LECTURES ON THE GEOMETRY OF MANIFOLDS

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Thinking of Iaşi, my hometown

Introduction

Shape is a fascinating and intriguing subject which has stimulated the imagination of many people. It suffices to look around to become curious. Euclid did just that and came up with the first pure creation. Relying on the common experience, he created an abstract world that had a life of its own. As the human knowledge progressed so did the ability of formulating and answering penetrating questions. In particular, mathematicians started wondering whether Euclid's "obvious" absolute postulates were indeed obvious and/or absolute. Scientists realized that Shape and Space are two closely related concepts and asked whether they really look the way our senses tell us. As Felix Klein pointed out in his Erlangen Program, there are many ways of looking at Shape and Space so that various points of view may produce different images. In particular, the most basic issue of "measuring the Shape" cannot have a clear cut answer. This is a book about Shape, Space and some particular ways of studying them.

Since its inception, the differential and integral calculus proved to be a very versatile tool in dealing with previously untouchable problems. It did not take long until it found uses in geometry in the hands of the Great Masters. This is the path we want to follow in the present book.

In the early days of geometry nobody worried about the natural context in which the methods of calculus "feel at home". There was no need to address this aspect since for the particular problems studied this was a non-issue. As mathematics progressed as a whole the "natural context" mentioned above crystallized in the minds of mathematicians and it was a notion so important that it had to be given a name. The geometric objects which can be studied using the methods of calculus were called smooth manifolds. Special cases of manifolds are the curves and the surfaces and these were quite well understood. B. Riemann was the first to note that the low dimensional ideas of his time were particular aspects of a higher dimensional world.

The first chapter of this book introduces the reader to the concept of smooth manifold through abstract definitions and, more importantly, through many we believe relevant examples. In particular, we introduce at this early stage the notion of Lie group. The main geometric and algebraic properties of these objects will be gradually described as we progress with our study of the geometry of manifolds. Besides their obvious usefulness in geometry, the Lie groups are academically very friendly. They provide a marvelous testing ground for abstract results. We have consistently taken advantage of this feature throughout this book. As a bonus, by the end of these lectures the reader will feel comfortable manipulating basic Lie theoretic concepts.

To apply the techniques of calculus we need "things to derivate and integrate". These "things" are introduced in Chapter 2. The reason why smooth manifolds have many differentiable objects attached to them is that they can be locally very well approximated by linear spaces called tangent spaces . Locally, everything looks like traditional calculus. Each point has a tangent space attached to it so that we obtain a "bunch of tangent spaces" called the tangent bundle. We found it appropriate to introduce at this early point the notion of vector bundle. It helps in structuring both the language and the thinking.

Once we have "things to derivate and integrate" we need to know how to explicitly perform these operations. We devote the Chapter 3 to this purpose. This is perhaps one of the most unattractive aspects of differential geometry but is crucial for all further developments. To spice up the presentation, we have included many examples which will found applications in later chapters. In particular, we have included a whole section devoted to the representation theory of compact Lie groups essentially describing the equivalence between representations and their characters.

The study of Shape begins in earnest in Chapter 4 which deals with Riemann manifolds. We approach these objects gradually. The first section introduces the reader to the notion of geodesics which are defined using the Levi-Civita connection. Locally, the geodesics play the same role as the straight lines in an Euclidian space but globally new phenomena arise. We illustrate these aspects with many concrete examples. In the final part of this section we show how the Euclidian vector calculus generalizes to Riemann manifolds.

The second section of this chapter initiates the local study of Riemann manifolds. Up to first order these manifolds look like Euclidian spaces. The novelty arises when we study "second order approximations " of these spaces. The Riemann tensor provides the complete measure of how far is a Riemann manifold from being flat. This is a very involved object and, to enhance its understanding, we compute it in several instances: on surfaces (which can be easily visualized) and on Lie groups (which can be easily formalized). We have also included Cartan's moving frame technique which is extremely useful in concrete computations. As an application of this technique we prove the celebrated Theorema Egregium of Gauss. This section concludes with the first global result of the book, namely the Gauss-Bonnet theorem. We present a proof inspired from [21] relying on the fact that all Riemann surfaces are Einstein manifolds. The Gauss-Bonnet theorem will be a recurring theme in this book and we will provide several other proofs and generalizations.

