \mathfrak{sl}_2(\mathbb{K})$ we obtain with Lemma 2.4.1 the relation

$$\sigma_V \rho(z) \sigma_V^{-1} = \rho(\sigma z \sigma^{-1}).$$

This implies that

$$\sigma_V(V_\alpha(\rho(h))) = V_{-\alpha}(\rho(h))$$

because we have for $v \in V_\alpha(\rho(h))$:

$$\rho(h)(\sigma_V(v)) = \sigma_V\big(\sigma_V^{-1}\rho(h)\sigma_V\big)(v) = \sigma_V \rho(-h)(v) = -\alpha \sigma_V(v).$$

This completes the proof. □

Exercises for Section 5.2

Exercise 5.2.1. We consider the 2-dimensional nonabelian complex Lie algebra $\mathfrak{g}$ in which we choose a basis (h, e) satisfying $[h, e] = e$. In the following V denotes a $\mathfrak{g}$-module and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ the corresponding representation. Classify all finite-dimensional $\mathfrak{g}$-modules V for which $\rho(h)$ is diagonalizable.

Exercise 5.2.2. Let $A = \mathbb{K}[Z, Z^{-1}]$ be the algebra of Laurent polynomials with coefficients in the field $\mathbb{K}$. Show that every derivation of A is of the form $D := f\frac{d}{dZ}$ for some $f \in A$.

Exercise 5.2.3. On the space $V = C^\infty(\mathbb{R}^n)$ we consider the operators

$$e := \frac{1}{2}\sum_j \frac{\partial^2}{\partial x_j^2}, \quad \text{and} \quad f = \frac{1}{2}m_{r^2}$$

(multiplication operator with $r^2 := \sum_j x_j^2$), and $h := \frac{n}{2}\mathbf{1} + \sum_j x_j \frac{\partial}{\partial x_j}$. Show that (h, e, f) satisfy the $\mathfrak{sl}_2$-relations

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

5.3 Root Decompositions of Semisimple Lie Algebras

The technique of root decompositions is particularly fruitful for semisimple Lie algebras $\mathfrak{g}$ because, for this class of Lie algebras, all elements h of a Cartan subalgebra $\mathfrak{h}$ turn out to be ad-semisimple, i.e., $\operatorname{ad} h$ is a semisimple endomorphism of $\mathfrak{g}$. For complex Lie algebras we thus obtain a root space decomposition diagonalizing $\operatorname{ad} \mathfrak{h}$.

Proposition 5.3.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a splitting Cartan subalgebra of $\mathfrak{g}$ and $m_\lambda := \dim \mathfrak{g}^\lambda$.*

(i) $\kappa_{\mathfrak{g}}(h, h') = \sum_{\lambda \in \Delta(\mathfrak{g},\mathfrak{h})} m_\lambda \lambda(h)\lambda(h')$ *for $h, h' \in \mathfrak{h}$.*
(ii) *If $\lambda + \mu \neq 0$, then $\mathfrak{g}^\lambda$ and $\mathfrak{g}^\mu$ are orthogonal with respect to the Cartan–Killing form.*

Proof. (i) Both sides of the equation define a symmetric bilinear form on $\mathfrak{h}$. By polarization, it suffices to verify the equality for $h = h'$. But $\lambda(h)$ is the only eigenvalue of $\operatorname{ad} h$ on $\mathfrak{g}^\lambda$. Therefore, the Jordan canonical form, together with the root decomposition (5.1), shows that $\operatorname{tr}(\operatorname{ad}(h)^2) = \sum_\lambda m_\lambda \lambda(h)^2$.
(ii) By Proposition 5.1.5, we have $\operatorname{ad}(x)\operatorname{ad}(y)\mathfrak{g}^\nu \subseteq \mathfrak{g}^{\lambda+\mu+\nu}$ for $x \in \mathfrak{g}^\lambda$ and $y \in \mathfrak{g}^\mu$. Choose a basis for $\mathfrak{g}$ consisting of elements in the $\mathfrak{g}^\nu$. Then since $\nu + \lambda + \mu \neq \nu$, the trace of $\operatorname{ad}(x)\operatorname{ad}(y)$ equals zero. □

Proposition 5.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ be a splitting Cartan subalgebra.*

(i) *The Cartan–Killing form* κ *of* $\mathfrak{g}$ *induces a nondegenerate* pairing *of* $\mathfrak{g}^\alpha$ *and* $\mathfrak{g}^{-\alpha}$*, i.e., for* $x \in \mathfrak{g}^\alpha$ *and* $y \in \mathfrak{g}^{-\alpha}$*,*

$$\kappa(x, \mathfrak{g}^{-\alpha}) = \{0\} \Rightarrow x = 0 \quad \textit{and} \quad \kappa(\mathfrak{g}^\alpha, y) = \{0\} \Rightarrow y = 0.$$

In particular, $m_\alpha = m_{-\alpha}$ *and* $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ *is nondegenerate.*

(ii) *The Lie algebra* $\mathfrak{h}$ *is a toral Cartan subalgebra. In particular,* $\mathfrak{h}$ *is abelian and* $\mathfrak{g}^\alpha = \mathfrak{g}_\alpha$ *for each root* α*.*

(iii) $\Delta(\mathfrak{g}, \mathfrak{h})$ *spans* $\mathfrak{h}^*$*.*

Proof. (i) This claim immediately follows from Proposition 5.3.1(ii), since the Cartan–Killing form is nondegenerate.
(ii) Since the adjoint representation of $\mathfrak{g}$ is injective ($\mathfrak{z}(\mathfrak{g}) = \{0\}$) and the set $(\operatorname{ad}\mathfrak{h})_s$ is commutative because it is diagonalized by the root decomposition, it suffices to show that for each $h \in \mathfrak{h}$, the operator $\operatorname{ad} h$ is semisimple.

In view of Proposition 4.3.10, the nilpotent Jordan component $(\operatorname{ad} h)_n$ is a derivation of $\mathfrak{g}$, and since all derivations of $\mathfrak{g}$ are inner (Theorem 4.5.14), there exists an $h_n \in \mathfrak{g}$ with $\operatorname{ad}(h_n) = (\operatorname{ad} h)_n$. Since the derivation $(\operatorname{ad} h)_n$ commutes with $(\operatorname{ad}\mathfrak{h})_s$, we have

$$\operatorname{ad}\big((\operatorname{ad}\mathfrak{h})_s h_n\big) = [(\operatorname{ad}\mathfrak{h})_s, \operatorname{ad} h_n] = \{0\},$$

so that the injectivity of ad entails that $h_n \in \mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$ (cf. Proposition 4.1.10 and Lemma 5.1.13). As $\operatorname{ad}(h_n)$ is nilpotent, all roots vanish on h_n, and the formula in Proposition 5.3.1(i) implies that $\kappa(h_n, \mathfrak{h}) = \{0\}$. Now $h_n = 0$ follows from (i). This proves that $\operatorname{ad} h$ is diagonalizable for each $h \in \mathfrak{h}$ and since ad is injective, $\mathfrak{h} \cong \operatorname{ad}\mathfrak{h}$ is abelian.
(iii) As a consequence of (ii) and the injectivity of the adjoint representation, $\Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$ separates the points of $\mathfrak{h}$, and this is equivalent to (iii). □

Since the Cartan–Killing form is nondegenerate on the Cartan subalgebra, we can assign to every root α a uniquely determined element $h'_\alpha \in \mathfrak{h}$ via the equation

$$\kappa(h, h'_\alpha) = \alpha(h). \tag{5.8}$$

Further, we can introduce a bilinear form on $\mathfrak{h}^*$ via

$$(\alpha, \beta) := \kappa(h'_\alpha, h'_\beta) = \alpha(h'_\beta) = \beta(h'_\alpha). \tag{5.9}$$

Lemma 5.3.3. *For* $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$*,*

$$[x, y] = \kappa(x, y) h'_\alpha \quad \textit{for} \quad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}. \tag{5.10}$$

Proof. Recall that Proposition 5.3.2(ii) implies that $[h, x] = \alpha(h)x$ for $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$. Both sides of the equation are in $\mathfrak{h}$, hence (5.10) follows from

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(h, \kappa(x, y) h'_\alpha),$$

since the Cartan–Killing form is nondegenerate on $\mathfrak{h}$ by Proposition 5.3.2. □

5.3.1 $\mathfrak{sl}_2$-Triples

The following theorem is the starting point of a complete classification of simple Lie algebras. It emphasizes the special role which the algebra $\mathfrak{sl}_2(\mathbb{K})$ plays in the theory.

Theorem 5.3.4 ($\mathfrak{sl}_2$-Theorem). *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a splitting Cartan subalgebra and $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$.*

(i) *For every root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, we have $(\alpha, \alpha) \neq 0$ and there are elements $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that*

$$[h_\alpha, e_\alpha] = 2e_\alpha,\ [h_\alpha, f_\alpha] = -2f_\alpha \quad \textit{and} \quad [e_\alpha, f_\alpha] = h_\alpha.$$

In particular, $\mathfrak{g}(\alpha) := \operatorname{span}\{h_\alpha, e_\alpha, f_\alpha\}$ is a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{K})$.

(ii) $m_\alpha = \dim \mathfrak{g}_\alpha = \dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = 1$ *and* $\mathbb{Z}\alpha \cap \Delta = \{\pm\alpha\}$.

(iii) $\alpha(h_\alpha) = 2$.

Proof. (i) From $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq \{0\}$ and Lemma 5.3.3 we obtain elements $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ with $[e_\alpha, e_{-\alpha}] = h'_\alpha$. To see that $(\alpha, \alpha) = \alpha(h'_\alpha)$ is nonzero, let us assume the contrary and consider some $\beta \in \Delta$. Then the subspace

$$V := \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$$

of $\mathfrak{g}$ is invariant under $\operatorname{ad}(e_{\pm\alpha})$, so that

$$\begin{aligned} 0 = \operatorname{tr}([\operatorname{ad} e_\alpha|_V, \operatorname{ad} e_{-\alpha}|_V]) &= \operatorname{tr}(\operatorname{ad} h'_\alpha|_V) = \sum_{k \in \mathbb{Z}} (\beta + k\alpha)(h'_\alpha) \cdot m_{\beta + k\alpha} \\ &= \beta(h'_\alpha) \cdot \sum_{k \in \mathbb{Z}} m_{\beta + k\alpha}. \end{aligned}$$

Since $\sum_{k \in \mathbb{Z}} m_{\beta + k\alpha} \geq m_\beta > 0$, we get $\beta(h'_\alpha) = 0$ for all roots β. But since the roots span $\mathfrak{h}^*$ (Proposition 5.3.2), this contradicts $h'_\alpha \neq 0$. We conclude that $(\alpha, \alpha) = \alpha(h'_\alpha) \neq 0$.

Set $h_\alpha := 2\frac{h'_\alpha}{\alpha(h'_\alpha)}$. This proves (iii). From Proposition 5.3.2 we get an element $f_\alpha \in \mathfrak{g}_{-\alpha}$ with $\kappa(e_\alpha, f_\alpha) = \frac{2}{\alpha(h'_\alpha)}$, so that Lemma 5.3.3 implies that $[e_\alpha, f_\alpha] = h_\alpha$. Now (i) follow from (iii).

(ii) We consider the subspace

$$V := \mathbb{K}f_\alpha + \mathfrak{h} + \sum_{n=1}^{\infty} \mathfrak{g}_{n\alpha}$$

of $\mathfrak{g}$. One verifies easily that this subspace is invariant under $\operatorname{ad}\big(\mathfrak{g}(\alpha)\big)$ because it is invariant under $\operatorname{ad}\mathfrak{h}$, $[f_\alpha, \mathfrak{h}] = \mathbb{K}f_\alpha$, and $[e_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\beta+\alpha}$. According to Lemma 5.2.9, we therefore have

$$\dim V_m(\operatorname{ad} h_\alpha) = \dim V_{-m}(\operatorname{ad} h_\alpha)$$

for all $m \in \mathbb{Z}$. This leads to

$$\dim \mathfrak{g}_\alpha = \dim V_2(\operatorname{ad} h_\alpha) = \dim V_{-2}(\operatorname{ad} h_\alpha) = 1$$

and $\dim \mathfrak{g}_{n\alpha} = 0$ for $n > 1$. Now (ii) follows from (i). ☐

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a splitting Cartan subalgebra. Then $\mathfrak{h}$ is splitting and we obtain a root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For brevity we put $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$. For a root $\alpha \in \Delta$, we have already seen that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one-dimensional subspace of $\mathfrak{h}$ on which α does not vanish. Hence there is a unique element

$$\check{\alpha} = h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \quad \text{with} \quad \alpha(\check{\alpha}) = 2,$$

called the *coroot* corresponding to α (cf. Theorem 5.3.4).

Lemma 5.3.5 (Root String Lemma). *Let $\alpha, \beta \in \Delta$.*

(i) *For $\beta \in \Delta \setminus \{\pm\alpha\}$ the set $\{k \in \mathbb{Z} \colon \beta + k\alpha \in \Delta\}$ is an interval in $\mathbb{Z}$. If it is of the form $[-p, q] \cap \mathbb{Z}$ with $p, q \in \mathbb{Z}$, then $p - q = \beta(\check{\alpha})$. In particular, $\beta(\check{\alpha}) \in \mathbb{Z}$.*
(ii) *If $\beta(\check{\alpha}) < 0$, then $\beta + \alpha \in \Delta$ and if $\beta(\check{\alpha}) > 0$, then $\beta - \alpha \in \Delta$.*
(iii) *If $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$, then $\beta(\check{\alpha}) \geq 0$.*
(iv) *If $\alpha + \beta \neq 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. (i) We consider the subspace $V := \sum_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$. Note that $\beta \neq \{\pm\alpha\}$ implies that 0 is not contained in $\beta + \mathbb{Z}\alpha$ (Theorem 5.3.4(ii)). From $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \subseteq \mathfrak{g}_{\gamma+\delta}$, we derive that V is a $\mathfrak{g}(\alpha)$-submodule of $\mathfrak{g}$. The eigenvalues of $\check{\alpha} = h_\alpha$ on V are given by

$$\mathcal{P}_V := \{(\beta + k\alpha)(\check{\alpha}) \colon \beta + k\alpha \in \Delta\} = \beta(\check{\alpha}) + 2\{k \colon \beta + k\alpha \in \Delta\}.$$

Hence the string property of $\mathfrak{sl}_2$-modules (Proposition 5.2.8) implies the string property of the root system.

Next we note that $\beta \in \Delta$ leads to $p \geq 0$. In view of Proposition 5.2.8, we have $\mathcal{P}_V = -\mathcal{P}_V$. Therefore

$$\beta(\check{\alpha}) - 2p = (\beta - p\alpha)(\check{\alpha}) = -(\beta + q\alpha)(\check{\alpha}) = -\beta(\check{\alpha}) - 2q.$$

(ii) If $\beta(\check{\alpha}) < 0$, then (i) leads to $q > 0$ and hence to $\beta + \alpha \in \Delta$. The second assertion follows similarly.
(iii) As all multiplicities of the eigenvalues of $\check{\alpha}$ on V are 1 (Theorem 5.3.4(ii)), the $\mathfrak{sl}_2(\mathbb{K})$-module V is simple and isomorphic to $L(\beta(\check{\alpha}) + 2q)$ (apply Proposition 5.2.4 to a nonzero element of $\mathfrak{g}_{\beta+q\alpha}$). This immediately shows that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{\beta+k\alpha}] = \mathfrak{g}_{\beta+(k+1)\alpha} \quad \text{for} \quad k = -p, -p+1, \ldots, q-1. \tag{5.11}$$

If $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$, then $V := \sum_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$ is invariant under $\mathfrak{g}(\alpha) \cong \mathfrak{sl}_2(\mathbb{K})$ and $\beta(\check{\alpha})$ is the maximal eigenvalue of $\mathrm{ad}(\check{\alpha})$ on V. Hence Proposition 5.2.8 shows that $\beta(\check{\alpha}) \geq 0$.
(iv) We may assume that $\beta + \alpha \in \Delta$ (otherwise $\mathfrak{g}_{\alpha+\beta} = \{0\}$), so that $q \geq 1$. Then (iv) follows from (5.11). □

Lemma 5.3.6. *Each root $\alpha \in \Delta$ satisfies $\mathbb{K}\alpha \cap \Delta = \{\pm\alpha\}$.*

Proof. Suppose that α and $c\alpha$ lie in Δ for some $c \in \mathbb{K}$. By Lemma 5.3.5, $2c = c\alpha(\check{\alpha}) \in \mathbb{Z}$, so that $c \in \frac{1}{2}\mathbb{Z}$. For symmetry reasons, also $c^{-1} \in \frac{1}{2}\mathbb{Z}$, so that $c \in \{\pm 2, \pm 1, \pm\frac{1}{2}\}$. From the $\mathfrak{sl}_2$-Theorem 5.3.4 we know already that $\mathbb{Z}\alpha \cap \Delta = \{\pm\alpha\}$, which rules out the cases $c = \pm 2$. The cases $c = \pm\frac{1}{2}$ are likewise ruled out by applying the same argument to $c\alpha$ instead. □

Lemma 5.3.7. *The subspace $\mathfrak{h}_\mathbb{R} := \mathrm{span}\{\check{\alpha} \colon \alpha \in \Delta\}$ has the following properties:*

(i) *$\alpha(\mathfrak{h}_\mathbb{R}) = \mathbb{R}$ for every root $\alpha \in \Delta$.*
(ii) *κ is positive definite on $\mathfrak{h}_\mathbb{R}$.*
(iii) *$\mathfrak{h}_\mathbb{R}$ spans the space $\mathfrak{h}$ ($\mathfrak{h}_\mathbb{R} = \mathfrak{h}$ for $\mathbb{K} = \mathbb{R}$).*
(iv) *$\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$ if $\mathbb{K} = \mathbb{C}$.*

Proof. (i) This follows from $\beta(\check{\alpha}) \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$, which is a consequence of the Root String Lemma 5.3.5.
(ii) This follows from Proposition 5.3.1 and (i).
(iii) If $\beta \in \mathfrak{h}^*$ vanishes on $\mathfrak{h}_\mathbb{R}$, then $0 = \beta(\check{\alpha}) = \kappa(h'_\beta, \check{\alpha})$ for each root α, and since $\check{\alpha}$ is a nonzero multiple of h'_α, this implies that $\alpha(h'_\beta) = 0$. As Δ spans $\mathfrak{h}^*$, we get $\beta = 0$, and this implies that $\mathrm{span}_\mathbb{C}\, \mathfrak{h}_\mathbb{R} = \mathfrak{h}$ for $\mathbb{K} = \mathbb{C}$.
(iv) $0 \leq \kappa(x, x)$ and $0 \leq \kappa(ix, ix) = -\kappa(x, x)$ for $x \in \mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R}$. Hence $\kappa(x, x) = 0$, and therefore, $x = 0$ by (ii). □

5.3.2 Examples

The following lemma is a useful tool to see that the examples discussed below are indeed semisimple Lie algebras.

Lemma 5.3.8. *Suppose that $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ is a Lie algebra and $\mathfrak{h}$ a toral Cartan subalgebra, such that*

(i) *$\mathfrak{g}(\alpha) := \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2(\mathbb{K})$ for each root α, and*
(ii) *$\mathfrak{z}(\mathfrak{g}) = \{0\}$.*

Then $\mathfrak{g}$ is semisimple.

Proof. Let $\mathfrak{r} := \mathrm{rad}(\mathfrak{g})$ be the solvable radical of $\mathfrak{g}$. As an ideal, it is $\mathfrak{h}$-invariant, hence adapted to the root space decomposition: $\mathfrak{r} = \mathfrak{r}_0 + \sum_\alpha \mathfrak{r}_\alpha$ (Exercise 1.1.1). Since all semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ intersect $\mathfrak{r}$ trivially (otherwise $\mathfrak{s} \cap \mathfrak{r}$ would be a nontrivial solvable ideal of $\mathfrak{s}$), $\mathfrak{r}_\alpha \subseteq \mathfrak{g}(\alpha) \cap \mathfrak{r} = \{0\}$. Hence $\mathfrak{r} \subseteq \mathfrak{h}$, and, in view of $[\mathfrak{r}, \mathfrak{g}_\alpha] \subseteq \mathfrak{r} \cap \mathfrak{g}_\alpha = \{0\}$, we get $\mathfrak{r} \subseteq \bigcap_{\alpha \in \Delta} \ker \alpha = \mathfrak{z}(\mathfrak{g}) = \{0\}$. □

Example 5.3.9 (The special linear Lie algebra). Let

$$\mathfrak{g} := \mathfrak{sl}_n(\mathbb{K}) := \{x \in \mathfrak{gl}_n(\mathbb{K})\colon \operatorname{tr} x = 0\}.$$

Then the subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$ is abelian, and for the matrix units E_{ij} with a single nonzero entry 1 in position (i, j), we have

$$[\operatorname{diag}(h), E_{ij}] = (h_i - h_j)E_{ij} \quad \text{for} \quad h \in \mathbb{K}^n.$$

We conclude that $\mathfrak{h}$ is maximal abelian, $\mathfrak{g}^0(\mathfrak{h}) = \mathfrak{g}_0(\mathfrak{h}) = \mathfrak{h}$, so that $\mathfrak{h}$ is a Cartan subalgebra, and that the one-dimensional subspace $\mathbb{K}E_{ij}$ is a root space corresponding to the root $\varepsilon_i - \varepsilon_j$, where $\varepsilon_i(\operatorname{diag}(h)) := h_i$. The corresponding root system is

$$A_{n-1} := \{\varepsilon_j - \varepsilon_k : 1 \leq j \neq k \leq n\}.$$

Example 5.3.10 (The orthogonal Lie algebras). Let

$$I_{n,n} := \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \in M_{2n}(\mathbb{K}) \cong M_2(M_n(\mathbb{K}))$$

and recall the Lie algebra

$$\mathfrak{o}_{n,n}(\mathbb{K}) = \{x \in \mathfrak{gl}_{2n}(\mathbb{K})\colon x^\top I_{n,n} + I_{n,n}x = 0\}.$$

In terms of block matrices, we then have

$$\mathfrak{o}_{n,n}(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(M_n(\mathbb{K}))\colon a^\top = -a, d^\top = -d, b^\top = c \right\}.$$

In this matrix presentation, it is quite inconvenient to describe a root decomposition of this Lie algebra. It is much simpler to use an equivalent description, based on the following observation. For

$$g := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}$$

we have

$$g^\top I_{n,n} g = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} =: S.$$

Hence $x \in \mathfrak{o}_{n,n}(\mathbb{K})$ is equivalent to $g^{-1}xg$ being contained in

$$\mathfrak{g} := \mathfrak{o}(\mathbb{K}^{2n}, S) := \{x \in \mathfrak{gl}_{2n}(\mathbb{K})\colon x^\top S + Sx = 0\} = g^{-1}\mathfrak{o}_{n,n}(\mathbb{K})g.$$

From

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{o}(\mathbb{K}^{2n}, S) \quad \Leftrightarrow \quad d = -a^\top, b^\top = -b, c^\top = -c,$$

we immediately derive that $\mathfrak{g}$ has a root decomposition with respect to the maximal abelian subalgebra

$$\mathfrak{h} = \operatorname{span}\{E_{jj} - E_{n+j,n+j} : j = 1, \ldots, n\} = \{\operatorname{diag}(h, -h) \colon h \in \mathbb{K}^n\}.$$

The corresponding root system is

$$D_n := \{\pm\varepsilon_j \pm \varepsilon_k : j, k = 1, \ldots, n, j \neq k\},$$

where $\varepsilon_j \colon \mathfrak{h} \to \mathbb{K}$ is the linear functional defined by $\varepsilon_k(\operatorname{diag}(h, -h)) := h_k$. Here the roots in the subsystem A_{n-1} of D_n correspond to the root spaces in the image of the embedding

$$\mathfrak{sl}_n(\mathbb{K}) \to \mathfrak{o}(\mathbb{K}^{2n}, S), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & -x^\top \end{pmatrix}.$$

Further, $\mathfrak{g}_{\varepsilon_j+\varepsilon_k} = \mathbb{K}(E_{j,n+k} - E_{k,n+j})$ and $\mathfrak{g}_{-\varepsilon_j-\varepsilon_k} = \mathbb{K}(E_{n+k,j} - E_{n+j,k})$.

For the symmetric matrix

$$T := \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_{2n+1}(\mathbb{K}),$$

we also obtain a Lie algebra

$$\mathfrak{o}(\mathbb{K}^{2n+1}, T) := \{x \in \mathfrak{gl}_{2n+1}(\mathbb{K}) \colon x^\top T + Tx = 0\}.$$

Then

$$\begin{pmatrix} a & b & x \\ c & d & y \\ \widetilde{x} & \widetilde{y} & z \end{pmatrix} \in \mathfrak{o}(\mathbb{K}^{2n+1}, T) \quad \Leftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{o}(\mathbb{K}^{2n}, S), \widetilde{x} = -y^\top, \widetilde{y} = -x^\top, z = 0$$

implies that this Lie algebra has a root decomposition with respect to the maximal abelian subalgebra

$$\mathfrak{h} = \operatorname{span}\{E_{jj} - E_{n+j,n+j} : j = 1, \ldots, n\} = \{\operatorname{diag}(h, -h, 0) \colon h \in \mathbb{K}^n\}.$$

The corresponding root system is

$$B_n := \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k = 1, \ldots, n, j \neq k\},$$

where $\varepsilon_j \colon \mathfrak{h} \to \mathbb{K}$ is the linear functional defined by $\varepsilon_k(\operatorname{diag}(h, -h, 0)) := h_k$. Here the root spaces corresponding to roots in the subsystem D_n of B_n correspond to root spaces in the subalgebra $\mathfrak{o}(\mathbb{K}^{2n}, S)$ (corresponding to $x = y = 0$), and

$$\mathfrak{g}_{\varepsilon_j} = \mathbb{K}(E_{j,2n+1} - E_{2n+1,n+j}) \quad \text{and} \quad \mathfrak{g}_{-\varepsilon_j} = \mathbb{K}(E_{j+n,2n+1} - E_{2n+1,j}).$$

Example 5.3.11 (The symplectic Lie algebra). For the skew symmetric matrix

$$J := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \in M_{2n}(\mathbb{K}),$$

we obtain the symplectic Lie algebra

$$\mathfrak{sp}_n(\mathbb{K}) = \{x \in \mathfrak{gl}_{2n}(\mathbb{K}) \colon x^\top J + Jx = 0\}.$$

Using

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp}_n(\mathbb{K}) \quad \Leftrightarrow \quad d = -a^\top, b^\top = b, c^\top = c,$$

we see that $\mathfrak{g} := \mathfrak{sp}_n(\mathbb{K})$ has a root decomposition with respect to the maximal abelian subalgebra

$$\mathfrak{h} = \operatorname{span}\{E_{jj} - E_{n+j,n+j} : j = 1, \ldots, n\} = \{\operatorname{diag}(h, -h) \colon h \in \mathbb{K}^n\}.$$

The corresponding root system is

$$C_n := \{\pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k = 1, \ldots, n, j \neq k\},$$

where $\varepsilon_j \colon \mathfrak{h} \to \mathbb{K}$ is the linear functional defined by $\varepsilon_k(\operatorname{diag}(h, -h)) := h_k$. Again, the roots in the subsystem A_{n-1} of C_n correspond to the root spaces in the image of the embedding

$$\mathfrak{sl}_n(\mathbb{K}) \to \mathfrak{sp}_n(\mathbb{K}), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & -x^\top \end{pmatrix}.$$

Further,

$$\mathfrak{g}_{\varepsilon_j + \varepsilon_k} = \mathbb{K}(E_{j,n+k} + E_{k,n+j}) \quad \text{and} \quad \mathfrak{g}_{-\varepsilon_j - \varepsilon_k} = \mathbb{K}(E_{n+k,j} + E_{n+j,k}).$$

Example 5.3.12 (The odd symplectic Lie algebra). There is also an "odd-dimensional" version of the symplectic Lie algebra, which is not semisimple, due to the fact that no skew-symmetric form on an odd-dimensional space is nondegenerate.

For the skew symmetric matrix

$$J := \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_{2n+1}(\mathbb{K}),$$

we define the *odd symplectic Lie algebra*

$$\mathfrak{sp}_{n+\frac{1}{2}}(\mathbb{K}) := \{x \in \mathfrak{gl}_{2n+1}(\mathbb{K}) \colon x^\top J + Jx = 0\}.$$

Then

$$\begin{pmatrix} a & b & \widetilde{x} \\ c & d & \widetilde{y} \\ x & y & z \end{pmatrix} \in \mathfrak{sp}_{n+\frac{1}{2}}(\mathbb{K}) \quad \Leftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp}_n(\mathbb{K}), \widetilde{x} = \widetilde{y} = 0,$$

we see that $\mathfrak{g} := \mathfrak{sp}_{n+\frac{1}{2}}(\mathbb{K})$ has a root decomposition with respect to the maximal abelian subalgebra

$$\mathfrak{h} = \{\mathrm{diag}(h, -h, z) \colon h \in \mathbb{K}^n, z \in \mathbb{K}\}.$$

The corresponding root system is

$$\{\widetilde{\varepsilon}_j^{\pm}, \pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k = 1, \ldots, n, j \neq k\},$$

where $\varepsilon_j \colon \mathfrak{h} \to \mathbb{K}$ is the linear functional defined by $\varepsilon_k(\mathrm{diag}(x, -x, z)) := x_k$ and $\widetilde{\varepsilon}_k^{\pm}(\mathrm{diag}(x, -x, z)) := z \pm x_k$. Here the subsystem C_n corresponds to the subalgebra $\mathfrak{sp}_n(\mathbb{K})$ (defined by $\widetilde{x} = \widetilde{y} = 0$). Further,

$$\mathfrak{g}_{\widetilde{\varepsilon}_j} = \mathbb{K}E_{2n+1,n+j} \quad \text{and} \quad \mathfrak{g}_{-\widetilde{\varepsilon}_j} = \mathbb{K}E_{2n+1,j}.$$

Note that

$$\mathfrak{r} := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & z \end{pmatrix} : x, y \in \mathbb{K}^n, z \in \mathbb{K} \right\}$$

is a nilpotent ideal and that

$$\mathfrak{sp}_{n+\frac{1}{2}}(\mathbb{K}) \cong \mathfrak{r} \rtimes \mathfrak{sp}_n(\mathbb{K})$$

is a Levi decomposition. Restricting the roots to the Cartan subalgebra $\mathfrak{h}_s := \mathfrak{h} \cap \mathfrak{sp}_n(\mathbb{K})$, we obtain the (nonreduced) root system

$$BC_n := \{\pm\varepsilon_j, \pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k = 1, \ldots, n, j \neq k\}.$$

5.4 Abstract Root Systems and their Weyl Groups

In the previous section we proved a number of results on the set of roots associated with a given splitting Cartan subalgebra of a semisimple Lie algebra. In this section we distill some of these properties into the definition of an abstract root system and show how to derive further properties using this concept.

Definition 5.4.1. Let E be a *euclidian space*, i.e., a finite-dimensional real vector space with an inner product $(\cdot, \cdot)$, i.e., a positive definite symmetric bilinear form. A *reflection* in E is a linear map $\sigma \in \mathrm{End}(E)$ which induces the map $-\,\mathrm{id}$ on a line $\mathbb{R}\alpha$ and the identity on the hyperplane

$$\alpha^{\perp} = \{\beta \in E \mid (\beta, \alpha) = 0\}.$$

If $\alpha \in E \setminus \{0\}$ is given, then we write $P_\alpha := \alpha^\perp$ and σ_α for σ. Then

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

for all $\beta \in E$.

Lemma 5.4.2. *Let E be a Euclidian space and $\Phi \subseteq E$ be a finite subset which spans E, and which is invariant under all reflections σ_α with $\alpha \in \Phi$. Suppose that $\sigma \in \mathrm{GL}(E)$ fixes a hyperplane P pointwise and maps some nonzero $\alpha \in \Phi$ to $-\alpha$. If σ leaves Φ invariant, then $\sigma = \sigma_\alpha$.*

Proof. Let $\tau := \sigma\sigma_\alpha \in \mathrm{GL}(E)$. Then $\tau(\Phi) = \Phi$ (since Φ is finite) and $\tau(\alpha) = \alpha$. The linear automorphism $\widetilde{\tau} \in \mathrm{GL}(E/\mathbb{R}\alpha)$ induced by τ coincides with the linear automorphism $\widetilde{\sigma} \in \mathrm{GL}(E/\mathbb{R}\alpha)$ induced by σ. Since $E = P + \mathbb{R}\alpha$, our assumption yields $\widetilde{\sigma} = \mathrm{id}$. Together, we obtain that τ is split and 1 is its only eigenvalue. Therefore $\tau_s = \mathrm{id}$, so that $\tau - \mathrm{id}$ is nilpotent, i.e., τ is unipotent. On the other hand, the τ-invariance of the finite set Φ shows that there has to be a power τ^k which keeps Φ pointwise fixed. But Φ spans E, so that $\tau^k = \mathrm{id}$. As τ is unipotent, it follows that $\tau = \mathrm{id}$. □

Definition 5.4.3. Let E be a Euclidian space and $\Delta \subseteq E \setminus \{0\}$ be a finite subset which spans E. Then Δ is called a *reduced root system* if it satisfies the following conditions

(R1) $\Delta \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Delta$,
(R2) $\sigma_\alpha(\Delta) \subseteq \Delta$ for all $\alpha \in \Delta$,
(R3) For $\alpha \in \Delta$ the coroot $\check{\alpha} := \frac{2\alpha}{(\alpha,\alpha)}$ satisfies $(\beta, \check{\alpha}) \in \mathbb{Z}$ for all $\beta \in \Delta$.

It is called *root system* if it only satisfies (R2) and (R3). If Δ is a root system, then we call the group $W = W(\Delta)$ generated by the reflections σ_α, $\alpha \in \Delta$, the *Weyl group* of the root system.

Remark 5.4.4. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a splitting Cartan subalgebra. Recall the Euclidian space $\mathfrak{h}_\mathbb{R}$ from Lemma 5.3.7 together with the finite subset $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ of all roots with respect to $\mathfrak{h}$. Then Lemma 5.3.6 shows that $\Delta \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Delta$. Moreover, Lemma 5.3.5 yields $(\beta, \check{\alpha}) \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$. The same lemma also shows

$$\beta - (\beta, \check{\alpha})\alpha \in \Delta,$$

since, in the notation of Lemma 5.3.5, we have $r \leq -(\beta, \check{\alpha}) \leq s$. Thus Δ is a reduced root system. In particular we see that to every pair $(\mathfrak{g}, \mathfrak{h})$ consisting of a semisimple complex Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$, a *Weyl group* $W(\mathfrak{g}, \mathfrak{h}) := W(\Delta)$ is assigned in a canonical way.

Remark 5.4.5. If Δ is a nonreduced root system and $\alpha, c\alpha \in \Delta$ for some $c > 1$, then

$$c \cdot (\alpha, \check{\alpha}) = 2c \in \mathbb{Z}$$

implies that $c \in \frac{1}{2}\mathbb{Z}$. Further, $(c\alpha)\check{} = \frac{1}{c}\check{\alpha}$ leads to $\frac{1}{c}(\alpha, \check{\alpha}) = \frac{2}{c} \in \mathbb{Z}$, so that $c = 2$. We therefore get

$$\Delta \cap \mathbb{R}\alpha = \{\pm\alpha, \pm 2\alpha\}.$$

Remark 5.4.6. The angle $\theta \in [0, \pi]$ between α and β is defined by the identity

$$\|\alpha\| \, \|\beta\| \cos\theta = (\alpha, \beta),$$

where $\|\alpha\| = \sqrt{(\alpha, \alpha)}$ is the norm of the Euclidian space E. We have

$$2\frac{\|\beta\|}{\|\alpha\|}\cos\theta = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = (\beta, \check{\alpha}) \quad \text{and} \quad (\alpha, \check{\beta})(\beta, \check{\alpha}) = 4\cos^2\theta.$$

Hence, if $(\beta, \check{\alpha}), (\alpha, \check{\beta}) \in \mathbb{Z}$, then we also have $4\cos^2\theta \in \mathbb{Z}$, and there are only the following possibilities, provided $\|\alpha\| \leq \|\beta\|$:

$(\alpha, \check{\beta})$	0	1	−1	1	−1	1	−1	1	−1	2	−2
$(\beta, \check{\alpha})$	0	1	−1	2	−2	3	−3	4	−4	2	−2
θ	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	0	π	0	π
$\frac{\|\beta\|^2}{\|\alpha\|^2}$	arb.	1	1	2	2	3	3	4	4	1	1

Proposition 5.4.7. *Let $\Delta \subseteq E$ be a root system with Weyl group W. If $\tau \in \mathrm{GL}(E)$ leaves Δ invariant, then*

(i) $\tau\sigma_\alpha\tau^{-1} = \sigma_{\tau\alpha}$ *for all* $\alpha \in \Delta$.
(ii) $(\beta, \check{\alpha}) = \big(\tau(\beta), \tau(\alpha)\check{}\big)$ *for all* $\alpha, \beta \in \Delta$.

Proof. (i) Note that $\tau\sigma_\alpha\tau^{-1}(\tau(\beta)) = \tau\sigma_\alpha(\beta) \in \tau(\Delta) \subseteq \Delta$. Since Δ is finite and $\tau \colon \Delta \to \Delta$ is injective, we have $\tau(\Delta) = \Delta$. Hence we also have $\tau\sigma_\alpha\tau^{-1}(\Delta) \subseteq \Delta$. Further, $\tau\sigma_\alpha\tau^{-1}$ keeps the hyperplane $\tau(\alpha^\perp)$ pointwise fixed, and it maps $\tau\alpha$ to $-\tau\alpha$. Hence Lemma 5.4.2 shows that $\tau\sigma_\alpha\tau^{-1} = \sigma_{\tau\alpha}$.
(ii) In view of (i), this follows by comparison of the formulas

$$\tau\sigma_\alpha\tau^{-1}(\tau(\beta)) = \tau(\beta - (\beta, \check{\alpha})\alpha) = \tau(\beta) - (\beta, \check{\alpha})\tau(\alpha)$$

and $\sigma_{\tau(\alpha)}(\tau(\beta)) = \tau(\beta) - \big(\tau(\beta), \tau(\alpha)\check{}\big)\tau(\alpha)$. □

Lemma 5.4.8. *Let Δ be a root system, and suppose that $\alpha, \beta \in \Delta$ are not proportional. If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Delta$.*

Proof. Since (α, β) is positive if and only if $(\alpha, \check{\beta})$ is positive, Remark 5.4.6 shows that we have $(\alpha, \check{\beta}) = 1$ or $(\beta, \check{\alpha}) = 1$. If $(\alpha, \check{\beta}) = 1$, then $\alpha - \beta = \sigma_\beta(\alpha) \in \Delta$. Similarly, for $(\beta, \check{\alpha}) = 1$, we have $\beta - \alpha = \sigma_\alpha(\beta) \in \Delta$, hence $\alpha - \beta \in \Delta$. □

5.4.1 Simple Roots

Definition 5.4.9. Let $\Delta \subseteq E$ be a root system and $\Pi \subseteq \Delta$. Then Π is called a *basis* for Δ if Π is a basis for the vector space E, and if every root $\beta \in \Delta$ is of the form $\beta = \sum_{\alpha\in\Pi} k_\alpha \alpha$, where either all $k_\alpha \in \mathbb{N}_0$, or all $k_\alpha \in -\mathbb{N}_0$. In this case, the elements of Π are called *simple roots*. The *height* of the root $\beta = \sum_{\alpha\in\Pi} k_\alpha \alpha$ is the number $\sum_{\alpha\in\Pi} k_\alpha$. Roots with positive height are called *positive* and roots with negative height are called *negative*. The set of positive roots is denoted Δ^+, the set of negative roots by Δ^-. We define a partial order $\prec$ on E by

$$\alpha \prec \beta \quad :\Longleftrightarrow \quad \beta - \alpha \in \sum_{\gamma\in\Delta^+} \mathbb{N}_0 \gamma.$$

Lemma 5.4.10. *Let Δ be a root system and Π be a basis for Δ. Suppose that $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$. Then $(\alpha, \beta) \leq 0$ and $\alpha - \beta$ is not a root.*

Proof. Since Π is a basis for E, α and β cannot be proportional. If we have $(\alpha, \beta) > 0$, then Lemma 5.4.8 shows that $\alpha - \beta \in \Delta$. But this contradicts the definition of a basis for Δ. □

Lemma 5.4.11. *Let $M \subseteq E$ be contained in an open half space of E, i.e., there is a $\lambda \in E$ with $(\lambda, \alpha) > 0$ for all $\alpha \in M$, and $(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in M$ with $\alpha \neq \beta$. Then M is linearly independent.*

Proof. Suppose that $\sum_{\alpha\in M} r_\alpha \alpha = 0$ with $r_\alpha \in \mathbb{R}$, and set

$$M_\pm := \{\alpha \in M \mid \pm r_\alpha > 0\}.$$

Then

$$\nu := \sum_{\alpha\in M_+} |r_\alpha|\alpha = \sum_{\beta\in M_-} |r_\beta|\beta,$$

and therefore

$$(\nu, \nu) = \sum_{\alpha\in M_+, \beta\in M_-} |r_\alpha r_\beta|(\alpha, \beta) \leq 0$$

which leads to $\nu = 0$. But we then have $0 = (\lambda, \nu) = \sum_{\alpha\in M_\pm} |r_\alpha|(\lambda, \alpha)$, which implies $M_\pm = \emptyset$ since we otherwise arrive at the contradiction $r_\alpha = 0$ for some $\alpha \in M_\pm$. □

Definition 5.4.12. (a) Let $\Delta \subseteq E$ be a root system and $\lambda \in E$. Then λ is called *regular* if $\lambda \notin \alpha^\perp$ holds for all $\alpha \in \Delta$. Otherwise, λ is called *singular*. The connected components of the set of the regular elements are called *Weyl chambers*. The Weyl chamber which contains the regular element $\lambda \in E$ is denoted by $\mathcal{C}(\lambda)$.

(b) For any regular element $\lambda \in E$, the set

$$\Delta^+(\lambda) := \{\alpha \in \Delta \mid (\lambda, \alpha) > 0\}$$

is called the corresponding *positive system*. An element $\alpha \in \Delta^+(\lambda)$ is called *decomposable* if there are $\beta_1, \beta_2 \in \Delta^+(\lambda)$ with $\alpha = \beta_1 + \beta_2$, otherwise, it is called *indecomposable*.

Theorem 5.4.13. *For each regular element $\lambda \in E$, the set $\Pi := \Pi(\lambda)$ of indecomposable elements in $\Delta^+(\lambda)$ is a basis for Δ. Conversely, every basis is of this form.*

Proof. **Claim 1**: $\Pi(\lambda)$ is a basis for Δ.

First, we show that every element of $\Delta^+(\lambda)$ can be written as a linear combination of elements of $\Pi(\lambda)$ with coefficients in $\mathbb{N}_0$. For this, we suppose that $\alpha \in \Delta^+(\lambda)$ cannot be written in this form, and that it has the smallest (α, λ) of all elements with this property. In particular, $\alpha \notin \Pi(\lambda)$, and there exist $\beta_1, \beta_2 \in \Delta^+(\lambda)$ with $\alpha = \beta_1 + \beta_2$, hence, $(\alpha, \lambda) = (\beta_1, \lambda) + (\beta_2, \lambda)$. But since $(\beta_i, \lambda) > 0$ for $i = 1, 2$, the minimality of (α, λ) shows that the β_i can be written as linear combination of elements of $\Pi(\lambda)$ with coefficients in $\mathbb{N}_0$. Then this also holds for α, contradicting our assumption.

As a consequence, we see that every $\beta \in \Delta$ is of the form $\beta = \sum_{\alpha \in \Pi(\lambda)} k_\alpha \alpha$, where either all $k_\alpha \in \mathbb{N}_0$ or all $k_\alpha \in -\mathbb{N}_0$. Since Δ spans the space E, it remains to be shown that $\Pi(\lambda)$ is linearly independent.

Next we show that $(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in \Pi(\lambda)$ with $\alpha \neq \beta$. In fact, if $(\alpha, \beta) > 0$, then $\alpha \notin -\mathbb{R}^+\beta$. By Remark 5.4.5, α and β can only be proportional if $\alpha = 2\beta$ or $\beta = 2\alpha$ which would contradict the indecomposability of these elements. Hence we can apply Lemma 5.4.8, and we get $\alpha - \beta \in \Delta = \Delta^+(\lambda) \cup -\Delta^+(\lambda)$. If $\alpha - \beta \in \Delta^+(\lambda)$, then $\alpha = \beta + (\alpha - \beta)$ which contradicts the assumption that α is indecomposable. Similarly, $\beta - \alpha \in \Delta^+(\lambda)$ gives a contradiction by $\beta = \alpha + (\beta - \alpha)$. Now Claim 1 follows by Lemma 5.4.11, applied to $M = \Pi(\lambda)$.

Claim 2: Every basis Π for Δ is of the form $\Pi(\lambda)$ for some regular element $\lambda \in E$.

We arrange Π in the form $\alpha_1, \ldots, \alpha_n$, and consider the dual basis $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n$ for E which is given by $(\alpha_i, \widetilde{\alpha}_j) = \delta_{ij}$. Set $\lambda := \sum_{j=1}^n \widetilde{\alpha}_j$. Then $(\lambda, \alpha_j) = 1$ for all $j = 1, \ldots, n$, so that $(\lambda, \alpha) > 0$ for all $\alpha \in \Pi$. Since every $\beta \in \Delta$ can be written as a linear combination of the $\alpha \in \Pi$ with coefficients of the same sign, λ is regular. Then the set Δ^+ of positive roots defined by Π satisfies $\Delta^+ \subseteq \Delta^+(\lambda)$ which leads to $\Delta^+ = \Delta^+(\lambda)$ because of $\Delta^+ \cup -\Delta^+ = \Delta = \Delta^+(\lambda) \cup -\Delta^+(\lambda)$. From the definition of a basis for Δ, we see that Π consists of indecomposable elements of $\Delta^+ = \Delta^+(\lambda)$, and therefore it is contained in $\Pi(\lambda)$. On the other hand, the cardinalities of Π and $\Pi(\lambda)$ are both equal to $n = \dim E$, since both sets are bases for E. This proves that $\Pi = \Pi(\lambda)$. □

5.4.2 Weyl Chambers

Remark 5.4.14. Let $\Delta \subseteq E$ be a root system and $\lambda, \lambda' \in E$ be regular elements.

(i) $\mathcal{C}(\lambda) = \mathcal{C}(\lambda') \quad \Leftrightarrow \quad \Delta^+(\lambda) = \Delta^+(\lambda') \quad \Leftrightarrow \quad \Pi(\lambda) = \Pi(\lambda')$.

(ii) By (i) and Theorem 5.4.13, there is a bijection between the set of Weyl chambers and the set of bases for Δ.

(iii) If $\Pi = \Pi(\lambda)$, then we call $\mathcal{C}(\Pi) := \mathcal{C}(\lambda)$ the *fundamental chamber* associated with the basis Π. It is given by

$$\begin{aligned}\mathcal{C}(\Pi) &= \{\beta \in E \mid (\beta, \alpha) > 0 \text{ for all } \alpha \in \Pi\} \\ &= \{\beta \in E \mid (\beta, \alpha) > 0 \text{ for all } \alpha \in \Delta^+\}.\end{aligned}$$

Lemma 5.4.15. *Let $\Delta \subseteq E$ be a root system and Π be a basis for Δ.*

(i) *For $\alpha \in \Delta^+ \setminus \Pi$, there exists a $\beta \in \Pi$ with $\alpha - \beta \in \Delta^+$.*

(ii) *If Δ is reduced and $\alpha \in \Pi$, then σ_α permutes the set $\Delta^+ \setminus \{\alpha\}$.*

(iii) *Let $\alpha_1, \ldots, \alpha_r \in \Pi$ and set $\sigma_i := \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_{r-1}(\alpha_r) \in \Delta^-$, then there is an $s \in \{1, \ldots, r-1\}$ such that*

$$\sigma_1 \cdots \sigma_r = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{r-1}.$$

Proof. (i) Suppose for all $\beta \in \Pi$, we have $(\alpha, \beta) \leq 0$. Then the set $\Pi \cup \{\alpha\}$ satisfies the assumptions of Lemma 5.4.11, hence is linearly independent. Since Π is a basis for E, this cannot be the case, i.e., there is a $\beta \in \Pi$ with $(\alpha, \beta) > 0$.

Case 1: α and β are not proportional. Then Lemma 5.4.8 shows that $\alpha - \beta$ is a root.

Case 2: α and β are proportional. Then Remark 5.4.5 shows that $\alpha = 2\beta$ since β is indecomposable. But then $\alpha - \beta = \beta$ is a root.

Since $\alpha \in \Delta^+ \setminus \Pi$, it is a linear combination of elements in Π with at least two positive (integral) coefficients. Subtracting β leaves at least one positive coefficient, so $\alpha - \beta$, being a root, has to be positive.

(ii) Let $\beta \in \Delta^+ \setminus \{\alpha\}$ and $\beta = \sum_{\gamma \in \Pi} k_\gamma \gamma$ with $k_\gamma \in \mathbb{N}_0$. Since Δ is reduced, β is not proportional to α. Hence there is a $\gamma \neq \alpha$ with $k_\gamma > 0$. Since

$$\sigma_\alpha(\beta) = \beta - (\beta, \check{\alpha})\alpha = (k_\alpha - (\beta, \check{\alpha}))\alpha + \sum_{\gamma \in \Pi \setminus \{\alpha\}} k_\gamma \gamma,$$

$\sigma_\alpha(\beta)$ has a positive coefficient k_γ in its representation as a linear combination of simple roots. Thus, all coefficients are nonnegative, and $\sigma_\alpha(\beta) \in \Delta^+$. By $\sigma_\alpha(\alpha) = -\alpha$, we also have $\sigma_\alpha(\beta) \neq \alpha$, i.e., $\sigma_\alpha(\beta) \in \Delta^+ \setminus \{\alpha\}$. Since the latter set is finite, the claim follows.

(iii) Set

$$\beta_i := \begin{cases} \sigma_{i+1} \cdots \sigma_{r-1}(\alpha_r) & \text{for } i = 0, \ldots, r-2, \\ \alpha_r & \text{for } i = r-1. \end{cases}$$

Then $\beta_0 \prec 0 \prec \beta_{r-1}$, and there is a minimal $s \in \{1, \ldots, r-1\}$ with $0 \prec \beta_s$. For this s, we have $\sigma_s(\beta_s) = \beta_{s-1} \prec 0$. In view of (ii), this shows $\beta_s = \alpha_s$. By Proposition 5.4.7, for $\sigma := \sigma_{s+1} \cdots \sigma_{r-1}$, we have

$$\sigma_s = \sigma_{\beta_s} = \sigma_{\sigma\alpha_r} = \sigma\sigma_r\sigma^{-1} = (\sigma_{s+1}\cdots\sigma_{r-1})\sigma_r(\sigma_{r-1}\cdots\sigma_{s+1}),$$

which shows the claim. □

Proposition 5.4.16. *Let $\Delta \subseteq E$ be a root system and Π be a basis for Δ.*

(i) *Every $\beta \in \Delta^+$ can be written in the form $\alpha_1 + \ldots + \alpha_m$ with $\alpha_j \in \Pi$ such that $\sum_{j=1}^k \alpha_j \in \Delta^+$ for each $k \in \{1, \ldots, m\}$.*
(ii) *Let Δ be reduced and $\rho := \frac{1}{2}\sum_{\beta\in\Delta^+}\beta$. Then $\sigma_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in \Pi$.*
(iii) *Let $\sigma = \sigma_1 \cdots \sigma_r$, where the $\sigma_j = \sigma_{\alpha_j}$ are reflections associated with the simple roots $\alpha_j \in \Pi$, and where r is the minimal number of factors needed to represent σ as such a product. Then $\sigma(\alpha_r) \in \Delta^-$.*

Proof. (i) This immediately follows by induction using Lemma 5.4.15(i).
(ii) In view of $\sigma_\alpha(\alpha) = -\alpha$, this is a direct consequence of Lemma 5.4.15(ii).
(iii) This immediately follows from $\sigma(\alpha_r) = \sigma_1 \cdots \sigma_r(\alpha_r) = \sigma_1 \cdots \sigma_{r-1}(-\alpha_r)$ and Lemma 5.4.15(iii). □

Theorem 5.4.17. *Let Δ be a reduced root system, W be the corresponding Weyl group, and Π a basis for Δ.*

(i) *For every regular element $\lambda \in E$, there is a $\sigma \in W$ such that*

$$(\sigma\lambda, \alpha) > 0 \quad \textit{for} \quad \alpha \in \Pi,$$

i.e., $\sigma(\mathcal{C}(\lambda)) = \mathcal{C}(\Pi)$. In particular, W acts transitively on the set of the Weyl chambers.
(ii) *Let Π' be another basis for Δ. Then there is a $\sigma \in W$ with $\sigma(\Pi') = \Pi$, i.e., the Weyl group also acts transitively on the set of the bases.*
(iii) *For every root $\alpha \in \Delta$, there is a $\sigma \in W$ with $\sigma(\alpha) \in \Pi$.*
(iv) *W is generated by the σ_α with $\alpha \in \Pi$.*
(v) *If $\sigma(\Pi) = \Pi$ for $\sigma \in W$, then $\sigma = \mathbf{1}$.*

Proof. Let W' be the subgroup of W, generated by the σ_α with $\alpha \in \Pi$. Suppose, (iii) holds for W' instead of W. Then for $\alpha \in \Delta$, we can find a $\sigma \in W'$ with $\sigma\alpha \in \Pi$. Then by $\sigma_{\sigma\alpha} = \sigma\sigma_\alpha\sigma^{-1}$ (cf. Proposition 5.4.7), we obtain that $\sigma_\alpha = \sigma^{-1}\sigma_{\sigma\alpha}\sigma \in W'$. Since W is generated by the σ_α with $\alpha \in \Delta$, we get $W = W'$, and hence (iv). Then (v) is an immediate consequence of Proposition 5.4.16(iii) applied to σ. Therefore, we see that it suffices to show (i)-(iii) for W' instead of W.
(i) Choose $\sigma \in W'$ such that the number $(\sigma\lambda, \rho)$ is maximal for the given λ and $\rho = \frac{1}{2}\sum_{\beta\in\Delta^+}\beta$ as above. For $\alpha \in \Pi$, we have $\sigma_\alpha\sigma \in W'$, hence, by Proposition 5.4.16(ii), we get

$$(\sigma\lambda, \rho) \geq (\sigma_\alpha\sigma\lambda, \rho) = (\sigma\lambda, \sigma_\alpha\rho) = (\sigma\lambda, \rho - \alpha) = (\sigma\lambda, \rho) - (\sigma\lambda, \alpha).$$

This gives $(\lambda, \sigma^{-1}\alpha) = (\sigma\lambda, \alpha) \geq 0$ for all $\alpha \in \Pi$, hence, also for all $\alpha \in \Delta^+$. Since λ is regular, all inequalities are strict. By Remark 5.4.14, the claim follows.

(ii) By (i), this is an immediate implication of Remark 5.4.14.
(iii) Because of (ii), it suffices to show that α is an element of *some* basis. If $\beta \neq \pm\alpha$ is a root, then α and β are not proportional since Δ is reduced. Thus, $\alpha^\perp \neq \beta^\perp$, and we can find a

$$\lambda \in \alpha^\perp \setminus \bigcup_{\beta \in \Delta \setminus \{\pm\alpha\}} \beta^\perp.$$

By a small modification of λ (add e.g. $\varepsilon\alpha$ with a small $\varepsilon > 0$), we obtain a $\lambda' \in E$ with

$$|(\lambda', \beta)| > (\lambda', \alpha) > 0 \quad \text{for} \quad \beta \in \Delta \setminus \{\pm\alpha\}.$$

Then λ' is regular, and we have $\alpha \in \Pi(\lambda')$ (cf. Theorem 5.4.13). □

Remark 5.4.18 (The dual root system). If $\Delta \subseteq E$ is a root system, then we put

$$\check{\Delta} := \{\check{\alpha} \colon \alpha \in \Delta\}.$$

We claim that $\check{\Delta}$ also is a root system, called the *dual root system*. It is reduced if and only if Δ is reduced.

To verify (R1) for $\check{\Delta}$ (if Δ is reduced), we note that $\check{\beta} \in \mathbb{R}\check{\alpha}$ implies $\beta \in \mathbb{R}\alpha$ and hence $\beta = \pm\alpha$, which in turn leads to $\check{\beta} = \pm\check{\alpha}$.

Since $\sigma_{\check{\alpha}}$ is the orthogonal reflection in $\alpha^\perp = \check{\alpha}^\perp$, we have $\sigma_\alpha = \sigma_{\check{\alpha}}$. As σ_α is an isometry, it satisfies $\sigma_\alpha(\check{\beta}) = \sigma_\alpha(\beta)\check{}$, so that $\check{\Delta}$ satisfies (R2). Finally we note that for $\alpha, \beta \in \Delta$, we have

$$(\check{\alpha}, \check{\alpha}) = \frac{4}{(\alpha, \alpha)},$$

so that $(\check{\alpha})\check{} = \alpha$. Therefore

$$(\check{\alpha}, (\check{\beta})\check{}) = (\beta, \check{\alpha}) \in \mathbb{Z},$$

and we conclude that $\check{\Delta}$ also is a root system.

Now let

$$\Delta^+ = \Delta^+(\lambda) = \{\alpha \in \Delta \colon (\lambda, \alpha) > 0\}$$

be a positive system of Δ. From the definition of the dual root system, it follows that Δ and $\check{\Delta}$ define the same set of regular elements. Therefore

$$\check{\Delta}^+ := \{\check{\alpha} \in \check{\Delta} \colon (\lambda, \check{\alpha}) > 0\}$$

also is a positive system of the dual root system $\check{\Delta}$.

The elements α of the basis $\Pi = \Pi(\lambda)$ for the positive system Δ^+ are uniquely determined by the property that they generate extremal rays of the convex cone

$$D(\Delta^+) := \sum_{\alpha \in \Delta^+} \mathbb{R}^+ \alpha$$

and satisfy $\frac{\alpha}{2} \notin \Delta$ (cf. Remark 5.4.5).

We define a map $\varphi\colon \Pi \to \check{\Delta}$ by $\varphi(\alpha) = \check{\alpha}$ if $2\alpha \notin \Delta$ and $\varphi(\alpha) := \frac{\check{\alpha}}{2} = (2\alpha)\check{}$ otherwise. Then $D(\Delta^+) = D(\check{\Delta}^+)$ immediately shows that $\varphi(\Pi)$ is a root basis for $\check{\Delta}$. If, in addition, Δ is reduced, then

$$\check{\Pi} := \{\check{\alpha}\colon \alpha \in \Pi\}$$

is a root basis for $\check{\Delta}^+$.

Notes on Chapter 5

Cartan subalgebras actually occur first in the work of W. Killing who classified the finite-dimensional simple complex Lie algebras (cf. [Kil89]). Unfortunately, Killing's work contained some serious gaps, concerning the basic properties of Cartan subalgebras which were cleaned up later by Élie Cartan in his thesis [Ca94].

Serre's Theorem on the presentation of semisimple Lie algebras with a splitting Cartan subalgebra can be extended to a construction of a semisimple Lie algebra from an abstract root system Δ with a basis $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Then we put $a_{ij} := \alpha_j(\check{\alpha}_i)$ and consider the Lie algebra $L(X, R)$ defined by the generators h_i, e_i, f_i, $i = 1, \ldots, r$ and the relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i$$

and for $i \neq j$:

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0.$$

In this context the main point is to show that $L(X, R)$ is a semisimple Lie algebra with the Cartan subalgebra $\mathfrak{h} = \operatorname{span}\{h_1, \ldots, h_r\}$ and a root system isomorphic to Δ. In the 1960s this description of the finite-dimensional semisimple Lie algebras was the starting point for the theory of Kac–Moody–Lie algebras, which are defined by the same set of generators and relations for more general matrices $(a_{ij}) \in M_r(\mathbb{Z})$, called generalized Cartan matrices.

6

Representation Theory of Lie Algebras

Even though representation theory is not in the focus of this book, we provide in the present chapter the basic theory for Lie algebras as it repeatedly plays an important role in structural questions. In this chapter, we first introduce the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. This is a unital associative algebra containing $\mathfrak{g}$ as a Lie subalgebra and is generated by $\mathfrak{g}$. It has the universal property that each representation of $\mathfrak{g}$ extends uniquely to $\mathcal{U}(\mathfrak{g})$, so that any $\mathfrak{g}$-module becomes a $\mathcal{U}(\mathfrak{g})$-module. We may thus translate freely between Lie algebra modules and algebra modules, which is convenient for several representation theoretic constructions. The Poincaré–Birkhoff–Witt (PBW) Theorem 6.1.9 provides crucial information on the structure on $\mathcal{U}(\mathfrak{g})$, including the injectivity of the natural map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$.

As a first application of the PBW Theorem, we prove Serre's Theorem 6.2.10, which shows how to construct semisimple Lie algebras from root systems. The second application is the Highest Weight Theorem 6.3.15 providing a classification of irreducible finite-dimensional representations of (split) semisimple Lie algebras. This is the main result of the Cartan–Weyl Theory of simple modules of semisimple complex Lie algebras. In view of Weyl's Theorem that any module over such a Lie algebra is semisimple, the classification of the simple modules provides a complete picture of the finite-dimensional representation theory of complex semisimple Lie algebras.

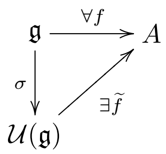
A third application of the PBW Theorem is Ado's Theorem 6.4.1 which says that each finite-dimensional Lie algebra has an injective finite-dimensional representation, i.e., can be viewed as a Lie algebra of matrices. Finally, we introduce basic cohomology theory for Lie algebras which has many applications. Here we use it to describe extensions of Lie algebras.

In general, modules of a Lie algebra $\mathfrak{g}$ are not semisimple, so that submodules need not have module complements. This leads to the concept of a nontrivial extension of modules which is most naturally dealt with in the context of Lie algebra cohomology which we develop in Section 6.5. We also explain how Lie algebra cohomology provides a tool to deal with the nontriviality of other extension problems such as central and abelian extensions of Lie algebras, for which it also provides a classifying parameter space. Several of the structure theoretic results encountered in Section 4.6 have natural cohomological interpretations, such as Weyl's Theorem and Levi's Theorem which are reflected in the two Whitehead Lemmas.

6.1 The Universal Enveloping Algebra

Representing a Lie algebra by linear maps leads to a mapping of the Lie algebra into an associative algebra such that the Lie bracket turns into the commutator bracket. A priori it is not clear that an injective map of this type exists, not even, if we allow the associative algebra to be infinite-dimensional. The point of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is that every representation of $\mathfrak{g}$ on V factors through a homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ of associative algebras.

Definition 6.1.1. Let $\mathfrak{g}$ be a Lie algebra. A pair $(\mathcal{U}(\mathfrak{g}), \sigma)$, consisting of a unital associative algebra $\mathcal{U}(\mathfrak{g})$ and a homomorphism $\sigma \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})_L$ of Lie algebras, is called a (universal) *enveloping algebra* of $\mathfrak{g}$ if it has the following universal property. For each homomorphism $f \colon \mathfrak{g} \to A_L$ of $\mathfrak{g}$ into the Lie algebra A_L, where A is a unital associative algebra, there exists a unique homomorphism $\widetilde{f} \colon \mathcal{U}(\mathfrak{g}) \to A$ of unital associative algebras with $\widetilde{f} \circ \sigma = f$.



The universal property determines a universal enveloping algebra uniquely in the following sense:

Lemma 6.1.2 (Uniqueness of the enveloping algebra). *If $(\mathcal{U}(\mathfrak{g}), \sigma)$ and $(\widetilde{\mathcal{U}}(\mathfrak{g}), \widetilde{\sigma})$ are two enveloping algebras of the Lie algebra $\mathfrak{g}$, then there exists an isomorphism $f \colon \mathcal{U}(\mathfrak{g}) \to \widetilde{\mathcal{U}}(\mathfrak{g})$ of unital associative algebras satisfying $f \circ \sigma = \widetilde{\sigma}$.*

Proof. Since $\widetilde{\sigma} \colon \mathfrak{g} \to \widetilde{\mathcal{U}}(\mathfrak{g})_L$ is a homomorphism of Lie algebras, the universal property of the pair $(\mathcal{U}(\mathfrak{g}), \sigma)$ implies the existence of a unique algebra homomorphism

$$f \colon \mathcal{U}(\mathfrak{g}) \to \widetilde{\mathcal{U}}(\mathfrak{g}) \quad \text{with} \quad f \circ \sigma = \widetilde{\sigma}.$$

Similarly, the universal property of $(\widetilde{\mathcal{U}}(\mathfrak{g}), \widetilde{\sigma})$ implies the existence of an algebra homomorphism

$$g \colon \widetilde{\mathcal{U}}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \quad \text{with} \quad g \circ \widetilde{\sigma} = \sigma.$$

Then $g \circ f : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is an algebra homomorphism with $(g \circ f) \circ \sigma = \sigma$, so that the uniqueness part of the universal property of $(\mathcal{U}(\mathfrak{g}), \sigma)$ yields $g \circ f = \mathrm{id}_{\mathcal{U}(\mathfrak{g})}$. We likewise get $f \circ g = \mathrm{id}_{\widetilde{\mathcal{U}}(\mathfrak{g})}$, showing that f is an isomorphism of unital algebras. □

To prove the existence of an enveloping algebra, we recall some basic algebraic concepts. Let $\mathcal{A}$ be an associative algebra. A subspace J of $\mathcal{A}$ is called an *ideal* if

$$\mathcal{A}J \cup J\mathcal{A} \subseteq J.$$

Let M be a subset of $\mathcal{A}$. Since the intersection of a family of ideals is again an ideal, the intersection J_M of all ideals of $\mathcal{A}$ containing M is the smallest ideal of $\mathcal{A}$ containing M. It is called the *ideal generated by* M. If J is an ideal of $\mathcal{A}$, then the *factor algebra* $\mathcal{A}/J$ is the quotient vector space, endowed with the associative multiplication

$$(a_1 + J)(a_2 + J) := a_1 a_2 + J \quad \text{for} \quad a_1, a_2 \in \mathcal{A}.$$

6.1.1 Existence

Proposition 6.1.3 (Existence of an enveloping algebra). *Each Lie algebra* $\mathfrak{g}$ *has an enveloping algebra* $(\mathcal{U}(\mathfrak{g}), \sigma)$.

Proof. Let $\mathcal{T}(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$ (Definition B.1.7) and consider the subset

$$M := \{x \otimes y - y \otimes x - [x, y] \in \mathcal{T}(\mathfrak{g}) \colon x, y \in \mathfrak{g}\}.$$

Then

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/J_M$$

is a unital associative algebra and

$$\sigma \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g}), \quad \sigma(x) := x + J_M,$$

is a linear map, satisfying

$$\sigma([x, y]) = [x, y] + J_M = x \otimes y - y \otimes x + J_M = \sigma(x)\sigma(y) - \sigma(y)\sigma(x),$$

so that σ is a homomorphism of Lie algebras $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})_L$.

To verify the universal property for $(\mathcal{U}(\mathfrak{g}), \sigma)$, let $f \colon \mathfrak{g} \to A_L$ be a homomorphism of Lie algebras, where A is a unital associative algebra. In view of the universal property of $\mathcal{T}(\mathfrak{g})$ (Lemma B.1.8), there exists an algebra homomorphism $\widehat{f} \colon \mathcal{T}(\mathfrak{g}) \to A$ with $\widehat{f}(x) = f(x)$ for all $x \in \mathfrak{g}$. Then $M \subseteq \ker \widehat{f}$, and since $\ker \widehat{f}$ is an ideal of $\mathcal{T}(\mathfrak{g})$, we also have $J_M \subseteq \ker \widehat{f}$, so that $\widehat{f}$ factors through an algebra homomorphism

$$\widetilde{f} \colon \mathcal{U}(\mathfrak{g}) \to A \quad \text{with} \quad \widetilde{f} \circ \sigma = f.$$

To see that $\widetilde{f}$ is unique, it suffices to note that $\sigma(\mathfrak{g})$ and $\mathbf{1}$ generate $\mathcal{U}(\mathfrak{g})$ as an associative algebra because $\mathfrak{g}$ and $\mathbf{1}$ generate $\mathcal{T}(\mathfrak{g})$ as an associative algebra. □

Remark 6.1.4. The universal property of $(\mathcal{U}(\mathfrak{g}), \sigma)$ implies that each representation (π, V) of $\mathfrak{g}$ defined a representation $\widetilde{\pi} \colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$, which is uniquely determined by $\widetilde{\pi} \circ \sigma = \pi$. From the construction of $\mathcal{U}(\mathfrak{g})$ we also know that the algebra $\mathcal{U}(\mathfrak{g})$ is generated by $\sigma(\mathfrak{g})$. This implies that for each $v \in V$, the subspace

$$\mathcal{U}(\mathfrak{g}) \cdot v \subseteq V$$

is the smallest subspace containing v and invariant under $\mathfrak{g}$, i.e., the $\mathfrak{g}$-submodule of V generated by V. Hence the enveloping algebra provides a tool to understand $\mathfrak{g}$-submodules of a $\mathfrak{g}$-module. But before we are able to use this tool effectively, we need some more information on the structure of $\mathcal{U}(\mathfrak{g})$.

6.1.2 The Poincaré–Birkhoff–Witt Theorem

Note that the canonical map $\sigma \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})_L$ by definition is a homomorphism of Lie algebras. But we do not know yet if it is injective, so that we obtain an embedding of $\mathfrak{g}$ into an associative algebra. Our next goal is the Poincaré–Birkhoff–Witt Theorem which entails in particular that σ is injective.

Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathfrak{g}$ and set $\xi_i := \sigma(x_i)$. For a finite sequence $I = (i_1, \ldots, i_k)$ of natural numbers between 1 and n, we set $\xi_I := \xi_{i_1} \cdots \xi_{i_k}$. If $i \in \mathbb{N}$, then we write $i \leq I$ if $i \leq i_j$ for all $j = 1, \ldots, k$. We write $\mathcal{U}_p(\mathfrak{g}) := \sum_{k \leq p} \sigma(\mathfrak{g})^k$ and note that these subspaces are all finite-dimensional and satisfy

$$\mathcal{U}_p(\mathfrak{g})\mathcal{U}_q(\mathfrak{g}) \subseteq \mathcal{U}_{p+q}(\mathfrak{g}) \quad \text{for} \quad p, q \in \mathbb{N}_0.$$

Next we construct a suitable basis for $\mathcal{U}(\mathfrak{g})$.

Lemma 6.1.5. *Let $y_1, \ldots, y_p \in \mathfrak{g}$ and π be a permutation of $\{1, \ldots, p\}$, then*

$$\sigma(y_1) \cdots \sigma(y_p) - \sigma(y_{\pi(1)}) \cdots \sigma(y_{\pi(p)}) \in \mathcal{U}_{p-1}(\mathfrak{g}).$$

Proof. Since every permutation is a composition of transpositions of neighboring elements, it suffices to prove the claim for $\pi(j) = j$ for $j \notin \{i, i+1\}$ and $\pi(i) = i + 1$. But then we have

$$\begin{aligned}
&\sigma(y_1) \cdots \sigma(y_p) - \sigma(y_{\pi(1)}) \cdots \sigma(y_{\pi(p)}) \\
&= \sigma(y_1) \cdots \sigma(y_{i-1})\big(\sigma(y_i)\sigma(y_{i+1}) - \sigma(y_{i+1})\sigma(y_i)\big)\sigma(y_{i+2}) \cdots \sigma(y_p) \\
&= \sigma(y_1) \cdots \sigma(y_{i-1})\sigma([y_i, y_{i+1}])\sigma(y_{i+2}) \cdots \sigma(y_p) \in \mathcal{U}_{p-1}(\mathfrak{g}).
\end{aligned}$$

□

Lemma 6.1.6. *The vector space $\mathcal{U}_p(\mathfrak{g})$ is spanned by the ξ_I with increasing sequences I of length less than or equal to p.*

Proof. It is clear that $\mathcal{U}_p(\mathfrak{g})$ is spanned by the ξ_I for *all* sequences I of length less that or equal to p. By induction on p, we have the claim for $\mathcal{U}_{p-1}(\mathfrak{g})$. But since for a rearrangement I' of the sequence I, we have $\xi_I - \xi_{I'} \in \mathcal{U}_{p-1}(\mathfrak{g})$ by Lemma 6.1.5, we also obtain the claim for $\mathcal{U}_p(\mathfrak{g})$. □

Let $\mathcal{P} := \mathbb{K}[z_1, \ldots, z_n]$ be the associative algebra of polynomials over $\mathbb{K}$ in the commuting variables $z_1, \ldots, z_n$. For $i \in \mathbb{N} \cup \{0\}$, let $\mathcal{P}_i$ be the set of all polynomials of degree less or equal to i. As in $\mathcal{U}(\mathfrak{g})$, we write $z_I := z_{i_1} \cdots z_{i_k}$ for a finite sequence I of natural numbers between 1 and n. For the empty sequence, we set $z_\emptyset = 1$.

Lemma 6.1.7. *There exists* $\mathfrak{g}$*-module structure on* $\mathcal{P}$ *with*

$$x_i \cdot z_I = z_i z_I \quad \textit{for} \quad i \leq I.$$

Proof. By induction on $k \in \mathbb{N}_0$, we construct linear maps

$$\rho_k \colon \mathfrak{g} \to \mathrm{Hom}(\mathcal{P}_k, \mathcal{P}_{k+1})$$

with the following properties:

(a_k) $\rho_k(x_i)z_I = z_i z_I$ for $i \leq I$ and $z_I \in \mathcal{P}_k$.
(b_k) $\rho_k(x_i)z_I - z_i z_I \in \mathcal{P}_j$ for $z_I \in \mathcal{P}_j$ and $j \leq k$.
(c_k) $\rho_k(x_i)\rho_k(x_j)z_J - \rho_k(x_j)\rho_k(x_i)z_J = \rho_k([x_i, x_j])z_J$ for $z_J \in \mathcal{P}_{k-1}$.
(d_k) $\rho_k(x)|_{\mathcal{P}_{k-1}} = \rho_{k-1}(x)$ for $k > 0$ and $x \in \mathfrak{g}$.

For $k = 0$ we put $\rho_0(x_i)1 := z_i$ for $i = 1, \ldots, k$. Then (a_0) and (b_0) are satisfied because $x_1, \ldots, x_k$ is a basis for $\mathfrak{g}$, and (c_0) and (d_0) are empty conditions.

Now we assume that we already have ρ_{k-1}. In view of (d_k), we have to show that the maps $\rho_{k-1}(x_i)$ can be extended to maps $\mathcal{P}_k \to \mathcal{P}_{k+1}$ so that $(a_k)-(c_k)$ are satisfied. Since $\mathcal{P}$ is commutative, the z_I with increasing I form a basis for $\mathcal{P}$. Thus, let $I := (i_1, \ldots, i_k)$ be increasing. For $I_1 := (i_2, \ldots, i_k)$, by (a_{k-1}), we have

$$z_I = z_{i_1} z_{I_1} = \rho_{k-1}(x_{i_1}) z_{I_1}.$$

By (b_{k-1}),

$$w(I, i) := \rho_{k-1}(x_i) z_{I_1} - z_i z_{I_1} \in \mathcal{P}_{k-1}.$$

We set

$$\rho_k(x_i) z_I := \begin{cases} z_i z_I & \text{if } i \leq I \\ z_i z_I + \rho_{k-1}(x_{i_1}) w(I, i) + \rho_{k-1}([x_i, x_{i_1}]) z_{I_1} & \text{otherwise.} \end{cases}$$

For this definition, (a_k) and (b_k) are obviously satisfied. But we still have to verify (c_k). Two cases occur:

Case 1: $i \neq j$, and one of them is less than J.

In this case, by $[x_i, x_j] = -[x_j, x_i]$, we may assume that $j < i$ and $j \leq J$. Then with (a_{k-1}) and (b_{k-1}), we calculate

$$\begin{aligned} &\rho_k(x_i)\rho_{k-1}(x_j)z_J - \rho_k(x_j)\rho_{k-1}(x_i)z_J \\ &= \rho_k(x_i) z_j z_J - \rho_k(x_j) z_i z_J - \rho_{k-1}(x_j)\big(\rho_{k-1}(x_i) z_J - z_i z_J\big) \\ &= z_i z_j z_J + \rho_{k-1}(x_j)\big(\rho_{k-1}(x_i) z_J - z_i z_J\big) + \rho_{k-1}([x_i, x_j]) z_J \\ &\quad - z_i z_j z_J - \rho_{k-1}(x_j)\big(\rho_{k-1}(x_i) z_J - z_i z_J\big) \\ &= \rho_{k-1}([x_i, x_j]) z_J = \rho_k([x_i, x_j]) z_J. \end{aligned}$$

Case 2: $J = (j_1, \ldots, j_{k-1})$ and $j_1 < i, j$.

We set $l := j_1$, $L := (j_2, \ldots, j_{k-1})$, and abbreviate $\rho_k(x_i)z_I$ by $x_i(z_I)$. Then it follows from (a_{k-1}) and Case 1 that

$$x_j(z_J) = x_j\big(x_l(z_L)\big) = x_l\big(x_j(z_L)\big) + [x_j, x_l](z_L)$$

and, using (c_{k-1}) and Case 1,

$$\begin{aligned} &x_i\big(x_j(z_J)\big) = x_i\big(x_l(x_j(z_L))\big) + x_i\big([x_j, x_l](z_L)\big) \\ &= x_l\big(x_i(x_j(z_L))\big) + [x_i, x_l](x_j(z_L)) + [x_j, x_l](x_i(z_L)) + [x_i, [x_j, x_l]](z_L). \end{aligned}$$

Combining this with (c_{k-1}) and the Jacobi identity, we obtain

$$\begin{aligned} &x_i\big(x_j(z_J)\big) - x_j\big(x_i(z_J)\big) \\ &= x_l\big(x_i\big(x_j(z_L)\big)\big) - x_l\big(x_j\big(x_i(z_L)\big)\big) + [x_i, [x_j, x_l]](z_L) - [x_j, [x_i, x_l]](z_L) \\ &= x_l\big([x_i, x_j](z_L)\big) + [x_i, [x_j, x_l]](z_L) + [x_j, [x_l, x_i]](z_L) \\ &= [x_i, x_j]\big(x_l(z_L)\big) + [x_l, [x_i, x_j]](z_L) + [x_i, [x_j, x_l]](z_L) + [x_j, [x_l, x_i]](z_L) \\ &= [x_i, x_j]\big(x_l(z_L)\big) = [x_i, x_j]z_J. \end{aligned}$$

This completes our induction. In view of (d_k), we obtain a well-defined map

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathcal{P}) \quad \text{by} \quad \rho(x)|_{\mathcal{P}_k} = \rho_k(x),$$

and (c_k) implies that it is a homomorphism of Lie algebras, so that we obtain on $\mathcal{P}$ a $\mathfrak{g}$-module structure, and (a_k) ensures that it has the required property. □

Proposition 6.1.8. *The ξ_I with increasing I form a basis for $\mathcal{U}(\mathfrak{g})$. In particular, the canonical map $\sigma\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.*

Proof. Let $\rho\colon \mathfrak{g} \to \operatorname{End}(\mathcal{P})_L$ be the Lie algebra homomorphism defining the module structure constructed in Lemma 6.1.7 and $\widetilde{\rho}\colon \mathcal{U}(\mathfrak{g}) \to \operatorname{End}(\mathcal{P})$ the algebra homomorphism determined by $\widetilde{\rho}\circ\sigma = \rho$. Then the property $\rho(x_i)z_I = z_i z_I$ for $i \leq I$ ensures that for $i_1 \leq \ldots \leq i_k$, we have

$$\widetilde{\rho}(\xi_{i_1} \cdots \xi_{i_k})(\mathbf{1}) = z_{i_1} \cdots z_{i_k}.$$

Then the linear map

$$\varphi\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{P}, \quad \xi \mapsto \widetilde{\rho}(\xi)(\mathbf{1})$$

maps the set $\mathcal{B}$ of the ξ_I with increasing I to the (linearly independent) set of the z_I with increasing I. Therefore $\mathcal{B}$ is linearly independent, and the claim now follows from Lemma 6.1.6. □

Theorem 6.1.9 (Poincaré–Birkhoff–Witt Theorem (PBW)). *Let $\mathfrak{g}$ be a Lie algebra and $\{x_1, \ldots, x_n\}$ be a basis for $\mathfrak{g}$. Then*

$$\{x_1^{\mu_1} \cdots x_n^{\mu_n} \in \mathcal{U}(\mathfrak{g}) \mid \mu_k \in \mathbb{N} \cup \{0\}\}$$

is a basis for $\mathcal{U}(\mathfrak{g})$.

Proof. We apply Proposition 6.1.8 and identify the x_i with the $\xi_i = \sigma(x_i)$. □

Exercises for Section 6.1

Exercise 6.1.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and β be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Suppose that $x_1, \ldots, x_n$ is a basis for $\mathfrak{g}$ and let $x^1, \ldots, x^n$ be the dual basis w.r.t. β, i.e., $\beta(x_i, x^j) = \delta_{ij}$.

(i) Show that the *Casimir element* $\Omega := \sum_{i=1}^n x^i x_i$ lies in the center of $\mathcal{U}(\mathfrak{g})$.
(ii) Show that there is a nondegenerate symmetric invariant bilinear form on the oscillator algebra. Hence such forms do not only exist on semisimple Lie algebras.
(iii) Let $\mathfrak{g} = \mathfrak{so}_n(\mathbb{R})$. Show that $\beta(x, y) = -\operatorname{tr}(xy)$ defines an invariant scalar product on $\mathfrak{so}_n(\mathbb{R})$. For an orthonormal basis $x_1, \ldots, x_n$, we therefore have $\Omega := \sum_{i=1}^n x_i^2 \in Z\big(\mathcal{U}(\mathfrak{so}_n(\mathbb{R}))\big)$.
(iv) Show that the *operators of angular momentum*

$$x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \qquad i, j = 1, \ldots, n,$$

generate a Lie algebra which is isomorphic to $\mathfrak{so}_n(\mathbb{R})$.
(v) The *Laplace operator* $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ commutes with the operators of angular momentum.

Exercise 6.1.2. A function $f \in C^\infty(\mathbb{R}^n)$ is called *harmonic* if $\Delta(f) = 0$ for the Laplace operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Show that the subspace $H \subseteq C^\infty(\mathbb{R}^n)$ of the harmonic functions is invariant under the angular momentum operators (cf. Exercise 6.1.1).

6.2 Generators and Relations for Semisimple Lie Algebras

In this section we shall use the root decomposition of a semisimple Lie algebras to find a description by generators and relations (Serre's Theorem).

6.2.1 A Generating Set for Semisimple Lie Algebras

Proposition 6.2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a splitting Cartan subalgebra. Fix a positive system $\Delta^+ \subseteq \Delta$ and let $\Pi \subseteq \Delta^+$ be the set of simple roots. For each $\alpha \in \Pi$, we fix a corresponding $\mathfrak{sl}_2$-triple $(h_\alpha, e_\alpha, f_\alpha)$. Then the following assertions hold:*

(i) *The subspace $\mathfrak{n} := \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$ is a nilpotent subalgebra generated by $\{e_\alpha \colon \alpha \in \Pi\}$.*

(ii) *The Lie algebra* $\mathfrak{g}$ *is generated by the* $\mathfrak{sl}_2$*-subalgebras* $\mathfrak{g}(\alpha)$*,* $\alpha \in \Pi$*, i.e.,* $\{h_\alpha, e_\alpha, f_\alpha \colon \alpha \in \Pi\}$ *generates* $\mathfrak{g}$*. These elements satisfy the relations*

$$[h_\alpha, h_\beta] = 0, \quad [h_\alpha, e_\beta] = \beta(\check{\alpha})e_\beta, \quad [h_\alpha, f_\beta] = -\beta(\check{\alpha})f_\beta, \quad [e_\alpha, f_\beta] = \delta_{\alpha,\beta}h_\alpha \tag{6.1}$$

and for $\alpha \neq \beta$ *in* Π *we further have*

$$(\operatorname{ad} e_\alpha)^{1-\beta(\check{\alpha})}e_\beta = 0, \quad (\operatorname{ad} f_\alpha)^{1-\beta(\check{\alpha})}f_\beta = 0. \tag{6.2}$$

The relations (6.1) and (6.2) are called the *Serre relations.*

Proof. (i) If $\beta, \gamma \in \Delta^+$, then either $\beta + \gamma \in \Delta^+$ or $\beta + \gamma$ is not a root. Hence $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}$. Pick $x_0 \in \mathfrak{h}_\mathbb{R}$ with $\Delta^+ = \{\beta \in \Delta \colon \beta(x_0) > 0\}$ and let

$$m := \min \Delta^+(x_0) \quad \text{and} \quad M := \max \Delta^+(x_0),$$

where $\Delta^+(x_0) = \{\beta(x_0) : \beta \in \Delta^+\}$. For any $N \in \mathbb{N}$ with $Nm > M$ we then have $C^N(\mathfrak{n}) = \{0\}$, showing that $\mathfrak{n}$ is nilpotent.

In view of Proposition 5.4.16(i), every $\beta \in \Delta^+$ can be written in the form $\alpha_1 + \ldots + \alpha_m$ with $\alpha_j \in \Pi$ such that $\sum_{j=1}^k \alpha_j \in \Delta^+$ for each $k \in \{1, \ldots, m\}$. Then Lemma 5.3.5 implies that

$$\mathfrak{g}_\beta = [\mathfrak{g}_{\alpha_m}, [\mathfrak{g}_{\alpha_{m-1}}, [\cdots, [\mathfrak{g}_{\alpha_2}, \mathfrak{g}_{\alpha_1}] \cdots]]],$$

and this proves (i).

(ii) From (i) we also derive that $\overline{\mathfrak{n}} := \sum_{\alpha \in -\Delta^+} \mathfrak{g}_\alpha$ is generated by $\{f_\beta \colon \beta \in \Pi\}$. Therefore the Lie algebra generated by the $\mathfrak{g}(\alpha), \alpha \in \Pi$, contains all root spaces and

$$\operatorname{span}\{h_\alpha \colon \alpha \in \Pi\} = \operatorname{span}\{h'_\alpha \colon \alpha \in \Pi\} = \mathfrak{h}$$

follows from $\operatorname{span} \Pi = \mathfrak{h}^*$.

It remain to verify the Serre relation. Since $h_\alpha = \check{\alpha}$, the first three relations are trivial, and the fact that $\alpha - \beta \notin \Delta$ for $\alpha \neq \beta$ in Π implies that $[e_\alpha, f_\beta] = \{0\}$ in this case.

If $\alpha \neq \beta$, then we consider the $\mathfrak{g}(\alpha)$-submodule of $\mathfrak{g}$ generated by f_β. Since $[e_\alpha, f_\beta] = 0$, this is a highest weight module M with highest weight $-\beta(\check{\alpha})$, so that Proposition 5.2.4 implies that $\dim M = 1 - \beta(\check{\alpha})$ and $(\operatorname{ad} f_\alpha)^{\dim M} f_\beta = 0$.

The first relation in (6.2) is obtained with similar arguments, applied to the $\mathfrak{sl}_2$-triple $(-h_\alpha, f_\alpha, e_\alpha)$ and the $\mathfrak{g}(\alpha)$-submodule generated by e_β. □

Example 6.2.2. We have seen in Example 5.3.9 how to find a natural root decomposition of the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$ with respect to the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. In the root system

$$\Delta = \{\varepsilon_j - \varepsilon_k : 1 \leq j \neq k \leq n\},$$

the subset

$$\Delta^+ = \{\varepsilon_j - \varepsilon_k : 1 \leq j < k \leq n\}$$

is a natural positive system with root basis

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n\}.$$

Then

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha = \operatorname{span}\{E_{jk} \colon j < k\}$$

is the Lie algebra of strictly upper triangular matrices. It is generated by the root vectors $E_{j,j+1}$, $j = 1, \ldots, n-1$. For each pair of indices $j \neq k$, we have

$$\mathfrak{g}(\varepsilon_j - \varepsilon_k) = \operatorname{span}\{E_{jk}, E_{kj}, E_{jj} - E_{kk}\} \cong \mathfrak{sl}_2(\mathbb{K}),$$

and the subalgebras

$$\mathfrak{g}(\varepsilon_1 - \varepsilon_2), \quad \ldots, \quad \mathfrak{g}(\varepsilon_{n-1} - \varepsilon_n)$$

sitting on the diagonal, generate $\mathfrak{sl}_n(\mathbb{K})$. Moreover, $\mathfrak{sl}_n(\mathbb{K})$ is also generated by the $2(n-1)$ element: $E_{j,j+1}, E_{j+1,j}$, $j = 1, \ldots, n-1$.

6.2.2 Free Lie Algebras

Free Lie algebras are defined via universal properties. Their existence then has to be proved separately. The point of introducing them is the possibility to describe Lie algebras via generators and relations.

Definition 6.2.3. Let X be a set. A pair (L, η) of a Lie algebra L and a map $\eta \colon X \to L$ is said to be a *free Lie algebra over* X if it has the following universal property. For each map $\alpha \colon X \to \mathfrak{g}$ into a Lie algebra $\mathfrak{g}$, there exists a unique morphism $\widetilde{\alpha} \colon L \to \mathfrak{g}$ of Lie algebras with $\widetilde{\alpha} \circ \eta = \alpha$.

Remark 6.2.4. (a) For two free Lie algebras (L_1, η_1) and (L_2, η_2) over X there exists a unique isomorphism $\varphi \colon L_1 \to L_2$ with $\varphi \circ \eta_1 = \eta_2$. In fact, we choose φ as the unique morphism of Lie algebras with $\varphi \circ \eta_1 = \eta_2$. Then the unique morphism of Lie algebras $\psi \colon L_2 \to L_1$ with $\psi \circ \eta_2 = \eta_1$ satisfies $\psi \circ \varphi \circ \eta_1 = \eta_1$, so that the uniqueness in the universal property implies $\psi \circ \varphi = \mathrm{id}_{L_1}$, and likewise $\varphi \circ \psi = \mathrm{id}_{L_2}$.

(b) If (L, η) is a free Lie algebra over X, then L is generated as a Lie algebra by $\eta(X)$. To see this, let $L_0 \subseteq L$ be the Lie subalgebra generated by $\eta(X)$ and $\eta_0 \colon X \to L_0$ be the corestriction of η. Then (L_0, η_0) also has the universal property, so that (a) implies the existence of an isomorphism $\varphi \colon L_0 \to L$ with $\varphi \circ \eta_0 = \eta$. Since L_0 is generated by $\eta_0(X) = \eta(X)$, the Lie algebra L is generated by $\eta(X)$, which leads to $L = L_0$.

Proposition 6.2.5. *For each set X, there exists a free Lie algebra $(L(X), \eta)$ over X.*

Proof. Let X be a set. We put $X_1 = X$ and define inductively

$$X_n := \bigcup_{p+q=n, p,q\in\mathbb{N}} X_p \times X_q.$$

The *free magma* $M(X)$ is defined as the disjoint union

$$M(X) := \dot{\bigcup_{i=1}^{\infty}} X_n.$$

Elements of the free magma are expressions of the form

$$((((x_1, x_2), x_3), x_4)), \quad ((x_1, x_2), (x_3, x_4)), \quad ((x_1, x_2), ((x_3, x_4), x_5)).$$

One should think of it as a set containing all possible nonassociative products of elements of X. The maps $X_n \times X_m \to X_{n+m}, (x, y) \mapsto x \cdot y := (x, y)$ can be put together to a multiplication

$$M(X) \times M(X) \to M(X), \quad (x, y) \mapsto x \cdot y.$$

The free vector space $AM(X) := \mathbb{K}[M(X)]$, with basis $M(X)$, thus inherits a bilinear multiplication extending the multiplication on $M(X)$.

Let $\mathfrak{n} \subseteq AM(X)$ denote the two sided ideal generated by the expressions of the form

$$Q(x) := x \cdot x, \quad x \in AM(X)$$

and

$$J(x, y, z) := x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y), \quad x, y, z \in M(X).$$

We claim that the quotient algebra $L(X) := AM(X)/\mathfrak{n}$ is a free Lie algebra over X with respect to the map $\eta\colon X \to L(X), x \mapsto x + \mathfrak{n}$. First we note that the quotient space $L(X)$ inherits a bilinear multiplication turning it into a Lie algebra. We have to show that for each map $\varphi\colon X \to \mathfrak{g}$ into a Lie algebra $\mathfrak{g}$ there exists a unique Lie algebra homomorphism $L(\varphi)\colon L(X) \to \mathfrak{g}$ with $L(\varphi)(x + \mathfrak{n}) = \varphi(x)$ for each $x \in X$. In fact, one inductively defines mappings

$$\varphi_n\colon X_n \to \mathfrak{g}, \quad (x, y) \mapsto [\varphi_p(x), \varphi_q(x)], \quad (x, y) \in X_p \times X_q, p + q = n$$

and puts them together to a map $\varphi'\colon M(X) \to \mathfrak{g}$ satisfying $\varphi(x \cdot y) = [\varphi(x), \varphi(y)]$. This map extends uniquely to a linear map $\varphi''\colon AM(X) \to \mathfrak{g}$ with the same property. Since $\mathfrak{g}$ is a Lie algebra, the map φ'' factors to a Lie algebra homomorphism $L(\varphi)\colon L(X) \to \mathfrak{g}$ which is uniquely determined by $L(\varphi)(x + \mathfrak{n}) = \varphi(x)$. □

Definition 6.2.6. (cf. Exercise 6.2.1) (a) In the following we denote a free Lie algebra over X by $(L(X), \eta)$.

(b) Let X be a set, $R \subseteq L(X)$ a subset and $I_R \subseteq L(X)$ the ideal generated by R. Then $L(X, R) := L(X)/I_R$ is called the *Lie algebra defined by the generators X and the relations R.*

Example 6.2.7. (a) If $X = \{p, q, z\}$ and $R = \{[p, q] - z, [p, z], [q, z]\}$, then $L(X, R)$ is the 3-dimensional Heisenberg–Lie algebra. It is defined by the generators p, q, z and the relations $[p, q] = z, [p, z] = [q, z] = 0$.
(b) If $X = \{h, e, f\}$ and $R = \{[e, f] - h, [h, e] - 2e, [h.f] + 2f\}$, then $L(X, R) \cong \mathfrak{sl}_2(\mathbb{K})$.
(c) If $X = \{x\}$ is a one-element set, then $L(X) = \mathbb{C}x$ is one-dimensional. For a two-element set $X = \{x, y\}$, the free Lie algebra $L(X)$ is infinite-dimensional (Exercise 6.2.2).

6.2.3 Serre's Theorem

Now we return to the situation of Proposition 6.2.1. Let

$$X := \{\widehat{h}_\alpha, \widehat{e}_\alpha, \widehat{f}_\beta \colon \alpha \in \Pi\}$$

and consider in the free Lie algebra $L(X)$ the relations

$$\begin{aligned} R := \{&[\widehat{h}_\alpha, \widehat{h}_\beta], [\widehat{h}_\alpha, \widehat{e}_\beta] - \beta(\check{\alpha})\widehat{e}_\beta, [\widehat{h}_\alpha, \widehat{f}_\beta] + \beta(\check{\alpha})\widehat{f}_\beta, [\widehat{e}_\alpha, \widehat{f}_\beta] - \delta_{\alpha,\beta}\widehat{h}_\alpha\} \\ \cup \{&(\operatorname{ad}\widehat{e}_\alpha)^{1-\beta(\check{\alpha})}\widehat{e}_\beta, (\operatorname{ad}\widehat{f}_\alpha)^{1-\beta(\check{\alpha})}\widehat{f}_\beta \colon \alpha \neq \beta \in \Pi\}. \end{aligned}$$

Then Proposition 6.2.1(ii) implies the existence of a unique surjective Lie algebra homomorphism

$$q \colon \widehat{\mathfrak{g}} := L(X, R) \to \mathfrak{g} \quad \text{with} \quad q(\widehat{h}_\alpha) = h_\alpha, \quad q(\widehat{e}_\alpha) = e_\alpha \quad \text{and} \quad q(\widehat{f}_\alpha) = f_\alpha.$$

The goal of this subsection is to prove Serre's Theorem that q is an isomorphism, i.e., that the Lie algebra $\mathfrak{g}$ is defined by the generating set X and the relations R. Let

$$\widehat{\mathfrak{g}}_+ := \langle\{\widehat{e}_\alpha \colon \alpha \in \Pi\}\rangle_{\text{Lie alg}} \quad \text{and} \quad \widehat{\mathfrak{g}}_- := \langle\{\widehat{f}_\alpha \colon \alpha \in \Pi\}\rangle_{\text{Lie alg}}$$

denote the subalgebras of $\widehat{\mathfrak{g}}$ generated by the elements $\widehat{e}_\alpha$, resp., $\widehat{f}_\alpha$. Since the elements $h_\alpha, \alpha \in \Pi$, are linearly independent in $\mathfrak{h}$, this also holds for the elements $\widehat{h}_\alpha$ in $\widehat{\mathfrak{h}}$, so that $q|_{\widehat{\mathfrak{h}}} \colon \widehat{\mathfrak{h}} \to \mathfrak{h}$ is a linear isomorphism. For $\gamma \in \mathfrak{h}^*$, we put $\widehat{\gamma} := \gamma \circ q \in \widehat{\mathfrak{h}}^*$.

Lemma 6.2.8. *Let $\mathfrak{b}$ be a subalgebra of a Lie algebra $\mathfrak{g}$ and $M \subseteq \mathfrak{g}$ be a subset. Then $\mathfrak{b}_M := \{x \in \mathfrak{b} \colon [M, x] \subseteq \mathfrak{b}\}$ is a subalgebra of $\mathfrak{b}$.*

Proof. This follows from the following calculation for $m \in M$ and $x, y \in \mathfrak{b}_M$:

$$\big[m, [x, y]\big] = \big[[m, x], y\big] + \big[x, [m, y]\big] \subseteq [\mathfrak{b}, \mathfrak{b}] + [\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{b}.$$

□

Lemma 6.2.9. *The Lie algebra $\widehat{\mathfrak{g}}$ has the following properties:*

(i) *$\widehat{\mathfrak{g}}$ is the direct sum of $\widehat{\mathfrak{h}}$-weight spaces.*

(ii) $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{g}}_-$ *is a direct sum of subalgebras and* $\widehat{\mathfrak{h}} = \widehat{\mathfrak{g}}_0$. *Moreover,*

$$\widehat{\mathfrak{g}}_\pm \subseteq \sum_{\gamma \in \widehat{\Delta}_\pm} \widehat{\mathfrak{g}}_\gamma \quad \text{for} \quad \widehat{\Delta}_\pm := \pm \mathrm{span}_{\mathbb{N}} \widehat{\Pi}.$$

(iii) $\widehat{\mathfrak{g}}_{\widehat{\alpha}} = \mathbb{K}\widehat{e}_\alpha$ *and* $\widehat{\mathfrak{g}}_{-\widehat{\alpha}} = \mathbb{K}\widehat{f}_\alpha$ *for* $\alpha \in \Pi$.
(iv) *The derivations* $\mathrm{ad}\, \widehat{e}_\alpha$ *and* $\mathrm{ad}\, \widehat{f}_\alpha$ *of* $\widehat{\mathfrak{g}}$ *are locally nilpotent, i.e.,* $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^0(\mathrm{ad}\, \widehat{e}_\alpha) = \widehat{\mathfrak{g}}^0(\mathrm{ad}\, \widehat{f}_\alpha)$ *for each* $\alpha \in \Pi$.
(v) *The set* $\widehat{\Delta}$ *of roots of* $\widehat{\mathfrak{g}}$ *is invariant under the group* $\widehat{\mathcal{W}}$ *generated by the reflections defined by* $\sigma_{\widehat{\alpha}}\widehat{\beta} := \widehat{\beta} - \widehat{\beta}(\widehat{h}_\alpha)\widehat{\alpha}$. *This is a finite group isomorphic to* $\mathcal{W}$.
(vi) $\widehat{\Delta} = \widehat{\mathcal{W}}\widehat{\Pi}$.

Proof. (i) Let $\widehat{\mathfrak{g}}_f := \sum_{\gamma \in \widehat{\mathfrak{h}}^*} \widehat{\mathfrak{g}}_\gamma$ be the sume of all $\widehat{\mathfrak{h}}$-weight spaces in $\widehat{\mathfrak{g}}$. In view of $[\widehat{\mathfrak{g}}_\gamma, \widehat{\mathfrak{g}}_\delta] \subseteq \widehat{\mathfrak{g}}_{\gamma+\delta}$, this is a subalgebra of $\widehat{\mathfrak{g}}$. We also have $\widehat{\mathfrak{h}} \subseteq \widehat{\mathfrak{g}}_0$ and

$$\widehat{e}_\alpha \in \widehat{\mathfrak{g}}_{\widehat{\alpha}} \quad \text{and} \quad \widehat{f}_\alpha \in \widehat{\mathfrak{g}}_{-\widehat{\alpha}},$$

and since $\widehat{\mathfrak{g}}$ is generated by these elements, it follows that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_f = \sum_{\gamma \in \widehat{\mathfrak{h}}^*} \widehat{\mathfrak{g}}_\gamma$ is a direct sum of $\widehat{\mathfrak{h}}$-weight spaces. The directness of the sum of the weight spaces can be obtained from Lemma 5.1.3 and the observation that each finite subset of $\widehat{\mathfrak{g}}$ is contained in a finite $\widehat{\mathfrak{h}}$-invariant subspace, which follows from $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_f$.

(ii) Clearly, $\widehat{\mathfrak{h}} + \widehat{\mathfrak{g}}_\pm$ are subalgebras of $\widehat{\mathfrak{g}}$ because $\widehat{\mathfrak{g}}_\pm$ is generated by $\widehat{\mathfrak{h}}$-eigenvectors. For $\alpha \in \Pi$, we consider the subalgebra

$$\mathfrak{b}_\alpha := \{x \in \widehat{\mathfrak{h}} + \widehat{\mathfrak{g}}_+ \colon [x, \widehat{f}_\alpha] \subseteq \widehat{\mathfrak{h}} + \widehat{\mathfrak{g}}_+\}$$

(Lemma 6.2.8). For $\beta \in \Pi$ we have $[\widehat{f}_\alpha, \widehat{e}_\beta] = 0$ for $\alpha \neq \beta$ and $[\widehat{f}_\alpha . \widehat{e}_\alpha] \subseteq \widehat{\mathfrak{h}}$. Hence the subalgebra $\mathfrak{b}_\alpha$ contains each $\widehat{e}_\beta$ and therefore $\widehat{\mathfrak{g}}_+$. From that we derive

$$[\widehat{f}_\alpha, \widehat{\mathfrak{h}} + \widehat{\mathfrak{g}}_+] \subseteq \widehat{\mathfrak{h}} + \widehat{\mathfrak{g}}_+ + \mathbb{K}\widehat{f}_\alpha,$$

which in turn implies that each $\widehat{f}_\alpha$ normalizes the subspace $\widehat{\mathfrak{g}}_d := \widehat{\mathfrak{g}}_- + \widehat{\mathfrak{h}} + \widehat{\mathfrak{g}}_+$. Similarly, we obtain $e_\alpha \in \mathfrak{n}_{\widehat{\mathfrak{g}}}(\widehat{\mathfrak{g}}_d)$, so that $\widehat{\mathfrak{g}}_d$ is a subalgebra of $\widehat{\mathfrak{g}}$. Since it contains all generators, we obtain $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_d$.

Next we observe that the set of all $\widehat{\mathfrak{h}}$-weights in $\widehat{\mathfrak{g}}_\pm$ is contained in $\widehat{\Delta}_\pm$, so that we obtain in particular that $\widehat{\mathfrak{g}}_0 = \widehat{\mathfrak{h}}$.

(iii) Since $\widehat{\mathfrak{g}}_+$ is generated by the elements $\widehat{e}_\alpha \in \widehat{\mathfrak{g}}_{\widehat{\alpha}}$ and $\widehat{\Pi}$ is linearly independent, the $\widehat{\alpha}$-weight space is spanned by $\widehat{e}_\alpha$. Similarly, $\widehat{\mathfrak{g}}_{\widehat{\beta}} = \mathbb{K}\widehat{f}_\beta$.

(iv) For any derivation $D \in \mathrm{der}(\widehat{\mathfrak{g}})$, the nilspace $\widehat{\mathfrak{g}}^0(D)$ is a subalgebra because we have

$$D^n[a, b] = \sum_{k=0}^{n} \binom{n}{k} [D^k a, D^{n-k} b].$$

For $D = \operatorname{ad} \widehat{e}_\alpha$, our relations imply that this subalgebra contains each $\widehat{h}_\beta$, each $\widehat{e}_\beta$ and also each $\widehat{f}_\beta$, because $(\operatorname{ad} \widehat{e}_\alpha)^3 \widehat{f}_\alpha = 0$. This leads to $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^0(\operatorname{ad} \widehat{e}_\alpha)$, and a similar argument shows that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^0(\operatorname{ad} \widehat{f}_\alpha)$.

(v) Let $\widehat{\mathfrak{g}}(\alpha) := \operatorname{span}\{\widehat{e}_\alpha, \widehat{h}_\alpha, \widehat{f}_\alpha\}$ and note that our relations imply that $\widehat{\mathfrak{g}}(\alpha) \cong \mathfrak{sl}_2(\mathbb{K})$. Since $\operatorname{ad} \widehat{e}_\alpha$ and $\operatorname{ad} \widehat{f}_\alpha$ are locally nilpotent, for each $x \in \widehat{\mathfrak{g}}$, the $\widehat{\mathfrak{g}}(\alpha)$-submodule

$$U(\widehat{\mathfrak{g}}(\alpha))x = \sum_{\ell,m,n} (\operatorname{ad} \widehat{f}_\alpha)^\ell (\operatorname{ad} \widehat{h}_\alpha)^m (\operatorname{ad} \widehat{e}_\alpha)^n x$$

generated by x is finite-dimensional (here we use the Poincaré–Birkhoff–Witt Theorem 6.1.9 below). We therefore obtain a well-defined linear operator

$$\widehat{\sigma}_\alpha := e^{\widehat{\operatorname{ad}} e_\alpha} e^{-\widehat{\operatorname{ad}} f_\alpha} e^{\widehat{\operatorname{ad}} e_\alpha} \in \operatorname{GL}(\widehat{\mathfrak{g}}) \tag{6.3}$$

which clearly commutes with $\ker \widehat{\alpha} \subseteq \widehat{\mathfrak{h}}$ and satisfies

$$\widehat{\sigma}_\alpha(\widehat{\mathfrak{g}}_\lambda(\widehat{h}_\alpha)) = \widehat{\mathfrak{g}}_{-\lambda}(\widehat{h}_\alpha)$$

(cf. Lemma 5.2.9). Next we observe that $\sigma_{\widehat{\alpha}}$ fixes $\widehat{h}_\alpha^\perp \subseteq \widehat{\mathfrak{h}}^*$ pointwise and satisfies $\sigma_{\widehat{\alpha}} \widehat{\alpha} = -\widehat{\alpha}$. We therefore obtain $\widehat{\sigma}_\alpha(\widehat{\mathfrak{g}}_\gamma) = \widehat{\mathfrak{g}}_{\sigma_\alpha \gamma}$ for each $\gamma \in \widehat{\Delta}$. Clearly, $q \circ \sigma_{\widehat{\alpha}} = \sigma_\alpha \circ q$, and since $q|_{\widehat{\mathfrak{h}}}$ is a linear isomorphism to $\mathfrak{h}$, we see that $\widehat{\mathcal{W}}$ is a finite group isomorphic to $\mathcal{W}$.

(vi) Since $-\widehat{\Pi} \subseteq \widehat{\mathcal{W}}\widehat{\Pi}$, it suffices to show that each positive root $\gamma \in \widehat{\Delta}_+$ lies in $\widehat{\mathcal{W}}\widehat{\Pi}$. For $\gamma = \sum_{\alpha \in \Pi} n_\alpha \widehat{\alpha} \in \widehat{\Delta}_+$ we define its height by

$$\operatorname{ht}(\gamma) := \sum_{\alpha \in \Pi} n_\alpha.$$

Pick $\delta \in \widehat{\mathcal{W}}\gamma \cap \widehat{\Delta}_+$ with minimal height. If $\delta \notin \widehat{\Pi}$, then $\delta = \sum_{\alpha \in \Pi} n_\alpha \widehat{\alpha}$ with $n_{\alpha_1}, n_{\alpha_2} > 0$ for two roots $\alpha_1 \neq \alpha_2 \in \Pi$. Then

$$\sigma_{\widehat{\beta}} \delta = \delta - \delta(\widehat{h}_\beta)\widehat{\beta} = (n_\beta - \delta(\widehat{h}_\beta))\widehat{\beta} + \sum_{\alpha \in \Pi \setminus \beta} n_\alpha \widehat{\alpha} \in \widehat{\Delta}_+,$$

because $n_{\alpha_j} > 0$ for $j = 1, 2$ and one of these two roots is different from β. Therefore the minimality of the height of δ implies $\delta(\widehat{h}_\beta) \leq 0$ for each $\beta \in \Pi$. For the corresponding linear functional $\overline{\delta} \in \mathfrak{h}^*$ with $\delta = \overline{\delta} \circ q$, we then have $\overline{\delta} \in \sum_{\alpha \in \Pi} \mathbb{N}_0 \alpha$ and $\overline{\delta}(\check{\alpha}) \leq 0$ for each $\alpha \in \Pi$. This leads to

$$(\overline{\delta}, \overline{\delta}) = \sum_{\alpha \in \Pi} n_\alpha (\overline{\delta}, \alpha) = \sum_{\alpha \in \Pi} n_\alpha \overline{\delta}(h'_\alpha) \leq 0$$

because h'_α is a positive multiple of h_α (Theorem 5.3.4). As the scalar product on $\mathfrak{h}^*$ is positive definite, we arrive at a contradiction. This proves that $\delta \in \widehat{\Pi}$, and hence that $\gamma \in \widehat{\mathcal{W}}\widehat{\Pi}$. □

Theorem 6.2.10 (Serre's Theorem). *The homomorphism*

$$q \colon \widehat{\mathfrak{g}} = L(X,R) \to \mathfrak{g}$$

is an isomorphism of Lie algebras.

Proof. We know already that q is surjective, so that it remains to see that it is also injective. Let $\mathfrak{n} := \ker q$. Then $\mathfrak{n}$ is an ideal of $\widehat{\mathfrak{g}}$, hence in particular invariant under $\widehat{\mathfrak{h}}$, which in turn implies that it is adapted to the $\widehat{\mathfrak{h}}$-weight decomposition of $\widehat{\mathfrak{g}}$:

$$\mathfrak{n} = (\mathfrak{n} \cap \widehat{\mathfrak{h}}) \oplus \sum_{\gamma \in \widehat{\Delta}} (\mathfrak{n} \cap \widehat{\mathfrak{g}}_\gamma).$$

Since q is injective on $\widehat{\mathfrak{h}} + \sum_{\alpha \in \Pi} (\widehat{\mathfrak{g}}_{\widehat{\alpha}} + \widehat{\mathfrak{g}}_{-\widehat{\alpha}})$ (Lemma 6.2.9(iii)), we have $\mathfrak{n} \cap \widehat{\mathfrak{g}}_\gamma = \{0\}$ for $\gamma \in \{0\} \cup \pm\widehat{\Pi}$.

On the other hand, the invariance of the ideal $\mathfrak{n}$ under $\widehat{\mathfrak{g}}(\alpha)$ implies that it is also invariant under the linear maps $\widehat{\sigma}_\alpha \in \mathrm{GL}(\widehat{\mathfrak{g}})$, defined in (6.3), so that

$$\{\gamma \in \widehat{\Delta} \colon \mathfrak{n} \cap \widehat{\mathfrak{g}}_\gamma \neq \{0\}\}$$

is invariant under $\widehat{\mathcal{W}}$ (Lemma 6.2.9). Now $\widehat{\Delta} = \widehat{\mathcal{W}}\widehat{\Pi}$ implies that this set is empty and this proves that $\mathfrak{n} = \{0\}$. □

Corollary 6.2.11. *There exists an automorphism $\varphi \colon \mathfrak{g} \to \mathfrak{g}$ with $\varphi|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$.*

Proof. Let X be as above and define the map

$$\widehat{\varphi} \colon X \to \mathfrak{g}, \quad \widehat{\varphi}(\widehat{h}_\alpha) := -h_\alpha, \quad \widehat{\varphi}(\widehat{e}_\alpha) := -f_\alpha, \quad \widehat{\varphi}(\widehat{f}_\alpha) := -e_\alpha,$$

which defines a unique homomorphism $L(X) \to \mathfrak{g}$, and the Serre relations (6.1) and (6.2) imply that this homomorphism actually factors through a homomorphism $\widetilde{\varphi} \colon L(X,R) \to \mathfrak{g}$. Now $\varphi := \widetilde{\varphi} \circ q^{-1} \colon \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ with $\varphi|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. □

Exercises for Section 6.2

Exercise 6.2.1. Let $\mathfrak{g}$ be a Lie algebra and $(b_j)_{j \in J}$ a basis for $\mathfrak{g}$. Then the Lie bracket on $\mathfrak{g}$ is determined by the numbers c_{ij}^k (called structure constants), defined by

$$[b_i, b_j] = \sum_{k \in J} c_{ij}^k b_k.$$

Show that $\mathfrak{g} \cong L(B,R)$, where

$$B = \{b_j \colon j \in J\} \quad \text{and} \quad R = \Big\{[b_i, b_j] - \sum_{k \in J} c_{ij}^k b_k \colon i,j \in J\Big\}.$$

Exercise 6.2.2. Let $X = \{x, y\}$ be a two element set. Show that the free Lie algebra $L(X)$ is infinite-dimensional. Hint: Consider the semidirect product $\mathfrak{g} = \mathbb{K}[X] \rtimes_M \mathbb{K}$, where $\mathbb{K}[X]$ is considered as an abelian Lie algebra and $Mf(X) := Xf(X)$ is the multiplication with X.

6.3 Highest Weight Representations

We know already from Weyl's Theorem 4.5.21 on Complete Reducibility that any finite-dimensional module over a semisimple Lie algebra $\mathfrak{g}$ is semisimple. This reduces the classification of finite-dimensional modules to the classification of simple ones. In this section, we address this problem for the class of those semisimple Lie algebras which are *split*, i.e., contain a toral Cartan subalgebra. Note that $\mathfrak{sl}_n(\mathbb{K})$ and in particular any complex semisimple Lie algebra is split.

Throughout this section, $\mathfrak{g}$ denotes a semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a toral Cartan subalgebra. For $\lambda \in \mathfrak{h}^*$ and a representation (π, V) of $\mathfrak{g}$, we write

$$V_\lambda := \{v \in V : (\forall h \in \mathfrak{h})\ \pi(h)(v) = \lambda(h)v\}$$

for the corresponding weight space in V and $\mathcal{P}(V) := \{\lambda \in \mathfrak{h}^* : V_\lambda \neq \{0\}\}$ for the set of $\mathfrak{h}$-weights of V. We simply write $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$ for the set of roots and $\mathfrak{g}_\alpha := \mathfrak{g}_\alpha(\mathfrak{h})$, $\alpha \in \Delta$, for the root spaces.

Proposition 6.3.1. *If* $\dim V < \infty$*, then* $\mathfrak{h}$ *acts by diagonalizable operators on* V *and* V *is the direct sum of its weight spaces. All weights take real values on* $\mathfrak{h}_\mathbb{R}$*.*

Proof. In view of Lemma 5.3.7, $\mathfrak{h}$ is spanned by the coroots $h_\alpha = \check{\alpha}$. Therefore it suffices to see that for each root $\alpha \in \Delta$, the element $h_\alpha \in \mathfrak{h}$ is diagonalizable on V. Since the $\mathfrak{g}$-representation on V restricts to a representation of

$$\mathfrak{g}(\alpha) = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathbb{K}h_\alpha \cong \mathfrak{sl}_2(\mathbb{K})$$

on V, it suffices to apply Proposition 5.2.8. Moreover, we see that all eigenvalues of h_α are integral, which, since $\mathfrak{h}$ is abelian, implies that each weight takes only real values on the real subspace $\mathfrak{h}_\mathbb{R}$. □

Definition 6.3.2. Let $\Delta^+ \subseteq \Delta$ be a positive system of roots. (cf. Theorem 5.4.13). Then

$$\mathfrak{b} := \mathfrak{h} + \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$$

is a solvable subalgebra of $\mathfrak{g}$ because it is of the form $\mathfrak{b} = \mathfrak{n} \rtimes \mathfrak{h}$ for a nilpotent Lie algebra $\mathfrak{n}$ (Proposition 6.2.1). Subalgebras of this type are called *standard Borel subalgebras* with respect to $\mathfrak{h}$.

6.3.1 Highest Weights

Definition 6.3.3. A $\mathfrak{g}$-module V is called a *module with highest weight* $\lambda \in \mathfrak{h}^*$ if there is a $\mathfrak{b}$-invariant line $\mathbb{K}v \in V$ with

$$h \cdot v = \lambda(h)v \quad \text{for} \quad h \in \mathfrak{h},$$

and v generates the $\mathfrak{g}$-module V (i.e., V is the smallest submodule containing v). Then λ is called the *highest weight* and the nonzero elements of the generating line $\mathbb{K}v$ are called *highest weight vectors*.

Proposition 6.3.4. *Each finite-dimensional simple $\mathfrak{g}$-module is a highest weight module.*

Proof. Let V be a simple $\mathfrak{g}$-module. In view of Proposition 6.3.1, V is a direct sum of its weight spaces $V = \bigoplus_{\alpha\in\mathcal{P}(V)} V_\alpha$. Since V is finite-dimensional, the set $\mathcal{P}(V)$ of weights is finite.

Pick $x_0 \in \mathfrak{h}_\mathbb{R}$ with

$$\Delta^+ = \{\alpha \in \Delta\colon \alpha(x_0) > 0\}.$$

Then $\mathcal{P}(V)(x_0) \subseteq \mathbb{R}$ (Proposition 6.3.1) and we can pick a $\lambda \in \mathcal{P}(V)$ such that $\lambda(x_0)$ is maximal. Let $v \in V_\lambda \setminus \{0\}$. For each $\alpha \in \Delta^+$, we then have $\mathfrak{g}_\alpha \cdot v \subseteq V_{\lambda+\alpha}$, but the choice of λ implies that $V_{\lambda+\alpha} = \{0\}$. Hence v is a $\mathfrak{b}$-eigenvector of weight λ. Since V is simple, it is generated by v. □

Remark 6.3.5. If $\mathbb{K} = \mathbb{C}$, then we can also use Lie's Theorem 4.4.8 to see that a simple $\mathfrak{g}$-module V contains a $\mathfrak{b}$-eigenvector, hence is a highest weight module.

For $\beta \in \Delta^+ = \{\beta_1, \ldots, \beta_m\}$, we choose an $\mathfrak{sl}_2$-triple $(h_\beta, e_\beta, f_\beta)$ as in the $\mathfrak{sl}_2$-Theorem 5.3.4. As for abstract root systems (see p. 153), we define a partial order $\prec$ on $\mathfrak{h}^*$ by

$$\lambda \prec \mu \quad :\Longleftrightarrow \quad \mu - \lambda \in \mathbb{N}_0[\Delta^+] := \sum_{\beta\in\Delta^+} \mathbb{N}_0\beta.$$

Let $\Pi \subseteq \Delta^+$ be the corresponding set of simple roots (Theorem 5.4.13).

The following theorem describes some properties of highest weight modules which are not necessarily finite-dimensional.

Theorem 6.3.6. *Let V be a $\mathfrak{g}$-module with highest weight λ and $0 \neq v \in V_\lambda$ a highest weight vector. Then*

(i) $V = \operatorname{span}\{f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} \cdot v \mid i_j \in \mathbb{N}_0\}$. *In particular, V is the direct sum of its weight spaces.*
(ii) $\mathcal{P}(V) \subseteq \lambda - \mathbb{N}_0[\Delta^+] = \lambda - \sum_{\beta\in\Pi} \mathbb{N}_0\beta$.
(iii) $\dim V_\mu < \infty$ *for all* $\mu \in \mathcal{P}(V)$.
(iv) $\dim V_\lambda = 1$.
(v) *Every $\mathfrak{g}$-submodule of V is the direct sum of its weight spaces.*
(vi) *V contains exactly one maximal proper $\mathfrak{g}$-submodule $V_{\max}$ and $V/V_{\max}$ is the unique simple quotient module of V.*
(vii) *Every nonzero module quotient of V is a module with highest weight λ.*

Proof. (i) Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Then

$$f_{\beta_1}, \ldots, f_{\beta_m}, h_{\alpha_1}, \ldots, h_{\alpha_r}, x_{\beta_1}, \ldots, x_{\beta_m}$$

is a basis for $\mathfrak{g}$, to which we apply the Poincaré-Birkhoff-Witt Theorem 6.1.9. Then the claim follows from

$$h_{\alpha_1}^{j_1} \cdots h_{\alpha_r}^{j_r} x_{\beta_1}^{\ell_1} \cdots x_{\beta_m}^{\ell_m} \cdot v \subseteq \mathbb{K}v$$

and $V = \mathcal{U}(\mathfrak{g}) \cdot v$ (cf. Remark 6.1.4).
(ii) In view of (i), V is spanned by vectors of the form

$$f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} \cdot v,$$

which are weight vectors of weight $\lambda - \sum_{\ell=1}^m i_\ell \beta_\ell$ by Lemma 5.2.3. Assertion (ii) now follows, since the positive roots can be written as sums of simple roots.
(iii) For $\mu \in \mathfrak{h}^*$, there are only finitely many vectors of the form $f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} \cdot v$ for which $\lambda - \sum_{\ell=1}^m i_\ell \beta_\ell$ equals μ. In fact, if $x_0 \in \mathfrak{h}_{\mathbb{R}}$ satisfies $\beta_j(x_0) > 0$ for each j, then $\mu = \lambda - \sum_{l=1}^m i_\ell \beta_\ell$ yields

$$\sum_{\ell=1}^m i_\ell \beta_\ell(x_0) = (\lambda - \mu)(x_0),$$

and there are only finitely many solutions $(i_1, \ldots, i_m) \in \mathbb{N}_0^m$ of this equation.
(iv) The equality $\lambda = \lambda - \sum_{l=1}^m i_l \beta_l$ is only possible for $i_1 = \ldots = i_m = 0$.
(v) Let W be a $\mathfrak{g}$-submodule of V. Then Exercise 1.1.1(b) implies that each element of $\mathfrak{h}$ is diagonalizable on W, so that $\mathfrak{h}$ is simultaneously diagonalizable on W.
(vi) By (v), every proper submodule of V is contained in $\sum_{\mu \in \mathcal{P}(V) \setminus \{\lambda\}} V_\mu$. Therefore the union of all proper submodules is still proper, and hence, a maximal proper submodule $V_{\max}$ exists. The quotient module $V/V_{\max}$ is simple, since for every nontrivial submodule $W \subseteq V/V_{\max}$, its inverse image W' in V would be a proper submodule of V, strictly containing $V_{\max}$. Conversely, every submodule W of V, for which V/W is simple (and nonzero), is a maximal submodule, hence equal to $V_{\max}$.
(vii) This is obvious. □

Corollary 6.3.7. *If V is a simple highest weight module, then V contains only one $\mathfrak{b}$-invariant line.*

Proof. Let $\mathbb{K}v$ be a $\mathfrak{b}$-invariant line. Then v is a weight vector for some weight μ and v generates the simple module V (each simple module is generated by each nonzero element). Hence $\mathcal{P}(V) \subseteq \mu - \mathbb{N}_0[\Delta^+]$. If λ is the highest weight of V, we also have $\mathcal{P}(V) \subseteq \lambda - \mathbb{N}_0[\Delta^+]$, which leads to

$$\lambda \prec \mu \prec \lambda,$$

and hence to $\lambda = \mu$. Finally, Theorem 6.3.6(iv) implies that V_λ is one-dimensional, which completes the proof. □

Proposition 6.3.8. *Two simple* $\mathfrak{g}$*-modules with the same highest weight* λ *are isomorphic.*

Proof. Let V and W be two such modules. We choose nonzero elements $v \in V_\lambda$ and $w \in W_\lambda$. Set $M := V \oplus W$ and $m := v + w$. Then $\mathbb{K}m$ is a $\mathfrak{b}$-invariant line and the submodule $M' := \mathcal{U}(\mathfrak{g}) \cdot m$ of M generated by m is a module with highest weight λ. Let $\mathrm{pr}_V : M' \to V$ and $\mathrm{pr}_W : M' \to W$ be the canonical projections with respect to the direct sum $V \oplus W$. Then both, pr_V and pr_W, are homomorphisms of $\mathfrak{g}$-modules. From $\mathrm{pr}_V(m) = v$ and $\mathrm{pr}_W(m) = w$ we see that pr_V and pr_W are surjective. Therefore, we must have $\ker \mathrm{pr}_V = M'_{\max} = \ker \mathrm{pr}_W$ by Theorem 6.3.6(vi), and this implies $V \cong M'/M'_{\max} \cong W$. □

Definition 6.3.9 (Verma modules). Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a splitting Cartan subalgebra, and $\mathfrak{b} = \mathfrak{h} + \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$ the Borel subalgebra of $\mathfrak{g}$ corresponding to the positive system Δ^+ of Δ. For $\lambda \in \mathfrak{h}^*$, we extend λ to a linear functional, also called λ, on $\mathfrak{b}$ vanishing on all root spaces $\mathfrak{g}_\alpha$. Then

$$[\mathfrak{b}, \mathfrak{b}] = \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta \subseteq \ker \lambda$$

implies that $\lambda \colon \mathfrak{b} \to \mathbb{K} \cong \mathfrak{gl}_1(\mathbb{K})$ is a homomorphism of Lie algebras, hence defines a one-dimensional representation of $\mathfrak{b}$. We write $\mathbb{K}_\lambda$ for the corresponding $\mathfrak{b}$-module. The Lie algebra homomorphism λ further extends to an algebra homomorphism $\widetilde{\lambda} \colon \mathcal{U}(\mathfrak{b}) \to \mathbb{K}$, turning $\mathbb{K}_\lambda$ into a $\mathcal{U}(\mathfrak{b})$-module.

In the following we shall use the notation

$$AB := \mathrm{span}\{ab \colon a \in A, n \in B\}$$

for subsets A, B of an associative algebra. In this sense, we write

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) / \big(\mathcal{U}(\mathfrak{g})\{b - \lambda(b)\mathbf{1} \colon b \in \mathfrak{b}\}\big).$$

The module $M(\lambda)$ is called the *Verma module* of highest weight λ. We write $[D], D \in \mathcal{U}(\mathfrak{g})$, for its elements. Since $M(\lambda)$ is a quotient by a subspace invariant under the natural (left multiplication) action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$, it carries a natural $\mathfrak{g}$-module structure.

It is indeed a highest weight module of highest weight λ because $v := [\mathbf{1}]$ satisfies for $b \in \mathfrak{b}$:

$$b \cdot [\mathbf{1}] = [b] = [\lambda(b)\mathbf{1}] = \lambda(b)[\mathbf{1}],$$

and

$$\mathcal{U}(\mathfrak{g}) \cdot [\mathbf{1}] = [\mathcal{U}(\mathfrak{g})] = M(\lambda).$$

Using Theorem 6.3.6, we conclude that $L(\lambda) := M(\lambda)/M(\lambda)_{\max}$ is a simple highest weight module with highest weight λ. In particular, such a highest weight module exists for each $\lambda \in \mathfrak{h}^*$.

6.3.2 Classification of Finite-Dimensional Simple Modules

We want to characterize the linear functionals on $\mathfrak{h}$ which occur as highest weights of simple $\mathfrak{g}$-modules.

Definition 6.3.10. A linear functional $\lambda \in \mathfrak{h}^*$ is said to be *integral* if

$$\lambda(\check{\alpha}) \in \mathbb{Z} \quad \text{for} \quad \alpha \in \Delta,$$

and it is called *dominant* with respect to the positive system Δ^+ if

$$\lambda(\check{\alpha}) \geq 0 \quad \text{for} \quad \alpha \in \Delta^+.$$

We denote the set of all integral functionals on $\mathfrak{h}$ by Λ, and the set of all dominant integral functionals by Λ^+.

Let $\Pi \subseteq \Delta^+$ be the set of simple roots, which is a basis for $\mathfrak{h}^*$. Since the set $\check{\Pi} = \{\check{\alpha} \colon \alpha \in \Pi\}$ is a basis for the dual root system $\check{\Delta} \subseteq \mathfrak{h}_{\mathbb{R}}$ (Remark 5.4.18),

$$\Lambda = \{\lambda \in \mathfrak{h}^* \colon (\forall \alpha \in \Pi)\ \lambda(\check{\alpha}) \in \mathbb{Z}\} \cong \mathbb{Z}^r$$

is a lattice in the real vector space $\mathfrak{h}^*_{\mathbb{R}}$, called the *weight lattice* and

$$\Lambda^+ = \{\lambda \in \mathfrak{h}^* \colon (\forall \alpha \in \Pi)\ \lambda(\check{\alpha}) \in \mathbb{N}_0\} \cong \mathbb{N}_0^r.$$

Remark 6.3.11. The Weyl group of the root system Δ is the subgroup $W \subseteq \mathrm{GL}(\mathfrak{h}^*_{\mathbb{R}})$ generated by the reflections

$$\sigma_\alpha(\lambda) = \lambda - \lambda(\check{\alpha})\alpha,$$

and this formula immediately implies that the weight lattice Λ is invariant under W (Theorem 5.4.17). Moreover, Theorem 5.4.17 implies that for each integral element $\nu \in \mathfrak{h}^*$, there exists a $w \in W$ with $w(\nu) \in \Lambda^+$.

Proposition 6.3.12. *If V is a finite-dimensional module with highest weight λ, then λ is dominant integral.*

Proof. Let $\alpha \in \Delta^+$ and $\mathfrak{g}(\alpha) \cong \mathfrak{sl}_2(\mathbb{K})$ be the corresponding 3-dimensional subalgebra of $\mathfrak{g}$. If $v \in V$ is a highest weight vector, then we obtain with Proposition 5.2.4 that $\lambda(h_\alpha) = \lambda(\check{\alpha}) \in \mathbb{N}_0$. □

Lemma 6.3.13. *Let V be a $\mathfrak{g}$-module and $W = W(\Delta)$ the Weyl group of Δ. If, for each $\alpha \in \Pi$, V is a locally finite $\mathfrak{g}(\alpha)$-module, i.e., a union of finite-dimensional submodules, then each weight μ of V satisfies*

$$\dim V_\mu = \dim V_{w(\mu)}.$$

Proof. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ denote the representation of $\mathfrak{g}$ on V and $(h_\alpha, e_\alpha, f_\alpha)$ an $\mathfrak{sl}_2$-triple corresponding to $\alpha \in \Pi$. According to our hypothesis, each $v \in V$ is contained in a finite-dimensional submodule, so that

$$\Phi_\alpha v := e^{\rho(e_\alpha)} \circ e^{-\rho(f_\alpha)} \circ e^{\rho(e_\alpha)} v$$

is defined. Moreover, its restriction to each finite-dimensional $\mathfrak{g}(\alpha)$-submodule is invertible, and this implies that Φ_α actually is an element of $\mathrm{GL}(V)$. Let $\mu \in \mathfrak{h}^*$ be a weight of V. From Lemma 5.2.9 we know that Φ_α maps the h_α-eigenspace for the eigenvalue $\mu(h_\alpha)$ to the h_α-eigenspace for the eigenvalue $-\mu(h_\alpha)$. If $h \in \mathfrak{h}$ satisfies $\alpha(h) = 0$, then $\rho(h)$ commutes with Φ_α, so that Φ_α preserves the h-eigenspaces. This implies that

$$\Phi_\alpha(V_\mu) = V_{\sigma_\alpha(\mu)},$$

where $\sigma_\alpha(\mu) = \mu - \mu(\check{\alpha})\alpha$ is the reflection in W corresponding to α. Composing the various Φ_α for $\alpha \in \Pi$, we see that the set of weights of V is invariant under the Weyl group W, which is generated by $\{\sigma_\alpha\colon \alpha \in \Pi\}$ (Theorem 5.4.17). Moreover it is now clear that $\dim V_\mu = \dim V_{w(\mu)}$ for all $w \in W$. □

Proposition 6.3.14. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be a splitting Cartan subalgebra of $\mathfrak{g}$. If $\lambda \in \mathfrak{h}^*$ is dominant integral with respect to Δ^+, then the simple highest weight module $V = L(\lambda)$ of highest weight λ is finite-dimensional.*

Proof. For each simple root $\alpha \in \Pi$, using Theorem 5.3.4, we choose an $\mathfrak{sl}_2$-triple $(h_\alpha, e_\alpha, f_\alpha)$, so that $\mathfrak{g}(\alpha) = \mathrm{span}\{h_\alpha, e_\alpha, f_\alpha\} \cong \mathfrak{sl}_2(\mathbb{K})$. Let $v^+ \in V$ be a highest weight vector. For $\alpha, \beta \in \Pi$, we put $m_\alpha := \lambda(h_\alpha) = \lambda(\check{\alpha}) \in \mathbb{N}_0$ and observe that

$$e_\beta f_\alpha^{m_\alpha+1} v^+ = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ (m_\alpha + 1) f_\alpha^{m_\alpha}(h_\alpha - m_\alpha \mathbf{1}) v^+ = 0 & \text{if } \alpha = \beta. \end{cases}$$

Here we use that $[e_\beta, f_\alpha] \in \mathfrak{g}_{\beta-\alpha} = \{0\}$ for $\alpha \neq \beta$, and for $\alpha = \beta$ we use the formulas from Lemma 5.2.2(ii) and $e_\beta \cdot v^+ = 0$. Since the e_α, $\alpha \in \Pi$, generate the subalgebra $\sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$ (Proposition 6.2.1), the vector $f_\alpha^{m_\alpha+1} v^+$ generates a proper highest weight submodule of V (cf. Theorem 6.3.6 and Corollary 6.3.7), so that the irreducibility of V yields $f_\alpha^{m_\alpha+1} v^+ = 0$. Therefore

$$\mathrm{span}\{v^+, f_\alpha v^+, \ldots, f_\alpha^{m_\alpha} v^+\}$$

is a finite-dimensional $\mathfrak{g}(\alpha)$-module with highest weight $\lambda(\check{\alpha}) = m_\alpha$. Therefore the subspace V'_α of V spanned by all finite-dimensional $\mathfrak{g}(\alpha)$-submodules is nonzero.

Let $E \subseteq V$ be a finite-dimensional $\mathfrak{g}(\alpha)$-submodule. Then $\mathrm{span}(\mathfrak{g} \cdot E)$ is finite-dimensional and $\mathfrak{g}(\alpha)$-stable, so it is contained in V'. This implies that V'_α is a $\mathfrak{g}$-submodule. Since V is simple, we conclude that $V = V'_\alpha$. Thus

Lemma 6.3.13 is applicable and yields $\dim V_{w\cdot\mu} = \dim V_\mu$ for all $w \in W$ as well as the W-invariance of $\mathcal{P}(\lambda)$.

It follows from the definition of the ordering $\prec$ that the set of dominant integral elements $\mu \in \mathfrak{h}^*$ with $\mu \prec \lambda$ is finite. Since all weights of V are integral, Remark 6.3.11, together with the above, shows that

$$\mathcal{P}(\lambda) \subseteq W\big(\{\mu \in \Lambda^+ \mid \mu \prec \lambda\}\big)$$

is finite because W is finite and the set $\{\mu \in \Lambda^+ \mid \mu \prec \lambda\}$ is finite. As all weight spaces V_μ are finite dimensional by Theorem 6.3.6, this concludes the proof. □

Theorem 6.3.15 (Highest Weight Theorem). *Let $\mathfrak{g}$ be a split semisimple Lie algebra and $\mathfrak{h}$ a splitting Cartan subalgebra of $\mathfrak{g}$. The assignment $\lambda \mapsto L(\lambda)$ defines a bijection between the set Λ^+ of dominant integral functionals and the set of isomorphism classes of finite-dimensional simple $\mathfrak{g}$-modules.*

Proof. First we recall from Proposition 6.3.4 that each finite-dimensional simple $\mathfrak{g}$-module is a highest weight module with some highest weight λ. In view of Proposition 6.3.12, λ is dominant integral, and Proposition 6.3.8 shows that it only depends on the isomorphism class of the module. Finally, Proposition 6.3.14 shows that any dominant integral functional is the highest weight of some finite-dimensional simple $\mathfrak{g}$-module. □

6.3.3 The Eigenvalue of the Casimir Operator

In this section we construct a special element $C_\mathfrak{g}$ in the center of the enveloping algebra of $\mathfrak{g}$ and calculate its (scalar) action in a highest weight module. In special cases this allows to identify a given simple $\mathfrak{g}$-module.

Definition 6.3.16 (Universal Casimir element). Let $\mathfrak{g}$ be a finite-dimensional split semisimple Lie algebra with Cartan–Killing form κ. As before, we choose for each $\beta \in \Delta^+$ an $\mathfrak{sl}_2$-triple $(h_\beta, e_\beta, f_\beta)$ and $e_\beta^* \in \mathfrak{g}_{-\beta}$, $f_\beta^* \in \mathfrak{g}_\beta$ with

$$\kappa(e_\beta, e_\beta^*) = 1 = \kappa(f_\beta, f_\beta^*).$$

We further choose a basis $h_1, \ldots, h_r$ for $\mathfrak{h}$, and we write $h^1, \ldots, h^r$ for the dual basis with respect to the nondegenerate restriction of κ to $\mathfrak{h} \times \mathfrak{h}$. Then

$$\{h_i, e_\beta, f_\beta \colon i = 1, \ldots, r; \beta \in \Delta^+\}$$

is a basis for $\mathfrak{g}$ and

$$\{h^i, e_\beta^*, f_\beta^* \colon i = 1, \ldots, r; \beta \in \Delta^+\}$$

is the dual basis with respect to κ. We therefore obtain a central element of $\mathcal{U}(\mathfrak{g})$ by

$$C_{\mathfrak{g}} = \sum_{i=1}^{k} h_i h^i + \sum_{\beta \in \Delta^+} e_\beta e_\beta^* + f_\beta f_\beta^* \tag{6.4}$$

(Lemma 4.5.16). It is called the *universal Casimir element*.

Lemma 6.3.17. *For a positive system Δ^+, we put $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. If (ρ_V, V) is a highest weight module with highest weight λ, also considered as a $\mathcal{U}(\mathfrak{g})$-module, then*

$$\rho_V(C_{\mathfrak{g}}) = (\lambda, \lambda + 2\rho)\mathbf{1} = \big(\|\lambda+\rho\|^2 - \|\rho\|^2\big)\mathbf{1}.$$

If λ is dominant and nonzero, then $\rho_V(C_{\mathfrak{g}}) \neq 0$.

Proof. We write the Casimir element $C_{\mathfrak{g}}$ in the form (6.4) as described in Definition 6.3.16. We compute the action of $C_{\mathfrak{g}}$ on V. Let v^+ be a highest weight vector in V. Then $e_\beta \cdot v^+ = f_\beta^* \cdot v^+ = 0$ for each $\beta \in \Delta^+$, and $[e_\beta, e_\beta^*] = h'_\beta$ (cf. Lemma 5.3.3) implies

$$e_\beta e_\beta^* \cdot v^+ = [e_\beta, e_\beta^*]\cdot v^+ + e_\beta^* e_\beta \cdot v^+ = \lambda(h'_\beta)v^+ = (\lambda,\beta)v^+,$$

so that $\sum_{\beta\in\Delta^+}(e_\beta e_\beta^* + f_\beta f_\beta^*)\cdot v^+ = 2(\lambda,\rho)v^+$. On the other hand, we calculate

$$\sum_{i=1}^{k}\lambda(h_i)\lambda(h^i) = \lambda\Big(\sum_{i=1}^{k}\kappa(h'_\lambda, h^i)h_i\Big) = \lambda(h'_\lambda) = (\lambda,\lambda).$$

Putting these facts together yields

$$C_{\mathfrak{g}}v^+ = (\lambda,\lambda+2\rho)v^+ = \big(\|\lambda+\rho\|^2 - \|\rho\|^2\big)v^+.$$

Since $C_{\mathfrak{g}}$ is central in $\mathcal{U}(\mathfrak{g})$ (Exercise 6.1.1), $C_{\mathfrak{g}}$ acts by the same scalar on the entire $\mathcal{U}(\mathfrak{g})$-module $V = \mathcal{U}(\mathfrak{g})v^+$.

Finally, we assume that λ is dominant and nonzero. Then $\lambda(\check{\alpha}) \geq 0$ for all $\alpha \in \Delta^+$ implies that $(\lambda,\alpha) \geq 0$, and hence that $(\lambda,\rho) \geq 0$. This leads to $(\lambda,\lambda+2\rho) \geq (\lambda,\lambda) > 0$. □

Exercises for Section 6.3

Exercise 6.3.1. A $\mathfrak{g}$-module V is said to be *cyclic* if it is generated by some element $v \in V$. If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is the module structure and $\widetilde{\rho}\colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ the canonical extension, then the annihilator

$$I := \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(v) := \{D \in \mathcal{U}(\mathfrak{g})\colon \widetilde{\rho}(D)v = 0\}$$

of v is a left ideal. Show that:

(a) If $I \subseteq \mathcal{U}(\mathfrak{g})$ is a left ideal, then the quotient $\mathcal{U}(\mathfrak{g})/I$ carries a natural $\mathfrak{g}$-module structure, defined by $x \cdot (D + I) := \sigma(x)D + I$, and this $\mathfrak{g}$-module is cyclic.

(b) Every cyclic $\mathfrak{g}$-module is isomorphic to one of the form $\mathcal{U}(\mathfrak{g})/I$, as in (a).

Exercise 6.3.2. Simple $\mathfrak{g}$-modules are particular examples of cyclic $\mathfrak{g}$-modules. Show that:

(a) If $I \subseteq \mathcal{U}(\mathfrak{g})$ is a maximal (proper) left ideal, then the quotient $\mathcal{U}(\mathfrak{g})/I$ is a simple $\mathfrak{g}$-module.
(b) Every simple $\mathfrak{g}$-module is isomorphic to one of the form $\mathcal{U}(\mathfrak{g})/I$, where I is a maximal left ideal in $\mathcal{U}(\mathfrak{g})$.

Exercise 6.3.3. Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a toral Cartan subalgebra. Let V be a simple $\mathfrak{g}$-module on which $\mathfrak{h}$ acts by diagonalizable operators (such modules are called *weight modules*. Identifying the root system of $\mathfrak{g}$ with that of its semisimple commutator algebra, the notion of a positive system makes also sense for $\mathfrak{g}$. Show that, in this sense, for each positive system Δ^+ of Δ, V is a highest weight module.

6.4 Ado's Theorem

In this section we return to the question when a finite-dimensional Lie algebra can be written as a subalgebra of a finite-dimensional associative algebra equipped with the commutator bracket. Ado's Theorem in fact says that this is always the case.

Theorem 6.4.1 (Ado's Theorem). *Every finite-dimensional Lie algebra has a faithful, i.e., injective, finite-dimensional representation whose restriction to the maximal nilpotent ideal is nilpotent.*

Ado's Theorem is one of the cornerstones of finite-dimensional Lie theory. For any Lie algebra $\mathfrak{g}$, we have the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ which is faithful if and only if $\mathfrak{z}(\mathfrak{g})$ is trivial. Therefore the main point of Ado's Theorem is to find a representation of $\mathfrak{g}$ which is nontrivial on the center. A crucial first case is to see how to obtain faithful representations of nilpotent Lie algebras.

Proposition 6.4.2. *Let $\mathfrak{n}$ be a nilpotent Lie algebra with $C^k(\mathfrak{n}) = \{0\}$. For $j \in \mathbb{N}_0$, let $\mathcal{U}(\mathfrak{n})_j \subseteq \mathcal{U}(\mathfrak{n})$ be the ideal spanned by $\sum_{i \geq j} \sigma(\mathfrak{n})^i$, where $\sigma\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the canonical embedding. Then the $\mathcal{U}(\mathfrak{n})/\mathcal{U}(\mathfrak{n})_j$ are finite-dimensional associative unital algebras, and the canonical homomorphism*

$$\sigma_k\colon \mathfrak{n} \to \mathcal{U}(\mathfrak{n})/\mathcal{U}(\mathfrak{n})_k$$

is injective.

Proof. We have to show that $\mathcal{U}(\mathfrak{n})_k \cap \sigma(\mathfrak{n}) = \{0\}$.

Let $d_j := \dim C^j(\mathfrak{n})$ for $j = 1, \ldots, k$ and choose a basis e_i for $\mathfrak{n}$ with the property that for each $j < k$ the e_i, $i \leq d_j$, form a basis for $C^j(\mathfrak{n})$.

For a nonzero element $x \in \mathfrak{n}$, we defined its *weight* $w(x)$ as the number h with $x \in C^h(\mathfrak{n}) \setminus C^{h+1}(\mathfrak{n})$ and $w(0) := \infty$. Since $[C^a(\mathfrak{n}), C^b(\mathfrak{n})] \subseteq C^{a+b}(\mathfrak{n})$ (Exercise 4.2.7), we have for any two elements $x, y \in \mathfrak{n}$ the relation

$$w([x,y]) \geq w(x) + w(y).$$

Moreover, if $x = \sum_j c_j e_j \in C^a(\mathfrak{n})$, then the choice of basis tells us that all $c_j e_j$ are in $C^a(\mathfrak{n})$ as well. For any monomial $e_{i_1} e_{i_2} \cdots e_{i_s} \in \mathcal{U}(\mathfrak{n})$, we define the weight as

$$w(e_{i_1} e_{i_2} \cdots e_{i_s}) := \sum_{j=1}^{s} w(e_{i_j}).$$

Now take a monomial and rewrite it, in the sense of the Poincaré–Birkhoff–Witt Theorem, as a linear combination of monomials of the form

$$e_1^{m_1} \cdots e_j^{m_j}, \quad m_1, \ldots, m_j \in \mathbb{N}_0.$$

Each time we have to substitute a product $e_a e_b$, $b < a$, by $e_b e_a + [e_a, e_b]$, we obtain in the sum a monomial of lesser degree, but when we write $[e_a, e_b] = \sum_j c_j e_j$, the weight of each of the resulting monomials is at least as big as the weight of the monomial we started with. This shows that when we write an element of the ideal $\mathcal{U}(\mathfrak{n})_k$ in the PBW basis, we have a linear combination of elements of weight at least k, but all elements of $\sigma(\mathfrak{n})$ have weight at most $k - 1$. This completes the proof. □

It follows immediately from the PBW-Theorem that all ideals $\mathcal{U}(\mathfrak{n})_j$ have finite codimension. Since left multiplication defines a faithful finite-dimensional representation for each unital finite-dimensional associative algebra, the preceding proposition already proves Ado's Theorem for the special class of nilpotent Lie algebras.

Lemma 6.4.3. *Let $\mathfrak{g}$ be a Lie algebra and $\sigma\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})_L$ be the canonical homomorphism. For every derivation $D \in \operatorname{der}(\mathfrak{g})$, there exists a unique derivation $\widetilde{D} \in \operatorname{der}(\mathcal{U}(\mathfrak{g}))$ with $\sigma \circ D = \widetilde{D} \circ \sigma$.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \forall D\ } & \mathfrak{g} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathcal{U}(\mathfrak{g}) & \xrightarrow[\exists \widetilde{D}]{} & \mathcal{U}(\mathfrak{g}) \end{array}$$

Proof. Let $\mathcal{A} := M_2(\mathcal{U}(\mathfrak{g}))$ be the algebra of the 2×2-matrices with entries in $\mathcal{U}(\mathfrak{g})$. Then the map

$$\varphi\colon \mathfrak{g} \to \mathcal{A}_L, \quad \varphi(x) = \begin{pmatrix} x & D(x) \\ 0 & x \end{pmatrix}$$

is a homomorphism of Lie algebras. From the universal property of $(\mathcal{U}(\mathfrak{g}), \sigma)$, we now obtain a homomorphism $\widetilde{\varphi}\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ of unital algebras with $\widetilde{\varphi} \circ \sigma = \varphi$. Since $\mathcal{U}(\mathfrak{g})$ is generated by $\mathbf{1}$ and $\sigma(\mathfrak{g})$, we have

$$\widetilde{\varphi}(x) = \begin{pmatrix} x & \widetilde{D}(x) \\ 0 & x \end{pmatrix}$$

for some linear map $\widetilde{D}\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, and since $\widetilde{\varphi}$ is an algebra homomorphism, $\widetilde{D}$ is a derivation. It immediately follows from the construction that $\widetilde{D} \circ \sigma = D$. The uniqueness of $\widetilde{D}$ follows from the fact that $\sigma(\mathfrak{g})$ and $\mathbf{1}$ generate $\mathcal{U}(\mathfrak{g})$ because a derivation annihilates the unit and it is determined by its values on a generating set. ◻

Proposition 6.4.4. *If $\mathfrak{g} = \mathfrak{n} \rtimes_\gamma \mathfrak{l}$ is a semidirect sum of a nilpotent ideal and a Lie algebra $\mathfrak{l}$, then $\mathfrak{g}$ has a finite-dimensional representation which is faithful and nilpotent on $\mathfrak{n}$.*

Proof. Suppose that $C^k(\mathfrak{n}) = \{0\}$ and put $\mathcal{A} := \mathcal{U}(\mathfrak{n})/\mathcal{U}(\mathfrak{n})_k$. Then Proposition 6.4.2 implies that the natural map $\sigma_k\colon \mathfrak{n} \to \mathcal{A}$ is injective. Clearly, this defines a nilpotent representation of $\mathfrak{n}$ on $\mathcal{A}$.

In view of the preceding Lemma 6.4.3, the representation $\gamma\colon \mathfrak{l} \to \operatorname{der}(\mathfrak{n})$ induces a representation $\widetilde{\gamma}\colon \mathfrak{l} \to \operatorname{der}(\mathcal{U}(\mathfrak{n}))$, and it is clear that the derivations $\widetilde{\gamma}(x)$ preserve the ideal $\mathcal{U}(\mathfrak{n})_k$, hence induce derivations $\overline{\gamma}(x)$ on $\mathcal{A}$. Write $L_x\colon \mathcal{A} \to \mathcal{A}, a \mapsto xa$ for the left multiplications on $\mathcal{A}$. Then

$$\pi\colon \mathfrak{g} = \mathfrak{n} \rtimes_\gamma \mathfrak{l} \to \mathfrak{gl}(\mathcal{A}), \quad (n, x) \mapsto L_{\sigma_k(n)} + \overline{\gamma}(x)$$

defines a representation of $\mathfrak{g}$ on $\mathcal{A}$ whose restriction to $\mathfrak{n}$ is faithful and nilpotent. ◻

Lemma 6.4.5. *Let $\mathfrak{n} \subseteq \mathfrak{g}$ be a nilpotent ideal and $x \in \mathfrak{g}$. If there exists an $n \in \mathfrak{n}$ such that $\operatorname{ad}(x+n)$ is nilpotent, then $\operatorname{ad} x$ is nilpotent.*

Proof. Let $\mathcal{F} = (\mathfrak{g}_0, \ldots, \mathfrak{g}_k)$ be a maximal flag in $\mathfrak{g}$ consisting of ideals. This means in particular that the quotient algebras $\mathfrak{q}_j := \mathfrak{g}_j/\mathfrak{g}_{j-1}$, $j = 1, \ldots, k$, are simple $\mathfrak{g}$-modules. As $\mathfrak{g}$ is a nilpotent $\mathfrak{n}$-module, the same holds for all the simple quotients $\mathfrak{g}_{j+1}/\mathfrak{g}_j$, so that $\mathfrak{n}$ acts trivially on them. This means that $[\mathfrak{n}, \mathfrak{g}_{j+1}] \subseteq \mathfrak{g}_j$, i.e., $\operatorname{ad} \mathfrak{n} \subseteq \mathfrak{g}_n(\mathcal{F})$. But $\operatorname{ad} x$ is nilpotent if and only if all the induced maps $\operatorname{ad}_{\mathfrak{g}_{j+1}/\mathfrak{g}_j}(x)$ are nilpotent. In view of

$$\operatorname{ad}_{\mathfrak{g}_{j+1}/\mathfrak{g}_j}(x) = \operatorname{ad}_{\mathfrak{g}_{j+1}/\mathfrak{g}_j}(x+n),$$

all these maps are nilpotent, so that $\operatorname{ad} x$ is nilpotent. ◻

Lemma 6.4.6. *Let $\mathfrak{r}$ be a solvable Lie algebra and (ρ, V) a finite-dimensional representation of $\mathfrak{r}$. Then the set*

$$\mathfrak{n}_\rho := \{x \in \mathfrak{r}\colon (\exists N \in \mathbb{N})\ \rho_V(x)^N = 0\}$$

of those elements in $\mathfrak{r}$ for which $\rho_V(x)$ is nilpotent is an ideal of $\mathfrak{r}$.

Proof. After complexification, we may assume that $\mathbb{K} = \mathbb{C}$. Then we use Lie's Theorem to find a complete flag $\mathcal{F}$ in V with $\rho_V(\mathfrak{r}) \subseteq \mathfrak{g}(\mathcal{F})$. Then $\mathfrak{n}_\rho = \rho_V^{-1}(\mathfrak{g}_n(\mathcal{F}))$ is the inverse image of the ideal $\mathfrak{g}_n(\mathcal{F})$ of $\mathfrak{g}(\mathcal{F})$, hence an ideal of $\mathfrak{r}$. □

Remark 6.4.7. Applying the preceding lemma to the adjoint representation, we see that for a finite-dimensional Lie algebra $\mathfrak{g}$, the set

$$\mathfrak{n}_{\mathrm{ad}} := \{x \in \mathrm{rad}(\mathfrak{g}) \colon (\exists N \in \mathbb{N})\ (\mathrm{ad}\, x)^N = 0\}$$

is an ideal of $\mathfrak{r}$, whose nilpotency follows from Engel's Theorem. Moreover, $\mathfrak{n}_{\mathrm{ad}}$ contains each nilpotent ideal of $\mathfrak{g}$, hence is the maximal nilpotent ideal of $\mathfrak{g}$.

Proof of Ado's Theorem. We shall prove Ado's Theorem by embedding $\mathfrak{g}$ into a semidirect sum of a nilpotent ideal and a reductive Lie algebra, to which we apply Proposition 6.4.4.

Let $\mathfrak{r} := \mathrm{rad}(\mathfrak{g})$ and $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$. Applying Proposition 4.5.17 to the semisimple $\mathfrak{s}$-module $\mathfrak{r}$, we see that

$$\mathfrak{r} = [\mathfrak{s}, \mathfrak{r}] + \mathfrak{z}_{\mathfrak{r}}(\mathfrak{s}).$$

Let $\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{r}}(\mathfrak{s})$ be a Cartan subalgebra. Then $\mathfrak{d} := (\mathrm{ad}\, \mathfrak{h})_s \subseteq \mathfrak{gl}(\mathfrak{g})$ is a commutative subalgebra of derivations of $\mathfrak{g}$ (Lemma 5.1.13). We consider the Lie algebra

$$\widehat{\mathfrak{g}} := \mathfrak{g} \rtimes \mathfrak{d}.$$

Our goal is to show that $\widehat{\mathfrak{g}}$ is the semidirect sum of a reductive Lie algebra and a nilpotent ideal.

Since $\mathrm{ad}\, \mathfrak{h}$ annihilates the subspace $\mathfrak{s}$ of $\mathfrak{g}$, the same is true for $\mathfrak{d}$, so that $\mathfrak{l} := \mathfrak{s} \oplus \mathfrak{d}$ is a reductive Lie algebra with center $\mathfrak{d}$. We also note that

$$\widehat{\mathfrak{g}} = (\mathfrak{r} \rtimes \mathfrak{s}) \rtimes \mathfrak{d} = \widehat{\mathfrak{r}} \rtimes \mathfrak{s},$$

where $\widehat{\mathfrak{r}} := \mathfrak{r} \rtimes \mathfrak{d}$ is the solvable radical of $\widehat{\mathfrak{g}}$.

Next we recall that the quotient map $\mathfrak{z}_{\mathfrak{r}}(\mathfrak{s}) \to \mathfrak{z}_{\mathfrak{r}}(\mathfrak{s})/[\mathfrak{z}_{\mathfrak{r}}(\mathfrak{s}), \mathfrak{z}_{\mathfrak{r}}(\mathfrak{s})]$ maps $\mathfrak{h}$ surjectively onto the abelian quotient algebra (Proposition 5.1.11(ii)), so that

$$\mathfrak{z}_{\mathfrak{r}}(\mathfrak{s}) = [\mathfrak{z}_{\mathfrak{r}}(\mathfrak{s}), \mathfrak{z}_{\mathfrak{r}}(\mathfrak{s})] + \mathfrak{h} \subseteq [\mathfrak{r}, \mathfrak{r}] + \mathfrak{h},$$

and thus

$$\mathfrak{r} \subseteq [\mathfrak{s}, \mathfrak{r}] + \mathfrak{z}_{\mathfrak{r}}(\mathfrak{s}) \subseteq [\mathfrak{g}, \mathfrak{r}] + \mathfrak{h}.$$

Let $x \in \mathfrak{r}$ and write it as $x = n + h$ with $n \in [\mathfrak{g}, \mathfrak{r}]$ and $h \in \mathfrak{h}$. Then $d := (\mathrm{ad}\, h)_s \in \mathfrak{d}$, and thus $x - d \in \widehat{\mathfrak{r}}$. Now $[\mathfrak{g}, \mathfrak{r}]$ is a nilpotent ideal of $\widehat{\mathfrak{g}}$ and $\mathrm{ad}_{\widehat{\mathfrak{g}}}(h - d)$ is nilpotent on $\mathfrak{g} \supseteq [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$, where it coincides with $\mathrm{ad}_{\mathfrak{g}}(h) - d$. We thus conclude with Lemma 6.4.5 that $\mathrm{ad}_{\widehat{\mathfrak{g}}}(x - d)$ is nilpotent, hence contained in the maximal nilpotent ideal

$$\widehat{\mathfrak{n}} := \{x \in \widehat{\mathfrak{r}} \colon (\exists N \in \mathbb{N})\, (\operatorname{ad} x)^N = 0\}$$

of $\widehat{\mathfrak{g}}$ (Remark 6.4.7). We conclude that $\widehat{\mathfrak{r}} \subseteq \widehat{\mathfrak{n}} + \mathfrak{d}$. On the other hand, $\widehat{\mathfrak{n}} \cap \mathfrak{d} = \{0\}$ follows from the fact that the nonzero elements of $\mathfrak{d}$ act by semisimple endomorphisms on $\mathfrak{g}$. Therefore

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}} \rtimes \mathfrak{l}$$

is a semidirect sum of the nilpotent ideal $\widehat{\mathfrak{n}}$ and the reductive Lie algebra $\mathfrak{l}$. Note that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{z}(\widehat{\mathfrak{g}}) \subseteq \widehat{\mathfrak{n}}$. Finally, Proposition 6.4.4 provides a representation $\widehat{\pi} \colon \widehat{\mathfrak{g}} \to \mathfrak{gl}(V)$ which is faithful and nilpotent on $\widehat{\mathfrak{n}}$. Hence the restriction $\pi := \widehat{\pi}|_{\mathfrak{g}}$ is faithful on $\mathfrak{z}(\mathfrak{g})$. Now the direct sum representation $\pi \oplus \operatorname{ad}$ of $\mathfrak{g}$ on $V \oplus \mathfrak{g}$ is a faithful finite-dimensional representation of $\mathfrak{g}$. Since the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ is clearly contained in $\widehat{\mathfrak{n}}$, the representation is nilpotent on $\mathfrak{n}$. □

Exercises for Section 6.4

Exercise 6.4.1. Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{b}$ be a characteristic ideal in $\mathfrak{g}$ and $\mathfrak{a}$ be a characteristic ideal in $\mathfrak{b}$. Then $\mathfrak{a}$ is a characteristic ideal in $\mathfrak{g}$.

Exercise 6.4.2. Find an example of an ideal of an ideal of a Lie algebra $\mathfrak{g}$ which is not an ideal of $\mathfrak{g}$.

Exercise 6.4.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, let $\mathcal{U}(\mathfrak{g})$ be its enveloping algebra and $\sigma : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the canonical embedding. By $\mathcal{U}_i(\mathfrak{g})$, we denote the subspace spanned by the products of degree less or equal to i (cf. Lemma 6.1.6). Show:

(i) For every automorphism α of the Lie algebra $\mathfrak{g}$, there is precisely one automorphism $\mathcal{U}(\alpha)$ of $\mathcal{U}(\mathfrak{g})$ with $\mathcal{U}(\alpha) \circ \sigma = \sigma \circ \alpha$. This automorphism leaves $\mathcal{U}_i(\mathfrak{g})$ invariant for every $i \in \mathbb{N}$.

(ii) If $\gamma : \mathbb{R} \to \operatorname{Aut}\big(\mathcal{U}(\mathfrak{g})\big)$ is a one-parameter group of automorphisms of $\mathcal{U}(\mathfrak{g})$ for which

$$\gamma(t)\mathcal{U}_i(\mathfrak{g}) \subseteq \mathcal{U}_i(\mathfrak{g}) \qquad \text{for} \quad i \in \mathbb{N},$$

then $D := \frac{d}{dt}|_{t=0}\gamma(t)$ exists and it is a derivation of $\mathcal{U}(\mathfrak{g})$.

(iii) Apply (i) and (ii) to obtain a new proof of Lemma 6.4.3.
By (i), the map $\mathcal{U}(e^{\operatorname{ad} x})$ is a well defined automorphism of $\mathcal{U}(\mathfrak{g})$ for every $x \in \mathfrak{g}$, and the derivation $\operatorname{ad} x$ can directly be continued to $\mathcal{U}(\mathfrak{g})$ by

$$\operatorname{ad} x(z) := xz - zx \qquad \text{for} \quad x \in \mathfrak{g}, z \in \mathcal{U}(\mathfrak{g}).$$

(iv) For $z \in \mathcal{U}(\mathfrak{g})$, the following statements are equivalent:
 (i) $z \in Z\big(\mathcal{U}(\mathfrak{g})\big)$.
 (ii) $\operatorname{ad} x(z) = 0$ for all $x \in \mathfrak{g}$.
 (iii) $\mathcal{U}(e^{\operatorname{ad} x})z = z$ for all $x \in \mathfrak{g}$.

Exercise 6.4.4. Let $\alpha : \mathfrak{g} \to \operatorname{End}(V)$ be a representation of the Lie algebra $\mathfrak{g}$ and $\widetilde{\alpha} : \mathcal{U}(\mathfrak{g}) \to \operatorname{End}(V)$ be the corresponding representation of the enveloping algebra. Show: Let $z \in Z\big(\mathcal{U}(\mathfrak{g})\big)$ and V_α be an eigenspace of $\widetilde{\alpha}(z)$, then V_α is invariant under $\alpha(\mathfrak{g})$.

6.5 Lie Algebra Cohomology

The cohomology of Lie algebras is the natural tool to understand how we can build new Lie algebras $\widehat{\mathfrak{g}}$ from given Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$ in such a way that $\mathfrak{n} \trianglelefteq \widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}/\mathfrak{n} \cong \mathfrak{g}$. An important special case of this situation arises if $\mathfrak{n}$ is assumed to be abelian, so that $\mathfrak{n}$ is simply a $\mathfrak{g}$-module. We shall see, in particular, how the abelian extensions of Lie algebras can be parameterized by a certain cohomology space.

We shall also deal with the extension problem for $\mathfrak{g}$-modules, i.e., the problem to determine, for a pair (V, W) of $\mathfrak{g}$-modules, how many nonisomorphic modules $\widehat{V}$ exist which contain W as a submodule and satisfy $\widehat{V}/W \cong V$.

Throughout this section $\mathfrak{g}$ denotes a Lie algebra over the field $\mathbb{K}$. We do not have to make any assumption on the dimension of $\mathfrak{g}$ or the nature of the field $\mathbb{K}$.

6.5.1 Basic Definitions and Properties

Definition 6.5.1. Let V and W be vector spaces and $p \in \mathbb{N}$. A multilinear map $f\colon W^p \to V$ is called *alternating* if

$$f(w_{\sigma_1}, \ldots, w_{\sigma_p}) = \operatorname{sgn}(\sigma) f(w_1, \ldots, w_p)$$

for $w_i \in W$ and $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_p$.

Definition 6.5.2. Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module.
(a) We write $C^p(\mathfrak{g}, V)$ for the space of alternating p-linear mappings $\mathfrak{g}^p \to V$ (the *p-cochains*) and put $C^0(\mathfrak{g}, V) := V$. We also define

$$C(\mathfrak{g}, V) := \bigoplus_{k=0}^{\infty} C^k(\mathfrak{g}, V).$$

On $C^p(\mathfrak{g}, V)$ we define the *(Chevalley–Eilenberg) differential* $\mathtt{d}$ by

$$\begin{aligned}\mathtt{d}\omega(x_0, \ldots, x_p) := &\sum_{j=0}^{p} (-1)^j x_j \cdot \omega(x_0, \ldots, \widehat{x}_j, \ldots, x_p)\\ &+ \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p),\end{aligned}$$

where $\widehat{x}_j$ means that x_j is omitted. Observe that the right hand side defines for each $\omega \in C^p(\mathfrak{g}, V)$ an element of $C^{p+1}(\mathfrak{g}, V)$ because it is alternating. To see that this is the case, it suffices to show that it vanishes if $x_i = x_{i+1}$ for $i = 0, \ldots, p-1$. Since ω is alternating, we only to note that for $x_i = x_{i+1}$, we have

$$\begin{aligned} 0 &= (-1)^i x_i \cdot \omega(x_0, \ldots, \widehat{x}_i, x_{i+1} \ldots, x_p) \\ &\qquad + (-1)^{i+1} x_i \cdot \omega(x_0, \ldots, x_i, \widehat{x}_{i+1}, \ldots, x_p) \\ 0 &= (-1)^{i+j} \omega([x_i, x_j], \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p) \\ &\qquad + (-1)^{i+j+1} \omega([x_{i+1}, x_j], \ldots, \widehat{x}_{i+1}, \ldots, \widehat{x}_j, \ldots, x_p) \\ 0 &= (-1)^{j+i} \omega([x_j, x_i], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\ &\qquad + (-1)^{j+i+1} \omega([x_j, x_{i+1}], \ldots, \widehat{x}_j, \ldots, \widehat{x}_{i+1}, \ldots, x_p). \end{aligned}$$

Putting the differentials on all the spaces $C^p(\mathfrak{g}, V)$ together, we obtain a linear map $\mathtt{d} = \mathtt{d}_{\mathfrak{g}} \colon C(\mathfrak{g}, V) \to C(\mathfrak{g}, V)$.

The elements of the subspace

$$Z^k(\mathfrak{g}, V) := \ker(\mathtt{d}|_{C^k(\mathfrak{g}, V)})$$

as called *k-cocycles*, and the elements of the spaces

$$B^k(\mathfrak{g}, V) := \mathtt{d}\big(C^{k-1}(\mathfrak{g}, V)\big) \quad \text{and} \quad B^0(\mathfrak{g}, V) := \{0\}$$

are called *k-coboundaries*. We will see below that $\mathtt{d}^2 = 0$, which implies that $B^k(\mathfrak{g}, V) \subseteq Z^k(\mathfrak{g}, V)$, so that it makes sense to define the k^{th} *cohomology space of* $\mathfrak{g}$ *with values in the module* V:

$$H^k(\mathfrak{g}, V) := Z^k(\mathfrak{g}, V)/B^k(\mathfrak{g}, V).$$

(b) We further define for each $x \in \mathfrak{g}$ and $p > 0$ the *insertion map* or *contraction*

$$i_x \colon C^p(\mathfrak{g}, V) \to C^{p-1}(\mathfrak{g}, V), \quad \big(i_x\omega\big)(x_1, \ldots, x_{p-1}) = \omega(x, x_1, \ldots, x_{p-1}).$$

We further define i_x to be 0 on $C^0(\mathfrak{g}, V)$.

Note 6.5.3. The names (co)cycle, (co)boundary, and (co)boundary operator are derived from simplicial homology theory which has large formal similarities with the cohomology theory for Lie algebras. In simplicial homology one considers *simplices*, such as triangles or tetrahedra. A triangle is described by its vertices x_1, x_2, x_3. Its faces with inherent orientation given by the ordering of the indices are denoted by x_{12}, x_{23}, x_{31}. The entire triangle is denoted by x_{123}. Chains are formal linear combinations of the type $\sum a_{ij} x_{ij}$. Cycles are chains which have no boundary, for instance $x_{12} + x_{23} + x_{31}$ or $2x_{12} + 2x_{23} + 2x_{31}$. For the latter one, the cycle $x_{12} + x_{23} + x_{31}$ is passed through twice. The boundary of x_{12} is $x_2 - x_1$, the boundary of x_{123} is $x_{12} + x_{23} + x_{31}$. Chains which are themselves boundaries of something have no boundary, i.e., they are cycles. This corresponds to the relation $\mathtt{d}^2 = 0$ in Lie algebra cohomology.

Remark 6.5.4. For elements of low degree we have in particular:

$$\begin{aligned}
p=0: &\quad \mathtt{d}\omega(x) &&= x\cdot\omega\\
p=1: &\quad \mathtt{d}\omega(x,y) &&= x\cdot\omega(y) - y\cdot\omega(x) - \omega([x,y])\\
p=2: &\quad \mathtt{d}\omega(x,y,z) &&= x\cdot\omega(y,z) - y\cdot\omega(x,z) + z\cdot\omega(x,y)\\
& && \quad -\omega([x,y],z) + \omega([x,z],y) - \omega([y,z],x)\\
& && = x\cdot\omega(y,z) + y\cdot\omega(z,x) + z\cdot\omega(x,y)\\
& && \quad -\omega([x,y],z) - \omega([y,z],x) - \omega([z,x],y).
\end{aligned}$$

Example 6.5.5. (a) This means that

$$Z^0(\mathfrak{g},V) = V^{\mathfrak{g}} := \{v\in V\colon \mathfrak{g}\cdot v = \{0\}\}$$

is the maximal trivial submodule of V. Since $B^0(\mathfrak{g},V)$ is trivial by definition, we obtain

$$H^0(\mathfrak{g},V) = V^{\mathfrak{g}}.$$

(b) The elements $\alpha\in Z^1(\mathfrak{g},V)$ are also called *crossed homomorphisms*. They are defined by the condition

$$\alpha([x,y]) = x\cdot\alpha(y) - y\cdot\alpha(x),\quad x,y\in\mathfrak{g}.$$

The elements $\alpha(x)\cdot v := x\cdot v$ of the subspace $B^1(\mathfrak{g},V)$ are also called *principal crossed homomorphisms*. It follows immediately from the definition of a $\mathfrak{g}$-module that each principal crossed homomorphism is a crossed homomorphism.

If V is a trivial module, then it is not hard to compute the cohomology spaces in degree one. In view of $\{0\} = \mathtt{d}V = \mathtt{d}C^0(\mathfrak{g},V) = B^1(\mathfrak{g},V)$, we have $H^1(\mathfrak{g},V) = Z^1(\mathfrak{g},V)$, and the condition that $\alpha\colon\mathfrak{g}\to V$ is a crossed homomorphism reduces to $\alpha([x,y]) = \{0\}$ for $x,y\in\mathfrak{g}$. This leads to

$$H^1(\mathfrak{g},V)\cong \mathrm{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],V)\cong \mathrm{Hom}_{\mathrm{Liealg}}(\mathfrak{g},V).$$

Example 6.5.6. Let $\mathfrak{g}$ be a Lie algebra and $V := \mathfrak{g}$, considered as a trivial $\mathfrak{g}$-module. Then the Lie bracket

$$\beta\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},\quad \beta(x,y) := [x,y]$$

is a 2-cocycle. In fact, the two Lie algebra axioms mean that $\beta\in C^2(\mathfrak{g},\mathfrak{g})$ and $\mathtt{d}\beta = 0$.

Example 6.5.7. Let $\mathfrak{g}$ be an abelian Lie algebra and V a trivial $\mathfrak{g}$-module. Then $\mathtt{d} = 0$, so that $H^p(\mathfrak{g},V) = C^p(\mathfrak{g},V)$ holds for each $p\in\mathbb{N}_0$.

Our first goal will be to show that $\mathtt{d}^2 = 0$. This can be proved directly by an awkward computation (Exercise 6.5.7). We will follow another way which is more conceptual and leads to additional insights and tools which are useful in other situations: Let (ρ_V, V) be a $\mathfrak{g}$-module. Then the representations on the space $\mathrm{Mult}^p(\mathfrak{g}, V)$ of p-linear maps $\mathfrak{g}^p \to V$ defined by

$$\rho_+(x)\omega := \rho_V(x) \circ \omega$$

and

$$\big(\rho_j(x)\omega\big)(x_1, \ldots, x_p) := -\omega(x_1, \ldots, x_{j-1}, \operatorname{ad} x(x_j), x_{j+1}, \ldots, x_p)$$

commute pairwise. Therefore the sum of these representations is again a representation which implies the following lemma, by restricting to the subspace $C^p(\mathfrak{g}, V) \subseteq \mathrm{Mult}^p(\mathfrak{g}, V)$.

Lemma 6.5.8. *There exists a representation $\mathcal{L} = \rho_+ + \rho_0 = \rho_+ + \sum_{j=1}^p \rho_j$ of $\mathfrak{g}$ on $C(\mathfrak{g}, V)$ given on the subspace $C^p(\mathfrak{g}, V)$ by*

$$\begin{aligned}(\mathcal{L}_x\omega)(x_1, \ldots, x_p) \\ &= x \cdot \omega(x_1, \ldots, x_p) - \sum_{j=1}^p \omega(x_1, \ldots, x_{j-1}, [x, x_j], x_{j+1}, \ldots, x_p) \\ &= x \cdot \omega(x_1, \ldots, x_p) + \sum_{j=1}^p (-1)^j \omega([x, x_j], x_1, \ldots, \widehat{x}_j, \ldots, x_p).\end{aligned}$$

Note that ρ_0 is the representation on $C(\mathfrak{g}, V)$ corresponding to the trivial module structure on V.

Lemma 6.5.9 (Cartan Formula). *The representation $\mathcal{L} \colon \mathfrak{g} \to \mathfrak{gl}\big(C(\mathfrak{g}, V)\big)$ satisfies, for $x \in \mathfrak{g}$, the Cartan formula*

$$\mathcal{L}_x = \mathtt{d} \circ i_x + i_x \circ \mathtt{d}. \tag{6.5}$$

Proof. Using the insertion map i_{x_0}, we can rewrite the formula for the differential as

$$\begin{aligned}
&\big(i_{x_0}\mathtt{d}\omega\big)(x_1,\ldots,x_p)\\
=&x_0\cdot\omega(x_1,\ldots,x_p)-\sum_{j=1}^{p}(-1)^{j-1}x_j\cdot\omega(x_0,\ldots,\widehat{x}_j,\ldots,x_p)\\
&+\sum_{j=1}^{p}(-1)^{j}\omega([x_0,x_j],x_1,\ldots,\widehat{x}_j,\ldots,x_p)\\
&+\sum_{1\leq i<j}(-1)^{i+j}\omega([x_i,x_j],x_0,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_p)\\
=&x_0\cdot\omega(x_1,\ldots,x_p)-\sum_{j=1}^{p}\omega(x_1,\ldots,x_{j-1},[x_0,x_j],x_{j+1},\ldots,x_p)\\
&-\sum_{j=1}^{p}(-1)^{j-1}x_j\cdot\omega(x_0,\ldots,\widehat{x}_j,\ldots,x_p)\\
&-\sum_{1\leq i<j}(-1)^{i+j}\omega(x_0,[x_i,x_j],\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_p)\\
=&(\mathcal{L}_{x_0}\omega)(x_1,\ldots,x_p)-\mathtt{d}\big(i_{x_0}\omega\big)(x_1,\ldots,x_p).
\end{aligned}$$

This proves the Cartan formula. □

Lemma 6.5.10. *Any two elements $x, y \in \mathfrak{g}$ satisfy $i_{[x,y]} = [i_x, \mathcal{L}_y]$.*

Proof. The explicit formula for $\mathcal{L}_y$ (Lemma 6.5.8) yields for $x = x_1$ the relation $i_x\mathcal{L}_y = \mathcal{L}_y i_x - i_{[y,x]}$. □

Lemma 6.5.11. *For each $x \in \mathfrak{g}$, the Lie derivative $\mathcal{L}_x$ commutes with $\mathtt{d}$.*

Proof. In view of Lemma 6.5.10, we obtain with the Cartan formula

$$\begin{aligned}
[\mathcal{L}_x,\mathcal{L}_y] &= [\mathtt{d}\circ i_x,\mathcal{L}_y]+[i_x\circ\mathtt{d},\mathcal{L}_y]\\
&= [\mathtt{d},\mathcal{L}_y]\circ i_x+\mathtt{d}\circ i_{[x,y]}+i_{[x,y]}\circ\mathtt{d}+i_x\circ[\mathtt{d},\mathcal{L}_y]\\
&= [\mathtt{d},\mathcal{L}_y]\circ i_x+\mathcal{L}_{[x,y]}+i_x\circ[\mathtt{d},\mathcal{L}_y],
\end{aligned}$$

so that the fact that $\mathcal{L}$ is a representation leads to

$$[\mathtt{d},\mathcal{L}_y]\circ i_x+i_x\circ[\mathtt{d},\mathcal{L}_y]=0. \tag{6.6}$$

We now prove by induction over k that $[\mathtt{d},\mathcal{L}_y]$ vanishes on $C^k(\mathfrak{g},V)$. For $\omega\in C^0(\mathfrak{g},V)\cong V$, we have

$$\begin{aligned}
&\big([\mathtt{d},\mathcal{L}_y]\omega\big)(x)=\mathtt{d}(y\cdot\omega)(x)-\big(y\cdot(\mathtt{d}\omega)\big)(x)\\
&=x\cdot(y\cdot\omega)-\big(y\cdot(x\cdot\omega)-\mathtt{d}\omega([y,x])\big)=[x,y]\cdot\omega+[y,x]\cdot\omega=0.
\end{aligned}$$

Suppose that $[\mathtt{d},\mathcal{L}_y]C^k(\mathfrak{g},V)=\{0\}$. Then (6.6) implies that

$$i_x[\mathtt{d},\mathcal{L}_y]C^{k+1}(\mathfrak{g},V) = -[\mathtt{d},\mathcal{L}_y]i_xC^{k+1}(\mathfrak{g},V) \subseteq [\mathtt{d},\mathcal{L}_y]C^k(\mathfrak{g},V) = \{0\}$$

for each $x \in \mathfrak{g}$. Hence $[\mathtt{d},\mathcal{L}_y]C^{k+1}(\mathfrak{g},V) = \{0\}$. By induction, this leads to $[\mathtt{d},\mathcal{L}_y] = 0$ for each $y \in \mathfrak{g}$. □

Proposition 6.5.12. $\mathtt{d}^2 = 0$.

Proof. We put Lemma 6.5.11 into the Cartan Formula (6.5) and get

$$0 = [\mathtt{d},\mathcal{L}_x] = \mathtt{d}^2 \circ i_x - i_x \circ \mathtt{d}^2. \tag{6.7}$$

We use this formula to show by induction over k that $\mathtt{d}^2$ vanishes on $C^k(\mathfrak{g},V)$. For $\omega \in C^0(\mathfrak{g},V) \cong V$, we have $\mathtt{d}\omega(x) = x \cdot \omega$ and

$$\mathtt{d}^2\omega(x,y) = x\cdot\mathtt{d}\omega(y) - y\cdot\mathtt{d}\omega(x) - (\mathtt{d}\omega)([x,y]) = x\cdot(y\cdot\omega) - y\cdot(x\cdot\omega) - [x,y]\cdot\omega = 0.$$

If $\mathtt{d}^2(C^k(\mathfrak{g},V)) = \{0\}$, we use (6.7) to see that

$$i_x\mathtt{d}^2(C^{k+1}(\mathfrak{g},V)) = \mathtt{d}^2 i_x C^{k+1}(\mathfrak{g},V) \subseteq \mathtt{d}^2(C^k(\mathfrak{g},V)) = \{0\}$$

for all $x \in \mathfrak{g}$, and hence that $\mathtt{d}^2(C^{k+1}(\mathfrak{g},V)) = \{0\}$. By induction on k, this proves $\mathtt{d}^2 = 0$. □

Since the differential commutes with the action of $\mathfrak{g}$ on the graded vector space $C(\mathfrak{g},V)$ (Lemma 6.5.11), the space of k-cocycles and of k-coboundaries is $\mathfrak{g}$-invariant, so that we obtain a natural representation of $\mathfrak{g}$ on the quotient spaces $H^k(\mathfrak{g},V)$.

Lemma 6.5.13. *The action of $\mathfrak{g}$ on $H^k(\mathfrak{g},V)$ is trivial, i.e., $\mathcal{L}_\mathfrak{g} Z^k(\mathfrak{g},V) \subseteq B^k(\mathfrak{g},V)$.*

Proof. In view of Lemma 6.5.9, we have for $\omega \in Z^k(\mathfrak{g},V)$ the relation

$$\mathcal{L}_x\omega = i_x\mathtt{d}\omega + \mathtt{d}\big(i_x\omega\big) = \mathtt{d}\big(i_x\omega\big) \in B^k(\mathfrak{g},V).$$

Hence the $\mathfrak{g}$-action induced on the cohomology space $H^k(\mathfrak{g},V)$ is trivial. □

Remark 6.5.14. Let M be a smooth manifold and $\mathfrak{g} := \mathcal{V}(M)$ the Lie algebra of smooth vector fields on M (cf. Chapter 8 below). We consider the $\mathfrak{g}$-module $V := C^\infty(M,\mathbb{R})$ of smooth functions on M. Then we can identify the space $\Omega^p(M,\mathbb{R})$ of alternating $C^\infty(M,\mathbb{R})$-multilinear maps $\mathfrak{g}^p \to C^\infty(M,\mathbb{R})$ with a subspace of $C^p(\mathfrak{g},V)$. The elements of the space $\Omega^p(M,\mathbb{R})$ are called *smooth p-forms on M*, and

$$\Omega(M,\mathbb{R}) := \bigoplus_{p\in\mathbb{N}_0} \Omega^p(M,\mathbb{R})$$

is the space of *exterior forms on M*. The restriction of $\mathtt{d}$ to these spaces is called the *exterior differential.* The space $\Omega(M,\mathbb{R})$ is invariant under the differential $\mathtt{d}$ and the $\mathfrak{g}$-action given by the *Lie derivative.* Together with the exterior derivative, the spaces $\Omega^p(M,\mathbb{R})$ now form the so called *de Rham complex*

$$\Omega^0(M,\mathbb{R}) = C^\infty(M,\mathbb{R}) \xrightarrow{\mathtt{d}} \Omega^1(M,\mathbb{R}) \xrightarrow{\mathtt{d}} \Omega^2(M,\mathbb{R}) \xrightarrow{\mathtt{d}} \dots$$

The cohomology groups of this subcomplex are the *de Rham cohomology groups* of M:

$$H^0_{\mathrm{dR}}(M,\mathbb{R}) := \ker\big(\mathtt{d}|_{\Omega^0(M,\mathbb{R})}\big), \quad H^p_{\mathrm{dR}}(M,\mathbb{R}) := \frac{\ker\big(\mathtt{d}|_{\Omega^p(M,\mathbb{R})}\big)}{\mathtt{d}(\Omega^{p-1}(M,\mathbb{R}))} \quad \text{for} \quad p > 0.$$

6.5.2 Extensions and Cocycles

In this section we interprete the cohomology spaces in low degrees in terms of extensions of modules and Lie algebras.

Definition 6.5.15. (a) Each element $\omega \in Z^1(\mathfrak{g}, V)$ defines a $\mathfrak{g}$-module

$$V_\omega := V \times \mathbb{K} \quad \text{with} \quad x \cdot (v,t) = (x \cdot v + t\omega(x), 0).$$

Then the inclusion $j \colon V \hookrightarrow V_\omega$ is a module homomorphism, and we thus obtain a short exact sequence

$$\mathbf{0} \to V \hookrightarrow V_\omega \twoheadrightarrow \mathbb{K} \to \mathbf{0} \tag{6.8}$$

of $\mathfrak{g}$-modules, where $\mathbb{K}$ is considered as a trivial $\mathfrak{g}$-module.
(b) In the following we write $\mathfrak{aff}(V) := V \rtimes \mathfrak{gl}(V)$ for the *affine Lie algebra of* V with the Lie bracket

$$[(v,A),(v',A')] = (A \cdot v' - A' \cdot v, [A,A']).$$

An *affine representation* of a Lie algebra $\mathfrak{g}$ on V is identified with a homomorphism $\pi \colon \mathfrak{g} \to \mathfrak{aff}(V)$. We associate with each pair $(v,A) \in \mathfrak{aff}(V)$ the affine map $w \mapsto Aw + v$. The Lie algebra $\mathfrak{aff}(V)$ acts linearly on the space $V \times \mathbb{K}$ by $(v,A) \cdot (w,t) := (Aw + tv, 0)$.

Proposition 6.5.16. *Let (ρ, V) be a $\mathfrak{g}$-module. An element $\omega \in C^1(\mathfrak{g}, V)$ is in $Z^1(\mathfrak{g}, V)$ if and only if the map*

$$\rho_\omega \colon \mathfrak{g} \to \mathfrak{aff}(V) \cong V \rtimes \mathfrak{gl}(V), \quad x \mapsto \big(\omega(x), \rho(x)\big)$$

is a homomorphism of Lie algebras.

Let $e^{\operatorname{ad} V} := \mathbf{1} + \operatorname{ad} V \subseteq \operatorname{Aut}(\mathfrak{aff}(V))$ denote the group of automorphisms defined by the abelian ideal $V \trianglelefteq \mathfrak{aff}(V)$. Then the space $H^1(\mathfrak{g}, V)$ parameterizes the $e^{\operatorname{ad} V}$-conjugacy classes of affine representations of $\mathfrak{g}$ on V whose corresponding linear representation is ρ. The coboundaries correspond to those affine representations which are conjugate to a linear representation, i.e., which have a fixed point $p \in V$ in the sense that $\rho(x)p + \omega(x) = 0$ for all $x \in \mathfrak{g}$.

Proof. The first assertion is easily checked. For $v \in V$ we consider the automorphism of $\mathfrak{aff}(V)$ given by $\eta_v = e^{\operatorname{ad} v} := \mathbf{1} + \operatorname{ad} v$. Then $\eta_v(w, A) = (w - A \cdot v, A)$, so that

$$\eta_v \circ \rho_\omega = \rho_{\omega - \mathtt{d}v},$$

where $(\mathtt{d}v)(x) = x \cdot v$. Thus two affine representations ρ_ω and $\rho_{\omega'}$ are conjugate under some η_v if and only if the cohomology classes of ω and ω' coincide. In this sense, $H^1(\mathfrak{g}, V)$ parameterizes the $e^{\operatorname{ad} V}$-conjugacy classes of affine representations of $\mathfrak{g}$ on V whose corresponding linear representation coincides with ρ. The coboundaries correspond to those affine representations which are conjugate to a linear representation. Moreover, it is clear that an affine representation ρ_ω is conjugate to a linear representation, if and only if there exists a *fixed point* $v \in V$, i.e., $\omega = -\mathtt{d}v$. □

Definition 6.5.17. (a) Let $\mathfrak{g}$ and $\mathfrak{n}$ be Lie algebras. A *short exact sequence*

$$\mathbf{0} \to \mathfrak{n} \xrightarrow{\iota} \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to \mathbf{0}$$

(this means ι injective, q surjective, and $\operatorname{im} \iota = \ker q$) is called an *extension of* $\mathfrak{g}$ *by* $\mathfrak{n}$. If we identify $\mathfrak{n}$ with its image in $\widehat{\mathfrak{g}}$, this means that $\widehat{\mathfrak{g}}$ is a Lie algebra containing $\mathfrak{n}$ as an ideal such that $\widehat{\mathfrak{g}}/\mathfrak{n} \cong \mathfrak{g}$. If $\mathfrak{n}$ is abelian (central) in $\widehat{\mathfrak{g}}$, then the extension is called *abelian (central)*. Two extensions $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_1 \twoheadrightarrow \mathfrak{g}$ and $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_2 \twoheadrightarrow \mathfrak{g}$ are called *equivalent* if there exists a Lie algebra homomorphism $\varphi\colon \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the diagram

$$\begin{array}{ccccc} \mathfrak{n} & \xrightarrow{\iota_1} & \widehat{\mathfrak{g}}_1 & \xrightarrow{q_1} & \mathfrak{g} \\ \downarrow{\scriptstyle \operatorname{id}_{\mathfrak{n}}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \operatorname{id}_{\mathfrak{g}}} \\ \mathfrak{n} & \xrightarrow{\iota_2} & \widehat{\mathfrak{g}}_2 & \xrightarrow{q_2} & \mathfrak{g} \end{array}$$

is commutative. It is easy to see that this implies that φ is an isomorphism of Lie algebras (Exercise).
(b) We call an extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q = \mathfrak{n}$ *trivial*, or say that the extension *splits* if there exists a Lie algebra homomorphism $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \operatorname{id}_{\mathfrak{g}}$. In this case the map

$$\mathfrak{n} \rtimes_\delta \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (n, x) \mapsto n + \sigma(x)$$

is an isomorphism, where the semidirect sum is defined by the homomorphism

$$\delta\colon \mathfrak{g} \to \operatorname{der}(\mathfrak{n}), \quad \delta(x)(n) := [\sigma(x), n].$$

For a trivial central extension we have $\delta = 0$ and therefore $\widehat{\mathfrak{g}} \cong \mathfrak{n} \times \mathfrak{g}$.
(c) A particular important case arises if $\mathfrak{n}$ is abelian. Then each Lie algebra extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{n}$ leads to a $\mathfrak{g}$-module structure on $\mathfrak{n}$ defined by $q(x) \cdot n := [x, n]$, which is well defined because $[\mathfrak{n}, \mathfrak{n}] = \{0\}$. It is easy to see that equivalent extensions lead to the same module structure (Exercise). Therefore it makes sense to write $\operatorname{Ext}_\rho(\mathfrak{g}, \mathfrak{n})$ for the set of equivalence classes

of extensions of $\mathfrak{g}$ by $\mathfrak{n}$ corresponding to the module structure given by the representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n}) = \mathrm{der}(\mathfrak{n}).$$

For a $\mathfrak{g}$-module V, we also write $\mathrm{Ext}(\mathfrak{g}, V) := \mathrm{Ext}_{\rho_V}(\mathfrak{g}, V)$, where ρ_V is the representation of $\mathfrak{g}$ on V corresponding to the module structure.

Proposition 6.5.18. *For an element $\omega \in C^2(\mathfrak{g}, V)$, the formula*

$$[(v, x), (v', x')] = (x \cdot v' - x' \cdot v + \omega(x, x'), [x, x'])$$

defines a Lie bracket on $V \times \mathfrak{g}$ if and only if $\omega \in Z^2(\mathfrak{g}, V)$. For a cocycle $\omega \in Z^2(\mathfrak{g}, V)$ we write $\mathfrak{g}_\omega := V \oplus_\omega \mathfrak{g}$ for the corresponding Lie algebra. Then we obtain for each cocycle ω an extension of $\mathfrak{g}$ by the abelian ideal V:

$$\mathbf{0} \to V \hookrightarrow \mathfrak{g}_\omega \twoheadrightarrow \mathfrak{g} \to \mathbf{0}.$$

This extension splits if and only if ω is a coboundary.

The map $Z^2(\mathfrak{g}, V) \to \mathrm{Ext}(\mathfrak{g}, V)$ defined by assigning to ω the equivalence class of the extension $\mathfrak{g}_\omega$ induces a bijection

$$H^2(\mathfrak{g}, V) \to \mathrm{Ext}(\mathfrak{g}, V).$$

Therefore $H^2(\mathfrak{g}, V)$ classifies the abelian extensions of $\mathfrak{g}$ by V for which the corresponding representation of $\mathfrak{g}$ on V is given by the module structure on V.

Proof. An easy calculation shows that $\mathfrak{g}_\omega = V \oplus_\omega \mathfrak{g}$ is a Lie algebra if and only if ω is a 2-cocycle, i.e., an element of $Z^2(\mathfrak{g}, V)$.

To see that every abelian Lie algebra extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q = V$ (as a $\mathfrak{g}$-module) is equivalent to some $\mathfrak{g}_\omega$, let $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be a linear map with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then the map

$$V \times \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (v, x) \mapsto v + \sigma(x)$$

is a bijection, and it becomes an isomorphism of Lie algebras if we endow $V \times \mathfrak{g}$ with the bracket of $\mathfrak{g}_\omega$ for

$$\omega(x, y) := [\sigma(x), \sigma(y)] - \sigma([x, y]).$$

This implies that $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is equivalent to $\mathfrak{g}_\omega$, and therefore that the map $Z^2(\mathfrak{g}, V) \to \mathrm{Ext}(\mathfrak{g}, V)$ is surjective.

Two Lie algebras $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega'}$ are equivalent as V-extensions of $\mathfrak{g}$ if and only if there exists a linear map $\varphi\colon \mathfrak{g} \to V$ such that the map

$$\widetilde{\varphi}\colon \mathfrak{g}_\omega = V \times \mathfrak{g} \to \mathfrak{g}_{\omega'} = V \times \mathfrak{g}, \quad (v, x) \mapsto (v + \varphi(x), x)$$

is a Lie algebra homomorphism. This means that

$$\begin{aligned} \widetilde{\varphi}([(v, x), (v', x')]) &= \widetilde{\varphi}(x \cdot v' - x' \cdot v + \omega(x, x'), [x, x']) \\ &= \big(x \cdot v' - x' \cdot v + \omega(x, x') + \varphi([x, x']), [x, x']\big) \end{aligned}$$

equals

$$\begin{aligned}[\widetilde{\varphi}(v,x),\widetilde{\varphi}(v',x')] &= (x\cdot(v'+\varphi(x')) - x'\cdot(v+\varphi(x)) + \omega'(x,x'),[x,x']) \\ &= (x\cdot v' - x'\cdot v + x\cdot\varphi(x') - x'\cdot\varphi(x) + \omega'(x,x'),[x,x']),\end{aligned}$$

which is equivalent to

$$\omega'(x,x') - \omega(x,x') = \varphi([x,x']) - x\cdot\varphi(x') + x'\cdot\varphi(x) = -(\mathtt{d}\varphi)(x,x').$$

Therefore $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega'}$ are equivalent abelian extensions of $\mathfrak{g}$ if and only if $\omega'-\omega$ is a coboundary. Hence the map

$$Z^2(\mathfrak{g},V)\to \mathrm{Ext}(\mathfrak{g},V), \quad \omega\mapsto[\mathfrak{g}_\omega]$$

induces a bijection $H^2(\mathfrak{g},V)\to\mathrm{Ext}(\mathfrak{g},V)$. □

Example 6.5.19. If $\mathfrak{g}=\mathbb{K}$ is the one-dimensional Lie algebra, then the $\mathfrak{g}$-module structures on a vector space V are in one-to-one correspondence with endomorphisms $D\in\mathrm{End}(V)$.

In this case $C^p(\mathfrak{g},V)$ vanishes for $p>1$, and we have

$$C^0(\mathfrak{g},V)=V \quad\text{and}\quad C^1(\mathfrak{g},V)\cong V.$$

With respect to this identification, the map $\mathtt{d}$ corresponds to the endomorphism $D\in\mathrm{End}(V)$, so that

$$H^0(\mathfrak{g},V)=V^{\mathfrak{g}}=\ker D \quad\text{and}\quad H^1(\mathfrak{g},V)\cong V/D(V)=\mathrm{coker}\,(D).$$

Example 6.5.20. Let $\mathfrak{g}:=\mathbb{K}^2$ be the abelian two-dimensional Lie algebra with the canonical basis e_1, e_2. Then a $\mathfrak{g}$-module structure on a vector space V corresponds to a pair (D_1,D_2) of commuting endomorphisms of V.

We have

$$C^1(\mathfrak{g},V)=\mathrm{Hom}(\mathbb{K}^2,V)\cong V^2 \quad\text{and}\quad C^2(\mathfrak{g},V)\cong V,$$

where the isomorphism $C^2(\mathfrak{g},V)\to V$ is given by $\beta\mapsto\beta(e_1,e_2)$. As $\dim\mathfrak{g}=2$, we have $C^3(\mathfrak{g},V)=\{0\}$, so that $C^2(\mathfrak{g},V)=Z^2(\mathfrak{g},V)$. For $(v,w)\in V^2\cong C^1(\mathfrak{g},V)$ we have

$$\mathtt{d}(v,w)(e_1,e_2)=e_1\cdot w-e_2\cdot v=D_1(w)-D_2(v),$$

and thus

$$H^2(\mathfrak{g},V)\cong V/(D_1(V)+D_2(V)).$$

Proposition 6.5.21. *If $V=\mathfrak{g}$ with respect to the adjoint representation, then $Z^1(\mathfrak{g},\mathfrak{g})\cong\mathrm{der}\,\mathfrak{g}$, $B^1(\mathfrak{g},\mathfrak{g})\cong\mathrm{ad}\,\mathfrak{g}$ and*

$$H^1(\mathfrak{g},\mathfrak{g})\cong\mathrm{out}(\mathfrak{g}):=\mathrm{der}\,\mathfrak{g}/\,\mathrm{ad}\,\mathfrak{g}$$

is the space of outer derivations of $\mathfrak{g}$.

Proof. Let $V = \mathfrak{g}$ with respect to the adjoint representation. For $c \in C^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{End}(\mathfrak{g})$ we then have

$$\mathtt{d}c(x, y) = [x, c(y)] - [y, c(x)] - c([x, y]),$$

showing that $Z^1(\mathfrak{g}, \mathfrak{g}) = \operatorname{der} \mathfrak{g}$. For $c \in C^0(\mathfrak{g}, \mathfrak{g}) \cong \mathfrak{g}$, we have $\mathtt{d}c(x) = [x, c]$, showing that $B^1(\mathfrak{g}, \mathfrak{g}) = \operatorname{ad} \mathfrak{g}$. □

Definition 6.5.22. Let $\mathfrak{g}$ be a Lie algebra and V and W modules of $\mathfrak{g}$. A *short exact sequence*

$$\mathbf{0} \to W \xrightarrow{\iota} \widehat{V} \xrightarrow{q} V \to \mathbf{0} \tag{6.9}$$

(this means ι injective, q surjective, and $\operatorname{im} \iota = \ker q$) is called an *extension of V by W*. If we identify W with its image in $\widehat{V}$, this means that $\widehat{V}$ is a $\mathfrak{g}$-module containing W as a submodule such that $\widehat{V}/W \cong V$.

Two extensions $W \hookrightarrow \widehat{V}_1 \twoheadrightarrow V$ and $W \hookrightarrow \widehat{V}_2 \twoheadrightarrow V$ are called *equivalent* if there exists a module homomorphism $\varphi\colon \widehat{V}_1 \to \widehat{V}_2$ such that the diagram

$$\begin{array}{ccccc} W & \xrightarrow{\iota_1} & \widehat{V}_1 & \xrightarrow{q_1} & V \\ \big\downarrow{\scriptstyle \mathrm{id}_W} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \mathrm{id}_V} \\ W & \xrightarrow{\iota_2} & \widehat{V}_2 & \xrightarrow{q_2} & V \end{array}$$

commutes. It is easy to see that this implies that φ is an isomorphism of $\mathfrak{g}$-modules (Exercise) and that we thus obtain an equivalence relation on the class of extensions of V by W. We write $\mathrm{Ext}(V, W)$ for the set of equivalence classes of module extensions of V by W. We call an extension (6.9) *trivial*, or say that the extension *splits*, if there exists a module homomorphism $\sigma\colon V \to \widehat{V}$ with $q \circ \sigma = \mathrm{id}_V$. In this case the map

$$W \oplus V \to \widehat{V}, \quad (w, x) \mapsto \iota(w) + \sigma(x)$$

is a module isomorphism.

The following proposition gives a cohomological interpretation of the set $\mathrm{Ext}(B, A)$ for two $\mathfrak{g}$-modules B and A. In particular it shows that this set carries a natural vector space structure.

Proposition 6.5.23. *For $\mathfrak{g}$-modules (ρ_A, A) and (ρ_B, B),*

$$\mathrm{Ext}(B, A) \cong H^1(\mathfrak{g}, \mathrm{Hom}(B, A)),$$

where the representation of $\mathfrak{g}$ on $\mathrm{Hom}(B, A)$ is given by

$$x \cdot \varphi = \rho_A(x)\varphi - \varphi\rho_B(x).$$

Proof. First we check that a module extension $q\colon C \twoheadrightarrow B$ by the module A can be written as a space $C = A \times B$ on which the $\mathfrak{g}$-module representation is given by

$$x \cdot (a, b) = (x \cdot a + \omega(x)(b), x \cdot b), \tag{6.10}$$

where $\omega \in \operatorname{Hom}(\mathfrak{g}, \operatorname{Hom}(B, A)) = C^1(\mathfrak{g}, \operatorname{Hom}(B, A))$. To see this, we simply choose a linear map $\sigma\colon B \to C$ with $q \circ \sigma = \mathrm{id}_B$ and define

$$\omega(x)(b) := x \cdot \sigma(b) - \sigma(x \cdot b).$$

Then the linear bijection $A \times B \to C, (a, b) \mapsto \sigma(b) + a$ is a module isomorphism with respect to the above module structure.

If $\omega \in C^1(\mathfrak{g}, \operatorname{Hom}(B, A))$ is given, then the condition that (6.10) defines a $\mathfrak{g}$-module structure on C means that

$$\begin{aligned}
&\big([y, x] \cdot a + \omega([y, x])(b), [y, x] \cdot b\big) \\
&= [y, x] \cdot (a, b) \overset{!}{=} y \cdot (x \cdot (a, b)) - x \cdot (y \cdot (a, b)) \\
&= \Big(y \cdot (x \cdot a) + y \cdot \omega(x)(b) + \omega(y)(x \cdot b), y \cdot (x \cdot b)\Big) \\
&\quad - \Big(x \cdot (y \cdot a) + x \cdot \omega(y)(b) + \omega(x)(y \cdot b), x \cdot (y \cdot b)\Big) \\
&= \Big([y, x] \cdot a + y \cdot \omega(x)(b) + \omega(y)(x \cdot b) - x \cdot \omega(y)(b) - \omega(x)(y \cdot b), [y, x] \cdot b\Big) \\
&= \Big([y, x] \cdot a + (y \cdot \omega(x))(b) - (x \cdot \omega(y))(b), [y, x] \cdot b\Big)
\end{aligned}$$

This is equivalent to

$$\omega([y, x]) = y \cdot \omega(x) - x \cdot \omega(y),$$

which in turn means that $\omega \in Z^1(\mathfrak{g}, \operatorname{Hom}(B, A))$. The different parameterizations of C as $B \times A$ correspond to linear maps $\sigma\colon B \to C$ with $q \circ \sigma = \mathrm{id}_B$, where $q\colon C \to B$ is the quotient map. In this sense we have

$$\omega_\sigma(x)(b) = x \cdot \sigma(b) - \sigma(x \cdot b).$$

For a linear map $\gamma \in \operatorname{Hom}(B, A)$, we therefore have

$$\omega_{\sigma+\gamma}(x)(b) = \omega_\sigma(x)(b) + (x \cdot \gamma)(b),$$

i.e., $\omega_{\sigma+\gamma} = \omega_\sigma + \mathtt{d}\gamma$. We conclude that the different sections lead to cohomologous cocycles and this observation leads to a bijection $\operatorname{Ext}(B, A) \cong H^1\big(\mathfrak{g}, \operatorname{Hom}(B, A)\big)$. ☐

6.5.3 Invariant Volume Forms and Cohomology

In this subsection we characterize the existence of volume forms in terms of cohomology. We refer the reader to Appendix B for background material on exterior products of alternating maps.

Proposition 6.5.24. *For a finite-dimensional real Lie algebra* $\mathfrak{g}$*, the following are equivalent:*

(i) $\operatorname{tr}(\operatorname{ad} x) = 0$ *for all* $x \in \mathfrak{g}$.
(ii) $\mathfrak{g}$ *carries a volume form* $\mu \in C^n(\mathfrak{g}, \mathbb{R})$ *invariant under the adjoint action.*
(iii) $H^n(\mathfrak{g}, \mathbb{R}) \neq \{0\}$ *for* $n = \dim \mathfrak{g}$.
(iv) $\dim H^n(\mathfrak{g}, \mathbb{R}) = 1$.

Proof. Let $x_1, \ldots, x_n$ be a basis for $\mathfrak{g}$ and $x_1^*, \ldots, x_n^*$ the dual basis for $\mathfrak{g}^*$. Then

$$\mu = x_1^* \wedge \cdots \wedge x_n^*$$

is a nonzero volume form on $\mathfrak{g}$, so that $C^n(\mathfrak{g}, \mathbb{R}) = \mathbb{R}\mu$ (cf. Definition B.2.27).

(i) $\Leftrightarrow$ (ii): For any $x \in \mathfrak{g}$ we have

$$\mathcal{L}_x \mu = -\operatorname{tr}(\operatorname{ad} x)\mu.$$

In fact, let $[x, x_j] = \sum_{k=1}^n a_{kj} x_k$. Then

$$(\mathcal{L}_x \mu)(x_1, \ldots, x_n) = -\sum_{j=1}^n \mu(x_1, \ldots, x_{j-1}, [x, x_j], x_{j+1}, \ldots, x_n)$$

$$= -\sum_{j=1}^n a_{jj} \mu(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) = -\sum_{j=1}^n a_{jj} = -\operatorname{tr}(\operatorname{ad} x).$$

(iii) $\Leftrightarrow$ (iv) follows from $\dim C^n(\mathfrak{g}, \mathbb{R}) = 1$.
(ii) $\Leftrightarrow$ (iii): For each j we have

$$i_{x_j} \mu = (-1)^{j-1} x_1^* \wedge \cdots x_{j-1}^* \wedge x_{j+1}^* \cdots \wedge x_n^*,$$

and these elements form a basis for $C^{n-1}(\mathfrak{g}, \mathbb{R})$. Therefore $\mathtt{d}\mu = 0$ and the Cartan Formula imply that

$$B^n(\mathfrak{g}, \mathbb{R}) = \mathtt{d}(i_{\mathfrak{g}} \mu) = \mathcal{L}_{\mathfrak{g}} \mu.$$

This space vanishes if and only if μ is invariant if and only if $H^n(\mathfrak{g}, \mathbb{R})$ is nonzero. ☐

For reasons that we shall understand later in our discussion of invariant measures on Lie groups, Lie algebras satisfying the equivalent conditions from the preceding proposition are called *unimodular*.

Example 6.5.25. (a) Each perfect real Lie algebra $\mathfrak{g}$ is unimodular. In fact,

$$\operatorname{tr} \circ \operatorname{ad} \colon \mathfrak{g} \to \mathbb{R}$$

is a homomorphism of Lie algebras, so that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \subseteq \ker(\operatorname{tr} \circ \operatorname{ad})$. In particular, each semisimple Lie algebra is unimodular.

(b) Each nilpotent Lie algebra is unimodular because $\operatorname{ad} x$ is nilpotent for each $x \in \mathfrak{g}$, which implies in particular that $\operatorname{tr}(\operatorname{ad} x) = 0$.

(c) The 2-dimensional nonabelian Lie algebra $\mathfrak{g} = \operatorname{span}\{e_1, e_2\}$ with $[e_1, e_2] = e_2$ is not unimodular because $\operatorname{tr}(\operatorname{ad} e_1) = 1 \neq 0$.

6.5.4 Cohomology of Semisimple Lie Algebras

In this subsection, we discuss the connection between Weyl's and Levi's Theorems and a more general theorem of J. H. C. Whitehead, concerning the cohomology of finite-dimensional modules of semisimple Lie algebras.

If $\mathfrak{s}$ is a semisimple finite-dimensional Lie algebra, then Weyl's Theorem states that every finite-dimensional $\mathfrak{g}$-module V is semisimple. This means that every submodule has a module complement, and therefore that all module extensions are trivial. In view of Proposition 6.5.23, this is equivalent to

$$H^1(\mathfrak{g}, \mathrm{Hom}(A, B)) = \{0\}$$

for each pair of finite-dimensional $\mathfrak{g}$-modules A and B. For $A = \mathbb{K}$, the trivial module, we have $\mathrm{Hom}(A, B) \cong B$ as $\mathfrak{g}$-modules, and therefore we obtain $H^1(\mathfrak{g}, B) = \{0\}$. This argument proves the

Lemma 6.5.26 (First Whitehead Lemma). *If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, then each finite-dimensional $\mathfrak{g}$-module V satisfies*

$$H^1(\mathfrak{g}, V) = \{0\}.$$

From the above argument it is easy to see that the First Whitehead Lemma says essentially the same as Weyl's Theorem.

Now let $\omega \in Z^2(\mathfrak{g}, V)$ be a 2-cocycle and $\mathfrak{g}_\omega := V \oplus_\omega \mathfrak{g}$ the corresponding abelian Lie algebra extension of $\mathfrak{g}$ by V. Then $V = \mathrm{rad}(\mathfrak{g}_\omega)$ because V is an abelian ideal of $\mathfrak{g}_\omega$ and the quotient $\mathfrak{g}_\omega/V \cong \mathfrak{g}$ is semisimple. Therefore Levi's Theorem implies the existence of a Levi complement in $\widehat{\mathfrak{g}}_\omega$, which means that there exists a Lie algebra homomorphism $\sigma\colon \mathfrak{g} \to \mathfrak{g}_\omega$, splitting the exact sequence $V \hookrightarrow \mathfrak{g}_\omega \xrightarrow{q} \mathfrak{g}$, i.e., satisfying $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Now Proposition 6.5.18 implies that ω is a coboundary, which leads to the

Lemma 6.5.27 (Second Whitehead Lemma). *If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, then each finite-dimensional $\mathfrak{g}$-module V satisfies*

$$H^2(\mathfrak{g}, V) = \{0\}.$$

Again we see from the argument above that the Second Whitehead Lemma is essentially the case of Levi's Theorem where $\alpha\colon \mathfrak{g} \to \mathfrak{s}$ is a surjective homomorphism onto a semisimple Lie algebra $\mathfrak{s}$ with abelian kernel, but this was the crucial case in the proof of Levi's Theorem.

We now aim at more general results concerning also the higher degree cohomology of modules of semisimple Lie algebras.

Lemma 6.5.28. *If $\omega \in C^p(\mathfrak{g}, V)^{\mathfrak{g}}$ is a $\mathfrak{g}$-invariant element with respect to the action ρ from Lemma 6.5.8, and $\mathtt{d}^0$ is the Chevalley–Eilenberg differential with respect to the trivial $\mathfrak{g}$-module structure on V, then $\mathtt{d}\omega = -\mathtt{d}^0\omega$.*

Proof. First we note that for $x_0, x_1, \ldots, x_p \in \mathfrak{g}$, the invariance of ω implies that

$$\begin{aligned}
&x_i \cdot \omega(x_0, \ldots, \widehat{x}_i, \ldots, x_p) \\
&= \omega([x_i, x_0], x_1, \ldots, \widehat{x}_i, \ldots, x_p) + \ldots + \omega(x_0, \ldots, \widehat{x}_i, \ldots, [x_i, x_p]) \\
&= \sum_{j=0}^{i-1} (-1)^j \omega([x_i, x_j], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&\qquad + \sum_{j=i+1}^{p} (-1)^{j+1} \omega([x_i, x_j], \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p).
\end{aligned}$$

This leads to

$$\begin{aligned}
&\sum_{i=0}^{p} (-1)^i x_i \cdot \omega(x_0, \ldots, \widehat{x}_i, \ldots, x_p) \\
&= \sum_{j<i} (-1)^{j+i} \omega([x_i, x_j], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&\quad + \sum_{j>i} (-1)^{i+j+1} \omega([x_i, x_j], \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p) \\
&= \sum_{j<i} (-1)^{j+i} \omega([x_i, x_j], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&\quad + \sum_{j<i} (-1)^{i+j+1} \omega([x_j, x_i], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&= \sum_{j<i} (-1)^{j+i} \omega([x_i, x_j], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&\quad + \sum_{j<i} (-1)^{i+j} \omega([x_i, x_j], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&= 2 \sum_{j<i} (-1)^{i+j} \omega([x_i, x_j], \ldots, \widehat{x}_j, \ldots, \widehat{x}_i, \ldots, x_p) \\
&= -2(\mathtt{d}^0 \omega)(x_0, \ldots, x_p).
\end{aligned}$$

We conclude that $\mathtt{d}\omega = -2\mathtt{d}^0\omega + \mathtt{d}^0\omega = -\mathtt{d}^0\omega$. □

Lemma 6.5.29. *If V is a finite-dimensional module over the semisimple Lie algebra $\mathfrak{g}$, then*

$$H^p(\mathfrak{g}, V) \cong Z^p(\mathfrak{g}, V)^{\mathfrak{g}} / B^p(\mathfrak{g}, V)^{\mathfrak{g}},$$

i.e., each cohomology class can be represented by an invariant cocycle.

Proof. From Proposition 4.5.17 we obtain for each $p \in \mathbb{N}_0$ the decomposition

$$Z^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V)^{\mathfrak{g}} \oplus \mathfrak{g} \cdot Z^p(\mathfrak{g}, V).$$

Since $\mathfrak{g}$ acts trivially on the quotient space $H^p(\mathfrak{g}, V)$, we have $\mathfrak{g} \cdot Z^p(\mathfrak{g}, V) \subseteq B^p(\mathfrak{g}, V)$, and the assertion follows. □

Proposition 6.5.30. *If V is a trivial $\mathfrak{g}$-module, then*

$$H^p(\mathfrak{g}, V) \cong Z^p(\mathfrak{g}, V)^{\mathfrak{g}} = C^p(\mathfrak{g}, V)^{\mathfrak{g}} \quad \textit{for each } p \in \mathbb{N}_0.$$

Proof. The preceding Lemma 6.5.29 shows that each cohomology class is represented by an invariant cocycle. Further, Lemma 6.5.28 and the triviality of V implies that each $\mathfrak{g}$-invariant cochain ω satisfies $\mathtt{d}^0\omega = \mathtt{d}\omega = -\mathtt{d}^0\omega$, so that ω is a cocycle. This proves $C^p(\mathfrak{g}, V)^{\mathfrak{g}} = Z^p(\mathfrak{g}, V)^{\mathfrak{g}}$.

To see that $B^p(\mathfrak{g}, V)^{\mathfrak{g}}$ vanishes, we note that we have the direct sum decomposition

$$C^{p-1}(\mathfrak{g}, V) = C^{p-1}(\mathfrak{g}, V)^{\mathfrak{g}} \oplus \mathfrak{g} \cdot C^{p-1}(\mathfrak{g}, V),$$

and likewise

$$B^p(\mathfrak{g}, V) = B^p(\mathfrak{g}, V)^{\mathfrak{g}} \oplus \mathfrak{g} \cdot B^p(\mathfrak{g}, V),$$

so that

$$B^p(\mathfrak{g}, V)^{\mathfrak{g}} = \mathtt{d}(C^{p-1}(\mathfrak{g}, V)^{\mathfrak{g}}) = \{0\},$$

since Lemma 6.5.11 shows that $\mathtt{d}\big(\mathfrak{g}\cdot C^{p-1}(\mathfrak{g}, V)\big) \subseteq \mathfrak{g}\cdot\mathtt{d}\big(C^{p-1}(\mathfrak{g}, V)\big)$. Therefore $H^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V)$, and the proof is complete. □

Example 6.5.31. For each semisimple Lie algebra $\mathfrak{g}$, we have

$$H^3(\mathfrak{g}, \mathbb{K}) \neq \{0\}.$$

In fact, the invariance of the Cartan–Killing form κ implies that

$$\Gamma(\kappa)(x, y, z) := \kappa([x, y], z)$$

is an invariant alternating 3-form on $\mathfrak{g}$, hence a nonzero element of $C^3(\mathfrak{g}, \mathbb{K})^{\mathfrak{g}} \cong H^3(\mathfrak{g}, \mathbb{K})$.

Lemma 6.5.32. *If V is a real module over the real Lie algebra $\mathfrak{g}$ and $V_{\mathbb{C}}$ the corresponding complex $\mathfrak{g}_{\mathbb{C}}$-module, then*

$$H^p(\mathfrak{g}, V)_{\mathbb{C}} \cong H^p(\mathfrak{g}_{\mathbb{C}}, V_{\mathbb{C}}) \quad \textit{for} \quad p \in \mathbb{N}_0.$$

Proof. Since any alternating p-form $\omega \in C^p(\mathfrak{g}, V)$ extends uniquely to a complex p-linear map $\omega_{\mathbb{C}} \in C^p(\mathfrak{g}_{\mathbb{C}}, V_{\mathbb{C}})$, we have an embedding

$$C^p(\mathfrak{g}, V) \hookrightarrow C^p(\mathfrak{g}_{\mathbb{C}}, V_{\mathbb{C}}).$$

The image of this map is the set of all elements $\eta \in C^p(\mathfrak{g}_{\mathbb{C}}, V_{\mathbb{C}})$, whose values on $\mathfrak{g}^p$ lie in the real subspace $V \cong 1 \otimes V \subseteq V_{\mathbb{C}}$. Since any $\eta \in C^p(\mathfrak{g}_{\mathbb{C}}, V_{\mathbb{C}})$ can be written in a unique fashion as a $\eta_1 + i\eta_2$ with $\eta_j(\mathfrak{g}^p) \subseteq V$, we see that

$$C^p(\mathfrak{g}_{\mathbb{C}}, V_{\mathbb{C}}) \cong C^p(\mathfrak{g}, V)_{\mathbb{C}}.$$

The subspaces of cocycles and coboundaries inherit the corresponding property, and from that the assertion follows. □

Theorem 6.5.33 (Whitehead's Vanishing Theorem). *If V is a finite-dimensional module over the semisimple Lie algebra $\mathfrak{g}$ with $V^{\mathfrak{g}} = \{0\}$, then $H^p(\mathfrak{g}, V) = \{0\}$ for any $p \in \mathbb{N}_0$.*

Proof. In view of Weyl's Theorem, the condition $V^{\mathfrak{g}} = \{0\}$ implies that V is a direct sum of simple nontrivial $\mathfrak{g}$-modules $V = \bigoplus_{i=1}^k V_i$. Then

$$H^p(\mathfrak{g}, V) \cong \bigoplus_{i=1}^{k} H^p(\mathfrak{g}, V_i)$$

(Exercise), so that we may w.l.o.g. assume that V is a simple nontrivial module. By Lemma 6.5.32, we may further assume that $\mathfrak{g}$ and V are complex.

In view of Lemma 6.5.29, we have to show that $Z^p(\mathfrak{g}, V)^{\mathfrak{g}} \subseteq B^p(\mathfrak{g}, V)^{\mathfrak{g}}$. For $p = 0$, this follows from our assumption $Z^0(\mathfrak{g}, V) = V^{\mathfrak{g}} = \{0\}$. Let $x_1, \ldots, x_m$ be a basis for $\mathfrak{g}$ and $x^1, \ldots, x^m$ the dual basis with respect to the Cartan–Killing form. For $p > 0$, we then define a linear map

$$\Gamma\colon C^p(\mathfrak{g}, V) \to C^{p-1}(\mathfrak{g}, V), \quad \Gamma(\omega) := \sum_{j=1}^{m} \rho_V(x_j) \circ i(x^j)\omega.$$

First we show that Γ is $\mathfrak{g}$-equivariant. For $x \in \mathfrak{g}$, we have

$$\begin{aligned} x \cdot \Gamma(\omega) &= \sum_{j=1}^{m} x \cdot \big(\rho_V(x_j) \circ i(x^j)\omega\big) \\ &= \sum_{j=1}^{m} \rho_V([x, x_j]) \circ i(x^j)\omega + \rho_V(x_j) \circ i([x, x^j])\omega + \Gamma(x \cdot \omega). \end{aligned}$$

If

$$\operatorname{ad} x(x_j) = \sum_{k=1}^{n} a_{kj} x_k \quad \text{and} \quad \operatorname{ad} x(x^j) = \sum_{k=1}^{n} a^{kj} x^k,$$

then $a_{kj} = \kappa([x, x_j], x^k) = -\kappa(x_j, [x, x^k]) = -a^{jk}$, and this leads to

$$\begin{aligned} &\sum_{j=1}^{m} \rho_V([x, x_j]) \circ i(x^j)\omega + \rho_V(x_j) \circ i([x, x^j])\omega \\ &= \sum_{j,k=1}^{m} a_{kj} \rho_V(x_k) \circ i(x^j)\omega + a^{kj} \rho_V(x_j) \circ i(x^k)\omega \\ &= \sum_{j,k=1}^{m} a_{kj} \rho_V(x_k) \circ i(x^j)\omega + a^{jk} \rho_V(x_k) \circ i(x^j)\omega = 0. \end{aligned}$$

We conclude that Γ is $\mathfrak{g}$-equivariant, hence maps $\mathfrak{g}$-invariant cochains to $\mathfrak{g}$-invariant cochains.

For $\omega \in Z^p(\mathfrak{g}, V)^{\mathfrak{g}}$, we thus have $\Gamma(\omega) \in C^{p-1}(\mathfrak{g}, V)^{\mathfrak{g}}$. With Lemma 6.5.28 we obtain $\mathtt{d}^0\omega = -\mathtt{d}\omega = 0$. We also note that the representation $\mathcal{L}$ of $\mathfrak{g}$ on $C(\mathfrak{g}, V)$ can be written as $\mathcal{L} = \rho_0 + \rho_+$, where ρ_0 is the representation corresponding to the trivial $\mathfrak{g}$-module structure on V. Since ω is $\mathcal{L}_{\mathfrak{g}}$-invariant, we therefore have

$$\rho_0(x)\omega = -\rho_V(x) \circ \omega.$$

Next we calculate with the Cartan formulas for the trivial $\mathfrak{g}$-module V and the corresponding representation ρ_0 of $\mathfrak{g}$ on $C(\mathfrak{g}, V)$:

$$\begin{aligned}
\mathtt{d}(\Gamma(\omega)) &= -\mathtt{d}^0(\Gamma(\omega)) = -\sum_{j=1}^m \rho_V(x_j) \circ \mathtt{d}^0\big(i_{x^j}\omega\big) \\
&= -\sum_{j=1}^m \rho_V(x_j) \circ \big(\rho_0(x^j)\omega - i_{x^j}\mathtt{d}^0\omega\big) = \sum_{j=1}^m \rho_V(x_j)\rho_V(x^j) \circ \omega \\
&= \rho_V(C_{\mathfrak{g}}) \circ \omega,
\end{aligned}$$

where $C_{\mathfrak{g}} \in \mathcal{U}(\mathfrak{g})$ is the universal Casimir operator of $\mathfrak{g}$. Since V is a simple $\mathfrak{g}$-module, $\rho_V(C_{\mathfrak{g}})$ is a nonzero multiple of id_V (Lemma 6.3.17), so that ω is a coboundary. □

6.5.5 Cohomology of Nilpotent Lie Algebras

We conclude this section with a short subsection in which we show that a certain vanishing result for the cohomology of nilpotent Lie algebras implies that all Cartan subalgebras of a solvable Lie algebra are conjugate.

Proposition 6.5.34. *Let V be a finite-dimensional module over the nilpotent Lie algebra $\mathfrak{h}$. If $V^0(\mathfrak{h}) = \{0\}$, then*

$$H^p(\mathfrak{h}, V) = \{0\} \quad \textit{for} \quad p \in \mathbb{N}_0.$$

Proof. If V and $\mathfrak{h}$ are real, then $V_{\mathbb{C}}$ carries a natural $\mathfrak{h}_{\mathbb{C}}$-module structure with $V^0(\mathfrak{h}_{\mathbb{C}}) = V^0(\mathfrak{h})_{\mathbb{C}} = \{0\}$ (Exercise). In view of Lemma 6.5.32, we may therefore assume that V and $\mathfrak{h}$ are complex, so that our assumption implies that

$$V = \bigoplus_{0 \neq \lambda \in \mathfrak{h}^*} V^\lambda(\mathfrak{h})$$

(Lemma 5.1.3). We therefore have

$$H^p(\mathfrak{h}, V) = \bigoplus_{0 \neq \lambda \in \mathfrak{h}^*} H^p(\mathfrak{h}, V^\lambda(\mathfrak{h})),$$

so that we may assume that $V = V^\lambda(\mathfrak{h})$ for some nonzero $\lambda \in \mathfrak{h}^*$.

Then the action of $\mathfrak{h}$ on $C^p(\mathfrak{g}, V)$ is given by

$$\mathcal{L}_x\omega = \rho_V(x) \circ \omega + \rho_0(x)\omega = \rho_+(x)\omega + \rho_0(x)\omega,$$

where ρ_0 is the action corresponding to the trivial module structure on V. The operator $\rho_0(x)$ is on the larger space $\mathrm{Mult}^p(\mathfrak{g}, V)$ a sum of $p+1$ pairwise commuting linear maps, all of which are nilpotent (cf. Lemma 6.5.8). These in turn commute with $\rho_+(x)$, which is a sum of $\lambda(x)\mathbf{1}$ and the nilpotent linear map $\rho_V(x) - \lambda(x)\mathbf{1}$. This implies that

$$C^p(\mathfrak{h}, V) = C^p(\mathfrak{h}, V)^\lambda(\mathfrak{h})$$

under the action of $\mathfrak{h}$. We therefore have

$$Z^p(\mathfrak{h}, V) = Z^p(\mathfrak{h}, V)^\lambda(\mathfrak{h}) \subseteq \mathrm{span}(\mathcal{L}_\mathfrak{h} Z^p(\mathfrak{h}, V)) \subseteq B^p(\mathfrak{h}, V),$$

which implies that $H^p(\mathfrak{h}, V)$ vanishes. □

Theorem 6.5.35 (Conjugacy Theorem for Cartan Subalgebras of Solvable Lie Algebras). *Let $\mathfrak{h}$ and $\mathfrak{h}'$ be Cartan subalgebras of the solvable Lie algebra $\mathfrak{g}$. Then there exists an element $x \in C^\infty(\mathfrak{g}) = \bigcap_{n\in\mathbb{N}} C^n(\mathfrak{g})$ with $e^{\mathrm{ad}\,x}\mathfrak{h} = \mathfrak{h}'$.*

Proof. We use induction on the dimension of $\mathfrak{g}$. The case $\mathfrak{g} = \{0\}$ is trivial. Now assume that $\mathfrak{g} \neq \{0\}$. The last nonzero term of the derived series of $\mathfrak{g}$ is an abelian ideal of $\mathfrak{g}$, so that $\mathfrak{g}$ possesses a nonzero minimal abelian ideal $\mathfrak{n}$. Let $\varphi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ denote the quotient homomorphism. Then $\varphi(\mathfrak{h})$ and $\varphi(\mathfrak{h}')$ are Cartan subalgebras of $\mathfrak{g}/\mathfrak{n}$ (Proposition 5.1.11), so that the induction hypothesis implies the existence of an element $y = \varphi(x) \in C^\infty(\mathfrak{g}/\mathfrak{n}) = \varphi(C^\infty(\mathfrak{g}))$ with $e^{\mathrm{ad}\,y}\varphi(\mathfrak{h}) = \varphi(\mathfrak{h}')$. Replacing $\mathfrak{h}$ by $e^{\mathrm{ad}\,x}\mathfrak{h}$, we may therefore assume that $\varphi(\mathfrak{h}) = \varphi(\mathfrak{h}')$, i.e.,

$$\mathfrak{h} + \mathfrak{n} = \mathfrak{h}' + \mathfrak{n}.$$

Then $\mathfrak{h}$ and $\mathfrak{h}'$ are Cartan subalgebras of the subalgebra $\mathfrak{h} + \mathfrak{n}$ (Lemma 5.1.11(iii)). If $\mathfrak{h} + \mathfrak{n} \neq \mathfrak{g}$, the assertion follows from the induction hypothesis. We may thus assume that $\mathfrak{g} = \mathfrak{h} + \mathfrak{n} = \mathfrak{h}' + \mathfrak{n}$.

In view of the minimality of $\mathfrak{n}$, we either have $\mathfrak{n} = [\mathfrak{g}, \mathfrak{n}]$ or $[\mathfrak{g}, \mathfrak{n}] = \{0\}$. In the latter case, $\mathfrak{n}$ is central in $\mathfrak{g}$, so that $\mathfrak{n} \subseteq \mathfrak{h} \cap \mathfrak{h}'$, so that $\mathfrak{h} = \mathfrak{h} + \mathfrak{n} = \mathfrak{h}' + \mathfrak{n} = \mathfrak{h}'$. This leaves the first case $\mathfrak{n} = [\mathfrak{g}, \mathfrak{n}]$. In this case $\mathfrak{n} \subseteq C^\infty(\mathfrak{g})$ and $\mathfrak{n}$ is a simple $\mathfrak{g}$-module. As $\mathfrak{n}$ is abelian and $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$, the $\mathfrak{g}$-action on $\mathfrak{n}$ factors through an action of $\mathfrak{h}$, so that $\mathfrak{n}$ is a simple $\mathfrak{h}$-module. If $\mathfrak{n} \cap \mathfrak{h} \neq \{0\}$, the simplicity of the $\mathfrak{h}$-module $\mathfrak{n}$ yields $\mathfrak{n} \subseteq \mathfrak{h}$, and therefore $\mathfrak{h} = \mathfrak{g}$, which leads to $\mathfrak{h} = \mathfrak{h}'$.

