Transformation Groups acting on the Real Plane and their Differential Invariants

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Abstract

Realizations of finite-dimensional Lie algebras on the real plane are reviewed. A complete set of differential invariants and Lie determinants of continuous transformation groups acting on the real plane is constructed.

1 Introduction

Differential invariants emerged as one of the most important tools in investigation of differential equations in the works of S. Lie. In 1884 [8] he proved that any non-singular invariant system of differential equations can be expressed in terms of differential invariants of the corresponding symmetry group. In the same paper he also applied differential invariants to integration of ODEs. If differential invariants of a Lie group are known, the differential equations admitting this group can be easily described and the special representation (so-called group foliation) of such differential equations can be constructed.

Differential invariants of all finite-dimensional local transformation groups on a space of two complex variables were described by S. Lie himself in [10]. A modern treatment of these results was adduced in [15]. Namely, functional bases of differential invariants, operators of invariant differentiation and Lie determinants were constructed for all inequivalent realizations of point and contact finite-dimensional transformation groups on the complex plane. The real finite-dimensional Lie algebras of contact vector fields and their differential invariants were completely classified in [2]. Differential invariants of an one-parameter group of local transformations in the case of arbitrary number of dependent and independent variables were studied in [18].

The subject of this paper is exhaustive description of differential invariants and Lie determinants of finite-dimensional Lie groups acting on the real plane. A necessary prerequisite to do it is the classification of Lie algebra realizations in vector fields on the real plane up to local diffeomorphisms.

The plan of the paper is following. In Section 2 we discuss and compare different classifications of realizations of finite-dimensional Lie algebras on the real and complex planes, which are available in literature. In particular, we thoroughly study the question of parametrization and equivalence in series of realizations. In Section 3 some definitions and results concerning differential invariants are adduced. At the appendix of the paper the classification of realizations of finite-dimensional Lie algebras of vector fields on the real plane, the complete set of differential invariants, operators of invariant differentiation, Lie determinants and transformations that reduce real Lie algebras to complex ones are arranged in the form of Tables 1–3.

2 Realizations of Lie algebras on the real and complex planes

There exist two important classification problems in the classical theory of Lie algebras.

The first one is classification of Lie algebra structures, i.e. classification of possible commutation relations between basis elements. All the possible complex Lie algebras of dimensions no greater than four were listed by S. Lie himself [9]. The semisimple Lie algebras [3] and the Lie algebras of dimensions no greater than six [12, 13, 14, 11, 21] over the complex and real fields were classified later. Unfortunately, this problem can not be solved in the case of arbitrary fixed dimension, since it is wild, i.e. it contains the problem of reduction of two matrices to a canonical form as a subproblem. A wider review on the subject can be found e.g. in [19].

The other problem established by S. Lie is the problem of description of different Lie algebra representations and realizations, particularly, by vector fields up to local diffeomorphisms. Realizations by vector fields are widely applicable in the general theory of differential equations, integration of differential equations and their systems, group classification of ODEs and PDEs, classification of gravity fields of a general form under the motion groups etc.

Complete classifications of realizations of Lie algebras by vector fields in one real, one and two complex variables were obtained by S. Lie [9]. In 1990 A. Gonzalez-Lopez and coauthors ordered the Lie's classification of complex Lie algebras [6] and extended it to the the real case [7]. A complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables was constructed in [19].

Starting from the above results, we detailed and amended the classification of realizations of finite-dimensional Lie algebras on the real plane. The obtained classification is compared with real [7] and complex [9] classifications and arranged in the Table 1.

The nontrivial transformations over the complex field that reduce realizations from [7] to realizations from [15] are adduced in Table 2.

Notations. Below we denote ∂/∂_x, ∂/∂_y, ... as ∂_x, ∂_y, ... The indices i and j run from 1 to r. The label N_0 consists of two parts which denote the page (from 57 to 73) and realization numbers in [9] correspondingly. The labels N_1 and N_2 coincide with the numerations of real and complex realizations in [7, 15]. N_3 corresponds to the numeration of realizations introduced in [19] in accordance with the Mubarakzyanov's classification of real low-dimensional Lie algebras [12]. The symbol N without subscripts correspond to the numeration used in the present paper.

Remark 1. The case $N = 4$ is missed in [7] but it can be simply joined to the case $N_1 = 24$ after replacement the condition $r \geq 1$ by $r \geq 0$.

Remark 2. There are two different approaches to classification of Lie algebra realizations by vector fields. According to the first approach, one should start from classification of Lie algebras of fixed dimension and then look for basis vector fields that satisfy the given commutation relations. The second approach consists in the direct construction of finite-dimensional spaces of vector fields, which are closed with respect to the standard Lie bracket. If a complete list of realizations of a fixed dimension is constructed then the problem of separation of the realizations for a given Lie algebra from others arises and becomes nontrivial in the case of parameterized series of realizations.

Example 1. Consider, for example, the series $\{A^b_{4.8}\}$ [12] of real four-dimensional Lie algebras parameterized with the parameter $|b| \le 1$. For a fixed value of b, the basis elements of $A^b_{4.8}$ satisfy the canonical commutation relations

$$[e_1, e_3] = e_1, \quad [e_1, e_4] = (1+b)e_1, \quad [e_2, e_3] = e_2, \quad [e_3, e_4] = be_3.$$

In the framework of the first approach we obtain two inequivalent realizations in vector fields on a space of two variables

$$\langle \partial_x,\ \partial_y,\ y\partial_x,\ (1+b)x\partial_x + y\partial_y \rangle \quad \text{and} \quad \langle \partial_x,\ y\partial_x,\ -\partial_y,\ (1+b)x\partial_x + by\partial_y \rangle \tag{1}$$

of the algebra $A^b_{4.8}$ if $|b| < 1$. There is a unique inequivalent realization in the case $b = \pm 1$ since under this condition realizations (1) are equivalent and we have to choose only one of them.

