Realizations of real semisimple low-dimensional Lie algebras

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Abstract

A complete set of inequivalent realizations of three- and four-dimensional real unsolvable Lie algebras in vector fields on a space of an arbitrary (finite) number of variables is obtained.

Representations of Lie algebras by vector fields are widely applicable e.g. in integrating of ordinary differential equations, group classification of partial differential equations, the theory of differential invariants, general relativity and other physical problems. There exist many papers devoted to the problem of construction of realizations of Lie algebras. All possible realizations of Lie algebras in vector fields on the two-dimensional complex and real spaces were first classified by S.Lie himself [1, 2].

In this paper we construct a complete set of inequivalent faithful realizations of unsolvable real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables.

A necessary step to classify realizations of Lie algebras is classification of these algebras, i.e. classification of possible commutative relations between basis elements. Complete classification of Lie algebras of dimension up to and including six can be found in the papers of V.V. Morozov [3], G.M. Mubarakzyanov [4, 5, 6] and P. Turkowski [7]. The problem of classification of Lie algebras of higher orders is solved only for some classes e.g. the simple and semi-simple algebras.

There are exist four unsolvable real Lie algebras of dimension no greater than four (here $q = 1, 2, 3$):

$$sl(2,\mathbb{R})\colon \quad [e_1,e_2]=e_1, \quad [e_1,e_3]=2e_2, \quad [e_2,e_3]=e_3;$$

$$so(3)\colon \quad [e_1,e_2]=e_3, \quad [e_3,e_1]=e_2, \quad [e_2,e_3]=e_1;$$

$$sl(2,\mathbb{R})\oplus A_1\colon \quad [e_1,e_2]=e_1, \quad [e_1,e_3]=2e_2, \quad [e_2,e_3]=e_3, \quad [e_q,e_4]=0;$$

$$so(3)\oplus A_1\colon \quad [e_1,e_2]=e_3, \quad [e_3,e_1]=e_2, \quad [e_2,e_3]=e_1, \quad [e_q,e_4]=0.$$

Remark 1. *Notations and conventions.* Below $\partial_a = \partial/\partial x_a$, $x = (x_1, \ldots, x_n)$, $\check{x} = (x_3, \ldots, x_n)$, $\hat{x} = (x_4, \ldots, x_n)$, $a = \overline{1,n}$, $j, k = \overline{4,n}$. We use convention on summation over repeat indexes. We denote the N-th realization of an algebra A as $R(A, N)$.

To classify realizations of a m-dimensional Lie algebra A in the most direct way, we have to take m linearly independent vector fields of the general form $e_s = \xi^{sa}(x)\partial_a$, $s = \overline{1,m}$, and require them to satisfy the commutation relations of A. As a result, we obtain a system of first-order PDEs for the coefficients ξ^{sa} and then we integrate it, considering all the possible cases. For each case we transform the solution into the simplest form, using either local diffeomorphisms of the space of x and automorphisms of A if we looking for the weakly inequivalent classification or only local diffeomorphisms of the space of x if the strong inequivalence is meant. A disadvantage of this method is the necessity to solve a complicated nonlinear system of PDEs. Another way is to classify sequentially realizations of a series of nested subalgebras of A, starting with a one-dimensional subalgebra or other subalgebra with known realizations and ending up with A. Thus, to prove the following theorem, we apply the above method, starting from the algebra $A_{2.1}$ formed by e_1 and e_2.

Theorem 1. *Let first-order differential operators satisfy the commutation relations of $sl(2,\mathbb{R})$. Then there exist transformations reducing these operators to one of the forms:*

1) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2 + x_2\partial_3;$

2) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ (x_1^2 + x_2^2)\partial_1 + 2x_1x_2\partial_2;$

3) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ (x_1^2 - x_2^2)\partial_1 + 2x_1x_2\partial_2;$

4) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2;$

5) $\partial_1,\ x_1\partial_1,\ x_1^2\partial_1.$

Theorem 2. *A complete list of inequivalent realizations of $sl(2,\mathbb{R}) \oplus A_1$ is exhausted by the following ones:*

1) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2 + x_2\partial_3, \partial_4;$

2) $\partial_1, x_1\partial_1 + x_2\partial_2, x_1^2\partial_1 + 2x_1x_2\partial_2 + x_2\partial_3, x_2\partial_1 + 2x_2x_3\partial_2 + (x_3^2 + x_4)\partial_3;$

3) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2 + x_2\partial_3,\ x_2\partial_1 + 2x_2x_3\partial_2 + (x_3^2 + c)\partial_3,$ $c \in \{-1; 0; 1\};$

4) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ (x_1^2 + x_2^2)\partial_1 + 2x_1x_2\partial_2,\ \partial_3;$

5) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ (x_1^2 - x_2^2)\partial_1 + 2x_1x_2\partial_2,\ \partial_3;$

6) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2,\ \partial_3;$

7) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2,\ x_2x_3\partial_2;$

8) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2,\ x_2\partial_2;$

9) $\partial_1,\ x_1\partial_1,\ x_1^2\partial_1,\ \partial_2.$

Proof. The automorphism group of the algebra $sl(2,\mathbb{R}) \oplus A_1$ is a direct product of the automorphism groups of $sl(2,\mathbb{R})$ and A_1. We extend the realizations of $sl(2,\mathbb{R})$ to realizations of $sl(2,\mathbb{R}) \oplus A_1$ with the operator e_4, beginning from the most general form $e_4 = \eta^a(x)\partial_a$.

Consider the realization $R(sl(2,\mathbb{R}),1)$. The general form of the operator e_4 which commutates with the basis elements of $R(sl(2,\mathbb{R}),1)$ is as follows:

$$e_4 = \xi^1 x_2\partial_1 + (2\xi^1 x_3 + \xi^2)x_2\partial_2 + (\xi^1 x_3^2 + \xi^2 x_3 + \xi^3)\partial_3 + \xi^j\partial_j,$$

where ξ^a are arbitrary functions of $\hat{x}$. The form of operators e_1, e_2 and e_3 is preserved by the transformation:

$$\tilde{x}_1 = x_1 + \frac{f^1 x_2}{1-f^1x_3}, \quad \tilde{x}_2 = \frac{f^2 x_2}{(1-f^1x_3)^2}, \quad \tilde{x}_3 = \frac{f^2x_3}{1-f^1x_3} + f^3, \quad \tilde{x}_j = f^j,$$

where f^a are arbitrary functions of $\hat{x}$. After action of this transformation the operator e_4 turns into the operator $\tilde{e}_4$ of the same form with following functions $\tilde{\xi^a}$:

$$\tilde{\xi^1} = \frac{1}{f^2}(\xi^1 + \xi^2 f^1 + \xi^3(f^1)^2 + \xi^j f^1_j), \quad \tilde{\xi^2} = \xi^2 + 2\xi^3 f^1 - 2\tilde{\xi^1}(f^3)^2 + \xi^j\frac{f^2_j}{f^2},$$
$$\tilde{\xi^3} = \xi^3 f^2 - \tilde{\xi^1}(f^3)^2 - \tilde{\xi^2}f^3 + \xi^3 f^2 + \xi^j f^3_j, \quad \tilde{\xi^j} = \xi^k f^j_k.$$

Here and below subscripts mean differentiation with respect to the corresponding variables x_a.

There are two possible cases.

1) $\exists j$: $\xi^j \neq 0$. Then the operator e_4 can be transformed to the form $\tilde{e}_4 = \partial_4$ and we obtain the realization $R(sl(2,\mathbb{R})\oplus A_1,1)$.

2) $\tilde{\xi^j} = 0$. The expression $I = (\xi^2)^2 - 4\xi^1\xi^3$ is an invariant of the above transformations of ξ. Therefore, we can make $\tilde{\xi_1} = 1$, $\tilde{\xi_2} = 0$, $\tilde{\xi_3} = I$. If $I = \mathrm{const}$ then we obtain the realization $R(sl(2,\mathbb{R})\oplus A_1,3)$, otherwise we can choose new variable $\tilde{x}_4 = I$ and obtain the realization $R(sl(2,\mathbb{R})\oplus A_1,2)$.

We omit calculations on the realizations $R(sl(2,\mathbb{R})\oplus A_1, 4\text{–}9)$, because they are simpler than the adduced ones and are made in the same way, starting from three other realizations of the algebra $sl(2,\mathbb{R})$.

Inequivalence of the obtained realizations can be easily proved by means of technics proposed in [9]. □

Theorem 3. *There are only two inequivalent realizations of the algebra* $so(3)$*:*

1) $-\sin x_1 \tan x_2\partial_1 - \cos x_1\partial_2,\ -\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2,\ \partial_1$;

2) $-\sin x_1\tan x_2\partial_1 - \cos x_1\partial_2 + \sin x_1 \sec x_2\partial_3,$
$-\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2 + \cos x_1\sec x_2\partial_3,\ \partial_1.$

Remark 2. The realizations $R(so(3),1)$ and $R(so(3),2)$ are well-known. At the best of our knowledge, completeness of the list of these realizations was first proved in [8]. We do not assert that the adduced forms of realizations are optimal for all applications and the classification from Theorem 3 is canonical.