We may therefore assume that $\mathfrak{h} \cap \mathfrak{n} = \{0\}$, so that $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathfrak{h}$ is a semidirect sum. Since $\mathfrak{h}$ and $\mathfrak{h}'$ have the same dimension, we also have $\mathfrak{h}' \cap \mathfrak{n} = \{0\}$, and there exists a linear map $f\colon \mathfrak{h} \to \mathfrak{n}$ for which

$$\mathfrak{h}' = \{h + f(h)\colon h \in \mathfrak{h}\}$$

is the corresponding graph. Since $\mathfrak{h}'$ is a subalgebra, we have

$$[h + f(h), h' + f(h')] = [h, h'] + [h, f(h')] - [h', f(h)] \in \mathfrak{h}',$$

showing that

$$f([h, h']) = [h, f(h')] - [h', f(h)],$$

i.e., $f \in Z^1(\mathfrak{h}, \mathfrak{n})$. As $\mathfrak{n}$ is a simple nontrivial $\mathfrak{h}$-module, we have $\mathfrak{n}^0(\mathfrak{h}) = \{0\}$, and Proposition 6.5.34 implies that $H^1(\mathfrak{h}, \mathfrak{n}) = \{0\}$. Hence there exists a $v \in \mathfrak{n}$ with $f(h) = [h, v]$ for all $h \in \mathfrak{h}$, and then

$$e^{\operatorname{ad} v}\mathfrak{h} = (\mathbf{1} + \operatorname{ad} v)(\mathfrak{h}) = \{h + [x, h] \colon h \in \mathfrak{h}\} = \mathfrak{h}'. \qquad \square$$

Exercises for Section 6.5

Exercise 6.5.1. (a) A central extension $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with kernel $\mathfrak{z}$ is trivial if and only if $\mathfrak{z} \cap [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = \{0\}$.
(b) For a vector space V, the central extension

$$\mathbb{K}\mathbf{1} \hookrightarrow \mathfrak{gl}(V) \twoheadrightarrow \mathfrak{pgl}(V) := \mathfrak{gl}(V)/\mathbb{K}\mathbf{1}$$

is trivial if and only if $\dim V < \infty$ and the characteristic $\operatorname{char}(\mathbb{K})$ of the field $\mathbb{K}$ is either 0 or does not divide $\dim(V)$. If this is not the case, then there exist endomorphisms $P, Q \in \operatorname{End}(V)$ with $[P, Q] = \mathbf{1}$.

Exercise 6.5.2. (a) Let V be a module over the Lie algebra $\mathfrak{g}$ and V^* be the dual space. Then V^* becomes a $\mathfrak{g}$-module with

$$(x \cdot f)(v) := -f(x \cdot v) \qquad \text{for} \qquad x \in \mathfrak{g}, f \in V^*, v \in V.$$

(b) If $V_1, \ldots, V_n$ and W are $\mathfrak{g}$-modules, then the space L of n-linear maps $V_1 \times \cdots \times V_n \to W$ carries a $\mathfrak{g}$-module structure defined by

$$(x \cdot f)(v_1, \ldots, v_n) := x \cdot f(v_1, \ldots, v_n) - \sum_{i=1}^{n} f(v_1, \ldots, v_{i-1}, x \cdot v_i, v_{i+1}, \ldots, v_n).$$

Exercise 6.5.3. Let $V_1, \ldots, V_n$ be modules of the Lie algebra $\mathfrak{g}$. Then the tensor product $E := V_1 \otimes \ldots \otimes V_n$ becomes a $\mathfrak{g}$-module with

$$x \cdot (v_1 \otimes \ldots \otimes v_n) := x \cdot v_1 \otimes \ldots \otimes v_n + v_1 \otimes x \cdot v_2 \otimes \ldots \otimes v_n + v_1 \otimes \ldots \otimes x \cdot v_n.$$

Exercise 6.5.4. Let V and W be two finite-dimensional $\mathfrak{g}$-modules and consider the linear isomorphism

$$\Phi \colon V^* \otimes W \to \operatorname{Hom}(V, W), \quad \Phi(f \otimes w)(v) := f(v)w, \qquad f \in V^*, w \in W, v \in V.$$

Show that Φ is an isomorphism of $\mathfrak{g}$-modules if the $\mathfrak{g}$-module structure on $\operatorname{Hom}(V, W)$, defined in Proposition 6.5.23, and the $\mathfrak{g}$-module structure which is defined by Exercises 6.5.2 and 6.5.3.

Exercise 6.5.5. $\mathbb{R}^2$ becomes an $\mathfrak{sl}_2(\mathbb{R})$-module by $x \cdot v := x(v)$. Show:

(i) $\mathbb{R}^2$ is a simple $\mathfrak{sl}_2(\mathbb{R})$-module.
(ii) The $\mathfrak{sl}_2(\mathbb{R})$-module $\mathbb{R}^2 \otimes \mathbb{R}^2$ splits into the direct sum of two simple $\mathfrak{sl}_2(\mathbb{R})$-modules.

Exercise 6.5.6. Let $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an invariant bilinear form and put

$$\Gamma(\kappa)(x, y, z) := \kappa([x, y], z).$$

Show that:

(a) If κ is skew-symmetric, then $\Gamma(\kappa) = 0$.
(b) If κ is symmetric, then $\Gamma(\kappa) \in Z^3(\mathfrak{g}, \mathbb{K})$ is a 3-cocycle.

Exercise 6.5.7. Show directly by computation that the Chevalley–Eilenberg differential $\mathtt{d}\colon C(\mathfrak{g}, V) \to C(\mathfrak{g}, V)$ satisfies $\mathtt{d}^2 = 0$.

6.6 General Extensions of Lie Algebras

In this section we discuss a method to classify extensions of a Lie algebra $\mathfrak{g}$ by a Lie algebra $\mathfrak{n}$ in terms of data associated to these two Lie algebras. This generalizes the classification of abelian extension of $\mathfrak{g}$ by a $\mathfrak{g}$-module V in terms of the second cohomology space $H^2(\mathfrak{g}, V)$.

Throughout this section, we shall denote an extension of $\mathfrak{g}$ by $\mathfrak{n}$ by a short exact sequence

$$\mathbf{0} \to \mathfrak{n} \xrightarrow{\iota} \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to \mathbf{0},$$

where we identify $\mathfrak{n}$ with the ideal $\iota(\mathfrak{n})$ of $\widehat{\mathfrak{g}}$ to simplify notation.

6.6.1 $\mathfrak{g}$-Kernels

One of the major difficulties of extensions by nonabelian Lie algebras is that $\mathfrak{n}$ acts nontrivially on itself by the adjoint action of $\widehat{\mathfrak{g}}$, so that the action of $\widehat{\mathfrak{g}}$ on $\mathfrak{n}$ does not factor through an action of $\mathfrak{g}$. To overcome this problem, we introduce the concept of a $\mathfrak{g}$-kernel:

Definition 6.6.1. Recall the Lie algebra $\operatorname{out}(\mathfrak{n}) := \operatorname{der}(\mathfrak{n})/\operatorname{ad}(\mathfrak{n})$ of outer derivations of $\mathfrak{n}$ and write $[D]$ for the image of a derivation D in $\operatorname{out}(\mathfrak{n})$. A $\mathfrak{g}$-*kernel for* $\mathfrak{n}$ is a homomorphism

$$s\colon \mathfrak{g} \to \operatorname{out}(\mathfrak{n})$$

of Lie algebras. As $\operatorname{ad}(\mathfrak{n})$ acts trivially on the center $\mathfrak{z}(\mathfrak{n})$, we have a natural $\operatorname{out}(\mathfrak{n})$-module structure on $\mathfrak{z}(\mathfrak{n})$, so that each $\mathfrak{g}$-kernel $s\colon \mathfrak{g} \to \operatorname{out}(\mathfrak{n})$ defines in particular a $\mathfrak{g}$-module structure on $\mathfrak{z}(\mathfrak{n})$.

Remark 6.6.2. If $\mathfrak{n}$ is abelian, then a $\mathfrak{g}$-kernel for $\mathfrak{n}$ is simply a homomorphism $s\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$ since $\mathrm{ad}(\mathfrak{n}) = \{0\}$.

Lemma 6.6.3. *For each extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of $\mathfrak{g}$ by the Lie algebra $\mathfrak{n}$, the adjoint action of $\widehat{\mathfrak{g}}$ on the ideal $\mathfrak{n}$ induces a homomorphism*

$$s\colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n}), \quad q(x) \mapsto [\mathrm{ad}_{\widehat{\mathfrak{g}}}(x)|_{\mathfrak{n}}],$$

called the corresponding $\mathfrak{g}$-kernel. *Equivalent extensions define the same* $\mathfrak{g}$*-kernel.*

Proof. Since the adjoint representation

$$\mathrm{ad}_{\mathfrak{n}}\colon \widehat{\mathfrak{g}} \to \mathrm{der}(\mathfrak{n}), \quad x \mapsto \mathrm{ad}(x)|_{\mathfrak{n}}$$

maps $\mathfrak{n}$ onto $\mathrm{ad}(\mathfrak{n})$, it factors through a homomorphism $s\colon \mathfrak{g} \cong \widehat{\mathfrak{g}}/\mathfrak{n} \to \mathrm{out}(\mathfrak{n})$, satisfying $s(q(x)) = [\mathrm{ad}_{\widehat{\mathfrak{g}}}(x)|_{\mathfrak{n}}]$.

If $\varphi\colon \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ is an equivalence of extensions of $\mathfrak{g}$ by $\mathfrak{n}$, inducing the identity on $\mathfrak{n}$, then $q_2 \circ \varphi = q_1$, and we have for each $x \in \widehat{\mathfrak{g}}_1$ the relation

$$\mathrm{ad}_{\widehat{\mathfrak{g}}_1}(x)|_{\mathfrak{n}} = \varphi \circ \mathrm{ad}_{\widehat{\mathfrak{g}}_1}(x)|_{\mathfrak{n}} = \mathrm{ad}_{\widehat{\mathfrak{g}}_2}(\varphi(x))|_{\mathfrak{n}}.$$

Therefore the corresponding $\mathfrak{g}$-kernels satisfy

$$s_1(q_1(x)) = [\mathrm{ad}_{\widehat{\mathfrak{g}}_1}(x)|_{\mathfrak{n}}] = [\mathrm{ad}_{\widehat{\mathfrak{g}}_2}(\varphi(x))|_{\mathfrak{n}}] = s_2(q_2(\varphi(x))) = s_2(q_1(x)).$$

This proves that $s_2 = s_1$. □

For a given $\mathfrak{g}$-kernel s for $\mathfrak{n}$, we write $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ for the set of equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{n}$ for which s is the corresponding $\mathfrak{g}$-kernel in the sense of Lemma 6.6.3. The classification of all extensions of $\mathfrak{g}$ by $\mathfrak{n}$ now decomposes into two different problems:

(P1) For a fixed $\mathfrak{g}$-kernel s we have to parameterize the set $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$.
(P2) We need a method to decide for which $\mathfrak{g}$-kernels the set $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ is nonempty.

Let $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ be an extension corresponding to the $\mathfrak{g}$-kernel s. To obtain a suitable description of $\widehat{\mathfrak{g}}$ in terms of $\mathfrak{g}$ and $\mathfrak{n}$, we choose a linear section $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ of q. Then the linear map

$$\Phi\colon \mathfrak{n} \times \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (n, x) \mapsto n + \sigma(x)$$

is an isomorphism of vector spaces. To express the Lie bracket of $\widehat{\mathfrak{g}}$ in terms of the corresponding product coordinates, we define the linear map

$$S\colon \mathfrak{g} \to \mathrm{der}\,\mathfrak{n}, \quad S(x) := \mathrm{ad}_{\mathfrak{n}}(\sigma(x)) := (\mathrm{ad}\,\sigma(x))|_{\mathfrak{n}}$$

and the alternating bilinear map

$$\omega\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{n},\quad \omega(x,y):=[\sigma(x),\sigma(y)]-\sigma([x,y]).$$

Then Φ is an isomorphism of Lie algebras if we endow $\mathfrak{n}\times\mathfrak{g}$ with the Lie bracket

$$[(n,x),(n',x')]:=([n,n']+S(x)n'-S(x')n+\omega(x,x'),[x,x']). \tag{6.11}$$

Remark 6.6.4. It is easy to verify directly that if $\mathfrak{n}$ and $\mathfrak{g}$ are Lie algebras, $\omega\in C^2(\mathfrak{g},\mathfrak{n})$ and $S\in C^1(\mathfrak{g},\mathrm{der}(\mathfrak{n}))$, then (6.11) defines a Lie bracket on $\mathfrak{n}\times\mathfrak{g}$ if and only if

$$[S(x),S(y)]-S([x,y])=\mathrm{ad}(\omega(x,y))\quad \text{for}\quad x,y\in\mathfrak{g} \tag{6.12}$$

and

$$(\mathtt{d}_S\omega)(x,y,z):=\sum_{\text{cycl.}}S(x)\omega(y,z)-\omega([x,y],z)=0. \tag{6.13}$$

Definition 6.6.5. If $\mathfrak{g}$ and $\mathfrak{n}$ are Lie algebras, then a *factor system for* $(\mathfrak{g},\mathfrak{n})$ is a pair (S,ω) with $S\in C^1(\mathfrak{g},\mathrm{der}\,\mathfrak{n})$ and $\omega\in C^2(\mathfrak{g},\mathfrak{n})$, satisfying (6.12) and (6.13).

We write $\widehat{\mathfrak{g}}:=\mathfrak{n}\times_{(S,\omega)}\mathfrak{g}$ for the corresponding Lie algebra and note that $q\colon\widehat{\mathfrak{g}}\to\mathfrak{g}, q(n,x)=x$, is a surjective homomorphism with kernel $\mathfrak{n}\cong\mathfrak{n}\times\{0\}$. The corresponding $\mathfrak{g}$-kernel is given by $s\colon\mathfrak{g}\to\mathrm{out}(\mathfrak{n}), s(x)=[S(x)]$. In order to keep track of the $\mathfrak{g}$-action on $\mathfrak{z}(\mathfrak{n})$ induced by s we write $Z^k(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))_s$, $B^k(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))_s$, and $H^k(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))_s$ instead of $Z^k(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))$, $B^k(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))$, and $H^k(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))$.

Theorem 6.6.6. *Let $s\colon\mathfrak{g}\to\mathrm{out}(\mathfrak{n})$ be a $\mathfrak{g}$-kernel for $\mathfrak{n}$. If $\mathrm{Ext}(\mathfrak{g},\mathfrak{n})_s\neq\emptyset$ and $S\in C^1(\mathfrak{g},\mathrm{der}\,\mathfrak{n})$ satisfies $s(x)=[S(x)]$ for each $x\in\mathfrak{g}$, then each extension of $\mathfrak{g}$ by $\mathfrak{n}$ corresponding to s is equivalent to some $\mathfrak{n}\times_{(S,\omega)}\mathfrak{g}$, where (S,ω) is a factor system. For any such factor system we obtain a bijection*

$$\Gamma\colon H^2(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))_s\to\mathrm{Ext}(\mathfrak{g},\mathfrak{n})_s,\quad [\eta]\mapsto[\mathfrak{n}\times_{(S,\omega+\eta)}\mathfrak{g}].$$

Proof. Let $q\colon\widehat{\mathfrak{g}}\to\mathfrak{g}$ be any extension of $\mathfrak{g}$ by $\mathfrak{n}$ corresponding to s. If $\sigma\colon\mathfrak{g}\to\widehat{\mathfrak{g}}$ is a linear section, then we have $[S(x)]=s(x)=[\mathrm{ad}_{\mathfrak{n}}(\sigma(x))]$, so that there exists a linear map $\gamma\colon\mathfrak{g}\to\mathfrak{n}$ with $S(x)=\mathrm{ad}_{\mathfrak{n}}(\sigma(x))+\mathrm{ad}(\gamma(x))$ for each $x\in\mathfrak{g}$. Then the new section $\widetilde{\sigma}:=\sigma+\gamma$ satisfies $S(x)=\mathrm{ad}_{\mathfrak{n}}(\widetilde{\sigma}(x))$, so that

$$\omega(x,y):=[\widetilde{\sigma}(x),\widetilde{\sigma}(y)]-\widetilde{\sigma}([x,y])$$

leads to a factor system (S,ω) with $\widehat{\mathfrak{g}}\cong\mathfrak{n}\times_{(S,\omega)}\mathfrak{g}$.

For any other factor system $(S,\widetilde{\omega})$, we have $\eta:=\widetilde{\omega}-\omega\in C^2(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))$ with

$$\mathtt{d}_{\mathfrak{g}}\eta=\mathtt{d}_S\widetilde{\omega}-\mathtt{d}_S\omega=0.$$

Here we use that $\mathfrak{g}$-module structure on $\mathfrak{z}(\mathfrak{n})$ is given by $x\cdot z=S(x)z$, so that the corresponding Chevalley–Eilenberg differential $\mathtt{d}_{\mathfrak{g}}$ coincides with $\mathtt{d}_S$ on $\mathfrak{z}(\mathfrak{n})$-valued cochains. We conclude that the map

$$Z^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s \to \mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s, \quad [\eta] \mapsto [\mathfrak{n} \times_{(S,\omega+\eta)} \mathfrak{g}]$$

is surjective.

If $\varphi\colon \mathfrak{n} \times_{(S,\omega)} \mathfrak{g} \to \mathfrak{n} \times_{(S,\omega')} \mathfrak{g}$ is an equivalence of extensions, then $\varphi(n,x) = (n + \gamma(x), x)$ for some linear map $\gamma\colon \mathfrak{g} \to \mathfrak{n}$. In view of

$$\begin{aligned}(S(x)n + [\gamma(x), n], 0) &= [(\gamma(x), x), (n, 0)] = [\varphi(0, x), (n, 0)] = \varphi(S(x)n, 0)\\ &= (S(x)n, 0),\end{aligned}$$

we have $\gamma(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{n})$. Further,

$$\begin{aligned}(\omega(x,y) + \gamma([x,y]), [x,y]) &= \varphi([(0,x),(0,y)]) = [(\gamma(x),x),(\gamma(y),y)]\\ &= (S(x)\gamma(y) - S(y)\gamma(x) + \omega'(x,y), [x,y])\end{aligned}$$

implies that $\omega = \mathsf{d}_{\mathfrak{g}}\gamma + \omega'$. If, conversely, $\omega - \omega' = \mathsf{d}_{\mathfrak{g}}\gamma$ for some $\gamma \in C^1(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))$, then $\varphi(n,x) = (n + \gamma(x), x)$ defines an equivalence of extensions

$$\varphi\colon \mathfrak{n} \times_{(S,\omega)} \mathfrak{g} \to \mathfrak{n} \times_{(S,\omega')} \mathfrak{g},$$

as a straight forward calculation shows. We conclude that the extensions defined by two factor systems (S,ω) and (S,ω') are equivalent if and only if $\omega - \omega' \in B^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))$, and this implies the theorem. □

Remark 6.6.7. The preceding theorem shows that the set $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$, if nonempty, has the structure of an affine space whose translation group is $H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s \cong \mathrm{Ext}(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s$ (cf. Proposition 6.5.18).

Remark 6.6.8. The group $H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s$ very much depends on the $\mathfrak{g}$-kernel s. Let $\mathfrak{g} = \mathbb{K}^2$ and $\mathfrak{n} = \mathbb{K}$. Then $C^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))$ is 1-dimensional. Further, $\dim \mathfrak{g} = 2$ implies $C^3(\mathfrak{g}, \mathfrak{z}(\mathfrak{n})) = \{0\}$, so that each 2-cochain is a cocycle. Since $B^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s$ vanishes if the module $\mathfrak{z}(\mathfrak{n})$ is trivial and coincides with $Z^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s$ otherwise, we have

$$H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s \cong \begin{cases} \mathbb{K} & \text{for } \ \mathfrak{g} \cdot \mathfrak{z}(\mathfrak{n}) = \{0\} \\ \{0\} & \text{for } \ \mathfrak{g} \cdot \mathfrak{z}(\mathfrak{n}) \neq \{0\}. \end{cases}$$

Example 6.6.9. If $\mathfrak{g}$ is finite-dimensional semisimple, then the Second Whitehead Lemma 6.5.27 implies that $H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s = \{0\}$ for each $\mathfrak{g}$-kernel $s\colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n})$. Further, Corollary 4.6.10 implies that s lifts to a homomorphism $S\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$. Therefore any extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by a finite-dimensional Lie algebra $\mathfrak{n}$ is a semidirect product $\mathfrak{n} \rtimes_S \mathfrak{g}$ and all the sets $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ consist of only one element.

Remark 6.6.10. If $\mathfrak{z}(\mathfrak{n}) = \{0\}$, then $H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s = \{0\}$ for each $\mathfrak{g}$-kernel s, so that all sets $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ contain at most one element (Theorem 6.6.6). On the other hand, we obtain for each $\mathfrak{g}$-kernel $s\colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n})$ the pullback extension

$$\widehat{\mathfrak{g}} := s^* \,\mathrm{der}(\mathfrak{n}) := \{(x, D) \in \mathfrak{g} \times \mathrm{der}(\mathfrak{n})\colon [D] = s(x)\}$$

of $\mathfrak{g}$ by $\mathrm{ad}(\mathfrak{n}) \cong \mathfrak{n}$. Since this extension obviously corresponds to the $\mathfrak{g}$-kernel s, it follows that $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s \neq \emptyset$ for any $\mathfrak{g}$-kernel and further that $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s = \{[s^* \operatorname{der}(\mathfrak{n})]\}$.

Remark 6.6.11 (Split extensions). An extension $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{n}$ splits if and only if there exists a homomorphism $\sigma \colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then $S(x) := \mathrm{ad}(\sigma(x))|_{\mathfrak{n}}$ defines a homomorphism $S \colon \mathfrak{g} \to \operatorname{der}(\mathfrak{n})$ with

$$\widehat{\mathfrak{g}} \cong \mathfrak{n} \rtimes_S \mathfrak{g} \cong \mathfrak{n} \times_{(S,0)} \mathfrak{g}.$$

(a) For a $\mathfrak{g}$-kernel $s \colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n})$, the existence of a split extension in the set $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ therefore is equivalent to the existence of a homomorphism $S \colon \mathfrak{g} \to \operatorname{der}(\mathfrak{n})$ with $[S(x)] = s(x)$ for each $x \in \mathfrak{g}$. Such a lift always exists if the extension $\operatorname{der}(\mathfrak{n})$ of $\mathrm{out}(\mathfrak{n})$ by $\mathrm{ad}\,\mathfrak{n}$ splits, but if this is not the case, then $\mathfrak{g} = \mathrm{out}(\mathfrak{n})$ and $s = \mathrm{id}_{\mathfrak{g}}$ is a $\mathfrak{g}$-kernel for which a homomorphic lift does not exist.

(b) If $\mathfrak{n}$ is abelian, then each set $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ contains exactly one split extension, namely $\mathfrak{n} \rtimes_s \mathfrak{g}$, but in general $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ may contain different classes of split extensions. To understand this phenomenon, consider two homomorphic lifts $S, S' \colon \mathfrak{g} \to \operatorname{der}(\mathfrak{n})$ of the $\mathfrak{g}$-kernel s. Then $\gamma := S' - S \colon \mathfrak{g} \to \mathrm{ad}\,\mathfrak{n}$ is a linear map with $S' = S + \gamma$, and the requirement that S' also is a homomorphism of Lie algebras is equivalent to γ being a *crossed homomorphism*, i.e.,

$$[S(x), \gamma(y)] - [S(y), \gamma(x)] - \gamma([x, y]) + [\gamma(x), \gamma(y)] = 0 \quad \text{for} \quad x, y \in \mathfrak{g}. \tag{6.14}$$

A particularly simple case arises for $s = 0$. Then $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s$ contains the class of the split extension $\mathfrak{n} \oplus \mathfrak{g}$ (Lie algebra direct sum). For the lift $S = 0$, the other lifts simply correspond to homomorphisms $\gamma \colon \mathfrak{g} \to \mathrm{ad}\,\mathfrak{n}$. For any such homomorphism, there exists a linear lift $\widetilde{\gamma} \colon \mathfrak{g} \to \mathfrak{n}$ with $\mathrm{ad} \circ \widetilde{\gamma} = \gamma$, but in general $\widetilde{\gamma}$ is not a homomorphism of Lie algebras. Then

$$\omega(x, y) := [\widetilde{\gamma}(x), \widetilde{\gamma}(y)] - \widetilde{\gamma}([x, y])$$

is a Lie algebra cocycle in $Z^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))$, corresponding to the central extension

$$\gamma^* \mathfrak{n} = \{(x, n) \in \mathfrak{g} \times \mathfrak{n} \colon \gamma(x) = \mathrm{ad}\, n\}$$

of $\mathfrak{g}$ by $\mathfrak{z}(\mathfrak{n})$. We thus obtain a well defined map

$$\mathrm{Hom}(\mathfrak{g}, \mathrm{ad}\,\mathfrak{n}) \to H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n})), \quad \gamma \mapsto [\gamma^* \mathfrak{n}] = [\omega].$$

We are interested in a criterion for the split extension $\mathfrak{n} \rtimes_\gamma \mathfrak{g}$ to be equivalent to the trivial one.

For any lift $\widetilde{\gamma}$ as above, the map

$$\varphi \colon \mathfrak{n} \times_{(\gamma, \omega)} \mathfrak{g} \to \mathfrak{n} \oplus \mathfrak{g}, \quad (n, x) \mapsto (n + \widetilde{\gamma}(x), x)$$

is a homomorphism of Lie algebras, hence an equivalence of extensions because we have for the $\mathfrak{n}$-components:

$$[n + \widetilde{\gamma}(x), n' + \widetilde{\gamma}(x')] = [n, n'] + \gamma(x)n' - \gamma(x')n + [\widetilde{\gamma}(x), \widetilde{\gamma}(x')]$$
$$= \big([n, n'] + \gamma(x)n' - \gamma(x')n + \omega(x, x')\big) + \widetilde{\gamma}([x, x']).$$

With Theorem 6.6.6 we therefore see that $\mathfrak{n} \rtimes_\gamma \mathfrak{g}$ is equivalent to the trivial extension if and only if the cohomology class $[\omega] \in H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))$ vanishes, which in turn is equivalent to the existence of a homomorphic lift $\widetilde{\gamma}$ of γ.

If $\mathfrak{g}$ and $\operatorname{ad}\mathfrak{n}$ are abelian, then we have a bracket map

$$\eta\colon \operatorname{ad}\mathfrak{n} \times \operatorname{ad}\mathfrak{n} \to \mathfrak{z}(\mathfrak{n}), \quad \eta(\operatorname{ad} n, \operatorname{ad} n') := [n, n'],$$

and for a homomorphism $\gamma\colon \mathfrak{g} \to \operatorname{ad}\mathfrak{n}$ the extension $\gamma^*\mathfrak{n}$ is trivial if and only if $\gamma^*\eta$ vanishes, i.e., $\gamma(\mathfrak{g})$ is isotropic for η.

Example 6.6.12. We have seen above, that the theory of extensions of a Lie algebra $\mathfrak{g}$ by a Lie algebra $\mathfrak{n}$ simplifies significantly if the extension $\operatorname{der}(\mathfrak{n})$ of $\operatorname{out}(\mathfrak{n})$ by $\operatorname{ad}(\mathfrak{n})$ splits. Here we describe a series of finite-dimensional nilpotent Lie algebras, for which this is not the case.

For each $n \geq 3$, we consider the $(n+1)$-dimensional *filiform Lie algebra* L_n with a basis $e_0, \ldots, e_n$, where all nonzero brackets between basis elements are

$$[e_0, e_i] = e_{i+1}, \quad i = 1, \ldots, n-1.$$

Clearly, L_n is generated as a Lie algebra by e_0 and e_1, and $L_n \cong V \rtimes_A \mathbb{K}$, where $V = \operatorname{span}\{e_1, \ldots, e_n\}$ and $A \in \mathfrak{gl}(V)$ is the nilpotent shift corresponding to the action of $\operatorname{ad} e_0$ on V. Further, $\mathfrak{z}(L_n) = \mathbb{K}e_n$, so that $\operatorname{ad}(L_n) \cong L_{n-1}$ is an n-dimensional Lie algebra. One easily verifies that the following maps define derivations of L_n:

$$h_k := (\operatorname{ad} e_0)^k \quad \text{and} \quad k = 2, \ldots, n-1,$$

$$t_1(e_i) := \begin{cases} 0 & \text{for } i = 0 \\ e_i & \text{for } i > 0, \end{cases} \qquad t_2(e_i) := \begin{cases} e_0 & \text{for } i = 0 \\ (i-1)e_i & \text{for } i > 0, \end{cases}$$

and

$$t_3(e_i) := \begin{cases} e_1 & \text{for } i = 0 \\ 0 & \text{for } i > 0 \end{cases}$$

(cf. [GK96]). For the brackets of these derivations, we find

$$[t_1, t_2] = 0, \ [t_1, t_3] = t_3, \ [t_2, t_3] = -t_3, \quad [h_i, h_j] = 0,$$

$$[t_1, h_k] = 0, \quad [t_2, h_k] = kh_k, \quad \text{and} \quad [t_3, h_k] = \operatorname{ad}(e_k).$$

Considering the action on the hyperplane ideal V, we see that $e_0, h_2, \ldots, h_{k-1}$, t_1 and t_2 lead to a linearly independent set of endomorphisms of V. Taking also the values on e_0 into account, we see that the $t_i, i = 1, 2, 3$, h_k, $k = 2, \ldots, n-1$ and $\operatorname{ad}(e_i)$, $i = 0, \ldots, n-1$, are linearly independent. Therefore

$$\operatorname{span}\{[h_k], [t_i] \colon i = 1, 2, 3; k = 2, \ldots, n-1\}$$

is an $(n+1)$-dimensional Lie subalgebra of $\mathrm{out}(L_n)$ with the bracket relations

$$[[t_1],[h_k]] = 0, \quad [[t_2],[h_k]] = k[h_k] \quad \text{and} \quad [[t_3],[h_k]] = 0.$$

We consider the 3-dimensional Lie subalgebra

$$\mathfrak{g} := \mathrm{span}\{[t_1],[t_3],[h_2]\} \subseteq \mathrm{out}(L_n)$$

and claim that the Lie algebra extension

$$\mathbf{0} \to \mathrm{ad}(\mathfrak{n}) \subseteq \widehat{\mathfrak{g}} := \mathrm{ad}(\mathfrak{n}) + \mathrm{span}\{t_1, t_3, h_2\} \twoheadrightarrow \mathfrak{g} \to \mathbf{0}$$

does not split. This implies in particular that $\mathrm{der}(\mathfrak{n})$ does not split as an extension of $\mathrm{out}(\mathfrak{n})$.

Suppose that $\widehat{\mathfrak{g}}$ does split, i.e., that there are 3 elements $z_1, z_3, x_2 \in L_n$ such that the derivations

$$\widetilde{t}_1 := t_1 + \mathrm{ad}(z_1), \quad \widetilde{t}_3 := t_3 + \mathrm{ad}(z_3) \quad \text{and} \quad \widetilde{h}_2 := h_2 + \mathrm{ad}(x_2)$$

satisfy

$$[\widetilde{t}_1, \widetilde{t}_3] = \widetilde{t}_3 \quad \text{and} \quad [\widetilde{t}_1, \widetilde{h}_2] = [\widetilde{t}_3, \widetilde{h}_2] = 0.$$

The first relation is equivalent to

$$\mathrm{ad}(t_1 z_3 - t_3 z_1 + [z_1, z_3]) = \mathrm{ad}(z_3).$$

Writing $z_3 = z_3^0 e_0 + z_3'$ with $z_3' \in V$, this leads to

$$\mathrm{ad}(z_3' - z_1^0 e_1 + [z_1, z_3]) = \mathrm{ad}(z_3^0 e_0 + z_3'),$$

and therefore to

$$\mathrm{ad}(-z_1^0 e_1 + [z_1, z_3]) = \mathrm{ad}(z_3^0 e_0).$$

Applying this identity to e_1, we get $z_3^0 = 0$, so that $z_3 = z_3' \in V$.

Next we analyze the relation $[\widetilde{t}_3, \widetilde{h}_2] = 0$, which is equivalent to

$$0 = \mathrm{ad}(e_2 + t_3 x_2 - h_2 z_3 + [z_3, x_2]).$$

Since $e_2 - h_2 z_3 + [z_3, x_2] \in \mathrm{span}\{e_2, \ldots, e_n\}$ and $t_3 x_2 \in \mathbb{K} e_1$, it follows that $t_3 x_2 = 0$ and hence $x_2 \in V$. Then $[z_3, x_2] = 0$, and we derive $\mathrm{ad}(e_2 - h_2 z_3) = 0$, which leads to the contradiction $e_2 \in h_2(V) + \mathbb{K} e_n \subseteq \mathrm{span}\{e_3, \ldots, e_n\}$.

6.6.2 Integrability of $\mathfrak{g}$-Kernels

We call a $\mathfrak{g}$-kernel $s \colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n})$ *integrable* if there exists a factor system (S, ω) with $s(x) = [S(x)]$ for each $x \in \mathfrak{g}$ which is equivalent to $\mathrm{Ext}(\mathfrak{g}, \mathfrak{n})_s \neq \emptyset$ (cf. Theorem 6.6.6).

Remark 6.6.13. Let

$$Q_{\mathfrak{n}} \colon \operatorname{der}(\mathfrak{n}) \to \operatorname{out}(\mathfrak{n}) := \operatorname{der}(\mathfrak{n})/\operatorname{ad}\mathfrak{n}$$

denote the quotient homomorphism and $s \colon \mathfrak{g} \to \operatorname{out}(\mathfrak{n})$. Then there exists a linear map $S \colon \mathfrak{g} \to \operatorname{der}\mathfrak{n}$ with $Q_{\mathfrak{n}} \circ S = s$. Since s and $Q_{\mathfrak{n}}$ are homomorphisms,

$$[S(x), S(y)] - S([x,y]) \in \operatorname{ad}(\mathfrak{n})$$

for $x, y \in \mathfrak{g}$, and from that we derive the existence of an alternating bilinear map $\omega \in C^2(\mathfrak{g}, \mathfrak{n})$ satisfying (6.12). Indeed, for any linear section $\alpha \colon \operatorname{ad}(\mathfrak{n}) \to \mathfrak{n}$, we may put $\omega(x,y) := \alpha([S(x), S(y)] - S([x,y]))$. In general, the relation $\mathtt{d}_S\omega = 0$ will not be satisfied.

Definition 6.6.14. For the following calculations, we introduce a product structure on $C(\mathfrak{g}, \mathfrak{n})$, defined for $\alpha \in C^p(\mathfrak{g}, \mathfrak{n})$ and $\beta \in C^q(\mathfrak{g}, \mathfrak{n})$ by

$$\begin{aligned} &[\alpha, \beta](x_1, \ldots, x_{p+q}) \\ &\quad := \sum_{\sigma \in \operatorname{Sh}(p,q)} \operatorname{sgn}(\sigma)[\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}), \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)})], \end{aligned}$$

where $\operatorname{Sh}(p,q)$ denotes the set of all (p,q)-shuffles in S_{p+q}, i.e., all permutations with

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

For $p = 1$ and $q = 2$ we then have

$$[\alpha, \alpha](x,y) = [\alpha(x), \alpha(y)] - [\alpha(y), \alpha(x)] = 2[\alpha(x), \alpha(y)]$$

and

$$[\alpha, \beta](x,y,z) = \sum_{\text{cycl.}} [\alpha(x), \beta(y,z)],$$

as well as

$$[\beta, \alpha](x,y,z) = \sum_{\text{cycl.}} [\beta(x,y), \alpha(z)] = -[\alpha, \beta](x,y,z).$$

In particular, the Jacobi identity implies that

$$[\alpha, [\alpha, \alpha]] = 0.$$

We also put for $\alpha \in C^1(\mathfrak{g}, \mathfrak{n})$:

$$(\mathtt{d}_S\alpha)(x,y) := S(x)\alpha(y) - S(y)\alpha(x) - \alpha([x,y]).$$

Lemma 6.6.15. *Every* $\gamma \in C^1(\mathfrak{g}, \mathfrak{n})$ *satisfies* $\frac{1}{2}\mathtt{d}_S[\gamma, \gamma] = [\mathtt{d}_S\gamma, \gamma] = -[\gamma, \mathtt{d}_S\gamma]$.

Proof. We calculate

$$\begin{aligned}\frac{1}{2}(\mathtt{d}_S[\gamma,\gamma])(x,y,z) &= \sum_{\text{cycl.}} S(x)[\gamma(y),\gamma(z)] - [\gamma([x,y]),\gamma(z)] \\ &= \sum_{\text{cycl.}} [S(x)\gamma(y),\gamma(z)] + [\gamma(y),S(x)\gamma(z)] - [\gamma([x,y]),\gamma(z)] \\ &= \sum_{\text{cycl.}} [S(x)\gamma(y),\gamma(z)] + [\gamma(z),S(y)\gamma(x)] - [\gamma([x,y]),\gamma(z)] \\ &= \sum_{\text{cycl.}} [(\mathtt{d}_S\gamma)(x,y),\gamma(z)] = [\mathtt{d}_S\gamma,\gamma](x,y,z).\end{aligned}$$

□

Lemma 6.6.16. *If* $S\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$ *is a linear map and* $\omega \in C^2(\mathfrak{g},\mathfrak{n})$ *satisfies* (6.12)*, then* $\mathtt{d}_S\omega \in Z^3(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))_s$*, where* $s = Q_{\mathfrak{n}} \circ S$*.*

Proof. First we show that $\mathrm{im}\,(\mathtt{d}_S\omega) \subseteq \mathfrak{z}(\mathfrak{n})$:

$$\begin{aligned}&\mathrm{ad}(\mathtt{d}_S\omega(x,y,z)) \\ &= \sum_{\text{cycl.}} \mathrm{ad}(S(x)\omega(y,z)) - \mathrm{ad}(\omega([x,y],z)) \\ &= \sum_{\text{cycl.}} [S(x),\mathrm{ad}(\omega(y,z))] - [S([x,y]),S(z)] - S([[x,y],z]) \\ &= \sum_{\text{cycl.}} [S(x),[S(y),S(z)]] - [S(x),S([y,z])] - [S([x,y]),S(z)] - S([[x,y],z]) \\ &= \sum_{\text{cycl.}} [S(x),[S(y),S(z)]] - S([[x,y],z]) = 0,\end{aligned}$$

where the last equality follows from the Jacobi identities in $\mathrm{der}(\mathfrak{n})$ and $\mathfrak{g}$. We conclude that $\mathrm{im}\,(\mathtt{d}_S\omega) \subseteq \mathfrak{z}(\mathfrak{n})$.

It remains to see that $\mathtt{d}_S\omega$ is a 3-cocycle. Let

$$\widetilde{\mathfrak{g}} := s^*\,\mathrm{der}(\mathfrak{n}) = \{(x,D) \in \mathfrak{g} \times \mathrm{der}(\mathfrak{n})\colon [D] = s(x)\}$$

and note that $q(x,D) := x$ defines a Lie algebra extension of $\mathfrak{g}$ by $\mathrm{ad}\,\mathfrak{n}$. Further, $\rho(x,D) := D$ defines an action of $\widetilde{\mathfrak{g}}$ on $\mathfrak{n}$ by derivations, satisfying

$$[\rho(x,D)] = s(x) = s(q(x,D)).$$

Since $\mathfrak{z}(\mathfrak{n})$ is $\widetilde{\mathfrak{g}}$-invariant and the action of $\widetilde{\mathfrak{g}}$ on $\mathfrak{z}(\mathfrak{n})$ factors through the given action of $\mathfrak{g}$ on $\mathfrak{z}(\mathfrak{n})$, it suffices to see that the pullback $q^*(\mathtt{d}_S\omega) \in C^3(\widetilde{\mathfrak{g}},\mathfrak{z}(\mathfrak{n}))$ is a cocycle. For $\widetilde{S}(x,D) := S(x)$ and $\widetilde{\omega} := q^*\omega$, we have

$$\begin{aligned}\big(q^*(\mathtt{d}_S\omega)\big)(x,y,z) &= \sum_{\text{cycl.}} S(q(x))\omega(q(y),q(z)) - \omega([q(x),q(y)],q(z)) \\ &= \sum_{\text{cycl.}} \widetilde{S}(x)\widetilde{\omega}(y,z) - \widetilde{\omega}([x,y],z) = (\mathtt{d}_{\widetilde{S}}\widetilde{\omega})(x,y,z).\end{aligned}$$

Since $\widetilde{S}(x,D) - D = S(x) - D \in \mathrm{ad}(\mathfrak{n})$ for each $x \in \mathfrak{g}$, there exists a linear map $\beta\colon \widetilde{\mathfrak{g}} \to \mathfrak{n}$ with

$$\widetilde{S} = \rho + \mathrm{ad} \circ \beta\colon \widetilde{\mathfrak{g}} \to \mathrm{der}(\mathfrak{n}).$$

With respect to the $\widetilde{\mathfrak{g}}$-module structure on $\mathfrak{n}$, we thus have

$$\mathtt{d}_{\widetilde{S}}\widetilde{\omega} = \mathtt{d}_{\widetilde{\mathfrak{g}}}\widetilde{\omega} + [\beta, \widetilde{\omega}].$$

Since

$$\begin{aligned}
\mathrm{ad}(\widetilde{\omega}(x,y)) &= [\widetilde{S}(x), \widetilde{S}(y)] - \widetilde{S}([x,y]) \\
&= [\rho(x) + \mathrm{ad}\,\beta(x), \rho(y) + \mathrm{ad}\,\beta(y)] - \rho([x,y]) - \mathrm{ad}(\beta([x,y])) \\
&= \mathrm{ad}\big(\rho(x)\beta(y) - \rho(y)\beta(x) + [\beta(x), \beta(y)] - \beta([x,y])\big) \\
&= \mathrm{ad} \circ \big(\mathtt{d}_{\widetilde{\mathfrak{g}}}\beta + \frac{1}{2}[\beta,\beta]\big)(x,y),
\end{aligned}$$

we further get with Lemma 6.6.15 and $[\beta, [\beta, \beta]] = 0$:

$$[\beta, \widetilde{\omega}] = \big[\beta, \mathtt{d}_{\widetilde{\mathfrak{g}}}\beta + \frac{1}{2}[\beta,\beta]\big] = [\beta, \mathtt{d}_{\widetilde{\mathfrak{g}}}\beta] = -2\mathtt{d}_{\widetilde{\mathfrak{g}}}[\beta,\beta].$$

This proves that $\mathtt{d}_{\widetilde{S}}\widetilde{\omega} = \mathtt{d}_{\widetilde{\mathfrak{g}}}(\widetilde{\omega} - 2[\beta,\beta])$ is an element of $B^3(\widetilde{\mathfrak{g}}, \mathfrak{n})_s$, hence in particular a cocycle, and therefore $\mathtt{d}_S\omega$ also is a cocycle. □

Lemma 6.6.17. *The cohomology class $\chi(s) := [\mathtt{d}_S\omega] \in H^3(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s$ only depends on s, as long as $s(x) = [S(x)]$ holds for each $x \in \mathfrak{g}$.*

Proof. If S is fixed and $\omega' \in C^2(\mathfrak{g}, \mathfrak{n})$ also satisfies (6.12), then $\omega' - \omega \in C^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))$, so that

$$\mathtt{d}_S\omega' = \mathtt{d}_S\omega + \mathtt{d}_S(\omega' - \omega) = \mathtt{d}_S\omega + \mathtt{d}_{\mathfrak{g}}(\omega' - \omega) \in \mathtt{d}_S\omega + B^3(\mathfrak{g}, \mathfrak{z}(\mathfrak{n}))_s,$$

and thus $[\mathtt{d}_S\omega] = [\mathtt{d}_S\omega']$.

If $S'\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$ is another linear map with $[S'(x)] = s(x)$ for each $x \in \mathfrak{g}$, then $S' = S + \mathrm{ad} \circ \gamma$ for some linear map $\gamma\colon \mathfrak{g} \to \mathfrak{n}$. A direct calculation shows that

$$\omega' := \omega + \mathtt{d}_S\gamma + \frac{1}{2}[\gamma, \gamma]$$

satisfies

$$\mathrm{ad}(\omega'(x,y)) = [S'(x), S'(y)] - S'([x,y]).$$

We claim that $\mathtt{d}_{S'}\omega' = \mathtt{d}_S\omega$, and this will complete the proof. In fact, we obtain with Lemma 6.6.15

$$\begin{aligned}
\mathtt{d}_{S'}\omega' &= \mathtt{d}_S\omega' + [\gamma, \omega'] = \mathtt{d}_S\omega + \mathtt{d}_S^2\gamma + \frac{1}{2}\mathtt{d}_S[\gamma,\gamma] - [\omega', \gamma] \\
&= \mathtt{d}_S\omega + \mathtt{d}_S^2\gamma + [\mathtt{d}_S\gamma, \gamma] - [\omega', \gamma].
\end{aligned}$$

Further,

$$
\begin{aligned}
(\mathtt{d}_S^2\gamma)(x,y,z) &= \sum_{\text{cycl.}} S(x)\big(\mathtt{d}_S\gamma(y,z)\big) - \mathtt{d}_S\gamma([x,y],z) \\
&= \sum_{\text{cycl.}} S(x)\big(S(y)\gamma(z) - S(z)\gamma(y) - \gamma([y,z])\big) \\
&\qquad - S([x,y])\gamma(z) + S(z)\gamma([x,y]) + \gamma([[x,y],z]) \\
&= \sum_{\text{cycl.}} S(x)S(y)\gamma(z) - S(y)S(x)\gamma(z) - S(x)\gamma([y,z]) \\
&\qquad - S([x,y])\gamma(z) + S(x)\gamma([y,z]) \\
&= \sum_{\text{cycl.}} [\omega(x,y),\gamma(z)] - S(x)\gamma([y,z]) + S(x)\gamma([y,z]) \\
&= [\omega,\gamma](x,y,z).
\end{aligned}
$$

We thus obtain

$$\mathtt{d}_{S'}\omega' = \mathtt{d}_S\omega + [\omega + \mathtt{d}_S\gamma - \omega', \gamma] = \mathtt{d}_S\omega - \frac{1}{2}[[\gamma,\gamma],\gamma] = \mathtt{d}_S\omega. \qquad \square$$

Theorem 6.6.18. *For a* $\mathfrak{g}$*-kernel* $s\colon \mathfrak{g} \to \operatorname{out}(\mathfrak{n})$*, the set* $\operatorname{Ext}(\mathfrak{g},\mathfrak{n})_s$ *is nonempty if and only if the cohomology class* $\chi(s)$ *vanishes.*

Proof. The set $\operatorname{Ext}(\mathfrak{g},\mathfrak{n})_s$ is nonempty if and only if there exists a factor system (S,ω) corresponding to s with $\mathtt{d}_S\omega = 0$. If this is the case, then $\chi(s) = [\mathtt{d}_S\omega] = 0$. If, conversely, $\chi(s) = 0$ and $\beta \in C^2(\mathfrak{g},\mathfrak{z}(\mathfrak{n}))$ satisfies $\mathtt{d}_S\omega = \mathtt{d}_{\mathfrak{g}}\beta$, then $\widetilde{\omega} := \omega - \beta$ also satisfies

$$\operatorname{ad}(\widetilde{\omega}(x,y)) = \operatorname{ad}(\omega(x,y)) = [S(x),S(y)] - S([x,y])$$

for $x,y \in \mathfrak{g}$, and further $\mathtt{d}_S\widetilde{\omega} = \mathtt{d}_S\omega - \mathtt{d}_S\beta = \mathtt{d}_S\omega - \mathtt{d}_{\mathfrak{g}}\beta = 0$. Hence $(S,\widetilde{\omega})$ is a factor system corresponding to s. $\square$

Exercises for Section 6.6

Definition 6.6.19. A *Lie superalgebra* (over a field $\mathbb{K}$ with $2,3 \in \mathbb{K}^\times$) is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ with a bilinear map $[\cdot,\cdot]$ satisfying

(LS1) $[\alpha,\beta] = (-1)^{pq+1}[\beta,\alpha]$ for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$.
(LS2) $(-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{qp}[[\beta,\gamma],\alpha] + (-1)^{qr}[[\gamma,\alpha],\beta] = 0$ for $\alpha \in \mathfrak{g}_p$, $\beta \in \mathfrak{g}_q$ and $\gamma \in \mathfrak{g}_r$.

Note that (LS1) implies that $[\alpha,\alpha] = 0 = [\beta,[\beta,\beta]]$ for $\alpha \in \mathfrak{g}_{\overline{0}}$ and $\beta \in \mathfrak{g}_{\overline{1}}$.

Exercise 6.6.1. Let $\mathfrak{g}$ and $\mathfrak{n}$ be Lie algebras. Then the bracket defined in Definition 6.6.14 defines on the $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$C(\mathfrak{g},\mathfrak{n}) := \bigoplus_{p\in\mathbb{N}_0} C^p(\mathfrak{g},\mathfrak{n}) := C^{\text{even}}(\mathfrak{g},\mathfrak{n}) \oplus C^{\text{odd}}(\mathfrak{g},\mathfrak{n})$$

the structure of a Lie superalgebra.

Notes on Chapter 6

The finite dimension of the Lie algebra $\mathfrak{g}$ was not essential for the proof of the Poincaré–Birkhoff–Witt Theorem 6.1.9. Only slight changes yield the same theorem for any Lie algebra (cf. [Bou89, Ch. 1]).

The origin of Lie algebra cohomology lies in the work of É. Cartan who reduced the determination of the rational (singular) cohomology of compact Lie groups to a purely algebraic problem which later led to the invention of Lie algebra cohomology by C. Chevalley and S. Eilenberg [CE48]. Computational methods to evaluate Lie algebra cohomology spaces have been developed in [Kos50], where the cohomology of semisimple Lie algebras is studied in detail.

The proof of Ado's Theorem we present in Section 6.4 follows an idea of Y. A. Neretin [Ner02].

In [Ho54a], G. Hochschild shows that, for each $\mathfrak{g}$-module V of a Lie algebra $\mathfrak{g}$, each element of $H^3(\mathfrak{g}, V)$ arises as an obstruction class for a homomorphism $s\colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n})$, where $\mathfrak{n}$ is a Lie algebra with $V = \mathfrak{z}(\mathfrak{n})$. In [Ho54b] he analyzes for a finite-dimensional Lie algebra $\mathfrak{g}$ and a finite-dimensional $\mathfrak{g}$-module V, the question of the existence of a finite-dimensional Lie algebra $\mathfrak{n}$ with the above properties. In this case the answer is affirmative if $\mathfrak{g}$ is solvable, but if $\mathfrak{g}$ is semisimple, then all obstructions of homomorphism $s\colon \mathfrak{g} \to \mathrm{out}(\mathfrak{n})$ are trivial because s lifts to a homomorphism $S\colon \mathfrak{g} \to \mathrm{der}\, \mathfrak{n}$ by Levi's Theorem 4.6.6). The general result is that a cohomology class $[\omega] \in H^3(\mathfrak{g}, V)$ arises as an obstruction $\chi(s)$ of a $\mathfrak{g}$-kernel s if and only if its restriction to a Levi complement $\mathfrak{s}$ in $\mathfrak{g}$ vanishes.

Part III

Manifolds and Lie Groups

7

Smooth Manifolds

Even though it is possible to prove that each Lie group, up to covering, is isomorphic to a linear Lie group of the type discussed in Part I, the natural setting for Lie groups is the category of smooth manifolds. in which Lie groups can be viewed as the group objects. Thus we will use linear Lie groups rather as a source of examples and start in Chapter 8 to build the theory of Lie groups from scratch defining them as groups which are smooth manifolds for which the group operations are smooth.

We start in the present chapter by reviewing same basic features of differential analysis on open domains in $\mathbb{R}^n$. Then we introduce smooth manifolds and their tangent bundles, which allows us to define the Lie algebra vector fields, thus preparing the grounds for the definition of the Lie algebra of a general Lie group. Relating vector fields to ordinary differential equations leads to integral curves and local flows. These are used later on to define the exponential function of a Lie group. We conclude the chapter with a discussion of various concepts of submanifolds which naturally play are role in the study of subgroups of Lie groups.

In basic calculus courses one mostly deals with (differentiable) functions on open subsets of $\mathbb{R}^n$, but as soon as one wants to solve equations of the form $f(x) = y$, where $f\colon U \to \mathbb{R}^m$ is a differentiable function and U is open in $\mathbb{R}^n$, one observes that the set $f^{-1}(y)$ of solutions behaves in a much more complicated manner than one is used to from linear algebra, where f is linear and $f^{-1}(y)$ is the intersection of U with an affine subspace. One way to approach differentiable manifolds is to think of them as the natural objects arising as solutions of nonlinear equations as above (under some nondegeneracy condition on f, made precise by the Implicit Function Theorem). For submanifolds of $\mathbb{R}^n$, this is a quite natural approach, which immediately leads to the method of Lagrange multipliers to deal with extrema of differentiable functions under differentiable constraints. This is the external perspective on differentiable manifolds, which has the serious disadvantage that it depends very much on the surrounding space $\mathbb{R}^n$.

It is much more natural to adopt a more intrinsic perspective: an n-dimensional manifold is a topological space which locally looks like $\mathbb{R}^n$. More precisely, it arises by gluing open subsets of $\mathbb{R}^n$ in a smooth (differentiable) way. Below we shall make this more precise.

The theory of smooth manifolds has three levels:

(1) The **infinitesimal level**, where one deals with tangent spaces, tangent vectors and differentials of maps,
(2) the **local level**, which is analysis on open subsets of $\mathbb{R}^n$, and
(3) the **global level**, where one studies the global behavior of manifolds and other related structures.

These three levels are already visible in one-variable calculus: Suppose we are interested in the global maximum of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ which is a question about the global behavior of this function. The necessary condition $f'(x_0) = 0$ belongs to the infinitesimal level because it says something about the behavior of f infinitesimally close to the point x_0. The sufficient criterion for a local maximum: $f'(x_0) = 0$, $f''(x_0) < 0$ provides information on the local level. Of course, this is far from being the whole story and one really has to study global properties of f, such as $\lim_{x\to\pm\infty} f(x) = 0$, to guarantee the existence of global maxima.

7.1 Smooth Maps in Several Variables

First we recall some facts and definitions from calculus in several variables, formulated in a way that will be convenient for us in the following.

Definition 7.1.1 (Differentiable maps).
(a) Let $n, m \in \mathbb{N}$ and $U \subseteq \mathbb{R}^n$ be an open subset. A function $f\colon U \to \mathbb{R}^m$ is called *differentiable at* $x \in U$ if there exists a linear map $L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ such that for one norm (and hence for all norms) on $\mathbb{R}^n$ we have

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - L(h)}{\|h\|} = 0. \tag{7.1}$$

If f is differentiable in x, then for each $h \in \mathbb{R}^n$ we have

$$\lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big) = \lim_{t\to 0} \frac{1}{t} L(th) = L(h),$$

so that $L(h)$ is the directional derivative of f in x in the direction h. It follows in particular that condition (7.1) determines the linear map L uniquely. We therefore write

$$\mathtt{d}f(x)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big) = L(h)$$

and call the linear map $\mathtt{d}f(x)$ the *differential of* f *in* x.

(b) Let $e_1, \ldots, e_n$ denote the canonical basis vectors in $\mathbb{R}^n$. Then

$$\frac{\partial f}{\partial x_i}(x) := \mathtt{d}f(x)(e_i)$$

is called the *i-th partial derivative of f in x*. If f is differentiable in each $x \in U$, then the partial derivatives are functions

$$\frac{\partial f}{\partial x_i} \colon U \to \mathbb{R}^m,$$

and we say that f is *continuously differentiable*, or a *C^1-map*, if all its partial derivatives are continuous. For $k \geq 2$, the map f is said to be a *C^k-map* if it is C^1 and all its partial derivatives are C^{k-1}-maps. We say that f is *smooth* or a *C^∞-map* if it is C^k for each $k \in \mathbb{N}$. We denote the the space of C^k-maps $U \to \mathbb{R}^m$ by $C^k(U, \mathbb{R}^m)$.

(c) If $I \subseteq \mathbb{R}$ is an interval and $\gamma \colon I \to \mathbb{R}^n$ is a differentiable curve, we also write

$$\dot{\gamma}(t) = \gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

This is related to the notation from above by

$$\gamma'(t) = \mathtt{d}\gamma(t)(e_1),$$

where $e_1 = 1 \in \mathbb{R}$ is the canonical basis vector.

Definition 7.1.2. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. A map $f \colon U \to V$ is called C^k if it is C^k as a map $U \to \mathbb{R}^m$.

For $n \geq 1$ a C^k-map $f \colon U \to V$ is called a *C^k-diffeomorphism* if there exists a C^k-map $g \colon V \to U$ with

$$f \circ g = \mathrm{id}_V \quad \text{and} \quad g \circ f = \mathrm{id}_U .$$

Obviously, this is equivalent to f being bijective and f^{-1} being a C^k-map. Whenever such a diffeomorphism exists, we say that the domains U and V are *C^k-diffeomorphic*. For $k = 0$ we thus obtain the notion of a *homeomorphism*.

Theorem 7.1.3 (Chain Rule). *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. Further let $f \colon U \to V$ be a C^k-map and $g \colon V \to \mathbb{R}^d$ a C^k-map. Then $g \circ f$ is a C^k-map, and for each $x \in U$ we have in* $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^d)$*:*

$$\mathtt{d}(g \circ f)(x) = \mathtt{d}g(f(x)) \circ \mathtt{d}f(x).$$

The Chain Rule is an important tool which permits to "linearize" nonlinear information. The following proposition is an example.

Proposition 7.1.4 (Invariance of the Dimension). *If the nonempty open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are C^1-diffeomorphic, then $m = n$.*

Proof. Let $f\colon U \to V$ be a C^1-diffeomorphism and $g\colon V \to U$ its inverse. Pick $x \in U$. Then the Chain Rule implies that

$$\mathrm{id}_{\mathbb{R}^n} = \mathtt{d}(g \circ f)(x) = \mathtt{d}g(f(x)) \circ \mathtt{d}f(x)$$

and

$$\mathrm{id}_{\mathbb{R}^m} = \mathtt{d}(f \circ g)(f(x)) = \mathtt{d}f(x) \circ \mathtt{d}g(f(x)),$$

so that $\mathtt{d}f(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism. This implies that $m = n$. □

Theorem 7.1.5 (Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^n$ be an open subset, $x_0 \in U$, $k \in \mathbb{N} \cup \{\infty\}$, and $f\colon U \to \mathbb{R}^n$ a C^k-map for which the linear map $\mathtt{d}f(x_0)$ is invertible. Then there exists an open neighborhood V of x_0 in U for which $f|_V\colon V \to f(V)$ is a C^k-diffeomorphism onto an open subset of $\mathbb{R}^n$.*

Corollary 7.1.6. *Let $U \subseteq \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}^n$ be an injective C^k-map ($k \geq 1$) for which $\mathtt{d}f(x)$ is invertible for each $x \in U$. Then $f(U)$ is open and $f\colon U \to f(U)$ is a C^k-diffeomorphism.*

Proof. First we use the Inverse Function Theorem to see that for each $x \in U$ the image $f(U)$ contains a neighborhood of $f(x)$, so that $f(U)$ is an open subset of $\mathbb{R}^n$. Since f is injective, the inverse function $g = f^{-1}\colon f(U) \to U$ exists. Now we apply the Inverse Function Theorem again to see that for each $x \in U$ there exists a neighborhood of $f(x)$ in $f(U)$ on which g is C^k. Therefore g is a C^k-map, and this means that f is a C^k-diffeomorphism. □

Example 7.1.7. That the injectivity assumption in Corollary 7.1.6 is crucial is shown by the following example, which is a real description of the complex exponential function. We consider the smooth map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2).$$

Then the matrix of $\mathtt{d}f(x)$ with respect to the canonical basis is

$$[\mathtt{d}f(x)] = \begin{pmatrix} e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\ e^{x_1} \sin x_2 & e^{x_1} \cos x_2 \end{pmatrix}.$$

Its determinant is $e^{2x_1} \neq 0$, so that $\mathtt{d}f(x)$ is invertible for each $x \in \mathbb{R}^2$.

Polar coordinates immediately show that $f(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{(0,0)\}$, which is an open subset of $\mathbb{R}^2$, but the map f is not injective because it is 2π-periodic in x_2:

$$f(x_1, x_2 + 2\pi) = f(x_1, x_2).$$

Therefore the Inverse Function Theorem applies to each $x \in \mathbb{R}^2$, but f is not a global diffeomorphism.

Remark 7.1.8. The best way to understand the Implicit Function Theorem is to consider the linear case first. Let $g\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. We are interested in conditions under which the equation $g(x, y) = 0$ can be solved for x, i.e., there is a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $g(x,y) = 0$ is equivalent to $x = f(y)$.

Since we are dealing with linear maps, there are matrices $A \in M_m(\mathbb{R})$ and $B \in M_{m,n}(\mathbb{R})$ with

$$g(x,y) = Ax + By \quad \text{for} \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

The unique solvability of the equation $g(x,y) = 0$ for x is equivalent to the unique solvability of the equation $Ax = -By$, which is equivalent to the invertibility of the matrix A. If $A \in \mathrm{GL}_m(\mathbb{R})$, we thus obtain the linear function

$$f\colon \mathbb{R}^n \to \mathbb{R}^m, \quad f(y) = -A^{-1}By$$

for which $x = f(y)$ is equivalent to $g(x,y) = 0$.

Theorem 7.1.9 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be an open subset and $g\colon U \to \mathbb{R}^m$ be a C^k-function, $k \in \mathbb{N} \cup \{\infty\}$. Further let $(x_0, y_0) \in U$ with $g(x_0, y_0) = 0$ such that the linear map*

$$\mathtt{d}_1 g(x_0,y_0)\colon \mathbb{R}^m \to \mathbb{R}^m, \quad v \mapsto \mathtt{d}g(x_0,y_0)(v,0)$$

is invertible. Then there exist open neighborhoods V_1 of x_0 in $\mathbb{R}^m$ and V_2 of y_0 in $\mathbb{R}^n$ with $V_1 \times V_2 \subseteq U$, and a C^k-function $f\colon V_2 \to V_1$ with $f(y_0) = x_0$ such that

$$\{(x,y) \in V_1 \times V_2 \colon g(x,y) = 0\} = \{(f(y), y)\colon y \in V_2\}.$$

Definition 7.1.10 (Higher derivatives). For $k \geq 2$, a C^k-map $f\colon U \to \mathbb{R}^m$ and $U \subseteq \mathbb{R}^n$ open, higher derivatives are defined inductively by

$$\begin{aligned}&\mathtt{d}^k f(x)(h_1, \ldots, h_k)\\ &:= \lim_{t \to 0} \frac{1}{t}\big(\mathtt{d}^{k-1} f(x + th_k)(h_1, \ldots, h_{k-1}) - \mathtt{d}^{k-1} f(x)(h_1, \ldots, h_{k-1})\big).\end{aligned}$$

We thus obtain continuous maps

$$\mathtt{d}^k f\colon U \times (\mathbb{R}^n)^k \to \mathbb{R}^m.$$

In terms of concrete coordinates and the canonical basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, we then have

$$\mathtt{d}^k f(x)(e_{i_1}, \ldots, e_{i_k}) = \frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}}(x).$$

Let V and W be vector spaces. We recall that a map $\beta\colon V^k \to W$ is called *k-linear* if all the maps

$$V \to W, \quad v \mapsto \beta(v_1, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_k)$$

are linear. It is said to be *symmetric* if

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \beta(v_1, \ldots, v_k)$$

holds for all permutations $\sigma \in S_k$.

Proposition 7.1.11. *If $f \in C^k(U,\mathbb{R}^m)$ and $k \geq 2$, then the functions $(h_1,\ldots,h_k) \mapsto \mathtt{d}^k f(x)(h_1,\ldots,h_k)$, $x \in U$, are symmetric k-linear maps.*

Proof. From the definition it follows inductively that $(\mathtt{d}^k f)(x)$ is linear in each argument h_i, because if all other arguments are fixed, it is the differential of a C^1-function.

To verify the symmetry of $(\mathtt{d}^k f)(x)$, we may also proceed by induction. It suffices to show that for $h_1,\ldots,h_{k-2}$ fixed, the map

$$\beta(v,w) := (\mathtt{d}^k f(x))(h_1,\ldots,h_{k-2},v,w)$$

is symmetric. This map is the second derivative $\mathtt{d}^2 F(x)$ of the function

$$F(x) := (\mathtt{d}^{k-2} f)(x)(h_1,\ldots,h_{k-2}).$$

We may therefore assume that $k = 2$.

In view of the bilinearity, it suffices to observe that the Schwarz Lemma implies

$$(\mathtt{d}^2 F)(x)(e_j,e_i) = \Big(\frac{\partial^2}{\partial x_i x_j} F\Big)(x) = \Big(\frac{\partial^2}{\partial x_j x_i} F\Big)(x) = (\mathtt{d}^2 F)(x)(e_i,e_j). \qquad \square$$

Theorem 7.1.12 (Taylor's Theorem). *Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ of class C^{k+1}. If $x + [0,1]h \subseteq U$, then we have the Taylor Formula*

$$f(x+h) = f(x) + \mathtt{d}f(x)(h) + \ldots + \frac{1}{k!}\mathtt{d}^k f(x)(h,\ldots,h) + \frac{1}{k!}\int_0^1 (1-t)^k \big(\mathtt{d}^{k+1} f(x+th)\big)(h,\ldots,h)\, dt.$$

Proof. For each $i \in \{1,\ldots,m\}$ we consider the C^{k+1}-maps

$$F\colon [0,1] \to \mathbb{R}, \quad F(t) := f_i(x+th) \quad \text{with} \quad F^{(k)}(t) = \mathtt{d}^k f_i(x+th)(h,\ldots,h)$$

and apply the Taylor Formula for functions $[0,1] \to \mathbb{R}$ to get

$$F(1) = F(0) + \ldots + \frac{F^{(k)}(0)}{k!} + \frac{1}{k!}\int_0^1 (1-t)^k F^{(k+1)}(t)\, dt. \qquad \square$$

7.2 Smooth Manifolds and Smooth Maps

Before we turn to the concept of a smooth manifold, we recall the concept of a Hausdorff space. We assume, however, some familiarity with basic topological constructions and concepts, such as the quotient topology. A topological space (X,τ) is called a *Hausdorff space* if for two different points $x, y \in X$ there exist disjoint open subsets O_x, O_y with $x \in O_x$ and $y \in O_y$. Recall that each metric space (X,d) is Hausdorff.

Definition 7.2.1. Let M be a topological space.

(a) A pair (φ, U), consisting of an open subset $U \subseteq M$ and a homeomorphism $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$ of U onto an open subset of $\mathbb{R}^n$ is called an *n-dimensional chart of M*.

(b) Two n-dimensional charts (φ, U) and (ψ, V) of M are said to be *C^k-compatible* ($k \in \mathbb{N} \cup \{\infty\}$) if $U \cap V = \emptyset$ or the map

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)}\colon \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. Since $\varphi\colon U \to \varphi(U)$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, $\varphi(U \cap V)$ is an open subset of $\varphi(U)$ and hence of $\mathbb{R}^n$.

(c) An *n-dimensional C^k-atlas of M* is a family $\mathcal{A} := (\varphi_i, U_i)_{i\in I}$ of n-dimensional charts of M with the following properties:

(A1) $\bigcup_{i\in I} U_i = M$, i.e., $(U_i)_{i\in I}$ is an open covering of M.
(A2) All charts (φ_i, U_i), $i \in I$, are pairwise C^k-compatible. For $U_{ij} := U_i \cap U_j$, this means that all maps

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})}\colon \varphi_i(U_{ij}) \to \varphi_j(U_{ij})$$

are C^k-maps because $\varphi_{ji}^{-1} = \varphi_{ij}$.

(d) A chart (φ, U) is called *compatible* with a C^k-atlas $(\varphi_i, U_i)_{i\in I}$ if it is C^k-compatible with all charts of the atlas $\mathcal{A}$. A C^k-atlas $\mathcal{A}$ is called *maximal* if it contains all charts compatible with it. A maximal C^k-atlas is also called a *C^k-differentiable structure on M*. For $k = \infty$ we also call it a *smooth structure*.

Remark 7.2.2. (a) In Definition 7.2.1(b) we required that the map

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)}\colon \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. Since φ and ψ are homeomorphisms, this map always is a homeomorphism between open subsets of $\mathbb{R}^n$. The differentiability is an additional requirement.

(b) For $M = \mathbb{R}$ the maps (M, φ) and (M, ψ) with $\varphi(x) = x$ and $\psi(x) = x^3$ are 1-dimensional charts. These charts are not C^1-compatible: the map

$$\psi \circ \varphi^{-1}\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3$$

is smooth, but not a diffeomorphism, since its inverse $\varphi \circ \psi^{-1}$ is not differentiable.

(c) Every atlas $\mathcal{A}$ is contained in a unique maximal atlas: We simply add all charts compatible with $\mathcal{A}$, and thus obtain a maximal atlas. This atlas is unique (Exercise 7.2.2).

Definition 7.2.3. An *n-dimensional C^k-manifold* is a pair $(M, \mathcal{A})$ of a Hausdorff space M and a maximal n-dimensional C^k-atlas $\mathcal{A}$ for M. For $k = \infty$ we call it a smooth manifold.

To specify a manifold structure, it suffices to specify a C^k-atlas $\mathcal{A}$ because this atlas is contained in a unique maximal one (Exercise 7.2.2). In the following we shall never describe a maximal atlas. We shall always try to keep the number of charts as small as possible. For simplicity, we always assume in the following that $k = \infty$.

7.2.1 Examples

Example 7.2.4 (Open subsets of $\mathbb{R}^n$). Let $U \subseteq \mathbb{R}^n$ be an open subset. Then U is a Hausdorff space with respect to the induced topology. The inclusion map $\varphi\colon U \to \mathbb{R}^n$ defines a chart (φ, U) which already defines a smooth atlas of U, turning U into an n-dimensional smooth manifold.

Example 7.2.5 (The n-dimensional sphere). We consider the unit sphere

$$\mathbb{S}^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}$$

in $\mathbb{R}^n$, endowed with the subspace topology, turning it into a compact space.

(a) To specify a smooth manifold structure on $\mathbb{S}^n$, we consider the open subsets

$$U_i^\varepsilon := \{x \in \mathbb{S}^n \colon \varepsilon x_i > 0\}, \quad i = 0, \ldots, n, \quad \varepsilon \in \{\pm 1\}.$$

These $2(n+1)$ subsets form a covering of $\mathbb{S}^n$. We have homeomorphisms

$$\varphi_i^\varepsilon \colon U_i^\varepsilon \to B := \{x \in \mathbb{R}^n \colon \|x\|_2 < 1\}$$

onto the open unit ball in $\mathbb{R}^n$, given by

$$\varphi_i^\varepsilon(x) = (x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

and with continuous inverse map

$$(y_1, \ldots, y_n) \mapsto \Big(y_1, \ldots, y_i, \varepsilon\sqrt{1 - \|y\|_2^2}, y_{i+1}, \ldots, y_n\Big).$$

This leads to charts $(\varphi_i^\varepsilon, U_i^\varepsilon)$ of $\mathbb{S}^n$.

It is easy to see that these charts are pairwise compatible. We have $\varphi_i^\varepsilon \circ (\varphi_i^{\varepsilon'})^{-1} = \mathrm{id}_B$, and for $i < j$, we have

$$\varphi_i^\varepsilon \circ (\varphi_j^{\varepsilon'})^{-1}(y) = \Big(y_1, \ldots, y_i, y_{i+2}, \ldots, y_j, \varepsilon'\sqrt{1 - \|y\|_2^2}, y_{j+1}, \ldots, y_n\Big),$$

which is a smooth map

$$\varphi_j^{\varepsilon'}(U_i^\varepsilon \cap U_j^{\varepsilon'}) \to \varphi_i^\varepsilon(U_i^\varepsilon \cap U_j^{\varepsilon'}).$$

(b) There is another atlas of $\mathbb{S}^n$ consisting only of two charts, where the maps are slightly more complicated.

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_+ \colon U_+ := \mathbb{S}^n \setminus \{e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 - y_0} y$$

and

$$\varphi_- \colon U_- := \mathbb{S}^n \setminus \{-e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 + y_0} y.$$

Both maps are bijective with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2} \Big)$$

(Exercise 7.2.8). This implies that (φ_+, U_+) and (φ_-, U_-) are charts of $\mathbb{S}^n$. That both are smoothly compatible, hence a smooth atlas, follows from

$$(\varphi_+ \circ \varphi_-^{-1})(x) = (\varphi_- \circ \varphi_+^{-1})(x) = \frac{x}{\|x\|^2}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

which is the inversion at the unit sphere.

Example 7.2.6. Let E be an n-dimensional real vector space. We know from Linear Algebra that E is isomorphic to $\mathbb{R}^n$, and that for each ordered basis $B := (b_1, \ldots, b_n)$ for E, the linear map

$$\varphi_B \colon \mathbb{R}^n \to E, \quad x = (x_1, \ldots, x_n) \mapsto \sum_{j=1}^n x_j b_j$$

is a linear isomorphism. Using such a linear isomorphism φ_B, we define a topology on E in such a way that φ_B is a homeomorphism, i.e., $O \subseteq E$ is open if and only if $\varphi_B^{-1}(O)$ is open in $\mathbb{R}^n$.

For any other choice of a basis $C = (c_1, \ldots, c_m)$ in E we recall from linear algebra that $m = n$ and that the map

$$\varphi_C^{-1} \circ \varphi_B \colon \mathbb{R}^n \to \mathbb{R}^n$$

is a linear isomorphism, hence a homeomorphism. This implies that for a subset $O \subseteq E$ the condition that $\varphi_B^{-1}(O)$ is open is equivalent to $\varphi_C^{-1}(O) = \varphi_C^{-1} \circ \varphi_B \circ \varphi_B^{-1}(O)$ being open. We conclude that the topology introduced on E by φ_B does not depend on the choice of a basis.

We thus obtain on E a natural topology for which it is homeomorphic to $\mathbb{R}^n$, hence in particular a Hausdorff space. From each coordinate map $\kappa_B := \varphi_B^{-1}$ we obtain a chart (κ_B, E) which already defines an atlas of E. We thus obtain on E the structure of an n-dimensional smooth manifold. That all these charts are compatible follows from the smoothness of the linear maps $\kappa_C \circ \kappa_B^{-1} = \varphi_C^{-1} \circ \varphi_B \colon \mathbb{R}^n \to \mathbb{R}^n$.

Example 7.2.7 (Submanifolds of $\mathbb{R}^n$). A subset $M \subseteq \mathbb{R}^n$ is called a *d-dimensional submanifold* if for each $p \in M$ there exists an open neighborhood U of p in $\mathbb{R}^n$ and a diffeomorphism

$$\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$$

onto an open subset $\varphi(U)$ with

$$\varphi(U \cap M) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}). \tag{7.2}$$

Whenever this condition is satisfied, we call (φ, U) a *submanifold chart.*

A submanifold of *codimension* 1, i.e., $\dim M = n - 1$, is called a *smooth hypersurface.*

We claim that M carries a natural d-dimensional manifold structure when endowed with the topology inherited from $\mathbb{R}^n$, which obviously turns it into a Hausdorff space.

In fact, for each submanifold chart (φ, U), we obtain a d-dimensional chart

$$(\varphi|_{U\cap M}, U \cap M),$$

where we have identified $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\}$. For two such charts coming from (φ, U) and (ψ, V), we have

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap M)} = (\psi|_{V\cap M}) \circ (\varphi|_{U\cap M})^{-1}|_{\varphi(U\cap V\cap M)},$$

which is a smooth map onto an open subset of $\mathbb{R}^d$. We thus obtain a smooth atlas of M.

The following proposition provides a particularly handy criterion to verify that the set of solutions of a nonlinear equation is a submanifold.

Definition 7.2.8. Let $f\colon U \to \mathbb{R}^m$ be a C^1-map. We call $y \in \mathbb{R}^m$ a *regular value of* f if for each $x \in U$ with $f(x) = y$ the differential $\mathtt{d}f(x)$ is surjective. Otherwise y is called a *singular value of* f. Note that, in particular, each $y \in \mathbb{R}^m \setminus f(U)$ is a regular value.

Proposition 7.2.9 (Regular Value Theorem–Local Version). *Let $U \subseteq \mathbb{R}^n$ be an open subset, $f\colon U \to \mathbb{R}^m$ a smooth map and $y \in \mathbb{R}^m$ a regular value of f. Then $M := f^{-1}(y)$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$, hence in particular a smooth manifold.*

Proof. Let $d := n - m$ and observe that $d \geq 0$ because $\mathtt{d}f(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective for each $x \in M$. We have to show that for each $x_0 \in M$ there exists an open neighborhood V of x_0 in $\mathbb{R}^n$ and a diffeomorphism

$$\varphi\colon V \to \varphi(V) \subseteq \mathbb{R}^n$$

with

$$\varphi(V \cap M) = \varphi(V) \cap (\mathbb{R}^d \times \{0\}).$$

After a permutation of the coordinates, we may w.l.o.g. assume that the vectors

$$\mathtt{d}f(x_0)(e_{d+1}), \ldots, \mathtt{d}f(x_0)(e_n)$$

form a basis for $\mathbb{R}^m$. Then we consider the map

$$\varphi\colon U \to \mathbb{R}^n, \quad x = (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d, f_1(x) - y_1, \ldots, f_m(x) - y_m).$$

In view of

$$\mathtt{d}\varphi(x_0)(e_j) = \begin{cases} (e_j, \mathtt{d}f(x_0)e_j) & \text{for } j \leq d \\ \mathtt{d}f(x_0)(e_j) & \text{for } j > d, \end{cases}$$

it follows that the linear map $\mathtt{d}\varphi(x_0)\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Hence the Inverse Function Theorem implies the existence of an open neighborhood $V \subseteq U$ of x_0 for which $\varphi|_V\colon V \to \varphi(V)$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$.

Since

$$M = \{p \in U\colon \varphi(p) = (\varphi_1(p), \ldots, \varphi_d(p), 0, \ldots, 0)\} = \varphi^{-1}(\mathbb{R}^d \times \{0\}),$$

it follows that $\varphi(M \cap V) = \varphi(V) \cap (\mathbb{R}^d \times \{0\})$. □

Example 7.2.10. The preceding proposition is particularly easy to apply for hypersurfaces, i.e., to the case $m = 1$. Then $f\colon U \to \mathbb{R}$ is a smooth function and the condition that $\mathtt{d}f(x)$ is surjective simply means that $\mathtt{d}f(x) \neq 0$, i.e., that there exists some j with $\frac{\partial f}{\partial x_j}(x) \neq 0$.

(a) Let $A = A^\top \in M_n(\mathbb{R})$ be a symmetric matrix and

$$f(x) := x^\top A x = \sum_{i,j=1}^n a_{ij} x_i x_j$$

the corresponding quadratic form. We want to show that the corresponding *quadric*

$$Q := \{x \in \mathbb{R}^n\colon f(x) = 1\}$$

is a submanifold of $\mathbb{R}^n$. To verify the criterion from Proposition 7.2.9, we assume that $f(x) = 1$ and note that

$$\mathtt{d}f(x)v = v^\top A x + x^\top A v = 2v^\top A x$$

(Exercise; use Exercise 7.2.11). Therefore $\mathtt{d}f(x) = 0$ is equivalent to $Ax = 0$, which is never the case if $x^\top A x = 1$. We conclude that all level surfaces of f are smooth hypersurfaces of $\mathbb{R}^n$.

For $A = E_n$ (the identity matrix), we obtain the $(n-1)$-dimensional unit sphere $Q = \mathbb{S}^{n-1}$.

For $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and nonzero λ_i we obtain the *hyperboloids*

$$Q = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i x_i^2 = 1 \right\}$$

which degenerate to hyperbolic cylinders if some λ_i vanish.

(b) For singular values the level sets may or may not be submanifolds: For the quadratic form

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = x_1 x_2$$

the value 0 is singular because $f(0,0) = 0$ and $\mathtt{d}f(0,0) = 0$. The inverse image is

$$f^{-1}(0) = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}),$$

which is not a submanifold of $\mathbb{R}^2$ (Exercise).

For the quadratic form

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = x_1^2 + x_2^2$$

the value 0 is singular because $f(0,0) = 0$ and $\mathtt{d}f(0,0) = 0$. The inverse image is

$$f^{-1}(0) = \{(0,0)\},$$

which is a zero-dimensional submanifold of $\mathbb{R}^2$.

For the quadratic form

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

the value 0 is singular because $f(0) = 0$ and $f'(0) = 0$. The inverse image is

$$f^{-1}(0) = \{0\},$$

which is a submanifold of $\mathbb{R}$.

(c) On $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ we consider the quadratic function

$$f : M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) : A^\top = A\}, \quad X \mapsto XX^\top.$$

Then

$$f^{-1}(\mathbf{1}) = \mathrm{O}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) : g^\top = g^{-1}\}$$

is the *orthogonal group*.

To see that this is a submanifold of $M_n(\mathbb{R})$, we note that

$$\mathtt{d}f(X)(Y) = XY^\top + YX^\top$$

(Exercise 7.2.11). If $f(X) = \mathbf{1}$, we have $X^\top = X^{-1}$, so that for any $Z \in \mathrm{Sym}_n(\mathbb{R})$ the matrix $Y := \frac{1}{2}ZX$ satisfies

$$XY^\top + YX^\top = \frac{1}{2}(XX^\top Z + ZXX^\top) = Z.$$

Therefore $\mathtt{d}f(X)$ is surjective in each orthogonal matrix X, and Proposition 7.2.9 implies that $\mathrm{O}_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$ of dimension

$$d = n^2 - \dim\big(\mathrm{Sym}_n(\mathbb{R})\big) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Remark 7.2.11 (The gluing picture). Let M be an n-dimensional manifold with an atlas $\mathcal{A} = (\varphi_i, U_i)_{i\in A}$ and $V_i := \varphi_i(U_i)$ the corresponding open subsets of $\mathbb{R}^n$.

Note that we have used the topology of M to define the notion of a chart. We now explain how the topological space M can be reconstructed from the atlas $\mathcal{A}$. First we consider the set

$$S := \bigcup_{i\in I} \{i\} \times V_i,$$

which we consider as the disjoint union of the open subset $V_i \subseteq \mathbb{R}^n$. We endow S with the topology of the disjoint sum, i.e., a subset $O \subseteq S$ is open if and only if all its intersections with the subsets $\{i\} \times V_i \cong V_i$ are open.

Then we consider the surjective map

$$\Phi\colon S \to M, \quad (i, x) \mapsto \varphi_i^{-1}(x).$$

On each subset $\{i\} \times V_i$ this map is a homeomorphism onto U_i. Hence Φ is continuous, surjective and open, which implies that it is a quotient map, i.e., that the topology on M coincides with the quotient topology on $S/\sim$, where

$$(i, x) \sim (j, y) \quad \Longleftrightarrow \quad \varphi_i(\varphi_j^{-1}(y)) = x.$$

In this sense we can think of M as obtained by gluing of the patches $U_i \cong V_i$, where $U_{ij} = U_i \cap U_j$ and $x_i \in \varphi_i(U_{ij}) \subseteq V_i$ is identified with the point $x_j = \varphi_j(\varphi_i^{-1}(x_i)) \in V_j$.

Example 7.2.12. We discuss an example of a "non-Hausdorff manifold". We endow the set $S := (\{1\} \times \mathbb{R}) \cup (\{2\} \times \mathbb{R})$ with the disjoint sum topology and define an equivalence relation on S by

$$(1, x) \sim (2, y) \quad \Longleftrightarrow \quad x = y \neq 0.$$

If $[i, x]$ denotes the class of (i, x) we see that all classes except $[1, 0]$ and $[2, 0]$ contain 2 points. The topological quotient space

$$M := S/\sim\, = \{[1, x] \colon x \in \mathbb{R}\} \cup \{[2, 0]\} = \{[2, x] \colon x \in \mathbb{R}\} \cup \{[1, 0]\}$$

is the union of a real line with an extra point, but the two points $[1, 0]$ and $[2, 0]$ have no disjoint open neighborhoods.

The subsets $U_j := \{[j, x] \colon x \in \mathbb{R}\}$, $j = 1, 2$, of M are open, and the maps

$$\varphi_j \colon U_j \to \mathbb{R}, [j, x] \mapsto x,$$

are homeomorphisms defining a smooth atlas on M (Exercise 7.2.12).

Remark 7.2.13. One can also define manifold structures on sets carrying no a priori topology. To this end one proceeds as follows. We start with a set M.

An n-dimensional chart of M is a pair (φ, U), where $U \subseteq M$ is a subset and $\varphi\colon U \to \mathbb{R}^n$ an injection with an open image. Two charts (φ, U) and (ψ, V) are called C^k-compatible if both $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open and

$$\psi^{-1} \circ \varphi\colon \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. The notion of a C^k-atlas is introduced as in Definition 7.2.1(c).

Let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an n-dimensional C^k-atlas on M. We call a subset $O \subseteq M$ open if for each $i \in I$ the subset $\varphi_i(O \cap U_i) \subseteq \mathbb{R}^n$ is open. It is easy to see that we thus obtain a topology on M. For any subset $O_j \subseteq U_j$, we have

$$\varphi_i(O_j \cap U_i) = (\varphi_i \circ \varphi_j^{-1})(\varphi_j(O_j) \cap \varphi_j(U_i \cap U_j)),$$

showing that O_j is open in M if and only if $\varphi_j(O_j)$ is open in $\mathbb{R}^n$. Hence each chart (φ_j, U_j) defines a homeomorphism $U_j \to \varphi(U_j)$. As Example 7.2.12 shows, the topology on M need not be Hausdorff, but if it is, then the arguments above show that the pair $(M, \mathcal{A})$ is an n-dimensional C^k-manifold.

A sufficient condition for the Hausdorff property to hold is that for any pair $x \neq y$ in M either

(H1) there exists an $i \in I$ with $x, y \in U_i$,
or
(H2) there exist $i, j \in I$ with $x \in U_i$, $y \in U_j$ and $U_i \cap U_j = \emptyset$.

In fact, in the second case x and y are separated by the open sets U_i and U_j, and in the first case the fact that U_i is Hausdorff implies the existence of disjoint open subsets of U_i, and hence of M, separating x and y.

As an application of the preceding remark, we discuss the Graßmann manifolds of k-dimensional subspaces of an n-dimensional vector space.

Example 7.2.14 (Graßmannians). Let $V = \mathbb{R}^n$ and write $\mathrm{Gr}_k(V) = \mathrm{Gr}_k(\mathbb{R}^n)$ for the set of all k-dimensional linear subspaces of V. We now use the approach described in Remark 7.2.13 to define on $\mathrm{Gr}_k(\mathbb{R}^n)$ the structure of a smooth manifold of dimension $k(n-k)$.

For an $(n-k)$-dimensional subspace $F \subseteq V$, we consider the subset

$$U_F := \{E \in \mathrm{Gr}_k(V)\colon F \oplus E = V\}$$

of all subspaces complementing F. Fixing an element $E_0 \in U_F$, we have $V = F \oplus E_0$, and each $E \in U_F$ can be written as the graph

$$\Gamma(f) = \{(x, f(x))\colon x \in E_0\} \quad \text{of some} \quad f \in \mathrm{Hom}(E_0, F).$$

In terms of the projections $\mathrm{pr}^F_{E_0}\colon V \to E_0$ and $\mathrm{pr}^{E_0}_F\colon V \to F$ along the decomposition $V = F \oplus E_0$, the linear map f can be expressed as

$$f = \mathrm{pr}^{E_0}_F \circ (\mathrm{pr}^F_{E_0}|_E)^{-1}.$$

Therefore

$$\varphi_{F,E_0} : U_F \to \mathrm{Hom}(E_0, F), \quad E \mapsto \mathrm{pr}_F^{E_0} |_E \circ (\mathrm{pr}_{E_0}^F |_E)^{-1}$$

is a bijection. Replacing E_0 by another subspace $E_1 \in U_F$, the relation

$$\Gamma(f) = \Gamma(\widetilde{f}) = E \quad \text{for} \quad f \in \mathrm{Hom}(E_0, F), \widetilde{f} \in \mathrm{Hom}(E_1, F),$$

leads to the relation $x + f(x) = y + \widetilde{f}(y)$, where $x \in E_0$ and $y = \mathrm{pr}_{E_1}^F(x) \in E_1$. But then

$$\widetilde{f}(y) = f(x) + x - y = f((\mathrm{pr}_{E_1}^F |_{E_0})^{-1}(y)) + (\mathrm{pr}_{E_1}^F |_{E_0})^{-1}(y) - y.$$

This means that

$$\varphi_{F,E_1}(E) = \widetilde{f} = f \circ (\mathrm{pr}_{E_1}^F |_{E_0})^{-1} + \varphi_{F,E_1}(E_0).$$

Therefore the transition between the charts φ_{F,E_1} and φ_{F,E_0} is given by an invertible affine map.

If two sets U_F and $U_{F'}$ intersect nontrivially, then the set of those $f \in \mathrm{Hom}(E_0, F)$ for which $F' \oplus \Gamma(f) = V$ is open in subset of $\mathrm{Hom}(E_0, F)$. In fact, if $b_1, \ldots, b_k$ is a basis for E_0 and $c'_1, \ldots, c'_{n-k}$ is a basis for F', then the condition on f is that

$$\det(f(b_1), \ldots, f(b_k), c'_1, \ldots, c'_{n-k}) \neq 0,$$

which specifies an open subset of $\mathrm{Hom}(E_0, F) \cong \mathbb{R}^{k(n-k)}$.

We further note that, for $E_0 \in U_F \cap U_{F'}$, we have

$$\begin{aligned}\varphi_{F',E_0}(E) &= \mathrm{pr}_{F'}^{E_0} \circ (\mathrm{pr}_{E_0}^{F'} |_E)^{-1} = (\mathrm{pr}_{F'}^E |_F) \circ \mathrm{pr}_F^{E_0} \circ (\mathrm{pr}_{E_0}^F |_E)^{-1} \\ &= (\mathrm{pr}_{F'}^E |_F) \circ \varphi_{F,E_0}(E).\end{aligned}$$

Hence all these coordinate changes are given by restrictions of invertible linear maps. This proves that the collection $(\varphi_{F,E_0}, U_F)_{F \oplus E_0 = V}$ yields a smooth $k(n-k)$-dimensional atlas of $\mathrm{Gr}_k(V)$ if we identify all the spaces $\mathrm{Hom}(E_0, F)$ with $\mathbb{R}^{n(k-n)}$ by some linear isomorphism.

In view of Remark 7.2.13, it now suffices to verify that the topology on $\mathrm{Gr}_k(V)$ defined by our smooth atlas is Hausdorff and (H1) shows that it suffices to find for $E_0, E_1 \in \mathrm{Gr}_k(V)$ a subspace F with $E_0, E_1 \in U_F$.

Writing $V = F_1 \oplus (E_0 \cap E_1)$ for some subspace F_1, we have to find a subspace F_2 of $F_1 \cap (E_0 + E_1)$ complementing $E_0 \cap F_1$ and $E_1 \cap F_1$. Since the latter two spaces intersect trivially, we have (Exercise)

$$F_1 \cap (E_0 + E_1) = (E_0 \cap F_1) \oplus (E_1 \cap F_1).$$

Because of $(E_0 \cap F_1) \oplus (E_0 \cap E_1) = E_0$, using standard dimension formulas from linear algebra, one obtains a linear isomorphism

$$\varphi : E_0 \cap F_1 \to E_1 \cap F_1.$$

Then its graph $\Gamma(\varphi)$ is a linear subspace of $F_1 \cap (E_0 + E_1)$ complementing both subspaces $E_i \cap F_1$.

Example 7.2.15 (Projective space). As an important special case of a Graßmann manifold we obtain the *real projective space*

$$\mathbb{P}(\mathbb{R}^n) := \mathrm{Gr}_1(\mathbb{R}^n)$$

of all 1-dimensional subspaces of $\mathbb{R}^n$. It is a smooth manifold of dimension $n-1$. For $n=2$, this space is called the *projective line* and for $n=3$ it is called the *projective plane* (it is a 2-dimensional manifold, thus also called a *surface*).

We write $[x] := \mathbb{R}x \in \mathbb{P}(\mathbb{R}^n)$ for the subspace generated by a nonzero element $x \in \mathbb{R}^n$. Since each one-dimensional subspace $\mathbb{R}x$ of $\mathbb{R}^n$ is not contained in some of the hyperplanes $F_i := \mathrm{span}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$, the manifold $\mathbb{P}(\mathbb{R}^n)$ is covered by the n open sets

$$U_i := U_{F_i} = \{[x] \colon x_i \neq 0\}, \quad i = 1, \ldots, n.$$

The simplest charts are obtained by picking the complement $E_0 = \mathbb{R}e_i$ of the hyperplane F_i and identifying $\mathrm{Hom}(E_0, F_i) \cong F_i \cong \mathbb{R}^{n-1}$. For any $[x] \in U_i$, we then have

$$\mathbb{R}x = \Gamma(f) \quad \text{with} \quad f(e_i) = \Big(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\Big),$$

which leads to the coordinates

$$\varphi_i([x]) = \Big(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\Big).$$

These charts are called *homogeneous coordinates*. They play a fundamental role in projective geometry.

Example 7.2.16. We discuss an example of a manifold which is not separable. In particular, its topology has no countable basis. We define a new topology on $\mathbb{R}^2$ by defining $O \subseteq \mathbb{R}^2$ to be open if and only if for each $y \in \mathbb{R}$ the set

$$O^y := \{x \in \mathbb{R} \colon (x, y) \in O\}$$

is open in $\mathbb{R}$. This defines a topology on $\mathbb{R}^2$ for which all sets $U_y := \mathbb{R} \times \{y\}$ are open and the maps

$$\varphi_y \colon U_y \to \mathbb{R}, \quad (x, y) \mapsto x$$

are homeomorphisms. We thus obtain a smooth 1-dimensional manifold structure on $\mathbb{R}^2$ for which it has uncountably many connected components, namely the subsets U_y, $y \in \mathbb{R}$.

Remark 7.2.17 (Coordinates versus parameterizations). (a) Let (φ, U) be an n-dimensional chart of the smooth manifold M. Then $\varphi \colon U \to \mathbb{R}^n$ has n components $\varphi_1, \ldots, \varphi_n$ which we consider as coordinate functions on

U. Sometimes it is convenient to write $x_i(p) := \varphi_i(p)$ for $p \in U$, so that $(x_1(p), \dots, x_n(p))$ are the coordinates of $p \in U$ w.r.t. the chart (φ, U).

If we have another chart (ψ, V) of M with $U \cap V \neq \emptyset$, then any $p \in U \cap V$ has a second tuple of coordinates, $x_i'(p) := \psi_i(p)$, given by the components of ψ. Now the change of coordinates is given by

$$x'(x) = \psi(\varphi^{-1}(x)) \quad \text{and} \quad x(x') = \varphi(\psi^{-1}(x')).$$

In this sense the maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ describe how we translate between the x-coordinates and the x'-coordinates.

(b) Instead of putting the focus on coordinates, which are functions on open subsets of the manifold, one can also parameterize open subset of M. This is done by maps $\varphi \colon V \to M$, where V is an open subsets of some $\mathbb{R}^n$ and $(\varphi^{-1}, \varphi(V))$ is a chart of M. Then the point $p \in M$ corresponding to the parameter values $(x_1, \dots, x_n) \in V$ is $p = \varphi(x)$. In this picture the lines

$$t \mapsto \varphi(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$$

are curves on M, called the *parameter lines*.

7.2.2 New Manifolds from Old Ones

Definition 7.2.18 (Open subsets are manifolds). Let M be a smooth manifold and $N \subseteq M$ an open subset. Then N carries a natural smooth manifold structure.

Let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an atlas of M. Then $V_i := U_i \cap N$ and $\psi_i := \varphi_i|_{V_i}$ define a smooth atlas $\mathcal{B} := (\psi_i, V_i)_{i \in I}$ of N (Exercise).

Definition 7.2.19 (Products of manifolds). Let M and N be smooth manifolds of dimensions d, resp., k and

$$M \times N = \{(m, n) \colon m \in M, n \in N\}$$

the product set, which we endow with the product topology.

We show that $M \times N$ carries a natural structure of a smooth $(d+k)$-dimensional manifold. Let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an atlas of M and $\mathcal{B} = (\psi_j, V_j)_{j \in J}$ an atlas of N. Then the product sets $W_{ij} := U_i \times V_j$ are open in $M \times N$ and the maps

$$\gamma_{ij} := \varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^d \times \mathbb{R}^k \cong \mathbb{R}^{d+k}, \quad (x, y) \mapsto (\varphi_i(x), \psi_j(y))$$

are homeomorphisms onto open subsets of $\mathbb{R}^{d+k}$. On $\gamma_{i'j'}(W_{ij} \cap W_{i'j'})$ we have

$$\gamma_{ij} \circ \gamma_{i'j'}^{-1} = (\varphi_i \circ \varphi_{i'}^{-1}) \times (\psi_j \circ \psi_{j'}^{-1}),$$

which is a smooth map. Therefore $(\varphi_{ij}, W_{ij})_{(i,j) \in I \times J}$ is a smooth atlas on $M \times N$.

Definition 7.2.20. Let M and N be smooth manifolds.

(a) A continuous map $f\colon M \to N$ is called *smooth*, if for each chart (φ, U) of M and each chart (ψ, V) of N the map

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is smooth. Note that $\varphi(f^{-1}(V) \cap U)$ is open because f is continuous.

We write $C^\infty(M, N)$ for the set of smooth maps $M \to N$ and abbreviate $C^\infty(M, \mathbb{R})$ by $C^\infty(M)$.

(b) A map $f\colon M \to N$ is called a *diffeomorphism*, or a *smooth isomorphism*, if there exists a smooth map $g\colon N \to M$ with

$$f \circ g = \mathrm{id}_N \quad \text{and} \quad g \circ f = \mathrm{id}_M .$$

This condition obviously is equivalent to f being bijective and its inverse f^{-1} being a smooth map.

We write $\mathrm{Diff}(M)$ for the set of all diffeomorphisms of M.

Lemma 7.2.21. *Compositions of smooth maps are smooth. In particular, the set* $\mathrm{Diff}(M)$ *is a group (with respect to composition) for each smooth manifold* M. [1]

Proof. Let $f\colon M \to N$ and $g\colon N \to L$ be smooth maps. Pick charts (φ, U) of M and (γ, W) of L. To see that the map $\gamma \circ (g \circ f) \circ \varphi^{-1}$ is smooth on $\varphi((g \circ f)^{-1}(W))$, we have to show that each element $x = \varphi(p)$ in this set has a neighborhood on which it is smooth. Let $q := f(p)$ and note that $g(q) \in W$. We choose a chart (ψ, V) of N with $q \in V$. We then have

$$\gamma \circ (g \circ f) \circ \varphi^{-1} = (\gamma \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1})$$

on the open neighborhood $\varphi\big(f^{-1}(V) \cap (g \circ f)^{-1}(W)\big)$ of x. Since compositions of smooth maps on open domains in $\mathbb{R}^n$ are smooth by the Chain Rule (Theorem 7.1.3), $\gamma \circ (g \circ f) \circ \varphi^{-1}$ is smooth on $\varphi((g \circ f)^{-1}(W))$. This proves that $g \circ f\colon M \to L$ is a smooth map. □

Remark 7.2.22. (a) If $I \subseteq \mathbb{R}$ is an open interval, then a smooth map $\gamma\colon I \to M$ is called a *smooth curve*.

For a not necessarily open interval $I \subseteq \mathbb{R}$, a map $\gamma\colon I \to \mathbb{R}^n$ is called smooth if all derivatives $\gamma^{(k)}$ exist in all points of I and define continuous functions $I \to \mathbb{R}^n$. Based on this generalization of smoothness for curves, a curve $\gamma\colon I \to M$ is said to be smooth, if for each chart (φ, U) of M the curves

[1] For each manifold M the identity $\mathrm{id}_M\colon M \to M$ is a smooth map, so that this lemma leads to the "category of smooth manifolds". The objects of this category are smooth manifolds and the morphisms are the smooth maps. In the following we shall use consistently category theoretical language, but we shall not go into the formal details of category theory.

$$\varphi \circ \gamma \colon \gamma^{-1}(U) \to \mathbb{R}^n$$

are smooth.

A curve $\gamma\colon [a,b] \to M$ is called *piecewise smooth* if γ is continuous and there exists a subdivision $x_0 = a < x_1 < \ldots, < x_N = b$ such that $\gamma|_{[x_i,x_{i+1}]}$ is smooth for $i = 0, \ldots N-1$.

(b) Smoothness of maps $f\colon M \to \mathbb{R}^n$ can be checked more easily. Since the identity is a chart of $\mathbb{R}^n$, the smoothness condition simply means that for each chart (φ, U) of M the map

$$f \circ \varphi^{-1}\colon \varphi(f^{-1}(V) \cap U) \to \mathbb{R}^n$$

is smooth.

(c) If U is an open subset of $\mathbb{R}^n$, then a map $f\colon U \to M$ to a smooth m-dimensional manifold M is smooth if and only if for each chart (φ, V) of M the map

$$\varphi \circ f\colon f^{-1}(V) \to \mathbb{R}^n$$

is smooth.

(d) Any chart (φ, U) of a smooth n-dimensional manifold M defines a diffeomorphism $U \to \varphi(U) \subseteq \mathbb{R}^n$, when U is endowed with the canonical manifold structure as an open subset of M.

In fact, by definition, we may use (φ, U) as an atlas of U. Then the smoothness of φ is equivalent to the smoothness of the map $\varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$, which is trivial. Likewise, the smoothness of $\varphi^{-1}\colon \varphi(U) \to U$ is equivalent to the smoothness of $\varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$.

Exercises for Section 7.2

Exercise 7.2.1. Let $M := \mathbb{R}$, endowed with its standard topology. Show that C^k-compatibility of 1-dimensional charts is not an equivalence relation.

Exercise 7.2.2. Show that each n-dimensional C^k-atlas is contained in a unique maximal one.

Exercise 7.2.3. Let If M_i, $i = 1, \ldots, n$, be smooth manifolds of dimension d_i. Show that the product space $M := M_1 \times \ldots \times M_n$ carries the structure of a $(d_1 + \ldots + d_n)$-dimensional manifold.

Exercise 7.2.4 (Relaxation of the smoothness definition). Let M and N be smooth manifolds. Show that a map $f\colon M \to N$ is smooth if and only if for each point $x \in M$ there exists a chart (φ, U) of M with $x \in U$ and a chart (ψ, V) of N with $f(x) \in V$ such that the map

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(f^{-1}(V)) \to \psi(V)$$

is smooth.

Exercise 7.2.5. Show that the set $A := C^\infty(M, \mathbb{R})$ of smooth real-valued functions on M is a real algebra. If $g \in A$ is nonzero and $U := g^{-1}(\mathbb{R}^\times)$, then $\frac{1}{g} \in C^\infty(U, \mathbb{R})$.

Exercise 7.2.6. Let $f_1 \colon M_1 \to N_1$ and $f_2 \colon M_2 \to N_2$ be smooth maps. Show that the map

$$f_1 \times f_2 \colon M_1 \times M_2 \to N_1 \times N_2, \quad (x, y) \mapsto (f_1(x), f_2(y))$$

is smooth.

Exercise 7.2.7. Let $f_1 \colon M \to N_1$ and $f_2 \colon M \to N_2$ be smooth maps. Show that the map

$$(f_1, f_2) \colon M \to N_1 \times N_2, \quad x \mapsto (f_1(x), f_2(x))$$

is smooth.

Exercise 7.2.8. (a) Verify the details in Example 7.2.5, where we describe an atlas of $\mathbb{S}^n$ by stereographic projections.

(b) Show that the two atlasses of $\mathbb{S}^n$ constructed in Example 7.2.5 and the atlas obtained from the realization of $\mathbb{S}^n$ as a quadric in $\mathbb{R}^{n+1}$ define the same differentiable structure.

Exercise 7.2.9. Let N be an open subset of the smooth manifold M. Show that if $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ is a smooth atlas of M, $V_i := U_i \cap N$ and $\psi_i := \varphi_i|_{V_i}$, then $\mathcal{B} := (\psi_i, V_i)_{i \in I}$ is a smooth atlas of N.

Exercise 7.2.10. Smoothness is a local property: Show that a map $f \colon M \to N$ between smooth manifolds is smooth if and only if for each $p \in M$ there is an open neighborhood U such that $f|_U$ is smooth.

Exercise 7.2.11. Let $V_1, \ldots, V_k$ and V be finite-dimensional real vector space and

$$\beta \colon V_1 \times \ldots \times V_k \to V$$

be a k-linear map. Show that β is smooth with

$$\mathtt{d}\beta(x_1, \ldots, x_k)(h_1, \ldots, h_k) = \sum_{j=1}^{k} \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_k).$$

Exercise 7.2.12. Show that the space M defined in Example 7.2.12 is not Hausdorff, but that the two maps $\varphi_j([j, x]) := x$, $j = 1, 2$, define a smooth atlas of M.

Exercise 7.2.13. A map $f \colon X \to Y$ between topological spaces is called a *quotient map* if a subset $O \subseteq Y$ is open if and only if $f^{-1}(O)$ is open. Show that:

(1) If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are open quotient maps, then the cartesian product

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2, \quad (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$$

is a quotient map.

(2) If $f : X \to Y$ is a quotient map and we define on X an equivalence relation by $x \sim y$ if $f(x) = f(y)$, then the map $\overline{f} : X/\sim \to Y$ is a homeomorphism if $X/\sim$ is endowed with the quotient topology.

(3) The map $q : \mathbb{R}^n \to \mathbb{T}^n, x \mapsto (e^{2\pi i x_j})_{j=1,\ldots,n}$ is a quotient map.

(4) The map $\overline{q} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n, [x] \mapsto (e^{2\pi i x_j})_{j=1,\ldots,n}$ is a homeomorphism.

Exercise 7.2.14. Let M be a compact smooth manifold containing at least two points. Then each atlas of M contains at least two charts. In particular the atlas of $\mathbb{S}^n$ obtained from stereographic projections is minimal.

Exercise 7.2.15. Let X and Y be topological spaces and $q : X \to Y$ a quotient map, i.e., q is surjective and $O \subseteq Y$ is open if and only if $q^{-1}(O)$ is open in X. Show that a map $f : Y \to Z$ (Z a topological space) is continuous if and only if the map $f \circ q : X \to Z$ is continuous.

Exercise 7.2.16. Let M and B be smooth manifolds. A smooth map $\pi : M \to B$ is said to defined a *(locally trivial) fiber bundle* with typical fiber F over the base manifold B if each $b_0 \in B$ has an open neighborhood U for which there exists a diffeomorphism

$$\varphi : \pi^{-1}(U) \to U \times F,$$

satisfying $\mathrm{pr}_U \circ \varphi = \pi$, where $\mathrm{pr}_U : U \times F \to U, (u, f) \mapsto u$ is the projection onto the first factor. Then the pair (φ, U) is called a *local trivialization.*

Show that:

(1) If $(\varphi, U), (\psi, V)$ are local trivializations, then

$$\varphi \circ \psi^{-1}(b, f) = (b, g_{\varphi\psi}(b)(f))$$

holds for a function $g_{\varphi\psi} : U \cap V \to \mathrm{Diff}(F)$.

(2) If (γ, W) is another local trivialization, then

$$g_{\varphi\varphi} = \mathrm{id}_F \quad \text{and} \quad g_{\varphi\psi} g_{\psi\gamma} = g_{\varphi\gamma} \quad \text{on} \quad U \cap V \cap W.$$

Exercise 7.2.17. Show that a function $f : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism if and only if either

(1) $f' > 0$ and $\lim_{x \to \pm\infty} f(x) = \pm\infty$.

(2) $f' < 0$ and $\lim_{x \to \pm\infty} f(x) = \mp\infty$.

Exercise 7.2.18. Let $B \in \mathrm{GL}_n(\mathbb{R})$ be an invertible matrix which is symmetric or skew-symmetric. Show that:

(1) $G := \{g \in \mathrm{GL}_n(\mathbb{R})\colon g^\top B g = B\}$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.
(2) If $B = B^\top$, then B is a regular value of the smooth function

$$f\colon M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R}), \quad x \mapsto x^\top B x.$$

(3) If $B = -B^\top$, then B is a regular value of the smooth function

$$f\colon M_n(\mathbb{R}) \to \mathrm{Skew}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R})\colon A^\top = -A\}, \quad x \mapsto x^\top B x.$$

(4) G is a submanifold of $M_n(\mathbb{R})$.
(5) For $B = I_{p,q} := \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$ the *indefinite orthogonal group*

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R})\colon g^\top I_{p,q} g = I_{p,q}\}$$

is a submanifold on $M_n(\mathbb{R})$ of dimension $\frac{n(n-1)}{2}$.
(6) For $J = \begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix} \in M_2(M_n(\mathbb{R})) \cong M_{2n}(\mathbb{R})$ the *symplectic group*

$$\mathrm{Sp}_{2n}(\mathbb{R}) := \{g \in \mathrm{GL}_{2n}(\mathbb{R})\colon g^\top J g = J\}$$

is a submanifold on $M_{2n}(\mathbb{R})$ of dimension $n(2n+1)$.

7.3 The Tangent Bundle

The real strength of the theory of smooth manifolds is due to the fact that it permits to analyze differentiable structures in terms of their derivatives. To model these derivatives appropriately, we introduce the tangent bundle TM of a smooth manifold, tangent maps of smooth maps and smooth vector fields.

We start with the definition of a tangent vector of a smooth manifold. The subtle point of this definition is that tangent vectors and the vector space structure can only be defined rather indirectly. The most straight forward way is to construct tangent vectors as "tangents" to smooth curves.

7.3.1 Tangent Vectors and Tangent Maps

Definition 7.3.1. Let M be a smooth manifold, $p \in M$ and (φ, U) a chart of M with $p \in U$. Let $\gamma\colon I \to M$ be a smooth curve, where $I \subseteq \mathbb{R}$ is an interval containing 0 and $\gamma(0) = p$. We call two such curves $\gamma_i\colon I_i \to M$, $i = 1, 2$, *equivalent*, denoted $\gamma_1 \sim \gamma_2$, if

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Clearly, this defines an equivalence relation. The equivalence classes are called *tangent vectors in* p. We write $T_p(M)$ for the set of all tangent vectors in p and $[\gamma] \in T_p(M)$ for the equivalence class of the curve γ. The disjoint union

$$T(M) := \coprod_{p \in M} T_p(M)$$

is called the *tangent bundle of* M and we write $\pi_{TM} \colon TM \to M$ for the projection, mapping $T_p(M)$ to $\{p\}$.

Remark 7.3.2. (a) The equivalence relation defining tangent vectors does not depend on the chart (φ, U). If (ψ, V) is a second chart with $p \in V$ and $\gamma \colon I \to M$ a smooth curve with $\gamma(0) = p$, then

$$(\psi \circ \gamma)'(0) = \mathtt{d}(\psi \circ \varphi^{-1})(\varphi(p))(\varphi \circ \gamma)'(0),$$

so that we obtain the same equivalence relation on curves through p.

(b) If $U \subseteq \mathbb{R}^n$ is an open subset and $p \in U$, then each smooth curve $\gamma \colon I \to U$ with $\gamma(0) = p$ is equivalent to the curve $\eta_v(t) := p + tv$ for $v = \gamma'(0)$. Hence each equivalence class contains exactly one curve η_v. We may therefore think of a tangent vector in $p \in U$ as a vector $v \in \mathbb{R}^n$ attached to the point p, and the map

$$\mathbb{R}^n \to T_p(U), \quad v \mapsto [\eta_v]$$

is a bijection. In this sense, we identify all tangent spaces $T_p(U)$ with $\mathbb{R}^n$, so that we obtain a bijection

$$T(U) \cong U \times \mathbb{R}^n.$$

As an open subset of the product space $T(\mathbb{R}^n) \cong \mathbb{R}^{2n}$, the tangent bundle $T(U)$ inherits a natural manifold structure.

(c) For each $p \in M$ and any chart (φ, U) with $p \in U$, the map

$$T_p(\varphi) \colon T_p(M) \to \mathbb{R}^n, \quad [\gamma] \mapsto (\varphi \circ \gamma)'(0)$$

is well-defined by the definition of the equivalence relation. Moreover, the curve

$$\gamma(t) := \varphi^{-1}(\varphi(p) + tv),$$

which is smooth and defined on some neighborhood of 0, satisfies $(\varphi \circ \gamma)'(0) = v$. Hence $T_p(\varphi)$ is a bijection.

Definition 7.3.3. (a) Each tangent space $T_p(M)$ carries the unique structure of an n-dimensional vector space with the property that for each chart (φ, U) of M with $p \in U$, the map

$$T_p(\varphi) \colon T_p(M) \to \mathbb{R}^n, \quad [\gamma] \mapsto (\varphi \circ \gamma)'(0)$$

is a linear isomorphism.

In fact, since $T_p(\varphi)$ is a bijection, we may define a vector space structure on $T_p(M)$ by

$$v + w := T_p(\varphi)^{-1}(T_p(\varphi)v + T_p(\varphi)w) \quad \text{and} \quad \lambda v := T_p(\varphi)^{-1}(\lambda T_p(\varphi)v)$$

for $\lambda \in \mathbb{R}$, $v, w \in T_p(M)$. For any other chart (ψ, V) with $p \in V$ we then have

$$T_p(\psi) = \mathtt{d}(\psi \circ \varphi^{-1})(\varphi(p)) \circ T_p(\varphi),$$

and since $\mathtt{d}(\psi \circ \varphi^{-1})(\varphi(p))$ is a linear automorphism of $\mathbb{R}^n$, the vector space structure on $T_p(M)$ does not depend on the chart we use for its definition.

(b) If $f\colon M \to N$ is a smooth map and $p \in M$, then we obtain a linear map

$$T_p(f)\colon T_p(M) \to T_{f(p)}(N), \quad [\gamma] \mapsto [f \circ \gamma].$$

In fact, we only have to observe that for any chart (φ, U) of N with $f(p) \in U$ and any chart (ψ, V) of M with $p \in V$, we have

$$\begin{aligned} T_{f(p)}(\varphi)[f \circ \gamma] &= (\varphi \circ f \circ \gamma)'(0) = \mathtt{d}(\varphi \circ f \circ \psi^{-1})(\psi(p))(\psi \circ \gamma)'(0) \\ &= \mathtt{d}(\varphi \circ f \circ \psi^{-1})(\psi(p))T_p(\psi)[\gamma]. \end{aligned}$$

This relation shows that $T_p(f)$ is well-defined, and a linear map.

The collection of all these maps defines a map

$$T(f)\colon T(M) \to T(N) \quad \text{with} \quad T_p(f) = T(f)|_{T_p(M)}, p \in M.$$

It is called the *tangent map of f*.

(c) If $M \subseteq \mathbb{R}$ is an open subset, then $f\colon M \to N$ is a smooth curve in N, and its tangent vector is $f'(t) := T_t(f)(1)$, where $1 \in T_t(\mathbb{R}) \cong \mathbb{R}$ is considered as a tangent vector.

(d) If N is a vector space, then we identify $T(N)$ in a natural way with $N \times N$. Accordingly we have

$$T_p(f)(v) = (f(p), \mathtt{d}f(p)v),$$

for a map $\mathtt{d}f\colon T(M) \to N$ with $\mathtt{d}f(p) := \mathtt{d}f|_{T_p(M)}$.

Remark 7.3.4. (a) For an open subset $U \subseteq \mathbb{R}^n$ and $p \in U$, the vector space structure on $T_p(U) = \{p\} \times \mathbb{R}^n$ is simply given by

$$(p, v) + (p, w) := (p, v + w) \quad \text{and} \quad \lambda(p, v) := (p, \lambda v)$$

for $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

(b) If $f\colon U \to V$ is a smooth map between open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, $p \in U$, and $\eta_v(t) = p + tv$, then the tangent map satisfies

$$T(f)(p, v) = [f \circ \eta_v] = (f \circ \eta_v)'(0) = (f(p), \mathtt{d}f(p)\eta_v'(0)) = (f(p), \mathtt{d}f(p)v).$$

The main difference to the map $\mathtt{d}f$ is the book keeping; here we keep track of what happens to the point p and the tangent vector v. We may also write

$$T(f) = (f \circ \pi_{TU}, \mathtt{d}f)\colon TU \cong U \times \mathbb{R}^n \to TV \cong V \times \mathbb{R}^n,$$

where $\pi_{TU}\colon TU \to U, (p, v) \mapsto p$, is the projection map.

(c) If (φ, U) is a chart of M and $p \in U$, then we identify $T(\varphi(U))$ with $\varphi(U) \times \mathbb{R}^n$ and obtain for $[\gamma] \in T_p(M)$:

$$T(\varphi)([\gamma]) = (\varphi(p), [\varphi \circ \gamma]) = (\varphi(p), (\varphi \circ \gamma)'(0)),$$

which is consistent with our previously introduced notation $T_p(\varphi)$ (Remark 7.3.2).

Lemma 7.3.5 (Chain Rule for Tangent Maps). *For smooth maps $f\colon M \to N$ and $g\colon N \to L$, the tangent maps satisfy*

$$T(g \circ f) = T(g) \circ T(f).$$

Proof. We recall from Lemma 7.2.21 that $g \circ f\colon M \to L$ is a smooth map, so that $T(g \circ f)$ is defined. For $p \in M$ and $[\gamma] \in T_p(M)$, we further have

$$T_p(g \circ f)[\gamma] = [g \circ f \circ \gamma] = T_{f(p)}(g)[f \circ \gamma] = T_{f(p)}(g)T_p(f)[\gamma].$$

Since p was arbitrary, this implies the lemma. □

So far we only considered the tangent bundle $T(M)$ of a smooth manifold M as a set, but this set also carries a natural topology and a smooth manifold structure.

Definition 7.3.6 (Manifold structure on $T(M)$). Let M be a smooth manifold. First we introduce a topology on $T(M)$.

For each chart (φ, U) of M, we have a tangent map

$$T(\varphi)\colon T(U) \to T(\varphi(U)) \cong \varphi(U) \times \mathbb{R}^n,$$

where we consider $T(U) = \bigcup_{p \in U} T_p(M)$ as a subset of $T(M)$. We define a topology on $T(M)$ by declaring a subset $O \subseteq T(M)$ to be open if for each chart (φ, U) of M, the set $T(\varphi)(O \cap T(U))$ is an open subset of $T(\varphi(U))$. It is easy to see that this defines indeed a Hausdorff topology on $T(M)$ for which all the subsets $T(U)$ are open and the maps $T(\varphi)$ are homeomorphisms onto open subsets of $\mathbb{R}^{2n}$ (Exercise 7.3.1; see also Remark 7.2.13).

Since for two charts (φ, U), (ψ, V) of M, the map

$$T(\varphi \circ \psi^{-1}) = T(\varphi) \circ T(\psi)^{-1}\colon T(\psi(V)) \to T(\varphi(U))$$

is smooth, for each atlas $\mathcal{A}$ of M, the collection $(T(\varphi), T(U))_{(\varphi, U) \in \mathcal{A}}$ is a smooth atlas of $T(M)$. We thus obtain on $T(M)$ the structure of a smooth manifold.

Lemma 7.3.7. *If $f\colon M \to N$ is a smooth map, then its tangent map $T(f)$ is smooth.*

Proof. Let $p \in M$ and choose charts (φ, U) and (ψ, V) of M, resp., N with $p \in U$ and $f(p) \in V$. Then the map

$$T(\psi) \circ T(f) \circ T(\varphi)^{-1} = T(\psi \circ f \circ \varphi^{-1}) \colon T(\varphi(f^{-1}(V) \cap U)) \to T(V)$$

is smooth, and this implies that $T(f)$ is a smooth map. □

Remark 7.3.8. For smooth manifolds $M_1, \ldots, M_n$, the projection maps

$$\pi_i \colon M_1 \times \cdots \times M_n \to M_i, \quad (p_1, \ldots, p_n) \mapsto p_i$$

induce a diffeomorphism

$$(T(\pi_1), \ldots, T(\pi_n)) \colon T(M_1 \times \cdots \times M_n) \to TM_1 \times \cdots \times TM_n$$

(Exercise 7.3.2).

Definition 7.3.9. Let M be a differentiable manifold and V a finite-dimensional vector space. A V-valued *Pfaffian form*, also called simply a 1-*form*, on M is a smooth map $\omega : TM \to V$ whose restrictions $\omega_p \colon T_p(M) \to V$ are linear for each $p \in M$. The set of all V-valued 1-forms on M is denoted by $\Omega^1(M, V)$ and for $V = \mathbb{R}$ we abbreviate $\Omega^1(M) := \Omega^1(M, \mathbb{R})$.

Recall the bundle projection $\pi_{TM} \colon TM \to M$. Then for any smooth function $f \colon M \to \mathbb{R}$, the pointwise product

$$f \cdot \omega := (f \circ \pi_{TM}) \cdot \omega : TM \to V$$

also is a V-valued 1-form. Combined with the obvious vector space structure on $\Omega^1(M, V)$, defined by pointwise addition and scalar multiplication, we thus obtain on $\Omega^1(M, V)$ the structure of a $C^\infty(M)$-module.

Example 7.3.10. If $f \colon M \to V$ is a smooth function, then $Tf \colon TM \to TV$ is also smooth. Identifying TV in the canonical fashion with $V \times V$ and writing $\mathrm{pr}_2 \colon TV \to V, (x, v) \mapsto v$ for the projection, the map

$$\mathtt{d}f := \mathrm{pr}_2 \circ Tf \colon TM \to V$$

is a V-valued 1-form and the tangent map Tf can now be written $Tf(v) = (f(p), \mathtt{d}f(p)v)$ for $v \in T_p(M)$.

Remark 7.3.11. Now let $\omega \in \Omega^1(M)$ and $X \in \mathcal{V}(M)$. For every $p \in M$, we can apply ω_p to $X(p)$. We thus get a smooth function $\omega(X) = \omega \circ X : M \to \mathbb{R}$.

Definition 7.3.12. Let $\omega \in \Omega^1(M, V)$ be a V-valued 1-form and $f \colon N \to M$ a smooth map. Then we obtain a V-valued 1-form

$$f^*\omega := \omega \circ Tf \colon TN \to V,$$

called the *pull-back* of ω to N.

Remark 7.3.13. We clearly have the following rules for pull-backs of 1-forms. For $\omega \in \Omega^1(M,V)$ and smooth maps $\psi\colon W \to N$ and $\varphi\colon N \to M$, we have

$$\mathrm{id}_M^* \omega = \omega \quad \text{and} \quad (\varphi \circ \psi)^*\omega = \psi^*(\varphi^*\omega).$$

Moreover,

$$\varphi^*\colon \Omega^1(M,V) \to \Omega^1(N,V)$$

is a linear map satisfying $\varphi^*(f \cdot \omega) = (f \circ \varphi) \cdot \varphi^*\omega$ for $f \in C^\infty(M)$.

Definition 7.3.14. Let $f\colon M_1 \to M_2$ be a smooth map and $m \in M_1$. The map f is called *submersive in* m if the differential $T_m(f)$ is surjective. Otherwise m is called a *critical point of* f.

The map f is said to be a *submersion* if $T_m(f)$ is surjective for each $m \in M_1$.

Lemma 7.3.15. *If $f\colon M_1 \to M_2$ is a smooth map which is submersive in $m \in M_1$, then there exists an open neighborhood $U \subseteq M_2$ of $p := f(m)$ and a smooth map $\sigma\colon U \to M_1$ with $f \circ \sigma = \mathrm{id}_U$ and $\sigma(p) = m$.*

Proof. Since the assertion is purely local, we may w.l.o.g. assume that M_1 is an open subset of $\mathbb{R}^n$ and M_2 is an open subset of $\mathbb{R}^k$. That f is submersive in m means that $\mathtt{d}f(m)\colon \mathbb{R}^n \to \mathbb{R}^k$ is surjective. Let $V \subseteq \mathbb{R}^n$ be a linear subspace for which $\mathtt{d}f(m)|_V\colon V \to \mathbb{R}^k$ is a linear isomorphism. Then $F := f|_{M_1 \cap V}\colon M_1 \cap V \to M_2$ is a smooth map whose differential $T_m(F)$ is invertible. Hence the Inverse Function Theorem implies the existence of a smooth inverse σ, defined on an open neighborhood of $p = F(m)$. □

Proposition 7.3.16 (Universal Property of Submersions). *Suppose that $f\colon M_1 \to M_2$ is a surjective submersion and N a smooth manifold. Then a map $h\colon M_2 \to N$ is smooth if and only if the map $h \circ f\colon M_1 \to N$ is smooth. In particular, for each smooth map $g\colon M_1 \to N$ which is constant on all fibers of f, there exists a unique smooth map $h\colon M_2 \to N$ with $h \circ f = g$.*

Proof. If h is smooth, then also the composition $h \circ f$ is smooth. Assume, conversely, that $h \circ f$ is smooth. To see that h is smooth, pick $p \in M_2$ and an open neighborhood $U \subseteq M_2$ of p on which there exists a smooth section $\sigma\colon U \to M_1$ with $f \circ \sigma = \mathrm{id}_U$ (Lemma 7.3.15). Then $h|_U = h \circ f \circ \sigma$ is smooth. Hence h is smooth on a neighborhood of p, and since $p \in M_2$ was arbitrary, h is smooth.

The second assertion immediately follows from the first one if we define the map $h\colon M_2 \to N$ by $h(f(x)) := g(x)$, which works if and only if g is constant on the fibers of f. □

Corollary 7.3.17. *If $f\colon M_1 \to M_2$ is a bijective submersion, then f is a diffeomorphism.*

Proof. Apply the preceding proposition with $N := M_1$ and $g = \mathrm{id}_{M_1}$. □

Remark 7.3.18. The smooth map $f\colon M_1 := \mathbb{R} \to M_2 := \mathbb{R}, x \mapsto x^3$ is submersive in all points $x \neq 0$. The map $g = \mathrm{id}_{\mathbb{R}}\colon M_1 = \mathbb{R} \to N := \mathbb{R}$ is smooth and bijective, hence constant on the fibers of f, but the map $\overline{g}\colon M_2 \to N, x \mapsto x^{\frac{1}{3}}$ is not smooth in 0. This shows that the assumption in Proposition 7.3.16 that f is submersive is really needed.

7.3.2 Algebraic Tangent Spaces

An alternative algebraic approach to the tangent space is to consider not the velocity vectors of curves, but to study their actions on smooth functions: Let $v \in T_p(M)$ and U be an open neighborhood of p in M. For a smooth function $f : U \to \mathbb{R}$ we then put (cf. Definition 7.3.3(d))

$$v(f) := \mathtt{d}f(p)v. \tag{7.3}$$

From the product rule (see Exercise 7.3.7), one immediately derives

$$v(f \cdot g) = v(f)g(p) + f(p)v(g).$$

We also observe that $v(f) = v(g)$ if f and g coincide on a neighborhood of p. This motivates the following definition:

Definition 7.3.19. Let M be a differentiable manifold. For $p \in M$, let

$$C^\infty(M,p) = \coprod_{p\in U} C^\infty(U)$$

be the set of smooth functions which are defined on an open subset U containing p. On $C^\infty(M,p)$, we consider the equivalence relation defined by $f \sim g$ if there exists an open neighborhood U of p in M such that $f|_U = g|_U$. We denote the equivalence class of $f \in C^\infty(M,p)$ with respect to $\sim$ by f_p. It is called the *germ of f in p*. The set $C^\infty(M)_p := C^\infty(M,p)/_\sim$ of equivalence classes inherits an algebra structure (given by pointwise addition and multiplication of functions)

$$f_p + g_p := (f+g)_p \quad \text{and} \quad f_pg_p := (fg)_p.$$

A linear functional $v : C^\infty(M)_p \to \mathbb{R}$ is called a *derivation in p* if

$$v(f_p \cdot g_p) = v(f_p) \cdot g(p) + f(p) \cdot v(g_p)$$

for $f_p, g_p \in C^\infty(M)_p$. The *algebraic tangent space* $T_p^{\mathrm{alg}}(M)$ of M in p is *defined* to be the set of all derivations $v : C^\infty(M)_p \to \mathbb{R}$.

Note that we have assigned a derivation to every $v \in T_p(M)$, but we have not assigned a tangent vector to every derivation. This is nevertheless possible, and will be done in Proposition 7.3.24.

Proposition 7.3.20. *Let M be a differentiable manifold. Then for $p \in M$ the following is true*

(i) $T_p^{\mathrm{alg}}(M)$ *is a vector space with respect to the pointwise operations.*
(ii) *If $f_p \in C^\infty(M)_p$ is such that f is constant on a neighborhood of p, then $v(f_p) = 0$ for all $v \in T_p^{\mathrm{alg}}(M)$.*

Proof. The first statement is obvious, and for the second, it suffices to prove that $v(1_p) = 0$ for all $v \in T_p^{\mathrm{alg}}(M)$, where 1 is the constant function with value 1. For this, we compute:

$$v(1_p) = v(1_p \cdot 1_p) = v(1_p) \cdot 1 + v(1_p) \cdot 1 = 2v(1_p),$$

which implies $v(1_p) = 0$. □

Note that at this point it is not clear what the size of the tangent space is. We cannot even say whether it is finite-dimensional. In order to clarify the structure of $T_p^{\mathrm{alg}}(M)$, we describe it in local coordinates.

Definition 7.3.21. Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart with $p \in U$. We define n elements $\frac{\partial}{\partial x_j}\big|_p \in T_p^{\mathrm{alg}}(M)$ for $j = 1, \ldots, n$ by

$$\frac{\partial}{\partial x_j}\Big|_p (f_p) := \frac{\partial (f \circ \varphi^{-1})}{\partial x_j}\big(\varphi(p)\big),$$

where the points in $\mathbb{R}^n$ are denoted by $x = (x_1, \ldots, x_n)$. Note that this definition depends on the choice of the chart.

It turns out that the $\frac{\partial}{\partial x_j}\big|_p$ form a basis for T_pM, which we call the *φ-basis* for $T_p^{\mathrm{alg}}(M)$. To prove this, we start with a lemma:

Lemma 7.3.22 (Hadamard). *Let B be an open ball in $\mathbb{R}^n$ with center a and $f \in C^\infty(B)$. Then there exist smooth functions $g_1, \ldots, g_n \in C^\infty(B)$ such that*

(i) $f(x) = f(a) + \sum\limits_{j=1}^{n} (x_j - a_j) g_j(x)$ *for all $x \in B$.*
(ii) $g_j(a) = \frac{\partial f}{\partial x_j}(a)$.

Proof. For fixed $x \in B$, we consider the function

$$\xi : [-1, 1] \to \mathbb{R}, \quad \xi(t) := f\big(a + t(x - a)\big).$$

Then, for $g_j(x) := \int\limits_0^1 \frac{\partial f}{\partial x_j}\big(a + t(x - a)\big) dt$, we have

$$\begin{aligned} f(x) &= \xi(0) + \int_0^1 \xi'(t)\,dt = \xi(0) + \int_0^1 \sum_{j=1}^n \tfrac{\partial f}{\partial x_j}\big(a+t(x-a)\big)\cdot(x_j-a_j)\,dt \\ &= f(a) + \sum_{j=1}^n (x_j-a_j)\Big(\int_0^1 \tfrac{\partial f}{\partial x_j}\big(a+t(x-a)\big)dt\Big) \\ &= f(a) + \sum_{j=1}^n (x_j-a_j)g_j(x). \end{aligned}$$

Since $f \in C^\infty(B)$ by assumption, we also have $g_j \in C^\infty(B)$ (Exercise). □

Proposition 7.3.23. *Let M be a smooth manifold and (φ, U) be a chart on M with* coordinate functions $x_j : U \to \mathbb{R}$ *given by $\varphi(q) = \big(x_1(q), \ldots, x_n(q)\big)$, see Remark 7.2.17. Then the tangent vectors $\frac{\partial}{\partial x_j}\big|_p$, $j = 1, \ldots, n$, form a basis for $T_p^{\rm alg}(M)$ and for every $v \in T_p^{\rm alg}(M)$, the formula holds*

$$v = \sum_{j=1}^n v\big((x_j)_p\big)\cdot \frac{\partial}{\partial x_j}\Big|_p.$$

Proof. Fix $v \in T_p^{\rm alg}(M)$ and $f \in C^\infty(\widetilde{V})$, where $\widetilde{V} \subseteq U$ is an open subset containing p. Let $V = \varphi(\widetilde{V})$ and put $F = f \circ \varphi^{-1} : V \to \mathbb{R}$. Shrinking the neighborhoods, we may assume that V is an open ball with center $\varphi(p)$. We apply Lemma 7.3.22 to F and get functions $g_j : V \to \mathbb{R}$ satisfying $F(x) = F(a) + \sum_{j=1}^n (x_j - a_j)g_j(x)$. Then, using Proposition 7.3.20, we compute

$$\begin{aligned} v(f_p) &= v\big((F\circ\varphi)_p\big) = v\Big(F(\varphi(p)) + \sum_{j=1}^n \big(x_j - x_j(p)\big)_p (g_j\circ\varphi)_p\Big) \\ &= v\big(f(p)\cdot \mathbf{1}_p\big) + v\Big(\sum_{j=1}^n \big(x_j - x_j(p)\cdot \mathbf{1}\big)_p \cdot \big(g_j\circ\varphi\big)_p\Big) \\ &= 0 + \sum_{j=1}^n \Big(v\big((x_j)_p\big)\cdot g_j\big(\varphi(p)\big) + v\big((g_j\circ\varphi)_p\big)\cdot\big(x_j(p) - x_j(p)\cdot \mathbf{1}\big)\Big) \\ &= \sum_{j=1}^n v\big((x_j)_p\big)\cdot \tfrac{\partial(f\circ\varphi^{-1})}{\partial x_j}\big(\varphi(p)\big) = \sum_{j=1}^n v\big((x_j)_p\big)\cdot \tfrac{\partial}{\partial x_j}\big|_p (f_p). \end{aligned}$$

Hence, every derivation $v \in T_p^{\rm alg}(M)$ is a linear combination of the $\frac{\partial}{\partial x_j}\big|_p$, $j = 1, \ldots, n$. It remains to show that the $\frac{\partial}{\partial x_j}\big|_p$ for $j = 1, \ldots, n$ are linearly independent. But this is immediate from $\frac{\partial}{\partial x_j}\big|_p\big((x_i)_p\big) = \delta_{ij}$. □

Proposition 7.3.24. *Let M be a smooth manifold and $p \in M$. Then*

$$\Gamma\colon T_p(M) \to T_p^{\rm alg}(M), \quad \Gamma(v)(f_p) := v(f_p) = \mathtt{d}f(p)v$$

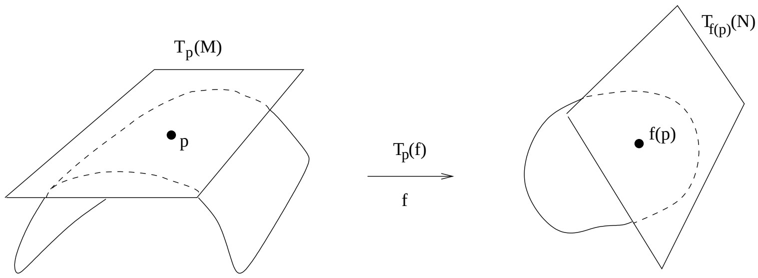
is a linear isomorphism.

Proof. It follows from (7.3) and the subsequent calculation that Γ is a linear map into the algebraic tangent space.

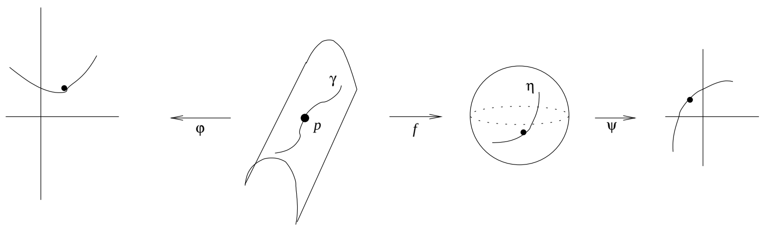
Consider a chart (φ, U) with $p \in U$. In view of Proposition 7.3.23, the condition $\Gamma(v) = 0$ is equivalent to $0 = v((x_j)_p) = \mathtt{d}x_j(p)v$ for all coordinate functions, which in turn is equivalent to $T_p(\varphi)v = 0$, i.e., to $v = 0$. Thus Γ is injective. Since both spaces are of the same dimension, Γ is a linear bijection. □

In view of Proposition 7.3.24, we will no longer distinguish between the geometric and the algebraic tangent spaces.

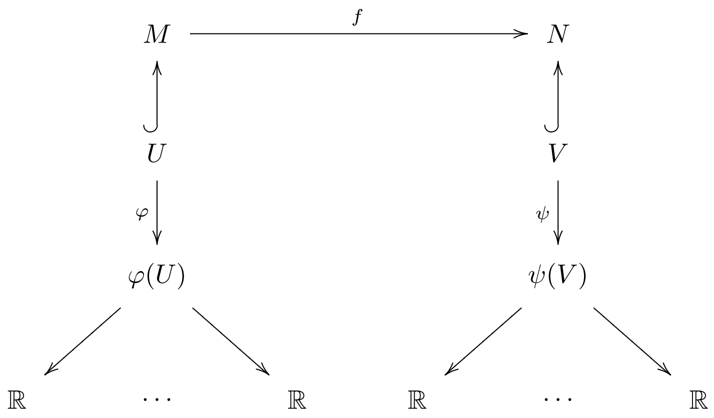
Recall that with any differential map $f : M \to N$ between two manifolds we associate a linear map $T_p(f) : T_pM \to T_{f(p)}N$ as derivative.



Remark 7.3.25. For charts (φ, U) of M and (ψ, V) of N, we obtained distinguished bases for T_pM and $T_{f(p)}N$. We want to describe the derivative of f with respect to these bases.



Let $F := \psi \circ f \circ \varphi^{-1} : \; : \varphi\big(U \cap f^{-1}(V)\big) \to \psi(V)$ and write F_i for its components.



Then we find (exercise)

$$T_p(f)\Big(\frac{\partial}{\partial x_j}\Big|_p\Big) = \sum_{i=1}^{\dim N} \frac{\partial F_i(\varphi(p))}{\partial x_j} \frac{\partial}{\partial y_i}\Big|_{f(p)}$$

Hence the Jacobian of F in $\varphi(p)$ is exactly the matrix of $T_p(f)$ with respect to the bases of T_pM and $T_{f(p)}N$, associated with φ and ψ, respectively.

Exercises for Section 7.3

Exercise 7.3.1. Let M be a smooth manifold. We call a subset $O \subseteq T(M)$ open if for each chart (φ, U) of M, the set $T(\varphi)(O \cap T(U))$ is an open subset of $T(\varphi(U))$. Show that:

(1) This defines a topology on $T(M)$.
(2) All subsets $T(U)$ are open (Remark 7.3.4(b)).
(3) The maps $T(\varphi)\colon TU \to T(\varphi(U)) \cong \varphi(U) \times \mathbb{R}^n$ are homeomorphisms onto open subsets of $\mathbb{R}^{2n} \cong T(\mathbb{R}^n)$.
(4) The projection $\pi_{TM}\colon T(M) \to M$ is continuous.
(5) $T(M)$ is Hausdorff.

Exercise 7.3.2. For smooth manifolds $M_1, \ldots, M_n$, the projection maps

$$\pi_i\colon M_1 \times \cdots \times M_n \to M_i, \quad (p_1, \ldots, p_n) \mapsto p_i$$

induce a diffeomorphism

$$(T(\pi_1), \ldots, T(\pi_n))\colon T(M_1 \times \cdots \times M_n) \to TM_1 \times \cdots \times TM_n.$$

Exercise 7.3.3. Let N and $M_1, \ldots, M_n$ be a smooth manifolds. Show that a map

$$f\colon N \to M_1 \times \cdots \times M_n$$

is smooth if and only if all its component functions $f_i\colon N \to M_i$ are smooth.

Exercise 7.3.4. Let $f\colon M \to N$ be a smooth map between manifolds, $\pi_{TM}\colon TM \to M$ the tangent bundle projection and $\sigma_M\colon M \to TM$ the zero section. Show that for each smooth map $f\colon M \to N$ we have

$$\pi_{TN} \circ Tf = f \circ \pi_{TM} \quad \text{and} \quad \sigma_N \circ f = Tf \circ \sigma_M.$$

Exercise 7.3.5 (Inverse Function Theorem for manifolds). Let $f\colon M \to N$ be a smooth map and $p \in M$ such that $T_p(f)\colon T_p(M) \to T_p(N)$ is a linear isomorphism. Show that there exists an open neighborhood U of p in M such that the restriction $f|_U\colon U \to f(U)$ is a diffeomorphism onto an open subset of N.

Exercise 7.3.6. Let $\mathbb{P}_n(\mathbb{R})$ be the n-dimensional projective space over $\mathbb{R}$. For a point $x \in \mathbb{P}_n(\mathbb{R})$, let L_x be the line in $\mathbb{R}^{n+1}$ represented by x. Show that $L := \coprod_{x\in M} L_x$ carries the structure of a *line bundle*, i.e., a vector bundle of rank 1.

Exercise 7.3.7. Let $\mu\colon E \times F \to W$ be a bilinear map and M a smooth manifold. For $f \in C^\infty(M,E)$, $g \in C^\infty(M,F)$ and $p \in M$ set $h(p) := \mu\big(f(p), g(p)\big)$. Show that h is smooth with

$$T(h)v = \mu\big(T(f)v, g(p)\big) + \mu\big(f(p), T(g)v\big) \quad \text{for } v \in T_p(M).$$

7.4 Vector Fields

Vector fields are maps which associate with each point in a manifold a tangent vector at this point. They can be interpreted as a geometric way to formulate first order differential equations on a manifold, a point of view we will elaborate on in Section 7.5. First we introduce the Lie algebra structure on the space of smooth vector fields.

7.4.1 The Lie Algebra of Vector Fields

Definition 7.4.1. (a) Let $\pi_{TM}\colon TM \to M$ denote the canonical projection mapping $T_p(M)$ to p. A *(smooth) vector field* X on M is a smooth section of the tangent bundle TM, i.e., a smooth map $X\colon M \to TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M.

(b) If $U \subseteq M$ is an open subset and $f \in C^\infty(U,V)$ is a smooth function on U with values in some finite-dimensional vector space V and $X \in \mathcal{V}(M)$, then we obtain a smooth function on U via

$$\mathcal{L}_X f := \mathtt{d}f \circ X|_U\colon U \to TU \to V.$$

We thus obtain for each $X \in \mathcal{V}(M)$ a linear operator $\mathcal{L}_X$ on $C^\infty(U,V)$. The function $\mathcal{L}_X f$ is also called the *Lie derivative* of f with respect to X.

Remark 7.4.2. (a) If U is an open subset of $\mathbb{R}^n$, then $TU = U \times \mathbb{R}^n$ with the bundle projection

$$\pi_{TU} \colon U \times \mathbb{R}^n \to U, \quad (x, v) \mapsto x.$$

Therefore each smooth vector field is of the form $X(x) = (x, \widetilde{X}(x))$ for some smooth function $\widetilde{X} \colon U \to \mathbb{R}^n$, and we may thus identify $\mathcal{V}(U)$ with the space $C^\infty(U, \mathbb{R}^n)$ of smooth $\mathbb{R}^n$-valued functions on U.

(b) The space $\mathcal{V}(M)$ carries a natural vector space structure given by

$$(X + Y)(p) := X(p) + Y(p), \quad (\lambda X)(p) := \lambda X(p)$$

(Exercise 7.4.1).

More generally, we can multiply vector fields with smooth functions

$$(fX)(p) := f(p)X(p), \quad f \in C^\infty(M, \mathbb{R}), X \in \mathcal{V}(M).$$

Before we turn to the Lie bracket on the space $\mathcal{V}(M)$ of smooth vector fields on a manifold M, we take a closer look at the local level.

Lemma 7.4.3. *Let $U \subseteq \mathbb{R}^n$ be an open subset. Then we obtain a Lie bracket on the space $C^\infty(U, \mathbb{R}^n)$ by*

$$[X, Y](p) := \mathtt{d}Y(p)X(p) - \mathtt{d}X(p)Y(p) \quad \textit{for} \quad p \in U.$$

With respect to this Lie bracket, the map

$$C^\infty(U, \mathbb{R}^n) \to \mathrm{End}(C^\infty(U, \mathbb{R})), \quad X \mapsto \mathcal{L}_X, \quad \mathcal{L}_X(f)(p) := \mathtt{d}f(p)X(p)$$

is an injective homomorphism of Lie algebras, i.e., $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$.

Proof. (L1) and (L2) are obvious from the definition. To verify the Jacobi identity, we first observe that the map $X \mapsto \mathcal{L}_X$ is injective. In fact, if $\mathcal{L}_X = 0$, then we have for each linear function $f \colon \mathbb{R}^n \to \mathbb{R}$ the relation $0 = (\mathcal{L}_X f)(p) = \mathtt{d}f(p)X(p) = f(X(p))$, and therefore $X(p) = 0$.

Next we observe that

$$\begin{aligned} \mathcal{L}_X \mathcal{L}_Y(f)(p) &= \mathtt{d}(\mathcal{L}_Y f)(p)X(p) = \mathtt{d}(\mathtt{d}f \circ Y)(p)X(p) \\ &= (\mathtt{d}^2 f)(p)(X(p), Y(p)) + \mathtt{d}f(p)\mathtt{d}Y(p)X(p), \end{aligned}$$

so that the Schwarz Lemma implies $\mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f = \mathcal{L}_{[X,Y]} f$. Since $\mathrm{End}(C^\infty(U, \mathbb{R}))$ is a Lie algebra with respect to the commutator bracket (Lemma 3.1.2) and $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$, the Jacobi identity in $\mathrm{End}(C^\infty(U, \mathbb{R}))$ implies the Jacobi identity in $C^\infty(U, \mathbb{R}^n)$. $\square$

Remark 7.4.4. For any open subset $U \subseteq \mathbb{R}^n$, the map

$$\mathcal{V}(U) \to C^\infty(U, \mathbb{R}^n), \quad X \to \widetilde{X}$$

with $X(p) = (p, \widetilde{X}(p))$ is a linear isomorphism. We use this map to transfer the Lie bracket on $C^\infty(U, \mathbb{R}^n)$, defined in Lemma 7.4.3, to a Lie bracket on $\mathcal{V}(U)$, determined by

$$[X, Y]\widetilde{\ }(p) := [\widetilde{X}, \widetilde{Y}](p) = \mathtt{d}\widetilde{Y}(p)\widetilde{X}(p) - \mathtt{d}\widetilde{X}(p)\widetilde{Y}(p).$$

Our goal is to use the Lie brackets on the space $\mathcal{V}(U)$ and local charts to define a Lie bracket on $\mathcal{V}(M)$. The following lemma will be needed to ensure consistency in this process.

First, we introduce the concept or related vector fields. If $\varphi\colon M \to N$ is a smooth map, then we call two vector fields $X' \in \mathcal{V}(M)$ and $X \in \mathcal{V}(N)$ *φ-related* if

$$X \circ \varphi = T\varphi \circ X' \colon M \to TN. \tag{7.4}$$

With respect to the pullback map

$$\varphi^*\colon C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \quad f \mapsto f \circ \varphi,$$

the φ-relatedness of X and X' implies that

$$\mathcal{L}_{X'}(\varphi^* f) = \mathcal{L}_{X'}(f \circ \varphi) = \mathtt{d}(f \circ \varphi) \circ X' = \mathtt{d}f \circ T\varphi \circ X' = \mathtt{d}f \circ X \circ \varphi = \varphi^*(\mathcal{L}_X f),$$

i.e.,

$$\mathcal{L}_{X'} \circ \varphi^* = \varphi^* \circ \mathcal{L}_X. \tag{7.5}$$

Lemma 7.4.5. *Let $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ be open subsets. Suppose that X', resp., $Y' \in \mathcal{V}(M)$ is φ-related to X, resp., $Y \in \mathcal{V}(N)$. Then $[X', Y']$ is φ-related to $[X, Y]$.*

Proof. In view of (7.5), we have

$$\mathcal{L}_{X'} \circ \varphi^* = \varphi^* \circ \mathcal{L}_X \quad \text{and} \quad \mathcal{L}_{Y'} \circ \varphi^* = \varphi^* \circ \mathcal{L}_Y$$

as linear maps $C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$. Therefore

$$\begin{aligned}[\mathcal{L}_{X'}, \mathcal{L}_{Y'}] \circ \varphi^* &= \mathcal{L}_{X'} \circ \mathcal{L}_{Y'} \circ \varphi^* - \mathcal{L}_{Y'} \circ \mathcal{L}_{X'} \circ \varphi^* \\ &= \varphi^* \circ \mathcal{L}_X \circ \mathcal{L}_Y - \varphi^* \circ \mathcal{L}_Y \circ \mathcal{L}_X = \varphi^* \circ [\mathcal{L}_X, \mathcal{L}_Y].\end{aligned}$$

For any $f \in C^\infty(N, \mathbb{R})$, we thus obtain

$$\mathtt{d}f \circ T\varphi \circ [X', Y'] = \mathcal{L}_{[X', Y']}(f \circ \varphi) = (\mathcal{L}_{[X,Y]} f) \circ \varphi = \mathtt{d}f \circ [X, Y] \circ \varphi.$$

Since each linear functional on the space $T_x(N) \cong \mathbb{R}^m$ is of the form $\mathtt{d}f(x)$ for some linear map $f\colon \mathbb{R}^m \to \mathbb{R}$, the assertion follows. □

Proposition 7.4.6. *For a vector field $X \in \mathcal{V}(M)$ and a chart (φ, U) of M, we write $X_\varphi := T\varphi \circ X \circ \varphi^{-1}$ for the corresponding vector field on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.*

For $X, Y \in \mathcal{V}(M)$, there exists a vector field $[X, Y] \in \mathcal{V}(M)$ which is uniquely determined by the property that for each chart (φ, U) of M, the following equation holds

$$[X, Y]_\varphi = [X_\varphi, Y_\varphi]. \tag{7.6}$$

Proof. If (φ, U) and (ψ, V) are charts of M, the vector fields X_φ on $\varphi(U)$ and X_ψ on $\psi(V)$ are $(\psi \circ \varphi^{-1})$-related. Therefore Lemma 7.4.5 implies that $[X_\varphi, Y_\varphi]$ is $(\psi \circ \varphi^{-1})$-related to $[X_\psi, Y_\psi]$. This in turn is equivalent to

$$T(\varphi)^{-1} \circ [X_\varphi, Y_\varphi] \circ \varphi = T(\psi)^{-1} \circ [X_\psi, Y_\psi] \circ \psi,$$

which is an identity of vector fields on the open subset $U \cap V$.

Hence there exists a unique vector field $[X, Y] \in \mathcal{V}(M)$, satisfying

$$[X, Y]|_U = T(\varphi)^{-1} \circ [X_\varphi, Y_\varphi] \circ \varphi$$

for each chart (φ, U), i.e., $[X, Y]_\varphi = [X_\varphi, Y_\varphi]$ on $\varphi(U)$. □

Lemma 7.4.7. *For $f \in C^\infty(M, \mathbb{R})$ and $X, Y \in \mathcal{V}(M)$, the following equation holds*

$$\mathcal{L}_{[X,Y]} f = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f).$$

Proof. It suffices to show that this relation holds on U for any chart (φ, U) of M. For $f_\varphi := f \circ \varphi^{-1}$, we then obtain with (7.5)

$$\begin{aligned}
\mathcal{L}_{[X,Y]} f &= \mathtt{d}f \circ [X, Y] = \mathtt{d}f \circ T(\varphi^{-1}) \circ [X, Y]_\varphi \circ \varphi \\
&= \mathtt{d}f_\varphi \circ [X_\varphi, Y_\varphi] \circ \varphi = \varphi^*\big(\mathcal{L}_{[X_\varphi, Y_\varphi]} f_\varphi\big) \\
&= \varphi^*\big(\mathcal{L}_{(X_\varphi)}\mathcal{L}_{(Y_\varphi)} f_\varphi - \mathcal{L}_{(Y_\varphi)}\mathcal{L}_{(X_\varphi)} f_\varphi\big) \\
&= \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f),
\end{aligned}$$

because $\varphi^* f_\varphi = f$. □

Proposition 7.4.8. $(\mathcal{V}(M), [\cdot, \cdot])$ *is a Lie algebra.*

Proof. Clearly (L1) and (L2) are satisfied. To verify the Jacobi identity, let $X, Y, Z \in \mathcal{V}(M)$ and (φ, U) be a chart of M. For the vector field $J(X, Y, Z) := \sum_{\text{cycl.}} [X, [Y, Z]] \in \mathcal{V}(M)$ we then obtain from the definition of the bracket, Remark 7.4.4 and Proposition 7.4.6:

$$J(X, Y, Z)_\varphi = J(X_\varphi, Y_\varphi, Z_\varphi) = 0$$

because $[\cdot, \cdot]$ is a Lie bracket on $\mathcal{V}(\varphi(U))$. This means that $J(X, Y, Z)$ vanishes on U, but since the chart (φ, U) was arbitrary, $J(X, Y, Z) = 0$. □

We shall see later that the following lemma is an extremely important tool.

Lemma 7.4.9 (Related Vector Field Lemma). *Let M and N be smooth manifolds, $\varphi\colon M \to N$ a smooth map, $X, Y \in \mathcal{V}(N)$ and $X', Y' \in \mathcal{V}(M)$. If X' is φ-related to X and Y' is φ-related to Y, then the Lie bracket $[X', Y']$ is φ-related to $[X, Y]$.*

Proof. We have to show that for each $p \in M$ we have

$$[X, Y](\varphi(p)) = T_p(\varphi)[X', Y'](p).$$

Let (ρ, U) be a chart of M with $p \in U$ and (ψ, V) a chart of N with $\varphi(p) \in V$. Then the vector fields X'_ρ and X_ψ are $\psi \circ \varphi \circ \rho^{-1}$-related on the domain $\rho(\varphi^{-1}(V) \cap U)$:

$$\begin{aligned} &T(\psi \circ \varphi \circ \rho^{-1})X'_\rho = T(\psi \circ \varphi \circ \rho^{-1})(T(\rho) \circ X' \circ \rho^{-1}) \\ &= T(\psi) \circ T(\varphi) \circ X' \circ \rho^{-1} = T(\psi) \circ X \circ \varphi \circ \rho^{-1} = X_\psi \circ (\psi \circ \varphi \circ \rho^{-1}), \end{aligned}$$

and the same holds for the vector fields Y'_ρ and Y_ψ, hence for their Lie brackets (Lemma 7.4.5).

Now the definition of the Lie bracket on $\mathcal{V}(N)$ and $\mathcal{V}(M)$ implies that

$$\begin{aligned} &T(\psi) \circ T(\varphi) \circ [X', Y'] = T(\psi \circ \varphi \circ \rho^{-1}) \circ [X', Y']_\rho \circ \rho \\ &= T(\psi \circ \varphi \circ \rho^{-1}) \circ [X'_\rho, Y'_\rho] \circ \rho = [X_\psi, Y_\psi] \circ \psi \circ \varphi \circ \rho^{-1} \circ \rho \\ &= [X_\psi, Y_\psi] \circ \psi \circ \varphi = [X, Y]_\psi \circ \psi \circ \varphi = T(\psi) \circ [X, Y] \circ \varphi, \end{aligned}$$

and since $T(\psi)$ is injective, the assertion follows. □

Example 7.4.10. Let (φ, U) be a chart of M and $x_1, \ldots, x_n \colon U \to \mathbb{R}$ the corresponding coordinate functions. Then we obtain on U the vector fields X_j, $j = 1, \ldots, n$, defined by

$$X_j(p) := T_p(\varphi)^{-1} e_j := \frac{\partial}{\partial x_j}(p) := \frac{\partial}{\partial x_j}\Big|_p,$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. We call these vector fields the *φ-basic vector fields* on U. The expression basic vector field is doubly justified. On the one hand, $\big(X_1(p), \ldots, X_n(p)\big)$ is a basis for $T_p(M)$ for every $p \in U$. On the other hand, the definition shows that every $X \in \mathcal{V}(U)$ can be written as

$$X = \sum_{j=1}^n a_j \cdot X_j \quad \text{for} \quad a_j \in C^\infty(U).$$

Since basic vector fields are φ-related with the constant vector fields on $\mathbb{R}^n$, they commute (Related Vector Field Lemma 7.4.9), i.e., $[X_j, X_k] = 0$.

Remark 7.4.11. Let M be a smooth manifold of dimension n, (φ, U) be a chart on M and $x_j \colon U \to \mathbb{R}$ be the j-th component of φ. Then, for $p \in U$, the differentials $\big(\mathtt{d}x_j(p)\big)_{j=1,\ldots,n}$ form the dual basis to $\big(\frac{\partial}{\partial x_j}\big|_p\big)_{j=1,\ldots,n}$ for T_pM^*. Indeed $x_j \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{R}$ is simply the projection to the j-th coordinate, so

$$\mathtt{d}x_i(p)\Big(\frac{\partial}{\partial x_j}\Big|_p\Big) = \frac{\partial x_i}{\partial x_j}(p) = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

(Kronecker delta).

Example 7.4.12. Let (φ, U) be a chart of M and $x_1, \ldots, x_n \colon U \to \mathbb{R}$ the corresponding coordinate functions. Then we obtain on U the 1-forms $\mathtt{d}x_j$, $j = 1, \ldots, n$. We call these 1-forms the *φ-basic forms* on U. As in the case of φ-basic vector fields the expression basic form is doubly justified. On the one hand, $\big(\mathtt{d}x_1(p), \ldots, \mathtt{d}x_n(p)\big)$ is a basis for the dual space $T_p(M)^*$ for every $p \in U$. On the other hand, setting $a_j := \omega\Big(\frac{\partial}{\partial x_j}\Big)$, the formula $\mathtt{d}x_k\Big(\frac{\partial}{\partial x_j}\Big) = \delta_{kj}$ implies that every $\omega \in \Omega^1(U)$ can be written as

$$\omega = \sum_{j=1}^{n} a_j \cdot \mathtt{d}x_j \quad \text{for} \quad a_j \in C^\infty(U).$$

7.4.2 Vector Fields as Derivations

Let M be a smooth manifold with tangent bundle TM, and $\pi_{TM} \colon TM \to M$ the canonical projection. Recall from Definition 7.4.1 that a *vector field* on M is a smooth section of the tangent bundle and that it induces a map

$$\mathcal{L}_X \colon C^\infty(M) \to C^\infty(M), \quad f \mapsto \mathcal{L}_X f = \mathtt{d}f \circ X. \tag{7.7}$$

Then the Product Rule $\mathtt{d}(fg) = f\,\mathtt{d}g + g\,\mathtt{d}f$ implies

$$\mathcal{L}_X(f \cdot g) = \mathcal{L}_X f \cdot g + f \cdot \mathcal{L}_X g. \tag{7.8}$$

Definition 7.4.13. Let A be an associative algebra over $\mathbb{K}$. A linear map $D : A \to A$ is called a *derivation* of A, if it satisfies the following property:

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g) \quad \text{for } f, g \in A.$$

The set of all derivations of A is denoted by $\operatorname{der}(A)$.

Remark 7.4.14. The discussion above shows that for a vector field $X \in \mathcal{V}(M)$ the map $\mathcal{L}_X$ is a derivation of the algebra $C^\infty(M)$. We thus obtain a map

$$\mathcal{V}(M) \to \operatorname{der}\big(C^\infty(M)\big), \quad X \mapsto \mathcal{L}_X.$$

Below, we will show that this map is a bijection, and after that, we will also write Xf for $\mathcal{L}_X f$. Note that the set $\operatorname{der}\big(C^\infty(M)\big)$ carries several algebraic structures. It is obvious that we can add derivations and multiply them with scalars. Derivations can also be multiplied with functions by

$$(g \cdot D)(f) := g \cdot \big(D(f)\big) \quad \forall f, g \in C^\infty(M). \tag{7.9}$$

Moreover, it is easy to check directly that $\operatorname{der}\big(C^\infty(M)\big)$ is closed under the commutator bracket (this also follows from Lemma 7.4.7 once $\operatorname{der}\big(C^\infty(M)\big)$ is identified with $\mathcal{V}(M)$). One thus obtains that $\operatorname{der}\big(C^\infty(M)\big)$ is at the same time a Lie algebra and a $C^\infty(M)$-module (cf. Exercise 4.1.4).

Lemma 7.4.15. *Let M be a smooth manifold.*

(i) *Let C be a compact subset of M and $V \supseteq C$ be an open subset. Then there exists a smooth function f with $f|_C = 1$ and $f|_{M \setminus V} = 0$.*
(ii) *For each $p \in M$, the map $C^\infty(M) \to C^\infty(M)_p, f \mapsto f_p$ is surjective.*

Proof. To prove (i), we first claim that for each $p \in M$ and each open subset B containing p, there exists a smooth function $f \in C^\infty(B)$ with $f = 0$ on $M \setminus B$ and $f = 1$ on a neighborhood A of p.

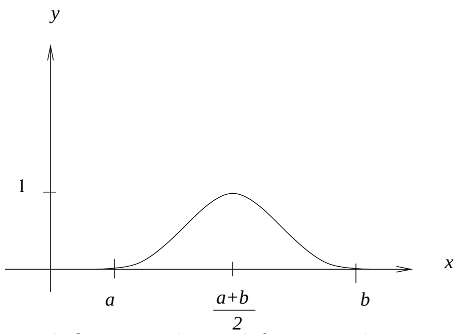
Since this is local property, it suffice s to prove it for balls in $\mathbb{R}^n$. So let $A = B_a(0)$ and $B = B_b(0)$ be the open balls of radius a and b around 0, where $0 < a < b$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ which is defined by

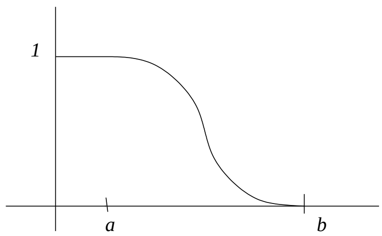
$$f(x) = \begin{cases} \exp\left(\frac{1}{x-b} - \frac{1}{x-a}\right) & \text{for } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

It is elementary to check (Exercise) that f and the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \frac{\int_x^b f(t)dt}{\int_a^b f(t)dt}$$

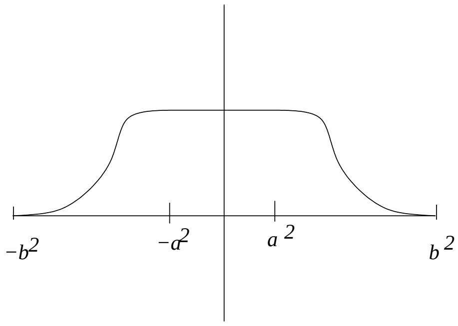
are smooth.





We define our desired function by

$$\zeta(x_1, \ldots, x_n) := F\left(\sqrt{x_1^2 + \ldots + x_n^2}\right)$$



One immediately checks that $\zeta|_A \equiv 1$ and $\zeta|_{\mathbb{R}^n \setminus B} \equiv 0$, so that the claim is proved.

Now cover C by finitely many open sets $U_i \subseteq V$ such that one has smooth functions $f_i \in C^\infty(M)$ with $f_i = 0$ on $M \setminus V$ and $f_i = 1$ on U_i. Then $f := 1 - \prod_i (1 - f_i)$ has the desired properties which completes the proof of (i).

Finally, let U be an open neighborhood of $p \in M$ and $h \in C^\infty(U)$. Further, let W be an open neighborhood of p with compact closure $\overline{W} \subseteq U$. Let $f \in C^\infty(M)$ be a smooth function vanishing on $M \setminus W$ and which is 1 on a neighborhood of p. Then

$$H(x) := \begin{cases} f(x)h(x) & \text{for } x \in U \\ 0 & \text{for } x \notin \overline{W} \end{cases}$$

defines a smooth function on M whose germ in p is $H_p = h_p$. □

Lemma 7.4.16. *The map* $\mathcal{V}(M) \to \operatorname{der}\big(C^\infty(M)\big),\ X \mapsto \mathcal{L}_X$ *is injective.*

Proof. Let $X \in \mathcal{V}(M)$ with $\mathcal{L}_X = 0$. Then by (7.7) we have

$$X(p)(f_p) = (\mathcal{L}_X f)(p) = 0$$

for all $f \in C^\infty(M)$ and all $p \in M$. Since each germ in p comes from a global smooth function on M by Lemma 7.4.15, this implies $X(p) = 0$ for each $p \in M$, hence $X = 0$. □

Lemma 7.4.17. *Let M be a smooth manifold and* $D \in \operatorname{der}\big(C^\infty(M)\big)$.

(i) $D(f) = 0$ *for all constant functions* $f \in C^\infty(M)$.
(ii) *Let $V \subset M$ be an open subset and $f \in C^\infty(M)$ with $f|_V = 0$. Then* $D(f)|_V = 0$ *as well.*

Proof. Just as Proposition 7.3.20, the assertion (i) is a direct consequence of the defining equation of a derivation. To show (ii), pick $p \in V$. By Lemma 7.4.15, there exists a function $g \in C^\infty(M)$ which is 1 outside V and which vanishes in p. Then $g \cdot f = f$, and therefore,

$$D(f)(p) = D(f \cdot g)(p) = f(p)\big(D(g)(p)\big) + g(p)\big(D(f)(p)\big) = 0.$$

Since $p \in V$ was chosen arbitrarily, the lemma is proved. □

To finish the program outlined in Remark 7.4.14, we still have to show that the assignment $X \mapsto \mathcal{L}_X$ is also surjective. So, let $D \in \operatorname{der}\big(C^\infty(M)\big)$ be given. By Lemma 7.4.17 $D(f)(p)$ only depends on f_p, so, in view of Lemma 7.4.16, we can define an element $X_D(p) \in T_p(M)$ via

$$X_D(p)(f_p) := D(f)(p) \tag{7.10}$$

for all $f \in C^\infty(M)$ and for all $p \in M$.

All that remains to show to complete the proof of the bijectivity of $X \mapsto \mathcal{L}_X$ is the smoothness of X_D in each $p \in M$. This is a local property, so it suffices to verify the smoothness of X_D on some neighborhood of p. So let (φ, U) be a chart of M with $p \in U$. Let $\widehat{x}_j \in C^\infty(M)$ be functions which coincide with x_j on a neighborhood V of p (Lemma 7.4.15). Then Proposition 7.3.23 implies for $q \in V$

$$X_D(q) = \sum_{j=1}^{n} X_D(q)\big((x_j)_q\big) \cdot \tfrac{\partial}{\partial x_j}\big|_q = \textstyle\sum_{j=1}^n D(\widehat{x}_j)(q) \cdot \tfrac{\partial}{\partial x_j}\big|_q.$$

Therefore X_D is smooth on V. This completes the proof of the following theorem.

Theorem 7.4.18. *Let M be a smooth manifold. Then the map*

$$\mathcal{V}(M) \to \operatorname{der}\big(C^\infty(M)\big), \quad X \mapsto \mathcal{L}_X,$$

defined by $\mathcal{L}_X(f) = \mathtt{d}f \circ X$, is a bijection.

Theorem 7.4.18 is interesting not only in its own right, but also because of the methods that were used to prove it. We collect some of the facts we essentially derived in the course of the proof:

Remark 7.4.19. Let M be a smooth manifold and $p \in M$.

(i) Let $C \subseteq M$ be a compact subset, $U \supseteq C$ an open neighborhood and $X \in \mathcal{V}(U)$ a vector field on U. Let $W \subseteq U$ be a compact neighborhood of C in U and $f \in C^\infty(M)$ vanishing on $M \setminus W$ with $f|_C = 1$ (Lemma 7.4.15(i)). Then fX defines a smooth vector field on M which coincides with X on C.

(ii) For any chart (φ, U) with $p \in U$ a vector field $X \in \operatorname{der}\big(C^\infty(M)\big) \cong \mathcal{V}(M)$ can be written on U as

$$X|_U = \sum_{j=1}^{n} a_j \cdot X_j \tag{7.11}$$

with $a_j \in C^\infty(U)$ and $X_j \in \mathcal{V}(U)$, given by $\frac{\partial}{\partial x_j}\big|_q$ for all $q \in U$.

(iii) For every $v \in T_pM$, there exists a vector field $X \in \mathcal{V}(M)$ with $X(p) = v$.

Exercises for Section 7.4

Exercise 7.4.1. Let M be a smooth manifold and $q\colon \mathbb{V} \to M$ a smooth $\mathbb{K}$-vector bundle. Show that

$$(s_1 + s_2)(p) := s_1(p) + s_2(p) \quad (\lambda s)(p) := \lambda s(p), \quad \lambda \in \mathbb{K},$$

defines on the space $\Gamma\mathbb{V}$ of smooth sections of $\mathbb{V}$ the structure of a $\mathbb{K}$-vector space. Show also that the multiplication with smooth $\mathbb{K}$-valued functions defined by

$$(fs)(p) := f(p)s(p)$$

satisfies for $s, s_1, s_2 \in \Gamma\mathbb{V}$ and $f, g \in C^\infty(M, \mathbb{K})$:

(1) $f(s_1 + s_2) = fs_1 + fs_2$.
(2) $f(\lambda s) = \lambda \cdot fs = (\lambda f)s$ for $\lambda \in \mathbb{K}$.
(3) $(f + g)s = fs + gs$.
(4) $f(gs) = (fg)s$.

Exercise 7.4.2. Let M be a smooth manifold, $X, Y \in \mathcal{V}(M)$ and $f, g \in C^\infty(M, \mathbb{R})$. Show that
(1) $\mathcal{L}_X(f \cdot g) = \mathcal{L}_X(f) \cdot g + f \cdot \mathcal{L}_X(g)$, i.e., the map $f \mapsto \mathcal{L}_X(f)$ is a derivation.
(2) $\mathcal{L}_{fX}(g) = f \cdot \mathcal{L}_X(g)$.

Exercise 7.4.3. Let M be a smooth manifold of dimension n and $p \in M$. Pick a vector $v_\varphi \in \mathbb{R}^n$ for each chart (φ, U) with $p \in U$. We call the family (v_φ) of all such vectors a *tangent vector* (physicists' definition), if

$$v_\psi = \mathtt{d}(\psi \circ \varphi^{-1})_{\varphi(p)} v_\varphi.$$

It is obvious that these tangent vectors form a vector space isomorphic to $\mathbb{R}^n$. Fix a chart (φ, U) with $p \in U$. Consider the map

$$\mathbb{R}^n \to T_pM, \; v_\varphi \mapsto \sum_{j=1}^n (v_\varphi)_j \frac{\partial}{\partial x_j}\Big|_p$$

and show that it defines a linear isomorphism between the physicists and the algebraic version of the tangent space.

Exercise 7.4.4. Let M be a smooth manifold, and let (φ, U) and (ψ, V) be charts on M. Further, let $p \in U \cap V$ and $v \in T_pM$. Express v in the φ-basis for T_pM as well as in the ψ-basis for T_pM. How are the coefficients related?

Exercise 7.4.5. Let $f\colon M \to N$ be a smooth map between smooth manifolds. Pick charts (φ, U) and (ψ, V) of M and N, respectively, with $p \in U$ and $f(p) \in V$. Show that the derivative $T_pf\colon T_pM \to T_{f(p)}N$ of f in p in the physicists' definition of tangent space is given by

$$\big(T_pf(v_\varphi)\big)_\psi = \mathtt{d}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(v_\varphi).$$

Exercise 7.4.6. Show that $\mathrm{der}\big(C^\infty(M)\big)$ is a $C^\infty(M)$-module with respect to the multiplication (7.9). This means that for all $D, E \in \mathrm{der}\big(C^\infty(M)\big)$, $g, f \in C^\infty(M)$, and $r, s \in \mathbb{R}$ the following properties are satisfied:

$$\begin{aligned} g \cdot (rD + sE) &= r(g \cdot D) + s(g \cdot E), \\ g \cdot (f \cdot D) &= (g \cdot f) \cdot D, \\ (g + f) \cdot D &= g \cdot D + f \cdot D. \end{aligned}$$

7.5 Integral Curves and Local Flows

In this section we turn to the geometric nature of vector fields as infinitesimal generators of local flows on manifolds. This provides a natural perspective on (autonomous) ordinary differential equations.

7.5.1 Integral Curves

Throughout this subsection M denotes an n-dimensional smooth manifold.

Definition 7.5.1. Let $X \in \mathcal{V}(M)$ and $I \subseteq \mathbb{R}$ an open interval containing 0. A differentiable map $\gamma\colon I \to M$ is called an *integral curve* of X if

$$\gamma'(t) = X(\gamma(t)) \qquad \text{for each} \quad t \in I.$$

Note that the preceding equation implies that γ' is continuous and further that if γ is C^k, then γ' is also C^k. Therefore integral curves are automatically smooth.

If $J \supseteq I$ is an interval containing I, then an integral curve $\eta\colon J \to M$ is called an *extension* of γ if $\eta|_I = \gamma$. An integral curve γ is said to be *maximal* if it has no proper extension.

Remark 7.5.2. (a) If $U \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, then we write a vector field $X \in \mathcal{V}(U)$ as $X(x) = (x, F(x))$, where $F\colon U \to \mathbb{R}^n$ is a smooth function. A curve $\gamma\colon I \to U$ is an integral curve of X if and only if it satisfies the ordinary differential equation

$$\gamma'(t) = F(\gamma(t)) \quad \text{for all} \quad t \in I.$$

(b) If (φ, U) is a chart of the manifold M and $X \in \mathcal{V}(M)$, then a curve $\gamma\colon I \to M$ is an integral curve of X if and only if the curve $\eta := \varphi \circ \gamma$ is an integral curve of the vector field $X_\varphi := T(\varphi) \circ X \circ \varphi^{-1} \in \mathcal{V}(\varphi(U))$ because

$$X_\varphi(\eta(t)) = T_{\gamma(t)}(\varphi)X(\gamma(t)) \quad \text{and} \quad \eta'(t) = T_{\gamma(t)}(\varphi)\gamma'(t).$$

Remark 7.5.3. A curve $\gamma\colon I \to M$ is an integral curve of X if and only if $\widetilde{\gamma}(t) := \gamma(-t)$ is an integral curve of the vector field $-X$.

More generally, for $a, b \in \mathbb{R}$, the curve $\eta(t) := \gamma(at+b)$ is an integral curve of the vector field aX.

Definition 7.5.4. Let $a < b \in [-\infty, \infty]$. For a continuous curve $\gamma\colon]a,b[\to M$ we say that

$$\lim_{t \to b} \gamma(t) = \infty$$

if for each compact subset $K \subseteq M$ there exists a $c < b$ with $\gamma(t) \notin K$ for $t > c$. Similarly, we define

$$\lim_{t \to a} \gamma(t) = \infty.$$

Theorem 7.5.5 (Existence and Uniqueness of Integral Curves). *Let $X \in \mathcal{V}(M)$ and $p \in M$. Then there exists a unique maximal integral curve $\gamma_p\colon I_p \to M$ with $\gamma_p(0) = p$. If $a := \inf I_p > -\infty$, then $\lim_{t\to a} \gamma_p(t) = \infty$ and if $b := \sup I_p < \infty$, then $\lim_{t\to b} \gamma_p(t) = \infty$.*

Proof. We have seen in Remark 7.5.2 that in local charts, integral curves are solutions of an ordinary differential equation with a smooth right hand side. We now reduce the proof to the Local Existence- and Uniqueness Theorem for ODE's.

Uniqueness: Let $\gamma, \eta\colon I \to M$ be two integral curves of X with $\gamma(0) = \eta(0) = p$. The continuity of the curves implies that

$$0 \in J := \{t \in I \colon \gamma(t) = \eta(t)\}$$

is a closed subset of I. In view of the Local Uniqueness Theorem for ODE's, for each $t_0 \in J$ there exists an $\varepsilon > 0$ with $[t_0, t_0+\varepsilon] \subseteq J$, and likewise $[t_0-\varepsilon, t_0] \subseteq J$ (Remark 7.5.3). Therefore J is also open. Now the connectedness of I implies $I = J$, so that $\gamma = \eta$.

Existence: The Local Existence Theorem implies the existence of some integral curve $\gamma\colon I \to M$ on some open interval containing 0. For any other integral curve $\eta\colon J \to M$, the intersection $I \cap J$ is an interval containing 0, so that the uniqueness assertion implies that $\eta = \gamma$ on $I \cap J$.

Let $I_p \subseteq \mathbb{R}$ be the union of all open intervals I_j containing 0 on which there exists an integral curve $\gamma_j\colon I_j \to M$ of X with $\gamma_j(0) = p$. Then the preceding argument shows that

$$\gamma(t) := \gamma_j(t) \quad \text{for} \quad t \in I_j$$

defines an integral curve of X on I_p, which is maximal by definition. The uniqueness of the maximal integral curve also follows from its definition.

Limit condition: Suppose that $b := \sup I_p < \infty$. If $\lim_{t\to b} \gamma(t) = \infty$ does not hold, then there exists a compact subset $K \subseteq M$ and a sequence $t_m \in I_p$ with $t_m \to b$ and $\gamma(t_m) \in K$. As K can be covered with finitely many closed subsets homeomorphic to a closed subset of a ball in $\mathbb{R}^n$, after passing to a suitable subsequence, we may w.l.o.g. assume that K itself is homeomorphic to a compact subset of $\mathbb{R}^n$. Then a subsequence of $(\gamma(t_m))_{m\in\mathbb{N}}$ converges, and we may replace the original sequence by this subsequence, hence assume that $q := \lim_{m\to\infty} \gamma(t_m)$ exists.

The Local Existence Theorem for ODE's implies the existence of a compact neighborhood $V \subseteq M$ of q and $\varepsilon > 0$ such that the initial value problem

$$\eta(0) = x, \quad \eta' = X \circ \eta$$

has a solution on $[-\varepsilon, \varepsilon]$ for each $x \in V$. Pick $m \in \mathbb{N}$ with $t_m > b - \varepsilon$ and $\gamma(t_m) \in V$. Further let $\eta \colon [-\varepsilon, \varepsilon] \to M$ be an integral curve with $\eta(0) = \gamma(t_m)$. Then

$$\gamma(t) := \eta(t - t_m) \quad \text{for} \quad t \in [t_m - \varepsilon, t_m + \varepsilon],$$

defines an extension of γ to the interval $I_p \cup]t_m, t_m + \varepsilon[$ strictly containing $]a, b[$, hence contradicting the maximality of I_p. This proves that $\lim_{t \to b} \gamma(t) = \infty$. Replacing X by $-X$, we also obtain $\lim_{t \to a} \gamma(t) = \infty$. □

If $q = \gamma_p(t)$ is a point on the unique maximal integral curve of X through $p \in M$, then $I_q = I_p - t$ and

$$\gamma_q(s) := \gamma_p(t + s)$$

is the unique maximal integral curve through q. Here I_p is the domain of definition of the maximal integral curve through p and I_q is the domain of definition of the maximal integral curve through q.

Example 7.5.6. (a) On $M = \mathbb{R}$ we consider the vector field X given by the function $F(s) = 1 + s^2$, i.e., $X(s) = (s, 1 + s^2)$. The corresponding ODE is

$$\gamma'(s) = X(\gamma(s)) = 1 + \gamma(s)^2.$$

For
$\gamma(0) = 0$ the function $\gamma(s) := \tan(s)$ on $I :=] - \frac{\pi}{2}, \frac{\pi}{2}[$ is the unique maximal solution because

$$\lim_{t \to \frac{\pi}{2}} \tan(t) = \infty \quad \text{and} \quad \lim_{t \to -\frac{\pi}{2}} \tan(t) = -\infty.$$

(b) Let $M :=] - 1, 1[$ and $X(s) = (s, 1)$, so that the corresponding ODE is $\gamma'(s) = 1$. Then the unique maximal solution is

$$\gamma(s) = s, \quad I =] - 1, 1[.$$

Note that we also have in this case

$$\lim_{s \to \pm 1} \gamma(s) = \infty$$

if we consider γ as a curve in the noncompact manifold M.

For $M = \mathbb{R}$ the same vector field has the maximal integral curve

$$\gamma(s) = s, \quad I = \mathbb{R}.$$

(c) For $M = \mathbb{R}$ and $X(s) = (s, -s)$, the differential equation is $\gamma'(t) = -\gamma(t)$, so that we obtain the maximal integral curves $\gamma(t) = \gamma_0 e^{-t}$. For $\gamma_0 = 0$ this curve is constant, and for $\gamma_0 \neq 0$ we have $\lim_{t\to\infty} \gamma(t) = 0$, hence $\lim_{t\to\infty} \gamma(t) \neq \infty$. This shows that maximal integral curves do not always leave every compact subset of M if they are defined on an interval unbounded from above.

The preceding example shows in particular that the global existence of integral curves can also be destroyed by deleting parts of the manifold M, i.e., by considering $M' := M \setminus K$ for some closed subset $K \subseteq M$.

Definition 7.5.7. A vector field $X \in \mathcal{V}(M)$ is said to be *complete* if all its maximal integral curves are defined on all of $\mathbb{R}$.

Corollary 7.5.8. *All vector fields on a compact manifold M are complete.*

7.5.2 Local Flows

Definition 7.5.9. Let M be a smooth manifold. A *local flow on M* is a smooth map

$$\Phi\colon U \to M,$$

where $U \subseteq \mathbb{R} \times M$ is an open subset containing $\{0\} \times M$, such that for each $x \in M$ the intersection $I_x := U \cap (\mathbb{R} \times \{x\})$ is an interval containing 0 and

$$\Phi(0, x) = x \quad \text{and} \quad \Phi\big(t, \Phi(s, x)\big) = \Phi(t + s, x)$$

hold for all t, s, x for which both sides are defined. The maps

$$\alpha_x\colon I_x \to M, \qquad t \mapsto \Phi(t, x)$$

are called the *flow lines*. The flow Φ is said to be *global* if $U = \mathbb{R} \times M$.

Lemma 7.5.10. *If $\Phi\colon U \to M$ is a local flow, then*

$$X^\Phi(x) := \frac{d}{dt}\Big|_{t=0} \Phi(t, x) = \alpha_x'(0)$$

defines a smooth vector field.

It is called the *velocity field* or the *infinitesimal generator* of the local flow Φ.

Lemma 7.5.11. *If $\Phi\colon U \to M$ is a local flow on M, then the flow lines are integral curves of the vector field X^Φ. In particular, the local flow Φ is uniquely determined by the vector field X^Φ.*

Proof. Let $\alpha_x \colon I_x \to M$ be a flow line and $s \in I_x$. For sufficiently small $t \in \mathbb{R}$ we then have

$$\alpha_x(s+t) = \Phi(s+t,x) = \Phi\big(t, \Phi(s,x)\big) = \Phi\big(t, \alpha_x(s)\big),$$

so that taking derivatives in $t = 0$ leads to $\alpha_x'(s) = X^{\Phi}(\alpha_x(s))$.

That Φ is uniquely determined by the vector field X^{Φ} follows from the uniqueness of integral curves (Theorem 7.5.5). □

Theorem 7.5.12. *Each smooth vector field X is the velocity field of a unique local flow defined by*

$$\mathcal{D}_X := \bigcup_{x \in M} I_x \times \{x\} \quad \textit{and} \quad \Phi(t,x) := \gamma_x(t) \quad \textit{for} \quad (t,x) \in \mathcal{D}_X,$$

where $\gamma_x \colon I_x \to M$ is the unique maximal integral curve through $x \in M$.

Proof. If $(s,x), \big(t, \Phi(s,x)\big)$ and $(s+t,x) \in \mathcal{D}_X$, the relation

$$\Phi(s+t,x) = \Phi\big(t, \Phi(s,x)\big) \quad \text{and} \quad I_{\Phi(s,x)} = I_{\gamma_x(s)} = I_x - s$$

follow from the fact that both curves

$$t \mapsto \Phi(t+s,x) = \gamma_x(t+s) \quad \text{and} \quad t \mapsto \Phi\big(t, \Phi(s,x)\big) = \gamma_{\Phi(s,x)}(t)$$

are integral curves of X with the initial value $\Phi(s,x)$, hence coincide.

We claim that all maps

$$\Phi_t \colon M_t := \{x \in M \colon (t,x) \in \mathcal{D}_X\} \to M, \quad x \mapsto \Phi(t,x)$$

are injective. In fact, if $p := \Phi_t(x) = \Phi_t(y)$, then $\gamma_x(t) = \gamma_y(t)$, and on $[0,t]$ the curves $s \mapsto \gamma_x(t-s), \gamma_y(t-s)$ are integral curves of $-X$, starting in p. Hence the Uniqueness Theorem 7.5.5 implies that they coincide in $s = t$, which mans that $x = \gamma_x(0) = \gamma_y(0) = y$. From this argument it further follows that $\Phi_t(M_t) = M_{-t}$ and $\Phi_t^{-1} = \Phi_{-t}$.

It remains to show that $\mathcal{D}_X$ is open and Φ smooth. The local Existence Theorem provides for each $x \in M$ an open neighborhood U_x diffeomorphic to a cube and some $\varepsilon_x > 0$, as well as a smooth map

$$\varphi_x \colon]-\varepsilon_x, \varepsilon_x[\times U_x \to M, \quad \varphi_x(t,y) = \gamma_y(t) = \Phi(t,y).$$

Hence $]-\varepsilon_x, \varepsilon_x[\times U_x \subseteq \mathcal{D}_X$, and the restriction of Φ to this set is smooth. Therefore Φ is smooth on a neighborhood of $\{0\} \times M$ in $\mathcal{D}_X$.

Now let J_x be the set of all $t \in [0,\infty[$, for which $\mathcal{D}_X$ contains a neighborhood of $[0,t] \times \{x\}$ on which Φ is smooth. The interval J_x is open in $\mathbb{R}^+ := [0,\infty[$ by definition. We claim that $J_x = I_x \cap \mathbb{R}^+$. This entails that $\mathcal{D}_X$ is open because the same argument applies to $I_x \cap]-\infty, 0]$.

We assume the contrary and find a minimal $\tau \in I_x \cap \mathbb{R}^+ \setminus J_x$, because this interval is closed. Put $p := \Phi(\tau,x)$ and pick a product set $I \times W \subseteq \mathcal{D}_X$, where W is an open neighborhood of p and $I =]-2\varepsilon, 2\varepsilon[$ a 0-neighborhood, such that $2\varepsilon < \tau$ and $\Phi : I \times W \to M$ is smooth. By assumption, there exists an open neighborhood V of x such that Φ is smooth on $[0, \tau - \varepsilon] \times V \subseteq \mathcal{D}_X$. Then $\Phi_{\tau-\varepsilon}$ is smooth on V and

$$V' := \Phi_{\tau-\varepsilon}^{-1}\big(\Phi_\varepsilon^{-1}(W)\big) \cap V$$

is a neighborhood of $[0, \tau + \varepsilon] \times \{x\}$ in $\mathcal{D}_X$. Further,

$$V' = \Phi_{\tau-\varepsilon}^{-1}\big(\Phi_\varepsilon^{-1}(W)\big) \cap V = \Phi_\tau^{-1}(W) \cap V,$$

and Φ is smooth on V', because it is a composition of smooth maps:

$$]\tau - 2\varepsilon, \tau + 2\varepsilon[\times V' \to M, \quad (t, y) \mapsto \Phi\big(t - \tau, \Phi(\varepsilon, \Phi(\tau - \varepsilon, y))\big).$$

We thus arrive at the contradiction $\tau \in J_x$.

This completes the proof of the openness of $\mathcal{D}_X$ and the smoothness of Φ. The uniqueness of the flow follows from the uniqueness of the integral curves. □

Remark 7.5.13. Let $X \in \mathcal{V}(M)$ be a complete vector field. If

$$\Phi^X : \mathbb{R} \times M \to M$$

is the corresponding global flow, then the maps $\Phi_t^X : x \mapsto \Phi^X(t, x)$ satisfy

(A1) $\Phi_0^X = \mathrm{id}_M$.
(A2) $\Phi_{t+s}^X = \Phi_t^X \circ \Phi_s^X$ for $t, s \in \mathbb{R}$.

It follows in particular that $\Phi_t^X \in \mathrm{Diff}(M)$ with $(\Phi_t^X)^{-1} = \Phi_{-t}^X$, so that we obtain a group homomorphism

$$\gamma_X : \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \Phi_t^X.$$

With respect to the terminology introduced below, (A1) and (A2) mean that Φ^X defines a *smooth action of* $\mathbb{R}$ *on* M. As Φ^X is determined by the vector field X, we call X the infinitesimal generator of this action. In this sense the smooth $\mathbb{R}$-actions on a manifold M are in one-to-one correspondence with the complete vector fields on M.

Remark 7.5.14. Let $\Phi^X : \mathcal{D}_X \to M$ be the maximal local flow of a vector field X on M. Let $M_t = \{x \in M : (t, x) \in \mathcal{D}_X\}$, and observe that this is an open subset of M. We have already seen in the proof of Theorem 7.5.12 above, that all the smooth maps $\Phi_t^X : M_t \to M$ are injective with $\Phi_t^X(M_t) = M_{-t}$ and $(\Phi_t^X)^{-1} = \Phi_{-t}^X$ on the image. It follows in particular, that $\Phi_t^X(M_t) = M_{-t}$ is open, and that

$$\Phi_t^X : M_t \to M_{-t}$$

is a diffeomorphism whose inverse is Φ_{-t}^X.

Proposition 7.5.15 (Smooth Dependence Theorem). *Let M be a smooth manifold, V a finite-dimensional vector space, $V_1 \subseteq V$ an open subset, and $\Psi\colon V_1 \to \mathcal{V}(M)$ a map for which the map*

$$\widehat{\Psi}\colon V_1 \times M \to T(M), \quad (v,p) \mapsto \Psi(v)(p)$$

is smooth (the vector field $\Psi(v)$ depends smoothly on v). Then there exists for each $(p_0, v_0) \in M \times V_1$ an open neighborhood U of p_0 in M, an open interval $I \subseteq \mathbb{R}$ containing 0, an open neighborhood W of v_0 in V_1, and a smooth map

$$\Phi\colon I \times U \times W \to M$$

such that for each $(p,v) \in U \times W$ the curve

$$\Phi_p^v\colon I \to M, \quad t \mapsto \Phi(t,p,v)$$

is an integral curve of the vector field $\Psi(v)$ with $\Phi_p^v(0) = p$.

Proof. The parameters do not cause any additional problems, which can be seen by the following trick: On the product manifold $N := V_1 \times M$ we consider the smooth vector field Y, given by

$$Y(v,p) := (0, \Psi(v)(p)).$$

Then the integral curves of Y are of the form

$$\gamma(t) = (v, \gamma_v(t)),$$

where γ_v is an integral curve of the smooth vector field $\Psi(v)$ on M. Therefore the assertion is an immediate consequence on the smoothness of the local flow of Y on $V_1 \times M$ (Theorem 7.5.12). □

7.5.3 Lie Derivatives

We take a closer look at the interaction of local flows and vector fields. It will turn out that this leads to a new concept of a directional derivative which works for general tensor fields.

Let $X \in \mathcal{V}(M)$ and $\Phi^X : \mathcal{D}_X \to M$ its maximal local flow. For $f \in C^\infty(M)$ and $t \in \mathbb{R}$ we set

$$(\Phi_t^X)^* f := f \circ \Phi_t^X \in C^\infty(M_t).$$

Then we find

$$\lim_{t \to 0} \frac{1}{t}((\Phi_t^X)^* f - f) = \mathtt{d}f(X) = \mathcal{L}_X f \in C^\infty(M).$$

For a second vector field $Y \in \mathcal{V}(M)$, we define a smooth vector field on the open subset $M_{-t} \subseteq M$ by

$$(\Phi_t^X)_*Y := T(\Phi_t^X) \circ Y \circ \Phi_{-t}^X = T(\Phi_t^X) \circ Y \circ (\Phi_t^X)^{-1}$$

(cf. Remark 7.5.14) and define the *Lie derivative* by

$$\mathcal{L}_X Y := \lim_{t \to 0} \frac{1}{t}((\Phi_{-t}^X)_*Y - Y) = \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^X)_*Y,$$

which is defined on all of M since for each $p \in M$ the vector $((\Phi_t^X)_*Y)(p)$ is defined for sufficiently small t and depends smoothly on t.

Theorem 7.5.16. *$\mathcal{L}_X Y = [X, Y]$ for $X, Y \in \mathcal{V}(M)$.*

Proof. Fix $p \in M$. It suffices to show that $\mathcal{L}_X Y$ and $[X, Y]$ coincide in p. We may therefore work in a local chart, hence assume that $M = U$ is an open subset of $\mathbb{R}^n$.

Identifying vector fields with smooth $\mathbb{R}^n$-valued functions, we then have

$$[X, Y](x) = \mathtt{d}Y(x)X(x) - \mathtt{d}X(x)Y(x), \quad x \in U.$$

On the other hand,

$$\begin{aligned}((\Phi_{-t}^X)_*Y)(x) &= T(\Phi_{-t}^X) \circ Y \circ \Phi_t^X(x) \\ &= \mathtt{d}(\Phi_{-t}^X)(\Phi_t^X(x))Y(\Phi_t^X(x)) = \big(\mathtt{d}(\Phi_t^X)(x)\big)^{-1} Y(\Phi_t^X(x)).\end{aligned}$$

To calculate the derivative of this expression with respect to t, we first observe that it does not matter if we first take derivatives with respect to t and then with respect to x or vice versa. This leads to

$$\frac{d}{dt}\Big|_{t=0} \mathtt{d}(\Phi_t^X)(x) = \mathtt{d}\Big(\frac{d}{dt}\Big|_{t=0} \Phi_t^X\Big)(x) = \mathtt{d}X(x).$$

Next we note that for any smooth curve $\alpha\colon [-\varepsilon, \varepsilon] \to \mathrm{GL}_n(\mathbb{R})$ with $\alpha(0) = \mathbf{1}$ we have

$$(\alpha^{-1})'(t) = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1},$$

and in particular $(\alpha^{-1})'(0) = -\alpha'(0)$. Combining all this, we obtain with the Product Rule

$$\mathcal{L}_X(Y)(x) = -\mathtt{d}X(x)Y(x) + \mathtt{d}Y(x)X(x) = [X, Y](x). \qquad \square$$

Corollary 7.5.17. *If $X, Y \in \mathcal{V}(M)$ are complete vector fields, then their global flows $\Phi^X, \Phi^Y : \mathbb{R} \to \mathrm{Diff}(M)$ commute if and only if X and Y commute, i.e., $[X, Y] = 0$.*

Proof. (1) Suppose first that Φ^X and Φ^Y commute, i.e.,

$$\Phi^X(t) \circ \Phi^Y(s) = \Phi^Y(s) \circ \Phi^X(t) \quad \text{for } t, s \in \mathbb{R}.$$

Let $p \in M$ and $\gamma_p(s) := \Phi_s^Y(p)$ the Y-integral curve through p. We then have

$$\gamma_p(s) = \Phi_s^Y(p) = \Phi_t^X \circ \Phi_s^Y \circ \Phi_{-t}^X(p),$$

and passing to the derivative in $s = 0$ yields

$$Y(p) = \gamma_p'(0) = T(\Phi_t^X)Y(\Phi_{-t}^X(p)) = \big((\Phi_t^X)_*Y\big)(p).$$

Passing now to the derivative in $t = 0$, we arrive at $[X, Y] = \mathcal{L}_X(Y) = 0$.

(2) Now we assume $[X, Y] = 0$. First we show that $(\Phi_t^X)_*Y = Y$ holds for all $t \in \mathbb{R}$. For $t, s \in \mathbb{R}$ we have

$$(\Phi_{t+s}^X)_*Y = (\Phi_t^X)_*(\Phi_s^X)_*Y,$$

so that

$$\frac{d}{dt}(\Phi_t^X)_*Y = -(\Phi_t^X)_*\mathcal{L}_X(Y) = 0$$

for each $t \in \mathbb{R}$. Since for each $p \in M$ the curve

$$\mathbb{R} \to T_p(M), \quad t \mapsto \big((\Phi_t^X)_*Y\big)(p)$$

is smooth, and its derivative vanishes, it is constant $Y(p)$. This shows that $(\Phi_t^X)_*Y = Y$ for each $t \in \mathbb{R}$.

For $\gamma(s) := \Phi_t^X \Phi_s^Y(p)$ we now have $\gamma(0) = \Phi_t^X(p)$ and

$$\gamma'(s) = T(\Phi_t^X) \circ Y(\Phi_s^Y(p)) = Y(\Phi_t^X \Phi_s^Y(p)) = Y(\gamma(s)),$$

so that γ is an integral curve of Y. We conclude that $\gamma(s) = \Phi_s^Y(\Phi_t^X(p))$, and this means that the flows of X and Y commute. □

Remark 7.5.18. Let $X, Y \in \mathcal{V}(M)$ be two complete vector fields and Φ^X, resp., Φ^Y their global flows. We then consider the commutator map

$$F\colon \mathbb{R}^2 \to \mathrm{Diff}(M), \quad (t, s) \mapsto \Phi_t^X \circ \Phi_s^Y \circ \Phi_{-t}^X \circ \Phi_{-s}^Y.$$

We know from Corollary 7.5.17 that it vanishes if and only if $[X, Y] = 0$, but there is also a more direct way from F to the Lie bracket. In fact, we first observe that

$$\frac{\partial F}{\partial s}(t, 0) = (\Phi_t^X)_*Y - Y,$$

and hence that

$$\frac{\partial^2 F}{\partial t \partial s}(0, 0) = [Y, X].$$

Here we use that if $I \subseteq \mathbb{R}$ is an interval and

$$\alpha\colon I \to \mathrm{Diff}(M) \quad \text{and} \quad \beta\colon I \to \mathrm{Diff}(M)$$

are maps for which

$$\widehat{\alpha}\colon I \times M \to M, \quad (t, x) \mapsto \alpha(t)(x) \quad \text{and} \quad \widehat{\beta}\colon I \times M \to M, \quad (t, x) \mapsto \beta(t)(x)$$

are smooth, then the curve $\gamma(t) := \alpha(t) \circ \beta(t)$ also has this property (by the Chain Rule), and if $\alpha(0) = \beta(0) = \mathrm{id}_M$, then γ satisfies

$$\gamma'(0) = \alpha'(0) \circ \beta(0) + T(\alpha(0)) \circ \beta'(0) = \alpha'(0) + \beta'(0).$$

Exercises for Section 7.5

Exercise 7.5.1. Let $M := \mathbb{R}^n$. For a matrix $A \in M_n(\mathbb{R})$, we consider the linear vector field $X_A(x) := Ax$. Determine the maximal flow Φ^X of this vector field.

Exercise 7.5.2. Let M be a smooth manifold and $Y \in \mathcal{V}(M)$ a smooth vector field on M. Suppose that Y generates a local flow which is defined on an entire box of the form $[-\varepsilon, \varepsilon] \times M$. Show that this implies the completeness of X.

7.6 Submanifolds

In this section we describe subsets of a smooth manifold which themselves carry manifold structures. In applications there occur subsets with various degrees of compatibility of their smooth structures with the ambient manifold. These give rise to different concepts of submanifolds.

Definition 7.6.1 (Submanifolds). (a) Let M be a smooth n-dimensional manifold. A subset $S \subseteq M$ is called a *d-dimensional submanifold* if for each $p \in S$ there exists a chart (φ, U) of M with $p \in U$ such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}). \tag{7.12}$$

A submanifold of codimension 1, i.e., $\dim S = n - 1$, is called a *smooth hypersurface*.

(b) An *immersed submanifold of M* is a subset $S \subseteq M$, endowed with a smooth manifold structure, such that the inclusion map $i_S \colon S \to M$ is a smooth *immersion*, i.e., its tangent map $T(i_S)$ is injective on each tangent space of S.

(c) An *initial submanifold of M* is an immersed submanifold[2] such that for each other smooth manifold N a map $f \colon N \to S$ is smooth if and only if $i_S \circ f \colon N \to M$ is smooth. The latter condition means that a map into S is smooth if and only if it is smooth, when considered as a map into M.

Remark 7.6.2. For any two initial submanifold structures $\iota_1 \colon S_1 \to M$ and $\iota_2 \colon S_2 \to M$ on a subset $S = \iota_1(S_1) = \iota_2(S_2)$ of a smooth manifold M, there exists a diffeomorphism $\varphi \colon S_2 \to S_1$ with $\iota_1 \circ \varphi = \iota_2$. In particular, the smooth manifold structures on M coincide.

In fact, by the definition of an initial submanifold, the maps

$$\iota_1^{-1} \circ \iota_2 \colon S_2 \to S_1 \quad \text{and} \quad \iota_2^{-1} \circ \iota_1 \colon S_1 \to S_2$$

are smooth, Since they are mutually inverse, $\varphi := \iota_1^{-1} \circ \iota_2 \colon S_2 \to S_1$ is a diffeomorphism with the required properties.

[2] Note that the assumption that M be immersed is not redundant (see [KM97], §27.11)

We shall see below that submanifolds are initial submanifolds. Later (see Example 8.3.12) we will see an example of an initial submanifold, which is not a submanifold. The following example shows that not all immersed submanifolds are initial.

Example 7.6.3 (Figure Eight). Consider the immersed submanifold $S \subseteq \mathbb{R}^2$ defined by the immersion $\varphi\colon]0, 2\pi[\to \mathbb{R}^2,\ t \mapsto (\sin^3 t, \sin t\, \cos t)$. Define a map $\psi\colon]0, 2\pi[\to S$ via

$$\psi\big(\varphi(t)\big) := \begin{cases} \varphi(t+\pi) & \text{for } t < \pi, \\ 0 & \text{for } t = \pi, \\ \varphi(t-\pi) & \text{for } t > \pi. \end{cases}$$

Then we have $\psi(t) = (-\sin^3 t, \sin t\, \cos t)$, so that $i_S \circ \psi\colon]0, 2\pi[\to \mathbb{R}^2$ is smooth. But the limit $\lim_{t\to\pi} \psi(t)$ does not exist in S, so ψ is not even continuous.

Remark 7.6.4. (a) Any discrete subset S of M is a 0-dimensional submanifold.

(b) If $n = \dim M$, any open subset $S \subseteq M$ is an n-dimensional submanifold. If, conversely, $S \subseteq M$ is an n-dimensional submanifold, then the definition immediately shows that S is open.

Lemma 7.6.5. *Any submanifold S of a manifold M has a natural manifold structure, turning it into an initial submanifold.*

Proof. (a) We endow S with the subspace topology inherited from M, which turns it into a Hausdorff space. For each chart (φ, U) satisfying (7.12), we obtain a d-dimensional chart

$$(\varphi|_{U\cap S}, U \cap S)$$

of S. For two such charts coming from (φ, U) and (ψ, V), we have

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap S)} = (\psi|_{V\cap S}) \circ (\varphi|_{U\cap S})^{-1}|_{\varphi(U\cap V\cap S)},$$

which is a smooth map onto an open subset of $\mathbb{R}^d$. We thus obtain a smooth atlas on S.

(b) To see that i_S is smooth, let $p \in S$ and (φ, U) be a chart satisfying (7.12). Then

$$\varphi \circ i_S \circ (\varphi|_{S\cap U})^{-1}\colon \varphi(U) \cap (\mathbb{R}^d \times \{0\}) \to \varphi(U) \subseteq \mathbb{R}^n$$

is the inclusion map, hence smooth. This implies that i_S is smooth.

(c) If $f\colon N \to S$ is smooth, then the composition $i_S \circ f$ is smooth (Lemma 7.2.21). Suppose, conversely, that $i_S \circ f$ is smooth. Let $p \in N$ and choose a chart (φ, U) of M satisfying (7.12) with $f(p) \in U$. Then the map

$$\varphi \circ i_S \circ f|_{f^{-1}(U)} \colon f^{-1}(U) \to \varphi(U) \subseteq \mathbb{R}^n$$

is smooth, but its values lie in

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}).$$

Therefore $\varphi \circ i_S \circ f|_{f^{-1}(U)}$ is also smooth as a map into $\mathbb{R}^d$, which means that

$$\varphi|_{U \cap S} \circ f|_{f^{-1}(U)} \colon f^{-1}(U) \to \varphi(U \cap S) \subseteq \mathbb{R}^d$$

is smooth, and hence that f is smooth as a map $N \to S$. □

Remark 7.6.6 (Tangent spaces of submanifolds). From the construction of the manifold structure on S, it follows that for each $p \in S$ and each chart (φ, U) satisfying (7.12), we may identify the tangent space $T_p(S)$ with the subspace $T_p(\varphi)^{-1}(\mathbb{R}^d)$ mapped by $T_p(\varphi)$ onto the subspace $\mathbb{R}^d$ of $\mathbb{R}^n$.

Lemma 7.6.7. *A subset S of a smooth manifold M carries at most one initial submanifold structure, i.e., for any two smooth manifold structures S_1, S_2 on the same set S the map $\varphi = \mathrm{id}_S \colon S_1 \to S_2$ is a diffeomorphism.*

Proof. Since $i_{S_2} \circ \varphi = i_{S_1} \colon S_1 \to M$ is smooth, the map φ is smooth. Likewise, we see that the inverse map $\varphi^{-1} \colon S_2 \to S_1$ is smooth, showing that φ is a diffeomorphism $S_1 \to S_2$. □

Definition 7.6.8. Let $f \colon M \to N$ be a smooth map. We call $n \in N$ *regular value of* f if for each $x \in f^{-1}(n)$ the tangent map $T_x(f) \colon T_x(M) \to T_n(N)$ is surjective. Otherwise n is called a *singular value* of f. Note that, in particular, each $n \in N \setminus f(M)$ is a regular value.

We are now ready to prove a manifold version of the fact that inverse images of regular values are submanifolds.

Theorem 7.6.9 (Regular Value Theorem–Global Version). *Let M and N be smooth manifolds of dimensions m, resp., n, and $f \colon M \to N$ a smooth map. If $y \in N$ is a regular value of f, then $S := f^{-1}(y)$ is a submanifold of M of dimension $(m - n)$.*

Proof. We may assume that $y \in f(M)$ and note that then $d := m - n \geq 0$ because $T_x(f) \colon T_x(M) \cong \mathbb{R}^m \to \mathbb{R}^n \cong T_{f(x)}(N)$ is surjective for some $x \in M$.

Let $p \in S$ and choose charts (φ, U) of M with $p \in U$ and (ψ, V) of N with $f(p) \in V$. Then the map

$$F := \psi \circ f \circ \varphi^{-1} \colon \varphi(U \cap f^{-1}(V)) \to \mathbb{R}^n$$

is a smooth map, and for each $x \in F^{-1}(\psi(y)) = \varphi(S \cap U)$ the linear map

$$T_x(F) = T_{f \circ \varphi^{-1}(x)}(\psi) \circ T_{\varphi^{-1}(x)}(f) \circ T_x(\varphi^{-1}) \colon \mathbb{R}^m \to \mathbb{R}^n$$

is surjective. Therefore Proposition 7.2.9 implies the existence of an open subset $U' \subseteq \varphi(U \cap f^{-1}(V))$ containing $\varphi(p)$ and a diffeomorphism $\gamma\colon U' \to \gamma(U') \subseteq \mathbb{R}^m$ with

$$\gamma(U' \cap \varphi(U \cap S)) = (\mathbb{R}^d \times \{0\}) \cap \gamma(U').$$

Then $(\gamma \circ \varphi, \varphi^{-1}(U'))$ is a chart of M with

$$(\gamma \circ \varphi)(S \cap \varphi^{-1}(U')) = \gamma(\varphi(S \cap U) \cap U') = (\mathbb{R}^d \times \{0\}) \cap \gamma(U').$$

This shows that S is a d-dimensional submanifold of M. □

Remark 7.6.10. If $S \subseteq M$ is a submanifold, then we may identify the tangent spaces $T_p(S)$ with the subspaces $\operatorname{im}(T_p(i_S))$ of $T_p(M)$, where $i_S\colon S \to M$ is the smooth inclusion map (cf. Remark 7.6.6). If, in addition, $S = f^{-1}(y)$ for some regular value y of the smooth map $f\colon M \to N$, then we have

$$T_p(S) = \ker T_p(f) \quad \text{for} \quad p \in S.$$

To verify this relation, we recall that we know already that

$$\dim S = n - m = \dim T_p(S).$$

On the other hand, $f \circ i_S = y\colon S \to N$ is the constant map, so that $T_p(f \circ i_S) = T_p(f) \circ T_p(i_S) = 0$, which leads to $T_p(S) \subseteq \ker T_p(f)$. Since $T_p(f)$ is surjective by assumption, equality follows by comparing dimensions.

The following theorem which we mention without proof, implies in particular the existence of regular values for surjective smooth maps (cf. [La99, Thm. XVI.1.4]).

Theorem 7.6.11 (Sard's Theorem). *Let M_1 and M_2 be smooth second countable manifolds, $f\colon M_1 \to M_2$ a smooth map and M_1^c the set of critical points of f. Then $f(M_1^c)$ is a set of measure zero in M_2, i.e., for each chart (φ, U) of M_2 the set $\varphi(U \cap f(M_1^c))$ is of Lebesgue measure zero.*

Here we recall that a topological space X is said to be *second countable* if its topology has a countable basis, i.e., there exists a countable family $(O_n)_{n\in\mathbb{N}}$ of open subsets such that any open subset is the union of some of the O_n.

Notes on Chapter 7

The notion of a smooth manifold is more subtle than one may think on the surface. One of these subtleties arises from the fact that a topological space may carry different smooth manifold structures which are not diffeomorphic. Important examples of low dimension are the 7-sphere $\mathbb{S}^7$ and $\mathbb{R}^4$. Actually 4 is the only dimension n for which $\mathbb{R}^n$ carries two nondiffeomorphic smooth structures. At this point it is instructive to observe that two smooth structures might be diffeomorphic without having the same maximal atlas: The charts $(\varphi, \mathbb{R})$ and $(\psi, \mathbb{R})$ on $\mathbb{R}$ given by

$$\varphi(x) = x \quad \text{and} \quad \psi(x) = x^3$$

define two different smooth manifold structures $\mathbb{R}_\varphi$ and $\mathbb{R}_\psi$, but the map

$$\gamma\colon \mathbb{R}_\psi \to \mathbb{R}_\varphi, \quad x \mapsto x^3$$

is a diffeomorphism.

Later we shall see that there are also purely topological subtleties due to the fact that the topology might be "too large". The regularity assumption which is needed in many situations is the paracompactness of the underlying Hausdorff space.

Vector fields and their zeros play an important role in the topology of manifolds. To each manifold M we associate the maximal number $\alpha(M) = k$ for which there exist smooth vector fields $X_1, \ldots, X_k \in \mathcal{V}(M)$ which are linearly independent in each point of M. A manifold is called *parallelizable* if $\alpha(M) = \dim M$ (which is the maximal value). Clearly $\alpha(\mathbb{R}^n) = n$, so that $\mathbb{R}^n$ is parallelizable, but it is a deep theorem that the the n-sphere $\mathbb{S}^n$ is only parallelizable if $n = 0, 1, 3$ or 7. This in turn has important applications on the existence of real division algebras, namely that they only exist in dimensions 1, 2, 4 or 8 (this is the famous 1-2-4-8-Theorem). Another important result in topology is $\alpha(\mathbb{S}^2) = 0$, i.e., each vector field on the 2-sphere has a zero (Hairy Ball Theorem).

As we have seen in Theorem 7.5.16, the Lie bracket on the space $\mathcal{V}(M)$ of vector fields is closely related to the commutator in the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. This fact is part of a more general correspondence in the theory of Lie groups which associates to each Lie group G a Lie algebra $\mathbf{L}(G)$ given by a suitable bracket on the tangent space $T_{\mathbf{1}}(G)$ which is defined in terms of the Lie bracket of vector fields.

8

Basic Lie Theory

This chapter is devoted to the subject proper of this book: Lie groups, defined as smooth manifolds with a group structure such that all structure maps (multiplication and inversion) are smooth. Here we use vector fields to build the key tools of Lie theory. The Lie functor which associates a Lie algebra with a Lie group and the exponential function from the Lie algebra to the Lie group. They provide the means to translate global problems into infinitesimal ones and to lift infinitesimal solutions to local ones. To pass from the local to the global level, usually requires tools from covering theory, resp., topology. In the process we introduce smooth group actions and the adjoint representation, and provide a number of topological facts about Lie groups.

Further, we prove the Baker–Campbell–Dynkin–Hausdorff (BCDH) formula which expresses the group multiplication locally in terms of a universal power series involving Lie brackets. As a first set of applications of the translation mechanisms, we identify the Lie group structures of closed subgroups of Lie groups and show how to construct Lie groups from local and infinitesimal data. The key result is the Integral Subgroup Theorem 8.4.8 describing the subgroup generated by the exponential image of a Lie subalgebra as a Lie group. Combined with Ado's Theorem 6.4.1, it yields Lie's Third Theorem 8.4.11 saying that each finite-dimensional real Lie algebra is the Lie algebra of a Lie group. Then we systematically study coverings of Lie groups, thus providing the means to classify Lie groups with a given Lie algebra. Finally, we prove Yamabe's Theorem 8.6.1 asserting that any arcwise connected subgroup of a Lie group carries a natural Lie group structure and allows to equip *any* subgroup of a Lie group with a Lie group structure.

8.1 Lie Groups and their Lie Algebras

In the context of smooth manifolds, the natural class of groups are those endowed with a manifold structure compatible with the group structure. Such groups will be called Lie groups.

8.1.1 Lie Groups, First Examples and the Tangent Group

Definition 8.1.1. A *Lie group* is a group G, endowed with the structure of a smooth manifold, such that the group operations

$$m_G\colon G \times G \to G, \quad (x,y) \mapsto xy \quad \text{and} \quad \iota_G\colon G \to G, \quad x \mapsto x^{-1}$$

are smooth.

Throughout this section, G denotes a Lie group with multiplication map $m_G\colon G \times G \to G, (x,y) \mapsto xy$, inversion map $\iota_G\colon G \to G, x \mapsto x^{-1}$, and neutral element $\mathbf{1}$. For $g \in G$ we write $\lambda_g\colon G \to G, x \mapsto gx$ for the left multiplication map, $\rho_g\colon G \to G, x \mapsto xg$ for the right multiplication map, and $c_g\colon G \to G, x \mapsto gxg^{-1}$ for the conjugation with g. A *morphism of Lie groups* is a smooth homomorphism of Lie groups $\varphi\colon G_1 \to G_2$.

Remark 8.1.2. All maps λ_g, ρ_g and c_g are smooth. Moreover, they are bijective with $\lambda_{g^{-1}} = \lambda_g^{-1}$, $\rho_{g^{-1}} = \rho_g^{-1}$ and $c_{g^{-1}} = c_g^{-1}$, so that they are diffeomorphisms of G onto itself.

In addition, the maps c_g are automorphisms of G, so that we obtain a group homomorphism

$$C\colon G \to \mathrm{Aut}(G), \quad g \mapsto c_g,$$

where $\mathrm{Aut}(G)$ stands for the *group of automorphisms of the Lie group* G, i.e., the group automorphisms which are diffeomorphisms. The automorphisms of the form c_g are called *inner automorphisms* of G. The group of inner automorphisms of G is denoted by $\mathrm{Inn}(G)$.

One can show that the requirement of ι_G being smooth is redundant (Exercise 8.1.4).

Example 8.1.3. We consider the additive group $G := (\mathbb{R}^n, +)$, endowed with the natural n-dimensional manifold structure. A corresponding chart is given by $(\mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n)$, which shows that the corresponding product manifold structure on $\mathbb{R}^n \times \mathbb{R}^n$ is given by the chart $(\mathrm{id}_{\mathbb{R}^n} \times \mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n \times \mathbb{R}^n) = (\mathrm{id}_{\mathbb{R}^{2n}}, \mathbb{R}^{2n})$, hence coincides with the natural manifold structure on $\mathbb{R}^{2n}$. Therefore the smoothness of addition and inversion follows from the smoothness of the maps

$$\mathbb{R}^{2n} \to \mathbb{R}^n, \quad (x,y) \mapsto x+y \quad \text{and} \quad \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto -x.$$

Example 8.1.4. Next we consider the group $G := \mathrm{GL}_n(\mathbb{R})$ of invertible $(n \times n)$-matrices. If $\det\colon M_n(\mathbb{R}) \to \mathbb{R}$ denotes the determinant function

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

then det is a polynomial, hence in particular continuous, and therefore $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R}^\times)$ is an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Hence G carries a natural manifold structure.

We claim that G is a Lie group. The smoothness of the multiplication map follows directly from the smoothness of the bilinear multiplication map

$$M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad (A, B) \mapsto \Big(\sum_{k=1}^{n} a_{ik} b_{kj}\Big)_{i,j=1,\ldots,n},$$

which is given in each component by a polynomial function in the $2n^2$ variables a_{ij} and b_{ij} (cf. Exercise 7.2.11 on the smoothness of bilinear maps).

The smoothness of the inversion map follows from Cramer's Rule

$$g^{-1} = \frac{1}{\det g}(b_{ij}), \quad b_{ij} = (-1)^{i+j} \det(G_{ji}),$$

where $G_{ij} \in M_{n-1}(\mathbb{R})$ is the matrix obtained by erasing the i-th row and the j-th column in g.

Example 8.1.5. (a) (The circle group) We have already seen how to endow the circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$$

with a manifold structure (Example 7.2.5). Identifying it with the unit circle

$$\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$$

in $\mathbb{C}$, it also inherits a group structure, given by

$$(x, y) \cdot (x', y') := (xx' - yy', xy' + x'y) \quad \text{and} \quad (x, y)^{-1} = (x, -y).$$

With these explicit formulas, it is easy to verify that $\mathbb{T}$ is a Lie group (Exercise 8.1.1).

(b) (The n-dimensional torus) In view of (a), we have a natural manifold structure on the *n-dimensional torus* $\mathbb{T}^n := (\mathbb{S}^1)^n$. The corresponding direct product group structure

$$(t_1, \ldots, t_n)(s_1, \ldots, s_n) := (t_1 s_1, \ldots, t_n s_n)$$

turns $\mathbb{T}^n$ into a Lie group (Exercise 8.1.2).

Lemma 8.1.6. (a) *As usual, we identify $T(G \times G)$ with $T(G) \times T(G)$. Then the tangent map*

$$T(m_G) \colon T(G \times G) \cong T(G) \times T(G) \to T(G), \quad (v, w) \mapsto v \cdot w := Tm_G(v, w)$$

defines a Lie group structure on $T(G)$ with identity element $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$ and inversion $T(\iota_G)$. The canonical projection $\pi_{T(G)} \colon T(G) \to G$ is a morphism of Lie groups with kernel $(T_{\mathbf{1}}(G), +)$ and the zero section $\sigma \colon G \to T(G)$, $g \mapsto 0_g \in T_g(G)$ is a homomorphism of Lie groups with $\pi_{T(G)} \circ \sigma = \mathrm{id}_G$.

(b) *The map*

$$\Phi \colon G \times T_{\mathbf{1}}(G) \to T(G), \quad (g, x) \mapsto g.x := 0_g \cdot x = T(\lambda_g)x$$

is a diffeomorphism.

Proof. (a) Since the multiplication map $m_G\colon G\times G\to G$ is smooth, the same holds for its tangent map

$$Tm_G\colon T(G\times G)\cong T(G)\times T(G)\to T(G).$$

Let $\varepsilon_G\colon G\to G, g\mapsto \mathbf{1}$ be the constant homomorphism. Then the group axioms for G are encoded in the relations

(1) $m_G\circ(m_G\times \mathrm{id}_G)=m_G\circ(\mathrm{id}_G\times m_G)$ (associativity),
(2) $m_G\circ(\iota_G,\mathrm{id}_G)=m_G\circ(\mathrm{id}_G,\iota_G)=\varepsilon_G$ (inversion), and
(3) $m_G\circ(\varepsilon_G,\mathrm{id}_G)=m_G\circ(\mathrm{id}_G,\varepsilon_G)=\mathrm{id}_G$ (unit element).

Using the functoriality (cf. Lemma 7.3.5) of T and its compatibility with products, we see that these properties carry over to the corresponding maps on $T(G)$:

(1) $T(m_G)\circ T(m_G\times \mathrm{id}_G)=T(m_G)\circ(T(m_G)\times \mathrm{id}_{T(G)})$
$=T(m_G)\circ(\mathrm{id}_{T(G)}\times T(m_G))$ (associativity),
(2) $T(m_G)\circ(T(\iota_G),\mathrm{id}_{T(G)})=T(m_G)\circ(\mathrm{id}_{T(G)},T(\iota_G))=T(\varepsilon_G)$ (inversion), and
(3) $T(m_G)\circ(T(\varepsilon_G),\mathrm{id}_{T(G)})=T(m_G)\circ(\mathrm{id}_{T(G)},T(\varepsilon_G))=\mathrm{id}_{T(G)}$ (unit element).

Here we only have to observe that the tangent map $T(\varepsilon_G)$ maps each $v\in T(G)$ to $0_{\mathbf{1}}\in T_{\mathbf{1}}(G)$, which is the neutral element of $T(G)$. We conclude that $T(G)$ is a Lie group with multiplication $T(m_G)$, inversion $T(\iota_G)$, and unit element $0_{\mathbf{1}}\in T_{\mathbf{1}}(G)$.

The definition of the tangent map implies that the zero section $\sigma\colon G\to T(G)$ satisfies

$$Tm_G\circ(\sigma\times\sigma)=\sigma\circ m_G,\quad Tm_G(0_g,0_h)=0_{m_G(g,h)}=0_{gh},$$

which means that it is a morphism of Lie groups. That $\pi_{T(G)}$ also is a morphism of Lie groups follows likewise from the relation

$$\pi_{T(G)}\circ Tm_G=m_G\circ(\pi_{T(G)}\times\pi_{T(G)}),$$

which also is an immediate consequence of the definition of the tangent map T_{m_G}: it maps $T_g(G)\times T_h(G)$ into $T_{gh}(G)$.

For $v\in T_g(G)$ and $w\in T_h(G)$ the linearity of $T_{(g,h)}(m_G)$ implies that

$$\begin{aligned}Tm_G(v,w)=T_{(g,h)}(m_G)(v,w)&=T_{(g,h)}(m_G)(v,0)+T_{(g,h)}(m_G)(0,w)\\&=T_g(\rho_h)v+T_h(\lambda_g)w,\end{aligned}$$

and in particular $T_{(\mathbf{1},\mathbf{1})}(m_G)(v,w)=v+w$, so that the multiplication on the normal subgroup $\ker\pi_{T(G)}=T_{\mathbf{1}}(G)$ is simply given by addition.

(b) The smoothness of Φ follows from the smoothness of the multiplication of $T(G)$ and the smoothness of the zero section $\sigma\colon G\to T(G), g\mapsto 0_g$. That Φ is a diffeomorphism follows from the following explicit formula for its inverse: $\Phi^{-1}(v)=(\pi_{T(G)}(v),\pi_{T(G)}(v)^{-1}.v)$, so that its smoothness follows from the smoothness of $\pi_{T(G)}$ (its first component), and the smoothness of the multiplication on $T(G)$. □

8.1.2 The Lie Functor

The Lie functor assigns a Lie algebra to each Lie group and a Lie algebra homomorphism to each morphism of Lie groups. It is the key tool to translate Lie group problems into problems in linear algebra.

Definition 8.1.7 (The Lie algebra of G). A vector field $X \in \mathcal{V}(G)$ is called *left invariant* if

$$X = (\lambda_g)_* X := T(\lambda_g) \circ X \circ \lambda_g^{-1}$$

holds for each $g \in G$, i.e., $(\lambda_g)_* X = X$. We write $\mathcal{V}(G)^l$ for the set of left invariant vector fields in $\mathcal{V}(G)$. Clearly $\mathcal{V}(G)^l$ is a linear subspace of $\mathcal{V}(G)$.

Writing the left invariance as $X = T(\lambda_g) \circ X \circ \lambda_g^{-1}$, we see that it means that X is λ_g-related to itself (cf. Exercise 8.1.6). Therefore the Related Vector Field Lemma 7.4.9 implies that if X and Y are left-invariant, their Lie bracket $[X, Y]$ is also λ_g-related to itself for each $g \in G$, hence left invariant. We conclude that the vector space $\mathcal{V}(G)^l$ is a Lie subalgebra of $(\mathcal{V}(G), [\cdot,\cdot])$.

Next we observe that the left invariance of a vector field X implies that for each $g \in G$ we have $X(g) = g.X(\mathbf{1})$ (Lemma 8.1.6(b)), so that X is completely determined by its value $X(\mathbf{1}) \in T_{\mathbf{1}}(G)$. Conversely, for each $x \in T_{\mathbf{1}}(G)$, we obtain a left invariant vector field $x_l \in \mathcal{V}(G)^l$ with $x_l(\mathbf{1}) = x$ by $x_l(g) := g.x$. That this vector field is indeed left invariant follows from

$$x_l \circ \lambda_h(g) = x_l(hg) = (hg).x = h.(g.x) = T(\lambda_h)x_l(g)$$

for all $h, g \in G$. Hence

$$T_{\mathbf{1}}(G) \to \mathcal{V}(G)^l, \quad x \mapsto x_l$$

is a linear bijection. We thus obtain a Lie bracket $[\cdot,\cdot]$ on $T_{\mathbf{1}}(G)$ satisfying

$$[x, y]_l = [x_l, y_l] \quad \text{for all} \quad x, y \in T_{\mathbf{1}}(G). \tag{8.1}$$

The Lie algebra

$$\mathbf{L}(G) := (T_{\mathbf{1}}(G), [\cdot,\cdot]) \cong \mathcal{V}(G)^l$$

is called *the Lie algebra of G*.

Proposition 8.1.8 (Functoriality of the Lie algebra). *If $\varphi \colon G \to H$ is a morphism of Lie groups, then the tangent map*

$$\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi) \colon \mathbf{L}(G) \to \mathbf{L}(H)$$

is a homomorphism of Lie algebras.

Proof. Let $x, y \in \mathbf{L}(G)$ and x_l, y_l be the corresponding left invariant vector fields. Then $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ for each $g \in G$ implies that

$$T(\varphi) \circ T(\lambda_g) = T(\lambda_{\varphi(g)}) \circ T(\varphi),$$

and applying this relation to $x, y \in T_{\mathbf{1}}(G)$, we get

$$T\varphi \circ x_l = \big(\mathbf{L}(\varphi)x\big)_l \circ \varphi \quad \text{and} \quad T\varphi \circ y_l = \big(\mathbf{L}(\varphi)y\big)_l \circ \varphi, \tag{8.2}$$

i.e., x_l is φ-related to $\big(\mathbf{L}(\varphi)x\big)_l$ and y_l is φ-related to $\big(\mathbf{L}(\varphi)y\big)_l$. Therefore the Related Vector Field Lemma implies that

$$T\varphi \circ [x_l, y_l] = [\big(\mathbf{L}(\varphi)x\big)_l, \big(\mathbf{L}(\varphi)y\big)_l] \circ \varphi.$$

Evaluating at $\mathbf{1}$, we obtain $\mathbf{L}(\varphi)[x, y] = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$, showing that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. □

Remark 8.1.9. We obviously have $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$, and for two morphisms $\varphi_1 \colon G_1 \to G_2$ and $\varphi_2 \colon G_2 \to G_3$ of Lie groups, we obtain

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1),$$

from the Chain Rule:

$$T_{\mathbf{1}}(\varphi_2 \circ \varphi_1) = T_{\varphi_1(\mathbf{1})}(\varphi_2) \circ T_{\mathbf{1}}(\varphi_1) = T_{\mathbf{1}}(\varphi_2) \circ T_{\mathbf{1}}(\varphi_1).$$

The preceding lemma implies that the assignments $G \mapsto \mathbf{L}(G)$ and $\varphi \mapsto \mathbf{L}(\varphi)$ define a functor, called the *Lie functor*,

$$\mathbf{L} \colon \underline{LieGrp} \to \underline{LieAlg}$$

from the category $\underline{LieGrp}$ of Lie groups to the category $\underline{LieAlg}$ of (finite-dimensional) Lie algebras.

Corollary 8.1.10. *For each isomorphism of Lie groups $\varphi \colon G \to H$, the map $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras, and for each $x \in \mathbf{L}(G)$, the following equation holds*

$$\varphi_* x_l := T(\varphi) \circ x_l \circ \varphi^{-1} = \big(\mathbf{L}(\varphi)x\big)_l. \tag{8.3}$$

Proof. Let $\psi \colon H \to G$ be the inverse of φ. Then $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$ leads to $\mathbf{L}(\varphi) \circ \mathbf{L}(\psi) = \mathrm{id}_{\mathbf{L}(H)}$ and $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi) = \mathrm{id}_{\mathbf{L}(G)}$ (Remark 8.1.9). Further (8.3) follows from (8.2) in the proof of Proposition 8.1.8. □

8.1.3 Smooth Actions of Lie Groups

We already encountered smooth flows on manifolds in Chapter 7. These can be viewed as actions of the one-dimensional Lie group $(\mathbb{R}, +)$. In particular, we have seen that these actions are in one-to-one correspondence with complete vector fields, which is the corresponding Lie algebra picture. Now we describe the corresponding concept for general Lie groups.

Definition 8.1.11. Let M be a smooth manifold and G a Lie group. A *(smooth) action of* G *on* M is a smooth map

$$\sigma\colon G \times M \to M$$

with the following properties:

(A1) $\sigma(\mathbf{1}, m) = m$ for all $m \in M$.
(A2) $\sigma(g_1, \sigma(g_2, m)) = \sigma(g_1 g_2, m)$ for $g_1, g_2 \in G$ and $m \in M$.

We also write

$$g.m := \sigma(g, m), \quad \sigma_g(m) := \sigma(g, m), \quad \sigma^m(g) := \sigma(g, m) = g.m.$$

The map σ^m is called the *orbit map*.

For each smooth action σ, the map

$$\widehat{\sigma}\colon G \to \mathrm{Diff}(M), \quad g \mapsto \sigma_g$$

is a group homomorphism and any homomorphism $\gamma\colon G \to \mathrm{Diff}(M)$ for which the map

$$\sigma_\gamma\colon G \times M \to M, \quad (g, m) \mapsto \gamma(g)(m)$$

is smooth defines a smooth action of G on M.

Remark 8.1.12. What we call an action is sometimes called a *left action*. Likewise one defines a *right action* as a smooth map $\sigma_R\colon M \times G \to M$ with

$$\sigma_R(m, \mathbf{1}) = m, \quad \sigma_R(\sigma_R(m, g_1), g_2) = \sigma_R(m, g_1 g_2).$$

For $m.g := \sigma_R(m, g)$, this takes the form

$$m.(g_1 g_2) = (m.g_1).g_2$$

of an associativity condition.

If σ_R is a smooth right action of G on M, then

$$\sigma_L(g, m) := \sigma_R(m, g^{-1})$$

defines a smooth left action of G on M. Conversely, if σ_L is a smooth left action, then

$$\sigma_R(m, g) := \sigma_L(g^{-1}, m)$$

defines a smooth left action. This translation is one-to-one, so that we may freely pass from one type of action to the other.

Examples 8.1.13. (a) If $X \in \mathcal{V}(M)$ is a complete vector field (cf. Definition 7.5.7) and $\Phi\colon \mathbb{R} \times M \to M$ its global flow, then Φ defines a smooth action of $G = (\mathbb{R}, +)$ on M.
(b) If G is a Lie group, then the multiplication map $\sigma := m_G\colon G \times G \to G$ defines a smooth left action of G on itself. In this case the $(m_G)_g = \lambda_g$ are the left multiplications.