S. Lie [9] used the second approach to construct all possible realization of finite-dimensional Lie algebras on the plane. The algebras from the series $\{A^b_{4.8}\}$ are represented in the obtained list by the following realizations

$$\langle \partial_y, x\partial_y, \partial_x, x\partial_x + \tilde b y\partial_y \rangle,\ \tilde b \in \mathbb{R}, \quad \text{and} \quad \langle \partial_y, x\partial_y, y\partial_y, \partial_x \rangle. \tag{2}$$

In fact, the sets of realizations (1) and (2) coincide. To show it, we redenote the variables x and y in (2) (namely, $x \leftrightarrow y$) at first and shift the parameter $\tilde b$: $\tilde b = 1 + b'$. After reordering the basis in the first realization from (2) in the case $|b'| \le 1$, we obtain the first realization from (1), where $b = b'$. If $|b'| > 1$, the first realization from (2) is reduced to the second realization from (1) with $b = 1/b'$ by the additional simultaneous transformations of the basis and realization variables: $\tilde e_1 = b'e_1$, $\tilde e_2 = e_3$, $\tilde e_3 = -b'e_2$, $\tilde e_4 = be_4$; $\tilde x = bx$, $\tilde y = by$. The second realization from (2) coincides with the second one from (1), where $b = 0$.

The above consideration explains in some way why the parameter values $b = \pm 1$ are singular for the Lie algebra series $\{A^b_{4.8}\}$ from the viewpoint of number of realizations.

Remark 3. It is clear that realizations from different series adduced in Table 1 are inequivalent each to other but there can exist equivalent realizations belonging to the same series.

Example 2. Consider the series of realizations

$$N = 48: \quad \langle \xi_1(x)\partial_y,\ \xi_2(x)\partial_y,\ \ldots,\ \xi_r(x)\partial_y \rangle, \quad r \ge 5, \tag{3}$$

$$N = 49: \quad \langle y\partial_y,\ \xi_1(x)\partial_y,\ \xi_2(x)\partial_y,\ \ldots,\ \xi_r(x)\partial_y \rangle, \quad r \ge 4 \tag{4}$$

parameterized with arbitrary linearly independent real-valued functions ξ_i.

Any realization from series (3) or (4) pass into realizations from the same series under the basis transformations with non-singular constant matrices (c_{ij}) and the non-singular variable transformations $\tilde x = \varphi(x)$, $\tilde y = \psi(x)y$. By means of these equivalence transformations the parameter-functions ξ_i change in the following way $\tilde\xi_i(\tilde x) = c_{ij}\psi(x)\xi_j(x)|_{\tilde x = \varphi(x)}$. Consequently, without loss of generality we can put $\tilde\xi_1 = 1$ and $\tilde\xi_2 = \tilde x$. Hence, the series of realizations (3) and (4) takes the form adduced in the Table 1, namely

$$\langle \partial_{\tilde y},\ \tilde x\partial_{\tilde y},\ \tilde\xi_3(\tilde x)\partial_{\tilde y},\ \ldots,\ \tilde\xi_r(\tilde x)\partial_{\tilde y} \rangle \quad \text{and} \quad \langle \tilde y\partial_{\tilde y},\ \partial_{\tilde y},\ \tilde x\partial_{\tilde y},\ \tilde\xi_3(\tilde x)\partial_{\tilde y},\ \ldots,\ \tilde\xi_r(\tilde x)\partial_{\tilde y} \rangle. \tag{5}$$

Accurately speaking, the series with normalized forms (5) also contain equivalent realizations, and the corresponding equivalence transformations are restrictions of the aforesaid ones.

Example 3. Another example is given by two series of realizations

$$N=50: \quad \langle \partial_x, \eta_1(x)\partial_y, \ \dots, \ \eta_r(x)\partial_y\rangle, \quad r\geq 4, \tag{6}$$

$$N=51: \quad \langle \partial_x, y\partial_y, \eta_1(x)\partial_y, \ \dots, \ \eta_r(x)\partial_y\rangle, \quad r\geq 3 \tag{7}$$

parameterized with real functions η_i which form a fundamental system of solutions for an r-order ordinary differential equation with constant coefficients

$$\eta^{(r)}(x)+c_1\eta^{(r-1)}(x)+\dots+c_r\eta(x)=0.$$

The transformations that reduce any realization from the series $N=50$ and $N=51$ to a realization from the same series are generated by the changes of basis with non-singular constant matrices (c_{ij}) and the variable transformations $\tilde{x}=a_1x+a_0$, $\tilde{y}=by+f(x)$, where $f(x)=b_0\eta_0(x)+b_1\eta_1(x)+\dots+b_r\eta_r(x)$, $a_1,a_0,b,b_0,\dots,b_r\in\mathbb{R}$, $a_1b\neq 0$. The function $\eta_0(x)$ is a solution of the ODE $\eta_0^{(r)}(x)+c_1\eta_0^{(r-1)}(x)+\dots+c_r\eta_0(x)=1$ and in the case $N=51$ additionally $b_0=0$. These equivalence transformations act on the functions η_i as follows: $\tilde{\eta}_i(a_1x+a_0)=c_{ij}\eta_j(x)$.