Consider the realization $R(so(3),1)$ of rank 2 in more details. It acts transitively on the manifold S^2. With the stereographic projection $\tan x_1 = t/x$, $\operatorname{cotan} x_2 = \sqrt{x^2+t^2}$ it can be reduced to the well known realization on the plane [10]:

$$(1+t^2)\partial_t + xt\partial_x, \quad x\partial_t - t\partial_x, \quad -xt\partial_t - (1+x^2)\partial_x$$

If dimension of the x-space is not smaller than 3, the variables x_1, x_2 and the implicit variable x_3 in $R(so(3),1)$ can be interpreted as the angles and the radius of the spherical coordinates (imbedding S^2 in $\mathbb{R}^3$). Then in the corresponding Cartesian coordinates $R(so(3),1)$ has the well-known form:

$$x_2\partial_3 - x_3\partial_2, \qquad x_3\partial_1 - x_1\partial_3, \qquad x_1\partial_2 - x_2\partial_1,$$

which is generated by the standard representation of $SO(3)$ in $\mathbb{R}^3$.

Theorem 4. *A list of inequivalent realizations of the algebra $so(3)\oplus A_1$ in vector fields on a space of an arbitrary (finite) number of variables is exhausted by the following ones:*

1) $-\sin x_1\tan x_2\partial_1 - \cos x_1\partial_2$, $-\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2$, ∂_1, ∂_3;
2) $-\sin x_1\tan x_2\partial_1 - \cos x_1\partial_2 + \sin x_1\sec x_2\partial_3$, $-\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2 + \cos x_1\sec x_2\partial_3$, ∂_1, ∂_3;
3) $-\sin x_1\tan x_2\partial_1 - \cos x_1\partial_2 + \sin x_1\sec x_2\partial_3$, $-\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2 + \cos x_1\sec x_2\partial_3$, ∂_1, $x_4\partial_3$;
4) $-\sin x_1\tan x_2\partial_1 - \cos x_1\partial_2 + \sin x_1\sec x_2\partial_3$, $-\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2 + \cos x_1\sec x_2\partial_3$, ∂_1, ∂_4.

Proof. The automorphism group of $so(3)\oplus A_1$ is the direct product of the automorphism groups of $so(3)$ and A_1. To classify realizations of $so(3)\oplus A_1$, we start from the realizations $R(so(3),1)$ and $R(so(3),2)$. For convenience we rewrite them as a realization parameterized with $\alpha\in\{0;1\}$:

$$\begin{aligned} e_1 &= -\sin x_1\tan x_2\partial_1 - \cos x_1\partial_2 + \alpha\sin x_1\sec x_2\partial_3,\\ e_2 &= -\cos x_1\tan x_2\partial_1 + \sin x_1\partial_2 + \alpha\cos x_1\sec x_2\partial_3,\\ e_3 &= \partial_1 \end{aligned}$$

The values $\alpha=0$ and $\alpha=1$ correspond to the realizations $R(so(3),1)$ and $R(so(3),2)$.

We take the operator e_4 in the most general form $e_4 = \xi^a(x)\partial_a$ and obtain the equations for $\xi^a(x)$ from condition of vanishing commutators of e_4 with the other basis elements:

$$\xi^a_1 = 0, \quad \xi^2_2 = 0, \quad \xi^j_2 = 0, \tag{1a}$$

$$\alpha\xi^2_3-\xi^1\cos x_2=0,\quad \alpha\xi^1_3\cos x_2+\xi^2=0,\quad \xi^3_2\cos x_2+\alpha\xi^1=0,\qquad\text{(1b)}$$
$$\xi^1_2-\xi^1\tan x_2=0,\quad \alpha\xi^3_3-\alpha\xi^2\tan x_2=0,\quad \alpha\xi^j_3=0.\qquad\text{(1c)}$$

It follows from (1a) that $\xi^2=\xi^2(\check x)$ and $\xi^j=\xi^j(\check x)$.

In the case $\alpha=0$ we obtain from (1b) that $\xi^1=0$, $\xi^2=0$ and $\xi^3=\xi^3(\check x)$. Then e_4 has the form $e_4=\xi^p(\check x)\partial_p$, where $p=\overline{3,n}$, and one of the coefficients ξ^p does not vanish. Using allowable transformations of variables $\tilde x_1=x_1$, $\tilde x_2=x_2$, $\tilde x_p=f^p(\check x)$, we can can make $\xi^3=1$ and $\xi^j=0$. ("Allowable" means that such transformations preserve the form of e_1, e_2 and e_3.) As a result, we obtain realization $R(so(3)\oplus A_1,1)$.