One of the most fascinating aspects of Riemann geometry is the intimate correlation "local-global". The Riemann tensor is a local object with global effects. There are currently many techniques of capturing this correlation. We have already described one in the proof of Gauss-Bonnet theorem. In Chapter 5 we describe another such technique which relies on the study of the global behavior of geodesics. We felt we had the moral obligation to present the natural setting of this technique and we briefly introduce the reader to the wonderful world of the calculus of variations. The ideas of the calculus of variations produce remarkable results when applied to Riemann manifolds. For example, we explain in rigorous terms why "very curved manifolds" cannot be "too long" .

In Chapter 6 we leave for a while the "differentiable realm" and we briefly discuss the fundamental group and covering spaces. These notions shed a new light on the results of Chapter 5. As a simple application we prove Weyl's theorem that the semisimple Lie groups with definite Killing form are compact and have finite fundamental group.

Chapter 7 is the topological core of the book. We discuss in detail the cohomology of smooth manifolds relying entirely on the methods of calculus. In writing this chapter we could not, and would not escape the influence of the beautiful monograph [14], and this explains the frequent overlaps. In the first section we introduce the DeRham cohomology and the Mayer-Vietoris technique. Section 2 is devoted to the Poincaré duality, a feature which sets the manifolds apart from many other types of topological spaces. The third section offers a glimpse at homology theory. We introduce the notion of (smooth) cycle and then present some applications: intersection theory, degree theory, Thom isomorphism and we prove a higher dimensional version of the Gauss-Bonnet theorem at the cohomological level. The fourth section analyzes the role of symmetry in restricting the topological type of a manifold. We prove Élie Cartan's old result that the cohomology of a symmetric space is given by the linear space of its bi-invariant forms. We use this technique to compute the lower degree cohomology of compact semisimple Lie groups. We conclude this section by computing the cohomology of complex grassmannians relying on Weyl's integration formula and Schur polynomials. The chapter ends with a fifth section containing a concentrated description of Čech cohomology.

Chapter 8 is a natural extension of the previous one. We describe the Chern-Weil construction for arbitrary principal bundles and then we concretely describe the most important examples: Chern classes, Pontryagin classes and the Euler class. In the process, we compute the ring of invariant polynomials of many classical groups. Usually, the connections in principal bundles are defined in a global manner, as horizontal distributions. This approach is geometrically very intuitive but, at a first contact, it may look a bit unfriendly in concrete computations. We chose a local approach build on the reader's experience with connections on vector bundles which we hope will attenuate the formalism shock. In proving the various identities involving characteristic classes we adopt an invariant theoretic point of view. The chapter concludes with the general Gauss-Bonnet-Chern theorem. Our proof is a variation of Chern's proof.

Chapter 9 is the analytical core of the book. Many objects in differential geometry are defined by differential equations and, among these, the elliptic ones play an important role. This chapter represents a minimal introduction to this subject. After presenting some basic notions concerning arbitrary partial differential operators we introduce the Sobolev spaces and describe their main functional analytic features. We then go straight to the core of elliptic theory. We provide an almost complete proof of the elliptic a priori estimates (we left out only the proof of the Calderon-Zygmund inequality). The regularity results are then deduced from the a priori estimates via a simple approximation technique. As a first application of these results we consider a Kazhdan-Warner type equation which recently found applications in solving the Seiberg-Witten equations on a Kähler manifold. We adopt a variational approach. The uniformization theorem for compact Riemann surfaces is then a nice bonus. This may not be the most direct proof but it has an academic advantage. It builds a circle of ideas with a wide range of applications. The last section of this chapter is devoted to Fredholm theory. We prove that the elliptic operators on compact manifolds are Fredholm and establish the homotopy invariance of the index. These are very general Hodge type theorems. The classical one follows immediately from these results. We conclude with a few facts about the spectral properties of elliptic operators.

The last chapter is entirely devoted to a very important class of elliptic operators namely the Dirac operators. The important role played by these operators was singled out in the works of Atiyah and Singer and, since then, they continue to be involved in the most dramatic advances of modern geometry. We begin by first describing a general notion of Dirac operators and their natural geometric environment, much like in [10]. We then isolate a special subclass we called *geometric Dirac operators*. Associated to each such operator is a very concrete Weitzenböck formula which can be viewed as a bridge between geometry and analysis, and which is often the source of many interesting applications. The abstract considerations are backed by a full section describing many important concrete examples.

In writing this book we had in mind the beginning graduate student who wants to specialize in global geometric analysis in general and gauge theory in particular. The second half of the book is an extended version of a graduate course in differential geometry we taught at the University of Michigan during the winter semester of 1996.