3 Differential invariants

Consider a local r-parametric transformation group G acting on $M\subset X\times Y=\mathbb{R}\times\mathbb{R}$ and denote a prolonged transformation group acting on the subset of jet space $M^{(n)}=M\times\mathbb{R}^n$ as $\mathrm{pr}^{(n)}G$. Let $\mathfrak{g}$ be the r-dimensional Lie algebra with basis of infinitesimal operators $\{e_i=\xi_i(x,y)\partial_x+\eta_i(x,y)\partial_y\}$ which corresponds to G. Then the prolonged algebra $\mathrm{pr}^{(n)}\mathfrak{g}$ is generated by the prolonged first-order differential operators [16, 17]:

$$e_i^{(n)}=\xi_i(x,y)\partial_x+\eta_i(x,y)\partial_y+\sum_{k=1}^{n}\eta_i^k(x,y_{(k)})\partial_{y^{(k)}}.$$

Hereafter $n\geq 0$, $i=1,\dots,r$, the symbol $y_{(k)}$ denotes the tuple $(y,y',\dots,y^{(k)})$ of the dependent variable y and its derivatives with respect to x of order no greater than k.

Definition 1 ([17]). A smooth function $I=I(x,y_{(n)})\colon M^{(n)}\to\mathbb{R}$ is called a *differential invariant* of order n of the group G if I is an invariant of the prolonged group $\mathrm{pr}^{(n)}G$, namely

$$I(\mathrm{pr}^{(n)}g\cdot(x,y_{(n)}))=I(x,y_{(n)}), \quad (x,y_{(n)})\in M^{(n)}$$

for all $g\in G$ for which $\mathrm{pr}^{(n)}g\cdot(x,y_{(n)})$ is defined.

In infinitesimal terms, $I(x,y_{(n)})$ is an n-th order differential invariant of the group G if $e_i^{(n)}I(x,y_{(n)})=0$ for any prolonged basis infinitesimal generators $e_i^{(n)}$ of $\mathrm{pr}^{(n)}\mathfrak{g}$.

Consider the series of the ranks $r_k=\mathrm{rank}\{(\xi_i,\eta_i,\eta_i^1,\dots,\eta_i^k),\ i=1,\dots,r\}$. For further statements we introduce the number $\nu=\min\{k\in\mathbb{Z}\,|\,r_k=r\}$. Since the sequence $\{r_k\}$ is nondecreasing, bounded by r and reachs the value r, the number ν exists and the relation $r_\nu=r_{\nu+1}=\dots=r$ holds true.

Definition 2 ([16]). Let $\mathrm{pr}^{(\nu)}\mathfrak{g}$ is generated by the set of the prolonged infinitesimal operators $\{e_i^{(\nu)}\}$ and L is the matrix formed by their coefficients:

$$L = \begin{pmatrix} \xi_1 & \eta_1 & \eta_1^1 & \dots & \eta_1^{(\nu)} \\ \xi_2 & \eta_2 & \eta_2^1 & \dots & \eta_2^{(\nu)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_r & \eta_r & \eta_r^1 & \dots & \eta_r^{(\nu)} \end{pmatrix}.$$

A maximal minor of L, which does not vanish identically, is called a *Lie determinant*.

Importance of the adduced notions follows from the fact that if a system of ordinary differential equations is invariant under action of the prolonged group $\mathrm{pr}^{(n)}G$ then it can be locally presented as a union of conditions of vanishing Lie determinants and equations written in terms of differential invariants of G.

A natural question is whether it is possible to choose a minimal set of differential invariants that allows us to obtain all differential invariants of the given order by a finite number of certain operations. The answer to this question is affirmative.

Below we will briefly state several results concerning differential invariants of Lie groups acting on the plane. The presented statements are well known and adduced on purpose to complete only the picture of differential invariants on the plane. Detailed definitions and statements on the theory of differential invariants in general cases, review of main results and approaches to differential invariants that are different from the presented one (such as differential one-forms and moving coframes) and possibly more convenient for other applications could be found in [4, 5, 16, 17].

Definition 3 ([17]). A maximal set I_n of functionally independent differential invariants of order no greater than n (i.e. invariants of the prolonged group $\mathrm{pr}^{(n)}G$) is called a *universal differential invariant* of order n of the group G.

Note, that dimension of the jet space $M^{(n)}$ is $\dim M^{(n)} = n+2$ and the number of functionally independent differential invariants of order n is $d_n = n+2-r_n$. Any n-th order differential invariant I of G is necessarily an $n+l$-th order differential invariant of G, $l \geq 0$. Therefore for any $n, l \geq 0$ a universal differential invariant I_{n+l} can be obtained by extension of a universal differential invariant I_n.

Definition 4 ([17]). A vector field (or a differential operator) δ on the infinitely prolonged jet space $M^{(\infty)}$ is called an *operator of invariant differentiation* of the group G if for any differential invariant I of G the expression δI is also a differential invariant of G.

Any operator δ commuting with all formally infinitely prolonged basis infinitesimal generators e_i^{∞} of the corresponding Lie algebra is an operator of invariant differentiation of G. For any Lie group acting on the real or complex planes, there exists exactly one independent (over the field of invariants of this group) operator of invariant differentiation.

For any Lie group G there exists a finite *basis of differential invariants*, i.e. a finite set of functionally independent differential invariants such that any differential invariant of G can be obtained from it via a finite number of functional operations and operations of invariant differentiation. A basis of differential invariants of the group G is always contained in a universal differential invariant $I_{\nu+1}$.

To describe completely differential invariants of all transformation groups acting on the real plane, we obtain a functional basis of differential invariants and operators of invariant differentiation for each algebra from the known list of inequivalent Lie algebras of vector fields on the plane.

