Cartan's equivalence method and non-commutative derivations

Sylvain Neut and Michel Petitot

University of Lille, France

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Plan

- Necessary and sufficient conditions for the linearization of a third order ODE,
- Classification results for some PDE systems.
- Main optimizations of our implementation of the Cartan's equivalence method.

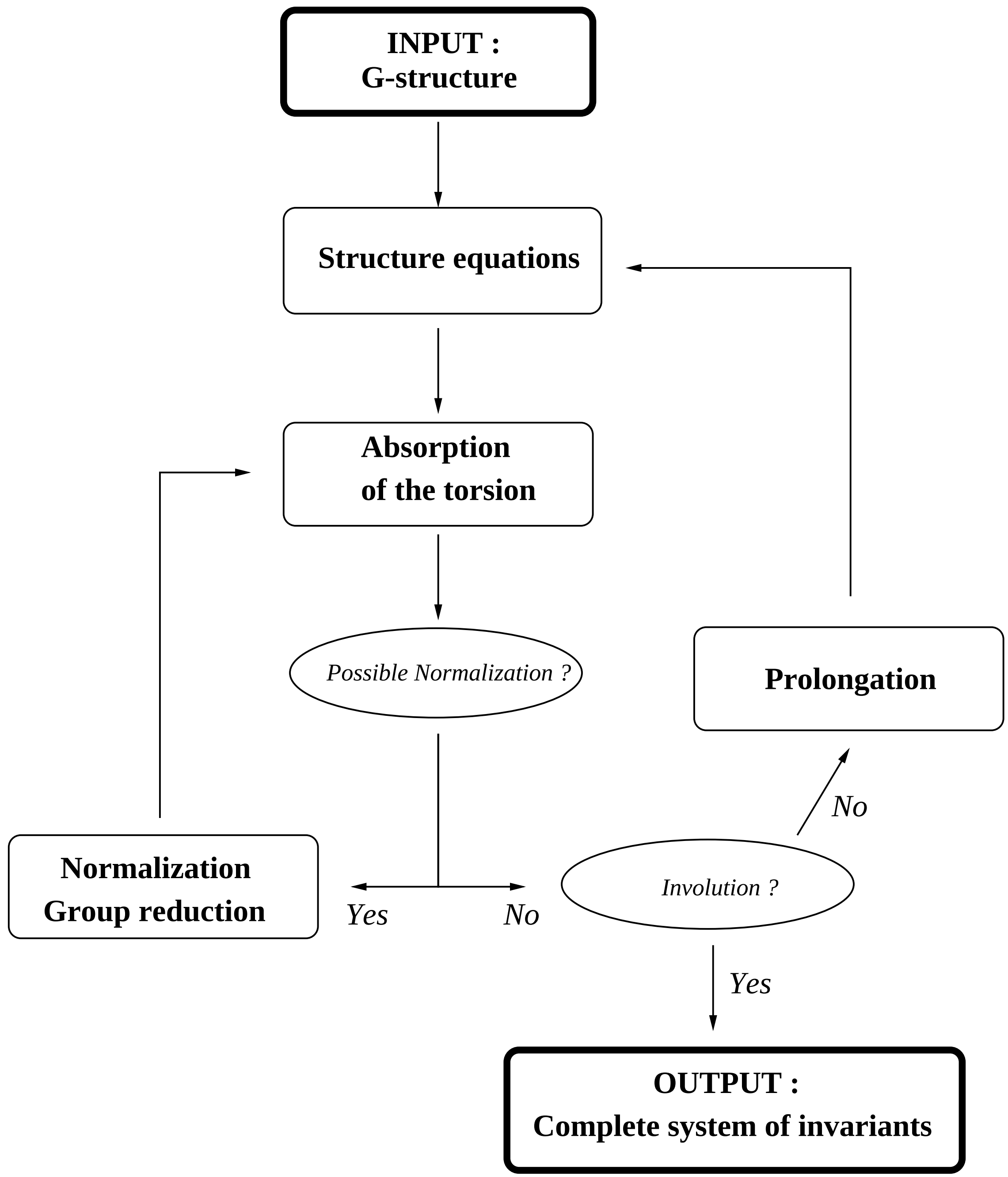
Introduction

- The Cartan's equivalence method allows to decide if two differential equations (or systems of differential equations) are equivalent under the action of a given group of transformation.

- Input of the algorithm : Two differential equations and a group of transformations.