The minimal background needed to successfully go through this book is a good knowledge of vector calculus and real analysis, some basic elements of point set topology and linear algebra. A familiarity with some basic facts about the differential geometry of curves of surfaces would ease the understanding of the general theory, but this is not a must. Some parts of Chapter 9 may require a more advanced background in functional analysis.

The theory is complemented by a large list of exercises. Quite a few of them contain technical results we did not prove so we would not obscure the main arguments. There are however many non-technical results which contain additional information about the subjects discussed in a particular section. We left hints whenever we believed the solution is not straightforward.

Personal note It has been a great personal experience writing this book and I sincerely hope I could convey some of the magic of the subject. Having access to the remarkable science library of the University of Michigan and its computer facilities certainly made my job a lot easier and improved the quality of the final product.

I learned differential equations from Professor Viorel Barbu, very generous and enthusiastic person who guided my first steps in this field of research. He stimulated my curiosity by his remarkable ability of unveiling the hidden beauty of this highly technical subject. My thesis advisor, Professor Tom Parker, introduced me to more than the fundamentals of modern geometry. He played a key role in shaping the manner in which I regard mathematics. In particular, he convinced me that behind each formalism there must be a picture and uncovering it is a very important part of the creation process. Although I did not directly acknowledge it, their influence is present throughout this book. I only hope the filter of my mind captured the full richness of the ideas they so generously shared with me.

My friends Louis Funar and Gheorghe Ionesei read parts of the manuscript. I am grateful to them for their effort, their suggestions and for their friendship. I want to thank Arthur Greenspoon for his advice, enthusiasm and relentless curiosity which boosted my spirits when I most needed it. Also, I appreciate very much the input I received from the graduate students of my “Special topics in differential geometry” course at the University of Michigan which had a beneficial impact on the style and content of this book.

At last, but not the least, I want to thank my family who supported me from the beginning to the completion of this project.

Ann Arbor, 1996.

Contents

Chapter 1

Manifolds

1.1 Preliminaries

1.1.1 Space and Coordinatization

Mathematics is a natural science with a special modus operandi. It replaces concrete natural objects with mental abstractions which serve as intermediaries. One studies the properties of these abstractions in the hope they reflect facts of life. So far, this approach proved to be very productive.

The most visible natural object is the Space, the place where all things happen. The first and most important abstraction of math is the notion of number. Loosely speaking, the aim of this book is to illustrate how these two concepts, Space and Number, fit together.

It is safe to say that geometry as a rigorous science is a creation of ancient Greeks. Euclid proposed a method of research which was later adopted by the entire mathematics. We refer of course to the axiomatic method. He viewed the space as a collection of points and distinguished some basic objects in the space such as lines, planes etc. He then postulated certain (natural) relations between them. All other properties were derived from these simple axioms.

Euclid's work is a masterpiece of mathematics and has produced many interesting results but it has its own limitations. For example, the most complicated figures one could reasonably study using this method are the conics and/or quadrics and the Greeks certainly did this. A major breakthrough in geometry was the discovery of *coordinates* by René Descartes in the 17th century. Numbers were put to work in the study of space. Their idea of producing what is now commonly referred to as Cartesian coordinates is familiar to any undergraduate. These coordinates are obtained using a very special method (in this case using three concurrent, pairwise perpendicular lines each one endowed with an identification with $\mathbb{R}$). What is important here is that they produced an one-to-one mapping

$$\text{Euclidian Space} \to \mathbb{R}^3 \qquad\qquad P \mapsto (x(P), y(P), z(P)).$$

We call such a process *coordinatization.* The corresponding map is called (in this case) *Cartesian system of coordinates.* A line or a plane becomes via coordinatization an algebraic object (more precisely an equation).

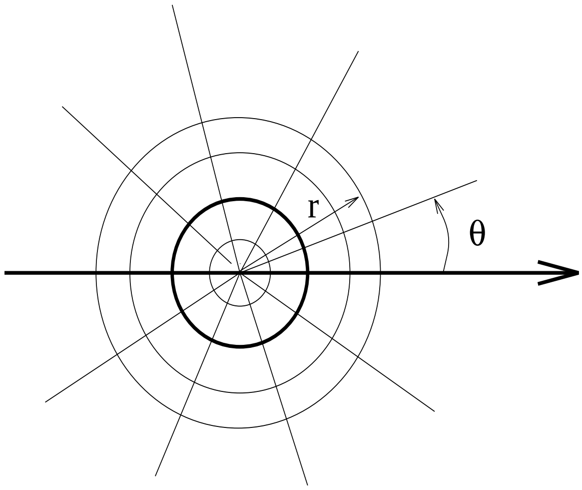

Figure 1.1: *Polar coordinates*

In general, any coordinatization replaces geometry by algebra and we get a two-way correspondence