Bases of differential invariants are constructed as a part of $(\nu+1)$-th order universal differential invariants using the infinitesimal approach. The constructive procedure for finding invariant differentiation operators is directly derived from the condition of their commutation with formally infinitely prolonged elements of the algebra. Namely, we look for an operator of invariant differentiation as the operator of total differentiation D_x with a multiplier λ depending on x and $y_{(\nu)}$:

$$X = \lambda(x, y_{(\nu)})D_x, \quad \text{where} \quad \lambda\colon M^{(\nu)} \to \mathbb{R}.$$

The function λ is implicitly determined by the equation $\varphi(x, y_{(\nu)}, \lambda) = 0$, where φ satisfies the condition:

$$\overline{\zeta_i^\nu}\varphi = 0, \qquad \overline{\zeta_i^\nu} = \xi_i\partial_x + \eta_i\partial_y + \eta_i^1\partial_{y'} + \cdots + \eta_i^\nu\partial_{y^{(\nu)}} + (\lambda D_x)\xi_i\partial_\lambda.$$

In other words, $\varphi(x, y_\nu, \lambda)$ should be an invariant of the vector fields $\overline{\zeta_i^\nu}$. Let us note that $\operatorname{rank}\{(\overline{\zeta_i^\nu}), i = 1, \dots r\} = r$. A universal invariant $\overline{I}$ of $\overline{\zeta_i^\nu}$ can be presented as $\overline{I} = (I_\nu, \hat{I})$, where $\hat{I}\colon M^{(\nu)} \times \mathbb{R} \to \mathbb{R}$, $\partial\hat{I}/\partial\lambda \neq 0$. So, the unknown function λ can be find from the condition $\hat{I}(x, y_\nu, \lambda) = C$ for a fixed constant C.

All the results obtained are presented in the form of Table 3 and may be used for group classification of ODEs of any finite order. So, in the future we plan to review and to generalize results of group classification of the third and fourth order ODEs that were obtained in [1, 20]. In a similar way one can describe the differential invariants of the transformations groups acting in the spaces of more than two variables and having a low number of parameters by means of using the classification of realizations of real low-dimensional Lie algebras [19] and then apply them to investigation of systems of ODEs or PDEs.

Remark 4. The form of differential invariants essentially depends on explicit form of realizations. For example, for the realization

$$\langle e^{-bx}\sin x\partial_y,\ e^{-bx}\cos x\partial_y,\ \partial_x\rangle,$$

where $b \geq 0$ ($N = 17$), a fundamental differential invariant, an operator of invariant differentiation and a Lie determinant are

$$I_2 = y'' + 2by' + (b^2+1)y, \quad X = D_x, \quad L = -e^{-2bx}.$$

For the equivalent form

$$\langle \partial_y,\ x\partial_y,\ (b-x)y\partial_y - (1+x^2)\partial_x\rangle$$

of this realization adduced in [19] the corresponding invariant values has the more complicated form:

$$I_2 = y''(1+x^2)^{3/2}e^{b\arctan x}, \quad X = (1+x^2)D_x, \quad L = -(1+x^2).$$

Remark 5. The cases marked with * in Table 3 are differ each from other by change of dependent and independent variables. They are adduced simultaneously because different forms may be convenient for different applications.

Table 1. Realizations of Lie algebras on the plane.