Consider the case $\alpha=1$. The general solution of system (1a)–(1c) is:

$$\xi^1=\frac{\varphi^1\sin x_3+\varphi^2\cos x_3}{\cos x_2},\quad \xi^2=\varphi^2\sin x_3-\varphi^1\cos x_3,$$
$$\xi_3=\varphi^3-(\varphi^1\sin x_3+\varphi^2\cos x_3)\tan x_2,\quad \xi^j=\varphi^j,$$

where φ^a are arbitrary functions of $\hat x$. Therefore, the operator e_4 can be presented in the form:

$$e_4=\varphi^1e'_1+\varphi^2e'_2+\varphi^3e'_3+\varphi^j\partial_j,$$

where operators e'_1–e'_3 are obtained from e_1–e_3 with transposition of the variables x_1 and x_3.

The next step is to simplify the operator e_4. Since in this case the direct method of finding allowable transformations of variables is too cumbersome and complicated, we use the infinitesimal approach.

An one-parametric group of local transformations in the space of variables x preserves the form of operators e_1–e_3 if its infinitesimal generator Q commutes with these operators. Therefore, Q has the same form as e_4:

$$Q=\rho^1e'_1+\rho^2e'_2+\rho^3e'_3+\rho^j\partial_j,$$

where ρ^a are arbitrary functions of $\hat x$.

There are two possible cases: $\xi^j=0$ or $\exists j$: $\xi^j\neq0$. In any case the operator e_4 can be transformed by means of allowable transformations $\tilde x_1=x_1$, $\tilde x_2=x_2$, $\tilde x_3=x_3$, $\tilde x_j=f^j(\hat x)$ to the form:

$$e_4=\varphi^1e'_1+\varphi^2e'_2+\varphi^3e'_3+\beta\partial_4,\quad \beta\in\{0,1\}.$$

Below we use only transformations preserving $\hat x$ and, therefore, assume $\rho^j=0$.

Introducing the vector notations $\bar\varphi=(\varphi^1,\varphi^2,\varphi^3)$, $\bar\rho=(\rho^1,\rho^2,\rho^3)$, $\bar\rho_4=(\rho^1_4,\rho^2_4,\rho^3_4)$, and $\bar e'=(e'_1,e'_2,e'_3)$, we can present the commutator $[e_4,Q]$ as follows:

$$[e_4,Q]=(\bar\rho\times\bar\varphi-\beta\bar\rho_4)\cdot\bar e',$$

where "$\times$" and "$\cdot$" denote the vector and scalar products. The finite transformations $\widetilde{\bar{\varphi}} = \bar{\gamma}(\varepsilon, \bar{\varphi}, \hat{x})$ generated by Q are found by integrating the Lie equations:

$$\frac{d\bar{\gamma}}{d\varepsilon} = \bar{\rho} \times \bar{\gamma} - \beta \bar{\rho}_4, \quad \bar{\gamma}|_{\varepsilon=0} = \bar{\varphi}, \tag{2}$$

where ε is a group parameter and $\hat{x}$ are assumed constants. Therefore,

$$\bar{\gamma} = OJ(\varepsilon)O^{\mathrm{T}}\bar{\varphi} - \beta O \int_0^{\varepsilon} J(\varepsilon) d\varepsilon \, O^{\mathrm{T}} \bar{\rho}_4,$$

where O is an orthogonal matrix having $\bar{\rho}/|\bar{\rho}|$ as the third column,

$$J(\varepsilon) = \begin{pmatrix} \cos|\bar{\rho}|\varepsilon & -\sin|\bar{\rho}|\varepsilon & 0 \\ \sin|\bar{\rho}|\varepsilon & \cos|\bar{\rho}|\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the case $\beta = 0$ φ^1 and φ^2 can be made to vanish by means of transformations $\widetilde{\bar{\varphi}} = \bar{\gamma}(\varepsilon, \bar{\varphi}, \hat{x})$ hence $\tilde{e}_4 = \tilde{\varphi}^3(\hat{x})\partial_3$. As a result we obtain the realizations $R(so(3) \oplus A_1, 2)$ and $R(so(3) \oplus A_1, 3)$ if $\varphi_3 = \mathrm{const}$ or $\varphi_3 \neq \mathrm{const}$ correspondingly.

In the case $\beta = 1$ we choose $\bar{\rho}$ as a solution of the system

$$\int_0^{\varepsilon} J(\varepsilon) d\varepsilon \, O^{\mathrm{T}} \bar{\rho}_4 = J(\varepsilon) O^{\mathrm{T}} \bar{\varphi}, \tag{3}$$

where ε is fixed such that there exist $(\int_0^{\varepsilon} J(\varepsilon) d\varepsilon)^{-1}$. Then the transformed $\bar{\varphi}$ vanishes, i.e. $\tilde{e}_4 = \partial_4$ and we have the realization $R(so(3) \oplus A_1, 4)$. $\square$

The complete classification of realizations of all Lie algebras of dimension up to and including 4 is adduced in [9].

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