- Output of the algorithm : A complete system of fundamental invariants associated to the equivalence problem.

- See Olver (Equivalence, Invariants and Symmetry). See Cartan, Chern (Third order ODE), Gardner, Olver, Kamran, Fels.


INPUT :
G-structure
Structure equations
Absorption
of the torsion
Possible Normalization ?
Prolongation
No
Normalization
Group reduction
Yes
No
Involution ?
Yes
OUTPUT :
Complete system of invariants

Equivalence of third order ODE's

• Let $\mathsf{x} := (x,\, y,\, p = y',\, q = y'') \in \mathbb{R}^4$ be a system of local coordinates of J^2. Two equations $y_{xxx} = f(x, y, p, q)$ and $\bar{y}''' = \bar{f}(\bar{x},\, \bar{y},\, \bar{y}',\, \bar{y}'')$ are equivalent by a contact transformation $\bar{\mathsf{x}} = \phi(\mathsf{x})$ when there exist functions $(a_1, \ldots, a_9)$ from J^2 to $\mathbb{R}$ such that

$$\phi^* \begin{pmatrix} d\bar{q} - \bar{f}(\bar{\mathsf{x}})d\bar{x} \\ d\bar{y} - \bar{p}d\bar{x} \\ d\bar{p} - \bar{q}d\bar{x} \\ d\bar{x} \end{pmatrix} = \underbrace{\begin{pmatrix} a_1(\mathsf{x}) & a_2(\mathsf{x}) & a_3(\mathsf{x}) & 0 \\ 0 & a_4(\mathsf{x}) & 0 & 0 \\ 0 & a_5(\mathsf{x}) & a_6(\mathsf{x}) & 0 \\ 0 & a_7(\mathsf{x}) & a_8(\mathsf{x}) & a_9(\mathsf{x}) \end{pmatrix} \underbrace{\begin{pmatrix} dq - f(\mathsf{x})dx \\ dy - pdx \\ dp - qdx \\ dx \end{pmatrix}}_{\omega(\mathsf{x})}}_{\theta(\mathsf{x})}$$

• The computation splits into two branches depending on the value of

$$I = -\frac{1}{3} f_p f_q + \frac{1}{2} D_x f_p - \frac{1}{6} D_x^2 f_q - f_y + \frac{1}{3} f_q D_x f_q - \frac{2}{27} f_q^3$$

where $D_x := \partial/\partial x + p\, \partial/\partial y + q\, \partial/\partial p + f(x, y, p, q)\, \partial/\partial q$.

Third order ODE's : the case $I = 0$

The final structure equations containing 9 fundamental invariants are :

$$\left\{ \begin{array}{lll} d\theta^1 & = & -\theta^1 \wedge \theta^5 - \theta^3 \wedge \theta^6, \\ d\theta^2 & = & -\theta^2 \wedge \theta^5 - 2\theta^2 \wedge \theta^9 - \theta^3 \wedge \theta^4, \\ d\theta^3 & = & -\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^6 - \theta^3 \wedge \theta^5 - \theta^3 \wedge \theta^9, \\ d\theta^4 & = & -\theta^2 \wedge \theta^7 - \theta^3 \wedge \theta^8 - \theta^4 \wedge \theta^9, \\ d\theta^5 & = & -2\theta^1 \wedge \theta^8 - \theta^3 \wedge \theta^7 - \theta^4 \wedge \theta^6, \\ d\theta^6 & = & -\theta^1 \wedge \theta^7 - \theta^3 \wedge \theta^{10} + \theta^6 \wedge \theta^9, \\ d\theta^7 & = & I_1\theta^1 \wedge \theta^2 + I_2\theta^1 \wedge \theta^3 + I_3\theta^2 \wedge \theta^3 - \theta^4 \wedge \theta^{10} - \theta^5 \wedge \theta^7 \\ & & -\theta^6 \wedge \theta^8 + \theta^7 \wedge \theta^9, \\ d\theta^8 & = & I_4\theta^1 \wedge \theta^2 + I_5\theta^1 \wedge \theta^3 + I_6\theta^2 \wedge \theta^3 - \theta^4 \wedge \theta^7 - \theta^5 \wedge \theta^8, \\ d\theta^9 & = & \theta^1 \wedge \theta^8 - \theta^2 \wedge \theta^{10} + \theta^4 \wedge \theta^6, \\ d\theta^{10} & = & I_7\theta^1 \wedge \theta^2 + I_8\theta^1 \wedge \theta^3 + I_9\theta^2 \wedge \theta^3 - \theta^5 \wedge \theta^{10} \\ & & -\theta^6 \wedge \theta^7 - 2\theta^9 \wedge \theta^{10}. \end{array} \right. \tag{1}$$