N	Realizations	N_1	N_0	N_3
1	∂_x	9	$57, (1)$	$R(A_1, 1)$
2	$\partial_x,\ \partial_y$	22	$57, (2)$	$R(2A_1, 1)$
3	$\partial_x,\ y\partial_x$	20	$57, (4)$	$R(2A_1, 2)$
4	$\partial_x,\ x\partial_x + y\partial_y$	—	$57, (3)$	$R(A_{2.1}, 1)$
5	$\partial_x,\ x\partial_x$	10	$57, (5)$	$R(A_{2.1}, 2)$
6	$\partial_y,\ x\partial_y,\ \varphi(x)\,\partial_y$	20	$57, (14)$	$R(3A_1, 5)$
7	$\partial_y, y\partial_y,\ \partial_x$	23	$73, (10)$	$R(A_{2.1} \oplus A_1, 3)$
8	$e^{-x}\partial_y,\ \partial_x,\ \partial_y$	22	$57, (8)$	$R(A_{2.1} \oplus A_1, 4)$
9	$\partial_y,\ \partial_x,\ x\partial_y$	22	$57, (9)$	$R(A_{3.1}, 3)$
10	$\partial_y,\ \partial_x,\ x\partial_x + (x+y)\partial_y$	25	$57, (11)$	$R(A_{3.2}, 2)$
11	$e^{-x}\partial_y,\ -xe^{-x}\partial_y,\ \partial_x$	22	$57, (7)$	$R(A_{3.2}, 3)$
12	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y$	12	$57, (10)$	$R(A_{3.3}, 2)$
13	$\partial_y,\ x\partial_y,\ y\partial_y$	21	$57, (15)$	$R(A_{3.3}, 4)$
14	$\partial_x,\ \partial_y,\ x\partial_x + ay\partial_y,\ 0 < \lvert a\rvert < 1$	12	$57, (10)$	$R(A^a_{3.4}, 2)$
15	$e^{-x}\partial_y,\ e^{-ax}\partial_y,\ \partial_x,\ 0 < \lvert a\rvert < 1$	22	$57, (6)$	$R(A^a_{3.4}, 3)$
16	$\partial_x,\ \partial_y,\ (bx+y)\partial_x + (by-x)\partial_y,\ b \geq 0$	1	$\overset{\mathbb{C}}{\sim} 57,\ (10)$	$R(A^b_{3.5}, 2)$
17	$e^{-bx}\sin x\partial_y,\ \ e^{-bx}\cos x\partial_y,\ \ \partial_x,\ \ b \geq 0$	22	$\overset{\mathbb{C}}{\sim} 57,\ (6)$	$R(A^b_{3.5}, 3)$
18	$\partial_x,\ x\partial_x + y\partial_y,\ (x^2-y^2)\partial_x + 2yx\partial_y$	2	$\overset{\mathbb{C}}{\sim} 57,\ (13);\ 73,\ (4)$	$R(sl(2,\mathbb{R}), 2)$
19	$\partial_x + \partial_y,\ x\partial_x + y\partial_y,\ x^2\partial_x + y^2\partial_y$	17	$57, (13); 73, (4)$	$R(sl(2,\mathbb{R}), 3)$
20	$\partial_x,\ 2x\partial_x + y\partial_y,\ x^2\partial_x + xy\partial_y$	18	$57, (16); 72, (10)$	$R(sl(2,\mathbb{R}), 4)$
21	$\partial_x,\ x\partial_x,\ x^2\partial_x$	11	$\overset{\mathbb{C}}{\sim} 57,\ (16);\ 72,\ (10)$	$R(sl(2,\mathbb{R}), 5)$
22	$y\partial_x - x\partial_y,\ \ (1+x^2-y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (1+y^2-x^2)\partial_y$	3	$\overset{\mathbb{C}}{\sim} 57,\ (13);\ 73,\ (4)$	$R(so(3), 1)$
23	$\partial_y,\ x\partial_y,\ \varphi(x)\partial_y,\ \psi(x)\partial_y$	20	$58, (8)$	$R(4A_1, 11)$
24	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y$	13	$58, (6)$	$R(2A_{2.1}, 5)$
25	$e^{-x}\partial_y,\ \partial_x,\ \partial_y\ y\partial_y$	23	$58, (1)$	$R(2A_{2.1}, 7)$
26	$e^{-x}\partial_y,\ -xe^{-x}\partial_y,\ \partial_x,\ \partial_y$	22	$57, (21)$	$R(A_{3.2}{\oplus}A_1, 9)$
27	$e^{-x}\partial_y,\ e^{-ax}\partial_y,\ \partial_x,\ \partial_y,\ 0 < \lvert a\rvert < 1$	22	$57, (20)$	$R(A^a_{3.4}{\oplus}A_1, 9)$
28	$e^{-bx}\sin x\partial_y,\ \ e^{-bx}\cos x\partial_y,\ \ \partial_x,\ \ \partial_y,\ \ b \geq 0$	22	$\overset{\mathbb{C}}{\sim} 57,\ (20)$	$R(A^b_{3.5}{\oplus}A_1, 8)$
29	$\partial_x,\ x\partial_x,\ y\partial_y,\ x^2\partial_x + xy\partial_y$	19	$58, (7)$	$R(sl(2,\mathbb{R}){\oplus}A_1, 8)$
30	$\partial_x,\ \partial_y,\ x\partial_x,\ x^2\partial_x$	14	$58, (3)$	$R(sl(2,\mathbb{R}){\oplus}A_1, 9)$

Table 1. (Continued.)