$$\text{Study of Space} \longleftrightarrow \text{Study of Equations.}$$

The shift from geometry to numbers is beneficial to geometry as long as one has efficient tools do deal with numbers and equations. Fortunately, about the same time with the introduction of coordinates Newton created the differential and integral calculus which opened new horizons in the study of equations.

The Cartesian system of coordinates is by no means the unique or the most useful coordinatization. Concrete problems dictate other choices. For example, the polar coordinates represent another coordinatization of (a piece of the plane) (see Figure 1.1).

$$P \mapsto (r(P), \theta(P)) \in (0, \infty) \times (-\pi, \pi).$$

This choice is related to the Cartesian choice by the well known formulae

$$x = r\cos\theta \quad y = r\sin\theta. \tag{1.1.1}$$

A remarkable feature of (1.1.1) is that $x(P)$ and $y(P)$ depend smoothly upon $r(P)$ and $\theta(P)$.

As science progressed so did the notion of Space. One can think of Space as a *configuration set*, i.e. the collection of all possible states of a certain phenomenon. For example, we know from the principles of dynamics that the motion of a particle in the ambient space can be completely described if we know the position and the velocity of the particle at a given moment. The space associated with this problem consists of all pairs *(position, velocity)* a particle can possibly have. We can coordinatize this space using 6 functions: three of them will describe the position and the other three of them will describe the velocity. We say the configuration space is 6-dimensional. We cannot visualize this space but it helps to think of it as an Euclidian space, only "roomier".

There are many ways to coordinatize the configuration space of a motion of a particle and for each choice of coordinates we get a different description of the motion. Clearly, all these descriptions must "agree" in some sense sense since they all reflect the same phenomenon. In other words, these descriptions should be *independent of coordinates.* Differential geometry studies the objects which are independent of coordinates.

The coordinatization process had been used by people centuries before mathematicians accepted it as a method. For example, sailors used it to travel from one point to another on Earth. Each point has a latitude and a longitude which completely determines its position on Earth. This coordinatization is not a global one. There exist four domains delimited by the Equator and the Greenwich meridian and each of them is then naturally coordinatized. The points on the Equator for example admit two different coordinatizations which are smoothly related.

The manifolds are precisely those spaces which can be piecewise coordinatized (with smooth correspondence on overlaps) and the intention of this book is to introduce the reader to the problems and the methods which arise in the study of manifolds. The next section is a technical interlude. We will review the implicit function theorem which will be one of the basic tools for detecting manifolds.

1.1.2 The implicit function theorem

We gather here, without proofs, a collection of classical analytical facts. For more details one can consult [22].

Let X and Y be two Banach spaces and denote by $L(X,Y)$ the space of bounded linear operators $X \to Y$.

Definition 1.1.1. *Let $F : U \subset X \to Y$ be a continuous function (U is an open subset of X). F is said to be (Frechet) differentiable at $u \in U$ if there exists $T \in L(X,Y)$ such that*

$$\|F(u_0 + h) - F(u_0) - Th\|_Y = o(\|h\|_X) \text{ as } h \to 0.$$

Loosely speaking, a continuous function is differentiable at a point if near that point it admits a " best approximation " by a linear map.

When F is differentiable at $u_0 \in U$, the operator T in the above definition is uniquely determined by

$$Th = \frac{d}{dt}\,|_{t=0}\, F(u_0 + th) = \lim_{t\to 0} \frac{1}{t}\,(F(u_0 + th) - F(u_0))\,.$$

We will use the notation $T = D_{u_0}F$ and we will call T the *Frechet derivative* of F at u_0. Assume $F : U \to Y$ is differentiable at each point $u \in U$. Then F is said to be of class C^1 if the map $u \mapsto D_uF \in L(X,Y)$ is continuous. F is said to be of class C^2 if $u \mapsto D_uF$ is of class C^1. One can define inductively C^k and C^∞ (or *smooth*) maps.