The multiplication map also defines a smooth right action of G on itself. The corresponding left action is

$$\sigma\colon G \times G \to G, \quad (g,h) \mapsto hg^{-1} \quad \text{with} \quad \sigma_g = \rho_g^{-1}.$$

There is a third action of G on itself, the *conjugation action*:

$$\sigma\colon G \times G \to G, \quad (g,h) \mapsto ghg^{-1} \quad \text{with} \quad \sigma_g = c_g.$$

(c) We have a natural smooth action of the Lie group $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$:

$$\sigma\colon \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma(g,x) := gx.$$

We further have an action of $\mathrm{GL}_n(\mathbb{R})$ on $M_n(\mathbb{R})$:

$$\sigma\colon \mathrm{GL}_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad \sigma(g,A) = gAg^{-1}.$$

(d) On the set $M_{p,q}(\mathbb{R})$ of $(p \times q)$-matrices we have an action of the direct product Lie group $G := \mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R})$ by $\sigma((g,h),A) := gAh^{-1}$.

The following proposition generalizes the passage from flows of vector fields to actions of general Lie groups.

Proposition 8.1.14. *Let G be a Lie group and $\sigma\colon G \times M \to M$ a smooth action of G on M. Then the assignment*

$$\dot\sigma\colon \mathbf{L}(G) \to \mathcal{V}(M), \qquad \dot\sigma(x)(m) := \mathbf{L}(\sigma)(x)(m) := -T_{\mathbf{1}}(\sigma^m)(x)$$

is a homomorphism of Lie algebras.

Proof. First we observe that for each $x \in \mathbf{L}(G)$ the map $\mathbf{L}(\sigma)(x)$ defines a smooth map $M \to T(M)$, and since $\mathbf{L}(\sigma)(x)(m) \in T_{\sigma(\mathbf{1},m)}(M) = T_m(M)$, it is a smooth vector field on M.

To see that $\dot\sigma$ is a homomorphism of Lie algebras, we pick $m \in M$ and write

$$\varphi^m := \sigma^m \circ \iota_G\colon G \to M, \quad g \mapsto g^{-1}.m$$

for the reversed orbit map. Then

$$\varphi^m(gh) = (gh)^{-1}.m = h^{-1}.(g^{-1}.m) = \varphi^{g^{-1}.m}(h),$$

which can be written as

$$\varphi^m \circ \lambda_g = \varphi^{g^{-1}.m}.$$

Taking the differential in $\mathbf{1} \in G$, we obtain for each $x \in \mathbf{L}(G) = T_{\mathbf{1}}(G)$:

$$\begin{aligned} T_g(\varphi^m)x_l(g) &= T_g(\varphi^m)T_{\mathbf{1}}(\lambda_g)x = T_{\mathbf{1}}(\varphi^m \circ \lambda_g)x = T_{\mathbf{1}}(\varphi^{g^{-1}.m})x \\ &= T_{\mathbf{1}}(\sigma^{g^{-1}.m})T_{\mathbf{1}}(\iota_G)x = -T_{\mathbf{1}}(\sigma^{\varphi^m(g)})x = \dot{\sigma}(x)(\varphi^m(g)). \end{aligned}$$

This means that the left invariant vector field x_l on G is φ^m-related to the vector field $\mathbf{L}(\sigma)(x)$ on M. Therefore the Related Vector Field Lemma 7.4.9 implies that for $x, y \in \mathbf{L}(G)$ the vector field $[x_l, y_l]$ is φ^m-related to $[\mathbf{L}(\sigma)(x), \mathbf{L}(\sigma)(y)]$, which leads for each $m \in M$ to

$$\begin{aligned} \mathbf{L}(\sigma)([x,y])(m) &= T_{\mathbf{1}}(\varphi^m)[x,y]_l(\mathbf{1}) = T_{\mathbf{1}}(\varphi^m)[x_l, y_l](\mathbf{1}) \\ &= [\mathbf{L}(\sigma)(x), \mathbf{L}(\sigma)(y)](\varphi^m(\mathbf{1})) = [\mathbf{L}(\sigma)(x), \mathbf{L}(\sigma)(y)](m). \quad \square \end{aligned}$$

8.1.4 Basic Topology of Lie Groups

In this subsection we collect some basic topological properties of Lie groups.

Proposition 8.1.15. *The topology of a Lie group G has the following properties:*

(i) *G is a locally compact space, i.e., each neighborhood of an element of g contains a compact one.*
(ii) *The identity component G_0 of G is an open normal subgroup which coincides with the arc-component of $\mathbf{1}$.*
(iii) *For a subgroup H of G the following are equivalent:*
  (a) *H is a neighborhood of $\mathbf{1}$.*
  (b) *H is open.*
  (c) *H is open and closed.*
  (d) *H contains G_0.*
(iv) *If the set $\pi_0(G) := G/G_0$ of connected components of G is countable, then, in addition, the following statements hold:*
  (a) *G is countable at infinity, i.e., a countable union of compact subsets.*
  (b) *For each $\mathbf{1}$-neighborhood U in G there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G with $G = \bigcup_{n\in\mathbb{N}} g_n U$.*
  (c) *G is second countable, i.e., the topology of G has a countable basis.*
  (d) *If $(U_i)_{i\in I}$ is a pairwise disjoint collection of open subsets of G, then I is countable.*

Proof. (i) This is true for any smooth n-dimensional manifold M. If $m \in M$, V is a neighborhood of m and (φ, U) is a chart with $m \in M$, then $\varphi(U \cap V)$ is a neighborhood of $\varphi(m)$ in $\mathbb{R}^n$. If $B \subseteq \varphi(U \cap V)$ is a closed ball around $\varphi(m)$, which is compact due to the Heine–Borel Theorem, its inverse image $\varphi^{-1}(B)$ is a compact neighborhood of m, contained in V. Here we use that M is Hausdorff to see that $\varphi^{-1}(B)$ is compact.

(ii) Since G is a smooth manifold, each point has an open neighborhood U homeomorphic to an open ball in some $\mathbb{R}^n$. Then U is in particular arcwise connected. This implies that the arc-components of G are open, hence that they coincide with the connected components.

To see that the identity component G_0 of G is a subgroup, we first note that G_0G_0 is the image of the connected set $G_0 \times G_0$ under the multiplication map, hence connected. Since it contains $\mathbf{1}$, we find $G_0G_0 \subseteq G_0$. Similarly, we see that the inversion preserves G_0, i.e., $G_0^{-1} \subseteq G_0$, showing that G_0 is a subgroup of G. Each conjugation $c_g(x) := gxg^{-1}$ fixes the identity element $\mathbf{1}$, hence maps the identity component G_0 into itself. Thus G_0 is normal.

(iii) (a) $\Rightarrow$ (b): If H is a neighborhood of $\mathbf{1}$, then each coset gH is a neighborhood of g because the left multiplication maps $\lambda_g \colon G \to G$ are homeomorphism. Hence all left cosets of H are open. In particular, H is open.

(b) $\Rightarrow$ (c): If H is an open subgroup, then its complement is the union of all cosets gH, $g \notin H$, hence also open. Therefore H is also closed.

(c) $\Rightarrow$ (d): If H is open and closed, then the connectedness of G_0 implies $G_0 \subseteq H$.

(d) $\Rightarrow$ (a) is trivial.

(iv) (a) In view of (i), there exists a compact identity neighborhood U in G. Replacing U by $U \cap U^{-1}$, we may w.l.o.g. assume that $U = U^{-1}$. Then each set

$$U^n := \{u_1 \cdots u_n \colon u_i \in U\}$$

is also compact, because it is the image of the compact topological product space $U^{\times n}$ under the n-fold multiplication map which is continuous.

Now $H := \bigcup_{n\in\mathbb{N}} U^n$ is a subgroup of G which is a $\mathbf{1}$-neighborhood, and (iii) implies $G_0 \subseteq H$. Hence the set of H-cosets is countable, and since each coset gH is a union of the countably many compact subsets gU^n, we see that G also is a countable union of compact subsets.

(b) In view of (a), we have $G = \bigcup_{n\in\mathbb{N}} K_n$, where each K_n is a compact subset. For each n, the open sets kU°, $k \in K_n$, cover the compact set K_n, so that there exist finitely many $k_{n,1}, \ldots, k_{n,m_n}$ with $K_n \subseteq \bigcup_{j=1}^{m_n} k_{n,j}U$. Then $G \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{m_n} k_{n,j}U$.

(c) Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of open $\mathbf{1}$-neighborhoods, we may take $U_n = \varphi(\frac{1}{n}B)$, where $B \subseteq \mathbf{L}(G)$ is an open ball with respect to some norm and $\varphi\colon B \to G$ is a diffeomorphism onto an open subset of G with $\varphi(0) = \mathbf{1}$. In view of (b), there exists for each $n \in \mathbb{N}$ a sequence $(g_{n,k})_{k\in\mathbb{N}}$ in G with $G = \bigcup_{k\in\mathbb{N}} g_{n,k}U_n$. We claim that $\{g_{n,k}U_n \colon n, k \in \mathbb{N}\}$ is a basis for the topology of G. In fact, if $O \subseteq G$ is an open subset and $g \in O$, then there exists some n with $gU_n \subseteq O$. Next we pick m such that $U_m^{-1}U_m \subseteq U_n$ and some $k \in \mathbb{N}$ with $g \in g_{m,k}U_m$. Then $g_{m,k}U_m \subseteq gU_m^{-1}U_m \subseteq gU_n \subseteq O$, and this proves our claim.

(d) follows immediately from (c). □

Exercises for Section 8.1

Exercise 8.1.1. Show that the natural group structure on $\mathbb{T} \cong \mathbb{S}^1 \subseteq \mathbb{C}^\times$ turns it into a Lie group.

Exercise 8.1.2. Let $G_1, \ldots, G_n$ be Lie groups and $G := G_1 \times \ldots \times G_n$, endowed with the direct product group structure

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') := (g_1 g_1', \ldots, g_n g_n')$$

and the product manifold structure. Show that G is a Lie group with

$$\mathbf{L}(G) \cong \mathbf{L}(G_1) \times \ldots \times \mathbf{L}(G_n).$$

Exercise 8.1.3. Let V and W be finite-dimensional real vector spaces and $\beta \colon V \times V \to W$ a bilinear map. Show that $G := W \times V$ is a Lie group with respect to

$$(w, v)(w', v') := (w + w' + \beta(v, v'), v + v').$$

For $(w, v) \in \mathbf{L}(G) \cong T_{(0,0)}(G)$, find a formula for the corresponding left invariant vector field $(w, v)_l$, considered as a smooth function $G \to W \times V$.

Exercise 8.1.4 (Automatic smoothness of the inversion). Let G be an n-dimensional smooth manifold, endowed with a group structure for which the multiplication map m_G is smooth. Show that:

(1) $T_{(g,h)}(m_G) = T_g(\rho_h) + T_h(\lambda_g)$ for $\lambda_g(x) = gx$ and $\rho_h(x) = xh$.
(2) $T_{(\mathbf{1},\mathbf{1})}(m_G)(v, w) = v + w$.
(3) The inverse map $\iota_G \colon G \to G, g \mapsto g^{-1}$ is smooth if it is smooth in a neighborhood of $\mathbf{1}$.
(4) The inverse map ι_G is smooth.

Exercise 8.1.5. Let A be a finite-dimensional unital real algebra and $A^\times$ its group of units. We write $\lambda_a(b) := ab$ for the left multiplication with $a \in A$. Show that:

(1) $A^\times = \{a \in A \colon \det(\lambda_a) \neq 0\}$.
(2) $A^\times$ is an open subset of A and with respect to the corresponding manifold structure it is a Lie group.
(3) Identifying vector fields on the open subset $A^\times$ with smooth A-valued functions, a vector field $X \in \mathcal{V}(A^\times) \cong C^\infty(A^\times, A)$ is left invariant if and only if there exists an element $x \in A$ with $X(a) = ax$ for $a \in A^\times$.

Exercise 8.1.6. Let G be a Lie group and X a vector field on G, viewed as a derivation of $C^\infty(G)$. Show that X is left invariant if and only if

$$(\mathrm{id} \otimes X) \circ m_G^* = m_G^* \circ X,$$

where $m_G^* f(g, h) = f(gh)$ and $(\mathrm{id} \otimes X)F(g, h) = (XF_g)(h)$ for $f \in C^\infty(G)$, $F \in C^\infty(G \times G)$ and $F_g(h) = F(g, h)$.

Exercise 8.1.7. Let $G = \mathbb{R}^n$. Show that the vector fields $X_i := \frac{\partial}{\partial x_i}$ form a basis for the space of (left) invariant vector fields.

Exercise 8.1.8. Consider the three-dimensional Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Determine the space of (left) invariant vector fields in the coordinates (x, y, z).

8.2 The Exponential Function of a Lie Group

In the preceding section we have introduced the Lie functor which assigns to a Lie group G its Lie algebra $\mathbf{L}(G)$ and to a morphism φ of Lie groups its tangent morphism $\mathbf{L}(\varphi)$ of Lie algebras. In this section, we introduce a key tool of Lie theory which will allow us to also go in the opposite direction: the exponential function $\exp_G\colon \mathbf{L}(G) \to G$. It is a natural generalization of the matrix exponential map, which is obtained for $G = \mathrm{GL}_n(\mathbb{R})$ and its Lie algebra $\mathbf{L}(G) = \mathfrak{gl}_n(\mathbb{R})$. We conclude this section with a discussion of the naturality of the exponential function (Proposition 8.2.10) and the Lie group versions of the Trotter Product Formula and the Commutator Formula.

8.2.1 Basic Properties of the Exponential Function

Proposition 8.2.1. *Each left invariant vector field X on G is complete.*

Proof. Let $g \in G$ and $\gamma\colon I \to G$ be the unique maximal integral curve (cf. Theorem 7.5.5) of $X \in \mathcal{V}(G)^l$ with $\gamma(0) = g$.

For each $h \in G$ we have $(\lambda_h)_* X = X$, which implies that $\eta := \lambda_h \circ \gamma$ also is an integral curve of X. Put $h = \gamma(s)g^{-1}$ for some $s > 0$. Then

$$\eta(0) = (\lambda_h \circ \gamma)(0) = h\gamma(0) = hg = \gamma(s),$$

and the uniqueness of integral curves implies that $\gamma(t + s) = \eta(t)$ for all t in the interval $I \cap (I - s)$ which is nonempty because it contains 0. In view of the maximality of I, it now follows that $I - s \subseteq I$, and hence that $I - ns \subseteq I$ for each $n \in \mathbb{N}$, so that the interval I is unbounded from below. Applying the same argument to some $s < 0$, we see that I is also unbounded from above. Hence $I = \mathbb{R}$, which means that X is complete. □

Definition 8.2.2. We now define the *exponential function*

$$\exp_G\colon \mathbf{L}(G) \to G, \quad \exp_G(x) := \gamma_x(1),$$

where $\gamma_x\colon \mathbb{R} \to G$ is the unique maximal integral curve of the left invariant vector field x_l, satisfying $\gamma_x(0) = \mathbf{1}$. This means that γ_x is the unique solution of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = x_l(\gamma(t)) = \gamma(t).x \quad \text{for all} \quad t \in \mathbb{R}.$$

Example 8.2.3. (a) Let $G := (V, +)$ be the additive group of a finite-dimensional vector space. The left invariant vector fields on V are given by

$$x_l(w) := \frac{d}{dt}\Big|_{t=0} w + tx = x,$$

so that they are simply the constant vector fields. Hence (cf. Lemma 7.4.3)

$$[x_l, y_l](0) = \mathtt{d}y_l(x_l(0)) - \mathtt{d}x_l(y_l(0)) = \mathtt{d}y_l(x) - \mathtt{d}x_l(y) = 0.$$

Therefore $\mathbf{L}(V)$ is an abelian Lie algebra.

For each $x \in V$, the flow of x_l is given by $\Phi^{x_l}(t, v) = v + tx$, so that

$$\exp_V(x) = \Phi^{x_l}(1, 0) = x, \quad \text{i.e.,} \quad \exp_V = \mathrm{id}_V.$$

(b) Now let $G := \mathrm{GL}_n(\mathbb{R})$ be the Lie group of invertible $(n \times n)$-matrices, which inherits its manifold structure from the embedding as an open subset of the vector space $M_n(\mathbb{R})$.

The left invariant vector field A_l corresponding to a matrix A is given by

$$A_l(g) = T_{\mathbf{1}}(\lambda_g)A = gA$$

because $\lambda_g(h) = gh$ extends to a linear endomorphism of $M_n(\mathbb{R})$. The unique solution $\gamma_A \colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = A_l(\gamma(t)) = \gamma(t)A$$

is the curve describing the fundamental system of the linear differential equation defined by the matrix A:

$$\gamma_A(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k.$$

It follows that $\exp_G(A) = e^A$ is the matrix exponential function.

The Lie algebra $\mathbf{L}(G)$ of G is determined from

$$\begin{aligned}[A, B] = [A_l, B_l](\mathbf{1}) &= \mathtt{d}B_l(\mathbf{1})A_l(\mathbf{1}) - \mathtt{d}A_l(\mathbf{1})B_l(\mathbf{1}) \\ &= \mathtt{d}B_l(\mathbf{1})A - \mathtt{d}A_l(\mathbf{1})B = AB - BA.\end{aligned}$$

Therefore the Lie bracket on $\mathbf{L}(G) = T_{\mathbf{1}}(G) \cong M_n(\mathbb{R})$ is given by the commutator bracket. This Lie algebra is denoted $\mathfrak{gl}_n(\mathbb{R})$, to express that it is the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$.

(c) If V is a finite-dimensional real vector space, then $V \cong \mathbb{R}^n$, so that we can immediately use (b) to see that $\mathrm{GL}(V)$ is a Lie group with Lie algebra $\mathfrak{gl}(V) := (\mathrm{End}(V), [\cdot,\cdot])$ and exponential function

$$\exp_{\mathrm{GL}(V)}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Lemma 8.2.4. (a) *For each* $x \in \mathbf{L}(G)$*, the curve* $\gamma_x \colon \mathbb{R} \to G$ *is a smooth homomorphism of Lie groups with* $\gamma_x'(0) = x$*.*

(b) *The global flow of the left invariant vector field* x_l *is given by*

$$\Phi(t, g) = g\gamma_x(t) = g \exp_G(tx).$$

(c) *If* $\gamma \colon \mathbb{R} \to G$ *is a smooth homomorphism of Lie groups and* $x := \gamma'(0)$*, then* $\gamma = \gamma_x$*. In particular, the map*

$$\mathrm{Hom}(\mathbb{R}, G) \to \mathbf{L}(G), \quad \gamma \mapsto \gamma'(0)$$

is a bijection, where $\mathrm{Hom}(\mathbb{R}, G)$ *stands for the set of morphisms, i.e., smooth homomorphisms, of Lie groups* $\mathbb{R} \to G$*.*

Proof. (a), (b) Since γ_x is an integral curve of the smooth vector field x_l, it is a smooth curve. Hence the smoothness of the multiplication in G implies that $\Phi(t, g) := g\gamma_x(t)$ defines a smooth map $\mathbb{R} \times G \to G$. In view of the left invariance of x_l, we have for each $g \in G$ and $\Phi^g(t) := \Phi(t, g)$ the relation

$$(\Phi^g)'(t) = T(\lambda_g)\gamma_x'(t) = T(\lambda_g)x_l(\gamma_x(t)) = x_l(g\gamma_x(t)) = x_l(\Phi^g(t)).$$

Therefore Φ^g is an integral curve of x_l with $\Phi^g(0) = g$, and this proves that Φ is the unique maximal flow of the complete vector field x_l.

In particular, we obtain for $t, s \in \mathbb{R}$:

$$\gamma_x(t + s) = \Phi(t + s, \mathbf{1}) = \Phi(t, \Phi(s, \mathbf{1})) = \Phi(s, \mathbf{1})\gamma_x(t) = \gamma_x(s)\gamma_x(t). \qquad (8.4)$$

Hence γ_x is a group homomorphism $(\mathbb{R}, +) \to G$.

(c) If $\gamma \colon (\mathbb{R}, +) \to G$ is a smooth group homomorphism, then

$$\Phi(t, g) := g\gamma(t)$$

defines a global flow on G whose infinitesimal generator is the vector field given by

$$X(g) = \frac{d}{dt}\Big|_{t=0} \Phi(t, g) = T(\lambda_g)\gamma'(0).$$

We conclude that $X = x_l$ for $x = \gamma'(0)$, so that X is a left invariant vector field. Since γ is its unique integral curve through 0, it follows that $\gamma = \gamma_x$. In view of (a), this proves (c). ☐

Proposition 8.2.5. *For a Lie group* G*, the exponential function*

$$\exp_G \colon \mathbf{L}(G) \to G$$

is smooth and satisfies

$$T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}.$$

In particular, $\exp_G$ *is a local diffeomorphism in* 0 *in the sense that it maps some* 0*-neighborhood in* $\mathbf{L}(G)$ *diffeomorphically onto some* $\mathbf{1}$*-neighborhood in* G*.*

Proof. Let $n \in \mathbb{N}$. In view of Lemma 8.2.4(c), we have

$$\exp_G(nx) = \gamma_x(n) = \gamma_x(1)^n = \exp_G(x)^n \tag{8.5}$$

for each $x \in \mathbf{L}(G)$. Since the n-fold multiplication map

$$G^n \to G, \quad (g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$$

is smooth, the n-th power map $G \to G, g \mapsto g^n$ is smooth. Therefore it suffices to verify the smoothness of $\exp_G$ in some 0-neighborhood W. Then (8.5) immediately implies smoothness in nW for each n, and hence on all of $\mathbf{L}(G)$.

The map $\Psi\colon \mathbf{L}(G) \to \mathcal{V}(G), x \mapsto x_l$ satisfies the assumptions of Proposition 7.5.15 because the map

$$\mathbf{L}(G) \times G \to T(G), \quad (x, g) \mapsto x_l(g) = g.x$$

is smooth (Lemma 8.1.6). In the terminology of Proposition 7.5.15, it now follows that the map

$$\Phi\colon \mathbb{R} \times \mathbf{L}(G) \times G \to G, \quad (t, x, g) \mapsto g\gamma_x(t) = g \exp_G(tx)$$

is smooth on a neighborhood of $(0, 0, \mathbf{1})$. In particular, for some $t > 0$, the map $x \mapsto \exp_G(tx)$ is smooth on a 0-neighborhood of $\mathbf{L}(G)$, and this proves that $\exp_G$ is smooth in some 0-neighborhood.

Finally, we observe that

$$T_0(\exp_G)(x) = \frac{d}{dt}\Big|_{t=0} \exp_G(tx) = \gamma_x'(0) = x,$$

so that $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$. □

Lemma 8.2.6 (Canonical Coordinates). *Let G be a Lie group and $b_1, \ldots, b_n$ be a basis for its Lie algebra $\mathbf{L}(G)$. Then the following maps restrict to diffeomorphisms of some 0-neighborhood in $\mathbb{R}^n$ to some open $\mathbf{1}$-neighborhood in G:*

(i) $x \mapsto \exp_G(x_1 b_1 + \ldots + x_n b_n)$ (*Canonical coordinates of the first kind*).
(ii) $x \mapsto \exp_G(x_1 b_1) \cdot \ldots \cdot \exp_G(x_n b_n)$ (*Canonical coordinates of the second kind*).

Proof. (i) This is immediate from Proposition 8.2.5.

(ii) In view of Proposition 8.2.5, $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$, and further $T_{\mathbf{1}}(m_G)(x, y) = x + y$ by Lemma 8.1.6. Therefore

$$\Phi\colon \mathbb{R}^n \to G, \quad x \mapsto \exp_G(x_1 b_1) \cdot \ldots \cdot \exp_G(x_n b_n)$$

satisfies $T_0(\Phi)(x) = \sum_{i=1}^n x_i b_i$. Hence the claim follows from the Inverse Function Theorem. □

Lemma 8.2.7. *If* $\sigma\colon G \times M \to M$ *is a smooth action and* $x \in \mathbf{L}(G)$*, then the global flow of the vector field* $\dot{\sigma}(x)$ *is given by* $\Phi^x(t,m) = \exp_G(-tx).m$*. In particular,*

$$\dot{\sigma}(x)(m) = \frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m.$$

Proof. In the proof of Proposition 8.1.14, we have seen that

$$T_g(\varphi^m)x_l(g) = \dot{\sigma}(x)(\varphi^m(g))$$

holds for the map $\varphi^m(g) = g^{-1}.m$. In view of Proposition 8.2.5 this yields

$$\frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m = T_{\mathbf{1}}(\varphi^m)T_0(\exp_G)x = T_{\mathbf{1}}(\varphi^m)x = \dot{\sigma}(x)(m),$$

and hence proves the lemma. □

Lemma 8.2.8. *If* $x, y \in \mathbf{L}(G)$ *commute, i.e.,* $[x,y] = 0$*, then*

$$\exp_G(x+y) = \exp_G(x)\exp_G(y).$$

Proof. If x and y commute, then the corresponding left invariant vector fields commute, and Corollary 7.5.17 implies that their flows commute. We conclude that for all $t, s \in \mathbb{R}$ we have

$$\exp_G(tx)\exp_G(sy) = \exp_G(sy)\exp_G(tx). \tag{8.6}$$

Therefore

$$\gamma(t) := \exp_G(tx)\exp_G(ty)$$

is a smooth group homomorphism. In view of

$$\gamma'(0) = T_{(\mathbf{1},\mathbf{1})}(m_G)(x,y) = x + y$$

(Lemma 8.1.6), Lemma 8.2.4(c) leads to $\gamma(t) = \exp_G(t(x+y))$, and for $t = 1$ we obtain the lemma. □

Lemma 8.2.9. *The subgroup* $\langle \exp_G(\mathbf{L}(G))\rangle$ *of* G *generated by* $\exp_G(\mathbf{L}(G))$ *coincides with the identity component* G_0 *of* G*, i.e., the connected component containing* $\mathbf{1}$*.*

Proof. Since $\exp_G$ is a local diffeomorphism in 0 (Proposition 8.2.5), the Inverse Function Theorem (see Exercise 7.3.5) implies that $\exp_G(\mathbf{L}(G))$ is a neighborhood of $\mathbf{1}$. We conclude that the subgroup $H := \langle \exp_G(\mathbf{L}(G))\rangle$ generated by the exponential image is a $\mathbf{1}$-neighborhood, hence contains G_0 (Proposition 8.1.15(iii)(d)). On the other hand, $\exp_G$ is continuous, so that it maps the connected space $\mathbf{L}(G)$ into the identity component G_0 of G, which leads to $H \subseteq G_0$, and hence to equality. □

8.2.2 Naturality of the Exponential Function

In this subsection we study how the exponential functions is related to the Lie functor.

Proposition 8.2.10. *Let $\varphi\colon G_1 \to G_2$ be a morphism of Lie groups and $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ its differential in $\mathbf{1}$. Then*

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}, \tag{8.7}$$

i.e., the following diagram commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \uparrow{\scriptstyle \exp_{G_1}} & & \uparrow{\scriptstyle \exp_{G_2}} \\ \mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(G_2). \end{array}$$

Proof. For $x \in \mathbf{L}(G_1)$ we consider the smooth homomorphism

$$\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1), \quad \gamma_x(t) = \exp_{G_1}(tx).$$

According to Lemma 8.2.4, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for $y = (\varphi \circ \gamma_x)'(0) = \mathbf{L}(\varphi)x$, because $\varphi \circ \gamma_x\colon \mathbb{R} \to G_2$ is a smooth group homomorphism. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x)),$$

which we had to show. □

Corollary 8.2.11. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1 \to G_2$ be a group homomorphism. Then the following are equivalent:*

(a) *φ is smooth in an identity neighborhood of G_1.*
(b) *φ is smooth.*
(c) *There exists a linear map $\psi\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ satisfying*

$$\exp_{G_2} \circ \psi = \varphi \circ \exp_{G_1}. \tag{8.8}$$

Proof. (a) ⇒ (b): Let U be an open $\mathbf{1}$-neighborhood of G_1 such that $\varphi|_U$ is smooth. Since each left translation λ_g is a diffeomorphism, $\lambda_g(U) = gU$ is an open neighborhood of g, and we have

$$\varphi(gx) = \varphi(g)\varphi(x), \quad \text{i.e.,} \quad \varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi.$$

Hence the smoothness of φ on U implies the smoothness of φ on gU, and therefore that φ is smooth.

(b) ⇒ (c): If φ is smooth, then $\psi := \mathbf{L}(\varphi)$ satisfies (8.8).

(c) ⇒ (a): If ψ is a linear map satisfying (8.8), then the fact that the exponential functions $\exp_{G_1}$ and $\exp_{G_2}$ are local diffeomorphisms and the smoothness of the linear map ψ implies (a). □

Corollary 8.2.12. *If $\varphi_1, \varphi_2 \colon G_1 \to G_2$ are morphisms of Lie groups with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$, then φ_1 and φ_2 coincide on the identity component of G_1.*

Proof. In view of Proposition 8.2.10, we have for $x \in \mathbf{L}(G_1)$:

$$\varphi_1(\exp_{G_1}(x)) = \exp_{G_2}(\mathbf{L}(\varphi_1)x) = \exp_{G_2}(\mathbf{L}(\varphi_2)x) = \varphi_2(\exp_{G_1}(x)),$$

so that φ_1 and φ_2 coincide on the image of $\exp_{G_1}$, hence on the subgroup generated by this set. Now the assertion follows from Lemma 8.2.9. ☐

Proposition 8.2.13. *For a morphism $\varphi \colon G_1 \to G_2$ of Lie groups, the following assertions hold:*

(1) $\ker \mathbf{L}(\varphi) = \{x \in \mathbf{L}(G_1) \colon \exp_{G_1}(\mathbb{R}x) \subseteq \ker \varphi\}$.
(2) *φ is an open map if and only if $\mathbf{L}(\varphi)$ is surjective.*
(3) *If $\mathbf{L}(\varphi)$ is a linear isomorphism and φ is bijective, then φ is an isomorphism of Lie groups.*

Proof. (1) The condition $x \in \ker \mathbf{L}(\varphi)$ is equivalent to

$$\{\mathbf{1}\} = \exp_{G_2}(\mathbb{R}\,\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(\mathbb{R}x)).$$

(2) Suppose first that φ is an open map. Since $\exp_{G_i}$, $i = 1, 2$, are local diffeomorphisms,

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1} \tag{8.9}$$

implies that there exists some 0-neighborhood in $\mathbf{L}(G_1)$ on which $\mathbf{L}(\varphi)$ is an open map, hence that $\mathbf{L}(\varphi)$ is surjective.

If, conversely, $\mathbf{L}(\varphi)$ is surjective, then $\mathbf{L}(\varphi)$ is an open map, so that the relation (8.9) implies that there exists an open $\mathbf{1}$-neighborhood U_1 in G_1 such that $\varphi|_{U_1}$ is an open map. We claim that this implies that φ is an open map. In fact, suppose that $O \subseteq G_1$ is open and $g \in O$. Then there exists an open $\mathbf{1}$-neighborhood U_2 of G_1 with $gU_2 \subseteq O$ and $U_2 \subseteq U_1$. Then

$$\varphi(O) \supseteq \varphi(gU_2) = \varphi(g)\varphi(U_2),$$

and since $\varphi(U_2)$ is open in G_2, we see that $\varphi(O)$ is a neighborhood of $\varphi(g)$, hence that $\varphi(O)$ is open because $g \in O$ was arbitrary.

(3) From the relation $\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}$ and the bijectivity of φ we derive that the group homomorphism φ^{-1} satisfies

$$\varphi^{-1} \circ \exp_{G_2} = \exp_{G_1} \circ \mathbf{L}(\varphi)^{-1},$$

so that Corollary 8.2.11 implies that φ^{-1} is also smooth. ☐

Proposition 8.2.14. *Let G be a Lie group with Lie algebra $\mathbf{L}(G)$. Then for $x, y \in \mathbf{L}(G)$ the following equations hold:*

(1) (Product Formula)

$$\exp_G(x+y) = \lim_{k\to\infty} \left(\exp_G(\tfrac{1}{k}x)\exp_G(\tfrac{1}{k}y)\right)^k.$$

(2) (Commutator Formula)

$$\exp_G([x,y]) = \lim_{k\to\infty} \Big(\exp_G\big(\tfrac{1}{k}x\big)\exp_G\big(\tfrac{1}{k}y\big)\exp_G\big(-\tfrac{1}{k}x\big)\exp_G\big(-\tfrac{1}{k}y\big)\Big)^{k^2}.$$

Proof. Let $U \subseteq \mathbf{L}(G)$ be an open 0-neighborhood for which

$$\exp_U := \exp_G|_U : U \to \exp_G(U)$$

is a diffeomorphism onto an open subset of G. Put

$$U^1 := \{(x,y) \in U \times U \colon \exp_G(x)\exp_G(y) \in \exp_G(U)\}$$

and observe that this is an open subset of $U \times U$ containing $(0,0)$ because $\exp_G(U)$ is open and $\exp_G$ is continuous.

For $(x,y) \in U_1$ we then define

$$x * y := \exp_U^{-1}(\exp_G(x)\exp_G(y))$$

and thus obtain a smooth map

$$m \colon U^1 \to \mathbf{L}(G), \quad (x,y) \mapsto x * y.$$

(1) In view of $m(0,x) = m(x,0) = x$, we have

$$\mathtt{d}m(0,0)(x,y) = \mathtt{d}m(0,0)(x,0) + \mathtt{d}m(0,0)(0,y) = x + y.$$

This implies that

$$\lim_{k\to\infty} k \cdot \big(\tfrac{1}{k}x * \tfrac{1}{k}y\big) = \lim_{k\to\infty} k \cdot \Big(m\big(\tfrac{1}{k}x, \tfrac{1}{k}y\big) - m(0,0)\Big) = \mathtt{d}m(0,0)(x,y) = x + y.$$

Applying $\exp_G$, it follows that

$$\begin{aligned}\exp_G(x+y) &= \lim_{k\to\infty} \exp_G\Big(k \cdot \big(\tfrac{1}{k}x * \tfrac{1}{k}y\big)\Big) = \lim_{k\to\infty} \exp_G\Big(\tfrac{1}{k}x * \tfrac{1}{k}y\Big)^k \\ &= \lim_{k\to\infty} \Big(\exp_G(\tfrac{1}{k}x)\exp_G(\tfrac{1}{k}y)\Big)^k.\end{aligned}$$

(2) Now let $x_l^* := T(\exp_U)^{-1} \circ x_l \circ \exp_U$ be the smooth vector field on U corresponding to the left invariant vector field x_l on $\exp_G(U)$. Then x_l^* and x_l are $\exp_U$-related, so that the Related Vector Field Lemma 7.4.9 leads to

$$[x_l^*, y_l^*](0) = T_0(\exp_U)[x_l^*, y_l^*](0) = [x_l, y_l](\exp_G(0)) = [x_l, y_l](\mathbf{1}) = [x,y].$$

The local flow of x_l through a point $\exp_U(y)$ is given by the curve $t \mapsto \exp_G(y)\exp_G(tx)$, which implies that the integral curve of x_l^* through $y \in U$ is given for t close to 0 by $\Phi_t^{x_l^*}(y) = y * tx$. We therefore obtain with Remark 7.5.18 (on commutators of flows):

$$\frac{\partial^2}{\partial s \partial t}\Big|_{t=s=0} tx * sy * (-tx) * (-sy) = \frac{\partial^2}{\partial s \partial t}\Big|_{t=s=0} \Phi_{-s}^{y_l^*} \circ \Phi_{-t}^{x_l^*} \circ \Phi_s^{y_l^*} \circ \Phi_t^{x_l^*}(0)$$
$$= [-x_l^*, -y_l^*](0) = [x_l^*, y_l^*](0) = [x, y].$$

Note that $F(t, s) := tx * sy * (-tx) * (-sy)$ vanishes for $t = 0$ and $s = 0$.

For $f(t) := F(t, t)$ we have

$$f'(t) = \frac{\partial F}{\partial t}(t, t) + \frac{\partial F}{\partial s}(t, t),$$

$$f''(t) = \frac{\partial^2 F}{\partial t^2}(t, t) + 2\frac{\partial^2 F}{\partial t \partial s}(t, t) + \frac{\partial^2 F}{\partial s^2}(t, t),$$

and

$$\frac{\partial^2 F}{\partial t^2}(0, 0) = \frac{\partial^2 F}{\partial s^2}(0, 0) = f'(0) = 0.$$

Therefore

$$\frac{1}{2} f''(0) = \frac{\partial^2 F}{\partial t \partial s}(0, 0) = [x, y]$$

leads to

$$\lim_{k \to \infty} k^2 \Big(\frac{1}{k}x * \frac{1}{k}y * (-\frac{1}{k}x) * (-\frac{1}{k}y)\Big) = \lim_{k \to \infty} k^2 f\Big(\frac{1}{k}\Big) = \frac{1}{2} f''(0) = [x, y].$$

Applying the exponential function, we obtain the Commutator Formula. □

Theorem 8.2.15 (One-parameter Group Theorem). *Let G be a Lie group. For each $x \in \mathfrak{g} := \mathbf{L}(G)$, the map $\gamma_x \colon (\mathbb{R}, +) \to G, t \mapsto \exp_G(tx)$ is a smooth group homomorphism. Conversely, every continuous one-parameter group $\gamma \colon \mathbb{R} \to G$ is of this form.*

Proof. The first assertion is an immediate consequence of Lemma 8.2.4(c). It therefore remains to show that each continuous one-parameter group γ of G is a γ_x for some $x \in \mathfrak{g}$. Let $U = -U$ be a convex 0-neighborhood in $\mathfrak{g}$ for which $\exp_G|_U$ is a diffeomorphism onto an open subset of G and put $U_1 := \frac{1}{2}U$. Since γ is continuous in 0, there exists an $\varepsilon > 0$ such that $\gamma([-\varepsilon, \varepsilon]) \subseteq \exp_G(U_1)$. Then $\alpha(t) := (\exp_G|_U)^{-1}(\gamma(t))$ defines a continuous curve $\alpha \colon [-\varepsilon, \varepsilon] \to U_1$ with $\exp(\alpha(t)) = \gamma(t)$ for $|t| \leq \varepsilon$. With the same arguments as in the proof of Theorem 2.2.6, we see that $\alpha(t) = tx$ for some $x \in \mathfrak{g}$. Hence $\gamma(t) = \exp_G(tx)$ for $|t| \leq \varepsilon$, but then $\gamma(nt) = \exp_G(ntx)$ for $n \in \mathbb{N}$ leads to $\gamma(t) = \exp_G(tx)$ for each $t \in \mathbb{R}$. □

Theorem 8.2.16 (Automatic Smoothness Theorem). *Each continuous homomorphism $\varphi \colon G \to H$ of Lie groups is smooth.*

Proof. From Theorem 8.2.15 we know that the map

$$\mathbf{L}(G) \to \mathrm{Hom}_c(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) := \exp_G(tx)$$

is a bijection, where $\mathrm{Hom}_c(\mathbb{R}, G)$ denotes the set of all continuous one-parameter groups of G. For $x \in \mathbf{L}(G_1)$ we consider the continuous homomorphism $\varphi \circ \gamma_x \in \mathrm{Hom}_c(\mathbb{R}, G_2)$. Since this one-parameter group is smooth (Theorem 8.2.15), it is of the form

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for $y = (\varphi \circ \gamma_x)'(0) \in \mathbf{L}(G_2)$. We define a map $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ by $\mathbf{L}(\varphi)x := (\varphi \circ \gamma_x)'(0)$. For $t = 1$ we then obtain

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}\colon \mathbf{L}(G_1) \to G_2. \tag{8.10}$$

Next we show that $\mathbf{L}(\varphi)$ is a linear map. Our definition immediately shows that $\mathbf{L}(\varphi)\lambda x = \lambda\,\mathbf{L}(\varphi)x$ for each $x \in \mathbf{L}(G_1)$. Further, the Product Formula (Proposition 8.2.14) yields

$$\begin{aligned}
&\exp_{G_2}(\mathbf{L}(\varphi)(x+y)) = \varphi\big(\exp_{G_1}(x+y)\big) \\
&= \lim_{k\to\infty} \varphi\Big(\exp_{G_1}\big(\tfrac{1}{k}x\big)\exp_{G_1}\big(\tfrac{1}{k}y\big)\Big)^k \\
&= \lim_{k\to\infty} \Big(\exp_{G_2}\big(\tfrac{1}{k}\,\mathbf{L}(\varphi)x\big)\exp_{G_2}\big(\tfrac{1}{k}\,\mathbf{L}(\varphi)y\big)\Big)^k \\
&= \exp_{G_2}\big(\mathbf{L}(\varphi)x + \mathbf{L}(\varphi)y\big).
\end{aligned}$$

This proves that $\mathbf{L}(\varphi)(x+y) = \mathbf{L}(\varphi)x + \mathbf{L}(\varphi)y$, so that $\mathbf{L}(\varphi)$ is indeed a linear map, hence in particular smooth. Since both maps $\exp_{G_i}$ are local diffeomorphisms, (8.10) implies that φ is smooth in an identity neighborhood of G_1, hence smooth by Corollary 8.2.11. □

Corollary 8.2.17. *A topological group G carries at most one Lie group structure.*

Proof. If G_1 and G_2 are two Lie groups which are isomorphic as topological groups, then the Automatic Smoothness Theorem applies to each topological isomorphism $\varphi\colon G_1 \to G_2$ and shows that φ is smooth. It likewise applies to φ^{-1}, so that φ is an isomorphism of Lie group. □

8.2.3 The Adjoint Representation

The Lie functor associates linear automorphisms of the Lie algebra with conjugations on the Lie group. The resulting representation of the Lie group is called the adjoint representation. Its interplay with the exponential function will be important in the entire theory.

Definition 8.2.18. (a) We know that for each finite-dimensional vector space V, the group $\mathrm{GL}(V)$ carries a natural Lie group structure. For a Lie group G, a smooth homomorphism $\pi\colon G \to \mathrm{GL}(V)$ is a called a *representation of* G *on* V (cf. Exercise 8.2.3).

Any representation defines a smooth action of G on V via

$$\sigma(g, v) := \pi(g)(v).$$

In this sense, representations are the same as *linear actions*, i.e., actions on vector spaces for which the σ_g are linear.

(b) If $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation of* $\mathfrak{g}$ *on* V (cf. Definition 4.1.4).

As a consequence of Proposition 8.1.8, we obtain

Proposition 8.2.19. *If* $\varphi\colon G \to \mathrm{GL}(V)$ *is a representation of* G*, then* $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ *is a representation of its Lie algebra* $\mathbf{L}(G)$*.*

The representation $\mathbf{L}(\varphi)$ obtained in Proposition 8.2.19 from the group representation φ is called the *derived representation.* This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)(x) = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)x} = \frac{d}{dt}\Big|_{t=0} \varphi(\exp_G tx).$$

Let G be a Lie group and $\mathbf{L}(G)$ its Lie algebra. For $g \in G$ we recall the conjugation automorphism $c_g \in \mathrm{Aut}(G), c_g(x) = gxg^{-1}$, and define

$$\mathrm{Ad}(g) := \mathbf{L}(c_g) \in \mathrm{Aut}(\mathbf{L}(G)).$$

Then

$$\mathrm{Ad}(g_1g_2) = \mathbf{L}(c_{g_1g_2}) = \mathbf{L}(c_{g_1}) \circ \mathbf{L}(c_{g_2}) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2)$$

shows that $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G))$ is a group homomorphism. It is called the *adjoint representation*. To see that it is smooth, we observe that for each $x \in \mathbf{L}(G)$ we have

$$\mathrm{Ad}(g)x = T_{\mathbf{1}}(c_g)x = T_{\mathbf{1}}(\lambda_g \circ \rho_{g^{-1}})x = T_{g^{-1}}(\lambda_g)T_{\mathbf{1}}(\rho_{g^{-1}})x = 0_g \cdot x \cdot 0_{g^{-1}}$$

in the Lie group $T(G)$ (Lemma 8.1.6). Since the multiplication in $T(G)$ is smooth, the representation Ad of G on $\mathbf{L}(G)$ is smooth (cf. Exercise 8.2.3), and

$$\mathbf{L}(\mathrm{Ad})\colon \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation.

Lemma 8.2.20. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$*, i.e.,* $\mathbf{L}(\mathrm{Ad})(x)(y) = [x, y]$*.*

Proof. Let $x, y \in \mathbf{L}(G)$ and x_l, y_l be the corresponding left invariant vector fields. Corollary 8.1.10 implies for $g \in G$ the relation

$$(c_g)_* y_l = \big(\mathbf{L}(c_g)y\big)_l = \big(\mathrm{Ad}(g)y\big)_l.$$

On the other hand, the left invariance of y_l leads to

$$(c_g)_* y_l = (\rho_g^{-1} \circ \lambda_g)_* y_l = (\rho_g^{-1})_* (\lambda_g)_* y_l = (\rho_g^{-1})_* y_l.$$

Next we observe that $\Phi_t^{x_l} = \rho_{\exp_G(tx)}$ is the flow of the vector field x_l (Lemma 8.2.4), so that Theorem 7.5.16 implies that

$$\begin{aligned}
[x_l, y_l] = \mathcal{L}_{x_l} y_l &= \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^{x_l})_* y_l = \frac{d}{dt}\Big|_{t=0} (c_{\exp_G(tx)})_* y_l \\
&= \frac{d}{dt}\Big|_{t=0} \big(\mathrm{Ad}(\exp_G(tx))y\big)_l.
\end{aligned}$$

Evaluating in $\mathbf{1}$, we get

$$[x, y] = [x_l, y_l](\mathbf{1}) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp_G(tx))y = \mathbf{L}(\mathrm{Ad})(x)(y). \qquad \square$$

Combining Proposition 8.2.10 with Lemma 8.2.20, we obtain the important formula

$$\mathrm{Ad} \circ \exp_G = \exp_{\mathrm{Aut}(\mathbf{L}(G))} \circ \operatorname{ad},$$

i.e.,

$$\mathrm{Ad}(\exp_G(x)) = e^{\operatorname{ad} x} \quad \text{for} \quad x \in \mathbf{L}(G). \tag{8.11}$$

Lemma 8.2.21. *For a Lie group G, the kernel of the adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G))$*, is given by*

$$Z_G(G_0) := \{g \in G \colon (\forall x \in G_0)\, gx = xg\},$$

where G_0 is the connected component of the identity in G. If, in addition, G is connected, then

$$\ker \mathrm{Ad} = Z(G).$$

Proof. Since G_0 is connected, the automorphism $c_g|_{G_0}$ of G_0 is trivial if and only if $\mathbf{L}(c_g) = \mathrm{Ad}(g)$ is trivial. This implies the lemma. $\square$

8.2.4 Semidirect Products

The easiest way to construct a new Lie group from two given Lie groups G and H, is to endow the product manifold $G \times H$ with the multiplication

$$(g_1, h_1)(g_2, h_2) := (g_1 g_2, h_1 h_2).$$

The resulting group is called the *direct product* of the Lie groups G and H. Here G and H can be identified with normal subgroups of $G \times H$ for which the multiplication map

$$(G \times \{\mathbf{1}\}) \times (\{\mathbf{1}\} \times H) \to G \times H, \quad ((g, \mathbf{1}), (\mathbf{1}, h)) \mapsto (g, \mathbf{1})(\mathbf{1}, h) = (g, h)$$

is a diffeomorphism. Relaxing this condition in the sense that only one factor is assumed to be normal, leads to the concept of a semidirect product of Lie groups, introduced below.

Definition 8.2.22. Let N and G be Lie groups and $\alpha\colon G \to \mathrm{Aut}(N)$ be a group homomorphism defining a smooth action $(g, n) \mapsto \alpha(g)(n)$ of G on N. Then the product manifold $N \times G$ is a group with respect to the product (cf. Lemma 1.2.3)

$$(n, g)(n', g') := (n\alpha(g)(n'), gg')$$

and the inversion

$$(n, g)^{-1} = (\alpha(g^{-1})(n^{-1}), g^{-1}).$$

Since multiplication and inversion are smooth, this group is a Lie group, called the *semidirect product of N and G with respect to α*. It is denoted by $N \rtimes_\alpha G$.

On the manifold $G \times N$ we also obtain a Lie group structure by

$$(g, n)(g', n') := (gg', \alpha(g')^{-1}(n)n'),$$

and this Lie group is denoted $G \ltimes_\alpha N$. It is easy to verify that the map

$$\Phi\colon N \rtimes_\alpha G \to G \ltimes_\alpha N, \quad (n, g) \mapsto (g, \alpha(g)^{-1}(n))$$

is an isomorphism of Lie groups.

Remark 8.2.23. If $\widehat{G} := N \rtimes_\alpha G$ is a semidirect product, then

$$\pi\colon \widehat{G} \to G, \quad (n, g) \mapsto g, \qquad \sigma\colon G \to \widehat{G}, \quad g \mapsto (\mathbf{1}, g)$$

and $\iota\colon N \to \widehat{G}, n \mapsto (n, \mathbf{1})$ are morphisms of Lie groups with $\pi \circ \sigma = \mathrm{id}_G$ and ι is an isomorphism of N onto the submanifold $\ker \pi$ of $\widehat{G}$ (cf. Remark 7.6.10).

Example 8.2.24. Let G be a Lie group and $T(G)$ its tangent Lie group (Lemma 8.1.6). We have already seen that the map $G \times \mathbf{L}(G) \to TG$, $(g, x) \mapsto g.x = 0_g \cdot x$ is a diffeomorphism, and for similar reasons, the map $\mathbf{L}(G) \times G \to TG, (x, g) \mapsto x.g := x \cdot 0_g$ is a diffeomorphism. In these coordinates, the multiplication is given by

$$(x.g) \cdot (x'.g') = x \cdot 0_g \cdot x' \cdot 0_{g'} = x \cdot \mathrm{Ad}(g)x' \cdot 0_g \cdot 0_{g'} = (x + \mathrm{Ad}(g)x').gg'.$$

This shows that the tangent bundle is a semidirect product

$$TG \cong \mathbf{L}(G) \rtimes_{\mathrm{Ad}} G.$$

Similarly, the calculation

$$(g.x) \cdot (g'.x') = 0_{gg'} \cdot \mathrm{Ad}(g')^{-1}x \cdot x' = (gg', \mathrm{Ad}(g')^{-1}x + x')$$

shows that also

$$TG \cong G \ltimes_{\mathrm{Ad}} \mathbf{L}(G).$$

Proposition 8.2.25. *The Lie algebra of the semidirect product group* $N \rtimes_\alpha G$ *is given by*

$$\mathbf{L}(N \rtimes_\alpha G) \cong \mathbf{L}(N) \rtimes_\beta \mathbf{L}(G),$$

where $\beta\colon \mathbf{L}(G) \to \mathrm{der}(\mathbf{L}(N))$ *is the derived representation of* $\mathbf{L}(G)$ *on* $\mathbf{L}(N)$ *corresponding to the representation of* G *on* $\mathbf{L}(N)$ *given by* $g.x := \mathbf{L}(\alpha(g))x$.

Proof. We identify $\mathbf{L}(N)$, resp., $\mathbf{L}(G)$, with a subspace of

$$T_{\mathbf{1}}(N) \oplus T_{\mathbf{1}}(G) = T_{(\mathbf{1},\mathbf{1})}(N \times G) \cong \mathbf{L}(N \rtimes_\alpha G).$$

Since N and G are subgroups, the functoriality of $\mathbf{L}$ implies that $\mathbf{L}(G)$ and $\mathbf{L}(N)$ are Lie subalgebras of $\mathbf{L}(N \rtimes_\alpha G)$. The normal subgroup N is the kernel of the projection $\pi\colon N \rtimes_\alpha G \to G$, so that our identification shows that $\mathbf{L}(N) = \ker \mathbf{L}(\pi)$ is an ideal of $\mathbf{L}(N \rtimes_\alpha G)$. This already implies

$$\mathbf{L}(N \rtimes_\alpha G) \cong \mathbf{L}(N) \rtimes_\beta \mathbf{L}(G)$$

for the homomorphism $\beta\colon \mathbf{L}(G) \to \mathrm{der}(\mathbf{L}(N))$, given by

$$(\beta(x)(y), 0) = [(0, x), (y, 0)].$$

To determine β in terms of α, we note that the smooth action of G on N by automorphisms induces a smooth action of G on the tangent bundle $T(N)$, hence in particular on $T_{\mathbf{1}}(N) \cong \mathbf{L}(N)$. We thus obtain a representation $\pi\colon G \to \mathrm{Aut}(\mathbf{L}(N))$. In $N \rtimes_\alpha G$ we have $(\mathbf{1}, g)(n, \mathbf{1})(\mathbf{1}, g)^{-1} = (\alpha(g)(n), \mathbf{1})$, so that

$$\pi(g)y = \mathbf{L}(\alpha(g))y = \mathrm{Ad}(\mathbf{1}, g)(y, 0).$$

Now Lemma 8.2.20 immediately shows that $\mathbf{L}(\pi)x = \mathrm{ad}(0, x) = \beta(x)$. □

To form semidirect products, we need smooth actions of a Lie group G on a Lie group N. The following lemma is a useful tool to verify smoothness in this context.

Lemma 8.2.26. *Let* G *and* N *be Lie groups and* $\alpha\colon G \to \mathrm{Aut}(N)$ *be a group homomorphism.*

(i) *The action* $\sigma(g, n) := \alpha(g)(n)$ *of* G *on* N *is smooth if and only if*
  (a) $\mathbf{L} \circ \alpha\colon G \to \mathrm{Aut}(\mathbf{L}(N))$ *is smooth as a map into* $\mathrm{GL}(\mathbf{L}(N))$.
  (b) *All orbit maps* $\sigma^n\colon G \to N, g \mapsto \alpha(g)(n)$ *are smooth.*

(ii) *If* N *is connected, then* (a) *implies* (b).

Proof. (i) If σ is smooth, then (b) is clearly satisfied. Moreover,

$$\exp_N \circ \mathbf{L}(\alpha(g)) = \alpha(g) \circ \exp_N$$

holds for each $g \in G$, so that the map

$$G \times \mathbf{L}(N) \to \mathbf{L}(N), \quad (g, x) \mapsto \mathbf{L}(\alpha(g))x$$

is smooth in an open neighborhood of $(\mathbf{1}, 0)$. This implies that $\mathbf{L} \circ \alpha \colon G \to \mathrm{GL}(\mathbf{L}(N))$ is smooth in an open neighborhood of $\mathbf{1}$, hence smooth because it is a group homomorphism.

Next we assume that (a) and (b) are satisfied. We have to show that for any pair $(g_1, n_1) \in G \times N$ the expression

$$\sigma(g_1 g_2, n_1 n_2) = \alpha(g_1)\big(\alpha(g_2)(n_1)\alpha(g_2)(n_2)\big),$$

is a smooth function of (g_2, n_2) in a neighborhood of $(\mathbf{1}, \mathbf{1})$. In view of the smoothness of $\alpha(g_1)$ and (b), it suffices to prove that σ is smooth in an open neighborhood of $(\mathbf{1}, \mathbf{1})$. According to (a), this follows from

$$\sigma(g, \exp_N x) = \alpha(g)(\exp_N(x)) = \exp_N(\mathbf{L}(\alpha(g))x). \tag{8.12}$$

(ii) Since G acts by group automorphisms, the set

$$S := \{n \in N \colon \sigma^n \in C^\infty(G, N)\}$$

is a subgroup. In view of (8.12), (a) implies that S contains the image of $\exp_N$, hence all of N (Lemma 8.2.9). □

8.2.5 The Baker–Campbell–Dynkin–Hausdorff Formula

In this subsection we show that the formula

$$\exp_G(x * y) = \exp_G x \exp_G y,$$

where $x * y$, for sufficiently small elements $x, y \in \mathfrak{g} = \mathbf{L}(G)$, is given by the Hausdorff series (cf. Proposition 2.4.5), also holds for the exponential function of a general Lie group G with Lie algebra $\mathfrak{g}$.

The *Maurer–Cartan form* $\kappa_G \in \Omega^1(G, \mathfrak{g})$ is the unique left invariant 1-form on G with $\kappa_{G,\mathbf{1}} = \mathrm{id}_{\mathfrak{g}}$, i.e., $\kappa_G(v) = g^{-1}.v$ for $v \in T_g(G)$. The *logarithmic derivative* of a smooth function $f \colon M \to G$ is now defined as the pull-back of the Maurer–Cartan form:

$$\delta(f) := f^* \kappa_G \in \Omega^1(M, \mathfrak{g}).$$

For each $v \in T_m(M)$ we then have

$$\delta(f)_m v = f(m)^{-1} \cdot T_m(f) v.$$

If $\alpha \in \Omega^1(M, \mathfrak{g})$ is a Lie algebra-valued 1-form and $f \colon M \to G$ a smooth function, then we define $\mathrm{Ad}(f)\alpha$ pointwise by $(\mathrm{Ad}(f)\alpha)_m := \mathrm{Ad}(f(m))\alpha_m$.

Lemma 8.2.27. *For two smooth maps $f, h\colon M \to G$, the logarithmic derivative of the pointwise products fh and fh^{-1} is given by the*

(1) *Product Rule:* $\delta(fh) = \delta(h) + \mathrm{Ad}(h^{-1})\delta(f)$, *and the*
(2) *Quotient Rule:* $\delta(fh^{-1}) = \mathrm{Ad}(h)(\delta(f) - \delta(h))$.

Proof. Writing $fg = m_G \circ (f, g)$, we obtain from

$$T_{(a,b)}(m_G)(v, w) = v \cdot b + a \cdot w$$

for $a, b \in G$ and $v, w \in \mathbf{L}(G) \subseteq TG$ the relation

$$T(fh) = T(m_G) \circ (T(f), T(h)) = T(f) \cdot h + f \cdot T(h)\colon T(M) \to T(G),$$

where $f \cdot T(h)$, resp., $T(f) \cdot h$ refers to the pointwise product in the group $T(G)$, containing G as the zero section (Lemma 8.1.6). This immediately leads to the Product Rule

$$\delta(fh) = (fh)^{-1} \cdot (T(f) \cdot h + f \cdot T(h)) = h^{-1} \cdot (\delta(f) \cdot h) + \delta(h) = \mathrm{Ad}(h)^{-1}\delta(f) + \delta(h).$$

For $h = f^{-1}$, we then obtain

$$0 = \delta(ff^{-1}) = \mathrm{Ad}(f)\delta(f) + \delta(f^{-1}),$$

hence $\delta(f^{-1}) = -\mathrm{Ad}(f)\delta(f)$. This in turn leads to

$$\delta(fh^{-1}) = \mathrm{Ad}(h)\delta(f) + \delta(h^{-1}) = \mathrm{Ad}(h)\delta(f) - \mathrm{Ad}(h)\delta(h),$$

which is the Quotient Rule. □

For any $g \in G$ and a smooth function $f\colon M \to G$, the function $\lambda_g \circ f$ has the same logarithmic derivative as f because $(\lambda_g \circ f)^*\kappa_G = f^*\lambda_g^*\kappa_G = f^*\kappa_G$. The following lemma provides a converse, which is a very convenient tool.

Lemma 8.2.28 (Uniqueness Lemma for Logarithmic Derivatives). *Let G be a Lie group and M a connected manifold. If $f, h \in C^\infty(M, G)$ satisfy $\delta(f) = \delta(h)$, then $h = \lambda_g \circ f$ holds for some $g \in G$.*

Proof. The Quotient Rule leads to

$$\delta(hf^{-1}) = \delta(f^{-1}) + \mathrm{Ad}(f)\delta(h) = -\mathrm{Ad}(f)\delta(f) + \mathrm{Ad}(f)\delta(f) = 0.$$

Hence $T_m(hf^{-1}) = 0$ in each $m \in M$, so that $h \cdot f^{-1}$ is constant equal to some $g \in G$. This means that $h = \lambda_g \circ f$. □

Proposition 8.2.29. *The logarithmic derivative of $\exp_G$ is given by*

$$\delta(\exp_G)(x) = \frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x}\colon \mathfrak{g} \to \mathfrak{g},$$

where the fraction on the right means $\Phi(\operatorname{ad} x)$ for the entire function

$$\Phi(z) := \frac{1 - e^{-z}}{z} = \sum_{k=1}^{\infty} \frac{(-z)^{k-1}}{k!}.$$

The series $\Phi(\operatorname{ad} x)$ converges for each $x \in \mathfrak{g}$.

Proof. Fix $t, s \in \mathbb{R}$. Then the smooth functions $f, f_t, f_s \colon \mathbf{L}(G) \to G$, given by

$$f(x) := \exp_G((t+s)x), \quad f_t(x) := \exp_G(tx) \quad \text{and} \quad f_s(x) := \exp_G(sx),$$

satisfy $f = f_t f_s$ pointwise on $\mathbf{L}(G)$. The Product Rule (Lemma 8.2.27) therefore implies that

$$\delta(f) = \delta(f_s) + \mathrm{Ad}(f_s)^{-1}\delta(f_t).$$

For the smooth curve $\psi \colon \mathbb{R} \to \mathbf{L}(G), \psi(t) := \delta(\exp_G)_{tx}(ty)$, we now obtain

$$\begin{aligned}\psi(t+s) = \delta(f)_x(y) &= \delta(f_s)_x(y) + \mathrm{Ad}(f_s)^{-1}\delta(f_t)_x(y) \\ &= \psi(s) + \mathrm{Ad}(\exp_G(-sx))\psi(t).\end{aligned}$$

We have $\psi(0) = 0$ and

$$\psi'(0) = \lim_{t\to 0} \delta(\exp_G)_{tx}(y) = \delta(\exp_G)_0(y) = y,$$

so that taking derivatives with respect to t in 0, leads to

$$\psi'(s) = \mathrm{Ad}(\exp_G(-sx))y = e^{-\mathrm{ad}(sx)}y.$$

Now the assertion follows by integration from

$$\delta(\exp_G)_x(y) = \psi(1) = \int_0^1 \psi'(s)\, ds$$

and $\int_0^1 e^{-s\,\mathrm{ad}\,x}\, ds = \sum_{k=0}^\infty \frac{(-\mathrm{ad}\,x)^k}{(k+1)!} = \Phi(\mathrm{ad}\,x)$, which we saw already in the proof of Proposition 2.4.2. □

Definition 8.2.30. We have seen in Proposition 8.2.29 that

$$\delta(\exp_G)(x) = \frac{\mathbf{1} - e^{-\mathrm{ad}\,x}}{\mathrm{ad}\,x}.$$

From Exercise 8.2.13 we now derive that

$$\mathrm{singexp}(\mathfrak{g}) := \{x \in \mathfrak{g} \colon \mathrm{Spec}\,(\mathrm{ad}\,x) \cap 2\pi i\mathbb{Z} \not\subseteq \{0\}\}$$

is the set of singular points of $\exp_G$, and

$$\mathrm{regexp}(\mathfrak{g}) := \{x \in \mathfrak{g} \colon \mathrm{Spec}\,(\mathrm{ad}\,x) \cap 2\pi i\mathbb{Z} \subseteq \{0\}\}$$

is the set of regular points of the exponential function. We note in particular, that these sets only depend on the Lie algebra $\mathfrak{g}$, and not on the group G.

Lemma 8.2.31. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the following assertions hold for $x \in \mathfrak{g}$:*

(a) *If $x \in \mathrm{regexp}(\mathfrak{g})$ and $\exp_G x = \exp_G y$ for some $y \in \mathfrak{g}$, then*

$$[x, y] = 0 \quad \textit{and} \quad \exp_G(x - y) = \mathbf{1}.$$

(b) *If* $x \in \operatorname{singexp}(\mathfrak{g})$*, then* $\exp_G$ *is not injective on any neighborhood of* x*.*

Proof. (a) The whole one-parameter group $\exp_G(\mathbb{R}y)$ commutes with $\exp_G y = \exp_G x$, which leads to

$$\exp_G(x) = c_{\exp_G ty}(\exp_G x) = \exp_G(e^{t \operatorname{ad} y}x) \quad \text{for} \quad t \in \mathbb{R}.$$

Consequently,

$$\frac{d}{dt}\Big|_{t=0} e^{t \operatorname{ad} y}x = [y, x] \in \ker T_x(\exp) = \{0\},$$

and thus $[x, y] = 0$. This in turn leads to

$$\exp_G(x - y) = \exp_G(x)\exp_G(-y) = \mathbf{1}$$

(Lemma 8.2.8).

(b) There exists some $0 \neq y \in \mathfrak{g}$ with $T_x(\exp)y = 0$, resp., $\Phi(\operatorname{ad} x)y = 0$ (Proposition 8.2.29). We consider the one-parameter group

$$\alpha\colon \mathbb{R} \to \operatorname{Aut}(G), \quad t \mapsto c_{\exp_G ty}$$

of automorphisms of G. It is generated by the vector field

$$\mathcal{X}(g) = \frac{d}{dt}\Big|_{t=0} \exp_G(ty)g\exp_G(-ty)$$

which has in $\exp_G x$ the value

$$\begin{aligned}\mathcal{X}(\exp x) &= \frac{d}{dt}\Big|_{t=0} c_{\exp_G(ty)}\exp_G(x) = \frac{d}{dt}\Big|_{t=0} \exp_G(e^{\operatorname{ad} ty}x) = T_x(\exp)[y, x] \\ &= T_{\mathbf{1}}(\lambda_{\exp_G x})\Phi(\operatorname{ad} x)(-\operatorname{ad} x)y = T_{\mathbf{1}}(\lambda_{\exp_G x})(-\operatorname{ad} x)\Phi(\operatorname{ad} x)y = 0\end{aligned}$$

(Proposition 8.2.29). This implies that $\alpha(t)(\exp_G x) = \exp_G(e^{\operatorname{ad} ty}x) = \exp_G x$ for every $t \in \mathbb{R}$. In view of $y \neq 0$ and $\Phi(\operatorname{ad} x)y = 0$, we have $[y, x] \neq 0$ (Exercise 8.2.12), so that the curve $e^{\operatorname{ad} ty}x$ is not constant. Hence $\exp_G$ is not injective on any neighborhood of x. □

Let $U \subseteq \mathfrak{g}$ be a convex 0-neighborhood for which $\exp_G|_U$ is a diffeomorphism onto an open subset of G and $V \subseteq U$ a smaller convex open 0-neighborhood with $\exp_G V \exp_G V \subseteq \exp_G U$. Put $\log_U := (\exp_G|_U)^{-1}$ and define for $x, y \in V$

$$x * y := \log_U(\exp_G x \exp_G y),$$

which defines a smooth map $V \times V \to U$. Fix $x, y \in V$. Then the smooth curve $F(t) := x * ty$ satisfies $\exp_G F(t) = \exp_G(x)\exp_G(ty)$, so that the logarithmic derivative of this curve is

$$\delta(\exp_G)_{F(t)}F'(t) = y.$$

We now choose U so small that the power series $\Psi(z) = \frac{z\log z}{z-1}$ from Lemma 2.4.3 satisfies

$$\Psi(e^{\operatorname{ad} z})\Phi(\operatorname{ad} z) = \mathrm{id}_{\mathfrak{g}} \quad \text{for} \quad z \in U$$

(cf. Proposition 2.1.6). For $z = F(t)$, we then arrive with Proposition 8.2.29 at

$$F'(t) = \Psi(e^{\operatorname{ad} F(t)})y.$$

Now the same arguments as in Propositions 2.4.4 and 2.4.5 imply that

$$x * y = F(1) = x + y + \frac{1}{2}[x,y] + \cdots$$

is given by the convergent *Hausdorff series*:

Proposition 8.2.32. *If G is a Lie group, then there exists a convex 0-neighborhood $V \subseteq \mathfrak{g}$ such that for $x, y \in V$ the Hausdorff series*

$$x * y := x + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1} \ldots (\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m}{p_1! q_1! \ldots p_k! q_k! m!} y.$$

converges and satisfies

$$\exp_G(x * y) = \exp_G(x)\exp_G(y).$$

Exercises for Section 8.2

Exercise 8.2.1. Let G be a connected Lie group and $x \in \mathfrak{g} = \mathbf{L}(G)$. Show that the corresponding left invariant vector field $x_l \in \mathcal{V}(G)$ is biinvariant, i.e., also invariant under all right multiplications, if and only if $x \in \mathfrak{z}(\mathfrak{g})$.

Exercise 8.2.2. Let $f_1, f_2 \colon G \to H$ be two group homomorphisms. Show that the pointwise product

$$f_1 f_2 \colon G \to H, \quad g \mapsto f_1(g) f_2(g)$$

is a homomorphism if and only if $f_1(G)$ commutes with $f_2(G)$.

Exercise 8.2.3. Let M be a manifold and V a finite-dimensional vector space with a basis $(b_1, \ldots, b_n)$. Let $f \colon M \to \mathrm{GL}(V)$ be a map. Show that the following are equivalent:

(1) f is smooth.
(2) For each $v \in V$ the map $f_v \colon M \to V, m \mapsto f(m)v$ is smooth.

(3) For each i, the map $f: M \to V, m \mapsto f(m)b_i$ is smooth.

Exercise 8.2.4. A vector field X on a Lie group G is called *right invariant* if for each $g \in G$ the vector field $(\rho_g)_*X = T(\rho_g) \circ X \circ \rho_g^{-1}$ coincides with X. We write $\mathcal{V}(G)^r$ for the set of right invariant vector fields on G. Show that:

(1) The evaluation map $\mathrm{ev}_{\mathbf{1}}: \mathcal{V}(G)^r \to T_{\mathbf{1}}(G)$ is a linear isomorphism.
(2) If X is right invariant, then there exists a unique $x \in T_{\mathbf{1}}(G)$ such that $X(g) = x_r(g) := T_{\mathbf{1}}(\rho_g)x = x \cdot 0_g$ (w.r.t. multiplication in $T(G)$).
(3) If X is right invariant, then $\widetilde{X} := (\iota_G)_*X := T(\iota_G) \circ X \circ \iota_G^{-1}$ is left invariant and vice versa.
(4) Show that $(\iota_G)_*x_r = -x_l$ and $[x_r, y_r] = -[x, y]_r$ for $x, y \in T_{\mathbf{1}}(G)$.
(5) Show that each right invariant vector field is complete and determine its flow.

Exercise 8.2.5. Let M be a smooth manifold, $\varphi \in \mathrm{Diff}(M)$ and $X \in \mathcal{V}(M)$. Show that the following are equivalent:

(1) φ commutes with the flow maps $\Phi_t^X: M_t \to M$ of X, i.e., each set M_t is φ-invariant and $\Phi_t^X \circ \varphi = \varphi \circ \Phi_t^X$ holds on M_t.
(2) For each integral curve $\gamma: I \to M$ of X the curve $\varphi \circ \gamma$ also is an integral curve of X.
(3) $X = \varphi_* X = T(\varphi) \circ X \circ \varphi^{-1}$, i.e., X is φ-invariant.

Exercise 8.2.6. Let G be a Lie group. Show that any map $\varphi: G \to G$ commuting with all left multiplications λ_g, $g \in G$, is a right multiplication.

Exercise 8.2.7. Let $X, Y \in \mathcal{V}(M)$ be two commuting complete vector fields, i.e., $[X, Y] = 0$. Show that the vector field $X + Y$ is complete and that its flow is given by

$$\Phi_t^{X+Y} = \Phi_t^X \circ \Phi_t^Y \quad \text{for all} \quad t \in \mathbb{R}.$$

Exercise 8.2.8. Let V be a finite-dimensional vector space and $\mu_t(v) := tv$ for $t \in \mathbb{R}^\times$. Show that:

(1) A vector field $X \in \mathcal{V}(V)$ is linear if and only if $(\mu_t)_*X = X$ holds for all $t \in \mathbb{R}^\times$.
(2) A diffeomorphism $\varphi \in \mathrm{Diff}(V)$ is linear if and only if it commutes with all the maps μ_t, $t \in \mathbb{R}^\times$.

Exercise 8.2.9. Let G be a connected Lie group and H a Lie group. For a smooth map $f: G \to H$, the following are equivalent:

(1) f is a homomorphism.
(2) $f(\mathbf{1}) = \mathbf{1}$ and there exists a homomorphism $\psi: \mathbf{L}(G) \to \mathbf{L}(H)$ of Lie algebras with $\delta(f) = \psi \circ \kappa_G$.

Exercise 8.2.10. Let G be a connected Lie group, H a Lie group and $\alpha: G \to H$ a homomorphism defining a smooth action of G on H. For a smooth map $f: G \to H$, the following are equivalent:

(1) f is a *crossed homomorphism*, i.e.,

$$f(xy) = f(x) \cdot \alpha(x)(f(y)) \quad \text{for} \quad x, y \in G.$$

(2) $f(\mathbf{1}) = \mathbf{1}$ and $\delta(f) \in \Omega^1(G, \mathbf{L}(H))$ is equivariant, i.e.,

$$\lambda_g^* \delta(f) = \mathbf{L}(\alpha(g)) \circ \delta(f)$$

for each $g \in G$.

(3) $(f, \mathrm{id}_G)\colon G \to H \rtimes_\alpha G$ is a homomorphism.

Exercise 8.2.11. No one-parameter group $\gamma : \mathbb{R} \to \mathrm{SU}_2(\mathbb{C})$ is injective, in particular, the image of $\gamma(\mathbb{R})$ is a circle group.

Exercise 8.2.12. (i) Let A be a semisimple endomorphism of the complex vector space V and let $h(z) := \sum_{n=0}^\infty a_n z^n$ be a complex power series converging on $\mathbb{C}$. We define $h(A) := \sum_{n=0}^\infty a_n A^n$ Then

$$\ker h(A) = \bigoplus_{z \in h^{-1}(0) \cap \mathrm{Spec}\,(A)} \ker(A - z\mathbf{1}).$$

(ii) Let A be a semisimple endomorphism of the real vector space V, and let $A_\mathbb{C}\colon V_\mathbb{C} \to V_\mathbb{C}$ be its complex linear extension. Then $\mathrm{Spec}\,(A) := \mathrm{Spec}\,(A_\mathbb{C})$ decomposes into the subsets

$$S_{\mathrm{re}} := \mathrm{Spec}\,(A) \cap \mathbb{R} \quad \text{and} \quad S_{\mathrm{im}} := \mathrm{Spec}\,(A) \setminus S_{\mathrm{re}}.$$

Now let h be as above and assume, in addition, that $h(\overline{z}) = \overline{h(z)}$. Then $h(A)V \subseteq V$ and

$$\ker h(A) = \bigoplus_{z \in h^{-1}(0) \cap S_{\mathrm{re}}} \ker(A - z\mathbf{1}) \oplus \bigoplus_{x+iy \in h^{-1}(0) \cap S_{\mathrm{im}}, y>0} \ker\big(A^2 - 2xA + (x^2 + y^2)\big).$$

Exercise 8.2.13. Let $A \in \mathrm{End}(V)$, where V is a finite dimensional real vector space and

$$f(z) := \frac{1 - e^{-z}}{z} = \sum_{k=0}^\infty \frac{(-1)^k z^k}{(k+1)!}.$$

Show that $f(A)$ is invertible if and only if

$$\mathrm{Spec}\,(A) \cap 2\pi i \mathbb{Z} \subseteq \{0\}.$$

8.3 Closed Subgroups of Lie Groups and their Lie Algebras

In this section, we show that closed subgroups of Lie groups are always Lie groups and that, for a closed subgroup H of G, its Lie algebra can be computed as

$$\mathbf{L}(H) = \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

This makes it particularly easy to verify that concrete groups of matrices are Lie groups and to determine their algebras.

8.3.1 The Lie Algebra of a Closed Subgroup

Definition 8.3.1. Let G be a Lie group and $H \leq G$ a closed subgroup. We define the set

$$\mathbf{L}^e(H) := \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}$$

and observe that $\mathbb{R}\,\mathbf{L}^e(H) \subseteq \mathbf{L}^e(H)$ follows immediately from the definition.

Note that, for each $x \in \mathbf{L}(G)$, the set

$$\{t \in \mathbb{R} \colon \exp_G(tx) \in H\} = \gamma_x^{-1}(H)$$

is a closed subgroup of $\mathbb{R}$, hence either discrete cyclic or equal to $\mathbb{R}$ (cf. Exercise 8.3.4).

Example 8.3.2. We consider the Lie group $G := \mathbb{R} \times \mathbb{T}$ (the cylinder) with Lie algebra $\mathbf{L}(G) \cong \mathbb{R}^2$ (Exercise 8.1.2) and the exponential function

$$\exp_G(x, y) = (x, e^{2\pi i y}).$$

For the closed subgroup $H := \mathbb{R} \times \{\mathbf{1}\}$, we then see that $(x, y) \in \mathbf{L}^e(H)$ is equivalent to $y = 0$, but $\exp_G^{-1}(H) = \mathbb{R} \times \mathbb{Z}$.

Proposition 8.3.3. *If $H \leq G$ is a closed subgroup of the Lie group G, then $\mathbf{L}^e(H)$ is a real Lie subalgebra of $\mathbf{L}(G)$.*

Proof. Let $x, y \in \mathbf{L}^e(H)$. For $k \in \mathbb{N}$ we then have $\exp_G \frac{1}{k}x, \exp_G \frac{1}{k}y \in H$, and with the Product Formula (Proposition 8.2.14), we get

$$\exp_G(x + y) = \lim_{k \to \infty} \Big(\exp_G \frac{x}{k} \exp_G \frac{y}{k}\Big)^k \in H$$

because H is closed. Therefore $\exp_G(x + y) \in H$, and $\mathbb{R}\,\mathbf{L}^e(H) = \mathbf{L}^e(H)$ now implies $\exp_G(\mathbb{R}(x + y)) \subseteq H$, hence $x + y \in \mathbf{L}^e(H)$.

Similarly, we use the Commutator Formula to get

$$\exp_G[x, y] = \lim_{k \to \infty} \Big(\exp_G \frac{x}{k} \exp_G \frac{y}{k} \exp_G -\frac{x}{k} \exp_G -\frac{y}{k}\Big)^{k^2} \in H,$$

hence $\exp_G([x, y]) \in H$, and $\mathbb{R}\,\mathbf{L}^e(H) = \mathbf{L}^e(H)$ yields $[x, y] \in \mathbf{L}^e(H)$. □

8.3.2 The Closed Subgroup Theorem and its Consequences

As we shall see below in the Initial Subgroup Theorem 8.6.13, $\mathbf{L}^e(H)$ is a Lie algebra for any subgroup H of G. We now address more detailed information on closed subgroups of Lie groups. We start with three key lemmas providing the main information for the proof of the Closed Subgroup Theorem.

Lemma 8.3.4. *Let $W \subseteq \mathbf{L}(G)$ be an open 0-neighborhood for which $\exp_G|_W$ is a diffeomorphism and $\log_W\colon \exp_G(W) \to W$ its inverse function. Further, let $H \subseteq G$ be a closed subgroup and $(g_k)_{k\in\mathbb{N}}$ be a sequence in $H \cap \exp_G(W)$ with $g_k \neq \mathbf{1}$ for all $k \in \mathbb{N}$ and $g_k \to \mathbf{1}$. We put $y_k := \log_W g_k$ and fix a norm $\|\cdot\|$ on $\mathbf{L}(G)$. Then every cluster point of the sequence $\left\{\frac{y_k}{\|y_k\|}\colon k \in \mathbb{N}\right\}$ is contained in $\mathbf{L}^e(H)$.*

Proof. Let x be such a cluster point. Replacing the original sequence by a subsequence, in view of the Bolzano–Weierstraß Theorem, we may assume that

$$x_k := \frac{y_k}{\|y_k\|} \to x \in \mathbf{L}(G).$$

Note that this implies $\|x\| = 1$. Let $t \in \mathbb{R}$ and put $p_k := \frac{t}{\|y_k\|}$. Then $tx_k = p_k y_k$ and $y_k \to \log_W \mathbf{1} = 0$, so that

$$\exp_G(tx) = \lim_{k\to\infty} \exp_G(tx_k) = \lim_{k\to\infty} \exp_G(p_k y_k)$$

and

$$\exp_G(p_k y_k) = \exp_G(y_k)^{[p_k]} \exp_G\big((p_k - [p_k])y_k\big),$$

where $[p_k] = \max\{l \in \mathbb{Z}\colon l \leq p_k\}$ is the *Gauß function*. We therefore have

$$\|(p_k - [p_k])y_k\| \leq \|y_k\| \to 0$$

and

$$\exp_G(tx) = \lim_{k\to\infty} (\exp_G y_k)^{[p_k]} = \lim_{k\to\infty} g_k^{[p_k]} \in H,$$

because H is closed. This implies $x \in \mathbf{L}^e(H)$. □

Lemma 8.3.5. *Let $H \subseteq G$ be a closed subgroup and $E \subseteq \mathbf{L}(G)$ be a vector subspace complementing $\mathbf{L}^e(H)$. Then there exists a 0-neighborhood $U_E \subseteq E$ with*

$$H \cap \exp_G(U_E) = \{\mathbf{1}\}.$$

Proof. We argue by contradiction. If a neighborhood U_E with the required properties does not exist, then for each compact convex 0-neighborhood $V_E \subseteq E$ we have for each $k \in \mathbb{N}$:

$$(\exp_G \tfrac{1}{k} V_E) \cap H \neq \{\mathbf{1}\}.$$

For each $k \in \mathbb{N}$ we therefore find $y_k \in V_E$ with $\mathbf{1} \neq g_k := \exp_G(\frac{y_k}{k}) \in H$. Now the compactness of V_E implies that the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded, so that $\frac{y_k}{k} \to 0$, which implies $g_k \to \mathbf{1}$. Now let $x \in E$ be a cluster point of the sequence $\frac{y_k}{\|y_k\|}$ which lies in the compact set $S_E := \{z \in E : \|z\| = 1\}$. According to Lemma 8.3.4, we have $x \in \mathbf{L}^e(H) \cap E = \{0\}$ because $g_k \in H \cap W$ for k sufficiently large. We arrive at a contradiction to $\|x\| = 1$. This proves the lemma. □

Lemma 8.3.6. *Let $E, F \subseteq \mathbf{L}(G)$ be vector subspaces with $E \oplus F = \mathbf{L}(G)$. Then the map*

$$\Phi\colon E \times F \to G, \quad (x, y) \mapsto \exp_G(x)\exp_G(y),$$

restricts to a diffeomorphism of a neighborhood of $(0,0)$ to an open $\mathbf{1}$-neighborhood in G.

Proof. The Chain Rule implies that

$$\begin{aligned} T_{(0,0)}(\Phi)(x,y) &= T_{(\mathbf{1},\mathbf{1})}(m_G) \circ (T_0(\exp_G)|_E, T_0(\exp_G)|_F)(x,y) \\ &= T_{(\mathbf{1},\mathbf{1})}(m_G)(x,y) = x + y, \end{aligned}$$

Since the addition map $E \times F \to \mathbf{L}(G) \cong T_{\mathbf{1}}(G)$ is bijective, the Inverse Function Theorem implies that Φ restricts to a diffeomorphism of an open neighborhood of $(0,0)$ in $E \times F$ onto an open neighborhood of $\mathbf{1}$ in G. □

Theorem 8.3.7 (Closed Subgroup Theorem). *Let H be a closed subgroup of the Lie group G. Then the following assertions hold:*

(i) *Each 0-neighborhood in $\mathbf{L}^e(H)$ contains an open 0-neighborhood V such that $\exp_G|_V : V \to \exp_G(V)$ is a homeomorphism onto an open subset of H.*
(ii) *H is a submanifold of G and $m_H := m_G|_{H\times H}$ induces a Lie group structure on H such that the inclusion map $\iota_H : H \to G$ is a morphism of Lie groups for which $\mathbf{L}(\iota_H)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is an isomorphism of $\mathbf{L}(H)$ onto $\mathbf{L}^e(H)$.*
(iii) *Let $E \subseteq \mathbf{L}(G)$ be a vector space complement of $\mathbf{L}^e(H)$. Then there exists an open 0-neighborhood $V_E \subseteq E$ such that*

$$\varphi\colon V_E \times H \to \exp_G(V_E)H, \quad (x, h) \mapsto \exp_G(x)h$$

is a diffeomorphism onto an open subset of G.

In view of (ii) above, we shall always identify $\mathbf{L}(H)$ with the subalgebra $\mathbf{L}^e(H)$ if H is a closed subgroup of G.

Proof. (i) Let $E \subseteq \mathbf{L}(G)$ be a vector space complement of the subspace $\mathbf{L}^e(H)$ of $\mathbf{L}(G)$ and define

$$\Phi\colon E \times \mathbf{L}^e(H) \to G, \quad (x, y) \mapsto \exp_G x \exp_G y.$$

According to Lemma 8.3.6, there exist open 0-neighborhoods $U_E \subseteq E$ and $U_H \subseteq \mathbf{L}^e(H)$ such that

$$\Phi_1 := \Phi|_{U_E \times U_H} \colon U_E \times U_H \to \exp_G(U_E)\exp_G(U_H)$$

is a diffeomorphism onto an open $\mathbf{1}$-neighborhood in G. In view of Lemma 8.3.5, we may even choose U_E so small that $\exp_G(U_E) \cap H = \{\mathbf{1}\}$.

Since $\exp_G(U_H) \subseteq H$, the condition

$$g = \exp_G x \exp_G y \in H \cap (\exp_G(U_E)\exp_G(U_H))$$

implies $\exp_G x = g(\exp_G y)^{-1} \in H \cap \exp_G U_E = \{\mathbf{1}\}$. Therefore

$$H \supseteq \exp_G(U_H) = H \cap (\exp_G(U_E)\exp_G(U_H))$$

is an open $\mathbf{1}$-neighborhood in H. This proves (i).
(ii) Let Φ_1, U_E and U_H be as in (i). For $h \in H$, the set $U_h := \lambda_h(\operatorname{im}(\Phi_1)) = h \operatorname{im}(\Phi_1)$ is an open neighborhood of h in G. Moreover, the map

$$\varphi_h\colon U_h \to E \oplus \mathbf{L}^e(H) = \mathbf{L}(G), \quad x \mapsto \Phi_1^{-1}(h^{-1}x)$$

is a diffeomorphism onto the open subset $U_E \times U_H$ of $\mathbf{L}(G)$, and we have

$$\begin{aligned}\varphi_h(U_h \cap H) &= \varphi_h(h \operatorname{im}(\Phi_1) \cap H) = \varphi_h(h(\operatorname{im}(\Phi_1) \cap H)) \\ &= \varphi_h(h \exp_G(U_H)) = \{0\} \times U_H = (U_E \times U_H) \cap (\{0\} \times \mathbf{L}^e(H)).\end{aligned}$$

Therefore the family $(\varphi_h, U_h)_{h \in H}$ provides a submanifold atlas for H in G. This defines a manifold structure on H for which $\exp_G|_{U_H}$ is a local chart (see Lemma 7.6.5).

The map $m_H\colon H \times H \to H$ is a restriction of the multiplication map m_G of G, hence smooth as a map $H \times H \to G$, and since H is an initial submanifold of G, Lemma 7.6.5 implies that m_H is smooth. With a similar argument we see that the inversion ι_H of H is smooth. Therefore H is a Lie group and the inclusion map $\iota_H\colon H \to G$ a smooth homomorphism. The corresponding morphism of Lie algebras $\mathbf{L}(\iota_H)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is injective, and from $\exp_G \circ \mathbf{L}(\iota_H) = \iota_H \circ \exp_H$ it follows that its image consists of the set $\mathbf{L}^e(H)$ of all elements $x \in \mathbf{L}(G)$ with $\exp_G(\mathbb{R}x) \subseteq H$ because each element of $L^e(H)$ defines a smooth one-parameter group of H (cf. Lemma 8.2.4).
(iii) Let E be as in the proof of (i) and consider the smooth map

$$\Psi\colon E \times H \to G, \quad (x, h) \mapsto \exp_G(x)h,$$

where H carries the submanifold structure from (ii). Since $\exp_H\colon \mathbf{L}^e(H) \to H$ is a local diffeomorphism in 0, the proof of (i) implies the existence of a 0-neighborhood $U_E \subseteq E$ and a $\mathbf{1}$-neighborhood $V_H \subseteq H$ such that

$$\Psi_1 := \Psi|_{U_E \times V_H} : U_E \times V_H \to \exp_G(U_E)V_H$$

is a diffeomorphism onto an open subset of G. We further recall from Lemma 8.3.5, that we may assume, in addition, that

$$\exp_G(U_E) \cap H = \{\mathbf{1}\}. \tag{8.13}$$

We now pick a small symmetric 0-neighborhood $V_E = -V_E \subseteq U_E$ such that $\exp_G(V_E)\exp_G(V_E) \subseteq \exp_G(U_E)V_H$. Its existence follows from the continuity of the multiplication in G. We claim that the map

$$\varphi := \Psi|_{V_E \times H} : V_E \times H \to \exp_G(V_E)H$$

is a diffeomorphism onto an open subset of G. To this end, we first observe that

$$\varphi \circ (\mathrm{id}_{V_E} \times \rho_h) = \rho_h \circ \varphi \quad \text{for each} \quad h \in H,$$

i.e., $\varphi(x, h'h) = \varphi(x, h')h$, so that

$$T_{(x,h)}(\varphi) \circ (\mathrm{id}_E \times T_{\mathbf{1}}(\rho_h)) = T_{\varphi(x,\mathbf{1})}(\rho_h) \circ T_{(x,\mathbf{1})}(\varphi).$$

Since $T_{(x,\mathbf{1})}(\varphi) = T_{(x,\mathbf{1})}(\Psi)$ is invertible for each $x \in V_E$, $T_{(x,h)}(\varphi)$ is invertible for each $(x,h) \in V_E \times H$. This implies that φ is a local diffeomorphism in each point (x,h). To see that φ is injective, we observe that

$$\exp_G(x)h = \varphi(x,h) = \varphi(x',h') = \exp_G(x')h'$$

implies that

$$\exp_G(x)^{-1}\exp_G(x') = h(h')^{-1} \in \exp_G(V_E)^2 \cap H \subseteq (\exp_G(U_E)V_H) \cap H = V_H,$$

where we have used (8.13). We thus obtain $\exp_G(x') \in \exp_G(x)V_H$, so that the injectivity of Ψ_1 yields $x = x'$, which in turn leads to $h = h'$. This proves that φ is injective and a local diffeomorphism, hence a diffeomorphism. $\square$

Example 8.3.8. We take a closer look at closed subgroups of the Lie group $(V,+)$, where V is a finite-dimensional vector space. From Example 8.2.3 we know that $\exp_V = \mathrm{id}_V$. Let $H \subseteq V$ be a closed subgroup. Then

$$\mathbf{L}(H) = \{x \in V : \mathbb{R}x \subseteq H\} \subseteq H$$

is the largest vector subspace contained in H. Let $E \subseteq V$ be a vector space complement for $\mathbf{L}(H)$. Then $V \cong \mathbf{L}(H) \times E$, and we derive from $\mathbf{L}(H) \subseteq H$ that

$$H \cong \mathbf{L}(H) \times (E \cap H).$$

Lemma 8.3.5 implies the existence of some 0-neighborhood $U_E \subseteq E$ with $U_E \cap H = \{0\}$, hence that $H \cap E$ is discrete because 0 is an isolated point of $H \cap E$. Now Exercise 8.3.4 implies the existence of linearly independent elements $f_1, \ldots, f_k \in E$ with

$$E \cap H = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_k.$$

We conclude that

$$H \cong \mathbf{L}(H) \times \mathbb{Z}^k \cong \mathbb{R}^d \times \mathbb{Z}^k \quad \text{for} \quad d = \dim \mathbf{L}(H).$$

Note that $\mathbf{L}(H)$ coincides with the connected component H_0 of 0 in H.

In view of Corollary 8.2.17, we may think of Lie groups as a special class of topological groups. We may therefore ask, which subgroups of a Lie group G are Lie groups with respect to the subspace topology:

Proposition 8.3.9. *A subgroup of a Lie group is a Lie group with respect to the induced topology if and only if it is closed.*

Proof. If H is closed, then the Closed Subgroup Theorem 8.3.7 implies that H is a submanifold of G which is a Lie group.

Suppose, conversely, that H is a Lie group. Since the inclusion map $\iota \colon H \to G$ is assumed to be a topological embedding, it is in particular continuous, hence smooth by the Automatic Smoothness Theorem 8.2.16 and since ι is injective, the same holds for $\mathbf{L}(\iota) \colon \mathbf{L}(H) \to \mathbf{L}(G)$ (Proposition 8.2.13).

Let $V \subseteq \mathbf{L}(G)$ be an open convex 0-neighborhood for which $\exp_G|_V$ is a diffeomorphism onto an open subset of G. Since $\mathbf{L}(\iota)$ is continuous, there exists a 0-neighborhood $U \subseteq \mathbf{L}(H)$ such that $\exp_H|_U$ is a diffeomorphism onto an open subset of H and $\mathbf{L}(\iota)U \subseteq V$. Since H is a topological subgroup of G, there exists an open subset $U_{\mathfrak{g}} \subseteq V$ with

$$\exp_G(U_{\mathfrak{g}}) \cap H = \iota(\exp_H(U)) = \exp_G(\mathbf{L}(\iota)U).$$

Since $\mathbf{L}(\iota)U$ is locally closed in $\mathbf{L}(G)$, it now follows that H is locally closed in G, hence closed by Exercise 8.3.3. □

Definition 8.3.10. Let G be a Lie group. A *Lie subgroup* of G is a closed subgroup H together with its Lie group structure provided by Proposition 8.3.9.

8.3.3 Examples

Example 8.3.11 (Closed Subgroups of $\mathbb{T}$). Let $H \subseteq \mathbb{T} \subseteq (\mathbb{C}^\times, \cdot)$ be a closed proper (=different from $\mathbb{T}$) subgroup. Since $\dim \mathbb{T} = 1$, it follows that $\mathbf{L}(H) = \{0\}$, so that the Identity Neighborhood Theorem implies that H is discrete, hence finite because $\mathbb{T}$ is compact.

If $q \colon \mathbb{R} \to \mathbb{T}$ is the covering projection, $q^{-1}(H)$ is a closed proper subgroup of $\mathbb{R}$, hence cyclic (this is a very simple case of Exercise 8.3.4), which implies that $H = q(q^{-1}(H))$ is also cyclic. Therefore H is one of the groups

$$C_n = \{z \in \mathbb{T} \colon z^n = 1\}$$

of n-th roots of unity.

Example 8.3.12 (Subgroups of $\mathbb{T}^2$). (a) Let $H \subseteq \mathbb{T}^2$ be a closed proper subgroup. Then $\mathbf{L}(H) \neq \mathbf{L}(\mathbb{T}^2)$ implies $\dim H < \dim \mathbb{T}^2 = 2$. Further, H is compact, so that the group $\pi_0(H)$ of connected components of H is finite.

If $\dim H = 0$, then H is finite, and for $n := |H|$ it is contained in a subgroup of the form $C_n \times C_n$, where $C_n \subseteq \mathbb{T}$ is the subgroup of n-th roots of unity (cf. Example 8.3.11).

If $\dim H = 1$, then H_0 is a compact connected 1-dimensional Lie group, hence isomorphic to $\mathbb{T}$ (Exercise 8.3.5). Therefore $H_0 = \exp_{\mathbb{T}^2}(\mathbb{R}x)$ for some $x \in \mathbf{L}(H)$ with $\exp_{\mathbb{T}^2}(x) = (e^{2\pi i x_1}, e^{2\pi i x_2}) = (1,1)$, which is equivalent to $x \in \mathbb{Z}^2$. We conclude that the Lie algebras of the closed subgroups are of the form $\mathbf{L}(H) = \mathbb{R}x$ for some $x \in \mathbb{Z}^2$.

(b) For each $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the image of the 1-parameter group

$$\gamma \colon \mathbb{R} \to \mathbb{T}^2, \quad t \mapsto (e^{i\theta t}, e^{it})$$

is not closed because γ is injective. Hence the closure of $\gamma(\mathbb{R})$ is a closed subgroup of dimension at least 2, which shows that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$. The subgroup $\gamma(\mathbb{R})$ is called a *dense wind*. We leave it as an exercise to the reader to verify that the dense wind is an initial submanifold of $\mathbb{T}^2$.

As we shall see later, in many situations it is important to have some information on the center of (simply) connected Lie groups. Below we shall use Lemma 8.2.21 to determine the kernel of the adjoint representation for various Lie groups. For that we have to know their center.

Example 8.3.13. (a) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. First we recall from Proposition 1.1.10 that $Z(\mathrm{GL}_n(\mathbb{K})) = \mathbb{K}^\times \mathbf{1}$ and from Exercise 1.2.14(v) that

$$Z(\mathrm{SL}_n(\mathbb{K})) = \{z\mathbf{1} \colon z \in \mathbb{K}^\times, z^n = 1\}.$$

In particular,

$$Z(\mathrm{SL}_n(\mathbb{C})) = \{z\mathbf{1} \colon z^n = 1\} \cong C_n$$

and

$$Z(\mathrm{SL}_n(\mathbb{R})) = \begin{Bmatrix} \mathbf{1} & \text{for } n \in 2\mathbb{N}_0 + 1 \\ \{\pm \mathbf{1}\} & \text{for } n \in 2\mathbb{N}. \end{Bmatrix}$$

(b) For $g \in Z(\mathrm{SU}_n(\mathbb{C})) = \ker \mathrm{Ad}$ we likewise have $gx = xg$ for all $x \in \mathfrak{su}_n(\mathbb{C})$. From

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) + i\mathfrak{u}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) + i\mathfrak{su}_n(\mathbb{C}) + \mathbb{C}\mathbf{1},$$

we derive that $g \in Z(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}^\times \mathbf{1}$. From that we immediately get

$$Z(\mathrm{SU}_n(\mathbb{C})) = \{z\mathbf{1} \colon z^n = 1\} \cong C_n$$

and similarly we obtain

$$Z(\mathrm{U}_n(\mathbb{C})) = \{z\mathbf{1} \colon |z| = 1\} \cong \mathbb{T}.$$

(c) (cf. also Exercise 1.2.16) Next we show that

$$Z(\mathrm{O}_n(\mathbb{R})) = \{\pm\mathbf{1}\} \quad \text{and} \quad Z(\mathrm{SO}_n(\mathbb{R})) = \begin{cases} \mathrm{SO}_2(\mathbb{R}) & \text{for } n = 2 \\ \mathbf{1} & \text{for } n \in 2\mathbb{N}+1 \\ \{\pm\mathbf{1}\} & \text{for } n \in 2\mathbb{N}+2. \end{cases}$$

If $g \in Z\big(\mathrm{O}_n(\mathbb{R})\big)$, then g commutes with each orthogonal reflection

$$\sigma_v \colon \mathbb{R}^n \to \mathbb{R}^n, \quad w \mapsto w - 2\langle v, w\rangle v$$

in the hyperplane $v^\perp$, where v is a unit vector. Since $\mathbb{R}v$ is the -1-eigenspace of σ_v, this space is invariant under g (Exercise 1.1.1). This implies that for each $v \in \mathbb{R}^n$ we have $g.v \in \mathbb{R}v$ which by an elementary argument leads to $g \in \mathbb{R}^\times\mathbf{1}$ (Exercise 8.3.11). We conclude that

$$Z(\mathrm{O}_n(\mathbb{R})) = \mathrm{O}_n(\mathbb{R}) \cap \mathbb{R}^\times\mathbf{1} = \{\pm\mathbf{1}\}.$$

To determine the center of $\mathrm{SO}_n(\mathbb{R})$, we consider for orthogonal unit vectors v_1, v_2 the map $\sigma_{v_1,v_2} := \sigma_{v_1}\sigma_{v_2} \in \mathrm{SO}_n(\mathbb{R})$ (a reflection in the subspace $v_1^\top \cap v_2^\top$). Since an element $g \in Z(\mathrm{SO}_n(\mathbb{R}))$ commutes with σ_{v_1,v_2}, it leaves the plane $\mathbb{R}v_1 + \mathbb{R}v_2 = \ker(\sigma_{v_1,v_2} + \mathbf{1})$ invariant. If a linear map preserves all two-dimensional planes and $n \geq 3$, then it preserves all one-dimensional subspaces. As above, we get $g \in \mathbb{R}^\times\mathbf{1}$, which in turn leads to

$$Z(\mathrm{SO}_n(\mathbb{R})) = \mathrm{SO}_n(\mathbb{R}) \cap \mathbb{R}^\times\mathbf{1},$$

and the assertion follows.

Exercises for Section 8.3

Exercise 8.3.1. If $(H_j)_{j\in J}$ is a family of subgroups of the Lie group G, then $\mathbf{L}(\bigcap_{j\in J} H_j) = \bigcap_{j\in J} \mathbf{L}(H_j)$.

Exercise 8.3.2. Let $\varphi \colon G \to H$ be a morphism of Lie groups. Show that

$$\mathbf{L}(\ker\varphi) = \ker\mathbf{L}(\varphi).$$

Exercise 8.3.3. (a) Show that each submanifold S of a manifold M is locally closed, i.e., for each point $s \in S$ there exists an open neighborhood U of s in M such that $U \cap S$ is closed.

(b) Show that any locally closed subgroup H of a Lie group G is closed.

Exercise 8.3.4. Let $D \subseteq \mathbb{R}^n$ be a discrete subgroup. Then there exist linearly independent elements $v_1, \ldots, v_k \in \mathbb{R}^n$ with $D = \sum_{i=1}^k \mathbb{Z}v_i$.

Exercise 8.3.5 (Connected abelian Lie groups). Let A be a connected abelian Lie group. Show that

(1) $\exp_A: (\mathbf{L}(A), +) \to A$ is a morphism of Lie groups.
(2) $\exp_A$ is surjective.
(3) $\Gamma_A := \ker \exp_A$ is a discrete subgroup of $(\mathbf{L}(A), +)$.
(4) $\mathbf{L}(A)/\Gamma_A \cong \mathbb{R}^k \times \mathbb{T}^m$ for some $k, m \geq 0$. In particular, it is a Lie group and the quotient map $q_A: \mathbf{L}(A) \to \mathbf{L}(A)/\Gamma_A$ is a smooth map.
(5) $\exp_A$ factors through a diffeomorphism $\varphi: \mathbf{L}(A)/\Gamma_A \to A$.
(6) $A \cong \mathbb{R}^k \times \mathbb{T}^m$ as Lie groups.

Exercise 8.3.6 (Divisible groups). An abelian group D is called *divisible* if for each $d \in D$ and $n \in \mathbb{N}$ there exists an $a \in D$ with $a^n = d$. Show that:

(1)* If G is an abelian group, H a subgroup and $f: H \to D$ a homomorphism into an abelian divisible group D, then there exists an extension of f to a homomorphism $\widetilde{f}: G \to D$.
(2) If G is an abelian group and D a divisible subgroup, then $G \cong D \times H$ for some subgroup H of G.

Exercise 8.3.7 (Nonconnected abelian Lie groups). Let A be an abelian Lie group. Show that:

(1) The identity component of A_0 is isomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ for some $k, m \in \mathbb{N}_0$.
(2) A_0 is divisible.
(3) $A \cong A_0 \times \pi_0(A)$, where $\pi_0(A) := A/A_0$.
(4) There exists a discrete abelian group D with $A \cong \mathbb{R}^k \times \mathbb{T}^m \times D$.

Exercise 8.3.8. If $q: G \to H$ is a surjective open morphism of topological groups, then the induced map $G/\ker q \to H$ is an isomorphism of topological groups, where $G/\ker q$ is endowed with the quotient topology.

Exercise 8.3.9. If G is a topological group and $\mathbf{1} \in U \subseteq G$ a connected subset. Then all sets $U^n := U \cdots U$ are connected and so is their union $\bigcup_n U^n$.

Exercise 8.3.10. Let G be a topological group. Then for each open subset $O \subseteq G$ and for each subset $S \subseteq G$ the product sets

$$OS = \{gh: g \in O, h \in S\} \quad \text{and} \quad SO = \{hg: g \in O, h \in S\}$$

are open.

Exercise 8.3.11. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$ with $Av \in \mathbb{K}v$ for all $v \in V$. Show that $A \in \mathbb{K}\,\mathrm{id}_V$.

8.4 Constructing Lie Group Structures on Groups

In this subsection we describe some methods to construct Lie group structures on groups, starting from a manifold structure on some "identity neighborhood" for which the group operations are smooth close to $\mathbf{1}$.

8.4.1 Group Topologies from Local Data

The following lemma describes how to construct a *group topology* on a group G, i.e., a Hausdorff topology for which the group multiplication and the inversion are continuous, from a filter basis of subsets which then becomes a filter basis of identity neighborhoods for the group topology.

Definition 8.4.1. Let X be a set. A set $\mathcal{F} \subseteq \mathbb{P}(X)$ of subsets of X is called a *filter basis* if the following conditions are satisfied:

(F1) $\mathcal{F} \neq \emptyset$.
(F2) Each set $F \in \mathcal{F}$ is nonempty.
(F3) $A, B \in \mathcal{F} \Rightarrow (\exists C \in \mathcal{F})\ C \subseteq A \cap B$.

Lemma 8.4.2. *Let G be a group and $\mathcal{F}$ a filter basis of subsets of G satisfying $\bigcap \mathcal{F} = \{\mathbf{1}\}$ and*

(U1) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ VV \subseteq U$.
(U2) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ V^{-1} \subseteq U$.
(U3) $(\forall U \in \mathcal{F})(\forall g \in G)(\exists V \in \mathcal{F})\ gVg^{-1} \subseteq U$.

Then there exists a unique group topology on G such that $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods in G. This topology is given by

$$\{U \subseteq G \colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

Proof. Let

$$\tau := \{U \subseteq G \colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

First we show that τ is a topology. Clearly $\emptyset, G \in \tau$. Let $(U_j)_{j\in J}$ be a family of elements of τ and $U := \bigcup_{j\in J} U_j$. For each $g \in U$, there exists a $j_0 \in J$ with $g \in U_{j_0}$ and a $V \in \mathcal{F}$ with $gV \subseteq U_{j_0} \subseteq U$. Thus $U \in \tau$ and we see that τ is stable under arbitrary unions.

If $U_1, U_2 \in \tau$ and $g \in U_1 \cap U_2$, there exist $V_1, V_2 \in \mathcal{F}$ with $gV_i \subseteq U_i$. Since $\mathcal{F}$ is a filter basis, there exists $V_3 \in \mathcal{F}$ with $V_3 \subseteq V_1 \cap V_2$, and then $gV_3 \subseteq U_1 \cap U_2$. We conclude that $U_1 \cap U_2 \in \tau$, and hence that τ is a topology on G.

We claim that the interior U° of a subset $U \subseteq G$ is given by

$$U_1 := \{u \in U \colon (\exists V \in \mathcal{F})\ uV \subseteq U\}.$$

In fact, if there exists a $V \in \mathcal{F}$ with $uV \subseteq U$, then we pick a $W \in \mathcal{F}$ with $WW \subseteq V$ and obtain $uWW \subseteq U$, so that $uW \subseteq U_1$. Hence $U_1 \in \tau$, i.e., U is open, and it clearly is the largest open subset contained in U, i.e., $U_1 = U^\circ$. It follows in particular that U is a neighborhood of g if and only if $g \in U^\circ$, and we see in particular that $\mathcal{F}$ is a neighborhood basis at $\mathbf{1}$. The property $\bigcap \mathcal{F} = \{\mathbf{1}\}$ implies that for $x \neq y$ there exists $U \in \mathcal{F}$ with $y^{-1}x \notin U$. For $V \in \mathcal{F}$ with $VV \subseteq U$ and $W \in \mathcal{F}$ with $W^{-1} \subseteq V$ we then obtain $y^{-1}x \notin VW^{-1}$, i.e., $xW \cap yV = \emptyset$. Thus (G, τ) is a Hausdorff space.

To see that G is a topological group, we have to verify that the map

$$f\colon G \times G \to G, \quad (x,y) \mapsto xy^{-1}$$

is continuous. So let $x, y \in G$, $U \in \mathcal{F}$ and pick $V \in \mathcal{F}$ with $yVy^{-1} \subseteq U$ and $W \in \mathcal{F}$ with $WW^{-1} \subseteq V$. Then

$$f(xW, yW) = xWW^{-1}y^{-1} = xy^{-1}y(WW^{-1})y^{-1} \subseteq xy^{-1}yVy^{-1} \subseteq xy^{-1}U$$

implies that f is continuous in (x, y). □

Before we turn to Lie group structures, it is illuminating to first consider the topological variant of Lemma 8.4.3 below.

Lemma 8.4.3. *Let G be a group and $U = U^{-1}$ a symmetric subset containing $\mathbf{1}$. We further assume that U carries a Hausdorff topology for which*

(T1) *$D := \{(x,y) \in U \times U \colon xy \in U\}$ is an open subset and the group multiplication $m_U \colon D \to U, (x,y) \mapsto xy$ is continuous,*
(T2) *the inversion map $\iota_U \colon U \to U, u \mapsto u^{-1}$ is continuous, and*
(T3) *for each $g \in G$, there exists an open $\mathbf{1}$-neighborhood U_g in U with $c_g(U_g) \subseteq U$, such that the conjugation map $c_g \colon U_g \to U, x \mapsto gxg^{-1}$ is continuous.*

Then there exists a unique group topology on G for which the inclusion map $U \hookrightarrow G$ is a homeomorphism onto an open subset of G.

If, in addition, U generates G, then (T1/2) *imply* (T3).

Proof. First we consider the filter basis $\mathcal{F}$ of $\mathbf{1}$-neighborhoods in U. Then (T1) implies (U1), (T2) implies (U2), and (T3) implies (U3). Moreover, the assumption that U is Hausdorff implies that $\bigcap \mathcal{F} = \{\mathbf{1}\}$. Therefore Lemma 8.4.2 implies that G carries a unique structure of a (Hausdorff) topological group for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods.

We claim that the inclusion map $U \to G$ is an open embedding. So let $x \in U$. Then

$$U_x := U \cap x^{-1}U = \{y \in U \colon (x,y) \in D\}$$

is open in U and λ_x restricts to a continuous map $U_x \to U$ with image $U_{x^{-1}}$. Its inverse is also continuous. Hence $\lambda_x^U \colon U_x \to U_{x^{-1}}$ is a homeomorphism. We conclude that the sets of the form xV, where V a neighborhood of $\mathbf{1}$, form a basis of neighborhoods of $x \in U$. Hence the inclusion map $U \hookrightarrow G$ is an open embedding.

Suppose, in addition, that G is generated by U. For each $g \in U$, there exists an open $\mathbf{1}$-neighborhood U_g with $gU_g \times \{g^{-1}\} \subseteq D$. Then $c_g(U_g) \subseteq U$, and the continuity of m_U implies that $c_g|_{U_g} \colon U_g \to U$ is continuous.

Hence, for each $g \in U$, the conjugation c_g is continuous in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing U, it contains U^n for each $n \in \mathbb{N}$, hence all of G because G is generated by $U = U^{-1}$. Therefore (T3) follows from (T1) and (T2). □

8.4.2 Lie Group Structures from Local Data

The following theorem, the smooth version of the preceding lemma, is an important tool to construct Lie group structures on groups. It is an important supplement to the Closed Subgroup Theorem 8.3.7.

Theorem 8.4.4. *Let G be a group and $U = U^{-1}$ a symmetric subset containing $\mathbf{1}$. We further assume that U is a smooth manifold and that*

(L1) $D := \{(x,y) \in U \times U \colon xy \in U\}$ *is an open subset and the multiplication* $m_U \colon D \to U, (x,y) \mapsto xy$ *is smooth,*
(L2) *the inversion map* $\iota_U \colon U \to U, u \mapsto u^{-1}$ *is smooth, and*
(L3) *for each $g \in G$ there exists an open $\mathbf{1}$-neighborhood $U_g \subseteq U$ with $c_g(U_g) \subseteq U$ and such that the conjugation map $c_g \colon U_g \to U, x \mapsto gxg^{-1}$ is smooth.*

Then there exists a unique structure of a Lie group on G such that the inclusion map $U \hookrightarrow G$ is a diffeomorphism onto an open subset of G.

If, in addition, U generates G, then (L1/2) *imply* (L3).

Proof. From the preceding Lemma 8.4.3, we immediately obtain the unique group topology on G for which the inclusion map $U \hookrightarrow G$ is an open embedding.

Now we turn to the manifold structure. Let $V = V^{-1} \subseteq U$ be an open $\mathbf{1}$-neighborhood with $VV \times VV \subseteq D$, for which there exists a chart (φ, V) of U. For $g \in G$ we consider the maps

$$\varphi_g \colon gV \to E, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V) \subseteq E$. We claim that $(\varphi_g, gV)_{g \in G}$ is a smooth atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = VV$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)} \colon \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $VV \times VV \to U$. This proves that the charts $(\varphi_g, gU)_{g \in G}$ form a smooth atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that for each $g \in G$ the conjugation map $c_g \colon G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since all left translations are smooth, and

$$c_g \circ \lambda_x = \lambda_{c_g(x)} \circ c_g,$$

the smoothness of c_g in a neighborhood of $x \in G$ follows. Therefore all conjugations and hence also all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally, the smoothness of the multiplication follows from the smoothness in $\mathbf{1} \times \mathbf{1}$ because

$$g_1 x g_2 y = g_1 g_2 c_{g_2^{-1}}(x) y.$$

Next we show that the inclusion $U \hookrightarrow G$ of U is a diffeomorphism. So let $x \in U$ and recall the open set $U_x = U \cap x^{-1}U$. Then λ_x restricts to a smooth map $U_x \to U$ with image $U_{x^{-1}}$. Its inverse is also smooth. Hence $\lambda_x^U : U_x \to U_{x^{-1}}$ is a diffeomorphism. Since $\lambda_x : G \to G$ also is a diffeomorphism, it follows that the inclusion $\lambda_x \circ \lambda_{x^{-1}}^U : U_{x^{-1}} \to G$ is a diffeomorphism. As x was arbitrary, the inclusion of U in G is a diffeomorphic embedding.

The uniqueness of the Lie group structure is clear because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism (Proposition 8.2.13(3)).

Finally, we assume that G is generated by U. We show that in this case (L3) is a consequence of (L1) and (L2); the argument is similar to the topological case. Indeed, for each $g \in U$, there exists an open $\mathbf{1}$-neighborhood U_g with $gU_g \times \{g^{-1}\} \subseteq D$. Then $c_g(U_g) \subseteq U$, and the smoothness of m_U implies that $c_g|_{U_g} : U_g \to U$ is smooth. Hence, for each $g \in U$, the conjugation c_g is smooth in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing U, it contains U^n for each $n \in \mathbb{N}$, hence all of G because G is generated by $U = U^{-1}$. Therefore (L3) is satisfied. □

Corollary 8.4.5. *Let G be a group and $N \trianglelefteq G$ a normal subgroup of G that carries a Lie group structure. Then there exists a Lie group structure on G for which N is an open subgroup if and only if for each $g \in G$ the restriction $c_g|_N$ is a smooth automorphism of N.*

Proof. If N is an open normal subgroup of the Lie group G, then clearly all inner automorphisms of G restrict to smooth automorphisms of N.

Suppose, conversely, that N is a normal subgroup of the group G which is a Lie group and that all inner automorphisms of G restrict to smooth automorphisms of N. Then we can apply Theorem 8.4.4 with $U = N$ and obtain a Lie group structure on G for which the inclusion $N \to G$ is a diffeomorphism onto an open subgroup of G. □

Corollary 8.4.6. *Let $\varphi : G \to H$ be a continuous homomorphism of topological groups which is a covering map. If G or H is a Lie group, then the other group carries a unique Lie group structure for which φ is a morphism of Lie groups which is a local diffeomorphism.*

Proof. Since $\ker \varphi$ is discrete, there exists an open symmetric identity neighborhood $U_G \subseteq G$ for which $U_G^3 := U_G U_G U_G$ intersects $\ker(\varphi)$ in $\{\mathbf{1}\}$. For $x, y \in U_G$ with $\varphi(x) = \varphi(y)$, we then have $x^{-1}y \in U_G^2 \cap \ker(\varphi) = \{\mathbf{1}\}$, so that $\varphi|_{U_G}$ is injective. Since φ is an open map, this implies that $\varphi|_{U_G}$ is a homeomorphism onto an open subset $U_H := \varphi(U_G)$ of H.

Suppose first that G is a Lie group. Then we apply Theorem 8.4.4 to U_H, endowed with the manifold structure for which $\varphi|_{U_G}$ is a diffeomorphism. Then (L2) follows from $\varphi(x)^{-1} = \varphi(x^{-1})$. To verify the smoothness of the multiplication map

$$m_{U_H} \colon D_H := \{(a,b) \in U_H \times U_H \colon ab \in U_H\} \to U_H,$$

we first observe that, if $x, y \in U_G$ satisfy $(\varphi(x), \varphi(y)) \in D_H$, i.e., $\varphi(xy) \in U_H$, then there exists a $z \in U_G$ with $\varphi(xy) = \varphi(z)$, and $xyz^{-1} \in {U_G}^3 \cap \ker(\varphi) = \{\mathbf{1}\}$ yields $z = xy$. We thus have $D_H = (\varphi \times \varphi)(D_G)$ for

$$D_G := \{(x,y) \in U_G \times U_G \colon xy \in U_G\}$$

and the smoothness of m_H follows from the smoothness of the multiplication $m_{U_G} \colon D_G \to U_G$ and

$$m_{U_H} \circ (\varphi \times \varphi) = \varphi \circ m_{U_G}.$$

To verify (L3), we note that the surjectivity of φ implies that for each $h \in H$ there is an element $g \in G$ with $\varphi(g) = h$. Now we choose an open $\mathbf{1}$-neighborhood $U_g \subseteq U_G$ with $c_g(U_g) \subseteq U_G$ and put $U_h := \varphi(U_g)$.

If, conversely, H is a Lie group, then we apply Theorem 8.4.4 to U_G, endowed with the manifold structure for which $\varphi|_{U_G}$ is a diffeomorphism onto U_H. Again, (L2) follows right away, and (L1) follows from $(\varphi \times \varphi)(D_G) \subseteq D_H$ and the smoothness of

$$m_{U_H} \circ (\varphi \times \varphi) = \varphi \circ m_{U_G}.$$

For (L3), we choose U_g as any open $\mathbf{1}$-neighborhood in U_G with $c_g(U_g) \subseteq U$. Then the smoothness of $c_g|_{U_g}$ follows from the smoothness of $\varphi \circ c_g = c_{\varphi(g)} \circ \varphi$. □

Corollary 8.4.7. *If G is a connected Lie group and $q_G \colon \widetilde{G} \to G$ its universal covering space, then $\widetilde{G}$ carries a unique Lie group structure for which q_G is a smooth covering map. We call this Lie group the simply connected covering group of G.*

Proof. We first have to construct a (topological) group structure on the universal covering space $\widetilde{G}$. Pick an element $\widetilde{\mathbf{1}} \in q_G^{-1}(\mathbf{1})$. Then the multiplication map $m_G \colon G \times G \to G$ lifts uniquely to a continuous map $\widetilde{m}_G \colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with $\widetilde{m}_G(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$. To see that the multiplication map $\widetilde{m}_G$ is associative, we observe that

$$\begin{aligned}
&q_G \circ \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) = m_G \circ (q_G \times q_G) \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) \\
&= m_G \circ (\mathrm{id}_G \times m_G) \circ (q_G \times q_G \times q_G) = m_G \circ (m_G \times \mathrm{id}_G) \circ (q_G \times q_G \times q_G) \\
&= q_G \circ \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}),
\end{aligned}$$

so that the two continuous maps

$$\widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G), \quad \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}) \colon \widetilde{G}^3 \to G,$$

are lifts of the same map $G^3 \to G$ and both map $(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}, \widetilde{\mathbf{1}})$ to $\widetilde{\mathbf{1}}$. Hence the uniqueness of lifts implies that $\widetilde{m}_G$ is associative. We likewise obtain that the unique lift $\widetilde{\iota}_G \colon \widetilde{G} \to \widetilde{G}$ of the inversion map $\iota_G \colon G \to G$ with $\widetilde{\iota}_G(\widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ satisfies

$$\widetilde{m}_G \circ (\iota_G, \mathrm{id}_{\widetilde{G}}) = \widetilde{\mathbf{1}} = \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}}, \iota_G).$$

Finally $\lambda_{\widetilde{\mathbf{1}}}$ lifts $\lambda_{\mathbf{1}} = \mathrm{id}_G$, so that $\lambda_{\widetilde{\mathbf{1}}}(\widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ leads to $\lambda_{\widetilde{\mathbf{1}}} = \mathrm{id}_{\widetilde{G}}$, and likewise one shows that $\rho_{\widetilde{\mathbf{1}}} = \mathrm{id}_{\widetilde{G}}$, so that $\widetilde{\mathbf{1}}$ is a neutral element for the multiplication on $\widetilde{G}$. Therefore $\widetilde{m}_G$ defines on $\widetilde{G}$ a topological group structure such that $q_G \colon \widetilde{G} \to G$ is a covering morphism of topological groups. Now Corollary 8.4.6 applies. □

8.4.3 Existence of Lie Groups for a given Lie Algebra

Theorem 8.4.8 (Integral Subgroup Theorem). *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Then the subgroup $H := \langle \exp \mathfrak{h} \rangle$ of G generated by $\exp_G \mathfrak{h}$ carries a Lie group structure with the following properties:*

(a) *There exists an open 0-neighborhood $W \subseteq \mathfrak{h}$ on which the Hausdorff series converges, and*

$$\exp_{\mathfrak{h}} \colon \mathfrak{h} \to H, \quad x \mapsto \exp_G x$$

maps W diffeomorphism onto its open image in H and satisfies

$$\exp_{\mathfrak{h}}(x * y) = \exp_{\mathfrak{h}}(x) \exp_{\mathfrak{h}}(y) \quad \textit{for} \quad x, y \in W.$$

(b) *The inclusion $i_H \colon H \to G$ is a smooth morphism of Lie groups and $\mathbf{L}(i_H) \colon \mathbf{L}(H) \to \mathfrak{h}$ an isomorphism of Lie algebras. These two properties determine the Lie group structure on H uniquely.*

(c) *If $H \subseteq H_1$ for some subgroup H_1 for which H_1/H is countable, then $\mathfrak{h} = \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H_1\}$. In particular,*

$$\mathfrak{h} = \mathbf{L}^e(H) := \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

(d) *H is connected.*

(e) *H is closed in G if and only if i_H is a topological embedding.*

Proof. (a) Let $V \subseteq \mathfrak{g}$ be an open convex symmetric 0-neighborhood for which the Hausdorff series for $x * y$ converges for $x, y \in V$ and satisfies

$$\exp_G(x * y) = \exp_G(x) \exp_G(y)$$

(Proposition 8.2.32). We further assume that $\exp_G|_V$ is a diffeomorphism onto an open subset of G.

Put $W := V \cap \mathfrak{h}$. Then $x * y \in \mathfrak{h}$ for $x, y \in W$ because each summand in the Hausdorff series is an iterated Lie bracket. Further, $x * y$ defines a smooth function $W \times W \to \mathfrak{h}$ because it is the restriction of a smooth function $V \times V \to \mathfrak{g}$. We consider the subset $U := \exp_G(W) \subseteq H$. From $W = -W$ we derive $U = U^{-1}$ and we note that $\varphi := \exp_G|_W$ is injective. We may thus endow $U \subseteq H$ with the manifold structure turning φ into a diffeomorphism.

Then

$$\widetilde{D} = \{(x, y) \in W \times W : x * y \in W\}$$

is an open subset of $W \times W$ on which the $*$-multiplication is smooth, so that the multiplication $D \to U$ is also smooth. We further observe that

$$\exp_G(-x) = \exp_G(x)^{-1},$$

from which it follows that the inversion on U is smooth. Since U generates H, (L3) follows from (L1) and (L2). Therefore U satisfies all assumptions of Theorem 8.4.4, so that we obtain a Lie group structure on H for which φ, resp., $\exp_{\mathfrak{h}}$ induces a local diffeomorphism in 0.

(b) Since the map $i_H \circ \exp_{\mathfrak{h}} : \mathfrak{h} \to G$ is smooth and $\exp_{\mathfrak{h}}$ is a local diffeomorphism in 0, the inclusion $i_H : H \to G$ is smooth. Now

$$\mathbf{L}(i_H) \colon \mathbf{L}(H) \to \mathbf{L}(G)$$

is injective, and by construction, its image contains $\mathfrak{h}$ because each element of x generates a one-parameter group of H. As $\dim \mathbf{L}(H) = \dim H = \dim \mathfrak{h}$, we have $\mathbf{L}(\iota_H)\,\mathbf{L}(H) = \mathfrak{h}$.

If $\widehat{H}$ denotes another Lie group structure on the subgroup H for which $i_{\widehat{H}} : \widehat{H} \to G$ is smooth and $\mathbf{L}(i_{\widehat{H}}) \colon \mathbf{L}(\widehat{H}) \to \mathfrak{h} \subseteq \mathbf{L}(G)$ is an isomorphism of Lie algebras, then the relation

$$i_{\widehat{H}} \circ \exp_{\widehat{H}} = \exp_G \circ \mathbf{L}(\iota_{\widehat{H}}) = \exp_{\mathfrak{h}} \circ \mathbf{L}(\iota_{\widehat{H}})$$

and (a) imply that the identical map

$$j := \iota_H^{-1} \circ \iota_{\widehat{H}} : \widehat{H} \to H$$

is a bijective morphism of connected Lie groups for which $\mathbf{L}(j)$ is an isomorphism, hence an isomorphism (Proposition 8.2.13(3)).

(c) Clearly, $\exp_G(\mathfrak{h}) \subseteq H \subseteq H_1$, so that it remains to show that any $x \in \mathbf{L}(G)$ satisfying $\exp_G(\mathbb{R}x) \subseteq H_1$ is contained in $\mathfrak{h}$. Let $U_H \subseteq H$ be a **1**-neighborhood with respect to the Lie group topology for which $U_H^{-1} U_H \subseteq U$, where $U = \exp_G(W)$ is chosen as in (a). In view of Proposition 8.1.15(iv) and the countability of the set H_1/H of H-cosets in H_1, there exists a sequence $(h_n)_{n \in \mathbb{N}}$ with $H_1 = \bigcup_{n \in \mathbb{N}} h_n U_H$. Choose $\varepsilon > 0$ with $\exp_G([-2\varepsilon, 2\varepsilon]x) \subseteq \exp_G(V)$. Since the interval $[0, \varepsilon]$ is uncountable, there exists some n for which $\{t \in [0, \varepsilon] \colon \exp_G(tx) \in h_n U_H\}$ is uncountable. In particular, there are two such elements $t_1 < t_2$ and $u_1, u_2 \in U_H$ with

$$\exp_G(t_1 x) = h_n u_1 \neq \exp_G(t_2 x) = h_n u_2.$$

Then

$$\begin{aligned}\exp_G((-t_1 + t_2)x) &= \exp_G(t_1 x)^{-1} \exp_G(t_2 x) \\ &= u_1^{-1} h_n^{-1} h_n u_2 \in U_H^{-1} U_H \subseteq U = \exp_G(W).\end{aligned}$$

In particular, there is a nonzero $y \in W \subseteq \mathfrak{h}$ with

$$\exp_G y = \exp_G((-t_1 + t_2)x).$$

Since y and $(t_2 - t_1)x$ are contained in V, it follows that $x = \frac{1}{t_2 - t_1} y \in \mathfrak{h}$.

(d) Since H is generated by $\exp_H(\mathbf{L}(H))$, this follows from Lemma 8.2.9.

(e) If H is closed, then the Closed Subgroup Theorem 8.3.7 shows that H is a Lie group with respect to the subspace topology, so that the uniqueness part of (b) implies that i_H is a topological embedding.

If, conversely, i_H is a topological embedding, then the subgroup H of G is locally compact, hence in particular locally closed and therefore closed by Exercise 8.3.3. □

Remark 8.4.9. Example 8.3.12, the dense wind in the 2-torus, shows that we cannot expect that the group $H = \langle \exp_G \mathfrak{h} \rangle$ is closed in G or that the inclusion map $H \to G$ (which is a smooth homomorphism) is a topological embedding.

Definition 8.4.10. Let G be a Lie group. An *integral* subgroup H of G is a subgroup that is generated by $\exp \mathfrak{h}$ for a subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G.

The Integral Subgroup Theorem implies in particular that each Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathbf{L}(G)$ of a Lie group G is *integrable* in the sense that it is the Lie algebra of some Lie group H.

Combining this with Ado's Theorem on the existence of faithful linear representations of a Lie algebra, we obtain one of the cornerstones of the theory of Lie groups:

Theorem 8.4.11 (Lie's Third Theorem). *Each finite-dimensional Lie algebra $\mathfrak{g}$ is the Lie algebra of a connected Lie group G.*

Proof. Ado's Theorem implies that $\mathfrak{g}$ is isomorphic to a subalgebra of some $\mathfrak{gl}_n(\mathbb{R})$, so that the assertion follows directly from the Integral Subgroup Theorem. □

Exercises for Section 8.4

Exercise 8.4.1. Let $\varphi\colon G \to H$ be a surjective morphism of topological groups. Show that the following conditions are equivalent:

(1) φ is open with discrete kernel.
(2) φ is a covering in the topological sense, i.e., each $h \in H$ has an open neighborhood U such that $\varphi^{-1}(U) = \bigcup_{i \in I} U_i$ is a disjoint union of open subsets U_i for which all restrictions $\varphi|_{U_i} : U_i \to U$ are homeomorphisms.

Exercise 8.4.2. (Refining Lemma 8.4.3) Show that the conclusion of Lemma 8.4.3 is still valid if the assumption (T1) is weakened as follows: There exists an open subset $D \subseteq U \times U$ with $xy \in U$ for all $(x, y) \in D$, containing all pairs (x, x^{-1}), $(x, \mathbf{1})$, $(\mathbf{1}, x)$ for $x \in U$, such that the group multiplication $m \colon D \to U$ is continuous.

Exercise 8.4.3. Let G be an abelian group and $N \leq G$ a subgroup carrying a Lie group structure. Then there exists a unique Lie group structure on G for which N is an open subgroup.

Exercise 8.4.4. Let G be a connected topological group and $\Gamma \trianglelefteq G$ a discrete normal subgroup. Then Γ is central.

Exercise 8.4.5. Let X be a topological space and $(X_i)_{i \in I}$ connected subspaces of X with $X = \bigcup_{i \in I} X_i$. If $\bigcap_{i \in I} X_i \neq \emptyset$, then X is connected.

Exercise 8.4.6. We consider the simply connected covering group $G := \widetilde{\mathrm{SL}}_2(\mathbb{R})$ with $\mathbf{L}(G) = \mathfrak{sl}_2(\mathbb{R})$ and we write $q : G \to \mathrm{SL}_2(\mathbb{R})$ for the covering homomorphism. The map

$$\alpha : \mathbb{R} \to G, \quad t \mapsto \exp_G(t2\pi u), \quad u =: \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is injective.

8.5 Covering Theory for Lie Groups

In this section we eventually turn to applications of covering theory to Lie groups. Our goal is to see to which extent the Lie algebra and the fundamental group determine a connected Lie group. From the preceding section we know that each connected Lie group G has a simply connected covering group $\widetilde{G}$ which also carries a Lie group structure. As we shall see below, the kernel of the covering morphism $q_G \colon \widetilde{G} \to G$ can be identified with the fundamental group $\pi_1(G)$. Since $\mathbf{L}(q_G)$ is an isomorphism of Lie algebras, we then have $\mathbf{L}(G) \cong \mathbf{L}(\widetilde{G})$. We further prove the Monodromy Principle which implies that any Lie algebra morphism $\mathbf{L}(G) \to \mathbf{L}(H)$ can be integrated to a group homomorphism, provided G is 1-*connected*, i.e., connected and simply connected. From that we shall derive that the Lie algebra $\mathbf{L}(G)$ determines the corresponding simply connected group up to isomorphy.

8.5.1 Simply Connected Coverings of Lie Groups

Proposition 8.5.1. *A surjective morphism $\varphi\colon G \to H$ of Lie groups is a covering if and only if $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a linear isomorphism.*

Proof. If φ is a covering, then it is an open homomorphism with discrete kernel (Exercise 8.4.1), so that $\mathbf{L}(\ker\varphi) = \{0\}$, and Proposition 8.2.13 implies that $\mathbf{L}(\varphi)$ is bijective, hence an isomorphism of Lie algebras.

If, conversely, $\mathbf{L}(\varphi)$ is bijective, then Proposition 8.2.13 implies that

$$\mathbf{L}(\ker\varphi) = \ker\mathbf{L}(\varphi) = \{0\},$$

and the Closed Subgroup Theorem 8.3.7 shows that $\ker\varphi$ is discrete. Since $\mathbf{L}(\varphi)$ is surjective, Proposition 8.2.13 implies that φ is an open map. Finally Exercise 8.4.1 shows that φ is a covering. □

Proposition 8.5.2. *For a covering $q\colon G_1 \to G_2$ of connected Lie groups, the following equalities hold*

$$q\big(Z(G_1)\big) = Z(G_2) \quad \textit{and} \quad Z(G_1) = q^{-1}(Z(G_2)).$$

Proof. Since q is a covering, $\mathbf{L}(q)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ is an isomorphism of Lie algebras, and the adjoint representations satisfy

$$\mathrm{Ad}_{G_2}(q(g_1)) \circ \mathbf{L}(q) = \mathbf{L}(q) \circ \mathrm{Ad}_{G_1}(g_1).$$

Hence

$$Z(G_1) = \ker\mathrm{Ad}_{G_1} = q^{-1}\ker\mathrm{Ad}_{G_2} = q^{-1}(Z(G_2)).$$

Now the claim follows from the surjectivity of q. □

Theorem 8.5.3 (Lifting Theorem for Groups). *Let $q\colon G \to H$ be a covering morphism of Lie groups. If $f\colon L \to H$ is a morphism of Lie groups, where L is 1-connected, then there exists a unique lift $\widetilde{f}\colon L \to G$ which is a morphism of Lie groups.*

Proof. Since Lie groups are locally arcwise connected, the Lifting Theorem A.2.9 implies the existence of a unique lift $\widetilde{f}$ with $\widetilde{f}(\mathbf{1}_L) = \mathbf{1}_G$. Then

$$m_G \circ (\widetilde{f} \times \widetilde{f})\colon L \times L \to G$$

is the unique lift of $m_H \circ (f \times f)\colon L \times L \to H$ mapping $(\mathbf{1}_L, \mathbf{1}_L)$ to $\mathbf{1}_G$. We also have

$$q \circ \widetilde{f} \circ m_L = f \circ m_L = m_H \circ (f \times f),$$

so that $\widetilde{f} \circ m_L$ is another lift mapping $(\mathbf{1}_L, \mathbf{1}_L)$ to $\mathbf{1}_G$. Therefore

$$\widetilde{f} \circ m_L = m_G \circ (\widetilde{f} \times \widetilde{f}),$$

which means that $\widetilde{f}$ is a group homomorphism.

Since q is a local diffeomorphism and $\widetilde{f}$ is a continuous lift of f, it is also smooth in an identity neighborhood of L, hence smooth by Corollary 8.2.11. □

Theorem 8.5.4. *Let G be a connected Lie group and $q_G\colon \widetilde{G} \to G$ a universal covering homomorphism. Then* $\ker q_G \cong \pi_1(G)$ *is a discrete central subgroup and* $G \cong \widetilde{G}/\ker q_G$.

Moreover, for any discrete central subgroup $\Gamma \subseteq \widetilde{G}$, the group $\widetilde{G}/\Gamma$ is a connected Lie group with the same universal covering group as G. We thus obtain a bijection from the set of all $\mathrm{Aut}(\widetilde{G})$*-orbits in the set of discrete central subgroups of $\widetilde{G}$ onto the set of isomorphy classes of connected Lie groups whose universal covering group is isomorphic to $\widetilde{G}$.*

Proof. First we note that $\ker q_G$ is a discrete normal subgroup of the connected Lie group $\widetilde{G}$, hence central by Exercise 8.4.4. Left multiplications by elements of $\ker q_G$ lead to deck transformations of the covering $\widetilde{G} \to G$, and this group of deck transformations acts transitively on the fiber $\ker q_G$ of $\mathbf{1}$. Proposition A.2.15 now shows that

$$\pi_1(G) \cong \ker q_G \tag{8.14}$$

as groups. Since $q_G\colon \widetilde{G} \to G$ is open and surjective, we have $G \cong \widetilde{G}/\ker q_G$ as topological groups (Exercise 8.3.8), hence as Lie groups (Theorem 8.2.16).

If, conversely, $\Gamma \subseteq \widetilde{G}$ is a discrete central subgroup, then the topological quotient group $\widetilde{G}/\Gamma$ is a Lie group (Corollary 8.4.6) whose universal covering group is $\widetilde{G}$. Two such groups $\widetilde{G}/\Gamma_1$ and $\widetilde{G}/\Gamma_2$ are isomorphic if and only if there exists a Lie group automorphism $\varphi \in \mathrm{Aut}(\widetilde{G})$ with $\varphi(\Gamma_1) = \Gamma_2$ (Theorem 8.5.3). Therefore the isomorphism classes of Lie groups with the same universal covering group G are parameterized by the orbits of the group $\mathrm{Aut}(\widetilde{G})$ in the set of discrete central subgroups of $\widetilde{G}$. □

Remark 8.5.5. (a) Since the normal subgroup $\mathrm{Inn}(\widetilde{G}) := \{c_g\colon g \in \widetilde{G}\}$ of inner automorphisms acts trivially on the center of $\widetilde{G}$, the action of $\mathrm{Aut}(\widetilde{G})$ on the set of all discrete normal subgroups factors through an action of the group $\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$.

(b) Since each automorphism $\varphi \in \mathrm{Aut}(G)$ lifts to a unique automorphism $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$ (Theorem 8.5.3), we have a natural embedding $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\widetilde{G})$, and the image of this homomorphism consists of the stabilizer of the subgroup $\ker q_G \subseteq Z(\widetilde{G})$ in $\mathrm{Aut}(\widetilde{G})$.

Example 8.5.6 (Connected abelian Lie groups). Let A be a connected abelian Lie group and $\exp_A\colon\ \mathbf{L}(A) \to A$ its exponential function. Then $\exp_A$ is a morphism of Lie groups with $\mathbf{L}(\exp_A) = \mathrm{id}_{\mathbf{L}(A)}$, hence a covering morphism. Since $\mathbf{L}(A)$ is simply connected, we have $\mathbf{L}(A) \cong \widetilde{A}$ and $\ker \exp_A \cong \pi_1(A)$ is the fundamental group of A.

As special cases we obtain in particular the finite-dimensional tori

$$\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d \quad \text{with} \quad \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n.$$

If we want to classify all connected abelian Lie groups A of dimension n, we can now proceed as follows. First we note that $\widetilde{A} \cong \mathbf{L}(A) \cong (\mathbb{R}^n, +)$ as abelian Lie groups. Then $\mathrm{Aut}(\widetilde{A}) \cong \mathrm{GL}_n(\mathbb{R})$ follows from the Automatic Smoothness Theorem 8.2.16. Further, Exercise 8.3.4 implies that the discrete subgroup $\pi_1(A)$ of $\widetilde{A} \cong \mathbb{R}^n$ can be mapped by some $\varphi \in \mathrm{GL}_n(\mathbb{R})$ onto

$$\mathbb{Z}^k \cong \mathbb{Z}^k \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n.$$

Therefore

$$A \cong \mathbb{R}^n/\mathbb{Z}^k \cong \mathbb{T}^k \times \mathbb{R}^{n-k},$$

and it is clear that the number k is an isomorphy invariant of the Lie group A, namely, the rank of its fundamental subgroup. Therefore connected abelian Lie groups A are determined up to isomorphism by the pair (n,k), where $n = \dim A$ and $k = \mathrm{rk}\,\pi_1(A)$. The case where $n = k$ gives the compact connected abelian Lie groups. The above argument shows that such groups are always of the form $A \cong \mathbf{L}(A)/\Gamma$, where Γ is a discrete subgroup of $\mathbf{L}(T)$ generated by a basis for $\mathbf{L}(T)$. Such discrete subgroups are called *lattices*.

Example 8.5.7. We show that

$$\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2 = \{\pm 1\}$$

by constructing a surjective homomorphism

$$\varphi\colon\ \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$$

with $\ker\varphi = \{\pm\mathbf{1}\}$. Since $\mathrm{SU}_2(\mathbb{C})$ is homeomorphic to $\mathbb{S}^3$, it is simply connected (Exercise A.1.3), so that we then obtain $\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2$ (Theorem 8.5.4).

We consider

$$\mathfrak{su}_2(\mathbb{C}) = \{x \in \mathfrak{gl}_2(\mathbb{C})\colon x^* = -x, \mathrm{tr}\,x = 0\} = \left\{ \begin{pmatrix} ai & b \\ -\overline{b} & -ai \end{pmatrix} : b \in \mathbb{C}, a \in \mathbb{R} \right\}$$

and observe that this is a three-dimensional real subspace of $\mathfrak{gl}_2(\mathbb{C})$. We obtain on $E := \mathfrak{su}_2(\mathbb{C})$ the structure of a euclidean vector space by the scalar product

$$\beta(x,y) := \mathrm{tr}(xy^*) = -\,\mathrm{tr}(xy).$$

Now we consider the adjoint representation

$$\mathrm{Ad}\colon\ \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}(E), \quad \mathrm{Ad}(g)(x) = gxg^{-1}.$$

Then we have for $x, y \in E$ and $g \in \mathrm{SU}_2(\mathbb{C})$ the relation

$$\begin{aligned} \beta\big(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y\big) &= \mathrm{tr}(gxg^{-1}(gyg^{-1})^*) = \mathrm{tr}(gxg^{-1}(g^{-1})^* y^* g^*) \\ &= \mathrm{tr}(gxg^{-1}gy^*g^{-1}) = \mathrm{tr}(xy^*) = \beta(x,y). \end{aligned}$$

This means that

$$\mathrm{Ad}(\mathrm{SU}_2(\mathbb{C})) \subseteq \mathrm{O}(E,\beta) \cong \mathrm{O}_3(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C})$ is connected, we further obtain $\mathrm{Ad}(\mathrm{SU}_2(\mathbb{C})) \subseteq \mathrm{SO}(E,\beta) \cong \mathrm{SO}_3(\mathbb{R})$, the identity component of $\mathrm{O}(E,\beta)$.

The derived representation is given by

$$\mathrm{ad}\colon \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}(E,\beta) \cong \mathfrak{so}_3(\mathbb{R}), \quad \mathrm{ad}(x)(y) = [x,y].$$

If $\mathrm{ad}\, x = 0$, then $\mathrm{ad}\, x(i\mathbf{1}) = 0$ implies that $\mathrm{ad}\, x(\mathfrak{u}_2(\mathbb{C})) = \{0\}$, so that $\mathrm{ad}\, x(\mathfrak{gl}_2(\mathbb{C})) = \{0\}$ follows from $\mathfrak{gl}_2(\mathbb{C}) = \mathfrak{u}_2(\mathbb{C}) + i\mathfrak{u}_2(\mathbb{C})$. This implies that $x \in \mathbb{C}\mathbf{1}$, so that $\mathrm{tr}\, x = 0$ leads to $x = 0$. Hence ad is injective, and we conclude with $\dim \mathfrak{so}(E,\beta) = \dim \mathfrak{so}_3(\mathbb{R}) = 3$ that

$$\mathrm{ad}(\mathfrak{su}_2(\mathbb{C})) = \mathfrak{so}(E,\beta)$$

Therefore

$$\mathrm{Im}\,\mathrm{Ad} = \langle \exp \mathfrak{so}(E,\beta)\rangle = \mathrm{SO}(E,\beta)_0 = \mathrm{SO}(E,\beta).$$

We thus obtain a surjective homomorphism

$$\varphi\colon\ \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C})$ is compact, the quotient group $\mathrm{SU}_2(\mathbb{C})/\ker\varphi$ is also compact, and the induced bijective morphism $\mathrm{SU}_2(\mathbb{C})/\ker\varphi \to \mathrm{SO}_3(\mathbb{R})$ is a homeomorphism, hence an isomorphism of topological groups.

We further have

$$\ker\varphi = Z(\mathrm{SU}_2(\mathbb{C})) = \{\pm\mathbf{1}\}$$

(Exercise 8.3.8), so that

$$\widetilde{\mathrm{SO}_3}(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C}) \quad \text{and} \quad \pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2.$$

8.5.2 The Monodromy Principle and its Applications

To round off the picture, we still have to provide the link between Lie algebras and covering groups. The main point is that, in general, one cannot integrate morphisms of Lie algebras $\mathbf{L}(G) \to \mathbf{L}(H)$ to morphisms of the corresponding groups $G \to H$ if G is not simply connected.

Proposition 8.5.8 (Monodromy Principle). *Let G be a connected simply connected Lie group and H a group. Let V be an open symmetric connected identity neighborhood in G and $f\colon V \to H$ a function with*

$$f(xy) = f(x)f(y) \quad \textit{for} \quad x,y,xy \in V.$$

Then there exists a unique group homomorphism extending f. If, in addition, H is a Lie group and f is smooth, then its extension is also smooth.

Proof. We consider the group $G\times H$ and the subgroup $S \subseteq G\times H$ generated by the subset $U := \{(x, f(x)) \colon x \in V\}$. We endow U with the topology for which $x \mapsto (x, f(x)), V \to U$ is a homeomorphism. Note that $f(\mathbf{1})^2 = f(\mathbf{1}^2) = f(\mathbf{1})$ implies $f(\mathbf{1}) = \mathbf{1}$, which further leads to $\mathbf{1} = f(xx^{-1}) = f(x)f(x^{-1})$, so that $f(x^{-1}) = f(x)^{-1}$. Hence $U = U^{-1}$.

We now apply Lemma 8.4.3 because S is generated by U, and (T1/2) directly follow from the corresponding properties of V and $(x, f(x))(y, f(y)) = (xy, f(xy))$ for $x, y, xy \in V$. This leads to a group topology on S, for which S is a connected topological group. Indeed, its connectedness follows from $S = \bigcup_{n\in\mathbb{N}} U^n$ and the connectedness of all sets U^n (Exercise 8.4.5). The projection $p_G \colon G \times H \to G$ induces a covering homomorphism $q \colon S \to G$ because its restriction to the open $\mathbf{1}$-neighborhood U is a homeomorphism (Exercise A.2.2(c)), and the connectedness of S and the simple connectedness of G imply that q is a homeomorphism (Corollary A.2.8). Now $F := p_H \circ q^{-1} \colon G \to H$ provides the required extension of f. In fact, for $x \in U$ we have $q^{-1}(x) = (x, f(x))$, and therefore $F(x) = f(x)$.

If, in addition, H is Lie and f is smooth, then the smoothness of the extension follows directly from Corollary 8.2.11. □

Theorem 8.5.9 (Integrability Theorem for Lie Algebra Homomorphisms). *Let G be a connected simply connected Lie group, H a Lie group and $\psi \colon \mathbf{L}(G) \to \mathbf{L}(H)$ a Lie algebra morphism. Then there exists a unique morphism $\varphi \colon G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. Let $U \subseteq \mathbf{L}(G)$ be an open connected symmetric 0-neighborhood on which the BCH-product is defined and satisfies $\exp_G(x*y) = \exp_G(x)\exp_G(y)$ and $\exp_H(\psi(x) * \psi(y)) = \exp_H(\psi(x))\exp_H(\psi(y))$ for $x, y \in U$ (Theorem 8.4.8). Assume further that $\exp_G|_U$ is a homeomorphism onto an open subset of G (cf. Proposition 8.2.5).

The continuity of ψ and the fact that ψ is a Lie algebra homomorphism imply that for $x, y \in U$ the element $\psi(x * y)$ coincides with the convergent Hausdorff series $\psi(x) * \psi(y)$. We define

$$f \colon \exp_G(U) \to H, \quad f(\exp_G(x)) := \exp_H(\psi(x)).$$

For $x, y, x * y \in U$, we then obtain

$$\begin{aligned} &f(\exp_G(x)\exp_G(y)) = f(\exp_G(x * y)) = \exp_H(\psi(x * y)) \\ &= \exp_H(\psi(x) * \psi(y)) = \exp_H(\psi(x))\exp_H(\psi(y)) = f(\exp_G(x))f(\exp_G(y)). \end{aligned}$$

Then $f \colon \exp(U) \to H$ satisfies the assumptions of Proposition 8.5.8, and we see that f extends uniquely to a group homomorphism $\varphi \colon G \to H$. Since $\exp_G$ is a local diffeomorphism, f is smooth in a $\mathbf{1}$-neighborhood, and therefore φ is smooth. We finally observe that φ is uniquely determined by $\mathbf{L}(\varphi) = \psi$ because G is connected (Corollary 8.2.12). □

The following corollary can be viewed as an integrability condition for ψ.

Corollary 8.5.10. *If G is a connected Lie group and H is a Lie group, then for a Lie algebra morphism $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$, there exists a morphism $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$ if and only if $\pi_1(G) \subseteq \ker \widetilde{\varphi}$, where $\pi_1(G)$ is identified with the kernel of the universal covering map $q_G\colon \widetilde{G} \to G$ and $\widetilde{\varphi}\colon \widetilde{G} \to H$ is the unique morphism with $\mathbf{L}(\widetilde{\varphi}) = \psi \circ \mathbf{L}(q_G)$.*

Proof. If φ exists, then

$$(\varphi \circ q_G) \circ \exp_{\widetilde{G}} = \varphi \circ \exp_G \circ \mathbf{L}(q_G) = \exp_H \circ \psi \circ \mathbf{L}(q_G)$$

and the uniqueness of $\widetilde{\varphi}$ imply that $\widetilde{\varphi} = \varphi \circ q_G$ and hence that $\pi_1(G) = \ker q_G \subseteq \ker \widetilde{\varphi}$.

If, conversely, $\ker q_G \subseteq \ker \widetilde{\varphi}$, then $\varphi(q_G(g)) := \widetilde{\varphi}(g)$ defines a continuous morphism $G \cong \widetilde{G}/\ker q_G \to H$ with $\varphi \circ q_G = \widetilde{\varphi}$ (Exercise 8.3.8) and

$$\varphi \circ \exp_G \circ \mathbf{L}(q_G) = \varphi \circ q_G \circ \exp_{\widetilde{G}} = \widetilde{\varphi} \circ \exp_{\widetilde{G}} = \exp_H \circ \psi \circ \mathbf{L}(q_G). \qquad \square$$

We recall that a Lie group G is called 1-connected if it is connected and simply connected.

Corollary 8.5.11. *If G is a 1-connected Lie group with Lie algebra $\mathfrak{g}$, then the map*

$$\mathbf{L}\colon \operatorname{Aut}(G) \to \operatorname{Aut}(\mathfrak{g})$$

is an isomorphism of groups. As a closed subgroup of $\operatorname{GL}(\mathfrak{g})$, the group $\operatorname{Aut}(\mathfrak{g})$ carries a natural Lie group structure, and we endow $\operatorname{Aut}(G)$ with the Lie group structure for which $\mathbf{L}$ is an isomorphism of Lie groups. Then the action of $\operatorname{Aut}(G)$ on G is smooth.

Proof. First, we recall from Corollary 8.1.10 that for each automorphism $\varphi \in \operatorname{Aut}(G)$ the endomorphism $\mathbf{L}(\varphi)$ of $\mathfrak{g}$ also is an automorphism. That $\mathbf{L}$ is injective follows from the connectedness of G (Corollary 8.2.12) and that $\mathbf{L}$ is surjective from the Integrability Theorem 8.5.9.

If we endow $\operatorname{Aut}(G)$ with the Lie group structure for which $\mathbf{L}$ is an isomorphism, then the smoothness of the action of $\operatorname{Aut}(\mathfrak{g})$ on $\mathfrak{g}$ and Lemma 8.2.26(i) imply that the action of $\operatorname{Aut}(G)$ on G is smooth. $\square$

Remark 8.5.12. With the Integrability Theorem 8.5.9 we can give a second proof of Lie's Third Theorem which does not require Ado's Theorem.

From the structure theory of Lie algebras, we know that $\mathfrak{g}$ is a semidirect sum $\mathfrak{g} = \mathfrak{r} \rtimes_\beta \mathfrak{s}$, where $\mathfrak{r}$ is solvable and $\mathfrak{s}$ is semisimple. Since $\mathfrak{s}$ is semisimple, $\mathfrak{z}(\mathfrak{s}) = \{0\}$ and the adjoint representation of $\mathfrak{s}$ is faithful. Therefore the Integral Subgroup Theorem 8.4.8 implies the existence of a connected Lie group S with $\mathbf{L}(S) = \mathfrak{s}$. Now Theorem 8.5.9 yields a homomorphism $\alpha\colon \widetilde{S} \to \operatorname{Aut}(\mathfrak{r})$ of the simply connected covering group $\widetilde{S}$ of S integrating the adjoint representation of $\mathfrak{s}$ on the ideal $\mathfrak{r}$ of $\mathfrak{g}$.

Next we recall that the solvable Lie algebra $\mathfrak{r}$ can be written as an iterated semidirect sum

$$\mathfrak{r} = \mathfrak{r}_1 \rtimes_{\beta_1} \mathbb{R}, \quad \mathfrak{r}_1 = \mathfrak{r}_2 \rtimes_{\beta_2} \mathbb{R} \quad \text{etc.}$$

If $r = \dim \mathfrak{r}$, it follows inductively that there exists a simply connected Lie group R_j, diffeomorphic to $\mathbb{R}^{r-j}$, with $\mathbf{L}(R_j) = \mathfrak{r}_j$, $j = r, r-1, \ldots, 1$. In view of $\mathrm{Aut}(R_j) \cong \mathrm{Aut}(\mathfrak{r}_j)$, the action β_j of $\mathbb{R}$ on $\mathfrak{r}_j$ integrates to a smooth action $\alpha_j \colon \mathbb{R} \to \mathrm{Aut}(R_j)$ (Corollary 8.5.11), so that $R_{j-1} := R_j \rtimes_{\alpha_j} \mathbb{R}$ is a Lie group diffeomorphic to $\mathbb{R}^{k-j+1}$ with Lie algebra $\mathfrak{r}_{j-1}$ (Proposition 8.2.25). For $j = 1$ we obtain a simply connected Lie group R with $\mathbf{L}(R) = \mathfrak{r}$. Again we use Corollary 8.5.11 to see that the smooth action of $\widetilde{S}$ on $\mathfrak{r}$ induces a smooth action of $\widetilde{S}$ on R, so that the semidirect product $G := R \rtimes \widetilde{S}$ is a simply connected Lie group with Lie algebra $\mathfrak{r} \rtimes \mathfrak{s} = \mathfrak{g}$ (Proposition 8.2.25).

8.5.3 Classification of Lie Groups with given Lie Algebra

Let G and H be linear Lie groups. If $\varphi \colon G \to H$ is an isomorphism, then the functoriality of $\mathbf{L}$ directly implies that $\mathbf{L}(\varphi) \colon \mathbf{L}(G) \to \mathbf{L}(H)$ is an isomorphism. In fact, if $\psi \colon H \to G$ is a morphism with $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$, then

$$\mathrm{id}_{\mathbf{L}(H)} = \mathbf{L}(\mathrm{id}_H) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$$

and likewise $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi) = \mathrm{id}_{\mathbf{L}(G)}$.

In this subsection we ask to which extent a Lie group G is determined by its Lie algebra $\mathbf{L}(G)$.

Theorem 8.5.13. *Two connected Lie groups G and H have isomorphic Lie algebras if and only if their universal covering groups $\widetilde{G}$ and $\widetilde{H}$ are isomorphic.*

Proof. If $\widetilde{G}$ and $\widetilde{H}$ are isomorphic, then we clearly have

$$\mathbf{L}(G) \cong \mathbf{L}(\widetilde{G}) \cong \mathbf{L}(\widetilde{H}) \cong \mathbf{L}(H)$$

(cf. Proposition 8.5.1).

Conversely, let $\psi \colon \mathbf{L}(G) \to \mathbf{L}(H)$ be an isomorphism. Using Theorem 8.5.9, we obtain a unique morphism $\varphi \colon \widetilde{G} \to \widetilde{H}$ with $\mathbf{L}(\varphi) = \psi$ and also a unique morphism $\widehat{\varphi} \colon \widetilde{H} \to \widetilde{G}$ with $\mathbf{L}(\widehat{\varphi}) = \psi^{-1}$. Then $\mathbf{L}(\varphi \circ \widehat{\varphi}) = \mathrm{id}_{\mathbf{L}(\widetilde{G})}$ implies $\varphi \circ \widehat{\varphi} = \mathrm{id}_{\widetilde{G}}$, and likewise $\widehat{\varphi} \circ \varphi = \mathrm{id}_{\widetilde{H}}$. Therefore $\widetilde{G}$ and $\widetilde{H}$ are isomorphic Lie groups. □

Combining the preceding theorem with Theorem 8.5.4, we obtain:

Corollary 8.5.14. *Let G be a connected Lie group and $q \colon \widetilde{G} \to G$ the universal covering morphism of connected Lie groups. Then for each discrete central subgroup $\Gamma \subseteq \widetilde{G}$, the group $\widetilde{G}/\Gamma$ is a connected Lie group with $\mathbf{L}(\widetilde{G}/\Gamma) \cong \mathbf{L}(G)$ and, conversely, each Lie group with the same Lie algebra as G is isomorphic to some quotient $\widetilde{G}/\Gamma$.*

Example 8.5.15. We now describe a pair of nonisomorphic Lie groups with isomorphism Lie algebras and isomorphic fundamental groups.

Let

$$\widetilde{G} := \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$$

whose center is $C_2 \times C_2$,

$$G := \widetilde{G}/(C_2 \times \{\mathbf{1}\}) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{C})$$

and

$$H := \widetilde{G}/\{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\} \cong \mathrm{SO}_4(\mathbb{R}),$$

where the latter isomorphy follows from Proposition 8.5.21 below. Then $\pi_1(G) \cong \pi_1(H) \cong C_2$, but there is no automorphism of $\widetilde{G}$ mapping $\pi_1(G)$ to $\pi_1(H)$.

Indeed, one can show that the two direct factors are the only nontrivial connected normal subgroups of $\widetilde{G}$, so that each automorphism of $\widetilde{G}$ either preserves both or exchanges them. Since $\pi_1(H)$ is not contained in any of them, it cannot be mapped to $\pi_1(G)$ by an automorphism of $\widetilde{G}$.

Examples 8.5.16. Here are some examples of pairs of linear Lie groups with isomorphic Lie algebras:

(1) $G = \mathrm{SO}_3(\mathbb{R})$ and $\widetilde{G} \cong \mathrm{SU}_2(\mathbb{C})$ (Example 8.5.7).

(2) $G = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and $H = \mathrm{SL}_2(\mathbb{R})$: In this case we actually have a covering morphism $\varphi\colon H \to G$ coming from the adjoint representation

$$\mathrm{Ad}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(\mathbf{L}(H)) \cong \mathrm{GL}_3(\mathbb{R}).$$

On $\mathbf{L}(H) = \mathfrak{sl}_2(\mathbb{R})$ we consider the symmetric bilinear form given by $\beta(x, y) := \frac{1}{2}\operatorname{tr}(xy)$ and the basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the matrix B of β with respect to this basis is

$$B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One easily verifies that

$$\operatorname{Im}\mathrm{Ad} \subseteq \mathrm{O}(\mathbf{L}(H), \beta) \cong \mathrm{O}_{2,1}(\mathbb{R}),$$

and since $\mathrm{ad}\colon \mathbf{L}(H) \to \mathfrak{o}_{2,1}(\mathbb{R})$ is injective between spaces of the same dimension 3 (Exercise), it is bijective. Therefore $\operatorname{im}\mathrm{Ad} = \langle \exp \mathfrak{o}_{2,1}(\mathbb{R})\rangle = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and Proposition 8.5.1 imply that

$$\mathrm{Ad}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_{2,1}(\mathbb{R})_0$$

is a covering morphism. Its kernel is given by $Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm\mathbf{1}\}$.

From the polar decomposition one derives that both groups are homeomorphic to $\mathbb{T} \times \mathbb{R}^2$, and topologically the map Ad is like $(z, x, y) \mapsto (z^2, x, y)$, a two-fold covering.

(3) $G = \mathrm{SL}_2(\mathbb{C})$ and $H = \mathrm{SO}_{3,1}(\mathbb{R})_0$: This will be explained in detail and greater generality in Example 16.2.6.

Example 8.5.17. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $H = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and recall that $\widetilde{G} \cong \widetilde{H}$ follows from $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R})$ (cf. Example 8.5.16).

We further have $q_G(Z(\widetilde{G})) \subseteq Z(G) = \{\pm\mathbf{1}\}$ and $\pi_1(G) = \ker q_G \subseteq Z(\widetilde{G})$ (cf. Proposition 8.5.2). Likewise $q_H(Z(\widetilde{G})) \subseteq Z(H) = \{\mathbf{1}\}$ implies

$$Z(\widetilde{G}) \cong \pi_1(H) \cong \pi_1(\mathrm{O}_2(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})) \cong \mathbb{Z},$$

where the latter is a consequence of the polar decomposition. This implies that $Z(\widetilde{G}) \cong \mathbb{Z}$, where

$$\pi_1(G) \cong 2\mathbb{Z} \quad \text{and} \quad \pi_1(H) \cong \mathbb{Z} = Z(\widetilde{G}).$$

Therefore G and H are not isomorphic, but they have isomorphic Lie algebras and isomorphic fundamental groups.

8.5.4 Nonlinear Lie Groups

We have already seen how to describe all connected Lie groups with a given Lie algebra. To determine all such groups which are, in addition, linear turns out to be a much more subtle enterprise. If $\widetilde{G}$ is a simply connected group with a given Lie algebra, it means to determine which of the groups $\widetilde{G}/D$ are linear. As the following examples show, the answer to this problem is not easy. In fact, a complete answer requires detailed knowledge of the structure of finite-dimensional Lie algebras.

Example 8.5.18. We show that the universal covering group $G := \widetilde{\mathrm{SL}}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ is not a linear Lie group. Moreover, we show that every continuous homomorphism $\varphi\colon G \to \mathrm{GL}_n(\mathbb{R})$ satisfies $D := \pi_1(\mathrm{SL}_2(\mathbb{R})) \subseteq \ker\varphi$, hence factors through $G/D \cong \mathrm{SL}_2(\mathbb{R})$.

We consider the Lie algebra homomorphism $\mathbf{L}(\varphi)\colon \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$. Then it is easy to see that

$$\mathbf{L}(\varphi)_{\mathbb{C}}(x + iy) := \mathbf{L}(\varphi)x + i\,\mathbf{L}(\varphi)y$$

defines an extension of $\mathbf{L}(\varphi)$ to a complex linear Lie algebra homomorphism

$$\mathbf{L}(\varphi)_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C}).$$

Since the group $\mathrm{SL}_2(\mathbb{C})$ is simply connected, there exists a unique group homomorphism $\psi\colon\ \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ with $\mathbf{L}(\psi) = \mathbf{L}(\varphi)_{\mathbb{C}}$.

Let $\alpha\colon G \to G/D \cong \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{C})$ be the canonical morphism. Then

$$\mathbf{L}(\varphi) = \mathbf{L}(\varphi)_{\mathbb{C}} \circ \mathbf{L}(\alpha) = \mathbf{L}(\psi) \circ \mathbf{L}(\alpha) = \mathbf{L}(\psi \circ \alpha)$$

implies $\varphi = \psi \circ \alpha$. We conclude that $\ker\varphi \supseteq \ker\alpha = D$. Therefore G has no faithful linear representation.

Lemma 8.5.19. *If A is a Banach algebra with unit $\mathbf{1}$ and $p, q \in A$ with $[p, q] = \lambda\mathbf{1}$, then $\lambda = 0$.*

Proof. ([Wie49]) By induction we obtain

$$[p, q^n] = \lambda n q^{n-1} \quad \text{for} \quad n \in \mathbb{N}. \tag{8.15}$$

In fact,

$$[p, q^{n+1}] = [p, q]q^n + q[p, q^n] = \lambda q^n + \lambda n q^n = \lambda(n + 1)q^n.$$

Therefore

$$|\lambda|n\|q^{n-1}\| \leq 2\|p\|\|q^n\| \leq 2\|p\|\|q\|\|q^{n-1}\|$$

for each $n \in \mathbb{N}$, which leads to

$$(|\lambda|n - 2\|p\|\|q\|)\|q^{n-1}\| \leq 0.$$

If $\lambda \neq 0$, then we obtain for sufficiently large n that $q^{n-1} = 0$. For $n > 1$ we derive from (8.15) that $q^{n-2} = 0$. Inductively we arrive at the contradiction $q = 0$. ☐

If A is a finite-dimensional algebra, we may w.l.o.g. assume that it is a subalgebra of some matrix algebra $M_n(\mathbb{K})$, and then $[p, q] = \lambda\mathbf{1}$ implies

$$n\lambda = \mathrm{tr}(\lambda\mathbf{1}) = \mathrm{tr}([p, q]) = 0$$

so that $\lambda = 0$.

Example 8.5.20. We consider the three-dimensional Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \quad \text{with} \quad \mathbf{L}(G) = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that $\exp_G\colon \mathbf{L}(G) \to G$ is a diffeomorphism whose inverse is given by

$$\log(g) = (g - \mathbf{1}) - \frac{1}{2}(g - \mathbf{1})^2$$

(Proposition 2.3.3). Let

$$z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $[p,q] = z$, $[p,z] = [q,z] = 0$, $\exp \mathbb{R}z = \mathbf{1} + \mathbb{R}z \subseteq Z(G)$ and $D := \exp(\mathbb{Z}z)$ is a discrete central subgroup of G. We claim that the group G/D is not a linear Lie group. This will be verified by showing that each homomorphism $\alpha : G \to \mathrm{GL}_n(\mathbb{C})$ with $D \subseteq \ker \alpha$ satisfies $\exp(\mathbb{R}z) \subseteq \ker \alpha$.

The map $\mathbf{L}(\alpha)\colon \mathbf{L}(G) \to \mathfrak{gl}_n(\mathbb{C})$ is a Lie algebra homomorphism and we obtain linear maps

$$P := \mathbf{L}(\alpha)(p), \quad Q := \mathbf{L}(\alpha)(q) \quad \text{and} \quad Z := \mathbf{L}(\alpha)(z)$$

with $[P,Q] = Z$. Now $\exp_G z \in D = \ker \alpha$ implies that $e^Z = \alpha(\exp z) = \mathbf{1}$ and hence that Z is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$ (Exercise 2.2.12). Let $V_\lambda := \ker(Z - \lambda\mathbf{1})$. Since z is central in $\mathbf{L}(G)$, the space V_λ is invariant under G (Exercise 1.1.1), hence also under $\mathbf{L}(G)$ (Exercise 3.2.3). Therefore the restrictions $P_\lambda := P|_{V_\lambda}$ and $Q_\lambda := Q|_{V_\lambda}$ satisfy $[P_\lambda, Q_\lambda] = \lambda\,\mathrm{id}$ in the Banach algebra $\mathrm{End}(V_\lambda)$. In view of the preceding lemma, we have $\lambda = 0$. Therefore the diagonalizability of Z entails that $Z = 0$ and hence that $\mathbb{R}z \subseteq \ker \mathbf{L}(\alpha)$. It follows in particular that the group G/D has no faithful linear representation.

8.5.5 The Quaternions, $\mathbf{SU_2}(\mathbb{C})$ and $\mathbf{SO_4}(\mathbb{R})$

In this subsection we shall use the quaternion algebra $\mathbb{H}$ (Section 1.3) to get some more information on the structure of the group $\mathrm{SO}_4(\mathbb{R})$. Here the idea is to identify $\mathbb{R}^4$ with $\mathbb{H}$.

Proposition 8.5.21. *There exists a covering homomorphism*

$$\varphi\colon \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{R}) \subseteq \mathrm{GL}(\mathbb{H}), \quad \varphi(a,b).x = axb^{-1}.$$

This homomorphism is a universal covering with $\ker \varphi = \{\pm(\mathbf{1},\mathbf{1})\}$.

Proof. Since $|a| = |b| = 1$, all the maps $\varphi(a,b)\colon \mathbb{H} \to \mathbb{H}$ are orthogonal, so that φ is a homomorphism

$$\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{O}_4(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ is connected, it further follows that $\mathrm{im}\,(\varphi) \subseteq \mathrm{SO}_4(\mathbb{R})$.

To determine the kernel of φ, suppose that $\varphi(a,b) = \mathrm{id}_{\mathbb{H}}$. Then $axb^{-1} = x$ for all $x \in \mathbb{H}$. For $x = b$ we obtain in particular $a = b$. Hence $ax = xa$ for all $x \in \mathbb{H}$. With $x = I$ and $x = J$ this leads to $a \in \mathbb{R}\mathbf{1}$, and hence to $(a,b) \in \{\pm(\mathbf{1},\mathbf{1})\}$. This proves the assertion on $\ker \varphi$.

The derived representation is given by

$$\mathbf{L}(\varphi)\colon \mathfrak{su}_2(\mathbb{C}) \times \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}_4(\mathbb{R}), \quad \mathbf{L}(\varphi)(x,y)(z) = xz - zy.$$

Since $\ker \varphi$ is discrete, it follows that $\ker \mathbf{L}(\varphi) \subseteq \mathbf{L}(\ker \varphi) = \{0\}$. Hence $\mathbf{L}(\varphi)$ is injective. Next $\dim \mathfrak{so}_4(\mathbb{R}) = 6 = 2\dim \mathfrak{su}_2(\mathbb{C})$ shows that $\mathbf{L}(\varphi)$ is surjective, and we conclude that

$$\operatorname{im}(\varphi) = \langle \exp \operatorname{im} \mathbf{L}(\varphi) \rangle = \mathrm{SO}_4(\mathbb{R}).$$

Therefore φ is a covering morphism (Proposition 8.5.1). Since $\mathrm{SU}_2(\mathbb{C})$ is simply connected, $\widetilde{\mathrm{SO}_4}(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C})^2$. □

Let $G := \mathrm{SU}_2(\mathbb{C})^2$. We have just seen that this is the universal covering group of $\mathrm{SO}_4(\mathbb{R})$. On the other hand $\mathrm{SU}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}_3}(\mathbb{R})$. From $Z(\mathrm{SU}_2(\mathbb{C})) = \{\pm\mathbf{1}\}$ we derive that

$$Z(G) = \{(\mathbf{1},\mathbf{1}),(\mathbf{1},-\mathbf{1}),(-\mathbf{1},\mathbf{1}),(-\mathbf{1},-\mathbf{1})\} \cong C_2^2.$$

We have

$$G/Z(G) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}),$$

and therefore

$$\mathrm{SO}_4(\mathbb{R})/\{\pm\mathbf{1}\} \cong G/Z(G) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}).$$

The group $\mathrm{SO}_4(\mathbb{R})$ is a twofold covering group of $\mathrm{SO}_3(\mathbb{R})^2$.

Exercises for Section 8.5

Exercise 8.5.1. Let G be a connected linear Lie group. Show that

$$\mathbf{L}(Z(G)) = \mathfrak{z}(\mathbf{L}(G)) := \{x \in \mathbf{L}(G) \colon (\forall y \in \mathbf{L}(G))\ [x,y] = 0\}.$$

Exercise 8.5.2. Let $q_G \colon \widetilde{G} \to G$ be a simply connected covering of the connected Lie group G.

(1) Show that each automorphism $\varphi \in \mathrm{Aut}(G)$ has a unique lift $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$.
(2) Derive from (1) that $\mathrm{Aut}(G) \cong \{\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G}) \colon \widetilde{\varphi}(\pi_1(G)) = \pi_1(G)\}$.
(3) Show that, in general, $\{\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G}) \colon \widetilde{\varphi}(\pi_1(G)) \subseteq \pi_1(G)\}$ is not a subgroup of $\mathrm{Aut}(\widetilde{G})$.

8.6 Arcwise Connected Subgroups and Initial Subgroups

The central result of this section is Yamabe's theorem characterizing integral subgroups of a Lie group by the purely topological property of being arcwise connected. Groups which are not arcwise connected can then be dealt with using the concept of initial subgroups.

8.6.1 Yamabe's Theorem

Theorem 8.6.1 (Yamabe). *A subgroup H of a connected Lie group G is arcwise connected if and only if it is an integral subgroup. More precisely, H is of the form $\langle \exp_G \mathfrak{h} \rangle$ for the Lie subalgebra $\mathfrak{h}$ of $\mathbf{L}(G)$, determined by*

$$\mathfrak{h} = \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

Clearly, each integral subgroup is arcwise connected, so that the main point of the proof is the converse. For that it is decisive to find the Lie algebra $\mathfrak{h}$ associated with an arcwise connected subgroup H. This is our first target.

Definition 8.6.2. Let $H \subseteq G$ be a subgroup and $x \in \mathbf{L}(G)$. We call x *H-attainable* if for each $\mathbf{1}$-neighborhood U in G there exists a continuous path $\gamma \colon [0, 1] \to H$ with $\gamma(0) = \mathbf{1}$ and

$$\gamma(t) \in \exp_G(tx)U$$

for $t \in [0, 1]$. We write $A(H) \subseteq \mathbf{L}(G)$ for the set of H-attainable elements of $\mathbf{L}(G)$.

Remark 8.6.3. For each $x \in A(H)$, each neighborhood of $\exp_G(x)$ is of the form $\exp_G(x)U$, U an identity neighborhood of G, and all these sets contain elements of H. This implies that $\exp_G(x) \in \overline{H}$.

Lemma 8.6.4. *Let (X, d) be a compact metric space, G a topological group and $f \colon X \to G$ a continuous map. Then the following assertions hold:*

(a) *f is uniformly continuous in the sense that for each $\mathbf{1}$-neighborhood $U \subseteq G$ there exists a $\delta > 0$ with*

$$f(y) \in f(x)U \quad \text{for} \quad d(x, y) < \delta.$$

(b) *If (γ_n) is a sequence of paths $\gamma_n \colon [0, 1] \to X$ converging uniformly to $\gamma \colon [0, 1] \to X$, then for each $\mathbf{1}$-neighborhood U in G, the relation*

$$f(\gamma_n(t)) \in f(\gamma(t))U$$

holds for all $t \in [0, 1]$ if n is sufficiently large.

(c) *Let $K \subseteq G$ be a compact subset and (X, d) a metric space. Then each continuous function $f \colon K \to X$ is uniformly continuous in the sense that for each $\varepsilon > 0$, there exists a $\mathbf{1}$-neighborhood U in G such that*

$$d(f(x), f(y)) < \varepsilon \quad \text{holds for} \quad x, y \in K, y \in xU.$$

Proof. (a) First, we pick a $\mathbf{1}$-neighborhood V in G with $V^{-1}V \subseteq U$. Since f is continuous, there exists for each $x \in X$ a δ_x such that $f(y) \in f(x)V$ whenever $d(x, y) < \delta_x$. Now the open balls $U_x := \{y \in X \colon 2d(x, y) < \delta_x\}$ form an open cover of X, and the compactness implies the existence of a finite subcover. Hence there exist $x_1, \ldots, x_n \in X$ with $X \subseteq \bigcup_{i=1}^n U_{x_i}$. Let $\delta := \frac{1}{2}\min\{\delta_{x_i} : i = 1, \ldots, n\}$ and $x, y \in X$ with $d(x,y) < \delta$. Then there exists an x_i with $2d(x, x_i) < \delta_{x_i}$, and then we also have

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + \frac{1}{2}\delta_{x_i} \leq \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i} = \delta_{x_i}.$$

Hence $f(x), f(y) \in f(x_i)V$, and thus $f(x)^{-1}f(y) \in V^{-1}V \subseteq U$.

(b) is an immediate consequence of (a).

(c) Since f is continuous, we find for each $x \in K$ a $\mathbf{1}$-neighborhood U_x in G with

$$f(K \cap xU_x) \subseteq U_{\frac{\varepsilon}{2}}(f(x)) := \Big\{z \in X \colon d(z, f(x)) < \frac{\varepsilon}{2}\Big\}.$$

We pick for each $x \in K$ another $\mathbf{1}$-neighborhood V_x in G with $V_x V_x \subseteq U_x$. Since K is compact, it is covered by finitely many $x_i V_{x_i}$, $i = 1, \ldots, n$. Let $U := \bigcap_{i=1}^n V_{x_i}$.

Now let $x, y \in K$ with $y \in xU$. Then $x \in x_i V_{x_i}$ for some i and therefore $y \in x_i V_{x_i} U \subseteq x_i V_{x_i} V_{x_i} \subseteq x_i U_{x_i}$. Therefore $f(x), f(y) \in U_{\frac{\varepsilon}{2}}(f(x_i))$ and thus $d(f(x), f(y)) < \varepsilon$. □

Remark 8.6.5. (a) Let $\gamma \colon [0,1] \to G$ be a C^1-curve with $\gamma(0) = \mathbf{1}$ and $x := \gamma'(0)$.

$U_{\mathfrak{g}} \subseteq \mathfrak{g} = \mathbf{L}(G)$ be an open 0-neighborhood for which $\exp_G|_{U_{\mathfrak{g}}}$ is a diffeomorphism onto the open $\mathbf{1}$-neighborhood $U_G = \exp_G(U_{\mathfrak{g}})$ in G. We assume that $\mathrm{im}\,(\gamma) \subseteq U_G$, so that

$$\beta := (\exp_G|_{U_{\mathfrak{g}}})^{-1} \circ \gamma \colon [0,1] \to \mathfrak{g}$$

is a C^1-curve starting in 0. We consider the sequence of curves

$$\beta_n(t) := n\beta\Big(\frac{t}{n}\Big) = \int_0^t \beta'\big(\frac{s}{n}\big)\, ds, \quad 0 \leq t \leq 1,$$

which converges uniformly to the curve $\beta(t) := tx$. Since $\exp_G$ is uniformly continuous on each compact neighborhood of $\mathrm{im}\,(\beta)$, Lemma 8.6.4 now implies that the sequence of curves

$$\gamma_n(t) := (\exp_G \circ \beta_n)(t) = \gamma\Big(\frac{t}{n}\Big)^n$$

has the property that for each $\mathbf{1}$-neighborhood U in $\mathbf{1}$, the relation

$$\gamma_n(t) \in \exp_G(tx)U$$

holds for all $t \in [0,1]$ whenever n is sufficiently large.

(b) If H is a subgroup of G and $\mathrm{im}\,(\gamma) \subseteq H$, then $\mathrm{im}\,(\gamma_n) \subseteq H$ for each n, so that (a) implies that $x = \gamma'(0) \in A(H)$.

Lemma 8.6.6. *$A(H)$ is a Lie subalgebra of* $\mathbf{L}(G)$

Proof. We split the proof into several steps.
Step 1: $x \in A(H) \Rightarrow -x \in A(H)$:

Let U be a **1**-neighborhood in G. Since $[0,1]$ is compact and the conjugation action of G on itself is continuous, we find with Exercise 8.6.2 a **1**-neighborhood V of **1** in G with $V = V^{-1}$ and

$$\exp_G(tx)V\exp_G(-tx) \subseteq U \quad \text{for} \quad t \in [0,1]. \tag{8.16}$$

Since $x \in A(H)$, there exists a continuous path $\gamma\colon [0,1] \to H$ starting in **1** with $\gamma(t) \in \exp_G(tx)V$ for $t \in [0,1]$. We put $\widetilde{\gamma}(t) = \gamma(t)^{-1}$. Then we find with (8.16)

$$\widetilde{\gamma}(t) \in V^{-1}\exp_G(-tx) \subseteq \exp_G(-tx)U.$$

Now $\widetilde{\gamma}(t) \in H$ implies $-x \in A(H)$.
Step 2: $\mathbb{R}x \subseteq A(H)$ for each $x \in A(H)$:

It is immediately clear from the definition that $sx \in A(H)$ for each $s \in [0,1]$. In view of Step 1, we further get $[-1,1]x \subseteq A(H)$ for $x \in A(H)$. Hence it suffices to show that $kx \in A(H)$ holds for $k \in \mathbb{N}$ and $x \in A(H)$. Let $U \subseteq G$ be a **1**-neighborhood. As above, we find a **1**-neighborhood V in G with

$$\prod_{j=1}^{k+1}\big(\exp_G(t_jx)V\big) \subseteq \exp_G\Big(\sum_{j=1}^{k+1} t_jx\Big)U \quad \text{for} \quad t_j \in [0,1]. \tag{8.17}$$

To see that such a V exists, we apply Exercise 8.6.2 to the continuous map

$$\mathbb{R}^{k+1} \times G^{k+1} \to G$$
$$(t_1,\ldots,t_{k+1},g_1,\ldots,g_{k+1}) \mapsto \exp_G\Big(\sum_{j=1}^{k+1} t_jx\Big)^{-1}\prod_{j=1}^{k+1}\big(\exp_G(t_jx)g_j\big),$$

mapping $[0,1]^{k+1} \times \{\mathbf{1}\}^{k+1}$ to **1**.

Now let $\gamma\colon [0,1] \to H$ be a continuous path starting in **1** with $\gamma(t) \in \exp_G(tx)V$ for $t \in [0,1]$. For $t \in [0,1]$, we put

$$\widetilde{\gamma}(t) = \gamma(1)^{[kt]}\gamma(kt - [kt]),$$

where $[kt] = \max\{b \in \mathbb{N}\colon b \leq kt\}$. Then $\widetilde{\gamma}$ is continuous and (8.17) leads to

$$\widetilde{\gamma}(t) \in \big(\exp_G(x)V\big)^{[kt]}\exp_G((kt - [kt])x)V \subseteq \exp_G(tkx)U$$

for $t \in [0,1]$, and hence to $kx \in A(H)$.
Step 3: $x + y \in A(H)$ for $x, y \in A(H)$:

Let $\beta(t) := \exp_G(tx)\exp_G(ty)$. Then $\beta\colon [0,1] \to G$ is a smooth curve with $\beta(0) = \mathbf{1}$ and $\beta'(0) = x + y$. In view of Remark 8.6.5, there exists for each **1**-neighborhood V in G an $N \in \mathbb{N}$, such that

$$\beta\Big(\frac{t}{k}\Big)^k = \Big(\big(\exp_G\frac{tx}{k}\big)\big(\exp_G\frac{ty}{k}\big)\Big)^k \in \exp_G\big(t(x+y)\big)V \tag{8.18}$$

for all $t \in [0,1]$ and $k > N$. For any such k, there exists a **1**-neighborhood W in G with

$$\Big(\big(\exp_G \frac{tx}{k}\big)W\big(\exp_G \frac{ty}{k}\big)W\Big)^k \subseteq \Big(\big(\exp_G \frac{tx}{k}\big)\big(\exp_G \frac{ty}{k}\big)\Big)^k V \qquad (8.19)$$

for all $t \in [0,1]$.

Now let U be a **1**-neighborhood in G, choose V such that $VV \subseteq U$, W as above. Further, let $\gamma_x, \gamma_y \colon [0,1] \to H$ be continuous curves starting in **1** with $\gamma_x(t) \in (\exp_G \frac{tx}{k})W$ and $\gamma_y(t) \in (\exp_G \frac{ty}{k})W$ for $t \in [0,1]$. The existence of such paths is due to $\frac{x}{k}, \frac{y}{k} \in A(H)$. Put

$$\widetilde{\gamma}(t) := \big(\gamma_x(t)\gamma_y(t)\big)^k, \qquad (8.20)$$

so that we obtain with (8.18) and (8.19)

$$\begin{aligned}\widetilde{\gamma}(t) &\in \Big(\big(\exp_G \frac{tx}{k}\big)W\big(\exp_G \frac{ty}{k}\big)W\Big)^k \subseteq \Big(\big(\exp_G \frac{tx}{k}\big)\big(\exp_G \frac{ty}{k}\big)\Big)^k V \\ &\subseteq \exp_G\big(t(x+y)\big)VV \subseteq \exp_G\big(t(x+y)\big)U.\end{aligned}$$

This implies that $x + y \in A(H)$.
Step 4: $\mathrm{Ad}(h)x \in A(H)$ for $h \in H$ and $x \in A(H)$:

Let $h \in H$ and $x \in A(H)$. For an identity neighborhood U in G, we then find an identity neighborhood U_h of G with $c_h(U_h) \subseteq U$. If $\gamma_x \colon [0,1] \to H$ satisfies $\gamma_x(t) \in \exp_G(tx)U$ for each $t \in [0,1]$, we then obtain

$$h\gamma_x(t)h^{-1} \in c_h(\exp_G(tx))c_h(U) \subseteq \exp_G(t\,\mathrm{Ad}(h)x)U,$$

so that $\mathrm{Ad}(h)x \in A(H)$.
Step 5: $[x,y] \in A(H)$ for $x,y \in A(H)$:

The normalizer $N := \{g \in G \colon \mathrm{Ad}(g)A(H) = A(H)\}$ of the subspace $A(H) \subseteq \mathbf{L}(G)$ in G is a closed subgroup, and we know from Step 4 that it contains H. In view of Remark 8.6.3, it also contains $\exp_G(A(H))$, so that $e^{t\,\mathrm{ad}\,x}y \in A(H)$ for $x, y \in A(H)$ and $t \in \mathbb{R}$. Taking derivatives in $t = 0$, we obtain $[x,y] \in A(H)$. □

Lemma 8.6.7. *If H is an arcwise connected topological subgroup of G, then for each identity neighborhood U of* **1** *in G, the arc-component U_a of $U \cap H$ generates H.*

Proof. Let $H_U \subseteq H$ denote the subgroup generated by U_a. Further, let $h \in H$ and $\gamma \colon [0,1] \to H$ be a continuous path from **1** to h. We consider the set

$$S := \{t \in [0,1] \colon \gamma(t) \in H_U\}.$$

Then the subset $\gamma^{-1}(U)$ of S is a neighborhood of 0. For any $t_0 \in S$, there exists an $\varepsilon > 0$ with

$$\gamma(t_0)^{-1}\gamma(t) \in U_a$$

for $|t-t_0| \leq \varepsilon$. This implies that S is an open subset of $[0,1]$. If $t_1 \in [0,1] \setminus S$, then we also find an $\varepsilon > 0$ with

$$\gamma(t_1)^{-1}\gamma(t) \in U_a \subseteq H_U$$

for $|t-t_1| \leq \varepsilon$, so that $\gamma(t_1) \notin H_U$ leads to $\gamma(t) \notin H_U$ for $|t-t_1| \leq \varepsilon$. Thus $[0,1] \setminus S$ is also open in $[0,1]$, and now the connectedness of $[0,1]$ implies that $S = [0,1]$, hence $h = \gamma(1) \in H_U$. ☐

It remains to show that the integral subgroup $\langle \exp_G A(H) \rangle$ coincides with H. In the following lemma we show one inclusion.

Lemma 8.6.8. *If H is an arcwise connected subgroup of the Lie group G, then $H \subseteq \langle \exp_G(A(H)) \rangle$.*

Proof. Let $(r_m)_{m\in\mathbb{N}}$ be any sequence of positive real numbers converging to 0 with $r_{m+1} < r_m$ for $m \in \mathbb{N}$. We pick a norm $\|\cdot\|$ on $\mathfrak{g}$ and put

$$B_m := \{x \in \mathfrak{g} \colon \|x\| < r_m\} \quad \text{and} \quad U_m := \exp_G(B_m).$$

Let $\mathfrak{p} \subseteq \mathfrak{g} = \mathbf{L}(G)$ be a vector space complement of the Lie subalgebra $A(H)$. Then there exist open convex symmetric 0-neighborhoods $V_m \subseteq A(H)$ and $W_m \subseteq \mathfrak{p}$, such that the map

$$\psi_m \colon V_m \times W_m \to U_m, \quad (x,y) \mapsto \exp_G(x)\exp_G(y)$$

is a diffeomorphism onto an open subset $U'_m := \exp_G(V_m)\exp_G(W_m)$ of U_m (cf. Lemma 8.3.6). An easy induction shows that we may choose the sets V_m and W_m in such a way that $V_{m+1} \subseteq V_m$ and $W_{m+1} \subseteq W_m$ for each $m \in \mathbb{N}$.

Let H_m denote the arc-component of $H \cap U'_m$ containing $\mathbf{1}$. Then H_m generates H by Lemma 8.6.7. Therefore it remains to show that $H_m \subseteq \exp_G(A(H))$ for some $m \in \mathbb{N}$. If this is not the case, then there exists a sequence

$$h_m = \exp_G(v_m)\exp_G(w_m) \in H_m \quad \text{with} \quad v_m \in V_m, 0 \neq w_m \in W_m.$$

Then the sequence $\widetilde{w}_m := \frac{w_m}{\|w_m\|}$ has a cluster point, and by passing to a suitable subsequence of $(r_m)_{m\in\mathbb{N}}$, we may assume that the sequence $(\widetilde{w}_m)_{m\in\mathbb{N}}$ converges to some $w \in \mathfrak{p}$ with $\|w\| = 1$.

To arrive at a contradiction, we claim that $w \in A(H)$. To this end, we fix some $m \in \mathbb{N}$ and let $U^{(m)}$ be a $\mathbf{1}$-neighborhood in G. Then there exists a smaller $\mathbf{1}$-neighborhood $U''_m \subseteq U^{(m)} \cap U_m$ with

$$h_m^{-1} U''_m h_m \subseteq U^{(m)}.$$

Further, $-v_m \in A(H)$ implies the existence of a continuous path $\gamma_m \colon [0,1] \to H$ starting in $\mathbf{1}$ with $\gamma_m(t) \in \exp_G(-tv_m)U''_m$ for $t \in [0,1]$. Since H_m is arcwise connected, there also exists a continuous path $\eta_m \colon [0,1] \to H_m$ from $\mathbf{1}$ to h_m. Now

$$\widetilde{\gamma}(t) := \gamma_m(t)\eta_m(t)$$

satisfies $\widetilde{\gamma}_m(0) = \mathbf{1}$ and

$$\widetilde{\gamma}_m(1) \in \exp_G(-v_m)U_m''h_m \subseteq \exp_G(-v_m)h_m U^{(m)} = (\exp_G w_m)U^{(m)}. \tag{8.21}$$

Moreover,

$$\widetilde{\gamma}_m(t) \in (\exp_G V_m)U_m''H_m \subseteq U_m^3.$$

If m is sufficiently large, then we have a smooth inverse $\log\colon U_m' \to \mathfrak{g}$ of the exponential function, so that we may put

$$z_m := \log\big(\widetilde{\gamma}_m(1)\big).$$

Then (8.21) yields

$$z_m = w_m * w \quad \text{with some} \quad w \in V^{(m)} := \log(U^{(m)}).$$

Next we choose $U^{(m)} = \exp_G(V^{(m)})$ so small that

$$\|w_m * w - w_m\| \leq \|w_m\|^2 \leq r_m \quad \text{for all} \quad w \in V^{(m)} \tag{8.22}$$

and

$$w_m * V^{(m)} \subseteq B_m. \tag{8.23}$$

Then

$$\lim_{m\to\infty} \frac{z_m}{\|w_m\|} = \lim_{m\to\infty} \Big(\frac{z_m - w_m}{\|w_m\|} + \frac{w_m}{\|w_m\|}\Big) = w.$$

For $p_m := \|w_m\|^{-1}$ and

$$\widehat{\gamma}_m(t) := \widetilde{\gamma}_m(1)^{[tp_m]}\widetilde{\gamma}_m(tp_m - [tp_m])$$

we also have

$$\begin{aligned}
\widehat{\gamma}_m(t) &= \exp_G([tp_m]z_m)\widetilde{\gamma}_m(tp_m - [tp_m]) \\
&= \exp_G(tp_m z_m)\exp_G\big(([tp_m] - tp_m)z_m\big)\widetilde{\gamma}_m(tp_m - [tp_m]) \\
&\in \exp_G(tp_m z_m)U_m U_m^3 \subseteq \exp_G(tp_m z_m)U_m^4.
\end{aligned}$$

Finally, let U be any $\mathbf{1}$-neighborhood in G. Then there exists a $k \in \mathbb{N}$ with $U_k^5 \subseteq U$ and the sequence $\exp_G(tp_m z_m)$ converges for $m \to \infty$ uniformly in $t \in [0,1]$ to $\exp_G(tw)$ (Lemma 8.6.4(b)). Hence there exists an $m > k$ with

$$\exp_G(tp_m z_m) \subseteq (\exp_G tw)U_k \quad \text{for all} \quad t \in [0,1].$$

Then we finally arrive at

$$\widehat{\gamma}_m(t) \in \exp_G(tp_m z_m)U_m^4 \subseteq (\exp_G tw)U_k U_m^4 \subseteq (\exp_G tw)U_k^5 \subseteq \exp_G(tw)U$$

and thus $w \in A(H)$; contradicting $w \in \mathfrak{p} \setminus \{0\}$. □

To prove also the converse of Lemma 8.6.8, we need a corollary to Brouwer's Fixed Point Theorem, saying that each continuous selfmap of the closed unit ball in $\mathbb{R}^m$ (with respect to any norm) has a fixed point (cf. [Hir76, p.73]).

Lemma 8.6.9. *Let* $\|x\| := \sqrt{\sum_i x_i^2}$ *denote the euclidean norm on* $\mathbb{R}^n$. *If* $f\colon [-1,1]^m \to \mathbb{R}^m$ *is a continuous map with*

$$\|f(x) - x\| \leq \frac{1}{2} \quad \text{for} \quad x \in [-1,1]^m,$$

then $\{x \in \mathbb{R}^m \colon \|x\| < \frac{1}{2}\} \subseteq f([-1,1]^m)$.

Proof. Let $y \in \mathbb{R}^m$ with $\|y\|_\infty \leq \|y\| < \frac{1}{2}$ and put $g(x) := x - f(x) + y$. Then

$$g\colon [-1,1]^m \to [-1,1]^m$$

is a continuous map and Brouwer's Fixed Point Theorem implies that g has some fixed point x_o. Then $f(x_o) = x_o - g(x_o) + y = y$. □

Lemma 8.6.10. *If* H *is an arcwise connected subgroup of the Lie group* G, *then* $\exp_G(A(H)) \subseteq H$.

Proof. It suffices to show that H contains an open $\mathbf{1}$-neighborhood in $H^\sharp := \langle \exp_G(A(H)) \rangle$ with respect to the intrinsic Lie group topology of this group (Integral Subgroup Theorem 8.4.8). We choose a basis $x_1, \ldots, x_m$ of $A(H)$, for which the map

$$\varphi\colon]-2,2[^m \to U, \quad t = (t_1, \ldots, t_m) \mapsto \prod_{j=1}^m \exp_G(t_j x_j)$$

is a homeomorphism onto an open subset U of $H^\sharp$. The existence of such a basis follows from the Inverse Function Theorem (cf. Lemma 8.3.6). In view of the compactness of $[-1,1]^m$, there exists a compact $\mathbf{1}$-neighborhood U_1 in $H^\sharp$ with

(a) $\varphi([-1,1]^m)U_1 \subseteq U$,
(b) $\|s-t\| \leq \frac{1}{2}$ if $s \in]-2,2[^m, t \in [-1,1]^m$ and $\varphi(s) \in \varphi(t)U_1$ (Lemma 8.6.4(c)).

We further find a $\mathbf{1}$-neighborhood U_2 in $H^\sharp$ with

$$\prod_{j=1}^m \big(\exp_G(t_j x_j)U_2\big) \subseteq \varphi(t_1, \ldots, t_m)U_1 \qquad \text{for} \quad t_1, \ldots, t_m \in [-1,1].$$

Since $\pm x_j \in A(H)$, there exist continuous curves $\gamma_j\colon [-1,1] \to H$ with $\gamma_j(0) = \mathbf{1}$ and $\gamma_j(t) \in (\exp_G(t x_j))U_2$ for all t. We then have

$$\prod_{j=1}^m \gamma_j(t_j) \in \prod_{j=1}^m \big(\exp_G(t_j x_j)U_2\big) \subseteq \varphi(t_1, \ldots, t_m)U_1 \subseteq U.$$

Using φ, we may thus write

$$\prod_{j=1}^{m} \gamma_j(t_j) = \prod_{j=1}^{m} \exp_G(f_j(t_1, \ldots, t_m)x_j),$$

and obtain a continuous function

$$f\colon [-1,1]^m \to \mathbb{R}^m, \quad (t_1, \ldots, t_m) \mapsto \big(f_1(t_1, \ldots, t_m), \ldots, f_m(t_1, \ldots, t_m)\big).$$

Note that

$$\varphi\big(f_1(t_1, \ldots, t_m), \ldots, f_m(t_1, \ldots, t_m)\big) = \prod_{j=1}^{m} \gamma_j(t_j) \in \varphi(t_1, \ldots, t_m)U_1 \subseteq U,$$

so that, in view of (b), $\|f(t) - t\| \leq \frac{1}{2}$. Finally we apply Lemma 8.6.9 and see that $f([-1,1]^m)$ contains a neighborhood of 0 in $]-2,2[^m$. Therefore

$$\gamma_1([-1,1]) \cdots \gamma_m([-1,1])$$

contains a $\mathbf{1}$-neighborhood of $H^\sharp$, and this completes the proof. □

Proof. (of Theorem 8.6.1) Combining Lemmas 8.6.8 and 8.6.10, we see that $H = \langle \exp_G(A(H)) \rangle$ is the integral subgroup of G, corresponding to the Lie subalgebra $A(H)$ of $\mathbf{L}(G)$. The definition of $A(H)$ implies in particular that

$$\{x \in \mathbf{L}(G)\colon \exp_G(\mathbb{R}x) \subseteq H\} \subseteq A(H),$$

which leads to the equality

$$A(H) = \{x \in \mathbf{L}(G)\colon \exp_G(\mathbb{R}x) \subseteq H\}.$$ □

8.6.2 Initial Lie Subgroups

Definition 8.6.11. An injective morphism $\iota\colon H \to G$ of Lie groups is called *an initial Lie subgroup* if $\mathbf{L}(\iota)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is injective, and for each smooth map $f\colon M \to G$ from a smooth manifold M to G with $\operatorname{im}(f) \subseteq H$, the corresponding map $\iota^{-1} \circ f\colon M \to H$ is smooth.

The following lemma shows that the existence of an initial Lie group structure only depends on the subgroup H, considered as a subset of G.

Lemma 8.6.12. *Any subgroup H of a Lie group G carries at most one structure of an initial Lie subgroup.*

Proof. If $\iota'\colon H' \hookrightarrow G$ is another initial Lie subgroup with the same range as $\iota\colon H \to G$, then $\iota^{-1} \circ \iota'\colon H' \to H$ and $\iota'^{-1} \circ \iota\colon H \to H'$ are smooth morphisms of Lie groups, so that H and H' are isomorphic. □

Theorem 8.6.13 (Initial Subgroup Theorem). *Each subgroup H of a Lie group G carries an initial Lie subgroup structure for which the identity component H_0 coincides with the arc-component of H with respect to the subspace topology and*

$$\mathbf{L}(H) \cong \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

Proof. Let $H_a \subseteq H$ be the arc-component of H, viewed as a topological subgroup of G. According to Yamabe's Theorem 8.6.1, H_a is an integral subgroup with Lie algebra

$$\mathfrak{h} = \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H_a\} = \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

Let H_a^L denote the group H_a, endowed with its intrinsic Lie group topology for which $\exp = \exp_G|_{\mathfrak{h}} \colon \mathfrak{h} \to H_a^L$ is a local diffeomorphism in 0. Then $H \subseteq \{g \in G \colon \operatorname{Ad}(g)\mathfrak{h} = \mathfrak{h}\}$ and $c_g \circ \exp = \exp \circ \operatorname{Ad}(g)|_{\mathfrak{h}}$ imply that for each $h \in H$, the conjugation c_h defines a smooth automorphisms on H_a^L, so that H carries a Lie group structure for which H_a^L is an open subgroup (Corollary 8.4.5). Let H^L denote this Lie group. Now the inclusion map $\iota \colon H^L \to G$ is an immersion whose differential in $\mathbf{1}$ is the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

We claim that $\iota \colon H^L \to G$ is an initial Lie subgroup. In fact, let $f \colon M \to G$ be a smooth map from the smooth manifold M to G with $f(M) \subseteq H$. We have to show that f is smooth, i.e., that f is smooth in a neighborhood of each point $m \in M$. Replacing f by $f \cdot f(m)^{-1}$ and observing that the group operations in H and G are smooth, we may w.l.o.g. assume that $f(m) = \mathbf{1}$.

Let $U \subseteq \mathfrak{h}$ be an open 0-neighborhood, $\mathfrak{m} \subseteq \mathfrak{g}$ be a vector space complement to $\mathfrak{h}$ and $V \subseteq \mathfrak{m}$ an open 0-neighborhood for which the map

$$\Phi \colon U \times V \to G, \quad (x, y) \mapsto \exp_G x \exp_G y$$

is a diffeomorphism onto an open subset of G. Then

$$H_a \cap (\exp_G U \exp_G V) = \dot{\bigcup}_{y \in V, \exp_G y \in H_a} \exp_G U \exp_G y,$$

where the union on the right is disjoint because φ is bijective, and each set $\exp_G U \exp_G y$ contained in H_a also is an open subset of H^L. Since the topology of H_a^L is second countable (Proposition 8.1.15(iv)), the set $\exp_G^{-1}(H_a) \cap V$ is countable. Every smooth arc $\gamma \colon I \to \exp_G U \exp_G V$ is of the form $\gamma(t) = \exp_G \alpha(t) \exp_G \beta(t)$ with smooth arcs $\alpha \colon I \to U$ and $\beta \colon I \to V$, and for every smooth arc contained in H_a, the countability of $\beta(I)$ implies that β is constant.

We conclude that if $W \subseteq M$ is an open connected neighborhood of m with $f(W) \subseteq \exp_G U \exp_G V$, then $f(W) \subseteq \exp_G U$. Then the map

$$\exp_G|_U^{-1} \circ f|_W \colon W \to \mathfrak{h}$$

is smooth, so that the corresponding map

$$\iota^{-1} \circ f|_W = \exp_{H^L} \circ \exp_G|_U^{-1} \circ f|_W : W \to H^L$$

is also smooth. This proves that the map $\iota^{-1} \circ f : M \to H^L$ is smooth, and hence that $\iota : H^L \to G$ is an initial Lie subgroup of G. □

Exercises for Section 8.6

Exercise 8.6.1. Let X and Y be topological spaces and $K_X \subseteq X$ and $K_Y \subseteq Y$ compact subsets. If $V \subseteq X \times Y$ is an open subset containing $K_X \times K_Y$, then there exist open subsets $U_X \subseteq X$ and $U_Y \subseteq Y$ with $U_X \times U_Y \subseteq U$.

Exercise 8.6.2. Let X, Y and Z be topological spaces and $f : X \times Y \to Z$ a continuous map. If $K_X \subseteq X$ and $K_Y \subseteq Y$ are compact and $U \subseteq Z$ open with $f(K_X \times K_Y) \subseteq U$, then there exist open subsets $U_X \subseteq X$ and $U_Y \subseteq Y$ with $f(U_X \times U_Y) \subseteq U$.

Exercise 8.6.3. Construct a subgroup of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is connected but not arcwise connected.

Notes on Chapter 8

In a way this chapter describes the essence of Lie theory, namely the translation process between global and infinitesimal objects. A subtle point in this matter is the correspondence of subobjects. Given a Lie group G with Lie algebra $\mathbf{L}(G)$, Lie subalgebras generate subgroups of G, via integral manifolds or the exponential function, but unless they are closed, these groups are Lie groups only if one changes the topology. On the other hand, the Initial Subgroup Theorem 8.6.13 shows that *any* subgroup H can be given a topology such that it becomes a Lie group whose connected component arises in this way. Thus in defining the concept of a *Lie subgroup* one has to decide whether one wants to include the topology into the structure or not. If not, *any* subgroup is a Lie subgroup and the concept is superfluous. If one makes the topology part of the structure, only closed subgroups will be Lie subgroups, which explains our Definition 8.3.10. Of course this does not mean that the nonclosed subgroups associated with Lie subgroups are not important. So they also get names. In the literature arcwise connected subgroups often go for instance under the name *analytic subgroups*. Singling out this class of subgroups is of course motivated by Yamabe's Theorem 8.6.1. We prefer to call them *integral subgroups* since they systematically arise as integral manifolds. If one wants to characterize the Lie group structure of general subgroups, in view of Lemma 8.6.12 and the Initial Subgroup Theorem 8.6.13, the initial submanifold property is the crucial one. Our terminology concerning different kinds of subgroup basically follows Bourbaki (cf. [Bou89], Chap. 3).

In the 1890s Sophus Lie developed his theory of differentiable groups (called continuous groups at a time when the concept of a topological space was not yet developed) to study symmetries of differential equations. The infinitesimal and local theory was worked out and further developed by Friedrich Engel (1861–1941), Wilhelm Killing (1847–1923), and Élie Cartan (1869–1951). The global theory of Lie groups was initiated by Hermann Weyl (1885–1955) in the seminal series [We25] of papers from 1925/26, motivated by representation theory and harmonic analysis. It inspired Élie Cartan to reconsider his approach to Lie groups and eventually lead to him to deep insights into the topology of Lie groups as well as to the invention of symmetric spaces (see [Ca30]). An important role in the circulation of Lie theoretic ideas was played by Claude Chevalley (1909–1984) through his papers and in particular his book [Ch46] (see also [BG81]). Later developments were often motivated by either representation theory or harmonic analysis, pioneered by Harish-Chandra (1923–1983) and Israil M. Gelfand (1913–2009).

9

Smooth Actions of Lie Groups

In many areas of mathematics Lie groups appear naturally as symmetry groups. Examples are groups of isometries of Riemannian manifolds, groups of holomorphic automorphisms of complex domains or groups of canonical transformations in hamiltonian mechanics. In all these cases one considers group actions on manifolds by smooth maps. Even though the focus of this book is the geometry and structure theory of Lie groups rather than their applications, we have to study the concept of a smooth action of a Lie group on a manifold in some detail since it is an essential tool in the smooth versions of group theoretic considerations like the study of quotient groups and conjugacy classes.

In the first section we collect some basic facts on orbits and stabilizers for smooth actions. Section 9.1.2 is devoted to the structure of homogeneous spaces. Its results are used to provide orbits of smooth actions with manifold structures. We also introduce frame bundles which turn out to be very useful in the description of bundles with symmetries. With this preparation it is easy to describe general tensor and density bundles as well as the corresponding sections. Using (invariant) densities and differential forms, one can then quickly describe a basic integration theory for manifolds together with its interaction with symmetries. In particular, we find the Haar measure on a Lie group and its basic properties, which allows us to use averaging techniques in Lie theory.

A smooth action of a Lie group on a manifold gives rise to a representation of the Lie algebra by vector fields on the manifold. In Section 9.5 we give a proof of Palais' Theorem which gives conditions under which, conversely, a finite dimensional Lie algebra of vector fields can be integrated to a smooth group action.

9.1 Homogeneous Spaces

In this section we begin our study of the structure of smooth actions of Lie groups on smooth manifolds (cf. Definition 8.1.11) with some elementary observations on orbits and stabilizers.

9.1.1 Orbits and Stabilizers

Definition 9.1.1. Let $\sigma\colon G\times M \to M, (g,m) \mapsto g.m$ be a group action. In the following we write $\sigma_g(m) := \sigma(g,m)$ for the corresponding diffeomorphisms of M and $\sigma^m(g) := \sigma(g,m)$ for the *orbit map* of m. For $m \in M$, the set

$$\mathcal{O}_m := G.m := \{g.m\colon g \in G\} = \{\sigma(g,m)\colon g \in G\}$$

is called the *orbit of* m. The action is said to be *transitive* if there exists only one orbit, i.e., for $x, y \in M$, there exists a $g \in G$ with $y = g.x$. We write $M/G := \{\mathcal{O}_m\colon m \in M\}$ for the set of G-orbits on M.

Remark 9.1.2. If $\sigma\colon G \times X \to X$ is an action of G on X, then the orbits form a partition of X (Exercise 9.1.1).

A subset $R \subseteq X$ is called a set of *representatives for the action* if each G-orbit in X meets R exactly once:

$$(\forall x \in X) \quad |R \cap \mathcal{O}_x| = 1.$$

Examples 9.1.3. (1) We consider the action of the circle group

$$\mathbb{T} = \{z \in \mathbb{C}^\times : |z| = 1\}$$

on $\mathbb{C}$ by

$$\sigma\colon \mathbb{T} \times \mathbb{C} \to \mathbb{C}, \quad t.z = tz.$$

The orbits of this action are concentric circles:

$$\mathcal{O}_z = \{tz\colon t \in \mathbb{T}\} = \{w \in \mathbb{C}\colon |w| = |z|\}.$$

A set of representatives is given by

$$R := [0,\infty[= \{r \in \mathbb{R}\colon r \geq 0\}.$$

(2) For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and the action

$$\sigma\colon \mathrm{GL}_n(\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^n, \quad (g,x) \mapsto gx,$$

there are only two orbits:

$$\mathcal{O}_0 = \{0\} \quad \text{and} \quad \mathcal{O}_x = \mathbb{K}^n \setminus \{0\} \quad \text{for} \quad x \neq 0.$$

Each non-zero vector $x \in \mathbb{K}^n$ can be complemented to a basis for $\mathbb{K}^n$, hence arises as a first column of an invertible matrix g. Then $ge_1 = x$ implies that $\mathcal{O}_x = \mathcal{O}_{e_1}$. We conclude that $\mathbb{K}^n \setminus \{0\} = \mathcal{O}_{e_1}$.

(3) For the conjugation action

$$\mathrm{GL}_n(\mathbb{K}) \times M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad (g, A) \mapsto gAg^{-1},$$

the orbits are the similarity classes of matrices $\mathcal{O}_A = \{gAg^{-1} \colon g \in \mathrm{GL}_n(\mathbb{K})\}$.

Definition 9.1.4. Let $\sigma \colon G \times M \to M, (g, m) \mapsto g.m$, be an action of the group G on M. For $m \in M$, the subset

$$G_m := \{g \in G \colon g.m = m\}$$

is called the *stabilizer of* m. For $g \in G$ we write

$$\mathrm{Fix}(g) := M^g := \{m \in M \colon g.m = m\}$$

for the set of fixed points of g in M. We then have

$$m \in M^g \quad \Longleftrightarrow \quad g \in G_m.$$

For a subset $S \subseteq M$ we write

$$G_S := \bigcap_{m \in S} G_m = \{g \in G \colon (\forall m \in S)\ g.m = m\}$$

and for $H \subseteq G$ we write

$$M^H := \{m \in M \colon (\forall h \in H)\ h.m = m\}$$

for the set of points in M fixed by H.

Lemma 9.1.5. *For each smooth action $\sigma \colon G \times M \to M$, the following assertions hold:*

(i) *For each $m \in M$, the stabilizer G_m of m is a closed subgroup of G.*
(ii) *For $m \in M$ and $g \in G$, the following equality holds: $G_{g.m} = gG_mg^{-1}$.*
(iii) *If $S \subseteq M$ is a G-invariant subset, then $G_S \trianglelefteq G$ is a normal subgroup.*

Proof. (i) That G_m is a subgroup is a trivial consequence of the action axioms. Its closedness follows from the continuity of the orbit map σ^m and the closedness of the points of M.

(ii) If $h \in G_m$, then

$$ghg^{-1}.(g.m) = ghg^{-1}g.m = gh.m = g.m,$$

hence $ghg^{-1} \in G_{g.m}$ and thus $gG_mg^{-1} \subseteq G_{g.m}$. Similarly, we get $g^{-1}G_{g.m}g \subseteq G_{g^{-1}.(g.m)} = G_m$ and therefore $gG_mg^{-1} = G_{g.m}$.

(iii) follows directly from (ii). □

The normal subgroup G_M consisting of all elements of G which do not move any element of M is called the *effectivity kernel* of the action. It is the kernel of the corresponding homomorphism $G \to \mathrm{Diff}(M), g \mapsto \sigma_g$.

Proposition 9.1.6. *Let $\sigma\colon G \times M \to M$ be a smooth action of G on M. Recall the associated Lie algebra homomorphism $\mathbf{L}(\sigma)\colon \mathbf{L}(G) \to \mathcal{V}(M)$ from Proposition 8.1.14. Then the following assertions hold:*

(i) $m \in M^G \Rightarrow \mathbf{L}(\sigma)(x)(m) = 0$ *for each* $x \in \mathbf{L}(G)$. *The converse holds if G is connected.*
(ii) *If* $\mathbf{L}(\sigma)(\mathbf{L}(G))(m) = T_m(M)$, *then the orbit* $\mathcal{O}_m$ *of* m *is open.*

Proof. (i) Suppose first that m is a fixed point and let $x \in \mathbf{L}(G)$. Then

$$\mathbf{L}(\sigma)(x)(m) = \frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m = \frac{d}{dt}\Big|_{t=0} m = 0.$$

If, conversely, all vector fields $\mathbf{L}(\sigma)(x)$ vanish in m, then m is a fixed point for all flows generated by these vector fields, which leads to $\exp_G(x).m = m$ for each $x \in \mathbf{L}(G)$. This means that $G_m \supseteq \langle \exp_G \mathbf{L}(G)\rangle$, which in turn is the identity component of G (Lemma 8.2.9). If G is connected, we get $G = G_m$.

(ii) Since the linear map $T_{\mathbf{1}}(\sigma^m)\colon \mathbf{L}(G) \to T_m(M), x \mapsto \mathbf{L}(\sigma)(x)(m)$ is surjective, the Implicit Function Theorem 7.1.9 implies that $G.m = \sigma^m(G)$ is a neighborhood of m. Since all maps σ_g are diffeomorphisms of M, $\sigma_g(G.m) = gG.m = G.m$ also is a neighborhood of $g.m$, so that $G.m$ is open. □

Corollary 9.1.7. *For each* $m \in M$,

$$\mathbf{L}(G_m) = \{x \in \mathbf{L}(G)\colon\ \mathbf{L}(\sigma)(x)(m) = 0\}.$$

The preceding proposition shows in particular that the orbit $\mathcal{O}_m$ is a submanifold if m is a fixed point (zero-dimensional case) and if $\mathcal{O}_m$ is open. Our next goal is to show that orbits of smooth actions always carry a natural manifold structure. This leads us to the geometry of homogeneous spaces in the next section.

Exercises for Section 9.1

Exercise 9.1.1. Show that the orbits of a group action $\sigma\colon G \times M \to M$ form a partition of M.

Exercise 9.1.2. Show that the following maps define group actions and determine their orbits by naming a representative for each orbit ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).

(a) $\mathrm{GL}_n(\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^n, (g, v) \mapsto gv$.
(b) $\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, (g, v) \mapsto gv$.
(c) $\mathrm{GL}_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C}), (g, x) \mapsto gxg^{-1}$.
(d) $\mathrm{U}_n(\mathbb{K}) \times \mathrm{Herm}_n(\mathbb{K}) \to \mathrm{Herm}_n(\mathbb{K}), (g, x) \mapsto gxg^{-1}$.

(e) $\mathrm{GL}_n(\mathbb{K}) \times \mathrm{Herm}_n(\mathbb{K}) \to \mathrm{Herm}_n(\mathbb{K}), (g,x) \mapsto gxg^*$.
(f) $\mathrm{O}_n(\mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n, (g,(x,y)) \mapsto (gx, gy)$.

Exercise 9.1.3. For a complex number $\lambda \in \mathbb{C}$, consider the smooth action

$$\sigma \colon \mathbb{R} \times \mathbb{C} \to \mathbb{C}, \quad \sigma(t,z) := e^{t\lambda} z.$$

(1) Sketch the orbits of this action in dependence of λ.
(2) Under which conditions are there compact orbits?
(3) Describe the corresponding vector field.

Exercise 9.1.4. For complex numbers $\lambda_1, \lambda_2 \in \mathbb{C}$, consider the smooth action

$$\sigma \colon \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2, \quad \sigma(t,(z_1,z_2)) := (e^{t\lambda_1} z_1, e^{t\lambda_2} z_2).$$

(1) For which pairs (λ_1, λ_2) are there bounded orbits?
(2) For which pairs (λ_1, λ_2) are there compact orbits?
(3) Describe a situation where the closure of some orbit is compact, but the orbit itself is not.

Exercise 9.1.5. Let G be a Lie group. For $x \in \mathfrak{g}$, let x_l be the left invariant vector field on G with $x_l(\mathbf{1}) = x$ and x_r the right invariant vector field on G with $x_r(\mathbf{1}) = x$. Show that

(i) the flow $\Phi_t^{x_l} \colon G \to G$ of x_l is given by

$$\Phi_t^{x_l}(g) = g \exp_G(tx) = \exp_G(t \operatorname{Ad}(g)x)g,$$

whereas the flow $\Phi_t^{x_r} \colon G \to G$ of x_r is given by $\Phi_t^{x_r}(g) = \exp_G(tx)g$,
(ii) $x_r(g) = T(c_g)x_l(g)$,
(iii) for the (left) actions $\sigma^l \colon G \times G \to G, (g,h) \mapsto \sigma^l(g,h) = \lambda_g(h) = gh$ and $\sigma^r \colon G \times G \to G, (g,h) \mapsto \sigma^r(g,h) = \rho_{g^{-1}}(h) = hg^{-1}$, we have $\mathbf{L}(\sigma^l)x = -x^r$ and $\mathbf{L}(\sigma^r)(x) = x_l$,
(iv) $x_l \mapsto -x_r$ yields a Lie algebra isomorphism between the Lie algebra of left invariant vector fields and the Lie algebra of right invariant vector fields on G.

Exercise 9.1.6. Let $\sigma : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. Show that $(\sigma_g)_*\big(\mathbf{L}(\sigma)(x)\big) = \mathbf{L}(\sigma)\big(\operatorname{Ad}(g)x\big)$ for all $g \in G$ and $x \in \mathfrak{g}$.

9.1.2 Transitive Actions and Homogeneous Spaces

The main result of this section is that for any smooth action of a Lie group G on a smooth manifold M, all orbits carry a natural manifold structure. First we take a closer look at transitive actions, i.e., actions with a single orbit.

Definition 9.1.8. (a) Let G be a group and H a subgroup. We write

$$G/H := \{gH\colon g \in G\}$$

for the set of *left cosets of H in G* and $q_{G/H}\colon G \to G/H, g \mapsto gH$ for the quotient map. Then

$$\sigma\colon G \times G/H \to G/H, \quad (g, xH) \mapsto gxH$$

defines a transitive action of G on the set G/H (easy exercise).

(b) Let G be a group and $\sigma_1\colon G \times M_1 \to M_1$ and $\sigma_2\colon G \times M_2 \to M_2$ two actions of the group G on sets. A map $f\colon M_1 \to M_2$ is called *G-equivariant* if

$$f(g.m) = g.f(m) \quad \text{holds for all} \quad g \in G, m \in M_1.$$

Remark 9.1.9. Let $\sigma\colon G \times M \to M$ be an action of the group G on the set M. Fix $m \in M$. Then the *orbit map* (cf. Definition 8.1.11)

$$\sigma^m\colon G \to \mathcal{O}_m \subseteq M, \quad g \mapsto g.m$$

factors through a bijective map

$$\overline{\sigma}^m\colon G/G_m \to \mathcal{O}_m, \quad gG_m \mapsto g.m$$

which is equivariant with respect to the G-actions on G/G_m and M (Exercise).

The preceding remark shows that if we want to obtain a manifold structure on orbits of smooth actions, it is natural to try to define a manifold structure on the coset spaces G/H for closed subgroups H of a Lie group G.

Theorem 9.1.10. *Let G be a Lie group and $H \leq G$ a closed subgroup. Then the coset space G/H, endowed with the quotient topology, carries a natural manifold structure for which the quotient map $q\colon G \to G/H, g \mapsto gH$ is a submersion.*

Moreover, $\sigma\colon G \times G/H \to G/H, (g, xH) \mapsto gxH$ defines a smooth action of G on G/H.

Proof. Let $E \subseteq \mathbf{L}(G)$ be a vector space complement of the subspace $\mathbf{L}(H)$ and V_E be as in the Closed Subgroup Theorem 8.3.7(iii).

Step 1 (The topology on G/H): We endow $M := G/H$ with the quotient topology. Since for each open subset $O \subseteq G$ the product OH is open (Exercise 8.3.10), the openness of $OH = q^{-1}(q(O))$ shows that q is an open map, i.e., maps open subsets to open subsets.

To see that G/H is a Hausdorff space, let $g_1, g_2 \in G$ with $g_1 H \neq g_2 H$, i.e., $g_1 \notin g_2 H$. Since H is closed, there exists a $\mathbf{1}$-neighborhood U_1 in G with $U_1 g_1 \cap g_2 H = \emptyset$, and further a symmetric $\mathbf{1}$-neighborhood U_2 with $U_2^{-1} U_2 \subseteq U_1$. Then $U_2 g_1 H$ and $U_2 g_2 H$ are disjoint q-saturated open subset of G, so that

$$q(U_2 g_1 H) = q(U_2 g_1) \quad \text{and} \quad q(U_2 g_2 H) = q(U_2 g_2)$$

are disjoint open subsets of G/H, separating g_1H and g_2H. This shows that G/H is a Hausdorff space.

We also observe that the action map σ is continuous because $\mathrm{id}_G \times q \colon G \times G \to G \times G/H$ is a quotient map since q is open (cf. Exercise 7.2.13) and

$$\sigma \circ (\mathrm{id}_G \times q) = q \circ m_G \colon G \times G \to G/H, \quad (g, x) \mapsto gxH$$

is continuous.

Step 2 (The atlas of G/H): Let $W := q(\exp_G(V_E))$ with V_E as above and define a smooth map

$$p_E \colon q^{-1}(W) = \exp_G(V_E)H \to V_E \quad \text{by} \quad \exp_G(x)h \mapsto x.$$

Since $q^{-1}(W)$ is open in G, W is open in G/H. Moreover, a subset $O \subseteq W$ is open if and only if $q^{-1}(O) \subseteq q^{-1}(W)$ is open. Since $q^{-1}(O) = \exp_G\big(p_E(q^{-1}(O))\big)H$ and $q^{-1}(W) \cong V_E \times H$, this is equivalent to $p_E(q^{-1}(O))$ being open in V_E. Therefore the map $\psi \colon W \to V_E, q(g) \mapsto p_E(g)$ is a homeomorphism and (ψ, W) is a chart of G/H.

For $g \in G$ we put $W_g := g.W$ and $\psi_g(x) = \psi(g^{-1}.x)$. Since all maps $\sigma_g \colon G/H \to G/H$ are homeomorphisms (by Step 1), we thus get charts $(\psi_g, W_g)_{g\in G}$, and it is clear that $\bigcup_{g\in G} W_g = G/H$.

We claim that this collection of homeomorphisms is a smooth atlas of G/H. Let $g_1, g_2 \in G$ and assume that $W_{g_1} \cap W_{g_2} \neq \emptyset$. We then have for $x \in V_E$:

$$\begin{aligned}\psi_{g_1} \circ \psi_{g_2}^{-1}(x) &= \psi({g_1}^{-1}g_2.\psi^{-1}(x)) = \psi\big({g_1}^{-1}g_2.q(\exp_G(x))\big) \\ &= \psi\big(q({g_1}^{-1}g_2 \exp_G(x))\big) = p_E\big({g_1}^{-1}g_2 \exp_G(x)\big).\end{aligned}$$

Since p_E is smooth, this map is smooth on its open domain, which shows that $(\psi_g, W_g)_{g\in G}$ is a smooth atlas of G/H.

Step 3 (Smoothness of the maps σ_g): For $g_1, g_2 \in G$ we have $\sigma_{g_1}(W_{g_2}) = W_{g_1g_2}$ and $\psi_{g_1g_2} \circ \sigma_{g_1} = \psi_{g_2}$, which immediately implies that $\sigma_{g_1}|_{W_{g_2}} \colon W_{g_2} \to W_{g_1g_2}$ is smooth. Since g_2 was arbitrary, all maps σ_g, $g \in G$, are smooth. From $\sigma_g \circ \sigma_{g^{-1}} = \mathrm{id}_M$ we further derive that they are diffeomorphisms.

Step 4 (q is a submersion): The smoothness of q on $q^{-1}(W)$ follows from $\psi(q(g)) = p_E(g)$ and the smoothness of p_E on $q^{-1}(W)$. Moreover, $T_{\mathbf{1}H}(\psi)T_{\mathbf{1}}(q) = T_{\mathbf{1}}(p_E) \colon \mathbf{L}(G) \to E$ is the linear projection onto E with kernel $\mathbf{L}(H)$, hence surjective. This proves that $T_{\mathbf{1}}(q)$ is surjective.

For each $g \in G$, we have $q \circ \lambda_g = \sigma_g \circ q$, so that Step 3 implies that q is smooth on all of G. Taking derivatives, we obtain

$$T_g(q) \circ T_{\mathbf{1}}(\lambda_g) = T_{\mathbf{1}H}(\sigma_g) \circ T_{\mathbf{1}}(q),$$

and since all σ_g are diffeomorphisms, this implies that all differentials $T_g(q)$ are surjective, hence that q is a submersion.

Step 5 (Smoothness of the action of G/H): Since q is a submersion, the product map $\mathrm{id}_G \times q\colon G \times G \to G \times G/H$ also is a submersion. In view of Proposition 7.3.16, it therefore suffices to show that

$$\sigma \circ (\mathrm{id}_G \times q)\colon G \times G \to G/H$$

is a smooth map, which follows from $\sigma \circ (\mathrm{id}_G \times q) = q \circ m_G$. □

For later use we collect some facts from Step 1 of the proof of Theorem 9.1.10 into a separate corollary.

Corollary 9.1.11. *Let G be a Lie group and $H \leq G$ a closed subgroup. Then for any $x \in G/H$ there exists an open neighborhood $U \subseteq G/H$ and a smooth section $\sigma\colon U \to G$ for the quotient map $q\colon G \to G/H$ such that*

$$m_\sigma\colon U \times H \to \sigma(U)H, \ (u,h) \mapsto \sigma(u)h$$

is a diffeomorphism onto an open subset of G.

The following corollary shows that for each smooth group action, all orbits carry natural manifold structures. Not all these manifold structures turn these orbits into submanifolds, as the dense wind (see Example 8.3.12) shows.

Corollary 9.1.12. *Let $\sigma\colon G \times M \to M$ be a smooth action of the Lie group G on M. Then for each $m \in M$ the orbit map $\sigma^m\colon G \to M, g \mapsto g.m$ factors through a smooth injective equivariant map*

$$\overline{\sigma}^m\colon G/G_m \to M, \quad gG_m \mapsto g.m,$$

whose image is the set $\mathcal{O}_m$.

Proof. The existence of the map $\overline{\sigma}^m$ is clear (Remark 9.1.9). Since the quotient map $q\colon G \to G/G_m$ is a submersion, the smoothness of $\overline{\sigma}^m$ follows from the smoothness of the map $\overline{\sigma}^m \circ q = \sigma^m$ (Proposition 7.3.16). □

The preceding corollary provides on each orbit $\mathcal{O}_m$ of a smooth Lie group action the structure of a smooth manifold. Its dimension is given by

$$\begin{aligned}\dim(G/G_m) &= \dim G - \dim G_m = \dim \mathbf{L}(G) - \dim \mathbf{L}(G_m)\\ &= \dim \mathbf{L}(\sigma)(\mathbf{L}(G))(m),\end{aligned}$$

because $\mathbf{L}(G_m)$ is the kernel of the linear map

$$\mathbf{L}(G) \to T_m(M), \quad x \mapsto \mathbf{L}(\sigma)(x)(m)$$

(Corollary 9.1.7). In this sense we may identify the subspace $\mathbf{L}(\sigma)(\mathbf{L}(G))(m) \subseteq T_m(M)$ with the tangent space of the orbit $\mathcal{O}_m$.

We want to show that if G has at most countably many connected components, then $\mathcal{O}_m$ is always an initial submanifold. For this we a lemma.

Lemma 9.1.13. *Let* $\sigma\colon G\times M\to M$ *be a smooth action and* $m\in M$*. Suppose that* N *is a closed submanifold of an open neighborhood of* m *in* M *such that* $m\in N$ *and* $T_m(M)=T_m(N)\oplus T_{\mathbf{1}}(\sigma^m)(\mathfrak{g})$*. Further suppose that* C *is a closed submanifold of an open neighborhood of* $\mathbf{1}$ *in* G *such that* $\mathfrak{g}=T_{\mathbf{1}}(C)\oplus\mathfrak{g}_m$*. Then there are open neighborhoods* C_o*,* N_o*, and* M_o *of* $\mathbf{1}$ *and* m *in* C*,* N *and* M*, respectively, such that*

$$\Phi:=\sigma|_{C_o\times N_o}\colon C_o\times N_o\to M_o:=\sigma(C_o\times N_o)\subseteq M$$

is a diffeomorphism.

Proof. We have $T_{(\mathbf{1},m)}(\Phi)(x,v)=v+T_{\mathbf{1}}(\sigma^m)x$. If this expression vanishes, then $v\in T_m(N)\cap T_{\mathbf{1}}(\sigma^m)(\mathfrak{g})=\{0\}$ also implies $T_{\mathbf{1}}(\sigma^m)x=0$. Therefore $T_{(\mathbf{1},m)}(\Phi)$ is injective. From

$$T_m(M)=T_m(N)+T_{\mathbf{1}}(\sigma^m)(\mathfrak{g})=T_m(N)+T_{\mathbf{1}}(\sigma^m)(T_{\mathbf{1}}(C)),$$

we further derive that it is bijective. Now the claim follows with the Inverse Function Theorem 7.1.5. □

Proposition 9.1.14. *Let* $\sigma\colon G\times M\to M$ *be a smooth action and* $m\in M$*. If* G *has at most countably many connected components, then the immersion* $\overline{\sigma}^m\colon G/G_m\to\mathcal{O}_m\subseteq M$ *defines on the orbit* $\mathcal{O}_m$ *the structure of an initial submanifold of* M*.*

Proof. Using Lemma 9.1.13 and Corollary 9.1.11, we find submanifolds N and C of M and G, and a neighborhood U of m in M such that the action gives a diffeomorphism $\Phi\colon C\times N\to U$. Each of the sets $\sigma(C\times\{n\})$ belongs a single G-orbit. We claim that, if that orbit happens to be $\mathcal{O}_m$, $\sigma(C\times\{n\})$ contains an open subset of $\mathcal{O}_m$. In fact, the manifold structure on $\mathcal{O}_m$ is the one inherited from G/G_m, so that $\sigma(C\times\{n\})$ corresponds to a set of the form $Cg_oG_m\subseteq G/G_m$, where $\sigma(g_o,m)=n$ and $T_{\mathbf{1}}(C)\cap\mathbf{L}(G_n)=\{0\}$. This implies that the map $C\to G/G_m, c\mapsto cgG_m$ has surjective differential in $\mathbf{1}$, so that its image CgG_m contains an open subset of G/G_m. Next we recall from Proposition 8.1.15(iv)(d) that every pairwise disjoint family of open subsets of G is countable, and this property is inherited by G/G_m which carries the quotient topology. Since the sets $\sigma(C\times\{n\})$, $n\in N$, are pairwise disjoint, the preceding argument implies that $N\cap\mathcal{O}_m$ is countable.