N	Realizations	N_1	N_0	N_3
31	$\partial_y,\, -x\partial_y,\, \frac{1}{2}x^2\partial_y,\, \partial_x$	22	$57, (23)$	$R(A_{4.1}, 8)$
32	$e^{-bx}\partial_y,\, e^{-x}\partial_y,\, -xe^{-x}\partial_y,\, \partial_x$	22	$57, (18)$	$R(A^b_{4.2}, 8), b \neq 1$
33	$e^{-x}\partial_y,\, -x\partial_y,\, \partial_y,\, \partial_x$	22	$57, (22)$	$R(A_{4.3}, 8)$
34	$e^{-x}\partial_y,\, -xe^{-x}\partial_y,\, \frac{1}{2}x^2e^{-x}\partial_y,\, \partial_x$	22	$57, (19)$	$R(A_{4.4}, 7)$
35	$\partial_y,\, x\partial_y,\, \varphi(x)\partial_y,\, y\partial_y$	21	$58, (9)$	$R(A^{1,1,1}_{4.5}, 10)$
36	$e^{-ax}\partial_y,\, e^{-bx}\partial_y,\, e^{-cx}\partial_y,\, \partial_x,\, a < b < c$	22	$57, (17)$	$R(A^{a,b,c}_{4.5}, 7)$
37	$e^{-ax}\partial_y,\; e^{-bx}\sin x\partial_y,\; e^{-bx}\cos x\partial_y,\; \partial_x$	22	$\overset{\mathbb{C}}{\sim} 57,\ (17)$	$R(A^{a,b}_{4.6}, 6)$
38	$\partial_x,\, \partial_y,\, x\partial_y,\, x\partial_x + (2y + x^2)\partial_y$	25	$58, (5)$	$R(A_{4.7}, 5)$
39	$\partial_y,\, \partial_x,\, x\partial_y,\, (1 + b)x\partial_y + x\partial_y,\, \lvert b\rvert \leq 1$	24	$58, (4)$	$R(A^b_{4.8}, 5)$
40	$\partial_y,\, -x\partial_y,\, \partial_x,\, y\partial_y$	23	$58, (2); 72, (7)$	$R(A^b_{4.8}, 7),\ b = 0$
41	$\partial_x,\, \partial_y,\, x\partial_x + y\partial_y,\, y\partial_x - x\partial_y$	4	$\overset{\mathbb{C}}{\sim} 58,\ (6)$	$R(A_{4.10}, 6)$
42	$\sin x\partial_y,\, x_2\cos x\partial_y,\, y\partial_y,\, \partial_x$	23	$\overset{\mathbb{C}}{\sim} 58,\ (1)$	$R(A_{4.10}, 7)$
43	$\partial_x,\, \partial_y,\, x\partial_x - y\partial_y,\, y\partial_x,\, x\partial_y$	5	$71, (3)$	$\dim A = 5$
44	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y,\, y\partial_x,\, x\partial_y$	6	$71, (2)$	$\dim A = 6$
45	$\partial_x,\; \partial_y,\; x\partial_x + y\partial_y,\; y\partial_x - x\partial_y,$ $(x^2 - y^2)\partial_x - 2xy\partial_y,\; 2xy\partial_x - (y^2 - x^2)\partial_y$	7	$\overset{\mathbb{C}}{\sim} 73,\ (3)$	$\dim A = 6$
46	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y,\, x^2\partial_x,\, y^2\partial_y$	16	$73, (3)$	$\dim A = 6$
47	$\partial_x,\; \partial_y,\; x\partial_x,\; y\partial_y,\; y\partial_x,\; x\partial_y,$ $x^2\partial_x + xy\partial_y,\; xy\partial_x + y^2\partial_y$	8	$71, (1)$	$\dim A = 8$
48	$\partial_y,\, x\partial_y,\, \xi_1(x)\partial_y,\, \ldots,\, \xi_r(x)\partial_y,\, r \geq 3$	20	$73, (2)$	$\dim A \geq 5$
49	$y\partial_y,\, \partial_y,\, x\partial_y,\, \xi_1(x)\partial_y,\, \ldots,\, \xi_r(x)\partial_y,\, r \geq 2$	21	$72, (8)$	$\dim A \geq 5$
50	$\partial_x,\, \eta_1(x)\partial_y,\, \ldots,\, \eta_r(x)\partial_y,\, r \geq 4$	22	$73, (1)$	$\dim A \geq 5$
51	$\partial_x,\, y\partial_y,\, \eta_1(x)\partial_y,\, \ldots,\, \eta_r(x)\partial_y,\, r \geq 3$	23	$72, (7)$	$\dim A \geq 5$
52	$\partial_x,\, \partial_y,\, x\partial_x + cy\partial_y,\, x\partial_y,\, \ldots,\, x^r\partial_y,\, r \geq 2$	24	$72, (5)$	$\dim A \geq 5$
53	$\partial_x,\, \partial_y,\, x\partial_y,\, \ldots,\, x^{r-1}\partial_y, x\partial_x + (ry + x^r)\partial_y,\, r \geq 3$	25	$72, (6)$	$\dim A \geq 5$
54	$\partial_x,\, x\partial_x,\, y\partial_y,\, \partial_y,\, x\partial_y,\, \ldots,\, x^r\partial_y,\, r \geq 1$	26	$72, (4)$	$\dim A \geq 5$
55	$\partial_x,\; \partial_y,\; 2x\partial_x + ry\partial_y,\; x^2\partial_x + rxy\partial_y,$ $x\partial_y,\; x^2\partial_y,\; \ldots,\; x^r\partial_y,\; r \geq 1$	27	$71, (4); 72, (1)$	$\dim A \geq 5$
56	$\partial_x, x\partial_x, y\partial_y, x^2\partial_x + rxy\partial_y,$ $\partial_y, x\partial_y, x^2\partial_y,\; \ldots, x^r\partial_y, r \geq 0$	15; 28	$73, (5); 72, (2)$	$\dim A \geq 5$

Here ξ_i are linearly independent real-valued parameter-functions and $\{\eta_i\}$ is a fundamental system of solutions for an r-order ordinary differential equation with constant coefficients $\eta^{(r)}(x) + c_1\eta^{(r-1)}(x) + \cdots + c_r\eta(x) = 0$.

Table 2. Transformations of real realizations to complex ones.

N_1	Transformation of space variables	Transformation of basis elements	N_2
1	$\tilde{x}=x-iy$, $\tilde{y}=x+iy$	$\tilde{e}_1=\frac{1+i}{2}(e_1+e_2)$, $\tilde{e}_2=\frac{1}{c+i}e_3$, $\tilde{e}_3=\frac{1-i}{2}(e_1-e_2)$	2.7, $k=1$
2	$\tilde{x}=x-iy$, $\tilde{y}=\frac{1}{2iy}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=e_2$, $\tilde{e}_3=e_3$	2.2
3	$\tilde{x}=-\frac{1}{ix+y}$, $\tilde{y}=\frac{ix+y}{1+x^2+y^2}$	$\tilde{e}_1=\frac{1}{2}(ie_2+e_3)$, $\tilde{e}_2=ie_1$, $\tilde{e}_3=\frac{1}{2}(e_3-ie_2)$	2.2
4	$\tilde{x}=\frac{y-ix}{2}$, $\tilde{y}=-\frac{y+ix}{2}$	$\tilde{e}_1=ie_1-e_2$, $\tilde{e}_2=ie_1+e_2$, $\tilde{e}_3=\frac{e_3+ie_4}{2}$, $\tilde{e}_4=\frac{e_3-ie_4}{2}$	2.9, $k=1$
7	$\tilde{x}=y+ix$, $\tilde{y}=y-ix$	$\tilde{e}_1=\frac{e_1+ie_2}{2i}$, $\tilde{e}_2=\frac{e_3-ie_4}{2}$, $\tilde{e}_3=\frac{e_6+ie_5}{2}$, $\tilde{e}_4=\frac{ie_2-e_1}{2i}$, $\tilde{e}_5=\frac{e_3+ie_4}{2}$, $\tilde{e}_6=\frac{e_6-ie_5}{2}$	2.4
17	$\tilde{x}=y$, $\tilde{y}=\frac{1}{x-y}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=e_2$, $\tilde{e}_3=e_3$	2.2
18	$\tilde{x}=x$, $\tilde{y}=\frac{1}{y^2}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=\frac{1}{2}e_2$, $\tilde{e}_3=e_3$	2.1
19	$\tilde{x}=x$, $\tilde{y}=\frac{1}{y}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=e_2$, $\tilde{e}_3=-e_3$, $\tilde{e}_4=e_4$	2.3

