Elie Cartan's geometrical vision or how to avoid expression swell

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Abstract

Present differential equations solver are often based on a list of equations the solutions of which one knows (e.g. as listed in Kamke [5]). Each equation of this list and its solution are ordered in a table. Significant progress would be made if it was possible to compute in advance the differential invariants that allow to decide if one equation to solve is equivalent to one of the list by a change a coordinates. We will attempt to show that, for the computation of these invariants, a geometrical approch (E. Cartan 1905) for the computation of the integrability conditions for ordinary or partial differential equations offers advantages over non geometrical approaches (e.g. such as that of Riquier, Ritt, Kolchin etc.).

1. Consider two systems of differential equations and a group of transformations Φ. The change of coordinates $\varphi \in \Phi$ under which the two systems are equivalent is solution of a PDE system that we can compute by an algorithm. A program like DIFFALG ou RIF can compute the integrability conditions of such a PDE system [4], then the existence of φ is decidable.

EXAMPLE Consider two ordinary differential equations (E) and $(\bar{E})$ of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f\left(x,\, y,\, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \quad \text{and} \quad \frac{d^2\bar{y}}{\mathrm{d}\bar{x}^2} = \bar{f}\left(\bar{x},\, \bar{y},\, \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}\right) \tag{1}$$

and Φ the following pseudo-group of local diffeomorphisms $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form (the constant C and the function η are arbitrary) :

$$(\bar{x},\, \bar{y}) = \varphi(x, y) = (x + C,\, \eta(x, y)). \tag{2}$$

Let $\mathrm{x} = (x,\, y,\, p = y') \in \mathbb{R}^3$ be a system of local coordinates on the jet space $\mathrm{J}^1(\mathbb{R}, \mathbb{R})$. Supppose that the function $\bar{f}$ is known, for example that $f = 0$. Then the constant C is arbitrary and $\eta(x, y)$ is solution of the following PDE system

$$(D_x)^2\eta = 0, \quad \eta_p = 0, \quad \eta_y \neq 0 \tag{3}$$

with $D_x := \dfrac{\partial}{\partial x} + p\,\dfrac{\partial}{\partial y} + f(x, y, p)\,\dfrac{\partial}{\partial p}$. By the elimination of η in (3) using DIFFALG, we obtain the condition (4) on f for the equation $y'' = f(x, y, y')$ to be equivalent to $\bar{y}'' = 0$.

$$\begin{cases} -\dfrac{1}{4}(f_p)^2 - f_y + \dfrac{1}{2} D_x f_p = 0 \\ f_{ppp} = 0. \end{cases} \tag{4}$$

Unfortunately, this method is rarely efficient due to expression swell.

2. On the above example, we note that the use of a non commutative derivations frame $(D_x, \frac{\partial}{\partial y}, \frac{\partial}{\partial p})$ on the manifold $\mathrm{J}^1(\mathbb{R}, \mathbb{R})$ allows a compression of size of the expressions generated during the computation. In the study of the third order ODE $y''' = f(x,\, y,\, y',\, y'')$, we obtain a compression factor of order 100 (see [3] page 67).

3. E. Cartan's method transforms an analytic PDE system into an equivalent *linear* Pfaffian system (with a condition that specifies the independant variables). In 1905, he gave a method to compute the integrability conditions of analytic linear Pfaffian systems and therefore of analytic PDE systems. This algorithm is based on the process of absorption of the torsion, which effect is to progressively adapt the frame in which the computations are done and to make the equations smaller.

4. The change of coordinates $\varphi \in \Phi$ is solution of a linear Pfaffian system where two set of variables play a symmetrical role. For example, in the problem (1), the Pfaffian system is

$$\begin{pmatrix} \bar{a}_1 & \bar{a}_2 & 0 \\ 0 & \bar{a}_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}\bar{p} - \bar{f}(\bar{\mathrm{x}})\,\mathrm{d}\bar{x} \\ \mathrm{d}\bar{y} - \bar{p}\,\mathrm{d}\bar{x} \\ \mathrm{d}\bar{x} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}p - f(\mathrm{x})\,\mathrm{d}x \\ \mathrm{d}y - p\,\mathrm{d}x \\ \mathrm{d}x \end{pmatrix}.$$