Now let $f\colon M'\to\mathcal{O}_m$ be a map for which the composition with the inclusion map $\iota\colon\mathcal{O}_m\cong G/G_m\to M$ is smooth. Let $m'\in M$. We have to show that $\iota^{-1}\circ f$ is smooth in a neighborhood of m'. Since G acts on $\mathcal{O}_m$ and M by smooth maps, we may w.l.o.g. assume that $f(m')=m$, that $f(M')\subseteq\sigma(C\times N)$ and that M' is connected. Now $\Phi^{-1}\circ f\colon M'\to C\times N$ is a smooth map and for the N-projection $p_N\colon C\times N\to N$, the image $(p_N\circ\Phi^{-1}\circ f)(M')$ is connected by smooth arcs. On the other hand, it is contained in the countable subset $N\cap\mathcal{O}_m$, hence trivial because every non-constant smooth arc is uncountable. This proves that $f(M')\subseteq\sigma(C\times\{m\})$, and the map $h := p_C \circ \Phi^{-1} \circ f\colon M' \to C$ is smooth. Now $f(x) = \sigma(h(x), m)$, and the smoothness of G on $\mathcal{O}_m$ implies that $\iota^{-1} \circ f$ is smooth. This proves that ι defines on the subset $\mathcal{O}_m$ of M the structure of a smooth initial submanifold. □

Remark 9.1.15. The assumption that G has at most countably many connected components is crucial for $\mathcal{O}_m$ to be an initial submanifold. To see this, consider the group $\mathbb{R}_d$, which is the additive group $\mathbb{R}$, endowed with the discrete topology. This is a 0-dimensional Lie group, and $\sigma(x, y) = x + y$ defines a transitive action of $G = \mathbb{R}_d$ on $M = \mathbb{R}$. In this case the spaces $G/G_m \cong G$ are discrete, but $\mathcal{O}_m = \mathbb{R}$ is not.

In some case the orbit $\mathcal{O}_m$ may already have another manifold structure, f.i., if it is a submanifold of M. In this case the following proposition says that this manifold structure coincides with the one induced by identifying it with G/G_m.

Corollary 9.1.16. *Let $\sigma\colon G \times M \to M$ be a smooth action and $m \in M$. We assume that G has at most countably many connected components. If $\mathcal{O}_m$ is a submanifold of M, then the map $\overline{\sigma}^m\colon G/G_m \to \mathcal{O}_m$ is a diffeomorphism.*

Proof. We recall from Lemma 7.6.5 that the submanifold $\mathcal{O}_m$ of M is initial. On the other hand, we have seen in Proposition 9.1.14 that the map $\overline{\sigma}^m\colon G/G_m \to M$ also defines an initial submanifold structure on $\mathcal{O}_m$. Therefore the assertion follows from the uniqueness of initial submanifold structures (Remark 7.6.2). □

Corollary 9.1.17. *If $\sigma\colon G \times M \to M$ is a transitive smooth action of the Lie group G on the manifold M and $m \in M$, then the orbit map $\overline{\sigma}^m\colon G/G_m \to M$ is a G-equivariant diffeomorphism.*

Definition 9.1.18. The manifolds of the form $M = G/H$, where H is a closed subgroup of a Lie group G, are called *homogeneous spaces.* We know already that the canonical action of G on G/H is smooth and transitive, and Corollary 9.1.17 shows the converse, i.e., that each transitive action is equivalent to the action on some G/H because there exists an equivariant diffeomorphism.

9.1.3 Examples

Example 9.1.19 (Graßmannians). Let $M := \mathrm{Gr}_k(\mathbb{R}^n)$ denote the set of all k-dimensional subspaces of $\mathbb{R}^n$, the *Graßmann manifold of degree k.* We know from linear algebra that the natural action

$$\sigma\colon\ \mathrm{GL}_n(\mathbb{R}) \times \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n), \quad (g, F) \mapsto g(F)$$

is transitive (Exercise). Let $F := \mathrm{span}\{e_1, \ldots, e_k\}$. Writing elements of $M_n(\mathbb{R})$ as 2×2-block matrices, according to

$$M_n(\mathbb{R}) = \begin{pmatrix} M_k(\mathbb{R}) & M_{k,n-k}(\mathbb{R}) \\ M_{n-k,k}(\mathbb{R}) & M_{n-k}(\mathbb{R}) \end{pmatrix},$$

the stabilizer of F in $\mathrm{GL}_n(\mathbb{R})$ is

$$\mathrm{GL}_n(\mathbb{R})_F := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \mathrm{GL}_k(\mathbb{R}), b \in M_{k,n-k}(\mathbb{R}), d \in \mathrm{GL}_{n-k}(\mathbb{R}) \right\},$$

which is a closed subgroup. Then the homogeneous space $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{R})_F$ carries a natural manifold structure, and since the orbit map of F induces a bijection

$$\overline{\sigma}^F\colon\ \mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{R})_F \to \mathrm{Gr}_k(\mathbb{R}^n), \quad g\,\mathrm{GL}_n(\mathbb{R})_F \mapsto g(F),$$

we obtain a manifold structure on $\mathrm{Gr}_k(\mathbb{R}^n)$ for which the natural action of $\mathrm{GL}_n(\mathbb{R})$ is smooth.

The dimension of $\mathrm{Gr}_k(\mathbb{R}^n)$ is given by

$$\dim \mathrm{GL}_n(\mathbb{R}) - \dim \mathrm{GL}_n(\mathbb{R})_F = n^2 - (k^2 + (n-k)^2 + k(n-k)) = k(n-k).$$

Note that for $k = 1$ we obtain the manifold structure on the projective space $\mathbb{P}(\mathbb{R}^n)$.

Example 9.1.20 (Flag manifolds). A *flag* in $\mathbb{R}^n$ is a tuple

$$\mathcal{F} = (F_1, \ldots, F_m)$$

of subspaces of $\mathbb{R}^n$ with

$$F_1 \subseteq F_2 \subseteq \ldots \subseteq F_m.$$

Let $k_i := \dim F_i$ and call $(k_1, \ldots, k_m)$ the *signature of the flag*. We write $\mathrm{Fl}(k_1, \ldots, k_m)$ for the set of all flags of signature $(k_1, \ldots, k_m)$ in $\mathbb{R}^n$. Clearly,

$$\mathrm{Fl}(k_1, \ldots, k_m) \subseteq \mathrm{Gr}_{k_1}(\mathbb{R}^n) \times \ldots \times \mathrm{Gr}_{k_m}(\mathbb{R}^n).$$

We also have a natural action of $\mathrm{GL}_n(\mathbb{R})$ on the product of the Graßmann manifolds by

$$g.(F_1, \ldots, F_m) := (g(F_1), \ldots, g(F_m)).$$

To describe a base point, let

$$F_i^0 := \mathrm{span}\{e_1, \ldots, e_{k_i}\}$$

and note that

$$\mathcal{F}^0 := (F_1^0, \ldots, F_m^0) \in \mathrm{Fl}(k_1, k_2, \ldots, k_m).$$

From basic linear algebra, it follows that the action of $\mathrm{GL}_n(\mathbb{R})$ on the subset $\mathrm{Fl}(k_1, \ldots, k_m)$ is transitive, which is shown by choosing for each flag $\mathcal{F}$ of the given signature a basis $(b_i)_{1 \leq i \leq n}$ for $\mathbb{R}^n$ such that

$$F_i := \mathrm{span}\{b_1, \ldots, b_{k_i}\} \quad \text{for} \quad i = 1, \ldots, m.$$

Writing elements of $M_n(\mathbb{R})$ as $(m \times m)$-block matrices according to the partition

$$n = k_1 + (k_2 - k_1) + (k_3 - k_2) + \ldots + (k_m - k_{m-1}) + (n - k_m),$$

the stabilizer of $\mathcal{F}^0$ is given by

$$\mathrm{GL}_n(\mathbb{R})_{\mathcal{F}^0} = \{(g_{ij})_{i,j=1,\ldots,m} \colon (i > j \Rightarrow g_{ij} = 0); g_{ii} \in \mathrm{GL}_{k_i - k_{i-1}}(\mathbb{R})\},$$

which is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. We now proceed as above to get a manifold structure on the set $\mathrm{Fl}(k_1, \ldots, k_m)$, turning it into a homogeneous space, called a *flag manifold* (Exercise: Calculate the dimension of $\mathrm{Fl}(1,2,3,4)(\mathbb{R}^6)$.)

Example 9.1.21. The orthogonal group $\mathrm{O}_n(\mathbb{R})$ acts smoothly on $\mathbb{R}^n$, and its orbits are the spheres

$$S(r) := \{x \in \mathbb{R}^n \colon \|x\| = r\}, \quad r \geq 0.$$

We know already that all these spheres carry natural manifold structures. Therefore Corollary 9.1.17 implies that for each $r > 0$ we have

$$S(r) \cong \mathbb{S}^{n-1} \cong \mathrm{O}_n(\mathbb{R}) / \mathrm{O}_n(\mathbb{R})_{e_1},$$

where

$$\mathrm{O}_n(\mathbb{R})_{e_1} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \{\pm 1\}, d \in \mathrm{O}_{n-1}(\mathbb{R}) \right\} \cong (\mathbb{Z}/2\mathbb{Z}) \times \mathrm{O}_{n-1}(\mathbb{R}).$$

9.2 Frame Bundles

Tensor bundles are obtained naturally from the tangent bundle using constructions from linear algebra. The sections of these bundles are tensor fields. Tensor fields of various kinds play an important role in differential geometry and its applications. In particular, we give a unified construction of tensor bundles as associated bundles for the frame bundle.

We start by introducing the general concept of a fiber bundle which we then specialize to vector and principal bundles This construction requires bundles whose fibers are not necessarily vector spaces.

9.2.1 Fiber Bundles

Definition 9.2.1. (a) A quadruple $(\mathcal{F}, M, F, \pi)$, where $\mathcal{F}$, M and F are smooth manifolds and $\pi \colon \mathcal{F} \to M$ is a smooth map, is called a *fiber bundle* with typical *fiber* F over M if there exists an open covering $(U_i)_{i \in I}$ of M

and diffeomorphisms $\Phi_i : \pi^{-1}(U_i) \to U_i \times F$, called *local trivializations*, such that the following diagram commutes

$$\begin{array}{ccc} U_i \times F & \xleftarrow{\Phi_i} & \pi^{-1}(U_i) \\ & \searrow^{\mathrm{pr}_{U_i}} & \downarrow \pi \\ & & U_i. \end{array}$$

Then $\mathcal{F}$ is called the *total space* of $(\mathcal{F}, \pi)$, and M is called the *base space* of $(\mathcal{F}, \pi)$. The preimage $\mathcal{F}_m := \pi^{-1}(m)$ is called the *fiber* over m. We also refer to the family $(\Phi_i)_{i \in I}$ as a *local trivialization* of the bundle.

(b) A *morphism of fiber bundles* $\varphi\colon (\mathcal{F}_1, M_1, F_1, \pi_1) \to (\mathcal{F}_2, M_2, F_2, \pi_2)$, is a smooth map $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ for which there exists a smooth map $\varphi_M\colon M_1 \to M_2$ with $\pi_2 \circ \varphi = \varphi_M \circ \pi_1$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\varphi} & \mathcal{F}_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ M_1 & \xrightarrow[\varphi_M]{} & M_2. \end{array}$$

A morphism $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ of fiber bundles is called an *isomorphism* if there exists a second morphism of fiber bundles $\psi\colon \mathcal{F}_2 \to \mathcal{F}_1$ with $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}_1}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{F}_2}$. It is easy to see that a morphism of fiber bundles is an isomorphism of fiber bundles if and only if it is a diffeomorphism (Exercise).

An isomorphism $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ of fiber bundles over M is called an *equivalence* if $\varphi_M = \mathrm{id}_M$.

(c) The pair $(M \times F, \mathrm{pr}_M)$ is called the *trivial fiber bundle* with fiber F.

(d) If $(\mathcal{F}, M, F, \pi)$ is a fiber bundle, then for each open subset $U \subseteq M$ and $\mathcal{F}_U := \pi^{-1}(U)$, we obtain the fiber bundle $(\mathcal{F}_U, U, F, \pi|_{\mathcal{F}_U})$. A *bundle chart* $(\psi, \mathcal{F}_U)$ is an equivalence $\varphi\colon \mathcal{F}_U \to U \times F$ followed by a chart $\rho = \rho_U \times \rho_F$ with $\rho_U\colon U \to \mathbb{R}^n$ and $\rho_F\colon F \to \mathbb{R}^m$. For $i, j \in I$ we put $U_{ij} := U_i \cap U_j$. Then the map $\Phi_j \circ \Phi_i^{-1} \in \mathrm{Diff}(U_{ij} \times F)$ is a self-equivalence of the trivial bundle. This implies that it has the form

$$(x, f) \mapsto (x, \Phi_{ji}(x)(f)),$$

for a function $\Phi_{ji} : U_{ij} \to \mathrm{Diff}(F)$, called the *transition function* for $\mathcal{F}$ from Φ_i to Φ_j.

The construction of the tangent bundle and the tangent map yields examples of special bundles with vectors spaces as fibers. These will be call vector bundles.

Definition 9.2.2. (a) Let M be a differentiable manifold and V a finite-dimensional vector space. A *smooth vector bundle* with *typical fiber* V on M is a differentiable manifold $\mathbb{V}$, together with a smooth map $\pi : \mathbb{V} \to M$ such that

(i) All fibers $\mathbb{V}_p := \pi^{-1}(p)$ carry the structure of a vector space.
(ii) For each $p \in M$, there is an open neighborhood U of p in M and a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times V$, called a *local trivialization*, with $\mathrm{pr}_U \circ \varphi_U = \pi$, and for all $q \in U$, the map

$$\mathrm{pr}_V \circ \varphi_U\big|_{\mathbb{V}_q} : \mathbb{V}_q \to V$$

is a linear isomorphism, where $\mathrm{pr}_U(u, v) = u$ and $\mathrm{pr}_V(u, v) = v$.

(b) A smooth map $\sigma : M \to \mathbb{V}$ with $\pi \circ \sigma = \mathrm{id}_M$ is called a *section* of the bundle. The set $\Gamma(\mathbb{V})$ of sections of $\mathbb{V}$ is a vector space with respect to the pointwise addition and scalar multiplication.

(c) The vector bundle $\mathbb{V} = M \times V$ with the projection $\mathrm{pr}_M(m, v) := m$ and the obvious vector space structure on the fibers $\{p\} \times V$, is called the *trivial vector bundle* with typical fiber V.

Definition 9.2.3. Let $\pi_{\mathbb{V}} : \mathbb{V} \to M$ and $\pi_{\mathbb{W}} : \mathbb{W} \to N$ be smooth vector bundles. A smooth map $\varphi\colon \mathbb{V} \to \mathbb{W}$ is called a *morphism of vector bundles* if there exists a smooth map $\varphi_M\colon M \to N$ with $\varphi_M \circ \pi_{\mathbb{V}} = \pi_{\mathbb{W}} \circ \varphi$ and the restrictions $\varphi_p\colon \mathbb{V}_p \to \mathbb{W}_{\varphi(p)}$ to the fibers are linear. We then have a commutative diagram

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{\varphi} & \mathbb{W} \\ {\scriptstyle \pi_{\mathbb{V}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbb{W}}} \\ M & \xrightarrow[\varphi_M]{} & N. \end{array}$$

Example 9.2.4. The tangent bundle $T(M)$ of a smooth n-dimensional manifold M is a smooth vector bundle with typical fiber $V = \mathbb{R}^n$ and the tangent map $T(f)\colon T(M) \to T(N)$ of a smooth map $f\colon M \to N$ is a morphism of vector bundles.

Remark 9.2.5. The fiberwise linearity of the local trivializations of a vector bundle (cf. Definition 9.2.2) show that if a fiber bundle $(\mathcal{F}, M, F, \pi)$ is a vector bundle, then

(a) The typical fiber F is a vector space.
(b) One can choose a local trivialization Φ_i in such a way that for $i, j \in I$ we always have $\Phi_{ji}(U_{ij}) \subseteq \mathrm{GL}(F)$.

If, conversely, (a) and (b) are satisfied, then

$$\Phi_i^{-1}(x, v) + \Phi_i^{-1}(x, w) := \Phi_i^{-1}(x, v + w), \quad \lambda\Phi_i^{-1}(x, v) := \Phi_i^{-1}(x, \lambda v)$$

defines on each fiber $\mathcal{F}_x$, $x \in U_i$, a vector space structure, which does not depend on the choice of i. We thus obtain on $\mathcal{F}$ the structure of a vector bundle.

Example 9.2.6. Let M be a manifold of dimension n and TM its tangent bundle. The construction of the manifold structure of TM in Definition 7.3.6 shows that TM is a vector bundle with typical fiber $\mathbb{R}^n$ (cf. Example 9.2.4). A local trivialization is obtained from a smooth atlas $(\varphi_i, U_i)_{i\in I}$ of M by

$$\Phi_i^{-1} \colon U_i \times \mathbb{R}^n \to T(U_i) = T(M)_{U_i}, \quad (p, v) \mapsto T_p(\varphi_i)^{-1} v.$$

Thus the transition function is given by

$$\Phi_{ji}(p) = T_p(\varphi_j) T_p(\varphi_i)^{-1} = \mathtt{d}(\varphi_j \circ \varphi_i^{-1})\big(\varphi_i(p)\big).$$

Using the atlas $(\varphi_i, U_i)_{i\in I}$ and the identity as a chart for $\mathbb{R}^n$, the local trivialization yields the atlas for TM described in Definition 7.3.6. If the φ_i-coordinates are denoted by $x_1, \ldots, x_n$ and the φ_j-coordinates are denoted by $y_1, \ldots, y_n$, the transition functions can be written in the form

$$\Phi_{ji}(p) = \Big(\frac{\partial y_r(\varphi_i(p))}{\partial x_s}\Big)_{1\leq r,s\leq n} \in \mathrm{GL}_n(\mathbb{R}) \cong \mathrm{GL}(\mathbb{R}^n).$$

Definition 9.2.7 (Principal bundles). Let $(\mathcal{F}, M, G, \pi)$ be a fiber bundle over a Lie group G. If $\sigma \colon \mathcal{F} \times G \to \mathcal{F}$ is a smooth right action of G on $\mathcal{F}$ (cf. Remark 8.1.12), then $(\mathcal{F}, M, G, \pi, \sigma)$ is called a *principal bundle* with *structure group* G if there exists a local trivialization $(\Phi_i)_{i\in I}$ consisting of maps

$$\Phi_i \colon \pi^{-1}(U_i) \to U_i \times G$$

which are equivariant with respect to the natural right action $(x, h)g := (x, hg)$ of G on $U_i \times G$, i.e.,

$$\Phi_i^{-1}(x, hg) = \Phi_i^{-1}(x, h)g, \quad x \in U_i, h, g \in G.$$

Remark 9.2.8. For each principal bundle $(\mathcal{F}, M, G, \pi, \sigma)$, the action σ of G on $\mathcal{F}$ is *free*, i.e., $xg = x$ implies $g = \mathbf{1}$, and transitive on the fibers of π (Exercise).

Example 9.2.9 (Frame bundles). Let $(\mathbb{V}, M, V, \pi)$ be a vector bundle. We set

$$\mathrm{GL}(\mathbb{V}) := \coprod_{p\in M} \mathrm{Iso}(V, \mathbb{V}_p),$$

where $\mathrm{Iso}(V, W)$ denotes the set of linear isomorphisms $V \to W$ of two vector spaces. Then the map

$$\widetilde{\pi} : \mathrm{GL}(\mathbb{V}) \longrightarrow M, \quad \varphi \longmapsto p \text{ if } \varphi \in \mathrm{Iso}(V, \mathbb{V}_p)$$

is surjective, and

$$\sigma \colon \mathrm{GL}(\mathbb{V}) \times \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{V}), \quad (\varphi, g) \mapsto \varphi \circ g$$

defines a right action of the Lie group $\mathrm{GL}(V)$ on $\mathrm{GL}(\mathbb{V})$ which is easily seen to be free.

We want to turn $\mathrm{GL}(\mathbb{V})$ into a $\mathrm{GL}(V)$-principal bundle. First, we have to construct a smooth manifold structure on $\mathrm{GL}(\mathbb{V})$. We start with a local trivialization $(\Phi_i)_{i\in I}$ of $\mathbb{V}$ so that $\Phi_i\colon \pi^{-1}(U_i) \to U_i \times V$. For each $i \in I$, we have a natural map

$$\widetilde{\Psi}_i\colon U_i \times \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{V}), \quad (x,g) \mapsto \Psi_{i,x} \circ g,$$

where $\Psi_{i,x}\colon V \to \mathbb{V}_x, v \mapsto \Phi_i^{-1}(x,v)$. Since the action of $\mathrm{GL}(V)$ on $\mathrm{GL}(\mathbb{V})$ is free and transitive on the fibers $\mathrm{Iso}(V,\mathbb{V}_p)$, the map $\widetilde{\Psi}_i$ is injective with image $\widetilde{\pi}^{-1}(U_i)$. We denote its inverse by $\widetilde{\Phi}_i$. For $i,j \in I$, the maps

$$\begin{aligned}\widetilde{\Psi}_j^{-1} \circ \widetilde{\Psi}_i\colon U_{ij} \times \mathrm{GL}(V) &\to U_{ij} \times \mathrm{GL}(V)\\ (x,g) &\mapsto \big(x, (\Psi_{j,x}^{-1} \circ \Psi_{i,x})g\big)\end{aligned}$$

are smooth. In fact, $\Psi_{j,x}^{-1} \circ \Psi_{i,x} = \Phi_{ji}(x)$ is the transition function of $\mathbb{V}$ with respect to the local trivialization $(\Phi_i)_{i\in I}$.

We now endow $\mathrm{GL}(\mathbb{V})$ with the topology for which a subset $O \subseteq \mathrm{GL}(\mathbb{V})$ is open if and only if $\widetilde{\Psi}_j^{-1}(O)$ is open for each j. Since the maps $\widetilde{\Psi}_j^{-1} \circ \widetilde{\Psi}_i$ are homeomorphisms, each $\widetilde{\Psi}_i$ is an open embedding. Moreover, the projection $\widetilde{\pi}$ is continuous because all compositions

$$\mathrm{pr}_{U_i} = \widetilde{\pi} \circ \widetilde{\Psi}_i\colon U_i \times \mathrm{GL}(V) \to U_i$$

are continuous. From the continuity of $\widetilde{\pi}$ and the fact that the $\widetilde{\Psi}_j$'s are open embeddings, it follows easily that $\mathrm{GL}(\mathbb{V})$ is a Hausdorff space.

From the smoothness of the maps $\widetilde{\Psi}_j^{-1} \circ \widetilde{\Psi}_i$ it further follows that $\mathrm{GL}(\mathbb{V})$ carries a unique smooth manifold structure for which the maps $\widetilde{\Psi}_i$ are diffeomorphisms onto open subsets. Thus the $\widetilde{\Phi}_i\colon \widetilde{\pi}^{-1}(U_i) \to U_i \times \mathrm{GL}(V)$ are $\mathrm{GL}(V)$-equivariant local trivializations, so that $(\mathrm{GL}(\mathbb{V}), M, \mathrm{GL}(V), \widetilde{\pi}, \sigma)$ is a principal bundle. It is called the *frame bundle* of $\mathbb{V}$. Note that the transition functions of the frame bundle with respect to the local trivialization $(\widetilde{\Phi}_i)_{i\in I}$ are given by

$$\widetilde{\Phi}_{ji}(x) = \lambda_{\Phi_{ji}(x)}\colon\ \mathrm{GL}(V) \to \mathrm{GL}(V),$$

where the Φ_{ji} are the transition functions of $\mathbb{V}$ with respect to the local trivialization $(\Phi_i)_{i\in I}$ and λ_g denotes left multiplication by g.

If $\mathbb{V} = TM$ is the tangent bundle of a smooth manifold M (Example 9.2.4) with typical fiber $V = \mathbb{R}^n$, we simply write $\mathrm{GL}(M) := \mathrm{GL}(TM)$ and call it the *frame bundle* of M. It is a principal bundle with structure group $\mathrm{GL}(\mathbb{R}^n) \cong \mathrm{GL}_n(\mathbb{R})$.

Example 9.2.10 (Associated bundles). Let (P, M, G, π, σ) be a principal bundle and F a smooth manifold with a smooth left action $\tau\colon G \times F \to F$. Then

$$(p, f) \cdot g := (p \cdot g, g^{-1} \cdot f)$$

defines a smooth right G-action on $P \times F$. For each $x \in M$, each G-orbit in the invariant subset $P_x \times F \subseteq P \times F$ meets each of the sets $\{p_0\} \times F$ exactly once because G acts freely on P.

Let

$$P \times_G F := P \times_\tau F := (P \times F)/G := \{G \cdot (p, f) \colon p \in P, f \in F\}$$

be the set of G-orbits in $P \times F$. We write $[p, f] := G.(p, f)$ for the orbit of (p, f) and $\widetilde{\pi}([p, f]) := \pi(p)$ for the projection to M.

To turn $P \times_G F$ into a smooth manifold, we start with a G-equivariant local trivialization $(\Phi_i)_{i \in I}$ of the principal bundle P and consider the maps

$$\widetilde{\Psi}_i \colon U_i \times F \to P \times_G F, \quad (x, f) \mapsto [\Phi_i^{-1}(x, \mathbf{1}), f].$$

As we have already seen above, all these maps are injective. Moreover, if $\Phi_j \circ \Phi_i^{-1}(x, g) = (x, \Phi_{ji}(x)g)$, then

$$\widetilde{\Psi}_j^{-1} \circ \widetilde{\Psi}_i(x, f) = \widetilde{\Psi}_j^{-1}[\Phi_i^{-1}(x, \mathbf{1}), f] = (x, \Phi_{ji}(x) \cdot f)$$

follows from

$$\widetilde{\Psi}_j(x, \Phi_{ji}(x) \cdot f) = [\Phi_j^{-1}(x, \mathbf{1}), \Phi_{ji}(x) \cdot f] = [\Phi_j^{-1}\big(x, \Phi_{ji}(x)\big), f] = [\Phi_i^{-1}(x, \mathbf{1}), f].$$

Now one argues as in Example 9.2.9 to see that $P \times_G F$, endowed with the quotient topology is a Hausdorff space and that it carries a unique smooth structure for which all the sets $\widetilde{\pi}^{-1}(U_i) \subseteq P \times_G F$ are open and the maps $\widetilde{\Psi}_i \colon U_i \times F \to \widetilde{\pi}^{-1}(U_i)$ are diffeomorphisms. We thus obtain a fiber bundle $(P \times_G F, M, F, \widetilde{\pi})$, for which the maps $\widetilde{\Phi}_i := \widetilde{\Psi}_i^{-1} \colon \widetilde{\pi}^{-1}(U_i) \to U_i \times F$ form local trivializations. It is called the *bundle associated* with P and the G-space F. Note that the transition functions of $P \times_G F$ with respect to the local trivialization $(\widetilde{\Phi}_i)_{i \in I}$ are given by

$$\widetilde{\Phi}_{ji}(x)(f) = \Phi_{ji}(x) \cdot f,$$

where the Φ_{ji} are the transition functions of P with respect to the local trivialization $(\Phi_i)_{i \in I}$.

Example 9.2.11. An interesting special case arises for the one point space $F = \{*\}$ (endowed with the trivial G-action). Then $P \times_G F \cong P/G$ is the set of G-orbits in P. Since the map $\pi \colon P \to M$ is a submersion whose fibers are the G-orbits in P, it follows that the map $\widetilde{\pi} \colon P \times_G \{*\} \to M$ is a diffeomorphism.

Example 9.2.12. Let $(\mathbb{V}, M, V, \pi)$ be a vector bundle as in Example 9.2.9. Then its frame bundle $\mathrm{GL}(\mathbb{V})$ is a $\mathrm{GL}(V)$-principal bundle and we have the canonical smooth action $\tau \colon \mathrm{GL}(V) \times V \to V, (g, v) \mapsto gv$. From the definition of the frame bundle $\mathrm{GL}(\mathbb{V})$, we immediately obtain a smooth map

$$\mathrm{ev}\colon \mathrm{GL}(\mathbb{V}) \times V \to \mathbb{V}, \quad (p,v) \mapsto p(v),$$

which is a morphism of bundles over M. This map is constant on the $\mathrm{GL}(V)$-orbits in $\mathrm{GL}(\mathbb{V}) \times V$ under the action $g \cdot (p,v) := (p \circ g^{-1}, gv)$, hence factors through a smooth bundle morphism

$$\overline{\mathrm{ev}}\colon \mathrm{GL}(\mathbb{V}) \times_{\mathrm{GL}(V)} V \to \mathbb{V}, \quad [p,v] \mapsto p(v)$$

(cf. Proposition 7.3.16). In each fiber, the map

$$(\mathrm{Iso}(V, \mathbb{V}_x) \times V)/\,\mathrm{GL}(V) \to \mathbb{V}_x, \quad [p,v] \mapsto p(v)$$

is bijective because for any fixed isomorphism $p_0\colon V \to \mathbb{V}_x$, each $\mathrm{GL}(V)$-orbit meets the set $\{p_0\} \times V$ exactly once (cf. Exercise 9.2.8). This implies that $\overline{\mathrm{ev}}$ is a bijective submersion, and this implies that it is an equivalence of vector bundles (cf. Definition 9.2.3).

Corollary 9.2.13. *Let G be a Lie group, H a closed subgroup and $q\colon G \to G/H$ the quotient map. We write $\mathfrak{g} := \mathbf{L}(G)$ and $\mathfrak{h} := \mathbf{L}(H)$. Let $p_o := \mathbf{1}H = q(\mathbf{1})$ be the canonical base point for G/H and σ_g be the left translation by g on G/H (see Definition 9.1.8). Then $T_{p_o}(G/H) \cong \mathfrak{g}/\mathfrak{h}$ and with the H-action on $\mathfrak{g}/\mathfrak{h}$ given by $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(x + \mathfrak{h}) = \mathrm{Ad}(h)x + \mathfrak{h}$, the map*

$$G \times_H \mathfrak{g}/\mathfrak{h} \to T(G/H), \quad [g, x + \mathfrak{h}] \mapsto T_{p_o}(\sigma_g)T_{\mathbf{1}}(q)x$$

is a G-equivariant bundle isomorphism.

Proof. The G-equivariance is clear once we have shown that the map is well-defined. Thus, in view of Theorem 9.1.10, it only remains to show that

$$T_{p_o}(\sigma_g) \circ T_{\mathbf{1}}(q) = T_{p_o}(\sigma_{gh}) \circ T_{\mathbf{1}}(q) \circ \mathrm{Ad}(h)^{-1}.$$

In view of $\sigma_g \circ q = q \circ \lambda_g$ and $q \circ \rho_h = q$, this follows from

$$\begin{aligned} T_{p_o}(\sigma_{gh}) \circ T_{\mathbf{1}}(q) \circ \mathrm{Ad}(h)^{-1} &= T_{\mathbf{1}}(q \circ \lambda_{gh} \circ c_h^{-1}) = T_{\mathbf{1}}(q \circ \lambda_g \circ \rho_h) \\ &= T_{\mathbf{1}}(q \circ \lambda_g) = T_{p_o}(\sigma_g) \circ T_{\mathbf{1}}(q). \end{aligned}$$

□

9.2.2 Sections

Sections of bundles can be viewed as generalizations of functions. In particular in the case of vector bundles they unify a large number of concepts well-known in classical analysis: Functions, vector fields, differential forms, and tensor fields.

Definition 9.2.14. Let $(\mathcal{F}, M, F, \pi)$ be a fiber bundle. A map $s : M \to \mathcal{F}$ is called a *section* of $\mathcal{F}$ if $\pi \circ s = \mathrm{id}_M$. The set of all smooth sections of $\mathcal{F}$ is denoted by $\Gamma(\mathcal{F})$. If $\mathcal{F}$ is a vector bundle, then $\Gamma_c(\mathcal{F})$ denotes the vector space of all smooth sections s with compact support, i.e., $s(m) = 0$ for all m outside of a compact set.

The following proposition explains how smooth sections are described in terms of local trivializations. It is an elementary consequence of the definitions.

Proposition 9.2.15. *Let* $(\mathcal{F}, M, F, \pi)$ *be a fiber bundle with local trivializations* $\Phi_i \colon \mathcal{F}_{U_i} \to U_i \times F$, *transition functions* Φ_{ji}, *and* $s \colon M \to \mathcal{F}$ *be a smooth section. Then we obtain for each* $i \in I$ *a smooth function* $s_i \colon U_i \to F$, *determined by*

$$\Phi_i(s(x)) = (x, s_i(x)) \quad \textit{for } x \in U_i, \tag{9.1}$$

and these functions satisfy the relation

$$s_j(x) = \Phi_{ji}(x)(s_i(x)) \quad \textit{for } x \in U_i \cap U_j. \tag{9.2}$$

If, conversely, $(s_i)_{i \in I}$ *is a family of smooth functions* $s_i \colon U_i \to F$ *satisfying* (9.2), *then* (9.1) *yields a well-defined smooth section of* $\mathcal{F}$.

Example 9.2.16 (Density bundles). Let M be a smooth manifold of dimension n and $\widetilde{\pi} \colon \mathrm{GL}(M) \to M$ be the frame bundle of M. For $r > 0$, consider the one-dimensional representation $\Delta_r \colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}_+^\times$, $\Delta_r(g) := |\det g|^{-r}$. Then the line bundle $|\Lambda|^r(M) := \mathrm{GL}(M) \times_{\Delta_r} \mathbb{R}$ associated with this representation via Example 9.2.10 is called the *r-density bundle* of M. Using the notation from Example 9.2.6, the transition functions of $|\Lambda|^r(M)$ can be written

$$\Phi_{ji}(p) = \big| \det \mathtt{d}(\varphi_j \circ \varphi_i^{-1})\big(\varphi_i(p)\big)\big|^{-r}.$$

The sections of $|\Lambda|^r(M)$ are called *r-densities*. For $r = 1$ we will simply call them *densities* and write $|\Lambda|(M)$. Their transformation properties under coordinate changes is reminiscent of the change of variables formula in multivariable calculus and indeed, they play an important role in the integration theory on manifolds.

We give an alternative description of the density bundles: Let $p \in M$ and $L_p^{(r)}$ be the space of *r-densities* on $T_p(M)$, i.e., all continuous functions $f \colon T_p(M)^n \to \mathbb{R}$ satisfying

$$f(Av_1, \ldots, Av_n) = |\det A|^r f(v_1, \ldots, v_n)$$

for all $v_1, \ldots, v_n \in T_p(M)$ and $A \in \mathrm{End}\big(T_p(M)\big)$. Note that $f \in L_p^{(r)}$ is uniquely determined by its values on a single basis for $T_p(M)$, so that $L_p^{(r)}$ is one-dimensional. Set $L^{(r)}(M) = \coprod_{p \in M} L_p^{(r)}$ and use a smooth atlas $(\varphi_i, U_i)_{i \in I}$ of M to define maps $\Psi_i \colon U_i \times \mathbb{R} \to L^{(r)}(M)$ via

$$\Psi_i(p, t)\big(\tfrac{\partial}{\partial x_1^{(i)}}\big|_p, \ldots, \tfrac{\partial}{\partial x_n^{(i)}}\big|_p\big) = t,$$

where the $x_k^{(i)}$ are the coordinate functions for the chart φ_i. Then Ψ_i is injective with image $\pi^{-1}(U_i)$, where $\pi \colon L^{(r)}(M) \to M$ is the obvious base point projection. If now $p \in U_i \cap U_j$, we have $\Psi_i(p, t) = \Psi_j(p, s)$ if and only if

$$\begin{aligned} t &= \Psi_j(p,s)\Big(\frac{\partial}{\partial x_1^{(i)}}\Big|_p, \ldots, \frac{\partial}{\partial x_n^{(i)}}\Big|_p\Big) \\ &= \Big|\det\Big(\frac{\partial x_l^{(j)}}{\partial x_k^{(i)}}(\varphi_i(p))\Big)\Big|^r \Psi_j(p,s)\Big(\frac{\partial}{\partial x_1^{(j)}}\Big|_p, \ldots, \frac{\partial}{\partial x_n^{(j)}}\Big|_p\Big) \\ &= \Big|\det\Big(\frac{\partial x_l^{(j)}}{\partial x_k^{(i)}}(\varphi_i(p))\Big)\Big|^r s \\ &= \Big|\det \mathtt{d}(\varphi_j \circ \varphi_i^{-1})\big(\varphi_i(p)\big)\Big|^r s. \end{aligned}$$

Thus we can equip $L^{(r)}(M)$ with a line bundle structure such that the $\Phi_i := \Psi_i^{-1}$ form a local trivialization. The corresponding transition functions are

$$\Phi_{ji}(p) = \big|\det \mathtt{d}(\varphi_j \circ \varphi_i^{-1})\big(\varphi_i(p)\big)\big|^{-r},$$

which shows that $L^{(r)}(M)$ is indeed the bundle of r-densities.

It is also possible to give an explicit isomorphism $|\Lambda|^r(M) \to L^{(r)}(M)$: For $\varphi \in \mathrm{GL}(M)$ and $t \in \mathbb{R}$, let $\mu_{\varphi,t}$ be the r-density on $T_{\widetilde{\pi}(\varphi)}(M)$ satisfying

$$\mu_{\varphi,t}\big(\varphi(e_1), \ldots, \varphi(e_n)\big) = t,$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. Then $\mu_{\varphi g,t} = \mu_{\varphi,\Delta_r(g)^{-1}t}$ and

$$\mathrm{GL}(M) \times_{\Delta_r} \mathbb{R} \to L^{(r)}(M), \quad [\varphi, t] \mapsto \mu_{\varphi,t} \tag{9.3}$$

is the desired bundle isomorphism.

Proposition 9.2.17. *Let (P, M, G, q, σ) be a principal bundle and $\tau\colon G \times F \to F$ be a smooth action. We consider the set*

$$C^\infty(P,F)^G := \{\alpha \in C^\infty(P,F)\colon (\forall p \in P)(\forall g \in G)\ \alpha(pg) = g^{-1} \cdot \alpha(p)\}$$

of smooth G-equivariant maps. Then each $\alpha \in C^\infty(P,F)^G$ defines a smooth section

$$s_\alpha\colon M \to P \times_G F, \quad q(p) \mapsto [p, \alpha(p)]$$

and we thus obtain a bijection

$$\Phi\colon C^\infty(P,F)^G \to \Gamma(P \times_G F), \quad \alpha \mapsto s_\alpha.$$

If, in addition, F is a vector space and τ a linear action, i.e., a representation, then Φ is linear.

Proof. First we note that s_α is well-defined because

$$[pg, \alpha(pg)] = [p, g \cdot \alpha(pg)] = [p, \alpha(p)]$$

for each $p \in P$. Since the map $P \to P \times_G F,\ p \mapsto [p, \alpha(p)]$ is smooth, the smoothness of s_α follows from the fact that $q\colon P \to M$ is a submersion (Proposition 7.3.16).

Φ is injective: Suppose that $\Phi(\alpha) = \Phi(\beta)$. Then, for each $p \in P$, we have $[p, \alpha(p)] = [p, \beta(p)]$, and this implies that $\alpha(p) = \beta(p)$ because the action of G on P is free.

Φ is surjective: Let $s\colon M \to P \times_G F$ be a smooth section. For each $p \in P$, there exists a unique element $\alpha(p)$ with $s(q(p)) = [p, \alpha(p)]$. For each $g \in G$, we then have

$$s(q(p)) = s(q(pg)) = [pg, \alpha(gp)] = [p, g \cdot \alpha(pg)],$$

so that $\alpha\colon P \to F$ is equivariant. It remains to show that α is smooth. It suffices to show that for each local trivialization $\varphi\colon P_U \to U \times G$, the map $\alpha \circ \varphi^{-1}$ is smooth. For $x = q(p) \in U$ we have

$$s(x) = [\varphi^{-1}(x, \mathbf{1}), \alpha(\varphi^{-1}(x, \mathbf{1})] = \widetilde{\varphi}^{-1}(x, \alpha(\varphi^{-1}(x, \mathbf{1}))).$$

Since $\widetilde{\varphi}\colon P \times_G F \to U \times F$ is a local trivialization, the map $U \to F$, $x \mapsto \alpha(\varphi^{-1}(x, \mathbf{1}))$ is smooth, so that $(x, g) \mapsto \alpha(\varphi^{-1}(x, g)) = g^{-1}\alpha(\varphi^{-1}(x, \mathbf{1}))$ is also smooth. □

9.2.3 Tensor Bundles and Tensor Fields

Recall the concept of a V-valued Pfaffian form $\omega\colon TM \to V$ from Definition 7.3.9. The map ω can also be interpreted as a map

$$M \to \coprod_{p \in M} \mathrm{Hom}(T_p(M), V), \quad p \mapsto \omega_p$$

such that $\omega_p \in \mathrm{Hom}(T_p(M), V)$. This suggests to define a suitable vector bundle for which ω is a section. We will do this in greater generality.

Definition 9.2.18. Let M be a smooth manifold of dimension n and view the tangent bundle TM as the vector bundle $\mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ associated with the frame bundle of M (cf. Example 9.2.9). For any representation of $\mathrm{GL}_n(\mathbb{R})$ on a vector space W, we obtain the vector bundle $\mathbb{W} := \mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} W \to M$. To make our notation less clumsy we write $E := \mathbb{R}^n$. If W is a $\mathrm{GL}_n(\mathbb{R})$-invariant subspace of the tensor algebra $\mathcal{T}(E \oplus E^*)$, we call $\pi_{\mathbb{W}}\colon \mathbb{W} \to M$ a *tensor bundle* over M.

On $W := T^{r,s}(E) := (E^*)^{\otimes r} \otimes E^{\otimes s}$ we have a representation of $\mathrm{GL}(E)$ by

$$g \cdot (\alpha_1 \otimes \cdots \otimes \alpha_r) \otimes (\alpha_1 \otimes \cdots \otimes v_s) = ((g^{-1})^* \alpha_1 \otimes \cdots \otimes (g^{-1})^* \alpha_r) \otimes (gv_1 \otimes \cdots \otimes gv_s).$$

We write $\mathcal{T}^{r,s}(M)$ for $\mathbb{W} := \mathrm{GL}(M) \times_{\mathrm{GL}(E)} W$ and call it the bundle of tensors of *type* (r, s). The smooth sections of $\mathcal{T}^{r,s}(M)$ are called the *tensor fields* of type (r, s). The bundle $T^*(M) := \mathcal{T}^{1,0}(M)$ is called the *cotangent bundle* of M.

Example 9.2.19. If, in the situation of Definition 9.2.18, $W := \mathrm{Hom}(E, V)$, then the resulting vector bundle $(\mathbb{W}, M, W, \pi_{\mathbb{W}})$ satisfies

$$\mathbb{W}_p \cong \mathrm{Hom}(T_p(M), V)$$

because the evaluation map

$$\mathrm{Iso}(E, T_p(M)) \times \mathrm{Hom}(E, V) \to \mathrm{Hom}(T_p(M), V), \quad (\varphi, \alpha) \mapsto \alpha \circ \varphi^{-1}$$

factors through a bijection

$$\big(\mathrm{Iso}(E, T_p(M)) \times \mathrm{Hom}(E, V)\big)/\,\mathrm{GL}(E) \to \mathrm{Hom}(T_p(M), V), [(\varphi, \alpha)] \mapsto \alpha \circ \varphi^{-1}.$$

If TM is trivial over an open subset U, then so is $\mathrm{GL}(M)$, and thus $\mathbb{W}_U \cong \mathrm{GL}(M)_U \times_{\mathrm{GL}(E)} W$ is also trivial. Therefore $\Gamma(\mathbb{W}_U) \cong C^\infty(U, W) = C^\infty(U, \mathrm{Hom}(E, V))$.

Similarly, the elements of $\Omega^1(U, V)$ are given by smooth functions $\omega \in C^\infty(U \times E, V)$ satisfying $\omega_p := \omega(p, \cdot) \in \mathrm{Hom}(E, V)$. This leads to a linear isomorphism

$$\Omega^1(U, V) \to \Gamma(\mathbb{W}_U), \qquad \omega \mapsto (p \mapsto \omega_p),$$

which we use to identify these two spaces. Thus the V-valued Pfaffian forms are simply the sections of $\mathbb{W}$. In particular, the (scalar valued) Pfaffian forms are the tensor fields of type $(0, 1)$.

Example 9.2.20. Let M be a smooth manifold of dimension n and (φ, U) be a chart on M. In $T_p(M)$ and $T_p(M)^*$ we consider the φ-bases $\big(\frac{\partial}{\partial x_j}|_p\big)_{j=1,\ldots,n}$ and $(\mathtt{d}x_j(p))_{j=1,\ldots,n}$, respectively, and we define n^{r+s} elements of $\mathcal{T}^{r,s}(M)_p$ by

$$\big(\mathtt{d}x \otimes \tfrac{\partial}{\partial x}\big)_{(\varrho,\sigma)}(p) := \mathtt{d}x_{\varrho(1)}(p) \otimes \ldots \otimes \mathtt{d}x_{\varrho(r)}(p) \otimes \tfrac{\partial}{\partial x_{\sigma(1)}}|_p \otimes \ldots \otimes \tfrac{\partial}{\partial x_{\sigma(s)}}|_p,$$

where $\varrho : \{1, \ldots, r\} \to \{1, \ldots, n\}$ and $\sigma : \{1, \ldots, s\} \to \{1, \ldots, n\}$ are arbitrary functions. The resulting tensor fields $\big(\mathtt{d}x \otimes \frac{\partial}{\partial x}\big)_{(\varrho,\sigma)}$ are called the *φ-basic fields* on U. As for vector fields and Pfaffian forms this terminology is doubly justified. On the one hand, the $\big(dx \otimes \frac{\partial}{\partial x}\big)_{(\varrho,\sigma)}(p)$ form a basis for $\mathcal{T}^{r,s}(M)_p$ for every $p \in U$. On the other hand, combining Example 7.4.12 and Example 7.4.10, we see that every local tensor field $T \in \Gamma\big(\mathcal{T}^{r,s}(U)\big)$ is of the form

$$T|_U = \sum_{(\varrho,\sigma) \in \{1,\ldots,n\}^{r+s}} c_{(\varrho,\sigma)} \cdot \big(\mathtt{d}x \otimes \tfrac{\partial}{\partial x}\big)_{(\varrho,\sigma)},$$

where $c_{(\varrho,\sigma)} \in C^\infty(U)$ are uniquely determined by T.

Fix a smooth manifold M. Let $\omega \in \Omega^1(M)$ and $X \in \mathcal{V}(M)$. For every $p \in M$, we can apply $\omega(p)$ to $X(p)$ and obtain a number $\omega(p)\big(X(p)\big)$. As usual, if we consider all $p \in M$ simultaneously, we get a function $\widetilde{\omega}(X) : M \to \mathbb{R}$

which is defined by $\widetilde{\omega}(X)(p) = \omega(p)\big(X(p)\big)$. This function is smooth, as we easily see in local coordinates (see Exercise 9.2.1).

The form ω assigns a smooth function to each vector field $X \in \mathcal{V}(M)$. Clearly, this assignment is an $\mathbb{R}$-linear map $\widetilde{\omega} : \mathcal{V}(M) \to C^\infty(M)$. Recall from Remark 7.4.14 that one can multiply vector fields by smooth functions and in this way turn $\mathcal{V}(M)$ into a $C^\infty(M)$-module. For $f \in C^\infty(M)$, $\omega \in \Omega^1(M)$ and $X \in \mathcal{V}(M)$, we calculate:

$$\widetilde{\omega}(fX)(p) = \omega(p)\big(f(p)X(p)\big) = f(p)\Big(\omega(p)\big(X(p)\big)\Big) = \big(f \cdot \widetilde{\omega}(X)\big)(p).$$

This just means that $\widetilde{\omega}$ defines a $C^\infty(M)$-linear function $\mathcal{V}(M) \to C^\infty(M)$. We denote the set of all these maps by

$$\mathcal{V}(M)^* := \mathrm{Hom}_{C^\infty(M)}(\mathcal{V}(M), C^\infty(M)).$$

In algebraic terminology, $\mathcal{V}(M)^*$ is the dual $C^\infty(M)$-module to $\mathcal{V}(M)$. Similarly as for the case of vector fields (see Theorem 7.4.18), we want to show that the map $\Omega^1(M) \to \mathcal{V}(M)^*$ is a bijection.

As before, injectivity is an easy verification:

Lemma 9.2.21. *The map $\Omega^1(M) \to \mathcal{V}(M)^*, \omega \mapsto \widetilde{\omega}$ is injective.*

Proof. Let $v \in T_p(M)$ and use Remark 7.4.19(iii) to find a smooth vector field $X \in \mathcal{V}(M)$ with $X(p) = v$. If $\widetilde{\omega} = 0$, then $\omega_p(v) = \widetilde{\omega}(X)(p) = 0$, which implies that $\omega_p = 0$. □

As in the case of vector fields, for a given $F \in \mathcal{V}(M)^*$, we want to construct a form ω with $\widetilde{\omega} = F$. For this, we need to have:

$$\omega(p)(v) = \widetilde{\omega}(X)(p) = F(X)(p) \tag{9.4}$$

for all $v \in T_pM$ and $X \in \mathcal{V}(M)$ with $X(p) = v$. We can *define* a map $\omega : M \to T^*M$ with $\pi \circ \omega = \mathrm{id}_M$ by (9.4) if we are able to show that $F(X)(p)$ depends only on the value $X(p)$. But, this follows from the following lemma.

Lemma 9.2.22. *Let $F \in \mathcal{V}(M)^*$ and $X \in \mathcal{V}(M)$ with $X(p) = 0$, then $F(X)(p) = 0$.*

Proof. **Step 1**: Suppose that V is an open neighborhood of p with $X|_V \equiv 0$. Remark 7.4.19(i) yields a function $\chi \in C^\infty(M)$ such that $\chi(p) = 1$ and $\chi|_{M \setminus V} \equiv 0$. Then $X = (1-\chi)X$ and hence

$$F(X)(p) = F\big((1-\chi)X\big)(p) = (1-\chi)(p)\, F(X)(p) = 0.$$

Step 2: Let (φ, U) be a chart on M and $p \in U$. Then there exists a vector field $\widetilde{X} \in \mathcal{V}(M)$ with compact support contained in U such that $X - \widetilde{X}$ vanishes on a neighborhood of p (Remark 7.4.19(i)). Then Step 1 implies that $F(X)(p) = F(\widetilde{X})(p)$, so that we may w.l.o.g. assume that $\mathrm{supp}(X)$ is a compact subset of U.

Step 3: Using Remark 7.4.19(ii), we write $X(q) = \sum a_j(q)\, X_j(q)$ for $q \in U$ with $a_j \in C^\infty(U)$ and $X_j \in \mathcal{V}(U)$, where $X_j(q) = \frac{\partial}{\partial x_j}|_q$ for all $q \in U$. Then the functions a_j are compactly supported in U, hence extend to smooth functions on M. Moreover, there exists a smooth function $f \colon M \to \mathbb{R}$ with $f(q) = 1$ if $X(q) \neq 0$ and $\mathrm{supp}(f)$ is a compact subset of U. Then $X = fX = \sum_{j=1}^n a_j(fX_j)$, and each vector field fX_j extends by 0 to a smooth vector field on M. Then $X(p) = 0$ and $X_j(p) = \frac{\partial}{\partial x_j}|_p$ imply $a_j(p) = 0$ for all $j = 1, \ldots, n$. We thus obtain

$$F(X)(p) = F\Big(\sum_{j=1}^n a_j\ fX_j\Big)(p) = \sum_{j=1}^n a_j(p)\,\big(F(fX_j)(p)\big) = 0,$$

which concludes the proof. □

Note that in the proof of Lemma 9.2.22 it was decisive that F is not only $\mathbb{R}$-linear, but even $C^\infty(M)$-linear.

To show the bijectivity of the map $\Omega^1(M) \to \mathrm{der}\big(C^\infty(M)\big)^*$, $\omega \mapsto \widetilde{\omega}$, it only remains to show that the map $\omega : M \to T^*M$ defined by (9.4) is smooth for every $F \in \mathcal{V}(M)^*$. For this, we represent the map in local coordinates. From Remark 7.4.19 we get a chart (φ, U) on M, as well as vector fields $X_j \in \mathrm{der}\big(C^\infty(M)\big)$ with $X_j(p) = \frac{\partial}{\partial x_j}\big|_p$ for all $p \in U$. We obtain

$$\omega(p)\big(\tfrac{\partial}{\partial x_j}\big|_p\big) = F(X_j)(p) = \big(\sum_{k=1}^n F(X_k)\mathtt{d}x_k\big)(p)\big(\tfrac{\partial}{\partial x_j}\big|_p\big)$$

for all $j = 1, \ldots, n$ and $p \in U$. Therefore, we have $\omega|_U = \sum_{k=1}^n F(X_k)\mathtt{d}x_k\big|_U$ so that ω is smooth. Thus, we have proved the following theorem:

Theorem 9.2.23. *Let M be a smooth manifold. Then the map*

$$\Omega^1(M) \to \mathcal{V}(M)^*, \quad \omega \mapsto \widetilde{\omega},$$

defined by $\widetilde{\omega}(X)(p) = \omega(p)\big(X(p)\big)$, is a bijection.

Just as in Remark 7.4.19, which deals with vector fields, the following remark expresses the corresponding facts for sections of any vector bundle. It applies in particular to Pfaffian forms, which are sections of the cotangent bundle T^*M.

Remark 9.2.24. Let M be a smooth manifold of dimension and $(\mathbb{V}, M, V, \pi)$ a vector bundle.

(i) Let C be a compact subset of M, and let $U \supseteq C$ be an open subset of M. Then, for every section $s \in \Gamma(\mathbb{V}_U)$, there exists a global section $\widetilde{s} \in \Gamma\mathbb{V}$ which coincides with s on C.

(ii) Suppose that $U \subseteq M$ is an open subset for which $\mathbb{V}_U$ is trivial. Then there exist sections $s_1, \ldots, s_k \in \Gamma(\mathbb{V}_U)$ such that $s_1(x), \ldots, s_k(x)$ form a basis for $\mathbb{V}_x$ for each $x \in U$. Then any section $s \in \Gamma(\mathbb{V}_U)$ is of the form

$$s = \sum_{i=1}^{k} a_i \cdot s_i$$

with uniquely determined $a_i \in C^\infty(U)$.

(iii) For every $v \in \mathbb{V}_x$, there exists a section $s \in \Gamma\mathbb{V}$ with with $s(x) = v$.

9.2.4 Lie Derivatives

Lie derivatives of tensor fields generalize directional derivatives of functions and Lie derivatives of vector fields. The describe the infinitesimal changes of tensor fields under flows of vector fields.

Remark 9.2.25. Let M be a smooth manifold, $\mathrm{GL}(M)$ be the corresponding frame bundle, and $\tau \colon G \times F \to F$ a smooth action of the structure group $G := \mathrm{GL}_n(\mathbb{R})$ of $\mathrm{GL}(M)$. Further, let $U \subseteq M$ be open and $\varphi \colon U \to M$ be a diffeomorphism onto an open subset $\varphi(U)$ of M. Then φ lifts to a diffeomorphism $\mathrm{GL}(\varphi)\colon \mathrm{GL}(U) \to \mathrm{GL}(\varphi(U))$ via

$$\mathrm{GL}(\varphi)(\beta) = T(\varphi) \circ \beta \in \mathrm{Iso}\big(\mathbb{R}^n, T_{\varphi(x)}(M)\big)$$

for $\beta \in \mathrm{Iso}\big(\mathbb{R}^n, T_x(M)\big) \subseteq \mathrm{GL}(U)$.

Applying this construction to a local flow Φ_t^X, we see that a vector field $X \in \mathcal{V}(M)$ automatically lifts to a vector field $\widetilde{X} \in \mathcal{V}(\mathrm{GL}(M))$ defined by

$$\widetilde{X}(\beta) = \frac{d}{dt}\Big|_{t=0} \mathrm{GL}(\Phi_t^X)(\beta).$$

$\mathrm{GL}(\varphi)$ also enables us to let φ act on sections. More precisely, suppose that $\alpha \in C^\infty(\mathrm{GL}(\varphi(U)), F)^G$ represents a section $s_\alpha \in \Gamma(\mathrm{GL}(\varphi(U)) \times_G F)$ as explained in Proposition 9.2.17. Then we set

$$\varphi^*\alpha := \alpha \circ \mathrm{GL}(\varphi) \in C^\infty(\mathrm{GL}(U), F)$$

and note that

$$\begin{aligned}(\varphi^*\alpha)(\beta g) &= \alpha\big(T(\varphi) \circ (\beta g)\big) = \alpha\big((T(\varphi) \circ \beta)\, g\big) \\ &= g^{-1} \cdot \alpha(T(\varphi) \circ \beta) = g^{-1} \cdot (\varphi^*\alpha)(\beta).\end{aligned}$$

Therefore $\varphi^*\alpha \in C^\infty(\mathrm{GL}(U), F)^G$ and $\varphi^*\alpha$ represents a section $\varphi^* s_\alpha := s_{\varphi^*\alpha}$.

Definition 9.2.26. Let M be a manifold, $G = \mathrm{GL}_n(\mathbb{R})$, and $U \subseteq M$ open. Given a linear representation $\tau\colon G \times F \to F$ and a vector field $X \in \mathcal{V}(M)$, the *Lie derivative* $\mathcal{L}_X\alpha$ of a section $\alpha \in C^\infty(\mathrm{GL}(U), F)^G$ is defined via the *pull-back* $(\Phi_t^X)^*\alpha$ of α by the local flow of X (Remark 9.2.25). We set

$$\mathcal{L}_X(\alpha)_p := \lim_{t\to 0} \frac{1}{t}\big((\Phi_t^X)^*\alpha - \alpha\big)_p\,,$$

whence $\mathcal{L}_X(\alpha)$ is again a section in $C^\infty(\mathrm{GL}(U), F)^G$.

Remark 9.2.27. It is possible to reformulate the Lie derivative $\mathcal{L}_X(Y)$ of a vector field $Y \in \mathcal{V}(M)$ with respect to $X \in \mathcal{V}(M)$ in an analogous way:

$$\mathcal{L}_X(Y)(p) = \lim_{t\to 0} \frac{1}{t}\big((\Phi_t^X)^*(Y) - Y\big)(p).$$

Here we define the *pull-back* $\varphi^*(X) \in \mathcal{V}(M)$ of a vector-field $X \in \mathcal{V}(N)$ by a diffeomorphism $\varphi\colon M \to N$ via the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{\quad T(\varphi)\quad} & TN \\ {\scriptstyle \varphi^* X}\uparrow & & \uparrow{\scriptstyle X} \\ M & \xrightarrow{\quad \varphi \quad} & N \end{array}$$

so that

$$\varphi^*(X) = T(\varphi^{-1}) \circ X \circ \varphi.$$

In order to verify that this is compatible with Remark 9.2.25, observe that, according to Example 9.2.12, the tangent map

$$T(\varphi)\colon\ \mathrm{GL}(M) \times_{\mathrm{GL}_m(\mathbb{R})} \mathbb{R}^m \to \mathrm{GL}(N) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$$

of a smooth map $\varphi\colon M \to N$ in the realization of the tangent bundles as associated bundles is given by

$$T(\varphi)[p, v] = [q_0, w],$$

where $p \in \mathrm{Iso}\big(\mathbb{R}^m, T_x(M)\big)$, $v \in \mathbb{R}^m$, and $q_0 \in \mathrm{Iso}\big(\mathbb{R}^n, T_{\varphi(x)}(N)\big)$, $w \in \mathbb{R}^n$ are such that $q_0(w) = (T(\varphi) \circ p)(v) = \mathrm{GL}(\varphi)(v)$. If now $\varphi\colon M \to N$ is a diffeomorphism and $\alpha \in C^\infty(\mathrm{GL}(N), \mathbb{R}^n)^{\mathrm{GL}_n(\mathbb{R})}$, then Proposition 9.2.17 yields

$$\begin{aligned} T(\varphi) \circ s_{\varphi^*\alpha}(x) &= T(\varphi)\big(\varphi^*\alpha(p)\big) = T(\varphi)[p, \alpha(T(\varphi) \circ p)] \\ &= [T(\varphi) \circ p, \alpha(T(\varphi) \circ p)] = s_\alpha\big(\varphi(x)\big) \end{aligned}$$

since $n = m$ and $T(\varphi) \circ p \in \mathrm{Iso}\big(\mathbb{R}^n, T_{\varphi(x)}(N)\big)$.

Proposition 9.2.28. *Let M be a differentiable manifold of dimension m and $\mu\colon E \times F \to W$ a bilinear equivariant map of $\mathrm{GL}_m(\mathbb{R})$-spaces. If $\alpha, \alpha_1, \alpha_2 \in C^\infty(\mathrm{GL}(M), E)^{\mathrm{GL}_m(\mathbb{R})}$, $\beta \in C^\infty(\mathrm{GL}(M), F)^{\mathrm{GL}_m(\mathbb{R})}$, and $X \in \mathcal{V}(M)$, then*

(i) $\mathcal{L}_X(\alpha_1+\alpha_2) = \mathcal{L}_X(\alpha_1) + \mathcal{L}_X(\alpha_2)$.
(ii) $\mathcal{L}_X\big(\mu(\alpha,\beta)\big) = \mu\big(\mathcal{L}_X(\alpha),\beta\big) + \mu\big(\alpha,\mathcal{L}_X(\beta)\big)$.

In particular, setting $E = \mathbb{R}$ with the trivial action and $F = \mathbb{R}^m$ with the natural action, we find $\mathcal{L}_X(fY) = (Xf)Y + f\mathcal{L}_X(Y)$ for $f \in C^\infty(M)$ and $Y \in \mathcal{V}(M)$.

Proof. We leave the verification of (i) as an easy exercise to the reader. For (ii) we calculate

$$\begin{aligned}
\mathcal{L}_X\big(\mu(\alpha,\beta)\big)(p) &= \lim_{t\to 0} \tfrac{1}{t}\big((\Phi_t^X)^*\mu(\alpha,\beta)(p) - \mu(\alpha,\beta)(p)\big)\\
&= \lim_{t\to 0} \tfrac{1}{t}\big(\mu(\alpha,\beta)(T(\Phi_t^X)\circ p) - \mu(\alpha,\beta)(p)\big)\\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\mu\big(\alpha(T(\Phi_t^X)\circ p),\beta(T(\Phi_t^X)\circ p)\big) - \mu\big(\alpha(p),\beta(p)\big)\Big)\\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\mu\big(\alpha(T(\Phi_t^X)\circ p) - \alpha(p),\beta(T(\Phi_t^X)\circ p)\big)\\
&\qquad + \mu\big(\alpha(p),\beta(T(\Phi_t^X)\circ p) - \beta(p)\big)\Big)\\
&= \mu\big(\mathcal{L}_X(\alpha),\beta\big)(p) + \mu\big(\alpha,\mathcal{L}_X(\beta)\big)(p).
\end{aligned}$$

□

Remark 9.2.29. One can use Proposition 9.2.28, applied to the natural pairing of $\mathbb{R}^m$ and $(\mathbb{R}^m)^*$, to calculate the Lie derivatives of a Pfaffian form in local coordinates: For a vector field of the form $X = \sum_k a_k \frac{\partial}{\partial x_k}$ we find

$$\begin{aligned}
\big(\mathcal{L}_X(\mathtt{d}x_j)\big)\big(\tfrac{\partial}{\partial x_i}\big) &= -\mathtt{d}x_j\big(\mathcal{L}_X \tfrac{\partial}{\partial x_i}\big) = -\mathtt{d}x_j\big([X,\tfrac{\partial}{\partial x_i}]\big)\\
&= \mathtt{d}x_j\big(\textstyle\sum_k \frac{\partial a_k}{\partial x_i}\frac{\partial}{\partial x_k}\big) = \frac{\partial a_j}{\partial x_i}
\end{aligned}$$

so that

$$\mathcal{L}_X(\mathtt{d}x_j) = \sum_i \frac{\partial a_j}{\partial x_i}\mathtt{d}x_i.$$

Example 9.2.30. Let M and N be smooth manifolds and $\varphi\colon M \to N$ a diffeomorphism. Then the linear isomorphisms $T_p(\varphi)\colon T_p(M) \to T_{\varphi(p)}(N)$ induce linear isomorphisms

$$\mathcal{T}_p^{r,s}(\varphi)\colon \mathcal{T}^{r,s}(M)_p \to \mathcal{T}^{r,s}(N)_{f(p)}, \quad \alpha \mapsto \varphi_*\alpha,$$

where

$$\begin{aligned}
&(\varphi_*\alpha)_x(\omega_1\otimes\cdots\otimes\omega_r\otimes v_1\otimes\cdots\otimes v_s)\\
&= \alpha_{\varphi(x)}(\omega_1\circ T_x(\varphi)^{-1},\ldots,\omega_r\circ T_x(\varphi)^{-1}, T_x(\varphi)v_1,\ldots,T_x(\varphi)v_s)
\end{aligned}$$

for $x \in M$, $v_j \in T_x(M)$ and $\omega_j \in T_x^*(M)$. Together they form a bundle isomorphism $\mathcal{T}^{r,s}(\varphi)\colon \mathcal{T}^{r,s}(M) \to \mathcal{T}^{r,s}(N)$. This is easily read off from the isomorphisms

$$\mathcal{T}^{r,s}(M) \cong \mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} \mathcal{T}^{r,s}(\mathbb{R}^n), \quad \mathcal{T}^{r,s}(N) \cong \mathrm{GL}(N) \times_{\mathrm{GL}_n(\mathbb{R})} \mathcal{T}^{r,s}(\mathbb{R}^n),$$

because in this picture

$$\mathcal{T}^{r,s}(\varphi)([\alpha, v]) = [T_p(\varphi) \circ \alpha, v], \quad \alpha \in \mathrm{GL}(M)_p = \mathrm{Iso}(\mathbb{R}^n, T_p(M)).$$

In this picture the *pull-back* $\varphi^*T \in \Gamma\big(\mathcal{T}^{r,s}(M)\big)$ of $T \in \Gamma\big(\mathcal{T}^{r,s}(N)\big)$ is obtained via the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}^{r,s}(M) & \xrightarrow{\mathcal{T}^{r,s}(\varphi)} & \mathcal{T}^{r,s}(N) \\ {\scriptstyle \varphi^*T}\Big\uparrow & & \Big\uparrow{\scriptstyle T} \\ M & \xrightarrow{\varphi} & N. \end{array}$$

The *Lie derivative* $\mathcal{L}_X(T)$ of T with respect to a vector field $X \in \mathcal{V}(M)$ is then given by

$$\mathcal{L}_X(T)(p) := \lim_{t\to 0} \frac{1}{t}\big((\Phi_t^X)^*T - T\big)(p)$$

for all $p \in M$, where Φ_t^X is the local flow of X.

Corollary 9.2.31. *Let M be a manifold and $X \in \mathcal{V}(M)$. Then for two tensor fields T and T' on M the following equality holds*

$$\mathcal{L}_X(T \otimes T') = \mathcal{L}_X T \otimes T' + T \otimes \mathcal{L}_X T'.$$

Proof. Apply Proposition 9.2.28(ii) to the bilinear map $(T, T') \mapsto T \otimes T'$. □

Remark 9.2.32. While the Lie derivative is a good notion of differentiation of a tensor field with respect to a vector field, it does not lead to a notion of differentiation of a tensor field with respect to a tangent vector at a point. The reason for this is that the value of the Lie derivative of a tensor field at a point depends on the value of the vector field not only at that point, but in a neighborhood. It is easy to see that from the algebraic point of view this is due to the fact that the map $X \mapsto \mathcal{L}_X$ is itself a differential operator and not a $C^\infty(M)$-linear map. For instance, if $f \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$, we have (cf. Proposition 9.2.28)

$$\begin{aligned} (\mathcal{L}_{fX}\alpha)(Y) &= fX\big(\alpha(Y)\big) - \alpha(\mathcal{L}_{fX}Y) = fX\big(\alpha(Y)\big) - \alpha([fX, Y]) \\ &= fX\big(\alpha(Y)\big) - \alpha(f[X,Y] - (Yf)X) \\ &= f\big(X\big(\alpha(Y)\big) - \alpha(\mathcal{L}_X Y)\big) + (Yf)\cdot\alpha(X) \\ &= f\mathcal{L}_X\alpha(Y) + \alpha(X)\,\mathtt{d}f(Y). \end{aligned}$$

Thus we have

$$\mathcal{L}_{fX}\alpha = f\mathcal{L}_X\alpha + \alpha(X)\,\mathtt{d}f. \tag{9.5}$$

Exercises for Section 9.2

Exercise 9.2.1. Show that the function $\widetilde{\omega}(X)$ is smooth for $\omega \in \Omega^1(M)$ and $X \in \mathcal{V}(M)$.

Exercise 9.2.2. Show that the bidual $C^\infty(M)$-module $(\mathcal{V}(M)^*)^*$ of $\mathcal{V}(M)$ is isomorphic to $\mathcal{V}(M)$ as a $C^\infty(M)$-module.

Exercise 9.2.3. Compute how the coordinates of $\omega \in T_p^*(M)$ change when one changes charts for M and uses the associated charts for $T^*(M)$ constructed from $T^*(M) \cong \mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} (\mathbb{R}^n)^*$ and local trivializations of $\mathrm{GL}(M)$.

Exercise 9.2.4. We denote by $\left(\mathtt{d}x \otimes \frac{\partial}{\partial x}\right)_{(\varrho,\sigma)}(p)$ the element of $\mathcal{T}^{r,s}(T_pM)$ which is determined by ϱ and σ. Show that the

$$\mathtt{d}x_{\varrho(1)}(p) \otimes \ldots \otimes \mathtt{d}x_{\varrho(r)}(p) \otimes \frac{\partial}{\partial x_{\sigma(1)}}\Big|_p \otimes \ldots \otimes \frac{\partial}{\partial x_{\sigma(s)}}\Big|_p$$

form a basis for $\mathcal{T}^{r,s}(T_pM)$.

Exercise 9.2.5. Compute, how the coordinates of $t \in \mathcal{T}^{r,s}(T_pM)$ change when one changes charts for M and uses the associated charts for $\mathcal{T}^{r,s}(M)$ constructed from $\mathcal{T}^{r,s}(M) \cong \mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} \mathcal{T}^{r,s}(\mathbb{R}^n)$ and local trivializations of $\mathrm{GL}(M)$.

Exercise 9.2.6. Generalize Theorem 9.2.23 to general tensor bundles $\mathcal{T}^{r,s}(M)$. More precisely, show that the space of sections $\Gamma\big(\mathcal{T}^{r,s}(M)\big)$ is naturally isomorphic to $\big(\mathcal{V}(M)^*\big)^{\otimes r} \otimes \mathcal{V}(M)^{\otimes s}$ as a $C^\infty(M)$-module.

Exercise 9.2.7. In Exercise 9.2.5, it was calculated, how the coordinates vary under a change of coordinates. Conversely, show that a family $(a^i_{(\varrho,\sigma)})_{i\in I}$ of coordinate functions transforming as in Exercise 9.2.5 define a tensor field, which has these functions as coordinate functions (*"a tensor is something that transforms like a tensor"*).

Exercise 9.2.8. Let G be a group and $\tau\colon G \times F \to F, (g,f) \mapsto g \cdot f$ be an action of G on F. Show that:

(a) The map $\tau\colon G \times F \to F$ factors through a bijection $(G \times F)/G$, where the G-action on $G \times F$ is defined by $g(h,f) := (hg^{-1}, g \cdot f)$.
(b) If V is a vector space, then the evaluation map $\mathrm{ev}\colon \mathrm{GL}(V) \times V \to V$ factors through a bijection $(\mathrm{GL}(V) \times V)/\mathrm{GL}(V)$, where the $\mathrm{GL}(V)$-action on $\mathrm{GL}(V) \times V$ is defined by $g(h,f) := (hg^{-1}, g \cdot f)$.
(c) If V is an n-dimensional vector space, then the evaluation map

$$\mathrm{ev}\colon \mathrm{Iso}(\mathbb{R}^n, V) \times \mathbb{R}^n \to V, \quad (\varphi, v) \mapsto \varphi(v)$$

factors through a bijection $(\mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n)/\mathrm{GL}_n(\mathbb{R})$, where the $\mathrm{GL}_n(\mathbb{R})$-action on $\mathrm{Iso}(\mathbb{R}^n, V) \times \mathbb{R}^n$ is defined by $g(h,f) := (hg^{-1}, g \cdot f)$. Give an interpretation of the fibers of ev in terms of bases of V and coordinates of a vector with respect to a basis.

Exercise 9.2.9. Let $M := \mathbb{R}^n$. For a matrix $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ we consider the affine vector field

$$X_{A,b}(x) := Ax + b.$$

(1) Calculate the maximal flow $\Phi\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ of this vector field.
(2) Let

$$\mathfrak{aff}_n(\mathbb{R}) := \begin{pmatrix} \mathfrak{gl}_n(\mathbb{R}) & \mathbb{R}^n \\ 0 & 0 \end{pmatrix} \subseteq \mathfrak{gl}_{n+1}(\mathbb{R})$$

be the affine Lie algebra on $\mathbb{R}^n$, realized as a Lie subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{R})$, endowed with the commutator bracket. Show that the map

$$\varphi\colon \mathfrak{aff}_n(\mathbb{R}) \to \mathcal{V}(\mathbb{R}^n), \quad \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mapsto -X_{A,b}$$

is a homomorphism of Lie algebras.

9.3 Integration on Manifolds

Integration on manifolds is built up from integration on coordinate patches. Since the integrals on coordinate patches are not invariant under coordinate changes, the naive approach to integrate functions does not work. What can be integrated are sections of bundles which transform in the same way as integrals, i.e., via the absolute value of the Jacobi determinants of the coordinate changes. There are two choices: One either uses densities which have precisely this transformation behavior or one uses differential forms of degree $\dim M$. In the latter case one has to make sure that all coordinate changes have positive Jacobi determinants. This leads to the concept of an orientable manifold.

9.3.1 Differential Forms

Differential forms are tensor fields with additional symmetry conditions. They are of central importance in the description of the topology of a manifold. Moreover, they can be used to build an integration theory for which one has generalizations of the classical Stokes Theorem.

Let M be a smooth manifold of dimension n and $\mathrm{GL}(M)$ its frame bundle. Realize the tangent bundle TM as the associated bundle $\mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ (cf. Definition 9.2.18)

Definition 9.3.1. For $k \in \mathbb{N}_0$, we consider the $\mathrm{GL}_n(\mathbb{R})$-module $\mathrm{Alt}^k(\mathbb{R}^n, \mathbb{R})$ of alternating k-linear forms on E, on which $\mathrm{GL}_n(\mathbb{R})$ acts by

$$(g \cdot \alpha)(v_1, \ldots, v_k) := \alpha(g^{-1}v_1, \ldots, g^{-1}v_k).$$

The resulting associated bundle

$$\mathrm{Alt}^k(TM) := \mathrm{GL}(M) \times_{\mathrm{GL}_n(\mathbb{R})} \mathrm{Alt}^k(\mathbb{R}^n, \mathbb{R})$$

is called the *k-form bundle* over M. Its sections are called *alternating k-forms* or simply *k-forms* on M. One also speaks of *differential forms* of *degree k*. The space of differential forms of degree k on M will be denoted by $\Omega^k(M)$. Note that there are no non-zero differential forms of degree greater than the dimension of M. We set $\Omega(M) = \oplus_{k=0}^{\dim M} \Omega^k(M) = \oplus_{k=0}^{\infty} \Omega^k(M)$.

Example 9.3.2. If $n = \dim M$, then $\mathrm{Alt}^n(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}$ and the $\mathrm{GL}_n(\mathbb{R})$-representation on $\mathrm{Alt}^n(\mathbb{R}^n, \mathbb{R})$ is the inverse of the determinant. Thus the transition functions of the line bundle $\mathrm{Alt}^n(TM)$ are the inverses of the Jacobi determinants of the coordinate changes.

Remark 9.3.3. (cf. Exercise 9.2.6) Theorem 9.2.23 generalizes from $T^*(M) = \mathrm{Alt}^1(TM)$ to $\mathrm{Alt}^k(TM)$. More precisely, the space of differential forms $\Omega^k(M)$ is naturally isomorphic to the space of alternating $C^\infty(M)$-multilinear maps $\mathcal{V}(M)^k \to C^\infty(M)$.

Definition 9.3.4. For each vector space E we have a natural bilinear product

$$\wedge\colon \ \mathrm{Alt}^p(E, \mathbb{R}) \times \mathrm{Alt}^q(E, \mathbb{R}) \to \mathrm{Alt}^{p+q}(E, \mathbb{R}),$$

called the *exterior* or *wedge product*, defined by

$$\begin{aligned}&(\alpha \wedge \beta)(v_1, \ldots, v_{p+q})\\ &= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \beta(v_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).\end{aligned}$$

Taking into account that α and β are alternating, this product can also be written with $\binom{p+q}{p}$ summands instead of $(p+q)!$:

$$\begin{aligned}&(\alpha \wedge \beta)(v_1, \ldots, v_{p+q})\\ &= \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),\end{aligned}$$

where $\mathrm{Sh}(p, q)$ denotes the set of all (p, q)-*shuffles* in S_{p+q}, i.e., all permutations with

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

This product induces a bilinear bundle map

$$\wedge\colon \ \mathrm{Alt}^p(TM) \times \mathrm{Alt}^q(TM) \to \mathrm{Alt}^{p+q}(TM), \quad ([\psi, \alpha], [\psi, \beta]) \mapsto [\psi, \alpha \wedge \beta],$$

and hence a $C^\infty(M)$-bilinear product on the spaces of smooth sections

$$\Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta,$$

where $(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$ for each $p \in M$.

Proposition 9.3.5. *If α, β are differential forms on a manifold M, then*

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta \tag{9.6}$$

for each vector field $X \in \mathcal{V}(M)$.

Proof. Take μ in Proposition 9.2.28(ii) to be $(\alpha, \beta) \mapsto \alpha \wedge \beta$. □

Remark 9.3.6. Let X be a smooth vector field on a manifold M. From Definition 9.2.26 it is easy to deduce that for any section α of a vector bundle $\mathbb{V}$ associated with the frame bundle $\mathrm{GL}(M)$ of M, the equality $\mathcal{L}_X A = 0$ implies that A is invariant under the flow Φ_t^X of X, i.e., $(\Phi_t^X)^*\alpha = \alpha$. Hence we say such a section is *invariant* under X if $\mathcal{L}_X A = 0$.

Remark 9.3.7. Proposition 9.3.5 can be used to compute $\mathcal{L}_X$ on differential forms. We have

$$(\mathcal{L}_X\omega)(Y) = X\,\omega(Y) - \omega([X,Y])$$

for a 1-form ω and

$$(\mathcal{L}_X\alpha)(X_1,\ldots,X_k) = X\,\alpha(X_1,\ldots,X_k) - \sum_{i=1}^{k} \alpha(X_1,\ldots,[X,X_i],\ldots,X_k)$$

for forms α of degree k (cf. Lemma 6.5.8). This formula follows directly from Proposition 9.2.28 and $\mathcal{L}_X Y = [X,Y]$ for $X, Y \in \mathcal{V}(M)$.

Definition 9.3.8. Let α be a differential form of degree k. Then we define the *exterior derivative* $\mathtt{d}\alpha$ of α to be the differential form of degree $k+1$ given by the formula

$$\begin{aligned}(\mathtt{d}\alpha)(X_0,\ldots,X_k) := &\sum_{i=0}^{k}(-1)^i X_i\alpha(X_0,\ldots,\widehat{X}_i,\ldots,X_k) + \\ &\sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_k),\end{aligned}$$

where a hat over a symbol means that the symbol does not occur (cf. Definition 6.5.2).

Clearly $\mathtt{d}\alpha$ defines a $(k+1)$-linear map $\mathcal{V}(M)^{k+1} \to C^\infty(M)$, and it is easily seen to be alternating (cf. Definition 6.5.2). We claim that it is also $C^\infty(M)$-linear. Since it is alternating, it suffices to verify the $C^\infty(M)$-linearity in X_0. The terms for which the $C^\infty(M)$-linearity is not obvious are

$$\begin{aligned}&\sum_{i=1}^{k}(-1)^i X_i\alpha(X_0,\ldots,\widehat{X}_i,\ldots,X_k)\\ &+\sum_{0<i\le k}(-1)^i\alpha([X_0,X_i],X_1,\ldots,\widehat{X}_i,\ldots,X_k).\end{aligned}$$

But the relations

$$X(fg) = Xf \cdot g + f \cdot Xg \quad \text{and} \quad [X, fY] = (Xf) \cdot Y + f[X, Y]$$

imply that $\mathtt{d}\alpha$ is in fact $C^\infty(M)$-linear in each argument.

Example 9.3.9. If α is a form of degree 0, i.e., a function f, then the definition of the exterior derivative gives $\mathtt{d}f(X) = Xf$. If α is of degree 1, then

$$\mathtt{d}\alpha(X, Y) = X\,\alpha(Y) - Y\,\alpha(X) - \alpha([X, Y]).$$

Definition 9.3.10. If X is a vector field and α is a differential form of degree k on a manifold, we define the *interior product* or *contraction* of α with respect to X to be a form of degree $k-1$ given by

$$(i_X\alpha)(X_1, \ldots, X_{k-1}) := \alpha(X, X_1, \ldots, X_{k-1})$$

for $k > 0$. For $k = 0$, i.e., for functions, we set $i_X f = 0$.

Proposition 9.3.11. *For forms α, β of degree k, m on a manifold*

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^k \alpha \wedge i_X\beta.$$

Proof. This is a simple consequence of the definitions. □

Proposition 9.3.12. *The exterior derivative* $\mathtt{d}$ *on a manifold has the following properties.*

(i) $\mathtt{d}\colon \Omega(M) \to \Omega(M)$ *is linear.*
(ii) *Any vector field X satisfies the* Cartan formula

$$\mathtt{d} \circ i_X + i_X \circ \mathtt{d} = \mathcal{L}_X \quad \text{on} \quad \Omega(M).$$

(iii) *If α and β are differential forms of degree p and q, respectively, then*

$$\mathtt{d}(\alpha \wedge \beta) = \mathtt{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathtt{d}\beta.$$

(iv) $\mathtt{d} \circ \mathtt{d} = 0$.

Proof. (i) This is obvious from the definition of $\mathtt{d}$.

(ii) follows from Lemma 6.5.9.

(iii) This is proved by induction on $p + q$. If one of p, q is 0, then the assertion is easily seen by direct calculation. So let us assume that $p + q > 0$. In order to prove the equality of the differential forms on both sides, it is enough to show that they are the same after applying i_X for any X. Now we may use (ii) and the induction hypothesis to complete the proof. More precisely,

$$\begin{aligned}
&i_X \mathtt{d}(\alpha \wedge \beta) \\
&= \mathcal{L}_X(\alpha \wedge \beta) - \mathtt{d}\big(i_X(\alpha \wedge \beta)\big) \\
&\overset{(9.6)}{=} \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta - \mathtt{d}\big(i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X\beta\big) \\
&\overset{\text{ind.}}{=} i_X(\mathtt{d}\alpha) \wedge \beta + (-1)^p (i_X\alpha) \wedge \mathtt{d}\beta + (-1)^{p+1} \mathtt{d}\alpha \wedge (i_X\beta) + \alpha \wedge (i_X \mathtt{d}\beta) \\
&= i_X(\mathtt{d}\alpha \wedge \beta) + (-1)^p i_X(\alpha \wedge \mathtt{d}\beta) = i_X(\mathtt{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathtt{d}\beta).
\end{aligned}$$

(iv) follows from Proposition 6.5.12. □

Remark 9.3.13. Using the Cartan Formula, we find

$$\begin{aligned}
\mathcal{L}_{fX}\alpha &= i_{fX}\mathtt{d}\alpha + \mathtt{d}(i_{fX}\alpha) = f i_X(\mathtt{d}\alpha) + \mathtt{d}(f i_X\alpha) \\
&= f i_X(\mathtt{d}\alpha) + f\mathtt{d}(i_X\alpha) + \mathtt{d}f \wedge i_X\alpha = f\mathcal{L}_X\alpha + \mathtt{d}f \wedge i_X\alpha,
\end{aligned}$$

i.e.,

$$\mathcal{L}_{fX}\alpha = f\mathcal{L}_X\alpha + \mathtt{d}f \wedge i_X\alpha.$$

If α is of degree $n = \dim_{\mathbb{R}}(M)$, we have

$$0 = i_X(\mathtt{d}f \wedge \alpha) = i_X(\mathtt{d}f) \wedge \alpha - \mathtt{d}f \wedge i_X\alpha = (Xf)\alpha - \mathtt{d}f \wedge i_X\alpha,$$

so that $\mathcal{L}_{fX}(\alpha) = f\mathcal{L}_X\alpha + (Xf) \cdot \alpha$.

Definition 9.3.14. A differential form $\omega \in \Omega^k(M)$ satisfying $\mathtt{d}\omega = 0$ is called *closed*. If $\omega = \mathtt{d}\nu$ for some $\nu \in \Omega^{k-1}(M)$, then ω is called an *exact* form. Then $\mathtt{d}^2 = 0$ implies that each exact form is closed. Putting $\Omega^{-1}(M) := \{0\}$, the quotient space

$$H^k_{\mathrm{dR}}(M) := \frac{\ker\big(\mathtt{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)\big)}{\mathrm{im}\,\big(\mathtt{d}\colon \Omega^{k-1}(M) \to \Omega^k(M)\big)}$$

is called the k-th *de Rham cohomology space of* M.

Definition 9.3.15. Let M and N be smooth manifolds and $f\colon M \to N$ be a smooth map. Given a differential form $\omega \in \Omega^k(N)$ we can define the *pull-back* $f^*\omega$ by $(f^*\omega)_p := T_p(f)^*\omega_{f(p)}$, i.e.,

$$(f^*\omega)_p(v_1, \ldots, v_p) := \omega_{f(p)}(T_p(f)v_1, \ldots, T_p(f)v_p).$$

The pull-back of ω is a differential form $f^*\omega \in \Omega^k(M)$.

Proposition 9.3.16. *The pull-back of differential forms is compatible with the exterior derivative, i.e.,*

$$\mathtt{d}(f^*\omega) = f^*(\mathtt{d}\omega).$$

Proof. In fact, if $\omega = \varphi \in C^\infty(N) = \Omega^0(N)$, then

$$f^*(\mathtt{d}\varphi) = \mathtt{d}\varphi \circ T(f) = \mathtt{d}(\varphi \circ f) = \mathtt{d}(f^*\varphi).$$

Now suppose that we can write ω as an exterior product of 1-forms:

$$\omega = \varphi \mathtt{d}y_{i_1} \wedge \ldots \wedge \mathtt{d}y_{i_k}. \tag{9.7}$$

Then $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$ (Exercise!) implies

$$\begin{aligned}
&\mathtt{d}(f^*\omega)\\
&= \mathtt{d}\left((\varphi \circ f) f^*\mathtt{d}y_{i_1} \wedge \ldots \wedge f^*\mathtt{d}y_{i_k}\right) = \mathtt{d}\left((\varphi \circ f)\mathtt{d}(f^*y_{i_1}) \wedge \ldots \wedge \mathtt{d}(f^*y_{i_k})\right)\\
&= \mathtt{d}(\varphi \circ f) \wedge \mathtt{d}(f^*y_{i_1}) \wedge \ldots \wedge \mathtt{d}(f^*y_{i_k}) = f^*\mathtt{d}\varphi \wedge \mathtt{d}(f^*y_{i_1}) \wedge \ldots \wedge \mathtt{d}(f^*y_{i_k})\\
&= f^*\mathtt{d}\varphi \wedge f^*\mathtt{d}y_{i_1} \wedge \ldots \wedge f^*\mathtt{d}y_{i_k} = f^*(\mathtt{d}\varphi \wedge \mathtt{d}y_{i_1} \wedge \ldots \wedge \mathtt{d}y_{i_k})\\
&= f^*(\mathtt{d}\omega).
\end{aligned}$$

Locally all differential forms can be written as sums of terms of the type (9.7). Moreover, if ω vanishes on an open subset U of N, then $\mathtt{d}\omega$ vanishes on U, and $f^*\omega$ vanishes on $f^{-1}(U)$. But then also $\mathtt{d}(f^*\omega)$ vanishes on U. Thus the claim can indeed be checked locally. □

As a consequence of the preceding argument, pull-backs of closed/exact forms are closed/exact. We conclude that each smooth map $f\colon M \to N$ leads to well-defined linear maps (functoriality of de Rham cohomology)

$$f^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M), \quad [\alpha] \mapsto [f^*\alpha].$$

Lemma 9.3.17 (Homotopy Lemma). *Let M and N be smooth manifolds and $F\colon M \times I \to N$ be a smooth map, where $I \subseteq \mathbb{R}$ is an open interval containing 0 and 1. Then the induced smooth maps $F_0, F_1\colon M \to N$ satisfy*

$$F_0^* = F_1^*\colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M), \quad k \in \mathbb{N}_0.$$

Proof. We write $i_t\colon M \to M \times \{t\} \subseteq M \times I$ for the canonical embeddings.

For $\omega \in \Omega^k(M \times I, \mathbb{R})$, $k \geq 0$, we define the fiber integral $I(\omega) \in \Omega^{k-1}(M)$ by

$$I(\omega)_x(v_1, \ldots, v_{k-1}) := \int_0^1 \omega_{(t,x)}\big(\tfrac{\partial}{\partial t}, v_1, \ldots, v_{k-1}\big)\, dt,$$

i.e.,

$$I(\omega)_x := \int_0^1 i_{\frac{\partial}{\partial t}} \omega_{(t,x)}\, dt.$$

Let $X_0, \ldots, X_{k-1} \in \mathcal{V}(M)$ and extend these vector fields in the canonical fashion to vector fields $\widetilde{X}_i$ on $M \times I$, constant in the second component. Then we have $[\widetilde{X}_i, \widetilde{X}_j] = [X_i, X_j]\widetilde{\ }$, and from Cartan's formula (see Proposition 9.3.12 we further get

$$
\begin{aligned}
&(\mathtt{d}_M I(\omega))(X_0, \ldots, X_{k-1}) \\
&= \mathtt{d}_M\Big(\int_0^1 i_{\frac{\partial}{\partial t}}\omega_{(t,x)}\,dt\Big)(X_0, \ldots, X_{k-1}) \\
&= \int_0^1 \big(\mathtt{d}_{M\times I} i_{\frac{\partial}{\partial t}}\omega\big)(\widetilde{X}_0, \ldots, \widetilde{X}_{k-1})\,dt \\
&= \int_0^1 (\mathcal{L}_{\frac{\partial}{\partial t}}\omega)(\widetilde{X}_0, \ldots, \widetilde{X}_k)\,dt - \int_0^1 i_{\frac{\partial}{\partial t}}(\mathtt{d}_{M\times I}\omega)(\widetilde{X}_0, \ldots, \widetilde{X}_{k-1})\,dt \\
&= (i_1^*\omega)(X_0, \ldots, X_{k-1}) - (i_0^*\omega)(X_0, \ldots, X_{k-1}) - I(\mathtt{d}_{M\times I}\omega)(X_0, \ldots, X_{k-1}).
\end{aligned}
$$

This means that we have the homotopy formula

$$
\mathtt{d}_M I(\omega) + I(\mathtt{d}_{M\times I}\omega) = i_1^*\omega - i_0^*\omega. \tag{9.8}
$$

We apply this formula to $\omega = F^*\alpha$ for a closed form α of degree $k \geq 1$ on N and obtain

$$
[F_1^*\alpha - F_0^*\alpha] = [i_1^*\omega - i_0^*\omega] = [I(\mathtt{d}_{M\times I}\omega)] = [I(F^*\mathtt{d}\alpha)] = 0.
$$

For degree $k = 0$, the space $H^0_{\mathrm{dR}}(N)$ consists of locally constant functions f, and since $F_t^* f = f \circ F_t$ does not depend on t, we also get $F_0^* = F_1^*$ for $k = 0$. □

Corollary 9.3.18. *If M is a smooth manifold and $n \in \mathbb{N}_0$, then the projection $p_M \colon M \times \mathbb{R}^n \to M$ defines an isomorphism*

$$
p_M^* \colon H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(M \times \mathbb{R}^n), \quad [\alpha] \mapsto [p_M^*\alpha]
$$

for each $k \in \mathbb{N}_0$.

Proof. For $I = \mathbb{R}$, we consider the smooth map

$$
F \colon M \times \mathbb{R}^n \times I \to M \times \mathbb{R}^n, \quad (m, x, t) \mapsto (m, tx).
$$

Then $F_1 = \mathrm{id}_{M\times\mathbb{R}^n}$, and F_0 is the projection onto $M \times \{0\}$.

The inclusion $i_0 \colon M \to M \times \mathbb{R}^n, m \mapsto (m, 0)$ satisfies $p_M \circ i_0 = \mathrm{id}_M$ and $i_0 \circ p_M = F_0$. In view of Lemma 9.3.17, we have $F_0^* = F_1^* = \mathrm{id}$, so that the pull-back maps p_M^* and i_0^* induce mutually inverse isomorphisms between the spaces $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(M \times \mathbb{R}^n)$. □

Definition 9.3.19. A smooth manifold M is called *smoothly contractible* to a point $p \in M$ if there exists a smooth map $F \colon M \times I \to M$ with

$$
F(x, 0) = x \quad \text{and} \quad F(x, 1) = p,
$$

for all $x \in M$, where $I \subseteq \mathbb{R}$ is an open interval containing $[0, 1]$.

Theorem 9.3.20 (Poincaré Lemma). *Suppose that M is a smoothly contractible manifold. Then any closed form of degree at least one is exact.*

Proof. Let $F\colon M \times I \to M$ be a smooth contraction of M to $p \in M$ and $\alpha \in \Omega^k(M)$, $k > 0$. Since $F_0 = \mathrm{id}_M$ and $F_1 = p$ is the constant map, the Homotopy Lemma implies that

$$[\alpha] = [F_0^*\alpha] = [F_1^*\alpha].$$

But since F_1 is constant and $k > 0$, we have $F_1^*\alpha = 0$. □

Corollary 9.3.21. *Suppose that $M \subseteq \mathbb{R}^n$ is open and star-shaped. Then any closed form of degree at least one is exact.*

Proof. Contract linearly to any point with respect to which M is star-shaped. □

9.3.2 Integration of Densities

Recall the density bundle $|\Lambda|(M) := |\Lambda|^1(M)$ from Example 9.2.16.

Remark 9.3.22 (Transformation formula for densities). Let $(\varphi_i, U_i)_{i\in I}$ be a smooth atlas of the n-dimensional smooth manifold M and $(\Phi_i)_{i\in I}$ be the corresponding trivialization of $\mathrm{GL}(M)$ constructed in Example 9.2.9. Then, according to Example 9.2.16, the transition functions of $|\Lambda|(M)$ with respect to the local trivialization $(\widetilde{\Phi}_i)_{i\in I}$ constructed from $(\Phi_i)_{i\in I}$ in Example 9.2.16 are given by

$$\widetilde{\Phi}_{ji}(p) = \big|\det \mathtt{d}(\varphi_j \circ \varphi_i^{-1})\big(\varphi_i(p)\big)\big|^{-1} = \big|\det \mathtt{d}(\varphi_i \circ \varphi_j^{-1})\big(\varphi_j(p)\big)\big|.$$

Let μ be a density on M. Then we obtain for each i a density $\mu_i := (\varphi_i^{-1})^*\mu$ on the open subset $\varphi_i(U_i) \subseteq \mathbb{R}^n$. On this set we write $|\mathtt{d}x_1 \cdots \mathtt{d}x_n|$ for the density assigning in each point the value 1 to the canonical basis for $\mathbb{R}^n$. Then we can write

$$\mu_i = (\varphi_i^{-1})^*\mu = f_i \cdot |\mathtt{d}x_1 \cdots \mathtt{d}x_n|$$

for a real-valued function $f_i\colon \varphi_i(U_i) \to \mathbb{R}$. In terms of the local trivializations, this means that

$$\widetilde{\Phi}_i \circ \mu \circ \varphi_i^{-1} = (\varphi_i^{-1}, f_i)\colon \varphi_i(U_i) \to U_i \times \mathbb{R}.$$

We say that the f_i represent μ in the given atlas, resp., the corresponding local trivialization. The $(f_i)_{i\in I}$ satisfy the following transformation property

$$(f_j \circ \varphi_j)(p) = \widetilde{\Phi}_{ji}(p) \cdot (f_i \circ \varphi_i)(p),$$

which we can rewrite as

$$f_j = (f_i \circ \varphi_i \circ \varphi_j^{-1})\,\big|\det\big(\mathtt{d}(\varphi_i \circ \varphi_j^{-1})\big)\big|$$

on $\varphi_j(U_i \cap U_j)$, which corresponds to

$$\mu_j = (\varphi_i \circ \varphi_j^{-1})^* \mu_i = (\varphi_i \circ \varphi_j^{-1})^* f_j \cdot \big((\varphi_i \circ \varphi_j^{-1})^* |\mathtt{d}x_1 \cdots \mathtt{d}x_n| \big)$$

with

$$(\varphi_i \circ \varphi_j^{-1})^* |\mathtt{d}x_1 \cdots \mathtt{d}x_n| = \big| \det \big(\mathtt{d}(\varphi_i \circ \varphi_j^{-1}) \big) \big| \cdot |\mathtt{d}x_1 \cdots \mathtt{d}x_n|.$$

Definition 9.3.23. The transformation behavior allows to define *positive densities* by requiring that they be represented by positive functions. Similarly, we call a density *measurable* if the representing functions are (Borel) measurable. Finally, we say that a positive measurable density μ is a *locally bounded Borel density* if the representing functions with respect to the charts are bounded on compact subsets.

If μ is a positive and measurable density, we set for any open subset $O \subseteq U_i$:

$$\int_O \mu := \int_{\varphi_i(O)} (\varphi_i^{-1})^* \mu = \int_{\varphi_i(O)} f_i(x)\, |\mathtt{d}x_1 \cdots \mathtt{d}x_n| := \int_{\varphi_i(O)} f_i(x)\, dx.$$

The change of variables formula from multi-variable calculus allows us to calculate

$$\begin{aligned}\int_{\varphi_j(U_i \cap U_j)} f_j(y)\, dy &= \int_{\varphi_j(U_i \cap U_j)} (f_i \circ \varphi_i \circ \varphi_j^{-1})(y) |\det \mathtt{d}(\varphi_i \circ \varphi_j^{-1})(y)|\, dy \\ &= \int_{\varphi_i(U_i \cap U_j)} f_i(x)\, dx\end{aligned}$$

which shows that the definition is independent of the choice of the chart.

Definition 9.3.24. Let M be a topological space and $(U_i)_{i \in I}$ be an open cover of M. An open cover $(V_j)_{j \in J}$ of M is called a *refinement* of $(U_i)_{i \in I}$ if for each $j \in J$ there exists an $i \in I$ such that $V_j \subseteq U_i$. An open cover $(U_i)_{i \in I}$ is called *locally finite* if for each $p \in M$ there exists a neighborhood U in M such that $U \cap U_i \neq \emptyset$ only for finitely many $i \in I$. The space M is called *paracompact* if it is Hausdorff and every open cover has a locally finite refinement. Further M is called *σ-compact*, if it is a countable union of compact subsets.

Proposition 9.3.25. *Let M be a second countable smooth manifold of dimension n. Then M is σ-compact and paracompact.*

Proof. Let $(W_n)_{n \in \mathbb{N}}$ be a countable basis for the topology of M. Since M is locally compact, we may w.l.o.g. assume that all the closures $\overline{W_i}$ are compact because the relatively compact subsets also form a basis for the topology.

Then choose an increasing family $(A_k)_{k \in \mathbb{N}}$ of compact sets inductively as follows: A_1 is the closure of W_1. If A_k is defined, set

$$j_k := \min \left\{ j \geq k+1 \mid A_k \subseteq \bigcup_{m=1}^{j} W_m \right\}$$

and let A_{k+1} be the closure of $\bigcup_{m=1}^{j_k} W_m$. Then $\bigcup_{k\in\mathbb{N}} A_k = M$ and A_k is contained in the interior A_{k+1}° of A_{k+1}. Since each A_k is compact, we see that M is σ-compact.

Now let $(V_j)_{j\in J}$ be an open cover of M. Then any $p \in A_{k+1} \setminus A_k^\circ$ is contained in some V_j and there exists an open neighborhood

$$U_{p,j} \subseteq V_j \cap (A_{k+2}^\circ \setminus A_{k-1})$$

of p. Then the sets $U_{p,j}$ cover the compact set $A_{k+1}\setminus A_k^\circ$. Pick a finite subcover, and write U_i for the elements of this subcover. Collecting these sets for all k's gives the desired covering. In fact, the relation $U_j \subseteq A_{k+2}^\circ \setminus A_{k-1}$ proves local finiteness of the cover. □

Definition 9.3.26. Let M be a topological space. A *partition of unity* is a collection $(\rho_i)_{i\in I}$ of continuous functions $\rho_i \colon M \to [0,1]$ such that each $p \in M$ has a neighborhood U on which only finitely many functions ρ_i are non-zero and

$$\sum_{i\in I} \rho_i(p) = 1.$$

If $(U_i)_{i\in I}$ is an open cover of M and $\mathrm{supp}(\rho_i) \subseteq U_i$, then the partition of unity is said to be *subordinate* to the cover $(U_i)_{i\in I}$.

Theorem 9.3.27. *If M is second countable and $(U_j)_{j\in J}$ is a locally finite open cover of M, then there exists a smooth partition of unity on M, which is subordinate to $(U_j)_{j\in J}$.*

Proof. The proof of Proposition 9.3.25 shows that we can find a sequence, $(K_i)_{i\in\mathbb{N}}$ of compact subsets of M with $\bigcup_{n\in\mathbb{N}} K_n = M$ and $K_n \subseteq K_{n+1}^\circ$. Put $K_0 := \emptyset$. For $p \in M$ let i_p be the largest integer with $p \in M \setminus K_{i_p}$. We then have $p \in K_{i_p+1} \subseteq K_{i_p+2}^\circ$. Choose a $j_p \in J$ with $p \in U_{j_p}$ and pick $\psi_p \in C^\infty(M,\mathbb{R})$ with $\psi_p(p) > 0$ and

$$\mathrm{supp}(\psi_p) \subseteq U_{j_p} \cap (K_{i_p+2}^\circ \setminus K_{i_p})$$

(Lemma 7.4.15). Then $W_p := \psi_p^{-1}(]0,\infty[)$ is an open neighborhood of p. For each $i \geq 1$, choose a finite set of points p in M whose corresponding neighborhoods W_p cover the compact set $K_i \setminus K_{i-1}^\circ$. We order the corresponding functions ψ_p in a sequence $(\psi_i)_{i\in\mathbb{N}}$. Their supports form a locally finite family of subsets of M because for only finitely many of them, the supports intersect a given set K_i. Moreover, the sets $\psi_i^{-1}(]0,\infty[)$ cover M. Therefore

$$\psi := \sum_j \psi_j$$

is a smooth function which is everywhere positive (Exercise 9.3.5). Therefore we obtain smooth functions $\varphi_i := \frac{\psi_i}{\psi}$, $i \in \mathbb{N}$. Then the functions φ_i form a smooth partition of unity on M.

We now define a modified partition of unity, which will be subordinate to the open cover $(U_j)_{j\in J}$: For each $i \in \mathbb{N}$ we pick a $j_i \in J$ with $\mathrm{supp}(\varphi_i) \subseteq U_{j_i}$ and define

$$\alpha_j := \sum_{j_i=j} \varphi_i.$$

As the sum on the right hand side is locally finite, the functions α_j are smooth and

$$\mathrm{supp}(\alpha_j) \subseteq \bigcup_{j_i=j} \mathrm{supp}(\varphi_i) \subseteq U_j$$

(Exercise 9.3.3). We further observe that only countably many of the α_j are non-zero, that $0 \leq \alpha_j$, $\sum_j \alpha_j = 1$, and that the supports form a locally finite family because the cover $(U_j)_{j\in J}$ is locally finite. □

The following corollary is a refinement of Lemma 7.4.15(i).

Corollary 9.3.28. *Let M be a paracompact smooth manifold, $K \subseteq M$ a closed subset and $U \subseteq M$ an open neighborhood of K. Then there exists a smooth function $f\colon M \to \mathbb{R}$ with*

$$0 \leq f \leq 1, \quad f|_K = 1 \quad \text{and} \quad \mathrm{supp}(f) \subseteq U.$$

Proof. In view of Theorem 9.3.27, there exists a smooth partition of unity subordinate to the open cover $\{U, M \setminus K\}$. This is a pair of smooth functions (f, g) with $\mathrm{supp}(f) \subseteq U$, $\mathrm{supp}(g) \subseteq M \setminus K$, $0 \leq f, g$, and $f + g = 1$. These properties immediately imply the claim. □

Corollary 9.3.29. *A second countable manifold M always admits a nowhere vanishing density.*

Proof. Recall from Example 9.2.16 how to construct a local trivialization $(\Phi_i)_{i\in I}$ of $|\Lambda|(M)$ from a smooth atlas $(\varphi_i, U_i)_{i\in I}$ of M. We assume that $(U_i)_{i\in I}$ is locally finite (Proposition 9.3.25). Pick a partition of unity $(\rho_i)_{i\in I}$ subordinate to $(U_i)_{i\in I}$ (Theorem 9.3.27). From the construction of $(\Phi_i)_{i\in I}$ it is clear that $\mu_i(p)\big(\frac{\partial}{\partial x_1^{(i)}}\big|_p, \ldots, \frac{\partial}{\partial x_n^{(i)}}\big|_p\big) = 1$ defines a smooth density over U_i. Therefore $\rho_i\mu_i$ is a density on M. We claim that $\mu := \sum_{i\in I} \rho_i\mu_i$ is a nowhere vanishing smooth density on M. First the local finiteness of the atlas guarantees that μ is a well defined density. To see that μ is nowhere vanishing it suffices to show that $\mu_j|_{U_j\cap U_i}$ with respect to the chart φ_i is represented by a non-negative function. But since μ_j is represented by the constant function 1 with respect to the chart φ_j, this is an immediate consequence of

$$f_j = (f_i \circ \varphi_i \circ \varphi_j^{-1})\,\big|\det\big(\mathtt{d}(\varphi_i \circ \varphi_j^{-1})\big)\big|$$

on $\varphi_j(U_i \cap U_j)$. □

Definition 9.3.30. Let $\mathfrak{B}_M$ be the σ-algebra generated by the open subsets of the locally compact Hausdorff space M. It is called the *Borel σ-algebra*, and its elements are called *Borel subsets* or *(Borel) measurable subsets.* A measure μ on $(M, \mathfrak{B}_M)$, for which we have $\mu(K) < \infty$ for all compact subsets $K \subseteq M$, is called a *Borel measure* on M.

Definition 9.3.31. Let M be a locally compact Hausdorff space and $C_c(M)$ be the space of continuous functions $f\colon M \to \mathbb{C}$ with compact support. A linear functional $I\colon C_c(M) \to \mathbb{C}$ is called *positive* if $I(f) \geq 0$ whenever $f \geq 0$.

Definition 9.3.32. Let M be a second countable smooth manifold and $(\varphi_\alpha, U_\alpha)_{\alpha \in A}$ be an atlas of M corresponding to a locally finite open cover. Further, assume that $(\rho_\alpha)_{\alpha \in A}$ is a subordinate partition of unity. Let $\mu \in |\Lambda^n|(M)$ be a positive measurable density on M. We define the *integral* of μ over M by

$$\int_M \mu := \sum_{\alpha \in A} \int_{U_\alpha} \rho_\alpha \mu.$$

We have to show that this definition is independent of the choice of partition of unity and the underlying atlas. We know already that the definition of the integral over U_α is independent of the choice of the coordinates, once the atlas is chosen. So suppose $(V_\lambda)_{\lambda \in \Lambda}$ and $(\eta_\lambda)_{\lambda \in \Lambda}$ is another pair of an atlas and a subordinate partition of unity. Then the calculation

$$\begin{aligned}\sum_{\alpha \in A} \int_{U_\alpha} \rho_\alpha \mu &= \sum_{\alpha \in A} \int_{U_\alpha} \Big(\sum_{\lambda \in \Lambda} \eta_\lambda\Big) \rho_\alpha \mu = \sum_{\alpha \in A} \sum_{\lambda \in \Lambda} \int_{U_\alpha} \eta_\lambda \rho_\alpha \mu \\ &= \sum_{\lambda \in \Lambda} \sum_{\alpha \in A} \int_{U_\alpha \cap V_\lambda} \rho_\alpha \eta_\lambda \mu = \sum_{\lambda \in \Lambda} \int_{V_\lambda} \Big(\sum_{\alpha \in A} \rho_\alpha\Big) \eta_\lambda \mu = \sum_{\lambda \in \Lambda} \int_{V_\lambda} \eta_\lambda \mu\end{aligned}$$

proves that the integral of μ over M is well-defined.

Proposition 9.3.33. *Fix a positive measurable density μ and for a Borel measurable subset $E \subseteq M$ set*

$$\widetilde{\mu}(E) := \int_E \mu := \int_M \chi_E \, \mu \in [0, \infty],$$

where χ_E is the characteristic function of E. Then $\widetilde{\mu}$ is a positive measure on M. If μ is a locally bounded Borel density, then $\widetilde{\mu}$ is a Borel measure.

Proof. Exercise. □

Remark 9.3.34. Let μ a strictly positive continuous locally bounded Borel density on M, then $\int_M f \cdot \mu > 0$ for each nonzero continuous function $f \geq 0$. In fact, the integral over any Borel subset of M is nonnegative. Moreover we can find a coordinate patch U_α on which f and the function representing μ are strictly positive, so that the integral $\int_{U_\alpha} f \cdot \mu$ is also strictly positive.

Remark 9.3.35. (i) Given a locally bounded Borel density μ, we can integrate functions $f \in C_c(M)$ with respect to $\widetilde{\mu}$ via

$$I_\mu(f) := \int_M f(x) \mathrm{d}\widetilde{\mu}(x) = \int_M f \cdot \mu.$$

Note that $I_\mu \colon C_c(M) \to \mathbb{R}$ defines a positive linear functional.

(ii) Once one has the measure $\widetilde{\mu}$ one can also integrate functions $f \colon M \to V$ with values in finite-dimensional vector spaces using either a basis for V and coordinates or linear functionals.

(iii) If $A \colon V \to W$ is a linear map, then A commutes with integration:

$$\int_M (A \circ f)\mu = A\Big(\int_M f\mu\Big) \quad \text{for} \quad f \in C_c(M, V).$$

Definition 9.3.36. Let $\varphi \colon M \to N$ be a smooth map between n-dimensional manifolds and μ an r-density on N. Then we define an r-density $f^*\mu$ on M by

$$(f^*\mu)(p)(v_1, \ldots, v_n) = \mu\big(T_p f(v_1), \ldots, T_p f(v_n)\big)$$

for $v_1, \ldots, v_n \in T_p(M)$. We call $f^*\mu$ the *pull-back* of μ by f.

Remark 9.3.37 (Global transformation formula for densities). The well-definedness of the integral associated with a locally bounded Borel density shows that, given such a density μ on N, for a diffeomorphism $\varphi \colon M \to N$, we have

$$\int_{\varphi^{-1}(U)} (f \circ \varphi) \cdot \varphi^*\mu = \int_U f \cdot \mu,$$

where $f \in C_c(N)$ and $U \subseteq N$ is a Borel set.

9.3.3 Some Technical Results on Integration

In this section we compile a few technical results on integration, which will be used in later chapters.

Lemma 9.3.38. *Let $I \colon C_c(M) \to \mathbb{C}$ be a positive linear functional, and let $K \subseteq M$ be compact. Then there exists a constant C_K such that*

$$|I(f)| \leq C_K \|f\|_\infty \quad \textit{for} \quad f \in C_c(M), \operatorname{supp} f \subseteq K.$$

Proof. First we assume that f is a real valued function. By Corollary 9.3.28, there is a function $\varphi \in C_c(M)$ with values in $[0, 1]$ which is identical to the constant map 1 on K. Now, if $f \in C_c(M)$ with $\operatorname{supp} f \subseteq K$ then $|f| \leq \|f\|_\infty \varphi$. This gives $\|f\|_\infty \varphi \pm f \geq 0$, hence, $\|f\|_\infty I(\varphi) \pm I(f) \geq 0$, by assumption, and therefore, $|I(f)| \leq I(\varphi)\|f\|_\infty$.

If $f = f_1 + if_2$ is complex-valued, we now obtain with $C := I(\varphi)$:

$$|I(f)| = |I(f_1) + iI(f_2)| \leq |I(f_1)| + |I(f_2)| \leq C(\|f_1\|_\infty + \|f_2\|_\infty) \leq 2C\|f\|_\infty.$$

□

Remark 9.3.39. If μ is a locally bounded Borel density on M, then by Proposition 9.3.33 it defines a Borel measure $\widetilde{\mu}$ on M and Lemma 9.3.38 yields the following estimate for $K \subseteq M$ compact and $f \in C_c(M)$:

$$\left|\int_M f\mu\right| \leq c \sup_{x\in K} |f(x)|.$$

Here the constant c depends only on K and μ.

Theorem 9.3.40. *Let M be a manifold and $h \in C_c(M \times M)$ with* $\mathrm{supp}(h) \subseteq V \times V$ *for a compact subset $V \subseteq M$. Then the following statements are true:*

(i) *The maps*

$$M \to C(V), \qquad g \mapsto \big(x \mapsto h(g,x)\big)$$

and

$$M \to C(V), \qquad g \mapsto \big(x \mapsto h(x,g)\big)$$

are continuous, where $C(V)$ is equipped with the supremum norm $\|\cdot\|_\infty$.

(ii) (Fubini) *Let μ and μ' be positive Borel measures on M. Then the functions $y \mapsto \int_M h(x,y)\ d\mu(x)$ and $y \mapsto \int_M h(y,x)\ d\mu'(x)$ are continuous with*

$$\int_M \int_M h(x,y)\ d\mu(x)d\mu'(y) = \int_M \int_M h(x,y)\ d\mu'(y)d\mu(x).$$

Proof. (i) It suffices to prove the first of the two assertions. Write $h_x(y) := h(x,y)$. Let $x_0 \in M$ and $\varepsilon > 0$. Then

$$U := \{(x,y) \in M \times V \colon |h(x,y) - h(x_0,y)| < \varepsilon\}$$

is an open subset of $M \times V$ containing the compact subset $\{x_0\} \times V$, hence also a set of the form $W \times V$, where W is a neighborhood of x_0 (Exercise 9.3.6), and this means that for $x \in W$, we $\|h_x - h_{x_0}\|_\infty \leq \varepsilon$.

(ii) By (i) and Lemma 9.3.38, we get the continuity of both functions. Therefore, since the supports of both functions lie in V, both double integrals are defined because the integrands are compactly supported continuous functions. By Lemma 9.3.38, the linear maps

$$\alpha\colon C(V \times V) \to \mathbb{R}, \qquad f \mapsto \int_M \int_M f(x,y)\ d\mu(x)d\mu'(y)$$

and

$$\beta\colon C(V \times V) \to \mathbb{R}, \qquad f \mapsto \int_M \int_M f(x,y)\ d\mu'(y)d\mu(x)$$

are both continuous with respect to $\|f\| := \max\{|f(x,y)|\colon x,y \in V\}$. As we immediately see, both linear maps coincide on functions of the form $f(x,y) = f_1(x)f_2(y)$. By the Stone–Weierstraß Theorem, the subspace which is spanned by these functions is dense in $C(V \times V)$. This implies $\alpha = \beta$. □

Lemma 9.3.41 (Integrals with Parameters). *Let $(M, \mathfrak{M}, \widetilde{\mu})$ be a measure space, I an interval of positive length, and let $f\colon M \times I \to \mathbb{C}$ be a function such that $f(\cdot, t)\colon M \to \mathbb{C}$ is integrable for every $t \in I$. We set $F(t) := \int_M f(x,t)\ \mathtt{d}\widetilde{\mu}(x)$.*

(i) *For each integrable, nonnegative function g on M such that $|f(x,t)| \leq g(x)$ for all $x \in M$ and $t \in I$, and $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for all $x \in M$, we have*

$$\lim_{t\to t_0} F(t) = F(t_0).$$

(ii) *Assume that the partial derivative $\frac{\partial f}{\partial t}$ exists for $t \in I^\circ$, and assume furthermore that there exists an integrable, nonnegative function g on M such that $|\frac{\partial f}{\partial t}(x,t)| \leq g(x)$ for all $x \in M$ and $t \in I^\circ$. Then F is differentiable in $]a,b[$, and*

$$F'(t) = \int_M \frac{\partial f}{\partial t}(x,t)\ \mathtt{d}\widetilde{\mu}(x).$$

Proof. (i) Set $f_n(x) = f(x,t_n)$ where $t_n \to t_0 \in I$ for $n \to \infty$. By Lebesgue's Dominated Convergence Theorem, we get

$$\begin{aligned} F(t_0) &= \int_M f(x,t_0)\ \mathtt{d}\widetilde{\mu}(x) = \int_M \lim_{n\to\infty} f_n(x)\ \mathtt{d}\widetilde{\mu}(x) \\ &= \lim_{n\to\infty} \int_M f_n(x)\ \mathtt{d}\widetilde{\mu}(x) = \lim_{n\to\infty} \int_M f(x,t_n)\ \mathtt{d}\widetilde{\mu}(x) = \lim_{n\to\infty} F(t_n). \end{aligned}$$

(ii) For $h_n(x) := \frac{f(x,t_n)-f(x,t_0)}{t_n-t_0}$ we obtain $\lim_{n\to\infty} h_n(x) = \frac{\partial f}{\partial t}(x,t_0)$, hence, $\frac{\partial f}{\partial t}(\cdot, t_0)$ is measurable. The Mean Value Theorem implies

$$|h_n(x)| \leq \sup_{t\in I^\circ} \left|\tfrac{\partial f}{\partial t}(x,t)\right| \leq g(x)$$

for every $x \in M$. Using the Dominated Convergence Theorem again, we get

$$\begin{aligned} &\int_M \frac{\partial f}{\partial t}(x,t_0)\ \mathtt{d}\widetilde{\mu}(x) = \lim_{n\to\infty} \int_M h_n(x)\, \mathtt{d}\widetilde{\mu}(x) \\ &= \lim_{n\to\infty} \frac{1}{t_n - t_0} \int_M (f(x,t_n) - f(x,t_0))\ \mathtt{d}\widetilde{\mu}(x) = \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0}, \end{aligned}$$

which implies the claim. □

Proposition 9.3.42. *Let M and N be smooth manifolds, μ a locally bounded Borel density on M, and $f\colon M \times N \to \mathbb{R}$ a continuous function which is smooth in the second argument. We make the following assumptions:*

(a) *Locally, all partial derivatives of f with respect to the N-variables are also continuous as functions on $M \times N$.*
(b) *There exists a subset $C \subseteq N$ such that $f|_{M\times C}$ is compactly supported.*

Then

$$y \mapsto \int_M f^y \mu, \qquad \textit{where} \qquad f^y(x) = f(x,y),$$

defines a smooth function on the interior C° of C such that integration over M and differentiation with respect to the N-variables commute.

Proof. From the hypotheses we see that there exists a compact subset $K \subseteq M$ such that $\operatorname{supp} f \subseteq K \times N$. Pick finitely many open coordinate neighborhoods $U_1, \ldots, U_k$ in M covering K and a partition of unity $(\rho_i)_{i=0,\ldots,k}$ subordinate to $(U_i)_{i=0,\ldots,k}$ with $U_0 := M \setminus K$. Then, for each $y \in C$, the function f^y is bounded and we can write

$$\int_M f^y \mu = \sum_{j=1}^k \int_{U_j} \rho_j f^y \mu.$$

To prove the claim we may now assume that $M \subseteq \mathbb{R}^m$ and $C^\circ \subseteq \mathbb{R}^n$ are open boxes. But then, by hypothesis, the functions $\frac{\partial f}{\partial y_j} : M \times C \to \mathbb{R}$ are compactly supported and smooth. Therefore one can find a dominating function g as required in Lemma 9.3.41(ii), whence

$$\frac{\partial}{\partial y_j} \int_M f(x,y) \mathrm{d}\widetilde{\mu}(x) = \int_M \tfrac{\partial}{\partial y_j} f(x,y) \mathrm{d}\widetilde{\mu}(x)$$

exists and is continuous for $j = 1, \ldots n$. In view of (a) Lemma 9.3.41 shows that $y \mapsto \int_M f^y \mu$ has continuous partial derivatives, hence is C^1. But (b) implies that we can apply the same argument to the functions $(x,y) \mapsto \frac{\partial}{\partial y_j} f(x,y)$, so we can use induction to complete the proof. □

9.3.4 Orientations

We have seen in Subsection 9.3.2 that, under suitable conditions like positivity of compact support, one can integrate densities. In this subsection we show how to make the connection to the integration of differential forms which is used for instance in Stokes' Theorem. The objects that make such a connection possible are the orientations.

Definition 9.3.43. Let V be an n-dimensional $\mathbb{R}$-vector space and $e_1, \ldots, e_n$, $\widetilde{e}_1, \ldots, \widetilde{e}_n$ be two bases for V. Define the base change matrix $A := (a_{ij})_{1 \le i,j \le n}$ in $\mathrm{GL}_n(\mathbb{R})$ via

$$e_i = \sum_{j=1}^n a_{ji} \widetilde{e}_j.$$

If $\det(A) > 0$, we say that the two bases have the same *orientation*, and if $\det(A) < 0$, we say that the two bases have opposite orientation.

Definition 9.3.44. Let M be an n-dimensional smooth manifold. Two charts (φ, U) and (ψ, V) of M are said to be *equally oriented* if

$$\det\big(\mathtt{d}(\psi \circ \varphi^{-1})(x)\big) > 0 \quad \text{for} \quad x \in \varphi(U \cap V).$$

An *orientation* of M is an atlas $(\varphi_\alpha, U_\alpha)_{\alpha \in J}$ which is maximal with respect to the property that any two charts in this atlas are equally oriented. The manifold M is called *orientable*, if it has an orientation (for connected M in that case there are precisely two orientations).

Remark 9.3.45. Suppose that x^α and x^β are the coordinate functions on U_α and U_β for two equally oriented charts of M. If $x^\alpha = f_{\alpha\beta}(x^\beta)$ are corresponding coordinate changes, then

$$\det\Big(\frac{\partial x_i^\beta}{\partial x_j^\alpha}\Big) > 0.$$

This means that for $p \in U_\alpha \cap U_\beta$ the bases

$$\frac{\partial}{\partial x_1^\alpha}\Big|_p, \ldots, \frac{\partial}{\partial x_n^\alpha}\Big|_p \quad \text{and} \quad \frac{\partial}{\partial x_1^\beta}\Big|_p, \ldots, \frac{\partial}{\partial x_n^\beta}\Big|_p$$

of $T_p(M)$ have the same orientation.

Definition 9.3.46 (Orientation bundle). Let M be a smooth manifold of dimension n and $\mathrm{GL}(M) \to M$ be the frame bundle of M. Consider the one-dimensional representation $g \mapsto \mathrm{sign}(\det g)$ of $G = \mathrm{GL}_n(\mathbb{R})$. Then the line bundle $\mathrm{OR}(M) := \mathrm{GL}(M) \times_G \mathbb{R}$ associated with this representation via Definition 9.2.10 is called the *orientation bundle* of M. The transition functions of $\mathrm{OR}(M)$ are the signs of the Jacobi determinants of the coordinate changes.

The orientation bundle $\mathrm{OR}(M)$ can be used to give an elegant characterization of the orientability of M.

Proposition 9.3.47. *A smooth manifold M is orientable if and only if $\mathrm{OR}(M)$ admits a nowhere vanishing smooth section.*

Proof. Given an orientation, we can cover M by charts such that the Jacobi determinants of all coordinate changes are positive. Thus taking the constant function 1 on all trivializations defines a smooth nowhere vanishing section of the orientation bundle (Proposition 9.2.15).

Conversely, suppose we have a nowhere vanishing smooth section σ of $\mathrm{OR}(M)$. Let $(U_\alpha)_{\alpha \in J}$ be a cover of M by connected coordinate neighborhoods with coordinate functions x_i^α on U_α such that $\mathrm{OR}(M)$ is trivial over each U_α. If $s_\alpha \colon \varphi_\alpha(U) \to \mathbb{R}$ is the function representing σ on U_α, then changing the first coordinate x_1^α to $-x_1^\alpha$ if necessary, we may assume that all s_α are strictly positive. But then

$$s_\alpha(x^\alpha) = s_\beta(x^\beta)\,\mathrm{sign}\Big(\det\Big(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\Big)\Big)$$

now implies that $\det\Big(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\Big) > 0$, i.e., the charts $(\varphi_\alpha, U_\alpha)$ and (φ_β, U_β) are equally oriented for all α, β. Hence the cover defines an orientation on M. $\square$

Remark 9.3.48. Let M be a smooth manifold of dimension n. Then the tensor product of the line bundles $\mathrm{OR}(M)$ and $|\Lambda^n|(M)$ is again a line bundle and it corresponds to the representation $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$. But then the transition functions are the Jacobi determinants, i.e., this is the bundle of n-forms. If M is orientable and σ is a nowhere vanishing smooth section of $\mathrm{OR}(M)$, then multiplication by σ defines isomorphisms between the spaces of densities (smooth, continuous, compactly supported etc.) and n-forms (with the corresponding properties). Thus, given σ we can integrate n-forms.

Definition 9.3.49. Let M be a smooth manifold of dimension n. An element $\mu \in \Omega^n(M)$ is called a *volume form* if it does not vanish anywhere.

Corollary 9.3.50. *A second countable manifold M of dimension n is orientable if and only if there exists a volume form $\mu \in \Omega^n(M)$.*

Example 9.3.51. (i) Consider the 2-torus $T = \mathbb{R}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2)$ with the 1-forms $\mathtt{d}x_1, \mathtt{d}x_2$, where x_1, x_2 are the coordinates on $\mathbb{R}^2$. Then $\mathtt{d}x_1 \wedge \mathtt{d}x_2 \in \Omega^2(T)$ is never zero.

(ii) Consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and define $\mu \in \Omega^n(\mathbb{S}^n)$ via $\mu(x) := i_x(\mathtt{d}x_1 \wedge \ldots \wedge \mathtt{d}x_{n+1})$, where we identify the tangent spaces $T_x(\mathbb{S}^n)$ with the corresponding subspaces $x^\perp \subseteq \mathbb{R}^{n+1}$. Then we find

$$\begin{aligned}\mu(x) &= i_{\sum x_i \frac{\partial}{\partial x_j}}(\mathtt{d}x_1 \wedge \ldots \wedge \mathtt{d}x_{n+1}) \\ &= x_1\mathtt{d}x_2 \wedge \ldots \wedge \mathtt{d}x_{n+1} + \ldots + (-1)^{n-1}x_n\mathtt{d}x_1 \wedge \ldots \wedge \mathtt{d}x_n \\ &= \sum_{j=1}^{n+1}(-1)^{j-1}x_j\,\mathtt{d}x_1 \wedge \ldots \wedge \widehat{\mathtt{d}x_j} \wedge \ldots \wedge \mathtt{d}x_{n+1}\end{aligned}$$

(iii) The real projective spaces $\mathbb{P}_n(\mathbb{R})$ defined in Example 7.2.15 are not orientable for n even. To prove this, assume to the contrary that n is even and that $\eta \in \Omega^n\big(\mathbb{P}_n(\mathbb{R})\big)$ is nowhere zero. Let $\pi\colon \mathbb{S}^n \to \mathbb{P}_n(\mathbb{R})$ denote the quotient map. Then also $\pi^*\eta$ is nowhere zero. Recall that $\mu(x) := i_x\mathtt{d}x_1 \wedge \ldots \wedge \mathtt{d}x_{n+1}$ defines an element $\mu \in \Omega^n(\mathbb{S}^n)$, which is nowhere zero. Therefore $\pi^*\eta = f\mu$ for some $f \in C^\infty(\mathbb{S}^n)$ satisfying $f(p) \neq 0$ for all $p \in \mathbb{S}^n$. Since $\mathbb{S}^n$ is connected, we may assume that $f(p) > 0$ for all p. Consider the map $\sigma : \mathbb{S}^n \to \mathbb{S}^n$ defined by $\sigma(x) := -x$. Then we have $\sigma^*\mu = (-1)^{n+1}\mu = -\mu$ (because n is even) and since

$$-\sigma^*f \cdot \mu = \sigma^*f \cdot \sigma^*\mu = \sigma^*(f\mu) = \sigma^*\pi^*\eta = (\pi \circ \sigma)^*\eta = \pi^*\eta = f \cdot \mu$$

implies $-f \circ \sigma = f$, we arrive at a contradiction to the positivity of f.

Remark 9.3.52. If $\mu \in \Omega^n(M)$ is a volume form and $\varphi \in \mathrm{Diff}(M)$ is a diffeomorphism of M, then φ is called *orientation preserving* if $\varphi^*\mu = f\mu$ for a strictly positive function $f \in C^\infty(M)$. If M is connected, then this definition does not depend on the choice of the volume form. In fact, if $\widetilde{\mu}$ is another volume form, then $\widetilde{\mu} = h\mu$ for a smooth function $h \colon M \to \mathbb{R}^\times$. Since M is assumed to be connected, we either have $h > 0$ or $h < 0$ which implies $\frac{\varphi^* h}{h} > 0$ and further

$$\varphi^*\widetilde{\mu} = (\varphi^* h)(\varphi^*\mu) = \frac{\varphi^* h}{h} f h\mu = \frac{\varphi^* h}{h} f\widetilde{\mu}$$

with $\frac{\varphi^* h}{h} f > 0$.

Proposition 9.3.53 (Orientation of homogeneous spaces). *Let G be a Lie group and H a closed subgroup of G. If there exists an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} H$-invariant volume form on $\mathfrak{g}/\mathfrak{h}$, then there exists a G-invariant volume form on G/H. In particular, G/H is orientable.*

Proof. Let $\omega_0 \in \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h}, \mathbb{R})$ be an $\mathrm{Ad}\, H$-invariant volume form. The description of $\mathrm{Alt}^k T(G/H)$ as the associated bundle

$$G \times_H \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h}, \mathbb{R})$$

(see Definition 9.3.1 and Corollary 9.2.13) immediately shows that

$$\omega(gH) = [g, \omega_0] \in G \times_H \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h}, \mathbb{R})$$

for $gH \in G/H$ defines a G-invariant volume form on G/H. □

Remark 9.3.54. Proposition 9.3.53 in particular shows that each Lie group is orientable and that for each non-zero $\omega_0 \in \mathrm{Alt}^k(\mathfrak{g}, \mathbb{R})$ there is a unique choice of orientation in such a way that the left-invariant volume form ω associated with ω_0 is positive.

Exercises for Section 9.3

Exercise 9.3.1. Let M be a manifold and $(\mathbb{E}, M, E, \pi)$ a vector bundle. A subset $\mathbb{F} \subseteq \mathbb{E}$ is called a *vector subbundle* of $\mathbb{E}$ if the sets $\mathbb{F}_p := \mathbb{F} \cap \mathbb{E}_p$ are vector subspaces of the fibers $\mathbb{E}_p$ such that the restriction $\pi|_{\mathbb{F}} \colon \mathbb{F} \to M$ defines a vector bundle in its own right.

Exercise 9.3.2. Let X be a topological space and $(K_n)_{n\in\mathbb{N}}$ a sequence of compact subsets of X with $\bigcup_{n\in\mathbb{N}} K_n = X$ and $K_n \subseteq K_{n+1}^0$ for each $n \in \mathbb{N}$. Show that for each compact subset $C \subseteq X$ there exists an $n \in \mathbb{N}$ with $C \subseteq K_n$.

Exercise 9.3.3. A family $(S_i)_{i\in I}$ of subsets of a topological space X is said to be *locally finite* if each point $p \in X$ has a neighborhood intersecting only finitely many S_i. Show that if $(S_i)_{i\in I}$ is a locally finite family of closed subsets of X, then $\bigcup_{i\in I} S_i$ is closed.

Exercise 9.3.4. Let $(S_i)_{i\in I}$ be a locally finite family of subsets of the topological space X. Show that each compact subset $K \subseteq X$ intersects only finitely many of the sets S_i.

Exercise 9.3.5. Let E be a finite-dimensional vector space. A family $(f_j)_{j\in J}$ of smooth E-valued functions on M is called *locally finite* if each point $p \in M$ has a neighborhood U for which the set $\{j \in J : f_j|_U \neq 0\}$ is finite. Show that this implies that $f := \sum_{j\in J} f_j$ defines a smooth E-valued function on M.

Exercise 9.3.6. Let X and Y be Hausdorff spaces and $K \subseteq X$, resp., $Q \subseteq Y$ be a compact subset. Then for each open subset $U \subseteq X \times Y$ containing $K \times Q$, there exist open subsets $U_K \subseteq X$ containing K and $U_Q \subseteq Y$ containing Y with

$$K \times Q \subseteq U_K \times U_Q \subseteq U.$$

9.4 Invariant Integration

In this section we show how to construct measures on Lie groups and certain of their homogeneous spaces which are invariant under the group action. Following the approach via densities laid out in Section 9.3, we start with a description of invariant densities which will then give rise to the desired measures.

9.4.1 Invariant Densities

Definition 9.4.1. Let M be a smooth manifold, G a Lie group, and $\sigma\colon G \times M \to M$ a smooth action. An r-density μ on M is called *G-invariant*, if $\sigma_g^*\mu = \mu$, where $\sigma_g^*\mu$ is the pull-back of μ by σ_g (see Definition 9.3.36).

If $M = G$, then using left- and right translations as actions we obtain the notions of *left-* and *right invariant densities* on G.

Remark 9.4.2. (i) If a group G acts linearly on a finite-dimensional vector space V, then this action induces a G-action on the space of r-densities on V via

$$(g \cdot \mu)(v_1, \ldots, v_n) := \mu(g^{-1}v_1, \ldots, g^{-1}v_n) = |\det g|^{-r}(\mu)(v_1, \ldots, v_n)$$

for $v_1, \ldots, v_n \in V$. Thus it makes sense to talk about G-invariant densities on V.

(ii) Let M be a smooth manifold, G a Lie group, and $\sigma\colon G \times M \to M$ a smooth action. Recall the two realizations $|\Lambda|^r(M)$ and $L^{(r)}(M)$ of the r-density bundle from Definition 9.2.16. The action of G lifts to actions on $|\Lambda|^r(M)$ and $L^{(r)}(M)$ in such a way that the isomorphism $[\varphi, t] \mapsto \mu_{\varphi,t}$ between the two spaces described in (9.3) is G-equivariant: The lift of σ to

$\mathrm{GL}(M)\times_{\Delta_r}\mathbb{R}$ is obtained by the lift of σ to $\mathrm{GL}(M)$ which in turn is provided by the tangent maps of the σ_g. On $L^{(r)}(M)$, we set

$$(g\cdot\mu)(v_1,\ldots,v_n)=\mu\big(T_p(\sigma_{g^{-1}})v_1,\ldots,T_p(\sigma_{g^{-1}})v_n\big)$$

for $v_1,\ldots,v_n\in T_{\sigma_g(p)}(M)$ and $\mu\in L_p^{(r)}(M)$. To see the G-equivariance, we have to show that $\mu_{\sigma_g(\varphi),t}=g\cdot\mu_{\sigma_g(\varphi),t}$ for $g\in G$. Denote the canonical projection $\mathrm{GL}(M)\to M$ by $\widetilde{\pi}$. For $\widetilde{\pi}(\varphi)=p$ and $v_j=T_p(\sigma_g)\varphi(e_j)=\sigma_g(\varphi)(e_j)$ we have $\mu_{\sigma_g(\varphi),t}(v_1,\ldots,v_n)=t$. On the other hand

$$(g\cdot\mu_{\varphi,t})(v_1,\ldots,v_n)=\mu_{\varphi,t}\big(\varphi(e_1),\ldots,\varphi(e_n)\big)=t,$$

so that the claim is proved. Note that the corresponding G-action on the sections of $L^{(r)}(M)$ is the pull-back of densities.

Lemma 9.4.3. *Let G be a Lie group and H a closed subgroup. Then the following conditions are equivalent:*

(1) *There exists a G-invariant smooth density μ on G/H.*
(2) *There exists an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H)$-invariant density μ_0 on $\mathfrak{g}/\mathfrak{h}$.*
(3) $\big|\det\circ\mathrm{Ad}_G|_H\big|=|\det\circ\mathrm{Ad}_H|$.

Proof. Recall from Corollary 9.2.13 that we have a G-equivariant bundle isomorphism $G\times_H\mathfrak{g}/\mathfrak{h}\to T(G/H)$. Therefore the frame bundle $\mathrm{GL}(G/H)$ can be obtained as $G\times_H\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$, where H acts on $\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ by $(h,A)\mapsto\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)A$. But then the density bundle is

$$(G\times_H\mathrm{GL}(\mathfrak{g}/\mathfrak{h}))\times_{\mathrm{GL}(\mathfrak{g}/\mathfrak{h})}\mathbb{R}=G\times_H\mathbb{R},$$

where the action of H on $\mathbb{R}$ is given by

$$(h,t)\mapsto\big|\det\big(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\big)\big|^{-1}t. \tag{9.9}$$

According to Proposition 9.2.17, the smooth sections of $G\times_H\mathbb{R}$ are of the form $gH\mapsto[g,F(g)]$, where $F\colon G\to\mathbb{R}$ is a smooth function such that $F(gh)=h^{-1}\cdot F(g)$ for all $g\in G$ and $h\in H$. Such a section is G-invariant if and only if F is constant since G-invariance implies

$$[gx,F(x)]=g\cdot[x,F(x)]=[gx,F(gx)].$$

Thus invariant densities on G/H are in one-to-one correspondence with H-fixed points in the fiber over the base point $\mathbf{1}H$, i.e. $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H)$-invariant densities on $\mathfrak{g}/\mathfrak{h}$. But then (9.9) implies the equivalence of (1), (2), and (3), since

$$\det\big(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)\big)\,\det\big(\mathrm{Ad}_H(h)\big)\;=\;\det\big(\mathrm{Ad}_G(h)\big). \qquad\square$$

In view of Proposition 9.3.33 and Remark 9.3.39, we see that any $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\,H$-invariant density μ defines a positive Borel measure $\widetilde{\mu}$ on G/H.

Proposition 9.4.4. *Let G be a Lie group, H a closed subgroup, and μ_0 an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H)$-invariant density on $\mathfrak{g}/\mathfrak{h}$ with positive values. Then the associated smooth G-invariant density μ is positive. The associated measure $\widetilde{\mu}$ (see Remark 9.3.32) is left invariant and satisfies*

$$\int_{G/H} \lambda_g^* f \mathtt{d}\widetilde{\mu} = \int_{G/H} f \mathtt{d}\widetilde{\mu} \quad \text{for} \quad f \in C_c(G/H).$$

Proof. The first claim follows immediately from the proof of Lemma 9.4.3. Thus it suffices to show that $\widetilde{\mu}$ is G-invariant if and only if μ is G-invariant. But this follows from the global transformation formula given in Remark 9.3.37, which allows us to write

$$\int_M \lambda_g^* f \mathtt{d}(\lambda_g^* \mu)^\sim = \int_M f\, \mathtt{d}\widetilde{\mu}.$$ □

Definition 9.4.5 (Haar measure). Let G be a Lie group. A positive Borel measure μ on G is called a *left Haar measure*, if it is invariant under left translations. Similarly, it is called a *right Haar measure*, if it is invariant under right translations. We will denote Haar measures on G by μ_G and write the corresponding integrals as

$$\int_G f \mathtt{d}\mu_G = \int_G f(g) \mathtt{d}\mu_G(g).$$

Remark 9.4.6. (i) Note that Proposition 9.4.4 guarantees that left Haar measures exist. The existence of right Haar measures is shown analogously using right invariant densities.

(ii) The question of uniqueness is more subtle. It can be shown that left/right Haar integrals $C_c(G) \to \mathbb{R}$ are unique up to positive scalars. On the level of measures one first has to observe that on Lie groups with at most countably many connected components Borel measures automatically satisfy additional regularity conditions ([Ru86, Thm. 2.17]) ensuring that they are uniquely determined by the corresponding integrals of compactly supported continuous functions (which are in particular supported by finitely many connected components). The uniqueness of Haar measure up to positive scalar multiples then follows from a general theorem on locally compact groups. Together with Proposition 9.4.4 this shows that the left/right Haar integrals on a Lie group are in one-to-one correspondence with the left/right invariant densities. In this book we shall avoid arguments using the uniqueness of Haar measures and replace them by arguments using concrete left/right invariant densities.

(iii) In view of Proposition 9.3.53 (see also Corollary 9.3.50), Remark 9.3.39 also shows that any $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} H$-invariant volume form ω_0 on $\mathfrak{g}/\mathfrak{h}$ defines a left invariant nonvanishing differential form of degree $\dim G/H$ and then a positive Borel measure on G/H which we denote by ω. As in Proposition 9.4.4 one shows that ω is a left Haar measure. In many texts on Lie groups this is the path taken for the construction of Haar measures.

9.4.2 Integration Formulas

Proposition 9.4.7. *If μ_l is a left invariant density on a Lie group G, then*

$$\mu_r(x) := \big|\det \mathrm{Ad}(x)\big|\, \mu_l(x)$$

is a right invariant density on G and

$$\rho_g^* \mu_l = \big|\det \mathrm{Ad}(g)\big|^{-1} \mu_l. \tag{9.10}$$

Proof. In view of $(\rho_g^* \mu_r)(x) = \big|\det \mathrm{Ad}(xg)\big|(\rho_g^* \mu_l)(x)$, it remains to verify (9.10). As both sides of this equation are left invariant densities, it suffices to show that their values in $\mathbf{1}$ coincide. With $T_{\mathbf{1}}(c_g) = \mathrm{Ad}(g)$, this follows from

$$(\rho_g^* \mu_l)(\mathbf{1}) = (c_{g^{-1}}^* \mu_l)(\mathbf{1}) = \big|\det \mathrm{Ad}(g^{-1})\big|\mu_l(\mathbf{1}). \qquad \square$$

Corollary 9.4.8. *If μ is a left invariant density on G, then*

$$\int_G f(g^{-1}) \mathtt{d}\widetilde{\mu}(g) = \int_G f(g)\, |\det \mathrm{Ad}(g)| \mathtt{d}\widetilde{\mu}(g).$$

Proof. Let $\eta_G(g) = g^{-1}$ be the inversion on G and note that $\eta_G^* \mu$ is a right invariant density with the same value as μ in $\mathbf{1}$. Now Proposition 9.4.7 implies that

$$\eta_G^* \mu = |\det \circ \mathrm{Ad}\,| \cdot \mu,$$

so that the assertion follows from the transformation formula for integrals with respect to densities:

$$\int_G f(g^{-1}) \mathtt{d}\widetilde{\mu}(g) = \int_G f(g) \mathtt{d}(\eta_G^* \mu)\widetilde{\ }(g) = \int_G f(g)\, |\det \mathrm{Ad}(g)|\, \mathtt{d}\widetilde{\mu}(g). \qquad \square$$

Definition 9.4.9. A Lie group G is called *unimodular* if $|\det \mathrm{Ad}(x)| = 1$ for all $x \in G$. The function $\Delta_G(g) := |\det \circ \mathrm{Ad}(g)|^{-1}$ on G is called the *modular function*. It is a continuous group homomorphism $\Delta_G \colon G \to (\mathbb{R}_+^\times, \cdot)$.

If G is unimodular, any left invariant density is also a right invariant density measure. Thus the corresponding left Haar measure is also a right Haar measure.

Remark 9.4.10. For a connected Lie group G, the following properties are equivalent.

(1) G is unimodular.
(2) $\det \circ \mathrm{Ad} \equiv 1$.
(3) $\mathrm{tr} \circ \mathrm{ad} \equiv 0$.

Proposition 9.4.11. *The following properties of a Lie group G imply that it is unimodular:*

(a) *G is compact.*
(b) *G is abelian.*
(c) *The commutator group G' of G is dense in G.*
(d) *G is connected with nilpotent Lie algebra.*

Proof. (a) $\Delta_G(G)$ is a compact subgroup of $(\mathbb{R}_+^\times, \cdot)$, hence trivial.

(b) If G is abelian, then $\mathrm{Ad}(g) = \mathbf{L}(c_g) = \mathrm{id}_{\mathfrak{g}}$ for each $g \in G$.

(c) Since $\mathbb{R}^\times$ is abelian, $G' \subseteq \ker \Delta_G$, so, by continuity, the hypothesis implies that $\Delta_G = 1$.

(d) Since $\mathfrak{g}$ is nilpotent, the operators $\mathrm{ad}(x)$ for $x \in \mathfrak{g}$ are nilpotent, so they have zero trace. Hence $\mathrm{Ad}(\exp x) = e^{\mathrm{ad}(x)}$ leads to $\det \mathrm{Ad}(\exp x) = e^{\mathrm{tr}\,\mathrm{ad}(x)} = 1$, which implies (d) because $\exp \mathfrak{g}$ generates G. □

Proposition 9.4.12. *Let G be a Lie group and H a closed subgroup. Suppose that μ is the G-invariant density on G/H associated with an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H)$-invariant density μ_0 on $\mathfrak{g}/\mathfrak{h}$. Then we can find Haar measures μ_G and μ_H such that*

$$\int_G f(g)\,\mathrm{d}\mu_G(g) = \int_{G/H} \Big(\int_H f(gh)\,\mathrm{d}\mu_H(h) \Big)\,\mathrm{d}\widetilde{\mu}(gH) \tag{9.11}$$

for all $f \in C_c(G)$.

Proof. Using partitions of unity, it suffices to show formula (9.11) for G replaced by open subsets of the form $q^{-1}(U_\alpha)$, where $(U_\alpha)_{\alpha \in A}$ is a locally finite open covering of G/H. Using Corollary 9.1.11, we can cover G/H by open sets U for which we have a smooth section $\sigma \colon U \to G$ for the quotient map $q \colon G \to G/H$ such that $m_\sigma \colon U \times H \to \sigma(U)H, (u,h) \mapsto \sigma(U)H$ is a diffeomorphism onto an open subset of G. Note that $\sigma(U)H = q^{-1}(U)$.

Pick a basis $v_{k+1}, \ldots, v_n$ for $\mathfrak{h}$ and complete it by vectors $v_1, \ldots, v_k$ to a basis for $\mathfrak{g}$. Then the $\overline{v}_j := v_j + \mathfrak{h}$, $j = 1, \ldots, k$, form a basis for $\mathfrak{g}/\mathfrak{h}$ which we identify with the tangent space $T_{p_o}(G/H)$ of G/H at the base point $p_o = q(\mathbf{1})$. We may assume that $\mu(p_o)(\overline{v}_1, \ldots, \overline{v}_k) = 1$. Let ν_H, resp., ν_G be a left invariant density on H, resp., G with $\mu_G = \widetilde{\nu}_G$, resp., $\mu_H = \widetilde{\nu}_H$. We normalize them in such a way that

$$\nu_H(\mathbf{1})(v_{k+1}, \ldots, v_n) = \nu_G(\mathbf{1})(v_1, \ldots, v_n) = 1.$$

Consider the product density $\mu \otimes \nu_H$ on $U \times H$.

Claim: $m_\sigma^* \nu_G = \mu \otimes \nu_H$.

Before we prove the claim, we show how it implies the proposition: Since $m_\sigma^{-1}(UH) = U \times H$ and $UH = q^{-1}(U)$, the transformation formula for densities shows that

$$\int_{U \times H} f(\sigma(u)h)\,\mathrm{d}\widetilde{\mu}(u)\mathrm{d}\mu_H(h) = \int_{q^{-1}(U)} f(g)\,\mathrm{d}\mu_G(g),$$

where μ_H and μ_G are the Haar measures corresponding to the densities ν_H and ν_G. But this is precisely what we had to show since $\int_H f(\sigma(u)h)\,\mathtt{d}\mu_H(h)$ is independent of the choice of $\sigma(u)$.

To prove the claim, pick $u \in U$ and write $g = \sigma(u)$. Then $m_\sigma(u,h) = gh$ and we may choose as a basis for $T_{(u,h)}(U \times H) = T_{p_o}(\overline{\lambda}_g)(\mathfrak{g}/\mathfrak{h}) \times T_{\mathbf{1}}(\lambda_h)(\mathfrak{h})$

$$T_{p_o}(\overline{\lambda}_{gh})\overline{v}_1, \ldots, T_{p_o}(\overline{\lambda}_{gh})\overline{v}_k, T_{\mathbf{1}}(\lambda_h)v_{k+1}, \ldots, T_{\mathbf{1}}(\lambda_h)v_n,$$

where $\overline{\lambda}_g$ is the (left) translation by g on G/H and $\overline{v}_j = v_j + \mathfrak{h}$. To calculate $m_\sigma^*\nu_G(u,h)$, we note first that

$$T_{gh}(\lambda_{(gh)^{-1}})T_{(u,h)}(m_\sigma)T_{p_o,\mathbf{1}}(\overline{\lambda}_{gh} \times \rho_h) = T_{p_o,\mathbf{1}}(\lambda_{(gh)^{-1}} \circ m_\sigma \circ (\overline{\lambda}_{gh} \times \lambda_h))$$

and

$$\lambda_{(gh)^{-1}} \circ m_\sigma \circ (\overline{\lambda}_{gh} \times \lambda_h)(u', h') = (gh)^{-1}\sigma(gh.u')hh' \in G.$$

Since $gh.p_o = g.p_o = u$, this implies

$$(p_o, \mathbf{1}) \mapsto c_{h^{-1}}(g^{-1}\sigma(gh.p_o)) = c_{h^{-1}}(g^{-1}\sigma(u)) = c_{h^{-1}}(\mathbf{1}) = \mathbf{1}.$$

We set

$$w_j := T_{gh}(\lambda_{(gh)^{-1}})T_{(u,h)}(m_\sigma)T_{p_o,\mathbf{1}}(\overline{\lambda}_{gh} \times \lambda_h)(\overline{v}_j, 0) \in \mathfrak{g}, \quad j = 1, \ldots, k$$
$$w_j := T_{gh}(\lambda_{(gh)^{-1}})T_{(u,h)}(m_\sigma)T_{p_o,\mathbf{1}}(\overline{\lambda}_{gh} \times \lambda_h)(0, v_j) \in \mathfrak{g}, \quad j = k+1, \ldots, n$$

and use

$$q \circ \lambda_{(gh)^{-1}} \circ m_\sigma \circ (\overline{\lambda}_{gh} \times \lambda_h)(u', h') = q\big((gh)^{-1}\sigma(gh.u')hh'\big) = u' \in G/H$$

to see that

$$T_{\mathbf{1}}(q)(w_j) = v_j + \mathfrak{h}, \quad j = 1, \ldots, k,$$

which implies the existence of $y_j \in \mathfrak{h}, j = 1, \ldots, k$ such that

$$w_j = v_j + y_j, \quad j = 1, \ldots, k.$$

Moreover,

$$w_j = \frac{d}{dt}\bigg|_{t=0} (gh)^{-1}\sigma(g.p_o)h \exp tv_j = \frac{d}{dt}\bigg|_{t=0} \exp tv_j = v_j, \quad j = k+1, \ldots, n.$$

Thus the invariance of ν_G implies

$$\begin{aligned} &m_\sigma^*\nu_G(u,h)(T_{p_o}(\overline{\lambda}_{gh})\overline{v}_1, \ldots, T_{p_o}(\overline{\lambda}_{gh})\overline{v}_k, T_{\mathbf{1}}(\lambda_h)v_{k+1}, \ldots, T_{\mathbf{1}}(\lambda_h)v_n) \\ &= \nu_G(\mathbf{1})(w_1, \ldots, w_n) = \nu_G(\mathbf{1})(v_1 + y_1, \ldots, v_k + y_k, v_{k+1}, \ldots, v_n) \\ &= \nu_G(\mathbf{1})(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n) = 1 \end{aligned}$$

On the other hand, the invariance properties of $\mu \otimes \nu_H$ imply

$$
\begin{aligned}
&(\mu \otimes \nu_H)(u,h)(T_{p_o}(\overline{\lambda}_{gh})\overline{v}_1, \ldots, T_{p_o}(\overline{\lambda}_{gh})\overline{v}_k, T_{\mathbf{1}}(\lambda_h)v_{k+1}, \ldots, T_{\mathbf{1}}(\lambda_h)v_n)\\
&= \mu(p_o)(T_u(\overline{\lambda}_{g^{-1}})T_{p_o}(\overline{\lambda}_{gh})\overline{v}_1, \ldots, T_u(\overline{\lambda}_{g^{-1}})T_{p_o}(\overline{\lambda}_{gh})\overline{v}_k)\\
&\quad \cdot \nu_H(\mathbf{1})(T_h(\lambda_{h^{-1}})T_{\mathbf{1}}(\lambda_h)v_{k+1}, \ldots, T_h(\lambda_{h^{-1}})T_{\mathbf{1}}(\lambda_h)v_n)\\
&= (\mu \otimes \nu_H)(p_o, \mathbf{1})(T_{p_o}(\overline{\lambda}_h)\overline{v}_1, \ldots, T_{p_o}(\overline{\lambda}_h)\overline{v}_k, v_{k+1}, \ldots, v_n)\\
&= \mu(p_o)(\overline{\mathrm{Ad}(h)v_1}, \ldots, \overline{\mathrm{Ad}(h)v_k}) \cdot \nu_H(\mathbf{1})(v_{k+1}, \ldots, v_n)\\
&= \mu(p_o)(\overline{v}_1, \ldots, \overline{v}_k) \cdot \nu_H(\mathbf{1})(v_{k+1}, \ldots, v_n) = 1
\end{aligned}
$$

and this proves the claim. □

Proposition 9.4.13. *Let G be a Lie group and A, B two integral subgroups with compact intersection such that $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ and the multiplication induces an open map $\mu\colon A \times B \to G$, $(a,b) \mapsto ab$. If $G \setminus AB$ has Haar measure zero, then, for any two left Haar measures μ_A and μ_B on A and B we obtain a left Haar measure on G by*

$$\int_G f(g)\, \mathrm{d}\mu_G(g) = \int_B \Big|\frac{\det \mathrm{Ad}_G(b)}{\det \mathrm{Ad}_B(b)}\Big| \int_A f(ab)\, \mathrm{d}\mu_A(a)\, \mathrm{d}\mu_B(b)$$

for $f \in C_c(G)$.

Proof. The group $A \times B$ acts transitively on AB via $(a,b)\cdot x = axb^{-1}$ and the stabilizer of $\mathbf{1}$ is the compact subgroup $K := \{(a,a) \in A \times B : a \in A \cap B\}$. Therefore the map

$$\varphi : (A \times B)/K \to AB, \quad (a,b)K \mapsto ab^{-1}$$

is a diffeomorphism and for any left invariant density μ on G, we obtain the formula

$$\int_G f\, \mathrm{d}\widetilde{\mu} = \int_{(A\times B)/K} (f \circ \varphi)\, \mathrm{d}(\varphi^*\mu)^{\sim}. \tag{9.12}$$

Since φ is a diffeomorphism onto an open subset of G satisfying $\varphi \circ \lambda_a = \lambda_a \circ \varphi$ and $\varphi \circ \lambda_b = \rho_b^{-1} \circ \varphi$ for $a \in A$ and $b \in B$, the density $\varphi^*\mu$ on $(A \times B)/K$ is left A-invariant and satisfies

$$\lambda_b^*\varphi^*\mu = (\varphi \circ \lambda_b)^*\mu = (\rho_b^{-1} \circ \varphi)^*\mu = \varphi^*(\rho_b^{-1})^*\mu = |\det(\mathrm{Ad}_G(b))| \cdot \varphi^*\mu$$

(Proposition 9.4.7). Since the subgroup $A \cap B$ of G is compact, $\Delta_G(A \cap B)$ is a compact subgroup of $\mathbb{R}_+^\times$, hence trivial, and therefore the function

$$s\colon (A \times B)/K \to \mathbb{R}_+^\times, \quad s((a,b)K) := |\det(\mathrm{Ad}_G(b))|$$

is well defined and the preceding calculations imply that

$$\omega := s^{-1} \cdot \varphi^*\mu$$

is an $(A \times B)$-invariant density on $(A \times B)/K$. Since $K \cong A \cap B$ is compact, there exists a normalized Haar measure $\mu_{A\cap B}$, and, for a suitable normalization of Haar measure on $A \times B$, we then have (cf. Corollary 9.4.8 and Prop. 9.4.12)

$$\begin{aligned}
&\int_{A\times B} f(ab)\, \mathtt{d}\mu_A(a)\, \mathtt{d}\mu_B(b) \\
&= \int_{A\times B} f(ab^{-1})\, |\det \circ \mathrm{Ad}_B(b)|\, \mathtt{d}\mu_A(a)\, \mathtt{d}\mu_B(b) \\
&= \int_{(A\times B)/K} \int_{A\cap B} f\big((ah)(bh)^{-1}\big) |\det \circ \mathrm{Ad}_B(b)|\, \mathtt{d}\mu_{A\cap B}(h)\, \mathtt{d}\widetilde{\omega}((a,b)K) \\
&= \int_{(A\times B)/K} f(ab^{-1}) |\det \circ \mathrm{Ad}_B(b)|\, \mathtt{d}\widetilde{\omega}((a,b)K) \\
&= \int_{(A\times B)/K} f(ab^{-1}) \frac{|\det(\mathrm{Ad}_B(b))|}{|\det(\mathrm{Ad}_G(b))|}\, \mathtt{d}(\varphi^*\mu)\widetilde{\ }((a,b)K).
\end{aligned}$$

Comparing with (9.12), the assertion follows. □

9.4.3 Averaging

In this subsection we give a few applications of invariant integration. We start with the construction of invariant inner products for representation spaces of compact Lie groups, which is a key step in the proof of Weyl's Trick which provides an equivalence between unitary representations of compact Lie groups and holomorphic representations of their complexifications.

Lemma 9.4.14 (Unitarity Lemma for Compact Groups). *Let $\pi\colon G \to \mathrm{GL}(V)$ be a representation of the compact Lie group on the finite-dimensional $\mathbb{K}$-vector space V. Then there exists a positive definite hermitian form β on V with $\pi(G) \subseteq \mathrm{U}(V, \beta)$.*

Proof. Let μ_G be a normalized Haar measure on G. Using a basis, we obtain a linear isomorphism $\varphi\colon V \to \mathbb{K}^n$, so that we can use the standard hermitian form on $\mathbb{K}^n$ to obtain on V a positive definite hermitian form $\widetilde{\beta}$. We now put

$$\beta(v, w) := \int_G \widetilde{\beta}\big(\pi(g)v, \pi(g)w\big)\, d\mu_G(g) \quad \text{for} \quad x, y \in V.$$

This integral is defined because the integrand is a continuous function. It is clear that β is sesquilinear and hermitian because

$$\begin{aligned}
\overline{\beta(v,w)} &= \int_G \overline{\widetilde{\beta}\big(\pi(g)v, \pi(g)w\big)}\, d\mu_G(g) \\
&= \int_G \widetilde{\beta}\big(\pi(g)w, \pi(g)v\big)\, d\mu_G(g) = \beta(w, v).
\end{aligned}$$

For $v = w \neq 0$, we obtain $\beta(v, v) > 0$ because $\widetilde{\beta}(\pi(g)v, \pi(g)v) > 0$ holds for each $g \in G$. Finally, the right invariance of μ_G implies that

$$\begin{aligned}\beta(\pi(g)v, \pi(g)w) &= \int_G \widetilde{\beta}\big(\pi(hg)v, \pi(hg)w\big)\ d\mu_G(h) \\ &= \int_G \widetilde{\beta}\big(\pi(h)v, \pi(h)w\big)\ d\mu_G(h) = \beta(v, w).\end{aligned}$$

This proves that $\pi(G) \subseteq \mathrm{U}(V, \beta)$. ☐

Next we turn to differential forms and de Rham cohomology classes.

Definition 9.4.15. Let M be a smooth manifold and G a Lie group acting smoothly on M. A differential form $\omega \in \Omega^k(M)$ is called *G-invariant*, if $\sigma_g^*\omega = \omega$ where $\sigma_g \colon M \to M$ is the diffeomorphism induced by g on M via the action, which we denote by σ. We write $\Omega^k(M)^G$ for the space of G-invariant k-forms on M.

Remark 9.4.16. If $\sigma \colon G \times M \to M$ is a smooth action, then Remark 9.3.16 implies that exterior differentiation commutes with the pullback maps σ_g^* so that we obtain a map $\mathtt{d} \colon \Omega^k(M)^G \to \Omega^{k+1}(M)^G$. Thus we can define an invariant version of de Rham cohomology (cf. Definition 9.3.14) via

$$H^k_{\mathrm{dR},G}(M) := \frac{\ker\big(\mathtt{d} \colon \Omega^k(M)^G \to \Omega^{k+1}(M)^G\big)}{\mathrm{im}\big(\mathtt{d} \colon \Omega^{k-1}(M)^G \to \Omega^k(M)^G\big)}$$

is called the k-th *invariant de Rham cohomology space of M*. It is clear that the inclusions $\Omega^k(M)^G \to \Omega^k(G)$ induce natural maps

$$i_* \colon H^k_{\mathrm{dR},G}(M) \to H^k_{\mathrm{dR}}(M).$$

Proposition 9.4.17. *Let $\sigma \colon G \times M \to M$ be a smooth action of a compact Lie group G on a smooth manifold M. Then*

$$(P_k\omega)_x(v_1, \ldots, v_k) := \int_G (\sigma_g^*\omega)_x(v_1, \ldots, v_k)\, \mathtt{d}\mu_G(g)$$

for $x \in M$, $k \in \mathbb{N}_0$, and $v_1, \ldots, v_k \in T_x(M)$ defines linear projections $P_k \colon \Omega^k(M) \to \Omega^k(M)^G$ commuting with exterior differentiation.

Proof. The definition of the pull-back of differential form and the chain rule, together with the right invariance of the Haar measure, allow us to calculate

$$\begin{aligned}\big(\sigma_h^*(P_k\omega)\big)_x(v_1, \ldots, v_k) &= (P_k\omega)_{\sigma_h(x)}\big(T_x(\sigma_h)v_1, \ldots, T_x(\sigma_h)v_k\big) \\ &= \int_G (\sigma_g^*\omega)_{\sigma_h(x)}\big(T_x(\sigma_h)v_1, \ldots, T_x(\sigma_h)v_k\big)\, \mathtt{d}\mu_G(g) \\ &= \int_G \omega_{\sigma_{gh}(x)}\big(T_x(\sigma_{gh})v_1, \ldots, T_x(\sigma_{gh})v_k\big)\, \mathtt{d}\mu_G(g) \\ &= \int_G \omega_{\sigma_g(x)}\big(T_x(\sigma_g)v_1, \ldots, T_x(\sigma_g)v_k\big)\, \mathtt{d}\mu_G(g) \\ &= (P_k\omega)_x(v_1, \ldots, v_k).\end{aligned}$$

This implies the first claim. The second claim follows by integration from the fact that the exterior derivative commutes with pullbacks (Remark 9.3.16). □

The averaging operators P_k can be used to show that the maps

$$i_*\colon H^k_{\mathrm{dR},G}(M) \to H^k_{\mathrm{dR}}(M)$$

are injective:

Theorem 9.4.18. *Let M be a smooth manifold and G a compact Lie group acting smoothly on M. Then the natural maps $i_*\colon H^k_{\mathrm{dR},G}(M) \to H^k_{\mathrm{dR}}(M)$ are linear and injective.*

Proof. Linearity is clear, so we assume that $\omega = \mathtt{d}\nu \in \Omega^k(M)^G$ with $\nu \in \Omega^{k-1}(M)$. Then $P_{k-1}\nu \in \Omega^{k-1}(M)^G$ and

$$\mathtt{d}(P_{k-1}\nu) = P_k(\mathtt{d}\nu) = P_k\omega = \omega.$$

Thus $\omega \in \mathrm{im}\,\big(\mathtt{d}\colon \Omega^{k-1}(M)^G \to \Omega^k(M)^G\big)$ induces the zero cohomology class in $H^k_{\mathrm{dR}}(M)^G$. □

Remark 9.4.19. If, in the situation of Theorem 9.4.18, G is connected, one can show that i_* is indeed an isomorphism (see [GHV73, p. 151]).

Definition 9.4.20. Let G be a Lie group with Lie algebra $\mathfrak{g}$, V a vector space and $\rho\colon G \to \mathrm{GL}(V)$ be a smooth representation of G, so that the derived representation yields on V the structure of a $\mathfrak{g}$-module.

We call a p-form $\omega \in \Omega^p(G,V)$ *equivariant* if we have for all $g \in G$ the relation

$$\lambda_g^*\omega = \rho(g) \circ \omega.$$

If V is a trivial module, then an equivariant p-form is a left invariant V-valued p-form on G. An equivariant p-form is uniquely determined by the corresponding element $\omega_{\mathbf{1}} \in C^p(\mathfrak{g},V)$ via

$$\omega_g(gx_1,\ldots,gx_p) = \rho(g)\omega_{\mathbf{1}}(x_1,\ldots,x_p) \tag{9.13}$$

for $g \in G, x_i \in \mathfrak{g} \cong T_{\mathbf{1}}(G)$, where $G \times T(G) \to T(G), (g,x) \mapsto gx$ denotes the natural action of G on its tangent bundle $T(G)$ obtained by restricting the tangent map of the group multiplication.

Conversely, (9.13) provides for each $\omega \in C^p(\mathfrak{g},V)$ a unique equivariant p-form ω^{eq} on G with $\omega^{\mathrm{eq}}_{\mathbf{1}} = \omega$.

Remark 9.4.21. If $\omega \in C^p(\mathfrak{g},V)$ and $\omega^{\mathrm{eq}} \in \Omega^p(G,V)$ is the corresponding left equivariant p-form on G, then each right multiplication $\rho_g\colon G \to G$ satisfies

$$\begin{aligned}(\rho^*\omega^{\mathrm{eq}})_{\mathbf{1}}(x_1,\ldots,x_p) &= \omega^{\mathrm{eq}}(x_1g,\ldots,x_pg) = \rho(g)\omega(g^{-1}x_1g,\ldots,g^{-1}x_pg)\\ &= \rho(g)(\mathrm{Ad}(g^{-1})^*\omega)(x_1,\ldots,x_p).\end{aligned}$$

Since $\rho_g^*\omega^{\mathrm{eq}}$ is also left equivariant, we see that pullback with right multiplications correspond to the action of G on $C^p(\mathfrak{g},V)$ by $(g,\omega) \mapsto \mathrm{Ad}(g^{-1})^*\omega$.

Proposition 9.4.22. *For each* $\omega \in C^p(\mathfrak{g}, V)$ *the equation* $\mathtt{d}(\omega^{\mathrm{eq}}) = (\mathtt{d}_{\mathfrak{g}}\omega)^{\mathrm{eq}}$ *holds. Accordingly, the evaluation map*

$$\mathrm{ev}_{\mathbf{1}} \colon \Omega^p(G,V)^{\mathrm{eq}} \to C^p(\mathfrak{g},V), \quad \omega \mapsto \omega_{\mathbf{1}} \quad \textit{satisfies} \quad \mathrm{ev}_{\mathbf{1}} \circ \mathtt{d} = \mathtt{d}_{\mathfrak{g}} \circ \mathrm{ev}_{\mathbf{1}}\,.$$

Proof. For $g \in G$, we have

$$\lambda_g^* \mathtt{d}\omega^{\mathrm{eq}} = \mathtt{d}\lambda_g^*\omega^{\mathrm{eq}} = \mathtt{d}(\rho(g) \circ \omega^{\mathrm{eq}}) = \rho(g) \circ (\mathtt{d}\omega^{\mathrm{eq}}),$$

showing that $\mathtt{d}\omega^{\mathrm{eq}}$ is also equivariant.

For $x \in \mathfrak{g}$ we write x_l for the corresponding left invariant vector field on G, i.e., $x_l(g) = gx$. It suffices to calculate the value of $\mathtt{d}\omega^{\mathrm{eq}}$ on $(p+1)$-tuples of left invariant vector fields in the identity element. In view of

$$\omega^{\mathrm{eq}}(x_{1,l}, \ldots, x_{p,l})(g) = \rho(g)\omega(x_1, \ldots, x_p),$$

we obtain

$$\big(x_{0,l}\omega^{\mathrm{eq}}(x_{1,l}, \ldots, x_{p,l})\big)(\mathbf{1}) = \mathbf{L}(\rho)(x_0)\omega(x_1, \ldots, x_p),$$

and therefore

$$\begin{aligned}
&\big(\mathtt{d}\omega^{\mathrm{eq}}(x_{0,l}, \ldots, x_{p,l})\big)(\mathbf{1}) \\
&= \sum_{i=0}^{p} (-1)^i\, \mathbf{L}(\rho)(x_{i,l})\omega^{\mathrm{eq}}(x_{0,l}, \ldots, \widehat{x_{i,l}}, \ldots, x_{p,l})(\mathbf{1}) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega^{\mathrm{eq}}([x_{i,l}, x_{j,l}], x_{0,l}, \ldots, \widehat{x_{i,l}}, \ldots, \widehat{x_{j,l}}, \ldots, x_{p,l})(\mathbf{1}) \\
&= \sum_{i=0}^{p} (-1)^i\, \mathbf{L}(\rho)(x_i)\omega(x_0, \ldots, \widehat{x_i}, \ldots, x_p) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_p) \\
&= (\mathtt{d}_{\mathfrak{g}}\omega)(x_0, \ldots, x_p).
\end{aligned}$$

This proves our assertion. □

Corollary 9.4.23. *Every differential form* $\omega \in \Omega^p(G, \mathbb{R})$ *on a Lie group* G *which is invariant under left and right translations is closed.*

Proof. The right invariance of ω implies that $\omega_{\mathbf{1}} \in C^p(\mathfrak{g}, \mathbb{R})$ is invariant under the adjoint action, i.e.,

$$\mathrm{Ad}(g)^*\omega_{\mathbf{1}} = \omega_{\mathbf{1}} \quad \text{for} \quad g \in G$$

(Remark 9.4.21). Taking derivatives, we see that $\omega \in C^p(\mathfrak{g}, \mathbb{R})^{\mathfrak{g}}$, i.e., ω is $\mathfrak{g}$-invariant. Now Lemma 6.5.28 implies that $\mathtt{d}_{\mathfrak{g}}\omega_{\mathbf{1}} = 0$, and Proposition 9.4.22 shows that ω^{eq} is closed. □

Corollary 9.4.24. *Every non-zero differential form $\omega \in \Omega^p(G,\mathbb{R})$ on a compact Lie group G which is invariant under left and right translations is closed, but not exact, so it defines a nonzero element in $H^p_{\mathrm{dR}}(G,\mathbb{R})$.*

Proof. In view of Corollary 9.4.23, it only remains to show that an exact form $\omega \in \Omega^p(G,\mathbb{R})$ which is left and right invariant vanishes. To this end, pick such a form and let $G \times G$ act on G by $(g_1, g_2).x = g_1 x g_2^{-1}$. This defines a smooth action for which ω is invariant, i.e., $\omega \in \Omega^p(G)^{G\times G}$. But then exactness by Theorem 9.4.18 implies that there exists a $(G\times G)$-invariant form $\nu \in \Omega^{p-1}(G)$ with $\omega = \mathtt{d}\nu$. On the other hand, Corollary 9.4.23 implies that ν is closed, so that $\omega = \mathtt{d}\nu = 0$. □

Corollary 9.4.25. *If G is a compact connected Lie group of dimension n, then $H^n_{\mathrm{dR}}(G,\mathbb{R}) \neq \{0\}$.*

Proof. The compactness of the Lie algebra $\mathfrak{g}$ implies in particular that G is unimodular, i.e., $|\det(\mathrm{Ad}(g))| = 1$ for each $g \in G$. Taking derivatives, it follows that $\mathrm{tr}(\mathrm{ad}\, x) = 0$ for each $x \in \mathfrak{g}$. Using Proposition 6.5.24, we find a $\mathfrak{g}$-invariant volume form $\omega \in C^n(\mathfrak{g},\mathbb{R})$. As G is connected, the $\mathfrak{g}$-invariance of ω implies that $\mathrm{Ad}(g)^*\omega = \omega$ holds for each $g \in G$. Then the corresponding left invariant differential form $\omega^{\mathrm{eq}} \in \Omega^n(G,\mathbb{R})$ is also invariant under right translations, and Corollary 9.4.24 shows that it defines a non-zero class in $H^n_{\mathrm{dR}}(G,\mathbb{R})$. □

Exercises for Section 9.4

Exercise 9.4.1. Let G be a Lie group, μ_G a left Haar measure. For $f, h \in C_c(G)$, define the *convolution* product

$$(f * h)(g) := \int_G f(gx^{-1})h(x)\, d\mu_G(x) \quad \text{and} \quad \|f\|_1 := \int_G |f(x)|\, d\mu_G(x).$$

Show that:

(i) The convolution product turns $C_c(G)$ into an associative $\mathbb{R}$-algebra.
(ii) For $f, h \in C_c(G)$ we have $\|f * h\|_1 \leq \|f\|_1 \|h\|_1$, i.e. $\big(C_c(G), *, \|\cdot\|_1\big)$ is a *normed algebra*.
(iii) The completion of $C_c(G)$ with respect to the norm $\|\cdot\|_1$ is denoted by $L^1(G)$. It is possible to extend the convolution to an associative multiplication on $L^1(G)$, which turns $L^1(G)$ into a Banach algebra.
(iv) Write down the convolution for a finite group G.
(v) What is $L^1(G)$ for a discrete group, e.g., for $\mathbb{Z}$?

Exercise 9.4.2. Let G be a locally compact group, and let $f \in C_c(G)$. Then f is uniformly continuous in the sense that for every $\varepsilon > 0$ there is a neighborhood $V \subseteq G$ of unity such that

$$|f(x) - f(y)| \leq \varepsilon \quad \text{für} \quad y \in xV.$$

Exercise 9.4.3. Let μ be a Haar measure on the Lie group G and $h \in C(G)$ with $\mu(fh) = 0$ for all $f \in C_c(G)$. Show that $h = 0$.

9.5 Integrating Lie Algebras of Vector Fields

In Proposition 8.1.14 we obtained for each smooth action of a Lie group G on a manifold M a representation of the Lie algebra $\mathbf{L}(G)$ in the Lie algebra $\mathcal{V}(M)$ of vector fields on M. In this section we study the converse problem of integrating such a Lie algebra representation to a smooth action.

9.5.1 Palais' Theorem

We start with the case where all representation vector fields are known to be complete, a condition which is hard to verify.

Theorem 9.5.1 (Palais). *Let G be a 1-connected Lie group with Lie algebra $\mathfrak{g}$, M a smooth manifold and $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ a morphism of Lie algebras whose image consists of complete vector fields. Then there exists a homomorphism $\beta\colon G \to \mathrm{Diff}(M)$ defining a smooth G-action σ on M with $\dot{\sigma} = \alpha$.*

If $X \in \mathcal{V}(M)$ is a complete vector field, then we write

$$\exp(X) := \Phi_1^X \in \mathrm{Diff}(M)$$

for the time-1-flow of X.

Proposition 9.5.2. *If M is finite-dimensional and $E \subseteq \mathcal{V}(M)$ is a finite-dimensional subspace consisting of complete vector fields, then the map*

$$E \times M \to M, \quad (X, m) \mapsto \exp(X)(m)$$

is smooth.

Proof. Let $f\colon \mathbb{R}^n \to E$ be a linear isomorphism. Clearly, the map

$$\mathbb{R}^n \times M \to T(M), \quad (x, m) \mapsto f(x)(m)$$

is smooth. We thus obtain on the manifold $\widehat{M} := \mathbb{R}^n \times M$ a smooth vector field $\widehat{X}(x, m) := (0, f(x)(m))$ with

$$\exp(\widehat{X})(x, m) = (x, \exp(f(x))(m)),$$

so that the assertion follows from the smoothness of $\exp(\widehat{X})$ on $\widehat{M}$ (Theorem 7.5.12). $\square$

We have a natural action of $\mathrm{Diff}(M)$ on $\mathcal{V}(M)$ by

$$\mathrm{Diff}(M) \times \mathcal{V}(M) \to \mathcal{V}(M), \quad (\varphi, X) \mapsto \varphi_* X := T(\varphi) \circ X \circ \varphi^{-1},$$

called the *adjoint action.*

Definition 9.5.3. Let M and N be smooth manifolds.

(a) Although $\mathrm{Diff}(M)$ is certainly not a finite-dimensional Lie group, we call a map $\varphi\colon N \to \mathrm{Diff}(M)$ *smooth* if the map

$$\widehat{\varphi}\colon N \times M \to M \times M, \quad (n, x) \mapsto (\varphi(n)(x), \varphi(n)^{-1}(x))$$

is smooth. If N is an interval in $\mathbb{R}$, we obtain in particular a notion of a smooth curve.

(b) Smooth curves $\varphi\colon J \subseteq \mathbb{R} \to \mathrm{Diff}(M)$ have (left) logarithmic derivatives

$$\delta(\varphi)\colon J \to \mathcal{V}(M), \quad \delta(\varphi)_t := T(\varphi(t)^{-1}) \circ \varphi'(t)$$

which are curves in the Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M, i.e., time-dependent vector fields. Here we consider $\varphi'(t)\colon M \to TM$ as the smooth map defined by

$$\varphi'(t)(m) = \frac{d}{dt}\varphi(t)(m), \quad m \in M, t \in J.$$

As we shall see below, for general N, the logarithmic derivatives are $\mathcal{V}(M)$-valued 1-forms on N which are smooth in the sense that composition with any evaluation map $\mathrm{ev}_p\colon \mathcal{V}(M) \to T_p(M)$ yields a $T_p(M)$-valued differential form.

If $\varphi\colon N \to \mathrm{Diff}(M)$ is smooth and $\widehat{\varphi}_1\colon N \times M \to M, (n, x) \mapsto \varphi(n)(x)$, then we have a smooth tangent map

$$T(\widehat{\varphi}_1)\colon T(N \times M) \cong T(N) \times T(M) \to T(M)$$

and obtain for each $v \in T_p(N)$ the partial map

$$T_p(\varphi)v\colon M \to T(M), \quad m \mapsto T_{(p,m)}(\widehat{\varphi}_1)(v, 0).$$

We define the *(left) logarithmic derivative of* φ *in* p by

$$\delta(\varphi)_p\colon T_p(N) \to \mathcal{V}(M), \quad v \mapsto T(\varphi(p)^{-1}) \circ T_p(\varphi)(v).$$

Then $\delta(\varphi)$ is a $\mathcal{V}(M)$-valued 1-form on N.

Lemma 9.5.4. *If* $\mathfrak{g}$ *is a finite-dimensional Lie algebra and* $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ *is a homomorphism, then each* $x \in \mathfrak{g}$*, for which* $\alpha(x)$ *is complete, satisfies the relation*

$$\exp(-\alpha(x))_* \circ \alpha = \alpha \circ e^{\operatorname{ad} x}. \tag{9.14}$$

Proof. We consider the smooth curve

$$\psi(t) := \exp(t\alpha(x))_* \alpha(e^{t \operatorname{ad} x} y).$$

Then the definition of the Lie derivative implies that

$$\begin{aligned}\psi'(t) &= \exp(t\alpha(x))_*\big(-\mathcal{L}_{\alpha(x)}(\alpha(e^{t\,\mathrm{ad}\,x}y))\big) + \exp(t\alpha(x))_*\alpha([x, e^{t\,\mathrm{ad}\,x}y]) \\ &= \exp(t\alpha(x))_*\big(-[\alpha(x), \alpha(e^{t\,\mathrm{ad}\,x}y)]\big) + \exp(t\alpha(x))_*\alpha([x, e^{t\,\mathrm{ad}\,x}y]) \\ &= \exp(t\alpha(x))_*\alpha\big([-x, e^{t\,\mathrm{ad}\,x}y] + [x, e^{t\,\mathrm{ad}\,x}y]\big) = 0.\end{aligned}$$

For $t = 1$ we thus obtain $\exp(\alpha(x))_*\alpha(e^{\mathrm{ad}\,x}y) = \psi(1) = \psi(0) = \alpha(y)$, and the assertion follows. □

For calculations, it is convenient to observe the Product- and Quotient Rules (cf. Lemma 8.2.27), both easy consequences of the Chain Rule:

Lemma 9.5.5. *For two smooth maps $f, g\colon N \to \mathrm{Diff}(M)$, define*

$$(fg)(n) := f(n) \circ g(n) \quad \text{and} \quad (g^{-1})(n) := g(n)^{-1}.$$

Then the following assertions hold:

(i) *Product Rule: $\delta(fg) = \delta(g) + (g^{-1})_*\delta(f)$, and the*
(ii) *Quotient Rule: $\delta(fg^{-1}) = g_*(\delta(f) - \delta(g))$,*
where we write $(f_*\alpha)_n := f(n)_* \circ \alpha_n$ for a $\mathcal{V}(M)$-valued 1-form α on N.
(iii) *If $\delta(f) = \delta(g)$, then there exists a $\varphi \in \mathrm{Diff}(M)$ with $f(n) = \varphi \circ g(n)$ for all $n \in N$.*

Proof. (i),(ii) The smoothness of f^{-1} and g^{-1} follows directly from the definitions. For the smoothness of the product map fg, we observe that the maps

$$N^2 \times M \to M, \quad (t, s, x) \mapsto f(t)\big(g(s)(x)\big) = (f(t) \circ g(s))(x)$$

and

$$N^2 \times M \to M, \quad (t, s, x) \mapsto (f(t) \circ g(s))^{-1}(x) = g(s)^{-1}\big(f(t)^{-1}(x)\big)$$

are smooth because they are compositions of smooth maps.

The Chain Rule now implies

$$T_p(fg)v = T(f(p)) \circ (T_p(g)v) + (T_p(f)v) \circ g(p),$$

which leads to

$$\begin{aligned}\delta(fg)_p &= T(g(p)^{-1})T_p(g) + T(g(p)^{-1})T(f(p)^{-1})T_p(f) \circ g(p) \\ &= \delta(g)_p + (g(p)^{-1})_*\delta(f)_p,\end{aligned}$$

For $g = f^{-1}$, we obtain in particular

$$0 = \delta(ff^{-1}) = \delta(f^{-1}) + f_*\delta(f).$$

Combining this with (i), we obtain the Quotient Rule.

(iii) From (ii) we immediately derive that the logarithmic derivative of fg^{-1} vanishes, hence that it is constant, and this implies (iii). □

Lemma 9.5.6. *Let $X, Y \in \mathcal{V}(M)$ be two smooth vector fields with the property that $E := \mathbb{R}X + \mathbb{R}Y$ consists of complete vector fields. Then* $\mathrm{Exp} := \exp|_E$ *is smooth and satisfies (in the pointwise sense)*

$$\delta(\mathrm{Exp})_X(Y) = \int_0^1 \mathrm{Exp}(-tX)_* Y\, dt \quad \textit{for} \quad X, Y \in E.$$

Proof. We claim that the smooth function

$$\psi\colon I \to \mathcal{V}(M), \quad \psi(t) := \delta(\mathrm{Exp})_{tX}(tY),$$

satisfies the functional equation

$$\psi(t+s) = \mathrm{Exp}(-sX)_*\psi(t) + \psi(s). \tag{9.15}$$

We consider the three smooth functions $F, G, H\colon E \to \mathrm{Diff}(M)$, given by $F(X) := \mathrm{Exp}((t+s)X)$, $G(X) := \mathrm{Exp}(tX)$ and $H(X) := \mathrm{Exp}(sX)$, satisfying $F = G \cdot H$ pointwise. The Product Rule (Lemma 9.5.5) implies that

$$\delta(F) = \delta(H) + (H^{-1})_*\delta(G).$$

Now (9.15) follows from

$$\delta(G)_X(Y) = \psi(t), \quad \delta(H)_X(Y) = \psi(s) \quad \text{and} \quad \delta(F)_X(Y) = \psi(s+t).$$

Clearly $\psi(0) = 0$ and

$$\psi'(0) = \lim_{t\to 0} \delta(\mathrm{Exp})_{tX}(Y) = \delta(\mathrm{Exp})_0(Y) = Y.$$

Taking derivatives with respect to t in 0, (9.15) thus leads to

$$\psi'(s) = \mathrm{Exp}(-sX)_* Y,$$

which implies the assertion by integration:

$$\delta(\mathrm{Exp})_X(Y) = \psi(1) = \int_0^1 \psi'(s)\, ds = \int_0^1 \mathrm{Exp}(-sX)_* Y\, ds. \qquad \square$$

Proposition 9.5.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $U \subseteq \mathfrak{g}$ a convex symmetric 0-neighborhood such that $\exp_G|_U$ is a diffeomorphism onto an open subset of G and $V \subseteq U$ an open convex 0-neighborhood with* $\exp_G(V)\exp_G(V) \subseteq \exp_G(U)$. *For the multiplication*

$$x * y := (\exp_G|_U)^{-1}(\exp_G(x)\exp_G(y)), \quad x, y \in V,$$

we then have in $\mathrm{Diff}(M)$

$$\exp(-\alpha(x * y)) = \exp(-\alpha(x))\exp(-\alpha(y))$$

for $x, y \in V$.

Proof. On $I := [0,1]$ we define the two smooth functions

$$\gamma_1 \colon I \to \mathrm{Diff}(M), \quad \gamma_1(t) := \exp(-\alpha(x))\exp(-t\alpha(y))$$

and

$$\gamma_2 \colon I \to \mathrm{Diff}(M), \quad \gamma_2(t) := \exp(-\alpha(x * ty)).$$

Then $\gamma_1(0) = \gamma_2(0) = \exp(-\alpha(x))$ and $\delta(\gamma_1)(t) = -\alpha(y)$. For the smooth map $\exp \circ \alpha \colon \mathfrak{g} \to \mathrm{Diff}(M)$ we obtain with Lemmas 9.5.4 and 9.5.6

$$\begin{aligned}\delta(\exp\circ\alpha)_x(y) &= \int_0^1 \exp(-t\alpha(x))_*\alpha(y)\,dt = \int_0^1 \alpha(e^{t\,\mathrm{ad}\,x}y)\,dt \\ &= \alpha\Big(\int_0^1 e^{t\,\mathrm{ad}\,x}y\,dt\Big),\end{aligned}$$

i.e.,

$$\delta(\exp\circ\alpha)_x(y) = \alpha(\kappa_{\mathfrak{g}}(-x)y) \quad \text{for} \quad \kappa_{\mathfrak{g}}(x) := \int_0^1 e^{-t\,\mathrm{ad}\,x}\,dt.$$

On the other hand, the relation $\exp_G(x * ty) = \exp_G(x)\exp_G(ty)$ leads with Proposition 8.2.29 to

$$y = \delta(\exp_G)_{x*ty}\frac{d}{dt}x * ty = \kappa_{\mathfrak{g}}(x * ty)\frac{d}{dt}x * ty,$$

so that

$$\frac{d}{dt}x * ty = \kappa_{\mathfrak{g}}(x * ty)^{-1}y.$$

We thus obtain

$$\begin{aligned}\delta(\gamma_2)(t) &= -\delta(\exp\circ\alpha)_{(-x*ty)}\alpha\Big(\frac{d}{dt}x * ty\Big) = -\alpha\Big(\kappa_{\mathfrak{g}}(x * ty)\kappa_{\mathfrak{g}}(x * ty)^{-1}y\Big) \\ &= -\alpha(y).\end{aligned}$$

This shows that γ_1 and γ_2 have the same logarithmic derivative, and since both curves start in the same point, they coincide (Lemma 9.5.5(iii)). □

Proof. (of Palais' Theorem 9.5.1) Let V be as in Proposition 9.5.7 and define

$$f \colon \exp_G(V) \to \mathrm{Diff}(M), \quad f(\exp_G(x)) := \exp(-\alpha(x)).$$

Then

$$f(gh) = f(g) \circ f(h) \quad \text{for} \quad g, h \in \exp_G(V),$$

so that the Monodromy Principle (Proposition 8.5.8) implies the existence of an extension of f to a homomorphism $G \to \mathrm{Diff}(M)$. Since f is smooth on the identity neighborhood $\exp_G(V)$, it follows easily that f defines a smooth action σ of G on M, and the definition of f implies that $\mathbf{L}(\sigma) = \alpha$. □

9.5.2 Lie Algebras Generated by Complete Vector Fields

It is part of the assumptions of Theorem 9.5.1 that the Lie algebra $\alpha(\mathfrak{g})$ consists of complete vector fields. In this subsection we show that this condition can be relaxed significantly to the requirement that $\alpha(\mathfrak{g})$ is merely generated by complete vector fields.

To this end, we consider a finite-dimensional Lie subalgebra $\mathfrak{g} \subseteq \mathcal{V}(M)$ and assume that $\mathfrak{g}$ is generated, as a Lie algebra, by complete vector fields. Let $C(\mathfrak{g})$ denote the subset of complete vector fields in $\mathfrak{g}$.

Lemma 9.5.8. *If $X, Y \in C(\mathfrak{g})$, then $e^{\operatorname{ad} X} Y \in C(\mathfrak{g})$.*

Proof. We consider the smooth flow on M defined by

$$\gamma(t) := (\exp -X) \circ \exp(tY) \circ (\exp X) \in \operatorname{Diff}(M)$$

and note that

$$\gamma'(0) = (\exp -X)_* Y = e^{\operatorname{ad} X} Y$$

by Lemma 9.5.4. Therefore $e^{\operatorname{ad} X} Y$ is a complete vector field. □

Lemma 9.5.9. $\mathfrak{g} = \operatorname{span} C(\mathfrak{g})$.

Proof. Let $V := \operatorname{span} C(\mathfrak{g}) \subseteq \mathfrak{g}$. By Lemma 9.5.8, we have $e^{\operatorname{ad} X} V = V$ for each $X \in C(\mathfrak{g})$, hence $[C(\mathfrak{g}), V] \subseteq V$, so that $C(\mathfrak{g}) \subseteq \mathfrak{n}_{\mathfrak{g}}(V)$. Since $\mathfrak{n}_{\mathfrak{g}}(V)$ is a Lie subalgebra of $\mathfrak{g}$ and $C(\mathfrak{g})$ generates $\mathfrak{g}$, it follows that $[\mathfrak{g}, V] \subseteq V$, i.e., that V is an ideal, hence in particular a subalgebra. Now the fact that $C(\mathfrak{g})$ generates $\mathfrak{g}$ yields $V = \mathfrak{g}$. □

Proposition 9.5.10. *If $\mathfrak{g} \subseteq \mathcal{V}(M)$ is a finite-dimensional Lie subalgebra generated by complete vector fields, then $\mathfrak{g}$ consists of complete vector fields.*

Proof. In view of Lemma 9.5.9, $\mathfrak{g}$ is spanned by complete vector fields, so that there exists a basis $X_1, \ldots, X_n$, consisting of complete vector fields. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $Y \in \mathfrak{g}$.

Using canonical coordinates of the second kind on G with respect to the basis $X_1, \ldots, X_n$ (Lemma 8.2.6), we obtain smooth functions α_i, defined on some 0-neighborhood $[-\varepsilon, \varepsilon] \subseteq \mathbb{R}$, satisfying

$$\exp_G(tY) = \exp_G(\alpha_1(t)X_1) \cdots \exp_G(\alpha_n(t)X_n)$$

for each $t \in [-\varepsilon, \varepsilon]$. We then have $\alpha_i(0) = 0$ for each i and $Y = \sum_{i=1}^n \alpha_i'(0) X_i$.

Since the curve $\gamma(t) := \exp_G(tY)$ in G satisfies $\delta(\gamma)_t = Y$ for each $t \in \mathbb{R}$, the Product Rule for logarithmic derivatives (Lemma 8.2.27) yields

$$Y = \delta(\exp_G)_{\alpha_n(t)X_n} + \sum_{i=1}^{n-1} \operatorname{Ad}(\exp_G(-\alpha_n(t)X_n)) \cdots \operatorname{Ad}(\exp_G(-\alpha_{i+1}(t)X_{i+1}))\delta(\exp_G)_{\alpha_i(t)X_i} \alpha_i'(t) X_i$$

Since

$$\delta(\exp_G)(\alpha_i(t)X_i)\alpha_i'(t)X_i = \alpha_i'(t)X_i$$

follows from the Chain Rule, this simplifies to the relation

$$Y = \sum_{i=1}^{n} \mathrm{Ad}(\exp_G(-\alpha_n(t)X_n)) \cdots \mathrm{Ad}(\exp_G(-\alpha_{i+1}(t)X_{i+1}))\alpha_i'(t)X_i.$$

We now consider the smooth curve

$$\beta(t) := \exp(-\alpha_1(t)X_1) \cdots \exp(-\alpha_n(t)X_n) \in \mathrm{Diff}(M)$$

and use the Product Formula from Lemma 9.5.5 to obtain

$$\delta(\beta)_t = -\alpha_n'(t)X_n - \sum_{i=1}^{n-1} \exp(\alpha_n(t)X_n)_* \cdots \exp(\alpha_{i+1}(t)X_{i+1})_*\alpha_i'(t)X_i.$$

Further, Lemma 9.5.4 implies that the inclusion homomorphism $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ satisfies

$$\begin{aligned} Y &= \sum_{i=1}^{n} \alpha \circ \mathrm{Ad}(\exp_G(-\alpha_n(t)X_n)) \cdots \mathrm{Ad}(\exp_G(-\alpha_{i+1}(t)X_{i+1}))\alpha_i'(t)X_i \\ &= \sum_{i=1}^{n} \exp_G(\alpha_n(t)X_n)_* \cdots \exp(-\alpha_{i+1}(t)X_{i+1})_*\alpha_i'(t)X_i = -\delta(\beta)_t. \end{aligned}$$

This shows that $\delta(\beta)_t = -Y$ for each $t \in [-\varepsilon, \varepsilon]$ and applying this to the two curves

$$t \mapsto \beta(s) \circ \beta(t), \beta(s+t) \quad \text{for} \quad |t|, |s|, |t+s| \leq \varepsilon,$$

we see that both have the same logarithmic derivative with respect to t and the same values for $t = 0$. Hence Lemma 9.5.5(iii) leads to

$$\beta(s) \circ \beta(t) = \beta(s+t) \quad \text{for} \quad |t|, |s|, |t+s| \leq \varepsilon.$$

We conclude that β defines a local flow with generator Y which is defined on $[-\varepsilon, \varepsilon] \times M$. This easily implies that Y is complete (see Exercise 7.5.2). □

Notes on Chapter 9

Historically, the origin of smooth actions of Lie groups lies in Felix Klein's "Erlanger Programm" from 1872 [1] , in which a *geometry on a space S* was

[1] Christian Felix Klein (1849–1925) held the chair of geometry in Erlangen for a few years and the "Erlanger Programm" was his "Programmschrift", where he formulated his research plans when he came to Erlangen. Later he was a professor for mathematics in Munich, Leipzig and eventually in Göttingen.

considered as the same structure as a *group acting on this space*. Then the *geometric properties*[2] are those invariant under the action of the group.

Nowadays smooth actions of Lie groups on manifolds form a separate field of mathematics under the name of *Lie transformation groups*. In this chapter we only collected what was needed in the further course of the book. This explains why there are quite different topics, none of which is explored systematically. One problem which occurs in this context is that smooth actions are needed to describe quotient structures and geometric properties of Lie groups, but conversely detailed knowledge of Lie groups is required for a systematic treatment of transformation groups. We refer to [Ko95] and [DK00] for more detailed information.

[2] A typical example is the group $\mathrm{Mot}(E_2)$ of motions (orientation preserving isometries) of the euclidian plane. The length of an interval or the area of a triangle are properties preserved by this group, hence geometric quantities. It was an important conceptual step to observe that changing the group means changing the geometry, resp., the notion of a geometric quantity. For example the automorphism group $\mathrm{Aff}(A_2)$ of the two-dimensional affine plane A_2 does not preserve the area of a triangle (it is larger than the euclidian group), hence the area of a triangle cannot be considered as an affine geometric quantity.

Part IV

Structure Theory of Lie Groups

10

Normal Subgroups, Nilpotent and Solvable Lie Groups

In this chapter we address structural aspects of Lie groups. Here an important issue is to see that for any closed normal subgroup N of a Lie group G, the quotient G/N carries a canonical Lie group structure, so that we may consider N and G/N as two pieces into which G decomposes. With this information we then address the canonical factorization of a morphism of Lie groups into a surjective, a bijective and an injective one. In particular, we describe some tools to calculate fundamental groups of Lie groups and homogeneous spaces.

We continue the structural aspects in Section 10.1.4, where we discuss extensions of a Lie group G by a Lie group N. In this context semidirect products are splitting extensions and the Smooth Splitting Theorem 10.1.21 asserts that, for any simply connected Lie group G, any connected normal integral subgroup N is closed and $G \cong (G/N) \times N$ as smooth manifolds. The last section of this chapter treats nilpotent and solvable Lie groups. These groups have a relatively simple structure because of their rich supply of normal subgroups.

Let us briefly recall the main tools that we have developed so far to deal with Lie groups. In the following, a smooth homomorphism of Lie groups is simply referred to as morphism of Lie groups.

(i) For a morphism $\alpha\colon G_1 \to G_2$ of Lie groups

$$\mathbf{L}(\alpha) := T_{\mathbf{1}}(\alpha)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$$

is a homomorphism of Lie algebras for which the diagram

$$\begin{array}{ccc} \mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\alpha)} & \mathbf{L}(G_2) \\ \downarrow{\scriptstyle \exp_{G_1}} & & \downarrow{\scriptstyle \exp_{G_2}} \\ G_1 & \xrightarrow{\alpha} & G_2 \end{array}$$

commutes (Proposition 8.1.8)

(ii) **Lie's Third Theorem:** For a finite dimensional Lie algebra $\mathfrak{g}$, there is, up to isomorphism, a unique 1-connected Lie group with Lie algebra $\mathfrak{g}$. We denote this group by $G(\mathfrak{g})$ (Theorem 8.4.11).

(iii) Let $\mathfrak{h}$ be a Lie algebra and $\beta\colon \mathfrak{h} \to \mathbf{L}(G)$ be a homomorphism into the Lie algebra of a Lie group G. Then there is a unique morphism $\alpha\colon G(\mathfrak{h}) \to G$ such that the diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\beta=\mathbf{L}(\alpha)} & \mathbf{L}(G) \\ \downarrow{\scriptstyle \exp_{G(\mathfrak{h})}} & & \downarrow{\scriptstyle \exp_G} \\ G(\mathfrak{h}) & \xrightarrow{\alpha} & G \end{array}$$

commutes (Proposition 8.2.10).

(iv) The homomorphic images of connected Lie groups H in a Lie group G are the subgroups of the form $\langle \exp_G \mathfrak{h} \rangle$ for a Lie subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$, i.e., the integral subgroups.

(v) Closed subgroups are submanifolds which are Lie groups (Closed Subgroup Theorem 8.3.7).

These five facts provide the basis of a correspondence which allows us to describe the structure of G by the structure of $\mathbf{L}(G)$, and to explore a connected Lie group by means of its Lie algebra.

10.1 Normalizers, Normal Subgroups, and Semidirect Products

In this section we study normalizers of subgroups and in particular normal subgroups. In particular, we show that the quotient by a closed normal subgroup is a Lie group and consider normal subgroups which give rise to semidirect product decompositions.

10.1.1 Normalizers and Centralizers

Lemma 10.1.1. *Let G be a Lie group, $\mathfrak{g} = \mathbf{L}(G)$ and $E \subseteq \mathfrak{g}$ be a subspace. We define its normalizer in G, resp., $\mathfrak{g}$ by*

$$N_G(E) := \{g \in G\colon \operatorname{Ad}(g)E = E\} \quad \textit{and} \quad \mathfrak{n}_{\mathfrak{g}}(E) := \{x \in \mathfrak{g}\colon [x, E] \subseteq E\}$$

and its centralizer in G, resp., $\mathfrak{g}$, by

$$Z_G(E) := \{g \in G\colon \operatorname{Ad}(g)|_E = \operatorname{id}_E\} \quad \textit{and} \quad \mathfrak{z}_{\mathfrak{g}}(E) := \{x \in \mathfrak{g}\colon [x, E] = \{0\}\}.$$

Then $N_G(E)$ and $Z_G(E)$ are closed subgroups of G with the Lie algebras

$$\mathbf{L}\big(N_G(E)\big) = \mathfrak{n}_{\mathfrak{g}}(E) \quad \textit{and} \quad \mathbf{L}\big(Z_G(E)\big) = \mathfrak{z}_{\mathfrak{g}}(E).$$

Proof. The closedness of $N_G(E)$ and $Z_G(E)$ follows from the continuity of the adjoint representation. Let $x \in \mathbf{L}(N_G(E)) = \{y \in \mathfrak{g}\colon \exp_G(\mathbb{R}y) \subseteq N_G(E)\}$ and $z \in E$. Then

$$[x,z] = \frac{d}{dt}\Big|_{t=0} e^{\operatorname{ad} tx} z \in E,$$

and therefore $x \in \mathfrak{n}_{\mathfrak{g}}(E)$. Conversely, if $x \in \mathfrak{n}_{\mathfrak{g}}(E)$, then we inductively get $(\operatorname{ad} x)^n E \subseteq E$, and therefore

$$\operatorname{Ad}(\exp_G tx)E = e^{\operatorname{ad} tx} E \subseteq E \quad \text{ for } t \in \mathbb{R}.$$

Replacing t by $-t$, we see that $\operatorname{Ad}(\exp_G tx) \in N_G(E)$, so that $x \in \mathbf{L}(N_G(E))$. The arguments for the centralizer are similar. □

Proposition 10.1.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H, N integral subgroups of G with the Lie algebras $\mathfrak{h}$, resp., $\mathfrak{n}$. Then the following are equivalent*

(i) $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$, *i.e., $\mathfrak{h}$ normalizes $\mathfrak{n}$.*
(ii) $e^{\operatorname{ad} x}\mathfrak{n} \subseteq \mathfrak{n}$ *for all $x \in \mathfrak{h}$.*
(iii) $H \subseteq N_G(\mathfrak{n})$, *i.e.,* $\operatorname{Ad}(H)\mathfrak{n} = \mathfrak{n}$.
(iv) $H \subseteq N_G(N) = \{g \in G \colon gNg^{-1} = N\}$, *i.e., H normalizes N.*

Proof. The equivalence of (i) and (iii) follows immediately from Lemma 10.1.1, applied to $E = \mathfrak{n}$. Further, (iii) implies (ii) because $\operatorname{Ad}(\exp_G x) = e^{\operatorname{ad} x}$. Since each $e^{\operatorname{ad} x}$ is a linear isomorphism of $\mathfrak{g}$ with $(e^{\operatorname{ad} x})^{-1} = e^{-\operatorname{ad} x}$, (ii) implies $e^{\operatorname{ad} x}\mathfrak{n} = \mathfrak{n}$ for each $x \in \mathfrak{h}$, and since H is generated by $\exp_G \mathfrak{h}$, we also see that (ii) implies (iii).
(iii) ⇒ (iv): For $h \in H$, we obtain with (iii) that

$$c_h(N) = \langle c_h(\exp_G \mathfrak{n})\rangle = \langle \exp_G(\operatorname{Ad}(h)\mathfrak{n})\rangle = \langle \exp_G \mathfrak{n}\rangle = N.$$

(iv) ⇒ (iii): From the Integral Subgroup Theorem 8.4.8(c), we recall that

$$\mathfrak{n} = \{x \in \mathfrak{g} \colon \exp_G(\mathbb{R}x) \subseteq N\}.$$

For $h \in H$ and $x \in \mathfrak{g}$, we have $c_h(\exp_G(\mathbb{R}x)) = \exp_G(\mathbb{R}\operatorname{Ad}(h)x)$, so that (iv) entails $\operatorname{Ad}(h)\mathfrak{n} = \mathfrak{n}$. □

Specializing the preceding proposition to $\mathfrak{h} = \mathfrak{g}$, i.e., to $H = G_0$, we obtain:

Corollary 10.1.3. *For a Lie group G with Lie algebra $\mathfrak{g}$ and the integral subgroup N with Lie algebra $\mathfrak{n}$, the following are equivalent*

(i) $\mathfrak{n}$ *is an ideal of $\mathfrak{g}$.*
(ii) $e^{\operatorname{ad} x}\mathfrak{n} \subseteq \mathfrak{n}$ *for all $x \in \mathfrak{g}$.*
(iii) $\operatorname{Ad}(G_0)\mathfrak{n} = \mathfrak{n}$.
(iv) *N is a normal subgroup of G_0.*

The following proposition is proved along the same lines as Proposition 10.1.2.

Proposition 10.1.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H, N integral subgroups of G with the Lie algebras $\mathfrak{h}$, resp., $\mathfrak{n}$. Then the following are equivalent*

(i) $[\mathfrak{h}, \mathfrak{n}] = \{0\}$, *i.e.,* $\mathfrak{h}$ *centralizes* $\mathfrak{n}$.
(ii) $e^{\operatorname{ad} x} y = y$ *for* $x \in \mathfrak{h}$ *and* $y \in \mathfrak{n}$.
(iii) $H \subseteq Z_G(\mathfrak{n})$, *i.e.,* $\operatorname{Ad}(H)|_{\mathfrak{n}} = \{\operatorname{id}_{\mathfrak{n}}\}$.
(iv) $H \subseteq Z_G(N)$, *i.e., the subgroups* H *and* N *commute.*

10.1.2 Quotients of Lie Groups and Canonical Factorization

In this subsection we turn our attention to quotients of Lie grous by closed normal subgroups.

Theorem 10.1.5 (Quotient Theorem). *If N is a closed normal subgroup of the Lie group G, then the quotient group G/N carries a unique Lie group structure for which the quotient homomorphism $q\colon G \to G/N, g \mapsto gN$, is a submersion. In particular, $\mathbf{L}(q)\colon \mathbf{L}(G) \to \mathbf{L}(G/N)$ is a surjective morphism of Lie algebras with kernel $\mathbf{L}(N)$, so that*

$$\mathbf{L}(G/N) \cong \mathbf{L}(G)/\mathbf{L}(N).$$

Proof. Theorem 9.1.10 provides the manifold structure on G/N for which q is a submersion. Let $m_{G/N}$ denote the multiplication map on G/N. Since

$$q \times q\colon G \times G \to G/N \times G/N$$

also is a submersion, the smoothness of $m_{G/N}$ follows from the smoothness of

$$m_{G/N} \circ (q \times q) = q \circ m_G\colon G \times G \to G/N$$

(Proposition 7.3.16). Likewise, the smoothness of the inversion $\iota_{G/N}$ follows from the smoothness of $\iota_{G/N} \circ q = q \circ \iota_G$. □

Corollary 10.1.6 (Canonical Factorization). *Let $\varphi\colon G \to H$ be a morphism of Lie groups and endow $G/\ker\varphi$ with its natural Lie group structure. Then φ factors through a smooth injective morphism of Lie groups $\overline{\varphi}\colon G/\ker\varphi \to H, g\ker\varphi \mapsto \varphi(g)$.*

If, in addition, $\mathbf{L}(\varphi)$ is surjective, then $\overline{\varphi}$ is a diffeomorphism onto an open subgroup of H.

Proof. First we note that the smoothness of φ implies that $\ker\varphi$ is a closed normal subgroup. Therefore the Quotient Theorem 10.1.5 provides a Lie group structure on the quotient $G/\ker\varphi$, for which the quotient map $q\colon G \to G/\ker\varphi$ is a submersive morphism of Lie groups. The existence of the map $\overline{\varphi}$ is clear. It is also easy to see that it is a group homomorphism. Since q is a submersion, the smoothness of $\overline{\varphi}$ follows from the smoothness of $\overline{\varphi} \circ q = \varphi$.

If, in addition, $\mathbf{L}(\varphi)$ is surjective, then $\overline{\varphi}$ is a morphism of Lie groups whose differential is an isomorphism. Since it is also injective, the Inverse Function Theorem implies that it is a diffeomorphism onto an open subgroup of H. □

Remark 10.1.7. The preceding corollary shows how a morphism $\varphi\colon G \to H$ of Lie groups induces an injective morphism $\overline{\varphi}\colon G/\ker\varphi \to H$. We further know from Theorem 8.6.13 that the subgroup $\varphi(G)$ of H carries an initial Lie subgroup structure, so that the corestriction $\varphi\colon G \to \varphi(G)$ also is a morphism of Lie groups. Accordingly, the corresponding morphism

$$\overline{\varphi}\colon G/\ker\varphi \to \varphi(G)$$

is a smooth bijection of Lie groups.

As we know from the dense wind in the 2-torus (Example 8.3.12), inclusions of initial Lie subgroups need not be topological embeddings. Moreover, smooth bijections of Lie groups need not be isomorphisms because we obtain on any Lie group G another Lie group structure by endowing G with the discrete topology. Writing G_d for this new 0-dimensional Lie group, the identity map $G_d \to G$ is smooth, but not an isomorphism of Lie groups if $\dim G > 0$. If $\dim G > 0$, then G is an uncountable set, so that G_d is neither second countable nor a countable union of compact subsets.

Theorem 10.1.8 (Open Mapping Theorem). *If $\varphi\colon G \to H$ is a surjective morphism of Lie groups and $\pi_0(G) := G/G_0$ is countable, then $\mathbf{L}(\varphi)$ is surjective and the induced bijective morphism $\overline{\varphi}\colon G/\ker\varphi \to H$ is an isomorphism of Lie groups.*

Proof. First we note that

$$\varphi(G_0) = \varphi(\langle \exp_G \mathbf{L}(G)\rangle) = \langle \exp_H \big(\mathbf{L}(\varphi)\,\mathbf{L}(G)\big)\rangle = \langle \exp_H(\operatorname{im} \mathbf{L}(\varphi))\rangle$$

is an integral subgroup of H and $H/\varphi(G_0) = \varphi(G)/\varphi(G_0)$ is countable, so that Part (c) of the Integral Subgroup Theorem 8.4.8 implies that

$$\operatorname{im}(\mathbf{L}(\varphi)) = \{x \in \mathbf{L}(H)\colon \exp_H(\mathbb{R}x) \subseteq H\} = \mathbf{L}(H).$$

Therefore $\mathbf{L}(\varphi)$ is surjective, and thus $\mathbf{L}(\overline{\varphi})$ is bijective. Since $\overline{\varphi}$ is a group homomorphism $\overline{\varphi} \circ \lambda_g = \lambda_{\overline{\varphi}(g)} \circ \overline{\varphi}$ yields

$$T_g(\overline{\varphi}) \circ T_{\mathbf{1}}(\lambda_g) = T_{\mathbf{1}}(\lambda_{\overline{\varphi}(g)}) \circ \mathbf{L}(\overline{\varphi}), \tag{10.1}$$

so that $T_g(\overline{\varphi})$ is bijective for every $g \in G/\ker\varphi$. We conclude that $\overline{\varphi}$ is a diffeomorphism, hence an isomorphism of Lie groups. □

Remark 10.1.9. Alternatively, one can also use Sard's Theorem to prove the preceding proposition. Indeed, if $\mathbf{L}(\varphi)$ is not surjective, then (10.1), applied to φ instead of $\overline{\varphi}$, implies that $T_g(\varphi)$ is not surjective for any $g \in G$. If φ is surjective and G is second countable, this contradicts Sard's Theorem 7.6.11.

Proposition 10.1.10. *Let $\varphi\colon G_1 \to G_2$ be a quotient morphism of Lie groups and $H_2 \subseteq G_2$ be a closed subgroup. Then $H_1 := \varphi^{-1}(H_2)$ is a closed subgroup of G_1, and the induced map*

$$\psi\colon G_1/H_1 \mapsto G_2/H_2, \quad g \mapsto \varphi(g)H_2$$

is a diffeomorphism.

Proof. Let $q_i\colon G_i \to G_i/H_i$ denote the quotient map and recall that it is a submersion. Clearly, ψ is well-defined, and since q_1 is a submersion, its smoothness follows from the smoothness of $\psi \circ q_1 = q_2 \circ \varphi$. It is also clear that ψ is a bijection. Further, ψ satisfies the equivariance condition $\psi(gxH_1) = \varphi(g)\psi(xH_1)$, which implies that the tangent maps $T_p(\psi)$ all have the same rank because G_i acts on G_i/H_i by diffeomorphisms. Identifying $T_{\mathbf{1}H_i}(G_i/H_i)$ with $\mathbf{L}(G_i)/\mathbf{L}(H_i)$, it follows that the tangent map $T_{\mathbf{1}H_1}(\psi)$ corresponds to the projection

$$\mathbf{L}(G_1)/\mathbf{L}(H_1) \to \mathbf{L}(G_2)/\mathbf{L}(H_2), \quad x \mapsto \mathbf{L}(\varphi)x + \mathbf{L}(H_2),$$

and since $\mathbf{L}(H_1) = \mathbf{L}(\varphi)^{-1}(\mathbf{L}(H_2))$, it is a linear isomorphism. Now the Inverse Function Theorem implies that ψ is a diffeomorphism. □

10.1.3 Fundamental Groups of Quotients and Homogeneous Spaces

In this subsection we develop some tools that are needed to calculate fundamental groups of Lie groups and to relate them to fundamental groups of homogeneous spaces.

Lemma 10.1.11. *Let H be a closed subgroup of the Lie group G and H_0 its identity component. Then any subgroup $H_1 \subseteq H$ containing H_0 is an open closed subgroup of H, and the map*

$$\alpha\colon G/H_1 \to G/H, \qquad xH_1 \mapsto xH$$

is a covering. The group H acts transitively on the fiber $\alpha^{-1}(H)$, which leads to an identification with H/H_1.

Proof. Since H is a Lie group by the Closed Subgroup Theorem 8.3.7, H_0 is open in H and any subgroup H_1 of H containing H_0 is open and closed (Proposition 8.1.15(iii)). We write $\pi^1\colon G \to G/H_1$ and $\pi\colon G \to G/H$ for the quotient maps which are both submersions. Since π^1 is a submersion, the smoothness of α follows from the smoothness of $\alpha \circ \pi^1 = \pi$ (Theorem 9.1.10).

In view of the Closed Subgroup Theorem 8.3.7(iii), for any vector space complement $E \subseteq \mathbf{L}(G)$ of $\mathbf{L}(H)$, there exists an open 0-neighborhood $V_E \subseteq E$ such that

$$\varphi\colon V_E \times H \to \exp_G(V_E)H, \quad (x,h) \mapsto \exp_G(x)h$$

is a diffeomorphism onto an open subset $U := \exp_G(V_E)H$ of G. In these product coordinates, the map $\alpha|_{UH_1} : U/H_1 \to U/H$ is equivalent to the projection map

$$V_E \times H/H_1 \to V_E,$$

which clearly is a covering map because H/H_1 is a discrete space.

If gH is an arbitrary point in G/H, gU/H is an open neighborhood of gH and

$$\alpha^{-1}(gU/H) = gU/H_1 \cong g \exp_G(V_E)H/H_1$$

is a disjoint union of the open subsets $g \exp_G(V_E)hH_1$, where the h are taken from a set of representatives for the quotient space H/H_1. Since α is G-equivariant, its restriction to all these open subsets is a diffeomorphism onto gU/H. This proves that α is a covering.

The inverse image of $H \in G/H$ under α is the subset H/H_1 of G/H_1, on which H obviously acts transitively. □

Remark 10.1.12. Since, in this book, we do not want to go so far into homotopy theory as to derive the long exact homotopy sequence of fiber bundles, we will systematically look for the information on the fundamental groups which we are able to obtain directly. We consider the following situation:

Let H be a closed subgroup of the connected Lie group G and $i \colon H \to G$ be the inclusion. We write $q_G \colon \widetilde{G} \to G$ and $q_H \colon \widetilde{H}_0 \to H_0$ for the respective simply connected covering groups. Via $\mathbf{L}(i) \colon \mathbf{L}(H) \to \mathbf{L}(G)$, this map induces a homomorphism $\widetilde{i} : \widetilde{H}_0 \to \widetilde{G}$ of the simply connected universal coverings, and a homomorphism of the homotopy groups $\pi_1(i) \colon \pi_1(H) \to \pi_1(G)$ which we may consider as subgroups of the respective coverings by Theorem 8.5.4. If $q \colon G \to G/H$ is the projection onto the left-cosets, q also induces a homomorphism $\pi_1(q) \colon \pi_1(G) \to \pi_1(G/H)$. The map $q \circ q_G \colon \widetilde{G} \to G/H$ factors to a covering $q_{G/H} \colon \widetilde{G}/H_1 \to G/H \cong \widetilde{G}/q_G^{-1}(H)$ (Proposition 10.1.10), where $H_1 := \widetilde{i}(\widetilde{H}_0) = q_G^{-1}(H)_0$ (Lemma 10.1.11). Here we use that, since H is closed, the subgroup $q_G^{-1}(H)$ and its identity component H_1 are also closed, and since they are Lie groups, H_1 is generated by the exponential image of $\mathbf{L}(H_1)$, hence equal to $\widetilde{i}(\widetilde{H}_0)$. By Theorem 8.5.4, the kernel of $\widetilde{i}$ can be identified with $\pi_1(H_1)$. We write $\widetilde{q} \colon \widetilde{G} \to \widetilde{G}/H_1$ for the corresponding quotient map and collect all that information in the following diagram.

$$\begin{array}{ccccc}
\pi_1(H) & \xrightarrow{\pi_1(i)} & \pi_1(G) & \xrightarrow{\pi_1(q)} & \pi_1(G/H) \\
\downarrow & & \downarrow & & \\
\widetilde{H}_0 & \xrightarrow{\widetilde{i}} & \widetilde{G} & \xrightarrow{\widetilde{q}} & \widetilde{G}/H_1 \\
\downarrow{\scriptstyle q_H} & & \downarrow{\scriptstyle q_G} & & \downarrow{\scriptstyle q_{G/H}} \\
H_0 & \xrightarrow{i} & G & \xrightarrow{q} & G/H_0
\end{array}$$

The commutativity of the diagram is clear if one recalls that the homomorphism $\pi_1(i)$ is obtained by restriction of $\widetilde{i}$ to the subgroup $\ker q_H \cong \pi_1(H)$

of $\widetilde{H}_0$. The group $\widetilde{H}_0$ is simply connected, so that $\widetilde{i}\colon \widetilde{H}_0 \to H_1$ is the universal covering of H_1 because $\mathbf{L}(\widetilde{i})\colon \mathbf{L}(\widetilde{H}_0) \to \mathbf{L}(H_1)$ is an isomorphism of Lie algebras.

Lemma 10.1.13. *If the group H is connected, the natural homomorphism $\pi_1(q)\colon \pi_1(G) \to \pi_1(G/H)$ is surjective*

Proof. Let $\gamma\colon [0,1] \to G/H$ be a continuous path starting in H. Since the quotient map $q\colon G \to G/H$ is a submersion, there exists an open neighborhood U of H in G/H and a smooth map $\sigma\colon U \to G$ with $\sigma(H) = \mathbf{1}$ and $q \circ \sigma = \mathrm{id}_U$.

To construct a continuous lift $\widetilde{\gamma}\colon [0,1] \to G$ of γ, starting in $\mathbf{1}$, we cover the compact subset $\gamma([0,1])$ by finitely many open sets of the form gU, $g \in G$, and use the Lebesgue Lemma A.2.3, applied to the open cover $\big(\gamma^{-1}(gU)\big)_{g\in G}$ of $[0,1]$, to find a subdivision

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = 1,$$

such that $\gamma([t_i, t_{i+1}]) \subseteq g_i U$ for some $g_i \in G$. We define $\widetilde{\gamma}$ on $[t_0, t_1]$ by $\widetilde{\gamma}(t) := \sigma\big(\gamma(t)\big)$. Then we proceed by induction. If $\widetilde{\gamma}$ is already defined in $[t_0, t_k]$ and $k < n$, then we define $\widetilde{\gamma}$ on $[t_k, t_{k+1}]$ by

$$\widetilde{\gamma}(t) := g_k \cdot \sigma(g_k^{-1}\gamma(t)) \cdot \sigma(g_k^{-1}\gamma(t_k))^{-1} g_k^{-1}\widetilde{\gamma}(t_k).$$

This defines a continuous extension of $\widetilde{\gamma}$ to $[t_0, t_{k+1}]$, and that it actually defines a lift of γ follows from

$$\sigma(g_k^{-1}\gamma(t_k))^{-1} g_k^{-1}\widetilde{\gamma}(t_k) \in H,$$

which in turn follows from

$$q(g_k\sigma(g_k^{-1}\gamma(t_k))) = g_k.(g_k^{-1}\gamma(t_k)) = \gamma(t_k) = q(\widetilde{\gamma}(t_k)).$$

We thus obtain a continuous lift $\widetilde{\gamma}\colon [0,1] \to G$ of γ.

If γ is a loop, i.e., $\gamma(1) = \gamma(0) = H$, then $\widetilde{\gamma}(1) \in q^{-1}(H) = H$. Since H is connected, there exists a path $\beta\colon [0,1] \to H$ with $\beta(0) = \widetilde{\gamma}(1)$ and $\beta(1) = \mathbf{1}$. Therefore, with the notation of Definition A.1.1 for concatenation of paths, $\widetilde{\gamma} * \beta$ is a loop in G with

$$\pi_1(q)[\widetilde{\gamma} * \beta] = [\gamma * (q \circ \beta)] = [\gamma * H] = [\gamma].$$

Hence, $\pi_1(q)$ is surjective. □

Corollary 10.1.14. *If H is a closed subgroup of a 1-connected Lie group G, then*

$$\pi_1(G/H) \cong \pi_0(H) := H/H_0.$$

In particular, H is connected if and only if G/H is simply connected.

Proof. Lemma 10.1.13 implies that G/H_0 is simply connected, and

$$q_{G/H} \colon G/H_0 \to G/H, \quad xH_0 \mapsto xH$$

is a simply connected covering of G/H (Lemma 10.1.11), where $q_{G/H}^{-1}(H) \cong H/H_0$. Therefore the group $\pi_0(H) = H/H_0$ acts by deck transformations from the right on G/H_0 via

$$G/H_0 \times \pi_0(H) \to G/H_0, \quad (gH_0, hH_0) \mapsto gH_0hH_0 = ghH_0,$$

and since it acts simply transitively on the fiber over H_0, it follows from Proposition A.2.15 that $\pi_0(H) \cong \mathrm{Deck}(G/H_0, q_{G/H}) \cong \pi_1(G/H)$. □

In view of $G/H \cong \widetilde{G}/q_G^{-1}(H)$ (Proposition 10.1.10), we can identify the fundamental group $\pi_1(G/H)$ with $\pi_0(q_G^{-1}(H)) = q_G^{-1}(H)/H_1$ (Corollary 10.1.14 and Remark 10.1.12). We thus obtain an inclusion $j \colon \pi_1(G/H) \to \widetilde{G}/H_1$. Adding j to the above diagram, we obtain:

Theorem 10.1.15 (Homotopy Group Theorem). *For a closed connected subgroup H of the connected Lie group G, the following diagram is commutative. In addition, all arrows pointing away from the boundary are injective, and all arrows pointing to the boundary are surjective. In all other positions, the image of the ingoing arrow coincides with the kernel of the outgoing arrow, resp., the inverse image of the canonical base point. In this sense, the diagram is "exact".*

$$\begin{array}{ccccccc}
\pi_1(H_1) & \xrightarrow{\pi_1(q_G|_{H_1})} & \pi_1(H) & \xrightarrow{\pi_1(i)} & \pi_1(G) & \xrightarrow{\pi_1(q)} & \pi_1(G/H) \\
\downarrow{\cong} & & \downarrow & & \downarrow & & \downarrow{j} \\
\pi_1(H_1) & \longrightarrow & \widetilde{H} & \xrightarrow{\widetilde{i}} & \widetilde{G} & \xrightarrow{\widetilde{q}} & \widetilde{G}/H_1 \\
 & & \downarrow{q_H} & & \downarrow{q_G} & & \downarrow{q_{G/H}} \\
 & & H & \xrightarrow{i} & G & \xrightarrow{q} & G/H
\end{array}$$

Proof. Since $q_G|_{H_1} \colon H_1 \to H$ is a covering, $\pi_1(H_1)$ can be identified with a subgroup of $\pi_1(H)$ (Corollary A.2.7). Thus, the injectivity of the arrows pointing away from the boundary is clear. The surjectivity statement follows from Lemma 10.1.13. The exactness of the columns is clear, as well as the exactness of the two bottom rows, since $\widetilde{i} \colon \widetilde{H} \to H_1$ is the universal covering of H_1. Because of

$$\mathrm{im}\,\big(\pi_1(q_G|_{H_1})\big) = \pi_1(H_1) = \ker \widetilde{i} = \ker\big(\pi_1(i)\big),$$

(cf. Corollary A.2.7 for the first equality) the upper row is exact at $\pi_1(H)$. We further have

$$\mathrm{im}\,\big(\pi_1(i)\big) = H_1 \cap \pi_1(G) = \ker\big(\pi_1(q)\big)$$

since $j \circ \pi_1(q) = \widetilde{q}|_{\pi_1(G)}$. This proves the exactness at $\pi_1(G)$. □

Remark 10.1.16. For every connected Lie group, one can show that the second homotopy group $\pi_2(G)$ vanishes. In view of $\pi_2(\widetilde{G}) \cong \pi_2(G)$, the Manifold Splitting Theorem 13.3.8, and the Structure Theorem for Groups with Compact Lie Algebra 11.1.18, it suffices to verify this for compact simple Lie groups. The long exact homotopy sequence for the fiber bundle $G \to G/H$ thus leads to an exact sequence

$$\{\mathbf{1}\} = \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) = \{\mathbf{1}\}$$

if H is connected, i.e., the group which appears above as $\pi_1(H_1)$ is isomorphic to $\pi_2(G/H)$. If H is a normal subgroup of G, then G/H is a Lie group (Theorem 10.1.5). Then $\pi_2(G/H) \cong \pi_1(H_1) = \{\mathbf{1}\}$, and therefore, $\pi_1(G/H) \cong \pi_1(G)/\pi_1(H)$. But the triviality of $\pi_1(H_1)$ also is a direct consequence of Theorem 10.1.21 below. Thus the Homotopy Group Theorem 10.1.15 implies that

$$\mathbf{1} \xrightarrow{\pi_1(q_G|_{H_1})} \pi_1(H) \xrightarrow{\pi_1(i)} \pi_1(G) \xrightarrow{\pi_1(q)} \pi_1(G/H)$$

is exact for H normal in G.

Remark 10.1.17. If the subgroup H is not connected, then we still have

$$G/H \cong \widetilde{G}/q_G^{-1}(H),$$

so that q_G induces a map

$$\pi_1(G/H) \cong \pi_0(q_G^{-1}(H)) \xrightarrow{\quad \overline{q_G} \quad} \pi_0(H)$$

which obviously is surjective. Further,

$$\ker \overline{q_G} = q_G^{-1}(H_0)/q_G^{-1}(H)_0 = (H_1 \cdot \ker q_G)/H_1.$$

Identifying $\ker q_G$ with $\pi_1(G)$, we see that the right hand side coincides with the image of the natural homomorphism $\pi_1(G) \to \pi_1(G/H)$, so that we obtain an exact sequence

$$\pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \mathbf{1}.$$

10.1.4 Semidirect Products and Smooth Splitting of Extensions

An *extension of Lie groups* is a short exact sequence

$$\mathbf{1} \to N \xrightarrow{\iota} G \xrightarrow{q} H \to \mathbf{1}$$

of Lie group morphisms for which $\iota \colon N \to \ker q$ is an isomorphism of Lie groups. A particularly important class of extensions of a Lie group H by a Lie group N are the semidirect products $N \rtimes_\alpha H$. We start this section with a criterion for a Lie group to decompose as a semidirect product and we also prove the important Splitting Theorem 10.1.21 on quotients of simply connected groups by normal integral subgroups.

Proposition 10.1.18. *Let G be a Lie group and N and H be closed subgroups with $H \subseteq N_G(N)$. Then $\alpha(h)(n) := hnh^{-1}$ defines a smooth action of H on N, so that we can form the semidirect product group $N \rtimes_\alpha H$. Further, the multiplication map*

$$\mu\colon N \rtimes_\alpha H \to G, \quad (n, h) \mapsto nh$$

is a morphism of Lie groups whose image is the subgroup NH. The map μ is bijective if and only if

$$N \cap H = \{\mathbf{1}\} \quad \textit{and} \quad NH = G.$$

If, in addition, $\pi_0(G)$ is countable, μ is an isomorphism of Lie groups.

Proof. Since N and H are closed subgroups, the Closed Subgroup Theorem 8.3.7 implies that they are submanifolds with a natural Lie group structure. Hence the smoothness of α follows by restriction from the smoothness of the conjugation action of G on itself. It is immediate from the definition of the multiplication in the semidirect product that μ is a group homomorphism whose smoothness follows from the smoothness of the multiplication on G. In view of

$$\ker \mu \cong \{(n, n^{-1})\colon n \in H \cap N\} \cong H \cap N,$$

the injectivity of μ is equivalent to $H \cap N = \{\mathbf{1}\}$. Its surjectivity means that $NH = G$. If both these conditions are satisfies, then μ is a bijective smooth morphism of Lie groups. If, in addition, $\pi_0(G)$ is countable, then μ is an isomorphism of Lie groups by the Open Mapping Theorem 10.1.8. □

Proposition 10.1.19. *Let G, H and N be simply connected Lie groups with the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{n}$. Suppose that $\mathfrak{g} \cong \mathfrak{n} \rtimes_\beta \mathfrak{h}$ is a semidirect sum of the two subalgebras $\mathfrak{n}$ and $\mathfrak{h}$. Then there is a unique smooth action $\gamma\colon H \to \mathrm{Aut}(N) \cong \mathrm{Aut}(\mathfrak{n})$ with $\mathbf{L}(\gamma) = \beta\colon \mathfrak{h} \to \mathrm{der}\,\mathfrak{n}$ and the two natural homomorphisms $\iota_H\colon H \to G, \iota_N\colon N \to G$ combine to an isomorphism*

$$\mu\colon N \rtimes_\gamma H \to G, \quad (n, h) \mapsto \iota_N(n)\iota_H(h).$$

Proof. First we use Theorem 8.5.9 to obtain a homomorphism $\alpha\colon H \to \mathrm{Aut}(\mathfrak{n})$ integrating the homomorphism $\beta\colon \mathfrak{h} \to \mathrm{der}(\mathfrak{n})$. Since N is simply connected,

$$\mathbf{L}\colon\ \mathrm{Aut}(N) \to \mathrm{Aut}(\mathfrak{n})$$

is a bijection, which leads to a smooth action of $\mathrm{Aut}(N) \cong \mathrm{Aut}(\mathfrak{n})$ on N (Corollary 8.5.11). Composing with α, we obtain a smooth action of H on N, defined by a homomorphism $\gamma\colon H \to \mathrm{Aut}(N)$ with $\mathbf{L} \circ \gamma = \alpha$. We now form the simply connected group $N \rtimes_\gamma H$ whose Lie algebra $\mathfrak{n} \rtimes_\beta \mathfrak{h}$ is isomorphic to $\mathfrak{g}$ (Proposition 8.2.25). Hence the isomorphism $s\colon \mathfrak{n} \rtimes_\beta \mathfrak{h} \to \mathfrak{g}, (x, y) \mapsto x + y$ integrates to an isomorphism of Lie groups $\mu\colon N \rtimes_\gamma H \to G$. □

Corollary 10.1.20. *Let G be a Lie group and $\mathfrak{h}, \mathfrak{n} \subseteq \mathfrak{g} = \mathbf{L}(G)$ be subalgebras with $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$. If H and N are the corresponding integral subgroups of G, then HN is the integral subgroup corresponding to the Lie subalgebra $\mathfrak{n} + \mathfrak{h}$ of $\mathfrak{g}$.*

Proof. We define $\beta\colon \mathfrak{h} \to \operatorname{der}(\mathfrak{n})$ by $\beta(x)(y) = [x, y]$. Then Proposition 10.1.19 implies that the simply connected covering group $\widetilde{H}$ of H acts smoothly on the simply connected covering group $\widetilde{N}$ of N, so that we obtain a Lie group $G_1 := \widetilde{N} \rtimes_\gamma \widetilde{H}$ with $\mathbf{L}(G_1) \cong \mathfrak{n} \rtimes_\beta \mathfrak{h}$. The summation map

$$s\colon \mathfrak{n} \rtimes \mathfrak{h} \to \mathfrak{g}, \quad (n, h) \mapsto n + h$$

is a morphism of Lie algebras whose image is $\mathfrak{n} + \mathfrak{h}$. Since G_1 is simply connected, s integrates to a smooth group homomorphism $S\colon G_1 \to G$, whose image is

$$\langle \exp_G(\mathfrak{n} + \mathfrak{h}) \rangle = \langle \exp_G(s(\mathfrak{g}_1)) \rangle = S(G_1) = S(\widetilde{N}) S(\widetilde{H}) = NH. \qquad \square$$

Theorem 10.1.21 (Smooth Splitting Theorem). *Let G be a simply connected Lie group and $N \subseteq G$ be a normal integral subgroup. Then N is closed and there exists a smooth section $\sigma\colon G/N \to G$, so that the map*

$$G/N \times N \to G, \quad (p, n) \mapsto \sigma(p) n$$

is a diffeomorphism, but in general not an isomorphism of Lie groups. In particular, the groups N and G/N are simply connected.