Below in the Table 3 we use the following notations:

$$S_{k+3}=(k+1)^2\big(y^{(k)}\big)^2y^{(k+3)}-3(k+1)(k+3)y^{(k)}y^{(k+1)}y^{(k+2)}+2(k+2)(k+3)\big(y^{(k+1)}\big)^3,$$

$$Q_{k+2}=(k+1)y^{(k)}y^{(k+2)}-(k+2)\big(y^{(k+1)}\big)^2,\qquad \tilde{Q}_3=y'''B_1-3y'(y'')^2,$$

$$B_0=2+x^2+y^2+x^2y^2+2x^4+2y^4,\qquad B_1=1+(y')^2,$$

$$P_{i,j}(\varphi,\psi)=\varphi^{(i)}\psi^{(j)}-\varphi^{(j)}\psi^{(i)},\qquad R_4=3y''y^{\text{IV}}-5(y''')^2,$$

$$\tilde{U}_5=4y^{\text{V}}B_1^3Q+10y^{\text{IV}}y''B_1^3\left(4y'''y'+3(y'')^2\right)-5(y^{\text{IV}})^2B_1^4+40(y''')^2(y'')^2\left((y')^2-2\right)B_1^2$$
$$-40(y''')^3y'B_1^3-180y'''y'(y'')^4\left((y')^2-1\right)B_1^2-(y'')^6\left(45(6(y')^2+1)-135(y')^4\right),$$

$$U_5=(y')^2\left(Q_3D_x^2Q_3-\tfrac{5}{4}(D_xQ_3)^2\right)+y'y''Q_3D_xQ_3-\left(2y'y'''-(y'')^2\right)Q_3^2,$$

$$V_7=(y'')^2\left(S_5D_x^2S_5-\tfrac{7}{6}(D_xS_5)^2\right)+y''y'''S_5D_xS_5-\tfrac{1}{2}\left(9y''y^{\text{IV}}-7(y''')^2\right)S_5^2,$$

$$W(f_1,f_2,\dots,f_r)=\begin{vmatrix} f_1'(x) & f_2'(x) & \dots & f_r'(x)\\ f_1''(x) & f_2''(x) & \dots & f_r''(x)\\ \dots & \dots & \dots & \dots\\ f_1^{(r)}(x) & f_2^{(r)}(x) & \dots & f_r^{(r)}(x)\end{vmatrix},$$

$$K_r(\eta_1,\eta_2,\dots,\eta_r)=y^{(r)}+c_1y^{(r-1)}+c_1y^{(r-1)}+\dots+c_ry,$$

where $c_1,\dots,c_r$ are the constant coefficients of the r-th-order ODE

$$\eta^{(r)}(x)+c_1\eta^{(r-1)}(x)+\dots+c_r\eta(x)=0$$

which is satisfied by the functions $\eta_1(x)$, …, $\eta_r(x)$.

Table 3. Differential invariants and Lie determinants of real Lie algebras.

N	Basis of differential invariants	Operator of invariant differentiation	Lie determinant
1	y	D_x	const
1*	x, y'	D_x	const
2	y', y''	D_x	const
3	y, $\frac{y'^3}{y''}$	$\frac{1}{y'}D_x$	$-(y')^2$
3*	x, y''	D_x	const
4	y'	yD_x	y
5	y, $\frac{(y')^2}{y''}$	$\frac{1}{y'}D_x$	y'
5*	x, $\frac{y''}{y'}$	D_x	y'
6	x, $y''\varphi'''(x) - y'''\varphi''(x)$	D_x	$\varphi''(x)$
7	$\frac{y'}{y''}$	D_x	y'
8	$y' + y''$	D_x	$-e^{-x}$
9	y''	D_x	const
10	$y''e^{y'}$	$e^{y'}D_x$	const
11	$y'' + 2y' + y$	D_x	$-e^{-2x}$
12	$\frac{y'''}{(y'')^2}$	$\frac{1}{y''}D_x$	$-y''$
13	x, $\frac{y''}{y'''}$	D_x	$-y''$
14	$y''y'^{\frac{2-a}{a-1}}$	$(y')^{\frac{1}{a-1}}D_x$	$(a-1)y'$
15	$y'' + (a+1)y' + ay$	D_x	$(1-a)e^{-(1+a)x}$
16	$y''e^{-c\arctan y'}B_1^{-3/2}$	$e^{-c\arctan y'}B_1^{-1/2}D_x$	B_1
17	$y'' + 2by' + (b^2+1)y$	D_x	$-e^{-2bx}$
18	$(y''y + (y')^2 + 1)B_1^{-3/2}$	$2yB_1^{-1/2}D_x$	$2y^2B_1$
19	$(2y'(1+y') + y''(x-y))(y')^{-3/2}$	$(x-y)(y')^{-1/2}D_x$	$2y'(x-y)^2$
20	y^3y''	y^2D_x	y^2
21	x, $(y')^{-2}Q_3$	D_x	$y(y-x)y'$
21*	y, $(3y''^2 - 2y'y''')(y')^{-4}$	$\frac{1}{y'}D_x$	y'
22	$y''(1+x^2+y^2)B_1^{-3/2} + 2(y-xy')B_1^{-1/2}$	$(1+x^2+y^2)B_1^{-1/2}D_x$	B_0B_1
23	x, $y''P_{4,3}(\varphi,\psi) + y'''P_{2,4}(\varphi,\psi) + y^{(4)}$	D_x	$P_{2,3}(\varphi,\psi)$
24	$\frac{y'y'''}{(y'')^2}$	$\frac{y'}{y''}D_x$	$y'y''$
25	$\frac{y''+y'}{y'''+y''}$	D_x	$-e^{-x}(y'+y'')$
26	$y''' + 2y'' + y'$	D_x	$-e^{-2x}$
27	$y''' + (1+a)y'' + ay'$	D_x	$a(a-1)e^{-(1+a)x}$