Example 1.1.2. Consider $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Using Cartesian coordinates $x = (x^1, \cdots, x^n)$ in $\mathbb{R}^n$ and $u = (u^1, \cdots, u^m)$ in $\mathbb{R}^m$ we can think of F as a collection of m functions on U

$$u^1 = u^1(x^1, \cdots, x^n), \cdots, u^m = u^m(x^1, \cdots, x^n).$$

F is differentiable at a point $x \in U$ if and only if the functions u^i are differentiable at x in the usual sense of calculus. The Frechet derivative of F at x is the linear operator

$D_xF : \mathbb{R}^n \to \mathbb{R}^m$ given by the Jacobian matrix

$$D_xF = \frac{\partial(u^1, \cdots, u^m)}{\partial(x^1, \cdots, x^n)} = \left(\frac{\partial u^i}{\partial x^j}\right)_{1\le i\le m,\, 1\le j\le n}.$$

F is smooth if and only if the functions $u^i(x)$ are smooth. □

Exercise 1.1.1. (a) Let $\mathcal{U} \subset L(\mathbb{R}^n, \mathbb{R}^n)$ denote the set of invertible $n \times n$ matrices. Show that $\mathcal{U}$ is an open set.
(b) Let $F : \mathcal{U} \to \mathcal{U}$ be defined as $A \to A^{-1}$. Show that $D_AF(H) = -A^{-1}HA^{-1}$ for any $n \times n$ matrix H.
(c) Show the Frechet derivative of the map $\det : L(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ ($A \mapsto \det A$) at $\mathbb{1}_{\mathbb{R}^n} \in L(\mathbb{R}^n, \mathbb{R}^n)$ is given by

$$D_A\,|_{A=1} \det(H) = \operatorname{tr} H, \;\; \forall H \in L(\mathbb{R}^n, \mathbb{R}^n).$$

□

Theorem 1.1.3. (The inverse function theorem) *Let X, Y be two Banach spaces and $F : U \subset X \to Y$ a smooth function. If at a point $u_0 \in U$ the derivative $D_{u_0}F \in L(X, Y)$ is invertible, then there exits a neighborhood U_1 of u_0 in U such that $F(U_1)$ is an open neighborhood of $v_0 = F(u_0)$ in Y and $F : U_1 \to F(U_1)$ is bijective, with smooth inverse.*

The spirit of the theorem is very clear: the invertibility of the derivative $D_{u_0}F$ "propagates" (locally) to F because $D_{u_0}F$ is a very good local approximation for F.

More formally, if we set $T = D_{u_0}F$, then

$$F(u_0 + h) = F(u_0) + Th + r(h)$$

where $r(h) = o(\|h\|)$ as $h \to 0$. The theorem states that for every v sufficiently close to v_0 the equation $F(u) = v$ has a unique solution $u = u_0 + h$ with h very small. To prove the theorem one has to show that for $\|v - v_0\|_Y$ sufficiently small the equation below

$$v_0 + Th + r(h) = v$$

has a unique solution. We can rewrite the above equation as

$$Th = v - v_0 - r(h)$$

or

$$h = T^{-1}(v - v_0 - r(h)).$$

The last equation is a fixed point problem which can be approached successfully via the Banach fixed point theorem.

Theorem 1.1.4. (The implicit function theorem) *Let X, Y, Z be Banach spaces and $F : X \times Y \to Z$ a smooth map. Let $(x_0, y_0) \in X \times Y$ and set $z_0 = F(x_0, y_0)$. Set $F_2 : Y \to Z$, $F_2(y) = F(x_0, y)$. Assume $D_{y_0}F_2 \in L(Y, Z)$ is invertible. Then there exist neighborhoods U of $x_0 \in X$, V of $y_0 \in Y$ and a smooth map $G : U \to V$ such that $S = F^{-1}(z_0) \cap (U \times V)$ is the graph of G, i.e.*

$$\{(x, y) \in U \times V \;;\; F(x, y) = z_0\} = \{(x, G(x)) \in U \times V \;;\; x \in U\}.$$

Proof Consider the map $H : X \times Y \to X \times Z$, $\xi = (x, y) \mapsto (x, F(x, y))$. H is a smooth map and at $\xi_0 = (x_0, y_0)$ its derivative $D_{\xi_0}H : X \times Y \to X \times Z$ has the block decomposition

$$D_{\xi_0}H = \begin{bmatrix} \mathbb{1}_X & 0 \\ -D_{\xi_0}F_1 & D_{\xi_0}F_2 \end{bmatrix}.$$