Proof. By Proposition 10.1.2, $\mathbf{L}(N)$ is an ideal in $\mathbf{L}(G)$. According to Remark 4.6.12, applied to the quotient Lie algebra $\mathbf{L}(G)/\mathbf{L}(N)$, there exists an increasing sequence of subalgebras

$$\mathbf{L}(N) = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \ldots \subseteq \mathfrak{g}_n = \mathbf{L}(G)$$

such that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i+1}$, and the quotients $\mathfrak{q}_i := \mathfrak{g}_i/\mathfrak{g}_{i-1}$ are either isomorphic to $\mathbb{R}$ or simple. Using Levi's Theorem 4.6.6, we conclude that

$$\mathfrak{g}_i \cong \mathfrak{g}_{i-1} \rtimes \mathfrak{q}_i \quad \text{for} \quad i = 1, \ldots, n.$$

Now we can inductively apply Proposition 10.1.19 to see that

$$G \cong \Big(\Big(\ldots\Big(\big(G(\mathfrak{g}_0) \rtimes G(\mathfrak{q}_1)\big) \rtimes G(\mathfrak{q}_2)\Big) \ldots\Big) \rtimes G(\mathfrak{q}_n)\Big).$$

This implies in particular that $N = \langle \exp_G \mathbf{L}(N) \rangle \cong G(\mathfrak{g}_0)$ is a closed simply connected subgroup of G. In particular, we obtain diffeomorphisms

$$G \to N \times G(\mathfrak{q}_1) \times \ldots \times G(\mathfrak{q}_n) \quad \text{and} \quad G/N \to G(\mathfrak{q}_1) \times \ldots \times G(\mathfrak{q}_n).$$

Hence the normal subgroup N is closed and there exists a smooth section $\sigma\colon G/N \to G$. Finally, the existence of a diffeomorphism $G/N \times N \to G$ implies that N and G/N are connected and simply connected (cf. Remark A.1.7(b)). $\square$

In the following remark we briefly discuss general extensions of Lie groups.

Remark 10.1.22. (a) Let $q\colon \widehat{G} \to G$ be a surjective morphism of Lie groups with kernel N and $\sigma\colon G \to \widehat{G}$ a map with $q \circ \sigma = \mathrm{id}_G$. Then the map

$$\mu\colon N \times G \to \widehat{G}, \quad (n,g) \mapsto n\sigma(g)$$

is a bijection. To express the group structure on $\widehat{G}$ in the corresponding "product coordinates", we define the maps

$$S\colon G \to \mathrm{Aut}(N), \quad S(g)(n) := \sigma(g)n\sigma(g)^{-1} = c_{\sigma(g)}|_N$$

and

$$\omega\colon G \times G \to N, \quad \omega(g,g') := \sigma(g)\sigma(g')\sigma(gg')^{-1}.$$

Then

$$n\sigma(g)n'\sigma(g') = nS(g)(n')\omega(g,g')\sigma(gg')$$

implies that μ is an isomorphism of groups if we define the multiplication on $N \times G$ by

$$(n,g)(n',g') := (nS(g)(n')\omega(g,g'), gg'). \tag{10.2}$$

(b) According to the Smooth Splitting Theorem 10.1.21, there always exists a smooth splitting map σ, provided the group $\widehat{G}$ is simply connected. Then S and ω are also smooth, and (10.2) defines a smooth group structure on the product manifold $N \times G$.

(c) In general, we can use the fact that q is a submersion to find a smooth section $\sigma\colon U \to \widehat{G}$ on some open identity neighborhood U in G. Then we may extend σ to a not necessarily continuous section $\sigma\colon G \to \widehat{G}$ by choosing for $g \notin U$ an arbitrary element $\sigma(g) \in q^{-1}(g)$. In this case the maps S and ω are still smooth on a neighborhood of $\mathbf{1}$, resp., $(\mathbf{1},\mathbf{1})$.

(d) If, conversely, we start with two Lie groups G and N, we may ask under which conditions two maps

$$S\colon G \to \mathrm{Aut}(N) \quad \text{with} \quad S(\mathbf{1}) = \mathrm{id}_N$$

and

$$\omega\colon G \times G \to N \quad \text{with} \quad \omega(g,\mathbf{1}) = \omega(\mathbf{1},g) = \mathbf{1} \quad \text{for} \quad g \in G,$$

define via (10.2) a group structure on the product set $N \times G$. The associativity condition is easily seen to be equivalent to the two conditions

$$S(g)S(g') = c_{\omega(g,g')} \circ S(gg') \quad \text{for} \quad g,g' \in G \tag{10.3}$$

and

$$S(g)(\omega(g',g''))\omega(g,g'g'') = \omega(g,g')\omega(gg',g'') \quad \text{for} \quad g,g',g'' \in G. \tag{10.4}$$

If these two conditions are satisfied, then we obtain on $N \times G$ an associative multiplication for which $(\mathbf{1}, \mathbf{1})$ is an identity element. Moreover, each element (n, g) is invertible with

$$(n,g)^{-1} = (S(g)^{-1}\big(n^{-1}\omega(g,g^{-1})^{-1}\big), g^{-1}) = (\omega(g^{-1},g)^{-1}S(g^{-1})(n^{-1}), g^{-1}),$$

so that we actually obtain a group. It is denoted by $N \times_{(S,\omega)} G$. If S and ω are smooth, then we thus obtain a Lie group structure on the product manifold $N \times G$.

If S and ω are only smooth on an open symmetric neighborhood U_G of $\mathbf{1}$, resp., $U_G \times U_G$ of $(\mathbf{1}, \mathbf{1})$, then the subset $U := N \times U_G$ of $\widehat{G} := N \times_{(S,\omega)} G$ is symmetric and carries a natural product manifold structure and on the set $D := \{(x, y) \in U \times U : xy \in U\}$, the group multiplication and the inversion are smooth.

For $(n, g) \in \widehat{G}$ let $V_g \subseteq U_G$ be an open identity neighborhood with $c_g(V_g) \subseteq U_G$. Then $c_{(n,g)}(q^{-1}(V_g)) \subseteq U$. That the conjugation map

$$c_{(n,g)} : q^{-1}(V_g) \to U$$

is smooth in an identity neighborhood follows immediately from

$$\begin{aligned}&(n,g)(n',g')(n,g)^{-1}\\ &= \Big(nS(g)(n')\omega(g,g')\omega(gg'g^{-1},g)^{-1}S(gg'g^{-1})^{-1}(n^{-1}), gg'g^{-1}\Big).\end{aligned}$$

Therefore Theorem 8.4.4 implies that there exists a unique Lie group structure on $\widehat{G}$ for which the inclusion of U is a diffeomorphism onto an open subset. For this Lie group structure, the map

$$q : \widehat{G} \to G, \quad (n, g) \mapsto g$$

clearly is a surjective morphism of Lie groups with kernel N, i.e., an extension of G by N.

(e) In general, the quotient map $q : \widehat{G} \to G$ does not have a smooth section. Typical examples arise if N is discrete, so that q is a covering morphism, the simplest example being the squaring map

$$q : \mathbb{T} \to \mathbb{T}, \quad z \mapsto z^2$$

with kernel $N = \{\pm 1\}$. The nonexistence of a smooth section follows from

$$\pi_1(q) : \pi_1(\mathbb{T}) \cong \mathbb{Z} \to \pi_1(\mathbb{T}) \cong \mathbb{Z}$$

being multiplication with 2, which is not surjective.

10.2 Commutators, Nilpotent and Solvable Lie Groups

In this section we study subgroups of Lie groups generated by commutators. This leads naturally to the concepts of nilpotent and solvable Lie groups. We then verify that, for connected Lie groups, these concepts are compatible with the corresponding ones on the Lie algebra level and derive detailed results on the structure of nilpotent and solvable Lie groups.

10.2.1 Subgroups Generated by Commutators

In this subsection we study how subgroups generated by commutators give rise to Lie subalgebras generated by Lie brackets.

Definition 10.2.1. Let G be a group. For two subgroups $A, B \subseteq G$, we define (A, B) as the subgroup generated by the commutators $xyx^{-1}y^{-1}$ for $x \in A$ and $y \in B$. If we set

$$C^1(G) := G \quad \text{and} \quad C^n(G) := (G, C^{n-1}(G)) \quad \text{for} \quad n > 1,$$

then $(C^n(G))_{n\in\mathbb{N}}$ is called the *lower central series* of G.

If we set $D^0(G) := G$ and $D^n(G) := (D^{n-1}(G), D^{n-1}(G))$, then the sequence $(D^n(G))_{n\in\mathbb{N}_0}$ is called the *derived series* of G.

A group G is called *nilpotent* if there is an $n \in \mathbb{N}$ with $C^n(G) = \{\mathbf{1}\}$ and it is called *solvable* if there is an $n \in \mathbb{N}_0$ with $D^n(G) = \{\mathbf{1}\}$. The subgroup $C^2(G) = D^1(G)$ is called the *commutator subgroup* of G and often denoted by G'.

Lemma 10.2.2. *If $A, B \subseteq G$ are integral subgroups with the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$, then (A, B) also is an integral subgroup and its Lie algebra contains $[\mathfrak{a}, \mathfrak{b}]$.*

Proof. Since A and B are arcwise connected, the set

$$S := \{aba^{-1}b^{-1} \colon a \in A, b \in B\}$$

is arcwise connected, as a continuous image of $A \times B$. This in turn implies that the set $T := S \cup S^{-1}$ is arcwise connected because $\mathbf{1} \in S \cap S^{-1}$. We further conclude that all cartesian products $T \times \ldots \times T$ are arcwise connected, and therefore the subgroup $(A, B) = \bigcup_{n\in\mathbb{N}} T^n$ is arcwise connected because it is an increasing union of arcwise connected subsets. Now Yamabe's Theorem 8.6.1 implies that the arcwise connected subgroup (A, B) is integral. Furthermore, the Initial Subgroup Theorem 8.6.13 provides on (A, B) the structure of an initial Lie subgroup with Lie algebra

$$\mathfrak{h} = \{x \in \mathfrak{g} \colon \exp_G(\mathbb{R}x) \subseteq (A, B)\}.$$

To see that $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{h}$, let $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. Then

$$\gamma\colon \mathbb{R} \to (A, B),$$
$$\gamma(t) := \exp_G(x)\exp_G(ty)\exp_G(-x)\exp_G(-ty) = \exp_G(x)\exp_G(-e^{t\,\mathrm{ad}\,y}x)$$

is a smooth curve in (A, B) with $\gamma(0) = \mathbf{1}$, so that Proposition 8.2.29 leads to

$$\mathfrak{h} \ni \gamma'(0) = \delta(\gamma)(0) = \delta(\exp_G)_{-x}(-[y, x]) = (e^{\mathrm{ad}\,x} - \mathbf{1})y.$$

We conclude that, for each $s \in \mathbb{R}$, $e^{s\,\mathrm{ad}\,x}y - y \in \mathfrak{h}$, and by taking the derivative at $s = 0$, we finally obtain $[x, y] \in \mathfrak{h}$. □

Proposition 10.2.3. *Let A, B and C be integral subgroups of the connected Lie group G with Lie algebras $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, satisfying*

$$[\mathfrak{a}, \mathfrak{c}] \subseteq \mathfrak{c}, \quad [\mathfrak{b}, \mathfrak{c}] \subseteq \mathfrak{c} \quad \textit{and} \quad [\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{c}.$$

Then $(A, B) \subseteq C$ and if, in addition, $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c}$, then $(A, B) = C$.

Proof. From our assumptions, it immediately follows that $\mathfrak{a} + \mathfrak{b} + \mathfrak{c}$ is a subalgebra of $\mathbf{L}(G)$, and we may assume that it coincides with $\mathbf{L}(G)$. Then $\mathfrak{c}$ is an ideal of $\mathbf{L}(G)$. First, we assume that G is simply connected. Now C is a closed subgroup of G by Theorem 10.1.21, so that G/C carries a natural Lie group structure. Let $q\colon G \to G/C$ be the quotient morphism with $\ker q = C$. The relation $\ker(\mathbf{L}(q)) = \mathfrak{c}$ yields

$$[\mathbf{L}(q)\mathfrak{a}, \mathbf{L}(q)\mathfrak{b}] \subseteq \mathbf{L}(q)[\mathfrak{a}, \mathfrak{b}] = \{0\}.$$

Therefore the two subalgebras $\mathbf{L}(q)\mathfrak{a}$ and $\mathbf{L}(q)\mathfrak{b}$ commute, which implies that the corresponding integral subgroups $q(A) = \langle \exp_{G/C} \mathbf{L}(q)\mathfrak{a} \rangle$ and $q(B) = \langle \exp_{G/C} \mathbf{L}(q)\mathfrak{b} \rangle$ commute (Proposition 10.1.4). This implies that $(A, B) \subseteq \ker q = C$.

Now let G be arbitrary, $q_G\colon \widetilde{G} \to G$ be the universal covering of G, and $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, resp., the integral subgroups of $\widetilde{G}$ corresponding to $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. By what we have just seen, $(\widetilde{A}, \widetilde{B}) \subseteq \widetilde{C}$, and this implies that

$$(A, B) = q_G\big((\widetilde{A}, \widetilde{B})\big) \subseteq q_G(\widetilde{C}) = C.$$

If, in addition, $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c}$, then Lemma 10.2.2 implies that $\mathfrak{c} = [\mathfrak{a}, \mathfrak{b}] \subseteq \mathbf{L}((A, B))$, so that the integral subgroups (A, B) and C must coincide. □

As an immediate consequence of Proposition 10.2.3, we obtain:

Proposition 10.2.4. *For any connected Lie group G with Lie algebra $\mathfrak{g}$, the groups $D^n(G)$ in the derived series and $C^n(G)$ in the lower central series are normal integral subgroups with the Lie algebras*

$$\mathbf{L}(D^n(G)) = D^n(\mathfrak{g}) \quad \textit{and} \quad \mathbf{L}(C^n(G)) = C^n(\mathfrak{g}) \quad \textit{for} \quad n \in \mathbb{N}.$$

Since an integral subgroup is trivial if and only if its Lie algebra is, we derive the following important theorem connecting nilpotency and solvability of Lie groups and Lie algebras.

Theorem 10.2.5. *A connected Lie group G is abelian, nilpotent, resp., solvable, if and only if its Lie algebra is abelian, nilpotent, resp., solvable.*

10.2.2 Nilpotent Lie Groups

In this subsection we have a closer look at the structure of nilpotent Lie groups.

Theorem 10.2.6 (Local-Global Theorem for Nilpotent Lie Groups). *If $\mathfrak{g}$ is a nilpotent Lie algebra, then the Dynkin series defines a polynomial map*

$$*\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (x, y) \mapsto x + y + \frac{1}{2}[x, y] + \ldots.$$

We thus obtain a Lie group structure $(\mathfrak{g}, *)$ with $\exp_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$ and $\mathbf{L}(\mathfrak{g}, *) = \mathfrak{g}$.

Proof. Since $\mathfrak{g}$ is nilpotent, there is an $n \in \mathbb{N}$ with $C^n(\mathfrak{g}) = \{0\}$. Hence all terms of order $\geq n$ in the Dynkin series vanish, so that only finitely many terms remain. Therefore $*\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a polynomial map. It is clear that

$$0 * x = x * 0 = x \quad \text{and} \quad x * (-x) = (-x) * x = 0 \quad \text{for} \quad x \in \mathfrak{g}.$$

Now let G be a simply connected Lie group with $\mathbf{L}(G) = \mathfrak{g}$ (Theorem 8.4.11) and $\exp_G\colon \mathfrak{g} \to G$ its exponential function. By Proposition 8.2.32, there exists an open convex 0-neighborhood $U_{\mathfrak{g}} \subseteq \mathfrak{g}$ with

$$\exp_G(x * y) = \exp_G(x) \exp_G(y) \quad \text{for} \quad x, y \in U_{\mathfrak{g}} \tag{10.5}$$

and for which $\exp_G|_{U_{\mathfrak{g}}}$ is a diffeomorphism onto an open subset of G.

Let $V \subseteq U_{\mathfrak{g}}$ be an open convex 0-neighborhood for which

$$(V * V) * V \cup V * (V * V) \subseteq U_{\mathfrak{g}}.$$

For $x, y, z \in V$ we then also have $x * y, y * z \in U_{\mathfrak{g}}$ and

$$\exp_G(x * (y * z)) = \exp_G(x) \exp_G(y) \exp_G(z) = \exp_G((x * y) * z),$$

so that the injectivity of $\exp_G$ on $U_{\mathfrak{g}}$ implies that

$$x * (y * z) = (x * y) * z \quad \text{for} \quad x, y, z \in V. \tag{10.6}$$

Since both maps

$$\mu_{1/2}\colon \mathfrak{g}^3 \to \mathfrak{g}, \quad \mu_1(x, y, z) := x * (y * z), \quad \mu_2(x, y, z) := (x * y) * z$$

are polynomial, they coincide with their Taylor series in the point $(0, 0, 0)$, which coincide because of (10.6). We conclude that $\mu_1 = \mu_2$, i.e., that $(\mathfrak{g}, *)$ is a Lie group with identity element 0 and $x^{-1} = -x$.

Identifying $T_0(\mathfrak{g})$ with $\mathfrak{g}$, we may now consider the exponential function

$$\exp_{(\mathfrak{g}, *)}\colon \mathbf{L}(\mathfrak{g}, *) \cong (\mathfrak{g}, [\cdot, \cdot]) \to (\mathfrak{g}, *).$$

The relation $tx * sx = (t+s)x$ for $t, s \in \mathbb{R}$ and $x \in \mathfrak{g}$, which is a direct consequence of the explicit form the Dynkin series, implies that

$$\exp_{(\mathfrak{g},*)}(x) = x \quad \text{for} \quad x \in \mathfrak{g},$$

so that $\exp_{(\mathfrak{g},*)} = \mathrm{id}_{\mathfrak{g}}$. Therefore the bracket in $\mathbf{L}(\mathfrak{g},*)$ can be calculated with the Commutator Formula

$$[x,y]_{\mathbf{L}(\mathfrak{g},*)} = \lim_{k\to\infty} k^2 \cdot ((x/k)*(y/k)*(-x/k)*(-y/k)) = [x,y].$$

This shows that $\mathfrak{g} \cong \mathbf{L}(\mathfrak{g},*)$. □

Corollary 10.2.7. *Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent and $\exp_G\colon (\mathfrak{g},*) \to G$ is the universal covering morphism of G. In particular, the exponential function of G is surjective.*

Proof. In view of Theorem 10.2.5, $\mathfrak{g}$ is nilpotent, so that Theorem 10.2.6 implies that $(\mathfrak{g},*)$ is a 1-connected Lie group with Lie algebra $\mathfrak{g}$. Let $q_G\colon (\mathfrak{g},*) \to G$ be the unique morphism of Lie groups with $\mathbf{L}(q_G) = \mathrm{id}_{\mathfrak{g}}$ (Proposition 8.2.10). Then

$$q_G(x) = q_G(\exp_{(\mathfrak{g},*)}(x)) = \exp_G(\mathbf{L}(q_G)x) = \exp_G(x)$$

implies that $\exp_G = q_G$. □

We know already that any connected Lie group G is isomorphic to $\widetilde{G}/\pi_1(G)$, where $\pi_1(G)$ is identified with a discrete central subgroup of the universal covering group $\widetilde{G}$ (Theorem 8.5.4). To understand the structure of connected nilpotent Lie groups, we therefore need more information on the center of the simply connected groups $(\mathfrak{g},*)$.

Lemma 10.2.8. *If $\mathfrak{g}$ is a nilpotent Lie algebra, then the center of the group $(\mathfrak{g},*)$ coincides with the center $\mathfrak{z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$.*

Proof. The inclusion $\mathfrak{z}(\mathfrak{g}) \subseteq Z(\mathfrak{g},*)$ follows immediately from the definition of $*$ in terms of the Dynkin series, which implies that $x*z = x+z$ for $z \in \mathfrak{z}(\mathfrak{g})$. If, conversely, $z \in Z(\mathfrak{g},*)$, then $\mathrm{id}_{\mathfrak{g}} = \mathrm{Ad}(\exp_{(\mathfrak{g},*)} z) = e^{\mathrm{ad}\, z}$, and thus, $\mathrm{ad}\, z = 0$ since $\mathrm{ad}\, z$ is nilpotent and the exponential function is injective on the set of nilpotent elements of $\mathrm{End}(\mathfrak{g})$ (Proposition 2.3.3). □

Proposition 10.2.9. *If G is a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, then $Z(G) = \exp_G(\mathfrak{z}(\mathfrak{g}))$ is connected.*

Proof. Since $\exp_G\colon (\mathfrak{g},*) \to G$ is the universal covering morphism of G, we obtain with Lemma 10.2.8

$$Z(G) = \ker \mathrm{Ad}_G = \exp_G(\ker \mathrm{Ad}_{(\mathfrak{g},*)}) = \exp_G(\mathfrak{z}(\mathfrak{g})).$$ □

We are now ready to extend the description of the structure of connected abelian Lie groups (Example 8.5.6) to connected nilpotent Lie groups.

Theorem 10.2.10 (Structure Theorem for Connected Nilpotent Lie Groups). *Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then there exists a discrete subgroup $\Gamma \subseteq (\mathfrak{z}(\mathfrak{g}), +)$ with*

$$G \cong (\mathfrak{g}, *)/\Gamma.$$

In particular, G is diffeomorphic to the abelian Lie group $\mathfrak{g}/\Gamma$. Moreover, $\mathfrak{t} := \operatorname{span} \Gamma \subseteq \mathfrak{z}(\mathfrak{g})$ is a central Lie subalgebra for which $T := \exp_G(\mathfrak{t})$ is a torus, and G is diffeomorphic to the product manifold $(G/T) \times T$.

Proof. Let $\exp_G \colon (\mathfrak{g}, *) \to G$ be the universal covering of G. Then $\Gamma := \ker \exp_G \subseteq Z(\mathfrak{g}, *) = \mathfrak{z}(\mathfrak{g})$ is a discrete subgroup with $G \cong (\mathfrak{g}, *)/\Gamma$ (Lemma 10.2.8). Since $x * z = x + z$ for $z \in \mathfrak{z}(\mathfrak{g})$ and $x \in \mathfrak{g}$, we may identify the manifold G with the abelian Lie group $\mathfrak{g}/\Gamma$, only the group structure may be different.

Let $\mathfrak{t} := \operatorname{span} \Gamma$. Then Γ is a discrete generating subgroup of $\mathfrak{t}$, and it follows from the discussion in Example 8.5.6 that $\exp_G(\mathfrak{t}) \cong \mathfrak{t}/\Gamma$ is a torus. Moreover, T is central and in particular a normal subgroup, so that $G/T \cong (\mathfrak{g}, *)/\mathfrak{t}$ also is a Lie group, which is obviously isomorphic to $(\mathfrak{g}/\mathfrak{t}, *)$.

Now, let $\mathfrak{m} \subseteq \mathfrak{g}$ be a vector space complement for $\mathfrak{t}$. Then $x * z = x + z$ for $x \in \mathfrak{m}$ and $z \in \mathfrak{t} \subseteq \mathfrak{z}(\mathfrak{g})$ implies that the map

$$\mathfrak{m} \times \mathfrak{t} \to \mathfrak{g}, \quad (x, z) \mapsto x * z = x + z$$

is a diffeomorphism. Hence

$$\mathfrak{m} \times T \to G, \quad (x, z) \mapsto x * (z + \Gamma) = x + z + \Gamma$$

also is a diffeomorphism. Since the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{t}$ induces a linear isomorphism $\mathfrak{m} \to \mathfrak{g}/\mathfrak{t}$, we obtain a diffeomorphism $G \cong \mathfrak{m} \times T \cong G/T \times T$. □

Corollary 10.2.11. *Any compact connected nilpotent Lie group is abelian.*

Proof. In terms of the preceding theorem, the compactness of G implies the compactness of the quotient $\mathfrak{g}/\Gamma$, and hence that $\mathfrak{g} = \operatorname{span} \Gamma \subseteq \mathfrak{z}(\mathfrak{g})$. Therefore $\mathfrak{g}$, and hence also G, is abelian. □

Remark 10.2.12. The major difference between the abelian and the nilpotent case is that the exact sequence of groups

$$\{\mathbf{1}\} \to T \to G \xrightarrow{q} G/T \to \{\mathbf{1}\}$$

does need not split in the nilpotent case, i.e., that there may be no smooth homomorphism $\alpha \colon G/T \to G$ with $q \circ \alpha = \mathrm{id}_{G/T}$ (cf. Example 8.5.6). We obtain an easy example from the 3-dimensional Heisenberg–Lie algebra $\mathfrak{g}$ spanned by basis elements p, q, z satisfying

$$[p, q] = z, \quad \text{and} \quad [z, p] = [z, q] = 0.$$

For the quotient Lie group $G := (\mathfrak{g}, *)/\mathbb{Z}z$ we obtain a 1-dimensional torus $T = \mathbb{R}z/\mathbb{Z}z \cong \mathbb{R}/\mathbb{Z} = \mathbb{T}$, so that we get a diffeomorphism

$$G \cong (G/T) \times T \cong \mathbb{R}^2 \times \mathbb{T}.$$

Since the quotient Lie group $G/T \cong \mathbb{R}^2$ is abelian and T is central, there is no morphism of Lie groups $\sigma\colon G/T \to G$ splitting the quotient map, because otherwise $G \cong (G/T) \times T$ would be abelian. Another argument for the nonexistence of such a homomorphism is that the short exact sequence of Lie algebras

$$\{0\} \to \mathbb{R}z \to \mathfrak{g} \to \mathfrak{g}/\mathbb{R}z \to \{0\}$$

does not split.

Lemma 10.2.13. *Let $\mathfrak{n}$ be a nilpotent Lie algebra. Suppose that $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ holds for subalgebras $\mathfrak{a}$ and $\mathfrak{b}$. Then the multiplication map*

$$\mu\colon \mathfrak{a} \times \mathfrak{b} \to \mathfrak{n}, \quad (x, y) \mapsto x * y$$

is surjective. If, in addition, $\mathfrak{a} \cap \mathfrak{b} = \{0\}$, then it is a diffeomorphism.

Proof. Let $N := (\mathfrak{n}, *)$, $A := (\mathfrak{a}, *)$ and $B := (\mathfrak{b}, *)$. We prove the assertion by induction on $\dim \mathfrak{n}$. If $\dim \mathfrak{n} = 0$, there is nothing to prove. Suppose that $\mathfrak{n} \neq \{0\}$. Then $\mathfrak{z} := \mathfrak{z}(\mathfrak{n}) \neq \{0\}$ (Proposition 4.2.3), we apply induction to $\mathfrak{n}_1 := \mathfrak{n}/\mathfrak{z}$. With $Z := (\mathfrak{z}, +)$ we then obtain $N_1 := (\mathfrak{n}_1, *) = A_1 B_1$ for $A_1 := AZ/Z$ and $B_1 := BZ/Z$. From this we derive $N = AZBZ = AZB$. Since A and B are subgroups, the surjectivity of μ will follow if we can show that $Z \subseteq AB$. To see this, let $z \in \mathfrak{z}$. We write $z = a + b$, where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then $[a, b] = [a, z] = 0$ implies that $z = a + b = a * b \in AB$. Therefore μ is surjective.

Now we assume, in addition, that $\mathfrak{a} \cap \mathfrak{b} = \{0\}$. Then μ is injective because $\mu(a_1, b_1) = \mu(a_2, b_2)$ implies that $a_1^{-1} a_2 = b_1 b_2^{-1} \in \mathfrak{a} \cap \mathfrak{b} = \{0\}$. Therefore μ is a smooth bijection. That it is a diffeomorphism follows from the observation that $(a, b).x := a * x * (-b)$ defines a smooth action of the product Lie group $A \times B$ on N, so that $\mu(a, b) = a * b = (a, -b).0$ is an orbit map. This implies that μ has constant rank. In $(0, 0)$ the differential $T_{(0,0)}(\mu)\colon \mathfrak{a} \times \mathfrak{b} \to \mathfrak{n}$ is the addition map, hence a linear isomorphism. We now conclude with the Inverse Function Theorem that μ^{-1} also is a smooth map. ◻

10.2.3 Solvable Lie Groups

The structure of solvable Lie groups is substantially more complicated than the structure of the nilpotent ones. In particular, the exponential function of a solvable Lie group need not be surjective. A simple example with this property is the simply connected covering of the motion group $\mathrm{Mot}_2(\mathbb{R}) \cong$

$\mathbb{R}^2 \rtimes \mathrm{SO}_2(\mathbb{R})$ of the euclidian plane (cf. Exercise 10.2.1). In Section 13.4, we will obtain a characterization of those simply connected solvable Lie groups with a surjective exponential function in terms of the condition $\mathfrak{g} = \mathrm{regexp}(\mathfrak{g})$ on their Lie algebras. Other difficulties arise from the fact that the center of a connected solvable Lie group is not connected (again $\mathrm{Mot}_2(\mathbb{R})$ provides an example). Therefore, the description of the center of the universal covering group and hence the classification of all solvable Lie groups with a given Lie algebra is more complicated than in the nilpotent case, where everything boils down to factorization of a discrete subgroup of $\mathfrak{z}(\mathfrak{g})$. In Theorem 13.2.8 below, we shall also see how to master this difficulty. Presently, we can only show that 1-connected solvable Lie groups are diffeomorphic to vector spaces by showing the existence of global canonical coordinates of the second kind (cf. Lemma 8.2.6).

Theorem 10.2.14. *If G is a 1-connected solvable Lie group, then there exists a basis $x_1, \ldots, x_n$ for its Lie algebra $\mathfrak{g}$ such that the map*

$$\Phi\colon \mathbb{R}^n \to G, \quad (t_1, \ldots, t_n) \mapsto \prod_{j=1}^{n} \exp_G(t_j x_j)$$

is a diffeomorphism, the subgroups $R_j := \exp_G(\mathbb{R}x_j)$ of G are closed, and

$$G \cong \Big(\Big(\ldots\Big((R_1 \rtimes R_2) \rtimes R_3\Big)\ldots\Big) \rtimes R_n\Big).$$

Proof. We proceed as in the proof of Theorem 10.1.21. First, we observe that there exist subalgebras

$$\mathfrak{g}_0 = \{0\} \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \ldots \subseteq \mathfrak{g}_n = \mathfrak{g} \quad \text{with} \quad \mathfrak{g}_i \trianglelefteq \mathfrak{g}_{i+1},$$

such that $\mathfrak{g}_{i+1}/\mathfrak{g}_i \cong \mathbb{R}$. Then we pick $x_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$ and obtain

$$\mathfrak{g} \cong \Big(\Big(\ldots\Big((\mathbb{R}x_1 \rtimes \mathbb{R}x_2) \rtimes \mathbb{R}x_3\Big)\ldots\Big) \rtimes \mathbb{R}x_n\Big).$$

Now the assertion follows from Theorem 10.1.21 and Corollary 10.1.19. $\square$

We have already seen that integral subgroups of connected Lie groups need not be closed, even for abelian groups, such as the 2-dimensional torus $\mathbb{T}^2$ (cf. Example 8.3.12). However, the situation is much better for simply connected solvable groups, as the following proposition shows.

Proposition 10.2.15. *Any integral subgroup H of a simply connected solvable Lie group G is closed and simply connected and G/H is diffeomorphic to $\mathbb{R}^{\dim G/H}$.*

Proof. (a) We use the notation of Theorem 10.2.14 and its proof. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra and $H := \langle \exp \mathfrak{h} \rangle$ the corresponding integral subgroup. Let $k = \dim \mathfrak{h}$, and let $i_1 < \ldots < i_k$ be the indices with $\mathfrak{g}_{i_j-1} \cap \mathfrak{h} \neq \mathfrak{g}_{i_j} \cap \mathfrak{h}$. Then we may choose the basis $x_1, \ldots, x_n$ in the proof of Theorem 10.2.14 in such a way that $x_{i_j} \in \mathfrak{h}$ for $j = 1, \ldots, k$. Since these elements are linearly independent, $\{x_{i_1}, \ldots, x_{i_k}\}$ is a basis for $\mathfrak{h}$. Let $q_H \colon \widetilde{H} \to H$ be the universal covering morphism and $\alpha \colon \widetilde{H} \to G, h \mapsto q_G(h)$ the morphism of solvable Lie groups for which $\mathbf{L}(\alpha) \colon \mathfrak{h} \to \mathbf{L}(G)$ is the inclusion map. Then

$$\mathfrak{h} \cong \Big(\Big(\ldots\Big(\big(\mathbb{R}x_{i_1} \rtimes \mathbb{R}x_{i_2}\big) \rtimes \mathbb{R}x_{i_3}\Big)\ldots\Big) \rtimes \mathbb{R}x_{i_k}\Big),$$

and we obtain accordingly $\widetilde{H} = \exp_{\widetilde{H}}(\mathbb{R}x_{i_1}) \cdots \exp_{\widetilde{H}}(\mathbb{R}x_{i_k})$. Applying α, this leads to

$$H = \alpha(\widetilde{H}) = \exp_G(\mathbb{R}x_{i_1}) \cdots \exp_G(\mathbb{R}x_{i_k}).$$

Since the map Φ in Theorem 10.2.14 is a diffeomorphism, it now follows that q_H is injective, so that H is simply connected, and we also see that in the global chart of G defined by Φ, the subgroup H corresponds to a vector subspace of $\mathbb{R}^n$, so that it is in particular a closed submanifold.

(b) Let $d := \dim(G/H)$. We show that G/H is diffeomorphic to $\mathbb{R}^d$ by induction over $\dim G$. For $\dim G = 0$, the assertion holds trivially. So let us assume that $\dim G > 0$. Then $D^1(G)$ is a proper integral normal subgroup. We distinguish two cases.

Case 1: $H \subseteq D^1(G)$. Then we find subgroup $R_1, \ldots, R_k \cong \mathbb{R}$, $k > 0$, with $G \cong \big(\cdots(D^1(G) \rtimes R_1) \cdots \rtimes R_k\big)$, and then

$$G/H \cong H\backslash G \cong (H\backslash D^1(G)) \times \mathbb{R}^k,$$

so that the assertion follows from the induction hypothesis, applied to the subgroup H of $D^1(G)$.

Case 2: $H \not\subseteq D^1(G)$. Then we pick some $x \in \mathbf{L}(H) \setminus \mathbf{L}(D^1(G))$. Let $\mathfrak{g}_1 \subseteq \mathfrak{g}$ be a hyperplane containing the commutator algebra $D^1(\mathfrak{g}) = \mathbf{L}(D^1(G))$ (Proposition 10.2.4) but not x. Then $\mathfrak{g} \cong \mathfrak{g}_1 \rtimes \mathbb{R}x$ is a semidirect product. Accordingly, the simply connected group G is of the form $G \cong G_1 \rtimes R$ with $R = \exp_G(\mathbb{R}x) \cong \mathbb{R}$. From $x \in \mathfrak{h}$ we derive that $\mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$ satisfies $\mathfrak{h} = \mathfrak{h}_1 \rtimes \mathbb{R}x$, so that $H \cong H_1 \rtimes R$ (Proposition 10.1.19). This implies that

$$G/H = (G_1 \rtimes R)/(H_1 \rtimes R) \cong G_1/H_1 \cong \mathbb{R}^d$$

by the induction hypothesis. □

Exercises for Section 10.2

Exercise 10.2.1. Let $\alpha \colon \mathbb{R} \to \mathrm{Aut}(\mathbb{C}) \cong \mathbb{C}^\times$ be defined by $\alpha(t)z = e^{it}z$. Show:

(i) $G := \mathbb{C} \rtimes_\alpha \mathbb{R}$ is a three-dimensional, simply connected, solvable Lie group.
(ii) $Z(G) = \{0\} \times 2\pi\mathbb{Z}$. In particular, $Z(G)$ is not connected.
(iii) $G/Z(G)$ is isomorphic to the group $\mathrm{Mot}_2(\mathbb{R})$ of motions of the Euclidian plane.
(iv) Identifying $\mathbf{L}(G)$ with the corresponding semidirect product $\mathbb{C} \rtimes \mathbb{R}$ of Lie algebras (Proposition 8.2.25), the exponential function of G is given by

$$\exp_G(z,t) = \begin{cases} (z, 0) & \text{for } t = 0 \\ \big(z\frac{e^{it}-1}{it}, t\big) & \text{for } t \neq 0. \end{cases}$$

(v) The exponential function of G is not surjective.

Exercise 10.2.2. Let A and B be subgroups of a topological group G. Then the following statements hold (here the overline denotes closure in G):

(i) $\overline{(\overline{A}, \overline{B})} = \overline{(A, B)}$.
(ii) $\overline{D^n(\overline{A})} = \overline{D^n(A)}$ and $\overline{C^n(\overline{A})} = \overline{C^n(A)}$ for each $n \in \mathbb{N}$.
(iii) A is nilpotent, resp., solvable if and only if this holds for $\overline{A}$.

Exercise 10.2.3. Let $\alpha\colon \mathbb{R} \to \mathrm{GL}(V)$ be a smooth homomorphism and $G := V \rtimes_\alpha \mathbb{R}$ the corresponding semidirect product. Show that the exponential function of G is given by

$$\exp_G(v,t) := (\beta(t)v, t) \quad \text{with} \quad \beta(t)v = \int_0^1 \alpha(st)v\, ds. \tag{10.7}$$

Show that, for $\alpha(t) = e^{tD}$, $D \in \mathfrak{gl}(V)$, we have

$$\beta(t) = \int_0^1 e^{stD}\, ds = \frac{e^{tD} - \mathbf{1}}{tD} \quad \text{for} \quad t \neq 0.$$

Exercise 10.2.4. Show that the exponential map for the Lie group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$$

is a diffeomorphism.

Exercise 10.2.5. Consider the Lie group

$$G = \left\{ \begin{pmatrix} \cos t & \sin t & a \\ -\sin t & \cos t & b \\ 0 & 0 & 1 \end{pmatrix} \Big| t, a, b \in \mathbb{R} \right\}.$$

(i) Show that the exponential map for G is surjective.
(ii) Show that the exponential map for the simply connected covering group of G is not surjective.

10.3 The Automorphism Group of a Lie Group

In this section we take a closer look at the automorphism group $\mathrm{Aut}(G)$ of a Lie group G. In particular, we are interested in a Lie group structure on this group for which a map $f\colon M \to \mathrm{Aut}(G)$, M a smooth manifold, is smooth if and only if the corresponding map $f^\wedge\colon M \times G \to G, (m, g) \mapsto f(m)(g)$ is smooth. It is easy to see that any such Lie group structure is unique, and our main result is its existence if the group $\pi_0(G)$ of connected components is finitely generated. We shall also see examples, where no such finite dimensional Lie group structure on $\mathrm{Aut}(G)$ exists.

10.3.1 A Lie Group Structure on Aut(G)

What we have already seen is that, if G is 1-connected, then $\mathrm{Aut}(G)$ carries a Lie group structure for which $\mathbf{L}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism of Lie groups, and in this case $\mathrm{Aut}(G)$ acts smoothly on G (Corollary 8.5.11). If G is connected and $q_G\colon \widetilde{G} \to G$ its simply connected covering group, then we have an injection $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\widetilde{G})$ whose image consists of the normalizer of the discrete subgroup $\ker q_G \cong \pi_1(G)$ (Remark 8.5.5). Since this normalizer is closed, we also obtain in this case a Lie group structure on $\mathrm{Aut}(G)$ for which the canonical action on $\widetilde{G}$, and hence also on G, is smooth.

Definition 10.3.1. Let G be a Lie group. We say that a Lie group structure on $\mathrm{Aut}(G)$ is *adapted* if for each smooth manifold M, a map $f\colon M \to \mathrm{Aut}(G)$ is smooth if and only if the corresponding map

$$f^\wedge\colon M \times G \to G, \quad (m, g) \mapsto f(m)(g)$$

is smooth.

Applying this condition to $M = \mathrm{Aut}(G)$, it follows that the action of $\mathrm{Aut}(G)$ on G is smooth. If, conversely, this action is smooth, then the smoothness of f implies the smoothness of $f^\wedge$.

Lemma 10.3.2 (Uniqueness of the adapted Lie group structure). *Let G be a Lie group. Then there exists at most one adapted Lie group structure on* $\mathrm{Aut}(G)$.

Proof. Let A_1 and A_2 denote two Lie group structures on $\mathrm{Aut}(G)$ satisfying our requirements. Applying the adaptedness condition with $M = A_j$ and $f = \mathrm{id}_{\mathrm{Aut}(G)}$, it follows that for both groups the action of A_j on G is smooth. With $M = A_1$, we thus obtain that the identity map $\alpha\colon A_1 \to A_2$ is smooth, and with $M = A_2$, we also see that α^{-1} is smooth. Therefore α is an isomorphism of Lie groups. □

Lemma 10.3.3. *If G is connected, then the canonical Lie group structure on* $\mathrm{Aut}(G)$ *for which* $\mathbf{L}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ *is an isomorphism onto a closed subgroup, is adapted.*

Proof. We have already argued that $\mathrm{Aut}(G)$ acts smoothly on G. This implies that for any smooth map $f\colon M \to \mathrm{Aut}(G)$, the corresponding map $f^\wedge\colon M \times G \to G$ is smooth.

Suppose, conversely, that $f^\wedge$ is smooth. Then so is the tangent map

$$T(f^\wedge)\colon TM \times TG \to TG.$$

Restricting to $M \times T_{\mathbf{1}}(G)$ implies that $\mathbf{L} \circ f\colon M \to \mathrm{Aut}(\mathfrak{g})$ is also smooth. As $\mathbf{L}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism onto a Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$, it follows that f is smooth. □

Definition 10.3.4. If G is a Lie group, we write $\mathrm{Aut}_1(G)$ for the normal subgroup consisting of all automorphisms acting trivially on $\pi_0(G)$, i.e., which preserve all connected components.

Theorem 10.3.5. *If G is a Lie group for which $\pi_0(G)$ is finitely generated, then $\mathrm{Aut}(G)$ carries a Lie group structure with the property that, for any smooth manifold M, a map $f\colon M \to \mathrm{Aut}(G)$ is smooth if and only if the corresponding map $f^\wedge\colon M \times G \to G$ is smooth.*

Proof. **Step 1:** Let $g_1, \ldots, g_n$ be elements of G whose connected components generate the group $\pi_0(G)$. Since the action of $\mathrm{Aut}(G_0)$ on G_0 is smooth, the same holds for the diagonal action on G_0^n, so that we may form the semidirect product Lie group

$$P := G_0^n \rtimes \mathrm{Aut}(G_0).$$

Then

$$\zeta\colon \mathrm{Aut}_1(G) \to P, \quad \zeta(\varphi) := \big((g_1^{-1}\varphi(g_1), \ldots, g_n^{-1}\varphi(g_n)), \varphi_0\big), \quad \varphi_0 := \varphi|_{G_0}$$

satisfies

$$\begin{aligned}
&\zeta(\varphi)\zeta(\psi)\\
&= \big((g_1^{-1}\varphi(g_1), \ldots, g_n^{-1}\varphi(g_n)), \varphi_0\big)\big((g_1^{-1}\psi(g_1), \ldots, g_n^{-1}\psi(g_n)), \psi_0\big)\\
&= \big((g_1^{-1}\varphi(g_1)\varphi(g_1^{-1}\psi(g_1)), \ldots, g_n^{-1}\varphi(g_n)\varphi(g_n^{-1}\psi(g_n)), \varphi_0\psi_0\big)\\
&= \big((g_1^{-1}(\varphi\psi)(g_1), \ldots, g_n^{-1}(\varphi\psi)(g_n)), (\varphi\psi)_0\big) = \zeta(\varphi\psi),
\end{aligned}$$

so that ζ is a group homomorphism. Further, our choice of $g_1, \ldots, g_n$ implies that ζ is injective. In particular, $\zeta(\varphi) = (w_1, \ldots, w_n, \varphi_0)$ implies the relation

$$\varphi(g_i) = g_i w_i, \quad i = 1, \ldots, n. \tag{10.8}$$

Step 2: We claim that the image of ζ is closed. In fact, suppose that

$$((w_1, \cdots, w_n), \alpha) \in P$$

is the limit of a sequence $\zeta(\varphi_n)$. Then the sequence $(\varphi_n)_0$ converges pointwise to α, and (10.8) implies that φ_n converges pointwise on all of G to a map

$\varphi\colon G \to G$. Then φ is an endomorphism of the Lie group G preserving all connected components and whose restriction to G_0 is an automorphism. This implies that $\varphi \in \mathrm{Aut}(G)$, and then $\zeta(\varphi) = ((w_1, \ldots, w_n), \alpha)$ shows that ζ has closed range.

We now endow $\mathrm{Aut}_1(G)$ with the unique Lie group structure for which ζ is an isomorphism onto a Lie subgroup of P. To see that the action of $\mathrm{Aut}_1(G)$ on G is smooth, we have to verify that the map $\mathbf{L}\colon \mathrm{Aut}_1(G) \to \mathrm{Aut}(\mathfrak{g})$ is smooth and that all orbit maps are smooth (Lemma 8.2.26). The first assertion follows from $\mathbf{L}(\varphi) = \mathbf{L}(\varphi_0)$ and the definition of the Lie group structure on $\mathrm{Aut}(G_0)$. For the smoothness of the orbit maps, we observe that the smoothness of the action of $\mathrm{Aut}(G_0)$ on G_0 implies that the orbit maps of elements of G_0 are smooth. Further, (10.8) implies that the orbit maps of $g_1, \ldots, g_n$ are smooth. Since the set of all elements with smooth orbit maps is a subgroup and G is generated by G_0 and $g_1, \ldots, g_n$, all orbit maps are smooth.

Step 3: Next we show that the Lie group structure on $\mathrm{Aut}_1(G)$ has the property that a map $f\colon M \to \mathrm{Aut}_1(G)$, M a smooth manifold, is smooth if and only if the corresponding map $f^\wedge\colon M \times G \to G$ is smooth. Since $\mathrm{Aut}_1(G)$ acts smoothly on G, the smoothness of f implies the smoothness of $f^\wedge$. Suppose, conversely, that $f^\wedge$ is smooth. We have to show that the map $\zeta \circ f\colon M \to P$ is smooth. Lemma 10.3.3 implies that the map $f_0\colon M \to \mathrm{Aut}(G_0), m \mapsto f(m)_0$ is smooth, and the maps $M \to G, m \mapsto g_j^{-1} f(m)(g_j) = g_j^{-1} f^\wedge(m, g_j)$ are also smooth. Therefore f is smooth.

Step 4: For any $\varphi \in \mathrm{Aut}(G)$, the conjugation map c_φ restricts to an automorphism of the group $\mathrm{Aut}_1(G)$. In view of the preceding step, the smoothness of c_φ follows from the smoothness of the map

$$c_\varphi^\wedge\colon \mathrm{Aut}_1(G) \times G \to G, \quad (\psi, g) \mapsto \varphi(\psi(\varphi^{-1}(g))).$$

Therefore Corollary 8.4.5 implies the existence of a unique Lie group structure on $\mathrm{Aut}(G)$ for which $\mathrm{Aut}_1(G)$ is an open subgroup.

With respect to this Lie group structure, the smoothness of the action of G on $\mathrm{Aut}(G)$ follows immediately from the smoothness of the action of $\mathrm{Aut}_1(G)$. If, moreover, $f\colon M \to \mathrm{Aut}(G)$ is a map for which $f^\wedge$ is smooth, then we claim that f is smooth. In fact, we may w.l.o.g. assume that M is connected. Then the continuity of $f^\wedge$ implies that $f(M) \subseteq \mathrm{Aut}_1(G)$, so that Step 3 implies the smoothness of f. □

10.3.2 Infinitesimal Automorphisms

So far we have not said anything on the Lie algebra of the group $\mathrm{Aut}(G)$. The adaptedness condition implies that the smooth one-parameter groups of $\mathrm{Aut}(G)$ are in one-to-one correspondence with the smooth actions of the group $\mathbb{R}$ by automorphisms of G. We therefore take a closer look at the vector fields generating such flows.

Definition 10.3.6. Let G be a Lie group. A vector field $X \in \mathcal{V}(G)$ is said to be an *infinitesimal automorphism* if $X : G \to TG$ is a morphism of Lie groups, i.e.,

$$X(gh) = X(g)h + gX(h) \quad \text{for} \quad g, h \in G, \tag{10.9}$$

with respect to the canonical Lie group structure on the tangent bundle TG. Note that (10.9) implies $X(\mathbf{1}) = 0$ and

$$X(g^{-1}) = -g^{-1}X(g)g^{-1} \quad \text{for} \quad g \in G.$$

We write $\mathrm{IAut}(G) \subseteq \mathcal{V}(G)$ for the set of infinitesimal automorphisms of G.

We now justify this terminology:

Proposition 10.3.7. *If $\Phi : \mathbb{R} \times G \to G$ is a smooth action of $\mathbb{R}$ on the Lie group G by automorphisms, then its infinitesimal generator $X \in \mathcal{V}(G)$ is an infinitesimal automorphism. Conversely, every infinitesimal automorphism of G generates a smooth global flow by automorphisms.*

Proof. The first part follows immediately by taking derivatives in $t = 0$ in the relation

$$\Phi(t, gh) = \Phi(t, g)\Phi(t, h), \quad t \in \mathbb{R}, g, h \in G.$$

Suppose, conversely, that $X \in \mathrm{IAut}(G)$ and that $\Phi : \mathcal{D} \to G$ is the corresponding local flow, where $\mathcal{D} \subseteq \mathbb{R} \times G$ is its open domain (Theorem 7.5.12). Let $\varepsilon > 0$ be such that

$$\mathcal{D}_\varepsilon := \{g \in G : [-\varepsilon, \varepsilon] \times \{g\} \subseteq \mathcal{D}\}$$

is a $\mathbf{1}$-neighborhood in G. If $\gamma, \eta : [-\varepsilon, \varepsilon] \to G$ are integral curves of X, then their pointwise product also is an integral curve:

$$(\gamma\eta)'(t) = \gamma'(t)\eta(t) + \gamma(t)\eta'(t) = X(\gamma(t))\eta(t) + \gamma(t)X(\eta(t)) = X(\gamma(t)\eta(t)).$$

Further, the relation $X(g^{-1}) = -g^{-1}X(g)g^{-1}$ leads to

$$(\gamma^{-1})'(t) = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1} = -\gamma(t)^{-1}X(\gamma(t))\gamma(t)^{-1} = X(\gamma(t)^{-1}),$$

so that the pointwise inverse γ^{-1} also is an integral curve of X. Therefore $\mathcal{D}_\varepsilon$ is a subgroup of G, and since it contains a $\mathbf{1}$-neighborhood of G, it is open.

Let $g \in G$. Choosing ε small enough, we may assume that $g \in \mathcal{D}_\varepsilon$. Then $\mathcal{D}_\varepsilon$ is an open submanifold on which the flow of X is defined on $[-\varepsilon, \varepsilon] \times \mathcal{D}_\varepsilon$, which implies that $X|_{\mathcal{D}_\varepsilon}$ is complete (Exercise 7.5.2). Therefore $\mathbb{R} \times \{g\} \subseteq \mathcal{D}$, and since $g \in G$ was arbitrary, it follows that X is complete.

Let $\Phi_t^X \in \mathrm{Diff}(G)$ denote the corresponding global flow. To see that each $\Phi_t^X \in \mathrm{Aut}(G)$, let $g_1, g_2 \in G$ and $\gamma_1, \gamma_2 : \mathbb{R} \to G$ be the corresponding integral curves with $\gamma_j(0) = g_j$. Then the pointwise product $\gamma_1\gamma_2$ is the integral curve through g_1g_2, and this implies that

$$\Phi_t^X(g_1g_2) = \gamma_1(t)\gamma_2(t) = \Phi_t^X(g_1)\Phi_t^X(g_2),$$

so that $\Phi_t^X \in \mathrm{Aut}(G)$. □

Corollary 10.3.8. *For an adapted Lie group structure on* $\mathrm{Aut}(G)$ *we have* $\mathbf{L}(\mathrm{Aut}(G)) \cong \mathrm{IAut}(G)$.

Remark 10.3.9. The set $\mathrm{IAut}(G)$ of infinitesimal automorphisms of G can be made more explicit by writing the tangent bundle TG as a semidirect product $\mathfrak{g} \rtimes_{\mathrm{Ad}} G$, where the isomorphism is simply obtained by restricting the multiplication of TG to the subset $\mathfrak{g} \times G = T_{\mathbf{1}}(G) \times G$ (Example 8.2.24).

Then any vector field $X \in \mathcal{V}(G)$ can be written as $X(g) = (\alpha(g), g)$, where $\alpha\colon G \to \mathfrak{g}$ is a smooth function. The condition that $X\colon G \to TG$ is a group homomorphism now is equivalent to α being a 1-cocycle (cf. Definition 11.1.5 below):

$$\alpha(gh) = \alpha(g) + \mathrm{Ad}(g)\alpha(h), \quad g, h \in G.$$

Clearly, the set $Z^1(G, \mathfrak{g})$ of all smooth 1-cocycles is a vector space. In view of the preceding proposition, it is the natural candidate for the Lie algebra of the group $\mathrm{Aut}(G)$.

For any smooth 1-cocycle α, we have

$$\alpha(ghg^{-1}) = \alpha(gh) + \mathrm{Ad}(gh)\alpha(g^{-1}) = \alpha(g) + \mathrm{Ad}(g)\alpha(h) + \mathrm{Ad}(gh)\alpha(g^{-1}),$$

so that $\mathbf{L}(\alpha) := T_{\mathbf{1}}(\alpha)$ satisfies

$$\begin{aligned}\mathbf{L}(\alpha) \circ \mathrm{Ad}(g) &= \mathrm{Ad}(g) \circ \mathbf{L}(\alpha) - \mathrm{Ad}(g) \circ \mathrm{ad}(\alpha(g^{-1})) \\ &= \mathrm{Ad}(g) \circ \mathbf{L}(\alpha) + \mathrm{Ad}(g) \circ \mathrm{ad}(\mathrm{Ad}(g)^{-1}\alpha(g)) \\ &= \mathrm{Ad}(g) \circ \mathbf{L}(\alpha) + \mathrm{ad}(\alpha(g)) \circ \mathrm{Ad}(g).\end{aligned}$$

Taking derivatives in $g = \mathbf{1}$, we thus find with $\alpha(\mathbf{1}) = 0$ that

$$\mathbf{L}(\alpha) \circ \mathrm{ad}\, x = \mathrm{ad}\, x \circ \mathbf{L}(\alpha) + \mathrm{ad}(\mathbf{L}(\alpha)x),$$

and this means that $\mathbf{L}(\alpha) \in \mathrm{der}(\mathfrak{g})$.

The passage from a derivation $D \in \mathrm{der}(\mathfrak{g}) = Z^1(\mathfrak{g}, \mathfrak{g})$ to a one-parameter group of infinitesimal automorphisms requires an integration process. The best way to see this, is to consider the corresponding homomorphism

$$\widetilde{D}\colon \mathfrak{g} \to \mathfrak{g} \rtimes_{\mathrm{ad}} \mathfrak{g}, \quad \widetilde{D}(x) := (Dx, x).$$

If G is 1-connected, it can be integrated to a unique morphism $\alpha\colon G \to TG$ with $\mathbf{L}(\alpha) = \widetilde{D}$, but in general it leads to a homomorphism $\pi_1(G) \to \mathfrak{g}$ whose triviality characterizes the integrability of D to an infinitesimal automorphism (cf. Corollary 8.5.10).

Examples 10.3.10. (a) If G is 1-connected, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$ and $\mathbf{L}(G) \cong \mathrm{der}(\mathfrak{g})$ (cf. Example 3.2.5).

(b) If G is not connected, then there may be infinitesimal automorphisms vanishing on G_0. They correspond to 1-cocycles $\alpha\colon G \to \mathfrak{g}$ with $\alpha(G_0) = \{0\}$. If G_0 is abelian, then G_0 acts trivially on $\mathfrak{g}$, so that the adjoint action factors through an action $\overline{\mathrm{Ad}}$ of $\pi_0(G)$ on $\mathfrak{g}$, and all 1-cocycles $\overline{\alpha}\colon \pi_0(G) \to \mathfrak{g}$ yield infinitesimal automorphisms vanishing on G_0.

(c) For a torus $G = \mathbb{T}^d$, we have $\widetilde{G} \cong \mathbb{R}^d$, and

$$\mathrm{Aut}(G) \cong \{\varphi \in \mathrm{GL}_d(\mathbb{R})\colon \varphi(\mathbb{Z}^d) = \mathbb{Z}^d\} \cong \mathrm{GL}_d(\mathbb{Z})$$

is a discrete group. Since $G \subseteq TG \cong \mathfrak{g} \times G$ is a maximal compact subgroup, the zero section of TG is the only homomorphism $G \to TG$ which is a vector field. Therefore $\mathrm{IAut}(\mathbb{T}^d) = \{0\}$.

(d) For the abelian Lie group $G = \mathbb{R} \times \mathbb{Z}^{(\mathbb{N})}$, the group $\pi_0(G) \cong \mathbb{Z}^{(\mathbb{N})}$ is not finitely generated. Since the adjoint action is trivial, we have

$$\mathrm{IAut}(G) \cong Z^1(G, \mathfrak{g}) = \mathrm{Hom}(G, \mathfrak{g}) \cong \mathrm{Hom}(G, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}^{(\mathbb{N})},$$

and this space is infinite dimensional. As a consequence of Corollary 10.3.8, we now see that $\mathrm{Aut}(G)$ carries no finite dimensional adapted Lie group structure.

We have seen in the preceding example that there exist Lie groups G with countably many connected components for which $\mathrm{Aut}(G)$ carries no adapted Lie group structure. In view of Corollary 10.3.8, the finite dimensionality of the space $\mathrm{IAut}(G)$ is necessary. However, we have the following

Theorem 10.3.11. *If* $\mathrm{IAut}(G)$ *is finite dimensional, then* $\mathrm{Aut}(G)$ *carries the structure of a Lie group with Lie algebra* $\mathrm{IAut}(G)$ *for which the action on* G *is smooth.*

Proof. Clearly, the subspace $\mathrm{IAut}(G) \subseteq \mathcal{V}(G)$ is invariant under all automorphisms of G, hence in particular under all flow maps Φ_t^X, $X \in \mathrm{IAut}(G)$ (Proposition 10.3.7). We therefore obtain for $X, Y \in \mathrm{IAut}(G)$:

$$[X, Y] = \mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^X)_* Y \in \mathrm{IAut}(G).$$

This means that $\mathrm{IAut}(G)$ is a finite dimensional Lie subalgebra of $\mathcal{V}(G)$. Since it consists of complete vector fields (Proposition 10.3.7), Palais' Theorem 9.5.1 ensures the existence of a smooth action σ of the 1-connected Lie group H with Lie algebra $\mathfrak{h} = \mathrm{IAut}(G)$. The smoothness of this action implies that its kernel is closed, so that we may use $H/\ker\sigma$ to turn the subgroup

$$\mathrm{Aut}_0(G) := \{\Phi_1^X : X \in \mathrm{IAut}(G)\} \subseteq \mathrm{Aut}(G)$$

into a Lie group whose action on G is smooth. For each $\varphi \in \mathrm{Aut}(G)$, the relation $c_\varphi(\Phi_t^X) = \Phi_t^{\varphi_* X}$ implies that

$$(c_\varphi \circ \exp_{\mathrm{Aut}_0(G)})(X) = \exp_{\mathrm{Aut}_0(G)}(\varphi_* X),$$

and hence that c_φ defines a smooth automorphism of $\mathrm{Aut}_0(G)$. Therefore Corollary 8.4.5 implies the existence of a unique Lie group structure on $\mathrm{Aut}(G)$ for which $\mathrm{Aut}_0(G)$ is an open subgroup. Since $\mathrm{Aut}_0(G)$ acts smoothly on G, the same hold for $\mathrm{Aut}(G)$. □

Notes on Chapter 10

Our discussion of the fundamental group of a homogeneous space in Section 10.1 that culminates in the Homotopy Group Theorem avoids the use of the long exact homotopy sequence of fiber bundles to keep the presentation more self-contained.

Most of the material covered in this chapter can also be found in Hochschild's book [Ho65] and in Bourbaki [Bou89, Ch. 3]. In particular, our discussion of the automorphism group of a Lie group closely follows Bourbaki.

11

Compact Lie Groups

As we have seen in Chapter 4, Levi's Theorem 4.6.6 is a central result in the structure theory of Lie algebras. It often allows to split problems: one separately considers solvable and semisimple Lie algebras, and one puts together the results for both types. Naturally, this strategy also works to some extent for Lie groups. After dealing with nilpotent and solvable Lie groups in Chapter 10, we turn to the other side of the spectrum, to groups with semisimple or reductive Lie algebras. Here an important subclass is the class of compact Lie groups and the slightly larger class of groups with compact Lie algebra. Many problems can be reduced to compact Lie groups, and they are much easier to deal with than noncompact ones. The prime reason for that is the existence of a finite Haar measure whose existence was shown in Section 9.4.

In Section 11.1 we introduce the class of compact Lie algebras $\mathfrak{k}$ as those carrying an invariant scalar product and show that this condition is equivalent to the existence of a compact Lie group K with $\mathbf{L}(K) = \mathfrak{k}$. The main result of that section is a structure theorem for connected groups with a compact Lie algebra, saying that any such group is a direct product of a compact group with a vector group. In Section 11.2 we then turn to the internal structure of compact groups, which is approached by studying maximal torus subgroups. Here one of the main insights is that all maximal tori are conjugate under inner automorphisms and that each element of a compact connected Lie group is contained in some maximal torus. In the third section of this chapter we show that any compact Lie group has a faithful unitary representation, so that it can be realized as a subgroup of some unitary group $\mathrm{U}_n(\mathbb{C})$. In the last section we provide some more topological information and use it to show that fixed point sets of automorphisms in 1-connected compact Lie groups are connected (Theorem 11.4.26).

11.1 Lie Groups with Compact Lie Algebra

In this section we study the structure of Lie algebras which occur as Lie algebras of compact groups. We use the results to prove a number of structure theoretic theorems on Lie groups with such Lie algebras. As technical tools in this context, we introduce 1-cocycles and certain averaging integrals for cosets.

Definition 11.1.1. A Lie algebra $\mathfrak{g}$ is called *compact* if there exists a positive definite, invariant and symmetric bilinear form on $\mathfrak{g}$.

Lemma 11.1.2. (i) *Every subalgebra of a compact Lie algebra is compact.*

(ii) *A direct sum* $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ *of Lie algebras* $\mathfrak{g}_i$ *is compact if and only if all the* $\mathfrak{g}_i$ *are compact.*

(iii) *If* $\mathfrak{a} \trianglelefteq \mathfrak{g}$ *is an ideal, then the orthogonal complement* $\mathfrak{a}^\perp$ *with respect to any invariant scalar product is also an ideal, and* $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}^\perp$ *is a Lie algebra direct sum.*

(iv) *Every compact Lie algebra is reductive.*

Proof. (i) Let $\mathfrak{g}$ be a compact Lie algebra, and let β be a positive definite, symmetric and invariant bilinear form on $\mathfrak{g}$. The restriction of β to any subalgebra also is invariant and positive definite. Hence every subalgebra of $\mathfrak{g}$ is compact.

(ii) Exercise.

(iii) Since β is positive definite, $\mathfrak{a}^\perp$ is a vector space complement for $\mathfrak{a}$. For $a \in \mathfrak{a}$, $x \in \mathfrak{g}$ and $y \in \mathfrak{a}^\perp$, we have

$$\beta(a, [x, y]) = \beta([a, x], y) = 0$$

since $[a, x] \in \mathfrak{a}$. Hence, $[\mathfrak{g}, \mathfrak{a}^\perp] \subseteq \mathfrak{a}^\perp$, and therefore, $\mathfrak{a}^\perp$ is an ideal.

(iv) This is a consequence of (iii) (Definition 4.7.1). ☐

Definition 11.1.3. Let $\mathfrak{g}$ be a Lie algebra, and let $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{g})$ be the adjoint representation. For a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$, we set

$$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{a}) := \langle e^{\mathrm{ad}\,\mathfrak{a}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g}) \quad \text{and} \quad \mathrm{I{\scriptsize NN}}_{\mathfrak{g}}(\mathfrak{a}) := \overline{\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{a})}.$$

We also write $\mathrm{Inn}(\mathfrak{g}) := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g})$ and recall that $\mathbf{L}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{der}(\mathfrak{g})$ (Example 3.2.5).

The following proposition gives two alternative characterizations of compact Lie algebras.

Proposition 11.1.4. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. Then the following are equivalent:*

(i) *There exists a compact Lie group* G *with* $\mathbf{L}(G) \cong \mathfrak{g}$.

(ii) $\mathrm{I{\scriptsize NN}}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$ *is compact.*

(iii) $\mathfrak{g}$ *is compact.*

Proof. (i) $\Rightarrow$ (ii): Let G be a compact connected Lie group with $\mathbf{L}(G) = \mathfrak{g}$. Then $\mathrm{Ad}(G) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ is compact, so that $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{g}) = \overline{\langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle} = \mathrm{Ad}(G)$ also is compact.

(ii) $\Rightarrow$ (iii): We apply the Unitarity Lemma 9.4.14. It implies the existence of an $\mathrm{INN}(\mathfrak{g})$-invariant scalar product β on $\mathfrak{g}$. Then $\mathrm{INN}(\mathfrak{g}) \subseteq \mathrm{O}(\mathfrak{g}, \beta)$ leads to $\mathrm{ad}(\mathfrak{g}) \subseteq \mathbf{L}(\mathrm{O}(\mathfrak{g}, \beta)) = \mathfrak{o}(\mathfrak{g}, \beta)$, from which we derive that

$$\begin{aligned}\beta([x, y], z) &= \beta\big(-\mathrm{ad}\, y(x), z\big) = -\beta\big(x, \mathrm{ad}\, y^\top(z)\big) \\ &= \beta\big(x, \mathrm{ad}\, y(z)\big) = \beta(x, [y, z]).\end{aligned}$$

for $x, y, z \in \mathfrak{g}$. Hence β is invariant.

(iii) $\Rightarrow$ (i): By Lemma 11.1.2(iv), $\mathfrak{g}$ is reductive, hence the direct sum of an abelian Lie algebra $\mathfrak{a}$ and a semisimple compact Lie algebra $\mathfrak{s}$. If $\mathfrak{a} \cong \mathbb{R}^n$, then we set $A := \mathbb{R}^n/\mathbb{Z}^n$, and we get a compact group with $\mathbf{L}(A) = \mathfrak{a}$. By Lemma 11.1.2(i), there is a positive definite symmetric invariant bilinear form β on $\mathfrak{s}$. Then

$$\mathrm{Inn}_{\mathfrak{s}}(\mathfrak{s}) \subseteq \mathrm{O}(\beta) \cap \mathrm{Aut}(\mathfrak{s}).$$

By the Integral Subgroup Theorem 8.4.8 and Theorem 4.5.14, we have

$$\mathbf{L}(\mathrm{Inn}_{\mathfrak{s}}(\mathfrak{s})) = \mathrm{ad}(\mathfrak{s}) = \mathrm{der}(\mathfrak{s}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{s})),$$

and therefore, $\mathrm{Inn}_{\mathfrak{s}}(\mathfrak{s})$ is the identity component of $\mathrm{Aut}(\mathfrak{s})$, in particular, it is closed, and thus compact. Since $\ker(\mathrm{ad}_{\mathfrak{s}}) = \{0\}$, it follows that

$$\mathbf{L}(A \times \mathrm{Aut}(\mathfrak{s})) = \mathbf{L}(A \times \mathrm{Inn}_{\mathfrak{s}}(\mathfrak{s}) \cong \mathfrak{a} \oplus \mathfrak{s} \cong \mathfrak{g}. \qquad \square$$

Now we turn to the structure theory of Lie groups with compact Lie algebra. As for Lie algebras in Chapter 4, we also obtain splitting theorems for Lie groups with cohomological methods.

Definition 11.1.5. Let G and N be Lie groups and $\alpha\colon G \to \mathrm{Aut}(N)$ be a homomorphism defining a smooth action of G on N.

A smooth function $f\colon G \to N$ is called a 1-*cocycle* or a *crossed homomorphism* with respect to α if

$$f(g_1 g_2) = f(g_1) \cdot \alpha(g_1)(f(g_2)) \quad \text{for} \quad g_1, g_2 \in G.$$

Note that this condition is equivalent to

$$(f, \mathrm{id}_G)\colon G \to N \rtimes_\alpha G$$

being a morphism of Lie groups.

Example 11.1.6. Let G be a Lie group and $\alpha\colon G \to \mathrm{Aut}(G)$ define a smooth action of G on G.

(a) If $\alpha(G) = \{\mathrm{id}_G\}$ is trivial, then id_G is a 1-cocycle.

(b) If $\alpha(g) = c_g$ is the conjugation action, then the inversion $\iota_G(g) = g^{-1}$ is a 1-cocycle because

$$\iota_G(g_1 g_2) = g_2^{-1} g_1^{-1} = g_1^{-1} g_1 g_2^{-1} g_1^{-1} = \iota_G(g_1) c_{g_1}(\iota_G(g_2)).$$

The connection to semidirect product decompositions is provided by the following lemma.

Lemma 11.1.7. *Let G be a Lie group and N a closed normal subgroup. Then G acts smoothly on N by $\alpha(g)(n) = gng^{-1}$, and the following are equivalent:*

(i) *The short exact sequence $\mathbf{1} \to N \to G \to G/N \to \mathbf{1}$ of Lie groups splits.*
(ii) *There exists a closed subgroup $H \subseteq G$ for which the multiplication map*

$$\mu\colon N \rtimes_\alpha H \to G, \quad (n,h) \mapsto nh$$

is an isomorphism of Lie groups.
(iii) *There exists a 1-cocycle $f\colon G \to N$ with $f(n) = n^{-1}$ for $n \in N$.*

Proof. (i) $\Rightarrow$ (ii): Let $q\colon G \to G/N$ be the quotient morphism and $\sigma\colon G/N \to G$ be a morphism of Lie groups with $q \circ \sigma = \mathrm{id}_{G/N}$. Then the map

$$\Phi\colon N \times G/N \to G, \quad (n, gN) \mapsto n \cdot \sigma(gN)$$

is a smooth bijection with $\Phi^{-1}(g) = (g\sigma(q(g))^{-1}, q(g))$. Hence Φ is a diffeomorphism. This implies that $H := \sigma(G/N)$ is a closed subgroup for which (ii) holds.

(ii) $\Rightarrow$ (iii): On $G \cong N \rtimes_\alpha H$ we consider the map $f\colon G \to N, (n,h) \mapsto n^{-1}$. Then f is a 1-cocycle because for $g_1 = n_1 h_1$ and $g_2 = n_2 h_2$ we have

$$\begin{aligned} f(g_1 g_2) &= f(n_1 h_1 n_2 h_2) = (n_1 h_1 n_2 h_1^{-1})^{-1} = h_1 n_2^{-1} h_1^{-1} n_1^{-1} \\ &= n_1^{-1} n_1 h_1 n_2^{-1} h_1^{-1} n_1^{-1} = f(g_1)(g_1 f(g_2) g_1^{-1}), \end{aligned}$$

i.e., f is a 1-cocycle.

(iii) $\Rightarrow$ (i): Let $f\colon G \to N$ be a 1-cocycle with $f(n) = n^{-1}$ for $n \in N$. Then the map $\widetilde{\sigma}\colon G \to G, g \mapsto gf(g^{-1})^{-1}$ is a morphism of Lie groups:

$$\begin{aligned} \widetilde{\sigma}(g_1 g_2) &= g_1 g_2 f(g_2^{-1} g_1^{-1})^{-1} = g_1 g_2 (g_2^{-1} f(g_1^{-1})^{-1} g_2) f(g_2^{-1})^{-1} \\ &= g_1 f(g_1^{-1})^{-1} g_2 f(g_2^{-1})^{-1} = \widetilde{\sigma}(g_1)\widetilde{\sigma}(g_2). \end{aligned}$$

Since $N \subseteq \ker \widetilde{\sigma}$, it factors through a smooth morphism $\sigma\colon G/N \to G$, $gN \mapsto \widetilde{\sigma}(g)$. Clearly, $q(\sigma(gN)) = q(gf(g^{-1})^{-1}) = q(g) = gN$ implies (i). $\square$

Lemma 11.1.8. *Let V be a finite-dimensional vector space and G a Lie group. Further, let $H \subseteq G$ be a closed subgroup, $f\colon G \to V$ be a compactly supported smooth function and μ_H be a left Haar measure on H. Then the function*

$$F\colon G \to V, \quad F(g) := \int_H f(gh)\, \mathtt{d}\mu_H(h)$$

is smooth.

Proof. For a fixed $x \in G$, pick a compact neighborhood C_x in G and consider the continuous map $\gamma\colon C_x \times H \to G, (x,h) \mapsto xh$. Then $f \circ \gamma$ is compactly supported and we can apply Proposition 9.3.42 to see that F is smooth on C_x°. Since $x \in G$ was arbitrary, we see that F is smooth. □

To prove existence of certain 1-cocycles, the following lemma is a key tool.

Lemma 11.1.9. *Let G be a Lie group and $H \subseteq G$ a closed subgroup for which G/H is compact. Then there is a nonnegative smooth function $w\colon G \to \mathbb{R}$ with compact support such that*

$$\int_H w(gh)\ \mathtt{d}\mu_H(h) = 1 \qquad \textit{for all} \quad g \in G,$$

where μ_H is a left Haar measure on H.

Proof. Let C be a compact $\mathbf{1}$-neighborhood in G, C° its interior, and $\pi\colon G \to G/H, g \mapsto gH$ be the quotient map. Then the sets $\pi(gC^\circ) = g\pi(C^\circ)$ form an open covering of the compact space G/H, so that there exist $g_1, \ldots, g_n \in G$ such that

$$G/H = \bigcup_{i=1}^n g_i\pi(C) = \bigcup_{i=1}^n \pi(g_iC) = \pi\Big(\bigcup_{i=1}^n g_iC\Big).$$

Therefore $D := \bigcup_{i=1}^n g_iC$ is a compact subset of G with $G = DH$. In view of Lemma 7.4.15, there exists a smooth nonnegative function $\chi\colon G \to \mathbb{R}$ with compact support such that $\chi(D) = \{1\}$. We set

$$\varphi(x) := \int_H \chi(xh)\ \mathtt{d}\mu_H(h) \qquad \text{for} \quad x \in G$$

and observe that φ is smooth (Lemma 11.1.8).

For each $x \in G = DH$, there are $h \in H$ and $d \in D$ with $x = dh$, so that $\chi(xh^{-1}) = \chi(d) = 1$ leads to $\varphi(x) > 0$. We put

$$w(x) := \chi(x)/\varphi(x) \qquad \text{for} \quad x \in G.$$

Then w has compact support since χ has compact support. For $y \in H$, the relation $\varphi(xy) = \varphi(x)$ (a consequence of the left invariance of μ_H) leads to

$$\int_H w(xh)\ \mathtt{d}\mu_H(h) = \frac{1}{\varphi(x)} \int_H \chi(xh)\ \mathtt{d}\mu_H(h) = 1$$

for $x \in G$. □

Remark 11.1.10. The function w in the preceding lemma can be used to define for each finite-dimensional vector space V a projection

$$P\colon C^\infty(G,V) \to C^\infty(G,V)^H, \quad P(f)(g) := \int_H w(gh)f(gh)\,\mathtt{d}\mu_H(h),$$

where $C^\infty(G,V)^H$ denotes the subspace of all smooth functions $f\colon G \to V$ which are constant on the H-left cosets, so that they correspond to smooth functions $\overline{f}\colon G/H \to V$ via $\overline{f}(gH) := f(g)$.

That $P(f)$ is smooth follows from Lemma 11.1.8 since $f \cdot w$ is smooth with compact support. The right H-invariance is a direct consequence of the left invariance of the measure μ_H. If f is already H-right invariant, then

$$P(f)(g) = \int_H w(gh)f(g)\,\mathtt{d}\mu_H(h) = \int_H w(gh)\,\mathtt{d}\mu_H(h) \cdot f(g) = f(g),$$

and this implies that P is indeed a projection.

Before we prove the key lemma for the splitting theorem, we observe some properties of cocycles:

Remark 11.1.11. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G and $N \trianglelefteq G$ a normal subgroup with $N \subseteq \ker\rho$.

(a) If $f\colon G \to V$ is a 1-cocycle with respect to ρ, then

$$f(gn) = f(g) + \rho(g)(f(n)) \quad \text{for} \quad g \in G, n \in N$$

and

$$f(ng) = f(n) + f(g) \quad \text{for} \quad g \in G, n \in N.$$

In particular, $f|_N\colon N \to (V,+)$ is a group homomorphism, and we also have

$$\begin{aligned} f(gng^{-1}) &= f(g) + \rho(g)(f(ng^{-1})) = f(g) + \rho(g)(f(n)) + \rho(g)\big(f(g^{-1})\big) \\ &= \rho(g)(f(n)) \end{aligned}$$

because $0 = f(\mathbf{1}) = f(gg^{-1}) = f(g) + \rho(g)\big(f(g^{-1})\big)$ (cf. Exercise 11.1.3).

Lemma 11.1.12. *Let G be a Lie group and $N \subseteq G$ a closed normal subgroup for which G/N is compact. Suppose that $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional smooth representation of G with $N \subseteq \ker\rho$ and $f\colon N \to V$ is a smooth homomorphism which is G-equivariant, i.e.,*

$$f(gng^{-1}) = \rho(g)\big(f(n)\big) \qquad \textit{for} \quad g \in G, n \in N.$$

Then there exists a 1-cocycle $f^*\colon G \to V$ with respect to ρ extending f.

Proof. We use $w \in C^\infty_c(G,\mathbb{R})$ from Lemma 11.1.9 to define the function

$$F\colon G \to V, \quad F(g) := \int_N w(g^{-1}n)f(n)\,\mathtt{d}\mu_N(n).$$

Its smoothness follows from Lemma 11.1.8. For $n \in N$, the left invariance of the Haar measure μ_N immediately implies that

$$\begin{aligned} F(ng) &= \int_N w(g^{-1}n^{-1}n')f(n')\, \mathtt{d}\mu_N(n') = \int_N w(g^{-1}n')f(nn')\, \mathtt{d}\mu_N(n') \\ &= \int_N w(g^{-1}n')\big(f(n) + f(n')\big)\, \mathtt{d}\mu_N(n') = f(n) + F(g). \end{aligned}$$

In this respect F already behaves like a cocycle (cf. Remark 11.1.11). From this we further derive

$$F(gn) = F(gng^{-1}g) = f(gng^{-1}) + F(g) = \rho(g)(f(n)) + F(g).$$

For each $x \in G$, we now define the smooth function $h_x \in C^\infty(G, V)$ by

$$h_x(g) := \rho(g)^{-1}(F(gx) - F(g))$$

and observe that, whenever F is a cocycle, this function is constant with value $F(x)$. Although h_x need not be constant, it is constant on the cosets of N:

$$\begin{aligned} h_x(ng) &= \rho(g)^{-1}(F(ngx) - F(ng)) = \rho(g)^{-1}(f(n) + F(gx) - f(n) - F(g)) \\ &= h_x(g). \end{aligned}$$

We thus obtain a smooth map $\overline{h}_x \colon G/N \to V$, defined by $\overline{h}_x(gN) := h_x(g)$. With the normalized Haar measure $\mu_{G/N}$ of the compact Lie group G/N, we now define

$$f^* \colon G \to V, \qquad f^*(x) := \int_{G/N} \overline{h}_x(gN)\, \mathtt{d}\mu_{G/N}(gN).$$

The smoothness of f^* follows from Proposition 9.3.42 since the map

$$G \times G/N \to V, \quad (x, y) \to \overline{h}_x(y)$$

is smooth as the quotient map $G \times G \to G \times (G/N)$ is a submersion.

For $n \in N$, the relation

$$h_n(g) := \rho(g)^{-1}(F(gn) - F(g)) = f(n)$$

implies that $f^*(n) = f(n)$. We further observe that

$$\begin{aligned} h_{xy}(g) &= \rho(g)^{-1}(F(gxy) - F(g)) = \rho(g)^{-1}(F(gxy) - F(gx) + F(gx) - F(g)) \\ &= \rho(x)h_y(gx) + h_x(g), \end{aligned}$$

so that integration leads to the cocycle identity

$$f^*(xy) = \rho(x)f^*(y) + f^*(x),$$

because

$$\int_{G/N} \rho(x) h_y(gxN) \, \mathrm{d}\mu_{G/N}(gN) = \rho(x) \int_{G/N} h_y(gN \cdot xN) \, \mathrm{d}\mu_{G/N}(gN)$$
$$= \rho(x) \int_{G/N} h_y(gN) \, \mathrm{d}\mu_{G/N}(gN) = \rho(x) f^*(y)$$

follows from Remark 9.3.39 and the right invariance of Haar measure on the compact Lie group G/N (cf. Proposition 9.4.11). This proves that f^* is a smooth 1-cocycle extending f. □

Theorem 11.1.13 (Splitting Theorem). *Let G be a Lie group and $V \subseteq G$ be a normal vector subgroup such that G/V is compact. Then there exists a compact subgroup $K \subseteq G$ such that $G \cong V \rtimes K$.*

Proof. This follows from Lemmas 11.1.7 and 11.1.12 for $N = V$, $\rho(g)v := gvg^{-1}$, and $f(v) := v^{-1} = -v$ (see also Theorem 10.1.15). □

One should perceive the Splitting Theorem as a theorem of the type of the structure theorem for nilpotent Lie groups (Theorem 10.2.10). Just as that theorem, the Splitting Theorem provides a product decomposition of a Lie group G into a compact group and a manifold, diffeomorphic to a vector space. If the groups G and N in the definition of a 1-cocycle are abelian and $N \subseteq G$ is a subgroup, then $f\colon G \to N$ is a 1-cocycle if and only if it is a homomorphism. In this sense, the following lemma also is an extension theorem for 1-cocycles.

Lemma 11.1.14 (Torus Splitting Lemma). *Let T be a torus and $A \subseteq T$ be a closed connected subgroup. Then there is a homomorphism $f\colon T \to A$ with $f|_A = \mathrm{id}_A$. This implies in particular, that for the closed subgroup $B := \ker f$, the multiplication map*

$$\varphi\colon A \times B \to T, \quad (a, b) \mapsto ab$$

is an isomorphism of Lie groups with inverse $\varphi^{-1}(t) = (f(t), f(t)^{-1}t)$.

Proof. We consider the exponential function $\exp_T\colon \mathfrak{t} = \mathbf{L}(T) \to T$. Then $D := \exp^{-1}(A)$ is a closed subgroup of the vector space $\mathfrak{t}$. Its identity component is the set of all elements $x \in \mathfrak{t}$ with $\exp_T(\mathbb{R}x) \subseteq A$, i.e., the Lie algebra $\mathfrak{a}$ of A. If $E \subseteq \mathfrak{t}$ is a complement of $\mathfrak{a}$, then $E \cap D$ is discrete by the Closed Subgroup Theorem 8.3.7, and Exercise 8.3.4 provides linearly independent elements $e_1, \ldots, e_k \in E$ with

$$D = \mathfrak{a} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \ldots + \mathbb{Z}e_k.$$

Since $\ker \exp_T \subseteq D$ and $T \cong \mathfrak{t}/\ker(\exp_T)$ is compact, $\mathfrak{t}$ is spanned by $\mathfrak{a}$ and $e_1, \ldots, e_k$. By assumption, $\exp_T(e_i) \in A = \exp_T(\mathfrak{a})$, so that there also exists an element $a_i \in \mathfrak{a}$ with $\exp_T(a_i) = \exp_T(e_i)$. Then we also have

$$D = \mathfrak{a} \oplus \mathbb{Z}(e_1 - a_1) \oplus \mathbb{Z}(e_2 - a_2) \oplus \ldots \oplus \mathbb{Z}(e_k - a_k),$$

and

$$\ker(\exp_T) = \big(\ker(\exp_T) \cap \mathfrak{a}\big) + \mathbb{Z}(e_1 - a_1) + \mathbb{Z}(e_2 - a_2) + \ldots + \mathbb{Z}(e_k - a_k).$$

Let $\beta\colon \mathfrak{t} \to \mathfrak{a}$ be a linear map with $\beta|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$ and $\beta(e_i - a_i) = 0$ for $i = 1, \ldots, k$. Then $\beta(\ker(\exp_T)) \subseteq \ker(\exp_T)$ implies the existence of a group homomorphism $f\colon T \to A$ with $\mathbf{L}(f) = \beta$. For $a = \exp_T x$ with $x \in \mathfrak{a}$, we then have $f(\exp_T(x)) = \exp_T(\beta(x)) = \exp_T(x)$, which means that $f|_A = \mathrm{id}_A$. □

Definition 11.1.15. We call a connected Lie group G *semisimple*, resp., *simple*[1] if its Lie algebra $\mathbf{L}(G)$ is semisimple, resp., simple.

We want to show that connected semisimple Lie groups with compact Lie algebra are always compact. The following lemma is an important step.

Lemma 11.1.16. *Let G be a connected locally compact group and $D \subseteq G$ a discrete central subgroup such that G/D is compact and the commutator group is dense in G/D. Then D is finite and G is compact.*

Proof. First we show that D is finitely generated. For a subset $C \subseteq G$, we write C° for its interior. As in the proof of Lemma 11.1.7, we find a compact subset $C \subseteq G$ with $G = C^\circ D$. We may assume that C generates the group G because the connectedness of G implies that it is generated by any compact identity neighborhood. Therefore the open sets dC°, $d \in D$, cover in particular the compact subset C^2, and there exist $s_1, \ldots, s_n \in D$ with $C^2 \subseteq \bigcup_{i=1}^n s_i C^\circ$. Let $\Gamma \subseteq D$ be the subgroup generated by $s_1, \ldots, s_n$. Then $C^2 \subseteq \Gamma C$, and one inductively obtains $C^n \subseteq \Gamma C$, which in turn leads to $G = \Gamma C$. Now every d in D can be written as $d = \gamma c$ with $\gamma \in \Gamma$ and $c \in C$. Then $\Gamma \subseteq D$ implies that $c \in D$, and thus $c \in C \cap D$. Consequently, D is generated by $s_1, \ldots, s_n$ and the finite set $C \cap D$. This shows that D is a finitely generated abelian group, and therefore isomorphic to $\mathbb{Z}^r \times F$ for a finite abelian group F.

It remains to show that $r = 0$. For this, we assume that $r > 0$. Then there exists a nonconstant homomorphism $f\colon D \to \mathbb{Z} \subseteq \mathbb{R}$. To apply Lemma 11.1.12, we put $\rho(g) = \mathrm{id}_{\mathbb{R}}$ for $g \in G$ and $N := D$. This lemma now provides a 1-cocycle $f^*\colon G \to \mathbb{R}$ extending f. As the action of G on $\mathbb{R}$ was trivial, f^* is a homomorphism. The subgroup $f^*(G)$ is connected and contains $\mathbb{Z}$, hence coincides with $\mathbb{R}$, and thus f^* is surjective. In view of $f^*(D) \subseteq \mathbb{Z}$, we obtain by factorization a surjective homomorphism $\widetilde{f^*}\colon G/D \to \mathbb{R}/\mathbb{Z}$. This contradicts the density of the commutator group in G/D because $\mathbb{R}/\mathbb{Z}$ is abelian. We conclude that $r = 0$, so that D is finite and $G = CD$ is compact. □

Theorem 11.1.17 (Weyl's theorem on Lie groups with simple compact Lie algebra). *If G is a connected semisimple Lie group with compact Lie algebra, then G is compact and $Z(G)$ is finite.*

[1] Note that with this definition a simple Lie group need not be simple as a group (consider e.g. $\mathrm{SL}_2(\mathbb{R})$ which has nontrivial center).

Proof. First we recall that the semisimplicity of $\mathfrak{g} := \mathbf{L}(G)$ implies that $\mathrm{ad}(\mathfrak{g}) = \mathrm{der}(\mathfrak{g}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{g}))$ (Theorem 4.5.14, Example 3.2.5), so that $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_0$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$. We therefore have $G/Z(G) \cong \mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g}) = \mathrm{INN}(\mathfrak{g})$, and Proposition 11.1.4 implies that $G/Z(G)$ is compact. Further, the center of the semisimple Lie algebra $\mathfrak{g}$ is trivial, so that $\mathbf{L}(Z(G)) = \mathfrak{z}(\mathfrak{g}) = \{0\}$, and thus $Z(G)$ is discrete. By Proposition 10.2.4, the semisimple, connected Lie group $\mathrm{Ad}(G)$ coincides with its commutator group. Hence we can apply Lemma 11.1.16 with $D = Z(G)$ to complete the proof. □

Theorem 11.1.18 (Structure Theorem for Groups with Compact Lie Algebra).

(i) *Every connected Lie group G with compact Lie algebra is a direct product of a vector group V and a uniquely determined maximal compact group K of G which contains all other compact subgroups.*
(ii) *If $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_m$ is the decomposition of the reductive Lie algebra $\mathfrak{k} := \mathbf{L}(K)$ into its center and simple ideals, then the corresponding integral subgroups $Z(K)_0$ and $K_1, \ldots, K_m$ are compact, and the multiplication map*

$$\Phi\colon Z(K)_0 \times K_1 \times \cdots \times K_m \to K, \quad (z, k_1, \ldots, k_m) \mapsto z k_1 \cdots k_m$$

is a covering morphism of Lie groups with finite kernel.
(iii) *The commutator subgroup G' of G is compact.*

Proof. The Lie algebra $\mathfrak{g} := \mathbf{L}(G)$ of G is compact, so that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$, where $\mathfrak{g}' = D^1(\mathfrak{g})$ is compact and semisimple (Lemma 11.1.2). The identity component $Z(G)_0$ of the center of G is a connected abelian Lie group. By Exercise 8.3.5, there exists a closed vector group $V \subseteq Z(G)_0$ and a torus $T \subseteq Z(G)_0$ with $Z(G)_0 \cong V \times T$. By Weyl's Theorem 11.1.17, the normal subgroup $G' = \langle \exp \mathfrak{g}' \rangle$ (cf. Proposition 10.2.4) is compact since it is a continuous image of a connected semisimple Lie group with compact Lie algebra. Consequently, $K := G'T$ is a compact subgroup of G. From $\mathbf{L}(V) + \mathbf{L}(K) = \mathbf{L}(G)$, we derive that $G = \langle \exp_G(\mathbf{L}(K)) \rangle \exp_G(\mathbf{L}(V)) = KV$, and since V contains no nontrivial compact subgroups, we have $V \cap K = \{\mathbf{1}\}$, thus $G \cong V \times K$ (Proposition 10.1.18). If $U \subseteq G$ is any compact subgroup, then the projection of U along K onto V is trivial, so that U is contained in K. Thus (i) and (iii) are proven.

To prove also (ii), we observe that $Z(K)_0$ is closed, hence compact, and that the compactness of the simple factors K_j follows from the simplicity of their Lie algebra and Weyl's Theorem. Since all groups K_j commute pairwise, Φ is a morphism of Lie groups which is surjective because the Lie algebra of its image exhausts $\mathfrak{k}$. Its kernel Γ is a discrete normal subgroup, hence central. Since it intersects $Z(K)_0$ trivially, the projection onto $\prod_{j=1}^m K_j$ maps it injectively into $\prod_{j=1}^m Z(K_j)$ which is finite again by Weyl's Theorem. This proves that Γ is finite. □

Remark 11.1.19. The preceding structure theorem often permits to reduce questions on connected Lie groups G with compact Lie algebra $\mathfrak{g}$ to questions on vector groups and compact Lie groups. For the latter, we have on the Lie algebra level the decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$, but the intersection Γ of the corresponding integral subgroups $Z(G)_0$ and the commutator group $G' := C^1(G)$ may be nontrivial. Since G' is compact semisimple and $\Gamma \subseteq Z(G')$, it is finite by Weyl's Theorem 11.1.17, and the multiplication homomorphism $Z(G)_0 \times G' \to G, (z,g) \mapsto zg$ is a finite covering homomorphism with kernel

$$\{(\gamma, \gamma^{-1})\colon \gamma \in \Gamma\} \cong \Gamma.$$

Here is a concrete example, where Γ is nontrivial.

Example 11.1.20. The group $G := \mathrm{U}_2(\mathbb{C})$ is a simple example. Its Lie algebra is $\mathfrak{g} = \mathfrak{u}_2(\mathbb{C})$ with $\mathfrak{z}(\mathfrak{g}) = i\mathbb{R}\mathbf{1}$ and $\mathfrak{g}' = \mathfrak{su}_2(\mathbb{C})$. Therefore

$$Z(G)_0 = \exp_G(i\mathbb{R}\mathbf{1}) = \{z\mathbf{1}\colon |z| = 1\} \cong \mathbb{T},$$

and $G' = \langle \exp_G \mathfrak{g}' \rangle = \mathrm{SU}_2(\mathbb{C})$. We now observe that

$$Z(G)_0 \cap \mathrm{SU}_2(\mathbb{C}) = \{z\mathbf{1}\colon 1 = \det(z\mathbf{1}) = z^2\} = \{\pm\mathbf{1}\}$$

is nontrivial.

Although it cannot be decomposed as a direct product, we will see below that a compact connected Lie group K always splits as a semidirect product of K' and a torus group A (in general not central), so that $K \cong K' \rtimes A$ (Hofmann–Scheerer Splitting Theorem 11.2.6). Before we can give a proof, we need some information on maximal tori in compact Lie groups.

Corollary 11.1.21. *Let G be a Lie group with finitely many connected components and $\mathbf{L}(G)$ compact. Then there exists a compact subgroup K and a vector group V with $G \cong V \rtimes K$.*

Proof. From Theorem 11.1.18 we know that the identity component G_0 is a product $W \times K_0$, where K_0 is the unique maximal compact subgroup of G_0. Then K_0 is in particular invariant under all automorphisms and therefore normal in G. Since W is central in G_0, we have $W \subseteq \ker \mathrm{Ad}$, so that the adjoint representation factors through a representation of the group G/W. As it has only finitely many connected components and its identity component is a quotient of K_0, hence compact, G/W is compact. Now the Unitarity Lemma 9.4.14 implies the existence of an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g} = \mathbf{L}(G)$. Since K_0 is a normal subgroup, $\mathbf{L}(K_0)$ is $\mathrm{Ad}(G)$-invariant, so that there exists an invariant vector subspace $\mathfrak{v} \subseteq \mathfrak{g}$. The group K_0 acts trivially on $G/K_0 \cong W$, so that $\mathrm{Ad}(G)|_{\mathfrak{v}}$ is a finite group, and thus $\mathfrak{v} \subseteq \mathfrak{z}(\mathfrak{g})$. Let $V := \exp_G(\mathfrak{v})$, $\mathfrak{w} := \mathbf{L}(W)$, and $\mathfrak{k} := \mathbf{L}(K_0)$, so that $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{k}$. The vector space complement $\mathfrak{v}$ of $\mathfrak{k}$ can be written as $\Gamma(\alpha) = \{(w, \alpha(w))\colon w \in \mathfrak{w}\}$, where $\alpha\colon \mathfrak{w} \to \mathfrak{k}$ is a linear map, and since $\mathfrak{v} \subseteq \mathfrak{z}(\mathfrak{g})$, α is a homomorphism of Lie algebras, hence integrates to a group homomorphism $\beta\colon W \to K$ with $\mathbf{L}(\beta) = \alpha$. Then

$$\Gamma(\beta) = \{(w, \beta(w)) \colon w \in W\} \cong W$$

is a closed subgroup of G_0 whose Lie algebra is $\Gamma(\alpha) = \mathfrak{v}$. This shows that $V = \Gamma(\beta)$ is a closed vector subgroup of $Z(G_0)$ satisfying $G_0 \cong V \times K_0$. Hence G/V is compact and the Splitting Theorem 11.1.13 implies the existence of a compact subgroup K of G with $G \cong V \rtimes K$. ◻

Remark 11.1.22. If G is not connected, then we cannot expect a direct decomposition, because the group G may be of the form $G = V \rtimes F$, where F is a finite group acting nontrivially on the vector space V.

Exercises for Section 11.1

Exercise 11.1.1. Let G and N be Lie groups, $\alpha\colon G \to \mathrm{Aut}(N)$ be a homomorphism defining a smooth action of G on N. Then

$$(g, n)(g', n') := (gg', \alpha(g')^{-1}(n)n')$$

defines a Lie group structure on the product set $G \times N$, and we denote it by $G \ltimes_\alpha N$ (a "right" semidirect product). Show that:

(1) The map $\Phi\colon N \rtimes_\alpha G \to G \ltimes_\alpha N, (n, g) \mapsto (g, \alpha(g)^{-1}(n))$ is an isomorphism of Lie groups.
(2) For a map $f\colon G \to N$, the corresponding map $\widetilde{f} := (\mathrm{id}_G, f)\colon G \to G \ltimes_\alpha N$ is a homomorphism if and only if f is a *right crossed homomorphism*, i.e.,

$$f(g_1 g_2) = \alpha(g_2)^{-1}\big(f(g_1)\big) \cdot f(g_2) \quad \text{for} \quad g_1, g_2 \in G.$$

Exercise 11.1.2. Let G and N be Lie groups, $\alpha\colon G \to \mathrm{Aut}(N)$ be a homomorphism defining a smooth action of G on N, and $N \rtimes_\alpha G$ be the corresponding semidirect product. Show that:

(1) If $f\colon G \to N$ is a crossed homomorphism, $\widetilde{f}(g) := (f(g), g)$, and $n \in N$, then $c_{(n,\mathbf{1})} \circ \widetilde{f} = \widetilde{h}$ holds for the crossed homomorphism defined by

$$h(g) := nf(g)\alpha(g)(n^{-1}).$$

(2) For each $n \in N$, the map $f_n\colon G \to N, f_n(g) := n\alpha(g)(n^{-1})$ is a crossed homomorphism. These crossed homomorphisms are called *trivial*.
(3) $(n * f)(g) := nf(g)\alpha(g)(n^{-1})$ defines an action of the group N on the set $\mathrm{Hom}_\alpha(G, N)$ of crossed homomorphisms. The set $H^1(G, N) := \mathrm{Hom}_\alpha(G, N)/N$ of orbits for this action is called the *first cohomology set of G with values in N*.

(4) Each crossed homomorphism $f\colon G \to N$ defines a smooth action of G on the manifold N by
$$g * n := f(g)\alpha(g)(n).$$
This action has a fixed point if and only if f is trivial.

(5) If $f\colon G \to N$ is a crossed homomorphism and $\mathfrak{n} = \mathbf{L}(N)$, then its logarithmic derivative $\delta(f) \in \Omega^1(G, \mathfrak{n})$ is a *left equivariant* 1*-form*, i.e.,
$$\lambda_g^*\delta(f) = \mathbf{L}(\alpha(g)) \circ \delta(f) \quad \text{for} \quad g \in G.$$

(6) Let $\dot{\alpha}\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n})$ denote the derived action of $\mathfrak{g}$ on $\mathfrak{n}$, corresponding to the G-action on $\mathfrak{n}$ by $(g, x) \mapsto \mathbf{L}(\alpha(g))x$. If $f\colon G \to N$ is a crossed homomorphism, then $\mathbf{L}(f) := T_{\mathbf{1}}(f)\colon \mathfrak{g} \to \mathfrak{n}$ is a *crossed homomorphism of Lie algebras*, i.e.,
$$\mathbf{L}(f)([x, y]) = \dot{\alpha}(x)(y) - \dot{\alpha}(y)(x) + [\mathbf{L}(f)x, \mathbf{L}(f)y] \quad \text{for} \quad x, y \in \mathfrak{g}.$$

Exercise 11.1.3. Let $f\colon G \to N$ be a 1-cocycle with respect to a homomorphism $\alpha\colon G \to \mathrm{Aut}(N)$. Show that $f(\mathbf{1}_G) = \mathbf{1}_N$.

Exercise 11.1.4. Let $\mathfrak{g}$ be a compact real Lie subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Show that $\mathfrak{g}$ is isomorphic to $\mathfrak{su}_2(\mathbb{C})$.

11.2 Maximal Tori in Compact Lie Groups

In accordance with our philosophy to study Lie groups by means of their Lie algebras, we approach maximal tori via their Lie algebras, which turn out to be the Cartan subalgebras. From this we derive that every compact connected Lie group is the union of its maximal tori, a fact which has many important consequences.

11.2.1 Basic Results on Maximal Tori

Lemma 11.2.1. *Let $\mathfrak{g}$ be a compact Lie algebra.*

(a) *A subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subalgebra if and only if it is maximal abelian.*

(b) *For any such subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ we have $\mathfrak{g} = \mathrm{Inn}(\mathfrak{g})\mathfrak{t}$, i.e., each element is conjugate to an element of $\mathfrak{t}$.*

(c) *Any other Cartan subalgebra of $\mathfrak{g}$ is conjugate under $\mathrm{Inn}(\mathfrak{g})$ to $\mathfrak{t}$.*

Proof. (a) Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. By Lemma 11.1.2, $\mathfrak{t}$ is compact as a subalgebra of $\mathfrak{g}$, hence in particular reductive. Since it is nilpotent, it is abelian. Since $\mathfrak{t}$ is self-normalizing, it is maximal abelian. Conversely, let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra. To show that $\mathfrak{t}$ is a Cartan subalgebra, we have to show that it is self-normalizing. For any $x \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{t})$, $\mathfrak{t} + \mathbb{R}x$ is a solvable subalgebra of $\mathfrak{g}$, and since it is also compact (Lemma 11.1.2), it is abelian. Now $x \in \mathfrak{t}$ follows from the maximality of $\mathfrak{t}$.

(b), (c) If $x \in \mathfrak{g}$, there exists a maximal abelian subalgebra containing x, simply take one of maximal dimension.

To complete the proof and (b) and (c), it remains to show that any two Cartan subalgebras $\mathfrak{h}$, $\mathfrak{h}'$ of $\mathfrak{g}$ are conjugate under $\mathrm{Inn}(\mathfrak{g})$. Using Proposition 11.1.4, we find a compact connected Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ and an $\mathrm{Ad}(G)$-invariant scalar product β on $\mathfrak{g}$. Let x and x' be regular elements with $\mathfrak{h} = \mathfrak{g}^0(\mathrm{ad}\, x)$ and $\mathfrak{h}' = \mathfrak{g}^0(\mathrm{ad}\, x')$ (Theorem 5.1.18). Since G is compact, there exists an element $g_0 \in G$ for which the function

$$G \to \mathbb{R}, \qquad g \mapsto \beta\big(\mathrm{Ad}(g)x, x'\big)$$

assumes its minimum in g_0. For $y \in \mathfrak{g}$, we then have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \beta\Big(\mathrm{Ad}\left(\exp_G(ty)g_0\right)x, x'\Big) = \frac{d}{dt}\Big|_{t=0} \beta\big(e^{\mathrm{ad}\, ty}\,\mathrm{Ad}(g_0)x, x'\big) \\ &= \beta([y, \mathrm{Ad}(g_0)x], x') = \beta(y, [\mathrm{Ad}(g_0)x, x']). \end{aligned}$$

Then $[\mathrm{Ad}(g_0)x, x'] = 0$ because β is positive definite, and thus $\mathrm{Ad}(g_0)x \in \mathfrak{g}^0(x') = \mathfrak{h}'$. Consequently,

$$\mathfrak{h}' \subseteq \mathfrak{g}^0\big(\mathrm{Ad}(g_0)x\big) = \mathrm{Ad}(g_0)\mathfrak{g}^0(x) = \mathrm{Ad}(g_0)\mathfrak{h},$$

and since $\mathfrak{h}'$ is maximal abelian, $\mathfrak{h}' = \mathrm{Ad}(g_0)\mathfrak{h}$. □

Theorem 11.2.2 (Main Theorem on Maximal Tori). *For a compact connected Lie group G, the following assertions hold:*

(i) *A subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is maximal abelian if and only if it is the Lie algebra of a maximal torus of G.*
(ii) *For two maximal tori T and T', there exists a $g \in G$ with $gTg^{-1} = T'$.*
(iii) *Every element of G is contained in a maximal torus.*

Proof. (i) Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra. Then $T := \langle \exp_G \mathfrak{t} \rangle$ is a connected abelian subgroup of G. Since its closure also centralizes $\mathfrak{t}$ and $\mathfrak{t}$ is maximal abelian, it follows that $T = Z_G(\mathfrak{t})_0$ is closed (cf. Lemma 10.1.1), hence a torus. As $\mathfrak{t}$ is maximal abelian, T is a maximal torus.

Conversely, let $T \subseteq G$ be a maximal torus. Then $\mathfrak{t} = \mathbf{L}(T)$ is an abelian subalgebra of $\mathfrak{g}$ and for any abelian subalgebra $\mathfrak{a} \supseteq \mathfrak{t}$, the closure $A := \overline{\exp_G(\mathfrak{a})}$ is a torus containing T. From the maximality of T we now derive $A \subseteq T$, so that $\mathfrak{a} \subseteq \mathbf{L}(T) = \mathfrak{t}$.

(ii) This immediately follows by (i) and Lemma 11.2.1.

(iii) Let T be a maximal torus. In view of (ii), $M := \bigcup_{g \in G} gTg^{-1}$ is the union of all maximal tori of G. Since M is the image of the continuous map $G \times T \to G, (g,t) \mapsto gtg^{-1}$ and $G \times T$ is compact, M is compact. As G is connected, we can complete the proof by showing that M is also open.

In view of the invariance of M under conjugation, it suffices to show that each element $a = \exp_G(z) \in T$, $z \in \mathfrak{t} = \mathbf{L}(T)$ is an interior point of M. We show this by induction on the dimension of G, and we distinguish two cases:

Case 1: $a \in Z(G)$**:** It suffices to show that $a\exp_G(y)$ belongs to M for all $y \in \mathfrak{g}$ because $a\exp_G(\mathfrak{g})$ is a neighborhood of a in G. Pick $y \in \mathfrak{g}$ and use Lemma 11.2.1 to find a maximal abelian subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$ containing y. By (i), $T' := \exp_G(\mathfrak{t}')$ is a maximal torus of G. Since $T' = gTg^{-1}$ for some $g \in G$, we also have $a = gag^{-1} \in T'$, and therefore, $a\exp_G(y) \in T' \subseteq M$.

Case 2: $a \notin Z(G)$**:** Let $H := Z_G(a)_0$ be the identity component of the centralizer of a. Then H is a proper subgroup of G containing T, so that $\dim H < \dim G$. Our induction hypothesis implies that each element of H is contained in a maximal torus of H, but this implies that it is also contained in a maximal torus of G, so that $H \subseteq M$. The Lie algebra of

$$H = \{g \in G \colon c_a(g) = g\}_0$$

coincides with the fixed space of the automorphism $\mathbf{L}(c_a) = \mathrm{Ad}(a) = e^{\mathrm{ad}\, z}$ of $\mathfrak{g}$, i.e.,

$$\mathfrak{h} := \mathbf{L}(H) = \ker(e^{\mathrm{ad}\, z} - \mathbf{1}).$$

Let $\mathfrak{b} := \mathrm{im}\,(e^{\mathrm{ad}\, z} - \mathbf{1}) \subseteq \mathfrak{g}$ and note that this subspace is $\mathrm{ad}(z)$-invariant and complements $\mathfrak{h}$ (Exercise 11.2.6). We consider the map

$$\Phi \colon G \times H \to G, \qquad (g, h) \mapsto c_g(h) = ghg^{-1}$$

whose range is entirely contained in M. In view of the Implicit Function Theorem, it suffices to show that the tangent map

$$T_{(\mathbf{1},a)}(\Phi) \colon \mathfrak{g} \oplus T_a(H) \to T_a(G)$$

is surjective to see that a is an interior point of M. It is clear that

$$T_a(G) = T_{\mathbf{1}}(\rho_a)\mathfrak{g} = T_{\mathbf{1}}(\rho_a)\mathfrak{h} \oplus T_{\mathbf{1}}(\rho_a)\mathfrak{b}.$$

Clearly, $T_a(H) = T_{\mathbf{1}}(\rho_a)\mathfrak{h}$ is contained in the image of $T_{(0,a)}(\Phi)$ because $\Phi(\mathbf{1}, h) = h$. With $\Phi^a(g) := gag^{-1}$, we further find

$$\begin{aligned} T_{(\mathbf{1},a)}(\Phi)(y,0) &= T_{\mathbf{1}}(\Phi^a)(y) = T_{\mathbf{1}}(\rho_a)y - T_{\mathbf{1}}(\lambda_a)y = T_{\mathbf{1}}(\rho_a)(y - T_{\mathbf{1}}(\rho_a^{-1}\lambda_a)y) \\ &= T_{\mathbf{1}}(\rho_a)(y - \mathbf{L}(c_a)y) = T_{\mathbf{1}}(\rho_a)(y - \mathrm{Ad}(a)y) = T_{\mathbf{1}}(\rho_a)((\mathbf{1} - e^{\mathrm{ad}\, z})y). \end{aligned}$$

From this formula we see that $T_{\mathbf{1}}(\rho_a)\mathfrak{b} = T_{(\mathbf{1},a)}(\Phi)(\mathfrak{g} \times \{0\})$. This proves that $T_{(\mathbf{1},a)}(\Phi)$ is surjective and hence that M is a neighborhood of a. □

With the main theorem on maximal tori, we now have a powerful tool to study compact Lie groups.

Corollary 11.2.3. *The exponential function of a connected Lie group with compact Lie algebra is surjective.*

Proof. Let G be a connected Lie group with compact Lie algebra. By the Structure Theorem 11.1.18, G is a direct product $V \times K$ of a vector group V and a compact Lie group K. Let $g = (v, k) \in G$. Since every element of K is contained in a maximal torus T (Theorem 11.2.2), $g \in V \times T$, and this group is a quotient of $V \times \mathbf{L}(T)$, so that its exponential function is surjective. This proves that $g \in \exp_G(\mathfrak{g})$. □

Corollary 11.2.4. *The center of a connected compact Lie group is the intersection of all maximal tori.*

Proof. Let G be a connected compact Lie group. If $z \in Z(G)$, then it is contained in some maximal torus T (Theorem 11.2.2). Since all other maximal tori are conjugate to T and z is fixed under conjugation, every maximal torus contains z. If, conversely, $g \in G$ is not central, then there is an element g' not commuting with g. Now if T is maximal torus containing g' (Theorem 11.2.2), then $g \notin T$. □

Corollary 11.2.5. *Let G be a compact connected Lie group and $g \in G$. Then g belongs to the connected component $Z_G(g)_0$ of its centralizer $Z_G(g)$. Moreover, $Z_G(g)_0$ is the union of all maximal tori of G containing g.*

Proof. By Theorem 11.2.2(iii) g is contained in some maximal torus, hence in $Z_G(g)_0$. Since $Z_G(g)_0$ is a compact connected Lie group, the same theorem shows that it is a union of its maximal tori. As g is central in $Z_G(g)_0$, these tori all contain g (see Corollary 11.2.4). Hence they are maximal not only in $Z_G(g)_0$, but also in G. This shows that $Z_G(g)_0$ is contained in the union of all maximal tori of G containing g. The converse inclusion is clear. □

11.2.2 Complementing the Commutator Group

Now we can prove the announced splitting theorem:

Theorem 11.2.6 (Hofmann–Scheerer Splitting Theorem). *Let G be a connected compact Lie group and G' be its commutator group. Then there exists a torus $B \subseteq G$ with*

$$G \cong G' \rtimes B.$$

Proof. Let $Z := Z(G)_0$ be the identity component of the center. Then $Z \cap G'$ is central in G', hence contained in any maximal torus $A \subseteq G'$ (Corollary 11.2.4). Now $T := ZA$ is a compact connected abelian Lie group, hence a torus, since A is closed and connected in T. By the Torus Splitting Lemma 11.1.14, there is a torus $B \subseteq T$ with $B \cap A = \{\mathbf{1}\}$ and $AB = T$. Therefore,

$$G = ZG' = ZAG' = BAG' = BG'$$

and

$$B \cap G' \subseteq BA \cap G' = ZA \cap G' = (Z \cap G')A = A$$

leads to $B \cap G' \subseteq B \cap A = \{\mathbf{1}\}$. Now the multiplication map

$$\mu\colon G' \rtimes B \to G, \quad (g, b) \mapsto gb$$

is a smooth bijection of connected Lie groups. Since both groups are compact, μ is a topological isomorphism, hence an isomorphism of Lie groups by the Automatic Smoothness Theorem 8.2.16. Alternatively, one can argue with the Open Mapping Theorem 10.1.8. □

Corollary 11.2.7. *For a compact connected Lie group G with* $\dim Z(G) = r$, *we have*

$$\pi_1(G) \cong \mathbb{Z}^r \times \pi_1(G'),$$

where $\pi_1(G')$ is finite.

Proof. From the Hofmann–Scheerer Splitting Theorem 11.2.6, we know that G is homeomorphic to $G' \times B$, where $\dim B = \dim Z(G)_0 = r$. Since

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

holds for products of pointed topological spaces (Remark A.1.7), we see that $\pi(G) \cong \pi_1(G') \times \pi_1(B)$, and $B \cong \mathbb{T}^r$ implies $\pi_1(B) \cong \mathbb{Z}^r$. Finally we note that $\pi_1(G')$ is finite because its simply connected covering group $\widetilde{G'}$ is semisimple with compact Lie algebra, so that its center is finite by Weyl's Theorem 11.1.17. As $\pi_1(G')$ is isomorphic to a subgroup of $Z(\widetilde{G'})$, it is finite. □

Example 11.2.8. The connectedness is an essential assumption in the preceding theorem. Here are some illustrative examples.

(a) In the algebra $\mathbb{H}$ of quaternions (Section 1.3), we consider the finite subgroup

$$Q := \{\pm\mathbf{1}, \pm I, \pm J, \pm K\},$$

called the *quaternion group*. It is easy to see that its commutator group is $Q' = \{\pm\mathbf{1}\}$ and that $Q/Q' \cong (\mathbb{Z}/2\mathbb{Z})^2$. In particular, Q' is central, but since Q is not abelian, the short exact sequence

$$\mathbf{1} \to Q' \to Q \to Q/Q' \cong (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbf{1}$$

does not split. This example shows that finite groups do not behave like compact connected Lie groups K, for which the short exact sequence

$$\mathbf{1} \to K' \to K \to K/K' \to \mathbf{1}$$

always splits by the Hofmann–Scheerer Theorem.

(b) To obtain examples with a nontrivial identity component, we note that for any reflection $\sigma \in \mathrm{O}_2(\mathbb{R})$, we have

$$\mathrm{O}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R}) \rtimes \{\mathbf{1}, \sigma\} \cong \mathbb{T} \rtimes C_2,$$

where $C_n := \{z \in \mathbb{C}^\times : z^n = 1\}$ is the group of n-th roots of unity. On $O_2(\mathbb{R})$ the involution acts by inversion, so that $\sigma t \sigma^{-1} t^{-1} = t^{-2}$ for $t \in O_2(\mathbb{R})_0$ implies that $O_2(\mathbb{R})' = SO_2(\mathbb{R})$.

To obtain a group whose commutator group does not split, we modify $O_2(\mathbb{R})$ a little by considering

$$G := (SO_2(\mathbb{R}) \rtimes_\gamma C_4)/\{\pm(\mathbf{1}, 1)\},$$

where $\gamma(i)(g) := \sigma g \sigma$ defines the action of $C_4 = \{\pm 1, \pm i\}$ on $SO_2(\mathbb{R})$. We write $[(g, h)]$ for the image of the pair (g, h) in G. Then $G' = G_0 \cong SO_2(\mathbb{R})$ and $G/G' \cong \pi_0(G) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $G \cong G' \rtimes \mathbb{Z}/2\mathbb{Z}$ holds if and only if there exists an element $[(g, i)] = [(-g, -i)]$ of order 2, but

$$[(g, i)]^2 = [(g\sigma g\sigma, -1)] = [(gg^{-1}, -1)] = [(\mathbf{1}, -1)] \neq [(\mathbf{1}, 1)]$$

shows that G contains no such element. [2]

(c) We now discuss a third example, the most complicated one, where $G' = G_0 \cong SU_2(\mathbb{C})$ and $G/G' = \pi_0(G) \cong (\mathbb{Z}/2\mathbb{Z})^4$ does not split.

To construct this example, we start with a finite group F which is an extension of $(\mathbb{Z}/2\mathbb{Z})^4$ by $C_2 = \{\pm 1\}$, where the inverse images e_i of the four generators of $(\mathbb{Z}/2\mathbb{Z})^4$ satisfy the relations

$$e_i^2 = -\mathbf{1} \quad \text{and} \quad e_i e_j = -e_j e_i \quad \text{for } i \neq j.$$ [3]

Now we form the group

$$G := (SU_2(\mathbb{C}) \times F)/\{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}$$

to obtain a 3-dimensional compact Lie group with

$$G' = G_0 \cong SU_2(\mathbb{C}) \quad \text{and} \quad G/G' \cong \pi_0(G) \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

We claim that there is no subgroup $A \subseteq G$ complementing G'. Suppose that A is such a subgroup. Then its generators can be written as $\widetilde{e}_i = [(g_i, e_i)] = [(-g_i, -e_i)]$. We then have the relations

$$\mathbf{1} = \widetilde{e}_i^2 = [(g_i^2, -\mathbf{1})] = [(-g_i^2, \mathbf{1})],$$

which are equivalent to $g_i^2 = -\mathbf{1}$, and for $i \neq j$:

$$[(g_i g_j, e_i e_j)] = \widetilde{e}_i \widetilde{e}_j = \widetilde{e}_j \widetilde{e}_i = [(g_j g_i, e_j e_i)] = [(-g_j g_i, e_i e_j)],$$

which is equivalent to $g_i g_j g_i^{-1} = -g_j$ for $i \neq j$. We now show that there is no quadruple (g_1, g_2, g_3, g_4) of elements of $SU_2(\mathbb{C})$, satisfying these relations.

[2] One can show that $G \cong \mathrm{Pin}_2(\mathbb{R})$ is the pin group in dimension 2, sitting in the Clifford algebra $C_2 \cong \mathbb{H}$ (cf. Definition B.3.20 and Example B.3.24, where i corresponds to $I \in \mathbb{H}$).

[3] One can find this group F as a subgroup of the 16-dimensional Clifford algebra C_4 (cf. Definition B.3.4).

Realizing $\mathrm{SU}_2(\mathbb{C})$ as the set $\{z \in \mathbb{H} \colon \|z\| = 1\}$ of unit quaternions, it is easy to see that the set

$$\{g \in \mathrm{SU}_2(\mathbb{C}) \colon g^2 = -\mathbf{1}\} = \{aI + bJ + cK \colon a^2 + b^2 + c^2 = 1\}$$

is a 2-sphere on which $\mathrm{SU}_2(\mathbb{C})$ acts transitively by conjugation. We may therefore w.l.o.g. assume that $g_1 = I$. Then

$$\{g \in \mathrm{SU}_2(\mathbb{C}) \colon g^2 = -\mathbf{1}, IgI = -g\} = \{bJ + cK \colon b^2 + c^2 = 1\}$$

is a circle, on which the subgroup $\{x\mathbf{1} + yI \colon x^2 + y^2 = 1\} = e^{\mathbb{R}I} \subseteq \mathbb{H}$ acts transitively by conjugation, so that we may w.l.o.g. assume that $g_2 = J$. Then

$$\{g \in \mathrm{SU}_2(\mathbb{C}) \colon g^2 = -\mathbf{1}, IgI = -g, JgJ = -g\} = \{\pm K\},$$

leads, w.l.o.g. to $g_3 = K$, but there is no element $g \in \mathrm{SU}_2(\mathbb{C})$ satisfying $g_j g g_j^{-1} = -g$ for $j = 1, 2, 3$.

11.2.3 Centralizers of Tori and the Weyl Group

In this subsection we show that centralizers of tori in compact connected Lie groups are connected, so that in particular maximal tori are maximal abelian. Moreover, we introduce the analytic Weyl group associated with a maximal torus and compare it to the algebraic Weyl groups form Chapter 5.

Lemma 11.2.9. *If G is a compact abelian Lie group such that G/G_0 is cyclic, then G contains a dense cyclic subsemigroup.*

Proof. Since G_0 is a torus, it contains a dense cyclic subsemigroup generated by some $g_0 \in G_0$ (Exercise 11.2.10). If $g_1 G_0$ generates G/G_0 and n is the order of G/G_0, then $g_1^n \in G_0$ and we find a $g_0' \in G_0$ such that $g_1^n (g_0')^n = g_0$. But then the cyclic semigroup generated by $g_1 g_0'$ contains g_0 and representatives of each coset in G/G_0, hence is dense in G. □

Theorem 11.2.10. *Let G be a compact connected Lie group, $T \subseteq G$ a torus, and $g \in Z_G(T)$. Then there exists a torus $T' \subseteq G$ containing g and T.*

Proof. Let A be the closure of $\bigcup_{n \in \mathbb{Z}} g^n T$ in G. Then A is a compact abelian subgroup of G such that $T \subseteq A_0$. Since the cyclic subgroup generated by gA_0 is dense in the (finite) group A/A_0, it is equal to A/A_0. Then Lemma 11.2.9 shows that there is an $h \in A$ whose powers are dense in A. If $h = \exp_G(x)$ with $x \in \mathfrak{g}$ and T' is the closure of $\exp_G(\mathbb{R}x)$, then T' is a torus containing A, hence g and T. □

Corollary 11.2.11. *Let G be a compact connected Lie group and T a torus in G.*

(i) *The centralizer $Z_G(T)$ of T in G is connected.*

(ii) *If T is a maximal torus, then $Z_G(T) = T$, i.e., maximal tori are maximal abelian.*

Definition 11.2.12. Let G be a compact connected Lie group and T a maximal torus in G. Then the group $W(G,T) := N_G(T)/Z_G(T) = N_G(T)/T$ is called the *analytic Weyl group* associated with (G,T).

Proposition 11.2.13. *Let G be a compact connected Lie group and $T \subseteq G$ a maximal torus.*

(i) *If $t_1, t_2 \in T$ are conjugate under G, then there exists a $g \in N_G(T)$ such that $gt_1g^{-1} = t_2$.*
(ii) *The set of conjugacy classes of G is parameterized by the set $T/W(G,T)$ of $W(G,T)$-orbits in T.*
(iii) *A continuous function $f\colon T \to \mathbb{C}$ extends to a continuous function $F\colon G \to \mathbb{C}$ invariant under conjugation if and only if it is $W(G,T)$-invariant.*

Proof. (i) Suppose that $gt_1g^{-1} = t_2$ and consider the closed subgroup $Z_G(t_2)$ of G. Then T and gTg^{-1} are both maximal tori in $Z_G(t_2)_0$. Thus by Theorem 11.2.2 there exists an $h \in Z_G(t_2)_0$ such that $T = hgT(hg)^{-1}$. But then $hg \in N_G(T)$ satisfying

$$hgt_1(hg)^{-1} = ht_2h^{-1} = t_2.$$

(ii) In view of Theorem 11.2.2, this follows from (i).

(iii) The restriction to T of a conjugation invariant continuous function $F\colon G \to \mathbb{C}$ clearly is $W(G,T)$-invariant. So assume, conversely, that $f\colon T \to \mathbb{C}$ is a $W(G,T)$-invariant continuous function. By (ii), we can define $F\colon G \to \mathbb{C}$ by $F(gtg^{-1}) = f(t)$ for $t \in T$ and it only remains to show that this function is continuous. So assume that $g_n \in G$ and $t_n \in T$ are such that $\lim_{n\to\infty} g_nt_ng_n^{-1} = x \in G$ exists. We have to show that $F(g_nt_ng_n^{-1}) = f(t_n) \to F(x)$. If this is not the case, there exists a neighborhood U of $F(x)$ such that $f(t_n) \notin U$ for infinitely many n. We may therefore assume that $f(t_n) \notin U$ for every $n \in \mathbb{N}$. Since G and T are compact, there exist convergent subsequences $(g_{n_k})_{k\in\mathbb{N}}$ in G and $(t_{n_k})_{k\in\mathbb{N}}$ in T with $g_{n_k} \to g$ and $t_{n_l} \to t$. Then $gtg^{-1} = x$, and this leads to $f(t_{n_k}) \to f(t) = F(x)$, contradicting our assumption. This proves that $f(t_n) \to F(x)$. □

In Lemma 11.2.16 below we shall show that the analytic Weyl group $W(G,T)$ from Definition 11.2.12 is isomorphic to the algebraic Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ introduced in Remark 5.4.4. We need some preparation.

Remark 11.2.14. The Killing form κ of a semisimple compact Lie algebra $\mathfrak{g}$ is negative definite. In fact, by Weyl's Theorem 11.1.17, the simply connected group G with Lie algebra $\mathfrak{g}$ is compact so the Unitarity Lemma 9.4.14 shows that $\mathfrak{g}$ admits an $\mathrm{Ad}(G)$-invariant inner product. Therefore, the linear endomorphisms $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$, being generators of orthogonal one-parameter groups are semisimple with purely imaginary spectrum (cf. Exercise 2.2.7). Thus $\operatorname{tr}(\operatorname{ad}(x)^2) \leq 0$, and κ is negative semidefinite. On the other hand, it is nondegenerate by Cartan's Criterion 4.5.9, so it has to be negative definite.

Lemma 11.2.15. *Let G be a semisimple compact connected Lie group and T a maximal torus in G. Further, let $\mathfrak{t}$ be the Lie algebra $\mathbf{L}(T)$ of T and denote the complexification $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{t}$ by $\mathfrak{h}$ so that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathbf{L}(G)_{\mathbb{C}}$.*

(i) *For each $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ we can choose $h_\alpha, e_\alpha, f_\alpha \in \mathfrak{g}_{\mathbb{C}}(\alpha)$ spanning a copy of $\mathfrak{sl}_2(\mathbb{C})$ with the following properties:*
   (a) *$\overline{e_\alpha} = -f_\alpha$, where $x \mapsto \overline{x}$ is the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$.*
   (b) *If*
$$x_\alpha := \frac{1}{2}(e_\alpha + \overline{e_\alpha}) \quad \textit{and} \quad y_\alpha := \frac{1}{2i}(e_\alpha - \overline{e_\alpha}),$$
   then
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto ih_\alpha, \quad \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto x_\alpha, \quad \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto y_\alpha$$
   defines an isomorphism $\zeta_\alpha \colon \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}(\alpha) \cap \mathfrak{g}$.

(ii) *In the notation of* Lemma 5.3.7 *we have $\mathfrak{t} = i\mathfrak{h}_{\mathbb{R}}$.*

Proof. (i) Choose $h_\alpha, e_\alpha, f_\alpha \in \mathfrak{g}_{\mathbb{C}}(\alpha)$ as in the $\mathfrak{sl}_2$-Theorem 5.3.4. Then $\alpha(h_\alpha) = 2$ and

$$2\kappa(e_\alpha, f_\alpha) = \kappa([h_\alpha, e_\alpha], f_\alpha) = \kappa(h_\alpha, [e_\alpha, f_\alpha]) = \kappa(h_\alpha, h_\alpha) > 0.$$

Remark 11.2.14 implies $\alpha \in i\mathfrak{t}^*$, so that have $\overline{\alpha(x)} = -\alpha(x)$ for $x \in \mathfrak{t}$, and hence $\overline{e_\alpha} \in \mathfrak{g}_{\mathbb{C}}^{-\alpha}$. Thus $f_\alpha = c\overline{e_\alpha}$ and

$$0 > \kappa(e_\alpha + \overline{e_\alpha}, e_\alpha + \overline{e_\alpha}) = 2\kappa(e_\alpha, \overline{e_\alpha})$$

(cf. Remarks 4.5.6 and 11.2.14) shows that $c < 0$. Now we can replace e_α and f_α by $\sqrt{-c}e_\alpha$ and $\sqrt{-c}^{-1} f_\alpha$ such that the conclusion of the $\mathfrak{sl}_2$-Theorem is still valid and, in addition, we have $f_\alpha = -\overline{e_\alpha}$. Note that

$$\operatorname{ad} x_\alpha(y_\alpha) = \tfrac{1}{4i}[e_\alpha + \overline{e_\alpha}, e_\alpha - \overline{e_\alpha}] = \tfrac{1}{2i}[\overline{e}_\alpha, e_\alpha] = \tfrac{1}{2i}[-f_\alpha, e_\alpha] = \tfrac{1}{2i}h_\alpha \in \mathfrak{t}. \quad (11.1)$$

Now it is easy to check the remaining claims.

(ii) Part (i) implies $i\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{g}$. Since $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} + i\mathfrak{h}_{\mathbb{R}}$ we obtain $\mathfrak{t} = \mathfrak{t}_{\mathbb{C}} \cap \mathfrak{g} = i\mathfrak{h}_{\mathbb{R}}$. □

We call a triple $(ih_\alpha, x_\alpha, y_\alpha)$ as constructed in Lemma 11.2.15(i) an $\mathfrak{su}_2$-*triple*.

Lemma 11.2.16. *Let G be a connected compact Lie group and T a maximal torus in G. Then $\widetilde{\iota} : N_G(T) \to \operatorname{Aut}(\mathfrak{t}^*)$, $g \mapsto \operatorname{Ad}^*(g) := \operatorname{Ad}(g^{-1})^*|_{\mathfrak{t}^*}$ induces an isomorphism $\iota \colon W(G,T) \cong N_G(T)/T \to W\big(\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\big)$.*

Proof. Note first that $\mathrm{Ad}^*(g)|_{\mathfrak{t}^*} = \mathrm{id}_{\mathfrak{t}^*}$ implies that $\mathrm{Ad}(g)|_{\mathfrak{t}} = \mathrm{id}_{\mathfrak{t}}$, and hence $c_g|_T = \mathrm{id}_T$. Thus, by Proposition 11.2.13, $\widetilde{\iota}$ indeed factors to an injective homomorphism $\iota\colon W(G,T) \to \mathrm{Aut}(\mathfrak{t}_{\mathbb{C}}^*)$.

Claim 1: $W\big(\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\big) \subseteq \iota\big(W(G,T)\big)$.

It suffices to show that for each $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, the reflection $\sigma_\alpha \in \mathrm{Aut}(i\mathfrak{t}^*) \subseteq \mathrm{Aut}(\mathfrak{t}_{\mathbb{C}}^*)$ is contained in the image of ι. To this end, we apply Lemma 11.2.15(ii) to α. For $h \in \mathfrak{t}_{\mathbb{C}}$ we have

$$\mathrm{ad}\, x_\alpha(h) = \tfrac{1}{2}[e_\alpha + \overline{e_\alpha}, h] = -\tfrac{\alpha(h)}{2}(e_\alpha - \overline{e_\alpha}) = -i\alpha(h) y_\alpha.$$

Together with (11.1), this implies $\mathrm{ad}(x_\alpha)^2(ih_\alpha) = -ih_\alpha$. Therefore,

$$\begin{aligned}
&e^{s\,\mathrm{ad}\, x_\alpha}(ih_\alpha)\\
&= \sum_{m=0}^{\infty} \frac{s^{2m}}{(2m)!}\,\mathrm{ad}(x_\alpha)^{2m}(ih_\alpha) + \sum_{m=0}^{\infty} \frac{s^{2m+1}}{(2m+1)!}[x_\alpha, \mathrm{ad}(x_\alpha)^{2m}(ih_\alpha)]\\
&= \sum_{m=0}^{\infty} \frac{(-1)^m s^{2m}}{(2m)!} ih_\alpha + \sum_{m=0}^{\infty} \frac{(-1)^m s^{2m+1}}{(2m+1)!}[x_\alpha, ih_\alpha]\\
&= \cos(s) ih_\alpha + \sin(s)[x_\alpha, ih_\alpha],
\end{aligned}$$

so for $s = \pi$ and $h \in i\mathfrak{t}$ we obtain

$$e^{\pi\,\mathrm{ad}\, x_\alpha}(h) = h - \alpha(h) h_\alpha.$$

Thus $\mathrm{Ad}^*(\exp \pi x_\alpha) = \sigma_\alpha$, and this proves Claim 1.

Claim 2: $W\big(\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\big) \supseteq \iota\big(W(G,T)\big)$.

Note first that the action of $g \in N_G(T)$ on $\mathfrak{t}_{\mathbb{C}}^*$ permutes the roots as the following calculation shows

$$\begin{aligned}
[h, \mathrm{Ad}(g)e_\alpha] &= \mathrm{Ad}(g)[\mathrm{Ad}(g)^{-1}(h), e_\alpha] = \alpha\big(\mathrm{Ad}(g)^{-1}(h)\big)\,\mathrm{Ad}(g)e_\alpha\\
&= \big(\mathrm{Ad}^*(g)\alpha\big)(h)\big(\mathrm{Ad}(g)e_\alpha\big).
\end{aligned}$$

In particular, $W(G,T)$ acts on the set of bases for the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Fix such a basis Π and an element $g \in N_G(T)$. By Theorem 5.4.17, there exists a $\sigma \in W\big(\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\big)$ such that $\sigma \circ \mathrm{Ad}^*(g)(\Pi) = \Pi$. By Claim 1 we can find an $h \in N_G(T)$ such that $\mathrm{Ad}^*(h)|_{\mathfrak{t}_{\mathbb{C}}^*} = \sigma$. Then we have $\mathrm{Ad}^*(hg)(\Pi) = \Pi$ and it suffices to show that $hg \in T$.

Since hg permutes Π, it fixes $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, where Δ^+ is the positive system associated with Π. Therefore $hg \in Z_G(S)$, where S is the closure of $\exp i\mathbb{R}h_0$, where $h_0 \in i\mathfrak{t}$ is defined by $\rho(h) = \kappa(h_0, h)$ for $h \in \mathfrak{t}$. Corollary 11.2.11 and Lemma 10.1.1 imply that $Z_G(S)$ is connected with Lie algebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, where $\mathfrak{s} := \mathbf{L}(S)$. Thus it only remains to show that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \subseteq \mathfrak{t}$.

Proposition 5.4.16 implies

$$\sigma_\alpha(\rho) = \rho - \alpha = \rho - \rho(\check{\alpha})\alpha$$

for each $\alpha \in \Pi$, so that $\rho(\check{\alpha}) = 1$. This leads to $\alpha(h_0) = (\rho, \alpha) > 0$ for all positive roots α. Thus

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{g} \cap \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{s}) \subseteq \mathfrak{g} \cap \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(h_0) = \mathfrak{g} \cap \mathfrak{t}_{\mathbb{C}} = \mathfrak{t}. \qquad \square$$

Exercises for Section 11.2

Exercise 11.2.1. Show: If G is a Lie group and $H \subseteq G$ is a compact subgroup for which G/H is compact, then G is compact.

Exercise 11.2.2. Let $\mathfrak{g}$ be a compact Lie algebra, β be an invariant scalar product on $\mathfrak{g}$, and let $x_1, \ldots, x_n$ be an orthonormal basis for β. Show that:

(i) The element $\Omega := \sum_{i=1}^n x_i^2$ lies in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ (cf. Exercise 6.1.1).

(ii) Let G be a compact Lie group with $\mathbf{L}(G) = \mathfrak{g}$ and $\alpha \colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G on V and consider the representation

$$\mathcal{U}(\mathbf{L}(\alpha)) \colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V),$$

defined by the derived representation $\mathbf{L}(\alpha)$ of $\mathfrak{g}$ on V. Then $\mathcal{U}(\mathbf{L}(\alpha))\Omega$ is a negative definite operator for every G-invariant scalar product on V.

(iii) Let $\varphi \colon \mathfrak{so}_3(\mathbb{R}) \to \mathfrak{gl}_3(\mathbb{R})$ be the canonical representation on $\mathbb{R}^3$. Compute $\mathcal{U}(\varphi)\Omega$ for an invariant scalar product on $\mathfrak{so}_3(\mathbb{R})$.

One can use the concepts of this section to prove some linear algebra results, resp., to put them into a broader conceptual framework.

Exercise 11.2.3. Show that the diagonal matrices in $\mathfrak{su}_n(\mathbb{C})$ and in $\mathfrak{u}_n(\mathbb{C})$, resp., form a maximal abelian subalgebra.

Exercise 11.2.4. Let $A \in \mathfrak{gl}_n(\mathbb{C})$ be a skew-hermitian (hermitian) matrix. Then there is a unitary matrix $g \in \mathrm{SU}_n(\mathbb{C})$ for which gAg^{-1} is a diagonal matrix.

Exercise 11.2.5. Let $g \in \mathrm{U}_n(\mathbb{C})$ be a unitary matrix. Then there is a unitary matrix $u \in \mathrm{U}_n(\mathbb{C})$ such that ugu^{-1} is a diagonal matrix.

Exercise 11.2.6. Let (V, β) be a euclidian vector space, i.e., β is a positive definite symmetric bilinear form on V. Show that each $g \in \mathrm{O}(V, \beta)$ and each $X \in \mathrm{Sym}(V, \beta)$ is a semisimple endomorphism of V. Conclude in particular that

$$V = (g - \mathbf{1})V \oplus \ker(g - \mathbf{1}) = XV \oplus \ker(X).$$

Exercise 11.2.7. We consider the connected compact Lie group $\mathrm{U}_n(\mathbb{C})$. Show that

$$\mathrm{U}_n(\mathbb{C})' = \mathrm{SU}_n(\mathbb{C}), \quad \big(Z(\mathrm{U}_n(\mathbb{C}))\big)_0 = \mathbb{T}\mathbf{1} \quad \text{and} \quad \mathrm{SU}_n(\mathbb{C}) \cap Z(\mathrm{U}_n(\mathbb{C}))_0 \cong C_n,$$

where $C_n = \{z \in \mathbb{C}^\times : z^n = 1\}$. Show also that the subgroup

$$A := \{\mathrm{diag}(z, 1, \ldots, 1) \colon z \in \mathbb{T}\} \cong \mathbb{T}$$

of $\mathrm{U}_n(\mathbb{C})$ satisfies

$$\mathrm{U}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C}) \rtimes A.$$

Exercise 11.2.8. If $\alpha \colon G_1 \to G_2$ is a quotient homomorphism of topological groups, i.e., an open surjective continuous group homomorphism, then a subgroup $H \subseteq G_2$ is dense if and only if $\alpha^{-1}(H)$ is dense in G_1.

Exercise 11.2.9. We consider the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ with the exponential function

$$\exp \colon \mathbb{R}^n \to \mathbb{T}^n, x \mapsto x + \mathbb{Z}^n.$$

We want to know for which subspaces $V \subseteq \mathbb{R}^n$, the subgroup $\exp(V)$ is dense in $\mathbb{T}^n$. Step by step, show that the following statements are equivalent:

(a) $\exp V$ is dense in $\mathbb{T}^n$.
(b) $V + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$.
(c) There exists no $0 \neq a \in \mathbb{Z}^n$ such that $a^\top V = \{0\}$.
(d) For any basis $\mathcal{B} = \{v_1, \ldots, v_m\}$ for V, the $\mathbb{Q}$-vector space $\sum_i v_i^\top \cdot \mathbb{Q}^n \subseteq \mathbb{R}$ has dimension n.

Exercise 11.2.10. For the special case $V = \mathbb{R}x$, Exercise 11.2.9 implies that $\exp \mathbb{R}x$ is dense in the torus $\mathbb{T}^n$ if and only if the components of x are linearly independent over $\mathbb{Q}$. Additionally, we now assume that they are linearly independent from 1, which can always be achieved by rescaling. In this case, show that the semigroup $S := \exp \mathbb{N}x$ is already dense in $\mathbb{T}^n$ by the following steps:

(a) $A := \overline{S}$ is compact and a subsemigroup of $\mathbb{T}^n$, i.e., $AA \subseteq A$.
(b) There exists a sequence $n_k \in \mathbb{N}$ with $\exp(n_k x) \to \mathbf{1}$.
(c) A is a group.
(d) There exists an $n_0 \in \mathbb{N}$ with $\exp(n_0 x) \in A_0$.
(e) If $A_0 \neq G$, then there exists a $z \in \mathbb{Z}^n \setminus \{0\}$ with $z^\top x \in \mathbb{Z}$. This would be a contradiction to the assumption.
(f) Show the converse: $\overline{S} \neq \mathbb{T}^n$ if $\{1, x_1, \ldots, x_n\}$ is linearly independent over $\mathbb{Q}$.

Exercise 11.2.11. With the notation in Exercise 11.2.9, show that $\dim \overline{\exp V}$ coincides with the dimension of $\sum_{i=1}^m v_i^\top \mathbb{Q}^n$ over $\mathbb{Q}$ in $\mathbb{R}$.

11.3 Linearity of Compact Lie Groups

The goal of this section is to show that each compact Lie group G has a faithful finite-dimensional unitary representation $\pi \colon G \to \mathrm{U}_n(\mathbb{C})$. In particular, each compact Lie group is linear. This is done in three steps. First we construct the regular representation on $L^2(G, \mu_G)$, where μ_G is a Haar measure on G, then we observe that this representation contains enough finite-dimensional subrepresentations to separate the points of G, and then we argue that finitely many of these can be combined to a faithful representation.

Definition 11.3.1. (a) Let $\mathcal{H}$ be a complex Hilbert space with the hermitian scalar product $\langle \cdot, \cdot \rangle$ which is linear in the first argument and antilinear in the second. Then $\|v\| := \sqrt{\langle v, v\rangle}$ is the corresponding norm. We write

$$\mathrm{U}(\mathcal{H}) := \{g \in \mathrm{GL}(\mathcal{H}) \colon (\forall v \in \mathcal{H})\, \|gv\| = \|v\|\}$$

for the *unitary group of* $\mathcal{H}$.

We write $B(\mathcal{H})$ for the set of all bounded operators on $\mathcal{H}$. An element $A \in B(\mathcal{H})$ is said to be *symmetric* if

$$\langle Av, w\rangle = \langle v, Aw\rangle \qquad \text{for} \qquad v, w \in \mathcal{H}.$$

(b) Let G be a topological group. A homomorphism

$$\pi \colon G \to \mathrm{U}(\mathcal{H})$$

is called a *(continuous) unitary representation* of G if for each $v \in \mathcal{H}$, the orbit map

$$\pi^v \colon G \to \mathcal{H}, \qquad g \mapsto \pi(g)v$$

is continuous. We often denote unitary representations as pairs $(\pi, \mathcal{H})$.

(c) An operator $A \in B(\mathcal{H})$ is called an *intertwining operator* of a unitary representation $(\pi, \mathcal{H})$ if

$$A \circ \pi(g) = \pi(g) \circ A \quad \text{for all} \quad g \in G.$$

We write $B_G(\mathcal{H}) \subseteq B(\mathcal{H})$ for the set of intertwining operators.

In the following, μ_G denotes a Haar measure on the Lie group G.

Definition 11.3.2. Let G be a Lie group. On the space $C_c(G) := C_c(G, \mathbb{C})$ of all complex-valued continuous functions on G with compact support

$$\langle f, g\rangle := \int_G f(x)\overline{g(x)}\, \mathtt{d}\mu_G(x)$$

defines a positive definite hermitian form because $0 \neq f$ implies that

$$\int_G |f(x)|^2\, \mathtt{d}\mu_G(x) > 0$$

(cf. Remark 9.3.34). The corresponding norm $\|f\|_2 := \sqrt{\langle f, f\rangle}$ is called the L^2-*norm.* We write $L^2(G) := L^2(G, \mu_G)$ for the completion of $C_c(G)$ with respect to the L^2-norm.

Remark 11.3.3. (a) Suppose that G is compact and μ_G is normalized by $\mu_G(G) = 1$. Then $|f(x)| \leq \|f\|_\infty := \sup\{|f(y)| \colon y \in G\}$ yields the estimate

$$\|f\|_2 \leq \|f\|_\infty.$$

(b) If $\mathcal{H}$ is a Hilbert space and $V \subseteq \mathcal{H}$ a dense subspace, then each isometry $g \colon V \to V$ extends to a unique isometry of $\mathcal{H}$. In fact, if $v \in \mathcal{H}$ is written as $v = \lim_n v_n$ with $v_n \in V$, then $\|gv_n - gv_m\| = \|v_n - v_m\|$ shows that $(gv_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, hence convergent in $\mathcal{H}$, and we put

$$\widetilde{g}v := \lim_n gv_n.$$

Since g is isometric, we obtain the same limit for any other sequence $(w_n)_{n\in\mathbb{N}}$ with $w_m \to v$ because $g(v_n - w_n) \to 0$. We also note that $\|gv_n\| = \|v_n\| \to \|v\|$ implies that $\|gv\| = \|v\|$, so that $\widetilde{g}$ is isometric.

If g is surjective, we apply this process also to g^{-1} and see that g^{-1} extends to an isometry inverting $\widetilde{g}$.

Lemma 11.3.4. *Let G be a topological group, $\mathcal{H}$ a complex Hilbert space and $\pi \colon G \to \mathrm{U}(\mathcal{H})$ a group homomorphism. If there exists a dense subspace $V \subseteq \mathcal{H}$ such that the orbit maps $\pi^v \colon G \to \mathcal{H}$ are continuous for each $v \in V$, then π is a continuous unitary representation.*

Proof. Let $v \in \mathcal{H}$, $g_0 \in G$ and $\varepsilon > 0$. We pick $w \in V$ with $\|v - w\| < \frac{\varepsilon}{3}$ and choose a neighborhood U of g_0 in G with $\|\pi(g)w - \pi(g_0)w\| < \frac{\varepsilon}{3}$ for $g \in U$. Then

$$\begin{aligned}
&\|\pi(g)v - \pi(g_0)v\| \\
&\leq \|\pi(g)v - \pi(g)w\| + \|\pi(g)w - \pi(g_0)w\| + \|\pi(g_0)w - \pi(g_0)v\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore the orbit map of v is continuous. □

Proposition 11.3.5. *Let G be a Lie group. For each $g \in G$, the operator $g * f := f \circ \lambda_{g^{-1}}$ on $C_c(G)$ extends to a unitary operator $\pi(g)$ on $L^2(G)$, and*

$$\pi \colon G \to \mathrm{U}(L^2(G))$$

is a faithful unitary representation of G.

This representation is called the *(left) regular representation of G.*

Proof. The relation

$$\|g * f\|_2^2 = \int_G |f(g^{-1}x)|^2 \, \mathrm{d}\mu_G(x) = \int_G |f(x)|^2 \, \mathrm{d}\mu_G(x) = \|f\|_2^2$$

follows for $f \in C_c(G)$ from the left invariance of μ_G. We also note that $g^{-1} * (g * f) = f$, so that we obtain bijective isometries of $(C_c(G), \|\cdot\|_2)$. Now we use Remark 11.3.3(b) to extend these isometries to unitary operators $\pi(g)$ on $L^2(G)$.

From the relation $g * (h * f) = (gh) * f$ we immediately derive that $\pi(g)\pi(h) = \pi(gh)$, so that π is a homomorphism of groups. To verify the continuity of all orbit maps, we use Lemma 11.3.4 to see that it suffices to do that for $f \in C_c(G)$.

Let $f \in C_c(G)$. We claim that f is uniformly continuous in the sense that for each $\varepsilon > 0$ there exists a $\mathbf{1}$-neighborhood U in G with $|f(x) - f(y)| < \varepsilon$ for $yx^{-1} \in U$. In fact, since f is continuous, there exists for each $x \in G$ an open $\mathbf{1}$-neighborhood U_x such that $|f(y) - f(x)| < \frac{\varepsilon}{2}$ for $y \in U_x^2 \cdot x$. Since the open sets $U_x \cdot x$ cover G, the compactness of the support of f implies the existence of a finite subcover of $\mathrm{supp}(f)$. Hence there exist $x_1, \ldots, x_n \in G$ with $\mathrm{supp}(f) \subseteq \bigcup_{i=1}^n U_{x_i} \cdot x_i$. Let $U := \bigcap_{i=1}^n U_{x_i}$ and $x, y \in G$ with $yx^{-1} \in U$. If both are not contained in the support of f, then $f(x) = f(y)$. So let us assume that x is contained in the support of f, hence in some $U_{x_i} \cdot x_i$. Then $y \in Ux \subseteq U_{x_i}^2 \cdot x_i$, which leads to $|f(y) - f(x_i)| \leq \frac{\varepsilon}{2}$. Since also $|f(x) - f(x_i)| \leq \frac{\varepsilon}{2}$, we obtain $|f(x) - f(y)| \leq \varepsilon$.

Fix $f \in C_c(G)$ and a compact neighborhood C of g in G. Then for $g' \in C$ we have the relation

$$\|g * f - g' * f\|_2 \leq \|g * f - g' * f\|_\infty \sqrt{\mu_G(C \,\mathrm{supp}\, f)}.$$

The uniform continuity of f implies that there exists a symmetric $\mathbf{1}$-neighborhood U in G such that $|f(x) - f(y)|\sqrt{\mu_G(C \,\mathrm{supp}\, f)} \leq \varepsilon$ for $y \in Ux$. Then $g' \in C \cap gU$ implies $\|g * f - g' * f\|_2 \leq \varepsilon$, and this proves that π is a unitary representation.

To see that it is faithful, let $\mathbf{1} \neq g \in G$. With Lemma 7.4.15 we find a continuous function $f \in C_c(G)$ with $f(\mathbf{1}) = 0$ and $f(g^{-1}) = 1$. Then $(\pi(g)f)(\mathbf{1}) = 1$ implies $\pi(g)f \neq f$, so that $\pi(g) \neq \mathrm{id}_{L^2(G)}$, and we see that π is injective. □

There are many possibilities to get to invariant subspaces. For instance, one obtains them as eigenspaces of intertwining operators.

Lemma 11.3.6. *If $A \in B_G(\mathcal{H})$ is an intertwining operator for the unitary representation $(\pi, \mathcal{H})$, then its eigenspaces* $\ker(A - \lambda\mathbf{1})$ *are invariant under $\pi(G)$.*

Proof. For $v \in \ker(A - \lambda\mathbf{1})$ and $g \in G$ we have

$$A\pi(g)v = \pi(g)Av = \pi(g)\lambda v = \lambda\pi(g)v. \qquad \square$$

We have already seen that each Lie group has a faithful unitary representation on $L^2(G)$, and we are looking for finite-dimensional representations. Our goal is to find them in finite-dimensional eigenspaces of intertwining operators. So we need intertwining operators with finite-dimensional eigenspaces, which exist if the operator is compact (see Definition C.1.4).

Lemma 11.3.7. *Let G be a compact Lie group and $\chi \in C(G,\mathbb{R})$ be real with $\chi(x) = \chi(x^{-1})$ for each $x \in G$. Then*

$$K_\chi \colon L^2(G) \to L^2(G), \qquad K_\chi(f)(g) := \langle f, \pi(g)\chi \rangle$$

defines a symmetric compact intertwining operator for the regular representation.

Proof. First we observe that the continuity of π implies that all functions $K_\chi(f)$ are continuous, hence in particular contained in $L^2(G)$.

To show the compactness of K_χ, we want to apply Ascoli's Theorem C.1.5. From the Cauchy–Schwarz inequality we obtain

$$\|K_\chi(f)\|_\infty \leq \|f\|_2 \|\chi\|_2 \leq \|f\|_2 \|\chi\|_\infty,$$

so that the image of any bounded subset of $L^2(G)$ under K_χ is bounded in $C(G)$. We claim that it is equicontinuous, which is the main assumption in Ascoli's Theorem: For $\varepsilon > 0$ and $g \in G$ the continuity of the orbit map π^χ implies the existence of a neighborhood U of g with

$$\|\pi(g')\chi - \pi(g)\chi\|_2 < \varepsilon \quad \text{for} \quad g' \in U.$$

For $f \in L^2(G)$ we then find

$$|K_\chi(f)(g) - K_\chi(f)(g')| = |\langle f, \pi(g)\chi - \pi(g')\chi \rangle| \leq \|f\|_2 \varepsilon.$$

Now Ascoli's Theorem implies that K_χ maps bounded subsets of $L^2(G)$ into relatively compact subsets of $(C(G), \|\cdot\|_\infty)$, and since the inclusion

$$(C(G), \|\cdot\|_\infty) \hookrightarrow L^2(G)$$

is continuous, K_χ is a compact operator.

To see that K_χ is symmetric, it suffices to verify $\langle K_\chi(f), f' \rangle = \langle f, K_\chi(f') \rangle$ for $f, f' \in C(G)$: With Fubini's Theorem 9.3.40, we obtain

$$\begin{aligned}
\langle f, K_\chi(f') \rangle &= \int_G f(g) \overline{\langle f', \pi(g)\chi \rangle}\, \mathtt{d}\mu_G(g) \\
&= \int_G \int_G f(g) \overline{f'(x)} \chi(g^{-1}x)\, \mathtt{d}\mu_G(x) \mathtt{d}\mu_G(g) \\
&\overset{\text{Fubini}}{=} \int_G \int_G f(g) \overline{f'(x)} \chi(x^{-1}g)\, \mathtt{d}\mu_G(g) \mathtt{d}\mu_G(x) \\
&= \int_G \overline{f'(x)} \langle f, \pi(x)\chi \rangle\, \mathtt{d}\mu_G(x) = \langle K_\chi(f), f' \rangle,
\end{aligned}$$

showing that K_χ is symmetric. Finally,

$$\big(\pi(g)K_\chi(f)\big)(x) = \langle f, \pi(g^{-1}x)\chi\rangle = \langle \pi(g)f, \pi(x)\chi\rangle = K_\chi\big(\pi(g)f\big)(x)$$

shows that K_χ is an intertwining operator. ☐

Proposition 11.3.8. *For every element $g \neq \mathbf{1}$ of a compact Lie group G, there exists a finite-dimensional unitary representation $(\rho, \mathcal{H})$ with $\rho(g) \neq \mathbf{1}$.*

Proof. Let $U \subseteq G$ be a symmetric identity neighborhood with $g \notin U^2$. Then $U \cap gU = \emptyset$ and there exists a nonnegative continuous real-valued function η with $\operatorname{supp}(\eta) \subseteq U$ and $\eta(\mathbf{1}) = 1$ (Lemma 7.4.15). Then $\chi(x) := \eta(x)\eta(x^{-1})$ is symmetric, $\operatorname{supp}(\chi) \subseteq U$, and $\operatorname{supp}\big(\pi(g)\chi\big) \subseteq gU$ implies that $K_\chi(\chi)(g) = 0$. But we also have $K_\chi\big(\pi(g)\chi\big)(g) = \|\pi(g)\chi\|^2 > 0$. This proves that $\pi(g)K_\chi \neq K_\chi$, so that Theorem C.3.2(4) implies that, for some $\lambda \neq 0$, $\pi(g)$ acts nontrivially on one of the eigenspaces $L^2(G)_\lambda$ of K_χ (cf. Lemma 11.3.6). Since $L^2(G)_\lambda$ is finite-dimensional (Lemma 11.3.7), $\rho(g) := \pi(g)|_{L^2(G)_\lambda}$ defines a finite-dimensional unitary representation of G with $g \notin \ker \rho$. ☐

At this point, one could go much deeper into the representation theory of compact Lie groups, but our focus in this book is on structural results. We have developed this machinery to prove that all compact Lie groups are linear, which we now derive from Proposition 11.3.8.

Theorem 11.3.9 (Linearity Theorem for Compact Lie Groups). *Each compact Lie group K has a faithful finite-dimensional unitary representation. So each compact Lie group is isomorphic to a matrix group.*

Proof. **Step 1:** We claim that K contains no properly decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed subgroups. We prove this claim by induction on the dimension of K. If $\dim K = 0$, K is a finite group, and our claim is trivial. If $\dim K > 0$ and $(F_n)_{n\in\mathbb{N}}$ is a properly decreasing sequence of subgroups, then not each F_n contains the identity component K_0, because we otherwise obtain a properly decreasing sequence of subgroups of the finite group $\pi_0(K) = K/K_0$. If F_N does not contain K_0, then $\dim F_N < \dim K$, so that the induction hypothesis, applied to F_N, yields a contradiction.

Step 2: We now show that K has a faithful finite-dimensional unitary representation by showing that its nonexistence leads to an infinite properly decreasing sequence of closed subgroups $(F_n)_{n\in\mathbb{N}}$. Let $F_0 := K$. Suppose we do already have a finite properly decreasing sequence of subgroups

$$F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n$$

and finite-dimensional unitary representations $(\rho_j, \mathcal{H}_j)$ of K with

$$F_j = F_{j-1} \cap \ker \rho_j \quad \text{ for } j = 1, \ldots, n.$$

Then

$$\rho := \bigoplus_{j=1}^{n} \rho_j \colon K \to \mathrm{U}(\oplus_{j=1}^{n} \mathcal{H}_j)$$

is a finite-dimensional unitary representation of K with

$$\ker \rho = \bigcap_{j=1}^{n} \ker \rho_j = F_n.$$

Therefore our hypothesis implies that F_n contains some element $k_{n+1} \neq \mathbf{1}$, and we use Proposition 11.3.8 again to find a finite-dimensional unitary representation $(\rho_{n+1}, \mathcal{H}_{n+1})$ of K with $k_{n+1} \not\in \ker \rho_{n+1}$ and put $F_{n+1} := F_n \cap \ker \rho_{n+1}$. This procedure produces a properly decreasing sequence of closed subgroups of K, contradicting Step 1. $\square$

Remark 11.3.10. Although each properly decreasing sequence of closed subgroups of a compact Lie group is finite, compact Lie groups may contain infinite properly increasing sequences of subgroups. A simple example is the sequence $(C_{2^n})_{n \in \mathbb{N}}$ in $\mathbb{T}$.

Exercises for Section 11.3

Exercise 11.3.1. Let $\mathcal{H}$ be a Hilbert space and $\mathrm{U}(\mathcal{H})$ be its unitary group.

(i) If (g_i) is a net in $\mathrm{U}(\mathcal{H})$ that converges weakly to some $g \in \mathrm{U}(\mathcal{H})$, i.e.,

$$\langle g_i v, w \rangle \to \langle gv, w \rangle \quad \text{for all} \quad v, w \in \mathcal{H},$$

then it also converges pointwise to g, i.e., $g_i v \to gv$ for each $v \in \mathcal{H}$.

(ii) The topology of pointwise convergence turns $\mathrm{U}(\mathcal{H})$ into a topological group.

(iii) A continuous unitary representation $(\pi, \mathcal{H})$ of a topological group is the same as a continuous homomorphism $\pi \colon G \to \mathrm{U}(\mathcal{H})$ of topological groups.

11.4 Topological Properties

We conclude this chapter with some results on the topology of compact Lie groups. As for the representation theory of compact groups, we will only have a brief look at the tip of the iceberg.

11.4.1 The Fundamental Group

We start with a description of the fundamental group of a connected compact Lie group in terms of its Lie algebra and the exponential function. It turns out that certain discrete additive subgroups of Cartan subalgebras and their dual spaces play an important role in this context.

Definition 11.4.1. Let T be a torus. Then $\Gamma(T) := \exp_T^{-1}(\mathbf{1})$ is a *lattice*, i.e. a discrete additive subgroup of full rank, in $\mathfrak{t} := \mathbf{L}(T)$, and

$$\mathbf{0} \to \Gamma(T) \to \mathfrak{t} \xrightarrow{\exp_G} T \to \mathbf{1}$$

is an exact sequence of groups. Thus $\Gamma(T)$ is canonically isomorphic to the fundamental group $\pi_1(T)$ of T. We call $\Gamma(T)$ the *unit lattice* of T. Functions on T can thus be lifted to functions on $\mathfrak{t}$ invariant under $\Gamma(T)$.

Recall that $\mathbf{L}\colon \operatorname{Hom}(\mathbb{R}, T) \to \operatorname{Hom}(\mathbb{R}, \mathfrak{t}) \cong \mathfrak{t}$ is a bijection (Lemma 8.2.4). Identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, it follows that the image of $\operatorname{Hom}(\mathbb{T}, T)$ under $\mathbf{L}$ corresponds to $\Gamma(T)$, so that

$$\Gamma(T) \cong \operatorname{Hom}(\mathbb{T}, T), \tag{11.2}$$

where the group structure on the right hand side is given by pointwise multiplication.

Remark 11.4.2. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $T \subseteq G$ a maximal torus with Lie algebra $\mathfrak{t}$. The dimension of T, and hence of any maximal torus (cf. Theorem 11.2.2), is called the *rank* of G. By Lemma 11.2.1 and Proposition 5.1.11, $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, which is reductive by Theorem 11.1.18 and Exercise 4.5.2 (cf. also Proposition 4.7.3). The restriction $\operatorname{Ad}|_T$ is compatible with the root decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$: If we extend the $\operatorname{Ad}(t)$ to complex linear endomorphisms of $\mathfrak{g}_{\mathbb{C}}$, we see that the root spaces $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ for $\alpha \in \Delta := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are $\operatorname{Ad}(T)$-invariant with

$$\operatorname{Ad}(\exp_G x)|_{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = e^{\alpha(x)} \operatorname{id}_{\mathfrak{g}_{\mathbb{C}}^{\alpha}}$$

for $x \in \mathfrak{t}$. Since T is compact, $e^{\mathbb{R}\alpha(x)} \operatorname{id}_{\mathfrak{g}_{\mathbb{C}}^{\alpha}}$ is relatively compact in $\operatorname{GL}(\mathfrak{g}_{\mathbb{C}}^{\alpha})$, which is impossible unless $\alpha(x)$ is purely imaginary. Thus we have $\alpha \in i\mathfrak{t}^* \cong \operatorname{Hom}(\mathfrak{t}, i\mathbb{R})$ (cf. Lemma 5.3.7). Therefore,

$$\Gamma(T) \subseteq \Gamma_c(T) := \{x \in \mathfrak{t} : (\forall \alpha \in \Delta)\, \alpha(x) \in 2\pi i \mathbb{Z}\}, \tag{11.3}$$

which is called the *central lattice* because it coincides with $\exp_T^{-1}(Z(G))$. This follows from Theorem 13.2.8 below, but can also be checked directly using the root decomposition. Note that Corollary 11.2.4 implies that $Z(G) \subseteq T = \exp_T(\mathfrak{t})$, whence

$$Z(G) \cong \Gamma_c(T)/\Gamma(T). \tag{11.4}$$

Definition 11.4.3. For a compact Lie group G with Lie algebra $\mathfrak{g}$ we write

$$X(G) := \operatorname{Hom}(G, \mathbb{T})$$

for the set of continuous homomorphisms $G \to \mathbb{T} \subseteq \mathbb{C}^\times$. Since $\mathbb{T}$ is maximal compact in $\mathbb{C}^\times$, the range of any continuous homomorphism $G \to \mathbb{C}^\times$ lies in $\mathbb{T}$. The set $X(G)$ is an abelian group under pointwise multiplication, called the *character group* of G. Identifying $\mathbf{L}(\mathbb{C}^\times)$ with $\mathbb{C}$ and, accordingly, $\mathbf{L}(\mathbb{T})$ with

$i\mathbb{R}$, the Automatic Smoothness Theorem 8.2.16 and Proposition 8.1.8 show that each element $\chi \in X(G)$ defines an element $\mathbf{L}(\chi) \in \mathrm{Hom}(\mathfrak{g}, i\mathbb{R}) = i\mathfrak{g}^* \subseteq \mathfrak{g}_{\mathbb{C}}^*$ and $\frac{1}{2\pi i}\mathbf{L}(\chi)$ may be viewed as an element of $\mathfrak{g}^*$.

If $G = T$ is a torus, then $\mathbf{L}(\chi)$ maps the unit lattice $\Gamma(T)$ into $2\pi i\mathbb{Z} = \ker \exp_{\mathbb{C}^\times}$. Thus we can define a $\mathbb{Z}$-bilinear form $X(T) \times \Gamma(T) \to \mathbb{Z}$ by

$$\langle \chi, x\rangle := \tfrac{1}{2\pi i}\mathbf{L}(\chi)(x)$$

for $\chi \in X(T)$ and $x \in \Gamma(T) \subseteq \mathfrak{t}$.

Note that $\mathrm{Hom}(\mathbb{T}, \mathbb{T}) \cong \Gamma(\mathbb{T}) \cong \mathbb{Z}$, where the endomorphism corresponding to $n \in \mathbb{Z}$ is simply the power map $p_n(z) = z^n$. Identifying $\Gamma(T)$ with $\mathrm{Hom}(\mathbb{T}, T)$, the pairing from above simply amounts to the composition map

$$\mathrm{Hom}(\mathbb{T}, T) \times \mathrm{Hom}(T, \mathbb{T}) \to \mathrm{Hom}(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}, \quad (\chi, \gamma) \mapsto \chi \circ \gamma$$

which is clearly biadditive.

Proposition 11.4.4. *Let T be a torus. Then the maps*

$$X(T) \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma(T), \mathbb{Z}) \quad \textit{and} \quad \Gamma(T) \to \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$$

induced from the $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot\rangle \colon X(T) \times \Gamma(T) \to \mathbb{Z}$ are isomorphisms of abelian groups.

Proof. Writing T as a product of one-dimensional tori, we see that it suffices to proof this proposition for the unit circle. But in this case it is immediate. ☐

Note that the map $\chi \mapsto \mathbf{L}(\chi)$ embeds $X(T)$ as a lattice into $\mathfrak{t}^*$. Viewing $X(T)$ in this way as a subset of $\mathfrak{t}^*$, Proposition 11.4.4 can be rewritten as

$$X(T) = \{\lambda \in \mathfrak{t}^* : (\forall x \in \Gamma(T))\ \lambda(x) \in 2\pi i\mathbb{Z}\}, \tag{11.5}$$

$$\Gamma(T) = \{x \in \mathfrak{t} : (\forall \lambda \in X(T))\ \lambda(x) \in 2\pi i\mathbb{Z}\}. \tag{11.6}$$

Definition 11.4.5. Let G be a compact connected Lie group and V a finite-dimensional $\mathbb{C}$-vector space. For a continuous homomorphism $\rho \colon G \to \mathrm{GL}(V)$, we set

$$V_\chi(\rho) := \{v \in V : (\forall g \in G)\ \rho(g)v = \chi(g)v\} \quad \text{for} \quad \chi \in X(G).$$

If T a maximal torus in G, the $\chi \in X(T)$ such that $V_\chi(\rho|_T) \neq \{0\}$ are called the *weights* of ρ with respect to T. For $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ let $\rho : G \to \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ be the natural extension by complex linear maps. Then the nontrivial weights of ρ are called *roots*. The set of roots will be denoted by $\Delta(G, T)$. The weight space for $\chi \in \Delta(G, T)$ is called *root space* and denoted by $\mathfrak{g}_{\mathbb{C}}^\chi$. The subgroup of $X(T)$ generated by $\Delta(G, T)$ is called the *root lattice* denoted by $\Gamma_r(T)$.

Lemma 11.4.6. *Let G be a compact connected Lie group and T a maximal torus in G. Further, let S be a closed subgroup of T. Then the identity component $H := Z_G(S)_0$ of $Z_G(S)$ contains T as a maximal torus and satisfies*

(i) $\Delta(H,T) = \{\chi \in \Delta(G,T) : \alpha(S) = \{1\}\}$.
(ii) $Z(H) = \bigcap_{\chi\in\Delta(H,T)} \ker\chi$.

Proof. $x \in \mathbf{L}\left(Z_G(S)\right)$ if and only if $\exp_G(\mathbb{R}x) \subseteq Z_G(S)$, so that

$$\mathbf{L}\left(Z_G(S)\right) = \mathbf{L}(H) = \{x \in \mathfrak{g} : (\forall g \in S)\ \mathrm{Ad}(g)(x) = x\}.$$

Therefore, $\mathbf{L}(H)_{\mathbb{C}}$ is the direct sum of $\mathfrak{t}_{\mathbb{C}}$ and all root spaces $\mathfrak{g}_{\mathbb{C}}^{\chi}$ with $\chi(g) = 1$ for all $g \in S$. This proves (i). To prove (ii) we recall from Lemma 8.2.21 that $Z(H)$ is the kernel of the adjoint representation. Since $Z(H) \subseteq T$ by Corollary 11.2.4, the claim follows. □

Remark 11.4.7. Let G be a compact connected Lie group and T a maximal torus in G. For $\chi \in \Delta(G,T)$ the derived homomorphism $\mathbf{L}(\chi)\colon \mathbf{L}(G) \to \mathbf{L}(\mathbb{T})$ can be viewed as an element $\mathbf{L}(\chi) \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Comparing the weight decompositions of $\mathrm{Ad}|_T$ and $\mathrm{ad}|_{\mathfrak{t}}$, we see that $\Delta(G,T) \to \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\chi \mapsto \mathbf{L}(\chi)$ is a bijection. For any $\chi \in \Delta(G,T)$, set

$$x_\chi := 2\pi i\, \mathbf{L}(\chi)^{\check{}},$$

where $\mathbf{L}(\chi)^{\check{}}$ is the coroot of $\mathbf{L}(\chi)$ (cf. the $\mathfrak{sl}_2$-Theorem 5.3.4). We call

$$\{2\pi i\check{\alpha} : \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\} = \{x_\chi \in \mathfrak{t} : \chi \in \Delta(G,T)\} \tag{11.7}$$

the set of *nodal vectors* for T and note that it only depends on the Lie algebra $\mathbf{L}(T)_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}$.

Proposition 11.4.8. *Let G be a compact connected Lie group and T a maximal torus in G. For any root $\chi \in \Delta(G,T)$ we have the following properties.*

(i) $Z_G(\ker\chi) = Z_G\big((\ker\chi)_0\big)$ *is a closed connected subgroup of dimension* $2 + \dim T$. *Its complexified Lie algebra is*

$$\mathbf{L}\left(Z_G(\ker\chi)\right)_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{\chi} + \mathfrak{g}_{\mathbb{C}}^{\chi^{-1}} + \mathfrak{t}_{\mathbb{C}}. \tag{11.8}$$

(ii) *The commutator group G_χ of $Z_G(\ker\chi)$ is a closed connected subgroup of rank* 1 *and dimension* 3.
(iii) $\Delta(Z_G(\ker\chi), T) = \{\chi, \chi^{-1}\}$.
(iv) $G_\chi \cap T$ *is a maximal torus in* G_χ.

Proof. From Corollary 11.2.11 we know that the group $Z_G\big((\ker\chi)_0\big)$ is connected. Lemma 11.4.6(i) implies that $\Delta\big(Z_G\big((\ker\chi)_0\big), T\big)$ consists of all $\eta \in \Delta(G,T)$ with $\eta\big((\ker\chi)_0\big) = \{1\}$. Therefore the linear functional $\mathbf{L}(\eta)$ vanishes on the kernel of $\mathbf{L}(\chi)$ and hence is a multiple of $\mathbf{L}(\chi)$. In view of Remark 11.4.7 it now follows from Remark 5.4.4 that $\eta \in \{\chi, \chi^{-1}\}$. Thus we have shown

$$\Delta\big(Z_G\big((\ker\chi)_0\big), T\big) = \{\chi, \chi^{-1}\},$$

so that Lemma 11.4.6(ii) implies

$$Z\big(Z_G\big((\ker\chi)_0\big)\big) = \ker\chi. \tag{11.9}$$

We clearly have $Z_G(\ker\chi) \subseteq Z_G((\ker\chi)_0)$ and (11.9) implies the converse inclusion, so that $Z_G\big((\ker\chi)_0\big) = Z_G(\ker\chi)$. This proves the connectedness of $Z_G(\ker\chi)$ and (iii). But then Theorem 11.1.18 implies that G_χ is closed, connected, and semisimple. Note that

$$\mathbf{L}\left(Z_G\big((\ker\chi)_0\big)\right)_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{\chi} + \mathfrak{g}_{\mathbb{C}}^{\chi^{-1}} + \mathfrak{t}_{\mathbb{C}}, \tag{11.10}$$

Theorem 11.2.2, and Corollary 11.2.11 imply that $(T\cap G_\chi)_0$ is a maximal torus in G_χ, hence maximal abelian. Therefore $T \cap G_\chi$ is connected, i.e., a maximal torus in in G_χ. The remaining claims are now clear. □

Remark 11.4.9. Let G be a compact connected Lie group and $H \leq G$ a closed connected subgroup. Then the rank of H is less or equal to the rank of G. If they are equal, then H is called a subgroup of *maximal rank* of G. This is equivalent to H containing a maximal torus of G. If $Z \leq G$ is a closed central subgroup, then by Corollary 11.2.4, Z is contained in every maximal torus. Therefore T/Z is a torus subgroup of G/Z for each maximal torus T in G. Now Theorem 11.1.18 shows that T/Z is indeed a maximal torus in G/Z and then the conjugacy of maximal tori (see Theorem 11.2.2) implies that the maximal tori of G are precisely the preimages of the maximal tori in G/Z under the canonical projection.

All closed connected subgroups of maximal rank in G contain Z and they are precisely the preimages of the closed connected subgroups of maximal rank in G/Z under the canonical projection. All these preimages are connected because Z is contained in any maximal torus T, so that any subgroup $H \subseteq G$ containing T for which $H/Z \subseteq G/Z$ is connected is connected because T and H/T are connected (Exercise 11.4.1).

Proposition 11.4.10. *Let G be a compact connected Lie group, T a maximal torus in G, and $\chi \in \Delta(G,T)$. Then there exists a morphism of Lie groups $\varphi\colon \mathrm{SU}_2(\mathbb{C}) \to G$ satisfying the following conditions.*

(a) *The image of φ commutes with the kernel of χ.*
(b) *For all $a \in \mathbb{T}$ we have $\varphi\begin{pmatrix} a & 0 \\ 0 & \overline{a}\end{pmatrix} \in T$ and $\chi\circ\varphi\begin{pmatrix} a & 0 \\ 0 & \overline{a}\end{pmatrix} = a^2$.*

Proof. Recall the isomorphism $\zeta_\alpha\colon \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}(\alpha)\cap\mathfrak{g}$ for $\alpha = \mathbf{L}(\chi) \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ from Lemma 11.2.15 (cf. Remark 11.4.7). Since $\mathrm{SU}_2(\mathbb{C})$ is simply connected (see Example 8.5.7), it follows from Proposition 11.4.8, the Integrability Theorem 8.5.9, and Proposition 8.5.1 that there exists a covering morphism $\varphi\colon \mathrm{SU}_2(\mathbb{C}) \to G_\chi$ with $\mathbf{L}(\varphi) = \zeta_\alpha$. Note that

$$\mathbf{L}(\chi\circ\varphi)\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} = \alpha\circ\zeta_\alpha\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} = i\alpha(h_\alpha) = 2i.$$

Therefore

$$\chi \circ \varphi \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = e^{2it}$$

and this proves the claim. □

Remark 11.4.11. Let G be a compact connected Lie group and T a maximal torus in G. Given $\chi \in \Delta(G,T)$ fix $\varphi\colon \mathrm{SU}_2(\mathbb{C}) \to G$ as in Proposition 11.4.10. Then $\varphi_T : \mathbb{T} \to T$, defined by

$$\varphi_T(a) = \varphi \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix},$$

induces a homomorphism $\Gamma(\varphi_T)\colon \Gamma(\mathbb{T}) \to \Gamma(T)$. Since $\Gamma(\mathbb{T}) = 2\pi i\mathbb{Z}$ we may consider $\widetilde{x}_\chi := \Gamma(\varphi_T)(2\pi i) \in \Gamma(T) \cap [\mathfrak{g}_\mathbb{C}^\chi, \mathfrak{g}_\mathbb{C}^{\chi^{-1}}] \subseteq \mathfrak{t}$. Then

$$\langle \chi, \widetilde{x}_\chi \rangle = \frac{1}{2\pi i}\,\mathbf{L}(\chi)\big(\Gamma(\varphi)(2\pi i)\big) = \mathbf{L}(\chi \circ \varphi_T)(1) = 2$$

and

$$\widetilde{x}_\chi = 2\pi i\,\mathbf{L}(\chi)^{\check{}}\,.$$

Thus in the terminology of Remark 11.4.7 we have $x_\chi = \widetilde{x}_\chi \in \Gamma(T)$ and we see that all nodal vectors for T are contained in $\Gamma(T)$. We denote the subgroup of $\Gamma(T)$ generated by the nodal vectors by $\Gamma_0(T)$ and call it the *nodal group* of T.

Remark 11.4.12. We collect some information on the various lattices we have encountered in this chapter. From Remark 11.4.11 and (11.3) we know

$$\Gamma_0(T) \subseteq \Gamma(T) \subseteq \Gamma_c(T). \tag{11.11}$$

These inclusions are not always strict. For instance, if the center of G is trivial, then the characterization $\Gamma_c(T) = \exp_T^{-1}\big(Z(G)\big)$ shows that $\Gamma(T) = \Gamma_c(T)$. Moreover, in this case $Z(G) \cong \Gamma(T)/\Gamma_0(T)$ (see Remark 11.4.2) also shows $\Gamma(T) = \Gamma_0(T)$. We define the *dual lattice* Γ^* of a lattice $\Gamma \subseteq \mathfrak{t}$ by

$$\Gamma^* := \{\nu \in i\mathfrak{t}^* : (\forall x \in \Gamma)\ \nu(x) \in 2\pi i\mathbb{Z}\}.$$

Equation (11.11) yields the following chain of lattices

$$\Gamma_0(T)^* \supseteq \Gamma(T)^* \supseteq \Gamma_c(T)^*. \tag{11.12}$$

Embedding $X(T)$ into $i\mathfrak{t}^*$ via $\chi \mapsto \mathbf{L}(\chi)$, Proposition 11.4.4 implies that $\Gamma(T)^* = X(T)$. Comparing Definition 6.3.10 with (11.7), we see that $\Gamma_0(T)^*$ is the weight lattice Λ of integral linear functionals in $i\mathfrak{t}^*$. Finally (11.3) shows that $\Gamma_c(T)^*$ is the *root lattice*

$$R(G,T) := \langle \Delta \rangle_{\mathrm{grp}}.$$

Thus (11.12) can be rewritten as

$$\Lambda \supseteq X(T) \supseteq R(G,T). \tag{11.13}$$

Proposition 11.4.13. *Let G be a compact connected Lie group and $H \leq G$ a closed connected subgroup of maximal rank.*

(i) *The homogeneous space G/H is 1-connected.*
(ii) *The natural homomorphism $\pi_1(H) \to \pi_1(G)$ induced by the inclusion $H \hookrightarrow G$ is surjective.*

Proof. According to the Homotopy Group Theorem 10.1.15 and Remark 10.1.17, we have an exact sequence $\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \mathbf{1}$, so (i) is equivalent to (ii). Note that, in order to prove (i), Remark 11.4.9 allows us to first replace G by the semisimple group $\mathrm{Ad}(G)$, and then $\mathrm{Ad}(G)$ by its compact simply connected covering (cf. Weyl's Theorem 11.1.17). In other words, we may assume that G is semisimple and 1-connected. But under these hypotheses (ii) is trivially true. □

Theorem 11.4.14. *If G is a compact connected Lie group and T a maximal torus in G, then the canonical homomorphism $\Gamma(T) \cong \pi_1(T) \to \pi_1(G)$ induces an isomorphism*

$$\pi_1(G) \cong \Gamma(T)/\Gamma_0(T).$$

Proof. Proposition 11.4.13 implies that the homomorphism

$$\Phi_{G,T} \colon \Gamma(T) \cong \pi_1(T) \to \pi_1(G),$$

induced by the inclusion $T \hookrightarrow G$, is surjective. We have to show that its kernel is the nodal group $\Gamma_0(T)$ of T. To do that, we prove the assertion

$\mathbf{A}(G,T)$**:** $\ker \Phi_{G,T}$ is generated by the nodal vectors for T

for all pairs (G,T) satisfying the hypotheses above. Before we can do this, we have to consider two special cases:

Case 1: G is 1-connected.

In this case $\mathbf{A}(G,T)$ amounts to showing that $\Gamma_0(T) = \Gamma(T)$. Since G is compact, the hypothesis implies that in this case $\mathfrak{g}$, and hence also $\mathfrak{g}_{\mathbb{C}}$, is semisimple (Theorem 11.1.18). Let $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ be any dominant integral weight (see Definition 6.3.10). Then by Proposition 6.3.14 there exists a finite-dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ having λ as highest weight. Since G is 1-connected, the Integrability Theorem 8.5.9 shows that this representation integrates to a representation of G, so that λ is of the form $\mathbf{L}(\chi)$ for some $\chi \in X(T)$. In view of Remark 11.4.12, in particular (11.13), this implies $\Lambda = X(T)$, and hence $\Gamma_0(T) = \Gamma(T)$.

Case 2: G is the direct product of a 1-connected group $G^\sharp$ with a torus S.

In this case $T = T^\sharp \times S$, where $T^\sharp$ is a maximal torus in $G^\sharp$, $\Gamma(T) = \Gamma(T^\sharp) \times \Gamma(S)$, and $\pi_1(G) \cong \pi_1(G^\sharp) \times \pi_1(S)$. Moreover, $\Phi_{G,T} = \Phi_{G^\sharp,T^\sharp} \times \Phi_{S,S}$. Since the map $\Phi_{S,S}$ is an isomorphism, $\mathbf{A}(G^\sharp, T^\sharp)$, which was shown in Case 1, implies $\mathbf{A}(G,T)$.

Case 3: General case: The Structure Theorem 11.1.18 shows that there is a covering morphism $p : G^\sharp \to G$ such that $G^\sharp$ is the direct product of a

1-connected group with a torus. In particular, we may assume that $\mathbf{L}(G^\sharp) = \mathbf{L}(G)$ and $\mathbf{L}(p)\,\mathrm{id}_{\mathbf{L}(G)}$. Corollary 11.2.4 implies that

$$\ker p \subseteq Z(G^\sharp) \subseteq T^\sharp := \exp_{G^\sharp}(\mathbf{L}(T)) = p^{-1}(T).$$

Therefore we have $T \cong T^\sharp/\ker p$ and $\Gamma(T) = \exp_{T^\sharp}^{-1}(\ker p)$, so that $\ker p \cong \Gamma(T)/\Gamma(T^\sharp)$. Further we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \Gamma(T^\sharp) & \longrightarrow & \mathbf{L}(T^\sharp) & \xrightarrow{\exp_{T^\sharp}} & T^\sharp & \longrightarrow & \mathbf{1}\\
 & & \big\downarrow & & \| & & \big\downarrow p & & \\
\mathbf{0} & \longrightarrow & \Gamma(T) & \longrightarrow & \mathbf{L}(T) & \xrightarrow{\exp_T} & T & \longrightarrow & \mathbf{1}
\end{array}$$

Since the inverse of the natural isomorphism $\ker p \to \Gamma(T)/\Gamma(T^\sharp)$ is induced by $\exp_{T^\sharp} \colon \Gamma(T) \to T^\sharp$, we have another commutative diagram

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \ker(\Phi_{G^\sharp,T^\sharp}) & \longrightarrow & \ker(\Phi_{G,T}) & \longrightarrow & \mathbf{0} & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\mathbf{0} & \longrightarrow & \Gamma(T^\sharp) & \hookrightarrow & \Gamma(T) & \longrightarrow & \ker p & \longrightarrow & \mathbf{1}\\
 & & \big\downarrow \Phi_{G^\sharp,T^\sharp} & & \big\downarrow \Phi_{G,T} & & \| & & \\
\mathbf{0} & \longrightarrow & \pi_1(G^\sharp) & \longrightarrow & \pi_1(G) & \longrightarrow & \ker p & \longrightarrow & \mathbf{1}
\end{array}$$

in which the second line is exact. Remark 10.1.17 and Corollary A.2.7 show that also the third line is exact. Then a simple diagram chase shows that the first line exact as well. In other words, $\ker(\Phi_{G^\sharp,T^\sharp})$ and $\ker(\Phi_{G,T})$ agree, and this implies the claim, because Case 2 applies to $(G^\sharp, T^\sharp)$. $\square$

11.4.2 Fixed Points of Automorphisms

We conclude this chapter with a topological result which is very useful in the study of symmetric spaces: The set of fixed points of an automorphism of a compact connected Lie group is connected. To prove this result we need substantial preparation. In particular, we have to take a closer look at regular and singular points in a compact connected Lie group. Further, we introduce the diagram $D(G,T)$ of a maximal torus and the resulting alcoves as technical tools.

Definition 11.4.15. A *pinning* of a compact connected Lie group G is a triple $E = (T, \Pi, \{v_\alpha\}_{\alpha\in\Pi})$, where T is a maximal torus of G, Π a basis for the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and $v_\alpha \in (\mathfrak{g}_{\mathbb{C}}^\alpha \oplus \mathfrak{g}_{\mathbb{C}}^{-\alpha}) \cap \mathfrak{g}$ satisfies $-\kappa(v_\alpha, v_\alpha) = 1$ for the Killing form κ on $\mathfrak{g}$.

It is clear that $\mathrm{Aut}(G)$ acts on the set $\mathcal{E}(G)$ of all pinnings.

Proposition 11.4.16. *Let G be a compact connected Lie group. Then the group of inner automorphisms of G acts simply transitively on $\mathcal{E}(G)$.*

Proof. Let $e = (T, \Pi, \{v_\alpha\}_{\alpha\in\Pi})$ and $e' = (T', \Pi', \{v'_\alpha\}_{\alpha\in\Pi'})$ be two pinnings of G. By Theorem 11.2.2(iii), there exists a $g \in G$ such that $c_g(T) = T'$. Thus we may assume that $T = T'$. By Theorem 5.4.17, there exists a $\sigma \in W\big(\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})\big)$ such that $\sigma(\Pi) = \Pi'$. According to Lemma 11.2.16, we can find an element $g \in N_K(T)$ such that $\mathrm{Ad}^*(g)|_{\mathfrak{t}^*} = \sigma$. Thus we also may assume that $\Pi = \Pi'$. So to show the transitivity, it remains to show that T acts transitively on the set of tuples $\{v_\alpha\}_{\alpha\in\Pi}$ with

$$v_\alpha \in \mathfrak{g}^{[\alpha]} := (\mathfrak{g}_\mathbb{C}^\alpha \oplus \mathfrak{g}_\mathbb{C}^{-\alpha}) \cap \mathfrak{g} \quad \text{and} \quad \kappa(v_\alpha, v_\alpha) = -1.$$

The spaces $\mathfrak{g}^{[\alpha]}$ have real dimension 2, by Lemma 11.2.15. We equip them with the scalar products coming from $-\kappa$. Then it suffices to show that the morphism $\iota : T \to \prod_{\alpha\in\Pi} \mathrm{SO}(\mathfrak{g}^{[\alpha]})$, obtained by the restriction of the adjoint action, is surjective. But $\mathbf{L}(\iota)\colon \mathfrak{t} \to \bigoplus_{\alpha\in\Pi} \mathfrak{so}(\mathfrak{g}^{[\alpha]}) \cong \mathbb{R}^\Pi$ is simply given by $x \mapsto \big(i\alpha(x)\big)_{\alpha\in\Pi}$. Since Π is a basis for $\mathfrak{t}^*$ this implies that $\mathbf{L}(\iota)$, and hence also ι, is surjective. Thus we have shown that G acts transitively on $\mathcal{E}(G)$.

To conclude the proof, we have to show that $g \cdot e = e$ implies $g \in Z(G)$. It is clear from $c_g(T) = T$ that $g \in N_G(T)$, and $g \cdot \Pi = \Pi$ implies that $g \in T$ (see Theorem 5.4.17). Pick $x \in \mathfrak{t}$ with $g = \exp_T x$. Then, for each $\alpha \in \Pi$, the relation $g \cdot v_\alpha = v_\alpha$ implies $e^{i\alpha(x)} = 1$. Therefore $\alpha(2\pi i x) \in \mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, whence $x \in \Gamma_c(T) = \exp_T^{-1}\big(Z(G)\big)$ by Remark 11.4.2. □

Lemma 11.4.17. *Let σ be an automorphism of the compact connected Lie group G and F its group of fixed points. If the identity component F_0 of F is central in G, then G is abelian, hence a torus.*

Proof. By Proposition 10.2.4, the group G is abelian if and only if its commutator subgroup $D^1(G)$ is commutative. Since $D^1(G)$ is invariant under σ, we may therefore assume that G is semisimple.

If F_0 is central in G, its Lie algebra $\mathbf{L}(F)$ is trivial, because the center of $\mathfrak{g}$ is trivial. It follows that $\mathbf{L}(\sigma) - \mathrm{id}$ is bijective, because its kernel is contained in $\mathbf{L}(F)$. The map $f\colon G \to G$ defined by $f(g) = \sigma(g)^{-1}g$ is a submersion since its differential, which is given in g by

$$T_g(f)(xg) = \sigma(g)^{-1}xg - \sigma(g)^{-1}T_\mathbf{1}(\sigma)(x)\sigma(g)\sigma(g)^{-1}g = \sigma(g)^{-1}\big(x - T_\mathbf{1}(\sigma)(x)\big)g$$

for $x \in \mathfrak{g}$ and $g \in G$, is bijective. In particular, $f(G)$ is open in G. Since G is compact and connected, f is surjective.

By Proposition 11.4.16, for $E \in \mathcal{E}(G)$ one finds a unique $h \in G$ such that $\sigma(E) = c_h(E)$. Pick $g \in G$ such that $h = f(g) = \sigma(g)^{-1}g$. Then

$$\sigma \circ c_g = c_{\sigma(g)} \circ \sigma = c_g \circ c_h^{-1} \circ \sigma.$$

Therefore, by the choice of h, the pinning $c_g(E)$ is stable under σ. In particular, $\mathbf{L}(\sigma) \sum_{\alpha\in B} v_\alpha = \sum_{\alpha\in B} v_\alpha$, so that $\sum_{\alpha\in B} v_\alpha = 0$. This implies that the v_α are zero as the root space decomposition is direct. This contradicts $\kappa(v_\alpha, v_\alpha) = -1$ unless $\mathfrak{g} = \{0\}$, proving the claim. $\square$

Definition 11.4.18. Let G be a compact connected Lie group and T a maximal torus in G. Each root $\alpha \in \Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ corresponds to a character $\chi_\alpha \colon T \to \mathbb{C}^\times$ satisfying $\chi_\alpha(\exp x) = e^{\alpha(x)}$ (Remark 11.4.7). The sets

$$U_\alpha := \ker \chi_\alpha = \{t \in T : t = \exp_G x; \alpha(x) \in 2\pi i\mathbb{Z}\}$$

are closed subgroups of T of codimension 1.

(a) We define the *diagram* $D(G, T)$ of G with respect to T as the union of the sets

$$\exp_T^{-1}(U_\alpha) = \{x \in \mathfrak{t} : \alpha(x) \in 2\pi i\mathbb{Z}\}$$

for $\alpha \in \Delta$. Each of these sets is a countable union of parallel affine hyperplanes, hence the diagram is a countable union of affine hyperplanes as well. The connected components of $\mathfrak{t} \setminus D(G, T)$ are called *alcoves*.

(b) For $\alpha \in \Delta$ and $k \in \mathbb{Z}$, the reflection $\sigma_{\alpha,k} \colon \mathfrak{t} \to \mathfrak{t}$ at the affine hyperplane

$$H_{\alpha,k} := \{x \in \mathfrak{t} : \alpha(x) = 2\pi i k\}$$

is given by $\sigma_{\alpha,k}(x) = x - (\alpha(x) - 2\pi i\, k)\check{\alpha}$. The group generated by all $\sigma_{\alpha,k}$ with $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$ is called the *affine Weyl group* of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and denoted by W_{aff}.

(c) For an alcove $A \subseteq \mathfrak{t}$, we call $H_{\alpha,k}$ a *bounding hyperplane* if

$$A \subseteq \{x \in \mathfrak{t} \colon (2\pi i)^{-1}\alpha(x) < k\}$$

and there exists a boundary point $a \in \overline{A}$ with $\alpha(a) = 2\pi i k$ and $\beta(a) \notin 2\pi i\mathbb{Z}$ for any root $\beta \neq \pm\alpha$.

Remark 11.4.19. (i) The hyperplanes of $D(G, T)$ containing the origin divide $\mathfrak{t}$ into the Weyl chambers (cf. Definition 5.4.12). In each Weyl chamber C we find precisely one alcove A containing 0 in its closure, which shows that W acts transitively on the set of all these alcoves.

(ii) Note that W_{aff} leaves the diagram invariant and hence acts on the set of alcoves. The $\sigma_{\alpha,0}$ coincide with the reflections in the linear hyperplanes generating the Weyl group of the dual root system $\check{\Delta}$ (cf. Remark 5.4.18). As a group this Weyl group is isomorphic to W, but here it is represented on $\mathfrak{t}$ rather than $\mathfrak{t}^*$. A simple calculation shows that

$$\sigma_{\alpha,k}(x) = \tau^k_{2\pi i\check{\alpha}} \circ \sigma_{\alpha,0}(x), \tag{11.14}$$

where $\tau_y \colon \mathfrak{t} \to \mathfrak{t}$ is the translation $x \mapsto x + y$. Multiplying by $\sigma_{\alpha,0}$ this implies that

$$\{\tau_y : y \in \Gamma_0(T)\} \cong \Gamma_0(T)$$

is contained in W_{aff}. Identifying $\Gamma_0(T)$ in this way with a subgroup of W_{aff} another short calculation shows

$$\sigma_{\alpha,k} \circ \tau_y \circ \sigma_{\alpha,k} = \tau_{\sigma_\alpha y},$$

so that $\Gamma_0(T)$ is normal in W_{aff}. In fact, (11.14) and

$$W = \{w \in W_{\text{aff}} : w(0) = 0\}$$

imply that

$$\Gamma_0(T) \rtimes W \to W_{\text{aff}}, \; (y, w) \mapsto \tau_y \circ w$$

is an isomorphism of groups.

(iii) Let A be an alcove and H_{α_j,k_j}, $j = 1, \ldots, n$, be the hyperplanes bounding A. Then

$$A = \{x \in \mathfrak{t} \colon (\forall j)\, (2\pi i)^{-1}\alpha_j(x) < k_j\}.$$

In fact, the right hand side is an open convex set $\widetilde{A}$ containing A. If it is larger than A, then $\widetilde{A}$ contains a boundary point $a \in \overline{A}$ which is contained in some hyperplane $H_{\beta,m}$. Replacing a by a nearby point, we may further assume that $\pm\beta$ are the only roots with $(2\pi i)^{-1}\beta(a) \in \mathbb{Z}$. Then $H_{\beta,m}$ is a bounding hyperplane of A, contradicting the definition of $\widetilde{A}$.

Proposition 11.4.20. *The affine Weyl group has the following properties:*

(i) *For any alcove $A \subseteq \mathfrak{t}$, the reflections in its boundary hyperplanes generate W_{aff}.*
(ii) *Let A be an alcove, σ_j, $j = 1, \ldots, n$ be the reflections in its bounding hyperplanes $H_1, \ldots, H_n$. For $w \in W_{\text{aff}}$ we define the* length $\ell(w)$ *to be the smallest number r such that w can be written in the form $w = \sigma_{j_1} \cdots \sigma_{j_r}$. Then $\ell(w)$ coincides with the number of hyperplanes in $D(G,T)$ separating A and $w(A)$.*
(iii) *W_{aff} acts simply transitively on the set of alcoves.*
(iv) *For any alcove A, the closure contains a unique element of $\Gamma_0(T)$.*

Proof. (i) Let $W(A) \subseteq W_{\text{aff}}$ be the subgroup generated by the boundary reflections $\sigma_1, \ldots, \sigma_n$ of A, B be some alcove, $x \in A$, and $y \in B$. Since the orbit $W(A)y$ intersects each compact subset of $\mathfrak{t}$ in a finite set, there exists an element $w \in W(A)$ for which $\|wy - x\|$ is minimal. This implies that wy lies on the same side of each hyperplane $H_{\alpha,k}$ bounding A because otherwise $\sigma_{\alpha,k}wy$ would be closer to x. Hence Remark 11.4.19 implies that $wy \in A$. Since wB is the unique alcove containing wy, we obtain $wB = A$. This shows that $W(A)$ acts transitively on the set of all alcoves.

Since each reflection $\sigma = \sigma_{\alpha,k} \in W_{\text{aff}}$ corresponds to a bounding hyperplane of some alcove B, we obtain with any $w \in W(A)$ satisfying $wB = A$, that $w\sigma w^{-1}$ is a reflection in some bounding hyperplane of A, hence contained in $W(A)$. Then $\sigma \in w^{-1}W(A)w = W(A)$ yields $W_{\text{aff}} = W(A)$. This proves (i).

(ii) Fix $j = 1, \ldots, n$ and $w \in W_{\text{aff}}$. Suppose that A and wA are on the same side of H_j. For any hyperplane H' in the diagram which is different from

H_j, the fact that H_j is a bounding hyperplane of A implies that A and $\sigma_j(A)$ lie on the same side of H'. Therefore H' separates A from $\sigma_j w(A)$ if and only if it separates $\sigma_j A$ from $\sigma_j w(A)$, which in turn means that $\sigma_j(H')$ separates A from $w(A)$. Therefore the set of hyperplanes in the diagram separating A and $\sigma_j wA$ consists of H_j and all other hyperplanes $\sigma_j(H')$, where H' separates A and $w(A)$.

We now assume that $w = \sigma_{j_1} \cdots \sigma_{j_r}$ with $\ell(w) = r$ and prove the following claim by induction on r.

Claim: The hyperplanes in the diagram separating A and $w(A)$ are precisely r pairwise different hyperplanes

$$\sigma_{j_1} \cdots \sigma_{j_{r-1}}(H_{j_r}), \sigma_{j_1} \cdots \sigma_{j_{r-2}}(H_{j_{r-1}}), \ldots, \sigma_{j_1}(H_{j_2}), H_{j_1}.$$

The case $r = 1$ is trivial, so assume the claim is true for $\ell(w) = r - 1$ and set $w' = \sigma_{j_2} \cdots \sigma_{j_r}$. Then clearly $\ell(w') = r - 1$. By induction we see that the hyperplanes

$$\sigma_{j_2} \cdots \sigma_{j_{r-1}}(H_{j_r}), \sigma_{j_2} \cdots \sigma_{j_{r-2}}(H_{j_{r-1}}), \ldots, \sigma_{j_2}(H_{j_3}), H_{j_2}$$

are pairwise different and are precisely the hyperplanes in the diagram separating A and $w'(A)$. But then also the hyperplanes

$$\sigma_{j_1} \cdots \sigma_{j_{r-1}}(H_{j_r}), \sigma_{j_1} \cdots \sigma_{j_{r-2}}(H_{j_{r-1}}), \ldots, \sigma_{j_1}(H_{j_2})$$

are pairwise different. Suppose that H_{j_1} is one of them, say

$$H_{j_1} = \sigma_{j_1} \cdots \sigma_{j_m}(H_{j_{m+1}}).$$

Then

$$\sigma_{j_1} = (\sigma_{j_1} \cdots \sigma_{j_m})\sigma_{j_{m+1}}(\sigma_{j_1} \cdots \sigma_{j_m})^{-1}$$

and hence

$$w = \sigma_{j_2} \cdots \sigma_{j_m}\sigma_{j_{m+1}}\sigma_{j_m} \cdots \sigma_{j_2}\sigma_{j_2} \cdots \sigma_{j_r} = \sigma_{j_2} \cdots \sigma_{j_m}\sigma_{j_{m+2}} \cdots \sigma_{j_r}.$$

This contradicts $\ell(w) = r$, so

$$\sigma_{j_1} \cdots \sigma_{j_{r-1}}(H_{j_r}), \sigma_{j_1} \cdots \sigma_{j_{r-2}}(H_{j_{r-1}}), \ldots, \sigma_{j_1}(H_{j_2}), H_{j_1} \tag{11.15}$$

are pairwise different and H_{j_1} does not separate A and $w'(A)$. Thus the observation made at the beginning of the proof shows that the hyperplanes from (11.15) are precisely the ones separating A and $\sigma_{i_1} w'(A) = w(A)$.

(iii) We have already seen in (i) that W_{aff} acts transitively on the set of alcoves. To see that it acts simply transitively, let A be an alcove and $w \in W_{\text{aff}}$ with $wA = A$. Then (ii) implies that $\ell(w) = \mathbf{1}$, and hence that $w = \text{id}$.

(iv) Since W_{aff} acts transitively on the set of alcoves and $w(\overline{A} \cap \mathit{\Gamma}_0(T)) = w\overline{A} \cap \mathit{\Gamma}_0(T)$, we may w.l.o.g. assume that A is an alcove containing 0. It remains to show that any other element $y \in \mathit{\Gamma}_0(T) \cap \overline{A}$ vanishes. Now $\tau_{-y}A$ is an alcove whose closure contains 0, so that Remark 11.4.19(i) implies the existence of some $w \in W$ with $w\tau_{-y}A = A$. This means that the element $w\tau_{-y} \in W_{\text{aff}}$ fixes A, and (iii) implies that $w = \tau_y$, hence $y = 0$. □

Definition 11.4.21. Let G be a connected compact Lie group. We call an element $g \in G$ *regular* if it is contained in exactly one maximal torus. If g is contained in more than one maximal torus, it is called *singular*.

Proposition 11.4.22. *Let G be a connected compact Lie group and T a maximal torus in G. An element $t \in T$ is regular if and only if it lies in no U_α with $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Thus the diagram $D(G,T)$ is the inverse image under $\exp_T$ of the set of singular elements of G which lie in T.*

Proof. Corollary 11.2.5 shows that that $t = \exp_T x =\in T$ is regular if and only if $Z_G(t)_0 = T$. This in turn is equivalent to

$$\mathfrak{t} = \mathrm{Fix}(\mathrm{Ad}(t)) = \mathfrak{t} + \sum_{\alpha(x) \in 2\pi i\mathbb{Z}} \mathfrak{g}^{[\alpha]},$$

and hence to $\chi_\alpha(t) = e^{\alpha(x)} \neq 1$ for each $\alpha \in \Delta$. □

Lemma 11.4.23. *Let $t \in T$ and S be a subtorus of T. If the identity component of $Z_G(t) \cap Z_G(S)$ equals T, then there exists an element $s \in S$ such that ts is regular.*

Proof. For $\alpha \in \Delta$ set $S_\alpha = \{s \in S : \chi_\alpha(ts) = 1\}$. Then the S_α cover S if and only if all ts with $s \in S$ are singular. Assume that this is the case. Then one of the S_α coincides with S, because S cannot be a finite union of proper closed submanifolds. Thus there is a root $\alpha \in \Delta$ such that $\chi_\alpha(ts) = 1$ for all $s \in S$, and in particular $\chi_\alpha(t) = 1$. Hence $S \cup \{t\} \subseteq \ker \chi_\alpha = U_\alpha$ and therefore $Z_G(U_\alpha) \subseteq Z_G(S) \cap Z_G(t)$. Since U_α is a subgroup of T, the equality $(Z_G(S) \cap Z_G(t))_0 = T$ implies $T = Z_G(U_\alpha)_0$.

Pick $x \in \mathfrak{g} \cap (\mathfrak{g}_{\mathbb{C}}^\alpha + \mathfrak{g}_{\mathbb{C}}^{-\alpha})$. Then $x = x_+ + x_-$ with $x_\pm \in \mathfrak{g}_{\mathbb{C}}^{\pm\alpha}$, so that for $g \in U_\alpha$ we have

$$\begin{aligned} c_g(\exp_G x) &= \exp_G(\mathrm{Ad}(g)(x)) = \exp_G(\chi_\alpha(g)x_+ + \chi_{-\alpha}(g)x_-) \\ &= \exp_G(x_+ + x_-) = \exp_G(x), \end{aligned}$$

since $\chi_\alpha(g) = \chi_\alpha^{-1}(g) = 1$. This shows that g and $\exp_G(x)$ commute for all $g \in U_\alpha$, i.e., $\exp_G(\mathbb{R}x) \in Z_G(U_\alpha)_0 = T$. Thus $x \in \mathfrak{t}$ which is a contradiction, proving the claim. □

Proposition 11.4.24. *Let G be a compact connected Lie group and σ an automorphism of G. Further, let G^σ the group of fixed points of σ and G_0^σ its identity component. Then for every $g \in G^\sigma$ there is an $s \in G_0^\sigma$ such that gs is regular.*

Proof. Let $g \in G^\sigma$. Let S be a maximal torus in $Z_{G^\sigma}(g)$, and R be the identity component of $Z_G(g) \cap Z_G(S)$. Then R is a connected compact group. As S is a torus in G and g centralizes S, by Theorem 11.2.10, there is a maximal torus of G containing g and S, which has to be contained in R. So R is of maximal rank in G. It is stable under σ, because σ leaves $Z_G(g) \cap Z_G(S)$ invariant and then also its identity component. Note that $S \subseteq (R^\sigma)_0$. Conversely, $(R^\sigma)_0$ is contained in the centralizer of S in $Z_{G^\sigma}(g)_0$, which is equal to S by Corollary 11.2.11. Thus we have $S = (R^\sigma)_0$. Since S is central in R, we can apply Lemma 11.4.17 to R and $\sigma|_R$ to conclude that R is abelian. As R is connected and of maximal rank, it is a maximal torus in G. It contains g and S and is equal to the identity component of $Z_G(g) \cap Z_G(S)$. We can apply Lemma 11.4.23 to the torus R with subtorus S and get an element $s \in S$ such that gs is regular. As $S = (R^\sigma)_0 \subseteq G_0^\sigma$, the element s lies in the identity component of G^σ. □

Lemma 11.4.25. *Let G be a compact 1-connected Lie group, T a maximal torus in G, and C the closure of an alcove in $\mathfrak{t}$. Then C contains precisely one element of $\Gamma(T)$.*

Proof. Since G is simply connected, we know from Theorem 11.4.14 that $\Gamma(T) = \Gamma_0(T)$, so that the assertion follows from Proposition 11.4.20(iv). □

Theorem 11.4.26. *Let G be a compact 1-connected Lie group and σ be an automorphism of G. Then the set G^σ of fixed points of σ is connected.*

Proof. Let G_0^σ be the identity component of G^σ. For any $g \in G^\sigma$ we show that $g \in G_0^\sigma$. First let $g \in G^\sigma$ be a regular element of G, so that there is a unique maximal torus T containing g. Then σ restricts to an automorphism of T, because $\sigma(T)$ is a maximal torus containing g, hence equal to T. Further, σ maps singular elements of T to singular elements, thus the corresponding automorphism $\mathbf{L}(\sigma)$ of $\mathfrak{t}$ leaves the unit lattice $\Gamma := \Gamma(T) = \Gamma_0(T)$ (cf. Theorem 11.4.14) and the diagram $D(G)$ invariant.

Pick a point $x \in \exp_G^{-1}(g)$ and let C be the closure of the alcove in $\mathfrak{t}$ which contains x. By Lemma 11.4.25 there exists an element $\gamma \in \Gamma \cap C$. Therefore $C - \gamma$ is the closure of an alcove containing 0 and $x' := x - \gamma \in \exp_G^{-1}(g)$. Replacing x by x', we may therefore assume that $0 \in C$. As $\sigma(g) = g$, Then $\eta := \mathbf{L}(\sigma)x - x \in \Gamma$ follows from the invariance of Γ under $\mathbf{L}(\sigma)$. Therefore $\mathbf{L}(\sigma)C$ is the closure of the unique alcove containing the regular $\mathbf{L}(\sigma)x$, which leads to $\mathbf{L}(\sigma)C = C + \eta$. Therefore $\mathbf{L}(\sigma)(C)$ contains both 0 and η, so that the uniqueness part of Lemma 11.4.25 leads to $\eta = 0$, i.e., x is fixed under $\mathbf{L}(\sigma)$. Then $g \in \exp \mathbb{R}x \subseteq G_0^\sigma$, proving the claim for regular g.

Now let $g \in G^\sigma$ be arbitrary. By Proposition 11.4.24, there is an $s \in G_0^\sigma$ such that gs is regular. We apply the first part of the proof to $gs \in G^\sigma$ and obtain $gs \in G_0^\sigma$ which implies $g \in G_0^\sigma$. □

Corollary 11.4.27. *Let G be a 1-connected compact Lie group.*

(i) *$Z_G(x)$ is connected for any $x \in G$.*
(ii) *If $x, y \in G$ are commuting elements, then there is a maximal torus $T \subseteq G$ containing x and y.*

Proof. (i) Apply Theorem 11.4.26 to the inner automorphism c_x to show that $Z_G(x)$ is a connected subgroup of G.

(ii) By (i), $Z_G(x)$ is connected, and it contains y by hypothesis. By Theorem 11.2.2(iii), we find a maximal torus $T' \subseteq Z_G(x) \subseteq G$ containing y. Since x is in the center of $Z_G(x)$, by Corollary 11.2.4 it is contained in T'. This implies the claim since every torus $T' \subseteq G$ is contained in some maximal torus T of G. □

Remark 11.4.28. (a) The simple connectedness is crucial in the preceding corollary. We claim that $\mathrm{SO}_3(\mathbb{R})$ contains a commuting pair which is not contained in a maximal torus. Indeed, let

$$g_1 := \mathrm{diag}(-1,-1,1) \quad \text{and} \quad g_2 := \mathrm{diag}(1,-1,-1).$$

Then the centralizer of g_1 and g_2 in $\mathrm{GL}_3(\mathbb{R})$ coincides with the subgroup of diagonal matrices, so that

$$Z_{\mathrm{SO}_3(\mathbb{R})}(\{g_1,g_2\}) = \langle g_1, g_2\rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

is finite, hence does not contain a non-trivial torus.

(b) For some compact Lie groups, all commuting n-tuples are contained in a maximal torus. For example, any commuting n-tuple in $\mathrm{SU}_m(\mathbb{C})$ is simultaneously diagonalizable, hence contained in a maximal torus.

Exercises for Section 11.4

Exercise 11.4.1. Let G be a topological group and H be a closed connected subgroup for which G/H is connected. Show that G is connected.

Notes on Chapter 11

Most of the material of compact groups discussed in this chapter is quite standard. For an in depth treatment of compact Lie groups in the context of topological groups, we refer to [HM06]. A quite systematic treatment of compact Lie groups and in particular of their diagrams and related concepts can be found in [Bou82].

In [BFM02], Borel, Friedman and Morgan determine the space of commuting ordered pairs and triples in compact connected semisimple Lie groups. In particular, they show that commuting triples in compact 1-connected Lie groups are in general not contained in a maximal torus because the moduli space

$$\{(x,y,z) \in G^3 \colon xyx^{-1}y^{-1} = xzx^{-1}z^{-1} = yzy^{-1}z^{-1} = \mathbf{1}\}/G,$$

where G acts by $g.(x_1,x_2,x_3) = (c_g(x_1), c_g(x_2), c_g(x_3))$, is not connected.

12

Semisimple Lie Groups

In the preceding chapter, we studied groups with a compact Lie algebra. For these groups, we have seen how to split them into a direct product of a compact and a vector group, how to complement the commutator group by an abelian Lie group, and that all compact Lie groups are linear. We now proceed with our program to obtain similar results for arbitrary Lie groups with finitely many connected components. First, we turn to the important special case of semisimple Lie groups.

The main results in this chapter will be the Cartan decomposition and the Iwasawa decomposition of a connected semisimple Lie group. Both provide diffeomorphisms of G with product manifolds $K \times V$, where K is a compact subgroup of G and V is diffeomorphic to a vector space. In the case of the Cartan decomposition, V has the advantage that it is invariant under conjugation by K, and in the case of the Iwasawa decomposition, V is a solvable subgroup of G.

12.1 Cartan Decompositions

In this section we study the properties of a Cartan decomposition defined by a given Cartan involution. The *existence* of Cartan involutions will be treated in Section 12.2. It is instructive to first have a closer look at the example $G = \mathrm{SL}_n(\mathbb{R})$. On this group, $\theta(g) := (g^\top)^{-1}$ defines an involution and its group of fixed points is

$$G^\theta = \{g \in G \colon g^\top = g^{-1}\} = \mathrm{SO}_n(\mathbb{R}).$$

If x is a symmetric matrix in $\mathfrak{sl}_n(\mathbb{R})$, then $\theta(e^x) = e^{-x}$. The differential $\tau := \mathbf{L}(\theta), x \mapsto -x^\top$ on $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ is also an involutive automorphism for which

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}_n(\mathbb{R}) \oplus \mathrm{Sym}_n(\mathbb{R}),$$

is the corresponding eigenspace decomposition. By Proposition 1.1.5, the polar map

$$\Phi\colon \mathrm{SO}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R}), \qquad (g, x) \mapsto ge^x$$

is a homeomorphism. We also note that the symmetric bilinear form $\beta(x, y) := \mathrm{tr}(xy)$ on $\mathfrak{sl}_n(\mathbb{R})$ is positive definite on $\mathrm{Sym}_n(\mathbb{R})$ and negative definite on $\mathfrak{so}_n(\mathbb{R})$ because for a symmetric matrix $x = x^\top$, the matrix x^2 is positive semidefinite, and for a skew-symmetric matrix $x = -x^\top$, the matrix x^2 is negative semidefinite. Finally, we recall that β is a positive multiple of the Cartan–Killing form (apply Exercise 4.5.10 to the complexification $\mathfrak{so}_n(\mathbb{C})$ of $\mathfrak{so}_n(\mathbb{R})$). After these introductory remarks, we now turn to general semisimple Lie groups.

Definition 12.1.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra and κ be the Cartan–Killing form of $\mathfrak{g}$. Since a complex Lie algebra is semisimple if and only if the underlying real Lie algebra is semisimple, all definitions below apply to the complex case as well (Exercise 12.1.3).

An automorphism τ of $\mathfrak{g}$ is called a *Cartan involution* if

(i) $\tau^2 = \mathrm{id}_{\mathfrak{g}}$,
(ii) κ is negative definite on $\mathfrak{k} := \{x \in \mathfrak{g} : \tau(x) = x\}$, and
(iii) κ is positive definite on $\mathfrak{p} := \{x \in \mathfrak{g} : \tau(x) = -x\}$.

The eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called a *Cartan decomposition* of $\mathfrak{g}$. It is a direct sum of vector spaces not of Lie algebras (cf. Lemma 12.1.2).

In the following $\mathfrak{g}$ always denotes a semisimple real Lie algebra and κ its Cartan–Killing form.

Lemma 12.1.2. *If τ is a Cartan involution of $\mathfrak{g}$, then the following assertions hold:*

(i) $\mathfrak{k}$ *and* $\mathfrak{p}$ *are orthogonal with respect to* κ.
(ii) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ *and* $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

Proof. (i) For $\alpha \in \mathrm{Aut}(\mathfrak{g})$ we have $\mathrm{ad}\,\alpha(x) = \alpha \circ \mathrm{ad}\,x \circ \alpha^{-1}$, which implies that

$$\begin{aligned}\kappa(\alpha(x), \alpha(y)) &= \mathrm{tr}(\mathrm{ad}(\alpha(x))\,\mathrm{ad}(\alpha(y))) \\ &= \mathrm{tr}(\alpha \circ \mathrm{ad}\,x \circ \alpha^{-1} \circ \alpha \circ \mathrm{ad}\,y \circ \alpha^{-1}) = \mathrm{tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y) = \kappa(x, y).\end{aligned}$$

For $\tau(x) = x$ and $\tau(y) = -y$, this immediately leads to $\kappa(x, y) = 0$.

(ii) This follows from $\tau \in \mathrm{Aut}(\mathfrak{g})$. □

Lemma 12.1.3. *If τ is a Cartan involution on $\mathfrak{g}$, then the symmetric bilinear form*

$$\kappa_\tau\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad (x, y) \mapsto -\kappa(x, \tau y),$$

has the following properties:

(i) κ_τ *is positive definite.*
(ii) $\operatorname{ad}\mathfrak{k} \subseteq \mathfrak{o}(\mathfrak{g}, \kappa_\tau) := \{A \in \mathfrak{gl}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\ \kappa_\tau(Ax, y) + \kappa_\tau(x, Ay) = 0\}$.
(iii) $\operatorname{ad}\mathfrak{p} \subseteq \operatorname{Sym}(\mathfrak{g}, \kappa_\tau) := \{A \in \mathfrak{gl}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\ \kappa_\tau(Ax, y) = \kappa_\tau(x, Ay)\}$.

Proof. (i) With Lemma 12.1.2(i), we get for $x = y + z$ with $y \in \mathfrak{k}$ and $z \in \mathfrak{p}$

$$\kappa_\tau(x, x) = -\kappa(y + z, y - z) = -\kappa(y, y) + \kappa(z, z).$$

Now (i) follows by the definition of a Cartan involution.

(ii), (iii) For $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}\kappa_\tau(\operatorname{ad} z(x), y) &= -\kappa([z, x], \tau y) = \kappa(x, [z, \tau y])\\ &= -\kappa_\tau\big(x, \tau([z, \tau y])\big) = -\kappa_\tau\big(x, \operatorname{ad}(\tau z)(y)\big).\end{aligned}$$ □

Recall the group $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{k})$ of inner automorphisms generated by $e^{\operatorname{ad}\mathfrak{k}}$ from Definition 11.1.3.

Lemma 12.1.4. *If* $\mathfrak{g}$ *is a semisimple Lie algebra, then* $\operatorname{Inn}(\mathfrak{g})$ *is the identity component* $\operatorname{Aut}(\mathfrak{g})_0$ *of* $\operatorname{Aut}(\mathfrak{g})$*. If* τ *is a Cartan involution and* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *the corresponding Cartan decomposition, then*

$$\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{k}) = \{\gamma \in \operatorname{Aut}(\mathfrak{g})\colon \gamma\tau = \tau\gamma\}_0.$$

Proof. Since $\mathfrak{g}$ is semisimple, $\operatorname{ad}\mathfrak{g} = \operatorname{der}\mathfrak{g} = \mathbf{L}(\operatorname{Aut}\mathfrak{g})$ (Theorem 4.5.14 and Example 3.2.5). Hence $\operatorname{Inn}(\mathfrak{g}) = \langle e^{\operatorname{ad}\mathfrak{g}} \rangle = \operatorname{Aut}(\mathfrak{g})_0$ implies that $\operatorname{Inn}(\mathfrak{g})$ is closed (see Definition 11.1.17).

The group

$$K := \{\gamma \in \operatorname{Aut}(\mathfrak{g})\colon \gamma\tau = \tau\gamma\} = \{\gamma \in \operatorname{Aut}(\mathfrak{g})\colon \tau\gamma\tau = \gamma\}$$

is closed, it contains $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{k})$, and its Lie algebra is

$$\mathbf{L}(K) = \{\operatorname{ad} x\colon x \in \mathfrak{g}, \tau \operatorname{ad} x \tau = \operatorname{ad}(\tau x) = \operatorname{ad} x\}$$

(Lemma 10.1.1), so that the injectivity of ad yields

$$\mathbf{L}(K) = \operatorname{ad}\{x \in \mathfrak{g}\colon \tau x = x\} = \operatorname{ad}\mathfrak{k},$$

and thus $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{k}) = K_0$. □

Proposition 12.1.5. *If* τ *is a Cartan involution of the semisimple Lie algebra* $\mathfrak{g}$ *and* $K := \operatorname{Aut}(\mathfrak{g})^\tau = \{\gamma \in \operatorname{Aut}(\mathfrak{g})\colon \tau\gamma = \gamma\tau\}$*, then* K *is compact and the map*

$$\Psi\colon K \times \mathfrak{p} \to \operatorname{Aut}(\mathfrak{g}), \qquad (k, x) \mapsto k e^{\operatorname{ad} x}$$

is a diffeomorphism.

Proof. Clearly, Ψ is a smooth map. To apply Proposition 3.3.3, we observe that $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$ is an algebraic subgroup, because it is defined by the quadratic equations

$$g([x,y]) = [g(x), g(y)], \quad x, y \in \mathfrak{g}.$$

Further, κ_τ is a scalar product on $\mathfrak{g}$, and the invariance of κ under $\gamma \in \mathrm{Aut}(\mathfrak{g})$ (Exercise 12.1.1) implies that

$$\kappa_\tau(\gamma(x), y) = -\kappa(\gamma(x), \tau(y)) = -\kappa(x, \gamma^{-1}\tau(y)) = \kappa_\tau(x, \tau\gamma^{-1}\tau(y)),$$

so that transposition with respect to κ_τ is given by

$$\gamma^\top = \tau\gamma^{-1}\tau \in \mathrm{Aut}(\mathfrak{g}). \tag{12.1}$$

In particular, $\mathrm{Aut}(\mathfrak{g})$ is invariant under transposition and

$$\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{O}(\mathfrak{g}, \kappa_\tau) = \{\gamma \in \mathrm{Aut}(\mathfrak{g}) \colon \tau\gamma^{-1}\tau = \gamma^{-1}\} = K.$$

Choosing an orthonormal basis for $\mathfrak{g}$ to identify it with $\mathbb{R}^n$, we can now apply Proposition 3.3.3, to see that $\mathrm{ad}\,\mathfrak{p} = \mathrm{ad}\,\mathfrak{g} \cap \mathrm{Sym}(\mathfrak{g}, \kappa_\tau)$ implies that Ψ is a homeomorphism. Its inverse is given by

$$\Psi^{-1}(\gamma) = \Big(\gamma e^{-\frac{1}{2}\,\mathrm{ad}\log(\gamma^\top\gamma)}, \mathrm{ad}^{-1}\Big(\tfrac{1}{2}\log(\gamma^\top\gamma)\Big)\Big),$$

and since $\log\colon \mathrm{Pd}(\mathfrak{g}, \kappa_\tau) \to \mathrm{Sym}(\mathfrak{g}, \kappa_\tau)$ is a diffeomorphism by Proposition 2.3.5, Ψ^{-1} is a smooth map and thus Ψ is a diffeomorphism. □

Lemma 12.1.6. *Let G be a real semisimple connected Lie group with Lie algebra $\mathfrak{g}$, and let τ be a Cartan involution of $\mathfrak{g}$. Let $K := \langle \exp \mathfrak{k} \rangle$. Then*

$$Z(G) \subseteq K \qquad \textit{and} \qquad K = \mathrm{Ad}^{-1}(\mathrm{Inn}_{\mathfrak{g}}\,\mathfrak{k}).$$

Proof. By Lemma 8.2.21, $Z(G) = \ker \mathrm{Ad}$ because G is connected. Let $H := \mathrm{Ad}^{-1}(\mathrm{Inn}_{\mathfrak{g}}\,\mathfrak{k})$. We consider a path $\gamma\colon [0,1] \to G$ with $\gamma(0) = 1$ and $\gamma(1) = h \in H$. By Proposition 12.1.5 and Lemma 12.1.4 we have

$$\mathrm{Ad}\big(\gamma(t)\big) = k(t) e^{\mathrm{ad}\,x(t)}$$

with paths $k\colon [0,1] \to \mathrm{Inn}_{\mathfrak{g}}\,\mathfrak{k}$ and $x\colon [0,1] \to \mathfrak{p}$. Further, $x(0) = x(1) = 0$ since $\mathrm{Ad}\big(\gamma(1)\big) \in \mathrm{Inn}_{\mathfrak{g}}\,\mathfrak{k}$. Now we set $\widetilde{\gamma}(t) := \gamma(t)\exp\big(-x(t)\big)$ and observe that $\widetilde{\gamma}$ is a path in H which connects $\mathbf{1}$ with h. Therefore H is connected. Finally,

$$\mathbf{L}(H) = \mathrm{ad}^{-1}(\mathbf{L}(\mathrm{Inn}_{\mathfrak{g}}\,\mathfrak{k})) = \mathrm{ad}^{-1}(\mathrm{ad}\,\mathfrak{k}) = \mathfrak{k}$$

leads to $H = K$. In particular, $Z(G) = \ker \mathrm{Ad} \subseteq K$. □

Theorem 12.1.7 (Cartan Decomposition). *Let G be a Lie group with semisimple Lie algebra $\mathfrak{g}$, τ a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. If $K := \{g \in G \colon \tau \operatorname{Ad}(g) = \operatorname{Ad}(g)\tau\}$, then $\mathbf{L}(K) = \mathfrak{k}$, and*

$$\Phi\colon K \times \mathfrak{p} \to G, \qquad (k, x) \mapsto k \exp_G x$$

is a diffeomorphism.

Proof. First we show that Φ is bijective. Let $g \in G$ and write $\operatorname{Ad}(g) = k_0 e^{\operatorname{ad} x_0} = k_0 \operatorname{Ad}(\exp_G x_0)$ with $x_0 \in \mathfrak{p}$ and k_0 commuting with τ (Proposition 12.1.5). Then $g \exp_G(-x_0) \in K$ implies that Φ is surjective. For $k \exp x = k' \exp x'$, we have that $\operatorname{Ad}(k)e^{\operatorname{ad} x} = \operatorname{Ad}(k')e^{\operatorname{ad} x'}$, and therefore $x = x'$, again by Proposition 12.1.5. This implies $k = k'$ and hence Φ is bijective. The commutativity of the diagram

$$\begin{array}{ccc} K \times \mathfrak{p} & \xrightarrow{\quad\Phi\quad} & G \\ \Big\downarrow{\scriptstyle \operatorname{Ad}|_K \times \operatorname{id}_{\mathfrak{p}}} & & \Big\downarrow{\scriptstyle \operatorname{Ad}} \\ \operatorname{Aut}(\mathfrak{g})^\tau \times \mathfrak{p} & \xrightarrow{\quad\Psi\quad} & \operatorname{Ad}(G) \end{array}$$

and the fact that all vertical maps and Ψ are local diffeomorphisms now imply that Φ is regular, and therefore a diffeomorphism. But then also $\mathbf{L}(K) = \mathfrak{k}$ follows. □

Corollary 12.1.8. *With the notation of* Theorem 12.1.7, *K is self-normalizing, i.e., $K = N_G(K)$.*

Proof. If $g = k \exp_G x \in N_G(K)$, then also $\exp_G x \in N_G(K)$, which leads to $e^{\operatorname{ad} x}\mathfrak{k} = \operatorname{Ad}(\exp_G x)\mathfrak{k} = \mathfrak{k}$. Since $e^{\operatorname{ad} x} \in \operatorname{Aut}(\mathfrak{g})$ preserves the Cartan–Killing form it also leaves $\mathfrak{p} = \mathfrak{k}^{\perp_\kappa}$ invariant (Lemma 12.1.2). Hence $e^{\operatorname{ad} x}$ preserves both eigenspaces of τ, i.e., it commutes with τ, and this means that

$$e^{\operatorname{ad} x} = \tau e^{\operatorname{ad} x}\tau = e^{\tau \circ \operatorname{ad} x \circ \tau} = e^{\operatorname{ad} \tau(x)} = e^{-\operatorname{ad} x}.$$

Next we use the injectivity of the exponential function on $\operatorname{ad} \mathfrak{p}$ to derive that $x = -x$, hence $x = 0$ (see Exercise 12.1.2 for an alternative argument). □

Lemma 12.1.9. *Let G be a Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition defined by the Cartan involution τ. Then there exists a unique involutive automorphism $\theta \in \operatorname{Aut}(G)$ with $\mathbf{L}(\theta) = \tau$ fixing $K = \{g \in G \colon \tau \operatorname{Ad}(g) = \operatorname{Ad}(g)\tau\}$ pointwise. It satisfies*

$$\theta(k \exp_G x) = k \exp_G(-x) \quad \textit{for} \quad k \in K, x \in \mathfrak{p}.$$

The involution θ of G is called a *Cartan involution*.

Proof. **Step 1:** First we assume that G is connected. Let $q_G\colon \widetilde{G} \to G$ be a simply connected covering of G with $\mathbf{L}(q_G) = \mathrm{id}_{\mathfrak{g}}$ and write τ for the Cartan involution of $\mathfrak{g}$, whose eigenspaces are $\mathfrak{k}$ and $\mathfrak{p}$. Then there exists a unique automorphism $\widetilde{\theta} \in \mathrm{Aut}(\widetilde{G})$ with $\mathbf{L}(\widetilde{\theta}) = \tau$ (Theorem 8.5.9). For the subgroup

$$\widetilde{K} := \{g \in \widetilde{G}\colon\ \mathrm{Ad}(g)\tau = \tau\,\mathrm{Ad}(g)\}$$

the relation $\mathrm{Ad}(\widetilde{\theta}(g)) = \tau\,\mathrm{Ad}(g)\tau^{-1}$ implies that the group

$$\widetilde{G}^{\widetilde{\theta}} := \{g \in \widetilde{G}\colon \widetilde{\theta}(g) = g\}$$

of fixed points of $\widetilde{\theta}$ is contained in $\widetilde{K}$. From the Cartan decomposition of $\widetilde{G}$ we obtain a diffeomorphism $\widetilde{G} \cong \widetilde{K} \times \mathfrak{p}$, so that $\widetilde{K}$ is seen to be connected. Its Lie algebra is

$$\{x \in \mathfrak{g}\colon\ \mathrm{ad}\,x \circ \tau = \tau \circ \mathrm{ad}\,x\} = \{x \in \mathfrak{g}\colon\ \mathrm{ad}\,x = \mathrm{ad}\,\tau(x)\} = \mathfrak{k},$$

so that $\mathbf{L}(\widetilde{G}^{\widetilde{\theta}}) = \mathfrak{k}$ implies that $\widetilde{K} = \widetilde{G}^{\widetilde{\theta}}$. By Lemma 12.1.6 $\widetilde{\theta}$ leaves $Z(\widetilde{G})$ pointwise fixed, and since $\ker q_G \subseteq Z(\widetilde{G})$ (Theorem 8.5.4), it factors through an automorphism $\theta\colon G \to G$ such that the diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\ \widetilde{\theta}\ } & \widetilde{G} \\ \Big\downarrow q_G & & \Big\downarrow q_G \\ G & \xrightarrow{\ \theta\ } & G \end{array}$$

commutes.