Third order ODE's : the case $I = 0$

Theorem 1 *Consider the equation $y_{xxx} = f(x,\, y,\, p,\, q)$. Then the three following conditions are equivalent :*

(i) *The equation is equivalent to $y''' = 0$ by a contact transformation.*

(ii) *The equation admits a 10 parameters contact symmetry group.*

(iii) $I = 0$ *et* $f_{qqqq} = 0$.

●The Poincaré's identity

$$d(d\theta^i) = R^i_{j,k,l}\theta^j \wedge \theta^k \wedge \theta^l = 0$$

gives the following differential relations between the invariants :

$$\begin{array}{lll} -I_2 + I_4 = 0, & I_1 + 2I_6 = 0, & \dfrac{\partial I_3}{\partial \theta^4} + I_9 = 0, \\ \dfrac{\partial I_4}{\partial \theta^4} - I_6 + I_1 = 0, & \dfrac{\partial I_5}{\partial \theta^4} + I_4 + I_2 = 0, & \dfrac{\partial I_6}{\partial \theta^4} + I_3 = 0, \\ -I_6 - I_8 = 0, & \dfrac{\partial I_8}{\partial \theta^4} + I_7 = 0. & \end{array}$$

● From the above relations we deduce that $I_5 = -f_{qqqq}/(6a_1^3 a_9)$ is a base of the *differential* algebra generated by the nine invariants.

Third order ODE's : the case $I \neq 0$

The final structure equations containing 16 fundamental invariants are :

$$\left\{\begin{array}{rcl} d\theta^1 &=& -\theta^1\wedge\theta^5 + I_1\theta^2\wedge\theta^3 + \theta^2\wedge\theta^4 + I_2\theta^3\wedge\theta^4, \\ d\theta^2 &=& I_3\theta^2\wedge\theta^3 - \theta^2\wedge\theta^5 - \theta^3\wedge\theta^4, \\ d\theta^3 &=& I_4\theta^1\wedge\theta^2 - \theta^1\wedge\theta^4 + I_5\theta^2\wedge\theta^3 + I_6\theta^2\wedge\theta^4 - \theta^3\wedge\theta^5, \\ d\theta^4 &=& I_7\theta^1\wedge\theta^2 + I_8\theta^1\wedge\theta^3 + I_9\theta^2\wedge\theta^3 + I_{10}\theta^2\wedge\theta^4, \\ d\theta^5 &=& I_{11}\theta^1\wedge\theta^2 + I_{12}\theta^1\wedge\theta^3 + I_{13}\theta^1\wedge\theta^4 + I_{14}\theta^2\wedge\theta^3 \\ && +I_{15}\theta^2\wedge\theta^4 + I_{16}\theta^3\wedge\theta^4. \end{array}\right. \tag{2}$$

Third order ODE's : the case $I \neq 0$

In this case, the differential algebra of the invariants $I_1, \ldots, I_{16}$ is generated by the four invariants : (I_8, I_3, I_{10}, I_2).

Theorem 2 *A third order differential equation $y_{xxx} = f(x, y, p, q)$ is equivalent to $y_{xxx} = y$ under the group of contact transformations if and only if :*

$$\begin{cases} J_{qq} = 0, \\ -4J_q f_q J + f_{qq}(J)^2 - 6J_p J + 6J_q D_x J = 0, \\ D_x J_p - D_x^2 J_q - J_y = 0, \\ -3(J)^2 f_p - (J)^2 f_q^2 + 3(J)^2 D_x f_q - 6J D_x^2 J + 9(D_x J)^2 = 0, \end{cases}$$

where $J^3 = I$.

Third order ODE's : linearization conditions

• We used the above invariants to find necessary and sufficient condition so that the equation $y''' = f(x,\, y,\, y',\, y'')$ is equivalent to the linear equation

$$y''' + a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$$

by a contact transformation.

• Case $I = 0$

For a linear equation, we have $f_{qqqq} = 0$. Therefore, every linear equation for which $I = 0$ is equivalent to $y''' = 0$. Then if $I = 0$, a differential equation is linearizable if and only if it is equivalent to $y''' = 0$, i.e $f_{qqqq} = 0$.