This system is of the form $\bar{\theta} = \theta$ where θ and $\bar{\theta}$ are the canonical 1-forms attached to two G-structures (see [3] page 40).
5. Every algorithm like DIFFALG based on the notion of "ranking" breaks this symmetry. Conversely, Cartan's algorithm [1,2] computes the integrability conditions by separately and symmetricaly treating the invariant 1-forms θ and $\bar{\theta}$. This divides the number of indeterminates by two. These integrability conditions symmetricaly presented under the form

$$\bar{\mathrm{I}}(\bar{\mathrm{a}}, \bar{\mathrm{x}}) = \mathrm{I}(\mathrm{a}, \mathrm{x}) \tag{5}$$

are called fundamental invariants. For the problem (1), one obtains three fundamental invariants (see [3] page 59)

$$\begin{cases} \mathrm{I}_1(\mathrm{a}, \mathrm{x}) = -\dfrac{1}{4}(f_p)^2 - f_y + \dfrac{1}{2} D_x f_p, \\ \mathrm{I}_2(\mathrm{a}, \mathrm{x}) = \dfrac{f_{ppp}}{2{a_3}^2}, \\ \mathrm{I}_3(\mathrm{a}, \mathrm{x}) = \dfrac{f_{yp} - D_x f_{pp}}{2a_3}. \end{cases} \tag{6}$$

6. The invariants algebra attached to a given equation (for a given group of transformation Φ) is a *differential* algebra closed by the so called *invariant* derivations. These derivations, which generally do *n*ot commute, allow to compute a *complete* system of invariants from fundamental invariants. One obtains most of the differential relations between the fundamental invariants using the fundamental identity $\mathrm{d} \circ \mathrm{d} = 0$ where d denotes the exterior derivation. This low cost computation does not require the expression of the invariants in local coordinates or any elimination process. This is particularely useful since the memory space taken up by the invariants can reach 1.1 Mbytes for some examples treated by our software (see [3] page 124).
7. Even when one has demonstrated the existence of a change of coordinates $\varphi \in \Phi$, the explicit computation of φ requires the solution of a Pfaffian system verifying the Frobenius complete integrability condition. One obtains the transformation φ *without* integrating any differential equation when the equations (E) and $(\bar{E})$ have a symmetry group $S \subset \Phi$ with finite cardinality. Indeed, if the function φ is the general solution of a differential system, it depend on at least one arbitrary constant, and then the symmetry group S is *continuous* (see FIG. 1) and it follows that $\mathrm{card}(S) = \infty$. In the other case, the transformation φ is *algebraic*

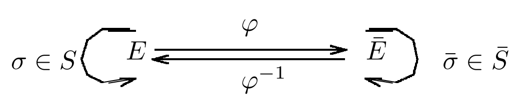

Figure 1: The symmetries σ and $\bar{\sigma}$ composed with φ

and is obtained by the elimination of the structural group coordinates a and $\bar{a}$ in the system of equations of the form (5) where appear the fundamental invariants and a finite number of derivated invariants.
Our software is available at `www.lifl.fr/_neut/logiciels`.

References :

[1] P. J. Olver ; *Equivalence, invariants and symetry*. Graduate Texts in Mathematics. Cambridge University Press, 1995.

[2] E. Cartan ; *Les problèmes d'équivalence*, volume 2 of *oeuvres complètes*, pages 1311–1334. Gauthiers–Villars, Paris, 1953.

[3] S. Neut ; *Implantation et nouvelles applications de la d'équivalence de Cartan*. Thèse de doctorat, univ. Lille I, 16 Oct 2003.
`http://www.lifl.fr/_neut/recherche/these.pdf`.

[4] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot ; Representation for the radical of a finitely generated differential ideal. In *proc. ISSAC'95, International Symposium on Symbolic and Algebraic Computation*, pages 158–166, Montréal, Canada, 1995.

[5] E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen*, Bd. 1: Gewöhnliche Differentialgleichungen, 9. Aufl. Stuttgart, Germany: Teubner, 1983.