Step 2: Now we consider the general case. We already have a Cartan involution θ on the identity component G_0 of G with $\mathbf{L}(\theta) = \tau$. For $k \in K$ we then observe that

$$\mathbf{L}(c_k \circ \theta) = \mathrm{Ad}(k) \circ \tau = \tau \circ \mathrm{Ad}(k) = \mathbf{L}(\theta \circ c_k),$$

showing that $\theta(kgk^{-1}) = k\theta(g)k^{-1}$ for $g \in G_0$ and $k \in K$ (Corollary 8.2.12). Now use the resulting Cartan decomposition $G = K\exp(\mathfrak{p})$ from Theorem 12.1.7 to extend θ to G via

$$\theta(k\exp_G x) := k\exp_G(-x).$$

It simply remains to see that θ is an automorphism of G. Writing $g \in G$ as kg_0 with $k \in K$, $g_0 \in G_0$ and likewise $g' = k'g_0'$, we find

$$\begin{aligned}\theta(gg') &= \theta(kg_0k'g_0') = k\theta(g_0k'g_0') = kk'\theta((k')^{-1}g_0k'g_0') \\ &= kk'\theta((k')^{-1}g_0k')\theta(g_0') = k\theta(g_0)k'\theta(g_0') = \theta(g)\theta(g'). \end{aligned}$$

□

Proposition 12.1.10. *For a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the semisimple Lie algebra $\mathfrak{g}$, the following assertions hold:*

(i) *If* $\mathfrak{m} \subseteq \mathfrak{p}$ *is a* $\mathfrak{k}$*-invariant subspace, then* $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] \trianglelefteq \mathfrak{g}$ *is an ideal.*
(ii) *If* $\mathfrak{g}$ *is simple noncompact, then* $\mathfrak{p}$ *is a simple* $\mathfrak{k}$*-module with* $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$*,* $\mathfrak{k}$ *is a maximal Lie subalgebra of* $\mathfrak{g}$*, and* $\dim \mathfrak{z}(\mathfrak{k}) \leq 1$*.*
(iii) *If* $\mathfrak{g}$ *is simple noncompact and* $\mathfrak{k}$ *is abelian, then* $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R})$*.*

Proof. (i) Clearly, the subspace $\mathfrak{h} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{g}$ is $\mathfrak{k}$-invariant, but we also have

$$[\mathfrak{p}, \mathfrak{h}] \subseteq [\mathfrak{p}, \mathfrak{m}] + [[\mathfrak{m}, \mathfrak{p}], \mathfrak{m}] \subseteq [\mathfrak{p}, \mathfrak{m}] + [\mathfrak{k}, \mathfrak{m}] \subseteq [\mathfrak{p}, \mathfrak{m}] + \mathfrak{m},$$

so that it suffices to show that $[\mathfrak{p}, \mathfrak{m}] \subseteq [\mathfrak{m}, \mathfrak{m}]$. Let $\mathfrak{m}^\perp \subseteq \mathfrak{p}$ be the orthogonal complement of $\mathfrak{m}$ with respect to the Cartan–Killing form κ, which is positive definite on $\mathfrak{p}$ and $\mathfrak{k}$-invariant, so that $\mathfrak{m}^\perp$ is also $\mathfrak{k}$-invariant. Now

$$\kappa(\mathfrak{k}, [\mathfrak{m}, \mathfrak{m}^\perp]) = \kappa([\mathfrak{k}, \mathfrak{m}], \mathfrak{m}^\perp) \subseteq \kappa(\mathfrak{m}, \mathfrak{m}^\perp) = \{0\},$$

and since κ is negative definite on $\mathfrak{k}$, we derive from $[\mathfrak{m}, \mathfrak{m}^\perp] \subseteq \mathfrak{k}$ that $[\mathfrak{m}, \mathfrak{m}^\perp] = \{0\}$, which in turn yields $[\mathfrak{p}, \mathfrak{m}] = [\mathfrak{m}, \mathfrak{m}]$.

(ii) From (i) we derive immediately that the $\mathfrak{k}$-module $\mathfrak{p}$ is simple if $\mathfrak{g}$ is a simple Lie algebra. We also obtain $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$, which proves that $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. If $\mathfrak{b} \supseteq \mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$, then $\mathfrak{b} = \mathfrak{k} \oplus (\mathfrak{b} \cap \mathfrak{p})$, where $\mathfrak{b} \cap \mathfrak{p}$ is a $\mathfrak{k}$-submodule, hence $\{0\}$ or $\mathfrak{p}$. This shows that $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$.

Let $\rho \colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ be the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$, i.e., $\rho(x) = \operatorname{ad} x|_{\mathfrak{p}}$. Each central element $z \in \mathfrak{z}(\mathfrak{k})$ acts by a skew-symmetric endomorphism on the euclidean vector space $\mathfrak{p}$, endowed with the Cartan–Killing form. Then $\rho(z)^2$ is symmetric and commutes with the $\mathfrak{k}$-action, so that all its eigenspaces are $\mathfrak{k}$-invariant. Now the simplicity of the $\mathfrak{k}$-module $\mathfrak{p}$ shows that $\rho(z)^2 = \lambda \mathbf{1}$ for some $\lambda \in \mathbb{R}$. As $\rho(z)$ is skew-symmetric, $\lambda \leq 0$, and if it is nonzero, we may w.l.o.g. assume that $I := \operatorname{ad} z|_{\mathfrak{p}}$ satisfies $I^2 = -\mathbf{1}$, hence defines a complex structure on $\mathfrak{p}$, turning $(\mathfrak{p}, I)$ into a complex $\mathfrak{k}$-module. For any other element $y \in \mathfrak{z}(\mathfrak{k})$, the endomorphism $\rho(y)$ commutes with I and $\rho(\mathfrak{k})$, so that Schur's Lemma (Exercise 12.1.4) implies $\rho(y) \in \mathbb{C}\mathbf{1}$, and the skew-symmetry therefore leads to $\rho(y) \in \mathbb{R}I = \rho(\mathbb{R}z)$. This proves that $\dim \rho(\mathfrak{z}(\mathfrak{k})) \leq 1$.

Finally we note that in view of $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ the relation $x \in \ker \rho$ implies $x \in \mathfrak{z}(\mathfrak{g}) = \{0\}$, so that ρ is injective.

(iii) If $\mathfrak{k}$ is abelian, then (ii) implies that $\dim \mathfrak{k} = 1$, so that the simple $\mathfrak{k}$-module $\mathfrak{p}$ must be 2-dimensional. This proves already that $\dim \mathfrak{g} = 3$. From the proof of (ii) we know that we may pick $u \in \mathfrak{k}$ with $\rho(u)^2 = -4 \cdot \mathbf{1}$. Then there exist $t, h \in \mathfrak{p}$ with

$$[u, t] = 2h \quad \text{and} \quad [u, h] = -2t.$$

In view of $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k} = \mathbb{R}u$, we have $[h, t] = \lambda u$ for some $\lambda \in \mathbb{R}^\times$. Then

$$(\operatorname{ad} h)^2 t = [h, \lambda u] = 2\lambda t,$$

and since $\operatorname{ad} h$ is symmetric with respect to the scalar product κ_τ, we have $\lambda > 0$. Normalizing h and t appropriately, we may now assume that $\lambda = 2$, which leads to the relations

$$[u,t] = 2h, \quad [h,u] = 2t, \quad [h,t] = 2u,$$

so that Example 4.1.23 implies that $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R})$. □

Exercises for Section 12.1

Exercise 12.1.1. Let $\mathfrak{g}$ be a Lie algebra and κ be its Cartan–Killing form. Show that β is invariant under $\mathrm{Aut}(\mathfrak{g})$, i.e.,

$$\kappa(\gamma x, \gamma y) = \kappa(x,y) \quad \text{for} \quad \gamma \in \mathrm{Aut}(\mathfrak{g}), x, y \in \mathfrak{g}.$$

Exercise 12.1.2. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}$. Let $x \in \mathfrak{p}$. Then

$$\{y \in \mathfrak{k} \colon e^{\mathrm{ad}\, x} y \in \mathfrak{k}\} = \mathfrak{k} \cap \ker(\mathrm{ad}\, x) \quad \text{and} \quad \{y \in \mathfrak{p} \colon e^{\mathrm{ad}\, x} y \in \mathfrak{p}\} = \mathfrak{p} \cap \ker(\mathrm{ad}\, x).$$

Exercise 12.1.3. Show that a complex Lie algebra $\mathfrak{g}$ is semisimple if and only if the underlying real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ is semisimple.

Exercise 12.1.4 (Schur's Lemma). Let V be a finite-dimensional complex vector space and $S \subseteq \mathrm{End}(V)$ be a set of operators for which $\{0\}$ and V are the only S-invariant subspaces of V. Then

$$\mathrm{End}_S(V) = \{A \in \mathrm{End}(V) \colon (\forall \varphi \in S)\varphi A = A\varphi\} = \mathbb{C}\mathbf{1}.$$

12.2 Compact Real Forms

In the preceding section, we derived consequences of the existence of a Cartan involution τ of a semisimple Lie algebra $\mathfrak{g}$. In particular, we proved the existence of a corresponding decomposition of any Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ and of an involutive automorphism θ of G with $\mathbf{L}(\theta) = \tau$. What we still have to show is that Cartan involutions always exist.

First we recall some concepts from Section 4.1 (cf. Definition 4.1.22). Let $\mathfrak{g}$ be complex Lie algebra. Then $\mathfrak{g}$ can be considered as real Lie algebra $\mathfrak{g}^{\mathbb{R}}$. A *real form* of $\mathfrak{g}$ is a real subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}^{\mathbb{R}}$ for which $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus_{\mathbb{R}} i\mathfrak{g}_0$, which is equivalent to $(\mathfrak{g}_0)_{\mathbb{C}} \cong \mathfrak{g}$. Then the map

$$\sigma \colon \mathfrak{g} \to \mathfrak{g}, \quad x + iy \mapsto x - iy$$

is an antilinear automorphism of the complex Lie algebra $\mathfrak{g}$.

Lemma 12.2.1. *Let $\mathfrak{g}$ be a complex Lie algebra. Then the Cartan–Killing form κ of $\mathfrak{g}$ and the Cartan–Killing form $\kappa^{\mathbb{R}}$ of $\mathfrak{g}^{\mathbb{R}}$ are related by*

$$\kappa^{\mathbb{R}}(x,y) = 2\,\mathrm{Re}\,\kappa(x,y) \qquad \textit{for} \quad x, y \in \mathfrak{g}^{\mathbb{R}}.$$

Proof. From Exercise 12.2.1 we obtain for $x, y \in \mathfrak{g}$ the relation

$$\begin{aligned}\kappa^{\mathbb{R}}(x,y) &= \operatorname{tr}_{\mathbb{R}}((\operatorname{ad} x)^{\mathbb{R}}(\operatorname{ad} y)^{\mathbb{R}}) = \operatorname{tr}_{\mathbb{R}}((\operatorname{ad} x \operatorname{ad} y)^{\mathbb{R}}) \\ &= 2\operatorname{Re}\operatorname{tr}_{\mathbb{C}}(\operatorname{ad} x \operatorname{ad} y) = 2\operatorname{Re}\kappa(x,y).\end{aligned}$$ □

Lemma 12.2.2. *A real form $\mathfrak{g}_0$ of a complex semisimple Lie algebra $\mathfrak{g}$ is semisimple. It is compact if and only if $\kappa^{\mathbb{R}} = 2\operatorname{Re}\kappa_{\mathfrak{g}}$ is negative definite on $\mathfrak{g}_0$.*

Proof. We recall from Remark 4.5.10 that the semisimplicity of the complex Lie algebra $\mathfrak{g} \cong (\mathfrak{g}_0)_{\mathbb{C}}$ implies that $\mathfrak{g}_0$ is semisimple.

For $x, y \in \mathfrak{g}_0$, the operators $\operatorname{ad} x$ and $\operatorname{ad} y$ preserve the two subspaces $\mathfrak{g}_0$ and $i\mathfrak{g}_0$ of $\mathfrak{g}$, which leads to

$$\kappa^{\mathbb{R}}(x,y) = \operatorname{tr}_{\mathbb{R}}(\operatorname{ad} x \operatorname{ad} y) = 2\kappa_{\mathfrak{g}_0}(x,y).$$

If $\mathfrak{g}_0$ is compact and β is an invariant scalar product on $\mathfrak{g}_0$, then the operators $\operatorname{ad} x$, $x \in \mathfrak{g}_0$, are β-skew symmetric, so that $(\operatorname{ad} x)^2$ is negative semidefinite, and thus

$$\kappa^{\mathbb{R}}(x,x) = 2\kappa_{\mathfrak{g}_0}(x,x) = 2\operatorname{tr}((\operatorname{ad} x)^2) \leq 0.$$

Therefore $\kappa^{\mathbb{R}}$ is negative definite on $\mathfrak{g}_0$.

If, conversely, $\kappa^{\mathbb{R}}$ is negative definite on $\mathfrak{g}_0$, then $-\kappa^{\mathbb{R}}$ is an invariant scalar product, whose existence implies that $\mathfrak{g}_0$ is compact. □

Lemma 12.2.3. *If $\mathfrak{g}_0$ is a compact real form of the complex semisimple Lie algebra $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is a Cartan decomposition.*

Proof. By definition, the complex conjugation σ w.r.t. $\mathfrak{g}_0$ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ if and only if the Cartan–Killing form $\kappa^{\mathbb{R}}$ of $\mathfrak{g}^{\mathbb{R}}$ is negative definite on $\mathfrak{g}_0$ and positive definite on $i\mathfrak{g}_0$. If $\kappa^{\mathbb{R}}$ is negative definite on $\mathfrak{g}_0$ (Lemma 12.2.2), we obtain

$$\kappa^{\mathbb{R}}(ix,iy) = 2\operatorname{Re}\kappa_{\mathfrak{g}}(ix,iy) = -2\operatorname{Re}\kappa_{\mathfrak{g}}(x,y) = -\kappa^{\mathbb{R}}(x,y)$$

for $x, y \in \mathfrak{g}_0$. Hence $\kappa^{\mathbb{R}}$ is positive definite on $i\mathfrak{g}_0$, and thus σ is a Cartan involution. □

Example 12.2.4. We consider the complex simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the complex basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With respect to this basis, the Cartan–Killing form κ is represented by the matrix

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{pmatrix}. \tag{12.2}$$

Therefore its real part is negative definite on the subalgebra $\mathfrak{u} := \mathfrak{su}_2(\mathbb{C}) = \mathbb{R}ih+\mathbb{R}it+\mathbb{R}u$, which implies that $\mathfrak{su}_2(\mathbb{C})$ is a compact real form of $\mathfrak{sl}_2(\mathbb{C})$. The corresponding Cartan involution τ leaves the real form $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$ invariant. Let $\tau_0 := \tau|_{\mathfrak{g}_0}$. Then $\mathfrak{k}_0 := \mathbb{R}u$ is the 1-eigenspace of τ_0, and $\mathfrak{p}_0 := \mathbb{R}h + \mathbb{R}t$ is the -1-eigenspace of τ_0. By (12.2), κ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$. Since the Cartan–Killing form κ_0 of $\mathfrak{g}_0$ is the restriction of κ to $\mathfrak{g}_0 \times \mathfrak{g}_0$ (Exercise 4.5.3), τ_0 is a Cartan involution of $\mathfrak{g}_0$. ∎

The construction of Example 12.2.4 works in general. It reduces the proof of the existence of a Cartan involution of $\mathfrak{g}$ to the existence of a compact real form of $\mathfrak{g}_{\mathbb{C}}$, for which $\mathfrak{g}$ is invariant under the corresponding involution.

Proposition 12.2.5. (i) *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{u}$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$ with the corresponding involution σ. If $\mathfrak{g}$ is invariant under σ, then $\tau := \sigma|_{\mathfrak{g}}$ is a Cartan involution of $\mathfrak{g}$.*

(ii) *If $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of the semisimple real Lie algebra $\mathfrak{g}$, then $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. (i) Since $\mathfrak{g}$ is σ-invariant, τ is an involutive automorphism of $\mathfrak{g}$ whose eigenspaces are $\mathfrak{k} := \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$. If $\kappa_{\mathfrak{g}_{\mathbb{C}}}$ is the Cartan–Killing form of $\mathfrak{g}_{\mathbb{C}}$, then $\kappa_{\mathfrak{g}} = \kappa_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{g}\times\mathfrak{g}}$ is the Cartan–Killing form of $\mathfrak{g}$ (Exercise 4.5.3), and thus

$$\kappa_{\mathfrak{g}}|_{\mathfrak{k}\times\mathfrak{k}} = \kappa_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{k}\times\mathfrak{k}} = \operatorname{Re}\kappa_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{k}\times\mathfrak{k}} = \tfrac{1}{2}\kappa^{\mathbb{R}}_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{k}\times\mathfrak{k}}$$

is negative definite. Similarly, we see that the restriction to $\mathfrak{p} \times \mathfrak{p}$ is positive definite, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition.

(ii) Let $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$ with $x + iy \neq 0$. Then

$$\kappa^{\mathbb{R}}_{\mathfrak{g}_{\mathbb{C}}}(x+iy, x+iy) = 2\operatorname{Re}\kappa_{\mathfrak{g}_{\mathbb{C}}}(x+iy, x+iy) = 2\kappa_{\mathfrak{g}}(x,x) - 2\kappa_{\mathfrak{g}}(y,y) < 0.$$

Hence the subalgebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ because it also satisfies $\mathfrak{u} \oplus i\mathfrak{u} = (\mathfrak{k} \oplus i\mathfrak{p}) \oplus (i\mathfrak{k} \oplus \mathfrak{p}) = \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$. □

Corollary 12.2.6. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{g}$ and $G^{\mathbb{C}}$ be a connected Lie group with $\mathbf{L}(G^{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}}$. Then $K := \langle \exp_{G^{\mathbb{C}}} \mathfrak{k}\rangle$ is a compact subgroup of $G^{\mathbb{C}}$.*

Moreover, for every homomorphism $\pi\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{R})$, the subgroup $\langle e^{\pi(\mathfrak{k})}\rangle \subseteq \mathrm{GL}_n(\mathbb{R})$ is compact, and the center of $\langle e^{\pi(\mathfrak{g})}\rangle$ is finite.

Proof. Let $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ be the corresponding compact real form of $\mathfrak{g}_{\mathbb{C}}$ (Proposition 12.2.5). Then $\mathfrak{u}$ is a semisimple compact Lie algebra, and thus $U := \langle \exp_{G_{\mathbb{C}}} \mathfrak{u}\rangle$ is a compact subgroup of $G^{\mathbb{C}}$ (Theorem 11.1.17). Therefore the subgroup K of U is relatively compact. Since K is contained in the normalizer $N_{G^{\mathbb{C}}}(\mathfrak{g})$ of $\mathfrak{g}$ and $\mathbf{L}\left(N_{G^{\mathbb{C}}}(\mathfrak{g})\right) = N_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{g}) = \mathfrak{g}$ (Lemma 10.1.1, Exercise 12.2.3), we have

$$\mathbf{L}(\overline{K}) \subseteq \mathfrak{u} \cap \mathfrak{g} = \mathfrak{k}.$$

Therefore the connectedness of $\overline{K}$ leads to $\overline{K} = \langle \exp_{G^{\mathbb{C}}} \mathbf{L}(\overline{K})\rangle \subseteq \langle \exp_{G^{\mathbb{C}}} \mathfrak{k}\rangle = K$, so that K is closed.

Now let $\pi\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{R})$ be a representation of $\mathfrak{g}$ and assume that the group $G^{\mathbb{C}}$ is simply connected (which we may after passing to a suitable covering group). Then π extends to a complex linear homomorphism $\pi_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}_n(\mathbb{C})$, so that there exists a unique morphism $\alpha\colon G^{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ of Lie groups with $\mathbf{L}(\alpha) = \pi_{\mathbb{C}}$. Since K is compact, the group

$$\alpha(K) = \langle e^{\pi(\mathfrak{k})} \rangle \subseteq \mathrm{GL}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{C})$$

is also compact. The finiteness of the center of $\langle e^{\pi(\mathfrak{g})} \rangle$ now follows from its discreteness ($\pi(\mathfrak{g})$ is semisimple), the compactness of $\alpha(K)$, the fact that $\pi(\mathfrak{k}) + \pi(\mathfrak{p})$ is a Cartan decomposition of $\pi(\mathfrak{g})$ (Exercise 12.2.4), and Lemma 12.1.6. ☐

12.2.1 Existence of a Compact Real Form

Now we take first steps towards the existence proof for compact real forms which we shall derive from a root decomposition of $\mathfrak{g}_{\mathbb{C}}$. In the following proposition we use the notation from Lemma 5.3.7, where $\mathfrak{g}$ is a complex semisimple Lie algebra with a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ and $\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{h}$ is the real form spanned by the coroots.

Proposition 12.2.7. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Then there exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ such that $i\mathfrak{h}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{u}$.*

Proof. By Corollary 6.2.11, there exists an automorphism φ of $\mathfrak{g}$ with $\varphi|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $x_\alpha \in \mathfrak{g}_\alpha$ with $\kappa_{\mathfrak{g}}(x_\alpha, x_{-\alpha}) = 1$. Then $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for every $\alpha \in \Delta$, and therefore $\varphi(x_\alpha) = c_\alpha x_{-\alpha}$ for some $c_\alpha \in \mathbb{C}^\times$. Since φ preserves the Cartan–Killing form (Exercise 12.1.1), $c_\alpha c_{-\alpha} = 1$. We choose complex numbers $a_\alpha, \alpha \in \Delta$, such that $a_\alpha a_{-\alpha} = 1$ and $a_\alpha^2 = -c_{-\alpha}$. Let $y_\alpha := a_\alpha x_\alpha$. We define $N_{\alpha,\beta} := 0$ for $\alpha, \beta \in \Delta$ with $\alpha + \beta \notin \Delta$, and via

$$[y_\alpha, y_\beta] = N_{\alpha,\beta} y_{\alpha+\beta},$$

otherwise. Using $c_\alpha c_{-\alpha} = 1$, we now find that $\varphi(y_\alpha) = -y_{-\alpha}$ for all roots α. Then $\varphi([y_\alpha, y_\beta]) = -N_{\alpha,\beta} y_{-\alpha-\beta}$ and

$$[\varphi(y_\alpha), \varphi(y_\beta)] = N_{-\alpha,-\beta} y_{-\alpha-\beta}$$

lead to

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta}.$$

From the definition of the y_α we see that $\kappa_{\mathfrak{g}}(y_\alpha, y_{-\alpha}) = 1$, and therefore

$$\kappa([y_\alpha, y_\beta], [y_{-\alpha}, y_{-\beta}]) = N_{\alpha,\beta} N_{-\alpha,-\beta} \kappa(y_{\alpha+\beta}, y_{-\alpha-\beta}) = N_{\alpha,\beta} N_{-\alpha,-\beta} = -N_{\alpha,\beta}^2.$$

To show that the $N_{\alpha,\beta}$ are real, we only have to verify

$$\kappa([y_\alpha, y_\beta], [y_{-\alpha}, y_{-\beta}]) \leq 0.$$

For this verification we need some information on $\mathfrak{sl}_2(\mathbb{C})$-modules. First we put

$$e := y_\alpha \quad \text{and} \quad f := \frac{2}{(\alpha, \alpha)} y_{-\alpha}$$

and note that $\check{\alpha} = h_\alpha = [e, f]$, so that (h, e, f) satisfies the commutation relations of $\mathfrak{sl}_2$. Since the α-string through β defines a simple module over the Lie algebra $\mathfrak{g}(\alpha) = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathbb{C}\check{\alpha}$, it follows from the calculation in the proof of Proposition 5.2.4 that

$$[y_{-\alpha}, [y_\alpha, y_\beta]] = c y_\beta$$

for some $c \geq 0$. This leads to

$$\kappa([y_\alpha, y_\beta], [y_{-\alpha}, y_{-\beta}]) = -\kappa(\big[y_{-\alpha}, [y_\alpha, y_\beta]\big], y_{-\beta}) = -c\kappa(y_\beta, y_{-\beta}) = -c \leq 0.$$

Now we set

$$\mathfrak{u} := i\mathfrak{h}_\mathbb{R} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R}(y_\alpha - y_{-\alpha}) \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R}i(y_\alpha + y_{-\alpha}).$$

Applying

$$\begin{aligned}
[y_\alpha + \varepsilon y_{-\alpha}, y_\beta + \delta y_{-\beta}] &= N_{\alpha,\beta} y_{\alpha+\beta} + \varepsilon N_{-\alpha,\beta} y_{-\alpha+\beta} \\
&\quad + \delta N_{\alpha,-\beta} y_{\alpha-\beta} + \varepsilon\delta N_{-\alpha,-\beta} y_{-\alpha-\beta} \\
&= N_{\alpha,\beta}(y_{\alpha+\beta} - \varepsilon\delta y_{-\alpha-\beta}) + N_{-\alpha,\beta}(\varepsilon y_{-\alpha+\beta} - \delta y_{\alpha-\beta}),
\end{aligned}$$

to the cases $\varepsilon = \delta = -1$, $\varepsilon = \delta = 1$ and $\varepsilon = -\delta = 1$, we see that $\mathfrak{u}$ is a real subalgebra of $\mathfrak{g}$. The relation $\mathfrak{g}^\mathbb{R} = \mathfrak{u} \oplus i\mathfrak{u}$ is obvious. By Lemma 12.2.1, it only remains to show that $\kappa(x, x) < 0$ for $0 \neq x \in \mathfrak{u}$. In view of Proposition 5.3.1, this follows from

$$\kappa(y_\alpha - y_{-\alpha}, y_\alpha + y_{-\alpha}) = 1 - 1 = 0$$

and

$$\kappa(y_\alpha - y_{-\alpha}, y_\alpha - y_{-\alpha}) = -2, \quad \kappa\big(i(y_\alpha + y_{-\alpha}), i(y_\alpha + y_{-\alpha})\big) = -2. \qquad \square$$

Lemma 12.2.8. *Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ with corresponding involution τ. Then for each $\gamma \in \mathrm{Aut}(\mathfrak{g})$, $\gamma(\mathfrak{u})$ is a compact real form of $\mathfrak{g}$ with involution $\gamma \circ \tau \circ \gamma^{-1}$.*

Proof. Since γ is an automorphism, $\gamma(\mathfrak{u})$ again is a real form of $\mathfrak{g}$ and its compactness follows from $\mathfrak{u} \cong \gamma(\mathfrak{u})$. Clearly $\gamma\tau\gamma^{-1}$ is the unique antilinear involution of $\mathfrak{g}$ fixing $\gamma(\mathfrak{u})$ pointwise. $\square$

12.2.2 Existence of a Cartan Involution

Now let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{u}$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$. We want to use $\mathfrak{u}$ to obtain a Cartan involution, resp., decomposition of $\mathfrak{g}$. If the antilinear involution τ of $\mathfrak{g}_{\mathbb{C}}$ with $(\mathfrak{g}_{\mathbb{C}})^\tau = \mathfrak{u}$ preserves the subalgebra $\mathfrak{g}$, then

$$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{u}) \oplus (\mathfrak{g} \cap i\mathfrak{u})$$

is a Cartan decomposition of $\mathfrak{g}$ (Proposition 12.2.5(i)), and

$$i\mathfrak{g} = i(\mathfrak{u} \cap \mathfrak{g}) \oplus_{\mathbb{R}} (\mathfrak{u} \cap i\mathfrak{g}).$$

This means that τ and the involution σ corresponding to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ have a simultaneous eigenspace decomposition, so that they commute. If, conversely, σ and τ commute, then they are simultaneously diagonalizable over $\mathbb{R}$, so that $\mathfrak{g}$ is τ-invariant and τ restricts to a Cartan involution on $\mathfrak{g}$.

We are therefore looking for a compact real form $\mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$ whose involution commutes with σ. To find such an involution, we have to replace $\mathfrak{u}$ by $\varphi(\mathfrak{u})$ for a suitable automorphism φ. To get an idea for how to find φ, consider the commutator $\rho := \sigma\tau\sigma^{-1}\tau^{-1} = \sigma\tau\sigma\tau$. Then ρ is complex linear, hence in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$, and we have

$$\rho\sigma = \sigma\rho^{-1} = (\rho\sigma)^{-1} \quad \text{and} \quad \rho\tau = \tau\rho^{-1} = (\rho\tau)^{-1}.$$

Now $\rho(\mathfrak{u})$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ and $\rho\tau\rho^{-1}$ is the corresponding involution. If

$$\sigma(\rho\tau\rho^{-1}) = (\rho\tau\rho^{-1})\sigma,$$

or, equivalently, $(\sigma\rho\tau\rho^{-1})^2 = \mathrm{id}_{\mathfrak{g}}$, we may take $\varphi := \rho$, but

$$(\sigma\rho\tau\rho^{-1})(\sigma\rho\tau\rho^{-1}) = \sigma\tau\rho^{-2}\sigma\rho\tau\rho^{-1} = \sigma\tau\sigma\rho^3\tau\rho^{-1} = \sigma\tau\sigma\tau\rho^{-4} = \rho^{-3}.$$

Thus $\rho(\mathfrak{u})$ still is not the desired compact real form. But if one can find an automorphism γ of $\mathfrak{g}$ with $\gamma^4 = \rho$, $\gamma\sigma = \sigma\gamma^{-1}$ and $\gamma\tau = \tau\gamma^{-1}$, then $\gamma(\mathfrak{u})$ is a compact real form of the desired kind because

$$\sigma(\gamma\tau\gamma^{-1})\sigma(\gamma\tau\gamma^{-1}) = \sigma\tau\gamma^{-2}\sigma\gamma^2\tau = \sigma\tau\sigma\gamma^4\tau = \sigma\tau\sigma\rho\tau = \mathbf{1}.$$

To find γ, we introduce a scalar product for which ρ is positive definite. Since the roots should also be complex linear, one needs a hermitian scalar product and the following lemma:

Lemma 12.2.9. *Let $\mathfrak{g}$ be a semisimple real Lie algebra and $\mathfrak{u}$ a compact real form of $\mathfrak{g}_{\mathbb{C}}$ with corresponding involution τ. Let κ be the Cartan–Killing form of $\mathfrak{g}_{\mathbb{C}}$ and σ be the involution of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the real form $\mathfrak{g}$. We then have:*

(i) *$\kappa_\tau(x, y) := -\kappa(x, \tau y)$ for $x, y \in \mathfrak{g}_{\mathbb{C}}$ defines a hermitian scalar product on $\mathfrak{g}_{\mathbb{C}}$.*

(ii) $\sigma\tau$ *is a hermitian automorphism of* $\mathfrak{g}_{\mathbb{C}}$.
(iii) $\rho := (\sigma\tau)^2$ *is positive definite.*
(iv) *There is a hermitian derivation* $\delta \in \mathrm{der}(\mathfrak{g}_{\mathbb{C}})$ *with* $e^\delta = \rho$, $\sigma\delta\sigma = -\delta$ *and* $\tau\delta\tau = -\delta$.

Proof. (i) In view of Exercise 12.2.2, we have

$$\kappa_\tau(x, y) = -\kappa(x, \tau y) = -\overline{\kappa(\tau x, \tau^2 y)} = -\overline{\kappa(y, \tau x)} = \overline{\kappa_\tau(y, x)}.$$

For $0 \neq x = a + ib$ with $a, b \in \mathfrak{u}$, we get

$$\begin{aligned}\kappa_\tau(x, x) &= -\kappa(a, a) - \kappa(ib, -ib) = -\kappa(a, a) - \kappa(b, b)\\ &= -\kappa_{\mathfrak{u}}(a, a) - \kappa_{\mathfrak{u}}(b, b) > 0.\end{aligned}$$

Hence κ_τ is positive definite.

(ii) $\kappa_\tau(\sigma\tau x, y) = -\kappa(\sigma\tau x, \tau y) = -\overline{\kappa(\tau x, \sigma\tau y)} = -\kappa(x, \tau\sigma\tau y) = \kappa_\tau(x, \sigma\tau y)$.

(iii) follows from (ii).

(iv) Since ρ is positive definite, $\delta := \log \rho$ is a hermitian endomorphism of $(\mathfrak{g}_{\mathbb{C}}, \kappa_\tau)$ and Proposition 12.1.5 on the polar decomposition of $\mathrm{Aut}\left((\mathfrak{g}_{\mathbb{C}})^{\mathbb{R}}\right)$ implies that $\delta \in \mathrm{der}\,\mathfrak{g}_{\mathbb{C}}$. From $\sigma\rho\sigma = \rho^{-1}$ we further derive that $\sigma\delta\sigma = -\delta$, and we likewise get $\tau\delta\tau = -\delta$. □

By the remarks above, we now have that $\sigma\tau_\gamma = \tau_\gamma\sigma$ for $\gamma = e^{\frac{1}{4}\delta}$ and $\tau_\gamma = \gamma\tau\gamma^{-1}$, i.e., the compact real form $\gamma(\mathfrak{u})$ is invariant under σ. We thus obtain:

Theorem 12.2.10. *Every real semisimple Lie algebra has a Cartan decomposition.*

12.2.3 Conjugacy of Cartan Decompositions

Next we turn to the variety of all Cartan decompositions of a real semisimple Lie algebra. The next theorem is the key to the fact that any two Cartan decompositions are conjugate.

Theorem 12.2.11. *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra,* $\tau \in \mathrm{Aut}(\mathfrak{g})$ *be a Cartan involution and* $U \subseteq \mathrm{Aut}(\mathfrak{g})$ *be a compact subgroup. Then there exists a* $\gamma \in \mathrm{Aut}(\mathfrak{g})_0$ *with*

$$\gamma U \gamma^{-1} \subseteq \mathrm{Aut}(\mathfrak{g})^\tau = \{\varphi \in \mathrm{Aut}(\mathfrak{g}) \colon \varphi\tau = \tau\varphi\}.$$

Proof. Let κ_τ be the scalar product on $\mathfrak{g}$ associated with τ. Then

$$U \times \mathrm{Sym}(\mathfrak{g}, \kappa_\tau) \to \mathrm{Sym}(\mathfrak{g}, \kappa_\tau), \qquad (u, A) \mapsto \pi(u)A := uAu^\top,$$

defines a linear action of the compact group U on the vector space $\mathrm{Sym}(\mathfrak{g}, \kappa_\tau)$.

Let φ be a positive definite automorphism of $\mathfrak{g}$ which is fixed under this action. Using the polar decomposition of $\operatorname{Aut}(\mathfrak{g})$ (Proposition 12.1.5), we write $\varphi = e^{\operatorname{ad} x}$ with $x \in \mathfrak{p}$. We set $\gamma := e^{-\frac{1}{2}\operatorname{ad} x} \in \operatorname{Aut}(\mathfrak{g})_0$. Then

$$u\gamma^{-2} = u\varphi = \varphi(u^\top)^{-1} = \gamma^{-2}(u^\top)^{-1},$$

and therefore

$$(\gamma u \gamma^{-1})^\top = (\gamma^{-1}u^\top\gamma) = \gamma^{-1}u^{-1}\gamma = (\gamma u\gamma^{-1})^{-1}.$$

In view of (12.1) this is equivalent to $\gamma u\gamma^{-1} \in \operatorname{Aut}(\mathfrak{g})^\tau$ so that $\gamma U\gamma^{-1} \subseteq \operatorname{Aut}(\mathfrak{g})^\tau$.

Hence it remains to find a positive definite fixed point of U in $\operatorname{Aut}(\mathfrak{g})$. As a first step, we define

$$\psi := \int_U \pi(u)\mathbf{1}\ \mathtt{d}\mu_U(u) = \int_U uu^\top\ \mathtt{d}\mu_U(u),$$

where $\mathtt{d}\mu_U$ is the normalized Haar measure on U. Then $\psi^\top = \psi$. For $x \in \mathfrak{g}\setminus\{0\}$, we have

$$\begin{aligned}\kappa_\tau(\psi x, x) &= \kappa_\tau\Big(\int_U uu^\top\ \mathtt{d}\mu_U(u)x, x\Big)\\ &= \int_U \kappa_\tau\big(uu^\top x, x\big)\ \mathtt{d}\mu_U(u) = \int_U \kappa_\tau\big(u^\top x, u^\top x\big)\ \mathtt{d}\mu_U(u) > 0,\end{aligned}$$

because $\kappa_\tau(x,x) > 0$. Thus ψ is positive definite, and ψ is fixed under the action of U because

$$\pi(u_0)\psi = \int_U \pi(u_0u)\mathbf{1}\ \mathtt{d}\mu_U(u) = \int_U \pi(u)\mathbf{1}\ \mathtt{d}\mu_U(u)$$

follows from the left invariance of μ_U.

For a linear action of a compact group which leaves a pointed convex cone with nonempty interior invariant, the above construction gives a fixed point in the interior of this cone. Hence we would have achieved our goal if ψ were a Lie algebra automorphism of $\mathfrak{g}$. But this will not be the case in general (cf. Example 12.2.12). However, we can use ψ to find φ as required. To this end, we define

$$H : \operatorname{Aut}(\mathfrak{g})_+ := \operatorname{Aut}(\mathfrak{g}) \cap \operatorname{Pds}(\mathfrak{g}, \kappa_\tau) = e^{\operatorname{ad}\mathfrak{p}} \to \mathbb{R}, \qquad g \mapsto \operatorname{tr}(\psi g^{-1} + g\psi^{-1}).$$

Note that $\operatorname{Aut}(\mathfrak{g})_+$ is a closed subset of $\operatorname{End}(\mathfrak{g})$ because $\operatorname{tr}(\operatorname{ad} x) = 0$ for $x \in \mathfrak{g}$ implies that $|\det g| = 1$ for $g \in \operatorname{Aut}(\mathfrak{g})_+$. For $u \in U$, we have $ugu^\top \in \operatorname{Aut}(\mathfrak{g})$ by (12.1), so that

$$\begin{aligned}H\big(\pi(u)g\big) &= \operatorname{tr}(\psi(u^{-1})^\top g^{-1}u^{-1} + ugu^\top\psi^{-1})\\ &= \operatorname{tr}(u\psi u^\top(u^{-1})^\top g^{-1}u^{-1} + ugu^\top(u^{-1})^\top\psi^{-1}u^{-1})\\ &= \operatorname{tr}(u\psi g^{-1}u^{-1} + ug\psi^{-1}u^{-1}) = \operatorname{tr}(\psi g^{-1} + g\psi^{-1}) = H(g).\end{aligned}$$

Because of

$$H(g) = \mathrm{tr}(\psi^{\frac{1}{2}} g^{-1} \psi^{\frac{1}{2}} + \psi^{-\frac{1}{2}} g \psi^{-\frac{1}{2}}),$$

we get $H(g) > 0$ for all $g \in e^{\mathrm{ad}\,\mathfrak{p}}$. Our strategy is to show that H has a unique minimum, which will be the required automorphism φ.

Existence of a minimum: We choose an orthonormal basis $x_1, \ldots, x_n$ for $\mathfrak{g}$ consisting of eigenvectors of ψ. Let $e_1, \ldots, e_n$ be the corresponding rank-1 operators on $\mathfrak{g}$ defined by $e_i(x_j) = \delta_{ij} x_i$. Then $\psi = \sum_{i=1}^n e^{r_i} e_i$ with $r_i \in \mathbb{R}$, and we have

$$H(g) = \sum_{i=1}^n \big(e^{r_i}\,\mathrm{tr}(e_i g^{-1}) + e^{-r_i}\,\mathrm{tr}(g e_i)\big).$$

For $M > 0$, we consider the set $C_M := \{g \in \mathrm{Aut}(\mathfrak{g})_+ \colon H(g) \leq M\}$. This is a closed subset of $\mathrm{Aut}(\mathfrak{g})_+$ and hence of $\mathrm{End}(\mathfrak{g})$ because $\mathrm{Aut}(\mathfrak{g})_+$ is closed. For $H(g) \leq M$, we have $e^{r_i}\,\mathrm{tr}(e_i g^{-1}) + e^{-r_i}\,\mathrm{tr}(g e_i) \leq M$ for all i, and there exists an $M' > 0$ with $\mathrm{tr}(g e_i) = \mathrm{tr}(e_i g e_i) \leq M'$ for all i and $g \in C_M$. In particular, $\mathrm{tr}(g^{\frac{1}{2}}(g^{\frac{1}{2}})^\top) = \mathrm{tr}(g) \leq nM'$, and this is the sum of the squares of the entries of $g^{\frac{1}{2}}$. Hence this closed set is bounded and therefore compact. If we choose $M := H(\mathrm{id})$, then the existence of a minimum of H immediately follows from the fact that we can restrict our attention to a compact set.

Uniqueness of the minimum: We suppose that there are two minima B and C. Then we find P and Q in $\mathrm{ad}\,\mathfrak{p}$ with $e^{2P} = B$ and $e^Q = e^{-P} C e^{-P}$. We set $\eta(t) := e^P e^{tQ} e^P$, so that $\eta(0) = B$ and $\eta(1) = C$, and show that $H \circ \eta$ is strictly convex. For this, we set $\psi' := e^{-P} \psi e^{-P}$, choose a basis as above such that Q has diagonal form $Q = \sum_{i=1}^n s_i e_i$, and calculate as follows:

$$\begin{aligned} H(\eta(t)) &= \mathrm{tr}(\psi e^{-P} e^{-tQ} e^{-P} + e^P e^{tQ} e^P \psi^{-1}) \\ &= \mathrm{tr}(e^P e^{-P} \psi e^{-P} e^{-tQ} e^{-P} + e^P e^{tQ} e^P \psi^{-1} e^P e^{-P}) \\ &= \mathrm{tr}(\psi' e^{-tQ} + e^{tQ} \psi'^{-1}) = \sum_{i=1}^n \big(e^{-ts_i}\,\mathrm{tr}(\psi' e_i) + e^{ts_i}\,\mathrm{tr}(e_i \psi'^{-1})\big). \end{aligned}$$

Since ψ' is positive definite, we have $\mathrm{tr}(\psi' e_i) = \kappa_\tau(\psi x_i, x_i) > 0$ for every i, and therefore the above function is strictly convex. By $H(\eta(0)) = H(\eta(1))$, it thus follows that $\eta(0) = B = \eta(1) = C$. This proves the existence of a unique minimum φ. For $u \in U$, we have $H(\pi(u)\varphi) = H(\varphi)$, so that the uniqueness of the minimum leads to $\pi(u)\varphi = \varphi$. □

Example 12.2.12. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ with the basis h, t, u as in Example 12.2.4. We set

$$\gamma := e^{\mathrm{ad}\,h}, \quad \theta := e^{\frac{\pi}{2}\,\mathrm{ad}\,u} \quad \text{and} \quad \alpha := \gamma\theta\gamma^{-1}.$$

Then $\theta^2 = \mathrm{id}_{\mathfrak{g}}$ and $\alpha^2 = \mathrm{id}_{\mathfrak{g}}$. In particular, $V := \{\mathrm{id}_{\mathfrak{g}}, \alpha\}$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$. We show that the linear map ψ in the proof of Theorem 12.2.11 is not an automorphism of $\mathfrak{g}$, in general. In our case, we have

$$\psi = \frac{1}{2}(1 + \alpha\alpha^\top) \quad \text{and} \quad \alpha\alpha^\top = \gamma\theta\gamma^{-2}\theta\gamma = \gamma^3\theta^2\gamma = \gamma^4.$$

Therefore $[h, t \pm u] = \pm 2(t \pm u)$ leads to

$$\psi(t \pm u) = \frac{1}{2}(1 + e^{\pm 8})(t \pm u).$$

On the other hand $[t + u, t - u] = 2[u, t] = 4h = \psi(4h)$, so that $(1 + e^8)(1 + e^{-8}) \neq 4$ implies that ψ is not an automorphism of $\mathfrak{g}$.

Corollary 12.2.13. *If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{p}}$ are two Cartan decompositions of $\mathfrak{g}$, then there exists a $\gamma \in \mathrm{Aut}(\mathfrak{g})_0$ such that*

$$\gamma(\mathfrak{k}) = \widetilde{\mathfrak{k}} \quad \textit{and} \quad \gamma(\mathfrak{p}) = \widetilde{\mathfrak{p}}.$$

Proof. Let θ and $\widetilde{\theta}$ be the corresponding Cartan involutions. If θ and $\widetilde{\theta}$ commute, then

$$\widetilde{\mathfrak{k}} = (\widetilde{\mathfrak{k}} \cap \mathfrak{k}) \oplus (\widetilde{\mathfrak{k}} \cap \mathfrak{p}),$$

but since all nonzero elements in $\mathfrak{p}$ are symmetric with respect to κ_θ (hence generate unbounded one-parameter groups) and all elements in $\widetilde{\mathfrak{k}}$ generate bounded one-parameter groups of automorphisms of $\mathfrak{g}$, it follows that $\widetilde{\mathfrak{k}} \cap \mathfrak{p} = \{0\}$, and likewise $\widetilde{\mathfrak{p}} \cap \mathfrak{k} = \{0\}$. This implies that $\widetilde{\mathfrak{k}} = \mathfrak{k}$ and $\widetilde{\mathfrak{p}} = \mathfrak{p}$.

Since $\widetilde{\theta}$ is an involution, Theorem 12.2.11 provides a $\gamma \in \mathrm{Aut}(\mathfrak{g})_0$ with $\theta' := \gamma\widetilde{\theta}\gamma^{-1} \subseteq \mathrm{Aut}(\mathfrak{g})^\theta$. Then θ and θ' are two commuting Cartan involutions, hence coincide. This leads to $\theta = c_\gamma(\widetilde{\theta})$, and the assertion follows. □

Proposition 12.2.14. *Let G be a Lie group with semisimple Lie algebra and τ a Cartan involution of its Lie algebra $\mathfrak{g}$, and*

$$K := \{g \in G \colon \tau\,\mathrm{Ad}(g) = \mathrm{Ad}(g)\tau\}.$$

Then for each subgroup $U \subseteq G$ for which $\mathrm{Ad}(U)$ is relatively compact there exists an element $g \in G_0$ with $gUg^{-1} \subseteq K$.

Proof. By Theorem 12.2.11, there exists a $g \in G_0$ with

$$\mathrm{Ad}(g)\,\mathrm{Ad}(U)\,\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(gUg^{-1}) \subseteq \mathrm{Aut}(\mathfrak{g})^\tau.$$

Then $gUg^{-1} \subseteq \mathrm{Ad}^{-1}(\mathrm{Aut}(\mathfrak{g})^\tau) = K$ □

Exercises for Section 12.2

Exercise 12.2.1. Let V be a finite-dimensional complex vector space and $A \in \mathrm{End}(V)$ be a complex linear endomorphism, which can also be considered as a real linear endomorphism $A^{\mathbb{R}}$ of the underlying real vector space. Show that

$$\mathrm{tr}_{\mathbb{R}}(A^{\mathbb{R}}) = 2\,\mathrm{Re}\,\mathrm{tr}_{\mathbb{R}}(A).$$

Exercise 12.2.2. Let $\mathfrak{g}$ be a complex Lie algebra, κ its Cartan–Killing form and $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ an antilinear automorphism of $\mathfrak{g}$. Then

$$\kappa\big(\sigma(x), \sigma(y)\big) = \overline{\kappa(x,y)} \qquad \text{for } x, y \in \mathfrak{g}.$$

Exercise 12.2.3. Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Show that

$$\mathfrak{n}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{g}) = \{z \in \mathfrak{g}_{\mathbb{C}} \colon [z, \mathfrak{g}] \subseteq \mathfrak{g}\} = \mathfrak{g}.$$

Exercise 12.2.4. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ the decomposition into simple ideal and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Show that

(i) $\mathfrak{k} = \sum_{i=1}^n \mathfrak{k} \cap \mathfrak{g}_i$.
(ii) $\mathfrak{p} = \sum_{i=1}^n \mathfrak{p} \cap \mathfrak{g}_i$, and that
(iii) $\mathfrak{g}_i = (\mathfrak{k} \cap \mathfrak{g}_i) \oplus (\mathfrak{p} \cap \mathfrak{g}_i)$ is a Cartan decomposition of $\mathfrak{g}_i$.

Exercise 12.2.5. Prove Weyl's theorem on complete reducibility 4.5.21 using representation theory of compact groups.

12.3 The Iwasawa Decomposition

In Section 12.1 we have seen how to use the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ to obtain a Cartan decomposition of semisimple Lie groups. Now we turn to another decomposition of this class of Lie groups, the Iwasawa decomposition. At the end of this section, we will see how the Cartan and the Iwasawa decomposition are connected.

In this section, $\mathfrak{g}$ denotes a real semisimple Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition with the involution τ. Further, G is a connected Lie group with $\mathbf{L}(G) = \mathfrak{g}$. We write κ for the Cartan–Killing form of $\mathfrak{g}$ and $\kappa_\tau(x,y) := -\kappa(x, \tau y)$ for the corresponding scalar product.

Definition 12.3.1. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra. With respect to κ_τ, all operators $\operatorname{ad} x$, $x \in \mathfrak{a}$, are symmetric, hence in particular diagonalizable. By Lemma 5.1.3, we thus have a simultaneous eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad \text{with} \quad \mathfrak{g}_\alpha := \mathfrak{g}_\alpha(\mathfrak{a}) = \{x \in \mathfrak{g} \colon (\forall a \in \mathfrak{a})[a, x] = \alpha(a)x\}$$

and we call

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{a}) := \big\{\alpha \in \mathfrak{a}^* \setminus \{0\} \colon \mathfrak{g}_\alpha \neq \{0\}\big\}$$

the *root system* of $\mathfrak{g}$ with respect to $\mathfrak{a}$.

Lemma 12.3.2. *For each $\alpha \in \Delta \cup \{0\}$, we have $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.*

Proof. For $x \in \mathfrak{g}_\alpha$, $\alpha \in \mathfrak{a}^*$ and $a \in \mathfrak{a}$, we have

$$[a, \tau(x)] = \tau[\tau(a), x] = -\tau[a, x] = -\alpha(a)\tau(x),$$

which implies the lemma. □

The root system Δ is finite, so that there exists some $a_0 \in \mathfrak{a}$ with $\alpha(a_0) \neq 0$ for all $\alpha \in \Delta$. We set

$$\Delta^+ := \Delta^+(a_0) := \{\alpha \in \Delta \colon \alpha(a_0) > 0\} \quad \text{and } \Delta^- := -\Delta^+.$$

Then Δ is the disjoint union of Δ^+ and Δ^-. We set

$$\mathfrak{n} := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{b} := \mathfrak{a} + \mathfrak{n}.$$

Theorem 12.3.3 (Iwasawa Decomposition of a Semisimple Lie Algebra). *The subspace* $\mathfrak{n}$ *is a nilpotent Lie subalgebra of* $\mathfrak{g}$, $\mathfrak{b} \cong \mathfrak{n} \rtimes \mathfrak{a}$ *is a solvable subalgebra, and* $\mathfrak{g}$ *is the vector space direct sum*

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}.$$

Proof. First we observe that for $\alpha, \beta \in \Delta^+$ with $\alpha + \beta \in \Delta$, the root $\alpha + \beta$ is positive, i.e., an element of Δ^+. Therefore $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (Proposition 5.1.5) implies that $\mathfrak{b}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$. If $c := \min \Delta^+(a_0)$, $c' := \max \Delta^+(a_0)$ and $kc > c'$, then no sum of k roots $\alpha_1, \ldots, \alpha_k$ in Δ^+ is a root, and this leads to $C^k(\mathfrak{n}) = \{0\}$, i.e., $\mathfrak{n}$ is nilpotent. As $\mathfrak{a} \cong \mathfrak{b}/\mathfrak{n}$ is abelian and $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{n}$, the Lie algebra $\mathfrak{b}$ is solvable.

Now let $x \in \mathfrak{g}$. According to the root decomposition with respect to $\mathfrak{a}$, it can be written as

$$x = a + k + \sum_{\alpha \in \Delta} x_\alpha$$

with $x_\alpha \in \mathfrak{g}_\alpha$, $a \in \mathfrak{a}$ and $k \in \mathfrak{k} \cap \mathfrak{g}_0$. Here we use that $\mathfrak{g}_0$ is τ-invariant (Lemma 12.3.2), which leads to

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}) = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}.$$

For $\alpha \in -\Delta^+$, we write

$$x_\alpha = (x_\alpha + \tau x_\alpha) - \tau x_\alpha.$$

Here the first summand is contained in $\mathfrak{k}$ because it is τ-invariant, and the second one is contained in $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{n}$. Hence $x \in \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. It remains to show that the sum $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is direct. So let $x + y + z = 0$ with $x \in \mathfrak{k}$, $y \in \mathfrak{a}$ and $z \in \mathfrak{n}$. Applying τ, we obtain $x - y + \tau(z) = 0$. Substituting x now gives $2y = \tau(z) - z$. By Lemma 5.1.3, the sum $\mathfrak{a} + \mathfrak{n} + \tau(\mathfrak{n})$ is direct, and therefore $y = z = 0$, which also implies $x = 0$. □

Lemma 12.3.4. *If $\alpha, \beta \in \mathfrak{a}^*$ with $\alpha + \beta \neq 0$, then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*

Proof. Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $a \in \mathfrak{a}$ with $(\alpha+\beta)(a) = 1$. Then

$$-\alpha(a)\kappa(x,y) = \kappa([x,a],y) = \kappa(x,[a,y]) = \beta(a)\kappa(x,y)$$

leads to $\kappa(x,y) = (\alpha+\beta)(a)\kappa(x,y) = 0$. □

Lemma 12.3.5. *There is a basis for $\mathfrak{g}$ such that the matrix representing $\operatorname{ad} x$ is:*

(i) *skew-symmetric for $x \in \mathfrak{k}$.*
(ii) *diagonal for $x \in \mathfrak{a}$.*
(iii) *strictly lower triangular for $x \in \mathfrak{n}$.*

Proof. We choose a κ_τ-orthonormal basis $h_1, \ldots, h_n$ for $\mathfrak{g}_0$ and an orthonormal basis $e_\alpha^1, \ldots, e_\alpha^{n_\alpha} \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta^+$. We show that

$$\{h_1, \ldots, h_n\} \cup \{e_\alpha^i, \tau e_\alpha^i \colon \alpha \in \Delta^+, i = 1, \ldots, n_\alpha\}$$

is a basis for $\mathfrak{g}$ with the desired properties. Since κ_τ is invariant under τ, all basis vectors have length 1. In view of Lemma 12.3.4, the basis is κ_τ-orthogonal. Clearly, (i) and (ii) are fulfilled. Now we order Δ^+ as follows:

$$\alpha_1, \ldots, \alpha_k \quad \text{and} \quad \alpha_i(a_0) \leq \alpha_j(a_0) \quad \text{for} \quad i \leq j$$

(with a_0 from the definition of Δ^+), and we set

$$\mathcal{B} := (\tau e_{\alpha_k}^{n_{\alpha_k}}, \ldots, \tau e_{\alpha_k}^1, \ldots, \tau e_{\alpha_1}^1, h_1, \ldots, h_n, e_{\alpha_1}^1, \ldots, e_{\alpha_k}^{n_{\alpha_k}})$$

If x is a basis vector in $\mathfrak{g}_\alpha$ and $n \in \mathfrak{g}_\beta \subseteq \mathfrak{n}$, then $[n,x] \in \mathfrak{g}_{\alpha+\beta}$ with $(\alpha+\beta)(a_0) > \alpha(a_0)$, which implies (iii). □

Example 12.3.6. An Iwasawa decomposition of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ can be obtained as follows. With respect to the Cartan involution $\tau(x) = -x^\top$ we have $\mathfrak{k} = \mathfrak{so}_n(\mathbb{R})$, the Lie algebra of skew-symmetric matrices and $\mathfrak{p} = \{x \in \mathrm{Sym}_n(\mathbb{R}) \colon \operatorname{tr} x = 0\}$. Therefore the subspace $\mathfrak{a}$ of all diagonal matrices in $\mathfrak{p}$ is maximal abelian and for a suitable ordering of the corresponding roots, $\mathfrak{n}$ is the subalgebra of strictly lower triangular matrices in $\mathfrak{g}$.

Lemma 12.3.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{k}$, $\mathfrak{b}$ be subalgebras with $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$ and $\mathfrak{k} \cap \mathfrak{b} = \{0\}$. For the integral subgroups $K := \langle \exp \mathfrak{k} \rangle$ and $B := \langle \exp \mathfrak{b} \rangle$, endowed with the intrinsic Lie group structure, the map*

$$\Phi \colon K \times B \to G, \qquad (k,b) \mapsto kb$$

is everywhere regular.

Proof. From $\Phi(kk', b'b) = k\Phi(k', b')b$ we obtain the relation

$$T_{(k,b)}(\Phi)\big(T_{\mathbf{1}}(\lambda_k)x, T_{\mathbf{1}}(\rho_b)y\big) = T_b(\lambda_k)T_{\mathbf{1}}(\rho_b)T_{(\mathbf{1},\mathbf{1})}(\Phi)(x, y)$$
$$= T_b(\lambda_k)T_{\mathbf{1}}(\rho_b)(x + y)$$

for $x \in \mathfrak{k}$ and $y \in \mathfrak{b}$. Since the addition map $\mathfrak{k} \times \mathfrak{b} \to \mathfrak{g}$ is a linear isomorphism, it follows that $T_{(k,b)}(\Phi)$ is an isomorphism for any $k \in K$ and $b \in B$. □

Theorem 12.3.8 (Iwasawa Decomposition of a Semisimple Lie Group). *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of the semisimple Lie algebra $\mathfrak{g}$ and G be a connected Lie group with $\mathbf{L}(G) = \mathfrak{g}$. Then the subgroups $K := \langle \exp_G \mathfrak{k} \rangle$, $A := \exp_G \mathfrak{a}$ and $N := \exp_G \mathfrak{n}$ are closed and the multiplication map*

$$\Phi\colon K \times A \times N \to G, \qquad (k, a, n) \mapsto kan$$

is a diffeomorphism. Moreover, the groups A and N are simply connected.

Proof. **Step 1:** First we assume that $Z(G) = \{\mathbf{1}\}$, so that $G \cong \mathrm{Ad}(G)$. We choose a basis for $\mathfrak{g}$ according to Lemma 12.3.5. Then the elements of K are represented by orthogonal matrices, the elements of A by diagonal matrices with positive entries, and the elements of N are represented by unipotent matrices.

We start with the injectivity of Φ. If $kan = k'a'n'$, then $k^{-1}k' = ann'^{-1}a'^{-1}$ is an orthogonal lower triangular matrix with positive diagonal entries, hence the identity matrix. This implies that $k' = k$, $a = a'$, and hence also $n' = n$.

Since $\mathfrak{n}$ is nilpotent, $N = e^{\mathrm{ad}\,\mathfrak{n}}$ is an integral subgroup of G (cf. Corollary 10.2.7), and Proposition 2.3.3 implies that $\mathfrak{n} \to N, x \mapsto e^{\mathrm{ad}\,x}$ is a homeomorphism onto a closed subset of G. It follows in particular that N is simply connected. That A is also simply connected and closed follows from the fact that the exponential function of

$$D := \{\mathrm{diag}(a_1, \ldots, a_d)\colon a_i \in \mathbb{R}^\times_+\} \subseteq \mathrm{GL}_d(\mathbb{R})$$

is a diffeomorphism. Since A normalizes N, $B := AN$ is a subgroup of G. Since the group of lower triangular matrices is a semidirect product of D and the group U of unipotent lower triangular matrices, the group $B = AN \cong A \times N$ is closed (as a product of two closed subsets of a product space) and the map $A \times N \to B, (a, n) \mapsto an$ is a diffeomorphism (Lemma 12.3.7). The subgroup K of G is compact by Proposition 12.1.5, and B is closed, so that the product KB also is closed (Exercise 12.3.5). In view of Lemma 12.3.7 and the Inverse Function Theorem, the relation $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ implies that the set KB is also open. Since G is connected, $KB = G$, i.e., Φ is surjective. This proves the assertion for the group $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})_0$.

Step 2: Now we turn to the general case. Then the groups A, and N, resp., coincide with the identity components of the preimages of $\mathrm{Ad}(A)$ and $\mathrm{Ad}(N)$, resp., so that they are closed. Note also that $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$ implies that A

normalizes N, so that AN is a subgroup. Since the group $\mathrm{Ad}(AN)$ is simply connected by Part 1, $Z(G) \cap AN = \ker(\mathrm{Ad}\,|_{AN}) = \{\mathbf{1}\}$, and therefore $\mathrm{Ad}\,|_{AN}$ is a diffeomorphism of AN onto $\mathrm{Ad}(AN)$. Further, $\mathrm{Ad}(K) \cap \mathrm{Ad}(AN) = \{\mathbf{1}\}$ leads to $K \cap AN \subseteq \ker \mathrm{Ad} = Z(G)$, and hence to $K \cap AN = \{\mathbf{1}\}$. This implies the injectivity of Φ. To see that Φ is also surjective, let $g \in G$ and write $\mathrm{Ad}(g) = \mathrm{Ad}(k)\,\mathrm{Ad}(a)\,\mathrm{Ad}(n)$ with $k \in K$, $a \in A$ and $n \in N$. Then $gn^{-1}a^{-1}k^{-1} \in Z(G) \subseteq K$ (Lemma 12.1.6), and thus Φ is surjective, too. The regularity of Φ follows again by applying Lemma 12.3.7 twice. □

The following result provides the connection between Cartan and Iwasawa decomposition.

Corollary 12.3.9. *With the notation of* Theorem 12.3.8*, let* $B = AN$*,* $P := \exp_G \mathfrak{p}$*, and* θ *be the corresponding Cartan involution of* G*. Then the map*

$$\psi \colon b \mapsto \theta(b)b^{-1}$$

is a diffeomorphism of B *onto* P*.*

Proof. Let $b \in B$ and write it as $b = \exp_G(x)k$ with $x \in \mathfrak{p}$ and $k \in K$ (Theorem 12.1.7). Then $\theta(b)b^{-1} = \exp_G(-2x) \in P$. In particular, $\psi(B) \subseteq P$.

Claim 1: ψ is injective. For $\theta(b)b^{-1} = \theta(b')b'^{-1}$, we have $\theta(b^{-1}b') = b^{-1}b'$, and thus $b^{-1}b' \in K \cap B = \{\mathbf{1}\}$.

Claim 2: ψ is surjective. Let $p = \exp_G x \in P$ with $x \in \mathfrak{p}$. Then there exist $b \in B$ and $k \in K$ with $\exp_G(\frac{1}{2}x) = kb^{-1}$ (Theorem 12.3.8), and then $\psi(b) = p$.

The smoothness of ψ^{-1} follows from the fact that it is a composition of the maps $(\exp_G|_{\mathfrak{p}})^{-1} \colon P \to \mathfrak{p}, e^x \mapsto x$, the multiplication with $\frac{1}{2}$ on $\mathfrak{p}$, the projection onto the B-component $G = KB \to B, g = kb \to b$, and the inversion $b \mapsto b^{-1}$. □

Exercises for Section 12.3

In this section, we encountered several methods to decompose noncompact semisimple Lie groups. To see how these decompositions look like in concrete cases, we will have a closer look at the most important example $\mathrm{SL}_2(\mathbb{R})$.

Exercise 12.3.1. Let $\exp \colon \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ be the exponential function, and let $k(x) := \frac{1}{2}\operatorname{tr}(x^2)$. Then we have

$$\exp x = C\big(k(x)\big)\mathbf{1} + S\big(k(x)\big)x \qquad \text{for} \qquad x \in \mathfrak{sl}_2(\mathbb{R}),$$

where the functions $C, S \colon \mathbb{R} \to \mathbb{R}$ are defined by

$$C(t) = \begin{cases} \cosh\sqrt{t}, & \text{for } t \geq 0, \\ \cos\sqrt{-t}, & \text{for } t < 0, \end{cases} \quad \text{and} \quad S(t) = \begin{cases} \frac{1}{\sqrt{|t|}}\sinh\sqrt{t}, & \text{for } t \geq 0, \\ \frac{1}{\sqrt{|t|}}\sin\sqrt{-t}, & \text{for } t < 0. \end{cases}$$

Exercise 12.3.2. A one-parameter group $\exp \mathbb{R}x$ of $\mathrm{SL}_2(\mathbb{R})$ is compact if and only if $\mathrm{tr}(x^2) < 0$.

As before, we use the following basis for $\mathfrak{sl}_2(\mathbb{R})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Exercise 12.3.3. With respect to this basis, we have:

(a) $k(\alpha h + \beta t + \gamma u) = \frac{1}{2} \mathrm{tr}\left((\alpha h + \beta t + \gamma u)^2\right) = \alpha^2 + \beta^2 - \gamma^2$.
(b) $\exp(\alpha h + \beta t) = \cosh \sqrt{\alpha^2 + \beta^2}\mathbf{1} + \frac{1}{\sqrt{\alpha^2+\beta^2}} \sinh \sqrt{\alpha^2 + \beta^2} \cdot (\alpha h + \beta t)$.
(c) $\det(\delta\mathbf{1} + \alpha h + \beta t + \gamma u) = \delta^2 + \gamma^2 - \beta^2 - \alpha^2$.
(d) $\mathrm{SL}_2(\mathbb{R})$ is a three-dimensional hyperboloid in $\mathfrak{gl}_2(\mathbb{R}) = \mathrm{span}\{\mathbf{1}, h, t, u\}$.
(e) Use the Cartan decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R}) \exp(\mathbb{R}h + \mathbb{R}t)$ to obtain a parameterization of this hyperboloid.

Exercise 12.3.4. With $e := \frac{1}{2}(t + u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have

(a) $\exp(\zeta e) = \mathbf{1} + \zeta e$.
(b) $\mathfrak{g} = \mathbb{R}u \oplus \mathbb{R}h \oplus \mathbb{R}e$ is an Iwasawa decomposition of $\mathfrak{sl}_2(\mathbb{R})$ with $\mathfrak{k} = \mathbb{R}u$, $\mathfrak{a} = \mathbb{R}h$ and $\mathfrak{n} = \mathbb{R}e$. Hint: Choose $a_0 := h \in \mathfrak{a}$ to define Δ^+.
(c) Compute the parameterization of $\mathrm{SL}_2(\mathbb{R})$ that one obtains from the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R}) \exp(\mathbb{R}h) \exp(\mathbb{R}e)$.

Exercise 12.3.5. Let G be a topological group, $K \subseteq G$ be a compact subgroup and $F \subseteq G$ be a closed subset. Show that the product set KF is closed.

Notes on Chapter 12

The existence of a Cartan decomposition is a crucial step in the understanding of the structure of semisimple Lie groups. The path we follow to obtain this result builds heavily on the theory of Lie algebras developed in Chapters 4 and 5 and in particular on Serre's Theorem. An alternative route to the existence of a Cartan decomposition, based on the Riemannian manifold of positive definite matrices (cf. [La99]) and, more generally, Riemannian manifolds with seminegative curvature, has been described by S. K. Donaldson in [Do07].

For a more detailed structure theory of semisimple Lie groups as it is needed for representation theory, we refer to [Wa88] and [Kn02].

13

General Structure Theory

In this chapter we shall reach our first main goal, namely the Manifold Splitting Theorem, that a Lie group G with finitely many connected components is diffeomorphic to $K \times \mathbb{R}^n$, where K is a maximal compact subgroup. Many results we proved so far for special classes of groups, such as nilpotent, compact and semisimple ones, will be used to obtain the general case. This approach is quite typical for Lie theory. To prove a theorem for general Lie groups, one first deals with special classes such as abelian, nilpotent and solvable groups, and then one turns to the other side of the spectrum, to semisimple Lie groups. Often the techniques required for semisimple and solvable groups are quite different. To obtain the Manifold Splitting Theorem for general Lie groups, Levi's Theorem is a crucial tool to combine the semisimple and the solvable pieces because the Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ of a Lie algebra implies a corresponding decomposition $G = R \rtimes S$ of the corresponding 1-connected Lie group G. This is already the central idea for the simply connected case. To deal with a general Lie group G, we need a better understanding of the center of a connected Lie group, because G is a quotient of its simply connected covering by a discrete central subgroup. We have to understand how this influences the splitting of $\widetilde{G}$ to obtain a solution of the general case.

Similar strategies can be applied to many problems in Lie theory. Often it is not easy to say on which level the difficulties will appear. For the splitting problem, most of the work has been done already in the semisimple case, where we had to prove the existence of Cartan decompositions and the Conjugacy Theorem 12.2.11.

13.1 Maximal Compact Subgroups

In this section, we show that every Lie group G with finitely many connected components contains a maximal compact subgroup K, and that any other compact subgroup $U \subseteq G$ is conjugate to a subgroup of K under inner automorphisms.

Lemma 13.1.1. *If the Lie group $G = V \rtimes_\gamma K$ is a semidirect product, where V is a vector group and K is compact, then for every compact subgroup $U \subseteq G$, there exists a $v \in V$ with $vUv^{-1} \subseteq K$.*

Proof. We write the elements $u \in U$ as $u = \sigma(u)\tau(u)$, where $\sigma(u) \in V$ and $\tau(u) \in K$. Then

$$\sigma(u_1u_2) = \sigma(u_1) + \gamma(\tau(u_1))\sigma(u_2),$$

so that

$$\alpha(u)v := \gamma(\tau(u))v - \sigma(u)$$

defines a smooth affine action of U on V. The condition $vUv^{-1} \subseteq K$ is equivalent to

$$(v, \mathbf{1})(\sigma(u), \tau(u))(-v, \mathbf{1}) = (v+\sigma(u)-\gamma(\tau(u))v, \tau(u)) = (v-\alpha(u)v, \tau(u)) \in K$$

for all $u \in U$, which is equivalent to $\alpha(u)v = v$. Therefore we have to show that the affine action α of U on V admits a fixed point.

Such a fixed point can easily be constructed with a normalized Haar measure μ_U on U. For some $v_0 \in V$, we consider the *center of mass* of its U-orbit

$$v := \int_U \alpha(u)v_0 \, d\mu_U(u)$$

and claim that v is a fixed point. In fact, since μ_U is normalized, we have for any $w \in V$ and $A \in \mathrm{End}(V)$ the relation

$$\int_U (Af(u) + w) \, d\mu_U(u) = A \int_U f(u) \, d\mu_U(u) + w,$$

which leads for $g \in U$ to

$$\alpha(g)v = \int_U \alpha(g)\alpha(u)v_0 \, d\mu_U(u) = \int_U \alpha(gu)v_0 \, d\mu_U(u) = v$$

by left invariance of the Haar measure. □

Lemma 13.1.2. *A Lie group G whose Lie algebra $\mathfrak{g}$ is not semisimple either contains a nontrivial closed normal vector subgroup or a nontrivial normal torus.*

Proof. Since $\mathfrak{g}$ is not semisimple, its radical $\mathfrak{r}$ is nonzero. The last nonzero element $\mathfrak{a} = D^n(\mathfrak{r})$ of its derived series is a nonzero abelian ideal of $\mathfrak{g}$ which is invariant under all automorphisms of $\mathfrak{g}$ (Exercise 13.1.1), hence in particular under $\mathrm{Ad}(G)$. Therefore $A := \overline{\langle \exp_G \mathfrak{a} \rangle}$ is a nontrivial connected closed normal abelian subgroup of G. If A is not a vector group, then $A \cong V \times K$, where V is a vector group and K is the unique maximal compact subgroup of A (cf. Example 8.5.6). In view of the uniqueness, K is invariant under all automorphisms of A, hence in particular under conjugation by elements of G, and therefore a normal subgroup of G. Since K is abelian and connected, it is a torus. □

Theorem 13.1.3 (Maximal Compact Subgroups). *Any Lie group G with finitely many connected components contains a compact subgroup C with the property that for every other compact subgroup U of G, there exists a $g \in G$ such that $c_g(U) \subseteq C$. This subgroup has the following properties:*

(i) *C is maximal compact.*
(ii) *$C \cap G_0$ is connected and C intersects each connected component of G. In particular, the inclusion $C \hookrightarrow G$ induces a group isomorphism $\pi_0(C) \to \pi_0(G)$.*
(iii) *Any other maximal compact subgroup of G is conjugate to C under some element of G_0.*
(iv) *$C_0 = C \cap G_0$ is maximal compact in G_0.*
(v) *If $U \subseteq G$ is a compact subgroup intersecting each connected component and for which $U \cap G_0$ is maximal compact in G_0, then U is maximal compact in G.*

Proof. First we deal with two special cases.

Case a: $\mathbf{L}(G)$ is a compact Lie algebra. Then Corollary 11.1.21 implies the existence of a vector subgroup V and a compact subgroup C with $G \cong V \rtimes C$. Since all compact subgroups of V are trivial, C is clearly maximal and Lemma 13.1.1 implies that any other compact subgroup U of G is conjugate under G_0 to a subgroup of C. Property (ii) follows from the semidirect product structure of G.

Case b: $\mathbf{L}(G)$ is semisimple. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and τ be the corresponding involution. Then $K := \{g \in G \colon \operatorname{Ad}(g)\tau = \tau \operatorname{Ad}(g)\}$ is a closed subgroup of G whose Lie algebra $\mathbf{L}(K) = \mathfrak{k}$ is compact (cf. Theorem 12.1.7). Hence Case (a) implies that K contains a maximal compact subgroup C. If $U \subseteq G$ is compact, then Proposition 12.2.14 implies that U is conjugate under G_0 to a subgroup of K, and conjugating with a suitable element in K_0 (Case (a)), we see that it is even conjugate to a subgroup of C. Property (ii) follows from Case (a) and the polar decomposition diffeomorphism $K \times \mathfrak{p} \cong G$ (Theorem 12.1.7).

Now we prove the theorem, including property (ii), by induction on the dimension n of G. If $\dim G = 0$, then $G = \pi_0(G)$ is finite, so that we may put $C := G$. Let us assume that the theorem holds for all Lie groups of dimension $< n$ and with finitely many connected components. The case where G is semisimple has already been treated above. If G is not semisimple, then one of the two cases from Lemma 13.1.2 occurs.

Case c.1: G contains a nontrivial normal torus T. Then our induction hypothesis implies the assertion for the quotient Lie group $Q := G/T$ and some compact subgroup C_Q. Let $q \colon G \to Q$ denote the quotient map and put $C := q^{-1}(C_Q)$.

This is a compact subgroup of G for which $C \cap G_0 = q^{-1}(C_Q \cap Q_0)$ is connected because T is connected and $C_Q \cap Q_0$ is connected. It is also clear that C intersects each connected component of G because C_Q intersects each connected component of Q. Let $U \subseteq G$ be another compact subgroup. Then

$q(U)$ is compact in Q, and there exists some $x \in Q$ with $c_x(q(U)) \subseteq C_Q = q(C)$. Writing $x = q(g)$ for some $g \in G$, we have $q(C) \supseteq q(c_g(U))$, which leads to $c_g(U) \subseteq q^{-1}(q(C)) = C$.

Case c.2: G contains a nontrivial normal vector subgroup V. Then the theorem holds for $Q := G/V$ and we write $C_Q \subseteq Q$ for the corresponding maximal compact subgroup. Then $M := q^{-1}(C_Q)$ is a closed subgroup of G containing the normal vector group V and for which $M/V \cong C_Q$ is compact. By Theorem 11.1.13, $M \cong V \rtimes C$ for a compact Lie group C. Since V is connected, $G_0 = q^{-1}(Q_0)$, which leads to $M \cap G_0 = q^{-1}(C_Q \cap Q_0)$, and the connectedness of $C_Q \cap Q_0$ and V imply that $M \cap G_0$ is connected. Hence the connectedness of $C \cap G_0$ follows from $C \cap G_0 \cong q(C \cap G_0) = q(M \cap G_0) = C_Q \cap Q_0$. Since V is connected and M intersects each connected component of G, the subgroup C also intersects each connected component of G.

Let $U \subseteq G$ be another compact subgroup. Then $q(U)$ is a compact subgroup of Q, so that there exists an element $x = q(g) \in Q$ with $q(c_g(U)) = c_x(q(U)) \subseteq C_Q$. This leads to $c_g(U) \subseteq M$. Now we use Lemma 13.1.1 to find an $m \in M$ with $c_m(c_g(U)) = c_{mg}(U) \subseteq C$.

(i) Let $U \supseteq C$ be a compact subgroup and pick $g \in G$ with $c_g(U) \subseteq C$. Then $c_g(U) \subseteq C$ is a compact subgroup of the same dimension, hence open. Next we observe that C and $c_g(C)$ have the same finite number of connected components, which leads to $C = c_g(C)$, and hence to $c_g(U) = c_g(C)$, and finally to $C = U$.

(ii) follows from the construction.

(iii) Let $U \subseteq G$ be another maximal compact subgroup of G. Then there exists a $g \in G$ with $c_g(U) \subseteq C$, so that the maximality of U implies that $c_g(U) = C$. Since $G = CG_0$ by (ii), we may write $g = kg_0$, so that $c_k(C) = C$ leads to $C = c_g(U) = c_{g_0}(U)$.

(iv) We have already seen that the subgroup $C_0 := C \cap G_0$ is connected, hence equal to the identity component of C. To see that C_0 is maximal compact in G_0, let $U \supseteq C_0$ be a compact subgroup of G_0 containing C_0. Then U is also a compact subgroup of G, so that there exists a $g \in G$ with $c_g(U) \subseteq C$. Then $c_g(U) \subseteq C \cap G_0 = C_0$, and $c_g(C_0) \subseteq C_0$ is a compact connected subgroup of the same dimension, hence equal to C_0. Therefore $U \subseteq c_{g^{-1}}(C_0) = C_0$ yields $U = C_0$, showing that C_0 is maximal compact in G_0.

(v) Since U is conjugate to a subgroup of C, we may w.l.o.g. assume that $U \subseteq C$. Then the maximality of $U \cap G_0 \subseteq C_0$ implies $U \cap G_0 = C_0$. Since U intersects each connected component of G, it contains all other connected components of C, which leads to the equality $C = U$. □

Corollary 13.1.4. *Any two maximal tori of a Lie group G are conjugate under the group of inner automorphisms.*

Proof. Since each maximal torus T of G is contained in the identity component G_0, we may w.l.o.g. assume that G is connected. Then Theorem 13.1.3 implies that T is contained in a maximal compact subgroup C, and that any other maximal torus T' is conjugate to a maximal torus of C. Now the assertion follows from Theorem 11.2.2. □

Exercises for Section 13.1

Exercise 13.1.1. We call an ideal $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ *fully characteristic* if it is invariant under all automorphisms and it is called *characteristic* if it is invariant under all derivations. Prove the following assertions:

(i) If $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is an ideal and $\mathfrak{b} \trianglelefteq \mathfrak{n}$ is characteristic, then $\mathfrak{b} \trianglelefteq \mathfrak{g}$ is an ideal.
(ii) The ideals $D^n(\mathfrak{g})$ and $C^n(\mathfrak{g})$ are characteristic and fully characteristic.
(iii) If $\mathfrak{g}$ is finite-dimensional, then its radical $\mathfrak{r}$ is fully characteristic and so are the ideals $D^n(\mathfrak{r})$ and $C^n(\mathfrak{r})$ for each n.
(iv) If $\mathfrak{g}$ is a finite-dimensional real Lie algebra, then each fully characteristic ideal of $\mathfrak{g}$ is characteristic.

Exercise 13.1.2. Let G be a compact Lie group and $K \subseteq G$ a subgroup isomorphic to G. Show that $K = G$.

13.2 The Center of a Connected Lie Group

In this section we show that the center of a connected Lie group G is contained in the exponential image. Further we explain how it can be described in terms of the Lie algebra $\mathfrak{g}$, which leads to the concept of a compactly embedded subalgebra.

Theorem 13.2.1. *For each connected Lie group G, we have*

$$Z(G) \subseteq \exp_G \mathfrak{g}.$$

Proof. If $q_G \colon \widetilde{G} \to G$ is the simply connected covering group, then $q_G(Z(\widetilde{G})) = Z(G)$ (Proposition 8.5.2), and since $q_G(\exp_{\widetilde{G}}(\mathfrak{g})) = \exp_G(\mathfrak{g})$, we may assume that G is 1-connected.

If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a Levi decomposition of $\mathfrak{g}$ (cf. Theorem 4.6.6 and Definition 4.6.7), we have $G \cong R \rtimes S$, where R and S are the 1-connected Lie groups with Lie algebras $\mathfrak{r}$ and s, respectively (Proposition 10.1.19). Let $z = (r, s) \in Z(G)$. Since the projection onto S is a group homomorphism, $s \in Z(S)$. By Lemma 12.1.6, s is contained in a connected subgroup K of S with compact Lie algebra. According to Corollary 11.2.3, there exists a $y \in \mathfrak{s}$ with $s = \exp_G y$. Hence it suffices to consider the group $G_1 := R \rtimes \exp_G(\mathbb{R}y)$ which is solvable and contains (r, s) in its center.

Replacing G_1 by its universal covering, we may thus assume that G is simply connected and solvable. If N is a nilpotent Lie group, then its center is connected (Proposition 10.2.9) and $Z(N) = \exp_N(\mathfrak{z}(\mathfrak{n}))$. The commutator group G' of G is nilpotent. Since the connected group G/G' is abelian, there exists an element $y \in \mathfrak{g}$ with $z \in \exp_G(y)G'$. Now it suffices to consider the subgroup $G' \rtimes \exp_G(\mathbb{R}y)$. Then, by Corollary 10.2.7, $z = \exp_G(y)\exp_G(x)$ with $x \in \mathfrak{g}'$, and $z \in Z(G)$ implies that $\exp_G x$ and $\exp_G y$ commute. Thus

$$\exp_G x = c_{\exp_G y}(\exp_G x) = \exp_G(e^{\operatorname{ad} y}x)$$

leads to $x = e^{\operatorname{ad} y}x$ because the exponential function of the simply connected nilpotent Lie group G' is injective (Corollary 10.2.7). As $\operatorname{ad} y$ is nilpotent, we conclude that $[y, x] = 0$, which finally gives

$$z = \exp_G y \exp_G x = \exp_G(x + y). \qquad \square$$

The preceding theorem is an important first step to access the center of a connected Lie group from its Lie algebra. However, it is not explicit enough to be really useful in practice. Below, we shall see that there even exists an abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ with $Z(G) \subseteq \exp_G \mathfrak{t}$.

Definition 13.2.2. A subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is called *compactly embedded* if the subgroup

$$\operatorname{INN}_{\mathfrak{g}}(\mathfrak{k}) = \overline{\langle \exp_G \operatorname{ad} \mathfrak{k} \rangle} \subseteq \operatorname{Aut}(\mathfrak{g})$$

is compact (cf. Definition 11.1.3). An element $x \in \mathfrak{g}$ is called *compact* if $\mathbb{R}x$ is compactly embedded. We write $\operatorname{comp}(\mathfrak{g})$ for the set of compact elements of $\mathfrak{g}$.

Lemma 13.2.3. *If $\mathfrak{k} \subseteq \mathfrak{g}$ is compactly embedded, then $\mathfrak{k}$ is a compact Lie algebra.*

Proof. Clearly, $\mathfrak{k} \subseteq \mathfrak{g}$ is invariant under the compact group $\operatorname{INN}_{\mathfrak{g}}(\mathfrak{k})$, so that $\operatorname{INN}_{\mathfrak{g}}(\mathfrak{k})|_{\mathfrak{k}} \supseteq e^{\operatorname{ad} \mathfrak{k}}$ implies that $\operatorname{INN}_{\mathfrak{k}}(\mathfrak{k})$ is also compact, i.e., $\mathfrak{k}$ is a compact Lie algebra. $\square$

Lemma 13.2.4. *If $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective homomorphism of Lie algebras and $\mathfrak{k}$ be compactly embedded in $\mathfrak{g}_1$, then $\varphi(\mathfrak{k})$ is compactly embedded in $\mathfrak{g}_2$.*

Proof. Since $\operatorname{INN}_{\mathfrak{g}_1}(\mathfrak{k})$ is compact, there exists an $\operatorname{Inn}_{\mathfrak{g}_1}(\mathfrak{k})$-invariant scalar product on $\mathfrak{g}_1$, and then the orthogonal complement E of $\ker \varphi$ in $\mathfrak{g}_1$ is invariant under $\operatorname{Inn}_{\mathfrak{g}_1}(\mathfrak{k})$. Since φ is surjective, $\varphi|_E\colon E \to \mathfrak{g}_2$ is a linear isomorphism. For $x \in \mathfrak{k}$ we have

$$\varphi \circ (\operatorname{ad} x|_E) = \operatorname{ad}(\varphi(x)) \circ \varphi|_E,$$

showing that

$$\operatorname{INN}_{\mathfrak{g}_2}(\varphi(\mathfrak{k})) = \varphi \circ \operatorname{INN}_{\mathfrak{g}_1}(\mathfrak{k}) \circ (\varphi|_E)^{-1}$$

is compact. $\square$

Definition 13.2.5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We fix a maximal compactly embedded subalgebra $\mathfrak{k}$, a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{k}$ (which is maximal abelian because $\mathfrak{k}$ is compact). Further, let $\mathfrak{t} \subseteq \mathfrak{h}$ be the Lie algebra of a maximal torus in $H := \exp_G \mathfrak{h}$. We set $K := \langle \exp_G \mathfrak{k} \rangle$ and $T := \exp_G \mathfrak{t}$.

Lemma 13.2.6. *The subgroups H and K of G are closed.*

Proof. Since $\mathrm{INN}_{\mathfrak{g}}\,\mathfrak{k}$ is compact, this also holds for $\mathrm{INN}_{\mathfrak{g}}\big(\mathbf{L}(\overline{K})\big)$ because $\mathrm{Ad}(\overline{K}) \subseteq \overline{\mathrm{Ad}(K)} = \mathrm{INN}_{\mathfrak{g}}\,\mathfrak{k}$. Hence the the maximality of $\mathfrak{k}$ shows that K is closed. The fact that H is closed in K (and hence in G) follows from the relation $H = N_K(\mathfrak{h})_0$, which in turn follows from $\mathfrak{h} = \mathfrak{n}_{\mathfrak{k}}(\mathfrak{h})$ and Lemma 10.1.1. □

Theorem 13.2.7. (a) *Two maximal compactly embedded subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{INN}_{\mathfrak{g}}\,\mathfrak{g}$.*
(b) *The Cartan subalgebras of the maximal compactly embedded subalgebras are the maximal compactly embedded abelian subalgebras.*
(c) *Two maximal compactly embedded abelian subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{INN}_{\mathfrak{g}}\,\mathfrak{g}$.*

Part (a) of this theorem will be sharpened in Exercise 13.5.1 below.

Proof. (a) By the Maximal Compact Subgroup Theorem 13.1.3, we may assume that $\mathrm{INN}_{\mathfrak{g}}\,\mathfrak{k}$ is contained in a maximal compact subgroup U of $\mathrm{INN}_{\mathfrak{g}}\,\mathfrak{g}$. Since $\mathfrak{k}$ is maximal compactly embedded, we then have $\mathfrak{k} = \mathrm{ad}^{-1}(\mathbf{L}(U))$. Let $\widetilde{\mathfrak{k}}$ be another maximal compactly embedded subalgebra of $\mathfrak{g}$. Theorem 13.1.3 implies the existence of $\gamma \in \mathrm{INN}_{\mathfrak{g}}\,\mathfrak{g}$ with $\mathrm{Inn}_{\mathfrak{g}}\,\gamma(\widetilde{\mathfrak{k}}) = c_\gamma(\mathrm{Inn}_{\mathfrak{g}}\,\widetilde{\mathfrak{k}}) \subseteq U$. Therefore

$$\gamma(\widetilde{\mathfrak{k}}) \subseteq \mathrm{ad}^{-1}(\mathbf{L}(U)) = \mathfrak{k}.$$

By the maximality of $\gamma(\widetilde{\mathfrak{k}})$, it now follows that $\gamma(\widetilde{\mathfrak{k}}) = \mathfrak{k}$.

(b) If $\mathfrak{h}$ is a maximal compactly embedded abelian subalgebra, then it is contained in some maximal compactly embedded subalgebra $\mathfrak{k}$, and by maximality it is maximal abelian in $\mathfrak{k}$, hence a Cartan subalgebra (Lemma 11.2.1).

Conversely, let $\mathfrak{h}$ be a Cartan subalgebra of a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. We have to show that every compactly embedded abelian subalgebra $\mathfrak{a}$ containing $\mathfrak{h}$ coincides with $\mathfrak{h}$. To this end, we choose a maximal compactly embedded subalgebra $\widetilde{\mathfrak{k}}$ containing $\mathfrak{a}$. Then $\mathfrak{a}$ is contained in a Cartan subalgebra of $\widetilde{\mathfrak{k}}$. By (a), $\widetilde{\mathfrak{k}}$ is isomorphic to $\mathfrak{k}$. In particular, $\mathfrak{k}$ and $\widetilde{\mathfrak{k}}$ have the same rank. Consequently, $\dim \mathfrak{a} \leq \dim \mathfrak{h}$, and therefore, $\mathfrak{a} = \mathfrak{h}$.

(c) This follows from (a) and Lemma 11.2.1. □

Theorem 13.2.8 (Fundamental Theorem on the Center). *The center of a connected Lie group G is given by*

$$Z(G) = \exp_G\{x \in \mathfrak{h}\colon\ \mathrm{Spec}\,(\mathrm{ad}\,x) \subseteq 2\pi i\mathbb{Z}\}$$

for every maximal compactly embedded abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Proof. In view of Proposition 8.5.2, it suffices to assume that G is simply connected. By Theorem 13.2.1, $Z(G)$ is contained in the image of the exponential function. Let $z \in Z(G)$. Then there exists an $x \in \mathfrak{g}$ with $\exp_G x = z$ and $\mathbb{R}x$ is a compactly embedded abelian subalgebra of $\mathfrak{g}$. Hence there exists a

$\gamma \in \mathrm{INN}_{\mathfrak{g}}\, \mathfrak{g}$ with $\gamma(x) \in \mathfrak{h}$ (Lemma 13.2.7). By the simple connectedness of G, the automorphism γ of $\mathfrak{g}$ induces an automorphism $\widetilde{\gamma}$ of G with $\mathbf{L}(\widetilde{\gamma}) = \gamma$. Since every such automorphism $\widetilde{\alpha}$ for $\alpha \in \mathrm{Inn}(\mathfrak{g}) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ fixes the center of G pointwise, this also holds for γ. Therefore

$$\exp_G\big(\gamma(x)\big) = \widetilde{\gamma}(\exp_G x) = \widetilde{\gamma}(z) = z \in \exp_G \mathfrak{h}.$$

Any $x \in \mathfrak{h}$ with $\exp_G x = z$ satisfies $\mathrm{Ad}(\exp_G x) = e^{\mathrm{ad}\,x} = \mathrm{id}_{\mathfrak{g}}$, and consequently $\mathrm{Spec}\,(\mathrm{ad}\,x) \subseteq 2\pi i \mathbb{Z} = \exp_{\mathbb{C}^\times}^{-1}(1)$ (cf. Exercise 2.2.7). The converse follows from $\mathrm{Ad}(\exp x) = e^{\mathrm{ad}\,x}$ because all operators $\mathrm{ad}\,x$, $x \in \mathfrak{h}$, are semisimple (cf. Exercise 2.2.7). □

Remark 13.2.9. If , then

We know already from Lemma 8.2.4 that for a Lie group G the map $\mathbf{L}(G) \to \mathrm{Hom}(\mathbb{R}, G), x \mapsto \gamma_x$, defined by $\gamma_x(t) := \exp_G(tx)$ is bijective. For $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the quotient map $q_{\mathbb{T}} \colon \mathbb{R} \to \mathbb{T}$ is the universal covering and

$$\mathrm{Hom}(\mathbb{T}, G) \to \mathrm{Hom}(\mathbb{R}, G), \quad \gamma \mapsto q_{\mathbb{T}}^* \gamma = \gamma \circ q_{\mathbb{T}}$$

is an injection. For an element $x \in \mathbf{L}(G)$ the homomorphism γ_x factors through a homomorphism $\overline{\gamma}_x \colon \mathbb{T} \to G$ if and only if $\gamma_x(1) = \exp_G x = \mathbf{1}$, which leads to a bijection

$$\{x \in \mathfrak{g} \colon \exp_G x = \mathbf{1}\} \to \mathrm{Hom}(\mathbb{T}, G).$$

Corollary 13.2.10. *For any connected Lie group G, the following assertions hold:*

(i) *Each homotopy class in $\pi_1(G)$ is represented by a curve of the form $\gamma_x(t) := \exp_G(tx)$, where $\exp_G x = \mathbf{1}$, i.e., γ_x defines a homomorphism $\mathbb{T} \to G$.*

(ii) *If G is connected abelian, then the assignment*

$$\pi_1(G) \cong \ker \exp_G \to \mathrm{Hom}(\mathbb{T}, G), \quad x \mapsto \gamma_x$$

is an isomorphism of abelian groups, where the group structure on the set $\mathrm{Hom}(\mathbb{T}, G)$ *is given by the pointwise product.*

(iii) *If $T \subseteq G$ is a maximal torus, then the inclusion $i : T \to G$ induces a surjective homomorphism $\pi_1(i) \colon \pi_1(T) \to \pi_1(G)$ and if $q_G \colon \widetilde{G} \to G$ is the universal covering morphism, then* $\ker q_G \subseteq \exp_{\widetilde{G}} \mathfrak{t}$*, where $\mathfrak{t} = \mathbf{L}(T)$.*

(iv) *$\pi_1(G)$ is a finitely generated abelian group.*

(v) *For a maximal torus $T \subseteq G$, we have $\pi_1(G/T) = \{\mathbf{1}\}$.*

Proof. (i) Let $[\alpha] \in \pi_1(G)$ be a homotopy class, and let $q_G \colon \widetilde{G} \to G$ be the universal covering of G. By Theorem 8.5.4 (cf. also Theorem A.2.5), there exists a lifting $\widetilde{\alpha} \colon [0,1] \to \widetilde{G}$ of α with $\widetilde{\alpha}(0) = \mathbf{1}$ and $d := \widetilde{\alpha}(1) \in Z(\widetilde{G})$. Using Theorem 13.2.8, we find an $x \in \mathbf{L}(G)$ with $\exp_{\widetilde{G}} x = d$. Since the paths

$\gamma\colon [0,1] \to \widetilde{G}, t \mapsto \exp(tx)$ and $\widetilde{\alpha}$ are homotopic in the simply connected group $\widetilde{G}$, we have $[\alpha] = [q_G \circ \gamma]$, where

$$q_G(\gamma(t)) = \exp_G(tx) \quad \text{and} \quad \exp_G(x) = q_G(d) = \mathbf{1}.$$

(ii) Since the exponential map $\exp_G\colon \mathfrak{g} \to G$ is the universal covering of G, we have a natural isomorphism $\pi_1(G) \cong \ker \exp_G$, so that (ii) follows from Remark 13.2.9 and $\gamma_x \cdot \gamma_y = \gamma_{x+y}$.

(iii) Let $\mathfrak{t} = \mathbf{L}(T)$ and observe that $\mathfrak{t}$ is abelian compactly embedded. Let $\mathfrak{h} \supseteq \mathfrak{t}$ be maximal compactly embedded abelian, and H be the corresponding integral subgroup which is closed (Lemma 13.2.6) and contains T as a maximal torus. In the proof of (i) we saw that each element of $\pi_1(G)$ is represented by a loop of the form $t \mapsto \exp_G(tx)$ with $x \in \mathfrak{h}$ satisfying $\exp_G x = \mathbf{1}$. By Theorem 11.1.18 any such x is contained in $\mathfrak{t}$, which implies that $\exp_G(\mathbb{R}x) \subseteq T$, and this proves (iii).

(iv) In view of (iii), $\pi_1(G)$ is a quotient of $\pi_1(T) \cong \mathbb{Z}^{\dim T}$.

(v) If T is the maximal torus in (iii), then $q_G^{-1}(T) = \exp_{\widetilde{G}}(\mathfrak{t})$ follows from $\ker q_G \subseteq Z(\widetilde{G}) \subseteq \exp_{\widetilde{G}} \mathfrak{t}$. Consequently, $G/T \cong \widetilde{G}/q_G^{-1}(T)$ is simply connected because $q_G^{-1}(T)$ is connected (Corollary 10.1.14). □

Exercises for Section 13.2

Exercise 13.2.1. Let G be a connected Lie group. Show that for $x \in \mathfrak{g}$ the following are equivalent:

(a) $e^{\operatorname{ad} x} = \mathrm{id}_{\mathfrak{g}}$.
(b) $\exp_G(x) \in Z(G)$.
(c) $\operatorname{ad} x$ is semisimple with $\operatorname{Spec}(\operatorname{ad} x) \subseteq 2\pi i\mathbb{Z}$.

Exercise 13.2.2. Find a finite-dimensional connected Lie group G and an element $x \in \mathbf{L}(G)$ with $\operatorname{Spec}(\operatorname{ad} x) \subseteq 2\pi i\mathbb{Z}$ for which $\exp_G x$ is not central in G.

Exercise 13.2.3. Show that in a connected Lie group G any discrete subgroup $\Gamma \subseteq Z(G)$ is finitely generated. Why is the discreteness needed for this conclusion?

It is natural to ask to which extent the condition that Γ is central is necessary, or whether one can even show that every discrete abelian (or even more general) subgroup of a connected Lie group is finitely generated.

Exercise 13.2.4 (Discrete subgroups which are not finitely generated). Show that the two matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

generate a free subgroup of $\mathrm{SL}_2(\mathbb{R})$, hence in particular a discrete subgroup (cf. [dlH00, Ex. II.5, 25]). This example shows that the free group F_2 with two generators embeds as a discrete subgroup into $\mathrm{SL}_2(\mathbb{R})$. The commutator group $N := (F_2, F_2)$ of F_2 is a normal subgroup whose quotient $F_2/N \cong \mathbb{Z}^2$ is infinite, and one can show that this implies that N, which also is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, is not finitely generated (cf. [dlH00, Ex. III.A.3]).

In view of the preceding example, general discrete subgroups of connected Lie groups need not be finitely generated. However, this is true for solvable discrete subgroups. More generally, for every closed solvable subgroup H of a connected Lie group G, the group $\pi_0(H)$ is finitely generated ([Ra72, Prop 3.8]). In particular, discrete abelian subgroups of connected Lie groups are finitely generated. Conversely, the Structure Theorem for finitely generated abelian groups implies that all these groups (see Appendix 13.6 below) occur as discrete subgroups of some $(\mathbb{C}^\times)^n$. For solvable groups the situation is more subtle. If a solvable group Γ occurs as a discrete subgroup of a connected Lie group, then all its subgroups also do, hence are finitely generated. This is not true for all finitely generated solvable groups:

Exercise 13.2.5 (Finitely generated solvable groups with non-finitely generated subgroups). Let $\Gamma \subseteq \mathrm{Aff}_1(\mathbb{Q}) \cong \mathbb{Q} \rtimes \mathbb{Q}^\times \subseteq \mathrm{GL}_2(\mathbb{R})$ be the subgroup generated by the two elements $a := (0, 2)$ and $b := (1, 0)$. Then

$$\Gamma \cong \Big(\frac{1}{2^\infty}\mathbb{Z}\Big) \rtimes \mathbb{Z},$$

and the abelian subgroup $\frac{1}{2^\infty}\mathbb{Z} := \bigcup_n \frac{1}{2^n}\mathbb{Z}$ is not finitely generated.

Exercise 13.2.6. Let

$$\mathbf{1} \to A \to B \to C \to \mathbf{1}$$

be a short exact sequence of groups. Show that A and C have the property that each subgroup is finitely generated if and only if B does.

Exercise 13.2.7. A finitely generated solvable group Γ has the property that all its subgroups are finitely generated if and only if Γ is *polycyclic*, i.e., if it has a normal series

$$\Gamma_0 = \{\mathbf{1}\} \trianglelefteq \Gamma_1 \trianglelefteq \Gamma_2 \cdots \trianglelefteq \Gamma_n = \Gamma$$

with cyclic factors Γ_i/Γ_{i-1} for $i = 1, \ldots, n$.

According to a theorem of Malcev and Auslander, a group is polycyclic if and only if it is isomorphic to a solvable subgroup of some $\mathrm{GL}_n(\mathbb{Z})$ (cf. Chapters 2 and 3 in [Seg83]), so that the preceding discussion yields a characterization of those finitely generated solvable groups that arise as discrete subgroups of connected Lie groups.

13.3 The Manifold Splitting Theorem

In Section 13.1 we proved the existence of a maximal compact subgroup K in a Lie group G with finitely many connected components. Now we want to prove that G is diffeomorphic to $K \times \mathbb{R}^n$ for some n. To obtain this result, we shall combine the Cartan decomposition of a semisimple Lie group with techniques related to maximal compactly embedded subalgebras. According to our guiding philosophy, we first decompose the Lie algebra in an appropriate way, then prove the theorem in the simply connected case, and finally, we derive the general case from the simply connected one. This technique in particular provides quite explicit information on how to find the manifold splitting in terms of the Lie algebra. But before we can address the decomposition of the Lie algebra, we have to provide a few basic tools from representation theory.

Theorem 13.3.1. *If $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional representation of the Lie algebra $\mathfrak{g}$ and $U := \overline{\langle e^{\pi(\mathfrak{g})}\rangle}$ is the closed subgroup of $\mathrm{GL}(V)$ generated by $e^{\pi(\mathfrak{g})}$, then a subspace $W \subseteq V$ is a $\mathfrak{g}$-submodule if and only if it is U-invariant.*

Proof. Any $\mathfrak{g}$-submodule W of V is closed, hence invariant under $e^{\pi(x)}$ for all $x \in \mathfrak{g}$, and then also under U. If, conversely, W is U-invariant and $x \in \mathfrak{g}$, then we obtain for each $w \in W$

$$\pi(x)w = \frac{d}{dt}\Big|_{t=0} e^{t\pi(x)}w \in W,$$

so that W is a $\mathfrak{g}$-submodule. □

Lemma 13.3.2. *Let $\pi\colon \mathfrak{k} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of a Lie algebra $\mathfrak{k}$ for which the group $K := \overline{\langle e^{\pi(\mathfrak{k})}\rangle}$ is compact. Then the following assertions hold:*

(a) *The $\mathfrak{k}$-module V is* semisimple*, i.e., for every submodule W of V, there exists a complementary submodule W' with $V \cong W \oplus W'$.*
(b) *$V = V_{\mathrm{fix}} \oplus V_{\mathrm{eff}}$ with*

$$V_{\mathrm{fix}} := \big\{v \in V \colon \pi(\mathfrak{k})v = \{0\}\big\} \quad \textit{and} \quad V_{\mathrm{eff}} := \operatorname{span} \pi(\mathfrak{k})V.$$

Proof. (a) Since the group K is compact, V carries a K-invariant scalar product (Proposition 9.4.14). Now if W is a $\mathfrak{k}$-submodule, then W, and thus $W^\perp$, is invariant under K and a $\mathfrak{k}$-submodule by Lemma 13.3.1.

(b) In view of (a), there exists a $\mathfrak{k}$-submodule $W \subseteq V$ complementing the submodule V_{eff}. Then $\pi(\mathfrak{k})W \subseteq W \cap V_{\mathrm{eff}} = \{0\}$ shows that $V = V_{\mathrm{fix}} + V_{\mathrm{eff}}$. If W_1 is a $\mathfrak{k}$-submodule of V_{eff} complementing $V_{\mathrm{fix}} \cap V_{\mathrm{eff}}$, then

$$\pi(\mathfrak{k})V \subseteq \pi(\mathfrak{k})V_{\mathrm{eff}} = \pi(\mathfrak{k})W_1 \subseteq W_1$$

implies that $W_1 = V_{\mathrm{eff}}$, and therefore that the sum of V_{eff} and V_{fix} is direct. □

Lemma 13.3.3. *For every maximal compactly embedded subalgebra* $\mathfrak{k}$ *of* $\mathfrak{g}$, *there exists a Levi decomposition* $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ *with the following properties:*

(1) $[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s}$.
(2) $[\mathfrak{k} \cap \mathfrak{r}, \mathfrak{s}] = \{0\}$.
(3) $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{r}) \oplus (\mathfrak{k} \cap \mathfrak{s})$.
(4) $\mathfrak{k}' \subseteq \mathfrak{s}$, *where* $\mathfrak{k}'$ *denotes the commutator algebra of* $\mathfrak{k}$.
(5) $\mathfrak{k} \cap \mathfrak{s}$ *is maximal compact in* $\mathfrak{s}$.

Proof. Set $\mathfrak{a} := \mathfrak{k} \cap \mathfrak{r}$. As a compact subalgebra of a solvable Lie algebra, $\mathfrak{a}$ is abelian. Now we decompose the semisimple $\mathfrak{a}$-module $\mathfrak{g}$ into $\mathfrak{g} = \mathfrak{g}_{\mathrm{fix}} \oplus \mathfrak{g}_{\mathrm{eff}}$ (Lemma 13.3.2). Note that $\mathfrak{g}_{\mathrm{fix}} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is a subalgebra of $\mathfrak{g}$, and because of $\mathfrak{g}_{\mathrm{eff}} = [\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{r}$, we have $\mathfrak{g} = \mathfrak{r} + \mathfrak{g}_{\mathrm{fix}}$, so that $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{g}_{\mathrm{fix}}/(\mathfrak{r} \cap \mathfrak{g}_{\mathrm{fix}})$ (Theorem 4.1.17). This Lie algebra is semisimple, whence $\mathfrak{r} \cap \mathfrak{g}_{\mathrm{fix}} = \mathrm{rad}(\mathfrak{g}_{\mathrm{fix}})$ is the maximal solvable ideal in $\mathfrak{g}_{\mathrm{fix}}$. Applying Levi's Theorem 4.6.6 to $\mathfrak{g}_{\mathrm{fix}}$, it follows that $\mathfrak{g}_{\mathrm{fix}}$ contains a Levi complement $\widetilde{\mathfrak{s}}$ of $\mathfrak{g}$. Now we choose a Cartan decomposition $\widetilde{\mathfrak{s}} = \mathfrak{k}_{\mathfrak{s}} + \mathfrak{p}$ of $\widetilde{\mathfrak{s}}$ (Theorem 12.2.10), and put $\widetilde{\mathfrak{k}} := \mathfrak{a} + \mathfrak{k}_{\mathfrak{s}}$. Then $[\mathfrak{a}, \mathfrak{k}_{\mathfrak{s}}] = \{0\}$ follows from $\widetilde{\mathfrak{s}} \subseteq \mathfrak{g}_{\mathrm{fix}}$. By Corollary 12.2.6, $\mathfrak{k}_{\mathfrak{s}}$ is compactly embedded in $\mathfrak{g}$, so that $[\mathfrak{a}, \mathfrak{k}_{\mathfrak{s}}] = \{0\}$ implies that $\widetilde{\mathfrak{k}}$ is a compactly embedded subalgebra of $\mathfrak{g}$ which satisfies (1)-(5) with respect to $\widetilde{\mathfrak{s}}$.

We still have to prove that $\widetilde{\mathfrak{k}}$ is maximal compactly embedded. Let $\widehat{\mathfrak{k}} \supseteq \widetilde{\mathfrak{k}}$ be compactly embedded. By Lemma 13.2.4, the projection of $\widehat{\mathfrak{k}}$ in $\widetilde{\mathfrak{s}}$ is compactly embedded in $\widetilde{\mathfrak{s}}$, hence cannot be strictly larger than $\mathfrak{k}_{\mathfrak{s}}$, which leads to $\widehat{\mathfrak{k}} \subseteq \mathfrak{r} \rtimes \mathfrak{k}_{\mathfrak{s}}$, and thus to $\widehat{\mathfrak{k}} = (\widehat{\mathfrak{k}} \cap \mathfrak{r}) \rtimes \mathfrak{k}_{\mathfrak{s}}$. Since $\widehat{\mathfrak{k}}$ and $\mathfrak{k}$ are conjugate under $\mathrm{Aut}(\mathfrak{g})$ and $\mathfrak{r}$ is $\mathrm{Aut}(\mathfrak{g})$-invariant, $\widehat{\mathfrak{k}} \cap \mathfrak{r}$ and $\mathfrak{a} = \mathfrak{k} \cap \mathfrak{r}$ have the same dimension, so that $\mathfrak{a} \subseteq \widehat{\mathfrak{k}}$ leads to $\mathfrak{a} = \widehat{\mathfrak{k}} \cap \mathfrak{r}$. This in turn implies that $\widetilde{\mathfrak{k}} = \widehat{\mathfrak{k}}$, i.e., $\widetilde{\mathfrak{k}}$ is maximal compactly embedded. Finally we use Theorem 13.2.7 again to see that there exists an automorphism $\gamma \in \mathrm{Aut}(\mathfrak{g})$ with $\gamma(\widetilde{\mathfrak{k}}) = \mathfrak{k}$. Then $\mathfrak{s} := \gamma(\widetilde{\mathfrak{s}})$ and $\mathfrak{k}$ satisfy (1)-(5). □

Lemma 13.3.4. *Let* $\mathfrak{n}$ *be a nilpotent Lie algebra with the Campbell-Hausdorff multiplication* $* : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$, *let* $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{n})$ *be a subspace, and* $\mathfrak{n}^+$ *be a vector space complement to* $\mathfrak{a}$ *in* $\mathfrak{n}$. *Then the map*

$$\Phi \colon \mathfrak{a} \times \mathfrak{n}^+ \to \mathfrak{n}, \quad (x, y) \mapsto x * y = x + y$$

is a diffeomorphism.

Proof. This is an easy exercise. □

Lemma 13.3.5. *A compactly embedded subalgebra* $\mathfrak{a}$ *of a nilpotent Lie algebra* $\mathfrak{n}$ *is central.*

Proof. For $x \in \mathfrak{a}$, the operator $\mathrm{ad}\, x$ is semisimple and nilpotent at the same time, so that $\mathrm{ad}\, x = 0$. □

Lemma 13.3.6. *Let R be a simply connected, solvable Lie group with Lie algebra $\mathfrak{r}$, $\mathfrak{n} \subseteq \mathfrak{r}$ a nilpotent ideal containing the commutator algebra $\mathfrak{r}'$, $\mathfrak{a} \subseteq \mathfrak{r}$ be a subspace with $\mathfrak{a} \cap \mathfrak{n} = \{0\}$, and $N := \langle \exp_R \mathfrak{n} \rangle$. Then the map*

$$\Phi\colon N \times \mathfrak{a} \to \langle \exp_R(\mathfrak{a} + \mathfrak{n}) \rangle, \quad (n, x) \mapsto \exp_R(x) n$$

is a diffeomorphism.

Proof. Let $M := \langle \exp_R(\mathfrak{a}+\mathfrak{n}) \rangle$ be the integral subgroup of R corresponding to the Lie subalgebra $\mathfrak{m} := \mathfrak{n} + \mathfrak{a}$. Since $\mathfrak{m}$ contains $\mathfrak{r}'$, it is an ideal, and therefore M is normal, hence simply connected and closed (Theorem 10.1.21). W.l.o.g., we may now assume that $R = M$, i.e., $\mathfrak{g} = \mathfrak{n} + \mathfrak{a}$.

First we note that Φ is smooth. If $\Phi(n, x) = \Phi(n', x')$, then

$$\exp_{R/N}(x) = \exp_R(x)N = \exp_R(x')N = \exp_{R/N}(x'),$$

and therefore, $x = x'$ since the exponential function of the vector group R/N is injective. The surjectivity of Φ also follows from the surjectivity of the exponential function of R/N. It remains to prove that the differential of Φ is injective in every point. So let $(n, x) \in N \times \mathfrak{a}$, $v \in \mathfrak{n}$ and $w \in \mathfrak{a}$ with $T_{(n,x)}(\Phi)(v, w) = 0$. If $\pi\colon R \to R/N$ denotes the quotient map, then

$$\pi \circ \Phi(n, x) = \exp_{G/N}(x)$$

implies that

$$T_x(\exp_{G/N})w = T_{\Phi(n,x)}(\pi)T_{(n,x)}(\Phi)(v, w) = 0,$$

so that the regularity of $\exp_{G/N}$ (which even is a diffeomorphism) leads to $w = 0$. This in turn implies that $0 = T_{(n,x)}(\Phi)(v, 0) = v$. Hence Φ is a local diffeomorphism, and since we know already that it is bijective, it is a diffeomorphism. □

In the following, $\mathfrak{n}$ denotes the maximal nilpotent ideal of $\mathfrak{g}$ and $\mathfrak{r}$ is the radical of $\mathfrak{g}$ (cf. Remark 6.4.7).

Theorem 13.3.7. *Let G be a 1-connected Lie group with Lie algebra $\mathfrak{g}$. We choose a maximal compactly embedded subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ and a Levi subalgebra $\mathfrak{s}$ as in* Lemma 13.3.3. *In addition, we choose $\mathfrak{k}$-submodules $\mathfrak{e}$ and $\mathfrak{n}^+$ of $\mathfrak{g}$ such that*

$$\mathfrak{k} \cap \mathfrak{r} = (\mathfrak{k} \cap \mathfrak{n}) \oplus \mathfrak{e} \quad \textit{and} \quad \mathfrak{n} = (\mathfrak{k} \cap \mathfrak{n}) \oplus \mathfrak{n}^+$$

are direct sums of $\mathfrak{k}$-modules. For any $\mathfrak{k}$-module complement $\mathfrak{f}$ of $\mathfrak{k} \cap \mathfrak{r}$ in $\mathfrak{r}$, we then have

$$\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{e} \oplus \mathfrak{f} \quad \textit{with} \quad [\mathfrak{k} \cap \mathfrak{r}, \mathfrak{f}] = \{0\},$$

and find a Cartan decomposition $\mathfrak{s} = \mathfrak{k}_\mathfrak{s} \oplus \mathfrak{p}$ with $\mathfrak{k}_\mathfrak{s} = \mathfrak{k} \cap \mathfrak{s}$ of $\mathfrak{s}$. If $K := \langle \exp_G \mathfrak{k} \rangle$, then the map

$$\Phi\colon \mathfrak{n}^+ \times \mathfrak{f} \times \mathfrak{p} \times K \to G, \quad (x, f, p, k) \mapsto \exp(x)\exp(f)\exp(p)k$$

is a diffeomorphism.

Proof. The existence of the required $\mathfrak{k}$-submodules follows from Lemma 13.3.2. Since $(\mathfrak{k} \cap \mathfrak{r}) + \mathfrak{n}$ contains the commutator algebra of $\mathfrak{r}$, this subspace is an ideal of $\mathfrak{r}$, hence is invariant under $\mathfrak{k} \cap \mathfrak{r}$. Let $\mathfrak{f}$ be a $\mathfrak{k}$-module complement of $(\mathfrak{k} \cap \mathfrak{r}) + \mathfrak{n} = \mathfrak{n} + \mathfrak{e}$ in $\mathfrak{r}$. Then we have

$$[\mathfrak{f}, \mathfrak{k} \cap \mathfrak{r}] \subseteq \big((\mathfrak{k} \cap \mathfrak{r}) + \mathfrak{n}\big) \cap \mathfrak{f} = \{0\}.$$

Now $K := \langle \exp \mathfrak{k} \rangle$ is a closed subgroup (Lemma 13.2.6) and $K \cap R = \exp(\mathfrak{k} \cap \mathfrak{r})$ is a closed vector group since R is a 1-connected solvable Lie group, hence does not contain a nontrivial torus (Theorem 10.2.15). We put $K_S := \langle \exp \mathfrak{k}_{\mathfrak{s}} \rangle$, to obtain an isomorphism of Lie groups

$$\Psi \colon (\mathfrak{k} \cap \mathfrak{n}) \times \mathfrak{e} \times K_S \to K, \quad (a_n, a_e, k) \mapsto \exp(a_n) \exp(a_e) k.$$

For $x \in \mathfrak{n}^+$, by $[\mathfrak{p}, \mathfrak{k} \cap \mathfrak{r}] = \{0\}$ and $[\mathfrak{f}, \mathfrak{k} \cap \mathfrak{n}] = \{0\}$, we obtain that

$$\begin{aligned} \Phi\big(x, f, p, \exp(a_n)\exp(a_e)k\big) &= \exp(x)\exp(f)\exp(p)\exp(a_n)\exp(a_e)k \\ &= \big(\exp(x)\exp(a_n)\big)\big(\exp(f)\exp(a_e)\big)\big(\exp(p)k\big). \end{aligned}$$

Since $G \cong R \rtimes S$ and $\mathfrak{p} \times K_S \to S, (p,k) \mapsto \exp(p)k$ is a diffeomorphism (Theorem 12.1.7), we may now assume that $G = R$ is solvable.

Applying Lemma 13.3.6 twice, we see that the map

$$(N \times \mathfrak{f}) \times \mathfrak{e} \to R, \quad (x, f, a_e) \mapsto \big(x \exp(f)\big) \exp(a_e)$$

is a diffeomorphism. Therefore it remains to show that

$$\mathfrak{n}^+ \times (\mathfrak{k} \cap \mathfrak{n}) \to N, \quad (x, k) \mapsto \exp(x) \exp(k)$$

is a diffeomorphism. But this follows from Lemma 13.3.4 and Lemma 13.3.5. □

To get this theorem for general connected Lie groups, we have to control the behavior of our decompositions under coverings. This is not very hard since we already know that the center is entirely contained in the subgroup $K = \langle \exp_G \mathfrak{k} \rangle$.

Theorem 13.3.8 (First Manifold Splitting Theorem). *Let G be a connected Lie group and keep the notation from* Theorem 13.3.7. *Then the map*

$$\Phi \colon \mathfrak{n}^+ \times \mathfrak{f} \times \mathfrak{p} \times K \to G, \quad (x, f, p, k) \mapsto \exp(x)\exp(f)\exp(p)k$$

is a diffeomorphism. It is K-equivariant with respect to the conjugation action on G and the action $k \cdot (x, f, p, g) := (\mathrm{Ad}(k)x, \mathrm{Ad}(k)f, \mathrm{Ad}(k)p, kgk^{-1})$ on the left.

Proof. This theorem holds for the simply connected covering $\widetilde{G}$ of G by Theorem 13.3.7. Let $q_G\colon \widetilde{G} \to G$ be the universal covering morphism and $\widetilde{K} := \langle \exp_{\widetilde{G}} \mathfrak{k}\rangle$. Then $\ker q_G \subseteq Z(\widetilde{G}) \subseteq Z(\widetilde{K})$ is discrete (Theorem 13.2.8). If $\widetilde{\Phi}$ is the respective map for $\widetilde{G}$, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{n}^+\times\mathfrak{f}\times\mathfrak{p}\times\widetilde{K} & \xrightarrow{\widetilde{\Phi}} & \widetilde{G} \\ {\scriptstyle \mathrm{id}_{\mathfrak{n}^+\times\mathfrak{f}\times\mathfrak{p}}\times q_G|_{\widetilde{K}}}\downarrow & & \downarrow{\scriptstyle q_G} \\ \mathfrak{n}^+\times\mathfrak{f}\times\mathfrak{p}\times K & \xrightarrow[\Phi]{} & G\end{array}$$

From this diagram, we immediately read off the bijectivity of Φ. Since both vertical maps and $\widetilde{\Phi}$ are local diffeomorphisms, this also follows for Φ. Hence Φ is a diffeomorphism. ☐

Remark 13.3.9. If $\mathfrak{g}$ is reductive, then $\mathfrak{n}^+ = \mathfrak{f} = \{0\}$, so that we have a diffeomorphism

$$\Phi\colon \mathfrak{p}\times K \to G, \quad (x,k)\mapsto \exp(x)k.$$

Moreover, $\mathfrak{g} = \mathfrak{z}(\mathfrak{g})\oplus\mathfrak{s}$, where $\mathfrak{s} = [\mathfrak{g},\mathfrak{g}]$ is semisimple, and $\mathfrak{k} = \mathfrak{z}(\mathfrak{g})\oplus\mathfrak{k}_{\mathfrak{s}}$, where $\mathfrak{s} = \mathfrak{k}_{\mathfrak{s}}\oplus\mathfrak{p}$ is a Cartan decomposition.

Corollary 13.3.10. *If G is a connected Lie group, then the following assertions hold:*

(i) *If $\mathfrak{t} \subseteq \mathfrak{g}$ is compactly embedded abelian, then the centralizer $Z_G(\mathfrak{t}) = \{g\in G\colon \mathrm{Ad}(g)|_{\mathfrak{t}} = \mathrm{id}_{\mathfrak{t}}\}$ is connected.*
(ii) *If $T\subseteq G$ is a torus, then its centralizer $Z_G(T)$ is connected.*

Proof. (i) Let $\mathfrak{k}\supseteq\mathfrak{t}$ be maximal compactly embedded and $K := \langle\exp_G\mathfrak{k}\rangle$. Then the First Manifold Splitting Theorem 13.3.8 implies that

$$\Phi\colon \mathfrak{n}^+\times\mathfrak{f}\times\mathfrak{p}\times K\to G, \quad (x,f,p,k)\mapsto \exp_G(x)\exp_G(f)\exp_G(p)k$$

is a K-equivariant diffeomorphism. For $T := \exp_G\mathfrak{t}\subseteq K$ and an $\mathrm{Ad}(T)$-invariant subspace $V\subseteq\mathfrak{g}$, we write $V^T := \{v\in V\colon \mathrm{Ad}(t)v = v\}$ for the subspace of T-invariant elements. Then Φ induces a diffeomorphism

$$(\mathfrak{n}^+)^T\times\mathfrak{f}^T\times\mathfrak{p}^T\times Z_K(\mathfrak{t})\to Z_G(\mathfrak{t}).$$

Therefore it suffices to show that $Z_K(\mathfrak{t})$ is connected.

Using the Structure Theorem 11.1.18 for groups with compact Lie algebra, we write $K = V\times C$, where V is a vector subgroup and C is compact. Then we accordingly have $\mathfrak{k} = \mathfrak{v}\oplus\mathfrak{c}$ and write $\mathfrak{a} := (\mathfrak{t}+\mathfrak{v})\cap\mathfrak{c}$ for the projection of $\mathfrak{t}$ to $\mathfrak{c}$. Since $\mathfrak{v}$ is central in $\mathfrak{k}$,

$$Z_K(\mathfrak{t}) = Z_K(\mathfrak{t}+\mathfrak{v}) = V\times Z_C(\mathfrak{a}) = V\times Z_C(\overline{\exp_C(\mathfrak{a})}),$$

and since $A := \overline{\exp_C(\mathfrak{a})}$ is a torus in C, the centralizer $Z_C(A)$ is connected by Corollary 11.2.11. This proves that $Z_K(\mathfrak{t})$ and therefore also $Z_G(\mathfrak{t})$ are connected.

(ii) follows from (i) because $Z_G(T) = Z_G(\mathbf{L}(T))$ and $\mathbf{L}(T)$ is compactly embedded abelian. □

Theorem 13.3.11 (Second Manifold Splitting Theorem). *Let G be a Lie group with finitely many connected components and $C \subseteq G$ be a maximal compact subgroup. Then there exists a closed submanifold M of G diffeomorphic to some $\mathbb{R}^n$, such that the map*

$$M \times C \to G, \quad (m, k) \mapsto mc$$

is a diffeomorphism.

Proof. Clearly, $\mathbf{L}(C)$ is a compactly embedded subalgebra of $\mathfrak{g}$, hence contained in a maximal compactly embedded subalgebra $\mathfrak{k}$. Let $K \subseteq G_0$ be the corresponding integral subgroup. In view of the Maximal Compact Subgroup Theorem 13.1.3, C_0 is maximal compact in G, and since it is contained in K, it is also maximal compact in K. As $\mathfrak{k} = \mathbf{L}(K)$ is a compact Lie algebra, there exists a vector group V with $K \cong C_0 \times V$ (Theorem 11.1.18).

Using the construction from Theorem 13.3.7 we put

$$M := \exp(\mathfrak{n}^+) \exp(\mathfrak{f}) \exp(\mathfrak{p}) V.$$

Then Theorem 11.1.18 and Theorem 13.3.8 imply that the map

$$\mu \colon M \times C \to G, \quad (m, c) \mapsto mc$$

restricts to a diffeomorphism $M \times C_0 \to G_0$.

Note that $\mu(m, cd) = \mu(m, c)d$ for $c, d \in C$. Since C by the Maximal Compact Subgroup Theorem intersects each connected component of G, μ is surjective. To see that it is also injective, note that $mc = m'c'$ implies that $c'c^{-1} \in G_0 \cap C = C_0$, so that the injectivity of the restriction to $M \times C_0$ implies that $c = c'$ and hence that $m = m'$. The regularity of μ on $M \times C$ follows from the regularity on $M \times C_0$. We conclude that μ is a regular bijection, i.e., a diffeomorphism. □

Theorem 13.3.12. *For a 1-connected Lie group G, the following are equivalent*

(i) *G is diffeomorphic to some $\mathbb{R}^n$.*
(ii) *G contains no nontrivial compact subgroup.*
(iii) *For any maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, the corresponding integral subgroup K of G is a vector group.*
(iv) *The maximal compactly embedded subalgebras $\mathfrak{k}$ of $\mathfrak{g}$ are abelian.*
(v) *The maximal compactly embedded subalgebras $\mathfrak{k}_{\mathfrak{s}}$ of the semisimple quotient $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ are abelian.*

(vi) $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})^n$ *for some* $n \in \mathbb{N}_0$.

Proof. (i) ⇒ (ii): Suppose that G is diffeomorphic to $\mathbb{R}^n$ for some $n > 0$. Then the Poincaré Lemma 9.3.20 implies that $H^j_{\mathrm{dR}}(G) = \{0\}$ for $j > 0$. If $C \subseteq G$ is a maximal compact subgroup, then the existence of a manifold splitting $G \cong \mathbb{R}^k \times C$ implies that $H^{n-k}_{\mathrm{dR}}(G) \cong H^{n-k}_{\mathrm{dR}}(C)$ (Corollary 9.3.18), and since C is compact and connected,

$$H^{n-k}_{\mathrm{dR}}(C) \neq \{0\}$$

(Corollary 9.4.25). This proves that $k = n$, so that $C = \{\mathbf{1}\}$.

(ii) ⇒ (i) is an immediate consequence of the manifold splitting.

(ii) ⇔ (iii): This follows from the fact that $K \cong C \times V$, where C is maximal compact in G and V is a vector group (Theorem 11.1.18), combined with the conjugacy of maximal compact subgroups and Theorem 13.3.8.

(iii) ⇔ (iv): Since K is 1-connected by Theorem 13.3.8, $\mathbf{L}(C) = \mathfrak{k}'$ is the commutator algebra of $\mathfrak{k}$, hence trivial if and only if $\mathfrak{k}$ is abelian.

(iv) ⇔ (v): With $\mathfrak{k}$ and $\mathfrak{s}$ as in Lemma 13.3.3, $\mathfrak{k} \cap \mathfrak{r}$ is central in $\mathfrak{k}$, so that $\mathfrak{k}' = \mathfrak{k}'_{\mathfrak{s}}$. Therefore $\mathfrak{k}$ is abelian if and only if $\mathfrak{k}_{\mathfrak{s}}$ has this property.

(v) ⇔ (vi): Since any Cartan decomposition of $\mathfrak{s}$ is adapted to the decomposition into simple ideals (Exercise 12.2.4) this follows from Proposition 12.1.10. □

Theorem 13.3.13 (Maximal Compact Subgroups and Normal Subgroups). *Let G be a Lie group, $N \subseteq G$ be a closed normal subgroup and $p\colon G \to G/N$ be the quotient map. If $\pi_0(G)$ and $\pi_0(N)$ are finite, then the following assertions hold:*

(i) *If $T \subseteq G$ is a maximal torus and $K \subseteq G$ a maximal compact subgroup, then*
   (a) *$K \cap N$, resp., $p(K)$, are maximal compact subgroups of N, resp., G/N.*
   (b) *$T \cap N_0$, resp., $p(T)$ are maximal tori in N, resp., G/N.*
   (c) *Suppose that G and N are connected. If G or the two groups N and G/N are 1-connected, then all these groups are 1-connected.*

(ii) *If N contains a maximal compact subgroup of G and V is an integral subgroup with $G = VN$, then $V \cap N$ is connected.*

Proof. (i) (c) Let $q_G\colon \widetilde{G} \to G$ be the universal covering homomorphism. The integral subgroup N_1 of $\widetilde{G}$ corresponding to $\mathfrak{n} := \mathbf{L}(N)$ is 1-connected (Theorem 5.1.11), so that the first row of the diagram in the Homotopy Group Theorem 10.1.15 and Remark 10.1.17 yield a short exact sequence

$$\mathbf{1} \to \pi_1(N) \to \pi_1(G) \to \pi_1(G/N) \to \mathbf{1}. \tag{13.1}$$

This proves (i)(c).

(i)(a) Let $K_N \subseteq N$ be a maximal compact subgroup and K_1 be a maximal compact subgroup of G containing K_N. Then there exists a $g \in G$ such that $gKg^{-1} = K_1$ (Theorem 13.1.3). Hence $g^{-1}K_N g$ is a maximal compact subgroup of N contained in $K \cap N$. Since the group $K \cap N$ is compact, we conclude that $g^{-1}K_N g = K \cap N$, hence that $K \cap N$ is maximal compact in N.

To see that $p(K)$ is maximal compact in G/N, we first note that, as a homomorphic image of a compact group, $p(K)$ is compact, so that it remains to prove maximality. Since $p(K)$ intersects each connected component of G/N, it suffices to show that $p(K_0)$ is maximal compact in $(G/N)_0 \cong G_0/(N \cap G_0)$ (Theorem 13.1.3(v)). For the proof of the maximal compactness of $p(K)$ in G/N, we may therefore assume that G is connected. The natural projection $q\colon G/N_0 \to G/N, gN_0 \mapsto gN$ is a finite covering, so that inverse images of compact subgroups of G/N are compact subgroups of G/N_0. Hence a subgroup $U \subseteq G/N_0$ is maximal compact if and only if $q(U)$ is maximal compact in G/N and $\ker q \subseteq U$. As K_N is maximal compact in N, it intersects each connected component of N, so that the image of K in G/N_0 contains the subgroup N/N_0. Therefore it is maximal compact if and only if $p(K)$ is maximal compact in G/N, and we may thus also assume that N is connected.

Let $K_1 \supseteq p(K)$ be a maximal compact subgroup of G/N. First we show that the commutator groups $p(K)'$ and K_1' are equal. Let $\mathfrak{k}_1 := \mathbf{L}(K_1)$. Then $\mathfrak{k}_1'$ is a compact semisimple Lie algebra, so that there exists a homomorphism $\beta\colon \mathfrak{k}_1' \to \mathfrak{g}$ with $\mathbf{L}(p) \circ \beta = \mathrm{id}_{\mathfrak{k}_1'}$ (Corollary 4.6.10). Let $Q_1 := \langle \exp_G \beta(\mathfrak{k}_1') \rangle$ denote the integral subgroup of G corresponding to $\beta(\mathfrak{k}_1')$. Then Q_1 is compact because $\mathfrak{k}_1'$ is semisimple. Then there exists an element $x \in G$ with $xQ_1x^{-1} \subseteq K$ (Theorem 13.1.3). Then $p(K) \supseteq p(x)p(Q_1)p(x)^{-1}$ contains the group $p(x)K_1'p(x)^{-1}$. So $p(K)$ contains the commutator group of the maximal compact subgroup $p(x)K_1p(x)^{-1}$ of G/N. Therefore $p(K)'$ is a maximal compact semisimple integral subgroup of G/N, so that $p(K)' = K_1'$.

It remains to show that $\dim Z\big(p(K)\big) = \dim Z(K_1)$. To see this, we recall that

$$\operatorname{rk} \pi_1(G/N) = \operatorname{rk} \pi_1(K_1) = \dim Z(K_1), \tag{13.2}$$

where the first equality follows from the Second Manifold Splitting Theorem 13.3.11 and the second one from Corollary 11.2.7. We likewise have $\operatorname{rk} \pi_1(G) = \operatorname{rk} \pi_1(K) = \dim Z(K)$. Next we note that (13.1) implies

$$\operatorname{rk} \pi_1(G) = \operatorname{rk} \pi_1(N) + \operatorname{rk} \pi_1(G/N_0) \tag{13.3}$$

(see Proposition 13.6.6 in the appendix to this chapter). Further, the homotopy sequence for the quotient $G/N = (G/N_0)/(N/N_0)$ (Remark 10.1.17) implies that $\pi_1(G/N_0)$ is of finite index $|N/N_0|$ in $\pi_1(G/N)$. In particular $\operatorname{rk} \pi_1(G/N_0) = \operatorname{rk} \pi_1(G/N)$. Combining all this, we now obtain

$$\dim Z(K) = \dim Z(K \cap N) + \dim Z(K_1).$$

Since $Z(K) \cap N \subseteq Z(K \cap N)$, we have $\dim Z(K) \cap N \leq \dim Z(K \cap N)$, hence

$$\dim p\big(Z(K)\big) \geq \dim Z(K) - \dim Z(K \cap N) = \dim Z(K_1).$$

On the other hand, $p\big(Z(K)\big)$ commutes with K_1' and therefore with K_1, hence is contained in $Z(K_1)$. In view of the estimate of the dimension, this proves that $Z(K_1)_0 = p\big(Z(K)_0\big)$. Finally, $K_1 = K_1' Z(K_1)_0 \subseteq p(K)$ shows that $p(K)$ is maximal compact in G/N.

(i)(b) Let $K \supseteq T$ be a maximal compact subgroup of G. Then $K \cap N$ is a normal subgroup of K, and (a) implies that it is a maximal compact subgroup of N. Therefore, in view of (a), it remains to show that $T \cap N_0$ is a maximal torus in $K \cap N$ and that $p(T)$ is a maximal torus in $p(K)$. Thus, we may w.l.o.g., assume that $G = K$ is compact and connected.

We have a direct decomposition $\mathfrak{k} = \mathfrak{n} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a complementary ideal. Since the Cartan subalgebra $\mathfrak{t} = \mathbf{L}(T)$ decomposes accordingly, we conclude that $\mathfrak{t} \cap \mathfrak{n}$ is a Cartan subalgebra of $\mathfrak{n}$ and that $\mathfrak{t} \cap \mathfrak{a} \cong \mathbf{L}(p)(\mathfrak{t}) \subseteq \mathfrak{k}/\mathfrak{n}$ is a Cartan subalgebra. Therefore $\exp_K(\mathfrak{t} \cap \mathfrak{n}) = (N \cap T)_0$ is a maximal torus in N and since maximal tori of compact connected Lie groups are maximal abelian (Corollary 11.2.11), we conclude that $N_0 \cap T$ is connected, hence a maximal torus of N_0 and therefore also of N.

To see that $p(T)$ is a maximal torus in G/N we first note that, as a homomorphic image of a torus, $p(T)$ is a torus. Since $p(T) = \exp_{G/N}(\mathbf{L}(p)\mathfrak{t})$ is of the same dimension as $\mathfrak{a} \cap \mathfrak{t}$, we conclude further that it is a maximal torus in K/N.

(ii) Let $K \subseteq N$ be a maximal compact subgroup of G. In view of (i), $q(K) = \{\mathbf{1}\}$ is a maximal compact subgroup of G/N. Therefore the group G/N is 1-connected by the Second Manifold Splitting Theorem 13.3.11. The mapping $q|_V \colon V \to G/N$ is surjective with kernel $V \cap N$, which leads to $G/N \cong V/(V \cap N)$ if V is endowed with its intrinsic Lie group topology. It follows in particular that $V/(V \cap N)$ is simply connected. The canonical map $V/(V \cap N)_0 \to V/(V \cap N)$ is a covering (Lemma 10.1.11), hence trivial since the image is simply connected. This proves that $V \cap N$ is connected. □

13.4 The Exponential Function of Solvable Groups

We return to the structure of connected solvable Lie groups in the light of the Manifold Splitting Theorem. For solvable groups, any maximal compactly embedded subalgebra is abelian. As a special case of Theorem 13.3.11, we thus have:

Theorem 13.4.1. *Let G be a connected solvable Lie group and $T \subseteq G$ be a maximal torus. Then T is maximal compact in G and there exists a closed submanifold $M \cong \mathbb{R}^n \subseteq G$ such that the map*

$$M \times T \to G, \quad (m,t) \mapsto mt$$

is a diffeomorphism.

If G is simply connected, this also holds for T, so that T is trivial. Theorem 13.3.8 provides a decomposition of G into three exponential manifold factors $\exp(\mathfrak{n}^+)\exp(\mathfrak{f})\exp(\mathfrak{k})$. Since $[\mathfrak{k},\mathfrak{f}] = \{0\}$, we have $\exp(f+x) = \exp(f)\exp(x)$ for $f \in \mathfrak{f}$ and $x \in \mathfrak{k}$. Hence $\exp(\mathfrak{f})\exp(\mathfrak{k}) = \exp(\mathfrak{f}+\mathfrak{k})$, and therefore,

$$G = \exp(\mathfrak{n}^+)\exp(\mathfrak{k}+\mathfrak{f}).$$

The question arises, under which circumstances a single exponential factor suffices. This in particular requires the exponential function to be regular.

Definition 13.4.2. A Lie algebra $\mathfrak{g}$ is called *exp-regular* if (cf. Definition 8.2.30)

$$\mathfrak{g} = \{x \in \mathfrak{g} \colon \operatorname{Spec}(\operatorname{ad} x) \cap i\mathbb{R} \subseteq \{0\}\} = \mathbb{R}\operatorname{regexp}(\mathfrak{g}).$$

If $\mathfrak{g} = \mathbf{L}(G)$ for a connected Lie group, then this condition is equivalent to $\exp_G$ being regular in each $x \in \mathfrak{g}$ (Definition 8.2.30).

Remark 13.4.3. Any exp-regular complex Lie algebra $\mathfrak{g}$ is nilpotent. In fact, if $\mathfrak{g}$ is not nilpotent, then Engel's Theorem implies the existence of some $x \in \mathfrak{g}$ for which $\operatorname{ad} x$ is not nilpotent. Multiplying x with a suitable nonzero scalar, we may assume that $i \in \operatorname{Spec}(\operatorname{ad} x)$, so that $\mathfrak{g}$ is not exp-regular.

Lemma 13.4.4. *For an exp-regular real Lie algebra $\mathfrak{g}$, the following assertions hold:*

(i) $\operatorname{comp}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$, *i.e., $\mathfrak{z}(\mathfrak{g})$ is maximal compactly embedded.*
(ii) *All subalgebras of $\mathfrak{g}$ are exp-regular.*
(iii) *All quotients of $\mathfrak{g}$ are exp-regular.*
(iv) *$\mathfrak{g}$ is solvable.*

Proof. (i) If $x \in \operatorname{comp}(\mathfrak{g})$, then $\operatorname{ad} x$ is semisimple with $\operatorname{Spec}(\operatorname{ad} x) \subseteq i\mathbb{R}$. Since $\mathbb{R}x \subseteq \mathbb{R}\operatorname{regexp}(\mathfrak{g})$, we obtain $\operatorname{Spec}(\operatorname{ad} x) = \{0\}$, so that $\operatorname{ad} x = 0$ by semisimplicity. This proves that $\operatorname{comp}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

(ii) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra. For $x \in \mathfrak{h}$ we then have $\operatorname{Spec}(\operatorname{ad}_{\mathfrak{h}} x) \subseteq \operatorname{Spec}(\operatorname{ad}_{\mathfrak{g}} x)$, so that the exp-regularity of $\mathfrak{g}$ implies that $\operatorname{Spec}(\operatorname{ad}_{\mathfrak{h}} x) \cap i\mathbb{R} = \{0\}$, hence that $\mathfrak{h}$ is exp-regular.

(iii) If $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ is a quotient of $\mathfrak{g}$ and $q \colon \mathfrak{g} \to \mathfrak{q}$ is the quotient map, then we also have for each $x \in \mathfrak{g}$ the relation $\operatorname{Spec}(\operatorname{ad} q(x)) \subseteq \operatorname{Spec}(\operatorname{ad} x)$, which immediately shows that $\mathfrak{q}$ is exp-regular.

(iv) Let $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ be the semisimple quotient $\mathfrak{g}$. In view of (iii), it is exp-regular, and (i) implies that $\{0\} = \mathfrak{z}(\mathfrak{s})$ is maximal compact. Hence the subalgebra $\mathfrak{k}_{\mathfrak{s}}$ in any Cartan decomposition $\mathfrak{s} = \mathfrak{k}_{\mathfrak{s}} \oplus \mathfrak{p}$ is trivial. This leads to $\mathfrak{s} = [\mathfrak{s},\mathfrak{s}] = [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}_{\mathfrak{s}} = \{0\}$, so that $\mathfrak{g}$ is solvable. □

The preceding lemma shows that all Lie groups for which the exponential function is a diffeomorphism, are solvable.

Example 13.4.5. If G is a solvable Lie group with Lie algebra $\mathfrak{g}$, then the condition that $\mathfrak{z}(\mathfrak{g})$ is maximal compactly embedded does not imply exp-regularity. A typical example is given by

$$\mathfrak{g} := \mathbb{C}^2 \rtimes_\alpha \mathbb{R} \quad \text{with} \quad \alpha(t) = t\begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix},$$

and $G := \mathbb{C}^2 \rtimes_\beta \mathbb{R}$ with $\beta(t) := e^{\alpha(t)}$. Then $\mathfrak{g}$ is solvable, $\mathfrak{z}(\mathfrak{g}) = \{0\}$ is maximal compactly embedded, and $\{0\} \oplus \mathbb{R}$ is a Cartan subalgebra of $\mathfrak{g}$ (Exercise 13.4.1). Despite these facts, $(0, 2\pi)$ is a singular point of the exponential function because $2\pi i \in \operatorname{Spec}(\operatorname{ad}(0, 2\pi))$.

Proposition 13.4.6. *A solvable real Lie algebra $\mathfrak{g}$ is exp-regular if and only if there exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that for all roots α of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$, we have*

$$\alpha(\mathfrak{h}) \cap i\mathbb{R} = \{0\}.$$

Proof. If there exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, an element $h \in \mathfrak{h}$ and a root $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ with $\alpha(h) \in i\mathbb{R}^\times$, then $0 \neq \alpha(h) \in \operatorname{Spec}(\operatorname{ad} h) \cap i\mathbb{R}$. Therefore $\mathfrak{g}$ is not exp-regular.

Assume, conversely, that $\mathfrak{g}$ is not exp-regular and that

$$\operatorname{Spec}(\operatorname{ad} x) \cap i\mathbb{R} \neq \{0\}.$$

We denote the maximal nilpotent ideal of $\mathfrak{g}$ by $\mathfrak{n}$ and let $\mathfrak{h} \subseteq \mathfrak{g}$ be any Cartan subalgebra (Theorem 5.1.18). Because of $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{g}' \subseteq \mathfrak{n}$, all root spaces of $\mathfrak{g}_\mathbb{C}$ are contained in $\mathfrak{n}_\mathbb{C}$, and therefore $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. In particular, there exists an $n \in \mathfrak{n}$ with $y := x + n \in \mathfrak{h}$. If we consider the action of $\operatorname{ad} y$ on the flag

$$\{0\} \subseteq C^k(\mathfrak{n}) \subseteq \ldots \subseteq C^2(\mathfrak{n}) \subseteq C^1(\mathfrak{n}) = \mathfrak{n},$$

then we see that $\operatorname{Spec}(\operatorname{ad} y) = \operatorname{Spec}(\operatorname{ad} x)$, since both induce the same maps on the quotient spaces $C^i(\mathfrak{n})/C^{i+1}(\mathfrak{n})$. Therefore $\operatorname{Spec}(\operatorname{ad} y) \cap i\mathbb{R} \neq \{0\}$. As the nonzero spectrum of $\operatorname{ad} y$ coincides with the values of the roots of $\mathfrak{g}_\mathbb{C}$ on y, there exists a root α with $\alpha(y) \in i\mathbb{R}^\times$. $\square$

As we already mentioned above, we want to know whether the exponential function of a simply connected Lie group with exp-regular Lie algebra is a diffeomorphism. For this, we need the following lemma.

Lemma 13.4.7. *If G is a connected Lie group with exp-regular Lie algebra, then the exponential function of G is surjective.*

Proof. If $q_G \colon \widetilde{G} \to G$ is the simply connected covering group, then $q_G(\operatorname{im} \exp_{\widetilde{G}}) = \operatorname{im}(\exp_G)$, so that we may w.l.o.g. assume that G is simply connected.

We prove the assertion by induction on $\dim G$. If $\dim G \leq 1$, then G is abelian, so that its exponential function is surjective. Now we assume that the exponential functions of all connected Lie groups H with exp-regular Lie algebra and $\dim H < \dim G$ are surjective. Let $\mathfrak{n}$ be the maximal nilpotent ideal of $\mathfrak{g} := \mathbf{L}(G)$ and let $g \in G$. By the induction hypothesis and Lemma 13.4.4, the exponential function of $G/\exp \mathfrak{z}(\mathfrak{n})$ is surjective since $\exp \mathfrak{z}(\mathfrak{n})$ is a closed vector group in G by Theorem 10.2.15. Hence there exists an element $z \in \mathfrak{z}(\mathfrak{n})$ and $x \in \mathfrak{g}$ with $g = \exp_G(x)\exp_G(z)$. The ideal $\mathfrak{z}(\mathfrak{n})$ of $\mathfrak{n}$ is invariant under all automorphisms of $\mathfrak{n}$, and since $\mathfrak{n}$ is invariant under all automorphisms of $\mathfrak{g}$, this property is inherited by $\mathfrak{z}(\mathfrak{n})$. We conclude that $\mathfrak{z}(\mathfrak{n})$ is an ideal of $\mathfrak{g}$. Hence $\mathfrak{a} := \mathfrak{z}(\mathfrak{n}) + \mathbb{R}x$ is a subalgebra of $\mathfrak{g}$ and therefore exp-regular by Lemma 13.4.4.

Now it suffices to show that the exponential function of the integral subgroup $A := \langle \exp \mathfrak{a} \rangle$ is surjective. If $x \in \mathfrak{z}(\mathfrak{n})$, then $\mathfrak{a} = \mathfrak{z}(\mathfrak{n})$ is abelian, so that $\exp_A(\mathfrak{a}) = A$ follows. If $x \notin \mathfrak{z}(\mathfrak{n})$, then $\mathfrak{a} \cong \mathfrak{z}(\mathfrak{n}) \rtimes_{\operatorname{ad} x} \mathbb{R}$ is a semidirect product of the abelian Lie algebra $V := \mathfrak{z}(\mathfrak{n})$ and $\mathbb{R}$, defined by the derivation $D := \operatorname{ad} x$ with $\operatorname{Spec}(D) \cap i\mathbb{R} = \{0\}$. It remains to show that the exponential function of the group $A \cong V \rtimes_\alpha \mathbb{R}$ with $\alpha(t) = e^{tD}$ is surjective. For this group it is given explicitly by

$$\exp_A(x,t) = \begin{cases} (x,0), & \text{for } t = 0, \\ \left(\frac{e^{tD}-\mathbf{1}}{tD}x, t\right), & \text{for } t \neq 0 \end{cases}$$

(Exercise 10.2.3). For every $t \neq 0$, the linear map $\frac{e^{tD}-\mathbf{1}}{tD}$ is invertible because $\operatorname{Spec}(D) \cap i\mathbb{R} \subseteq \{0\}$ (Exercise 8.2.13). This implies that $\exp_A$ is surjective. □

We can now prove the following theorem:

Theorem 13.4.8 (Dixmier's Theorem). *For a 1-connected Lie group G with Lie algebra $\mathfrak{g}$, the following are equivalent:*

(i) $\exp_G$ *is a diffeomorphism.*
(ii) $\exp_G$ *is injective.*
(iii) $\exp_G$ *is a regular map.*
(iv) $\mathfrak{g}$ *is an exp-regular Lie algebra.*

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) follows from Lemma 8.2.31(b).

(iii) ⇔ (iv) since $\mathbb{R}\operatorname{regexp}(\mathfrak{g}) = \mathfrak{g}$ if and only if $\operatorname{regexp}(\mathfrak{g}) = \mathfrak{g}$.

(iii) ⇒ (i): By assumption, $\exp_G$ is regular, and it is surjective by Lemma 13.4.7. If $\exp_G$ is not injective, then Lemma 8.2.31 implies the existence of a nontrivial torus in G. This contradicts Theorem 13.4.1 because G is simply connected. □

Remark 13.4.9. The characterization of exp-regular solvable Lie algebras is originally due to Saito [Sai57] and Dixmier [Dix57]. Saito even develops a testing device, characterizing the exp-regular Lie algebras as the solvable Lie algebras not containing a subalgebra isomorphic to $\mathfrak{mot}_2$, the Lie algebras of the motion group of the euclidian plane, or its four-dimensional central extension $\mathfrak{osc}$, the *oscillator algebra*. These Lie algebras can be described in terms of commutator relations as follows. The 3-dimensional Lie algebra $\mathfrak{mot}_2$ has a basis u, p, q with

$$[u,p] = q, \quad [u,q] = -p \quad \text{and} \quad [p,q] = 0,$$

whereas $\mathfrak{osc}$ has a basis u, p, q, z, where z is central with

$$[u,p] = q, \quad [u,q] = -p \quad \text{and} \quad [p,q] = z.$$

One implication of Saito's result is trivial, because in both Lie algebras we have $i \in \mathrm{Spec}\,(u)$, so that the occurrence of any such subalgebra in a Lie algebra $\mathfrak{g}$ implies that $\mathfrak{g}$ is not exp-regular.

Conversely, any finite-dimensional non-exp-regular Lie algebra $\mathfrak{g}$ contains a triple (u, p, q) satisfying

$$[u,p] = q \quad \text{and} \quad [u,q] = -p. \tag{13.4}$$

One finds more such pairs as follows: Put $p_1 := p$, $q_1 := q$ and $z_1 := [p,q]$ and, recursively, $p_{i+1} := [z_i, p_i]$, $q_{i+1} := [z_i, q_i]$, $z_{i+1} := [p_{i+1}, q_{i+1}]$. If $\mathfrak{g}$ is solvable, then it is easy to see that for some n the elements u, p_n, q_n, z_n span a subalgebra either isomorphic to $\mathfrak{mot}_2$ or $\mathfrak{osc}$ ([Sai57]). Here the main point is that $z_i \in D^i(\mathfrak{g})$ vanishes if i is large enough.

Exercises for Section 13.4

Exercise 13.4.1. Let $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathfrak{h}$, where $\mathfrak{n}$ and $\mathfrak{h}$ are nilpotent. If there exists no $0 \neq v \in \mathfrak{n}$ with $[\mathfrak{h}, v] = \{0\}$, then $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$.

Exercise 13.4.2. Find a solvable Lie group with a surjective exponential function whose Lie algebra is not exp-regular.

The following exercise shows that for 1-connected solvable Lie groups, the surjectivity of the exponential function already implies exp-regularity of the Lie algebra.

Exercise 13.4.3. Let G be a solvable Lie group whose exponential function $\exp_G \colon \mathfrak{g} \to G$ is surjective. Show that $\mathfrak{g}$ is exp-regular. For the proof, proceed along the following steps:

(a) We call a solvable Lie algebra $\mathfrak{g}$ *exponential* if the exponential function of the corresponding simply connected Lie group is surjective.
(b) If $\mathfrak{g}$ is an exponential Lie algebra, then this also holds for all homomorphic images of $\mathfrak{g}$.
(c) Let $\mathfrak{g}$ be exponential, and let $\mathfrak{a} \subseteq \mathfrak{g}$ be an ideal such that $\mathfrak{g}/\mathfrak{a}$ is exp-regular, then every subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{a}$ is also exponential.

(d) Let $\mathfrak{g} = \mathfrak{a} \rtimes \mathbb{R}x$ be exponential, where $\mathfrak{a}$ is abelian. Show that $\mathfrak{g}$ is exp-regular.
(e) Let $\mathfrak{g}$ be exponential, and let $\mathfrak{a} \subseteq \mathfrak{g}$ be an abelian ideal such that $\mathfrak{g}/\mathfrak{a}$ is exp-regular. Then $\mathfrak{g}$ is also exponential.
(f) Now show by induction that every exponential Lie algebra is exp-regular.

Exercise 13.4.4. Let G be a connected semisimple Lie group and $G = KAN$ be an Iwasawa decomposition of G (cf. Theorem 12.2.12). Show that the Lie algebra of $B := AN$ is exponential.

Exercise 13.4.5. We consider the solvable Lie group G from Exercise 10.2.1. Why is $\exp_G$ not surjective? How can this be concluded without computing the exponential function explicitly as in the proof of Dixmier's Theorem? Hint: Exercise 13.4.1.

13.5 Dense Integral Subgroups

We have already seen in the Integral Subgroup Theorem 8.4.8 that for each Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of a Lie group G, the subgroup $H := \langle \exp_G \mathfrak{h} \rangle$ of G carries a natural Lie group topology for which

$$\mathbf{L}(H) \cong \mathfrak{h} = \{x \in \mathfrak{g} \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

The corresponding inclusion morphism $\iota \colon H \to G$ is a topological isomorphism onto its image if and only if H is closed in G (Integral Subgroup Theorem 8.4.8). If this is not the case, then $\overline{H}$ is a closed connected subgroup of G, hence a Lie group. In this section we analyze the passage from H to its closure. In particular, we shall derive an important criterion for the integral subgroups to be closed. In the process, the results on maximal compactly embedded subalgebras and the center developed in Section 13.2 will be important tools. It is clear that we can restrict ourselves from the outset to the case where H is dense in G (otherwise we may replace G by the Lie group $\overline{H}$).

In this section, $H = \langle \exp_G \mathfrak{h} \rangle$ always denotes a dense subgroup of the Lie group G with Lie algebra $\mathfrak{g}$. In particular, G is connected.

Lemma 13.5.1. *Every normal integral subgroup of H is normal in G. In particular, H is a normal subgroup of G and $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. By Theorem 10.1.2, an integral subgroup $N \subseteq H$ is normal in H if and only if its Lie algebra $\mathfrak{n} := \mathbf{L}(N)$ is invariant under $\mathrm{Ad}(H)$, and it is normal in G if $\mathfrak{n}$ is invariant under $\mathrm{Ad}(G)$. Since $\mathrm{Ad}(H)$ is dense in $\mathrm{Ad}(G)$, every ideal of $\mathfrak{h}$ is also invariant under $\mathrm{Ad}(G)$. □

Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ by Lemma 13.5.1, Theorem 10.1.21 implies that the corresponding integral subgroup $\widehat{H} := \langle \exp_{\widetilde{G}} \mathfrak{h} \rangle$ of the universal covering group $\widetilde{G}$ of G is a closed normal subgroup. Recall that $G \cong \widetilde{G}/\Gamma$, where $\Gamma = \ker q_G$

is the discrete central kernel of the covering homomorphism $q_G \colon \widetilde{G} \to G$, and this implies that $q_G^{-1}(H) = \widehat{H} \cdot \Gamma$ is dense in $\widetilde{G}$ (Exercise 11.2.8). Therefore, Γ is nontrivial, so that G is not simply connected. We therefore have to take a closer look at how discrete central subgroups of $\widetilde{G}$ can be positioned with respect to $\widehat{H}$.

This is a situation that often occurs in Lie theory. One wants to solve a problem for general Lie groups, and one already controls it for simply connected Lie groups. Then one studies what happens after factorization of discrete central subgroups. As we already know from Lemma 13.2.10, the subgroups which appear here are always finitely generated, so that we shall need some of results in Appendix 13.6 on finitely generated abelian groups.

Lemma 13.5.2. *Let G be a 1-connected Lie group, $\mathfrak{k} \subseteq \mathfrak{g} = \mathbf{L}(G)$ be a maximal compactly embedded subalgebra, and D be a discrete central subgroup. Then*

$$\operatorname{rank}\big(D \cap \exp \mathfrak{z}(\mathfrak{k})\big) = \operatorname{rank} D.$$

Proof. Let $K := \langle \exp_G \mathfrak{k} \rangle$. First we recall that $D \subseteq Z(K)$ (Theorem 13.2.8), and note that K is 1-connected (Theorem 13.3.7), so that $K \cong K' \times Z(K)_0$, where K' is the commutator subgroup of K. The group K' is semisimple, hence compact with finite center (Theorem 11.1.17) and $Z(K)_0 = \exp \mathfrak{z}(\mathfrak{k})$ is a vector group. Now we apply Corollary 13.6.7 to the projection $\alpha \colon D \to K'$. Then $\alpha(D) \subseteq Z(K')$ is finite, so that

$$\operatorname{rank} D = \operatorname{rank} \alpha(D) + \operatorname{rank} \ker \alpha = \operatorname{rank}\big(D \cap Z(K)_0\big). \qquad \square$$

Theorem 13.5.3. *For a dense integral subgroup H of the Lie group G, the following assertions hold:*

1. *The commutator groups G' of G and H' of H coincide.*
2. *Let $\mathfrak{k}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$. Then there exists a subalgebra $\mathfrak{u} \subseteq \mathfrak{z}(\mathfrak{k})$ complementing $\mathfrak{h}$ such that $U := \exp_G \mathfrak{u}$ is a torus satisfying $G = UH$.*
3. *$\mathfrak{g} \cong \mathfrak{h} \rtimes \mathfrak{u}$.*
4. *$\overline{U \cap H} = U$.*
5. *If $S \subseteq G$ is a closed subgroup containing U, then $\overline{S \cap H} = S$.*
6. *There exists a vector group $V \subseteq H$ such that $\overline{V}$ is a torus with $\overline{V}H = G$.*

Proof. (1) Let $q_G \colon \widetilde{G} \to G$ be the universal covering morphism of G with $\mathbf{L}(q_G) = \mathrm{id}_{\mathfrak{g}}$, $D := \ker q_G \subseteq Z(\widetilde{G})$, and $\widetilde{H} := \langle \exp_{\widetilde{G}} \mathfrak{h} \rangle$, which by Lemma 13.5.1 is a closed simply connected normal subgroup by the Smooth Splitting Theorem 10.1.21. Then $B := \overline{D\widetilde{H}}$ is a closed subgroup of $\widetilde{G}$ containing D. Since q_G is a quotient map and B contains the kernel of q_G, $q_G(B)$ is closed in G. As it contains the dense subgroup $H = q_G(\widetilde{H})$, we obtain $q_G(B) = G$, and therefore $B = \widetilde{G}$. From this we derive that every commutator of elements of $\widetilde{G}$ is a limit of commutators of elements of $D\widetilde{H}$, but since D is central, all these commutators are contained in $\widetilde{H}'$. Using Theorem 10.1.21 and Proposition 10.2.4, we see that $\widetilde{H}'$ is closed in $\widetilde{G}$, so that the preceding argument leads to $\widetilde{G}' = \widetilde{H}'$. Another application of Proposition 10.2.4 now yields $\mathfrak{g}' = \mathfrak{h}'$, and thus $G' = H'$.

(2), (3) From $\mathfrak{g}' = \mathfrak{h}'$ it follows in particular that the quotient group $\widetilde{G}/\widetilde{H}$ is abelian. By Theorem 10.1.21 it is even isomorphic to a vector space. Let $p\colon \widetilde{G} \to \widetilde{G}/\widetilde{H}$ be the quotient map. Then the group $p(D) = p(D\widetilde{H})$ is dense in $p(\widetilde{G})$. We recall from Theorem 13.2.8 that $D \subseteq \exp_{\widetilde{G}}(\mathfrak{k})$. Hence the integral subgroup $p(\exp_{\widetilde{G}} \mathfrak{k}) = \exp_{\widetilde{G}/\widetilde{H}}(\mathbf{L}(p)\mathfrak{k})$ is dense in the vector space $\widetilde{G}/\widetilde{H}$, and since integral subgroups of vector spaces are linear subspaces, we have $p(\exp_{\widetilde{G}} \mathfrak{k}) = \widetilde{G}/\widetilde{H}$. This in turn implies that $\mathfrak{k} + \mathfrak{h} = \mathfrak{g}$.

Using $\mathfrak{k}' \subseteq \mathfrak{g}' \subseteq \mathfrak{h}$, we find that $\mathfrak{z}(\mathfrak{k}) + \mathfrak{h} = \mathfrak{k} + \mathfrak{h} = \mathfrak{g}$. Furthermore, the group $D_1 := D \cap \exp_{\widetilde{G}} \big(\mathfrak{z}(\mathfrak{k})\big)$ has finite index in D by Lemma 13.5.2. Therefore $p(D_1)$ spans the vector space $\widetilde{G}/\widetilde{H}$ (Corollary 13.6.8), and we thus find elements $x_1, \ldots, x_n \in \mathfrak{z}(\mathfrak{k})$ with $\exp_{\widetilde{G}}(x_i) \in D_1$ such that $p(\exp_{\widetilde{G}} x_1), \ldots, p(\exp_{\widetilde{G}} x_n)$ form a basis for $\widetilde{G}/\widetilde{H}$. Let $\mathfrak{u} := \operatorname{span}\{x_1, \ldots, x_n\}$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{u}$ is a direct vector sum, and $\mathfrak{g} \cong \mathfrak{h} \rtimes \mathfrak{u}$ follows from the fact that $\mathfrak{u}$ is a subalgebra and that $\mathfrak{h}$ is an ideal. The group $U := \exp_G \mathfrak{u}$ is a continuous image of the torus $\mathfrak{u}/(\mathbb{Z}x_1 + \ldots + \mathbb{Z}x_n)$, and therefore a torus, which clearly satisfies $G = HU = UH$.

(4) Let H be endowed with its intrinsic Lie group structure. Proposition 10.1.19 and (3) imply that $\widetilde{G} \cong \widetilde{H} \rtimes \widetilde{U}$, where $q_U\colon \widetilde{U} \to U$ denotes the simply connected covering of U. From the smoothness of the action of $\widetilde{U}$ on $\widetilde{H}$, we now derive that the conjugation action of the torus U on H is also smooth, so that we can form the semidirect product $H \rtimes U$ and obtain a surjective smooth homomorphism $\mu\colon H \rtimes U \to G, (h,u) \mapsto hu$ which induces an isomorphism

$$(H \rtimes U)/\ker\mu \cong G$$

by the Open Mapping Theorem 10.1.8. This entails that the inverse image $\mu^{-1}(H) = H \rtimes (U \cap H)$ is dense in $H \rtimes U$, which implies that $H \cap U$ is dense in U.

(5) S contains U, so in view of (4), $G = HU$ leads to $S = (H \cap S)U \subseteq (H \cap S)\overline{H \cap U} \subseteq \overline{H \cap S}$.

(6) Let T be a maximal torus of G containing U, and $\mathfrak{t} := \mathbf{L}(T)$ be its Lie algebra. Then $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{u}$ yields $\mathfrak{t} = \mathfrak{u} \oplus (\mathfrak{h} \cap \mathfrak{t})$. The abelian subgroup $\exp_H(\mathfrak{h} \cap \mathfrak{t})$ is a direct product of a torus T_1 with Lie algebra $\mathfrak{t}_1 \subseteq \mathfrak{h} \cap \mathfrak{t}$ and a vector group V. We set $\mathfrak{v} := \mathbf{L}(V) \subseteq \mathfrak{h} \cap \mathfrak{t}$. The normality of H implies that the product set $\overline{V}H$ is a subgroup of G. Therefore it suffices to show that it contains U, hence coincides with G.

Clearly $\overline{V}H$ contains $\overline{\exp_G(\mathfrak{h} \cap \mathfrak{t})} = \exp_G(\mathfrak{t}_1)\overline{\exp_G(\mathfrak{v})} = T_1\overline{V}$. Here we use that products of closed and compact subsets are closed (Exercise 12.3.5). Thus it remains to prove that $U \subseteq \overline{\exp_G(\mathfrak{h} \cap \mathfrak{t})}$. By (4), it suffices to show that $U \cap H \subseteq \exp_G(\mathfrak{h} \cap \mathfrak{t})$. So let $x \in \mathfrak{u}$ with $\exp_G x \in H$. Then there exists an element $d \in D$ with $\exp_{\widetilde{G}}(x)d \in \widetilde{H}$. By Corollary 13.2.10(iii), $D \subseteq \exp_{\widetilde{G}}(\mathfrak{t})$, so that there exists a $y \in \mathfrak{t}$ with $d = \exp_{\widetilde{G}}(y)$. Consequently, $\exp_{\widetilde{G}}(x + y) = \exp_{\widetilde{G}}(x)d \in \widetilde{H}$. Recall that we also have $\widetilde{G} = \widetilde{H} \rtimes \widetilde{U}$. Now projection onto $\widetilde{U}$, together with the injectivity of the exponential function of $\widetilde{U}$ implies $x+y \in \mathfrak{h}$. Hence $\exp_G(x) = \exp_G(x + y) \in U \cap H$. This proves our claim. □

Corollary 13.5.4. *If $H \subseteq G$ is a dense integral subgroup, then $HT = G$ holds for every maximal torus T of G.*

Proof. Since maximal tori of G are conjugate under $\mathrm{Aut}(G)$ (Corollary 13.1.4), and the torus $\overline{V}$ from Theorem 13.5.3(6) is contained in a maximal one, the claim follows. □

Corollary 13.5.5. *The maximal compact subgroups/tori of G are conjugate under H.*

Proof. We adopt the notation from Theorem 13.5.3. Let U be a maximal compact subgroup of G contained in $K := \langle \exp \mathfrak{k} \rangle$. Then we obtain all other maximal compact subgroups as gUg^{-1} with $g \in G$ (Theorem 13.1.3). From Theorem 13.5.3(2) we derive in particular $G = HZ(K)$, and conjugation with elements of $Z(K)$ fixes K. This proves that H acts transitively by conjugation on the set of all maximal compact subgroups.

The corresponding assertion for maximal tori follows similarly because K contains a maximal torus T of G, and this torus is centralized by $Z(K)$. □

From the preceding results we now derive the following criteria for integral subgroups to be closed:

Corollary 13.5.6 (Closed Integral Subgroups – Criteria). *Let G be a Lie group, H be an integral subgroup of G with Lie algebra $\mathfrak{h}$ and $T \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{t}$. Then the following conditions imply that H is closed:*

(a) *H contains T.*
(b) *$\mathfrak{h}$ intersects the Lie algebra of each torus in G trivially.*
(c) *H is normal and $(H \cap T)_0 = \exp_G(\mathfrak{h} \cap \mathfrak{t})$ is closed.*
(d) *For each $x \in \mathfrak{h}$, the closure of $\exp_G(\mathbb{R}x)$ is contained in H (Malcev's Criterion).*
(e) *There exists a maximal compactly embedded abelian subalgebra $\mathfrak{a}$ of $\mathfrak{h}$ for which $\exp_G(\mathfrak{a})$ is closed in G.*

Proof. (a) If $T \subseteq H$, then T also is a maximal torus of the closed subgroup $\overline{H}$ of G. Since maximal tori of $\overline{H}$ are conjugate under H (Corollary 13.5.5), and the torus $\overline{V}$ from Theorem 13.5.3(6) is contained in a maximal one, we obtain $H = HT \supseteq H\overline{V} = \overline{H}$.

(b) Let $V \subseteq H$ be as above an abelian integral subgroup for which $\overline{V}$ is a torus with $\overline{H} = H\overline{V}$. Since $\mathbf{L}(H)$ intersects the Lie algebras of all tori in

G trivially, we obtain $\mathbf{L}(V) \subseteq \mathbf{L}(H) \cap \mathbf{L}(\overline{V}) = \{0\}$, so that $V = \{\mathbf{1}\}$ leads to $\overline{H} = H$.

(c) Since all maximal tori are conjugate under the group of inner automorphisms (Corollary 13.1.4), H intersects *any* maximal torus of G in a closed subgroup. In view of Theorem 13.5.3(6), there exists a subgroup $V \subseteq H$ for which $\overline{V}$ is a torus and $\overline{H} = H\overline{V}$. Let $T \subseteq G$ be a maximal torus containing $\overline{V}$. Then $\mathbf{L}(V) \subseteq \mathbf{L}(H) \cap \mathbf{L}(T)$, and since $\exp_G(\mathbf{L}(H) \cap \mathbf{L}(T))$ is closed $\overline{V} \subseteq H \cap T$, which leads to $\overline{H} = H$.

(d) Suppose that H is not closed, and w.l.o.g. dense in G. We choose a vector group $V \subseteq H$ such that $\overline{V}$ is a torus with $G = \overline{H\overline{V}}$ (Theorem 13.5.3(6)). If $v_1, \ldots, v_n$ is a basis for V, then the subgroups $\overline{\exp_G(\mathbb{R}v_i)}$ are tori whose product is $\overline{V}$. We conclude that there exists some i for which $\overline{\exp_G(\mathbb{R}v_i)}$ is not contained in H.

(e) We may w.l.o.g. assume that H is dense in G. If H is not closed, then Theorem 13.5.3(6) implies the existence of an $x \in \mathfrak{h}$ such that $T := \overline{\exp_G(\mathbb{R}x)}$ is a torus with $\overline{T \cap H} \not\subseteq H$. Now

$$\mathrm{Inn}_{\mathfrak{h}}(\mathbb{R}x) \subseteq \mathrm{Inn}_{\mathfrak{g}}(T)|_{\mathfrak{h}} = \mathrm{Ad}(T)|_{\mathfrak{h}}$$

implies that $x \in \mathrm{comp}(\mathfrak{h})$. We extend $\mathbb{R}x$ to a maximal compactly embedded, abelian subalgebra $\mathfrak{a}'$ of $\mathfrak{h}$. By Exercise 13.5.1, there exists an $h \in H$ with $\mathrm{Ad}(h)\mathfrak{a} = \mathfrak{a}'$. Hence $\exp_G(\mathfrak{a}')$ is closed, a contradiction. ◻

Now we turn to some important consequences of these results.

Corollary 13.5.7. *If $\varphi\colon G \to \mathrm{GL}_n(\mathbb{R})$ is a representation of the connected semisimple Lie group G, then $\varphi(G)$ is closed.*

Proof. We may w.l.o.g. assume that φ is injective. Let $\mathfrak{a} \subseteq \mathfrak{g} = \mathbf{L}(G)$ be a maximal compactly embedded abelian subalgebra. Then there exists a maximal compactly embedded subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ containing $\mathfrak{a}$. By Corollary 12.2.6, the image $\varphi(K)$ of $K := \langle \exp_G \mathfrak{k} \rangle$ is compact. From that it follows that K is compact because K is closed in G by Lemma 13.2.6, so that $\varphi|_K$ is an isomorphism of Lie groups. Since $\exp_K \mathfrak{a}$ is closed in K (Lemma 13.2.6), also $\varphi(\exp_G \mathfrak{a})$ is closed, and we apply Corollary 13.5.6(e) to finish the proof. ◻

Lemma 13.5.8. *If H is a proper dense integral subgroup of the Lie group G, then the quotient group G/H is uncountable.*

Proof. Let $q_G\colon \widetilde{G} \to G$ be the universal covering of G. We set $\widetilde{H} := \langle \exp_{\widetilde{G}} \mathbf{L}(H) \rangle$. Then the groups G/H and $\widetilde{G}/q_G^{-1}(H)$ are isomorphic. We have $\widetilde{G} \cong \widetilde{H} \rtimes V$ for a vector group V (Theorem 13.5.3(3)) and $q_G^{-1}(H) = (\ker q_G)\widetilde{H}$. Therefore $\widetilde{G}/q_G^{-1}(H) \cong V/V \cap (\ker q_G\widetilde{H})$. The group $V \cap (\ker q_G\widetilde{H})$ is the projection of the countable group $\ker q_G \cong \pi_1(G)$ onto V (Corollary 13.2.10), hence countable. If $H \neq G$, then $V \neq \{0\}$, and the quotient G/H is uncountable because of the uncountability of V. ◻

Theorem 13.5.9. *If H is a semisimple integral subgroup of the Lie group G, and G is either simply connected or compact, then H is closed.*

Proof. First we assume that G is simply connected. Choose a Levi subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ containing $\mathfrak{h} := \mathbf{L}(H)$ (Corollary 4.6.14). Then $S := \langle \exp_G \mathfrak{s} \rangle$ is a semidirect factor by Proposition 10.1.19. In particular, S is closed and simply connected. Because of $H \subseteq S$, we may thus assume that $G = S$ is semisimple. By Corollary 13.5.7, $\mathrm{Ad}(H)$ is closed in $\mathrm{Aut}(\mathfrak{g})$, and therefore it is also closed in $\mathrm{Ad}(G)$. Hence the preimage $\mathrm{Ad}^{-1}\big(\mathrm{Ad}(H)\big) = HZ(G)$ is also closed in G, and consequently $\overline{H}/H$ is countable because $Z(G) \cong \pi_1\big(\mathrm{Ad}(G)\big)$ is countable (Corollary 13.2.10). Therefore H is closed by Lemma 13.5.8.

Now we assume that G is compact. Then $\mathbf{L}(G)$ is compact, and $\mathbf{L}(H)$ inherits this property. Since H is compact by Theorem 11.1.17, it is closed. □

Example 13.5.10. We describe an example of a connected Lie group whose commutator group is semisimple and dense. First, we recall that

$$Z := Z(\mathrm{SL}_2(\mathbb{R})\widetilde{\ }) \cong \mathbb{Z},$$

which follows easily by applying Theorem 13.2.8 to the maximal compactly embedded abelian subalgebra $\mathfrak{so}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{R})$. Let $\alpha\colon Z \cong \mathbb{Z} \to \mathbb{T}^2$ be a homomorphism with dense image (cf. Exercise 11.2.10). Consider the group $G := \big(\mathrm{SL}_2(\mathbb{R})\widetilde{\ } \times \mathbb{T}^2\big)/D$, where $D = \{(z, \alpha(z))\colon z \in Z\}$ is the graph of α, a discrete central subgroup of the product group. Then G is a 5-dimensional Lie group containing $G' \cong \mathrm{SL}_2(\mathbb{R})\widetilde{\ }$ as a dense integral subgroup.

Proposition 13.5.11. *If $\mathfrak{k} \subseteq \mathfrak{g}$ is a subalgebra contained in* $\mathrm{comp}(\mathfrak{g})$*, i.e., consisting of compact elements, then $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}$.*

Proof. Let $\mathfrak{k}_0 = \{0\} \subseteq \mathfrak{k}_1 \subseteq \ldots \subseteq \mathfrak{k}_n = \mathfrak{k}$ be a Jordan-Hölder series of $\mathfrak{k}$ (Remark 4.6.12). By induction on the dimension, we show that every ideal $\mathfrak{k}_i$ of $\mathfrak{k}$ is compactly embedded. For $i = 0$, there is nothing to show. Assume that $\mathfrak{k}_i$ is compactly embedded for some $i < n$. Then there exists a $\mathfrak{k}_i$-invariant complement $\mathfrak{b}_i$ in $\mathfrak{k}_{i+1}$, and this implies that $[\mathfrak{k}_i, \mathfrak{b}_i] \subseteq \mathfrak{b}_i \cap \mathfrak{k}_i = \{0\}$. Two cases occur:

Case 1: $\mathfrak{k}_{i+1}/\mathfrak{k}_i \cong \mathbb{R}$. Then $\mathfrak{k}_{i+1} \cong \mathfrak{k}_i \oplus \mathfrak{b}_i$, and therefore

$$\mathrm{Inn}_{\mathfrak{g}} \mathfrak{k}_{i+1} \subseteq (\mathrm{INN}_{\mathfrak{g}} \mathfrak{k}_i)(\mathrm{INN}_{\mathfrak{g}} \mathfrak{b}_i)$$

is relatively compact because $\mathfrak{b}_i \cong \mathbb{R}$ is compactly embedded by assumption.

Case 2: $\mathfrak{b} := \mathfrak{k}_{i+1}/\mathfrak{k}_i$ is simple. We claim that $\mathfrak{b}$ is compact. By Levi's Theorem 4.6.6, there exists a subalgebra $\mathfrak{s}$ of $\mathfrak{k}$ isomorphic to $\mathfrak{b}$ with $\mathfrak{k}_{i+1} = \mathfrak{k}_i \rtimes \mathfrak{s}$. If $\mathfrak{s} = \mathfrak{k}_{\mathfrak{s}} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{s}$, then no nonzero element of $\mathfrak{p}$ is compact (Proposition 12.1.5). Thus $\mathfrak{s} = \mathfrak{k}_{\mathfrak{s}}$ is compact, and therefore

$$\mathrm{INN}_{\mathfrak{g}} \mathfrak{k}_{i+1} \subseteq (\mathrm{INN}_{\mathfrak{g}} \mathfrak{k}_i)(\mathrm{Inn}_{\mathfrak{g}} \mathfrak{k}_{\mathfrak{s}})$$

is compact because both factors on the right are compact (Corollary 12.2.6). □

Theorem 13.5.12. *Let H be a dense integral subgroup of the Lie group G, and let $\overline{\exp(\mathbb{R}x)}$ be compact for all $x \in \mathfrak{h}$, then G is compact.*

Proof. By Proposition 13.5.11, $\mathfrak{h}$ is compactly embedded in $\mathfrak{g}$, and

$$\overline{\mathrm{Ad}(G)} = \overline{\mathrm{Ad}(H)}$$

thus is compact. Hence $\mathfrak{g}$ is a compact Lie algebra, and G is a direct product of a vector group V and a compact group K (Theorem 11.1.18). Let $p_V : G \to V$ be the projection onto the direct factor V. Then $p_V(H)$ is dense, and the closure of any one-parameter subgroup of H is compact in G. This implies $V = \{0\}$. ☐

Exercises for Section 13.5

Exercise 13.5.1. Prove the following strengthening of Theorem 13.2.7(a): Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be two maximal compactly embedded (abelian) subalgebras of $\mathfrak{g}$. Then there exists a $\gamma \in \mathrm{Inn}\,\mathfrak{g}$ with $\gamma(\mathfrak{k}_1) = \mathfrak{k}_2$. Hint: Corollary 13.5.5 and the proof of Theorem 13.2.7.

Exercise 13.5.2. Let $\gamma : \mathbb{R} \to \mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| = |z_2| = 1\}$ be a homomorphism with dense image. In the following, we consider $\mathbb{C}^2$ as an algebra with the componentwise multiplication $(z_1, z_2)(w_1, w_2) = (z_1w_1, z_2w_2)$. Show that:

(a) The map $\mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2 : (t, x) \mapsto \gamma(t)x$ defines an action of $\mathbb{R}$ on $\mathbb{C}^2$.
(b) $G := \mathbb{C}^2 \rtimes_\gamma \mathbb{R}$ is a solvable Lie group of dimension 5, and $Z(G) = \{\mathbf{1}\}$.
(c) $\mathfrak{k} := \{0\} \times \mathbb{R}$ is a maximal compactly embedded subalgebra and a Cartan subalgebra of $\mathbf{L}(G) = \mathbb{C}^2 \rtimes_{\mathbf{L}(\gamma)} \mathbb{R}$.
(d) The map $\alpha(z)(x, t) := (zx, t)$ defines a homomorphism $\alpha \colon \mathbb{T}^2 \to \mathrm{Aut}(G)$.
(e) $\alpha(\mathbb{T}^2) \subseteq \mathrm{INN}_\mathfrak{g}\,\mathfrak{k}$, but $\alpha(\mathbb{T}^2) \not\subseteq \mathrm{Inn}_\mathfrak{g}\,\mathfrak{k}$.

Exercise 13.5.3. Let H_1, H_2 be integral subgroups of the Lie group G such that H_2 normalizes H_1. Show that H_1H_2 is an integral subgroup of G with

$$\mathbf{L}(H_1H_2) = \mathbf{L}(H_1) + \mathbf{L}(H_2).$$

Exercise 13.5.4 (Non-closed commutator groups). Define a central extension G of $\mathbb{R}^2$ by $\mathbb{T}^2$ via

$$(t_1, t_2, x_1, x_2)(t_1', t_2', x_1', x_2') := (t_1t_1'e^{ix_1x_2'}, t_2t_2'e^{i\sqrt{2}x_1x_2'}, x_1 + x_1', x_2 + x_2')$$

and show that the commutator group G' of the nilpotent group G is a dense wind in $Z(G) \cong \mathbb{T}^2$.

13.6 Appendix: Finitely Generated Abelian Groups

Lemma 13.6.1. *Consider the action of the group* $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^n)$ *on* $\mathbb{Z}^n$ *by* $(g, \mathbf{z}) \mapsto g\mathbf{z}$*, where we write the elements* $\mathbf{z} \in \mathbb{Z}^n$ *as column vectors.*

If $d := \gcd(z_1, \ldots, z_n)$*, then the orbit* $\mathcal{O}_{\mathbf{z}} := \mathrm{GL}_n(\mathbb{Z})\mathbf{z}$ *contains the element* de_1 *and the orbit* $\mathcal{O}_{\mathbf{z}}$ *meets* $\mathbb{N}_0 e_1$ *only in this point, so that the orbits are classified by the invariant function*

$$\gcd\colon \mathbb{Z}^n \to \mathbb{N}_0.$$

Proof. For $0 \neq \mathbf{z} \in \mathbb{Z}^n$ we have

$$\gcd(\mathbf{z}) = \max\{m \in \mathbb{N}\colon \tfrac{1}{m}\mathbf{z} \in \mathbb{Z}^n\}.$$

For $g \in \mathrm{GL}_n(\mathbb{Z})$, the condition $\frac{1}{m}g\mathbf{z} \in \mathbb{Z}^n$ is equivalent to $\frac{1}{m}\mathbf{z} \in \mathbb{Z}^n$. We therefore have

$$\gcd(g\mathbf{z}) = \gcd(\mathbf{z}) \quad \text{for} \quad g \in \mathrm{GL}_n(\mathbb{Z}), \mathbf{z} \in \mathbb{Z}^n,$$

i.e., the function gcd is invariant.

For $i \neq j$ and $k \in \mathbb{Z}$, the matrix $g := \mathbf{1} + kE_{ij} \in \mathrm{GL}_n(\mathbb{Z})$ is invertible with $g^{-1} = \mathbf{1} - kE_{ij}$, and

$$\mathbf{z}' := g\mathbf{z} = \mathbf{z} + kE_{ij}\mathbf{z} = \mathbf{z} + kz_j e_i.$$

Therefore $z'_\ell = z_\ell$ for $\ell \neq i$ and $z'_i = z_i + kz_j$. This means that subtracting the k-fold multiple of one entry of $\mathbf{z}$ from another entry leads to an element in the same $\mathrm{GL}_n(\mathbb{Z})$-orbit.

If z_i is minimal and positive, then we may repeat this process to obtain a vector $\mathbf{z}'$ with $0 \leq z'_j < |z_i|$ for all $j \neq i$ and $z'_j = z_j$. If there exists a $j \neq i$ with $0 < z'_j$, then we repeat this procedure with j instead of i. Since the maximal absolute value of the entries decreases at least by one in each step, the procedure stops when we have achieved a vector $\mathbf{z}''$ with at most one nonzero entry. After multiplication with a permutation matrix we thus arrive at $\mathbf{z}'' = ke_1$, and since $-\mathbf{1} \in \mathrm{GL}_n(\mathbb{Z})$, we may assume that $k \in \mathbb{N}_0$. Then $k = \gcd(\mathbf{z})$, which shows that $\gcd(\mathbf{z})e_1 = \mathbf{z}'' \in \mathcal{O}_{\mathbf{z}}$.

For $\mathbf{z} = ke_1$ and $k \in \mathbb{N}$, we have $\gcd(\mathbf{z}) = k$, so that we obtain for different $k \in \mathbb{N}_0$ different orbits $\mathcal{O}_{ke_1}$. □

Proposition 13.6.2 (Subgroups of $\mathbb{Z}^n$). *For each subgroup* $\Gamma \leq \mathbb{Z}^n$*, there exists a* $g \in \mathrm{GL}_n(\mathbb{Z})$ *and a sequence of natural numbers*

$$d_1|d_2|\ldots|d_r, \quad \textit{i.e.,} \quad d_{i+1} \in d_i\mathbb{Z}, \quad r \leq n,$$

such that

$$g\Gamma = \mathbb{Z}d_1e_1 + \mathbb{Z}d_2e_2 + \ldots + \mathbb{Z}d_re_r.$$

In particular, $\Gamma \cong \mathbb{Z}^r$.

Proof. We argue by induction on n. For $n = 0$, we have $\mathbb{Z}^n = \{0\}$, and the assertion is trivial.

Let $n > 0$. If $\Gamma = \{0\}$, then we take $g = \mathbf{1}$ and put $r := 0$. If $\Gamma \neq \{0\}$, then let

$$d_1 := \min\{\gcd(\mathbf{z}) \colon 0 \neq \mathbf{z} \in \Gamma\}$$

and $\mathbf{z}_1 \in \Gamma$ with $d_1 = \gcd(\mathbf{z}_1)$. In view of Lemma 13.6.1, there exists a $g_1 \in \mathrm{GL}_n(\mathbb{Z})$ with $g_1\mathbf{z}_1 = d_1e_1$. We consider the subgroup $\Gamma_1 := g_1\Gamma$. It contains d_1e_1 and

$$d_1 = \min\{\gcd(\mathbf{z}) \colon 0 \neq \mathbf{z} \in \Gamma_1\}.$$

If $\mathbf{z} \in \Gamma_1 \setminus \mathbb{Z}d_1e_1$, then $\gcd(\mathbf{z} + kd_1e_1) \geq d_1$ for each $k \in \mathbb{Z}$, which implies that the first component $z_1 + kd_1$ of this vector never is contained in the set $\{1, \ldots, d_1 - 1\}$. Therefore, $z_1 \in \mathbb{Z}d_1$ and $z_1e_1 \in \Gamma_1$, hence

$$\mathbf{z} - z_1e_1 \in \Gamma_1 \cap \langle e_2, \ldots, e_n\rangle, \quad \langle e_2, \ldots, e_n\rangle \cong \mathbb{Z}^{n-1}.$$

For $\Gamma_2 := \Gamma_1 \cap \langle e_2, \ldots, e_n\rangle$ we thus obtain $\Gamma_1 = \mathbb{Z}d_1e_1 + \Gamma_2$, and $\mathbb{Z}d_1e_1 \cap \Gamma_2 = \{0\}$ implies that this sum is direct: $\Gamma_1 \cong \mathbb{Z}d_1e_1 \oplus \Gamma_2$. We now apply the induction hypothesis to the subgroup Γ_2 of $\mathbb{Z}^{n-1}$ and obtain a matrix $g' \in \mathrm{GL}_{n-1}(\mathbb{Z})$ and numbers $d_2, \ldots, d_r$ with

$$g'\Gamma_2 = \mathbb{Z}d_2e_2 + \ldots + \mathbb{Z}d_re_r \quad \text{and} \quad d_2|d_3|\cdots|d_r.$$

For the matrix

$$g_2 := \begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

we then obtain

$$g_2g_1\Gamma = g_2\Gamma_1 = g_2(\mathbb{Z}d_1e_1 + \Gamma_2) = \mathbb{Z}d_1e_1 + g'\Gamma_2 = \mathbb{Z}d_1e_1 + \mathbb{Z}d_2e_2 + \ldots + \mathbb{Z}d_re_r.$$

Our construction implies that $d_1 \leq \gcd(d_1e_1 + d_2e_2) = \gcd(d_1, d_2)$ and therefore $d_1 = \gcd(d_1, d_2)$ implies $d_1|d_2$. This completes the proof. □

Theorem 13.6.3 (Structure Theorem for Finitely Generated Abelian Groups). *For every finitely generated abelian group A, there exist $k, r \in \mathbb{N}_0$ and $d_1, \ldots, d_k \in \mathbb{N}$ with*

$$d_1|d_2|\ldots|d_k,$$

such that

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_k\mathbb{Z}.$$

Proof. Let $\{g_1, \ldots, g_n\} \subseteq A$ be a finite set of generators. Then

$$\Phi \colon \mathbb{Z}^n \to A, \quad (z_1, \ldots, z_n) \mapsto z_1g_1 + \ldots + z_ng_n$$

is a surjective group homomorphism. Its kernel $\Gamma := \ker\Phi$ is a subgroup of $\mathbb{Z}^n$, and

$$A = \mathrm{im}\,(\varphi) \cong \mathbb{Z}^n/\ker\Phi = \mathbb{Z}^n/\Gamma.$$

According to Proposition 13.6.2, there exists a $g \in \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}^n)$ and a sequence $d_1|d_2|\ldots|d_k$ of natural numbers with

$$g\Gamma = \mathbb{Z}d_1e_1 + \mathbb{Z}d_2e_2 + \ldots + \mathbb{Z}d_ke_k.$$

Then the map

$$\Psi \colon \mathbb{Z}^n/\Gamma \to \mathbb{Z}^n/g\Gamma, \quad z + \Gamma \mapsto g(z+\Gamma) = gz + g\Gamma$$

is an isomorphism of groups, and

$$\begin{aligned}\mathbb{Z}^n/g\Gamma &= (\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_n)/(\mathbb{Z}d_1e_1 + \mathbb{Z}d_2e_2 + \ldots + \mathbb{Z}d_ke_k)\\ &\cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \mathbb{Z}^{n-k}.\end{aligned}$$

This proves the assertion with $r = n - k$. □

Corollary 13.6.4. *For every finitely generated abelian group A, there exists an $r \in \mathbb{N}_0$ with $A \cong \mathbb{Z}^r \oplus \mathrm{tor}(A)$, where*

$$\mathrm{tor}(A) := \{a \in A \colon (\exists n \in \mathbb{N})\ na = 0\}$$

is a finite subgroup and

$$r = \mathrm{rk}(A) := \mathrm{rank}\,(A) := \max\{k \in \mathbb{N}_0 \colon \mathbb{Z}^k \text{ embeds into } A\}$$

does not depend on the decomposition.

Proof. From the Structure Theorem 13.6.3 we derive that $A \cong \mathbb{Z}^r \oplus F$, where F is finite. Then each element of F is of finite order, and since $\mathbb{Z}^r$ contains no nonzero element of finite order, $F = \mathrm{tor}(A)$. We then have $\mathbb{Z}^r \cong A/\mathrm{tor}(A)$, so that it remains to observe that if there exists an injection $\varphi \colon \mathbb{Z}^k \to \mathbb{Z}^r$, then $k \leq r$ (Proposition 13.6.2), so that r is the maximal number k for which $\mathbb{Z}^k$ embeds into A. □

Definition 13.6.5. The number $r = \mathrm{rk}(A) = \mathrm{rank}\,(A)$ in the preceding corollary is called the *rank of the finitely generated abelian group* A, and $\mathrm{tor}(A)$ is called the *torsion subgroup*.

Proposition 13.6.6. *Any subgroup S of a finitely generated abelian group A is finitely generated, and we have the rank formula*

$$\mathrm{rk}\, A = \mathrm{rk}\, S + \mathrm{rk}(A/S).$$

In particular, $\mathrm{rk}\, S = \mathrm{rk}\, A$ *is equivalent to A/S being finite.*

Proof. Let $r := \mathrm{rk}\, A$, $s := \mathrm{rk}\, S$, $q \colon \mathbb{Z}^n \to A$ be a surjective homomorphism and $\widehat{S} := q^{-1}(S)$. Then we use Proposition 13.6.2 to see that we may w.l.o.g. assume that there exist natural numbers $d_1|\cdots|d_m$, $m \leq n$, with

$$\widehat{S} = \mathbb{Z}d_1e_1 + \cdots + \mathbb{Z}d_me_m \subseteq \mathbb{Z}^n.$$

Note that

$$A/S = q(\mathbb{Z}^n)/q(\widehat{S}) \cong \mathbb{Z}^n/\widehat{S} \cong \operatorname{tor}(A/S) \oplus \mathbb{Z}^{n-m}.$$

Applying Proposition 13.6.2 to the subgroup $\Gamma := \ker q$ of $\widehat{S} \cong \mathbb{Z}^m$, we obtain a basis $e_1', \ldots, e_m'$ for $\widehat{S}$ and $d_1', \ldots, d_k'$ with

$$\Gamma = \mathbb{Z}d_1'e_1' + \ldots + \mathbb{Z}d_k'e_k'.$$

We conclude that $S = q(\widehat{S}) \cong \widehat{S}/\Gamma \cong \operatorname{tor}(S) \oplus \mathbb{Z}^{m-k}$, and finally, $A \cong \mathbb{Z}^n/\Gamma \cong \operatorname{tor}(A) \oplus \mathbb{Z}^{n-k}$. This immediately implies that

$$\operatorname{rk}(A) = n - k = n - m + m - k = \operatorname{rk}(A/S) + \operatorname{rk}(S). \qquad \square$$

Corollary 13.6.7. *For a surjective homomorphism $\alpha\colon A_1 \to A_2$ of finitely generated abelian groups, we have*

$$\operatorname{rank} A_1 = \operatorname{rank} A_2 + \operatorname{rank}(\ker \alpha).$$

Proof. In view of $A_2 \cong A_1/\ker \alpha$, this follows from Proposition 13.6.6. $\square$

Corollary 13.6.8. *If D is a discrete subgroup of a finite dimensional real vector space V, then D is finitely generated with*

$$\dim(\operatorname{span} D) = \operatorname{rank} D.$$

Proof. This follows from Exercise 8.3.4. $\square$

Notes on Chapter 13

In Theorem 13.3.12 we have seen a characterization of all Lie groups diffeomorphic to $\mathbb{R}^n$. Such Lie groups play an important role in Riemannian geometry and topology because many of them arise as simply connected coverings of compact manifolds with interesting properties. Such manifolds are of the form G/Γ, where G is a Lie group diffeomorphic to $\mathbb{R}^n$ and Γ is a discrete subgroup which is *cocompact*, i.e., for which G/Γ is compact. Such subgroups are called *lattices* ([Ra72]). In dimension ≤ 2 all such manifolds are tori, so that the situation starts to become interesting in dimension 3, for which a classification can be found in [RV81]. In this case the Lie group G is either the abelian $(\mathbb{R}^3, +)$, the nilpotent 3-dimensional Heisenberg group, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, or a solvable group with a very special structure, which is required for the existence of a lattice.

In view of the Second Manifold Splitting Theorem 13.3.11, more general homogeneous manifolds diffeomorphic to $\mathbb{R}^n$ arise as G/K, where G is a connected Lie group and K a maximal compact subgroup, and one obtains a more general class of compact manifolds by considering those whose universal covering is some G/K. These constructions lie at the heart of the solution of the proof of the Poincaré Conjecture, resp., Thurston's Geometrization Conjecture, because in dimension 3 they provide the geometric building blocks from which, according to Perelman's Theorem, 3-manifolds can be obtained (cf. [Mi03]).

The question whether the exponential function of a given Lie group is surjective has attracted attention even in recent years. Various authors provided new criteria not only for solvable, but also for semisimple and mixed groups (see [DH97], [Wu98], [Wu03], [Wu05], [MS08]). In particular, [Wu98] contains characterizations of connected solvable Lie groups which are not necessarily simply connected as in Dixmier's Theorem 13.4.8.

14

Complex Lie Groups

In this chapter we discuss complex Lie groups. Since we did not go into the theory of complex manifolds, we do this in a quite pedestrian fashion, but this will be enough for our purposes which are of a group theoretic nature. In particular, we define a complex Lie group as a real Lie group G whose Lie algebra $\mathbf{L}(G)$ is a complex Lie algebra and for which the adjoint representation maps into the group $\mathrm{Aut}_{\mathbb{C}}(\mathbf{L}(G))$ of complex linear automorphisms of $\mathbf{L}(G)$ (the latter condition is automatically satisfied if G is connected). One can show that this is equivalent to G carrying the structure of a complex manifold such that the group operations are holomorphic, but we shall never need this additional structure.

First we discuss the passage from real to complex Lie groups. In Section 14.1 we show that for every Lie group G, there exists a universal complexification $\eta_G\colon G \to G_{\mathbb{C}}$, i.e., $G_{\mathbb{C}}$ is a complex Lie group and all other morphisms of G to complex Lie groups factor uniquely through η_G.

One of the most important classes of complex Lie groups are the complexifications of compact Lie groups. These groups are also called linearly complex reductive because all their holomorphic representations are completely reducible, which is one instance of Weyl's Unitary Trick. The theory of these groups and their structure is studied in Section 14.2. Here an interesting issue is that linearly complex reductive Lie groups are reductive in the sense that their Lie algebra is reductive, but not every connected reductive Lie group is linearly complex reductive. A typical example is the additive group $\mathbb{C}$, which has no non-trivial compact subgroup.

As we shall see, the property of being linearly complex reductive is completely decided by the structure of the center. Therefore it is necessary to take a closer look at connected abelian complex Lie groups. Although connected real abelian Lie groups have a quite simple structure because they are products of a torus and a vector space, the structure is substantially richer in the complex case. In this situation we always have a product $G = T^* \times V$, where V is a complex vector space and T^* is the smallest complex integral subgroup containing the unique maximal torus. This factor has a relatively simple structure if $T^* \cong T_{\mathbb{C}}$, because then it is isomorphic to some $(\mathbb{C}^\times)^n \cong \mathbb{T}_{\mathbb{C}}^n$. In general the natural map $T_{\mathbb{C}} \to T^*$ is only surjective, but far from being injective. In Section 14.3 we study this situation in some detail. Interesting examples of complex one-dimensional groups are the elliptic curves $G = \mathbb{C}/\Gamma$, where $\Gamma \cong \mathbb{Z}^2$ is a lattice. Here G is compact, so that $G = T \ncong T_{\mathbb{C}}$. Finally we shall use the insights into the structure of abelian groups to obtain a characterization of the linearly complex reductive Lie groups among the connected reductive ones.

14.1 The Universal Complexification

Definition 14.1.1. (a) A *complex Lie group* is a real Lie group G whose Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ is a complex Lie algebra, and for which $\mathrm{Ad}(G) \subseteq \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$ (the group of complex linear automorphisms of $\mathfrak{g}$). Note that if G is connected, then $\mathrm{Ad}(G) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ automatically consists of complex linear automorphisms of $\mathfrak{g}$.

(b) A homomorphism $\alpha : G_1 \to G_2$ of complex Lie groups is called *holomorphic* if $\mathbf{L}(\alpha)$ is complex linear. It is called *antiholomorphic* if $\mathbf{L}(\alpha)$ is antilinear. If $G_2 = \mathrm{GL}(V)$ for a complex vector space, then a holomorphic homomorphism $\alpha : G_1 \to G_2$ is also called a *holomorphic representation* of G_1 on V.

(c) A subgroup H of a complex Lie group G is called a *complex Lie subgroup* if H is closed and its Lie algebra $\mathbf{L}(H)$ is a complex subspace of $\mathbf{L}(G)$. For a subset M of a complex Lie group G, we write $\langle M \rangle_{\mathbb{C}-\mathrm{grp}}$ for the smallest complex Lie subgroup of G containing M (Exercise 14.1.3).

Definition 14.1.2. Let G be a real Lie group. A pair $(\eta_G, G_{\mathbb{C}})$ of a complex Lie group $G_{\mathbb{C}}$ and a morphism $\eta_G \colon G \to G_{\mathbb{C}}$ of real Lie groups, is called a *universal complexification* of G if for every homomorphism $\alpha \colon G \to H$ to a complex Lie group H, there exists a unique holomorphic homomorphism

$$\alpha_{\mathbb{C}} \colon G_{\mathbb{C}} \to H \quad \text{with} \quad \alpha_{\mathbb{C}} \circ \eta_G = \alpha.$$

This is also called the *universal property* of $G_{\mathbb{C}}$ (compare, for instance, with the enveloping algebra).

Remark 14.1.3 (Uniqueness of Universal Complexifications). As we have already seen in many other contexts, the universal property determines the universal object up to isomorphism. More precisely, suppose that $(\eta_G, G_{\mathbb{C}})$ and $(\eta'_G, G'_{\mathbb{C}})$ are two universal complexifications of the Lie group G. Then the universal property implies the existence of a unique morphism $\alpha \colon G_{\mathbb{C}} \to G'_{\mathbb{C}}$ of complex Lie groups with $\alpha \circ \eta_G = \eta'_G$ and of a unique morphism $\alpha' \colon G'_{\mathbb{C}} \to G_{\mathbb{C}}$ with $\alpha' \circ \eta'_G = \eta_G$. Then $\alpha' \circ \alpha \circ \eta_G = \eta_G$, so that the uniqueness requirement leads to $\alpha' \circ \alpha = \mathrm{id}_{G_{\mathbb{C}}}$, and, likewise, $\alpha \circ \alpha' = \mathrm{id}_{G'_{\mathbb{C}}}$. Therefore $\alpha \colon G_{\mathbb{C}} \to G'_{\mathbb{C}}$ is an isomorphism of complex Lie groups.

Theorem 14.1.4 (Existence of a Universal Complexification). *For each Lie group G, there exists a universal complexification $(\eta_G, G_{\mathbb{C}})$. It has the following properties:*

(i) *If G is 1-connected, then $G_{\mathbb{C}}$ is also 1-connected, and η_G has discrete kernel.*
(ii) *If G is connected, then $\ker \eta_G$ is central in G.*
(iii) *The restriction $\eta_{G_0} := \eta_G|_{G_0} : G_0 \to G_{\mathbb{C},0}$ is a universal complexification of the identity component G_0 and η_G induces an isomorphism $\pi_0(G) \to \pi_0(G_{\mathbb{C}})$.*
(iv) *There exists a unique antiholomorphic involution σ on $G_{\mathbb{C}}$ with $\sigma \circ \eta_G = \eta_G$. The identity component of its group $G_{\mathbb{C}}^\sigma$ of fixed points is $(G_{\mathbb{C}}^\sigma)_0 = \eta_G(G_0)$.*
(v) *The inclusion $\eta_G(G_0) \hookrightarrow (G_{\mathbb{C}})_0$ is a universal complexification of $\eta_G(G_0)$.*

Proof. **Step 1:** First we assume that G is connected with Lie algebra $\mathfrak{g}$. Let $\iota : \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ be the inclusion, $q_G : \widetilde{G} \to G$ be a universal covering map with $\mathbf{L}(q_G) = \mathrm{id}_{\mathfrak{g}}$, and $\widetilde{G}_{\mathbb{C}}$ be a 1-connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then we have a canonical morphism $\eta_{\widetilde{G}} : \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ with $\mathbf{L}(\eta_{\widetilde{G}}) = \iota$. Let $D := \ker q_G$ and $A := \langle \eta_{\widetilde{G}}(D) \rangle_{\mathbb{C}-\mathrm{grp}} \subseteq \widetilde{G}_{\mathbb{C}}$ be the smallest complex Lie subgroup of $\widetilde{G}_{\mathbb{C}}$ containing $\eta_{\widetilde{G}}(D)$. Note that $\eta_{\widetilde{G}}(D) \subseteq \ker \mathrm{Ad}_{\widetilde{G}_{\mathbb{C}}} = Z(\widetilde{G}_{\mathbb{C}})$ implies that A is central, hence in particular normal, so that $G_{\mathbb{C}} := \widetilde{G}_{\mathbb{C}}/A$ carries a natural complex Lie group structure with Lie algebra $\mathfrak{g}_{\mathbb{C}}/\mathbf{L}(A)$. We claim that the induced map

$$\eta_G : G \to G_{\mathbb{C}}, \quad q_G(g) \mapsto \eta_{\widetilde{G}}(g)A$$

is a universal complexification of G.

Let $\alpha : G \to H$ be a homomorphism into a complex Lie group. Then $\mathbf{L}(\alpha) : \mathfrak{g} \to \mathbf{L}(H)$ induces a complex linear homomorphism $\mathbf{L}(\alpha)_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathbf{L}(H)$. This induces a homomorphism $\widetilde{\alpha}_{\mathbb{C}} : \widetilde{G}_{\mathbb{C}} \to H$ with $\mathbf{L}(\widetilde{\alpha}_{\mathbb{C}}) = \mathbf{L}(\alpha)_{\mathbb{C}}$. Therefore $\alpha \circ q_G$ and $\widetilde{\alpha}_{\mathbb{C}} \circ \eta_{\widetilde{G}}$ are two homomorphisms whose differential coincides with $\mathbf{L}(\alpha)$, hence they are equal. In particular, $\eta_{\widetilde{G}}(D) \subseteq \ker \widetilde{\alpha}_{\mathbb{C}}$, and therefore $\widetilde{\alpha}_{\mathbb{C}}$ factors through a homomorphism $\alpha_{\mathbb{C}} : G_{\mathbb{C}} \to H$ with $\alpha_{\mathbb{C}} \circ \pi = \widetilde{\alpha}_{\mathbb{C}}$, where $\pi : \widetilde{G}_{\mathbb{C}} \to G_{\mathbb{C}}$ is the canonical projection. We thus obtain a commutative diagram:

$$\begin{array}{ccccc} \widetilde{G} & \xrightarrow{\eta_{\widetilde{G}}} & \widetilde{G}_{\mathbb{C}} & \xrightarrow{\widetilde{\alpha}_{\mathbb{C}}} & H \\ \downarrow{\scriptstyle q_G} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{id}_H} \\ G & \xrightarrow{\eta_G} & G_{\mathbb{C}} & \xrightarrow{\alpha_{\mathbb{C}}} & H. \end{array}$$

Now

$$\mathbf{L}(\alpha_{\mathbb{C}} \circ \eta_G) = \mathbf{L}(\alpha_{\mathbb{C}} \circ \eta_G \circ q_G) = \mathbf{L}(\alpha_{\mathbb{C}} \circ \pi \circ \eta_{\widetilde{G}}) = \mathbf{L}(\widetilde{\alpha}_{\mathbb{C}} \circ \eta_{\widetilde{G}}) = \mathbf{L}(\alpha \circ q_G) = \mathbf{L}(\alpha),$$

and consequently $\alpha_{\mathbb{C}} \circ \eta_G = \alpha$. The uniqueness of $\alpha_{\mathbb{C}}$ immediately follows from the construction.

Step 2: If G is not connected, we proceed as follows, starting with a universal complexification $(\eta_{G_0}, G_{0,\mathbb{C}})$ of the identity component G_0.

Observe that G acts on G_0 by conjugation, and that this induces a homomorphism $\gamma\colon G \to \mathrm{Aut}_{\mathrm{hol}}(G_{0,\mathbb{C}})$, the group of holomorphic automorphisms of $G_{0,\mathbb{C}}$, determined uniquely by the relation

$$\gamma(g) \circ \eta_{G_0} = \eta_{G_0} \circ c_g|_{G_0}, \quad g \in G. \tag{14.1}$$

We consider the semidirect product group $S := G_{0,\mathbb{C}} \rtimes_\gamma G$ and the subset $B := \{(\eta_{G_0}(g), g^{-1})\colon g \in G_0\} \subseteq Z_S(\eta_{G_0}(G_0))$. Then

$$\begin{aligned}\big(\eta_{G_0}(x), x^{-1}\big)\big(\eta_{G_0}(y), y^{-1}\big) &= (\eta_{G_0}(y), \mathbf{1})\big(\eta_{G_0}(x), x^{-1}\big)(\mathbf{1}, y^{-1}) \\ &= \big(\eta_{G_0}(yx), x^{-1}y^{-1}\big)\end{aligned}$$

implies that B is a subgroup of S. We claim that it is normal. Since it commutes with $G_{0,\mathbb{C}} \times \{\mathbf{1}\}$, it remains to verify the invariance under $\{1\} \times G$. Using (14.1), this follows from

$$(\mathbf{1}, g)(\eta_{G_0}(x), x^{-1})(\mathbf{1}, g^{-1}) = (\gamma(g)(\eta_{G_0}(x)), c_g(x)^{-1}) = \big(\eta_{G_0}(c_g(x)), c_g(x)^{-1}\big).$$

Now it is easy to check that the map

$$\eta_G\colon G \to G_{\mathbb{C}} := S/B, \quad g \mapsto (\mathbf{1}, g)B$$

defines a universal complexification of G. The identity component of $G_{\mathbb{C}}$ is isomorphic to $G_{0,\mathbb{C}}$, hence a complex Lie group, and the action of $G_{\mathbb{C}}$ by conjugation induces an action by complex linear maps on $\mathbf{L}(G_{0,\mathbb{C}})$. Hence $G_{\mathbb{C}}$ is a complex Lie group.

(i) If G is 1-connected, then $D = \{\mathbf{1}\}$ and therefore $A = \{\mathbf{1}\}$. This leads to $\eta_G = \eta_{\widetilde{G}}$ and $G_{\mathbb{C}} = \widetilde{G}_{\mathbb{C}}$.

(ii) Since the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ factors through $\eta_G\colon G \to G_{\mathbb{C}}$, it is clear that $\ker \eta_G \subseteq \ker \mathrm{Ad} = Z(G)$.

(iii) follows immediately from our construction.

(iv) On the Lie algebra $\mathfrak{h} := \mathbf{L}(G_{\mathbb{C}})$, we define a new complex vector space structure by the scalar multiplication $\lambda * x := \overline{\lambda}x$ and write $\overline{\mathfrak{h}}$ for the complex Lie algebra obtained in this way. Accordingly, we obtain on $G_{\mathbb{C}}$ another complex Lie group structure, denoted $\overline{G_{\mathbb{C}}}$. From the universal property, we obtain a unique morphism $\widetilde{\sigma}\colon G_{\mathbb{C}} \to \overline{G_{\mathbb{C}}}$ of complex Lie groups with $\sigma \circ \eta_G = \eta_G$. We write $\sigma\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$ for the underlying morphism of real Lie groups. Since $\mathbf{L}(\widetilde{\sigma})\colon \mathfrak{h} \to \overline{\mathfrak{h}}$ is complex linear, $\mathbf{L}(\sigma)\colon \mathfrak{h} \to \mathfrak{h}$ is antilinear. Therefore $\sigma^2\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$ is holomorphic with $\sigma^2 \circ \eta_G = \eta_G$, so that the uniqueness in the universal property implies that $\sigma^2 = \mathrm{id}_{G_{\mathbb{C}}}$.

The relation $\sigma \circ \eta_G = \eta_G$ implies that $\eta_G(G)$ is contained in the fixed point group $G_{\mathbb{C}}^\sigma$, hence that $\mathrm{im}\,(\mathbf{L}(\eta_G))$ is contained in the set of fixed points of the antilinear involution $\mathbf{L}(\sigma)$, which is a real form of $\mathbf{L}(G_{\mathbb{C}})$. On the other hand, our construction of $G_{\mathbb{C}}$ implies that $\mathbf{L}(G_{\mathbb{C}}) = \mathrm{im}\,(\mathbf{L}(\eta_G)) + i\,\mathrm{im}\,(\mathbf{L}(\eta_G))$, which leads to the equality $\operatorname{im}(\mathbf{L}(\eta_G)) = \mathbf{L}(G_{\mathbb{C}})^{\mathbf{L}(\sigma)}$, and hence to $\eta_G(G_0) = (G_{\mathbb{C}}^{\sigma})_0$ because both groups are connected with the same Lie algebra.

(v) Let $\alpha\colon \eta_G(G_0) \to H$ be a morphism of Lie groups where H is complex. In view of (iii), the universal property of $(G_{\mathbb{C}})_0 \cong (G_0)_{\mathbb{C}}$ implies the existence of a unique morphism of complex Lie groups $\beta\colon (G_{\mathbb{C}})_0 \to H$ with $\beta \circ \eta_G = \alpha \circ \eta_G$. This shows that $\beta|_{\eta_G(G_0)} = \alpha$ and that β is uniquely determined by this property. ☐

Remark 14.1.5. Suppose that G is connected and that, in the notation of the proof of Theorem 14.1.4, $\eta_{\widetilde{G}}(D)$ is discrete. Then $A = \eta_{\widetilde{G}}(D)$ and therefore

$$G_{\mathbb{C}} \cong \widetilde{G}_{\mathbb{C}}/\eta_{\widetilde{G}}(D)$$

(cf. Exercises 14.1.3, 14.1.4). In particular, this is the case if G is semisimple because $\eta_{\widetilde{G}}(D) \subseteq Z(\widetilde{G}_{\mathbb{C}})$, and the latter group is discrete.

Examples 14.1.6. (a) If $(V, +)$ is the additive group of a real vector space, then Theorem 14.1.4(i) implies that the inclusion $\eta_V\colon V \to V_{\mathbb{C}}$ into the complexification is the universal complexification.

If $\Gamma \subseteq V$ is a discrete subgroup, then $\eta_V(\Gamma)$ is discrete in $V_{\mathbb{C}}$, so that Remark 14.1.5 implies that $(V/\Gamma)_{\mathbb{C}} \cong V_{\mathbb{C}}/\Gamma$.

For $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\} \subseteq \mathbb{C}^{\times}$ we have $\mathbb{T} \cong i\mathbb{R}/2\pi i\mathbb{Z}$ and, accordingly, $\mathbb{C}^{\times} \cong \mathbb{C}/2\pi i\mathbb{Z}$, so that the inclusion

$$\eta_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{C}^{\times}$$

is a universal complexification. Similarly, we obtain $(\mathbb{T}^n)_{\mathbb{C}} \cong (\mathbb{C}^{\times})^n$.

(b) If G and H are real Lie groups, then $G_{\mathbb{C}} \times H_{\mathbb{C}}$ is a complex Lie group, and it is easy to verify that

$$\eta_{G\times H} := \eta_G \times \eta_H\colon G \times H \to G_{\mathbb{C}} \times H_{\mathbb{C}}$$

is a universal complexification (cf. Exercise 14.1.6).

(c) If G is a connected semisimple Lie group and $\widetilde{G}_{\mathbb{C}}$ is the 1-connected Lie group with the semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, then $Z(\widetilde{G}_{\mathbb{C}}) \supseteq \eta_{\widetilde{G}}(\pi_1(G))$ is discrete, so that

$$G_{\mathbb{C}} \cong \widetilde{G}_{\mathbb{C}}/\eta_{\widetilde{G}}(\pi_1(G)),$$

where $\eta_{\widetilde{G}}\colon \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ is the canonical map and $G \cong \widetilde{G}/\pi_1(G)$.

If, in addition, $\mathfrak{g}$ is compact, then Proposition 12.2.5 implies that $\mathfrak{g}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ is a Cartan decomposition (Lemma 12.2.3). Then the Cartan Decomposition Theorem 12.1.7 now implies that the integral subgroup of $\widetilde{G}_{\mathbb{C}}$ corresponding to $\mathfrak{g}$ is simply connected, so that the complexification map $\eta_{\widetilde{G}}\colon \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ is injective. This implies that

$$\eta_G\colon G \to G_{\mathbb{C}} \cong \widetilde{G}_{\mathbb{C}}/\pi_1(G)$$

is also injective. The Cartan Decomposition Theorem also shows that the polar map

$$\Phi\colon G \times \mathfrak{g} \to G_{\mathbb{C}}, \quad (g, x) \mapsto g \exp_{G_{\mathbb{C}}}(ix)$$

is a diffeomorphism.

Remark 14.1.7. If σ is an antiholomorphic involution of the complex connected Lie group G, then it is in general false that the inclusion $(G^\sigma)_0 \to G$ is a universal complexification (cf. Theorem 14.1.4(v)). A simple example is the inclusion $\mathbb{R} \to \mathbb{C}^\times, t \mapsto e^t$ and $\sigma(z) = \overline{z}$.

We conclude this section with a simple observation which will be important in our discussion of linear Lie groups.

Proposition 14.1.8. *If G has a faithful representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ and $(\eta_G, G_{\mathbb{C}})$ is a universal complexification, then η_G is injective.*

Proof. The induced holomorphic morphism $\widetilde{\rho}\colon G_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ satisfies $\widetilde{\rho} \circ \eta_G = \rho$, and since ρ is injective, η_G is also injective. □

Exercises for Section 14.1

Exercise 14.1.1. The group $\mathrm{SL}_2(\mathbb{C})$ is simply connected.

Exercise 14.1.2. Let G be a connected real Lie group and $\eta_G\colon G \to G_{\mathbb{C}}$ be the universal complexification of G. Show that:

(a) Show that if η_G is injective, then, for every discrete central subgroup $D \subseteq G$, we have:
  (1) $\eta_G(D)$ is closed in $G_{\mathbb{C}}$.
  (2) $(G/D)_{\mathbb{C}} \cong G_{\mathbb{C}}/\eta_G(D)$.
  (3) The complexification map $\eta_{G/D}$ is injective.
  Hint: Consider the antiholomorphic involution σ on $\widetilde{G}_{\mathbb{C}}$ to see that $\eta_G(D)$ is closed.

(b) Let $q_G\colon \widetilde{G} \to G$ be the universal covering group of G. If $\eta_{\widetilde{G}}\colon \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ is injective, then $G_{\mathbb{C}} \cong \widetilde{G}_{\mathbb{C}}/\eta_{\widetilde{G}}(\ker q_G)$ and $\dim_{\mathbb{C}} G_{\mathbb{C}} = \dim_{\mathbb{R}} G$.

Exercise 14.1.3. Let G be a complex Lie group. Show that

(a) If $(H_i)_{i \in I}$ are complex Lie subgroups, then $\bigcap_{i \in I} H_i$ is a complex Lie subgroup, too.
(b) For every subset M of G, there is a smallest complex Lie subgroup $\langle M \rangle_{\mathbb{C}-\mathrm{grp}}$ containing M. If M is invariant under a holomorphic automorphism $\varphi \in \mathrm{Aut}_{\mathbb{C}}(G)$, then this also holds for $\langle M \rangle_{\mathbb{C}-\mathrm{grp}}$.
(c) If $N \subseteq G$ is a normal subgroup, then $\langle N \rangle_{\mathbb{C}-\mathrm{grp}}$ is also normal.
(d) Every discrete subgroup of a complex Lie group is a complex Lie subgroup.

Exercise 14.1.4. With the notation in Theorem 14.1.4, show that the group $\eta_{\widetilde{G}}(D)$ is not always closed, i.e., in general $\dim_{\mathbb{C}} G_{\mathbb{C}} < \dim_{\mathbb{R}} G$.

Exercise 14.1.5. Let G be a connected complex Lie group, H be a real Lie subgroup and D be central in H. If $\mathbf{L}(H) + i\,\mathbf{L}(H) = \mathbf{L}(G)$, then D is also central in G.

Exercise 14.1.6. If H and G are Lie groups, then the map

$$\eta_H \times \eta_G \colon H \times G \to H_{\mathbb{C}} \times G_{\mathbb{C}}$$

is a universal complexification.

Exercise 14.1.7. Let $\gamma \colon G \to \mathrm{Aut}(N)$ be a homomorphism defining a smooth action of G on N. From the universal property of $N_{\mathbb{C}}$, we thus obtain a homomorphism $\widehat{\gamma} \colon G \to \mathrm{Aut}_{\mathbb{C}}(N_{\mathbb{C}})$, defined by

$$\widehat{\gamma}(g) \circ \eta_N = \eta_N \circ \gamma(g) \quad \text{for} \quad g \in G.$$

Show that:

(a) $\widehat{\gamma}$ induces a unique homomorphism $\gamma_{\mathbb{C}} \colon G_{\mathbb{C}} \to \mathrm{Aut}_{\mathbb{C}}(N_{\mathbb{C}})$ for which

$$N_{\mathbb{C}} \rtimes_{\gamma_{\mathbb{C}}} G_{\mathbb{C}}$$

is a complex Lie group.

(b) $\eta_{N \rtimes G}(n, g) := (\eta_N(n), \eta_G(g))$ is a universal complexification of $N \rtimes_\gamma G$.

Exercise 14.1.8. Let G be a complex Lie group. Suppose that H is a closed subgroup of G such that $\mathbf{L}(G) = \mathbf{L}(H) \oplus_{\mathbb{R}} i\,\mathbf{L}(H)$ and H intersects each connected component of G. Show that

(a) $G = HG_0$, where G_0 is the identity component of G.
(b) $Z(H) = Z(G) \cap H$.

14.2 Linearly Complex Reductive Lie Groups

In this section we discuss the class of linearly complex reductive Lie groups which plays an important role in geometry and representation theory. There are many characterizations of this class of groups. One is that these are precisely the complexifications $K_{\mathbb{C}}$ of compact Lie groups K. Another is that these are precisely the complex linear Lie groups (with finitely many connected components) whose holomorphic representations are completely reducible.

14.2.1 Complexifications of Compact Lie Groups

Proposition 14.2.1. *If K is a compact Lie group, then the following assertions hold*

(a) η_K *is injective.*
(b) $\eta_K(K)$ *is a maximal compact subgroup of* $K_{\mathbb{C}}$*.*
(c) $\mathbf{L}(K_{\mathbb{C}}) \cong \mathfrak{k}_{\mathbb{C}}$ *and the polar map*

$$\Phi\colon K \times \mathfrak{k} \to K_{\mathbb{C}}, \quad (k, x) \mapsto k \exp_{K_{\mathbb{C}}}(ix)$$

is a diffeomorphism.
(d) *If* $\alpha\colon K \to \mathrm{GL}(V)$ *is a representation of the compact Lie group K on the complex vector space V, then the holomorphic extension* $\alpha_{\mathbb{C}}\colon K_{\mathbb{C}} \to \mathrm{GL}(V)$ *satisfies* $\ker(\alpha_{\mathbb{C}}) \cong (\ker \alpha)_{\mathbb{C}}$ *and* $\alpha_{\mathbb{C}}(K_{\mathbb{C}}) \cong \alpha(K)_{\mathbb{C}}$ *is a closed subgroup of* $\mathrm{GL}(V)$ *which is compact if and only if it is trivial.*
(e) $K_{\mathbb{C}}$ *admits a faithful holomorphic linear representation.*

Proof. (a) Since K has a faithful finite-dimensional unitary representation $\rho\colon K \to \mathrm{U}_n(\mathbb{C})$ (Corollary 11.3.9), the injectivity of η_K follows from Proposition 14.1.8.

(b), (c) **Step 1:** First we assume that K is connected. We follow the explicit construction of the universal complexification. Since K is compact, $\mathfrak{k} = \mathbf{L}(K)$ is reductive and thus $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}'$, where $\mathfrak{k}'$ is compact and semisimple. Therefore $\widetilde{K} \cong \widetilde{Z} \times \widetilde{K}'$, where $\widetilde{Z} \cong Z(\widetilde{K})_0$ is a vector group, and this leads to

$$\widetilde{K}_{\mathbb{C}} \cong \widetilde{Z}_{\mathbb{C}} \times \widetilde{K}'_{\mathbb{C}}$$

(Example 14.1.6(c)), which already implies that $\mathbf{L}(K_{\mathbb{C}}) \cong \mathfrak{k}_{\mathbb{C}}$. It also follows that the induced map $\eta_{\widetilde{K}}\colon \widetilde{K} \to \widetilde{K}_{\mathbb{C}}$ is injective because $\widetilde{Z}_{\mathbb{C}}$ is the complexification of the vector space $\widetilde{Z}$ and $\widetilde{K}'$ is compact (Theorem 11.1.17, Exercise 14.1.6). Since $\mathfrak{k}'$ is a compact real form of the complex semisimple Lie algebra $\mathfrak{k}'_{\mathbb{C}}$, $\mathfrak{k}'_{\mathbb{C}} = \mathfrak{k}' \oplus i\mathfrak{k}'$ is a Cartan decomposition (Lemma 12.2.3), and the Cartan Decomposition Theorem 12.1.7 implies that the polar map of $K'_{\mathbb{C}}$, and hence also the polar map

$$\Phi_{\widetilde{K}}\colon \widetilde{K} \times \mathfrak{k} \to \widetilde{K}_{\mathbb{C}}, \quad (g, x) \mapsto g \exp(ix),$$

is a diffeomorphism because the corresponding assertion for the central factor is trivial. It follows in particular that the image of $\pi_1(K)$ in $\widetilde{K}_{\mathbb{C}}$ is discrete, which implies that $K_{\mathbb{C}} = \widetilde{K}_{\mathbb{C}}/\eta_K(\pi_1(K))$ (Remark 14.1.5), and this in turn yields (c).

Step 2: If K is not connected, then the preceding argument implies that the polar map $\Phi_{K_0}\colon K_0 \times \mathfrak{k} \to (K_{\mathbb{C}})_0 \cong (K_0)_{\mathbb{C}}$ is a diffeomorphism. $\Phi\colon K \times \mathfrak{k} \to K_{\mathbb{C}}$ is a K-left equivariant map, and since the inclusion $K \to K_{\mathbb{C}}$ induces an isomorphism $\pi_0(K) \to \pi_0(K_{\mathbb{C}})$ (Theorem 14.1.4), Φ maps each connected component of $K \times \mathfrak{k}$ diffeomorphically onto the corresponding connected component of $K_{\mathbb{C}}$. Therefore Φ is a diffeomorphism.

This implies that $K \cong \eta_K(K)$ is a maximal compact subgroup of $K_{\mathbb{C}}$ because for each non-zero $x \in \mathfrak{k}$, the subgroup $\exp_{K_{\mathbb{C}}}(\mathbb{Z}x)$ is closed and non-compact.

(d) We use the polar decomposition $K_{\mathbb{C}} = K \exp_{K_{\mathbb{C}}}(i\mathfrak{k})$. In view of

$$\alpha_{\mathbb{C}}(k \exp_{K_{\mathbb{C}}} ix) = \alpha(k) e^{i\,\mathbf{L}(\alpha)x} \quad \text{for} \quad k \in K, x \in \mathfrak{k},$$

the uniqueness of the polar decomposition in $\mathrm{GL}_n(\mathbb{C})$ implies that the relation $\alpha_{\mathbb{C}}(k \exp_{K_{\mathbb{C}}} ix) = \mathbf{1}$ is equivalent to $\alpha(k) = \mathbf{1}$ and $\mathbf{L}(\alpha)x = 0$. Therefore

$$\ker(\alpha_{\mathbb{C}}) = (\ker \alpha) \exp_{K_{\mathbb{C}}}(i\,\mathbf{L}(\ker \alpha)) \cong (\ker \alpha)_{\mathbb{C}}.$$

We also derive from the polar decomposition that

$$\alpha_{\mathbb{C}}(K_{\mathbb{C}}) = \alpha(K) \exp(i\,\mathbf{L}(\alpha)\mathfrak{k})$$

is a closed subset of $\mathrm{GL}_n(\mathbb{C})$.

(e) From the Linearity Theorem for Compact Lie Groups 11.3.9, we obtain the existence of a faithful unitary representation $\alpha\colon K \to \mathrm{U}_n(\mathbb{C})$. Then (d) implies that its holomorphic extension $\alpha_{\mathbb{C}}\colon K_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ is faithful. □

Example 14.2.2. If G is a connected Lie group with compact Lie algebra, then the Structure Theorem 11.1.18 implies that $G \cong K \times V$ is a direct product of a compact Lie group K and a vector group V. Combining (a), (b) and (c) in Example 14.1.6 with Proposition 14.2.1, we see that the complexification map

$$\eta_G = \eta_K \times \eta_V \colon G = K \times V \to K_{\mathbb{C}} \times V_{\mathbb{C}}$$

is injective and that the polar map

$$\Phi\colon G \times \mathfrak{g} \to G_{\mathbb{C}}, \quad (g, x) \mapsto g \exp_{G_{\mathbb{C}}}(ix)$$

is a diffeomorphism because this holds for K by Proposition 14.2.1.

Remark 14.2.3. Proposition 14.2.1 shows that, for a compact Lie group K, the universal property of $K_{\mathbb{C}}$ can also be interpreted in such a way that every representation $\alpha\colon K \to \mathrm{GL}_n(\mathbb{C})$ can be extended to a holomorphic representation $\alpha_{\mathbb{C}}\colon K_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ because we can identify K with the subgroup $\eta_K(K)$ of $K_{\mathbb{C}}$.

The following proposition provides a kind of converse to Proposition 14.2.1.

Proposition 14.2.4 (Recognizing Complexifications). *Let K be a maximal compact subgroup of the complex Lie group G with finite $\pi_0(G)$ and assume that $\mathbf{L}(G) = \mathbf{L}(K)_{\mathbb{C}}$. Then the inclusion $K \hookrightarrow G$ is a universal complexification, in particular, $G \cong K_{\mathbb{C}}$.*

Proof. Let $\eta_K : K \to K_{\mathbb{C}}$ be the universal complexification of K. Because of the universal property of $K_{\mathbb{C}}$, there exists a unique holomorphic homomorphism $\beta : K_{\mathbb{C}} \to G$ with $\mathbf{L}(\beta) = \mathrm{id}_{\mathbf{L}(K)_{\mathbb{C}}}$. Since η_K and the inclusion $K \to G$ induce isomorphisms $\pi_0(K) \to \pi_0(K_{\mathbb{C}})$ (Theorem 14.1.4), resp., $\pi_0(K) \to \pi_0(G)$ (Second Manifold Splitting Theorem 13.3.11), β is clearly surjective and $\ker \beta \subseteq K_{\mathbb{C},0}$.

It remains to show injectivity. So let $d \in \ker \beta$. Then $d \in Z(K_{\mathbb{C},0})$ because β is a covering, and there exists an element $x \in \mathbf{L}(K)_{\mathbb{C}}$ with $\exp_{K_{\mathbb{C}}}(x) = d$ (Theorem 13.2.8). Then the subgroup $\beta(\exp_{K_{\mathbb{C}}} \mathbb{R}x) \subseteq G$ is compact, hence conjugate to a subgroup of K (Theorem 13.1.3). We may therefore assume that $x \in \mathbf{L}(K)$. From that it follows that $d \in \eta_K(K)$, and thus $d = \mathbf{1}$ because $\beta \circ \eta_K$ is injective. $\square$

Example 14.2.5. The polar decomposition implies that $\mathrm{U}_n(\mathbb{C})$ is a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{C})$, and since $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) \oplus i\mathfrak{u}_n(\mathbb{C}) \cong \mathfrak{u}_n(\mathbb{C})_{\mathbb{C}}$, we obtain $\mathrm{U}_n(\mathbb{C})_{\mathbb{C}} \cong \mathrm{GL}_n(\mathbb{C})$.

For any compact subgroup $K \subseteq \mathrm{U}_n(\mathbb{C})$, the polar decomposition of $K_{\mathbb{C}}$ implies that the holomorphic extension $K_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ of the inclusion of $K \to \mathrm{U}_n(\mathbb{C})$ maps $K_{\mathbb{C}}$ onto the subset $K \exp(i\mathfrak{k})$, and Proposition 14.2.1 yields

$$K_{\mathbb{C}} \cong K \exp(i\mathfrak{k}) \subseteq \mathrm{GL}_n(\mathbb{C}).$$

In particular, it follows that the product set $K \exp(i\mathfrak{k})$ actually is a subgroup.

This observation applies to many concrete groups with polar decomposition (cf. Proposition 3.3.3). In particular, we obtain

$$\mathrm{SU}_n(\mathbb{C})_{\mathbb{C}} \cong \mathrm{SL}_n(\mathbb{C}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{R})_{\mathbb{C}} \cong \mathrm{O}_n(\mathbb{C}).$$

Proposition 14.2.6. *Let G be a complex semisimple Lie group with finitely many connected components and Lie algebra $\mathfrak{g}$. Furthermore, $\mathfrak{k} \subseteq \mathfrak{g}$ be a compact real form and $K_0 := \langle \exp_G \mathfrak{k} \rangle$ be the corresponding integral subgroup. Then the following assertions hold:*

(a) *K_0 is maximal compact in G_0 and contained in a maximal compact subgroup K. The inclusion $K \hookrightarrow G$ is a universal complexification, so that $G \cong K_{\mathbb{C}}$.*
(b) *$Z(G_0)$ is a finite subgroup of K.*
(c) *G has a faithful finite-dimensional holomorphic representation.*

Proof. First we recall from Proposition 12.2.7 that $\mathfrak{g}$ has a compact real form $\mathfrak{k}$. This is a compact semisimple Lie algebra with $\mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}}$.