Table 3. (Continued.)

N	Basis of differential invariants	Operator of invariant differentiation	Lie determinant
28	$y''' + 2by'' + (1+b^2)y'$	D_x	$-(1+b^2)e^{-2bx}$
29	$S_3 Q_2^{-3/2}$	$\sqrt{\frac{y}{y''}} D_x$	$-2y^2 y''$
30	$Q_3 (y')^{-4}$	$\frac{1}{y'} D_x$	$2y'^2$
31	y'''	D_x	const
32	$y''' + (b+2)y'' + (2b+1)y' + by$	D_x	$(b-1)^2 e^{-(b+2)x}$
33	$y''' + y''$	D_x	$-e^{-x}$
34	$y''' + 3y'' + 3y' + y$	D_x	$-e^{-3x}$
35	$x,\ P_{2,3}(y,\varphi) P_{2,4}^{-1}(y,\varphi)$	D_x	$P_{3,2}(y,\varphi)$
36	$y''' + (a+b+c)y'' + (ab+ac+bc)y' + abcy$	D_x	$\frac{(b-a)(c-a)(c-b)}{e^{(a+b+c)x}}$
37	$y''' + (2b+a)y'' + (b^2+2ab+1)y' + a(b^2+1)y$	D_x	$((b-a)^2+1)e^{-(2b+a)}$
38	$y''' e^{\frac{y''}{2}}$	$e^{\frac{y''}{2}} D_x$	const
39	$b=1:\quad y'',\ y^{\text{IV}}(y''')^{-2}$	$\frac{1}{y'''} D_x$	y'''
	$b \neq 1:\quad (y'')^{\frac{2-b}{b-1}}$	$y''^{\frac{1}{b-1}} D_x$	$(1-b)y''$
40	$\frac{y''}{y'''}$	D_x	$-y''$
41	$(y'')^{-2} y''' B_1 - 3y'$	$\frac{B_1}{y''} D_x$	$3y'' B_1$
42	$\frac{y''+y}{y'''+y'}$	D_x	$y'' + x$
43	$(3y'' y^{\text{IV}} - 5(y''')^2)(y'')^{-8/3}$	$(y'')^{-1/3} D_x$	y''
44	$S_5 R_4^{-3/2}$	$y'' R_4^{-1/2} D_x$	$(y'')^2 R_4$
45	$\tilde{U}_5 \tilde{Q}_3^{-3}$	$B_1 \tilde{Q}_3^{-1/2} D_x$	$-16 B_1 \tilde{Q}_3^2$
46	$U_5 Q_3^{-3}$	$y' Q_3^{-1/2} D_x$	$-4y' Q_3^{-2}$
47	$V_7 S_5^{-8/3}$	$y'' S_5^{-1/3} D_x$	$-2y'' S_5^2$
48	$x,\ W(y', \xi_1', \xi_2', \ldots, \xi_r')$	D_x	$W(\xi_1', \xi_2', \ldots, \xi_r')$
49	$x,\ W(y', \xi_1', \xi_2', \ldots, \xi_r')/D_x W(y', \xi_1', \xi_2', \ldots, \xi_r')$	D_x	$W(y', \xi_1', \ldots, \xi_r')$
50	$K_r(\eta_1, \ldots, \eta_r)$	D_x	$W(\eta_1', \eta_2', \ldots, \eta_r')$
51	$K_r(\eta_1, \ldots, \eta_r)/D_x K_r(\eta_1, \ldots, \eta_r)$	D_x	$W(y', \eta_1', \ldots, \eta_r')$
52	$c \neq r+1:\quad (y^{(r+1)})^{\frac{2-c+r}{c-r-1}} y^{(r+2)}$	$(y^{(r+1)})^{\frac{1}{c-r-1}} D_x$	$y^{(r+1)}$
	$c = r+1:\quad y^{(r+1)},\ \frac{y^{(r+3)}}{(y^{(r+2)})^2}$	$\frac{1}{y^{(r+2)}} D_x$	$y^{(r+2)}$
53	$y^{(r+1)} e^{\frac{y^{(r)}}{r!}}$	$e^{\frac{y^{(r)}}{r!}} D_x$	const
54	$\frac{y^{(r+1)} y^{(r+3)}}{(y^{(r+2)})^2}$	$\frac{y^{(r+1)}}{y^{(r+2)}} D_x$	$y^{(r+1)} y^{(r+2)}$
55	$Q_{r+3} (y^{(r+1)})^{-\frac{2r+8}{r+2}}$	$(y^{(r+1)})^{-\frac{2}{r+2}} D_x$	$y^{(r+1)}$
56	$S_{r+4} Q_{r+3}^{-3/2}$	$y^{(r+1)} Q_{r+3}^{-1/2} D_x$	$y^{(r+1)} Q_{r+3}$

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