Third order ODE's : linearization conditions

- Case $I \neq 0$

Theorem 3 *Consider $y''' = f(x,\, y,\, y',\, y'')$, a third order ordinary differential equation such that $I \neq 0$. Then the five following conditions are equivalent :*

(i) *The equation is equivalent to the linear equation*

$$y''' + a(x)\, y'' + b(x)\, y' + c(x)\, y = 0.$$

by a contact transformation.

(ii) *The equation is equivalent to the canonical form $y''' + 2I_2(x)\, y' + \Big(1 + I_2'(x)\Big)\; y = 0$ by a contact transformation.*

(iii) *The invariants $(I_1, I_2, \ldots, I_{16})$ defined in (2), excepted eventually I_2, are zero.*

(iv) *The invariants $(I_1,\, I_3,\, I_8,\, I_{10},\, I_{16})$ defined in (2) are zero.*

(v) *The 2-forms $d\theta^4$ and $d\theta^5$ defined in (2) are zero.*

Second order PDE systems with two independant variables

• We have computed the invariants for the classification of second order pde systems

$$\begin{cases} u_{x^1x^1} &= f_{11}(x^1, x^2, u, u_{x^1}, u_{x^2}) \\ u_{x^1x^2} &= f_{12}(x^1, x^2, u, u_{x^1}, u_{x^2}) \\ u_{x^2x^2} &= f_{22}(x^1, x^2, u, u_{x^1}, u_{x^2}) \end{cases} \tag{3}$$

with one dependent variable u and two independent variables x^1, x^2 under the group of point transformations $\mathbb{R}^3 \ni (x^1, x^2, u) \to (\bar{x}^1, \bar{x}^2, \bar{u}) \in \mathbb{R}^3$.

• One can show that (3) is equivalent to the flat system

$$u_{x^1x^1} = u_{x^1x^2} = u_{x^2x^2} = 0$$

if and only if all the fundamental invariants are zero.

• These very large invariants occupy 1.1 Mega-bytes of memory. Using our software, we automatically prove that these equivalence conditions are reduced to the vanishing of a very small invariant.

Second order PDE systems with two independant variables

Theorem 4 *The three following conditions are equivalent :*

(i) *The system (3) can be mapped to the flat system* $u_{xx} = u_{xy} = u_{yy} = 0$ *by a point transformation.*

(ii) *The system (3) admits a* 15 *parameters point symmetry group*

(iii) *The functions* f_{11}, f_{12} *and* f_{22} *of (3) verify the following linear conditions :*

$$\begin{cases} \dfrac{\partial^2 f_{11}}{\partial u_2 \partial u_2} = 0, \\ \dfrac{\partial^2 f_{22}}{\partial u_1 \partial u_1} = 0, \\ \dfrac{\partial^2 f_{12}}{\partial u_2 \partial u_2} - \dfrac{\partial^2 f_{11}}{\partial u_1 \partial u_2} = 0, \\ \dfrac{\partial^2 f_{12}}{\partial u_1 \partial u_1} - \dfrac{\partial^2 f_{22}}{\partial u_1 \partial u_2} = 0, \\ \dfrac{\partial^2 f_{11}}{\partial u_1 \partial u_1} - 4\dfrac{\partial^2 f_{12}}{\partial u_1 \partial u_2} + \dfrac{\partial^2 f_{22}}{\partial u_2 \partial u_2} = 0. \end{cases}$$

Our Maple implementation of Cartan's equivalence method

• All the compputations are done using non-commutative derivations. These derivations are dual of the forms ω^i. Such derivations naturally compress the invariants.

Example	use of $\partial/\partial x^i$	use of $\partial/\partial\omega^i$
$y''' = f,\ I = 0$	18 Kb (35 sec)	8 Kb (20 sec)
$y''' = f,\ I \neq 0$	57 Kb (7 sec)	$7,5$ Kb ($3,5$ sec)
$y^{(4)} = f$	280 Kb (38 sec)	36 Kb (9 sec)
PDE system (3)	Does not finish	$1,1$ Mb (3 h)

• The Poincaré's identity, $d(d\theta^i) = 0$, is automatically performed. It generates a lot of differential relations between the invariants.

• These relations are used to compute a base of the differential ideal or differential algebra of the invariants. It allows a great simplification of the equivalence conditions.

• You can find this 4500 lines maple package at www.lifl.fr/ ~neut.