(a) We observe that $\mathfrak{k}$ by Lemma 12.1.3 is maximal compactly embedded. Since $\mathfrak{k}$ is semisimple, K_0 is compact (Theorem 11.1.17). Now Theorem 12.2.14 implies that K_0 is maximal compact in G_0.

Let U be a maximal compact subgroup of G. Then the Maximal Compact Subgroup Theorem 13.1.3 implies that U_0 is maximally compact in G_0 and that all maximal compact subgroups of G_0 are conjugate under inner automorphisms. We may therefore assume that $U_0 = K_0$ and put $K := U$. Now Proposition 14.2.4 yields $G \cong K_{\mathbb{C}}$.

(b) By Theorem 13.2.8, $Z(G_0)$ is contained in K_0, and it is finite since it is discrete in the compact semisimple Lie group K_0.

(c) follows from (a) and Proposition 14.2.1. □

14.2.2 Linearly Complex Reductive Lie Groups

In the following we call a complex Lie group G *reductive* if its Lie algebra $\mathfrak{g}$ is reductive. The notion of a reductive Lie group has to be handled with great care since there exist many notions of reductivity specifying certain classes of reductive Lie groups.

Definition 14.2.7. A complex Lie group G is said to be *linearly complex reductive* if there exists a compact Lie group K with $G \cong K_{\mathbb{C}}$. The complexifications $T_{\mathbb{C}}$ of torus groups T are called *complexified tori.* They are isomorphic to some $(\mathbb{C}^\times)^n$, $n = \dim T$ (Example 14.1.6(a)).

At this point it is not clear why one would call this class of groups *linearly* complex reductive. We will see in Theorem 14.3.11, however, that linearly complex reductive Lie groups are characterized by the fact that they admit faithful holomorphic representations and all holomorphic representations are completely reducible.

Remark 14.2.8. Clearly, the Lie algebra $\mathbf{L}(K_{\mathbb{C}}) \cong \mathfrak{k}_{\mathbb{C}}$ of any linearly reductive complex Lie group $G = K_{\mathbb{C}}$ is reductive, but $G = \mathbb{C}$ is a reductive complex Lie group which is not linearly reductive.

Theorem 14.2.9 (Characterization of Linearly Complex Reductive Groups 1). *A complex Lie group G with reductive Lie algebra and finite $\pi_0(G)$ is linearly complex reductive if and only if $Z(G_0)_0$ is a complexified torus.*

Proof. First, we assume that $G = K_{\mathbb{C}}$ for a compact Lie group K. Then $Z := Z(K_0)_0$ is a torus and $(K_0)'$ is compact semisimple. Moreover, the group $\Gamma := Z \cap (K_0)'$ is finite and the multiplication map $Z \times (K_0)' \to K_0, (z,k) \mapsto zk$ induces an isomorphism

$$(Z \times (K_0)')/\Gamma \cong K_0.$$

This in turn induces an isomorphism

$$(Z_{\mathbb{C}} \times (K_0)'_{\mathbb{C}})/\Gamma \cong (K_0)_{\mathbb{C}} \cong (K_{\mathbb{C}})_0$$

(Exercises 14.1.6 and 14.1.2(a)). In particular, $Z((K_{\mathbb{C}})_0)_0 \cong Z_{\mathbb{C}}$ is a complexified torus.

Suppose, conversely, that G is a complex Lie group with finite $\pi_0(G)$, reductive Lie algebra for which $Z(G_0)_0$ is a complexified torus.

Step 1: First we assume that G is connected. Then $Z(G)_0$ is a complexified torus and we consider a maximal compact subgroup $K \subseteq G$. In view of Proposition 13.3.13, $Z := K \cap Z(G)_0$ is a maximal torus of $Z(G)_0$ and $U := K \cap G'$ is a maximal compact subgroup of the connected complex semisimple Lie group G'. We conclude that $Z(G)_0 \cong Z_{\mathbb{C}}$ and $G' \cong U_{\mathbb{C}}$. Since $Z(G')$ is finite (Proposition 14.2.6), the intersection $\Gamma := Z(G)_0 \cap G'$ is finite, hence contained in Z. As Γ is fixed under conjugation and contained in some conjugate of U, we also obtain $\Gamma \subseteq U$, i.e., $\Gamma = Z \cap U$. The multiplication $Z(G)_0 \times G' \to G$ induces an isomorphism

$$\mu\colon (Z_{\mathbb{C}} \times U_{\mathbb{C}})/\Gamma \cong (Z(G)_0 \times G')/\Gamma \to G$$

mapping the maximal compact subgroup $Z \times U$ of $Z(G)_0 \times G'$ onto the subgroup $ZU \cong (Z \times U)/\Gamma$ of K. As the kernel of μ is finite, the maximal compactness of $Z \times U$ in $Z(G)_0 \times G'$ implies that $\mu^{-1}(K) = Z \times U$, which leads to $K = ZU$. This in turn implies that $Z = Z(K)_0$ and $U = K'$. We thus obtain

$$K_{\mathbb{C}} \cong (Z_{\mathbb{C}} \times U_{\mathbb{C}})/\Gamma \cong G.$$

Therefore G is linearly complex reductive.

Step 2: For the general case we pick a maximal compact subgroup $K \subseteq G$. Then $K \cap G_0$ is maximal compact in G_0 (Maximal Compact Subgroup Theorem 13.1.3), and the argument above implies that $G_0 \cong (K_0)_{\mathbb{C}}$. Therefore the natural map $\varphi\colon K_{\mathbb{C}} \to G$ induced by the inclusion $K \hookrightarrow G$ induces an isomorphism $(K_{\mathbb{C}})_0 \cong (K_0)_{\mathbb{C}} \to G_0$, and it also induces an isomorphism $\pi_0(K) \cong \pi_0(K_{\mathbb{C}}) \to \pi_0(G)$. Therefore φ is an isomorphism of complex Lie groups. □

Theorem 14.2.10 (Weyl's Unitary Trick). *Every holomorphic representation of a linearly complex reductive Lie group G is completely reducible.*

Proof. We write $G = K_{\mathbb{C}}$ for a compact Lie group K. Let $\rho\colon G \to \mathrm{GL}(V)$ be a holomorphic representation. We apply the Unitarity Lemma 9.4.14 to find a K-invariant positive definite hermitian form on V, so that we may identify V with $\mathbb{C}^n$ and assume that $\rho(K) \subseteq \mathrm{U}_n(\mathbb{C})$. Then

$$\rho(G) = \rho(K \exp_G(i\mathfrak{k})) = \rho(K)e^{i\,\mathbf{L}(\rho)(\mathfrak{k})}$$

implies that each K-invariant subspace $W \subseteq V$ is also G-invariant.

If $W \subseteq V$ is G-invariant, it is in particular K-invariant, hence $W^\perp$ is K-invariant, and therefore also G-invariant. As $V = W \oplus W^\perp$, the representation (ρ, V) is completely reducible. □

Definition 14.2.11. If H is a subgroup of the complex Lie group G and $\mathfrak{h} = \{x \in \mathbf{L}(G)\colon \exp_G(\mathbb{R}x) \subseteq H\}$ its Lie algebra, we write

$$H^* := H\langle \exp_G(\mathfrak{h} + i\mathfrak{h})\rangle$$

for the smallest subgroup of G containing H and $\exp_G(\mathfrak{h}+i\mathfrak{h})$. If H is arcwise connected, then $H = \langle \exp_G \mathfrak{h}\rangle$ by Yamabe's Theorem 8.6.1, so that H^* is the integral subgroup corresponding to the complex subalgebra $\mathfrak{h}+i\mathfrak{h} \subseteq \mathbf{L}(G)$ (cf. the Integral Subgroup Theorem 8.4.8).

The following lemma will be useful in providing the characterization of linearly complex reductive Lie groups in terms of holomorphic representations alluded to before.

Lemma 14.2.12. *Let G be a reductive complex Lie group with finitely many connected components and $K \subseteq G$ be a maximal compact subgroup. Then the following assertions hold:*

(i) *There exists a complex vector group V such that $Z(G_0)_0 \cong V \times T_Z^*$, where $T_Z \subseteq Z(G_0)_0$ is a maximal torus and V is normal in G.*
(ii) $G \cong V \times K^*$.
(iii) $K_0 = K_0' T_Z$.

Proof. (i) Since the identity component G_0 acts trivially on $\mathbf{L}(Z(G_0)) = \mathfrak{z}(\mathfrak{g})$, the group $\mathrm{Ad}(G)|_{\mathfrak{z}(\mathfrak{g})}$ is finite. Hence there exists an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{z}(\mathfrak{g})$, which implies the existence of an $\mathrm{Ad}(G)$-invariant complement $\mathfrak{v} \subseteq \mathfrak{z}(\mathfrak{g})$ of $\mathbf{L}(T_Z^*) = \mathbf{L}(T_Z) + i\,\mathbf{L}(T_Z)$. We put $V := \exp_G \mathfrak{v}$ and note that the $\mathrm{Ad}(G)$-invariance of $\mathfrak{v}$ implies that V is normal in G. We now have $Z := Z(G_0)_0 = T_Z^* V$.

Since T_Z^* contains a maximal torus of Z, Proposition 13.3.13(ii) implies that $T_Z^* \cap V$ is connected. Therefore $\mathbf{L}(T_Z^*) \cap \mathfrak{v} = \{0\}$ yields $V \cap T_Z^* = \{\mathbf{1}\}$ and that V is closed (Corollary 13.5.6). The subgroup T_Z^* of Z is also closed because it contains a maximal torus (Corollary 13.5.6), hence a Lie group. Finally we note that the multiplication map $V \times T_Z^* \to Z, (v,t) \mapsto vt$ is an isomorphism of complex Lie groups because it is bijective, holomorphic and open since its differential is bijective everywhere (cf. Proposition 10.1.18).

(ii) We note that Z is a complex normal Lie subgroup, so that G/Z is a complex semisimple Lie group, hence linearly complex reductive by Proposition 14.2.6. Let $q\colon G \to G/Z$ denote the projection morphism. Then $q(K)$ is maximal compact in G/Z (Proposition 13.3.13) and G/Z is linearly complex reductive. This implies that $q(K^*) = q(K)^* = G/Z$, which leads to $G = ZK^*$. In particular, K^* is a normal subgroup of G.

This implies in particular that $Z(K_0) \subseteq Z(G_0)$, so that the maximal torus T_Z of Z coincides with $Z(K_0)_0 = Z \cap K_0$ (cf. Proposition 13.3.13). Therefore $K_0 = K_0' Z(K_0)_0 = K_0' T_Z$. From (i) we know that $Z \cong T_Z^* \times V$ holds for a vector group V which is normal in G. In view of $T_Z^* \subseteq K^*$, we thus have $G = ZK^* = VK^*$. To see that the intersection of V and K^* is trivial, we use Proposition 13.3.13(ii) to see that it is connected and $\mathbf{L}(K^* \cap V) = \mathbf{L}(K^*) \cap \mathfrak{v} = \mathbf{L}(T_Z^*) \cap \mathfrak{v} = \{0\}$. Finally, we obtain $G \cong V \times K^*$ from (Proposition 10.1.18). ∎

We end this section by introducing the linearizer $\mathrm{Lin}_{\mathbb{C}}(G)$ of a complex Lie group G which measures how far G is from being complex linear.

Definition 14.2.13. For a complex Lie group G, we write $\mathrm{Lin}_{\mathbb{C}}(G)$ for the *linearizer of* G, i.e., for the common kernel of all finite-dimensional *holomorphic* representations of G. This subgroup is the obstruction for G to admit a faithful finite-dimensional holomorphic linear representation. We will see in Section 15.2 below that $\mathrm{Lin}_{\mathbb{C}}(G) = \{\mathbf{1}\}$ characterizes the groups admitting faithful finite-dimensional holomorphic linear representations.

Lemma 14.2.14. *Let G be a connected complex Lie group. Then the following assertions hold:*

(i) $\mathrm{Lin}_{\mathbb{C}}(G)$ *is a complex normal Lie subgroup of* G.
(ii) $\mathrm{Lin}_{\mathbb{C}}(G) \subseteq Z(G)$.
(iii) $\mathrm{Lin}_{\mathbb{C}}\big(G/\mathrm{Lin}_{\mathbb{C}}(G)\big) = \{\mathbf{1}\}$, *i.e., the finite-dimensional holomorphic representations of the group* $G/\mathrm{Lin}_{\mathbb{C}}(G)$ *separate the points.*

Proof. (i) Since $\mathrm{Lin}_{\mathbb{C}}(G)$ is an intersection of kernels, it is a closed normal subgroup. Its Lie algebra is the intersection of the kernels of the differentials of a set of complex linear representations, so it is a complex subalgebra. Hence $\mathrm{Lin}_{\mathbb{C}}(G)$ is a complex Lie subgroup of G.

(ii) The kernel of the adjoint representation is the center. Therefore $\mathrm{Lin}_{\mathbb{C}}(G) \subseteq Z(G) = \ker \mathrm{Ad}$.

(iii) Let $g \notin \mathrm{Lin}_{\mathbb{C}}(G)$. Then there exists a finite-dimensional holomorphic representation $\alpha \colon G \to \mathrm{GL}(V)$ with $\alpha(g) \neq \mathbf{1}$. Since $\mathrm{Lin}_{\mathbb{C}}(G) \subseteq \ker \alpha$, this representation factors through a holomorphic representation

$$\widetilde{\alpha} \colon G/\mathrm{Lin}_{\mathbb{C}}(G) \to \mathrm{GL}(V) \quad \text{with} \quad \widetilde{\alpha}\big(g\,\mathrm{Lin}_{\mathbb{C}}(G)\big) = \alpha(g) \neq \mathbf{1}. \qquad \square$$

Exercises for Section 14.2

Exercise 14.2.1. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{g}^{\mathbb{R}}$ the underlying real Lie algebra. Show that

(a) The space $\mathfrak{g}$, endowed with the same bracket and the opposite complex structure, defined by the scalar multiplication $z * x := \overline{z}x$ also is a complex Lie algebra. We write $\overline{\mathfrak{g}}$ for this Lie algebra.
(b) $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \overline{\mathfrak{g}}$, where the natural inclusion is given by $x \mapsto (x, x)$.

Exercise 14.2.2. Show that:

$$\mathrm{Sp}_{2n}(\mathbb{R})_{\mathbb{C}} \cong \mathrm{Sp}_{2n}(\mathbb{C}) \cong \mathrm{U}_n(\mathbb{H})_{\mathbb{C}} \quad \text{and} \quad \mathrm{O}_{p,q}(\mathbb{R})_{\mathbb{C}} \cong \mathrm{O}_{p+q}(\mathbb{C}) \cong \mathrm{O}_{p+q}(\mathbb{R})_{\mathbb{C}}$$

and that these complexifications are linearly complex reductive Lie groups. Determine a maximal compact subgroup.

Exercise 14.2.3. Let G be a complex Lie group. Then G is in particular a real Lie group, and we write $G^{\mathbb{R}}$ for the underlying real Lie group. We also write $\overline{G}$ for the *opposite complex Lie group*, which is determined by $\overline{G}^{\mathbb{R}} = G^{\mathbb{R}}$ and $\mathbf{L}(\overline{G}) = \overline{\mathbf{L}(G)}$ in the sense of Exercise 14.2.1. Show that:

(1) If G is connected and simply connected, then the embedding

$$\eta_G \colon G \to G \times \overline{G}, \quad g \mapsto (g, g)$$

is a universal complexification.

(2) If G is connected and $\pi_1(G) \subseteq \widetilde{G}_0$ is the kernel of the universal covering map, then

$$G_{\mathbb{C}} \cong (\widetilde{G} \times \overline{\widetilde{G}})/\{(g, g) \colon g \in \pi_1(G)\}.$$

(3) Show that in general $G_{\mathbb{C}} \not\cong G \times \overline{G}$.

14.3 Complex Abelian Lie Groups

In this section we take a look into the interesting world of abelian complex Lie groups. From a real perspective, one would not expect complicated phenomena because connected abelian real Lie groups are all isomorphic to some product $\mathbb{T}^n \times \mathbb{R}^m$, hence classified by two discrete parameters. As we shall see in the first subsection below, for complex Lie groups, the situation is drastically different, even in dimension 1, where one-dimensional compact connected abelian Lie groups are classified by a complex parameter, although the underlying real group is always the torus $\mathbb{T}^2$. In this section we study the structure of connected complex abelian Lie groups more systematically. In particular, we show that a complex vector space can always be split off and that all complications lie in the groups of the form T^*, where T is the maximal torus.

14.3.1 One-dimensional Complex Lie Groups

In this subsection we present the classification of one-dimensional connected complex Lie groups G. Then G is abelian, $\mathfrak{g} = \mathbf{L}(G) \cong \mathbb{C}$, as a complex vector space, and G is isomorphic to $\mathbb{C}/\Gamma$, where $\Gamma \subseteq \mathbb{C}$ is a finitely generated discrete subgroup isomorphic to $\pi_1(G)$ (see Corollaries 13.2.10 and 13.6.8). We distinguish 3 cases:

(1) G is simply connected: Then $G \cong \mathbb{C}$ is a one-dimensional complex vector space.

(2) $\Gamma \cong \mathbb{Z}$: Then there exists a $\lambda \in \mathbb{C}^\times$ with $\lambda\Gamma = 2\pi i\mathbb{Z}$, and then

$$\mathbb{C}/\Gamma \to \mathbb{C}^\times, \quad z \mapsto e^{\lambda z}$$

is an isomorphism of complex Lie groups.

(3) $\Gamma \cong \mathbb{Z}^2$. Then G is called an *elliptic curve* or a *complex torus* of dimension 1.

Clearly, the third case is the most interesting. In this case the underlying real group is $G^{\mathbb{R}} \cong \mathbb{T}^2$, but not all these groups are isomorphic. If

$$\varphi\colon \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$$

is an isomorphism of complex Lie groups, then $\mathbf{L}(\varphi)\colon \mathbb{C} \to \mathbb{C}$ is a complex linear isomorphism, hence multiplication with some $\lambda \in \mathbb{C}^\times$, mapping Γ_1 into Γ_2. If, conversely, $\Gamma_1, \Gamma_2 \subseteq \mathbb{C}$ are discrete subgroups and $\lambda \in \mathbb{C}^\times$ satisfies $\lambda\Gamma_1 = \Gamma_2$, then

$$\varphi\colon \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2, \quad z + \Gamma_1 \mapsto \lambda z + \Gamma_2 = \lambda(z + \Gamma_1)$$

is an isomorphism of complex Lie groups.

Therefore the isomorphism classes of compact complex one-dimensional Lie groups are parameterized by the orbits of $\mathbb{C}^\times$ in the set of all lattices $\Gamma \subseteq \mathbb{C}$. To describe this set, we note that, after multiplication with a complex number, we can move each lattice Γ into one generated by 1 and a complex number τ with $\operatorname{Im}\tau > 0$. It therefore suffices to consider lattices of the form

$$\Gamma = \Gamma(\tau) := \mathbb{Z} + \mathbb{Z}\tau \quad \text{with} \quad \operatorname{Im}\tau > 0.$$

We want to determine for which pairs (τ, τ') the groups $E(\tau) := \mathbb{C}/\Gamma(\tau)$ and $E(\tau')$ are isomorphic as complex Lie groups. This is the case if and only if there exists a complex number λ with $\lambda\Gamma(\tau) = \Gamma(\tau')$. If this is the case, then λ and $\lambda\tau$ form a basis of the lattice $\Gamma(\tau')$. Hence there exists a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ with

$$\lambda = c\tau' + d \quad \text{and} \quad \lambda\tau = a\tau' + b.$$

Then

$$\tau = g \cdot \tau' := \frac{a\tau' + b}{c\tau' + d}. \tag{14.2}$$

In view of

$$0 < \operatorname{Im}\tau = \frac{\operatorname{Im}\tau'}{|c\tau' + d|^2} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

it follows that $g \in \mathrm{SL}_2(\mathbb{Z})$. If, conversely, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau = g \cdot \tau'$, then $\lambda\Gamma(\tau) = \Gamma(\tau')$ for $\lambda = c\tau' + d$.

This completes the proof of the following proposition:

Proposition 14.3.1. *Let* $\mathrm{SL}_2(\mathbb{Z})$ *act on* $X := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ *by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

Then the isomorphism classes of one-dimensional compact compact connected complex Lie groups are parameterized by the space $X/\operatorname{SL}_2(\mathbb{Z})$ *of orbits where one associates the isomorphism class of the group* $E(\tau)$ *with an orbit* $\operatorname{SL}_2(\mathbb{Z})\tau$*.*

One can sharpen the preceding classification by showing that

$$D := \Big\{ z \in X : |z| \geq 1, |\operatorname{Re} z| \leq \frac{1}{2} \Big\}$$

meets each orbit of $\operatorname{SL}_2(\mathbb{Z})$ on X, and calculating which boundary points belong to the same orbit (cf. [Se88, p. 129]).

14.3.2 The Structure of Connected Complex Abelian Lie Groups

In view of Lemma 14.2.12(ii), the intricacies of complex abelian Lie groups all lie in the structure of the subgroup T^*. Therefore we now take a closer look at groups for which $G = T^*$. This means that G is the minimal complex integral subgroup (i.e., integral subgroup with complex Lie algebra) containing the maximal torus.

Lemma 14.3.2. *Let* T *be a torus and* $A \subseteq T$ *be a subtorus. Then there exists a torus* $B \subseteq T$ *for which the multiplication map* $A \times B \to T, (a, b) \mapsto ab$ *is an isomorphism of Lie groups.*

Proof. Let $\mathfrak{a} = \mathbf{L}(A)$ and $\mathfrak{t} = \mathbf{L}(T)$. Then $\ker \exp_T \cong \mathbb{Z}^n$ is a discrete generating subgroup of $\mathfrak{t} \cong \mathbb{R}^n$. Then $\mathfrak{a} \cap \mathbb{Z}^n$ is a subgroup, and Proposition 13.6.2 implies that there exists a basis $e_1, \ldots, e_n$ for $\mathfrak{t}$ such that $\ker \exp_T = \sum_{i=1}^n \mathbb{Z} e_i$ and $\ker \exp_A = \sum_{i=1}^k \mathbb{Z} e_i$. Then $\mathfrak{b} := \operatorname{span}\{e_{k+1}, \ldots, e_n\}$ is a subspace of $\mathfrak{t}$ for which $B := \exp_T(\mathfrak{b}) \cong \mathfrak{b}/(\mathbb{Z}e_{k+1} + \cdots + \mathbb{Z}e_n)$ is a torus with the required properties. □

Definition 14.3.3. We call a function $f\colon G \to \mathbb{C}$ on a complex connected abelian Lie group G *holomorphic* if the function $F := f \circ \exp_G\colon \mathfrak{g} \to \mathbb{C}$ is holomorphic in the sense that it is differentiable with complex linear differential $\mathtt{d}F(x)$ in each $x \in \mathfrak{g}$.

Proposition 14.3.4. *Let* G *be a connected complex abelian Lie group satisfying* $G = T^*$ *for the unique maximal torus* T *of* G*. Let* $A := \overline{\exp_G(\mathfrak{t} \cap i\mathfrak{t})}$ *and* $B \subseteq T$ *be a subtorus with* $T \cong A \times B$*. Then the following assertions hold:*

(i) $T^* \cong A^* \times B^*$ *and* $B^* \cong B_{\mathbb{C}}$.
(ii) *All holomorphic functions on* A^* *are constant.*
(iii) $\operatorname{Lin}_{\mathbb{C}}(G) = A^*$.
(iv) A^* *is the smallest complex Lie subgroup of* G *containing* $\exp_G(\mathfrak{t} \cap i\mathfrak{t})$.
(v) *There exists a torus* $C \subseteq T$ *such that* $C^* \cong C_{\mathbb{C}}$, $A^* = TC^*$, $T \cap C^* = C$ *and* A^*/C^* *is compact.*

Proof. (i) To avoid confusion between the complex structures of $\mathfrak{g} = \mathbf{L}(G)$ and $\mathfrak{t}_{\mathbb{C}}$, we write $\mathfrak{t}_{\mathbb{C}}$ as $\mathfrak{t} \times \mathfrak{t}$, where the complex structure is given by $I(x, y) := (-y, x)$. Accordingly, we have $T_{\mathbb{C}} \cong \mathfrak{t}_{\mathbb{C}}/D$, $D = \ker \exp_T$, for the universal complexification of T. We consider the induced homomorphisms $\gamma \colon T_{\mathbb{C}} \to T^*$ and $\gamma' := \gamma \circ \exp_{T_{\mathbb{C}}} \colon \mathfrak{t}_{\mathbb{C}} \to T^*$.

We claim that

$$\ker \gamma' = \{(x, iy) \colon y \in \mathfrak{t} \cap i\mathfrak{t}, x - y \in D\}. \tag{14.3}$$

If $y \in \mathfrak{t} \cap i\mathfrak{t}$ and $x - y \in D$, then $(x, iy) = (x - y, 0) + (y, iy)$ yields

$$\gamma'(x, iy) = \exp_T(x - y) \exp_T(y) \exp_T(-y) = \mathbf{1}.$$

If, conversely, $\gamma'(x, y) = \mathbf{1}$, then $\exp_G(\mathbb{R}(x + iy))$ is a compact one-parameter group of T^*, hence contained in T, so that $x + iy \in \mathfrak{t}$. Thus $x \in \mathfrak{t}$ implies that $iy \in \mathfrak{t}$ and therefore that $y \in \mathfrak{t} \cap i\mathfrak{t}$. Now $(x, y) = (x + iy, 0) + (-iy, y)$ and $\exp_T(x + iy) = \gamma'(x, y) = \mathbf{1}$, so that $x + iy \in D$. This proves (14.3).

Let $\mathfrak{e} \subseteq \mathfrak{t}$ be a subspace with $\mathfrak{e} \cap i\mathfrak{t} = \{0\}$. Then (14.3) implies that

$$\mathfrak{e}_{\mathbb{C}} \cap \ker \gamma' = D \cap \mathfrak{e}. \tag{14.4}$$

In fact, if $(x, y) \in \mathfrak{e}_{\mathbb{C}} = \mathfrak{e} \times \mathfrak{e}$ is contained in $\ker \gamma'$, then $y \in \mathfrak{e} \cap i\mathfrak{t} = \{0\}$ implies that $(x, y) = (x, 0) \in \mathfrak{e} \cap D$. Put $E := \exp_G(\mathfrak{e})$, so that $E^* = E \exp_G(i\mathfrak{e})$. For $x \in \mathfrak{e}$ with $\exp_G(ix) = \gamma'(0, x) \in T$, there exists a $y \in \mathfrak{t}$ with $\gamma'(0, x) = \gamma'(y, 0)$, so that $(-y, x) \in \ker \gamma'$ which implies $x = 0$. Therefore

$$E^* \cap T = E. \tag{14.5}$$

For $\mathfrak{e} = \mathfrak{b}$, this leads to $\mathfrak{b}_{\mathbb{C}} \cap \ker \gamma' = D \cap \mathfrak{b} = \ker \exp_B$, so that $B^* \cong B_{\mathbb{C}}$. In view of (14.5), we also have $B^* \cap T = B$, which is a closed subgroup. Since Corollary 13.5.6 implies that a subgroup H of G is closed if (and only if) $H \cap T$ is closed, it follows that B^* is closed.

We claim that A^* is also closed. Let $\mathfrak{a} := \mathbf{L}(A)$ and note that, since $A = \exp_G(\mathfrak{a})$ is a torus, $D_A := D \cap \mathfrak{a}$ is a generating discrete subgroup of $\mathfrak{a}$. If $q \colon \mathfrak{a} \to \mathfrak{a}/(\mathfrak{t} \cap i\mathfrak{t})$ denotes the quotient map, then $q(D_A)$ generates the vector space $\mathfrak{a}/(\mathfrak{t} \cap i\mathfrak{t})$. Hence there exist elements $d_1, \ldots, d_k \in D_A$ such that $q(d_1), \ldots, q(d_k)$ form a basis of $\mathfrak{a}/(\mathfrak{t} \cap i\mathfrak{t})$. Let $\mathfrak{c} := \operatorname{span}\{d_1, \ldots, d_k\} \subseteq \mathfrak{a}$. Then $\mathfrak{a} = (\mathfrak{t} \cap i\mathfrak{t}) \oplus \mathfrak{c}$ and $\mathfrak{a}_{\mathbb{C}} = (\mathfrak{t} \cap i\mathfrak{t})_{\mathbb{C}} \oplus \mathfrak{c}_{\mathbb{C}}$. From (14.4) and (14.5) we derive that $\ker \gamma' \cap \mathfrak{c}_{\mathbb{C}} = D \cap \mathfrak{c}$ and that the torus $C := \exp_G \mathfrak{c}$ satisfies $C^* \cap T = C$. Therefore C^* is a closed subgroup of G isomorphic to $C_{\mathbb{C}}$. Now $A^* = \exp\big((\mathfrak{t} \cap i\mathfrak{t}) + \mathfrak{c}\big) \exp(i\mathfrak{c}) = T \exp(i\mathfrak{c})$ implies that $A^* \cap T = T$, so that A^* is closed (Corollary 13.5.6).

Next we show that $A^* \cap B^* = \{\mathbf{1}\}$. If $g \in A^* \cap B^* = (A \exp i\mathfrak{c}) \cap B^*$, then there exist $x \in \mathfrak{a}$, $y \in \mathfrak{b}_{\mathbb{C}}$ and $z \in \mathfrak{c}$ such such that $g = \exp x \exp iz = \exp y$. Now $(x, z) - y \in \ker \gamma' \subseteq \mathfrak{t} \times (\mathfrak{t} \cap i\mathfrak{t})$. We conclude that $y \in \mathfrak{b}$, which leads to $\exp_G(iz) = \exp_G(y - x) \in T$. Therefore $iz \in \mathfrak{t} + \ker \gamma' \subseteq \mathfrak{t} \times (\mathfrak{t} \cap i\mathfrak{t})$ and consequently $z = 0$. Thus $g \in A \cap B = \{\mathbf{1}\}$. This proves that $A^* \cap B^* = \{\mathbf{1}\}$, so that $T^* \cong A^* \times B^*$ follows from Proposition 10.1.18.

(ii) Let $f\colon A^* \to \mathbb{C}$ be a holomorphic function. Then the holomorphic function

$$\widetilde{f}\colon \mathfrak{a}^* = \mathfrak{a} + i\mathfrak{a} \to \mathbb{C}, \quad x \mapsto f(\exp_{A^*} x)$$

is bounded on $\mathfrak{t} \cap i\mathfrak{t}$ because $\exp_G(\mathfrak{t} \cap i\mathfrak{t})$ is contained in the torus A. Therefore $\widetilde{f}$ is constant on $\mathfrak{t} \cap i\mathfrak{t}$ by Liouville's Theorem. Since f is continuous, it is constant on A, so that $\widetilde{f}$ is constant on $\mathfrak{a}$. Since $\mathfrak{a}$ contains a real form of the complex vector space $\mathfrak{a}^* = \mathfrak{a} + i\mathfrak{a}$, the Identity Theorem for Holomorphic Functions implies that $\widetilde{f}$ is constant, so that f is also constant.

(iii) Since all holomorphic functions on A^* are constant, all its holomorphic representations are trivial, i.e., $\mathrm{Lin}_{\mathbb{C}}(A^*) = A^*$. The direct product decomposition $G \cong A^* \times B^*$ with $B^* \cong B_{\mathbb{C}} \cong (\mathbb{C}^\times)^m$ for $m := \dim B$ implies that $\mathrm{Lin}_{\mathbb{C}}(G) = A^*$.

(iv) Since A^* is closed it is the smallest complex Lie subgroup of G containing $\exp_G(\mathfrak{t} \cap i\mathfrak{t})$. (v) follows from the proof of (i). □

Corollary 14.3.5. *A connected abelian complex Lie group G admits a faithful finite-dimensional holomorphic linear representation if and only if*

$$G \cong \mathbb{C}^n \times (\mathbb{C}^\times)^m \quad \textit{for some} \quad n, m \in \mathbb{N}_0.$$

Proof. It is clear that $\mathbb{C}^n \times (\mathbb{C}^\times)^m$ admits a faithful finite-dimensional linear representation (Exercise 14.3.1). If, conversely, G admits a faithful finite-dimensional linear representation, then $\mathrm{Lin}_{\mathbb{C}}(G) = \{\mathbf{1}\}$ and the assertion follows from Proposition 14.3.4(iii), combined with Lemma 14.2.12(i). □

Remark 14.3.6. As a consequence of Proposition 14.2.1(d), a complex torus $G := \mathbb{C}^n/(\mathbb{Z}^n + i\mathbb{Z}^n)$ has no faithful holomorphic representation. In this case $T = G$ and $\mathfrak{t} \cap i\mathfrak{t} = \mathfrak{g}$.

Proposition 14.3.7. *Any compact connected complex Lie group is abelian.*

Proof. If G is a compact connected complex Lie group, then Proposition 14.2.1 implies that each holomorphic representation of G is trivial. Therefore $\mathrm{Ad}(G) = \{\mathbf{1}\}$, and thus G is abelian. □

Definition 14.3.8. A complex connected Lie group G is called *toroidal* if it contains a connected normal subgroup H which is a complexified torus (i.e., isomorphic to $(\mathbb{C}^\times)^m$) such that G/H is compact.

Proposition 14.3.9. *For a connected complex Lie group G with the maximal torus T, the following assertions hold:*

(a) *G is toroidal if and only if $G = T^*$. In particular, toroidal groups are abelian.*

(b) *If G is toroidal and $C^* \subseteq G$ is a closed complexified torus such that G/C^* is compact, then $\mathfrak{t} = \mathfrak{c} \oplus (\mathfrak{t} \cap i\mathfrak{t})$.*

Proof. Let us first assume that G is a connected complex abelian Lie group with $G = T^*$. We recall the groups A, B and C from Proposition 14.3.4. Then

$$G/(C^* \times B^*) \cong A^*/C^* \cong T/(C^* \cap T)$$

is a torus. This means that the group A^* is an extension of a compact complex Lie group by a complexified torus.

Suppose, conversely, that G is toroidal and that $C^* \subseteq G$ is a closed complexified torus for which $M := G/C^*$ is compact. Then the fact that $C^* \cong C_{\mathbb{C}}$ is a complexified torus implies that $\mathfrak{g}$ is a semisimple C^*-module with respect to the adjoint action (Weyl's Unitary Trick 14.2.10). Therefore C^*, which acts trivially on $\mathbf{L}(C^*)$, being normal, also acts trivially on the quotient space, $\mathbf{L}(G)/\mathbf{L}(C^*) \cong \mathbf{L}(M)$. This implies that $C^* \subseteq \ker \mathrm{Ad} = Z(G)$ so that $\mathrm{Ad}(G)$ is a homomorphic image of M. But M has no nontrivial holomorphic representation. Thus $\mathrm{Ad}(G)$ is trivial, i.e., G is abelian.

(a),(b) Let $T \subseteq G$ be the unique maximal torus and C denote the maximal torus in C^*, so that $C \subseteq T$ and $C^* \cap T = C$. The latter relation implies in particular that $\mathbf{L}(C^*) \cong \mathfrak{c}_{\mathbb{C}}$ intersects $\mathfrak{t}$ only in $\mathfrak{c}$.

Since G/C^* is in particular a real torus, Proposition 13.3.13(a) implies that T maps surjectively onto G/C^*, so that $G = C^*T$. Hence $\mathfrak{g} = \mathfrak{t} + \mathfrak{c}^*$ implies that $\mathfrak{g} = \mathfrak{t}^*$ and therefore $G = T^*$. This completes the proof of (a).

To prove that $\mathfrak{t} = \mathfrak{c} \oplus (\mathfrak{t} \cap i\mathfrak{t})$, we note that $\mathfrak{g} = \mathfrak{t} + \mathfrak{c}^* = \mathfrak{t} \oplus i\mathfrak{c}$ implies $\mathfrak{g} = \mathfrak{c} \oplus i\mathfrak{t}$, so that we obtain for the subspace $\mathfrak{t}$ the relation $\mathfrak{t} = \mathfrak{c} \oplus (\mathfrak{t} \cap i\mathfrak{t})$. ☐

Example 14.3.10. In the additive group $\mathbb{C}^2$, we consider the discrete abelian group $\Gamma := \mathbb{Z}^2 \oplus \mathbb{Z}(i, \sqrt{2}i)$ and put $G := \mathbb{C}^2/\Gamma$. Then the Lie algebra of the maximal torus T is $\mathfrak{t} = \mathbb{R}^2 \oplus \mathbb{R}(i, \sqrt{2}i)$, so that $\mathfrak{t} \cap i\mathfrak{t} = \mathbb{C}(1, \sqrt{2})$. As $\exp_G(\mathfrak{t} \cap i\mathfrak{t})$ contains in particular the dense one-parameter group $\exp_G(\mathbb{R}(1, \sqrt{2}))$ in $\mathbb{R}^2/\mathbb{Z}^2$, it follows that $T = \overline{\exp_G(\mathfrak{t} \cap i\mathfrak{t})}$. In the terminology of Proposition 14.3.4, this means that $A = T$, so that we also have $G = T^* = A^*$, which implies that all holomorphic functions on G are constant. Since G is noncompact, actually isomorphic to $\mathbb{T}^3 \times \mathbb{R}$ as a real Lie group, this is a remarkable property.

Finally, we can prove the announced characterization of linearly complex reductive Lie groups in terms of their holomorphic representations.

Theorem 14.3.11 (Characterization of Linearly Complex Reductive Groups 2). *For a complex reductive Lie group G with finitely many connected components and with reductive Lie algebra $\mathfrak{g}$, the following are equivalent:*

(i) *G admits a faithful finite-dimensional holomorphic linear representation and all holomorphic representations of G are completely reducible.*
(ii) *$G = K^*$ for a maximal compact subgroup K and G admits a faithful finite-dimensional holomorphic linear representation.*

(iii) $Z(G_0)_0 \cong (\mathbb{C}^\times)^m$.
(iv) *G is the universal complexification $K_{\mathbb{C}}$ of a compact connected Lie group K, i.e., G is linearly complex reductive.*

Proof. (i) $\Rightarrow$ (ii): We have seen in Lemma 14.2.12 that $G \cong K^* \times V$, where $V \cong \mathbb{C}^n$ is a vector group. We claim that $V = \{\mathbf{1}\}$ if G is linearly complex reductive. For any linear functional α on $\mathfrak{v} = \mathbf{L}(V)$ and any linear operator $X \in \mathfrak{gl}_n(\mathbb{C})$, we obtain a holomorphic representation $\gamma\colon G \to \mathrm{GL}_n(\mathbb{C})$ by $\gamma(k, \exp z) = e^{\alpha(z)X}$ for $z \in \mathfrak{v}$ and $k \in K^*$. If X is not semisimple, then this representation is not completely reducible. Therefore $V = \{\mathbf{1}\}$, i.e., $G = K^*$ is a necessary condition for G to be linearly complex reductive.

(ii) $\Rightarrow$ (iii): Using Lemma 14.2.12, we conclude from $G = K^*$ that $Z := Z(G_0)_0 = T_Z^*$ holds for a maximal torus T_Z in Z. Since Z admits a faithful finite-dimensional holomorphic representation, Corollary 14.3.5 yields $Z \cong (\mathbb{C}^\times)^m$.

(iii) $\Rightarrow$ (iv): First we use Lemma 14.2.12(i) to see that (iii) implies that $G = K^*$. Write $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}'$. Then $\mathfrak{k}'$ is a compact real form of $\mathfrak{g}'$. Moreover, the facts that $Z(K_0)_0 \subseteq Z(K_0^*) = Z(G_0)$ and $T_Z := K \cap Z$ is a maximal torus in Z (Proposition 13.3.13(b)) imply that $\mathfrak{z}(\mathfrak{k}) = \mathbf{L}(T_Z) = \mathfrak{k} \cap \mathfrak{z}(\mathfrak{g})$. Therefore $\mathfrak{k} \cap i\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \cap i\mathfrak{z}(\mathfrak{k}) = \{0\}$ follows from (iii). Now Proposition 14.2.4 shows that the induced surjective map $K_{\mathbb{C}} \to K^* = G$ is injective, hence an isomorphism.

(iv) $\Rightarrow$ (i) is Weyl's Unitary Trick 14.2.10. □

Corollary 14.3.12. *If G is a connected complex reductive Lie group, then*

$$\mathrm{Lin}_{\mathbb{C}}(G) = \mathrm{Lin}_{\mathbb{C}}\big(Z(G)_0\big),$$

and this subgroup is connected.

Proof. In view of Proposition 14.3.4, the group $\mathrm{Lin}_{\mathbb{C}}\big(Z(G)_0\big)$ is connected. The inclusion $\mathrm{Lin}_{\mathbb{C}}\big(Z(G)_0\big) \subseteq \mathrm{Lin}_{\mathbb{C}}(G)$ is trivial.

Let $L := \mathrm{Lin}_{\mathbb{C}}\big(Z(G)_0\big)$. To complete the proof, it suffices to show that $G_1 := G/L$ admits a faithful finite-dimensional holomorphic linear representation. We have $Z(G_1)_0 \cong Z(G)_0/L$ and this group admits a faithful finite-dimensional holomorphic linear representation by Lemma 14.2.14. Now we write $G_1 = V_1 \times K_1^*$ according to Lemma 14.2.12(ii). Then $Z(K_1^*)_0$ is contained in $Z(G_1)_0$. Finally, Theorem 14.3.11 shows that K_1^* and therefore G_1 admits a faithful finite-dimensional holomorphic linear representation. □

Exercises for Section 14.3

Exercise 14.3.1. Show that the product $\mathbb{C}^n \times (\mathbb{C}^\times)^m$ is a linear complex Lie group.

14.4 The Automorphism Group of a Complex Lie Group

In this short section we use the results on automorphism groups of real Lie groups to show that whenever the automorphism group of the underlying real Lie group carries a Lie group structure, then we actually obtain a complex Lie group structure.

Proposition 14.4.1. *If G is a complex Lie group for which the space* $\mathrm{IAut}(G)$ *of infinitesimal automorphisms* (cf. Definition 10.3.6) *is finite-dimensional, then* $\mathrm{Aut}(G)$ *carries the structure of a complex Lie group with Lie algebra* $\mathrm{IAut}(G)$.

Proof. The description of $\mathrm{IAut}(G)$ as the space $Z^1(G, \mathfrak{g})$ of $\mathfrak{g}$-valued 1-cocycles given in Remark 10.3.9 shows immediately that the complex structure on $\mathfrak{g}$ also provides a complex structure on this space.

For $X \in \mathrm{IAut}(G)$ with $X(g) = \alpha(g)g$ for $g \in G$ and $\varphi \in \mathrm{Aut}(G)$, we find the transformation formula

$$(\varphi_* X)(g) = \alpha_\varphi(g)g \quad \text{with} \quad \alpha_\varphi(g) = \mathbf{L}(\varphi) \circ \alpha \circ \varphi^{-1}.$$

This formula shows that the adjoint action of $\mathrm{Aut}(G)$ on its Lie algebra $\mathrm{IAut}(G)$ is an action by complex linear maps. Since the Lie bracket on $\mathrm{IAut}(G)$ can be obtained from the formula for the Lie derivative, it follows that all operators $\mathrm{ad}\, X$ are also complex linear, so that the Lie bracket is complex bilinear. Therefore $\mathrm{IAut}(G)$ is a complex Lie algebra and $\mathrm{Aut}(G)$ is a complex Lie group. □

Notes on Chapter 14

The product decomposition in Proposition 14.3.4(ii) was first obtained by Morimoto ([Mo66, Thm. 3.2]). He also obtained far reaching classification results of groups of type $\mathrm{Lin}_{\mathbb{C}}(T^*)$ which are called *H.C. groups* in [Mo66] because every holomorphic function on such a group is constant (cf. Proposition 14.3.4(iii)). His results include in particular a classification of all connected complex abelian Lie groups up to dimension 3.

It is quite remarkable that the exponential function of a complex Lie group behaves quite differently from the one for real Lie groups. A particularly interesting fact is that the groups $\mathrm{PSL}_n(\mathbb{C})$ are the only simple complex Lie groups with a surjective exponential function ([Lai77, Lai78]). This result has recently been put in a much more general perspective by Moskowitz and Sacksteder who proved that the exponential function of a connected complex Lie group G is surjective if and only if its center $Z(G)$ is connected and the adjoint group $\mathrm{Ad}(G)$ has a surjective exponential function ([MS08]). That these two conditions are sufficient for the exponential function to be surjective is an easy exercise, and it is also clear that the surjectivity of the exponential function of G implies the same property for all quotient groups, hence in particular for $\mathrm{Ad}(G)$. In [Wu98] one finds a similar criterion for the surjectivity of the exponential function of a real solvable Lie group.

15

Linearity of Lie Groups

In this chapter we will make a connection to the topic of the first chapters by characterizing the connected Lie groups which admit faithful finite-dimensional representations. Eventually, it turns out that these are precisely the semidirect products of normal simply connected solvable Lie groups with linearly real reductive Lie groups, where the latter ones are, by definition, groups with reductive Lie algebra, compact center and a faithful finite dimensional representation. We complement this result by several other characterizations, e.g. in terms of linearizers or properties of a Levi decomposition. Moreover, we characterize the complex Lie groups which admit finite-dimensional *holomorphic* linear representations, thus completing the discussion from Chapter 14.

15.1 Linearly Real Reductive Lie Groups

In this section we carry out our program for the case of real Lie groups with reductive Lie algebra. We start, however, by introducing the real analog of the linearizer $\mathrm{Lin}_{\mathbb{C}}(G)$.

Definition 15.1.1. As for complex Lie groups, we also define for a real Lie group G its *linearizer* as the subgroup $\mathrm{Lin}_{\mathbb{R}}(G)$ which is the intersection of the kernels of all finite-dimensional continuous representations of G. This subgroup is the obstruction for G to admit a faithful finite-dimensional linear representation. In particular, we shall see that $\mathrm{Lin}_{\mathbb{R}}(G) = \{\mathbf{1}\}$ characterizes such groups.

Remark 15.1.2. Let G be a real Lie group and $\eta_G \colon G \to G_{\mathbb{C}}$ be its universal complexification.

(a) Every representation $\rho \colon G \to \mathrm{GL}(V)$ on a complex vector space V determines a holomorphic representation $\rho_{\mathbb{C}} \colon G_{\mathbb{C}} \to \mathrm{GL}(V)$ with $\rho_{\mathbb{C}} \circ \eta_G = \rho$. Therefore $\ker \eta_G \subseteq \ker \rho$, and this implies that

$$\ker \eta_G \subseteq \mathrm{Lin}_{\mathbb{R}}(G).$$

(b) If G is connected, then $\ker \mathrm{Ad} = Z(G)$ implies that

$$\mathrm{Lin}_{\mathbb{R}}(G) \subseteq Z(G).$$

In general, we have

$$\mathrm{Lin}_{\mathbb{R}}(G) \subseteq \ker \mathrm{Ad} = Z_G(G_0).$$

It is easy to determine the linearizer for semisimple Lie groups.

Proposition 15.1.3. *For a semisimple real Lie group G with finitely many connected components, we have* $\mathrm{Lin}_{\mathbb{R}}(G) = \ker \eta_G$.

Proof. In view of Remark 15.1.2, we have $\ker \eta_G \subseteq \mathrm{Lin}_{\mathbb{R}}(G)$. By Proposition 14.2.6 the universal complexification $G_{\mathbb{C}}$ admits a faithful finite-dimensional representation. Therefore we also have $\mathrm{Lin}_{\mathbb{R}}(G) \subseteq \ker \eta_G$. □

Definition 15.1.4. A connected real Lie group H is called *linearly real reductive*, if

(1) $\mathbf{L}(H)$ is reductive,
(2) $Z(H)$ is compact[1] and
(3) H admits a faithful finite-dimensional representation.

The following proposition characterizes the linearly real reductive Lie groups among linear Lie groups.

Proposition 15.1.5. *For a closed connected subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$, the following statements are equivalent:*

(i) *G is linearly real reductive.*
(ii) *There is a scalar product γ on $\mathbb{R}^n$ such that G is invariant under transposition and $Z(G)_0$ is a compact subgroup of $\mathrm{SO}(\mathbb{R}^n, \gamma)$.*

Proof. (i) ⇒ (ii): By definition, the Lie algebra $\mathfrak{g}$ of G is reductive, i.e., $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$, where $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$. Now we choose a Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}' = \mathfrak{k} + \mathfrak{p}$ (Theorem 12.2.10). Then $\mathfrak{u} := \mathfrak{z} + \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{gl}_n(\mathbb{C})$ is a subalgebra and $\mathfrak{u}' = \mathfrak{k} + i\mathfrak{p}$ is compact and semisimple. Hence the group $U := \langle \exp \mathfrak{u} \rangle \subseteq \mathrm{GL}_n(\mathbb{C})$ is compact and there exists a U-invariant positive definite hermitian form β on $\mathbb{C}^n$ (Theorem 9.4.14). Therefore the operators in $\mathfrak{u}$ are skew-hermitian, which implies that the elements of $\mathfrak{p}$ are hermitian. This entails that $\mathfrak{g}$ is invariant under $*$, which coincides with transposition with respect to the scalar product $\gamma = \mathrm{Re}\, \beta$ on $\mathbb{R}^n$. The invariance of G under $\top$ now follows from $(e^x)^\top = e^{x^\top}$ for $x \in \mathfrak{g}$. That $Z(G)_0$ is contained in $\mathrm{U}(\mathbb{C}^n, \beta)$ follows from the connectedness of $Z(G)_0$. Since $\mathrm{SO}(\mathbb{R}^n, \gamma) = \mathrm{O}(\mathbb{R}^n, \gamma)_0$ (Proposition 1.1.7), the claim follows.

[1] Compare this to Definition 14.2.7

(ii) ⇒ (i): We only have to show that $\mathfrak{g} := \mathbf{L}(G)$ is reductive if $\mathfrak{g}$ is invariant under transposition. We set

$$\mathfrak{k} := \{x \in \mathfrak{g} \colon x^\top = -x\} \quad \text{and} \quad \mathfrak{p} := \{x \in \mathfrak{g} \colon x^\top = x\}.$$

Since $\theta(x) := -x^\top$ is an automorphism of $\mathfrak{g}$, $\mathfrak{k}$ is a subalgebra,

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. We have to show that $\mathfrak{r} \subseteq \mathfrak{z}(\mathfrak{g})$. The radical, as a characteristic ideal, is also θ-invariant, so that $\mathfrak{r} = (\mathfrak{r} \cap \mathfrak{k}) \oplus (\mathfrak{r} \cap \mathfrak{p})$. Then $\widetilde{\mathfrak{r}} := \mathfrak{r} \cap \mathfrak{k} + i(\mathfrak{r} \cap \mathfrak{p}) \subseteq \widetilde{\mathfrak{g}} := \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{u}_n(\mathbb{C})$ is a solvable ideal of the compact Lie algebra $\widetilde{\mathfrak{g}}$, hence central. This implies that $\mathfrak{r} \subseteq \mathfrak{z}(\mathfrak{g})$. □

The problem of characterizing linearly real reductive Lie groups can be reduced to the semisimple case via the results in Section 14.1.

Proposition 15.1.6. *A connected real Lie group H with reductive Lie algebra $\mathfrak{h}$ for which $Z(H)_0$ is compact is linearly real reductive if and only if the commutator subgroup H' is linearly real reductive.*

Proof. If H is linearly real reductive, then H' also is linearly real reductive because it is semisimple and admits a faithful finite-dimensional representation, so that its center is finite (Corollary 12.2.6).

Conversely, assume that $\mathfrak{h}$ is reductive, H' is linearly real reductive and $Z := Z(H)_0$ is compact. Since H' admits a faithful finite-dimensional representation, $\eta_{H'} \colon H' \to H'_{\mathbb{C}}$ is injective (Proposition 14.1.8). This also holds for $\eta_Z \colon Z \to Z_{\mathbb{C}}$ (Proposition 14.2.1), and $\eta_Z(Z)$ is a maximal compact subgroup of $Z_{\mathbb{C}}$. Hence $\eta_{Z \times H'}$ is injective. This property is inherited by quotients with respect to discrete central subgroups (Exercise 14.1.2), and therefore also by $H = ZH'$ because the kernel of the multiplication map $Z \times H' \to H$ is discrete and central. Thus $\eta_H \colon H \to H_{\mathbb{C}}$ is injective.

Now it suffices to show that $H_{\mathbb{C}}$ admits a faithful finite dimensional representation. It is a connected complex reductive Lie group whose center $Z(H_{\mathbb{C}})_0 \cong Z_{\mathbb{C}}$ is a complexified torus. Therefore Theorem 14.2.9 implies that $H_{\mathbb{C}}$ is linearly complex reductive, hence by Proposition 14.2.1 admits a faithful finite-dimensional representation. □

Proposition 15.1.7. *Let G be a connected semisimple Lie group, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of its Lie algebra $\mathfrak{g}$ and $K := \langle \exp_G \mathfrak{k} \rangle$. Further, let $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ be the corresponding compact real form of $\mathfrak{g}_{\mathbb{C}}$ and U be the corresponding simply connected compact group. Then the following are equivalent:*

(i) *G admits a faithful finite-dimensional representation.*
(ii) *G is linearly real reductive.*
(iii) *The universal complexification $\eta_G \colon G \to G_{\mathbb{C}}$ is injective.*
(iv) *$K_U := \langle \exp_U \mathfrak{k} \rangle$ is a covering of K.*

This property is inherited by quotients G/D, where D is a discrete central subgroup of G.

Proof. The equivalence of (i) and (ii) follows from the fact that the center of semisimple Lie groups which admit faithful finite-dimensional representations is finite (Corollary 12.2.6). The equivalence of (i) and (iii) follows from Proposition 14.1.8 and the fact that $G_{\mathbb{C}}$ admits a faithful finite-dimensional representation (Proposition 14.2.6).

(iii) ⇒ (iv): From (iii) we derive that $K \cong \eta_G(K)$ is equal to $\langle \exp_{G_{\mathbb{C}}} \mathfrak{k} \rangle \subseteq \langle \exp_{G_{\mathbb{C}}}(\mathfrak{k} + i\mathfrak{p}) \rangle$, and the latter group is a quotient of U.

(iv) ⇒ (iii): Suppose that $\pi \colon K_U \to K$ is a covering, and let $\eta_G \colon G \to G_{\mathbb{C}}$ be the universal complexification of G. Further, let $\eta_{\widetilde{G}} \colon \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ be the universal complexification of the universal covering group $\widetilde{G}$ of G. We have to show that $\ker \eta_{\widetilde{G}}$ is contained in the kernel of the covering map $q_G \colon \widetilde{G} \to G$, because this implies that η_G is injective (Exercise 14.1.2). Recall that $\widetilde{G}_{\mathbb{C}}$ is isomorphic to the universal complexification $U_{\mathbb{C}}$ of U (Remark 14.1.5) since this group is simply connected and $\mathbf{L}(U_{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}} = \mathbf{L}(\widetilde{G}_{\mathbb{C}})$. Therefore, in view of Proposition 14.2.1, we can identify U with the corresponding integral subgroup of $\widetilde{G}_{\mathbb{C}}$. Now let $d \in \ker \eta_{\widetilde{G}}$, which is a discrete normal subgroup, hence central (Exercise 8.4.4). By Lemma 12.1.6, there is an $x \in \mathfrak{k}$ with $d = \exp_{\widetilde{G}} x$, so that

$$\exp_U x = \exp_{\widetilde{G}_{\mathbb{C}}} x = \eta_{\widetilde{G}}(\exp_{\widetilde{G}} x) = \eta_{\widetilde{G}}(d) = \mathbf{1}.$$

Therefore $q_G(d) = \exp_G x = \exp_K x = \pi(\exp_U x) = \mathbf{1}$. Hence $d \in \ker q_G$.

The last claim follows from the fact that the injectivity of η_G is inherited by quotients with respect to discrete central subgroups (Exercise 14.1.2). □

Example 15.1.8. The preceding criteria can be checked quite easily, provided the simply connected group U with Lie algebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ is known. For groups of the form $G = \widetilde{\mathrm{SL}_2(\mathbb{R})}/D$, D discrete central, we have $U \cong \mathrm{SU}_2(\mathbb{C})$ with $Z(U) = \{\pm \mathbf{1}\}$. Hence G admits a faithful finite-dimensional linear representation if and only if $|Z(\widetilde{G})/D| \leq 2$.

The following proposition generalizes the polar decomposition of $\mathrm{GL}_n(\mathbb{C})$ to linearly real reductive Lie groups. It will be an important tool later on.

Proposition 15.1.9 (Polar Decomposition of linearly real reductive Lie groups). *Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a subgroup which is a zero set of polynomials in the $2n^2$ real matrix entries, and which is invariant under the transposition $g \mapsto g^* = \overline{g}^\top$. We set*

$$\mathfrak{k} := \{x \in \mathfrak{g} \colon x^* = -x\}, \qquad \mathfrak{p} := \{x \in \mathfrak{g} \colon x^* = x\},$$

and $K := G \cap \mathrm{U}_n(\mathbb{C})$. Then $\mathbf{L}(K) = \mathfrak{k}$, the subgroup K is maximal compact in G, and the map

$$\Phi\colon K \times \mathfrak{p} \to G, \qquad (k,x) \mapsto k \exp x$$

is a diffeomorphism.

Proof. It follows from Proposition 3.3.3 that Φ is a homeomorphism. Clearly, Φ is smooth. Its inverse is given by

$$\Phi^{-1}(g) = \Big(ge^{-\frac{1}{2}\log(g^*g)}, \frac{1}{2}\log(g^*g)\Big),$$

and since $\log\colon \mathrm{Pd}_n(\mathbb{C}) \to \mathrm{Herm}_n(\mathbb{C})$ is a diffeomorphism by Proposition 2.3.5, Φ^{-1} is a smooth map and thus Φ is a diffeomorphism. □

Proposition 15.1.10. *A semisimple connected Lie group G is linearly real reductive if and only if all simple normal integral subgroups are linearly real reductive.*

Proof. For semisimple connected Lie groups, linear real reductivity is equivalent to the existence of faithful finite-dimensional linear representations since then the center is finite (Proposition 15.1.7). Evidently, this property is inherited by all simple integral subgroups.

Conversely, assume that all simple normal integral subgroups $G_1, \ldots, G_n$ are linearly real reductive. Then their complexification maps η_{G_i} are injective, a property that is inherited by the product group $H := G_1 \times \ldots \times G_n$. Since the injectivity of the complexification map is inherited by finite quotients (Exercise 14.1.2) and the multiplication map $H \to G$ is a finite covering, η_G is also injective. Now Proposition 15.1.7 shows that G is linearly real reductive. □

15.2 The Existence of Faithful Finite-dimensional Representations

In this section we characterize connected Lie groups with a faithful finite-dimensional representation in terms of semidirect product decompositions. At the same time we show that the existence of faithful finite-dimensional representation is characterized by $\mathrm{Lin}_{\mathbb{R}}(G) = \{\mathbf{1}\}$. Evidently, our task splits into two subproblems. First, we have to show that every Lie group with $\mathrm{Lin}_{\mathbb{R}}(G) = \{\mathbf{1}\}$ has a semidirect product structure as described at the beginning of this chapter. Here we shall make use of the theory of compactly embedded subalgebras, which has already been fruitful to obtain the manifold splitting. Conversely, we have to prove that every such semidirect product has a faithful finite-dimensional representation. Here the essential idea is to use Ado's Theorem to find a representation which is injective on the maximal normal nilpotent integral subgroup. This eventually leads to an inductive argument. We start with the solvable case.

Definition 15.2.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$, and $\mathfrak{n}$ be the maximal nilpotent ideal. We define the *radical* of G by $R := \langle \exp \mathfrak{r} \rangle$ and write $N := \langle \exp \mathfrak{n} \rangle$ for the maximal normal nilpotent integral subgroup of G.

Proposition 15.2.2. *The normal integral subgroups R and N of G are closed.*

Proof. We may w.l.o.g. assume that G is connected. Then $\overline{N}$ and $\overline{R}$ are normal subgroups. Moreover, $\overline{N}$ is nilpotent and $\overline{R}$ is solvable by Exercise 10.2.2. Hence the Lie algebra of $\overline{N}$ is a nilpotent ideal of $\mathfrak{g}$, so that the maximality of $\mathfrak{n}$ implies that $\overline{N} = N$. Similarly we argue that R is closed. ☐

Lemma 15.2.3. *Let R be a connected solvable Lie group and $T \subseteq R$ be a maximal torus.*

(i) $R' \cap T \subseteq \mathrm{Lin}_{\mathbb{R}}(R)$*, where R' is the commutator subgroup of R.*
(ii) *If* $\mathrm{Lin}_{\mathbb{R}}(R) = \{\mathbf{1}\}$*, then*

$$R \cong B \rtimes T,$$

holds for some simply connected normal subgroup B.

Proof. Let $\pi\colon R \to \mathrm{GL}_n(\mathbb{C})$ be a representation. Let $\mathfrak{r} := \mathbf{L}(R)$ and $\mathfrak{t} := \mathbf{L}(T)$. By Lie's Theorem 4.4.8, there exists a complete flag in $\mathbb{C}^n$ which is invariant under $\mathfrak{r}$. Therefore, we may assume that $\mathfrak{r}$, and thus also R, is represented by upper triangular matrices. Then $\mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}]$ is represented by strictly upper triangular matrices and $\pi(R')$ consists of unipotent upper triangular matrices. In particular, $\pi(R')$ cannot contain nontrivial compact subgroups. This implies that $R' \cap T \subseteq \ker \pi$, and hence proves (i).

Under the hypotheses of (ii) we have $R' \cap T \subseteq \mathrm{Lin}_{\mathbb{R}}(R) = \{\mathbf{1}\}$. Then Corollary 13.5.6(c) implies that R' is closed. Let $p\colon R \to R/R'$ be the quotient map. Then $p(T)$ is a maximal torus in the abelian group R/R' (Theorem 13.3.13). Hence there exists a vector subgroup V of R/R' with $R/R' \cong p(T) \times V$ (Theorem 10.2.10), and we set $B := p^{-1}(V)$. Then B is a closed normal subgroup of R with $BT = R$. If $g \in B \cap T$, then $p(g) \in p(B) \cap p(T) = V \cap p(T) = \{\mathbf{1}\}$, and therefore $g \in R' \cap T = \{\mathbf{1}\}$. Hence $B \cap T = \{\mathbf{1}\}$, which leads to $R \cong B \rtimes T$ (Proposition 10.1.18). Since R/T is simply connected (Theorem 13.3.8), this also holds for B. ☐

Proposition 15.2.4. *If a connected Lie group G satisfies* $\mathrm{Lin}_{\mathbb{R}}(G) = \{\mathbf{1}\}$*, then*

$$G \cong B \rtimes H$$

holds for some simply connected solvable group B and some linearly real reductive group H.

Proof. Let R be the radical of G and T be a maximal torus in R. By Lemma 15.2.3, we have a semidirect decomposition $R \cong M \rtimes T$, where M is simply connected and solvable. Now we choose a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing $\mathfrak{t} := \mathbf{L}(T)$, and further a Levi complement $\mathfrak{s}$ in $\mathfrak{g}$ with

$$[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s} \quad \text{and} \quad [\mathfrak{s}, \mathfrak{k} \cap \mathfrak{r}] = \{0\}$$

(Lemma 13.3.3). We then have $[\mathfrak{s}, \mathfrak{t}] = \{0\}$, and the subspace $\mathfrak{a} := \mathfrak{t} + \mathfrak{r}' \cong \mathfrak{t} \oplus \mathfrak{r}'$ of $\mathfrak{r}$ is invariant under $\mathfrak{s}$. Since the $\mathfrak{s}$-module $\mathfrak{r}$ is semisimple by Weyl's Theorem 4.5.21, there exists an $\mathfrak{s}$-submodule $\mathfrak{a}_1$ complementing $\mathfrak{a}$ (Proposition 4.5.18). Since $\mathfrak{b} := \mathfrak{r}' + \mathfrak{a}_1$ is invariant under $\mathfrak{s}$ and $\mathfrak{r}$, it is an ideal of $\mathfrak{g}$. We now have $\mathfrak{r} \cong \mathfrak{b} \rtimes \mathfrak{t}$ and put $B := \langle \exp \mathfrak{b} \rangle$ to obtain $BT = R$.

To show that B is closed, we consider for the subgroup $A := TR'$ the surjective homomorphism

$$\alpha \colon B/R' \to R/A \cong M/R', \qquad xR' \mapsto xA.$$

Note that R' is closed as a normal subgroup of the simply connected solvable Lie group M (Theorem 10.1.21). Thus $A = TR'$ is also closed. The group M/R' is abelian and simply connected (Theorem 10.1.21), hence a vector group. The kernel of α is discrete because $\mathfrak{b} \cap (\mathfrak{a} + \mathfrak{r}') = \mathfrak{r}'$. Thus α is a covering, and therefore a diffeomorphism since M/R' is simply connected. In particular, B/R' is simply connected and contains no nontrivial compact subgroups since each of these groups would be contained in a maximal torus, hence trivial (Theorem 13.4.1). We conclude that $B \cap T \subseteq R' \cap T = \{\mathbf{1}\}$. This implies in particular that B is closed in R because it is normal (Corollary 13.5.6), and hence it is also closed in G. Finally, $R \cong B \rtimes T$ follows from Proposition 10.1.18. Since B is homeomorphic to R/T, we see that B is simply connected (Theorem 13.4.1).

Set $S := \langle \exp \mathfrak{s} \rangle$ and $H := ST = \langle \exp \mathfrak{s} \oplus \mathfrak{t} \rangle$. Then

$$\mathrm{Lin}_{\mathbb{R}}(S) \subseteq S \cap \mathrm{Lin}_{\mathbb{R}}(G) = \{\mathbf{1}\}$$

in view of Proposition 15.1.3, and Theorem 14.3.11 (see also Corollary 12.2.6) implies that S is linearly real reductive and in particular has finite center. The normal subgroup $S \cap R$ is discrete in S, hence central, and thus finite. Therefore the group $(S \cap R)T$ is compact (Exercise 12.3.5). Since

$$BH = BTS = RS = G \quad \text{and} \quad H \cap B = (ST) \cap B = \big((S \cap R)T\big) \cap B,$$

this implies that $B \cap H$ is a compact subgroup of B, hence trivial. Consider the semidirect product $B \rtimes H$, where H acts on B by conjugation, and B and H are equipped with their natural Lie group structures as integral subgroups of G. Then the homomorphism

$$B \rtimes H \to BH = G, \quad (b, h) \to bh$$

is a smooth bijection, hence an isomorphism by the Open Mapping Theorem 10.1.8. In particular, H is closed in G.

To see that H is linearly real reductive, we observe first that $\mathbf{L}(H) = \mathfrak{s} \oplus \mathfrak{t}$ is reductive. Moreover, $Z(H) = Z(S)T$ is compact because $Z(S)$ is finite, as we have seen above. It remains to be seen that H admits a faithful finite-dimensional representation. But this follows from Proposition 15.1.6, since we have already seen that $S = H'$ is linearly real reductive. □

The preceding proposition constitutes one half of the characterization of connected Lie groups admitting faithful finite dimensional representations. For the other half, we have to construct representations of semidirect products. The following lemma describes an important method to find such representations. We recall that the conjugation with an element g is denoted by c_g, and the right multiplication is denoted by ρ_g.

Lemma 15.2.5. *Let $G = B \rtimes H$ be a semidirect product of Lie groups. Then the following assertions hold:*

(i) *For $g = (b,h) \in G$, the map $\alpha(b,h) := \lambda_b \circ c_h|_B$ defines a smooth action of G on B.*
(ii) *The maps*

$$\pi(g)\colon C(B) \to C(B), \quad f \mapsto f \circ \alpha(g^{-1})$$

define a representation of G on the space $C(B)$ of continuous functions on B.
(iii) *Let $\beta\colon G \to \mathrm{GL}(V)$ be a continuous representation of G and*

$$F := \{\omega \circ \beta|_B \colon \omega \in \mathrm{End}(V)^*\} \subseteq C(B).$$

Then F is a finite-dimensional G-invariant subspace of $C(B)$. Furthermore, $\pi(b)|_F \neq \mathrm{id}_F$ if $\beta(b) \neq \mathrm{id}_V$.

Proof. (i) It is clear that the map $(g,b) \mapsto \alpha(g)b$ is smooth and that each $\alpha(g)$ maps B into itself. Let $g = (b,h)$ and $g' = (b',h')$. Then

$$\begin{aligned} \alpha(gg') &= \lambda_{bhb'h^{-1}} \circ c_{hh'} = \lambda_b \circ c_h \circ \lambda_{b'} \circ c_h^{-1} \circ c_{hh'} \\ &= \lambda_b \circ c_h \circ \lambda_{b'} \circ c_{h'} = \alpha(b,h) \circ \alpha(b',h') = \alpha(g)\alpha(g'). \end{aligned}$$

(ii) follows immediately from (i).

(iii) Since V is finite-dimensional, so is $\mathrm{End}(V)$, and thus F is finite-dimensional. For $\omega \in \mathrm{End}(V)^*$, $h \in H$ and $x \in B$, we have

$$\big(\pi(h)(\omega \circ \beta)\big)(x) = (\omega \circ \beta)\big(c_{h^{-1}}(x)\big) = (\omega \circ c_{\beta(h^{-1})})\big(\beta(x)\big),$$

and $c_{\beta(h^{-1})}\colon \mathrm{End}(V) \to \mathrm{End}(V)$ is linear. For $b \in B$, we obtain

$$\big(\pi(b)(\omega \circ \beta)\big)(x) = \omega \circ \beta \circ \lambda_{b^{-1}}(x) = (\omega \circ \lambda_{\beta(b^{-1})})\big(\beta(x)\big).$$

Since $\lambda_{\beta(b)}\colon \mathrm{End}(V) \to \mathrm{End}(V)$ is also linear, we see that F is invariant under $\pi(G)$.

If $b \in B$ satisfies $\pi(b)|_F = \mathrm{id}_F$, then the above calculation shows that

$$\omega(\mathbf{1}) = \omega \circ \beta(\mathbf{1}) = (\omega \circ \lambda_{\beta(b)} \circ \beta)(\mathbf{1}) = \omega(\beta(b))$$

for all $\omega \in \mathrm{End}(V)^*$, and hence $\beta(b) = \mathbf{1}$. □

Proposition 15.2.6. *If $G = B \rtimes H$ is a semidirect product, where B is simply connected solvable and H is linearly real reductive, then G admits a faithful finite-dimensional representation.*

Proof. We argue by induction on $\dim B$. For $\dim B = 0$, the claim follows by the definition of a linearly real reductive group. So let $\dim B > 0$, and let N be the largest normal nilpotent integral subgroup of B. Then $N \neq \{\mathbf{1}\}$ and therefore $Z(N) \neq \{\mathbf{1}\}$. The subgroups N and $Z(N)$ are both characteristic in B, i.e., invariant under all automorphisms. Therefore both groups are normal in G, simply connected, and the quotient $B/Z(N)$ likewise is simply connected (Theorem 10.1.21). Now we apply the induction hypothesis to $G/Z(N) \cong \big(B/Z(N)\big) \rtimes H$ to obtain a faithful finite-dimensional representation $\widetilde{\alpha}_1 \colon G/Z(N) \to \mathrm{GL}_{n_1}(\mathbb{R})$. This leads to a representation

$$\alpha_1 \colon G \to \mathrm{GL}_{n_1}(\mathbb{R}) \qquad \text{with} \qquad \ker \alpha_1 = Z(N).$$

Ado's Theorem 6.4.1 provides a faithful finite dimensional representation $\alpha \colon \mathfrak{g} := \mathbf{L}(G) \to \mathfrak{gl}(V)$ such that V is a nilpotent module over the maximal nilpotent ideal of $\mathfrak{g}$. Since $\mathbf{L}\big(Z(N)\big)$ is a nilpotent ideal of $\mathfrak{g}$, $\alpha\big(\mathbf{L}\big(Z(N)\big)\big) \subseteq \mathfrak{gl}(V)$ consists of nilpotent endomorphisms. We now integrate α to a representation $\beta \colon \widetilde{G} \cong B \rtimes \widetilde{H} \to \mathrm{GL}(V)$ of the simply connected covering group of G, satisfying $\mathbf{L}(\beta) = \alpha$. Here we use Proposition 10.1.19 and Corollary 10.1.20 to see that $\widetilde{G} = B \rtimes \widetilde{H}$. For $0 \neq x \in \mathbf{L}\big(Z(N)\big)$, we then have

$$\beta(\exp x) = e^{\alpha(x)} \neq \mathrm{id}_V$$

since $\alpha(x) \neq 0$ is nilpotent. Now we apply Lemma 15.2.5 to find a finite-dimensional representation $\pi \colon \widetilde{G} \to \mathrm{GL}(F)$ on a space F of continuous functions on B such that the restriction to $Z(N)$ is injective, and

$$\pi(h)f = f \circ c_{h^{-1}} \quad \text{for} \quad h \in H.$$

Now let $q_G \colon \widetilde{G} = B \rtimes \widetilde{H} \to G = B \rtimes H$ be the universal covering map, and let $h \in \widetilde{H}$ be such that $q_G(h) = \mathbf{1}$. Then $h \in Z(\widetilde{G})$, and consequently, $c_h = \mathrm{id}_{\widetilde{G}}$. Therefore $\pi(\ker q_G) = \{\mathbf{1}\}$, and the representation π factors to a representation α_2 of G which is injective on $Z(N) \subseteq B$. We now define

$$\alpha \colon g \mapsto \alpha_1(g) \oplus \alpha_2(g) \in \mathrm{GL}\big(\mathbb{R}^n \oplus F\big).$$

The kernel of this representation is $\ker \alpha_1 \cap \ker \alpha_2 = \{\mathbf{1}\}$. □

Combining Propositions 15.2.4 and 15.2.6, we get:

Theorem 15.2.7 (Existence of Faithful Finite Dimensional Representations). *For a connected Lie group G the following statements are equivalent.*

(i) *G admits a faithful finite-dimensional representation.*
(ii) $\mathrm{Lin}_{\mathbb{R}}(G) = \{\mathbf{1}\}$.
(iii) *$G = B \rtimes H$, where B is simply connected solvable and H is linearly real reductive. Here $Z(H)_0$ is a maximal torus in the radical R of G and $R \cong B \rtimes Z(H)_0$.*

Corollary 15.2.8. *If a Lie group G admits a faithful finite-dimensional representation, then the commutator group G' is closed in G.*

Proof. In view of the Characterization Theorem 15.2.7, $G = B \rtimes H$ for a linearly real reductive group H and a simply connected solvable group B. Let R be the radical of G, and note that $S = H'$ is a maximal semisimple integral subgroup. On the Lie algebra level, we have a decomposition

$$\mathfrak{g} = \mathfrak{b} + \mathfrak{z} + \mathfrak{s},$$

where $\mathfrak{b} := \mathbf{L}(B)$, $\mathfrak{s} = \mathbf{L}(S)$, and $\mathfrak{z} = \mathfrak{z}\big(\mathbf{L}(H)\big)$. Hence

$$\mathfrak{g}' = [\mathfrak{b}+\mathfrak{z}+\mathfrak{s}, \mathfrak{b}+\mathfrak{z}+\mathfrak{s}] \subseteq [\mathfrak{b}+\mathfrak{z}, \mathfrak{b}+\mathfrak{z}]+[\mathfrak{s},\mathfrak{s}]+[\mathfrak{b}+\mathfrak{z},\mathfrak{s}] \subseteq \mathfrak{b}+[\mathfrak{b},\mathfrak{z}]+[\mathfrak{b},\mathfrak{s}]+\mathfrak{s} \subseteq \mathfrak{b}+\mathfrak{s},$$

and because $\mathfrak{s}$ is perfect, $\mathfrak{s} \subseteq \mathfrak{g}'$, so that $\mathfrak{g}' = \mathfrak{s} + (\mathfrak{b} \cap \mathfrak{g}')$. Therefore $G' = \langle \exp(\mathfrak{b} \cap \mathfrak{g}')\rangle S$. The factor on the left is closed in B by Theorem 10.2.15, and S is closed by Corollary 13.5.7. Therefore G' is closed in $G = R \rtimes H$. □

Theorem 15.2.9. *Let G be a connected Lie group, $R \trianglelefteq G$ be its solvable radical with the Lie algebra $\mathfrak{r} \trianglelefteq \mathfrak{g}$, and $S \subseteq G$ be an integral subgroup corresponding to a Levi complement $\mathfrak{s}$. Further, let $T_R \subseteq R$ be a maximal torus. Then G admits a faithful finite-dimensional representation if and only if*

(a) *S admits a faithful finite-dimensional representation, and*
(b) $\mathbf{L}(T_R) \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$.

The intersection $\mathbf{L}(T_R) \cap [\mathfrak{g}, \mathfrak{g}]$ is always central in $\mathfrak{g}$.

Proof. First we assume that G admits a faithful finite-dimensional representation, hence is a semidirect product $G = B \rtimes L$, where B is simply connected solvable and L is linearly real reductive (Theorem 15.2.7). Then S admits a faithful finite-dimensional representation, and it remains to verify (b). Since the quotient map $q\colon G \to L \cong G/B$ satisfies $\mathbf{L}(q)[\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{l}, \mathfrak{l}]$, we have

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{b} \rtimes [\mathfrak{l}, \mathfrak{l}].$$

We conclude that $\mathbf{L}(T_R) \cap [\mathfrak{g}, \mathfrak{g}] \subseteq \mathbf{L}(T_R) \cap \mathfrak{b} = \mathbf{L}(T_R \cap B)$. Since B is a simply connected solvable Lie group, it contains no nontrivial torus (Proposition 10.2.15), which leads to $\mathbf{L}(T_R \cap B) = \{0\}$ and hence to (b).

For the converse, assume that G satisfies (a) and (b). If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a Levi decomposition, then $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{r}] \rtimes \mathfrak{s}$ implies that $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$. In view of (b), $\mathbf{L}(T_R)$ intersects $[\mathfrak{g}, \mathfrak{r}]$ trivially, so that there exists a vector space complement $\mathfrak{b}$ of $\mathbf{L}(T_R)$ in $\mathfrak{r}$, containing $[\mathfrak{g}, \mathfrak{r}]$. Then $[\mathfrak{g}, \mathfrak{b}] \subseteq [\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{b}$ implies that $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, so that the corresponding integral subgroup B is normal in G. Since $\mathbf{L}(B) \cap \mathbf{L}(T_R) = \{0\}$, Corollary 13.5.6 implies that the normal integral subgroup B is closed in R, hence also in G. As $T_R \cap B$ is a maximal torus of B (Theorem 13.3.13), it is trivial, and therefore B is simply connected by the Second Manifold Splitting Theorem 13.3.11.

In view of Lemma 13.3.3, there exists a T_R-invariant Levi complement in $\mathfrak{g}$, and since all Levi complements are conjugate under inner automorphisms by Malcev's Theorem 4.6.13, we may w.l.o.g. assume that $\mathfrak{s} := \mathbf{L}(S)$ is T_R-invariant. Then $L := T_R S$ is an integral subgroup of G which is reductive, and $Z(L)_0 = T_R$ is compact. Since $S = L'$ admits a faithful finite-dimensional representation, it is linearly real reductive by Proposition 15.1.7, so that Proposition 15.1.6 implies that L is linearly real reductive.

In view of $\mathfrak{g} = \mathfrak{b} + \mathbf{L}(T_R) + \mathbf{L}(S) = \mathfrak{b} + \mathbf{L}(L)$, it follows from Corollary 10.1.20 that, endowing L with its intrinsic Lie group structure, the multiplication map $\mu \colon B \rtimes L \to G, (b, l) \mapsto bl$ is a surjective morphism of Lie groups. In view of $\mathfrak{b} \cap \mathbf{L}(L) = \{0\}$, its kernel, which is isomorphic to the intersection $B \cap L$, is discrete, hence central in both groups. $Z(S)$ is finite because S admits a faithful finite-dimensional representation (Corollary 12.2.6). Now $Z(L) = T_R Z(S)$ shows that each element of $B \cap L$ is contained in a compact subgroup of G. But since B is a simply connected solvable closed subgroup of G, all its compact subgroups are trivial, which leads to $B \cap L = \{\mathbf{1}\}$. Therefore μ is a bijective morphism of Lie groups whose differential $\mathbf{L}(\mu)$ is an isomorphism of Lie algebras, and this implies that μ is an isomorphism (Proposition 8.2.13). Now use Proposition 15.2.6 to obtain a faithful finite-dimensional representation of G.

To see that the intersection $\mathbf{L}(T_R) \cap [\mathfrak{g}, \mathfrak{g}]$ is always central in $\mathfrak{g}$, we recall from Proposition 4.4.14 that for each $x \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}$, the operator $\operatorname{ad} x$ is nilpotent. If, in addition, $x \in \mathbf{L}(T_R)$, then $e^{\mathbb{R} \operatorname{ad} x}$ is bounded, which implies that $\operatorname{ad} x$ is semisimple (cf. Exercise 2.2.7), and therefore $\operatorname{ad} x = 0$, which means that $x \in \mathfrak{z}(\mathfrak{g})$. □

We conclude this section with the observation that a Lie group which admits a faithful finite-dimensional representation also admits a faithful finite-dimensional representation with closed image. The proof of this nontrivial theorem is deferred to a sequence of exercises below. In a way this result justifies the fact that we only called closed subgroups of $\mathrm{GL}_n(\mathbb{K})$ *linear Lie groups* in Chapter 3.

Theorem 15.2.10. *For a real Lie group G the following properties are equivalent.*

(i) *G admits a faithful finite-dimensional representation.*

(ii) *G admits a faithful finite-dimensional representation with closed image.*

Exercises for Section 15.2

In the following, it is shown that a Lie group always which admits a faithful finite-dimensional representation also admits a faithful finite-dimensional representation as a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Exercise 15.2.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that G admits a faithful finite-dimensional representation.

(a) There is a maximal compactly embedded abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, and $\mathfrak{b}, \mathfrak{h}$ as in Theorem 15.2.4, such that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{h}), \quad \mathfrak{z}(\mathfrak{h}) \subseteq \mathfrak{a} \quad \text{and} \quad [\mathfrak{a} \cap \mathfrak{b}, \mathfrak{h}] = \{0\}.$$

From now on, let $\mathfrak{a}$ be as in (a).

(b) $\mathfrak{a} \cap \mathfrak{h}$ is maximal compactly embedded abelian in $\mathfrak{h}$.

(c) $\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{g}' \subseteq \mathfrak{z}(\mathfrak{n})$, where $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{g}$.

Exercise 15.2.2. Let G be linearly real reductive and $\mathfrak{t} \subseteq \mathbf{L}(G)$ be a maximal compactly embedded abelian subalgebra. Then $T := \exp_H \mathfrak{t}$ is a torus.

Exercise 15.2.3. Let $\gamma \colon \mathbb{R} \to G$ be a one-parameter subgroup of a Lie group. If $\gamma(\mathbb{R})$ is not relatively compact in G, then $\gamma(\mathbb{R})$ is closed and γ is a homeomorphism onto the image.

Exercise 15.2.4. Let $\mathfrak{e} = \mathbb{R}x_1 \oplus \mathfrak{t}$ be an abelian subalgebra of $\mathfrak{gl}_n(\mathbb{R})$, where $e^{\mathfrak{t}}$ is a torus. If $e^{\mathbb{R}x_1}$ is a noncompact closed subgroup, then this also holds for $e^{\mathbb{R}(\lambda x_1 + x_2)}$ if $\lambda \neq 0$ and $x_2 \in \mathfrak{t}$.

Exercise 15.2.5. Let $\mathfrak{a}$ be as in Exercise 15.2.1 and $\varphi_1 \colon G \to \mathrm{GL}_{n_1}(\mathbb{R})$ be a faithful representation of G. If $\varphi_1(G)$ is not closed, then there is an $x \in \mathfrak{a} \cap \mathfrak{b}$ such that $\varphi_1(\exp \mathbb{R}x)$ is not closed.

Exercise 15.2.6. Let $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}_1 \oplus (\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{g}')$ be a direct decomposition. Then there is a representation φ_2 of G on $\mathbb{R}^{n_2}$ such that $\varphi_2(\exp \mathfrak{a}_1)$ is a closed subgroup of $\mathrm{GL}_{n_2}(\mathbb{R})$ which is isomorphic to some $\mathbb{R}^n$.

Exercise 15.2.7. Let φ_1 and φ_2 be the two representations from Exercise 15.2.5 and Exercise 15.2.6. Then the representation

$$\varphi = (\varphi_1, \varphi_2) \colon G \to \mathrm{GL}_{n_1}(\mathbb{R}) \times \mathrm{GL}_{n_2}(\mathbb{R}) \subseteq \mathrm{GL}_{n_1+n_2}(\mathbb{R}), \quad g \mapsto (\varphi_1(g), \varphi_2(g))$$

has closed image.

15.3 Linearity of Complex Lie Groups

We are aiming at a characterization of the connected complex Lie groups which admit faithful finite-dimensional holomorphic representations as those which are semidirect products of simply connected solvable groups and linearly complex reductive groups. First we derive a structural condition which is necessary for the existence of a faithful finite-dimensional holomorphic representation and leads to the semidirect product structure. For the following proposition recall the notation from Definitions 10.2.1 and 14.2.11.

Proposition 15.3.1. *Let G be a connected complex Lie group, R its solvable radical, and K a maximal compact subgroup. If $K^* \cap (G,R) = \{\mathbf{1}\}$, then there exists a simply connected normal solvable subgroup B such that $G \cong B \rtimes K^*$.*

Proof. Since the normal subgroup (G,R) intersects K trivially, it intersects all maximal compact subgroups of G trivially. Therefore it is closed and simply connected (Corollary 13.5.6, Theorem 13.3.11). The quotient group $L := G/(G,R)$ is reductive. First we use Theorem 13.3.13 to see that, if $q\colon G \to L$ denotes the quotient map, then $q(K)$ is a maximal compact subgroup of L. Then we use Lemma 14.2.12 to find a vector group $V \subseteq L$ such that $L \cong V \times q(K)^* = V \times q(K^*)$.

Let $B := q^{-1}(V)$. Then B is closed and simply connected (Theorem 13.3.13(i)(c)). Moreover, $q(B \cap K^*) \subseteq V \cap q(K^*) = \{\mathbf{1}\}$ yields $B \cap K^* \subseteq (G,R) \cap K^* = \{\mathbf{1}\}$. Finally, we note that K^* is closed because it contains a maximal torus (Corollary 13.5.6), so that $G \cong B \rtimes K^*$ follows. □

Remark 15.3.2. We resume the situation of the proposition above. Let $U \subseteq \mathrm{Aut}_{\mathbb{C}}(G)$ be a group of holomorphic automorphisms of G such that $\mathfrak{g}$ is a semisimple U-module. Then the characteristic subgroup (G,R) is U-invariant so that U also acts on the quotient $G/(G,R)$ in such a way that $\mathfrak{g}/[\mathfrak{g},\mathfrak{r}]$ is a semisimple U-module. Therefore the complement for the image of K^* can be chosen in a U-invariant fashion (cf. proof of Lemma 14.2.12), and therefore the normal subgroup B obtained is invariant under U.

Remark 15.3.3. Let $G = B \rtimes K^*$ be a decomposition as in Proposition 15.3.1 and K_1 a second maximal compact subgroup of G. Then there exists an inner automorphism $c_g\colon x \mapsto gxg^{-1}$ such that $c_g(K) = K_1$ (Theorem 13.1.3(iii)). Since inner automorphisms of complex Lie groups are holomorphic, we conclude that $c_g(K^*) = K_1^*$. Using that B is also invariant under inner automorphisms, we see that $K_1^* \cap B = \{\mathbf{1}\}$, whence $G \cong B \rtimes K_1^*$. Therefore we obtain a similar decomposition for each maximal compact subgroup.

Lemma 15.3.4. *If G is a connected complex Lie group, R the solvable radical of G, and $K \subseteq G$ a maximal compact subgroup, then*

$$K^* \cap (G,R) \subseteq \mathrm{Lin}_{\mathbb{C}}(G).$$

Proof. Let $\alpha\colon G \to \mathrm{GL}(V)$ be a *holomorphic* representation of G and

$$V_0 = \{0\} \subseteq V_1 \subseteq \ldots \subseteq V_n = V$$

a Jordan–Hölder series for the G-module V, i.e., all subspaces V_i are submodules and the quotients V_{i+1}/V_i are simple modules.

Then V is a unipotent module over the group (G, R) because it is an integral subgroup with Lie algebra $[\mathfrak{g}, \mathfrak{r}]$, and V is a nilpotent $[\mathfrak{g}, \mathfrak{r}]$-module (Proposition 4.4.14). Therefore $[\mathfrak{g}, \mathfrak{r}]$, and hence also (G, R), acts trivially on the quotients V_{i+1}/V_i.

On the other hand the group $\alpha(K^*) = \alpha(K)^*$ is isomorphic to $\alpha(K)_{\mathbb{C}}$ (Proposition 14.2.1(d)), hence linearly complex reductive. Therefore, Theorem 14.3.11 implies that each V_{i+1} contains a K^*-invariant subspace complementary to V_i. Now the representation of $\alpha(K^*)$ on $\bigoplus_{i=1}^n (V_i/V_{i-1})$ is faithful. We conclude that

$$\alpha\big(K^* \cap (G, R)\big) \subseteq \alpha(K^*) \cap \alpha\big((G, R)\big) = \{\mathbf{1}\}.$$

Since α was arbitrary, $K^* \cap (G, R) \subseteq \mathrm{Lin}_{\mathbb{C}}(G)$. □

Proposition 15.3.5. *If G is a connected complex Lie group with* $\mathrm{Lin}_{\mathbb{C}}(G) = \{\mathbf{1}\}$*, then $G \cong B \rtimes K^*$, where B is simply connected solvable, K is a maximal compact subgroup of G, and $K^* \cong K_{\mathbb{C}}$ is linearly complex reductive.*

Proof. First we note that Lemma 15.3.4 implies that $(G, R) \cap K^* = \{\mathbf{1}\}$ and $K^* \cong K_{\mathbb{C}}$ holds for any maximal compact subgroup $K \subseteq G$ because $K^* \subseteq G$ is linear (Theorem 14.3.11). Therefore Proposition 15.3.1 provides the required semidirect decomposition $G \cong B \rtimes K^*$, where B is simply connected and solvable. □

So far we have seen that a connected complex Lie group which admits a faithful finite-dimensional holomorphic representation has some special structural features which, combining Lemma 15.3.4 with Proposition 15.3.1, lead to a semidirect decomposition into a simply connected solvable group and a linearly reductive Lie group $K^* \cong K_{\mathbb{C}}$. Next we prove a converse.

Proposition 15.3.6. *If G is a connected complex group which is a semidirect product $G = B \rtimes L$, where B is simply connected solvable and L is linearly complex reductive, then G admits a faithful finite dimensional holomorphic representation.*

Proof. The proof is almost the same as for Proposition 15.2.6. The only step which is different is that if $\beta\colon \widetilde{G} \cong B \rtimes \widetilde{L} \to \mathrm{GL}(V)$ is a holomorphic representation and

$$F := \{\omega \circ \beta|_B \colon \omega \in \mathrm{End}(V)^*\}$$

is the corresponding finite-dimensional subspace of $C(B)$ on which we have a representation of $B \rtimes \widetilde{L}$, given by

$$\pi_F(b,h)(f) = f \circ c_h^{-1} \circ \lambda_b^{-1}, \quad f \in F,$$

then π_F is holomorphic. This follows from the fact the evaluation maps $\mathrm{ev}_b \colon F \to \mathbb{C}, f \mapsto f(b)$, separate the points of F, and for each $x \in B$ and $f = \omega \circ \beta|_B \in F$, the map $B \rtimes \widetilde{L} \to \mathbb{C}$,

$$\begin{aligned}(b,h) \mapsto (\pi_F(b,h)f)(x) &= \omega(\beta(c_h^{-1}\lambda_b^{-1}x)) \\ &= \omega(\beta(hb^{-1}xh^{-1})) = \omega(\beta(h)\beta(b)^{-1}\beta(x)\beta(h)^{-1})\end{aligned}$$

has in $(\mathbf{1},\mathbf{1})$ the differential

$$(y,z) \mapsto -\omega(\mathbf{L}(\beta)(y)\cdot\beta(x)) + \omega([\mathbf{L}(\beta)(z),\beta(x)]),$$

which is complex linear. □

Summing up the results of this section we obtain the following characterization theorem:

Theorem 15.3.7. *For a connected complex Lie group G, the following are equivalent:*

(i) *G it is a semidirect product $G = B \rtimes L$, where B is simply connected solvable and L is linearly complex reductive.*
(ii) *G admits a faithful finite-dimensional holomorphic representation.*
(iii) $\mathrm{Lin}_{\mathbb{C}}(G) = \{\mathbf{1}\}$.
(iv) *If $K \subseteq G$ is maximal compact, then $G \cong B \rtimes K^*$, where B is simply connected solvable and K^* is linearly complex reductive.*
(v) *If $K \subseteq G$ is maximal compact, then K^* is linear and $K^* \cap (G,R) = \{\mathbf{1}\}$.*

For any such decomposition $G = B \rtimes L$, a maximal compact subgroup K of L is also maximal compact in G and $L = K^* \cong K_{\mathbb{C}}$.

Proof. (i) ⇒ (ii) follows from Proposition 15.3.6.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i): If $\mathrm{Lin}_{\mathbb{C}}(G) = \{\mathbf{1}\}$, then the existence of a semidirect decomposition with the required properties follows from Proposition 15.3.5.

(i) ⇔ (iv): Let $K \subseteq G$ be a maximal compact subgroup. Then its image K_L in $G/B \cong L$ is maximal compact (Theorem 13.3.13), so that any maximal compact subgroup K_L of L is also maximal compact in G. Since L is linearly complex reductive, we have $L = K_L^*$. Any other maximal compact subgroup of G is conjugate to K_L under some inner automorphism, which leads to $G \cong B \rtimes K^*$.

(iii) ⇒ (v): Lemma 15.3.4 implies $K^* \cap (G,R) = \{\mathbf{1}\}$. We know already that K^* is linearly complex reductive, hence by Theorem 14.3.11 it admits a faithful finite dimensional holomorphic representation.

(v) ⇒ (iv): Proposition 15.3.1 and Theorem 14.3.11(ii). □

Corollary 15.3.8. *A simply connected complex Lie group admits a faithful finite dimensional holomorphic representation.*

Proof. If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a Levi decomposition of the Lie algebra $\mathfrak{g}$ of G, then $G \cong R \rtimes S$. The group R is simply connected and solvable and S is simply connected and semisimple, hence linearly complex reductive (Proposition 14.2.6). Now the assertion follows from Theorem 15.3.7. $\square$

The preceding results contrast the real case, where the universal covering group of $\mathrm{SL}_2(\mathbb{R})$ constitutes the simplest example of a simply connected real Lie group which does not admit a faithful finite-dimensional representation.

Corollary 15.3.9. *Let G be a connected complex Lie group and $D \subseteq Z(G)$ a finite subgroup. Then the following statements are equivalent.*

(i) *G admits a faithful finite-dimensional holomorphic representation.*
(ii) *G/D admits a faithful finite-dimensional holomorphic representation.*

Proof. If G admits a faithful finite-dimensional holomorphic representation, we have $G \cong B \rtimes K^*$, where K is a maximal compact subgroup (Theorem 15.3.7). Then $D \subseteq K$ implies that $G/D \cong B \rtimes (K^*/D)$. The group B is simply connected solvable and $K^*/D \cong K_{\mathbb{C}}/D \cong (K/D)_{\mathbb{C}}$ is linearly complex reductive. Therefore G/D admits a faithful finite dimensional holomorphic representation by Theorem 15.3.7.

If, conversely, G/D admits a faithful finite-dimensional holomorphic representation and $G/D \cong B_1 \rtimes K_1^*$, where $K_1 \subseteq G_1 := G/D$ is a maximal compact subgroup, then the inverse image K of K_1 is a maximal compact subgroup in G. Let $B := \langle \exp_G \mathbf{L}(B_1) \rangle$ and write $q\colon G \to G/D$ for the quotient map. Then $q(K^* \cap B) \subseteq K_1^* \cap B = \{\mathbf{1}\}$ yields $K^* \cap B \subseteq D$. The mapping $q|_B\colon B \to B_1$ is a covering, hence injective because B_1 is simply connected and thus B is also simply connected. Therefore $B \cap D = \{\mathbf{1}\}$ which entails that $K^* \cap B = \{\mathbf{1}\}$. This proves that $G \cong B \rtimes K^*$. The group $Z(K)_0$ is a finite covering of $Z(K_1)_0 \cong (\mathbb{C}^\times)^m$, so it is also isomorphic to $(\mathbb{C}^\times)^m$. We conclude with Theorem 14.3.11 that K^* is linearly complex reductive so that the assertion follows from Theorem 15.3.7. $\square$

We conclude this section with some examples explaining some of the possible pathologies.

Example 15.3.10. We construct a solvable complex Lie group G which does not admit a faithful finite-dimensional holomorphic representation, whereas its maximal connected nilpotent normal subgroup N does. We put

$$G := (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}) \rtimes_\gamma \mathbb{C},$$

where the action of $\mathbb{C}$ on $\mathbb{C}^3$ is described by

$$\gamma(z)(a, e^b, c) := (e^z a, e^{b+zc}, c).$$

Then $N = \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}$ is the maximal nilpotent integral subgroup, which admits a faithful finite-dimensional representation. On the other hand, in the notation of Theorem 15.3.7, $K^* = \{\mathbf{1}\} \times \mathbb{C}^\times \times \{\mathbf{1}\}$ we have $K^* \subseteq G'$, so G does not admit a faithful finite-dimensional holomorphic representation.

Example 15.3.11. Let $A := (\mathbb{C}^\times)^2$, $T = \mathbb{T}^2$ be the maximal torus in A, $\mathfrak{t} := \mathbf{L}(T)$, and $x \in \mathfrak{a}$ with $\mathbb{C}x \cap \mathfrak{t} = \{0\}$. Then $B := \exp_A \mathbb{C}x$ is a complex Lie subgroup of A and $\mathfrak{b} \cap \mathfrak{t} = \mathfrak{b} \cap \mathbb{R}^2 = \{0\}$ implies that B is simply connected (Theorem 13.3.13). Therefore $B \cong \mathbb{C}$.

Now let $H = (\mathfrak{h}, *)$ denote the simply connected three-dimensional Heisenberg group which we identify with its Lie algebra $\mathfrak{h} = \operatorname{span}\{p, q, z\}$ endowed with the BCH multiplication $x * y := x + y + \frac{1}{2}[x, y]$.

Let $\alpha \colon Z(H) \to B$ be an isomorphism, put $G := H \times A$ and

$$D := \{(x, \alpha(x)) \colon x \in Z(H)\}.$$

Then $D \cong \mathbb{C}$ is a closed central subgroup of G. Let $G_1 := G/D$ and write $q \colon G \to G_1$ for the quotient homomorphism. Then $T_1 := q(T)$ is a maximal torus in G_1 (Theorem 13.3.13(b)) and $(G_1, G_1) = q((G, G)) = q(Z(H)) = q(B)$ together with $T_1^* = q(T^*) = q(A)$ yields

$$(G_1, G_1) \cap T_1^* = q(B) \cong \mathbb{C}.$$

Therefore G_1 does not admit a faithful finite-dimensional holomorphic representation but $T_1^* = q(A) \cong A$ does. So we have an example of a nilpotent group G_1 with

$$\operatorname{Lin}_{\mathbb{C}}(G_1) \neq \operatorname{Lin}_{\mathbb{C}}(T_1^*).$$

The quotient A/B is compact because the image of T in this group is a maximal torus which is two-dimensional. Therefore $q(A) \subseteq \operatorname{Lin}_{\mathbb{C}}(G)$ and since $G_1/q(A) \cong \mathbb{C}^2$, we see that $\operatorname{Lin}_{\mathbb{C}}(G_1) = q(A)$.

Example 15.3.12. We have seen above that complex Lie groups admitting faithful finite-dimensional holomorphic representations are characterized by the existence of a semidirect decomposition $G = B \rtimes K^*$, where B is simply connected and solvable. We construct an example, where we have two essentially different decompositions of this type. One where B is abelian and one where B is not even nilpotent.

Let $G := \mathbb{C} \times \operatorname{Aff}(1, \mathbb{C}) \cong \mathbb{C}^2 \rtimes \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by $z \cdot (a, b) = (a, zb)$. We fix a basis (e_1, e_2, x) for the Lie algebra $\mathfrak{g}$ with

$$[e_1, e_2] = 0, \quad [x, e_1] = 0, \quad \text{and} \quad [x, e_2] = e_2.$$

Then $\mathfrak{g}' = \mathbb{C}e_2$ and $\mathfrak{t} = ix$ is the Lie algebra of a maximal torus. Put $\mathfrak{b} := \operatorname{span}\{e_2, e_1 + x\}$. Then $\mathfrak{b}$ contains the commutator algebra, hence is an ideal of $\mathfrak{g}$. Moreover, $\mathfrak{b} \cap \mathfrak{t} = \{0\}$ which implies that $B := \exp \mathfrak{b}$ is closed and simply connected (Theorem 13.3.13). We have $T^* = \exp \mathbb{C}x \cong \mathbb{C}^\times$ and globally we find that $T^* \cap B = \{\mathbf{1}\}$. Thus we also have a semidirect decomposition $G \cong B \rtimes T^*$, where B is a two-dimensional nonabelian simply connected complex Lie group.

Notes on Chapter 15

The results described in this chapter are classical (cf. [Ho65]). The problem to characterize connected Lie groups which admit faithful finite-dimensional representations has various natural generalizations. Since these Lie groups are precisely the ones which can be embedded into a finite-dimensional associative algebra, it is natural to ask for a characterization of connected Lie groups injecting continuously into unit groups of Banach algebras. This problem has been solved by Luminet and Valette in [LV94], where it is shown that any connected Lie group for which the continuous homomorphisms into unit groups of Banach algebras separate the points is already a linear Lie group.

There is a natural class of topological algebras generalizing Banach algebras, the so-called *continuous inverse algebras*. A complete unital associative locally convex algebra $\mathcal{A}$ is said to be a continuous inverse algebra if its unit group $\mathcal{A}^\times$ is open and the inversion map $\iota_{\mathcal{A}} \colon \mathcal{A}^\times \to \mathcal{A}, a \mapsto a^{-1}$ is continuous. For this considerably larger class of algebras, the problem of characterizing connected finite-dimensional Lie groups G for which the homomorphisms into continuous inverse algebras separate the points is solved by Beltita and Neeb in [BN08], where it is shown that any such group is also a linear Lie group.

These two results show that linearity of a connected Lie group is a very strong property with many different characterizations which, a priori, look much weaker than the linearity requirement.

The criterion for the existence of faithful finite-dimensional representations in Theorem 15.2.9 is due to N. Nahlus (cf. [Na97]).

16

Classical Lie Groups

In this chapter we apply the general theory to classical matrix groups such as $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})$ and some of their real forms to provide explicit structural and topological information. We will start with compact real forms, i.e., $\mathrm{U}_n(\mathbb{K})$ and $\mathrm{SU}_n(\mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, since many results can be reduced to compact groups.

16.1 Compact Classical Groups

In this section we take a closer look at the groups $\mathrm{U}_n(\mathbb{K})$ and $\mathrm{SU}_n(\mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (cf. Definitions 1.1.3 and 1.3.3). Recall that $\mathrm{U}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$ and note that Proposition 1.3.4 implies that $\mathrm{U}_n(\mathbb{H}) \subseteq \mathrm{SL}_n(\mathbb{H})$, so that $\mathrm{SU}_n(\mathbb{H}) := \mathrm{U}_n(\mathbb{H}) \cap \mathrm{SL}_n(\mathbb{H})$, which we did not define explicitly in Section 1.3, coincides with $\mathrm{U}_n(\mathbb{H})$.

16.1.1 Unitary Groups

We recall from Lemma 1.1.4 that $\mathrm{U}_n(\mathbb{C})$ is compact and from Proposition 1.1.7 that it is connected.

Proposition 16.1.1. *The Lie algebra $\mathfrak{u}_n(\mathbb{C}) = \mathbf{L}(\mathrm{U}_n(\mathbb{C}))$ consists of the skew-hermitian matrices in $\mathfrak{gl}_n(\mathbb{C})$. It is compact, and*

$$[\mathfrak{u}_n(\mathbb{C}), \mathfrak{u}_n(\mathbb{C})] = \mathfrak{su}_n(\mathbb{C}), \quad \mathfrak{z}\big(\mathfrak{u}_n(\mathbb{C})\big) = \mathbb{R}i\mathbf{1} \quad \textit{and} \quad \mathfrak{u}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) \oplus \mathbb{R}i\mathbf{1}.$$

Proof. We have already seen in Example 3.2.4(b) that

$$\mathfrak{u}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \colon x^* = -x\}.$$

The compactness of the Lie algebra $\mathfrak{u}_n(\mathbb{C})$ follows from the compactness of $\mathrm{U}_n(\mathbb{C})$ and Theorem 11.1.4. By Lemma 11.1.2, the compact Lie algebra $\mathfrak{u}_n(\mathbb{C})$ is the direct sum of the semisimple Lie algebra $\mathfrak{u}_n(\mathbb{C})'$ and its center.

To determine the center, let $x \in \mathfrak{z}\big(\mathfrak{u}_n(\mathbb{C})\big)$. Then x also commutes with $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) \oplus i\mathfrak{u}_n(\mathbb{C})$, hence is a multiple of $\mathbf{1}$ (Exercise 1.1.16), which leads to $\mathfrak{z}(\mathfrak{u}_n(\mathbb{C})) = i\mathbb{R}\mathbf{1}$. It is clear that $\mathfrak{u}_n(\mathbb{C})' \subseteq \mathfrak{su}_n(\mathbb{C})$, and, in view of $\mathfrak{su}_n(\mathbb{C}) \cap i\mathbb{R}\mathbf{1} = \{0\}$, we have equality. □

Lemma 16.1.2. *For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $d := \dim_{\mathbb{R}} \mathbb{K}$, the group $\mathrm{U}_n(\mathbb{K})$ acts transitively on the unit sphere*

$$\mathbb{S}^{dn-1} = \{z \in \mathbb{K}^n \colon \|z\| = 1\}.$$

For $n > 1$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathrm{SU}_n(\mathbb{K})$ also acts transitively.

Proof. If $x \in \mathbb{K}^n$ with $\|x\| = 1$, then we extend x to an orthonormal basis $\{x, x_2, \ldots, x_n\}$ for $\mathbb{K}^n$ and define a $\mathbb{K}$-linear map $U \colon \mathbb{K}^n \to \mathbb{K}^n$ by $Ue_i := x_i$, $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical basis for $\mathbb{K}^n$. Then U is unitary and $Ue_1 = x$. This proves that $\mathrm{U}_n(\mathbb{K})$ acts transitively on the unit sphere in $\mathbb{K}^n$.

For $n > 1$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we replace x_n by $x_n' := (\det U)^{-1} x_n$ to obtain an element $U' \in \mathrm{SU}_n(\mathbb{C})$ with $U'e_1 = x$. □

Proposition 16.1.3. *The groups $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{U}_n(\mathbb{H})$ are 1-connected for each $n \in \mathbb{N}$.*

Proof. For $n \in \mathbb{N}$ and $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$, we put $G_n := \mathrm{SU}_n(\mathbb{C})$ for $\mathbb{K} = \mathbb{C}$ and $G_n := \mathrm{U}_n(\mathbb{H})$ for $\mathbb{K} = \mathbb{H}$. We also put $d := \dim_{\mathbb{R}} \mathbb{K}$. To see that G_n it is simply connected, we consider the surjective orbit map

$$\alpha \colon G_n \to \mathbb{S}^{dn-1}, \qquad g \mapsto ge_1$$

(Lemma 16.1.2). The stabilizer $H = (G_n)_{e_1}$ of e_1 is isomorphic to G_{n-1}. It only acts on the last $n-1$ components. The map α factors through a bijective continuous map

$$\beta \colon G_n/H \to \mathbb{S}^{dn-1},$$

and since G_n is compact, β is a homeomorphism. Now we prove the claim by induction on n. For $n = 1$ and $\mathbb{K} = \mathbb{C}$, the group $G_1 = \mathrm{SU}_1(\mathbb{C}) = \{\mathbf{1}\}$ is trivial, hence in particular 1-connected. For $\mathbb{K} = \mathbb{H}$, we have $\mathrm{U}_1(\mathbb{H}) \cong \mathrm{SU}_2(\mathbb{C}) \cong \mathbb{S}^3$ (by the map β), and we also see in this case that G_1 is 1-connected.

We now assume that $n > 1$ and that $G_{n-1} \cong H$ is 1-connected. Since $H \cong G_{n-1}$ and $G_n/H \cong \mathbb{S}^{dn-1}$ are connected, the group G_n is connected (Exercise 11.4.1). Next we use the Fundamental Group Diagram (Theorem 10.1.15) whose upper row provides an exact sequence

$$\{\mathbf{1}\} = \pi_1(H) \to \pi_1(G_n) \to \pi_1(G_n/H).$$

Since $\pi_1(G_n/H) = \pi_1(\mathbb{S}^{dn-1})$ vanishes for $n > 1$ (Exercise A.1.3), the homomorphism $\pi_1(H) \to \pi_1(G_n)$ is an isomorphism, showing that $\pi_1(G_n)$ also vanishes. □

Next we analyze how $\mathrm{U}_n(\mathbb{C})$ decomposes into its commutator group $\mathrm{SU}_n(\mathbb{C})$ and the center.

Proposition 16.1.4. *$Z\big(\mathrm{U}_n(\mathbb{C})\big) = \mathbb{T}\mathbf{1} = \{z\mathbf{1} \colon |z| = 1\}$ and $Z\big(\mathrm{SU}_n(\mathbb{C})\big) = \{z\mathbf{1} \colon z^n = 1\}$ is cyclic of order n.*

Proof. Let g be in the center of $\mathrm{U}_n(\mathbb{C})$. Then $\mathrm{Ad}(g)x = gxg^{-1} = x$ for each $x \in \mathfrak{u}_n(\mathbb{C})$ implies that g commutes with each $x \in \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) + i\mathfrak{u}_n(\mathbb{C})$, so that $g \in \mathbb{C}\mathbf{1}$ (Exercise 1.1.16). Along the same lines, we argue for $\mathrm{SU}_n(\mathbb{C})$, using that $\mathfrak{u}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) \oplus \mathbb{R}i\mathbf{1}$ (cf. Exercise 11.2.7, Proposition 16.1.1). □

Proposition 16.1.5. *The multiplication map*

$$Z\big(\mathrm{U}_n(\mathbb{C})\big) \times \mathrm{SU}_n(\mathbb{C}) \to \mathrm{U}_n(\mathbb{C}), \qquad (z, g) \mapsto zg$$

is a connected n-fold covering morphism of Lie groups. The subgroup

$$A := \{\mathrm{diag}(z, 1, \ldots, 1) \colon |z| = 1\} \cong \mathbb{T},$$

satisfies $\mathrm{U}_n(\mathbb{C}) = \mathrm{SU}_n(\mathbb{C}) \rtimes A$ (cf. Theorem 11.2.6).

Proof. Since $\mathrm{SU}_n(\mathbb{C})$ is connected, the first claim follows immediately from Proposition 16.1.4 and

$$Z\big(\mathrm{U}_n(\mathbb{C})\big) \cap \mathrm{SU}_n(\mathbb{C}) = Z\big(\mathrm{SU}_n(\mathbb{C})\big).$$

The second claim is a consequence of Proposition 10.1.18 and the triviality of the intersection $A \cap \mathrm{SU}_n(\mathbb{C}) = \{\mathbf{1}\}$. □

16.1.2 Orthogonal Groups

In this section, we write $s_i \in \mathrm{O}_n(\mathbb{R})$ for the orthogonal reflection in the hyperplane $e_i^\perp$, i.e.,

$$s_i(e_j) = \begin{cases} e_j & \text{for } j \neq i, \\ -e_i & \text{for } j = i, \end{cases}$$

Proposition 16.1.6. *$Z\big(\mathrm{O}_n(\mathbb{R})\big) = \{\pm\mathbf{1}\}$, $Z\big(\mathrm{SO}_2(\mathbb{R})\big) = \mathrm{SO}_2(\mathbb{R})$, and*

$$Z\big(\mathrm{SO}_n(\mathbb{R})\big) = \mathrm{SO}_n(\mathbb{R}) \cap Z\big(\mathrm{O}_n(\mathbb{R})\big) = \begin{cases} \{\mathbf{1}\} & \text{for } n \geq 3 \text{ odd}, \\ \{\pm\mathbf{1}\} & \text{for } n \geq 4 \text{ even}. \end{cases}$$

Proof. (cf. also Exercise 1.2.16) If $g \in Z\big(\mathrm{O}_n(\mathbb{R})\big)$, then g commutes with each s_i, hence is diagonal. Since g also commutes with all maps induced by permutations of the basis vectors, all diagonal entries of g are equal. As g is orthogonal, we obtain $g \in \{\pm\mathbf{1}\}$.

It is clear that the group $\mathrm{SO}_2(\mathbb{R})$ of rotations of the plane is abelian. Hence we may assume that $n > 2$. The products $s_i s_j$ are contained in $\mathrm{SO}_n(\mathbb{R})$. If $g \in$

$Z\big(\mathrm{SO}_n(\mathbb{R})\big)$, then g commutes with all of these maps, and a similar argument as above shows that g is diagonal. Further, g commutes with the maps induced by even permutations of the basis elements. But since the alternating group $A_n \subseteq \mathrm{SO}_n(\mathbb{R})$ acts transitively on the basis vectors (here we need again that $n > 2$), $g \in \{\pm\mathbf{1}\}$. In view of $\det(-\mathbf{1}) = (-1)^n$, $-\mathbf{1} \in \mathrm{SO}_n(\mathbb{R})$ is equivalent to n being even. □

Corollary 16.1.7. *For $n \geq 3$, the groups $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$ are semisimple. The Lie algebra $\mathfrak{so}_n(\mathbb{R})$ is semisimple and compact.*

Proof. Since $\mathrm{SO}_n(\mathbb{R})$ is a compact connected group (Lemma 1.1.4) with finite center (Theorem 16.1.6),

$$\mathbf{L}\left(\mathrm{O}_n(\mathbb{R})\right) = \mathbf{L}\left(\mathrm{SO}_n(\mathbb{R})\right) = \mathfrak{so}_n(\mathbb{R})$$

is a compact semisimple Lie algebra (Lemma 11.1.2 and Theorem 11.1.4). □

Corollary 16.1.8. *The group $\mathrm{SO}_n(\mathbb{R})$ is the identity component of $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R}) \rtimes \{\pm s_1\}$. There exists a direct decomposition $\mathrm{O}_n(\mathbb{R}) \cong \mathrm{SO}_n(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$ if and only if n is odd.*

Proof. The first claim follows from Proposition 1.1.7. If n is odd, then evidently $\mathrm{O}_n(\mathbb{R}) = \{\pm\mathbf{1}\} \times \mathrm{SO}_n(\mathbb{R})$. Conversely, if $\mathrm{O}_n(\mathbb{R}) = \{a, \mathbf{1}\} \times \mathrm{SO}_n(\mathbb{R})$ for some $a \in \mathrm{O}_n(\mathbb{R})$, then $a \in Z\big(\mathrm{O}_n(\mathbb{R})\big) = \{\pm\mathbf{1}\}$ (Proposition 16.1.6) implies that $a = -\mathbf{1}$. Since $a \notin \mathrm{SO}_n(\mathbb{R})$, n is odd. □

The relation between $\mathrm{SO}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{R})$ is much simpler than the one between $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{U}_n(\mathbb{C})$. However, it is more difficult to compute the fundamental group of $\mathrm{SO}_n(\mathbb{R})$.

Proposition 16.1.9. *For $n > 2$, the group $\pi_1\big(\mathrm{SO}_n(\mathbb{R})\big)$ has at most two elements.*

Proof. For $n = 3$, the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ is isomorphic to $\mathfrak{su}_2(\mathbb{C})$ (Example 4.1.23). Since the group $\mathrm{SU}_2(\mathbb{C})$ is simply connected

$$\widetilde{\mathrm{SO}_3(\mathbb{R})} \cong \mathrm{SU}_2(\mathbb{C}).$$

We saw already that $Z\big(\mathrm{SU}_2(\mathbb{C})\big) = \{\pm\mathbf{1}\}$, and that $Z\big(\mathrm{SO}_3(\mathbb{R})\big) = \{\mathbf{1}\}$, so that

$$\pi_1\big(\mathrm{SO}_3(\mathbb{R})\big) \cong \mathbb{Z}/2\mathbb{Z}.$$

Now let $n \geq 3$ and $e_1 \in \mathbb{R}^n$ be the first basis vector. The stabilizer subgroup $H := \{g \in \mathrm{SO}_n(\mathbb{R}) \colon ge_1 = e_1\}$ is isomorphic to $\mathrm{SO}_{n-1}(\mathbb{R})$. If we set $G := \mathrm{SO}_n(\mathbb{R})$, then we obtain a homeomorphism $\mathbb{S}^{n-1} \cong G/H$ (Lemma 16.1.2). Now we use the Fundamental Group Diagram (Theorem 10.1.15), whose upper row provides an exact sequence

$$\pi_1(H) \to \pi_1(G) \to \pi_1(G/H).$$

For $n > 2$, the group $\pi_1(G/H) \cong \pi_1(\mathbb{S}^{n-1})$ is trivial (Exercise A.1.3), so that the homomorphism $\pi_1(H) \to \pi_1(G)$ is surjective. Inductively, we thus conclude from $\pi_1\big(\mathrm{SO}_3(\mathbb{R})\big) \cong \mathbb{Z}/2\mathbb{Z}$ that $|\pi_1\big(\mathrm{SO}_n(\mathbb{R})\big)| \leq 2$ for $n \geq 3$. □

To show that $\pi_1\big(\mathrm{SO}_n(\mathbb{R})\big)$ actually has two elements, we will construct a connected 2-fold covering. We shall find these groups as subgroups of the unit groups of certain Clifford algebras (cf. Appendix B.3).

We recall from Remark B.3.22 the short exact sequence

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Spin}_n(\mathbb{R}) \xrightarrow{\Phi} \mathrm{SO}_n(\mathbb{R}) \to \mathbf{1}$$

and note that the definitions immediately imply that $\mathrm{Spin}_n(\mathbb{R})$ is a closed subgroup of the Clifford algebra C_n, hence in particular a Lie group, and that Φ is continuous.

Proposition 16.1.10. *The restriction of Φ to $\mathrm{Spin}_n(\mathbb{R})$ is a double covering*

$$\{\mathbf{1}\} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}_n(\mathbb{R}) \to \{\mathbf{1}\}$$

with kernel $\{\pm\mathbf{1}\}$. Further, $\mathrm{Spin}_n(\mathbb{R})$ is connected for $n > 1$.

Proof. Since $\mathrm{SO}_n(\mathbb{R})$ is connected (Corollary 16.1.8) and its fundamental group has at most two elements (Proposition 16.1.9), we only have to show that $\mathrm{Spin}_n(\mathbb{R})$ is connected, i.e., that $-\mathbf{1} \in \mathrm{Pin}_n(\mathbb{R})_0$. Identifying the basis elements $e_1, \ldots, e_n$ with the corresponding elements of the Clifford algebra C_n, we set

$$\gamma(t) := \cos(t)\mathbf{1} + \sin(t)e_1e_2.$$

Recall the grading automorphism ω from Definition B.3.7. From $(e_1e_2)^2 = -\mathbf{1}$, it follows that $\gamma(t) = e^{te_1e_2}$, which implies for $t \in \mathbb{R}$ that

$$\omega(\gamma(t)) = \gamma(t) \quad \text{and} \quad \gamma(t)^{-1} = \gamma(-t).$$

We now have

$$\begin{aligned} \omega\big(\gamma(t)\big)e_1\gamma(t)^{-1} &= \cos(2t)e_1 + \sin(2t)e_2, \\ \omega\big(\gamma(t)\big)e_2\gamma(t)^{-1} &= -\sin(2t)e_1 + \cos(2t)e_2, \\ \omega\big(\gamma(t)\big)e_i\gamma(t)^{-1} &= e_i \quad \text{for } i \geq 3. \end{aligned}$$

Hence $\gamma(t)$ is in $\Gamma(\mathbb{R}^n, \beta)$, the Clifford group, where $-\beta$ is the euclidian inner product on $\mathbb{R}^n$. Further, $\gamma(t)\gamma(t)^* = \gamma(t)\gamma(-t) = \mathbf{1}$, and consequently $\gamma(t) \in \mathrm{Pin}_n(\mathbb{R})$. Finally, $\gamma(\pi) = -\mathbf{1}$, which completes the proof. □

Combining Propositions 16.1.9 and 16.1.10, we obtain:

Theorem 16.1.11. *The group $\mathrm{Spin}_n(\mathbb{R})$ is the universal covering of $\mathrm{SO}_n(\mathbb{R})$ and, for $n > 2$, $\pi_1\big(\mathrm{SO}_n(\mathbb{R})\big) \cong \mathbb{Z}/2\mathbb{Z}$.*

Example 16.1.12. (a) From Example 8.5.7 we recall that

$$\mathrm{SU}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}_3}(\mathbb{R}) \cong \mathrm{Spin}_3(\mathbb{R}).$$

(b) Proposition 8.5.21 yields an isomorphism

$$\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}_4}(\mathbb{R}) \cong \mathrm{Spin}_4(\mathbb{R}).$$

16.1.3 Symplectic Groups

Viewing $M_n(\mathbb{H})$ as a real subalgebra of $M_{2n}(\mathbb{C})$ as in Proposition 1.3.2, it was shown in Proposition 1.3.4 that

$$\mathrm{U}_n(\mathbb{H}) = \mathrm{U}_{2n}(\mathbb{C}) \cap \mathrm{Sp}(\mathbb{C}^{2n}, \beta), \quad \text{where} \quad \beta(z, w) = z^\top J_n w,$$

and J_n is the skew-symmetric $(2n \times 2n)$-block diagonal matrix with entries $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since all nondegenerate skew-symmetric bilinear forms on $\mathbb{C}^{2n}$ are equivalent (Exercise 1.2.3), $\mathrm{Sp}(\mathbb{C}^{2n}, \beta) \cong \mathrm{Sp}_{2n}(\mathbb{C})$, and since $\mathrm{Sp}(\mathbb{C}^{2n}, \beta)$ is $*$-invariant, it follows that $\mathrm{U}_n(\mathbb{H})$ is a maximal compact subgroup. Moreover, the universal complexification $\mathrm{U}_n(\mathbb{H})_\mathbb{C}$ of $\mathrm{U}_n(\mathbb{H})$ is isomorphic to the group $\mathrm{Sp}_{2n}(\mathbb{C})$ (cf. Exercise 14.2.2, Proposition 3.3.3). We conclude in particular that $\mathrm{U}_n(\mathbb{H})$ is isomorphic to the maximal compact subgroup $\mathrm{U}_{2n}(\mathbb{C}) \cap \mathrm{Sp}_{2n}(\mathbb{C})$ of $\mathrm{Sp}_{2n}(\mathbb{C})$, which also called the *(compact) symplectic group*.

Remark 16.1.13. We have already seen in Proposition 16.1.3 that $\mathrm{U}_n(\mathbb{H})$ is connected. To determine its center, it is useful to use the realization as symplectic group. Then Exercise 14.1.8 together with Exercise 1.2.15 shows that

$$Z\big(\mathrm{Sp}_{2n}(\mathbb{C})\big) = Z\big(\mathrm{U}_n(\mathbb{H})\big) = \{\pm\mathbf{1}\}.$$

16.2 Noncompact Classical Groups

For noncompact groups information on connected components and the fundamental group will be derived from the corresponding information for the maximal compact subgroups via the polar decomposition (cf. Proposition 15.1.9).

Proposition 16.2.1. *For the complex semisimple Lie group* $\mathrm{SL}_n(\mathbb{C})$*,*

(1) $\mathrm{SU}_n(\mathbb{C})$ *is a maximal compact subgroup.*
(2) $\mathrm{SL}_n(\mathbb{C})$ *is simply connected.*
(3) $\mathfrak{su}_n(\mathbb{C})$ *is a compact real form of* $\mathfrak{sl}_n(\mathbb{C})$*.*
(4) $Z\big(\mathrm{SL}_n(\mathbb{C})\big) = \{z\mathbf{1}\colon z^n = 1\} \cong C_n$*.*

Proof. Since the condition $\det(g) = \mathbf{1}$ can also be written as two real polynomial equations $\mathrm{Re}(\det(g) - 1) = 0$ and $\mathrm{Im}(\det(g) - 1) = 0$, Proposition 15.1.9 applies, so that (1) follows immediately, and (2) follows since $\mathrm{SU}_n(\mathbb{C})$ is 1-connected (Proposition 16.1.3).

(3) From Proposition 16.1.1, we know that $\mathfrak{su}_n(\mathbb{C})$ is compact and semisimple. That $\mathfrak{su}_n(\mathbb{C})$ is a compact real form of $\mathfrak{sl}_n(\mathbb{C})$ follows from $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) \oplus i\mathfrak{su}_n(\mathbb{C})$.

(4) In view of $\mathrm{GL}_n(\mathbb{C}) = \mathbb{C}^\times \mathrm{SL}_n(\mathbb{C})$, we have

$$Z(\mathrm{SL}_n(\mathbb{C})) = Z(\mathrm{GL}_n(\mathbb{C})) \cap \mathrm{SL}_n(\mathbb{C}) = \{z\mathbf{1} \colon \det(z\mathbf{1}) = z^n = 1\} \cong C_n.$$

(Proposition 1.1.10). □

Proposition 16.2.2. *For* $\mathrm{GL}_n(\mathbb{C})$*,*

(1) $\mathrm{U}_n(\mathbb{C})$ *is a maximal compact subgroup.*
(2) $\mathrm{GL}_n(\mathbb{C})$ *is connected.*
(3) $\pi_1\big(\mathrm{GL}_n(\mathbb{C})\big) \cong \mathbb{Z}$.
(4) $Z\big(\mathrm{GL}_n(\mathbb{C})\big) = \mathbb{C}^\times \mathbf{1}$.
(5) *The multiplication map* $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}), (z, g) \mapsto zg$ *is an* n*-fold covering.*

Proof. (1) follows from Proposition 15.1.9.

(2) follows from Corollary 1.1.8.

(3) In view of the polar decomposition (Proposition 15.1.9), since $\mathrm{SU}_n(\mathbb{C})$ is 1-connected (Propositions 16.1.3 and 16.1.5), this follows from $\mathrm{U}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C}) \rtimes \mathbb{T}$.

(4) is a consequence of Proposition 1.1.10.

(5) follows from $\mathbb{C}^\times \mathbf{1} \cap \mathrm{SL}_n(\mathbb{C}) = Z\big(\mathrm{SL}_n(\mathbb{C})\big) \cong C_n$ (Proposition 16.2.1). □

Proposition 16.2.3. *For* $\mathrm{SL}_n(\mathbb{R}), n \geq 2$*,*

(1) $\mathrm{SO}_n(\mathbb{R})$ *is a maximal compact subgroup.*
(2) $\mathrm{SL}_n(\mathbb{R})$ *is connected.*
(3) $\pi_1\big(\mathrm{SL}_n(\mathbb{R})\big) \cong \begin{cases} \mathbb{Z} & \text{for } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n > 2. \end{cases}$
(4) $\mathrm{SL}_n(\mathbb{R})$ *is semisimple.*
(5) $Z\big(\mathrm{SL}_n(\mathbb{R})\big) = \begin{cases} \{\mathbf{1}\} & \text{for } n \geq 1 \text{ odd}, \\ \{\pm\mathbf{1}\} & \text{for } n \geq 2 \text{ even}. \end{cases}$

Proof. (1) is a consequence of the polar decomposition (Proposition 15.1.9).

Since $\mathrm{SO}_n(\mathbb{R})$ is connected (Proposition 1.1.7), (2) follows from (1) and the polar decomposition.

(3) follows from (1), the polar decomposition, Theorem 16.1.11 and $\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{T}$.

(4) follows from the fact that $\mathfrak{sl}_n(\mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{R})_{\mathbb{C}}$ is semisimple (Proposition 16.2.1).

(5) follows from Exercise 1.2.14(v). □

Proposition 16.2.4. *For* $\mathrm{GL}_n(\mathbb{R})$,

(1) $\mathrm{O}_n(\mathbb{R})$ *is a maximal compact subgroup.*
(2) $\mathrm{GL}_n(\mathbb{R})$ *has* 2 *connected components.*
(3) $\pi_1\big(\mathrm{GL}_n(\mathbb{R})\big) \cong \begin{cases} \mathbb{Z} & \text{for } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n > 2. \end{cases}$
(4) $Z\big(\mathrm{GL}_n(\mathbb{R})\big) = \mathbb{R}^\times \mathbf{1}$.
(5) *The multiplication map* $\mathbb{R}_+^\times \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})_0$ *is a homeomorphism.*

Proof. (1) follows from Proposition 15.1.9, (2) from Corollary 1.1.8, (4) and (5) from Proposition 1.1.10, and finally (3) is a consequence of (5) and Proposition 16.2.3. □

Proposition 16.2.5. *Let* $p, q \geq 1$. *For the pseudo-orthogonal group* $\mathrm{O}_{p,q}(\mathbb{R}) \subseteq \mathrm{GL}_{p+q}(\mathbb{R})$,

(1) $\mathrm{O}_p(\mathbb{R}) \times \mathrm{O}_q(\mathbb{R})$ *is a maximal compact subgroup, where* $\mathrm{O}_p(\mathbb{R})$ *acts on* $\mathbb{R}^p \times \{0\}$ *and* $\mathrm{O}_q(\mathbb{R})$ *on* $\{0\} \times \mathbb{R}^q$.
(2) $\mathrm{O}_{p,q}(\mathbb{R})$ *has* 4 *connected components,* 2 *of them are contained in* $\mathrm{SO}_{p,q}(\mathbb{R})$.
(3) $\pi_1\big(\mathrm{O}_{p,q}(\mathbb{R})\big) \cong \pi_1\big(\mathrm{SO}_p(\mathbb{R})\big) \times \pi_1\big(\mathrm{SO}_q(\mathbb{R})\big)$.
(4) $Z\big(\mathrm{O}_{p,q}(\mathbb{R})\big) = \{\pm\mathbf{1}\}$.
(5) *The group* $\mathrm{O}_{p,q}(\mathbb{R})$ *is semisimple.*

Proof. (1) follows from the polar decomposition (Proposition 15.1.9) and

$$\mathrm{O}_{p,q}(\mathbb{R}) \cap \mathrm{O}_{p+q}(\mathbb{R}) \cong \mathrm{O}_p(\mathbb{R}) \times \mathrm{O}_q(\mathbb{R}),$$

as asserted.

(2) and (3) follow from the polar decomposition and Corollary 16.1.8.

(4) (cf. also Exercise 1.2.16) Let $g \in Z\big(\mathrm{O}_{p,q}(\mathbb{R})\big)$. In view of (1), g commutes with the matrix

$$B = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix},$$

hence is a block diagonal matrix $g = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, where A commutes with $\mathrm{O}_p(\mathbb{R})$, and C commutes with $\mathrm{O}_q(\mathbb{R})$. There are four possibilities: $A = \pm\mathbf{1}_p$, and $C = \pm\mathbf{1}_q$ (Proposition 16.1.6). If $g \notin \{\pm\mathbf{1}\}$, then it has proper eigenspaces which are invariant under the whole group $\mathrm{O}_{p,q}(\mathbb{R})$, hence also under its Lie algebra $\mathfrak{o}_{p,q}(\mathbb{R})$ which contains all matrices of the form $\begin{pmatrix} \mathbf{0} & X \\ X^\top & \mathbf{0} \end{pmatrix}$ for $X \in M_{p,q}(\mathbb{R})$. We thus arrive at a contradiction which completes the proof of (4).

(5) The complexification of the Lie algebras $\mathfrak{so}_{p,q}(\mathbb{R})$ and $\mathfrak{so}_{p+q}(\mathbb{R})$ are isomorphic (Exercise 16.2.1). Hence $\mathfrak{so}_{p,q}(\mathbb{R})$ is a real form of $\mathfrak{so}_{p+q}(\mathbb{R})_{\mathbb{C}}$, and therefore semisimple. □

Example 16.2.6. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we consider the spaces

$$\mathrm{Herm}_2(\mathbb{K}) := \{A \in M_2(\mathbb{K}) \colon A^* = A\} = \left\{ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} : a, d \in \mathbb{R}, b \in \mathbb{K} \right\}$$

and

$$\mathrm{Aherm}_2(\mathbb{K}) := \{A \in M_2(\mathbb{K}) \colon A^* = -A\}$$

of hermitian and antihermitian 2×2-matrices. If $s := \dim_{\mathbb{R}} \mathbb{K}$, then

$$\dim \mathrm{Herm}_2(\mathbb{K}) = 2 + s.$$

If $a \neq 0$, then

$$\det_{\mathbb{R}} \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = \det_{\mathbb{R}} \begin{pmatrix} a & b \\ 0 & d - \frac{b^*}{a} b \end{pmatrix} = \det_{\mathbb{R}} \begin{pmatrix} a & b \\ 0 & d - \frac{|b|^2}{a} \end{pmatrix} = (ad - |b|^2)^s,$$

and this implies that

$$\det_{\mathbb{R}} \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = (ad - |b|^2)^s$$

for all elements in $\mathrm{Herm}_2(\mathbb{K})$. This means that the quadratic form

$$q \colon \mathrm{Herm}_2(\mathbb{K}) \to \mathbb{R}, \quad q \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} := ad - |b|^2$$

of signature $(1, s+1)$ satisfies $q^s = \det_{\mathbb{R}}$.

The group $\mathrm{GL}_2(\mathbb{K})$ acts linearly on $\mathrm{Herm}_2(\mathbb{K})$ by

$$\pi(g)A := gAg^*,$$

and we clearly have

$$\det_{\mathbb{R}}(\pi(g)A) = \det_{\mathbb{R}}(A) \det_{\mathbb{R}}(g) \det_{\mathbb{R}}(g^*) = \det_{\mathbb{R}}(A) \det_{\mathbb{R}}(g)^2.$$

For each $g \in \mathrm{GL}_2(\mathbb{K})$ we therefore have

$$q(\pi(g)A)^s = |\operatorname{der}_{\mathbb{R}}(g)|^2 q(A)^s, \quad A \in \mathrm{Herm}_2(\mathbb{K}),$$

which leads to

$$q(\pi(g)A) = \pm |\operatorname{der}_{\mathbb{R}}(g)|^{2/s} q(A).$$

In particular, g preserves the open subset

$$\mathrm{Herm}_2(\mathbb{K})^\times := \mathrm{GL}_2(\mathbb{K}) \cap \mathrm{Herm}_2(\mathbb{K}) = \{A \in \mathrm{Herm}_2(\mathbb{K}) : q(A) \neq 0\}.$$

Since q has Lorentzian signature, this set has three connected components, all of which are preserved under the action of the identity component $\mathrm{GL}_2(\mathbb{K})_0$.

It follows in particular that for $g \in \mathrm{GL}_2(\mathbb{K})_0$, $q(\pi(g)A)$ always has the same sign as $q(A)$, so that

$$q \circ \pi(g) = |\,\mathrm{der}_{\mathbb{R}}(g)|^{1/2} q \quad \text{for} \quad g \in \mathrm{GL}_2(\mathbb{K})_0.$$

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathrm{Herm}_2(\mathbb{K})$ thus defines a homomorphism

$$\pi\colon\ \mathrm{SL}_2(\mathbb{K}) \to \mathrm{SO}(\mathrm{Herm}_2(\mathbb{K}), q) \cong \mathrm{SO}_{1,s+1}(\mathbb{R}).$$

For $\mathbb{K} = \mathbb{H}$, we put

$$\mathrm{SL}_2(\mathbb{H}) := \{g \in \mathrm{GL}_2(\mathbb{H})\colon\ \det_{\mathbb{R}} g = 1\}$$

and note that, as a subgroup of $\mathrm{GL}_8(\mathbb{R})$, this group has a polar decomposition (cf. Proposition 15.1.9) with

$$\mathrm{SL}_2(\mathbb{H}) \cap \mathrm{O}_8(\mathbb{R}) = \mathrm{U}_2(\mathbb{H}),$$

where we use $\mathrm{Aherm}_2(\mathbb{H}) = \mathbf{L}(\mathrm{U}_2(\mathbb{H})) \subseteq \mathfrak{sl}_8(\mathbb{R})$ to see that $\mathrm{U}_2(\mathbb{H}) \subseteq \mathrm{SL}_2(\mathbb{H})$. Since $\mathrm{U}_2(\mathbb{H})$ is connected, the same holds for $\mathrm{SL}_2(\mathbb{H})$, and we also obtain a homomorphism

$$\pi\colon\ \mathrm{SL}_2(\mathbb{K}) \to \mathrm{SO}(\mathrm{Herm}_2(\mathbb{K}), q) \cong \mathrm{SO}_{1,s+1}(\mathbb{R})$$

for $\mathbb{K} = \mathbb{H}$.

To determine the kernel of π, we note that $\pi(g) = \mathbf{1}$ implies in particular that $\mathbf{1} = \pi(g)\mathbf{1} = gg^*$, so that $g \in \mathrm{U}_2(\mathbb{K})$. Hence $\pi(g) = \mathbf{1}$ means that g commutes with $\mathrm{Herm}_2(\mathbb{K})$, hence in particular with all real diagonal matrices. Therefore $g = \mathrm{diag}(a, d)$ is a diagonal matrix, where $a, d \in \mathrm{U}_1(\mathbb{K})$. For $A = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \in \mathrm{Herm}_2(\mathbb{K})$, the relation $\pi(g)A = A$ now implies that $abd^{-1} = b$ for all $b \in \mathbb{K}$. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, this is equivalent to $a = d$, and for $\mathbb{K} = \mathbb{H}$, we obtain, in addition, that $a = d \in Z(\mathbb{H}) = \mathbb{R}$. In all cases the requirement $\det(g) = 1$ leads to $g = \pm\mathbf{1}$. We therefore have

$$\ker \pi = \{\pm\mathbf{1}\}.$$

Next, we observe that

$$\dim_{\mathbb{R}} \mathrm{SL}_2(\mathbb{K}) = \begin{cases} 3 & \text{for } \mathbb{K} = \mathbb{R}, \\ 6 & \text{for } \mathbb{K} = \mathbb{C}, \\ 15 & \text{for } \mathbb{K} = \mathbb{H}. \end{cases}$$

This is in accordance with (cf. Exercise 16.2.1)

$$\dim \mathrm{SO}_{1,s+1}(\mathbb{R}) = \dim \mathrm{SO}_{s+2}(\mathbb{R}) = \frac{(s+2)(s+1)}{2} = \begin{cases} 3 & \text{for } s = 1\ (\mathbb{K} = \mathbb{R}), \\ 6 & \text{for } s = 2\ (\mathbb{K} = \mathbb{C}), \\ 15 & \text{for } s = 4\ (\mathbb{K} = \mathbb{H}). \end{cases}$$

Since $\ker\pi$ is discrete, it follows that $\mathbf{L}(\pi)\colon \mathfrak{sl}_2(\mathbb{K}) \to \mathfrak{so}_{1,s+1}(\mathbb{R})$ is an isomorphism of Lie algebras, so that $\pi\colon \mathrm{SL}_2(\mathbb{K}) \to \mathrm{SO}_{1,s+1}(\mathbb{R})_0$ is a covering morphism of Lie groups.

The maximal compact subgroup of $\mathrm{SO}_{1,s+1}(\mathbb{R})_0$ is isomorphic to

$$(\mathrm{O}_1(\mathbb{R}) \times \mathrm{O}_{s+1}(\mathbb{R}))_0 \cong \mathrm{SO}_{s+1}(\mathbb{R}),$$

so that the polar decomposition implies that

$$\pi_1(\mathrm{SO}_{1,s+1}(\mathbb{R})) \cong \begin{cases} \mathbb{Z} & \text{for } s = 1\ (\mathbb{K} = \mathbb{R}), \\ \mathbb{Z}/2 & \text{for } s = 2, 4\ (\mathbb{K} = \mathbb{C}, \mathbb{H}). \end{cases}$$

In each case π is a connected 2-fold covering, which is uniquely determined up to isomorphism. This implies that, for $s = 2, 4$,

$$\mathrm{SL}_2(\mathbb{K}) \cong \widetilde{\mathrm{SO}}_{1,s+1}(\mathbb{R})_0$$

is the simply connected covering group of $\mathrm{SO}_{1,s+1}(\mathbb{R})_0$.

A slight modification of the argument in the proof of Proposition 16.1.10 shows that for $p \geq 2$, we still have $-\mathbf{1} \in \mathrm{Pin}_{p,q}(\mathbb{R})_0 = \mathrm{Spin}_{p,q}(\mathbb{R})$. From the discussion in Remark B.3.22(c) we know that the homomorphism

$$\Phi\colon \widehat{\mathrm{Pin}}_{p,q}(\mathbb{R}) := \{\gamma \in \Gamma\colon N(\gamma)^2 = 1\} \to \mathrm{O}_{p,q}(\mathbb{R})$$

is surjective with kernel

$$\{\lambda \in \mathbb{R}^\times\colon N(\lambda)^2 = \lambda^4 = 1\} = \{\pm\mathbf{1}\}.$$

The partition of the group $\mathrm{O}_{p,q}(\mathbb{R})$ into four connected components is reflected in the fact that $\mathrm{Pin}_{p,q}(\mathbb{R})$ is a subgroup of index 2 in $\widehat{\mathrm{Pin}}_{p,q}(\mathbb{R})$ which is mapped onto a subgroup of index 2 of $\mathrm{O}_{p,q}(\mathbb{R})$. This group consists of all products of reflections $\sigma_{v_1} \cdots \sigma_{v_n}$, where an even number of factors corresponds to v_i with positive square length with respect to the quadratic form

$$\beta_{p,q}(x,x) = -x_1^1 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2.$$

For $p = s + 1$ and $q = 1$, we thus obtain a connected 2-fold covering

$$\Phi\colon \mathrm{Spin}_{s+1,1}(\mathbb{R}) \to \mathrm{SO}_{s+1,1}(\mathbb{R})_0.$$

Since $\pi_1(\mathrm{SO}_{s+1,1}(\mathbb{R})) \cong \pi_1(\mathrm{SO}_{s+1}(\mathbb{R}))$ is cyclic, this covering is unique, which leads to an isomorphism

$$\mathrm{SL}_2(\mathbb{K}) \cong \mathrm{Spin}_{s+1,1}(\mathbb{R})$$

is all cases $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. In particular, we obtain

$$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{Spin}_{2,1}(\mathbb{R}), \quad \mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}_{3,1}(\mathbb{R}) \quad \text{and} \quad \mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}_{5,1}(\mathbb{R})$$

(cf. Exercise 8.5.16).

Exercises for Section 16.2

Exercise 16.2.1. (a) Show that $\mathfrak{so}_{p,q}(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{so}_{p+q}(\mathbb{C})$.
(b) Show that $\dim \mathfrak{so}_{p,q}(\mathbb{R}) = \frac{(p+q)(p+q-1)}{2}$.

Exercise 16.2.2. Let ψ be the antilinear map of $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ defined by $\psi(z,w) = (\overline{w}, -\overline{z})$ for $z, w \in \mathbb{C}^n$. Define

$$\mathrm{SU}^*_{2n}(\mathbb{C}) := \{g \in \mathrm{SL}_{2n}(\mathbb{C}) \mid g \circ \psi = \psi \circ g\}.$$

Show that

(i) $\psi^2 = -\mathbf{1}$, so that $Jv := \psi(v)$ and $Iv := iv$ define a representation of the algebra $\mathbb{H}$ on $\mathbb{C}^{2n}$, turning $\mathbb{C}^{2n}$ into an $\mathbb{H}$-vector space isomorphic to $\mathbb{H}^n$.
(ii) $\mathrm{SU}^*_{2n}(\mathbb{C}) \cong \mathrm{SL}_n(\mathbb{H})$ as a Lie group.

Exercise 16.2.3. * Let G be one of the groups $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{U}_{p,q}(\mathbb{C})$, $\mathrm{U}_{p,q}(\mathbb{H})$.

(a) Determine a Cartan decomposition $K \exp \mathfrak{p}$ for G using Proposition 3.3.3, and conclude that G is connected.
(b) Determine a maximal torus T in K.
(c) Use Theorem 13.2.8 to determine $Z(G)$.

16.3 More Spin Groups

We have seen in Proposition 16.1.10 that, for each $n > 1$, the map

$$\Phi\colon \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}_n(\mathbb{R})$$

is a connected twofold covering which is universal for $n > 2$. For $n = 1$, $\mathrm{SO}_1(\mathbb{R}) = \{\mathbf{1}\}$, so that $\mathrm{Spin}_1(\mathbb{R}) = \{\pm\mathbf{1}\}$ is not connected. We now take a closer look at the corresponding homomorphism

$$\Phi\colon \mathrm{Spin}_{p,q}(\mathbb{R}) \to \mathrm{SO}_{p,q}(\mathbb{R})$$

obtained from the Clifford algebra $C_{p,q} = \mathrm{Cl}(\mathbb{R}^{p+q}, \beta_{p,q})$.

Proposition 16.3.1. *The image of Φ is the identity component $\mathrm{SO}_{p,q}(\mathbb{R})_0$, and*

$$\Phi\colon \mathrm{Spin}_{p,q}(\mathbb{R}) \to \mathrm{SO}_{p,q}(\mathbb{R})_0$$

is a twofold covering with kernel $\{\pm\mathbf{1}\}$.

We further have

$$\mathrm{Spin}_{0,q}(\mathbb{R}) \cong \mathrm{Spin}_q(\mathbb{R}),$$

and from the inclusions $C_{p,0}, C_{0,q} \hookrightarrow C_{p,q}$, we obtain the subgroup

$$\mathrm{Spin}_{p,0}(\mathbb{R}) \cdot \mathrm{Spin}_{0,q}(\mathbb{R}) \cong (\mathrm{Spin}_{p,0}(\mathbb{R}) \times \mathrm{Spin}_{0,q}(\mathbb{R}))/\{\pm\mathbf{1}\},$$

which is maximal compact in $\mathrm{Spin}_{p,q}(\mathbb{R})$. *In particular,* $\mathrm{Spin}_{p,q}(\mathbb{R})$ *is connected if and only if* $p > 1$ *or* $q > 1$*, and it is* 1*-connected for* $p > 2, q \leq 1$ *and for* $p \leq 1, q > 2$.

The restriction of Φ *to this maximal compact subgroup corresponds to the map*

$$\mathrm{Spin}_{p,0}(\mathbb{R}) \cdot \mathrm{Spin}_{0,q}(\mathbb{R}) \to \mathrm{SO}_p(\mathbb{R}) \times \mathrm{SO}_q(\mathbb{R}), \quad gh \mapsto \Phi(g)\Phi(h),$$

which specifies $\mathrm{Spin}_{p,q}(\mathbb{R})$ *as a covering group of* $\mathrm{SO}_{p,q}(\mathbb{R})_0$.

Proof. (a) From Remark B.3.22 we know that, in all cases, $\ker \Phi = \{\pm \mathbf{1}\}$. From Theorem B.3.16 we further derive that

$$\dim \mathrm{Spin}_{p,q}(\mathbb{R}) = \dim \mathrm{Pin}_{p,q}(\mathbb{R}) = \dim \Gamma(\mathbb{R}^{p+q}, \beta_{p,q}) - 1 = \dim \mathrm{O}_{p,q}(\mathbb{R}),$$

so that the image of Φ is open and Φ is a twofold covering of its image.

(b) First we consider the case $(p, q) = (0, 2)$. Then $C_{0,2}$ is 4-dimensional and $\dim \mathrm{Spin}_{0,2}(\mathbb{R}) = \dim \mathrm{SO}_2(\mathbb{R}) = 1$. To identify the Lie algebra of this group, we note that the basis elements $\mathbf{1}, e_1, e_2$ and $e_1 e_2$ of $C_{0,2}$ satisfy

$$e_1^2 = e_2^2 = \mathbf{1} \quad \text{and} \quad (e_1 e_2)^2 = -\mathbf{1}.$$

Then $I := e_1 e_2$ satisfies $\omega(I) = I$,

$$I^* = (e_1 e_2)^* = (-e_2)(-e_1) = e_2 e_1 = -e_1 e_2 = -I,$$

$$[I, e_1] = e_1 e_2 e_1 - e_1^2 e_2 = -2e_2, \quad \text{and} \quad [I, e_2] = e_1 e_2^2 - e_2 e_1 e_2 = 2e_1.$$

This implies that $\omega(e^{tI}) = e^{tI}$ for each $t \in \mathbb{R}$, and hence that $\mathrm{Ad}(e^{tI}) = e^{t \,\mathrm{ad}\, I}$ preserves $V = \mathrm{span}\{e_1, e_2\}$. Further, $(e^{tI})^* = e^{-tI}$ shows that $e^{\mathbb{R}I} \subseteq \mathrm{Pin}_{0,2}(\mathbb{R})$. Since the latter group is 1-dimensional, we have

$$e^{\mathbb{R}I} = \mathrm{Pin}_{0,2}(\mathbb{R})_0 \subseteq \mathrm{Spin}_{0,2}(\mathbb{R}).$$

Since $\mathrm{SO}_2(\mathbb{R})$ connected, $e^{\pi I} = -\mathbf{1}$ implies that $\mathrm{Spin}_{0,2}(\mathbb{R})$ is connected, and that

$$\Phi \colon \mathrm{Spin}_{0,2}(\mathbb{R}) \to \mathrm{SO}_2(\mathbb{R})$$

is a 2-fold covering.

(c) For $p = 0$ and $q > 2$, we recall from Remark B.3.22(b) that $\mathrm{Spin}_{0,q}(\mathbb{R}) = \mathrm{Pin}_{0,q}(\mathbb{R})$, and since $\mathrm{SO}_{0,q}(\mathbb{R}) \cong \mathrm{SO}_q(\mathbb{R})$ is connected,

$$\Phi \colon \mathrm{Spin}_{0,q}(\mathbb{R}) \to \mathrm{SO}_q(\mathbb{R})$$

is surjective. From the embedding $\mathrm{Spin}_{0,2}(\mathbb{R}) \hookrightarrow \mathrm{Spin}_{0,q}(\mathbb{R})$ and $-\mathbf{1} \in \mathrm{Spin}_{0,2}(\mathbb{R})$, we now derive that $\mathrm{Spin}_{0,q}(\mathbb{R})$ is connected. Since the fundamental group of $\mathrm{SO}_q(\mathbb{R})$ has two elements, it has a unique connected twofold covering, which leads to

$$\mathrm{Spin}_{0,q}(\mathbb{R}) \cong \mathrm{Spin}_q(\mathbb{R}) = \mathrm{Spin}_{q,0}(\mathbb{R}).$$

(d) Now we turn to the case where $p, q > 0$ and $(p, q) \neq (1, 1)$. From Remark B.3.22(c) we know that $\Phi(\mathrm{Spin}_{p,q}(\mathbb{R}))$ is a proper subgroup of $\mathrm{SO}_{p,q}(\mathbb{R})$. Since $\mathrm{SO}_{p,q}(\mathbb{R})$ has two connected components, $\Phi(\mathrm{Spin}_{p,q}(\mathbb{R})) = \mathrm{SO}_{p,q}(\mathbb{R})_0$.

As a consequence of the polar decomposition, the subgroup $\mathrm{SO}_p(\mathbb{R}) \times \mathrm{SO}_q(\mathbb{R})$ of $\mathrm{SO}_{p,q}(\mathbb{R})_0$ is maximal compact (Proposition 15.1.9). If $p = 1$, then $\mathrm{SO}_p(\mathbb{R})$ is trivial, and if $p > 1$, then the restriction of Φ to $\mathrm{Spin}_p(\mathbb{R}) \subseteq \mathrm{Spin}_{p,q}(\mathbb{R})$ is a 2-fold connected covering of $\mathrm{SO}_p(\mathbb{R})$ (Proposition 16.1.10). In view of (c), the same statements apply to the covering $\mathrm{Spin}_{0,q}(\mathbb{R}) \to \mathrm{SO}_q(\mathbb{R})$.

We conclude that $\mathrm{Spin}_{p,q}(\mathbb{R})$ is connected with maximal compact subgroup

$$\begin{aligned}\Phi^{-1}(\mathrm{SO}_p(\mathbb{R}) \times \mathrm{SO}_q(\mathbb{R})) &= \mathrm{Spin}_{p,0}(\mathbb{R})\,\mathrm{Spin}_{0,q}(\mathbb{R}) \\ &\cong (\mathrm{Spin}_p(\mathbb{R}) \times \mathrm{Spin}_q(\mathbb{R}))/\{\pm(\mathbf{1}, \mathbf{1})\}\end{aligned}$$

(cf. Exercise B.3.7). This picture shows that for $p \leq 1$, $q > 2$ or $q \leq 1$, $p > 2$,

$$\mathrm{Spin}_{p,q}(\mathbb{R}) \cong \widetilde{\mathrm{SO}}_{p,q}(\mathbb{R}).$$

For $p, q > 2$ the image of the homomorphism

$$\pi_1(\Phi) \colon \pi_1(\mathrm{Spin}_{p,q}(\mathbb{R})) \cong \mathbb{Z}/2 \to \pi_1(\mathrm{SO}_{p,q}(\mathbb{R}))$$

coincides with the subgroup

$$\{\pm(\mathbf{1}, \mathbf{1})\} \subseteq \pi_1(\mathrm{SO}_{p,q}(\mathbb{R})) \cong \{\pm\mathbf{1}\} \times \{\pm\mathbf{1}\}. \qquad \square$$

Remark 16.3.2. For $p = q = 1$, $\mathrm{SO}_{1,1}(\mathbb{R}) \cong \mathbb{R}^\times$, and thus $\mathrm{SO}_{1,1}(\mathbb{R})_0 \cong \mathbb{R}_0^\times$ is simply connected. Therefore $\mathrm{Spin}_{1,1}(\mathbb{R})$ is not connected and

$$\mathrm{Spin}_{1,1}(\mathbb{R}) \cong \{\pm\mathbf{1}\} \times \mathbb{R}_+^\times \cong \mathbb{R}^\times.$$

Example 16.3.3 (Complex spin groups). From Remark B.3.22(d) we recall the short exact sequence

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Spin}_n(\mathbb{C}) \xrightarrow{\Phi} \mathrm{SO}_n(\mathbb{C}) \to \mathbf{1}$$

and that $\mathrm{Spin}_n(\mathbb{C})$ is a closed subgroup of the Clifford algebra $C_n(\mathbb{C}) := \mathrm{Cl}(\mathbb{C}^n, \beta)$ with $\beta(z, w) = \sum_{j=1}^n z_j w_j$.

On the other hand, we know from the polar decomposition of $\mathrm{SO}_n(\mathbb{C})$ (Exercise 3.3.1) that its maximal compact subgroup is

$$\mathrm{SO}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{C}) \cap \mathrm{GL}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{C}) \cap \mathrm{U}_n(\mathbb{C}),$$

so that $\mathrm{SO}_n(\mathbb{C})$ is connected (Corollary 16.1.8) and

$$\pi_1(\mathrm{SO}_n(\mathbb{C})) \cong \pi_1(\mathrm{SO}_n(\mathbb{C})) \cong \mathbb{Z}/2 \quad \text{for} \quad n > 2$$

(Proposition 16.1.10).

Next we observe that the subspace $i\mathbb{R}^n \subseteq \mathbb{C}^n$ on which β restricts to the form $\beta_{n,0}$ leads to an inclusion $C_n \hookrightarrow C_n(\mathbb{C})$ of Clifford algebras. This in turn leads to an embedding $\mathrm{Spin}_n(\mathbb{R}) \hookrightarrow \mathrm{Spin}_n(\mathbb{C})$, so that $-\mathbf{1} \in \mathrm{Spin}_n(\mathbb{R})_0$ implies that $\mathrm{Spin}_n(\mathbb{C})$ is also connected. We thus obtain, as in the real case, that the map $\Phi\colon \mathrm{Spin}_n(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C})$ is a universal covering.

Example 16.3.4 (The isomorphism $\mathrm{SL}_4(\mathbb{C}) \cong \mathrm{Spin}_6(\mathbb{C})$). Let $V := \Lambda^2(\mathbb{C}^4)$ and pick a non-zero element $\omega \in \Lambda^4(\mathbb{C}^4) \cong \mathbb{C}$. Then the wedge product defines a symmetric complex bilinear map β on $\Lambda^2(\mathbb{C}^4)$ by

$$x \wedge y = \beta(x, y)\omega.$$

Then V is an orthogonal direct sum of the three subspaces

$$\mathrm{span}\{e_1 \wedge e_2, e_3 \wedge e_4\}, \quad \mathrm{span}\{e_1 \wedge e_3, e_2 \wedge e_4\} \quad \mathrm{span}\{e_1 \wedge e_4, e_2 \wedge e_3\}, \tag{16.1}$$

from which it easily follows that β is nondegenerate. As $g^*\omega = \det(g)\omega$ for $g \in \mathrm{GL}_4(\mathbb{C})$, we have

$$\mathrm{SL}_4(\mathbb{C}) = \{g \in \mathrm{GL}_4(\mathbb{C})\colon g^*\omega = \omega\},$$

so that the natural representation of $\mathrm{SL}_4(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^4)$ preserves β, which leads to a homomorphism

$$\gamma\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{O}(V, \beta) \cong \mathrm{O}_6(\mathbb{C}).$$

Next we observe that

$$\dim \mathrm{SL}_4(\mathbb{C}) = 15 = \frac{5 \cdot 6}{2} = \dim \mathrm{O}_6(\mathbb{C}).$$

Since $\mathrm{SL}_4(\mathbb{C})$ is connected and $\mathrm{SO}_6(\mathbb{C})$ is the identity component of $\mathrm{O}_6(\mathbb{C})$, we have $\gamma(\mathrm{SL}_4(\mathbb{C})) \subseteq \mathrm{SO}_6(\mathbb{C})$. We claim that $\ker\gamma = \{\pm\mathbf{1}\}$. That $-\mathbf{1} \in \ker\gamma$ follows that $\gamma(-\mathbf{1})(x \wedge y) = (-x) \wedge (-y) = x \wedge y$. If, conversely, $\gamma(g) = \mathrm{id}_V$, then g preserves each 2-dimensional subspace $E \subseteq \mathbb{C}^2$ because

$$E = \{v \in \mathbb{C}^4\colon v \wedge x \wedge y = 0\}$$

holds if (x, y) is a basis for E. Since every one-dimensional subspace is the intersection of two two-dimensional ones, we see that g also preserves each one-dimensional subspace, i.e., each vector is an eigenvector, and thus $g = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{C}$ with $\lambda^4 = 1$. Then $g(x \wedge y) = \lambda^2(x \wedge y)$ further leads to $\lambda^2 = \mathbf{1}$, so that $g \in \{\pm\mathbf{1}\}$.

Since $\ker\gamma$ is discrete, $\mathbf{L}(\gamma)$ is injective, hence surjective because both groups have the same dimension. This implies that $\gamma\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C})$ is a covering morphism (Proposition 8.5.1). Since $\mathrm{SL}_4(\mathbb{C})$ is simply connected (Proposition 16.2.1), we have

$$\mathrm{SL}_4(\mathbb{C}) \cong \widetilde{\mathrm{SO}_6}(\mathbb{C}) \cong \mathrm{Spin}_6(\mathbb{C}) \tag{16.2}$$

(cf. Example 16.3.3). As an isomorphism of noncompact connected groups induces an isomorphism of maximal compact subgroups, we further derive that

$$\mathrm{SU}_4(\mathbb{C}) \cong \widetilde{\mathrm{SO}}_6(\mathbb{R}) \cong \mathrm{Spin}_6(\mathbb{R}). \tag{16.3}$$

Example 16.3.5 (The isomorphism $\mathrm{SL}_4(\mathbb{R}) \cong \mathrm{Spin}_{3,3}(\mathbb{R})$). The same construction as in the preceding example can also be carried out for $\mathbb{K} = \mathbb{R}$. Here we obtain on $V = \Lambda^2(\mathbb{R}^4)$ a nondegenerate symmetric real-valued bilinear form β. As V is an orthogonal direct sum of the three two-dimensional subspaces on which the form is indefinite, it follows that $(V, \beta) \cong (\mathbb{R}^6, \beta_{3,3})$, so that $\mathrm{SO}(V, \beta) \cong \mathrm{SO}_{3,3}(\mathbb{R})$, and we obtain a covering

$$\gamma\colon\ \mathrm{SL}_4(\mathbb{R}) \to \mathrm{SO}_{3,3}(\mathbb{R})_0$$

with $\ker \gamma = \{\pm\mathbf{1}\}$, so that

$$\mathrm{SO}_{3,3}(\mathbb{R})_0 \cong \mathrm{SL}_4(\mathbb{R})/\{\pm\mathbf{1}\}. \tag{16.4}$$

For the maximal compact subgroups, we find accordingly

$$\mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SO}_4(\mathbb{R})/\{\pm\mathbf{1}\}, \tag{16.5}$$

which also follows from Proposition 8.5.21.

Recall that $\mathrm{SL}_4(\mathbb{R})$ is not simply connected, so that it is not the simply connected covering group of $\mathrm{SO}_{3,3}(\mathbb{R})$. However, the relation

$$\mathrm{SO}_4(\mathbb{R}) \cong (\mathrm{Spin}_3(\mathbb{R}) \times \mathrm{Spin}_3(\mathbb{R}))/\{\pm(\mathbf{1}, \mathbf{1})\}$$

and Proposition 16.3.1 imply that

$$\mathrm{SL}_4(\mathbb{R}) \cong \mathrm{Spin}_{3,3}(\mathbb{R}). \tag{16.6}$$

Example 16.3.6 (The isomorphism $\mathrm{SU}_{2,2}(\mathbb{C}) \cong \mathrm{Spin}_{2,4}(\mathbb{R})$). We endow the complex vector space $\mathbb{C}^4$ with the hermitian form defined by

$$\gamma(z, w) = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3} + z_4\overline{w_4}$$

whose isometry group is $\mathrm{U}_{2,2}(\mathbb{C})$. Now we use γ to define a hermitian form on the 6-dimensional space $\Lambda^2(\mathbb{C}^4)$ by

$$\beta(a \wedge b, c \wedge d) := \gamma(a, c)\gamma(b, d) - \gamma(b, c)\gamma(a, d).$$

The basis $e_j \wedge e_k$, $j < k$, for $\Lambda^2(\mathbb{C}^4)$ is β-orthogonal. The quadratic form $q(v) := \beta(v, v)$ satisfies

$$q(e_1 \wedge e_2) = q(e_3 \wedge e_4) = 1$$

and

$$q(e_1 \wedge e_3) = q(e_1 \wedge e_4) = q(e_2 \wedge e_3) = q(e_2 \wedge e_4) = -1,$$

so that β has signature $(2,4)$ and the action of $\mathrm{U}_{2,2}(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^4)$ leads to a homomorphism $\mathrm{U}_{2,2}(\mathbb{C}) \to \mathrm{U}_{2,4}(\mathbb{C})$.

Next we observe that there exists a unique antilinear map

$$*\colon \Lambda^2(\mathbb{C}^4) \to \Lambda^2(\mathbb{C}^4)$$

satisfying

$$v \wedge *w = \beta(v,w)\omega, \quad v,w \in \Lambda^2(\mathbb{C}^4),\ \omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

This map satisfies

$$*(e_1 \wedge e_2) = e_3 \wedge e_4, \quad *(e_1 \wedge e_3) = e_2 \wedge e_4, \quad *(e_1 \wedge e_4) = -e_2 \wedge e_3$$

and

$$*(e_2 \wedge e_3) = -e_1 \wedge e_4, \quad *(e_2 \wedge e_4) = e_1 \wedge e_3, \quad *(e_3 \wedge e_4) = e_1 \wedge e_2.$$

In particular, we see that $*^2 = \mathrm{id}$, so that

$$V := \{x \in \Lambda^2(\mathbb{C}^4)\colon\ * x = x\}$$

is a real form. Next we observe that

$$\beta(*v, *w)\omega = (*v) \wedge (* * w) = w \wedge *v = \beta(w,v)\omega$$

implies that $*$ is a β-isometry. This in turn implies that β is the unique hermitian extension of the real symmetric bilinear form $\beta_r := \beta|_{V \times V}$ and in particular β_r is non-degenerate of signature $(2,4)$ (Exercise 16.4.5).

The fact that the action of $\mathrm{U}_{2,2}(\mathbb{C})$ on $\Lambda(\mathbb{C}^4)$ is compatible with the wedge product and its restriction to $\Lambda^2(\mathbb{C}^4)$ preserves β implies that it also commutes with $*$, hence preserves V. We thus obtain a homomorphism

$$\alpha\colon\ \mathrm{SU}_{2,2}(\mathbb{C}) \to \mathrm{O}_{2,4}(\mathbb{R}),$$

and since $\mathrm{SU}_{2,2}(\mathbb{C})$ is connected (exercise), the image of α is contained in $\mathrm{SO}_{2,4}(\mathbb{R})_0$. From Example 16.3.4 we know already that $\ker \alpha = \{\pm \mathbf{1}\}$. Therefore

$$\dim \mathrm{SU}_{2,2}(\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{SL}_4(\mathbb{C}) = 15 = \dim_{\mathbb{C}} \mathrm{SO}_6(\mathbb{C}) = \dim \mathrm{SO}_{2,4}(\mathbb{R})$$

implies that α is a twofold covering. Since $\pi_1(\mathrm{SO}_{2,4}(\mathbb{R})) \cong \mathbb{Z} \times \mathbb{Z}/2$ has three different subgroups of index 2:

$$\mathbb{Z} \times \{0\}, \quad 2\mathbb{Z} \times \mathbb{Z}/2, \quad \langle (1, \overline{1}) \rangle, \tag{16.7}$$

the group $\mathrm{SO}_{2,4}(\mathbb{R})_0$ has three non-equivalent twofold coverings.

We therefore have to take a closer look at the restriction of α to the maximal compact subgroup

$$S(\mathrm{U}_2(\mathbb{C}) \times \mathrm{U}_2(\mathbb{C})) := \{(a,b) \in \mathrm{U}_2(\mathbb{C}) \times \mathrm{U}_2(\mathbb{C}) \colon \det a \det b = 1\}.$$

This group is a semidirect product

$$(\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})) \rtimes \mathbb{T},$$

and therefore its fundamental group is cyclic. A generator is given by the homomorphism

$$\sigma \colon \mathbb{T} \to \mathrm{SU}_{2,2}(\mathbb{C}), \quad t \mapsto \mathrm{diag}(t, 1, t^{-1}, 1).$$

The form β_r on V is positive definite on the subspace spanned by the elements of the form

$$ze_1 \wedge e_2 + \overline{z}e_3 \wedge e_4, \quad z \in \mathbb{C},$$

and negative definite on its orthogonal complement, spanned by the elements of the form

$$ze_1 \wedge e_3 + \overline{z}e_2 \wedge e_4, \quad ze_1 \wedge e_4 - \overline{z}e_2 \wedge e_3, \quad z \in \mathbb{C}.$$

We have

$$\sigma(z)(e_1 \wedge e_2 + e_3 \wedge e_4) = ze_1 \wedge e_2 + \overline{z}e_3 \wedge e_4,$$

$e_1 \wedge e_3 + e_2 \wedge e_4$ is fixed by $\sigma(z)$, and

$$\sigma(z)(e_1 \wedge e_4 - e_2 \wedge e_3) = ze_1 \wedge e_2 - \overline{z}e_3 \wedge e_4.$$

Therefore σ defines a homomorphism $\mathbb{T} \to \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_4(\mathbb{R}) \subseteq \mathrm{SO}_{2,4}(\mathbb{R})$ which projects on each factor to a generator of the fundamental group. From this observation we derive that the image of $\pi_1(\mathrm{SU}_{2,2}(\mathbb{C}))$ in $\pi_1(\mathrm{SO}_{2,4}(\mathbb{R}))$ is neither of the two factors. From Proposition 16.3.1 and (16.7) we now easily derive that

$$\mathrm{SU}_{2,2}(\mathbb{C}) \cong \mathrm{Spin}_{2,4}(\mathbb{R}). \tag{16.8}$$

16.4 Conformal Groups

In this section we briefly discuss the conformal completion of a quadratic space because it nicely illustrates the use of Lie theoretic techniques in a geometric context. Here we shall also encounter some of the exceptional isomorphisms discussed in the preceding section.

16.4.1 The Conformal Completion

Let (V, β) be a nondegenerate finite-dimensional quadratic space, i.e., a vector space V over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, endowed with a nondegenerate symmetric bilinear form β.

Definition 16.4.1. We write $\mathbb{P}(V)$ for the projective space of V. Then

$$Q(V,\beta) := \{[v] = \mathbb{K}v \in \mathbb{P}(V) \colon \beta(v,v) = 0\}$$

is called the *projective quadric of* (V,β). If β is nondegenerate, this is either empty or a submanifold of $\mathbb{P}(V)$ (Exercise 16.4.4).

Example 16.4.2. (a) If β is positive or negative definite, then $Q(V,\beta) = \emptyset$.

(b) We write $\mathbb{R}^{p,q}$ for the quadratic space $(V,\beta) = (\mathbb{R}^{p+q},\beta)$, where

$$\beta(x,y) := \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{p+q} x_j y_j$$

and we write $v = (v_+, v_-)$ with $v_+ \in \mathbb{R}^p$ and $v_- \in \mathbb{R}^q$. Then

$$Q_{p,q} := Q(V,\beta) = \{[(v_+,v_-)] \colon \|v_+\| = \|v_-\| = 1\}$$

implies that the map

$$q \colon \mathbb{S}^{p-1} \times \mathbb{S}^{q-1} \to Q(V,\beta), \quad (x,y) \mapsto [(x,y)]$$

is surjective. It is easy to see that this map is a twofold covering, which leads to a diffeomorphism

$$(\mathbb{S}^{p-1} \times \mathbb{S}^{q-1})/\{\pm\mathbf{1}\} \cong Q_{p,q}.$$

For $q = 1$ we obtain in particular $Q_{p,1} \cong \mathbb{S}^{p-1}$.

On the extended space $\widetilde{V} := V \oplus \mathbb{K}^2$, we consider the symmetric bilinear form

$$\widetilde{\beta}((v,s,t),(v',s',t')) := \beta(v,v') + ss' - tt'$$

and the corresponding quadric

$$Q := Q(\widetilde{V},\widetilde{\beta}) = \{[(v,s,t)] \colon \beta(v,v) + s^2 - t^2 = 0\}.$$

Then we have a smooth map

$$\eta : V \to Q, \quad \eta(v) = \Big[\big(v, \tfrac{1}{2}(1-\beta(v,v)), \tfrac{1}{2}(1+\beta(v,v))\big)\Big].$$

Lemma 16.4.3. *The map η has the following properties:*

(i) $\eta(V) = \{[(v,s,t)] \in Q \colon s + t \neq 0\} = \{[x] \in Q \colon \widetilde{\beta}(x,v_\infty) \neq 0\}$ *for* $v_\infty = (0,1,-1)$.

(ii) η *is injective.*

(iii) $\eta(V)$ *is open and dense in* Q.

Proof. (i) If $[\widetilde{v}] = \eta(v)$, then we obviously have $s + t \neq 0$. If, conversely, this is the case for an isotropic vector $\widetilde{v} = (v, s, t)$, then we may w.l.o.g. assume that $s + t = 1$, because $(s + t)^{-1}\widetilde{v}$ generates the same ray. Then

$$\beta(v, v) = t^2 - s^2 = (t - s)(t + s) = t - s$$

implies that

$$t = \tfrac{1}{2}(1 + \beta(v, v)) \quad \text{and} \quad s = \tfrac{1}{2}(1 - \beta(v, v)),$$

which leads to $[\widetilde{v}] = \eta(v)$.

(ii) Suppose that $\eta(v) = \eta(w)$ for some $v \neq 0$. Then $w = \lambda v$ for some $\lambda \in \mathbb{K}^\times$, and thus

$$\begin{aligned}\eta(v) &= \Big[\big(v, \tfrac{1}{2}(1 - \beta(v, v)), \tfrac{1}{2}(1 + \beta(v, v))\big)\Big] \\ &= \Big[\big(\lambda v, \tfrac{1}{2}(1 - \lambda^2\beta(v, v)), \tfrac{1}{2}(1 + \lambda^2\beta(v, v))\big)\Big],\end{aligned}$$

leads to

$$\lambda(1 \pm \beta(v, v)) = 1 \pm \lambda^2\beta(v, v), \quad \text{i.e.,} \quad \lambda - 1 = \pm(\lambda^2 - \lambda)\beta(v, v),$$

which leads to $\lambda = 1$, and hence to $v = w$.

(iii) From (i) we immediately derive that $\eta(V)$ is an open subset of Q. Suppose now that $[(v, s, t)] \in Q \setminus \eta(V)$. Then $s = -t$ and $\beta(v, v) = 0$. If $s = 0$, then $v \neq 0$ and

$$\eta(nv) = [(nv, \tfrac{1}{2}, \tfrac{1}{2})] = \Big[\big(v, \frac{1}{2n}, \frac{1}{2n}\big)\Big] \to [(v, 0, 0)].$$

If $s \neq 0$, then we may assume that $s = 1$ and pick some $w \in V$ with $\beta(v, w) \neq 0$. Then

$$\beta(v + xw) = x2\beta(v, w) + x^2\beta(w, w) \neq 0$$

for x sufficiently close to 0. Further, $\widetilde{v} := (v + xw, s + y, -s) \in \widetilde{V}$ is isotropic if and only if

$$0 = \beta(v + xw, v + xw) + (s + y)^2 - s^2 = \beta(v + xw, v + xw) + 2sy + y^2,$$

so that, if x is sufficiently small, there exists a $y(x) \in \mathbb{K}$ with $y(x) \to 0$ for $x \to 0$, such that $[(v + xw, s + y, -s)] \in Q$. This implies that $[\widetilde{v}]$ is contained in the closure of $\eta(V)$, so that $Q = \overline{\eta(V)}$. $\square$

Remark 16.4.4. For an isotropic vector $\widetilde{v} = (v, s, t)$ with $s + t = 0$, we have $[\widetilde{v}] \in Q \setminus \eta(V)$, and two cases occur. If $s \neq 0$, then a multiple of $\widetilde{v}$ has the form $(v, 1, -1)$, where $\beta(v, v) = 0$, and if $s = 0$, then $\widetilde{v} = (v, 0, 0)$ with $\beta(v, v) = 0$. Accordingly, the set $Q \setminus \eta(V)$ contains a copy of

$$Q(V, \beta) = \{[v] \in \mathbb{P}(V) \colon \beta(v, v) = 0\}$$

and of the subset of isotropic vectors in V.

The fact that the projective quadric $Q = Q(\widetilde{V}, \widetilde{\beta})$ contains V, resp., $\eta(V)$ as a dense open subset justifies the name *conformal completion of* V for Q. Here an interesting point is that Q is a homogeneous space of the orthogonal group $G := \mathrm{O}(\widetilde{V}, \widetilde{\beta})$ (Exercise 16.4.3), and since V is realized via η as an open subset of Q, we obtain a partially defined action of G on V by so-called *conformal maps*. The group G is also called the *conformal group of* Q.

Example 16.4.5. (a) For the euclidian space $V = \mathbb{R}^n = \mathbb{R}^{n,0}$ we have $\widetilde{V} = \mathbb{R}^{n+1,1}$ and the corresponding quadric $Q_{n+1,1} \cong \mathbb{S}^n$ is a sphere. In this case the map η can also be written as

$$\eta(v) = \Big[\Big(\frac{2v}{1+\|v\|^2}, \frac{1-\|v\|^2}{1+\|v\|^2}, 1\Big)\Big],$$

which is the inverse of the stereographic projection (cf. Example 7.2.5).

(b) We know from Proposition 8.5.21 that

$$\mathrm{SO}_4(\mathbb{R}) \cong (\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}))/\{\pm\mathbf{1}\} \cong (\mathbb{S}^3 \times \mathbb{S}^3)/\{\pm\mathbf{1}\},$$

which implies the projective completion $Q_{4,4}$ of $\mathbb{R}^{3,3}$ is diffeomorphic to $\mathrm{SO}_4(\mathbb{R})$.

(c) The multiplication map $\mu\colon \mathbb{T}\mathbf{1} \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{U}_2(\mathbb{C})$ induces a diffeomorphism

$$(\mathbb{S}^1 \times \mathbb{S}^3)/\{\pm\mathbf{1}\} \cong \mathrm{U}_2(\mathbb{C}),$$

so that the compact group $\mathrm{U}_2(\mathbb{C})$ is the conformal completion $Q_{2,4}$ of the Minkowski space $\mathbb{R}^{3,1}$.

16.4.2 Conformal Maps

We take a closer look at various types of conformal maps on V. First we observe that $\mathrm{O}(V, \beta)$ embeds canonically into G as the pointwise stabilizer of the plane $\{0\} \times \mathbb{K}^2$. From the definition of η it follows immediately that η is equivariant with respect to the action of $\mathrm{O}(V, \beta)$ on V and the corresponding action on Q, obtained by the embedding into G.

Dilations on V: For $r \in \mathbb{K}^\times$, we consider the element

$$g := \frac{1}{2r}\begin{pmatrix} 1+r^2 & 1-r^2 \\ 1-r^2 & 1+r^2 \end{pmatrix} \in \mathrm{SO}_{1,1}(\mathbb{R})$$

(Exercise 16.4.2). Identifying $\mathrm{O}_{1,1}(\mathbb{K})$ with a subgroup of G, acting on the last two components, we then find for $s := \frac{1}{2}(1 - \beta(v, v))$ the relation

$$\begin{aligned} g.\eta(v) &= g.[(v, s, 1-s)] \\ &= [(rv, \tfrac{1}{2}((1+r^2)s + (1-r^2)(1-s)), \tfrac{1}{2}((1-r^2)s + (1+r^2)(1-s))] \\ &= [(rv, \tfrac{1}{2}(1 - r^2 + 2r^2 s), \tfrac{1}{2}(1 + r^2 - 2r^2 s)] = \eta(rv). \end{aligned}$$

Note that, for $\mathbb{K} = \mathbb{R}$ and $t = -\log r$ we have

$$\frac{1+r^2}{2r} = \cosh t \quad \text{and} \quad \frac{1-r^2}{2r} = \sinh t.$$

Translations on V: We write linear operators on $\widetilde{V} = V \oplus \mathbb{K}^2$ as (2×2)-block matrices

$$X = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, \quad \text{where} \quad A \in \mathfrak{gl}(V), \quad b = (b_1, b_2) \in \operatorname{Hom}(\mathbb{K}^2, V) \cong V^2,$$

and

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \operatorname{Hom}(V, \mathbb{K}^2) \cong (V^*)^2 \quad \text{and} \quad d \in M_2(\mathbb{K}).$$

Evaluating the conditions that $X \in \mathfrak{o}(\widetilde{V}, \widetilde{\beta})$, leads to

$$A \in \mathfrak{o}(V,\beta), \quad d \in \mathfrak{o}_{1,1}(\mathbb{K}) = \mathbb{K}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_1 = -\beta(b_1, \cdot) \quad \text{and} \quad c_2 = \beta(b_2, \cdot).$$

To simplify notation, we put $b^\flat := \beta(b, \cdot)$.

For $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{o}_{1,1}(\mathbb{K}) \subseteq \mathfrak{o}(\widetilde{V}, \widetilde{\beta})$, we obtain the eigenspace decomposition of $\mathfrak{g} := \mathfrak{o}(\widetilde{V}, \widetilde{\beta})$:

$$\mathfrak{g}_0 = \mathfrak{o}(V,\beta) \oplus \mathfrak{o}_{1,1}(\mathbb{K}),$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & v & -v \\ -v^\flat & 0 & 0 \\ -v^\flat & 0 & 0 \end{pmatrix} : v \in V \right\} \quad \text{and} \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & v & v \\ -v^\flat & 0 & 0 \\ v^\flat & 0 & 0 \end{pmatrix} : v \in V \right\}.$$

For

$$X = \begin{pmatrix} 0 & v & v \\ -v^\flat & 0 & 0 \\ v^\flat & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \quad \text{we find} \quad X^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta(v,v) & -\beta(v,v) \\ 0 & \beta(v,v) & \beta(v,v) \end{pmatrix}$$

and $X^3 = 0$, so that

$$e^X = \mathbf{1} + X + \tfrac{1}{2}X^2 = \begin{pmatrix} \mathbf{1} & v & v \\ -v^\flat & \mathbf{1} - \frac{1}{2}\beta(v,v) & -\frac{1}{2}\beta(v,v) \\ v^\flat & \frac{1}{2}\beta(v,v) & \mathbf{1} + \frac{1}{2}\beta(v,v) \end{pmatrix}.$$

From

$$e^X \begin{pmatrix} w \\ \frac{1}{2}(1 - \beta(w,w)) \\ \frac{1}{2}(1 + \beta(w,w)) \end{pmatrix} = \begin{pmatrix} v + w \\ \frac{1}{2}(1 - \beta(v+w, v+w)) \\ \frac{1}{2}(1 + \beta(v+w, v+w)) \end{pmatrix},$$

it follows that $e^X.\eta(w) = \eta(v+w)$, so that the abelian subgroup $\exp(\mathfrak{g}_{-1}) \cong (V, +)$ acts by translations on $V \cong \eta(V)$.

Inversion in the sphere: For the element $\sigma := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{O}_{1,1}(\mathbb{K})$ we have $\sigma.[(v,s,t)] = [(v,-s,t)]$, so that $\sigma.\eta(v) \in \eta(V)$ is equivalent to $\beta(v,v) \neq 0$. A direct calculation shows that the corresponding induced map on

$$V^\times = \{v \in V \colon \beta(v,v) \neq 0\}$$

is given by $\sigma.\eta(v) = \eta(\beta(v,v)^{-1}v)$, which is the *inversion in the unit sphere* $\{v \in V \colon \beta(v,v) = 1\}$ of (V,β).

Exercises for Section 16.4

Exercise 16.4.1. Show that for any field $\mathbb{K}$ with $2 \in \mathbb{K}^\times$ the map

$$\alpha \colon \mathbb{K}^\times \to \mathrm{SL}_2(\mathbb{K}), \quad \alpha(r) := \frac{1}{2r}\begin{pmatrix} 1+r^2 & 1-r^2 \\ 1-r^2 & 1+r^2 \end{pmatrix}$$

is a group homomorphism.

Exercise 16.4.2. Show that for any field $\mathbb{K}$ with $2 \in \mathbb{K}^\times$, we have

$$\mathrm{SO}_{1,1}(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 - b^2 = 1 \right\} = \left\{ \frac{1}{2r}\begin{pmatrix} 1+r^2 & 1-r^2 \\ 1-r^2 & 1+r^2 \end{pmatrix} : r \in \mathbb{K}^\times \right\}$$

and

$$\mathrm{O}_{1,1}(\mathbb{K}) = \mathrm{SO}_{1,1}(\mathbb{K}) \dot{\cup} \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} : a^2 - b^2 = 1 \right\}.$$

Exercise 16.4.3. Let β be a nondegenerate symmetric bilinear form on the $\mathbb{K}$-vector space V, and $2 \in \mathbb{K}^\times$. Show that the group $\mathrm{O}(V,\beta)$ acts transitively on the set $\widetilde{Q} := \{v \in V \colon 0 \neq v, \beta(v,v) = 0\}$ of isotropic vectors, i.e., that for two elements $v, w \in \widetilde{Q}$, there exists a $g \in \mathrm{O}(V,\beta)$ with $gv = w$. Proceed according to the following steps:

(i) Prove the assertion for $V = \mathbb{K}^2$, endowed with the form $\beta((s,t),(s',t')) = ss' - tt'$.
(ii) If $w \in \mathbb{K}v$, then any vector $u \in V$ with $\beta(u,v) \neq 0$ leads to a nondegenerate subspace $W := \mathrm{span}\{v,u\}$, and the assertion follows from (i) and the embedding $\mathrm{O}(W,\beta_W) \hookrightarrow \mathrm{O}(V,\beta)$ for $\beta_W := \beta|_{W \times W}$.
(iii) If $\beta(v,w) \neq 0$ and v, w are linearly independent isotropic vectors, then $W := \mathrm{span}\{v,w\}$ is a nondegenerate subspace, and the assertion follows, as in (ii), from (i).
(iii) If $\beta(v,w) = 0$ and v, w are linearly independent, then use the nondegeneracy of β to find $u \in V$ with $\beta(u,v) = \beta(u,w) = 1$. Replacing u by $u + tv$ for a suitable $t \in \mathbb{K}$, we may assume, in addition, that u is isotropic. Applying (i) to the nondegenerate subspaces $\mathrm{span}\{v,u\}$ and $\mathrm{span}\{w,u\}$ now implies that v, w and u lie in the same orbit of $\mathrm{O}(V,\beta)$.

Exercise 16.4.4. Let (V, β) be a finite-dimensional nondegenerate quadratic space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show that

(i) For any affine hyperplane

$$H = \{v \in V : \alpha(v) = 1\}, \quad 0 \neq \alpha \in V^*,$$

the map $\varphi: H \to \mathbb{P}(V), v \mapsto [v]$, is a diffeomorphism onto an open subset of $\mathbb{P}(V)$.

(ii) The 0-level set of $q(v) := \beta(v, v)$ on H is a submanifold. Hint: Since $\mathtt{d}q(v) = 2\beta(v, \cdot)$, the singularity of q in $v \in H$ implies that $2\beta(v, \cdot) \in \mathbb{K}^\times \alpha$ because β is nondegenerate. Now $\alpha(v) = 1$ leads to $\beta(v, v) \neq 0$, so that all points in the 0-level set of q are regular.

(iii) The projective quadric

$$Q = \{[v] \in \mathbb{P}(V) : \beta(v, v) = 0\}$$

is a submanifold of $\mathbb{P}(V)$.

Exercise 16.4.5. (a) Let (V, β) be a real quadratic space and $V_{\mathbb{C}}$ be the complexification of V. Show that there exists a unique hermitian form $\beta_{\mathbb{C}}$ on V with

$$\beta_{\mathbb{C}}|_{V \times V} = \beta.$$

(b) Let γ be a hermitian form on the complex vector space V and $\sigma: V \to V$ be an antilinear involution leaving γ invariant, i.e.,

$$\gamma(\sigma v, \sigma w) = \gamma(w, v) \quad \text{for} \quad v, w \in V.$$

Show that the subspace $V^\sigma := \{v \in V : \sigma v = v\}$ is a real form of V, the restriction β of γ to V^σ is real-valued, and $\gamma = \beta_{\mathbb{C}}$ in the sense of (a).

Notes on Chapter 16

The discussion of concrete matrix groups in this chapter supplements the rather elementary discussion from Chapter 1. Here we focus on the quite easily accessible class of unitary groups over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, and some non-compact groups containing unitary groups as maximal compact subgroups.

A systematic discussion of the spin groups corresponding to indefinite forms and the exceptional isomorphisms between low-dimensional groups seems to be hard to find in the literature. We hope that Sections 16.3 and 16.4 close this gap to some extent. Far more details on classical Lie groups can be found in [GW09] or [Gr01].

17

Nonconnected Lie Groups

The examples from Chapter 16 show that many geometrically defined Lie groups have several connected components. While only the connected component of the identity is accessible to the methods built on the exponential function, there are still tools to analyze nonconnected Lie groups. In the present chapter, we present some of these tools. The key notion is that of an extension of a discrete group by a (connected) Lie group.

Any Lie group G is an extension of the discrete group $\pi_0(G)$ of connected components by the connected Lie group G_0. In addition, there is a characteristic homomorphism $s\colon \pi_0(G) \to \mathrm{Out}(G_0) = \mathrm{Aut}(G_0)/\,\mathrm{Inn}(G_0)$. One of the main results of Section 17.1 states that, up to equivalence of extensions, the Lie groups corresponding to the same s are parameterized by the cohomology group $H^2_s(\pi_0(G), Z(G_0))$. Some basic facts and definitions on group cohomology are provided in Appendix 17.3. In Section 17.2 we discuss coverings of nonconnected Lie groups and in particular the obstruction for the existence of a simply connected covering group.

17.1 Extensions of Discrete Groups by Lie Groups

In this section we discuss the basic concepts related to extensions of discrete groups by Lie groups. Since any Lie group G is an extension of the group $\pi_0(G) = G/G_0$ of connected components by its identity component, the techniques described in this section can be used to classify all Lie groups G for which $\pi_0(G)$ and G_0 are given.

Definition 17.1.1. Recall that an extension of Lie groups is a short exact sequence

$$\mathbf{1} \to N \xrightarrow{\iota} \widehat{G} \xrightarrow{q} G \to \mathbf{1}$$

of Lie group morphisms for which $\iota\colon N \to \ker q$ is an isomorphism of Lie groups. In the following we shall often identify N with the subgroup $\iota(N) \trianglelefteq \widehat{G}$. We write

$$C_N : \widehat{G} \to \mathrm{Aut}(N), \quad C_N(\widehat{g})(n) := \iota^{-1}(\widehat{g}\iota(n)\widehat{g}^{-1})$$

for the canonical homomorphism defined by the conjugation action of $\widehat{G}$ on N.

We call two extensions $N \hookrightarrow \widehat{G}_1 \twoheadrightarrow G$ and $N \hookrightarrow \widehat{G}_2 \twoheadrightarrow G$ of the Lie group G by the Lie group N *equivalent* if there exists a Lie group morphism $\varphi : \widehat{G}_1 \to \widehat{G}_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} N & \xrightarrow{\iota_1} & \widehat{G}_1 & \xrightarrow{q_1} & G \\ \downarrow \mathrm{id}_N & & \downarrow \varphi & & \downarrow \mathrm{id}_G \\ N & \xrightarrow{\iota_2} & \widehat{G}_2 & \xrightarrow{q_2} & G \end{array}$$

It is easy to see that any such φ is in particular an isomorphism of Lie groups. We write $\mathrm{Ext}(G, N)$ for the set of equivalence classes of Lie group extensions of G by N.

We call an extension $q : \widehat{G} \to G$ with $\ker q = N$ *split* or *trivial* if there exists a Lie group morphism $\sigma : G \to \widehat{G}$ with $q \circ \sigma = \mathrm{id}_G$.

Remark 17.1.2. (a) If G is a Lie group, then the short exact sequence

$$\mathbf{1} \to G_0 \longrightarrow G \longrightarrow \pi_0(G) \to \mathbf{1}$$

shows that G is an extension of the discrete group $\pi_0(G)$ by the connected Lie group G_0. Thus the study of Lie groups with given identity component G_0 reduces to the study of extensions of discrete groups by G_0 up to equivalence. The fact that we are interested primarily in extensions of discrete groups simplifies the discussion considerably because we do not have to worry about certain continuity and smoothness issues.

(b) Suppose that $\sigma : G \to \widehat{G}$ is a splitting homomorphism for the extension

$$\mathbf{1} \to N \longrightarrow \widehat{G} \xrightarrow{q} G \to \mathbf{1},$$

i.e., $q \circ \sigma = \mathrm{id}_G$. Then the homomorphism

$$S := C_N \circ \sigma : G \to \mathrm{Aut}(N)$$

defines a smooth action of the discrete Lie group G on N (cf. Proposition 10.1.18), so that $N \rtimes_S G$ carries a natural Lie group structure. It is easy to see that the multiplication map $N \rtimes_S G \to \widehat{G}, (n, g) \mapsto n\sigma(g)$ is an isomorphism.

Definition 17.1.3. Let $\mathbf{1} \to N \xrightarrow{\iota} \widehat{G} \xrightarrow{q} G \to \mathbf{1}$ be a Lie group extension and $\mathrm{Inn}(N) = \{c_n : n \in N\}$ be the group of inner automorphisms of N. Then

$$\mathrm{Out}(N) := \mathrm{Aut}(N)/\mathrm{Inn}(N) = \{[\varphi] := \varphi \cdot \mathrm{Inn}(N) : \varphi \in \mathrm{Aut}(N)\}$$

is called the group of *outer automorphisms of* N. Note that, in general, $\mathrm{Inn}(N)$ will not be closed in $\mathrm{Aut}(N)$ (Exercise 17.1.2), so we do not consider a topology on $\mathrm{Out}(N)$.

For $g \in G$ we pick an element $\widehat{g} \in q^{-1}(g)$, i.e., we consider a map $\sigma\colon G \to \widehat{G}$ satisfying $q \circ \sigma = \mathrm{id}_G$, and set $\widehat{g} = \sigma(g)$. We call σ a *section* of q. Then $[C_N \circ \sigma] \in \mathrm{Out}(N)$ does not depend on the choice of σ, so we can define

$$s(g) := [C_N(\widehat{g})] \in \mathrm{Out}(N).$$

One easily verifies that the resulting map

$$s\colon G \to \mathrm{Out}(N)$$

is a group homomorphism. We call it the *characteristic homomorphism of the extension.*

Remark 17.1.4. (a) If N is abelian, then $\mathrm{Aut}(N) = \mathrm{Out}(N)$, so that we simply have $s(g) = c_{\widehat{g}}|_N$, which does not depend on $\widehat{g}$. In this case $s\colon G \to \mathrm{Aut}(N)$ is a well-defined action of G on the abelian group N.

(b) If $\widehat{G} = N \rtimes_S G$ is a semidirect product defined by the homomorphism $S\colon G \to \mathrm{Aut}(N)$, then $s(g) = [S(g)]$ is the corresponding characteristic homomorphism.

Examples 17.1.5. (a) If G is a Lie group, then the short exact sequence

$$\mathbf{1} \to G_0 \longrightarrow G \longrightarrow \pi_0(G) \to \mathbf{1}$$

leads to a *characteristic homomorphism of G:*

$$s_G\colon \pi_0(G) \to \mathrm{Out}(G_0).$$

To obtain some extra information on the group $\mathrm{Out}(G_0)$, let $q_{G_0}\colon \widetilde{G}_0 \to G_0$ denote the universal covering map and recall that we have an embedding $\mathrm{Aut}(G_0) \hookrightarrow \mathrm{Aut}(\widetilde{G}_0)$ whose image consists of those automorphisms preserving $\ker(q_{G_0}) \cong \pi_1(G)$ (cf. Remark 8.5.5, Exercise 8.5.2). On the other hand, $\mathrm{Aut}(\widetilde{G}_0) \cong \mathrm{Aut}(\mathfrak{g})$ (Corollary 8.5.11), and the corresponding embedding is

$$\mathrm{Aut}(G_0) \to \mathrm{Aut}(\mathfrak{g}), \quad \varphi \mapsto \mathbf{L}(\varphi).$$

Clearly,

$$\mathrm{Inn}(\mathfrak{g}) = \langle \{e^{\mathrm{ad}\, x} \colon x \in \mathfrak{g}\}\rangle = \langle \{\mathrm{Ad}(\exp_G x) \colon x \in \mathfrak{g}\}\rangle = \mathrm{Ad}(G_0) \cong \mathrm{Inn}(G_0),$$

so that $\mathrm{Out}(G_0)$ can be identified with a subgroup of

$$\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\,\mathrm{Inn}(\mathfrak{g}) \cong \mathrm{Out}(\widetilde{G}_0).$$

(b) If G is a connected compact Lie group and $T \subseteq G$ a maximal torus, then $Z_G(T) = T$ (Corollary 11.2.11), and its normalizer is an extension

$$\mathbf{1} \to T \longrightarrow N_G(T) \longrightarrow W(G,T) \to \mathbf{1}$$

of the finite Weyl group $W(G,T)$ by the torus group T (cf. Definition 11.2.12).

Lemma 17.1.6. *Equivalent extensions of G by N define the same characteristic homomorphism. In particular, we have a map*

$$\mathrm{Ext}(G,N) \to \mathrm{Hom}\big(G, \mathrm{Out}(N)\big).$$

Proof. Let

$$\begin{array}{ccccc} N & \xrightarrow{\quad \iota_1 \quad} & \widehat{G}_1 & \xrightarrow{\quad q_1 \quad} & G \\ \big\downarrow \mathrm{id}_N & & \big\downarrow \varphi & & \big\downarrow \mathrm{id}_G \\ N & \xrightarrow{\quad \iota_2 \quad} & \widehat{G}_2 & \xrightarrow{\quad q_2 \quad} & G \end{array} \tag{17.1}$$

be an equivalence of two extensions of G by N and $s_j\colon G \to \mathrm{Out}(N)$, $j = 1, 2$, be the corresponding characteristic homomorphisms. For $\widehat{g} \in \widehat{G}_1$ we then have $q_1(\widehat{g}) = q_2(\varphi(\widehat{g}))$. Further, $\varphi \circ \iota_1 = \iota_2$ implies that $C_N(\varphi(\widehat{g})) = C_N(\widehat{g})$, and hence that $s_2(g) = [C_N(\varphi(\widehat{g}))] = [C_N(\widehat{g})] = s_1(g)$. □

Remark 17.1.7. According to Lemma 17.1.6, it makes sense to consider, for a given homomorphism $s\colon G \to \mathrm{Out}(N)$, the set $\mathrm{Ext}_s(G,N)$ of equivalence classes of all extensions whose characteristic homomorphism is s. The main questions to be asked in this chapter are then:

(Q1) When is $\mathrm{Ext}_s(G,N)$ nonempty?
(Q2) How can we parameterize the set $\mathrm{Ext}_s(G,N)$ if it is nonempty.

To approach an answer to these questions for the case where G is discrete, we first recall from Remark 10.1.22 how to introduce product coordinates on group extensions:

Remark 17.1.8. Let $\mathbf{1} \to N \xrightarrow{\iota} \widehat{G} \xrightarrow{q} G \to \mathbf{1}$ be a Lie group extension of the discrete group G by the Lie group N. Let $\sigma\colon G \to \widehat{G}$ be a section of q which is *normalized* in the sense that $\sigma(\mathbf{1}) = \mathbf{1}$. Then, according to Remark 10.1.22, the map

$$\Phi\colon N \times G \to \widehat{G}, \quad (n,g) \mapsto n\sigma(g)$$

is an isomorphism of Lie groups, where $N \times G$ is equipped with the multiplication

$$(n,g)(n',g') = (nS(g)(n')\omega(g,g'), gg'), \tag{17.2}$$

where $S := C_N \circ \sigma$, and

$$\omega\colon G \times G \to N, \quad (g,g') \mapsto \sigma(g)\sigma(g')\sigma(gg')^{-1}. \tag{17.3}$$

The maps S and ω satisfy the relations

$$\sigma(g)\sigma(g') = \omega(g,g')\sigma(gg'), \tag{17.4}$$

$$S(g)S(g') = C_N(\omega(g,g'))S(gg'), \tag{17.5}$$

and

$$\omega(g,g')\omega(gg',g'') = S(g)\big(\omega(g',g'')\big)\omega(g,g'g'') \tag{17.6}$$

(cf. Remark 10.1.22(a),(d)). Since G is discrete, Φ and its inverse $\widehat{g} \mapsto \big(\widehat{g}\sigma(g)^{-1}, q(\widehat{g})\big)$ are smooth so that Φ is a diffeomorphism.

Lemma 17.1.9. *Let G be a group and N a Lie group. Further, let (S, ω) be a pair of maps*

$$S\colon G \to \mathrm{Aut}(N), \quad \omega\colon G \times G \to N$$

with

$$S(\mathbf{1}) = \mathrm{id}_N \quad \text{and} \quad \omega(g, \mathbf{1}) = \omega(\mathbf{1}, g) = \mathbf{1}, \quad g \in G. \tag{17.7}$$

Then (17.2) *defines a Lie group structure on $N \times G$ if and only if* (17.5) *and* (17.6) *are satisfied.*

Proof. The associativity of the multiplication on $N \times G$ is equivalent to the equality of

$$\begin{aligned}\big((n,g)(n',g')\big)(n'',g'') &= (nS(g)(n')\omega(g,g'), gg')(n'',g'') \\ &= (nS(g)(n')\omega(g,g')S(gg')(n'')\omega(gg',g''), gg'g'')\end{aligned}$$

and

$$\begin{aligned}&(n,g)\big((n',g')(n'',g'')\big) = (n,g)(n'S(g')(n'')\omega(g',g''), g'g'') \\ &= (nS(g)\big(n'S(g')(n'')\omega(g',g'')\big)\omega(g,g'g''), gg'g'') \\ &= (nS(g)(n')(S(g)S(g')(n''))S(g)(\omega(g',g''))\omega(g,g'g''), gg'g'')\end{aligned}$$

for all $g, g', g'' \in G$ and $n, n', n'' \in N$. This means that

$$\omega(g,g')S(gg')(n'')\omega(gg',g'') = (S(g)S(g')(n''))S(g)(\omega(g',g''))\omega(g,g'g'').$$

For $n'' = 1$, this leads to (17.6). If, conversely, (17.6) is satisfied, then the associativity condition is equivalent to

$$\omega(g,g')S(gg')(n'')\omega(gg',g'') = (S(g)S(g')(n''))\omega(g,g')\omega(gg',g'')$$

and hence to (17.5). Therefore these two conditions are equivalent to the associativity of the multiplication on $N \times G$.

To see that we actually obtain a group, we first observe that $S(\mathbf{1}) = \mathrm{id}_N$ and $\omega(g, \mathbf{1}) = \omega(\mathbf{1}, g) = \mathbf{1}$ imply that $\mathbf{1} := (\mathbf{1}, \mathbf{1})$ is an identity element of $\widehat{G} := N \times G$, so that $(\widehat{G}, \mathbf{1})$ is a monoid. For $(n, g) \in \widehat{G}$, the element

$$(S(g)^{-1}\big(n^{-1}\omega(g,g^{-1})^{-1}\big), g^{-1})$$

is a right inverse and likewise $(\omega(g^{-1},g)^{-1}S(g^{-1})(n^{-1}), g^{-1})$ is a left inverse. Now the associativity of $\widehat{G}$ implies that left and right inverse are equal, hence an inverse of (n, g). Therefore $\widehat{G}$ is a group. □

Definition 17.1.10. The pairs (S, ω) satisfying (17.5), (17.6) and (17.7) are called *factor systems for* (G, N). For a factor system (S, ω), we write $N \times_{(S,\omega)} G$ for the set $N \times G$, endowed with the (twisted) group multiplication (17.2).

Remark 17.1.11. Combining the proof of Lemma 17.1.9 with the calculations in Remark 10.1.22(d), we see that, for any factor system (S,ω) for (G,N), the short exact sequence

$$\mathbf{1} \to N \xrightarrow{\iota} N \times_{(S,\omega)} G \xrightarrow{q} G \to \mathbf{1}$$

with $\iota(n) = (n,\mathbf{1})$ and $q(n,g) = g$, is an extension of Lie groups, when G is equipped with the discrete topology. Moreover,

$$\sigma \colon G \to N \times_{(S,\omega)} G, \quad g \mapsto (\mathbf{1}, g),$$

is a normalized section of q, so that the corresponding characteristic homomorphism $s : G \to \mathrm{Out}(N)$ is given $s(g) = [C_N \circ \sigma(g)] = [S(g)]$.

Definition 17.1.12. Since the group $\mathrm{Out}(N)$ acts naturally on $Z(N)$ by $[\varphi]z := \varphi(z)$, each homomorphism $s \colon G \to \mathrm{Out}(N)$ defines on $Z(N)$ the structure of a G-module. We write $\rho_s \colon G \to \mathrm{Aut}(Z(N))$ for the corresponding homomorphism.

The following theorem answers question (Q2) from Remark 17.1.7 in terms of the second cohomology of the G-module $Z(N)$. It is also an important step to a cohomological answer to question (Q1). We refer to the Appendix 17.3 for the relevant definitions from group cohomology.

Theorem 17.1.13. *Let G be discrete and $s \colon G \to \mathrm{Out}(N)$ a homomorphism and $S \colon G \to \mathrm{Aut}(N)$ be any normalized lift, i.e., $S(\mathbf{1}) = \mathrm{id}_N$ and $s(g) = [S(g)]$. We also fix some extension class $[\widehat{G}] \in \mathrm{Ext}_s(G,N)$. Then the following assertions hold:*

(a) *There exists an $\omega \colon G \times G \to N$ such that $\widehat{G} \cong N \times_{(S,\omega)} G$. In particular, (S,ω) is a factor system.*
(b) *For a map $\omega' \colon G \times G \to N$, the pair (S,ω') is a factor system if and only if there exists a 2-cocycle $\eta \in Z^2(G,Z(N))$ with $\omega' = \omega\eta$.*
(c) *The map*

$$\Gamma \colon H^2_{\rho_s}(G,Z(N)) \to \mathrm{Ext}_s(G,N), \quad [\eta] \mapsto [N \times_{(S,\omega\eta)} G]$$

is a bijection.

Proof. (a) Let $q \colon \widehat{G} \to G$ be the quotient map defining the extension. Then we choose a section $\sigma \colon G \to \widehat{G}$ in such a way that $S(g) = c_{\sigma(g)}|_N$ and define ω by (17.3). Now (a) follows from Remark 17.1.8.

(b) If (S,ω') is a factor system, then (17.5) implies that $C_N(\omega(g,g')) = C_N(\omega'(g,g'))$, so that

$$\eta(g,g') := \omega(g,g')^{-1}\omega'(g,g') \in Z(N) \quad \text{for} \quad g,g' \in G.$$

Comparing (17.6) for (S,ω) and (S,ω'), it follows that $\mathtt{d}_G\eta = 0$, i.e., that η is a 2-cocycle. If, conversely, $\eta \in Z^2(G,Z(N))$, then $\omega' := \omega\eta$ satisfies (17.5) because $\eta(g,g') \in Z(N)$, and finally (17.6) follows from $\mathtt{d}_G\eta = 0$.

(c) Let $\varphi\colon N \times_{(S,\omega')} G \to N \times_{(S,\omega)} G$ be an equivalence of extensions. Then there exists a function $h\colon G \to N$ with $h(\mathbf{1}) = \mathbf{1}$ such that φ is of the form $\varphi(n,g) = (nh(g),g)$, and since $(\mathbf{1},g)$ induces in both groups the same automorphism $S(g)$ on N, we have $h(g) \in Z(N)$. We now observe that

$$\begin{aligned}\varphi(n,g)\varphi(n',g') &= (nh(g),g)(n'h(g'),g') = (nh(g)S(g)(n'h(g'))\omega(g,g'),gg') \\ &= \big(nS(g)(n')\omega(g,g')h(gg')(\mathtt{d}_G h)(g,g'),gg'\big),\end{aligned}$$

and

$$\varphi((n,g)(n',g')) = (nS(g)(n')\omega'(g,g')h(gg'),gg').$$

This implies that $\omega' = \omega \cdot \mathtt{d}_G h$. If, conversely, $\omega^{-1}\omega' \in Z^2(G,Z(N))$ is a coboundary $\mathtt{d}_G h$, then φ, as defined above, is an equivalence of extensions. This proves that Γ is well-defined and a bijection from $H^2_{\rho_s}(G,Z(N))$ onto $\mathrm{Ext}_s(G,N)$. □

Remark 17.1.14 (Abelian extensions). Suppose that $N = A$ is an abelian Lie group. Then the adjoint representation of A is trivial and a factor system (S,ω) for (G,A) consists of a homomorphism $S\colon G \to \mathrm{Aut}(A)$ and an element $\omega \in Z^2(G,A)$ because (17.6) is the cocycle condition. In this case we write $A \times_\omega G$ for this Lie group, which is $A \times G$, endowed with the multiplication

$$(a,g)(a',g') = (a + ga' + \omega(g,g'),gg').$$

Here, we suppress S from the notation by writing ga' for $S(g)(a')$. According to Remark 17.1.11, the characteristic homomorphism $s\colon G \to \mathrm{Out}(A) = \mathrm{Aut}(A)$ in this case reduces to S. Theorem 17.1.13 now yields a bijection

$$H^2(G,A) \to \mathrm{Ext}_s(G,A), \quad [\omega] \mapsto [A \times_\omega G].$$

Remark 17.1.15. The parameterization of $\mathrm{Ext}_s(G,N)$ by $H^2_{\rho_s}(G,Z(N))$ given in Theorem 17.1.13(c) under the condition that $\mathrm{Ext}_s(G,N)$ is nonempty, depends on the choice of a base point $[N \times_{(S,\omega)} G]$. A more conceptual way to formulate this parameterization is to say that the map

$$H^2_{\rho_s}(G,Z(N))\times\mathrm{Ext}_s(G,N) \to \mathrm{Ext}_s(G,N),\ ([\eta],[N\times_{(S,\omega)}G]) \mapsto [N\times_{(S,\omega\eta)}G]$$

is a simply transitive group action (Exercise).

Remark 17.1.16. (a) If s lifts to a homomorphism $S\colon G \to \mathrm{Aut}(N)$, then $(S,\mathbf{1})$ is a factor system and $N \rtimes_S G$ is a split extension corresponding to s.

(b) The condition that s lifts to a homomorphism $S\colon G \to \mathrm{Aut}(N)$ is equivalent to the triviality of the group extension

$$\mathbf{1} \to \mathrm{Inn}(N) \to s^*\,\mathrm{Aut}(N) \to G \to \mathbf{1},$$

where

$$s^*\,\mathrm{Aut}(N) := \{(g,\varphi) \in G \times \mathrm{Aut}(N)\colon s(g) = [\varphi]\}.$$

Therefore the nontriviality of this extension is the obstruction to the existence of a homomorphic lift.

Now we turn to the answer of question (Q1). Starting with a homomorphism $s\colon G \to \mathrm{Out}(N)$, we can always lift it to a *map* $S\colon G \to \mathrm{Aut}(N)$ with $S(\mathbf{1}) = \mathrm{id}_N$, and Theorem 17.1.13 implies that that $\mathrm{Ext}_s(G,N) \neq \emptyset$ if and only if there exists a map $\omega\colon G \times G \to N$ for which (S,ω) is a factor system.

Proposition 17.1.17. *Let G be a discrete group and N be a Lie group. Further, let $s\colon G \to \mathrm{Out}(N)$ be a homomorphism and $S\colon G \to \mathrm{Aut}(N)$ be a lift of s with $S(\mathbf{1}) = \mathrm{id}_N$. Then there exists a map $\omega\colon G \times G \to N$ satisfying* (17.5) *and* (17.7)*, so that* (17.6) *is equivalent to the vanishing of*

$$(\mathtt{d}_S\omega)(g,g',g'') := S(g)(\omega(g',g''))\omega(g,g'g'')\omega(gg',g'')^{-1}\omega(g,g')^{-1} \in N.$$

Proof. For $g,g' \in G$, the automorphism $S(g)S(g')S(gg')^{-1}$ of N is inner, hence of the form $C_N(\omega(g,g'))$ for some $\omega(g,g') \in N$. If either g or g' is $\mathbf{1}$, then it is id_N, and we may put $\omega(g,g') := \mathbf{1}$. □

To understand the nature of the condition from Proposition 17.1.17, we first observe that:

Lemma 17.1.18. *The function $\mathtt{d}_S\omega\colon G \times G \times G \to N$ has values in $Z(N)$ and it is a 3-cocycle.*

Proof. First we show that $(\mathtt{d}_S\omega)(g,g',g'') \in Z(N)$. Let

$$G^\sharp := s^* \mathrm{Aut}(N) = \{(g,\varphi) \in G \times \mathrm{Aut}(N)\colon s(g) = [\varphi]\},$$

and note that this is a subgroup of $G \times \mathrm{Aut}(N)$. The projection

$$p_G\colon G^\sharp \to G, \quad (g,\varphi) \mapsto g$$

is a surjective group homomorphism with kernel $\mathrm{Inn}(N)$, so that we can think of it as an extension of G by $\mathrm{Inn}(N)$. The map

$$\sigma^\sharp\colon G \to G^\sharp, \quad \sigma(g) := (g, S(g))$$

clearly is a set-theoretic section satisfying

$$\omega^\sharp(g,g') := \sigma^\sharp(g)\sigma^\sharp(g')\sigma^\sharp(gg')^{-1} = S(g)S(g')S(gg')^{-1} = C_N(\omega(g,g')),$$

and for $\varphi \in \mathrm{Aut}(N)$ we have $c_{\sigma^\sharp(g)}\varphi = c_{S(g)}\varphi$. Putting $S^\sharp(g) := c_{S(g)}$, it follows that the pair $(S^\sharp,\omega^\sharp)$ is a factor system for $(G,\mathrm{Inn}(N))$, which implies that $\mathtt{d}_{S^\sharp}\omega^\sharp = 0$. We thus obtain

$$\begin{aligned}&C_N\big((\mathtt{d}_S\omega)(g,g',g'')\big)\\ &= S(g)C_N(\omega(g',g''))S(g)^{-1}C_N(\omega(g,g'g''))C_N(\omega(gg',g''))^{-1}C_N(\omega(g,g'))^{-1}\\ &= (\mathtt{d}_{S^\sharp}\omega^\sharp)(g,g',g'') = \mathbf{1},\end{aligned}$$

which means that $\mathtt{d}_S\omega(g,g',g'') \in Z(N)$.

In the following calculation we write the group structure on $Z(N)$ multiplicatively. We shall use several times that the values of $\mathtt{d}_S\omega$ are central in N, so that they commute with all values of ω. We have to show that for $g, g', g'', g''' \in G$, the following expression vanishes:

$$\begin{aligned}
&\mathtt{d}_G(\mathtt{d}_S\omega)(g, g', g'', g''') \\
&= S(g)\big((\mathtt{d}_S\omega)(g', g'', g''')\big)(\mathtt{d}_S\omega)(g, g'g'', g''')(\mathtt{d}_S\omega)(g, g', g'') \\
&\quad (\mathtt{d}_S\omega)(g, g', g''g''')^{-1}(\mathtt{d}_S\omega)(gg', g'', g''')^{-1}.
\end{aligned}$$

This will be achieved by calculating the expression

$$L := S(g)\big(S(g')(\omega(g'', g'''))\omega(g', g''g''')\big)\omega(g, g'g''g''')$$

in two ways. First we use

$$S(g)(\omega(g', g''))\omega(g, g'g'') = (\mathtt{d}_S\omega)(g, g', g'')\omega(g, g')\omega(gg', g'') \tag{17.8}$$

to evaluate the inner expression:

$$\begin{aligned}
L &= S(g)\big((\mathtt{d}_S\omega)(g', g'', g''')\omega(g', g'')\omega(g'g'', g''')\big)\omega(g, g'g''g''') \\
&= S(g)\big(\mathtt{d}_S\omega)(g', g'', g''')\big)S(g)(\omega(g', g''))S(g)(\omega(g'g'', g'''))\omega(g, g'g''g''') \\
&= S(g)\big(\mathtt{d}_S\omega)(g', g'', g''')\big)(\mathtt{d}_S\omega)(g, g', g'')\omega(g, g')\omega(gg', g'')\omega(g, g'g'')^{-1} \\
&\quad\ (\mathtt{d}_S\omega)(g, g'g'', g''')\omega(g, g'g'')\omega(gg'g'', g''') \\
&= S(g)\big(\mathtt{d}_S\omega)(g', g'', g''')\big)(\mathtt{d}_S\omega)(g, g', g'')(\mathtt{d}_S\omega)(g, g'g'', g''') \\
&\quad\ \omega(g, g')\omega(gg', g'')\omega(gg'g'', g''').
\end{aligned}$$

Then we evaluate L by using $S(g)S(g') = C_N(\omega(g, g'))S(gg')$ to obtain

$$\begin{aligned}
L &= \omega(g, g')S(gg')(\omega(g'', g'''))\omega(g, g')^{-1}S(g)(\omega(g', g''g'''))\omega(g, g'g''g''') \\
&= \omega(g, g')(\mathtt{d}_S\omega)(gg', g'', g''')\omega(gg', g'')\omega(gg'g'', g''')\omega(gg', g''g''')^{-1} \\
&\quad\ \omega(g, g')^{-1}(\mathtt{d}_S\omega)(g, g', g''g''')\omega(g, g')\omega(gg', g''g''') \\
&= (\mathtt{d}_S\omega)(gg', g'', g''')(\mathtt{d}_S\omega)(g, g', g''g''')\omega(g, g')\omega(gg', g'')\omega(gg'g'', g''').
\end{aligned}$$

Comparing the two expressions for L, we see that $\mathtt{d}_G(\mathtt{d}_S\omega)$ vanishes. □

Lemma 17.1.19. *The cohomology class* $\chi(s) := [\mathtt{d}_S\omega] \in H^3(G, Z(N))$ *does not depend on the choices of* ω *and* S.

Proof. First we show that the class $[\mathtt{d}_S\omega]$ does not depend on the choice of ω if S is fixed. So let $\widetilde{\omega}\colon G \times G \to N$ be another map with

$$\widetilde{\omega}(g, \mathbf{1}) = \widetilde{\omega}(\mathbf{1}, g') = \mathbf{1} \quad \text{and} \quad S(g)S(g') = C_N(\widetilde{\omega}(g, g'))S(gg'), \tag{17.9}$$

Then $\beta := \widetilde{\omega} \cdot \omega^{-1}$ has values in $Z(N)$. As all values of β commute with all values of ω, this leads to

$$\mathtt{d}_S\widetilde{\omega} = \mathtt{d}_S\beta + \mathtt{d}_S\omega = \mathtt{d}_G\beta + \mathtt{d}_S\omega \in \mathtt{d}_S\omega + B^3(G, Z(N)).$$

Therefore the choice of ω has no effect on the cohomology class $[\mathtt{d}_S\omega]$.

Now we consider another section $\widetilde{S}\colon G \to \mathrm{Aut}(N)$ of s with $\widetilde{S}(\mathbf{1}) = \mathrm{id}_N$. Then there exists a function $\alpha\colon G \to N$ with $\alpha(\mathbf{1}) = \mathbf{1}$ and $\widetilde{S} = (C_N \circ \alpha) \cdot S$. With the notation

$$\delta_S(g, g') := S(g)S(g')S(gg')^{-1} = C_N(\omega(g, g')),$$

we now obtain

$$\begin{aligned} \delta_{\widetilde{S}}(g, g') &= C_N(\alpha(g))S(g)C_N(\alpha(g'))S(g')S(gg')^{-1}C_N(\alpha(gg'))^{-1} \\ &= C_N(\alpha(g))c_{S(g)}(C_N(\alpha(g')))\delta_S(g, g')C_N(\alpha(gg'))^{-1} \\ &= C_N\big(\alpha(g)S(g)(\alpha(g'))\omega(g, g')\alpha(gg')^{-1}\big). \end{aligned}$$

Therefore the prescription

$$\widetilde{\omega}(g, g') = \alpha(g)S(g)(\alpha(g'))\omega(g, g')\alpha(gg')^{-1}$$

defines a function $\widetilde{\omega}\colon G^2 \to N$ with

$$\widetilde{\omega}(\mathbf{1}, g') = \widetilde{\omega}(g, \mathbf{1}) = \mathbf{1} \quad \text{and} \quad C_N \circ \widetilde{\omega} = \delta_{\widetilde{S}}.$$

We now calculate

$$\begin{aligned} &(\mathtt{d}_{\widetilde{S}}\widetilde{\omega})(g, g', g'') \\ &= \widetilde{S}(g)(\widetilde{\omega}(g', g''))\widetilde{\omega}(g, g'g'')\widetilde{\omega}(gg', g'')^{-1}\widetilde{\omega}(g, g')^{-1} \\ &= \underbrace{\alpha(g)}_{1} S(g)\Big(\alpha(g')S(g')(\alpha(g''))\omega(g', g'')\underbrace{\alpha(g'g'')^{-1}}_{5}\Big)\underbrace{\alpha(g)^{-1}}_{2} \\ &\quad \underbrace{\alpha(g)}_{2}\underbrace{S(g)(\alpha(g'g''))}_{5}\omega(g, g'g'')\underbrace{\alpha(gg'g'')^{-1}}_{3} \\ &\quad \underbrace{\alpha(gg'g'')}_{3}\omega(gg', g'')^{-1}S(gg')(\alpha(g''))^{-1}\underbrace{\alpha(gg')^{-1}}_{4} \\ &\quad \underbrace{\alpha(gg')}_{4}\omega(g, g')^{-1}S(g)(\alpha(g'))^{-1}\underbrace{\alpha(g)^{-1}}_{1} \\ &= \underbrace{S(g)\alpha(g')}_{1}[S(g)S(g')(\alpha(g''))]S(g)(\omega(g', g''))\omega(g, g'g'')\omega(gg', g'')^{-1} \\ &\quad S(gg')(\alpha(g''))^{-1}\omega(g, g')^{-1}\underbrace{S(g)(\alpha(g'))^{-1}}_{1}. \end{aligned}$$

With the relation $S(g)S(g') = C_N(\omega(g, g'))S(gg')$, we finally arrive at

$$\begin{aligned} (\mathtt{d}_{\widetilde{S}}\widetilde{\omega})(g, g', g'') &= \omega(g, g')(S(gg')\alpha(g''))\omega(g, g')^{-1}(\mathtt{d}_S\omega)(g, g', g'')\omega(g, g') \\ &\qquad S(gg')(\alpha(g''))^{-1}\omega(g, g')^{-1} \\ &= (\mathtt{d}_S\omega)(g, g', g''). \end{aligned}$$

□

Now we can answer question (Q2) from Remark 17.1.7:

Theorem 17.1.20. *Let G be a discrete group, N a Lie group, and $s\colon G \to \mathrm{Out}(N)$ be a homomorphism. Let $\chi(s) \in H^3_{\rho_s}(G, Z(N))$ be the cohomology class associated with s via* Lemma 17.1.19*. Then the following are equivalent.*

(i) *There exists a factor system (S', ω') such that $[N \times_{(S',\omega')} G] \in \mathrm{Ext}_s(G, N)$.*
(ii) *$\chi(s) \in H^3_{\rho_s}(G, Z(N))$ vanishes.*

Proof. In view of Proposition 17.1.17, (ii) is an immediate consequence of (i). For the converse, let (S, ω) be the maps associated with s in Proposition 17.1.17 and its proof. Then $\chi(s)$ is the cohomology class of $\mathtt{d}_S\omega$, i.e., (ii) implies that $\mathtt{d}_S\omega$ is a 3-coboundary. This means that there is a map $\beta \in C^2\big(G, Z(N)\big)$ such that $\mathtt{d}_S\omega = \mathtt{d}_G\beta$. Set $\omega' := \beta^{-1}\omega\colon G \times G \to Z(N)$. Then the definition of $C^2\big(G, Z(N)\big)$ implies that (S, ω') still satisfy (17.5) and (17.7). Further,

$$\mathtt{d}_S\omega' = -\mathtt{d}_S\beta + \mathtt{d}_S\omega = -\mathtt{d}_G\beta + \mathtt{d}_S\omega = 0,$$

so that Proposition 17.1.17 shows that (S', ω') is a factor system such that s is the characteristic homomorphism of $N \times_{(S',\omega')} G$. □

Remark 17.1.21. (a) The results developed in this section have the following implications for the classification of Lie groups G with a given identity component G_0 and given component group $\Gamma := \pi_0(G)$. We think of these groups as extensions of Γ by G_0 and want to classify them up to equivalence of extensions.

Step 1: First one has to describe the set of those homomorphisms $s\colon \Gamma \to \mathrm{Out}(G_0)$ for which $\chi(s) \in H^3_{\rho_s}(\Gamma, Z(G_0))$ vanishes.

Step 2: If $\chi(s) = 0$ is fixed, then by we know that $\mathrm{Ext}_s(\Gamma, G_0) \neq \emptyset$ is acted upon in a simply transitive fashion by the group $H^2_{\rho_s}(\Gamma, Z(G_0))$ (Theorem 17.1.20), so that we obtain the desired classification by constructing one extension G corresponding to s and then determining the group $H^2_{\rho_s}(\Gamma, Z(G_0))$ (Theorem 17.1.13).

(b) For the special case where s lifts to a homomorphism $S\colon \Gamma \to \mathrm{Aut}(G_0)$, we have in particular $\chi(s) = 0$ and $G := G_0 \rtimes_S \Gamma$ is an extension of Γ by G_0 corresponding to s. All other extensions are now determined by elements of the group $H^2_{\rho_s}(\Gamma, Z(N)) \cong \mathrm{Ext}_s(\Gamma, Z(N))$.

(c) For the 2-element group $\Gamma = \mathbb{Z}/2$, a homomorphism $s\colon \Gamma \to \mathrm{Out}(G_0)$ is specified by an involution $\tau \in \mathrm{Out}(G_0)$. If this involution lifts to an involution in $\mathrm{Aut}(G_0)$, then the corresponding set of extension classes is parameterized by

$$H^2_s(\Gamma, Z(G_0)) \cong Z(G_0)^\tau / (\mathbf{1} + \tau)(Z(G_0))$$

(cf. Example 17.3.5 below).

If τ does not lift to an involution of $\mathrm{Aut}(G_0)$, then we have to evaluate the obstruction class

$$\chi(s) \in H^3_s(\Gamma, Z(G_0)) \cong Z(G_0)^{-\tau}/(\mathbf{1} - \tau)(Z(G_0))$$

(see also Example 17.3.5 below).

Example 17.1.22. (a) For the group $\mathrm{GL}_n(\mathbb{R})$, we have

$$\mathrm{GL}_n(\mathbb{R})_0 = \mathrm{GL}_n(\mathbb{R})_+ := \{g \in \mathrm{GL}_n(\mathbb{R})\colon \det g > 0\} \cong \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^\times_+$$

and $\pi_0(\mathrm{GL}_n(\mathbb{R})) \cong \mathbb{Z}/2$. The matrix $\tau := \mathrm{diag}(-1, 1, \ldots, 1)$ satisfies $\det \tau = -1$ and $\tau^2 = \mathbf{1}$, so that

$$\mathrm{GL}_n(\mathbb{R}) \cong \mathrm{GL}_n(\mathbb{R})_0 \rtimes \{\mathbf{1}, \tau\}.$$

We have

$$Z := Z(\mathrm{GL}_n(\mathbb{R})_0) = Z(\mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^\times_+ = \begin{cases} \mathbb{R}^\times & \text{for } n \text{ even,} \\ \mathbb{R}^\times_+ & \text{for } n \text{ odd,} \end{cases}$$

and τ commutes with this group. This implies that the corresponding cohomology group is

$$H^2(\mathbb{Z}/2, Z) = Z/\{z^2 \colon z \in Z\} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } n \text{ even,} \\ \{\mathbf{1}\} & \text{for } n \text{ odd.} \end{cases}$$

We conclude that, for n odd, there are no twisted variants of $\mathrm{GL}_n(\mathbb{R})$ in the sense that they are extensions of $\mathbb{Z}/2$ by $\mathrm{GL}_n(\mathbb{R})_0$ with the same characteristic homomorphism, but for n even, there is one non-trivial twisted group G. This group can be described as

$$G := (\mathrm{GL}_n(\mathbb{R})_0 \rtimes_\alpha C_4)/\{(\mathbf{1}, 1), (-\mathbf{1}, -1)\},$$

where the generator $i \in C_4$ acts on $\mathrm{GL}_n(\mathbb{R})_0$ by $\alpha(i)(g) = \tau g \tau$. Then G is a Lie group with $G_0 = \mathrm{GL}_n(\mathbb{R})_0$, $\pi_0(G) \cong \mathbb{Z}/2\mathbb{Z}$, and the corresponding characteristic homomorphism is the same as for $\mathrm{GL}_n(\mathbb{R})$. However, any element of the form $[g, \pm i] \in G$ satisfies $[g, \pm i]^2 = [g\tau g\tau, -1]$, and this equals $[\mathbf{1}, 1] = [-\mathbf{1}, -1]$ if and only if $(g\tau)^2 = -\mathbf{1}$. This relation means that $g\tau$ is a complex structure on $\mathbb{R}^n$, which implies that $\det(g\tau) = 1$, so that this is never the case for $g \in \mathrm{GL}_n(\mathbb{R})_0$. We conclude that the extension of $\pi_0(G)$ by G_0 does not split.

(b) The situation changes if we replace $\mathrm{GL}_n(\mathbb{R})$ by a covering group. Let $\widetilde{\mathrm{GL}}_n(\mathbb{R})_0$ denote the simply connected covering of $\mathrm{GL}_n(\mathbb{R})_0$. Since the inclusion

$$\mathrm{SO}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$$

induces surjective homomorphisms

$$\pi_1(\mathrm{SO}_2(\mathbb{R})) \cong \mathbb{Z} \xrightarrow{\cong} \pi_1(\mathrm{GL}_2(\mathbb{R})) \cong \mathbb{Z} \twoheadrightarrow \pi_1(\mathrm{GL}_n(\mathbb{R})) \cong \pi_1(\mathrm{SL}_n(\mathbb{R}))$$

(cf. Proposition 16.2.4) and $\tau g\tau = g^{-1}$ for $g \in \mathrm{SO}_2(\mathbb{R})$, τ acts (via the automorphism induced by conjugation) on $\pi_1(\mathrm{GL}_n(\mathbb{R}))$ by inversion. From $\widetilde{\mathrm{GL}}_n(\mathbb{R})_0 \cong \widetilde{\mathrm{SL}}_n(\mathbb{R}) \times \mathbb{R}_+^\times$, we further obtain

$$Z(\widetilde{\mathrm{GL}}_n(\mathbb{R})_0) \cong Z(\widetilde{\mathrm{SL}}_n(\mathbb{R})) \times \mathbb{R}_+^\times \cong \begin{cases} \mathbb{Z} \times \mathbb{R}_+^\times & \text{for } n = 2, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}_+^\times & \text{for } n \in 2\mathbb{N}+1, \\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{R}_+^\times & \text{for } n \in 2\mathbb{N}+2, \end{cases}$$

and (the lift of) τ acts on this group by $\tau \cdot (x,y) = (-x,y)$. We thus obtain

$$\begin{aligned} H^2(\mathbb{Z}/2, Z(\widetilde{\mathrm{GL}}_n(\mathbb{R})_0)) &\cong Z(\widetilde{\mathrm{GL}}_n(\mathbb{R})_0)^\tau / \{z \cdot \tau.z \colon z \in Z(\widetilde{\mathrm{GL}}_n(\mathbb{R})_0)\} \\ &\cong \begin{cases} \{0\} & \text{for } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n > 2. \end{cases} \end{aligned}$$

(cf. Example 17.3.5).

Exercises for Section 17.2

Exercise 17.1.1 (Baer products). Let

$$N \xrightarrow{\iota_1} \widehat{G}_1 \xrightarrow{q_1} G$$

and $Z(N) \xrightarrow{\iota_2} \widehat{G}_2 \xrightarrow{q_2} G$ be two extensions of G, where the G-module structure on $Z(N)$ induced from $\widehat{G}_2$ coincides with the one induced from $\widehat{G}_1$. Show that:

(a) $G^\sharp := \widehat{G}_1 \times_G \widehat{G}_2 := \{(\widehat{g}_1, \widehat{g}_2) \in \widehat{G}_1 \times \widehat{G}_2 \colon q_1(\widehat{g}_1) = q_2(\widehat{g}_2)\}$ is an extension of G by $Z(N) \times N$.

(b) $Z := \{(z, z^{-1}) \colon z \in Z(N)\}$ is a normal subgroup of $G^\sharp$ and $\widehat{G}_3 := G^\sharp/Z$ is another extension of G by N.

(c) If we identify $H^2_s(G, Z(N))$ with $\mathrm{Ext}_s(G, Z(N))$, then the map

$$([\widehat{G}_1], [\widehat{G}_2]) \mapsto [\widehat{G}_3]$$

corresponds to the action of the group $H^2_s(G, Z(N))$ on the set $\mathrm{Ext}_s(G, N)$.

Exercise 17.1.2. (The Mautner group) Let $\theta \in \mathbb{R}$ be irrational and consider the semidirect product Lie group $G := \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$ for $\alpha(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{i\theta t} \end{pmatrix}$. Show that the group $\mathrm{Ad}(G) = \mathrm{Inn}(G)$ of inner automorphisms of G is not closed.

17.2 Coverings of Nonconnected Lie Groups

Let G be a nonconnected Lie group. Then it is a natural question whether G has a simply connected covering group $q_G \colon \widetilde{G} \to G$, i.e., q_G is a group homomorphism which is a covering and $\pi_1(\widetilde{G})$ is trivial. Any such group is an extension of $\pi_0(G) \cong \pi_0(\widetilde{G})$ by the simply connected group $\widetilde{G}_0$ for which the characteristic homomorphism coincides with the characteristic homomorphism

$$s_G\colon \pi_0(G) \to \mathrm{Out}(G_0) \subseteq \mathrm{Out}(\widetilde{G}_0),$$

if we consider $\mathrm{Out}(G_0)$ as a subgroup of $\mathrm{Out}(\widetilde{G}_0)$ (cf. Example 17.1.5). Accordingly, we have two cohomology classes $\chi(s_G) \in H^3(\pi_0(G), Z(G_0))$ and $\widetilde{\chi}(s_G) \in H^3(\pi_0(G), Z(\widetilde{G}_0))$. From Theorem 17.1.20, we conclude that $\widetilde{\chi}(s_G)$ vanishes if G has a simply connected covering group $\widetilde{G}$.

Assume, conversely, that $\widetilde{\chi}(s_G) = 0$, and let $G^\sharp$ be a $\widetilde{G}_0$-extension of $\pi_0(G)$ with $[G^\sharp] \in \mathrm{Ext}_{s_G}(\pi_0(G), \widetilde{G}_0)$. Then $G^\sharp$ is simply connected, and we know that the equivalence classes of all other simply connected covering groups of G are parameterized by the group $H^2(\pi_0(G), Z(\widetilde{G}_0))$ which acts simply transitively on $\mathrm{Ext}_{s_G}(\pi_0(G), \widetilde{G}_0)$ (Theorem 17.1.13). Since the image of s_G preserves the subgroup $\ker q_{G_0} \cong \pi_1(G) \subseteq Z(\widetilde{G}_0)$, the group $\ker q_G$ is normal in $G^\sharp$, so that the quotient group $G^\sharp/\ker q_G$ is an extension of $\pi_0(G)$ by G_0, and there exists a unique class $c \in H^2(\pi_0(G), Z(G_0))$ with

$$c \cdot [G] = [G^\sharp/\ker q_G] \in \mathrm{Ext}_{s_G}(\pi_0(G), G_0)$$

(see Remark 17.1.15).

Recall that the covering $q_{G_0}\colon \widetilde{G}_0 \to G_0$ induces a short exact sequence

$$\mathbf{1} \to \pi_1(G) \cong \ker(q_{G_0}) \hookrightarrow Z(\widetilde{G}_0) \xrightarrow{q_{G_0}} Z(G_0) \to \mathbf{1} \qquad (17.10)$$

(Proposition 8.5.2). Passing from $[G^\sharp]$ to another element of $\mathrm{Ext}_s(\pi_0(G), \widetilde{G}_0)$ changes c by an element of the image of the canonical homomorphism

$$H^2(\pi_0(G), Z(\widetilde{G}_0)) \to H^2(\pi_0(G), Z(G_0)).$$

On the other hand, the short exact sequence of abelian groups (17.10) leads to a long exact cohomology sequence

$$\cdots \to H^2(\pi_0(G), \pi_1(G)) \to H^2(\pi_0(G), Z(\widetilde{G}_0)) \to H^2(\pi_0(G), Z(G_0))$$
$$\xrightarrow{\delta} H^3(\pi_0(G), \pi_1(G)) \to H^3(\pi_0(G), Z(\widetilde{G}_0)) \to H^3(\pi_0(G), Z(G_0)) \to \cdots$$

(see Remark 17.3.4 below), and the preceding discussion now shows that the cohomology class

$$\chi_G := \delta(c) \in H^3(\pi_0(G), \pi_1(G))$$

does not depend on the choice of $G^\sharp$. It turns out to be the obstruction for the existence of a simply connected covering group of G:

Theorem 17.2.1. *The Lie group G has a simply connected covering group $\widetilde{G}$ if and only if the cohomology class $\chi_G \in H^3(\pi_0(G), \pi_1(G))$ vanishes.*

Proof. The group $G^\sharp$ from above is a simply connected covering group of G if and only if $c = 0$, i.e., $G^\sharp/\ker q_G$ is equivalent to G as an extension of $\pi_0(G)$ by G_0.

If $\widetilde{G} \to G$ is a simply connected covering group of G, then $G^\sharp := \widetilde{G}$ leads to $\chi_G = \delta(c) = 0$. If, conversely, $0 = \chi_G = \delta(c)$, then the exactness of the long exact cohomology sequence implies the existence of some $\widetilde{c} \in H^2(\pi(G_0), Z(\widetilde{G}_0))$ with $H^2(\pi(G_0), q_{G_0})\widetilde{c} = c$. This means that $-\widetilde{c}.[G^\sharp] = [\widetilde{G}]$ for some $\widetilde{G}_0$-extension $\widetilde{q}_G \colon \widetilde{G} \to G$ of $\pi_0(G)$ by $\widetilde{G}_0$ satisfying $\widetilde{G}/\ker \widetilde{q}_G \cong G$. Hence $\widetilde{G}$ is a simply connected covering group of G. □

Example 17.2.2. (a) If $\pi_0(G)$ is a free abelian group, then $H^n(\pi_0(G), A)$ vanishes for every $\pi_0(G)$-module A and $n > 0$ ([ML63, Thm. 7.3]), and this implies that G has a simply connected covering group.

(b) If $G = G_0 \rtimes_S \pi_0(G)$ is a semidirect product, then $\widetilde{G} := \widetilde{G}_0 \rtimes_{\widetilde{S}} \pi_0(G)$ is a simply connected covering group, where $\widetilde{S} \colon \pi_0(G) \to \mathrm{Aut}(\widetilde{G}_0)$ is the canonical lift of S. This group defines an extension class $[\widetilde{G}] \in \mathrm{Ext}_{s_G}(\pi_0(G), \widetilde{G}_0)$. We also know that the group $H^2_{s_G}(\pi_0(G), Z(\widetilde{G}_0))$ acts simply transitively on this set, and, for an extension $[G^\sharp] = c \cdot [\widetilde{G}]$, $c \in H^2_{s_G}(\pi_0(G), Z(\widetilde{G}_0))$, the extension $G^\sharp/\pi_1(G)$ of $\pi_0(G)$ by $\widetilde{G}_0/\pi_1(G) \cong G_0$ is equivalent to G if and only if the image of c in $H^2_{s_G}(\pi_0(G), Z(G_0))$ vanishes. According to the long exact cohomology sequence from above, this happens if and only if c is contained in the image of $H^2_{s_G}(\pi_0(G), \pi_1(G))$. This proves that the group $H^2_{s_G}(\pi_0(G), \pi_1(G))$ acts transitively on the fiber over $[G]$ in $\mathrm{Ext}_{s_G}(\pi_0(G), \widetilde{G}_0)$. This action need not be effective. In view of the long exact cohomology sequence, its kernel is the image of $H^1_{s_G}(\pi_0(G), Z(G_0))$ in $H^2_{s_G}(\pi_0(G), \pi_1(G))$.

Example 17.2.3. (a) An important 2-fold covering group of $\mathrm{GL}_n(\mathbb{R})$ is the *metalinear group*

$$\mathrm{ML}_n(\mathbb{R}) := \{(g, z) \in \mathrm{GL}_n(\mathbb{R}) \times \mathbb{C}^\times \colon \det g = z^2\}$$

(cf. [Bl73]). Then

$$q \colon \mathrm{ML}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (g, z) \mapsto g$$

is a covering morphism with kernel $\{(\mathbf{1}, 1), (\mathbf{1}, -1)\}$. On the identity component $\mathrm{GL}_n(\mathbb{R})_0 = \{g \in \mathrm{GL}_n(\mathbb{R}) \colon \det g > 0\}$, we have a homomorphism

$$\sqrt{\det} \colon \mathrm{GL}_n(\mathbb{R})_0 \to \mathbb{R}^\times \subseteq \mathbb{C}^\times, \quad g \mapsto \sqrt{\det(g)},$$

and then

$$\sigma \colon \mathrm{GL}_n(\mathbb{R})_0 \to \mathrm{ML}_n(\mathbb{R}), \quad g \mapsto (g, \sqrt{\det(g)})$$

is a homomorphism splitting the covering map q over $\mathrm{GL}_n(\mathbb{R})_0$. With $\tau := \mathrm{diag}(-1, 1, \ldots, 1)$ and

$$\Gamma := q^{-1}(\{\mathbf{1}, \tau\}) = \{(\mathbf{1}, 1), (\mathbf{1}, -1), (\tau, i), (\tau, -i)\} \cong C_4,$$

we then obtain an isomorphism

$$\mathrm{ML}_n(\mathbb{R}) \cong \mathrm{GL}_n(\mathbb{R})_0 \rtimes \Gamma \cong \mathrm{GL}_n(\mathbb{R})_0 \rtimes C_4.$$

(b) From the inclusion $\mathrm{O}_2(\mathbb{R}) \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ it follows that the nontrivial element $[\tau] \in \pi_0(\mathrm{GL}_n(\mathbb{R}))$ acts by inversion on the cyclic group $\pi_1(\mathrm{GL}_n(\mathbb{R}))$ (cf. Example 17.1.22). With Example 17.3.5(b) below, we conclude that

$$H^2(\pi_0(\mathrm{GL}_n(\mathbb{R})), \pi_1(\mathrm{GL}_n(\mathbb{R}))) \cong \begin{cases} \mathbf{1} & \text{for } n \leq 2 \\ \pi_1(\mathrm{GL}_n(\mathbb{R})) \cong \mathbb{Z}/2 & \text{for } n > 2. \end{cases}$$

Hence there is only one simply connected covering of $\mathrm{GL}_2(\mathbb{R})$, and, for $n > 2$, we have at most one twist. The same arguments apply to the subgroup $\mathrm{O}_n(\mathbb{R})$ which has the same groups π_0 and π_1.

However, there are twisted coverings of $\mathrm{GL}_2(\mathbb{R})$ that are not simply connected. If $\tau \in C_2 = \{\mathbf{1}, \tau\}$ acts by inversion on the cyclic group $C_m = \mathbb{Z}/m\mathbb{Z}$, then Example 17.3.5(b) below implies that

$$H^2(C_2, C_m) \cong \begin{cases} \mathbf{1} & \text{for } m \text{ odd} \\ \mathbb{Z}/2 & \text{for } m \text{ even.} \end{cases}$$

For $n = 2$, this implies, for each even integer m, the existence of a nontrivial covering of $\mathrm{GL}_2(\mathbb{R})$ with fundamental group $\mathbb{Z}/m\mathbb{Z}$. In particular, the twofold cover admits a nontrivial twist. For $\mathrm{O}_2(\mathbb{R})$, this twist is realized by the group $\mathrm{Pin}_2(\mathbb{R})$ (Example B.3.24).

Example 17.2.4. We consider the special case where $G_0 = T \cong \mathbb{T}^d$ is a torus group and $\Gamma = \pi_0(G)$ is finite, acting on T by $S\colon \Gamma \to \mathrm{Aut}(T)$. Then $Z(\widetilde{G}_0) = \widetilde{G}_0 \cong \mathbb{R}^d$ and $Z(G_0) = G_0$. In view of [ML63, Cor. IV.5.4], the cohomology groups $H^n(\Gamma, Z(\widetilde{G}_0))$ vanish for $n > 0$, so that the long exact cohomology sequence corresponding to the short exact sequence

$$\mathbf{1} \to \mathbb{Z}^d = \pi_1(T) \to \mathbb{R}^d = \widetilde{T} \to \mathbb{T}^d = T \to \mathbf{1}$$

of Γ-modules (Remark 17.3.4) leads to isomorphisms

$$\delta_n\colon H^n(\Gamma, T) \to H^{n+1}(\Gamma, \pi_1(T)) \quad \text{for} \quad n > 0.$$

Any Lie group G with $\pi_0(G) = \Gamma$ and $G_0 = T$ is determined, up to equivalence, by the corresponding cohomology class $[f] \in H^2(\Gamma, T) \cong H^3(\Gamma, \pi_1(T))$. Then

$$G = T \times_f \Gamma \quad \text{with} \quad (t, \gamma)(t', \gamma') = (t \cdot \gamma(t') \cdot f(\gamma, \gamma'), \gamma\gamma').$$

In this case we may choose $G^\sharp := \widetilde{T} \rtimes_{\widetilde{S}} \Gamma$, where $\widetilde{S}\colon \Gamma \to \mathrm{Aut}(\widetilde{T})$ is the canonical lift of S. Then $[G^\sharp/\pi_1(T)] = c \cdot [G]$ for $c = -[f]$ implies that

$$\chi_G = -\delta_2([f]) \in H^3(\Gamma, \pi_1(T)).$$

Since δ_2 is an isomorphism, G has a simply connected covering group if and only if $[f]=0$, i.e., if $G \cong T \rtimes_S \Gamma$ is a semidirect product.

The simplest example of a Lie group without a simply connected covering group is

$$G := (\mathbb{T} \rtimes_S C_4)/\{(1,1),(-1,-1)\},$$

where $C_4 = \{1, i, -1, -i\}$ acts on $\mathbb{T}$ by $S(i)(t) = t^{-1}$. In this case $G_0 \cong \mathbb{T}$, and $\pi_0(G) \cong C_2$ acts on T by inversion. Each element $g = [t,z]$ which is not contained in the identity component, satisfies $z^2 = -1$, so that $g^2 = [tS(z)(t), -1] = [1,-1] \neq [1,1]$. This implies that the short exact sequence

$$\mathbf{1} \to G_0 = \mathbb{T} \to G \to \pi_0(G) \cong C_2 \to \mathbf{1}$$

does not split. Since

$$(\mathbb{T} \times_S C_4)/\{(1,1),(1,-1)\} \cong \mathrm{O}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R}) \rtimes C_2,$$

G is a twisted form of $\mathrm{O}_2(\mathbb{R})$. The possible twists are parameterized by the group

$$H^2_S(C_2, \mathbb{T}) = \mathbb{T}^S = \{1,-1\} \cong C_2.$$

17.3 Appendix: Group Cohomology

In this appendix we provide definitions and basic facts concerning group cohomology. Here G denotes a group and A a *G-module*, i.e., an abelian group (written additively), endowed with an action of G, defined by a homomorphism $\rho\colon G \to \mathrm{Aut}(A)$. We write $ga := \rho(g)a$ for this action.

Definition 17.3.1. For $p \in \mathbb{N}_0$, we write $C^p(G,A)$ for the space of all maps $G^p \to A$ vanishing if at least one argument is $\mathbf{1}$. The elements of $C^p(G,A)$ are called *p-cochains*. The set $C^p(G,A)$ carries a natural group structure, defined by pointwise addition. We define additive maps

$$\mathtt{d}_G\colon C^p(G,A) \to C^{p+1}(G,A)$$

by

$$(\mathtt{d}_G f)(g_0,\ldots,g_p) := g_0 f(g_1,\ldots,g_p) + \sum_{j=1}^{p} (-1)^j f(g_0,\ldots,g_{j-1}g_j,\ldots,g_p) + (-1)^{p+1} f(g_0,\ldots,g_{p-1}).$$

A quick inspection shows that $\mathtt{d}_G f$ also vanishes if one of its arguments vanishes. We now combine them to an additive map

$$\mathtt{d}_G\colon C(G,A) := \bigoplus_{p=0}^{\infty} C^p(G,A) \to C(G,A).$$

It is not hard to verify that $\mathtt{d}_G^2 = 0$ (cf. Exercise 17.3.2(c)). This implies that the subgroup $Z^p(G, A) := \ker(\mathtt{d}_G|_{C^p(G,A)})$ of *p-cocycles* contains the subgroup $B^p(G, A) := \mathtt{d}_G(C^{p-1}(G, A))$ of *p-coboundaries*. The quotient group

$$H^p_\rho(G, A) := H^p(G, A) := Z^p(G, A)/B^p(G, A)$$

is the *pth cohomology group of G with values in the G-module A*. We write $[f]$ for the cohomology class of $f \in Z^p(G, A)$.

Example 17.3.2. (a) For any G-module A, we have $C^0(G, A) \cong A$, $Z^0(G, A) \cong A^G$, the set of fixed points, and

$$H^0(G, A) \cong A^G.$$

(b) A map $f : G \to A$ is a 1-cocycle if and only if

$$f(gg') = f(g) + gf(g') \quad \text{for} \quad g, g' \in G,$$

and the 1-coboundaries are of the form $(\mathtt{d}_G a)(g) = ga - a$.

A geometric interpretation of 1-cocycles is that each 1-cocycle f defines an "affine" action of G on A by

$$g * a := ga + f(g),$$

and this action has a fixed point if and only if the cohomology class $[f] \in H^1(G, A)$ vanishes (Exercise).

Another interpretation of 1-cocycles is that they provide splitting homomorphisms

$$\sigma_f := (f, \mathrm{id}_G) : G \to A \rtimes_\rho G,$$

and two such homomorphisms σ_f and $\sigma_{f'}$ are conjugate under the normal subgroup A of $A \rtimes_\rho G$ if and only if $[f] = [f']$ in $H^1(G, A)$.

(c) The group $H^2(G, A)$ parameterizes equivalence classes of abelian extensions of G by A by associating to the cocycle $\omega \in Z^2(G, A)$, the abelian extension $A \times_\omega G$, consisting of the set $A \times G$, endowed with the product

$$(a, g)(a', g') = (a + ga' + \omega(g, g'), gg')$$

(cf. Remark 17.1.14).

Definition 17.3.3. Let A and B be G-modules and $\varphi : A \to B$ be a *morphism of G-modules*, i.e., a G-equivariant homomorphism of abelian groups. Then we obtain corresponding homomorphisms

$$C^n(G, \varphi) : C^n(G, A) \to C^n(G, B), \quad f \mapsto \varphi \circ f,$$

with

$$C^{n+1}(G, \varphi) \circ \mathtt{d}_G = \mathtt{d}_G \circ C^n(G, \varphi). \tag{17.11}$$

This implies that $C^n(G, \varphi)$ maps cocycles into cocycles and coboundaries into coboundaries, hence induces a homomorphism

$$H^n(G, \varphi) : H^n(G, A) \to H^n(G, B), \quad [f] \mapsto [\varphi \circ f].$$

Remark 17.3.4. Let $\mathbf{0} \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to \mathbf{0}$ be a short exact sequence of G-modules. Then we obtain short exact sequences

$$\mathbf{0} \to C^n(G,A) \xrightarrow{C^n(G,\alpha)} C^n(G,B) \xrightarrow{C^n(G,\beta)} C^n(G,C) \to \mathbf{0},$$

which, in view of (17.11), combine to a short exact sequence of chain complexes. Now basic homological algebra provides *connecting maps*

$$\delta_n \colon H^n(G,C) \to H^{n+1}(G,A),$$

which lead to a long exact cohomology sequence

$$\cdots \to H^n(G,A) \xrightarrow{H^n(G,\alpha)} H^n(G,B) \xrightarrow{H^n(G,\beta)} H^n(G,C)$$
$$\xrightarrow{\delta_n} H^{n+1}(G,A) \xrightarrow{H^{n+1}(G,\alpha)} H^{n+1}(G,B) \cdots$$

(cf. [ML63, Thm. II.4.1]).

Example 17.3.5. (a) Let $C_n \cong \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with n elements and write $\rho \in C_n$ for a generator. Then a C_n-module structure on an abelian group is given by an automorphism $\varphi \in \mathrm{Aut}(A)$ with $\varphi^n = \mathrm{id}_A$. It defines an endomorphism $N \in \mathrm{End}(A)$ by

$$Na := a + \varphi(a) + \ldots + \varphi^{n-1}(a)$$

whose range lies in $A^\varphi = \{a \in A \colon \varphi(a) = a\}$ and which commutes with the C_n-action. We then have

$$H^{2k}(C_n,A) \cong A^\varphi/N(A), \quad H^{2k-1}(C_n,A) \cong \ker N/\operatorname{im}(\varphi - \mathbf{1}), \quad k > 0$$

(cf. [ML63, Thm. IV.7.1]).

(b) We take a closer look at the case $n = 2$. Then each $f \in C^p(C_2,A)$ vanishes in all tuples containing $\mathbf{1}$, hence is determined by $f(\rho,\ldots,\rho)$, so that we obtain an isomorphism

$$C^p(C_2,A) \to A, \quad f \mapsto f(\rho,\ldots,\rho)$$

of groups. Identifying $C^p(C_2,A)$ accordingly with A, it follows immediately from the formula for the group differential that, for $a \in A \cong C^p(C_2,A)$, we have

$$\mathtt{d}a = \varphi(a) + (-1)^{p+1}a = (\varphi + (-1)^{p+1}\mathbf{1})a.$$

We conclude that

$$Z^{2k}(C_2,A) = A^\varphi = \ker(\varphi - \mathbf{1}) \quad \text{and} \quad Z^{2k+1}(C_2,A) = A^{-\varphi} = \ker(\varphi + \mathbf{1}).$$

We also derive that

$$H^{2k}(C_2,A) = A^\varphi/\operatorname{im}(\varphi + \mathbf{1}) \quad \text{and} \quad H^{2k+1}(C_2,A) = A^{-\varphi}/\operatorname{im}(\varphi - \mathbf{1}).$$

For more interpretations of the cohomology groups in low degree, we refer to [We95] or [ML63].

Exercises for Section 17.3

Exercise 17.3.1. Let X be a set and A be an abelian group. For $p \in \mathbb{N}_0$, we consider the group $C^p_{AS}(X,A)$ of all functions $F\colon X^{p+1} \to A$. The group structure on $C^p_{AS}(X,A)$ is given by pointwise addition. We define an additive map $\mathtt{d}_{AS}\colon C^p_{AS}(X,A) \to C^{p+1}_{AS}(X,A)$ by

$$(\mathtt{d}_{AS}F)(x_0,\ldots,x_{p+1}) := \sum_{i=0}^{p+1}(-1)^i F(x_0,\ldots,\widehat{x_i},\ldots,x_{p+1}),$$

where $\widehat{\ }$ denotes omission. [1] Putting all these maps together, we obtain an additive map

$$\mathtt{d}_{AS}\colon C_{AS}(X,A) := \bigoplus_{p=0}^{\infty} C^p_{AS}(X,A) \to C_{AS}(X,A).$$

Show that $\mathtt{d}^2_{AS}F = 0$.

Exercise 17.3.2. Now let G be a group and A be a G-module. We define maps $\Phi_n\colon C^n(G,A) \to C^n_{AS}(G,A)$ by

$$\Phi_n(f)(g_0,\ldots,g_n) := g_0 f(g_0^{-1}g_1, g_1^{-1}g_2,\ldots,g_{n-1}^{-1}g_n).$$

Show that:

(a) Each Φ_n is injective.
(b) The image of Φ_n consists of all elements $F \in C^n_{AS}(G,A)$ with

$$F(x_0,\ldots,x_p) = 0 \quad \text{whenever} \quad x_i = x_{i+1} \quad \text{for some} \quad i \in \{0,\ldots,p-1\}$$

and

$$F(gg_0,\ldots,gg_n) = gF(g_0,\ldots,g_n), \quad g,g_0,\ldots,g_n \in G.$$

(c) The map $\Psi_n\colon \operatorname{im}(\Phi_n) \to C^n(G,A)$, define by

$$\Psi_n(F)(g_1,\ldots,g_n) := F(\mathbf{1}, g_1, g_1g_2,\ldots,g_1\cdots g_n).$$

in an inverse of Φ_n.
(d) $\mathtt{d}_{AS} \circ \Phi_n = \Phi_{n+1} \circ \mathtt{d}_G$.
(e) $\mathtt{d}^2_G = 0$.

Notes on Chapter A

For the extension theory of abstract groups, we refer to MacLane's classic [ML63]. In Section 17.1 we interpret this theory in the context of extensions of discrete groups by connected Lie groups. For a detailed discussion of the subtleties of extensions of arbitrary Lie groups by Lie groups, we refer to [Ne07], where these issues are treated for infinite-dimensional Lie groups.

[1] The subscript AS refers to Alexander–Spanier because this differential leads to the Alexander–Spanier cohomology in algebraic topology.

Part V

Appendices

A

Basic Covering Theory

In this appendix we provide the main results on coverings of topological spaces needed to develop coverings of Lie groups and manifolds. In particular, this material is needed to show that, for each finite-dimensional Lie algebra $\mathfrak{g}$, there exists a 1-connected Lie group G with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$ which is unique up to isomorphism.

A.1 The Fundamental Group

To define the notion of a simply connected space, we first have to define its fundamental group. The elements of this group are homotopy classes of loops. The present section develops this concept and provides some of its basic properties.

Definition A.1.1. Let X be a topological space, $I := [0,1]$, and $x_0, x_1 \in X$. We write

$$P(X, x_0) := \{\gamma \in C(I, X) \colon \gamma(0) = x_0\}$$

and

$$P(X, x_0, x_1) := \{\gamma \in P(X, x_0) \colon \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X, x_0, x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H \colon I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

(for $H_t(s) := H(t, s)$) and

$$(\forall t \in I) \quad H(t, 0) = x_0, \; H(t, 1) = x_1.$$

It is easy to show that $\sim$ is an equivalence relation (Exercise A.1.2), called *homotopy*. The homotopy class of α is denoted by $[\alpha]$.

We write $\Omega(X, x_0) := P(X, x_0, x_0)$ for the set of loops based at x_0. For $\alpha \in P(X, x_0, x_1)$ and $\beta \in P(X, x_1, x_2)$ we define a product $\alpha * \beta$ in $P(X, x_0, x_2)$ as the concatenation

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Lemma A.1.2. *If $\varphi\colon [0,1] \to [0,1]$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then for each $\alpha \in P(X, x_0, x_1)$ we have $\alpha \sim \alpha \circ \varphi$.*

Proof. Use $H(t, s) := \alpha(ts + (1-t)\varphi(s))$. □

Proposition A.1.3. *The following assertions hold:*

(1) *$\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ implies $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$, so that we obtain a well-defined product*
$$[\alpha] * [\beta] := [\alpha * \beta]$$
of homotopy classes.

(2) *If x also denotes the constant map $I \to \{x\} \subseteq X$, then*
$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \textit{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) (Associativity) *$[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ for $\alpha \in P(X, x_0, x_1)$, $\beta \in P(X, x_1, x_2)$ and $\gamma \in P(X, x_2, x_3)$.*

(4) (Inverse) *For $\alpha \in P(X, x_0, x_1)$ and $\overline{\alpha}(t) := \alpha(1-t)$ we have*
$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) (Functoriality) *For any continuous map $\varphi\colon X \to Y$ with $\varphi(x_0) = y_0$ we have*
$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta)$$
and $\alpha \sim \beta$ implies $\varphi \circ \alpha \sim \varphi \circ \beta$.

Proof. (1) If H^α is a homotopy from α_1 to α_2 and H^β a homotopy from β_1 to β_2, then we put

$$H(t, s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

(cf. Exercise A.1.1).

(2) For the first assertion we use Lemma A.1.2 and

$$x_0 * \alpha = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For the second, we have

$$\alpha * x_1 = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{4} \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(4)

$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \leq \frac{1-t}{2} \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \overline{\alpha}(2s-1) & \text{for } s \geq \frac{1+t}{2}. \end{cases}$$

(5) is trivial. □

Definition A.1.4. From the preceding definition, we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in x_0 carries a natural group structure. This group is called the *fundamental group of X with respect to x_0*.

A space X is called *simply connected* if $\pi_1(X, x_0)$ vanishes for all $x_0 \in X$. If X is pathwise connected it suffices to check this for a single $x_0 \in X$ (Exercise A.1.4).

Lemma A.1.5 (Functoriality of the Fundamental Group). *If $f\colon X \to Y$ is a continuous map with $f(x_0) = y_0$, then*

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$

is a group homomorphism. Moreover, we have

$$\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)} \quad \text{and} \quad \pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g).$$

Proof. This follows directly from Proposition A.1.3(5). □

Remark A.1.6. The map

$$\sigma\colon \pi_1(X, x_0) \times \big(P(X, x_0)/\sim\big) \to P(X, x_0)/\sim, \quad ([\alpha], [\beta]) \mapsto [\alpha * \beta] = [\alpha] * [\beta]$$

defines an action of the group $\pi_1(X, x_0)$ on the set $P(X, x_0)/\sim$ of homotopy classes of paths starting in x_0 (Proposition A.1.3).

Remark A.1.7. (a) Suppose that the topological space X is contractible, i.e., there exists a continuous map $H\colon I \times X \to X$ and $x_0 \in X$ with $H(0, x) = x$ and $H(1, x) = x_0$ for $x \in X$. Then $\pi_1(X, x_0) = \{[x_0]\}$ is trivial (Exercise).

(b) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(c) $\pi_1(\mathbb{R}^n, 0) = \{0\}$ because $\mathbb{R}^n$ is contractible.

More generally, if the open subset $\Omega \subseteq \mathbb{R}^n$ is starlike with respect to x_0, then $H(t,x) := x + t(x - x_0)$ yields a contraction to x_0, and we conclude that $\pi_1(\Omega, x_0) = \{\mathbf{1}\}$.

(d) If $G \subseteq \mathrm{GL}_n(\mathbb{R})$ is a linear Lie group with a polar decomposition, i.e., for $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$, the polar map

$$p \colon K \times \mathfrak{p} \to G, \quad (k,x) \mapsto ke^x$$

is a homeomorphism, then the inclusion $K \to G$ induces an isomorphism

$$\pi_1(K, \mathbf{1}) \to \pi_1(G, \mathbf{1})$$

because the vector space $\mathfrak{p}$ is contractible.

The following lemma implies in particular, that fundamental groups of topological groups are always abelian.

Lemma A.1.8. *Let G be a topological group and consider the identity element $\mathbf{1}$ as a base point. Then the path space $P(G, \mathbf{1})$ also carries a natural group structure given by the pointwise product $(\alpha \cdot \beta)(t) := \alpha(t)\beta(t)$ and we have*

(1) *$\alpha \sim \alpha'$, $\beta \sim \beta'$ implies $\alpha \cdot \beta \sim \alpha' \cdot \beta'$, so that we obtain a well-defined product*

$$[\alpha][\beta] := [\alpha] \cdot [\beta] := [\alpha \cdot \beta]$$

of homotopy classes, defining a group structure on $P(G, \mathbf{1})/\sim$.

(2) *$\alpha \sim \beta \iff \alpha \cdot \beta^{-1} \sim \mathbf{1}$, the constant map.*

(3) (Commutativity) *$[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$ for $\alpha, \beta \in \Omega(G, \mathbf{1})$.*

(4) (Consistency) *$[\alpha] \cdot [\beta] = [\alpha] * [\beta]$ for $\alpha \in \Omega(G, \mathbf{1})$, $\beta \in P(G, \mathbf{1})$.*

Proof. (1) follows by composing homotopies with the multiplication map m_G.

(2) follows from (1).

(3)

$$[\alpha][\beta] = [\alpha * \mathbf{1}][\mathbf{1} * \beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [(\mathbf{1} * \beta)(\alpha * \mathbf{1})] = [\mathbf{1} * \beta][\alpha * \mathbf{1}] = [\beta][\alpha].$$

(4) $[\alpha][\beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [\alpha * \beta] = [\alpha] * [\beta]$. □

As a consequence of (4), we can calculate the product of homotopy classes as a pointwise product of representatives and obtain:

Proposition A.1.9 (Hilton's Lemma). *For each topological group G, the fundamental group $\pi_1(G) := \pi_1(G, \mathbf{1})$ is abelian.*

Proof. We only have to combine (3) and (4) in Lemma A.1.8 for loops $\alpha, \beta \in \Omega(G, \mathbf{1})$. □

Exercises for Section A.1

Exercise A.1.1. If $f\colon X \to Y$ is a map between topological spaces and $X = X_1 \cup \ldots \cup X_n$ holds with closed subsets $X_1, \ldots, X_n$, then f is continuous if and only if all restrictions $f|_{X_i}$ are continuous.

Exercise A.1.2. Show that the homotopy relation on $P(X, x_0, x_1)$ is an equivalence relation.

Exercise A.1.3. Show that for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected. For the proof, proceed along the following steps:
(a) Let $\gamma : [0,1] \to \mathbb{S}^n$ be continuous. Then there exists an $m \in \mathbb{N}$ such that $\|\gamma(t) - \gamma(t')\| < \frac{1}{2}$ for $|t - t'| < \frac{1}{m}$.
(b) Define $\widetilde{\alpha} : [0,1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0, \ldots, m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0,1] \to \mathbb{S}^n$.
(c) $\alpha \sim \gamma$.
(d) α is not surjective. The image of α is the central projection of a polygonal arc on the sphere.
(e) If $\beta \in \Omega(\mathbb{S}^n, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map).
(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$.

Exercise A.1.4. Let X be a topological space, $x_0, x_1 \in X$ and $\alpha \in P(X, x_0, x_1)$ a path from x_0 to x_1. Show that the map

$$C\colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense the fundamental group does not depend on the base point if X is arcwise connected.

Exercise A.1.5. Let $\sigma\colon G \times X \to X$ be a continuous action of the topological group G on the topological space X and $x_0 \in X$. Then the orbit map $\sigma^{x_0}\colon G \to X, g \mapsto \sigma(g, x_0)$ defines a group homomorphism

$$\pi_1(\sigma^{x_0})\colon \pi_1(G) \to \pi_1(X, x_0).$$

Show that the image of this homomorphism is central, i.e., lies in the center of $\pi_1(X, x_0)$.

A.2 Coverings

In this section we discuss the concept of a covering map. Its main application in Lie theory is that it provides, for each connected Lie group G, a simply connected covering group $q_G\colon \widetilde{G} \to G$ and hence also a tool to calculate its fundamental group $\pi_1(G) \cong \ker q_G$. In the following chapter we shall investigate to which extent a Lie group is determined by its Lie algebra and its fundamental group.

Definition A.2.1. Let X and Y be topological spaces. A continuous map $q\colon X \to Y$ is called a *covering* if each $y \in Y$ has an open neighborhood U such that $q^{-1}(U)$ is a nonempty disjoint union of open subsets $(V_i)_{i\in I}$, such that for each $i \in I$ the restriction $q|_{V_i}\colon V_i \to U$ is a homeomorphism. We call any such U an *elementary* open subset of X.

Note that this condition implies in particular that q is surjective and that the fibers of q are discrete subsets of X.

Examples A.2.2. (a) The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z$ is a covering map.

(b) The map $q\colon \mathbb{R} \to \mathbb{T}, x \mapsto e^{ix}$ is a covering.

(c) The power maps $p_k\colon \mathbb{C}^\times \to \mathbb{C}^\times, \quad z \mapsto z^k$ are coverings.

(d) If $q\colon G \to H$ is a surjective open morphism of topological groups with discrete kernel, then q is a covering (Exercise A.2.2). All the examples (a)-(c) are of this type.

Lemma A.2.3 (Lebesgue Number). *Let (X,d) be a compact metric space and $(U_i)_{i\in I}$ an open cover. Then there exists a positive number $\lambda > 0$, called a Lebesgue number of the covering, such that any subset $S \subseteq X$ with diameter $\leq \lambda$ is contained in some U_i.*

Proof. Let us assume that such a number λ does not exist. Then for each $n \in \mathbb{N}$ there exists a subset S_n of diameter $\leq \frac{1}{n}$ which is not contained in some U_i. Pick a point $s_n \in S_n$. The sequence (s_n) has a subsequence converging to some $s \in X$ and s is contained in some U_i. Since U_i is open, there exists an $\varepsilon > 0$ with $U_\varepsilon(s) \subseteq U_i$. If $n \in \mathbb{N}$ is such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $d(s_n, s) < \frac{\varepsilon}{2}$, we arrive at the contradiction $S_n \subseteq U_{\varepsilon/2}(s_n) \subseteq U_\varepsilon(s) \subseteq U_i$. $\square$

Remark A.2.4. (1) If $(U_i)_{i\in I}$ is an open cover of the unit interval $[0,1]$, then there exists an $n > 0$ such that all subsets of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = 0, \ldots, n-1$, are contained in some U_i.

(2) If $(U_i)_{i\in I}$ is an open cover of the unit square $[0,1]^2$, then there exists an $n > 0$ such that all subsets of the form

$$\left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right], \quad k, j = 0, \ldots, n-1,$$

are contained in some U_i.

Theorem A.2.5 (Path Lifting Theorem). *Let $q\colon X \to Y$ be a covering map and $\gamma\colon [0,1] \to Y$ a path. Let $x_0 \in X$ be such that $q(x_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma}\colon [0,1] \to X$ such that*

$$q \circ \widetilde{\gamma} = \gamma \quad \textit{and} \quad \widetilde{\gamma}(0) = x_0.$$

Proof. Cover Y by elementary open sets $U_i, i \in I$. By Lemma A.2.3, there exists an $n \in \mathbb{N}$ such that all sets $\gamma(\left[\frac{k}{n}, \frac{k+1}{n}\right])$, $k = 0, \ldots, n-1$, are contained in some U_i. We now use induction to construct $\widetilde{\gamma}$. Let $V_0 \subseteq q^{-1}(U_0)$ be an open subset containing x_0 for which $q|_{V_0}$ is a homeomorphism onto U_0 and define $\widetilde{\gamma}$ on $\left[0, \frac{1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_0})^{-1} \circ \gamma(t).$$

Assume that we have already constructed a continuous lift $\widetilde{\gamma}$ of γ on the interval $\left[0, \frac{k}{n}\right]$ and that $k < n$. Then we pick an open subset $V_k \subseteq X$ containing $\widetilde{\gamma}(\frac{k}{n})$ for which $q|_{V_k}$ is a homeomorphism onto some U_i and define $\widetilde{\gamma}$ for $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_k})^{-1} \circ \gamma(t).$$

We thus obtain the required lift $\widetilde{\gamma}$ of γ.

If $\widehat{\gamma}\colon [0,1] \to X$ is any continuous lift of γ with $\widehat{\gamma}(0) = x_0$, then $\widehat{\gamma}(\left[0, \frac{1}{n}\right])$ is a connected subset of $q^{-1}(U_0)$ containing x_0, hence contained in V_0. This shows that $\widetilde{\gamma}$ coincides with $\widehat{\gamma}$ on $\left[0, \frac{1}{n}\right]$. Applying the same argument at each step of the induction, we obtain $\widehat{\gamma} = \widetilde{\gamma}$, so that the lift $\widetilde{\gamma}$ is unique. ☐

Theorem A.2.6 (Covering Homotopy Theorem). *Let $I := [0,1]$ and $q\colon X \to Y$ be a covering map and $H\colon I^2 \to Y$ be a homotopy with fixed endpoints of the paths $\gamma := H_0$ and $\eta := H_1$. For any lift $\widetilde{\gamma}$ of γ there exists a unique lift $G\colon I^2 \to X$ of H with $G_0 = \widetilde{\gamma}$. Then $\widetilde{\eta} := G_1$ is the unique lift of η starting in the same point as $\widetilde{\gamma}$ and G is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\eta}$. In particular, lifts of homotopic curves in Y starting in the same point are homotopic in X.*

Proof. Using the Path Lifting Property (Theorem A.2.5), for each $t \in I$ we find a unique continuous lift $I \to X, s \mapsto G(s,t)$, starting in $\widetilde{\gamma}(t)$ with $q(G(s,t)) = H(s,t)$. It remains to show that the map $G\colon I^2 \to X$ obtained in this way is continuous.

So let $s \in I$. Using Lemma A.2.3, we find a natural number n such that for each connected neighborhood W_s of s of diameter $\leq \frac{1}{n}$ and each $i = 0, \ldots, n$, the set $H\big(W_s \times \left[\frac{k}{n}, \frac{k+1}{n}\right]\big)$ is contained in some elementary subset U_k of Y. Assuming that G is continuous in $W_s \times \{\frac{k}{n}\}$, G maps this set into a connected subset of $q^{-1}(U_k)$, hence into some open subset V_k for which $q|_{V_k}$ is a homeomorphism onto U_k. But then the lift G on $W_s \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$ must be contained in V_k, so that it is of the form $(q|_{V_k})^{-1} \circ H$, hence continuous. This means that G is continuous on $U_s \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$. Now an inductive argument shows that G is continuous on $U_s \times I$ and hence on the whole square I^2.

Since the fibers of q are discrete and the curves $s \mapsto H(s,0)$ and $s \mapsto H(s,1)$ are constant, the curves $G(s,0)$ and $G(s,1)$ are also constant. Therefore $\widetilde{\eta}$ is the unique lift of η starting in $\widetilde{\gamma}(0) = G(0,0) = G(1,0)$ and G is a homotopy with fixed endpoints from $\widetilde{\gamma}$ to $\widetilde{\eta}$. ☐

Corollary A.2.7. *If $q\colon X \to Y$ is a covering with $q(x_0) = y_0$, then the corresponding homomorphism*

$$\pi_1(q)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [q \circ \gamma]$$

is injective.

Proof. If γ, η are loops in x_0 with $[q \circ \gamma] = [q \circ \eta]$, then the Covering Homotopy Theorem A.2.6 implies that γ and η are homotopic. Therefore $[\gamma] = [\eta]$ shows that $\pi_1(q)$ is injective. □

Corollary A.2.8. *If Y is simply connected and X is arcwise connected, then each covering map $q\colon X \to Y$ is a homeomorphism.*

Proof. Since q is an open continuous map, it remains to show that q is injective. So pick $x_0 \in X$ and $y_0 \in Y$ with $q(x_0) = y_0$. If $x \in X$ also satisfies $q(x) = y_0$, then there exists a path $\alpha \in P(X, x_0, x)$ from x_0 to x. Now $q \circ \alpha$ is a loop in Y, hence contractible because Y is simply connected. Now the Covering Homotopy Theorem implies that the unique lift α of $q \circ \alpha$ starting in x_0 is a loop, and therefore that $x_0 = x$. This proves that q is injective. □

The following theorem provides a powerful tool, from which the preceding corollary easily follows. We recall that a topological space X is called *locally arcwise connected* if each neighborhood U of a point $x \in X$ contains an arcwise connected neighborhood.

Theorem A.2.9 (Lifting Theorem). *Assume that $q\colon X \to Y$ is a covering map with $q(x_0) = y_0$, that W is arcwise connected and locally arcwise connected, and that $f\colon W \to Y$ is a given map with $f(w_0) = y_0$. Then a continuous map $g\colon W \to X$ with*

$$g(w_0) = x_0 \quad \textit{and} \quad q \circ g = f \tag{A.1}$$

exists if and only if

$$\pi_1(f)(\pi_1(W, w_0)) \subseteq \pi_1(q)(\pi_1(X, x_0)), \ \textit{i.e.,}\ \operatorname{im}(\pi_1(f)) \subseteq \operatorname{im}(\pi_1(q)). \tag{A.2}$$

If g exists, then it is uniquely determined by (A.1). *Condition* (A.2) *is in particular satisfied if W is simply connected.*

Proof. If g exists, then $f = q \circ g$ implies that the image of the homomorphism $\pi_1(f) = \pi_1(q) \circ \pi_1(g)$ is contained in the image of $\pi_1(q)$.

Let us, conversely, assume that this condition is satisfied. To define g, let $w \in W$ and $\alpha_w\colon I \to W$ be a path from w_0 to w. Then $f \circ \alpha_w\colon I \to Y$ is a path which has a continuous lift $\beta_w\colon I \to X$ starting in x_0. We claim that $\beta_w(1)$ does not depend on the choice of the path α_w. Indeed, if α'_w is another path from w_0 to w, then $\alpha_w * \overline{\alpha'_w}$ is a loop in w_0, so that $(f \circ \alpha_w) * (f \circ \overline{\alpha'_w})$ is a loop in y_0. In view of (A.2), the homotopy class of this loop is contained in the image of $\pi_1(q)$, so that it has a lift $\eta\colon I \to X$ which is a loop in x_0. Since the reverse of the second half $\eta|_{[\frac{1}{2},1]}$ of η is a lift of $f \circ \alpha'_w$, starting in x_0, it is β'_w, and we obtain

$$\beta'_w(1) = \eta\Big(\frac{1}{2}\Big) = \beta_w(1).$$

We now put $g(w) := \beta_w(1)$, and it remains to see that g is continuous. This is where we shall use the assumption that W is locally arcwise connected. Let

$w \in W$ and put $y := f(w)$. Further, let $U \subseteq Y$ be an elementary neighborhood of y and V be an arcwise connected neighborhood of w in W such that $f(V) \subseteq U$. Fix a path α_w from w_0 to w as before. For any point $w' \in V$ we choose a path $\gamma_{w'}$ from w to w' in V, so that $\alpha_w * \gamma_{w'}$ is a path from w_0 to w'. Let $\widetilde{U} \subseteq X$ be an open subset of X for which $q|_{\widetilde{U}}$ is a homeomorphism onto U and $g(w) \in \widetilde{U}$. Then the uniqueness of lifts implies that

$$\beta_{w'} = \beta_w * \big((q|_{\widetilde{U}})^{-1} \circ (f \circ \gamma_{w'})\big).$$

We conclude that

$$g(w') = (q|_{\widetilde{U}})^{-1}(f(w')) \in \widetilde{U},$$

hence that $g|_V$ is continuous.

Finally, we show that g is unique. In fact, if $h\colon W \to X$ is another lift of f satisfying $h(w_0) = x_0$, then the set $S := \{w \in W \colon g(w) = h(w)\}$ is nonempty and closed. We claim that it is also open. In fact, let $w_1 \in S$ and U be a connected open elementary neighborhood of $f(w_1)$ and V an arcwise connected neighborhood of w_1 with $f(V) \subseteq U$. If $\widetilde{U} \subseteq q^{-1}(U)$ is the open subset on which q is a homeomorphism containing $g(w_1) = h(w_1)$, then, since V is arcwise connected, we have that $g(V), h(V) \subseteq \widetilde{U}$, whence $V \subseteq S$. Therefore S is open, closed and nonempty. Since W is connected this implies that $S = W$, i.e., $g = h$. □

Corollary A.2.10 (Uniqueness of Simply Connected Coverings). *Suppose that Y is locally arcwise connected. If $q_1 \colon X_1 \to Y$ and $q_2 \colon X_2 \to Y$ are two simply connected arcwise connected coverings, then there exists a homeomorphism $\varphi\colon X_1 \to X_2$ with $q_2 \circ \varphi = q_1$.*

Proof. Since Y is locally arcwise connected, both covering spaces X_1 and X_2 also have this property. Pick points $x_1 \in X_1$, $x_2 \in X_2$ with $y := q_1(x_1) = q_2(x_2)$. According to the Lifting Theorem A.2.9, there exists a unique lift $\varphi\colon X_1 \to X_2$ of q_1 with $\varphi(x_1) = x_2$. We likewise obtain a unique lift $\psi\colon X_2 \to X_1$ of q_2 with $\psi(x_2) = x_1$. Then $\varphi \circ \psi\colon X_1 \to X_1$ is a lift of id_Y fixing x_1, so that the uniqueness of lifts implies that $\varphi \circ \psi = \mathrm{id}_{X_1}$. The same argument yields $\psi \circ \varphi = \mathrm{id}_{X_2}$, so that φ is a homeomorphism with the required properties. □

Definition A.2.11. A topological space X is called *semilocally simply connected* if each point $x_0 \in X$ has a neighborhood U such that each loop $\alpha \in \Omega(U, x_0)$ is homotopic to $[x_0]$ in X, i.e., the natural homomorphism

$$\pi(i_U)\colon \pi_1(U, x_0) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [i_U \circ \gamma]$$

induced by the inclusion map $i_U \colon U \to X$ is trivial.

Theorem A.2.12 (Existence of simply connected coverings). *Let Y be arcwise connected and locally arcwise connected. Then Y has a simply connected covering space if and only if Y is semilocally simply connected.*

Proof. If $q\colon X \to Y$ is a simply connected covering space and $U \subseteq Y$ is a pathwise connected elementary open subset. Then each loop γ in U lifts to a loop $\widetilde{\gamma}$ in X, and since $\widetilde{\gamma}$ is homotopic to a constant map in X, the same holds for the loop $\gamma = q \circ \widetilde{\gamma}$ in Y.

Conversely, let us assume that Y is semilocally simply connected. We choose a base point $y_0 \in Y$ and let

$$\widetilde{Y} := P(Y, y_0)/\sim\, := \bigcup_{y_1 \in Y} P(Y, y_0, y_1)/\sim$$

be the set of homotopy classes of paths starting in y_0. We shall provide $\widetilde{Y}$ with a topology such that the map

$$q\colon \widetilde{Y} \to Y, \quad [\gamma] \mapsto \gamma(1)$$

defines a simply connected covering of Y.

Let $\mathcal{B}$ denote the set of all arcwise connected open subsets $U \subseteq Y$ for which each loop in U is contractible in Y and note that our assumptions on Y imply that $\mathcal{B}$ is a basis of the topology of Y, i.e., each open subset is a union of elements of $\mathcal{B}$. If $\gamma \in P(Y, y_0)$ satisfies $\gamma(1) \in U \in \mathcal{B}$, let

$$U_{[\gamma]} := \{[\eta] \in q^{-1}(U)\colon (\exists \beta \in C(I, U))\ \eta \sim \gamma * \beta\}.$$

We shall now verify several properties of these definitions, culminating in the proof of the theorem.

(1) $[\eta] \in U_{[\gamma]} \Rightarrow U_{[\eta]} = U_{[\gamma]}$.

To prove this, let $[\zeta] \in U_{[\eta]}$. Then $\zeta \sim \eta * \beta$ for some path β in U. Further $\eta \sim \gamma * \beta'$ for some path β' in U. Now $\zeta \sim \gamma * \beta' * \beta$, and $\beta' * \beta$ is a path in U, so that $[\zeta] \in U_{[\gamma]}$. This proves $U_{[\eta]} \subseteq U_{[\gamma]}$. We also have $\gamma \sim \eta * \overline{\beta'}$, so that $[\gamma] \in U_{[\eta]}$, and the first part implies that $U_{[\gamma]} \subseteq U_{[\eta]}$.

(2) q maps $U_{[\gamma]}$ injectively onto U.

That $q(U_{[\gamma]}) = U$ is clear since U and Y are arcwise connected. To show that it is one-to-one, let $[\eta], [\eta'] \in U_{[\gamma]}$, which we know from (1) is the same as $U_{[\eta]}$. Suppose $\eta(1) = \eta'(1)$. Since $[\eta'] \in U_{[\eta]}$, we have $\eta' \sim \eta * \alpha$ for some loop α in U. But then α is contractible in Y, so that $\eta' \sim \eta$, i.e., $[\eta'] = [\eta]$.

(3) $U, V \in \mathcal{B}$, $\gamma(1) \in U \subseteq V$, implies $U_{[\gamma]} \subseteq V_{[\gamma]}$.

This is trivial.

(4) The sets $U_{[\gamma]}$ for $U \in \mathcal{B}$ and $[\gamma] \in \widetilde{Y}$ form a basis of a topology on $\widetilde{Y}$.

Suppose $[\gamma] \in U_{[\eta]} \cap V_{[\eta']}$. Let $W \subseteq U \cap V$ be in $\mathcal{B}$ with $\gamma(1) \in W$. Then $[\gamma] \in W_{[\gamma]} \subseteq U_{[\gamma]} \cap V_{[\gamma]} = U_{[\eta]} \cap V_{[\eta']}$.

(5) q is open and continuous.

We have already seen in (2) that $q(U_{[\gamma]}) = U$, and these sets form a basis of the topology on $\widetilde{Y}$, resp., Y. Therefore q is an open map. We also have for $U \in \mathcal{B}$ the relation

$$q^{-1}(U) = \bigcup_{\gamma(1)\in U} U_{[\gamma]},$$

which is open. Hence q is continuous.

(6) $q|_{U_{[\gamma]}}$ is a homeomorphism.

This is because it is bijective, continuous and open.

At this point we have shown that $q\colon \widetilde{Y} \to Y$ is a covering map. It remains to see that $\widetilde{Y}$ is arcwise connected and simply connected.

(7) Let $H\colon I \times I \to Y$ be a continuous map with $H(t,0) = y_0$. Then $h_t(s) := H(t,s)$ defines a path in Y starting in y_0. Let $\widetilde{h}(t) := [h_t] \in \widetilde{Y}$. Then $\widetilde{h}$ is a path in $\widetilde{Y}$ covering the path $t \mapsto h_t(1) = H(t,1)$ in Y. We claim that $\widetilde{h}$ is continuous. Let $t_0 \in I$. We shall prove continuity at t_0. Let $U \in \mathcal{B}$ be a neighborhood of $h_{t_0}(1)$. Then there exists an interval $I_0 \subseteq I$ which is a neighborhood of t_0 with $h_t(1) \in U$ for $t \in I_0$. Then $\alpha(s) := H(t_0 + s(t-t_0), 1)$ is a continuous curve in U with $\alpha(0) = h_{t_0}(1)$ and $\alpha(1) = h_t(1)$, so that $h_{t_0} * \alpha$ is curve with the same endpoint as h_t. Applying Exercise A.2.1 to the restriction of H to the interval between t_0 and t, we see that $h_t \sim h_{t_0} * \alpha$, so that $\widetilde{h}(t) = [h_t] \in U_{[h_{t_0}]}$ for $t \in I_0$. Since $q|_{U_{[h_{t_0}]}}$ is a homeomorphism, $\widetilde{h}$ is continuous in t_0.

(8) $\widetilde{Y}$ is arcwise connected.

For $[\gamma] \in \widetilde{Y}$ put $h_t(s) := \gamma(st)$. By (7), this yields a path $\widetilde{\gamma}(t) = [h_t]$ in $\widetilde{Y}$ from $\widetilde{y}_0 := [y_0]$ (the class of the constant path) to the point $[\gamma]$.

(9) $\widetilde{Y}$ is simply connected.

Let $\widetilde{\alpha} \in \Omega(\widetilde{Y}, \widetilde{y}_0)$ be a loop in $\widetilde{Y}$ and $\alpha := q \circ \widetilde{\alpha}$ its image in Y. Let $h_t(s) := \alpha(st)$. Then we have the path $\widetilde{h}(t) = [h_t]$ in $\widetilde{Y}$ from (7). This path covers α since $h_t(1) = \alpha(t)$. Further, $\widetilde{h}(0) = \widetilde{y}_0$ is the constant path. Also, by definition, $\widetilde{h}(1) = [\alpha]$. From the uniqueness of lifts we derive that $\widetilde{h} = \widetilde{\alpha}$ is closed, so that $[\alpha] = [y_0]$. Therefore the homomorphism

$$\pi_1(q)\colon \pi_1(\widetilde{Y}, \widetilde{y}_0) \to \pi_1(Y, y_0)$$

vanishes. Since it is also injective (Corollary A.2.7), $\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is trivial, i.e., $\widetilde{Y}$ is simply connected. □

Definition A.2.13. Let $q\colon X \to Y$ be a covering. A homeomorphism $\varphi\colon X \to X$ is called a *deck transformation* of the covering if $q \circ \varphi = q$. This means that φ permutes the elements in the fibers of q. We write $\mathrm{Deck}(X, q)$ for the group of deck transformations.

Example A.2.14. For the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, the deck transformations have the form

$$\varphi(z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$

Proposition A.2.15. *Let* $q\colon \widetilde{Y} = (P(Y, y_0)/\sim) \to Y$ *be the simply connected covering of* Y *with base point* $\widetilde{y}_0 = [y_0]$. *For each* $[\gamma] \in \pi_1(Y, y_0)$ *we write* $\varphi_{[\gamma]} \in \mathrm{Deck}(\widetilde{Y}, q)$ *for the unique lift of* id_Y *mapping* $[y_0]$ *to the endpoint* $[\gamma] = \widetilde{\gamma}(1)$ *of the canonical lift* $\widetilde{\gamma}$ *of* γ *starting in* $\widetilde{y}_0$. *Then the map*

$$\Phi\colon \pi_1(Y, y_0) \to \mathrm{Deck}(\widetilde{Y}, q), \quad \Phi([\gamma]) = \varphi_{[\gamma]}$$

is an isomorphism of groups.

Proof. For $\gamma, \eta \in \Omega(Y, y_0)$, the composition $\varphi_{[\gamma]} \circ \varphi_{[\eta]}$ is a deck transformation mapping $\widetilde{y}_0$ to the endpoint of $\varphi_{[\gamma]} \circ \widetilde{\eta}$ which coincides with the endpoint of the lift of η starting in $\widetilde{\gamma}(1)$. Hence it also is the endpoint of the lift of the loop $\gamma * \eta$. Therefore Φ is a group homomorphism.

To see that Φ is injective, we note that $\varphi_{[\gamma]} = \mathrm{id}_{\widetilde{Y}}$ implies that $\widetilde{\gamma}(1) = \widetilde{y}_0$, so that $\widetilde{\gamma}$ is a loop, and hence that $[\gamma] = [y_0] = \widetilde{y}_0$.

For the surjectivity, let φ be a deck transformation and $y := \varphi(\widetilde{y}_0)$. If α is a path from $\widetilde{y}_0$ to y, then $\gamma := q \circ \alpha$ is a loop in y_0 with $\alpha = \widetilde{\gamma}$, so that $\varphi_{[\gamma]}(\widetilde{y}_0) = y$, and the uniqueness of lifts implies that $\varphi = \varphi_{[\gamma]}$. □

Exercises for Section A.2

Exercise A.2.1. Let $F\colon I^2 \to X$ be a continuous map with $F(0, s) = x_0$ for $s \in I$ and define

$$\gamma(t) := F(t, 0), \quad \eta(t) := F(t, 1), \quad \alpha(t) := F(1, t), \quad t \in I.$$

Show that $\gamma * \alpha \sim \eta$.

Exercise A.2.2. Let $q\colon G \to H$ be an morphism of topological groups with discrete kernel Γ. Show that:

(1) If $V \subseteq G$ is an open $\mathbf{1}$-neighborhood with $(V^{-1}V) \cap \Gamma = \{\mathbf{1}\}$ and q is open, then $q|_V\colon V \to q(V)$ is a homeomorphism.
(2) If q is open and surjective, then q is a covering.
(3) If q is open and H is connected, then q is surjective, hence a covering.

Exercise A.2.3. A map $f\colon X \to Y$ between topological spaces is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood U such that $f|_U\colon U \to f(U)$ is a homeomorphism onto an open subset of Y.

(1) Show that each covering map is a local homeomorphism.
(2) Find a surjective local homeomorphism which is not a covering. Can you also find an example where X is connected?

B

Some Multilinear Algebra

In this appendix we provide some tools from multilinear algebra. Some are needed in Chapter 6 on Lie algebras, where we construct the universal enveloping algebra and some other tools are needed for the discussion of differential forms on manifolds in Chapter 9. Section B.3 on Clifford algebras plays a crucial role in Chapter 16 on classical groups. Throughout, $\mathbb{K}$ is an arbitrary field of characteristic zero.

B.1 Tensor Products and Tensor Algebra

Let V and W be vector spaces. A *tensor product* of V and W is a pair $(V \otimes W, \otimes)$ of a vector space $V \otimes W$ and a bilinear map

$$\otimes \colon V \times W \to V \otimes W, \quad (v, w) \mapsto v \otimes w$$

with the following universal property. For each bilinear map $\beta \colon V \times W \to U$ into a vector space U, there exists a unique linear map $\widetilde{\beta} \colon V \otimes W \to U$ satisfying

$$\widetilde{\beta}(v \otimes w) = \beta(v, w) \quad \text{for} \quad v \in V, w \in W.$$

Taking $(U, \beta) = (V \otimes W, \otimes)$, we conclude immediately that $\mathrm{id}_{V \otimes W}$ is the unique linear endomorphism of $V \otimes W$ fixing all elements of the form $v \otimes w$.

Before we turn to the existence of tensor products, we discuss their uniqueness. In category theory, one gives a precise meaning to the statement that objects with a universal property are determined up to isomorphism. The following lemma makes this precise for tensor products.

Lemma B.1.1 (Uniqueness of tensor products). *If $(V \otimes W, \otimes)$ and $(V \widetilde{\otimes} W, \widetilde{\otimes})$ are two tensor products of the vector spaces V and W, then there exists a unique linear isomorphism*

$$f \colon V \otimes W \to V \widetilde{\otimes} W \quad \textit{with} \quad f(v \otimes w) = v \widetilde{\otimes} w \quad \textit{for} \quad v \in V, w \in W.$$

Proof. Since $\widetilde{\otimes}$ is bilinear, the universal property of $(V \otimes W, \otimes)$ implies the existence of a unique linear map

$$f : V \otimes W \to V \widetilde{\otimes} W \quad \text{with} \quad f(v \otimes w) = v \widetilde{\otimes} w \quad \text{for} \quad v \in V, w \in W.$$

Similarly, the universal property of $(V \widetilde{\otimes} W, \widetilde{\otimes})$ implies the existence of a linear map

$$g : V \widetilde{\otimes} W \to V \otimes W \quad \text{with} \quad g(v \widetilde{\otimes} w) = v \otimes w \quad \text{for} \quad v \in V, w \in W.$$

Then $g \circ f \in \mathrm{End}(V \otimes W)$ is a linear map with $(g \circ f)(v \otimes w) = v \otimes w$ for $v \in V$ and $w \in W$, so that the uniqueness part of the universal property of $(V \otimes W, \otimes)$ yields $g \circ f = \mathrm{id}_{V \otimes W}$. We likewise get $f \circ g = \mathrm{id}_{V \widetilde{\otimes} W}$, showing that f is a linear isomorphism. □

Now we turn to the existence of the tensor product.

Definition B.1.2. Let S be a set. We write $F(S) := \mathbb{K}^{(S)}$ for the *free vector space on* S. It is the subspace of the cartesian product $\mathbb{K}^S$, the set of all functions $f : S \to \mathbb{K}$ for which the set $\{s \in S : f(s) \neq 0\}$ is finite.

For $s \in S$, we define $\delta_s(t) := \delta_{st}$, which is 1 for $s = t$, and 0 otherwise. Then $(\delta_s)_{s \in S}$ is a basis for the vector space $F(S)$ and we have a map

$$\delta : S \to F(S), \quad s \mapsto \delta_s.$$

Now the pair $(F(S), \delta)$ has the universal property that, for each map $f : S \to V$ to a vector space V, there exists a unique linear map $\widetilde{f} : F(S) \to V$ with $\widetilde{f} \circ \delta = f$.

Proposition B.1.3 (Existence of tensor products). *If V and W are vector spaces, then there exists a tensor product $(V \otimes W, \otimes)$.*

Proof. In the free vector space $F(V \times W)$ over $V \times W$, we consider the subspace N, generated by elements of the form

$$\delta_{(v_1+v_2,w)} - \delta_{(v_1,w)} - \delta_{(v_2,w)}, \quad \delta_{(v,w_1+w_2)} - \delta_{(v,w_1)} - \delta_{(v,w_2)},$$

and

$$\delta_{(\lambda v,w)} - \delta_{(v,\lambda w)}, \quad \lambda \delta_{(v,w)} - \delta_{(\lambda v,w)},$$

for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{K}$. We put

$$V \otimes W := F(V \times W)/N \quad \text{and} \quad v \otimes w := \delta_{(v,w)} + N.$$

The bilinearity of $\otimes$ follows from the definition of N. In particular, we have

$$(v_1 + v_2) \otimes w = \delta_{(v_1+v_2,w)} + N = \delta_{(v_1,w)} + \delta_{(v_2,w)} + N = v_1 \otimes w + v_2 \otimes w$$

and

$$(\lambda v) \otimes w = \delta_{(\lambda v, w)} + N = \lambda \delta_{(v,w)} + N = \lambda(v \otimes w).$$

The linearity in the second argument is verified similarly.

To show that $(V \otimes W, \otimes)$ has the required universal property, let $\beta\colon V \times W \to U$ be a bilinear map. We use the universal property of $(F(V \times W), \delta)$ to obtain a linear map

$$\gamma\colon F(V \times W) \to U \quad \text{with} \quad \gamma(\delta_{(v,w)}) = \beta(v, w)$$

for $v \in V, w \in W$. The bilinearity of β now implies that $N \subseteq \ker \gamma$, so that γ factors through a unique linear map

$$\widetilde{\beta}\colon V \otimes W = F(V \times W)/N \to U \quad \text{with} \quad \widetilde{\beta}(v \otimes w) = \gamma(\delta_{(v,w)}) = \beta(v, w).$$

That $\widetilde{\beta}$ is uniquely determined by this property follows from the fact that the elements of the form $v \otimes w$ generate $V \otimes W$ linearly, which in turn follows from $\delta(V \times W)$ being a linear basis for $F(V \times W)$. □

Tensor products of finitely many factors are defined in a similar fashion as follows.

Definition B.1.4. Let $V_1, \ldots, V_k$ be vector spaces. A *tensor product* of $V_1, \ldots, V_k$ is a pair

$$(V_1 \otimes V_2 \otimes \cdots \otimes V_k, \otimes)$$

of a vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ and a k-linear map

$$\otimes\colon V_1 \times \cdots \times V_k \to V_1 \otimes V_2 \otimes \cdots \otimes V_k, \quad (v_1, \ldots, v_k) \mapsto v_1 \otimes \cdots \otimes v_k,$$

with the following universal property. For each k-linear map

$$\beta\colon V_1 \times \cdots \times V_k \to U$$

into a vector space U, there exists a unique linear map $\widetilde{\beta}\colon V_1 \otimes \cdots \otimes V_k \to U$ satisfying

$$\widetilde{\beta}(v_1 \otimes \cdots \otimes v_k) = \beta(v_1, \ldots, v_k) \quad \text{for} \quad v_i \in V_i.$$

For $(U, \beta) = (V_1 \otimes \cdots \otimes V_k, \otimes)$, we conclude immediately that $\mathrm{id}_{V_1 \otimes \cdots \otimes V_k}$ is the unique linear endomorphism of $V_1 \otimes \cdots \otimes V_k$ fixing all elements of the form $v_1 \otimes \cdots \otimes v_k$.

Again, the universal property determines k-fold tensor products.

Lemma B.1.5 (Uniqueness of k-fold tensor products). *If*

$$(V_1 \otimes \cdots \otimes V_k, \otimes) \quad \text{and} \quad (V_1 \widetilde{\otimes} \cdots \widetilde{\otimes} V_k, \widetilde{\otimes})$$

are two tensor products of the vector spaces $V_1, \ldots, V_k$, then there exists a unique linear isomorphism

$$f\colon V_1 \otimes \cdots \otimes V_k \to V_1 \widetilde{\otimes} \cdots \widetilde{\otimes} V_k \quad \text{with} \quad f(v_1 \otimes \cdots \otimes v_k) = v_1 \widetilde{\otimes} \cdots \widetilde{\otimes} v_k$$

for $v_i \in V_i$.

We omit the simple proof of the uniqueness. The existence is easily reduced to the two-fold case:

Lemma B.1.6. *If $V_1, \ldots, V_k$ are vector spaces and $k \geq 2$, then the iterated two-fold tensor product*

$$V_1 \otimes \cdots \otimes V_k := (V_1 \otimes \cdots \otimes V_{k-1}) \otimes V_k$$

and

$$v_1 \otimes \cdots \otimes v_k := (v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k$$

is a tensor product of $V_1, \ldots, V_k$.

Proof. Since we know already that this is true for $k = 2$, we argue by induction and assume that the assertion holds for $(k-1)$-fold iterated tensor products. In this way we immediately see that $(v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k$ is k-linear.

To verify the universal property, let $\beta\colon V_1 \times \cdots \times V_k \to U$ be a k-linear map. We first use the induction hypothesis to obtain for each $v_k \in V_k$ a unique linear map $\widetilde{\beta}_{v_k}\colon V_1 \otimes \cdots \otimes V_{k-1} \to U$ with

$$\widetilde{\beta}_{v_k}(v_1 \otimes \ldots \otimes v_{k-1}) = \beta(v_1, \ldots, v_{k-1}, v_k) \quad \text{for} \quad v_i \in V_i, i \leq k-1.$$

From the uniqueness of $\widetilde{\beta}_{v_k}$ we further derive that

$$\widetilde{\beta}_{\lambda v_k + \lambda' v_k'} = \lambda \widetilde{\beta}_{v_k} + \lambda' \widetilde{\beta}_{v_k'}$$

for $\lambda, \lambda' \in \mathbb{K}$ and $v_k, v_k' \in V_k$. Hence the map

$$(V_1 \otimes \cdots \otimes V_{k-1}) \times V_k \to U, \quad (x, v_k) \mapsto \widetilde{\beta}_{v_k}(x)$$

is bilinear. Now the universal property of the two-fold tensor product provides a unique linear map

$$\widetilde{\beta}\colon (V_1 \otimes \cdots \otimes V_{k-1}) \otimes V_k \to U$$

with $\widetilde{\beta}((v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k) = \widetilde{\beta}_{v_k}(v_1 \otimes \cdots \otimes v_{k-1}) = \beta(v_1, \ldots, v_{k-1}, v_k)$. □

Definition B.1.7 (The tensor algebra of a vector space). Let V be a $\mathbb{K}$-vector space and $V^{\otimes n}$ the n-fold tensor product of V with itself. For $n = 0, 1$, we put $V^{\otimes 0} := \mathbb{K}$ and $V^{\otimes 1} := V$.

We claim that, for $n, m \in \mathbb{N}$, there exists a bilinear map

$$\mu_{n,m}\colon V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)}$$

with

$$\mu_{n,m}\big((v_1 \otimes \ldots \otimes v_n), (v_{n+1} \otimes \ldots \otimes v_{n+m})\big) = v_1 \otimes \ldots \otimes v_{n+m}$$

for $v_1, \ldots, v_{n+m} \in V$. In fact, for each $x \in V^{\otimes n}$, the map

$$\mu_x : V^m \to V^{\otimes(n+m)}, \quad (w_1, \ldots, w_m) \mapsto x \otimes w_1 \otimes \ldots \otimes w_m$$

is m-linear, hence determines a linear map

$$\widetilde{\mu}_x : V^{\otimes m} \to V^{\otimes(n+m)} \quad \text{with} \quad \widetilde{\mu}_x(w_1 \otimes \cdots \otimes w_m) = \mu_x(w_1, \ldots, w_m).$$

Since μ_x is also linear in x, we obtain a unique bilinear map

$$\mu_{n,m} : V^{\otimes n} \times V^{\otimes m} \to V^{\otimes(n+m)}$$

with

$$\begin{aligned} &\mu_{n,m}\big((v_1 \otimes \ldots \otimes v_n), (v_{n+1} \otimes \ldots \otimes v_{n+m})\big) \\ &= \widetilde{\mu}_{(v_1 \otimes \ldots \otimes v_n)}(v_{n+1} \otimes \ldots \otimes v_{n+m}) = v_1 \otimes \ldots \otimes v_n \otimes v_{n+1} \otimes \ldots \otimes v_{n+m}. \end{aligned}$$

We further define bilinear maps

$$\mu_{0,n} : V^{\otimes 0} \times V^{\otimes n} = \mathbb{K} \times V^{\otimes n} \to V^{\otimes n}, \quad (\lambda, v) \mapsto \lambda v$$

and

$$\mu_{n,0} : V^{\otimes n} \otimes V^{\otimes 0} = V^{\otimes n} \times \mathbb{K} \to V^{\otimes n}, \quad (v, \lambda) \mapsto \lambda v.$$

Putting all maps $\mu_{n,k}$, $n, k \in \mathbb{N}_0$, together, we obtain a bilinear multiplication on the vector space

$$\mathcal{T}(V) := \oplus_{n=0}^{\infty} V^{\otimes n}.$$

It is now easy to show that this multiplication is associative and has an identity element $\mathbf{1} \in V^{\otimes 0}$ (Exercise B.1.5). The algebra obtained in this way is called the *tensor algebra of* V.

Lemma B.1.8 (Universal property of the tensor algebra). *Let V be a vector space and $\eta : V \to \mathcal{T}(V)$ the canonical embedding of V as $V^{\otimes 1}$. Then the pair $(\mathcal{T}(V), \eta)$ has the following property. For any linear map $f : V \to A$ into a unital associative $\mathbb{K}$-algebra A, there exists a unique homomorphism $\widetilde{f} : \mathcal{T}(V) \to A$ of unital associative algebras with $\widetilde{f} \circ \eta = f$.*

Proof. For the uniqueness of $\widetilde{f}$ we first note that the requirement of being a homomorphism of unital algebras determines $\widetilde{f}$ on $\mathbf{1}$ via $\widetilde{f}(\mathbf{1}) = \mathbf{1}_A$. On $\eta(V) = V^{\otimes 1}$ it is determined by $\widetilde{f} \circ \eta = f$, and on $\mathcal{T}(V)$ it is thus determined since the algebra $\mathcal{T}(V)$ is generated by the subspace $\mathbb{K}\mathbf{1} + V$.

For the existence of $\widetilde{f}$, we note that, for each $n \in \mathbb{N}$, the map

$$V^n \to A, \quad (v_1, \ldots, v_n) \mapsto f(v_1) \cdots f(v_n)$$

is n-linear, so that there exists a unique linear map

$$\widetilde{f}_n : V^{\otimes n} \to A \quad \text{with} \quad \widetilde{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n)$$

for $v_i \in V$. We now combine these linear maps $\widetilde{f}_n$ to a linear map

$$\widetilde{f}\colon \mathcal{T}(V) \to A \quad \text{with} \quad \widetilde{f}_n(\mathbf{1}) = \mathbf{1}_A, \quad \widetilde{f}|_{V^{\otimes n}} = \widetilde{f}_n.$$

Then the construction implies that $\widetilde{f} \circ \eta = f$. That $\widetilde{f}$ is an algebra homomorphism follows from

$$\begin{aligned}\widetilde{f}\big((v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m)\big) &= f(v_1) \cdots f(v_n) f(w_1) \cdots f(w_m) \\ &= \widetilde{f}(v_1 \otimes \cdots \otimes v_n) \widetilde{f}(w_1 \otimes \cdots \otimes w_k)\end{aligned}$$

for $v_1, \ldots, v_n, w_1, \ldots, w_k \in V$. □

Exercises for Section B.1

Exercise B.1.1. Let U, V and W be finite-dimensional vector spaces. Show that there are isomorphisms:

(i) $U \otimes V \cong V \otimes U$.
(ii) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$.

Exercise B.1.2. The aim of this exercise is to get a more concrete picture of the tensor product of two vector spaces in terms of bases. Let V and W be vector spaces. We consider a basis $B_V = \{e_i \colon i \in I\}$ for V and a basis $B_W = \{f_j \colon j \in J\}$ for W. Show that:

(i) Each function $f\colon B_V \times B_W \to \mathbb{K}$ has a unique bilinear extension $\widetilde{f}\colon V \times W \to \mathbb{K}$.
(ii) The set $B_V \otimes B_W = \{e_i \otimes f_j \colon i \in I, j \in J\}$ is a basis for $V \otimes W$.
(iii) Each element $x \in V \otimes W$ has a unique representation as a finite sum $x = \sum_{i \in I} e_i \otimes w_i$ with $w_i \in W$.
(iv) If V_1 and V_2 are vector spaces, then $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$.

Exercise B.1.3. Let $V := \mathbb{K}^n$ and $W := \mathbb{K}^m$. Show that one can turn the space $M_{n,m}(\mathbb{K})$ of $(n \times m)$-matrices with entries in $\mathbb{K}$ into a tensor product $(\mathbb{K}^n \otimes \mathbb{K}^m, \otimes)$ satisfying

$$e_i \otimes e_j := E_{ij},$$

where $e_1, \ldots, e_n$ denotes the canonical basis vectors in $\mathbb{K}^n$ and E_{ij} is the matrix which has a single nonzero entry in the i-th row and the j-th column.

Exercise B.1.4. If V and W are finite-dimensional, then the map

$$\Phi\colon V^* \otimes W \to \operatorname{Hom}(V, W), \qquad \Phi(\alpha \otimes w)(v) := \alpha(v)w$$

is a linear isomorphism.

Exercise B.1.5. Let V be a vector space and $\mathcal{T}(V) = \bigoplus_{n \in \mathbb{N}_0} V^{\otimes n}$. Show that the multiplication on $\mathcal{T}(V)$ defined by Definition B.1.7 yields an associative $\mathbb{K}$-algebra.

Exercise B.1.6. Let V_i and W_i be $\mathbb{K}$-vector spaces (for $i = 1, 2$) and $A \in \mathrm{Hom}_{\mathbb{K}}(V_1, V_2)$, $B \in \mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$. Show that there exists a unique $\mathbb{K}$-linear map $C\colon V_1 \otimes V_2 \to W_1 \otimes W_2$ such that

$$C(v_1 \otimes v_2) = A(v_1) \otimes B(v_2)$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. The map C is usually denoted by $A \otimes B$.

Exercise B.1.7. Suppose that $V_1, \ldots, V_k$ are vector spaces and that a group G acts linearly on each of them. Show that

$$g \cdot (v_1 \otimes \ldots \otimes v_k) := g \cdot v_1 \otimes \ldots \otimes g \cdot v_k$$

for $g \in G$ and $v_j \in V_j$ defines a linear action on $V_1 \otimes \ldots \otimes V_2$.

B.2 Symmetric and Exterior Products

B.2.1 Symmetric and Exterior Powers

Definition B.2.1. Let V be a vector space and $n \geq 2$. We define

$$S^n(V) := V^{\otimes n}/U,$$

where U is the subspace spanned by all elements of the form

$$v_1 \otimes \ldots \otimes v_n - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}, \quad \sigma \in S_n.$$

The space $S^n(V)$ is called the *n-th symmetric power of* V. We put

$$v_1 \vee \cdots \vee v_n := v_1 \otimes \cdots \otimes v_n + U$$

and observe that this product is symmetric in the sense that

$$v_1 \vee \cdots \vee v_n = v_{\sigma(1)} \vee \cdots \vee v_{\sigma(n)}$$

for each $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. For $n = 1$, we put $S^1(V) := V$ and also $S^0(V) := \mathbb{K}$.

If X and Y are sets, then a map $f\colon X^n \to Y$ is said to be *symmetric* if, for each permutation $\sigma \in S_n$, we have

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \quad \text{for} \quad x \in X^n.$$

Lemma B.2.2 (Universal property of symmetric powers). *Let V and X be vector spaces and $f\colon V^n \to X$ be a symmetric n-linear map. Then there exists a unique linear map $\widetilde{f}\colon S^n(V) \to X$ with*

$$\widetilde{f}(v_1 \vee \cdots \vee v_n) = f(v_1, \ldots, v_n) \quad \textit{for} \quad v_1, \ldots, v_n \in V.$$

Proof. From the universal property of the n-fold tensor product $V^{\otimes n}$, we obtain a unique linear map $f_0 \colon V^{\otimes n} \to X$ with

$$f_0(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n) \quad \text{for} \quad v_1, \ldots, v_n \in V.$$

In view of the symmetry of f, the linear map f_0 vanishes on U, hence factors through a linear map $\widetilde{f} \colon S^n(V) \to X$ with the desired property. □

Definition B.2.3. Let V and W be $\mathbb{K}$-vector spaces, $n \in \mathbb{N}$, and

$$\mathrm{sgn} \colon S_n \to \{1, -1\}$$

be the *signature homomorphism* mapping all transpositions to -1. An n-linear map $f \colon V^n \to W$ is called *alternating* if

$$f(v_1, \ldots, v_n) = \mathrm{sgn}(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

holds for all $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$.

We write $\mathrm{Alt}^n(V, W)$ for the set of alternating n-linear maps $V^n \to W$. Clearly, sums and scalar multiples of alternating maps are alternating, so that $\mathrm{Alt}^n(V, W)$ carries a natural vector space structure. For $n = 0$, we shall follow the convention that $\mathrm{Alt}^0(V, W) := W$ is the set of constant maps, which are considered to be 0-linear.

Example B.2.4. From linear algebra, we know the n-linear map

$$(\mathbb{K}^n)^n \to \mathbb{K}, \quad \det(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_{1,\sigma(1)} \cdots v_{k,\sigma(k)}.$$

Here we identify the space $M_n(\mathbb{K})$ of $(n \times n)$-matrices with entries in $\mathbb{K}$ with the space $(\mathbb{K}^n)^n$ of n-tuples of (column) vectors ([La93, Sect. XIII.4]).

Definition B.2.5. Let V be a vector space and $n \geq 2$. We define

$$\Lambda^n(V) := V^{\otimes n}/W,$$

where W is the subspace spanned by the elements of the form

$$v_1 \otimes \cdots \otimes v_n - \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad \sigma \in S_n.$$

The space $\Lambda^n(V)$ is called the *n-th exterior power of* V. We put

$$v_1 \wedge \cdots \wedge v_n := v_1 \otimes \cdots \otimes v_n + W$$

and note that this product is alternating, i.e.,

$$v_1 \wedge \cdots \wedge v_n = \mathrm{sgn}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$$

for all $\sigma \in S_n$ and $(v_1, \ldots, v_n) \in V^n$. For $n = 2$, this means that

$$v_1 \wedge v_2 = -v_2 \wedge v_1.$$

We also put $\Lambda^1(V) := V$ and $\Lambda^0(V) := \mathbb{K}$.

Lemma B.2.6 (Universal property of the exterior power). *Let V and X be vector spaces and $f \in \mathrm{Alt}^n(V, X)$. Then there exists a unique linear map $\widetilde{f}\colon \Lambda^n(V) \to X$ with*

$$\widetilde{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \ldots, v_n) \quad \textit{for} \quad v_1, \ldots, v_n \in V.$$

We thus obtain a linear bijection

$$\mathrm{Alt}^n(V, X) \to \mathrm{Hom}(\Lambda^n(V), X), \quad f \mapsto \widetilde{f}.$$

Proof. The proof is completely analogous to the symmetric case. □

B.2.2 Symmetric and Exterior Algebra

Definition B.2.7. Let V be a vector space and $(\mathcal{T}(V), \eta)$ the tensor algebra of V (cf. Lemma B.1.8). We define the *symmetric algebra* $S(V)$ over V as the quotient $\mathcal{T}(V)/I_s$, where I_s is the ideal generated by the elements $\eta(v) \otimes \eta(w) - \eta(w) \otimes \eta(v)$. We write

$$\eta_s\colon V \to S(V), \quad v \mapsto \eta(v) + I_s$$

for the canonical map induced by η. The product in $S(V)$ is denoted by $\vee$.

Likewise, we define the *exterior algebra* $\Lambda(V)$ over V as the quotient $\mathcal{T}(V)/I_a$, where I_a is the ideal generated by the elements

$$\eta(v) \otimes \eta(w) + \eta(w) \otimes \eta(v), \quad v, w \in V.$$

We write

$$\eta_a\colon V \to \Lambda(V), \quad v \mapsto \eta(v) + I_a$$

for the canonical map induced by η. The product in $\Lambda(V)$ is denoted by $\wedge$.

Lemma B.2.8 (Universal property of the symmetric algebra). *Let V be a vector space and $(S(V), \eta_s)$ its symmetric algebra. Then $S(V)$ is a commutative unital algebra and for any linear map $f\colon V \to A$ into a unital commutative associative algebra A, there exists a unique homomorphism $\widetilde{f}\colon S(V) \to A$ of unital associative algebras with $\widetilde{f} \circ \eta_s = f$.*

Proof. Using the universal property of the tensor algebra $\mathcal{T}(V)$, we see that there exists a unique unital algebra homomorphism $\widehat{f}\colon \mathcal{T}(V) \to A$ with $\widehat{f} \circ \eta = f$. Since A is commutative, for any $v, w \in V$, the element $\eta(v) \otimes \eta(w) - \eta(w) \otimes \eta(v)$ is contained in $\ker \widehat{f}$, and therefore $I_s \subseteq \ker \widehat{f}$ shows that $\widehat{f}$ factors through an algebra homomorphism $\widetilde{f}\colon S(V) \to A$ with $\widetilde{f} \circ \eta_s = f$. The uniqueness of $\widetilde{f}$ follows from the fact that $\mathcal{T}(V)$ is generated, as a unital algebra, by $\eta(V)$, so that $S(V)$ is generated by the image of η_s. Since the generators $\eta_s(v)$, $v \in V$, of $S(V)$ commute, the algebra $S(V)$ is commutative. □

Remark B.2.9. (a) The structure of the symmetric algebra can be made more concrete as follows. Let $\mathcal{T}(V)_k := V^{\otimes k}$ and $U_2 \subseteq \mathcal{T}(V)_2$ the subspace spanned by the commutators $[\eta(v), \eta(w)], v, w \in V$. Then the ideal I_s is of the form

$$I_s = \mathcal{T}(V)U_2\mathcal{T}(V) = \sum_{p,q\in\mathbb{N}_0} \mathcal{T}(V)_p \otimes U_2 \otimes \mathcal{T}(V)_q = \bigoplus_{n=2}^{\infty} I_{s,n},$$

where $I_{s,n} := \sum_{p+q=n-2} \mathcal{T}(V)_p \otimes U_2 \otimes \mathcal{T}(V)_q$. This implies that the symmetric algebra $S(V)$ is a direct sum

$$S(V) = \bigoplus_{n=0}^{\infty} S(V)_n, \quad \text{where} \quad S(V)_n := \mathcal{T}(V)_n / I_{s,n}.$$

Let

$$\mu_n \colon V^n \to S(V)_n, \quad (v_1, \ldots, v_n) \mapsto \eta_s(v_1) \vee \cdots \vee \eta_s(v_n)$$

denote the n-fold multiplication map. Since $S(V)$ is commutative, this map is symmetric, hence induces a linear map

$$\widetilde{\mu}_n \colon S^n(V) \to S(V)_n,$$

determined by

$$\widetilde{\mu}_n(v_1 \vee \cdots \vee v_n) = \eta_s(v_1) \vee \cdots \vee \eta_s(v_n).$$

On the other hand, it is clear that the subspace $I_{s,n}$ of $V^{\otimes n}$ is contained in the kernel of the quotient map $V^{\otimes n} \to S^n(V)$, so that there exists a linear map $f_n \colon S(V)_n \to S^n(V)$, with

$$f_n(\eta_s(v_1) \vee \cdots \vee \eta_s(v_n)) = v_1 \vee \cdots \vee v_n.$$

Then $f_n \circ \widetilde{\mu}_n = \mathrm{id}_{S^n(V)}$ and, similarly, $\widetilde{\mu}_n \circ f_n = \mathrm{id}_{S(V)_n}$. This proves that $\widetilde{\mu}_n$ is a linear isomorphism. In the following we therefore identify $S^n(V)$ with the subspace $S(V)_n$ of the symmetric algebra and write $\eta_s(v)$ simply as v.

Note that $S^n(V) \vee S^m(V) \subseteq S^{n+m}(V)$, so that the direct sum

$$S(V) = \bigoplus_{n\in\mathbb{N}} S^n(V)$$

defines the structure of a *graded algebra* on $S(V)$ with $S^0(V) = \mathbb{K}\mathbf{1}$ containing the identity element.

(b) A similar argument applies to the exterior algebra and shows that the ideal I_a has the form $I_a = \bigoplus_{n=2}^{\infty}(I_a \cap V^{\otimes n})$, so that

$$\Lambda(V) = \bigoplus_{n=0}^{\infty} \Lambda(V)_n, \quad \text{where} \quad \Lambda(V)_n := \mathcal{T}(V)_n / I_{a,n}.$$

Let $\mu_n : V^n \to \Lambda(V)_n, (v_1, \ldots, v_n) \mapsto \eta_a(v_1) \wedge \cdots \wedge \eta_a(v_n)$ denote the n-fold multiplication map. Then the relation $\eta_a(v_i)\eta_a(v_j) + \eta_a(v_j)\eta_a(v_i) = 0$ and the fact that S_n is generated by transpositions imply that μ_n is alternating. Hence it induces a linear map $\widetilde{\mu}_n : \Lambda^n(V) \to \Lambda(V)_n$, determined by

$$\widetilde{\mu}_n(v_1 \wedge \cdots \wedge v_n) = \eta_a(v_1) \wedge \cdots \wedge \eta_a(v_n).$$

On the other hand, it is clear that the subspace $I_{a,n}$ of $V^{\otimes n}$ is contained in the kernel of the quotient map $V^{\otimes n} \to \Lambda^n(V)$, so that there is a linear map $f_n : \Lambda(V)_n \to \Lambda^n(V)$ with

$$f_n(\eta_a(v_1) \wedge \cdots \wedge \eta_a(v_n)) = v_1 \wedge \cdots \wedge v_n.$$

As in the symmetric case, we now see that $\widetilde{\mu}_n$ is a linear isomorphism. In the following we therefore identify $\Lambda^n(V)$ with the subspace $\Lambda(V)_n$ of the symmetric algebra and write $\eta_a(v)$ simply as v.

Each subspace $\Lambda^n(V)$ is spanned by elements of the form $v_1 \wedge \cdots \wedge v_n$, and this implies that for $\alpha \in \Lambda^n(V)$ and $\beta \in \Lambda^m(V)$ we have

$$\alpha \wedge \beta = (-1)^{mn} \beta \wedge \alpha. \tag{B.1}$$

In this sense the graded algebra $\Lambda(V)$ is *graded commutative*. The even part of this algebra is the subspace

$$\Lambda^{\text{even}}(V) := \bigoplus_{k=0}^{\infty} \Lambda^{2k}(V)$$

which is a central subalgebra, and the odd part is

$$\Lambda^{\text{odd}}(V) := \bigoplus_{k=0}^{\infty} \Lambda^{2k+1}(V).$$

For two elements α, β of this subspace we have $\alpha \wedge \beta = -\beta \wedge \alpha$.

Lemma B.2.10 (Universal property of the exterior algebra). *Let V be a vector space and $(\Lambda(V), \eta_a)$ be its exterior algebra. Then $\Lambda(V)$ is a graded commutative unital algebra and for any linear map $f : V \to A$ into a unital associative algebra A, satisfying*

$$f(v)f(w) = -f(w)f(v) \quad \text{for} \quad v, w \in V,$$

there exists a unique homomorphism $\widetilde{f} : \Lambda(V) \to A$ of unital associative algebras with $\widetilde{f} \circ \eta_a = f$.

Proof. Using the universal property of the tensor algebra $\mathcal{T}(V)$, we see that there exists a unique unital algebra homomorphism $\widehat{f} : \mathcal{T}(V) \to A$ with $\widehat{f} \circ \eta = f$. Then we have for $v, w \in V$

$$\widehat{f}\big(\eta(v)\otimes\eta(w)+\eta(w)\otimes\eta(v)\big)=f(v)f(w)+f(w)f(v)=0.$$

Therefore $I_a \subseteq \ker \widehat{f}$ shows that $\widehat{f}$ factors through a unital algebra homomorphism $\widetilde{f}\colon \Lambda(V)\to A$ with $\widetilde{f}\circ\eta_a = f$. The uniqueness of $\widetilde{f}$ follows from the fact that $\mathcal{T}(V)$ is generated, as a unital algebra, by $\eta(V)$, so that $\Lambda(V)$ is generated by the image of η_a. □

B.2.3 Exterior Algebra and Alternating Maps

Below we shall see how general alternating maps can be expressed in terms of determinants.

Proposition B.2.11. *For any $\omega \in \mathrm{Alt}^k(V,W)$ we have:*

(i) *For $b_1,\ldots,b_k \in V$ and linear combinations $v_j = \sum_{i=1}^k a_{ij}b_i$, we have*

$$\omega(v_1,\ldots,v_k)=\det(A)\omega(b_1,\ldots,b_k), \qquad \textit{and} \qquad A:=(a_{ij})\in M_k(\mathbb{K}).$$

(ii) $\omega(v_1,\ldots,v_k)=0$ *if $v_1,\ldots,v_k$ are linearly dependent.*
(iii) *For $b_1,\ldots,b_n \in V$ and linear combinations $v_j=\sum_{i=1}^n a_{ij}b_i$ we have*

$$\omega(v_1,\ldots,v_k)=\sum_I \det(A_I)\omega(b_{i_1},\ldots,b_{i_k}),$$

where $A := (a_{ij}) \in M_{n,k}(\mathbb{K})$, $I=\{i_1,\ldots,i_k\}$ is a k-element subset of $\{1,\ldots,n\}$, $1\le i_1<\ldots<i_k\le n$, and $A_I := (a_{ij})_{i\in I, j=1,\ldots,k}\in M_k(\mathbb{K})$.

Proof. (i) For the following calculation we note that if $\sigma\colon\{1,\ldots,k\}\to\{1,\ldots,k\}$ is a map which is not bijective, then the alternating property implies that $\omega(v_{\sigma(1)},\ldots,v_{\sigma(k)})=0$. We therefore get

$$\begin{aligned}
\omega(v_1,\ldots,v_k) &= \omega\Big(\sum_{i=1}^k a_{i1}b_i,\ldots,\sum_{i=1}^k a_{ik}b_i\Big)\\
&= \sum_{i_1,\ldots,i_k=1}^k a_{i_1 1}\cdots a_{i_k k}\cdot\omega(b_{i_1},\ldots,b_{i_k})\\
&= \sum_{\sigma\in S_k} a_{\sigma(1)1}\cdots a_{\sigma(k)k}\cdot\omega(b_{\sigma(1)},\ldots,b_{\sigma(k)})\\
&= \sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)a_{\sigma(1)1}\cdots a_{\sigma(k)k}\cdot\omega(b_1,\ldots,b_k)=\det(A)\cdot\omega(b_1,\ldots,b_k).
\end{aligned}$$

(ii) follows immediately from (i) because the linear dependence of $v_1,\ldots,v_k$ implies that $\det A = 0$.
(iii) First we expand

$$\omega(v_1, \ldots, v_k) = \omega\Big(\sum_{i=1}^n a_{i1} b_i, \ldots, \sum_{i=1}^n a_{ik} b_i\Big)$$
$$= \sum_{i_1, \ldots, i_k = 1}^n a_{i_1 1} \cdots a_{i_k k} \cdot \omega(b_{i_1}, \ldots, b_{i_k}).$$

If $|\{i_1, \ldots, i_k\}| < k$, then the alternating property implies that $\omega(b_{i_1}, \ldots, b_{i_k}) = 0$ because two entries coincide. If $|\{i_1, \ldots, i_k\}| = k$, there exists a permutation $\sigma \in S_k$ with $i_{\sigma(1)} < \ldots < i_{\sigma(k)}$. We therefore get

$$\omega(v_1, \ldots, v_k) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{\sigma \in S_k} a_{i_{\sigma(1)} 1} \cdots a_{i_{\sigma(k)} k} \cdot \omega(b_{i_{\sigma(1)}}, \ldots, b_{i_{\sigma(k)}})$$
$$= \sum_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) a_{i_{\sigma(1)} 1} \cdots a_{i_{\sigma(k)} k} \cdot \omega(b_{i_1}, \ldots, b_{i_k})$$
$$= \sum_I \det(A_I) \omega(b_{i_1}, \ldots, b_{i_k}),$$

where the sum is to be extended over all k-element subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, where $i_1 < \ldots < i_k$. □

Corollary B.2.12. (i) *If* $\dim V < k$*, then* $\mathrm{Alt}^k(V, W) = \{0\}$.
(ii) *Let* $\dim V = n$ *and* $b_1, \ldots, b_n$ *be a basis for* V*. Then the map*

$$\Phi\colon \mathrm{Alt}^k(V, W) \to W^{\binom{n}{k}}, \quad \Phi(\omega) = (\omega(b_{i_1}, \ldots, b_{i_k}))_{i_1 < \ldots < i_k}$$

is a linear isomorphism. We obtain in particular $\dim\big(\mathrm{Alt}^k(V, \mathbb{K})\big) = \binom{n}{k}$.
(iii) *If* $\dim V = k$ *and* $b_1, \ldots, b_k$ *is a basis for* V*, then the map*

$$\Phi\colon \mathrm{Alt}^k(V, W) \to W, \quad \Phi(\omega) = \omega(b_1, \ldots, b_k)$$

is a linear isomorphism.

Proof. (i) In Proposition B.2.11(i), we may choose $b_k = 0$.

(ii) First we show that Φ is injective. So let $\omega \in \mathrm{Alt}^k(V, W)$ with $\Phi(\omega) = 0$. We now write any k elements $v_1, \ldots, v_k \in V$ with respect to the basis elements as $v_j = \sum_{i=1}^n a_{ij} b_i$ and obtain with Proposition B.2.11:

$$\omega(v_1, \ldots, v_k) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \det(A_I) \omega(b_{i_1}, \ldots, b_{i_k}) = 0.$$

To see that Φ is surjective, we pick for each k-element subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $1 \leq i_1 < \ldots < i_k \leq n$ an element $w_I \in W$. Then the tuple (w_I) is a typical element of $W^{\binom{n}{k}}$.

Expressing k elements $v_1, \ldots, v_k$ in terms of the basis elements $b_1, \ldots, b_n$ via $v_j = \sum_{i=1}^n a_{ij} b_i$, we obtain an $(n \times k)$-matrix A. We now define an alternating k-linear map $\omega \in \mathrm{Alt}^k(V, W)$ by

$$\omega(v_1, \ldots, v_k) := \sum_I \det(A_I) w_I.$$

The k-linearity of ω follows directly from the k-linearity of the maps

$$(v_1, \ldots, v_k) \mapsto \det(A_I).$$

For $i_1 < \ldots < i_k$ we further have $\omega(b_{i_1}, \ldots, b_{i_k}) = w_I$ because in this case $A_I \in M_k(\mathbb{K})$ is the identity matrix and all other matrices $A_{I'}$ have some vanishing columns. This implies that $\Phi(\omega) = (w_I)$, and hence that Φ is surjective.

(iii) is a special case of (ii). □

Definition B.2.13 (Alternator). Let V and W be vector spaces. For a k-linear map $\omega\colon V^k \to W$, we define a new k-linear map by

$$\mathrm{Alt}(\omega)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Writing

$$\omega^\sigma(v_1, \ldots, v_k) := \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

we then have

$$\mathrm{Alt}(\omega) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\omega^\sigma.$$

The map $\mathrm{Alt}(\cdot)$ is called the *alternator*. We claim that it turns any k-linear map into an alternating k-linear map. To see this, we first note that for $\sigma, \pi \in S_k$, we have

$$\begin{aligned}(\omega^\sigma)^\pi(v_1, \ldots, v_k) &= (\omega^\sigma)(v_{\pi(1)}, \ldots, v_{\pi(k)}) \\ &= \omega(v_{\pi\sigma(1)}, \ldots, v_{\pi\sigma(k)}) = \omega^{\pi\sigma}(v_1, \ldots, v_k).\end{aligned}$$

This implies that

$$\begin{aligned}\mathrm{Alt}(\omega)^\pi &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)(\omega^\sigma)^\pi = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\omega^{\pi\sigma} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\pi^{-1}\sigma)\omega^\sigma \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma)\omega^\sigma = \mathrm{sgn}(\pi)\,\mathrm{Alt}(\omega) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma\pi^{-1})\omega^\sigma \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\omega^{\sigma\pi} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)(\omega^\pi)^\sigma = \mathrm{Alt}(\omega^\pi).\end{aligned}$$

In particular, we see that $\mathrm{Alt}(\omega)$ is alternating.

Remark B.2.14. (a) We observe that if ω is alternating, then $\omega^\sigma = \mathrm{sgn}(\sigma)\omega$ for each permutation σ, and therefore

$$\mathrm{Alt}(\omega) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma)\omega = \frac{1}{k!} \sum_{\sigma \in S_k} \omega = \omega.$$

(b) For $k = 2$ we have $\mathrm{Alt}(\omega)(v_1, v_2) = \frac{1}{2}(\omega(v_1, v_2) - \omega(v_2, v_1))$, and for $k = 3$:

$$\begin{aligned}\mathrm{Alt}(\omega)(v_1, v_2, v_3) = \frac{1}{6}\Big(&\omega(v_1, v_2, v_3) - \omega(v_2, v_1, v_3) + \omega(v_2, v_3, v_1) \\ &- \omega(v_3, v_2, v_1) + \omega(v_3, v_1, v_2) - \omega(v_1, v_3, v_2)\Big).\end{aligned}$$

Definition B.2.15. Let $p, q \in \mathbb{N}_0$. For two multilinear maps

$$\omega_1 \colon V_1 \times \ldots \times V_p \to \mathbb{K} \quad \text{and} \quad \omega_2 \colon V_{p+1} \times \ldots \times V_{p+q} \to \mathbb{K}$$

we define the *tensor product* $\omega_1 \otimes \omega_2 \colon V_1 \times \cdots \times V_{p+q} \to \mathbb{K}$ by

$$(\omega_1 \otimes \omega_2)(v_1, \ldots, v_{p+q}) := \omega_1(v_1, \ldots, v_p)\omega_2(v_{p+1}, \ldots, v_{p+q}).$$

It is clear that $\omega_1 \otimes \omega_2$ is a $(p + q)$-linear map.

For $\lambda \in \mathbb{K}$ (the set of 0-linear maps), and a p-linear map ω as above, we obtain in particular

$$\lambda \otimes \omega := \omega \otimes \lambda := \lambda\omega.$$

For two alternating maps $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$ and $\beta \in \mathrm{Alt}^q(V, \mathbb{K})$ we define their *exterior product*:

$$\alpha \wedge \beta := \frac{(p + q)!}{p!q!} \mathrm{Alt}(\alpha \otimes \beta) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)(\alpha \otimes \beta)^\sigma. \tag{B.2}$$

It follows from (B.2) that $\alpha \wedge \beta$ is alternating, so that we obtain a bilinear map

$$\wedge \colon \mathrm{Alt}^p(V, \mathbb{K}) \times \mathrm{Alt}^q(V, \mathbb{K}) \to \mathrm{Alt}^{p+q}(V, \mathbb{K}), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta.$$

On the direct sum

$$\mathrm{Alt}(V, \mathbb{K}) := \bigoplus_{p \in \mathbb{N}_0} \mathrm{Alt}^p(V, \mathbb{K})$$

we now obtain a bilinear product by putting

$$\Big(\sum_p \alpha_p\Big) \wedge \Big(\sum_q \beta_q\Big) := \sum_{p,q} \alpha_p \wedge \beta_q.$$

As before, we identify $\mathrm{Alt}^0(V, \mathbb{K})$ with $\mathbb{K}$ and obtain

$$\lambda\alpha = \lambda \wedge \alpha = \alpha \wedge \lambda$$

for $\lambda \in \mathrm{Alt}^0(V, \mathbb{K}) = \mathbb{K}$ and $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$.

We take a closer look at the structure of the algebra $(\mathrm{Alt}(V, \mathbb{K}), \wedge)$.

Lemma B.2.16. *For* $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$, $\beta \in \mathrm{Alt}^q(V, \mathbb{K})$ *and* $\gamma \in \mathrm{Alt}^r(V, \mathbb{K})$, *we have*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

In particular the algebra $(\mathrm{Alt}(V, \mathbb{K}), \wedge)$ *is associative.*

Proof. First we recall from Definition B.2.13 that for any n-linear map $\omega \colon V^n \to W$ and $\pi \in S_n$ we have

$$\mathrm{Alt}(\omega^\pi) = \mathrm{sgn}(\pi)\,\mathrm{Alt}(\omega). \tag{B.3}$$

We identify S_{p+q} in the natural way with the subgroup of S_{p+q+r} fixing the numbers $p+q+1, \ldots, p+q+r$. We thus obtain

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \frac{(p+q+r)!}{(p+q)!r!}\,\mathrm{Alt}((\alpha \wedge \beta) \otimes \gamma) \\
&= \frac{(p+q+r)!}{p!q!(p+q)!r!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\,\mathrm{Alt}((\alpha \otimes \beta)^\sigma \otimes \gamma) \\
&= \frac{(p+q+r)!}{p!q!(p+q)!r!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\,\mathrm{Alt}((\alpha \otimes \beta \otimes \gamma)^\sigma) \\
&\overset{(B.3)}{=} \frac{(p+q+r)!}{p!q!(p+q)!r!} \sum_{\sigma \in S_{p+q}} \mathrm{Alt}(\alpha \otimes \beta \otimes \gamma) \\
&= \frac{(p+q+r)!}{p!q!r!}\,\mathrm{Alt}(\alpha \otimes \beta \otimes \gamma) = \frac{(p+q+r)!}{p!q!r!}\,\mathrm{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
&= \ldots = \frac{(p+q+r)!}{p!(q+r)!}\,\mathrm{Alt}(\alpha \otimes (\beta \wedge \gamma)) = \alpha \wedge (\beta \wedge \gamma).
\end{aligned}$$

□

From the associativity asserted in the preceding lemma, it follows that the multiplication in $\mathrm{Alt}(V, \mathbb{K})$ is associative. We may therefore suppress brackets and define

$$\omega_1 \wedge \ldots \wedge \omega_n := (\ldots((\omega_1 \wedge \omega_2) \wedge \omega_3) \cdots \wedge \omega_n).$$

Remark B.2.17. (a) From the calculation in the preceding proof we know that for three elements $\alpha_i \in \mathrm{Alt}^{p_i}(V, \mathbb{K})$, the triple product in the associative algebra $\mathrm{Alt}(V, \mathbb{K})$ satisfies

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(p_1+p_2+p_3)!}{p_1!p_2!p_3!}\,\mathrm{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3).$$

Inductively this leads for n elements $\alpha_i \in \mathrm{Alt}^{p_i}(V, \mathbb{K})$ to

$$\alpha_1 \wedge \ldots \wedge \alpha_n = \frac{(p_1+\ldots+p_n)!}{p_1!\cdots p_n!}\,\mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

(Exercise B.2.2).

(b) For $\alpha_i \in \mathrm{Alt}^1(V,\mathbb{K}) \cong V^*$, we in particular obtain

$$\begin{aligned}(\alpha_1 \wedge \ldots \wedge \alpha_n)(v_1,\ldots,v_n) &= n!\,\mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n)(v_1,\ldots,v_n) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\alpha_1(v_{\sigma(1)})\cdots\alpha_n(v_{\sigma(n)}) = \det(\alpha_i(v_j)).\end{aligned}$$

Proposition B.2.18. *The exterior algebra is graded commutative, i.e., for* $\alpha \in \mathrm{Alt}^p(V,\mathbb{K})$ *and* $\beta \in \mathrm{Alt}^q(V,\mathbb{K})$ *we have*

$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha.$$

Proof. Let $\sigma \in S_{p+q}$ denote the permutation defined by

$$\sigma(i) := \begin{cases} i+p & \text{for } 1 \leq i \leq q \\ i-q & \text{for } q+1 \leq i \leq p+q \end{cases}$$

which moves the first q elements to the last q positions. Then we have

$$\begin{aligned}&(\beta \otimes \alpha)^\sigma(v_1,\ldots,v_{p+q}) = (\beta \otimes \alpha)(v_{\sigma(1)},\ldots,v_{\sigma(p+q)}) \\ &= \beta(v_{p+1},\ldots,v_{p+q})\alpha(v_1,\ldots,v_p) = (\alpha \otimes \beta)(v_1,\ldots,v_{p+q}).\end{aligned}$$

This leads to

$$\begin{aligned}\alpha \wedge \beta &= \frac{(p+q)!}{p!q!}\,\mathrm{Alt}(\alpha \otimes \beta) = \frac{(p+q)!}{p!q!}\,\mathrm{Alt}((\beta \otimes \alpha)^\sigma) \\ &= \mathrm{sgn}(\sigma)\frac{(p+q)!}{p!q!}\,\mathrm{Alt}(\beta \otimes \alpha) = \mathrm{sgn}(\sigma)(\beta \wedge \alpha).\end{aligned}$$

On the other hand $\mathrm{sgn}(\sigma) = (-1)^F$, where

$$\begin{aligned}F &:= |\{(i,j) \in \{1,\ldots,p+q\} : i < j, \sigma(j) < \sigma(i)\}| \\ &= |\{(i,j) \in \{1,\ldots,p+q\} : i \leq q, j > q\}| = pq\end{aligned}$$

is the number of inversions of σ. Putting everything together, the lemma follows. □

Corollary B.2.19. *If* $\alpha \in \mathrm{Alt}^p(V,\mathbb{K})$ *and* p *is odd, then* $\alpha \wedge \alpha = 0$.

Proof. In view of Proposition B.2.18, we have $\alpha \wedge \alpha = (-1)^{p^2}\alpha \wedge \alpha = -\alpha \wedge \alpha$, which leads to $\alpha \wedge \alpha = 0$. □

Corollary B.2.20. *If* $\alpha_1,\ldots,\alpha_k \in V^* = \mathrm{Alt}^1(V,\mathbb{K})$ *and* $\beta_j = \sum_{i=1}^k a_{ij}\alpha_i$, *then*

$$\beta_1 \wedge \ldots \wedge \beta_k = \det(A) \cdot \alpha_1 \wedge \ldots \wedge \alpha_k \quad \textit{for} \quad A = (a_{ij}) \in M_k(\mathbb{K}).$$

Proof. The k-fold multiplication map

$$\Phi\colon (V^*)^k \to \mathrm{Alt}^k(V,\mathbb{K}), \quad (\gamma_1,\ldots,\gamma_k) \mapsto \gamma_1 \wedge \ldots \wedge \gamma_k$$

is alternating by Proposition B.2.18 because S_k is generated by transpositions. Hence the assertion follows from Proposition B.2.11. □

Corollary B.2.21. *If* $\dim V = n$, $b_1,\ldots,b_n$ *is a basis for* V, *and* $b_1^*,\ldots,b_n^*$ *the dual basis for* V^*, *then the products*

$$b_I^* := b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*, \quad I = (i_1,\ldots,i_k), \quad 1 \leq i_1 < \ldots < i_k \leq n,$$

form a basis for $\mathrm{Alt}^k(V,\mathbb{K})$.

Proof. For $J = (j_1,\ldots,j_k)$ with $j_1 < \ldots < j_k$, we get with Remark B.2.17(b)

$$b_I^*(b_{j_1},\ldots,b_{j_k}) = \det(b_{i_l}^*(b_{j_m})_{l,m=1,\ldots,k}) = \begin{cases} 1 & \text{for } I = J \\ 0 & \text{for } I \neq J. \end{cases}$$

If follows in particular that the elements b_I are linearly independent, and since $\dim \mathrm{Alt}^k(V,\mathbb{K}) = \binom{n}{k}$ (Corollary B.2.12), the assertion follows. □

Remark B.2.22. (a) From Corollary B.2.12 it follows in particular that

$$\dim \mathrm{Alt}(V,\mathbb{K}) = \sum_{k=0}^{\dim V} \binom{\dim V}{k} = 2^{\dim V}$$

if V is finite-dimensional.

(b) If V is infinite-dimensional, then it has an infinite basis $(b_i)_{i\in I}$ (this requires Zorn's Lemma). In addition, the set I carries a linear order $\leq$ (this requires the Well Ordering Theorem), and for each k-element subset $J = \{j_1,\ldots,j_k\} \subseteq I$ with $j_1 < \ldots < j_k$, we thus obtain an element

$$b_J^* := b_{j_1}^* \wedge \ldots \wedge b_{j_k}^*.$$

Applying the b_J^* to k-tuples of basis elements shows that they are linearly independent, so that for each $k > 0$ the space $\mathrm{Alt}^k(V,\mathbb{K})$ is infinite-dimensional.

Definition B.2.23. Let $\varphi\colon V_1 \to V_2$ be a linear map and W a vector space. For each p-linear map $\alpha\colon V_2^p \to W$ we define its *pull-back by* φ:

$$(\varphi^*\alpha)(v_1,\ldots,v_p) := \alpha(\varphi(v_1),\ldots,\varphi(v_p))$$

for $v_1,\ldots,v_p \in V_1$. It is clear that $\varphi^*\alpha$ is a p-linear map $V_1^p \to W$ and that $\varphi^*\alpha$ is alternating if α has this property.

Remark B.2.24. If $\varphi\colon V_1 \to V_2$ and $\psi\colon V_2 \to V_3$ are linear maps and $\alpha\colon V_3^p \to W$ is p-linear, then

$$(\psi \circ \varphi)^*\alpha = \varphi^*(\psi^*\alpha).$$

Proposition B.2.25. *Let $\varphi\colon V_1 \to V_2$ be a linear map. Then the pull-back map*

$$\varphi^*\colon \operatorname{Alt}(V_2, \mathbb{K}) \to \operatorname{Alt}(V_1, \mathbb{K})$$

is a homomorphism of algebras with unit.

Proof. For $\alpha \in \operatorname{Alt}^p(V_2, \mathbb{K})$ and $\beta \in \operatorname{Alt}^q(V_2, \mathbb{K})$ we have

$$\begin{aligned}\varphi^*(\alpha \wedge \beta) &= \frac{(p+q)!}{p!q!}\varphi^*(\operatorname{Alt}(\alpha \otimes \beta)) = \frac{(p+q)!}{p!q!}\operatorname{Alt}(\varphi^*(\alpha \otimes \beta))\\ &= \frac{(p+q)!}{p!q!}\operatorname{Alt}(\varphi^*\alpha \otimes \varphi^*\beta) = \varphi^*\alpha \wedge \varphi^*\beta.\end{aligned}$$ □

Remark B.2.26. The results in this section remain valid for alternating forms with values in any commutative algebra A. Then

$$\operatorname{Alt}(V, A) = \bigoplus_{p \in \mathbb{N}_0} \operatorname{Alt}^p(V, A)$$

also carries an associative, graded commutative algebra structure defined by

$$\alpha \wedge \beta := \frac{(p+q)!}{p!q!}\operatorname{Alt}(\alpha \otimes \beta),$$

where

$$(\alpha \otimes \beta)(v_1, \ldots, v_{p+q}) := \alpha(v_1, \ldots, v_p) \cdot \beta(v_{p+1}, \ldots, v_{p+q})$$

for $\alpha \in \operatorname{Alt}^p(V, A)$, $\beta \in \operatorname{Alt}^q(V, A)$.

This applies in particular to the 2-dimensional real algebra $A = \mathbb{C}$.

B.2.4 Orientations on Vector Spaces

Throughout this subsection, all vector spaces are real and finite-dimensional.

Definition B.2.27. (a) Let V be an n-dimensional real vector space. Then $\operatorname{Alt}^n(V, \mathbb{R})$ is one-dimensional. Any nonzero element μ of this space is called a *volume form* on V.

(b) We define an equivalence relation on the set $\operatorname{Alt}^n(V, \mathbb{R}) \setminus \{0\}$ of volume forms setting $\mu_1 \sim \mu_2$ if there exists a $\lambda > 0$ with $\mu_2 = \lambda\mu_1$. We write $[\mu]$ for the equivalence class of μ. These equivalence classes are called *orientations of* V. If $O = [\mu]$ is an orientation, then we write $-O := [-\mu]$ for the *opposite orientation*.

An *oriented vector space* is a pair (V, O), where V is a finite-dimensional real vector space and $O = [\mu]$ an orientation on V.

(c) An ordered basis $(b_1, \ldots, b_n)$ for $(V, [\mu])$ is said to be *positively oriented* if $\mu(b_1, \ldots, b_n) > 0$, and *negatively oriented* otherwise.

(d) An invertible linear map $\varphi\colon (V,[\mu_V]) \to (W,[\mu_W])$ between oriented vector spaces is called *orientation preserving* if $[\varphi^*\mu_W] = [\mu_V]$. Otherwise φ is called *orientation reversing.*

(e) We endow $\mathbb{R}^n$ with the canonical orientation, defined by the determinant form

$$\mu(x_1,\ldots,x_n) := \det(x_1,\ldots,x_n) = \det(x_{ij})_{i,j=1,\ldots,n}.$$

Remark B.2.28. (a) If $B := (b_1,\ldots,b_n)$ is a basis for V, then Corollary B.2.21 implies that we obtain a volume form by

$$\mu_B := b_1^* \wedge \ldots \wedge b_n^*,$$

and since $\mu_B(b_1,\ldots,b_n) = \det(b_i^*(b_j)) = \det(\mathbf{1}) = 1$, the basis B is positively oriented with respect to the orientation $[\mu_B]$. We call $[\mu_B]$ the *orientation defined by the basis B.*

(b) The terminology "volume form" corresponds to the interpretation of $\mu(v_1,\ldots,v_n)$ as an "oriented" volume of the flat

$$[0,1]v_1 + \ldots + [0,1]v_n$$

generated by the n-tuple $(v_1,\ldots,v_n)$. Note that $\mu_B(v_1,\ldots,v_n) = \det(b_i^*(v_j))$.

Lemma B.2.29. *If μ_V is a volume form on V and $\varphi \in \mathrm{End}(V)$, then*

$$\varphi^*\mu_V = \det(\varphi)\mu_V.$$

In particular, φ is orientation preserving if and only if $\det(\varphi) > 0$.

Proof. Let $B = (b_1,\ldots,b_n)$ be a positively oriented basis for V and $A = (a_{ij})$ the matrix of φ with respect to B, i.e., $\varphi(b_j) = \sum_i a_{ij}b_i$. Then

$$\begin{aligned}(\varphi^*\mu_V)(b_1,\ldots,b_n) &= \mu_V(\varphi(b_1),\ldots,\varphi(b_n)) = \det(A)\mu_V(b_1,\ldots,b_n)\\ &= \det(\varphi)\mu_V(b_1,\ldots,b_n)\end{aligned}$$

follows from Proposition B.2.11(i), and this implies the assertion. □

Example B.2.30. (a) If $V = \mathbb{R}^2$ and $\varphi \in \mathrm{GL}(V)$ is the reflection in a line, then $\det(\varphi) < 0$ implies that φ is orientation reversing. The same holds for the reflection in a hyperplane in $\mathbb{R}^n$.

(b) Rotations of $\mathbb{R}^3$ around an axis are orientation preserving.

(c) In $V = \mathbb{C}$, considered as a real vector space, we have the natural basis $B = (1, i)$. A corresponding volume form is given by

$$\mu(z,w) := \mathrm{Im}(\overline{z}w) = \mathrm{Re}\, z\, \mathrm{Im}\, w - \mathrm{Im}\, z\, \mathrm{Re}\, w$$

because $\mu(1,i) = \mathrm{Im}(i) = 1 > 0$.

Each complex linear map $\varphi\colon \mathbb{C} \to \mathbb{C}$ is given by multiplication with some complex number $x + iy$, and the corresponding matrix with respect to B is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

so that $\det(\varphi) = x^2 + y^2 > 0$ whenever $\varphi \neq 0$. We conclude that each nonzero complex linear map $V \to V$ is orientation preserving.

Proposition B.2.31 (Real vs. complex determinant). *Let V be a complex vector space, viewed as a real one, and $\varphi\colon V \to V$ a complex linear map. Then $\det_{\mathbb{R}}(\varphi) = |\det_{\mathbb{C}}(\varphi)|^2$. In particular, each invertible complex linear map is orientation preserving.*

Proof. Let $B_{\mathbb{C}} = (b_1, \ldots, b_n)$ be a complex basis for V, so that

$$B = (b_1, \ldots, b_n, ib_1, \ldots, ib_n)$$

is a real basis for V. Further, let $b_j^* \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, $j = 1, \ldots, n$, denote the complex dual basis. In $\mathrm{Alt}^{2n}(V, \mathbb{C}) \cong \mathbb{C}$ we then consider the element

$$\mu := b_1^* \wedge \ldots \wedge b_n^* \wedge \overline{b_1^*} \wedge \ldots \wedge \overline{b_n^*}.$$

That μ is nonzero follows from

$$\mu(b_1, \ldots, b_n, ib_1, \ldots, ib_n) = \det \begin{pmatrix} \mathbf{1}_n & i\mathbf{1}_n \\ \mathbf{1}_n & -i\mathbf{1}_n \end{pmatrix} = \det \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^n = (-2i)^n \neq 0.$$

If $A = (a_{ij}) \in M_n(\mathbb{C})$ is the matrix of φ with respect to $B_{\mathbb{C}}$, then we have

$$\varphi^* b_j^* = \sum_{k=1}^n a_{jk} b_k^* \quad \text{and} \quad \varphi^* \overline{b_j^*} = \sum_{k=1}^n \overline{a_{jk}} \overline{b_k^*}.$$

As in the proof of Lemma B.2.29, we now see that

$$\varphi^*(b_1^* \wedge \ldots \wedge b_n^*) = \det_{\mathbb{C}}(A) \cdot b_1^* \wedge \ldots \wedge b_n^*$$

and

$$\varphi^*(\overline{b_1^*} \wedge \ldots \wedge \overline{b_n^*}) = \overline{\det_{\mathbb{C}}(A)} \cdot \overline{b_1^*} \wedge \ldots \wedge \overline{b_n^*},$$

which leads with Proposition B.2.25 and Lemma B.2.29 to

$$\det_{\mathbb{R}}(\varphi)\mu = \varphi^*\mu = \det_{\mathbb{C}}(A)\overline{\det_{\mathbb{C}}(A)}\mu = |\det_{\mathbb{C}}(A)|^2\mu = |\det_{\mathbb{C}}(\varphi)|^2\mu. \qquad \square$$

Exercises for Section B.2

Exercise B.2.1. Fix $n \in \mathbb{N}$. Show that:

(1) For each matrix $A \in M_n(\mathbb{K})$, we obtain a bilinear map

$$\beta_A\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}, \quad \beta_A(x, y) := \sum_{i,j=1}^n a_{ij} x_i y_j.$$

(2) A can be recovered from β_A via $a_{ij} = \beta_A(e_i, e_j)$.
(3) Each bilinear map $\beta\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is of the form $\beta = \beta_A$ for a unique matrix $A \in M_n(\mathbb{R})$.
(4) $\beta_{A^\top}(x, y) = \beta_A(y, x)$.
(5) β_A is skew-symmetric if and only if A is so.

Exercise B.2.2. Show that for $\alpha_i \in \mathrm{Alt}^{p_i}(V, \mathbb{K}), i = 1, \ldots, n$, the exterior product satisfies

$$\alpha_1 \wedge \ldots \wedge \alpha_n = \frac{(p_1 + \ldots + p_n)!}{p_1! \cdots p_n!} \mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

Exercise B.2.3. Show that $(\mathrm{Alt}(V, \mathbb{K}), \wedge)$ is an exterior algebra over V^*.

B.3 Clifford Algebras, Pin and Spin Groups

A *quadratic space* is a pair (V, β), where V is a vector space and $\beta\colon V \times V \to \mathbb{K}$ is a symmetric bilinear form. We write $q(v) := \beta(v, v)$ for the corresponding quadratic form. In this section, $\mathbb{K}$ can be any field of characteristic $\neq 2$.

Definition B.3.1. A *Clifford algebra* for (V, β) is a pair (C, ι) of a unital associative algebra C and a linear map $\iota\colon V \to C$ satisfying

$$\iota(x)\iota(y) + \iota(y)\iota(x) = 2\beta(x, y)\mathbf{1} \quad \text{for} \quad x, y \in V \tag{B.4}$$

and the universal property that for each linear map $f\colon V \to A$, A a unital algebra, satisfying

$$f(x)f(y) + f(y)f(x) = 2\beta(x, y)\mathbf{1} \quad \text{for} \quad x, y \in V, \tag{B.5}$$

there exists a unique algebra homomorphism $\widetilde{f}\colon C \to A$ with $\widetilde{f} \circ \iota = f$.

Remark B.3.2. If V is a vector space over a field $\mathbb{K}$ of characteristic $\neq 2$, then β can be reconstructed from q via

$$\beta(x, y) = \frac{1}{2}\big(q(x + y) - q(x) - q(y)\big).$$

Accordingly, the relation (B.5) is equivalent to

$$f(x)^2 = q(x)\mathbf{1} \quad \text{for all} \quad x \in V. \tag{B.6}$$

Proposition B.3.3. *For each quadratic space (V, β), there exists a Clifford algebra (C, ι). It is unique up to isomorphism in the sense that for any other Clifford algebra (C', ι') of (V, β), there exists an algebra isomorphism $\varphi\colon C \to C'$ with $\varphi \circ \iota = \iota'$.*

Proof. **Uniqueness:** As usual, the uniqueness follows from the universal property. If (C,ι) and (C',ι') are Clifford algebras for (V,β), there exist uniquely determined unital algebra morphisms $f\colon C \to C'$ with $f \circ \iota = \iota'$ and $f'\colon C' \to C$ with $f' \circ \iota' = \iota$. Then $f' \circ f\colon C \to C$ is an algebra endomorphism with $(f' \circ f) \circ \iota = \iota$, so that the uniqueness in the universal property of (C,ι) implies that $f' \circ f = \mathrm{id}_C$. We likewise obtain $f \circ f' = \mathrm{id}_{C'}$, so that f is an isomorphism of unital algebras.

Existence: Let $\mathcal{T}(V)$ be the tensor algebra of V (Definition B.1.7) and consider the subset

$$M := \{x \otimes x - \beta(x,x)\colon x \in V\}.$$

We write J_M for the ideal generated by M. Then

$$C := \mathcal{T}(V)/J_M$$

is a unital associative algebra and

$$\iota\colon V \to C, \quad \iota(v) := v + J_M$$

is a linear map satisfying

$$\iota(x)^2 = x \otimes x + J_M = \beta(x,x)\mathbf{1}.$$

To verify the universal property, let $f\colon V \to A$ be a linear map into the unital algebra A, satisfying (B.5). In view of the universal property of $\mathcal{T}(V)$ (Lemma B.1.8), there exists a unital algebra homomorphism $\widehat{f}\colon \mathcal{T}(V) \to A$ with $\widehat{f}(x) = f(x)$ for all $x \in V$. Then $M \subseteq \ker \widehat{f}$ by (B.5), and since $\ker \widehat{f}$ is an ideal of $\mathcal{T}(V)$, we also have $J_M \subseteq \ker \widehat{f}$, so that $\widehat{f}$ factors through an algebra homomorphism

$$\widetilde{f}\colon C \to A \quad \text{with} \quad \widetilde{f} \circ \iota = f.$$

To see that $\widetilde{f}$ is unique, it suffices to note that $\iota(V)$ and $\mathbf{1}$ generate C as an associative algebra because V and $\mathbf{1}$ generate $\mathcal{T}(V)$. □

Definition B.3.4. Justified by the existence and uniqueness assertion of the preceding proposition, we write $(\mathrm{Cl}(V,\beta),\iota)$ for a Clifford algebra of (V,β).

For $\mathbb{K} = \mathbb{R}$ we consider on $\mathbb{R}^{p+q}$ the nondegenerate symmetric bilinear forms

$$\beta_{p,q}(x,y) := -\sum_{j=1}^{p} x_j y_j + \sum_{j=p+1}^{p+q} x_j y_j$$

and write

$$C_{p,q} := \mathrm{Cl}(\mathbb{R}^{p+q}, \beta_{p,q})$$

for the corresponding real Clifford algebras. For the negative definite form $\beta_n := \beta_{n,0}$, we also put $C_n := C_{n,0}$.

Examples B.3.5. (a) If $\beta = 0$, then $\mathrm{Cl}(V, \beta) \cong \Lambda(V)$ is the exterior algebra, over V (Lemma B.2.10).

(b) We discuss the Clifford algebras associated to one-dimensional vector spaces. Let $0 \neq e \in V$ be a basis element. Then $\mathrm{Cl}(V, \beta)$ is generated as an associative algebra by $\mathbf{1}$ and $x := \iota(e)$, which satisfies $x^2 = a := \beta(e, e)$, so that $\mathrm{Cl}(V, \beta) \cong \mathbb{K}[x]/(x^2 - a)$. We also write $\mathrm{Cl}(\mathbb{K}, a)$ for this Clifford algebra.

For $\beta = 0$ we thus obtain the ring $\mathbb{K}[\varepsilon] = \mathbb{K}\mathbf{1} + \mathbb{K}\varepsilon$ of dual numbers over $\mathbb{K}$, defined by the relation $\varepsilon^2 = 0$.

For $\beta \neq 0$, two cases occur. If a is a square, i.e., $a = b^2$ for some $b \in \mathbb{K}$, then $f := b^{-1}e$ is a basis element of V satisfying $\beta(f, f) = 1$, so that $\mathrm{Cl}(V, \beta) \cong \mathbb{K}[x]/(x^2 - 1)$. For $c_1 := \frac{1}{2}(\mathbf{1} + x)$ and $c_2 := \frac{1}{2}(\mathbf{1} - x)$ we then obtain two idempotents in $\mathrm{Cl}(V, \beta)$ satisfying $c_1 c_2 = 0$ and $c_1 + c_2 = \mathbf{1}$, which leads to $\mathrm{Cl}(V, \beta) \cong \mathbb{K} \oplus \mathbb{K}$, as an associative algebra.

If a is not a square in $\mathbb{K}$, then $\mathrm{Cl}(V, \beta) \cong \mathbb{K}[x]/(x^2 - a)$ is field. In fact, if $\lambda_g(h) = gh$ denotes left multiplication in this algebra, then the norm function

$$N \colon \mathrm{Cl}(V, \beta) \to \mathbb{K}, \quad r + sx \mapsto r^2 - as^2 = \det(\lambda_{r+sx}),$$

is multiplicative and nonzero on nonzero elements, which easily leads to

$$(r + sx)^{-1} = N(r + sx)^{-1}(r - sx) \quad \text{for} \quad \alpha, \beta \neq 0.$$

(c) For $\mathbb{K} = \mathbb{R}$ and $n = 1$, we have

$$C_1 \cong \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$$

(cf. Definition B.3.4), and $C_{0,1} \cong \mathbb{R}[x]/(x^2 - 1) \cong \mathbb{R} \oplus \mathbb{R}$.

The key to a systematic understanding of Clifford algebras is the understanding of how the Clifford algebra of an orthogonal direct sum $(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$ of two quadratic spaces can be described in terms of the Clifford algebras $\mathrm{Cl}(V_1, \beta_1)$ and $\mathrm{Cl}(V_2, \beta_1)$. First, we have to observe that Clifford algebras are 2-graded:

Lemma B.3.6. *There exists a unique involutive automorphism* ω *of* $\mathrm{Cl}(V, \beta)$ *with* $\omega \circ \iota = -\iota$.

Proof. The map $-\iota \colon V \to \mathrm{Cl}(V, \beta)$ also satisfies

$$(-\iota(x))^2 = \iota(x)^2 = \beta(x, x)\mathbf{1} \quad \text{for} \quad x \in V.$$

Therefore, the universal property of $\mathrm{Cl}(V, \beta)$ implies the existence of a homomorphism $\omega \colon \mathrm{Cl}(V, \beta) \to \mathrm{Cl}(V, \beta)$ of unital algebras with $\omega \circ \iota = -\iota$. Then $\omega^2 \circ \iota = \iota$ and the uniqueness part in the universal property imply that $\omega^2 = \mathrm{id}_{\mathrm{Cl}(V,\beta)}$. □

Definition B.3.7. (a) Let $\mathbb{Z}/2 = \{\overline{0}, \overline{1}\}$, considered as an abelian group. The eigenspaces

$$\mathrm{Cl}(V,\beta)_{\overline{0}} := \ker(\omega - \mathbf{1}) \quad \text{and} \quad \mathrm{Cl}(V,\beta)_{\overline{1}} := \ker(\omega + \mathbf{1})$$

of the involution ω define a 2-grading on $\mathrm{Cl}(V,\beta)$, i.e.,

$$\mathrm{Cl}(V,\beta) = \mathrm{Cl}(V,\beta)_{\overline{0}} \oplus \mathrm{Cl}(V,\beta)_{\overline{1}}$$

and

$$\mathrm{Cl}(V,\beta)_a \,\mathrm{Cl}(V,\beta)_b \subseteq \mathrm{Cl}(V,\beta)_{a+b}, \quad a,b \in \mathbb{Z}/2.$$

The involution ω is also called the *grading automorphism* of $\mathrm{Cl}(V,\beta)$.

Since $\mathrm{Cl}(V,\beta)$ is generated by $\iota(V)$ as a unital algebra, it is spanned by elements of the form[1] $\iota(v_1)\cdots\iota(v_k)$. On these elements we have

$$\omega(\iota(v_1)\cdots\iota(v_k)) = (-1)^k\iota(v_1)\cdots\iota(v_k),$$

so that

$$\mathrm{Cl}(V,\beta)_{\overline{0}} = \mathrm{span}\{\iota(v_1)\cdots\iota(v_k) \colon v_i \in V, k \in 2\mathbb{N}_0\}$$

and

$$\mathrm{Cl}(V,\beta)_{\overline{1}} = \mathrm{span}\{\iota(v_1)\cdots\iota(v_k) \colon v_i \in V, k \in 2\mathbb{N}_0 + 1\}.$$

(b) If $A = A_{\overline{0}} \oplus A_{\overline{1}}$ and $B = B_{\overline{0}} \oplus B_{\overline{1}}$ are $\mathbb{Z}/2$-graded algebras, then we define their *graded tensor product* as the vector space $C := A \otimes B$, endowed with the multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb'$$

for homogeneous elements $a, a' \in A$ and $b, b' \in B$. It is easy to verify that we thus obtain an associative $\mathbb{Z}/2$-graded algebra, denoted $A \widehat{\otimes} B$.

Proposition B.3.8. *For an orthogonal direct sum* $(V,\beta) = (V_1,\beta_1) \oplus (V_2,\beta_2)$ *of quadratic spaces, we have*

$$\mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \cong \mathrm{Cl}(V_1,\beta_1) \widehat{\otimes} \mathrm{Cl}(V_2,\beta_2).$$

Proof. In view of the uniqueness assertion of Proposition B.3.3, it suffices to show that the pair (C,ι) with $C := \mathrm{Cl}(V_1,\beta_1) \widehat{\otimes} \mathrm{Cl}(V_2,\beta_2)$ and

$$\iota \colon V_1 \oplus V_2 \to C, \quad v_1 \oplus v_2 \mapsto \iota_1(v_1) \otimes \mathbf{1} + \mathbf{1} \otimes \iota_2(v_2)$$

defines a Clifford algebra for $(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$. First we observe that

$$\begin{aligned}
&\big(\iota_1(v_1) \otimes \mathbf{1} + \mathbf{1} \otimes \iota_2(v_2)\big)^2 \\
&= \iota_1(v_1)^2 \otimes \mathbf{1} + \iota_1(v_1) \otimes \iota_2(v_2) + (\mathbf{1} \otimes \iota_2(v_2))(\iota_1(v_1) \otimes \mathbf{1}) + \mathbf{1} \otimes \iota_2(v_2)^2 \\
&= \beta_1(v_1,v_1)\mathbf{1} \otimes \mathbf{1} + \iota_1(v_1) \otimes \iota_2(v_2) - \iota_1(v_1) \otimes \iota_2(v_2) + \beta_2(v_2,v_2)\mathbf{1} \otimes \mathbf{1} \\
&= (\beta_1(v_1,v_1) + \beta_2(v_2,v_2))\mathbf{1} \otimes \mathbf{1}.
\end{aligned}$$

[1] By convention, the empty product corresponding to $k = 0$ is $\mathbf{1}$.

To verify the universal property, let $f\colon V_1 \oplus V_2 \to A$ be a linear map into a unital associative algebra satisfying

$$f(v_1 \oplus v_2)^2 = (\beta_1(v_1, v_1) + \beta_2(v_2, v_2))\mathbf{1}.$$

Then the universal property of the Clifford algebras $(\mathrm{Cl}(V_j, \beta_j), \iota_j)$ implies the existence of unique algebra homomorphisms $\widetilde{f_j}\colon \mathrm{Cl}(V_j, \beta_j) \to A$ with $\widetilde{f_j} \circ \iota_j = \iota|_{V_j}$. We combine these two maps to a linear map

$$\widetilde{f}\colon C \to A, \quad c_1 \otimes c_2 \mapsto \widetilde{f_1}(c_1)\widetilde{f_2}(c_2).$$

For $v_1 \in V_1$ and $v_2 \in V_2$, we have

$$f(v_1, 0)f(0, v_2) = -f(0, v_2)f(v_1, 0).$$

From Definition B.3.7, we therefore derive that

$$\widetilde{f_1}(a_1)\widetilde{f_2}(a_2) = (-1)^{\deg(a_1)\deg(a_2)}\widetilde{f_2}(a_2)\widetilde{f_1}(a_1)$$

holds for homogeneous elements $a_j \in \mathrm{Cl}(V_j, \beta_j)$. For homogeneous elements $a_j, a_j' \in \mathrm{Cl}(V_j, \beta_j)$, we thus obtain

$$\begin{aligned}\widetilde{f}((a_1 \otimes a_2)(a_1' \otimes a_2')) &= (-1)^{\deg(a_2)\deg(a_1')}\widetilde{f}(a_1 a_1' \otimes a_2 a_2') \\ &= (-1)^{\deg(a_2)\deg(a_1')}\widetilde{f_1}(a_1 a_1')\widetilde{f_2}(a_2 a_2') \\ &= (-1)^{\deg(a_2)\deg(a_1')}\widetilde{f_1}(a_1)\widetilde{f_1}(a_1')\widetilde{f_2}(a_2)\widetilde{f_2}(a_2') \\ &= \widetilde{f_1}(a_1)\widetilde{f_2}(a_2)\widetilde{f_1}(a_1')\widetilde{f_2}(a_2') = \widetilde{f}(a_1 \otimes a_2)\widetilde{f}(a_1' \otimes a_2'),\end{aligned}$$

and therefore $\widetilde{f}$ is an algebra homomorphism. It remains to show that $\widetilde{f}$ is uniquely determined. But this follows from the uniqueness of $\widetilde{f_1}, \widetilde{f_2}$ and the fact that C is generated by $\mathbf{1} \otimes \mathrm{Cl}(V_2, \beta_2)$ and $\mathrm{Cl}(V_1, \beta_1) \otimes \mathbf{1}$. □

The preceding proposition has a number of interesting consequences:

Corollary B.3.9. *If (V, β) is a quadratic space, then the following assertions hold:*

(i) *If* $\dim V < \infty$*, then* $\dim \mathrm{Cl}(V, \beta) = 2^{\dim V}$.
(ii) *If $v_1, \ldots, v_n \in V$ is an orthogonal basis, then the ordered products*

$$\iota(v_{i_1}) \cdots \iota(v_{i_k}), \quad 1 \leq i_1 < \cdots < i_k \leq n$$

form a basis for $\mathrm{Cl}(V, \beta)$.
(iii) *The structure map $\iota\colon V \to \mathrm{Cl}(V, \beta)$ is injective.*

Proof. (i) First, we recall that V possesses an orthogonal basis $v_1, \ldots, v_n$. This can be shown by induction on $\dim V$: We may w.l.o.g. assume that $\dim V > 0$ and that $\beta \neq 0$. Then there exists some $v_1 \in V$ with $\beta(v_1, v_1) \neq 0$ and $V = \mathbb{K}v_1 \oplus v_1^\perp$ is an orthogonal decomposition, so that the induction hypothesis implies the existence of an orthogonal basis $v_2, \ldots, v_n$ in $v_1^\perp$.

Now $V = \mathbb{K}v_1 \oplus \cdots \oplus \mathbb{K}v_n$ is an orthogonal decomposition, so that Proposition B.3.8 implies that

$$\mathrm{Cl}(V, \beta) \cong \mathrm{Cl}(\mathbb{K}, \beta(v_1, v_1)) \widehat{\otimes} \cdots \widehat{\otimes} \mathrm{Cl}(\mathbb{K}, \beta(v_n, v_n)).$$

Since we know already that Clifford algebras of one-dimensional quadratic spaces are two-dimensional (Examples B.3.5), (i) follows.

(ii) follows immediately from $\mathrm{Cl}(\mathbb{K}v_j, \beta(v_j, v_j)) = \mathbb{K}\mathbf{1} \oplus \mathbb{K}\iota(v_j)$ and the tensor product decomposition in (i).

(iii) Let $0 \neq v \in V$. We have to show that $\iota(v) \neq 0$. We claim that there exists a subspace $V_1 \ni v$ of V with $\dim V_1 \leq 2$ and a subspace V_2, such that V is the orthogonal direct sum of V_1 and V_2. In fact, if $\beta(v, V) = \{0\}$, then we put $V_1 := \mathbb{K}v$ and let V_2 be any complementary hyperplane in V. If $\beta(V, v) \neq \{0\}$ and $\beta(v, v) = 0$, then we choose some $w \in V$ with $\beta(v, w) \neq 0$ and put $V_1 := \mathbb{K}v + \mathbb{K}w$ and $V_2 := V_1^\perp$. If $\beta(v, v) \neq 0$, we put $V_1 := \mathbb{K}v$ and $V_2 := V_1^\perp$. This proves our claim. Now we apply Proposition B.3.8 to obtain

$$\mathrm{Cl}(V, \beta) \cong \mathrm{Cl}(V_1, \beta_1) \widehat{\otimes} \mathrm{Cl}(V_2, \beta_2),$$

where $\beta_j := \beta|_{V_j \times V_j}$, and $\iota(v) \neq 0$ follows from (ii) because V_1 possesses an orthogonal basis. □

B.3.1 The Clifford Group

In the following we simply write v for $\iota(v)$, which is justified by the injectivity of ι. We also write $C := \mathrm{Cl}(V, \beta)$ for the Clifford algebra of (V, β).

Definition B.3.10. The *twisted adjoint action* of the unit group $\mathrm{Cl}(V, \beta)^\times$ on $\mathrm{Cl}(V, \beta)$ is defined by

$$\mathrm{Ad}(a)x := \omega(a)xa^{-1}.$$

This defines a representation because ω is an algebra isomorphism. We define the *Clifford group* as the stabilizer of the subspace V of $\mathrm{Cl}(V, \beta)$:

$$\Gamma(V, \beta) := \{a \in \mathrm{Cl}(V, \beta)^\times : \ \mathrm{Ad}(a)V = V\}$$

and thus obtain a representation $\Phi \colon \Gamma(V, \beta) \to \mathrm{GL}(V), \Phi(a) := \mathrm{Ad}(a)|_V$.

Lemma B.3.11. *There exists a unique antiautomorphism $x \mapsto x^*$ of $\mathrm{Cl}(V, \beta)$ satisfying $v^* = -v$ for each $v \in V$. This involution commutes with ω.*

Proof. Let $\mathrm{Cl}(V,\beta)^{\mathrm{op}}$ be the opposite algebra, endowed with the product $x\sharp y := yx$. Then

$$f\colon V \to \mathrm{Cl}(V,\beta)^{\mathrm{op}}, \quad f(v) := -v$$

satisfies $f(v)\sharp f(v) = (-v)^2 = v^2 = \beta(v,v)\mathbf{1}$, so that the universal property of $\mathrm{Cl}(V,\beta)$ implies the existence of a unital algebra homomorphism $\widetilde{f}\colon \mathrm{Cl}(V,\beta) \to \mathrm{Cl}(V,\beta)^{\mathrm{op}}$ with $\widetilde{f}(v) = -v$ for $v \in V$. This means that, considered as a linear endomorphism of $\mathrm{Cl}(V,\beta)$, $\widetilde{f}(xy) = \widetilde{f}(y)\widetilde{f}(x)$ for $x, y \in \mathrm{Cl}(V,\beta)$, and thus $\widetilde{f}^2$ is an algebra endomorphism with $\widetilde{f}^2(v) = v$ for each $v \in V$, hence $\widetilde{f}^2 = \mathrm{id}_{\mathrm{Cl}(V,\beta)}$. Therefore $x^* := \widetilde{f}(x)$ defines an involutive antiautomorphism of $\mathrm{Cl}(V,\beta)$.

To see that $*$ commutes with ω, we note that $x \mapsto (\omega(x^*))^*$ is an algebra automorphism fixing V pointwise, hence equal to the identity. We conclude that $\omega(x^*) = x^*$ for $x \in \mathrm{Cl}(V,\beta)$. □

Examples B.3.12. (a) On $C_1 = \mathrm{Cl}(\mathbb{R},-1) \cong \mathbb{C}$ (cf. Definition B.3.4) we have

$$(x+iy)^* = \omega(x+iy) = x - iy.$$

Therefore the adjoint action is given by

$$\mathrm{Ad}(z)w = \overline{z}wz^{-1} = \overline{z}z^{-1} \cdot w,$$

which immediately shows that the Clifford group is

$$\Gamma(\mathbb{R},-1) = \Big\{ z \in \mathbb{C}^\times : \overline{z}z^{-1} = \frac{\overline{z}^2}{|z|^2} \in \mathbb{R} \Big\} = \mathbb{R}^\times \mathbf{1} \,\dot{\cup}\, \mathbb{R}^\times i.$$

(b) The four-dimensional Clifford algebra C_2 has the linear basis

$$\mathbf{1}, \quad I := e_1, \quad J := e_2 \quad \text{and} \quad K := e_1 e_2,$$

satisfying

$$I^2 = J^2 = K^2 = -\mathbf{1} \quad \text{and} \quad K = IJ = -JI,$$

so that $C_2 \cong \mathbb{H}$ as associative algebras. Here $V = \mathbb{R}I + \mathbb{R}J$ implies that the involution $*$ satisfies

$$(\alpha\mathbf{1} + \beta I + \gamma J + \delta K)^* = \alpha\mathbf{1} - \beta I - \gamma J - \delta K,$$

which is the canonical involution on $\mathbb{H}$.

As I and J are odd and $\mathbf{1}, K$ are even elements of $\mathbb{H}$, it follows from $KIK^{-1} = -I$ and $KJK^{-1} = -J$ that

$$\omega(z) = KzK^{-1} \quad \text{for} \quad z \in \mathbb{H}.$$

Therefore

$$\mathrm{Ad}(g)v = \omega(g)vg^{-1} = KgK^{-1}vg^{-1}.$$

As $KV = K\operatorname{span}\{I, J\} = \operatorname{span}\{I, J\} = V$, the condition $g \in \Gamma(V, \beta)$ is equivalent to $gVg^{-1} \subseteq V$. We write $g = \|g\| \cdot u$ with $\|u\| = 1$, where the norm is the one on the quaternions. Observing that conjugation with u preserves the natural scalar product on $\mathbb{H}$, we derive from $V^\perp = \mathbb{R}\mathbf{1} + \mathbb{R}K$ that $c_u(V) = V$ is equivalent to $c_u(\mathbb{R}\mathbf{1} + \mathbb{R}K) = \mathbb{R}\mathbf{1} + \mathbb{R}K$. In the subalgebra $\mathbb{R}\mathbf{1} + \mathbb{R}K$, the identity is fixed by c_u and $c_u(K)^2 = -\mathbf{1}$. Hence $c_u(V) = V$ is equivalent to $c_u(K) \in \pm K$, i.e., $uKu^{-1}K^{-1} = \pm\mathbf{1}$, which can also be written as $KuK^{-1} = \pm u$. Therefore

$$\Gamma(V, \beta) = \mathbb{R}^\times \cdot \Big(\{\alpha\mathbf{1} + \delta K \colon \alpha^2 + \delta^2 = 1\} \dot\cup \{\beta I + \gamma J \colon \beta^2 + \gamma^2 = 1\}\Big).$$

(c) If $\dim V = 1$ and $\mathrm{Cl}(V, \beta) = \mathrm{Cl}(\mathbb{K}, a) = \mathbb{K}\mathbf{1} + \mathbb{K}x$ with $x^2 = a$, then $\omega(x) = -x = x^*$. For $a \neq 0$ and $g \in \mathrm{Cl}(V, \beta)$ invertible,

$$\mathrm{Ad}(g)x = \omega(g)xg^{-1} \in \mathbb{K}x$$

is equivalent to $\omega(g) \in \mathbb{K}g$, i.e., g is an eigenvector of ω. Therefore

$$\Gamma(V, \beta) = \mathbb{K}^\times\mathbf{1} \dot\cup \mathbb{K}^\times x \quad \text{if} \quad a \neq 0.$$

For $a = 0$, we have $\omega(g)x = xg$ for any $g \in \mathrm{Cl}(\mathbb{K}, 0)$, so that $\Gamma(V, \beta) = \mathbb{K}^\times\mathbf{1} + \mathbb{K}x = \mathrm{Cl}(\mathbb{K}, a)^\times$.

Lemma B.3.13. *The Clifford group $\Gamma(V, \beta)$ is invariant under ω and $*$.*

Proof. Let $g \in \Gamma(V, \beta)$ and $v \in V$. Then $\mathrm{Ad}(g)v = \omega(g)vg^{-1} \in V$ implies that

$$V \ni \mathrm{Ad}(g)v = -\omega(\mathrm{Ad}(g)v) = -g\omega(v)\omega(g)^{-1} = \mathrm{Ad}(\omega(g))v,$$

which leads to $\omega(g) \in \Gamma(V, \beta)$. We likewise obtain

$$V \ni \mathrm{Ad}(g)v = -(\mathrm{Ad}(g)v)^* = -(g^*)^{-1}v^*\omega(g^*) = \mathrm{Ad}(\omega(g^*)^{-1})v \in V,$$

which shows that $\omega(g^*) \in \Gamma(V, \beta)$, and hence that $g^* \in \Gamma(V, \beta)$. □

Proposition B.3.14. *If (V, β) is nondegenerate and finite-dimensional, then the kernel of the representation $\Phi\colon \Gamma(V, \beta) \to \mathrm{GL}(V)$ is $\mathbb{K}^\times\mathbf{1}$.*

Proof. For $\lambda \in \mathbb{K}^\times$ and $v \in V$ we clearly have $\Phi(\lambda)v = \lambda v\lambda^{-1} = v$, so that $\mathbb{K}^\times\mathbf{1} \subseteq \ker\Phi$.

For the converse, we argue by induction on $\dim V$. If $\dim V = 1$ and $\mathrm{Cl}(V, \beta) = \mathbb{K}\mathbf{1} \oplus \mathbb{K}v$, then Example B.3.12(c) shows that $\Gamma(V, \beta) = \mathbb{K}^\times\mathbf{1} \dot\cup \mathbb{K}^\times v$ and $\ker\Phi = \{g \in \Gamma(V, \beta) \colon \omega(g) = g\} = \mathbb{K}^\times\mathbf{1}$. Now we assume that $\dim V > 1$. Let $g \in \ker\Phi$ and write it as $g = g_+ + g_-$, according to the $\mathbb{Z}/2$-grading of $\mathrm{Cl}(V, \beta)$. Then $\Phi(g)v = v$ is equivalent to $\omega(g)v = vg$, and decomposing into homogeneous summands leads to

$$g_+v = vg_+ \quad \text{and} \quad g_-v = -vg_- \quad \text{for all} \quad v \in V.$$

Let $v_1 \in V$ be a nonisotropic vector. Then $V = \mathbb{K}v_1 \oplus V_1$ with $V_1 := v_1^\perp$ is an orthogonal decomposition, so that $\mathrm{Cl}(V, \beta)$ decomposes accordingly as

$$\mathrm{Cl}(V,\beta) \cong \mathrm{Cl}(\mathbb{K}, \beta(v_1, v_1)) \widehat{\otimes} \mathrm{Cl}(V_1, \beta_1) \cong (\mathbb{K}\mathbf{1} + \mathbb{K}v_1) \widehat{\otimes} \mathrm{Cl}(V_1, \beta_1),$$

where $\beta_1 = \beta|_{V_1 \times V_1}$. The grading is given by

$$\mathrm{Cl}(V,\beta)_{\overline{0}} = \mathbf{1} \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{0}} \oplus v_1 \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{1}}$$

and

$$\mathrm{Cl}(V,\beta)_{\overline{1}} = \mathbf{1} \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{1}} \oplus v_1 \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{0}}$$

(cf. Proposition B.3.8). Accordingly, we write

$$g_\pm = \mathbf{1} \otimes a_\pm + v_1 \otimes b_\mp.$$

As $\mathbf{1} \otimes a_+$ commutes with $v_1 \otimes \mathbf{1}$ and $v_1 \otimes b_-$ anticommutes with $v_1 \otimes \mathbf{1}$, from $g_+ v_1 = v_1 g_+$ we obtain

$$0 = v_1^2 \otimes b_- = \beta(v_1, v_1)\mathbf{1} \otimes b_-,$$

which leads to $b_- = 0$. Likewise $g_- v_1 = -v_1 g_-$ leads to $b_+ = 0$, so that $g = \mathbf{1} \otimes (a_+ + a_-) \in \mathrm{Cl}(V_1, \beta_1)$, and our induction hypothesis implies that $g \in \mathbb{K}^\times \mathbf{1}$. □

Corollary B.3.15. *For $g \in \Gamma(V,\beta)$ we have $gg^* \in \mathbb{K}^\times \mathbf{1}$ and*

$$N \colon \Gamma(V,\beta) \to \mathbb{K}^\times, \quad gg^* = N(g)\mathbf{1},$$

defines a group homomorphism, called the norm homomorphism, *satisfying*

$$N(\omega(g)) = N(g) \quad \text{and} \quad N(\mathrm{Ad}(g)h) = N(h) \quad \text{for} \quad g, h \in \Gamma(V,\beta).$$

Proof. Since $\Gamma(V,\beta)$ is invariant under $*$, we have $gg^* \in \Gamma(V,\beta)$. In view of the preceding proposition, $gg^* \in \mathbb{K}^\times \mathbf{1}$ will follow if we can show that $gg^* \in \ker \Phi$.

For $x \in \mathrm{Cl}(V,\beta)$, we put $S(x) := \omega(x^*)$ and note that this defines an involutive antiautomorphism fixing V pointwise (Lemma B.3.11). Since $\Gamma(V,\beta)$ is invariant under $*$, we have to show that $\Phi(g^{-1}) = \Phi(g^*)$ for $g \in \Gamma(V,\beta)$. For $g \in \Gamma(V,\beta)$ and $v \in V$, the element $\Phi(g^*)v = \omega(g^*)v(g^{-1})^* = S(g)v(g^{-1})^* \in V$ is fixed by S, which leads to

$$\Phi(g^*)v = S(S(g)v(g^{-1})^*) = S((g^{-1})^*)vg = \omega(g^{-1})vg = \Phi(g^{-1})v,$$

i.e., $\Phi(g^*) = \Phi(g^{-1})$. This proves that $gg^* \in \ker \Phi = \mathbb{K}^\times \mathbf{1}$, so that $N(g)$ is defined. To see that N is a group homomorphism, we calculate

$$N(gh)\mathbf{1} = ghh^*g^* = g(N(h)\mathbf{1})g^* = N(h)gg^* = N(h)N(g)\mathbf{1}.$$

Applying ω to $gg^* = N(g)\mathbf{1}$, we obtain $N(\omega(g))\mathbf{1} = \omega(g)\omega(g)^* = N(g)\mathbf{1}$, so that $N(\omega(g)) = N(g)$, and this further implies that $N(\mathrm{Ad}(g)h) = N(\omega(g)hg^{-1}) = N(h)$. □

Theorem B.3.16. *If* (V, β) *is nondegenerate and finite-dimensional, then* $\mathrm{im}\,(\Phi) = \mathrm{O}(V, \beta)$*, so that* Φ *defines a short exact sequence*

$$\mathbf{1} \to \mathbb{K}^\times \hookrightarrow \Gamma(V, \beta) \xrightarrow{\Phi} \mathrm{O}(V, \beta) \to \mathbf{1}, \tag{B.7}$$

where $\Phi(v) = \sigma_v$ *is the reflection in* $v^\perp$ *for each nonisotropic element* $v \in V \subseteq \mathrm{Cl}(V, \beta)$*.*

Proof. For $v \in V$ we have

$$vv^* = -v^2 = -\beta(v, v)\mathbf{1}, \tag{B.8}$$

and for $g \in \Gamma(V, \beta)$ we have

$$\begin{aligned}(\Phi(g)v)(\Phi(g)v)^* &= \omega(g)vg^{-1}(\omega(g)vg^{-1})^* = \omega(g)vg^{-1}(g^{-1})^*(-v)\omega(g)^* \\ &= -N(g^{-1})\beta(v, v)\omega(gg^*) = -N(g^{-1})\beta(v, v)\omega(N(g)\mathbf{1}) = -\beta(v, v)\mathbf{1}.\end{aligned}$$

In view of (B.8), this proves that $\Phi(g) \in \mathrm{O}(V, \beta)$.

To identify the image of Φ, we observe that for any nonisotropic element $v \in V$, we have $\omega(v) = -v$ and $v^{-1} = \beta(v, v)^{-1}v$, so that

$$\mathrm{Ad}(v)x = -\beta(v, v)^{-1}vxv = \beta(v, v)^{-1}v(vx - 2\beta(v, x)\mathbf{1}) = x - 2\frac{\beta(v, x)}{\beta(v, v)}v,$$

which is the orthogonal reflection σ_v in the hyperplane $v^\perp$. We conclude in particular that $\Gamma(V, \beta)$ contains all nonisotropic elements of V, considered as a subspace of $\mathrm{Cl}(V, \beta)$, and that $\mathrm{Im}(\Phi)$ contains all orthogonal reflections.

If (V, β) is finite-dimensional and nondegenerate, then $\mathrm{O}(V, \beta)$ is generated by reflections (Exercise B.3.1), so that this argument shows that $\mathrm{Im}(\Phi) = \mathrm{O}(V, \beta)$. □

Remark B.3.17. If $\beta = 0$, then $\mathrm{Cl}(V, \beta) \cong \Lambda(V)$ is the exterior algebra of V, and $\Lambda(V)^\times = \mathbb{K}^\times\mathbf{1} \oplus \sum_{k \geq 1} \Lambda^k(V)$. Since $\Lambda(V)$ is graded commutative (see Lemma B.2.10), the even part is central and any two odd elements anticommute. This shows that for each invertible element $g = g_+ + g_-$, with g_+ even and g_- odd, we have

$$g_+v = vg_+ \quad \text{and} \quad g_-v = -vg_-$$

for all $v \in V$. This means that $\omega(g)v = vg$, so that $\Lambda(V)^\times = \Gamma(V, \beta) = \ker \Phi$.

Remark B.3.18. The proof of Theorem B.3.16 has several interesting consequences:

(a) As the image of Φ is generated by orthogonal reflections, it follows that $\Gamma(V, \beta)$ is generated by $\mathbb{K}^\times\mathbf{1} = \ker \Phi$ and the set

$$V^\times := \{v \in V \colon \beta(v, v) \neq 0\}$$

of nonisotropic vectors in V. For $\lambda \in \mathbb{K}^\times$ and $v \in V^\times$, the relation $\lambda\mathbf{1} = (\lambda v)v^{-1}$ implies that $\Gamma(V,\beta)$ is actually generated by $V^\times$. It follows in particular that $\omega(g) = \pm g$ for each $g \in \Gamma(V,\beta)$ and that

$$\omega(g) = \det(\Phi(g))g \quad \text{for} \quad g \in \Gamma(V,\beta).$$

(b) From $N(v) = -\beta(v,v)$ for $v \in V^\times$ it follows that the image $N(\Gamma(V,\beta))$ of $\Gamma(V,\beta)$ under N is the subgroup generated by the square classes represented by $-\beta$.

Remark B.3.19. Alternatively, one can introduce the Clifford group more directly as the subgroup Γ' of $\mathrm{Cl}(V,\beta)$ generated by the subset $V^\times$ of nonisotropic vectors. Since $\mathrm{Ad}(v)V = V$ for each such element and, as we have seen in the proof of Theorem B.3.16, $\mathrm{Ad}(v)|_V$ is the orthogonal reflection in the hyperplane $v^\perp$, we directly obtain a homomorphism $\Phi\colon \Gamma' \to \mathrm{O}(V,\beta)$ whose image is the subgroup generated by all reflections, hence all of $\mathrm{O}(V,\beta)$ if (V,β) is nondegenerate. For any $v \in V^\times$ and $\lambda \in \mathbb{K}^\times$, we have $\lambda\mathbf{1} = v^{-1}\cdot(\lambda v) \in \Gamma'$, so that Γ' contains $\mathbb{K}^\times$, which immediately leads to the short exact sequence

$$\mathbf{1} \to \mathbb{K}^\times \to \Gamma' \to \mathrm{O}(V,\beta) \to \mathbf{1}.$$

However, the advantage of the approach via the Clifford group is that it is specified by equations, hence in particular closed if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

B.3.2 Pin and Spin Groups

Definition B.3.20. We define the *pin group*[2]

$$\mathrm{Pin}(V,\beta) := \{g \in \Gamma(V,\beta)\colon N(g) = 1\} = \ker(N\colon \Gamma(V,\beta) \to \mathbb{K}^\times)$$

and the *spin group*

$$\mathrm{Spin}(V,\beta) := \mathrm{Pin}(V,\beta) \cap \Phi^{-1}(\mathrm{SO}(V,\beta)).$$

Note that

$$\ker\Phi \cap \mathrm{Pin}(V,\beta) = \{\lambda\mathbf{1}\colon \lambda \in \mathbb{K}^\times, \lambda^2 = 1\} = \{\pm\mathbf{1}\}$$

consists of 2 elements and that

$$\omega(g) = g \quad \text{for} \quad g \in \mathrm{Spin}(V,\beta) \tag{B.9}$$

(cf. Remark B.3.18). We also write

$$\mathrm{Pin}_{p,q}(\mathbb{K}) := \mathrm{Pin}(\mathbb{K}^{p+q},\beta_{p,q}) \quad \text{and} \quad \mathrm{Spin}_{p,q}(\mathbb{K}) := \mathrm{Spin}(\mathbb{K}^{p+q},\beta_{p,q})$$

for $\beta_{p,q}(x,y) = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_{p+q}y_{p+q}$. For $q = 0$, we put

$$\mathrm{Pin}_n(\mathbb{K}) := \mathrm{Pin}(\mathbb{K}^n,\beta_{n,0}) \quad \text{and} \quad \mathrm{Spin}_n(\mathbb{K}) := \mathrm{Spin}(\mathbb{K}^n,\beta_{n,0}).$$

[2] The name "pin" is due to J. P. Serre; see [ABS64] for the first occurrence of these groups.

Remark B.3.21. Our definition of the pin group follows [ABS64], but one finds slightly different definitions in the literature, all of which lead to the same spin group. F.i., Scharlau [Sch85] uses a different homomorphism

$$\widetilde{N} \colon \Gamma(V,\beta) \to \mathbb{K}^\times,$$

defined by $g\omega(g)^* = \widetilde{N}(g)\mathbf{1}$. To see how $\widetilde{N}$ differs from N, we note that the group $\Gamma(V,\beta)$ decomposes into the two subsets

$$\Gamma(V,\beta)_\pm := \{\gamma \in \Gamma(V,\beta) \colon \omega(\gamma) = \pm\gamma\}$$

and that $\varepsilon \colon \Gamma(V,\beta) \to \{\pm 1\}, \omega(\gamma) = \varepsilon(\gamma)\gamma$ defines a group homomorphism satisfying

$$\widetilde{N} = N \cdot \varepsilon.$$

Accordingly, the pin group $\{\gamma \in \Gamma(V,\beta) \colon \widetilde{N}(\gamma) = 1\}$ defined in [Sch85] is the union of

$$\mathrm{Pin}(V,\beta)_+ = \mathrm{Spin}(V,\beta) \quad \text{and} \quad \{\gamma \in \Gamma(V,\beta)_- \colon N(\gamma) = -1\}.$$

We also note that $\varepsilon(\gamma) = \det(\Phi(\gamma))$ follows from the fact that it holds on the generators, the nonisotropic elements of V. Therefore $\Gamma(V,\beta)_+ = \Phi^{-1}(\mathrm{SO}(V,\beta))$, and both homomorphisms N and $\widetilde{N}$, lead to the same spin group

$$\mathrm{Spin}(V,\beta) = \ker(N|_{\Gamma(V,\beta)_+}) = \ker(\widetilde{N}|_{\Gamma(V,\beta)_+}).$$

Remark B.3.22. (a) If $\mathbb{K} = \mathbb{R}$ and β is negative definite, then $N(v) = -\beta(v,v) > 0$ for each $0 \neq v \in V$, and each nonzero vector v has a multiple v' normalized by $\beta(v',v') = -1$. Then $v' \in \mathrm{Pin}(V,\beta)$, and this implies that the restriction

$$\Phi \colon \mathrm{Pin}(V,\beta) \to \mathrm{O}(V,\beta)$$

is still surjective, which leads to an exact sequence

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Pin}(V,\beta) \to \mathrm{O}(V,\beta) \to \mathbf{1}$$

and, accordingly, to an exact sequence

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta) \to \mathbf{1}.$$

(b) If $\mathbb{K} = \mathbb{R}$ and β is positive definite, then $N(\Gamma(V,\beta)_-) \subseteq \mathbb{R}^\times_-$ implies that $\mathrm{Pin}(V,\beta) \subseteq \Gamma(V,\beta)_+$, and hence that $\mathrm{Pin}(V,\beta) = \mathrm{Spin}(V,\beta)$. However, the alternative definition of the pin group, based on $\widetilde{N}$, yields a larger group (cf. Remark B.3.21).

(c) In general, the homomorphism

$$\Phi \colon \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta)$$

is not surjective. In fact, if $\mathbb{K} = \mathbb{R}$ and β is indefinite, then pick $v_1, v_2 \in V$ with $\beta(v_1, v_1) = 1 = -\beta(v_2, v_2)$. Now the product $g := \sigma_{v_1}\sigma_{v_2}$ of the corresponding orthogonal reflections is an element of $\mathrm{SO}(V, \beta)$, and in $\Gamma(V, \beta)$ we have

$$N(v_1 v_2) = N(v_1)N(v_2) = \beta(v_1, v_1)\beta(v_2, v_2) = -1.$$

Then $\Phi(v_1 v_2) = g$, and for any element $\gamma \in \Phi^{-1}(g)$, we have $\gamma = \lambda v_1 v_2$, $\lambda \in \mathbb{K}^\times$, and therefore $N(\gamma) = -\lambda^2 < 0$. This implies that $\gamma \notin \mathrm{Spin}(V, \beta)$, and hence that $\Phi(\mathrm{Spin}(V, \beta))$ is a proper subgroup of $\mathrm{SO}(V, \beta)$.

(d) Suppose that $\mathbb{K} = \mathbb{C}$ and $(V, \beta) = (\mathbb{C}^n, \beta)$ with the standard form $\beta(z, w) = \sum_{j=1}^n z_j w_j$. Since every complex number is a square, $N(v) = -\beta(v, v)$ implies that each nonisotropic vector $v \in \mathbb{C}^n$ has a scalar multiple $\widetilde{v}$ with $N(\widetilde{v}) = 1$. This proves that the homomorphism

$$\Phi\colon \mathrm{Pin}_n(\mathbb{C}) \to \mathrm{O}_n(\mathbb{C})$$

is surjective, which leads to exact sequences

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Pin}_n(\mathbb{C}) \to \mathrm{O}_n(\mathbb{C}) \to \mathbf{1}$$

and

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Spin}_n(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C}) \to \mathbf{1}.$$

Example B.3.23. We recall from Example B.3.12(a) that for $n = 1$, we have $C_1 \cong \mathbb{C}$, $\omega(z) = \overline{z}$ and

$$\Gamma(V, \beta) = \mathbb{R}^\times \cup i\mathbb{R}^\times.$$

From $N(z) = |z|^2$ we derive that

$$\mathrm{Pin}_1(\mathbb{R}) = \{z \in \Gamma(V, \beta)\colon |z| = 1\} = \{\pm 1, \pm i\} \quad \text{and} \quad \mathrm{Spin}_1(\mathbb{R}) = \{\pm 1\}.$$

Example B.3.24. For $n = 2$, we have $C_2 \cong \mathbb{H}$ and we recall from Example B.3.12(b) that

$$\Gamma(V, \beta) = \{\alpha\mathbf{1} + \delta K\colon \alpha^2 + \delta^2 > 0\}\dot{\cup}\{\beta I + \gamma J\colon \beta^2 + \gamma^2 > 0\}.$$

Further $N(x) = \|x\|^2$ follows from $xx^* = \|x\|^2\mathbf{1}$. This implies that

$$\mathrm{Pin}_2(\mathbb{R}) = \{\alpha\mathbf{1} + \delta K\colon \alpha^2 + \delta^2 = 1\}\dot{\cup}\{\beta I + \gamma J\colon \beta^2 + \gamma^2 = 1\}$$

is a union of two circles and

$$\mathrm{Spin}_2(\mathbb{R}) = \{\alpha\mathbf{1} + \delta K\colon \alpha^2 + \delta^2 = 1\} = \mathrm{Pin}_2(\mathbb{R})_0 \cong \mathbb{T}.$$

The complement of the identity component in $\mathrm{Pin}_2(\mathbb{R})$ is the set

$$\{\beta I + \gamma J\colon \beta^2 + \gamma^2 = 1\},$$

and for all these elements we have $(\beta I + \gamma J)^2 = -\mathbf{1}$, so that the short exact sequence

$$\mathbf{1} \to \mathrm{Spin}_2(\mathbb{R}) \to \mathrm{Pin}_2(\mathbb{R}) \to \pi_0(\mathrm{Pin}_2(\mathbb{R})) \cong \mathbb{Z}/2 \to \mathbf{1}$$

does not split.

Remark B.3.25. For the negative definite form β_n on $\mathbb{R}^n$, we have a short exact sequence

$$\mathbf{1} \to \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{Pin}_n(\mathbb{R}) \to \{\pm 1\} \to \mathbf{1},$$

so that it makes sense to ask when this sequence splits, i.e., if there exists an involution $\tau \in \mathrm{Pin}_n(\mathbb{R})$, not contained in $\mathrm{Spin}_n(\mathbb{R})$.

In the preceding two examples, we have seen that this is not the case for $n = 1, 2$. However, it is true for $n > 2$, which can be seen as follows. Let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathbb{R}^n$, so that $\beta_n(e_i, e_j) = -\delta_{ij}$. For $n \geq 3$ we put $\tau := e_1 e_2 e_3$. As $N(e_j) = -\beta(e_j, e_j) = 1$ for each j, we have $\tau \in \mathrm{Pin}_n(\mathbb{R}) \setminus \mathrm{Spin}_n(\mathbb{R})$. Moreover,

$$\tau^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_1^2 e_2 e_3 e_2 e_3 = e_2 e_3^2 e_2 = \mathbf{1},$$

so that τ is an involution in $\mathrm{Pin}_n(\mathbb{R}) \setminus \mathrm{Spin}_n(\mathbb{R})$ (cf. Exercise B.3.5).

Exercises for Section B.3

Exercise B.3.1. Show that if (V, β) is a nondegenerate finite-dimensional quadratic space, then the orthogonal group $\mathrm{O}(V, \beta)$ is generated by orthogonal reflections.

Exercise B.3.2. Show that if β is a positive definite form on $\mathbb{R}^2$, then $C_{0,2} \cong \mathrm{Cl}(\mathbb{R}^2, \beta) \cong M_2(\mathbb{R})$.

Exercise B.3.3. Establish isomorphisms

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C}), \quad \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R}).$$

Exercise B.3.4. (a) Write $C'_n := C_{0,n}$ for the Clifford algebra of $\mathbb{R}^n$, endowed with the positive definite form $\beta_{0,n}$. Establish isomorphisms

$$C_{n+2} \cong C'_n \otimes_{\mathbb{R}} C_2 \quad \text{and} \quad C'_{n+2} \cong C_n \otimes_{\mathbb{R}} C'_2.$$

Hint: Let $e_1, \ldots, e_{n+2}$ be an orthonormal basis for $\mathbb{R}^{n+2}$, so that $e_i^2 = -\mathbf{1}$ in C_{n+2}. Then for the first isomorphism map $e_i \mapsto e'_i \otimes e_1 e_2$ for $i = 1, \ldots, n$ and map e_{n+1} and e_{n+2} on $\mathbf{1} \otimes e_1$ and $\mathbf{1} \otimes e_2$ respectively.

(b) Show that $C_{n+4} \cong C_n \otimes_{\mathbb{R}} M_2(\mathbb{H}) \cong M_2(C_n) \otimes_{\mathbb{R}} \mathbb{H}$,

(c) Prove that $C_{n+8} \cong C_n \otimes_{\mathbb{R}} M_{16}(\mathbb{R}) \cong M_{16}(C_n)$. This is called the *periodicity property*.

Exercise B.3.5. Let $D_n = \mathrm{Cl}(\mathbb{C}^n, \beta_n)$ with $\beta_n(z, w) = \sum_{j=1}^n z_j w_j$. Show that

$$D_1 \cong \mathbb{C} \oplus \mathbb{C}, \quad D_2 \cong M_2(\mathbb{C}),$$

and establish the periodicity property

$$D_{n+2} \cong D_n \otimes_{\mathbb{C}} M_2(\mathbb{C}) \cong M_2(D_n), \quad n \geq 1.$$

Exercise B.3.6. In the unit group $C_n^\times$ of the Clifford algebra C_n associated to the negative definite form β_n on $\mathbb{R}^n$, we consider the subgroup F, generated by the canonical basis vectors $e_1, \ldots, e_n$. Show that:

(a) $\Phi(F) \cong (\mathbb{Z}/2)^n$.
(b) Φ induces a short exact sequence $\mathbf{1} \to \mathbb{Z}/2 \to F \to (\mathbb{Z}/2)^n \to \mathbf{1}$.
(c) F is 2-step nilpotent, i.e., $C^3(F) = \{\mathbf{1}\}$, with $C^2(F) = F' = \{\pm\mathbf{1}\}$ (for $n > 1$). Here $C^n(F)$ denotes the central series for the group F.
(d) Each element $f \in F$ can be written as $f = \pm e_{i_1} \cdots e_{i_k}$ with $1 \leq i_1 < \ldots < i_k \leq n$. It satisfies $f^2 = (-\mathbf{1})^{\binom{k+1}{2}}$. Conclude that for $n \leq 2$ all elements $\neq \pm\mathbf{1}$ are of order 4.

Exercise B.3.7. Consider the decomposition

$$\mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \cong \mathrm{Cl}(V_1, \beta_1) \widehat{\otimes} \mathrm{Cl}(V_2, \beta_2)$$

from Proposition B.3.8, where β_1 and β_2 are non-degenerate. Show that the subgroups $\mathrm{Spin}(V_1, \beta_1)$ and $\mathrm{Spin}(V_2, \beta_2)$ commute.

C

Some Functional Analysis

C.1 Bounded Operators

Definition C.1.1. Let X and Y be a normed spaces and $A\colon X \to Y$ be a linear map; also called an operator in this context. We define the *(operator) norm of* A by

$$\|A\| := \sup\{\|Ax\|\colon x \in X, \|x\| \leq 1\} \in [0, \infty].$$

The linear operator A is said to be *bounded* if $\|A\| < \infty$. We write $B(X, Y)$ for the set of bounded linear operators from X to Y. It is easy to see that $(B(X, Y), \|\cdot\|)$ is a normed space. For $X = Y$ we simply write $B(X) := B(X, X)$.

Remark C.1.2. Note that boundedness of an operator A does not mean that its range $A(X)$ is a bounded subset of Y. If B_X denotes the closed unit ball in X, then A is bounded if and only if $A(B_X)$ is a bounded subset of Y and $\|A\|$ is the radius of the smallest ball in Y centered in 0 containing $A(B_X)$.

Lemma C.1.3. *For a linear map* $A\colon X \to Y$ *between normed spaces, the following are equivalent*

(1) A *is continuous.*
(2) A *is continuous in* 0.
(3) A *is bounded.*

Definition C.1.4. Let X and Y be Banach spaces. Then a linear map $A\colon X \to Y$ is called *compact* if the image of every bounded sequence in X under A has a convergent subsequence. Since convergent sequences are in particular bounded, one easily shows that any compact operator is continuous (Exercise C.3.3).

To determine whether a given operator is compact, one needs some tools to determine under which circumstances bounded sequences in Banach spaces possess convergent subsequences. One such tool is the following theorem:

Theorem C.1.5 (Ascoli's Theorem). *Let X be a compact space, $C(X)$ be the Banach space of all continuous functions $f\colon X \to \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, endowed with the sup-norm*

$$\|f\| := \sup\{|f(x)|\colon x \in X\}$$

and $M \subseteq C(X)$ a subset satisfying the following conditions:

(a) *M is pointwise bounded, i.e.,* $\sup\{|f(x)|\colon f \in M\} < \infty$ *for each $x \in X$.*
(b) *M is equicontinuous, i.e., for each $\varepsilon > 0$ and each $x \in X$ there exists a neighborhood U_x with*

$$|f(x) - f(y)| \leq \varepsilon \quad \textit{for} \quad f \in M, y \in U_x.$$

Then each sequence in M possesses a convergent subsequence.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in M. For each $k \in \mathbb{N}$ we find with (b) points $x_1^k, \ldots, x_{m_k}^k$ in X and neighborhoods $V_1^k, \ldots, V_{m_k}^k$ of these points, such that $X \subseteq \bigcup_{i=1}^{m_k} V_i^k$ and

$$|f(x) - f(x_i^k)| < \frac{1}{k} \qquad \text{for} \quad f \in M, x \in V_i^k, i = 1, \ldots, m_k.$$

We order the countable set $\{x_i^k : k \in \mathbb{N}, i = 1, \ldots, m_k\}$ as follows to a sequence $(y_m)_{m\in\mathbb{N}}$:

$$x_1^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$$

For each y_m, the set $\{f_n(y_m) : n \in \mathbb{N}\} \subseteq \mathbb{K}$ is bounded, hence contains a subsequence f_n^1, converging in y_1. This sequence has a subsequence f_n^2, converging in y_2, etc. The sequence $(f_n^n)_{n\in\mathbb{N}}$ is a subsequence of the original sequence, converging on the set $\{y_m : m \in \mathbb{N}\}$. To simplify notation, we may now assume that the sequence f_n converges pointwise on this set.

Next we show that the sequence (f_n) converges pointwise. Pick $x \in X$. In view of the completeness of $\mathbb{K}$, it suffices to show that the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy. So let $\varepsilon > 0$. Then there exists a $k \in \mathbb{N}$ with $\frac{3}{k} < \varepsilon$ and y_m, such that

$$|f_n(x) - f_n(y_m)| < \frac{1}{k} \quad \text{for} \quad n \in \mathbb{N}.$$

We choose $n_0 \in \mathbb{N}$, such that

$$|f_n(y_m) - f_{n'}(y_m)| < \frac{1}{k} \quad \text{for} \quad n, n' > n_0.$$

Then

$$\begin{aligned}|f_n(x) - f_{n'}(x)| &\leq |f_n(x) - f_n(y_m)| + |f_n(y_m) - f_{n'}(y_m)| + |f_{n'}(y_m) - f_{n'}(x)| \\ &\leq \frac{3}{k} \leq \varepsilon.\end{aligned}$$

Let $F(x) := \lim_{n\to\infty} f_n(x)$. It remains to show that f_n converges uniformly to F. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ with $\frac{3}{k} < \varepsilon$. We pick $n_0 \in \mathbb{N}$ so large that

$$|f_n(x_i^k) - F(x_i^k)| \leq \frac{1}{k} \qquad \text{for} \quad n \geq n_0, i = 1, \ldots, m_k.$$

Since each element $x \in X$ is contained in one of the sets V_i^k,

$$|f_n(x) - F(x)| \leq |f_n(x) - f_n(x_i^k)| + |f_n(x_i^k) - F(x_i^k)| + |F(x_i^k) - F(x)| \leq \frac{3}{k} \leq \varepsilon,$$

because $|F(x_i^k) - F(x)| = \lim_{n\to\infty} |f_n(x_i^k) - f_n(x)| \leq \frac{1}{k}$. This proves that f_n converges uniformly to F, and the proof is complete. □

C.2 Hilbert Spaces

Definition C.2.1. A Banach space X is called a *Hilbert space* if there exists a sesquilinear positive definite hermitian form $\langle \cdot, \cdot \rangle$ on X with $\|v\|^2 = \langle v, v \rangle$ for each $v \in X$.

Lemma C.2.2. *Let E be a closed subspace of the Hilbert space $\mathcal{H}$ and $E^\perp := \{x \in \mathcal{H} \colon \langle x, E \rangle = \{0\}\}$. Then $\mathcal{H} = E \oplus E^\perp$.*

Proof. Clearly, $E \cap E^\perp = \{0\}$, so we have to show that each $x \in \mathcal{H}$ can be written as a sum of an element $x_0 \in E$ and an element $x_1 \in E^\perp$. The idea is to find x_0 as the point in E minimizing the distance to x.

Pick $x_n \in E$ with $\|x_n - x\| \to d := \inf\{\|y - x\| \colon y \in E\}$. In the Parallelogram Equation (Exercise C.3.1)

$$\|x_n + x_m - 2x\|^2 + \|x_n - x_m\|^2 = 2\|x_n - x\|^2 + 2\|x_m - x\|^2,$$

the right hand side is arbitrarily close to $4d^2$ if n and m are large enough. On the other hand $\frac{1}{2}(x_n + x_m) \in E$ implies that

$$\|x_n + x_m - 2x\|^2 = 4\left\|\frac{1}{2}(x_n + x_m) - x\right\|^2 \geq 4d^2.$$

Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, hence converges to some $x_0 \in E$ with $\|x - x_0\| = d$. For $y \in E$, the function

$$\|x + \lambda y - x_0\|^2 = \|x - x_0\|^2 + |\lambda|^2\|x_0\|^2 + 2\operatorname{Re}\lambda\langle x - x_0, y\rangle$$

is minimal at $\lambda = 0$, which implies that $x - x_0 \in E^\perp$ (Exercise C.3.2). Therefore $x = x_0 + (x - x_0) \in E + E^\perp$. □

C.3 Compact Symmetric Operators on Hilbert Spaces

Definition C.3.1. Let $\mathcal{H}$ be a complex Hilbert space. A bounded operator A on $\mathcal{H}$ is said to be *symmetric* if

$$\langle Av, w \rangle = \langle v, Aw \rangle \qquad \text{for} \quad v, w \in \mathcal{H}.$$

Theorem C.3.2. *For a compact symmetric operator* $A\colon \mathcal{H} \to \mathcal{H}$ *and* $\mathcal{H}_\lambda := \ker(A - \lambda\mathbf{1})$*, the following assertions hold:*

(1) $\|A\| = \sup\{|\langle Ax, x\rangle| \colon x \in \mathcal{H}, \|x\| = 1\}$.
(2) $\|A\|$ *or* $-\|A\|$ *is an eigenvalue of* A.
(3) $\bigoplus_{|\lambda|>\varepsilon} \mathcal{H}_\lambda$ *is finite-dimensional for every* $\varepsilon > 0$.
(4) $\bigoplus_{\lambda\in\mathbb{R}} \mathcal{H}_\lambda$ *is dense in* $\mathcal{H}$.

Proof. (1) Let $M := \sup\{|\langle Ax, x\rangle| \colon x \in \mathcal{H}, \|x\| = 1\}$. Since

$$\langle Ax, x\rangle \leq \|Ax\|\|x\| \leq \|A\|\|x\|^2$$

follows from the Cauchy–Schwarz inequality, $M \leq \|A\|$.

It remains to verify $\|A\| \leq M$. For $A = 0$ there is nothing to show. So pick $x \in \mathcal{H}$ with $\|x\| = 1$ and $Ax \neq 0$ and put $y := \frac{1}{\|Ax\|}Ax$. Then $\langle Ax, y\rangle = \|Ax\| = \langle x, Ay\rangle$, leads to

$$4\|Ax\| = \big\langle A(x+y), x+y\big\rangle - \big\langle A(x-y), x-y\big\rangle \leq M(\|x+y\|^2 + \|x-y\|^2) = 4M.$$

(2) In view of (1), there exists a sequence x_n of unit vectors with $|\langle Ax_n, x_n\rangle| \to \|A\|$. Passing to a subsequence, we may assume that the sequence $\langle Ax_n, x_n\rangle$ converges in $\mathbb{R}$. If $A = 0$, the assertion is trivial, so that we may assume $A \neq 0$. Then either $\langle Ax_n, x_n\rangle \to \|A\|$ or $\langle Ax_n, x_n\rangle \to -\|A\|$. We assume the former, the other case is treated in a similar fashion. Now

$$\begin{aligned} 0 &\leq \|Ax_n - \|A\|\, x_n\|^2 = \|Ax_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle + \|A\|^2 \\ &\leq 2\|A\|\big(\|A\| - \langle Ax_n, x_n\rangle\big) \to 0. \end{aligned}$$

Because of the compactness of A, we may assume that Ax_n converges to some $x \in \mathcal{H}$. From the above calculation we infer that $\|A\|x_n \to x$ and in particular $\|x\| = \|A\| > 0$. For $y := \frac{1}{\|A\|}x$, we now find $x_n \to y$ and therefore

$$Ay - \|A\|y = \lim_{n\to\infty} Ax_n - \|A\|x_n = 0,$$

sot that $\|A\|$ is an eigenvalue of A.

(3) Let $\lambda \neq 0$ be an eigenvalue of A. Then $A|_{\mathcal{H}_\lambda}$ is a compact operator. Therefore the identical map on $\mathcal{H}_\lambda$ is compact, and thus $\dim \mathcal{H}_\lambda < \infty$. In fact, every infinite-dimensional Hilbert space $\mathcal{H}$ contains an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$. Then (e_n) is bounded, but $\|e_n - e_m\|^2 = 2$ for $n \neq m$ implies that it contains no convergent subsequence.

Next we observe that if $x \in \mathcal{H}_\lambda$ and $y \in \mathcal{H}_\mu$ with $\mu \neq \lambda$, then

$$(\lambda - \mu)\langle x, y\rangle = \langle Ax, y\rangle - \langle x, Ay\rangle = 0$$

implies that eigenspaces for different eigenvalues are orthogonal. If there are infinitely many different eigenvalues λ_n with $|\lambda_n| > \varepsilon$, then we pick unit vectors $x_n \in \mathcal{H}_{\lambda_n}$ and observe that the sequence (x_n) is bounded, but the sequence $Ax_n = \lambda_n x_n$ has no convergent subsequence because $\|Ax_n - Ax_m\|^2 = \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon^2$. This contradicts the compactness of A.

(4) If λ is an eigenvector of A and x a corresponding unit eigenvector, then $\lambda = \langle Ax, x\rangle \in \mathbb{R}$. Let $E := \overline{\bigoplus_{\lambda\in\mathbb{R}} \mathcal{H}_\lambda}$. Then E is A-invariant, and for $y \in E^\perp$ and $x \in E$ the relation $\langle Ay, x\rangle = \langle y, Ax\rangle = 0$ implies that $E^\perp$ is also A-invariant. If $E^\perp \neq \{0\}$, then $AE^\perp \neq \{0\}$ because $\mathcal{H}_0 \subseteq E$. Further, (2) implies the existence of an eigenvector in $E^\perp$, which is a contradiction. We conclude that $E^\perp = \{0\}$, and therefore $E = \mathcal{H}$ follows from (Lemma C.2.2). □

Exercises for Section C.3

Exercise C.3.1. Show that in each Hilbert space $\mathcal{H}$, we have for $x, y \in \mathcal{H}$ the Parallelogram Law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Exercise C.3.2. We consider a function $f\colon \mathbb{C} \to \mathbb{R}$, given by

$$f(z) = a + 2\,\mathrm{Re}(zb) + c^2|z|^2, \qquad a, c \in \mathbb{R}, b \in \mathbb{C}.$$

Show that $b = 0$ if and only if f attains a minimal value at $z = 0$.

Exercise C.3.3. Show that each compact operator $A\colon X \to Y$ between Banach spaces is continuous.

D

Hints to Exercises

Exercise 1.1.1:

(b) If $w \in W$ is written as a finite sum $w = \sum_\lambda v_\lambda$ with $Av_\lambda = \lambda v_\lambda$, then each v_λ is contained in W. This can be proved by induction on the length of the sum.

(e) Suppose that $\sum_{i=1}^n v_{\lambda_i} = 0$ with $v_{\lambda_i} \in V_{\lambda_i}(\mathcal{A})$ and show by induction on n that all summands vanish.

Exercise 1.1.6: The verification of the convexity is easy. To see that $\mathrm{Pd}_n(\mathbb{K})$ is open, show first that for each $r > 0$ we have $B_r(r\mathbf{1}) = rB_1(\mathbf{1}) \subseteq \mathrm{Pd}_n(\mathbb{K})$ (here $B_r(x)$ denotes the open ball of radius r around x) by considering the eigenvalues and using that for $A \in \mathrm{Herm}_n(\mathbb{K})$ we have

$$\|A\| = \max\{|\lambda| \colon \det(A - \lambda\mathbf{1}) = 0\}.$$

Now observe that $\mathrm{Pd}_n(\mathbb{K}) = \bigcup_{r>0} B_r(r\mathbf{1})$ because for $A \in \mathrm{Pd}_n(\mathbb{K})$ with maximal eigenvalue r we have $A \in B_r(r\mathbf{1})$.

Exercise 1.1.8: Show that for $g \in \mathrm{O}_n(\mathbb{C})$ with polar decomposition $g = up$ both components u and p are contained in $\mathrm{O}_n(\mathbb{C})$. Compare the polar decomposition of $(g^\top)^{-1}$ and g. Why is $(p^\top)^{-1} \in \mathrm{Pd}_n(\mathbb{C})$?

Exercise 1.1.9: For metric spaces compactness is equivalent to sequential compactness, which means that every sequence has a convergent subsequence.

Exercise 1.1.10: The hermitian form $b(x,y) := \langle A.x, y\rangle$ satisfies the *polarization identity*

$$b(x,y) = \tfrac{1}{4}\big(b(x+y,x+y) - b(x-y,x-y) + ib(x+iy,x+iy) - ib(x-iy,x-iy)\big)$$

for $\mathbb{K} = \mathbb{C}$, and for $\mathbb{K} = \mathbb{R}$ we have

$$b(x,y) = \tfrac{1}{4}\big(b(x+y,x+y) - b(x-y,x-y)\big).$$

Exercise 1.1.11(2): Write $A = B + iC$ with B, C hermitian and use Exercise 1.1.10 to show that $C = 0$ if (2) holds.

Exercise 1.1.13(c): The subalgebra $\mathbb{K}[A] := \mathrm{span}\{A^k \colon k \in \mathbb{N}_0\} \subseteq \mathrm{End}(V)$ is isomorphic to $\mathbb{K}^n$ with pointwise multiplication and the basis vectors $e_j \in \mathbb{K}^n$ (which are idempotents of this algebra) correspond to the projections onto the eigenspace of A.

Exercise 1.1.15: Interpret invertible $(n \times n)$-matrices as bases of $\mathbb{R}^n$. Use the Gram–Schmidt algorithm to see that μ is surjective and that it has a continuous inverse map.

Exercise 1.1.16: Argue as in the proof of Proposition 1.1.10.

Exercise 1.2.1:

(a) Use induction on $\dim V$. If $\beta \neq 0$, then there exists $v_1 \in V$ with $\beta(v_1, v_1) = 1$ (polarization identity). Now proceed with the space $v_1^\perp := \{v \in V \colon \beta(v_1, v) = 0\}$.

(b) Consider the symmetric bilinear form $\beta(x, y) = x^\top By$.

Exercise 1.2.3: Pick $v_1 \in V \setminus \{0\}$ and find $w_1 \in V$ with $\beta(v_1, w_1) = 1$. Then consider the restriction β_1 of β to the subspace

$$V_1 := \{v_1, w_1\}^\perp = \{x \in V \colon \beta(x, v_1) = \beta(x, v_2) = 0\}$$

and argue by induction. Why is β_1 nondegenerate?

Exercise 1.2.5(2): Exercise 1.2.4.

Exercise 1.2.6: Use the polarization identity $\beta(v, w) = \frac{1}{4}\big(q(v+w) - q(v-w)\big)$.

Exercise 1.2.10(b): Consider the homomorphism

$$\varphi \colon \mathbb{K}^\times \to \mathrm{GL}_n(\mathbb{K}), \quad \lambda \mapsto \mathrm{diag}(\lambda, 1, \ldots, 1).$$

Exercise 1.2.13(ii): Consider the characteristic polynomial.

Exercise 1.2.15(ii): For $v_\varphi \in V$, take $\alpha_\varphi(w) := \beta(v_\varphi, w)$.

Exercise 1.3.4: Consider the eigenvectors with respect to left multiplication.

Exercise 2.1.1(a): Choose a basis in each space X_j and expand β accordingly.

Exercise 2.1.3: Use Exercise 2.1.2(b).

Exercise 2.2.2: For $x := \mathbf{1} - g$ the *Neumann series* $y := \sum_{n=0}^\infty x^n$ converges in $M_n(\mathbb{K})$. Show that y is an inverse of g.

Exercise 2.2.5:
(a) Existence (Jordan normal form), Uniqueness (what can you say about nilpotent diagonalizable matrices?)
(c) If A commutes with X, it preserves the generalized eigenspaces of X (Exercise 1.1.1), and this implies that it commutes with X_s, which is diagonalizable and whose eigenspaces are the generalized eigenspaces of X.

Exercise 2.2.6(a): Existence: Put $g_u := \mathbf{1} + g_s^{-1} g_n$.

Exercise 2.2.7: Choose a matrix $g \in \mathrm{GL}_n(\mathbb{C})$ for which $A' := gAg^{-1}$ is in Jordan normal form $A' = D + N$ (D diagonal and N strictly upper triangular).

Then show that the boundedness of $e^{\mathbb{R}A}$ implies $N = 0$ and the boundedness of the subset $e^{\mathbb{R}D}$.

Exercise 2.2.8: Jordan decomposition.

Exercise 2.2.9:
(b) Use the Jordan normal form to derive some information on the eigenvalues of matrices of the form e^x which is not satisfied by all elements of $\mathrm{GL}_2(\mathbb{R})_+$. (Either the spectrum is contained in the positive axis or its consists of two mutually conjugate complex numbers). The matrix $g := \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ is not contained in the image of exp.
(c) e^X commutes with X.

Exercise 2.2.10: Use Proposition 2.2.1 and Exercise 2.2.7.

Exercise 2.2.12: Use Exercise 2.2.11.

Exercise 2.3.1: Write $x \in \mathbb{K}^n$ as a sum $x = \sum_j x_j$, where $Ax_j = \lambda_j x_j$ and calculate $\|Ax\|^2$ in these terms.

Exercise 2.3.2: Use the multiplicative Jordan decomposition: Each $g \in \mathrm{GL}_n(\mathbb{C})$ can be written in a unique way as $g = du$ with d diagonalizable and u unipotent with $du = ud$; see also Proposition 2.3.3.

Exercise 3.1.2(3): $(\mathbf{1} + tE_{ij})^{-1} = \mathbf{1} - tE_{ij}$.

Exercise 3.1.3: $\operatorname{ad} X = \lambda_X - \rho_X$ and both summands commute.

Exercise 3.1.4: (1) implies $\exp X \exp Y \exp -X = \exp(e^{\operatorname{ad} X}.Y) = \exp Y$. Now conclude that $e^{\operatorname{ad} X}.Y = Y$ (Proposition 2.3.3) and then use Exercise 3.1.3 and Corollary 2.3.4.

Exercise 3.1.7: Use the linearity of the integral to see that every linear functional vanishing on F vanishes on I_t. Why does this imply the assertion?

Exercise 3.1.8: If $\|\cdot\|_1$ is any norm on $\mathfrak{g}$, then the continuity of the bracket implies that $\|[x,y]\|_1 \leq C\|x\|_1\|y\|_1$. Modify $\|\cdot\|_1$ to obtain $\|\cdot\|$.

Exercise 3.1.9: Show that

$$\|x * y\| \leq \|x\| + e^{\|x\|}\|y\| \sum_{k>0} \frac{1}{k+1}(e^{\|x\|+\|y\|} - 1)^k.$$

Exercise 4.1.1(iii): If A is commutative with unit $\mathbf{1}$, then $\operatorname{der}(A_L) = \operatorname{End}(A)$, but every derivation D of A satisfies $D\mathbf{1} = 0$.

Exercise 4.1.7: Consider the matrix units E_{ij} with a single nonzero entry 1 in position (i,j) and calculate their brackets.

Exercise 4.2.1: Write $V = \bigoplus_{i=1}^n V_i$ with $V_i := V_{\lambda_i}(x)$ for the generalized eigenspace decomposition of x and write accordingly each $A \in \operatorname{End}(V)$ as a matrix $A = (A_{ij})$ with $A_{ij} \in \operatorname{Hom}(V_j, V_i)$. Then show that $\operatorname{ad} x - (\lambda_i - \lambda_j)\mathbf{1}$ acts nilpotently on each space $\operatorname{Hom}(V_j, V_i)$ (cf. Exercise 3.1.3).

Exercise 4.2.5: Show that $C^{2m+1}(\mathfrak{a}+\mathfrak{b}) \subseteq C^{m+1}(\mathfrak{a}) + C^{m+1}(\mathfrak{b})$ for each $m \in \mathbb{N}$ by writing each $(2m+1)$-fold bracket of elements of $\mathfrak{a}+\mathfrak{b}$ as a sum of iterated brackets of elements of $\mathfrak{a}$, resp., $\mathfrak{b}$.

Exercise 4.2.7: Use a suitable induction.

Exercise 4.3.2: Use Exercise 1.1.1(b) to find invariant complements.

Exercise 4.3.5: Use exercises 4.3.2, 4.3.3, and 4.3.4.

Exercise 4.3.6: Proceed by induction on the degree of f. If $f = g \cdot h$, then either $g(\lambda_1) = 0$ or $h(\lambda_1) = 0$. Then split off one factor $X - \lambda_1$.

Exercise 4.3.8: Use the Fitting decomposition (Lemma 4.3.11) to find an A-invariant complement of $V^0(A)$ on which A is invertible.

Exercise 4.4.6: Consider the polynomials $f_i(t) := \prod_{j \neq i} \frac{t - x_j}{x_i - x_j}$ of degree $n-1$.

Exercise 4.5.1: Example 4.4.2(iv) and Theorem 4.5.11.

Exercise 4.5.5(ii): Corollary 4.2.7 and Exercise 4.1.9.

Exercise 4.5.6: Use Exercise 4.5.5 to see that $\operatorname{ad} x \operatorname{ad} y$ is nilpotent for $x \in \mathfrak{g}$ and $y \in \mathfrak{n}$.

Exercise 4.5.7: Proceed by the following steps: Let $\mathfrak{a} := [\mathfrak{g}, \mathfrak{g}]^\perp$.

(i) $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{g}^\perp$.
(ii) Apply the Cartan criterion 4.4.17 to $[\mathfrak{a}, \mathfrak{a}]$ in order to show that $\mathfrak{a}$ is a solvable ideal.
(iii) $\mathfrak{a} \subseteq \operatorname{rad}(\mathfrak{g})$.
(iv) $[\operatorname{rad}(\mathfrak{g}), \mathfrak{g}] \subseteq \mathfrak{g}^\perp$ (Corollary 4.4.15 and Exercise 4.5.6).
(v) $\operatorname{rad}(\mathfrak{g}) \subseteq \mathfrak{a}$.

Exercise 4.5.8: Use that $\mathfrak{gl}(V)$ is abelian.

Exercise 4.5.10:

(i) Consider the eigenspaces of the linear endomorphism $\varphi \in \operatorname{End}(\mathfrak{g})$ defined by $\kappa(x, y) = \kappa(\varphi(x), y)$ for all $x, y \in \mathfrak{g}$. They are ideals in $\mathfrak{g}$.
(ii) Consider a simple Lie algebra over $\mathbb{C}$, viewed as a real Lie algebra. It is simple, but its complexification is not.

Exercise 4.7.2: Use a Levi decomposition of $\mathfrak{g}$.

Exercise 5.1.2: Consider the Taylor expansion of p in some point $x \in U$.

Exercise 5.1.3: For $p(x), p(y) \neq 0$, consider the affine line $x + \mathbb{C}(y - x)$ spanned by x and y and show that it intersects $p^{-1}(\mathbb{C}^\times)$ in a connected set.

Exercise 5.1.4: Show that the Taylor expansion of p vanishes if p vanishes on V.

Exercise 5.2.1: Proceed along the following steps:

(i) If V is generated by $v_0 \in V_\lambda(h)$, then there exists a basis $(v_0, \ldots, v_n)$ of V with

$$h \cdot v_k = (\lambda + k)v_k \quad \text{and} \quad e \cdot v_k = \begin{cases} v_{k+1} & \text{if } k < n, \\ 0 & \text{if } k = n \end{cases}.$$

We write $V(\lambda, n)$ for the $(n+1)$-dimensional $\mathfrak{g}$-module, defined by these relations.

(ii) If $k \leq n$, then $V(\lambda + k, n - k)$ is a submodule of $V(\lambda, n)$.

(iii) Use Lie's Theorem to show that each simple finite-dimensional $\mathfrak{g}$-module is isomorphic to some $V(\lambda, 0)$.

(iv) For each finite-dimensional representation (ρ, V) of $\mathfrak{g}$, the operator $\rho(e)$ is nilpotent and for each n the subspaces $\ker\big(\rho(e)^n\big)$ and $\operatorname{im}\big(\rho(e)^n\big)$ are invariant under $\rho(h)$, hence $\mathfrak{g}$-submodules.

(v) Show that each finite-dimensional representation (ρ, V) for which $\rho(h)$ is diagonalizable is a direct sum of modules of the form $V(\lambda, n)$. Hint: Derive a Jordan normal form of $\rho(e)$, adapted to the eigenspace decomposition of $\rho(h)$.

Exercise 6.1.1:

(ii) Let p, q, z, h be the basis in Example 4.1.19, then put

$$\beta(ap + bq + cz + dh, a'p + b'q + c'z + d'h) = aa' + bb' + cd' + c'd.$$

(v) With (iii) and (iv), conclude that Δ lies in the center of the associative subalgebra of $\operatorname{End}\big(C^\infty(\mathbb{R}^n)\big)$ generated by the angular momentum operators.

Exercise 6.2.2: Consider the semidirect product $\mathfrak{g} = \mathbb{K}[X] \rtimes_M \mathbb{K}$, where $\mathbb{K}[X]$ is considered as an abelian Lie algebra and $Mf(X) := Xf(X)$ is the multiplication with X.

Exercise 6.4.3(i): Universal property of $\mathcal{U}(\mathfrak{g})$.

Exercise 6.5.1(b): If $\dim V = \infty$, then V contains a copy of the polynomial algebra $\mathbb{K}[X]$. Then consider the operators

$$P(f) = f' \quad \text{and} \quad Q(f) = Xf.$$

If $\operatorname{char}(\mathbb{K})$ divides $n := \dim V < \infty$, then we think of V as $\mathbb{K}[X]/(X^n)$. Since $P(X^n) = nX^{n-1} = 0$, both operators P and Q preserve the ideal (X^n) and induce operators on $\mathbb{K}[X]/(X^n)$ with $[P, Q] = \mathbf{1}$.

Exercise 6.5.5(ii): Show that the $\mathfrak{sl}_2(\mathbb{R})$-module $\mathbb{R}^2 \otimes \mathbb{R}^2$ is isomorphic to the $\mathfrak{sl}_2(\mathbb{R})$-module $\mathfrak{gl}_2(\mathbb{R})$ with $x \cdot y := [x, y]$.

Exercise 7.2.1: Consider the two charts from Remark 7.2.2(b) and the chart (ζ, W) with $\zeta(x) = x$ and $W =]1, 2[$.

Exercise 7.2.2: Add all n-dimensional charts which are C^k-compatible with the atlas.

Exercise 7.2.18(2): $\mathtt{d}f(x)y = y^\top Bx + x^\top By = z$ can be solved with the Ansatz $y := \frac{1}{2}xz$.

Exercise 7.3.1(5): Use (4) to separate points in different tangent spaces by disjoint open sets.

Exercise 8.1.4(4): Apply the Inverse Function Theorem to the map

$$\Phi\colon G \times G \to G \times G, (x, y) \mapsto (x, xy).$$

Exercise 8.1.5(2): For $a \in A^\times$ we have $\lambda_{a^{-1}} = \lambda_a^{-1}$.

Exercise 8.2.9: Apply the Uniqueness Lemma to functions of the form $f \circ \lambda_x$, $x \in G$.

Exercise 8.2.10: Apply the Uniqueness Lemma to functions of the form $f \circ \lambda_x$, $x \in G$.

Exercise 8.2.11: Every skew-symmetric matrix $x \in \mathfrak{su}_2(\mathbb{C})$ is conjugate to a diagonal matrix $\lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Exercise 8.2.13: Use Exercise 8.2.12

Exercise 8.3.3(b): Let $g \in \overline{H}$ and U an open **1**-neighborhood in G for which $U \cap H$ is closed. Show that:

(1) $g \in HU^{-1}$, i.e., $g = hu^{-1}$ with $h \in H$, $u \in U$.
(2) $\overline{H}$ is a subgroup of G.
(3) $u \in \overline{H} \cap U = \overline{H \cap U} = H \cap U$.
(4) $g \in H$.

Exercise 8.3.4: Use induction on $\dim \operatorname{span} D$.

(1) Show that D is closed.
(2) Show that we may w.l.o.g. assume that $\operatorname{span} D = \mathbb{R}^n$.
(3) Every compact subset $C \subseteq \mathbb{R}^n$ intersects D in a finite subset.
(4) Assume that $\operatorname{span} D = \mathbb{R}^n$ and assume that there exists a basis $f_1, \ldots, f_n$ of $\mathbb{R}^n$, contained in D, such that the hyper-plane $F := \operatorname{span}\{f_1, \ldots, f_{n-1}\}$ satisfies $F \cap D = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_{n-1}$. Show that

$$\delta := \inf \Big\{ \lambda_n > 0 \colon (\exists \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}) \ \sum_{i=1}^n \lambda_i f_i \in D \Big\} > 0.$$

Hint: It suffices to assume $0 \leq \lambda_i \leq 1$ for $i = 1, \ldots, n$ and to observe (4).
(5) Apply induction on n to find $f_1, \ldots, f_n$ as in (4) and pick $f_n' := \sum_{i=1}^n \lambda_i f_i \in D$ with $\lambda_n = \delta$. Show that $D = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_{n-1} + \mathbb{Z}f_n'$.

Exercise 8.3.6:

(1) Use Zorn's Lemma to reduce the situation to the case where G is generated by H and one additional element.
(2) Extend $\mathrm{id}_D\colon D \to D$ to a homomorphism $f\colon G \to D$ and define $H := \ker f$.

Exercise 8.4.4: For $\gamma \in \Gamma$, consider the map $G \to \Gamma, g \mapsto g\gamma g^{-1}$.

Exercise 8.4.6: α is the unique lifting of $\gamma : [0,1] \to \mathrm{SL}_2(\mathbb{R}), t \mapsto e^{t2\pi u}$. If α is not injective and $\alpha(n) = \mathbf{1}$, then $q \circ \alpha|_{[0,n]}$ homotopic to the zero map, contradicting to the relation $[q \circ \alpha|_{[0,n]}] = n[\gamma]$ in $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$.

Exercise 8.5.2(3): Consider $q_G \colon \mathbb{R} \to \mathbb{T}, x \mapsto e^{2\pi i x}$.

Exercise 8.6.1: Show first that for each $y \in K_Y$ there exists an open subset U_y of Y with $K_X \times U_y \subseteq V$, then use the compactness of Y.

Exercise 9.2.1: S Note that $\widetilde{\omega}(X) = \omega \circ X$ if we consider ω as a function on TM.

Exercise 9.2.2: Use Theorems 7.4.18 and 9.2.23, as well as the fact that the double dual of a finite-dimensional vector space is isomorphic to this vector space.

Exercise 9.2.5: see Remark 7.3.25 for the case $\mathcal{T}^{0,1}(T_pM)$.

Exercise 9.2.9(1): For each $t \in \mathbb{R}$, the map Φ_t is affine and the translation part is $\frac{e^{tA}-\mathbf{1}}{A}b$.

Exercise 9.4.3: If $0 \leq f \in C_c(G)$, then $\mu(fh^2) = 0$, too. Now, for any $g \in G$, use Corollary 9.3.28 to find a function $0 \leq f \in C_c(G)$ with $f(g) = 1$. Then use Remark 9.3.34.

Exercise 10.2.1(iii): Identify $\mathbb{C}$ with $\mathbb{R}^2$ and define the homomorphism $\beta \colon G \to \mathrm{Mot}_2(\mathbb{R})$ by $\alpha(z,t)(x) := e^{it}x + z$.

Exercise 11.1.4: Pick a nonzero element $x \in \mathfrak{g}$ and note that $\mathbb{C}x$ is a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Then use the corresponding root decomposition and the complex conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 11.2.2(ii): Every element of the Lie algebra $\mathfrak{g}$ is represented by a skew-symmetric matrix E and E^2 is negative semi-definite. Now apply (i).

Exercise 11.2.4: Lemma 11.2.1 and Exercise 11.2.3.

Exercise 11.2.5: Main Theorem on Maximal Tori and Exercise 11.2.3.

Exercise 11.2.9(c): Lemma 11.1.14.

Exercise 11.2.10:

(b) Consider a limit point g of the sequence $\exp(nx)$ with $\exp(m_k x) \to g$. Then $\exp\big((m_{k+1} - m_k)x\big) \to \mathbf{1}$.
(c) For $n \in \mathbb{N}$ and $\exp(n_k x) \to \mathbf{1}$, we have $\exp\big((n_k - n)x\big) \to \exp(-nx)$.

Exercise 11.3.1:

(i) Show that $\|g_i v - gv\|^2 = 2\|v\|^2 - 2\,\mathrm{Re}\langle g_i v, gv\rangle$.
(ii) Show first that $g_i \to g$ and $h_i \to h$ implies that $g_i^{-1}h_i \to g^{-1}h$ holds weakly.

Exercise 11.4.1: If $G = U \cup V$ is a decomposition into disjoint open sets, then $UH = U$, $VH = V$, and the images of of these sets give a decomposition of G/H into disjoint open sets.

Exercise 12.1.2: Write $e^{\operatorname{ad} x} y = \cosh(\operatorname{ad} x)y + \sinh(\operatorname{ad} x)y$ and show that $\sinh(\operatorname{ad} x)y = 0$ implies $[y, x] = 0$. Use that $\operatorname{ad} x$ is diagonalizable with real eigenvalues.

Exercise 12.1.3: The critical implication is to see that if $\mathfrak{g}$ is complex and semisimple, then any real solvable ideal $\mathfrak{r}$ is trivial, which follows from the solvability of the complex ideal $\mathfrak{r} + i\mathfrak{r}$.

Exercise 12.1.4: Consider the generalized eigenspaces of elements $A \in \operatorname{End}_S(V)$ and $\ker(A - \lambda\mathbf{1})^n$ for $n \in \mathbb{N}$.

Exercise 12.2.1: Choose a complex basis $B = (b_1, \ldots, b_n)$ for V to write it as $W \oplus iW$ for $W := \operatorname{span}_{\mathbb{R}} B$. Then $A^{\mathbb{R}}$ can be represented by a block matrix

$$\begin{pmatrix} C & -D \\ D & C \end{pmatrix} \in M_2(M_n(\mathbb{R})).$$

Now verify that $\operatorname{tr}_{\mathbb{R}}(A^{\mathbb{R}}) = 2\operatorname{tr}(C)$ and $\operatorname{tr}_{\mathbb{C}}(A) = \operatorname{tr}(C) + i\operatorname{tr}(D)$.

Exercise 12.2.5: Given a representation $\pi\colon \mathfrak{g} \to \operatorname{End}(V)$ of a semisimple Lie algebra $\mathfrak{g}$, extend it to $\mathfrak{g}_{\mathbb{C}}$, then restrict to a compact real form $\mathfrak{u}$, and finally lift the resulting representation of $\mathfrak{u}$ to the simply connected group U with Lie algebra $\mathfrak{u}$.

Exercise 12.3.1: $x^2 = -(\det x)\mathbf{1} = k(x)\mathbf{1}$ for all $x \in \mathfrak{sl}_2(\mathbb{R})$ (Cayley–Hamilton).

Exercise 12.3.3(e): Exercise 12.3.1 and (b).

Exercise 12.3.4(b): Choose $a_0 := h \in \mathfrak{a}$ to define Δ^+.

Exercise 13.2.7: Use Exercise 13.2.6 for one direction and argue for the converse direction that all compositions factors $D^k(\Gamma)/D^{k+1}(\Gamma)$ of the derived series are finitely generated abelian groups, hence polycyclic.

Exercise 13.4.3:

(c) Consider the simply connected group G with $\mathbf{L}(G) = \mathfrak{g}$ and use that the exponential function of $G/\langle \exp_G \mathfrak{a} \rangle$ is bijective.
(d) Compute the exponential function as in Lemma 13.4.7.
(e) Show that $\exp_G$ is injective if G is simply connected. Assuming the contrary, there exists an $x \in \mathfrak{g}$ such that $\exp_G$ is not injective on the subalgebra $\mathfrak{b} := \mathfrak{a} \rtimes \mathbb{R}x$. By (c), $\mathfrak{b}$ is exponential, and one thus obtains a contradiction to (d).

Exercise 13.4.4: Exercise 13.4.1.

Exercise 13.4.5: Exercise 13.4.1.

Exercise 13.5.1: Corollary 13.5.5 and the proof of Theorem 13.2.7.

Exercise 14.1.1: Exercise A.1.3.

Exercise 14.1.2(a): Consider the antiholomorphic involution σ on $\widetilde{G}_{\mathbb{C}}$ to see that $\eta_G(D)$ is closed.

Exercise 14.1.4: Consider the group $G := \big(\widetilde{\mathrm{SL}_2(\mathbb{R})} \times \mathbb{R}\big)/D$, where

$$D = \langle (\exp \pi u, 1), (\exp(-\pi u), \sqrt{2}) \rangle \quad \text{and} \quad u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}).$$

In this case, $\widetilde{G} = \mathrm{SL}_2(\mathbb{R})^{\sim} \times \mathbb{R}$, $\widetilde{G}_{\mathbb{C}} \cong \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}$, and we have

$$\eta_G(D) = \langle (-\mathbf{1}, 1), (-\mathbf{1}, \sqrt{2}) \rangle \supseteq \{0\} \times (\mathbb{Z} + \sqrt{2}\mathbb{Z}) \quad \text{in} \quad Z(\widetilde{G}_{\mathbb{C}}) = \{\pm \mathbf{1}\} \times \mathbb{C}.$$

Exercise 14.1.5: $\mathrm{Ad}_G(D) = \{\mathrm{id}_{\mathbf{L}(G)}\}$.

Exercise 14.1.8: Note that $h \in Z(H)$ implies that $\mathrm{Ad}(h)|_{\mathbf{L}(H)} = \mathrm{id}_{\mathbf{L}(H)}$, which implies $h \in Z_H(G_0)$.

Exercise 15.2.1:

(a) Lemma 13.3.3 and the construction of $\mathfrak{b}$ in the proof of Proposition 15.2.4.
(c) Corollary 4.6.9, Remark 6.4.7, and Lemma 13.3.5.

Exercise 15.2.2: $\mathfrak{t}$ contains the center and is closed by Lemma 13.2.6.

Exercise 15.2.3: Without loss of generality, G may assumed to be connected and abelian. Then use Exercise 8.3.5.

Exercise 15.2.4: Exercise 15.2.3.

Exercise 15.2.5: Corollary 13.5.6(e) and Exercise 15.2.4.

Exercise 15.2.6: Use Corollary 15.2.8 to find a faithful representation of G/G' with closed range. To this end, it is useful to show that

$$G/G' \cong B/(B \cap G') \times H/H',$$

where $B/(B \cap G')$ is a vector group and H/H' is a torus. This follows from $G' \cong (B \cap G') \rtimes H'$.

Exercise 15.2.7: If this is not the case, then, by Exercise 15.2.5, one finds an $x \in \mathfrak{a} \cap \mathfrak{b}$ such that $\varphi(\exp_G \mathbb{R}x)$ is not contained in $\varphi(G)$. By Exercise 15.2.6, $x \in \mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{g}'$. But this element lies in the center of the maximal nilpotent ideal (Exercise 15.2.1(c)).

Exercise 16.2.1 (a): In $\mathrm{GL}_n(\mathbb{C})$, we have

$$\begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & i\mathbf{1}_q \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & i\mathbf{1}_q \end{pmatrix}^{\top} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{pmatrix}.$$

Exercise 16.4.4(ii): Since $\mathtt{d}q(v) = 2\beta(v, \cdot)$, the singularity of q in $v \in H$ implies that $2\beta(v, \cdot) \in \mathbb{K}^{\times}\alpha$ because β is nondegenerate. Now $\alpha(v) = 1$ leads to $\beta(v, v) \neq 0$, so that all points in the 0-level set of q are regular.

Exercise 17.3.2(e): Use (b) and Exercise 17.3.1.

Exercise A.1.2: Exercise A.1.1 helps to glue homotopies.

Exercise A.1.3:
(c) Consider $H(t, s) := \frac{(1-s)\gamma(t) + s\alpha(t)}{\|(1-s)\gamma(t) + s\alpha(t)\|}$.

(e) Let $p \in \mathbb{S}^n \setminus \operatorname{im} \beta$. Using stereographic projection, where p corresponds to the point at infinity, show that $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible.

Exercise A.1.5: Mimic the argument in the proof of Lemma A.1.8.

Exercise A.2.1: Consider the map

$$G\colon I^2 \to I^2, \quad G(t,s) := \begin{cases} (2t, s) & \text{for } 0 \le t \le \frac{1}{2}, s \le 1-2t, \\ (1, 2t-1) & \text{for } \frac{1}{2} \le t \le 1, s \le 2t-1, \\ (t + \frac{1-s}{2}, s) & \text{else} \end{cases}$$

and show that it is continuous. Take a look at the boundary values of $F \circ G$.

Exercise B.1.2(iii): Use (ii) and collect suitable terms. Conclude that

$$V \otimes W \cong \bigoplus_{i \in I} e_i \otimes W \cong W^{(I)}.$$

Exercise B.3.1: Use induction on the dimensional on V and compose $g \in \mathrm{O}(V, \beta)$ with a suitable reflection to set up the induction.

Exercise B.3.3: For the third one use the map

$$f\colon \mathbb{H} \times \mathbb{H} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H}), \; f(x,y)(z) := xzy^*.$$

Exercise B.3.7: Use (B.9) and the fact that each element of $\mathrm{Spin}(V, \beta)$ is a product of elements of $V^\times$.

Exercise C.3.3: It suffices to show that the image of the unit ball is bounded.

References

[Ad96] Adams, J. F., “Lectures on Exceptional Lie Groups.” Edited by Z. Mahmud and M. Mimura, Chicago Lectures in Mathematics, 1996

[ABS64] Atiyah, M. F., R. Bott and A. Shapiro, *Clifford modules*, Topology **3**, suppl. 1 (1964), 3–38

[Bak01] Baker, H. F., *On the exponential theorem for a simply transitive continuous group, and the calculation of the finite equations from the constants of structure*, J. London Math. Soc. **34** (1901), 91–127 contains: BCH formula for the adjoint representation

[Bak05] —, *Alternants and continuous groups*, Lond. M. S. Proc. (2) **3** (1905), 24–47

[BN08] Beltita, D., K.-H. Neeb, *Finite-dimensional Lie subalgebras of algebras with continuous inversion*, Studia Math. **185:3** (2008), 249–262

[Bl73] Blattner, R. J., *Quantization and representation theory*, in “Harmonic analysis on homogeneous spaces” (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), pp. 147–165, Amer. Math. Soc., Providence, R. I., 1973

[Bl86] Block, R., *On nilpotency in the Ado–Harish–Chandra Theorem on Lie algebra representations*, Proc. of thee Amer. Math. Soc. **98:3** (1986)), 406–410

[BG81] Borel, A., “Essays in the History of Lie Groups and Algebraic Groups,” Amer. Math. Soc., Providence, 2001

[BFM02] Borel, A., R. Friedman, and J. W. Morgan, *Almost commuting elements in compact Lie groups*, Mem. Amer. Math. Soc. **157** (2002), no. 747, x + 136pp

[BG81] Boutet de Monvel, K., and V. Guillemin, “The Spectral Theory of Toeplitz Operators,” Annals of Math. Studies, Princeton Univ. Press, 1981

[Bou70] Bourbaki, N., “Algèbre,” Hermann, Paris, 1970

[Bou89] —, “Lie Groups and Lie Algebras, Chs. 1-3”, Springer, 1989

[Bou90] —, *Groupes et algèbres de Lie, Chapitres 7 et 8*, Masson, Paris, 1990

[Bou82] —, *Groupes et algèbres de Lie, Chapitres 9*, Masson, Paris, 1982

[Br36] Brauer, R., *Eine Bedingung für vollständige Reduzibilität von Darstellungen gewöhnlicher und infinitesimaler Gruppen*, Math. Zeitschr. **61** (1936), 330–339

[Cam97] Campbell, J. E., *On a law of combination of operators bearing on the theory of continuous transformation groups*, Proc. of the London Math. Soc. **28** (1897), 381–390

[Cam98] —, *On a law of combination of operators. (Second paper)*, Proc. of the London Math. Soc. **29** (1898), 14–32

[Ca94] Cartan, É., "Sur la structure des groupes de transformations finis et continue", thèse, Paris, Nony, 1894

[Ca30] —, "La théorie des des groupes finis et continus et l'analysis situs", Mém. Sci. Math. XLII, Gauthier-Villars, Paris, 1930

[CW35] Casimir, H., and B. L. van der Waerden, *Algebraischer Beweis der vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen*, Math. Ann. **111** (1935), 1–12

[Ch46] Chevalley, C. "Theory of Lie grooups I", Princeton Univ. Press, 1946

[CE48] Chevalley, C. and S. Eilenberg, *Cohomology Theory of Lie Groups and Lie Algebras,* Transactions of the Amer. Math. Soc. **63** (1948) 85–124

[Dix57] Dixmier, J., *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Soc. Math. Fr. **85** (1957), 113–121

[DH97] Djokovic, D., and K. H. Hofmann, *The surjectivity question for the exponential function in real Lie groups: A status report*, J. Lie Theory **7** (1997), 171–199

[Do07] Donaldson, S. K., *Lie algebra theory without algebra*, arXiv:math.DG/0702016v2, 15 Mar 2007

[DK00] Duistermaat, J. J., and J. A. C. Kolk, "Lie groups". Springer, 2000

[Dyn47] Dynkin, E. B., *Calculation of the coefficients in the Campbell–Hausdorff formula* (Russian), Doklady Akad. Nauk. SSSR (N.S.) **57** (1947), 323–326

[Dyn53] —, "Normed Lie Algebras and Analytic Groups," Amer. Math. Soc. Translation 1953, no. 97, 66 pp

[GW09] Goodman, R., and N. R. Wallach, "Symmetry, Representations, and Invariants," Springer-Verlag, New York, 2009.

[GK96] Goze, M., and Y. Khakimdjanov, "Nilpotent Lie Algebras," Mathematics and Its Applications, Kluwer Acad. Pub., 1996

[Gr78] Greub, W., "Multilinear Algebra," Second ed., Universitext, Springer-Verlag, New York-Heidelberg, 1978

[GHV73] Greub, W., Halperin, S., and R. Vanstone, "Connections, Curvature, and Cohomology," Academic Press, New York and London, 1973

[Gr01] Grove, L. C., "Classical Groups and Geometric Algebra," Amer. Math. Soc., Providence, 2001

[GS84] Guillemin, V., and S. Sternberg, "Symplectic Techniques in Physics," Cambridge University Press, 1984

[dlH00] de la Harpe, P., "Topics in Geometric Group Theory," Chicago Lectures in Math., The Univ. of Chicago Press, 2000

[Hau06] Hausdorff, F., *Die symbolische Exponentialformel in der Gruppentheorie*, Leipziger Berichte **58** (1906), 19–48

[Haw00] Hawkins, Th., "Emergence of the Theory of Lie Groups," Springer, 2000

[He01] Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces", AMS Graduate Studies in Mathematics **34**, Providence, 2001

[HN91] Hilgert, J., and K.-H. Neeb, "Lie-Gruppen und Lie-Algebren", Vieweg, Braunschweig, 1991

[Hir76] Hirsch, M. W., "Differential Topology," Graduate Texts in Mathematics **33**, Springer-Verlag, 1976

[Ho54a] Hochschild, G., *Lie algebra kernels and cohomology*, Amer. J. Math. **76** (1954), 698–716

[Ho54b] —, *Cohomology classes of finite type and finite-dimensional kernels for Lie algebras*, Amer. J. Math. **76** (1954), 763–778

[Ho65] —, "The Structure of Lie Groups," Holden Day, San Francisco, 1965

[HM06] Hofmann, K. H., and S. A. Morris, "The Structure of Compact Groups," 2nd ed., Studies in Math., de Gruyter, Berlin, 2006

[Iwa49] Iwasawa, K., *On some types of topological groups*, Ann. of Math. **50** (1949), 507–558

[Kil89] Killing, W., *Die Zusammensetzung der stetigen endlichen Transformationsgruppen II*, Math. Ann. **33** (1889), 1–48

[Kn02] Knapp, A. W., " Lie groups beyond an introduction". Second edition. Progress in mathematics **140**, Birkhäuser, Boston, 2002

[Ko95] Kobayashi, S., "Transformation Groups in Differential Geometry". Springer, 1995

[Kos50] J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127

[KM97] Kriegl, A., and P. W. Michor, "The Convenient Setting of Global Analysis". AMS Surveys and Monographs **53**, Providence, 1997

[Lai77] Lai, H. L., *Surjectivity of the exponential map on semisimple Lie groups*, J. Math. Soc. Japan **29** (1977), 303–325

[Lai78] —, *Exponential map of a center-free complex simple Lie group*, Osaka J. Math. **27** (1978), 553–560

[La93] Lang, S., "Algebra," Addison Wesley Publ. Comp., London, 1993.

[La99] —, "Fundamentals of Differential Geometry", Grad. Texts Math. **191**, Springer-Verlag, Berlin, 1999

[Le05] Levi, E. E., *Sulla structura dei gruppi finiti e continui*, Atti Accad. Torino **40** (1905), 3–17

[LV94] D. Luminet and A. Valette, *Faithful uniformly continuous representations of Lie groups*, J. London Math. Soc. (2) **49** (1994), 100–108

[ML63] MacLane, S., "Homological Algebra," Springer-Verlag, 1963

[Ma42] Malcev, A. I., *On the representation of an algebra as a direct sum of the radical and a semi-simple subalgebra*, Dokl. Akad. Nauk SSSR, **36:2** (1942), 42–45

[Mi03] Milnor, J., *Towards the Poincaré conjecture and the classification of 3-manifolds*, Notices Amer. Math. Soc. **50:10** (2003), 1226–1233

[Mo64] Morimoto, A., *Non-compact complex Lie groups without non-constant holomorphic functions*, Proc. of Conf. on Complex analysis, Minneapolis, 1964, 256–272

[Mo66] —, *On the classification of non-compact abelian Lie groups*, Trans. of the Amer. Math. Soc.**123**(1966), 200–228

[MS08] Moskowitz, M., and R. Sacksteder, *On complex exponential groups*, Math. Res. Lett. **15** (2008), 1197-1210.

[Na97] Nahlus, N., *Note on faithful representations and a local property of Lie groups*, Proc. Amer. Math. Soc. **125:9** (1997), 2767–2769

[Ne07] Neeb, K.-H., *Nonabelian extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier **56** (2007), 209–271

[Ner02] Neretin, Y. A., *A construction of finite-dimensional faithful representation of Lie algebra*, in Proceedings of the 22nd Winter School "Geometry and Physics" (Srni 2002), Rend. Circ. Mat. Palermo (2) Suppl. **71** (2003), 159–161, arXiv:math/0202190 [math.RT]

[Ra72] Raghunathan, M. S., "Discrete Subgroups of Lie Groups," Ergebnisse der Math. **68**, Springer, 1972

[RV81] Raymond, F., and A. T. Vasquez, *3-manifolds whose universal coverings are Lie groups*, Topology and its Appl. **12** (1981), 161–179

[Ro61] Rosenlicht, M., *Toroidal algebraic groups*, Proceedings of the Amer. Math. Soc. **12**(1961), 984–988

[Ru86] Rudin, W., "Real and Complex Analysis," McGraw Hill, 1986

[Sai57] Saito, M., *Sur certains groupes de Lie résolubles*, Sci. Pap. Coll. Gen. Educ. Univ. Tokyo **7** (1957), 1–11.

[Sch85] Scharlau, W., "Quadratic and Hermitian Forms," Grundlehren der math. Wiss. **270**, Springer Verlag, 1985

[Sch95] Schottenloher, M., "Geometrie und Symmetrie in der Physik," Vieweg Lehrbuch Mathematische Physik, Vieweg Verlag, 1995

[Sch97] —, "A Mathematical Introduction to Conformal Field Theory," Lecture Notes in Physics **m 43**, Springer, 1997

[Seg83] Segal, D., "Polycyclic Groups," Cambridge Univ. Press, 1983

[Se88] Serre, J.-P., "Cours d'arithmétique", Le Mathématicien, Presse Universitaire de France, 3^e edition, 1988

[Wa88] Wallach, N. R., "Real reductive groups I", Academic Press, Boston, 1988

[We95] Weibel, C. A., "An Introduction to Homological Algebra", Cambridge Studies in Advanced Math. **38**, Cambridge Univ. Press, Cambridge, 1995

[We25] Weyl, H., *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen.*I,II, III und Nachtrag. Math. Zeitschr. **23** (1925), 271–309; **24** (1926), 328–376, 377–395, 789–791

[Wh36] Whitehead, J. H. C., *On the decomposition of an infinitesimal group*, Proc. Camb. Philos. Soc. **32** (1936), 229–237

[Wie49] Wielandt, H., *Über die Unbeschränktheit der Operatoren der Quantenmechanik*, Math. Ann. **121** (1949), 21

[Wu98] Wüstner, M., *On the surjectivity of the exponential function of solvable Lie groups*, Math. Nachrichten **192** (1998), 255–266

[Wu03] —, *Supplements on the theory of exponential Lie groups*, J. Algebra **265** (2003), 148-170.

[Wu05] —, *The classification of all simple Lie groups with surjective exponential map.* J. Lie Theory **15** (2005), 269–278.

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