Above, DF_1 (resp. DF_2) denotes the derivative of $x \mapsto F(x, y_0)$ (resp. the derivative of $y \mapsto F(x_0, y)$). $D_{\xi_0}H$ is invertible and its inverse has the block decomposition

$$(D_{\xi_0}H)^{-1} = \begin{bmatrix} \mathbb{1}_X & 0 \\ (D_{\xi_0}F_2)^{-1}D_{\xi_0}F_1() & (D_{\xi_0}F_2)^{-1} \end{bmatrix}.$$

Thus, by the inverse function theorem the equation

$$(x, F(x, y)) = (x, z_0)$$

has a unique solution $(\tilde{x}, \tilde{y}) = H^{-1}(x, z_0)$ in a neighborhood of (x_0, y_0). It obviously satisfies $\tilde{x} = x$ and $F(\tilde{x}, \tilde{y}) = z_0$. Hence, the set $\{(x, y) \;;\; F(x, y) = z_0\}$ is locally the graph of $x \mapsto H^{-1}(x, z_0)$.

$\square$

1.2 Smooth manifolds

1.2.1 Basic definitions

We now introduce the object which will be the main focus of this book, namely we will define the concept of (smooth) manifold. It formalizes the general principles outlined in Sec. 1.1.1.

Definition 1.2.1. *A* smooth manifold *of dimension m is a locally compact, paracompact Hausdorff space M together with the following collection of data (henceforth called* atlas *or* smooth structure*) consisting of:*
(a) an open cover $\{U_i\}_{i \in I}$ of M;
(b) continuous, injective maps $\Psi_i : U_i \to \mathbb{R}^m$ (called charts *or* local coordinates*) such that $\Psi_i(U_i)$ is open in $\mathbb{R}^m$ for every i, and if $U_i \cap U_j \neq \emptyset$ then the transition map*

$$\Psi_j \circ \Psi_i^{-1} : \Psi_i(U_i \cap U_j) \subset \mathbb{R}^m \to \Psi_j(U_i \cap U_j) \subset \mathbb{R}^m$$

is smooth. (We say the various charts are compatible*; see Figure 1.2).*

The chart Ψ_i can be viewed as a collection of m functions $(x^1, \ldots, x^m)$ on U_i and similarly, we can view Ψ_j as a collection of functions $(y^1, \ldots, y^m)$. The transition map $\Psi_j \circ \Psi_i^{-1}$ can then be interpreted as a collection of maps

$$(x^1, \ldots, x^m) \mapsto (y^1(x^1, \ldots, x^m), \cdots, y^m(x^1, \ldots, x^m)).$$

The first and the most important example of manifold is $\mathbb{R}^n$ itself. The natural smooth structure consists of an atlas with a single chart $\mathbf{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$. To construct more examples we will use the implicit function theorem .

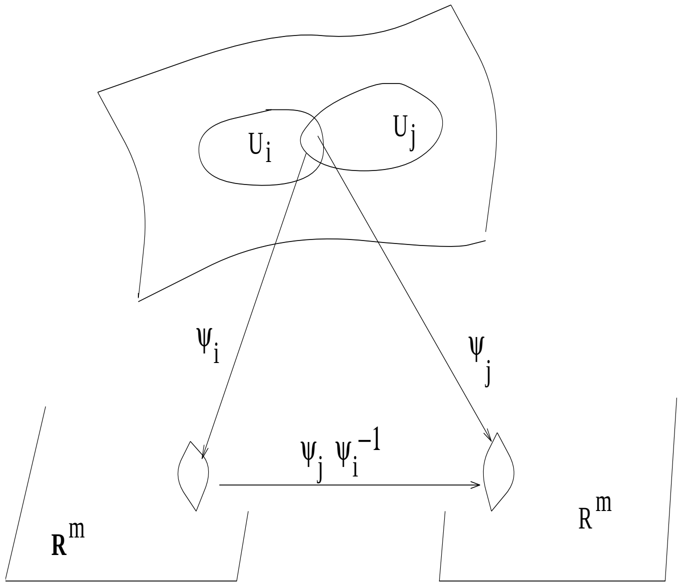

Figure 1.2: *Transition maps*

Definition 1.2.2. *(a) Let M, N be two smooth manifolds of dimensions m and respectively n. A continuous map $f : M \to N$ is said to be* smooth *if for any local chart ϕ on M and ψ on N the composition $\psi \circ f \circ \phi^{-1}$ (whenever this makes sense) is a smooth map $\mathbb{R}^m \to \mathbb{R}^n$.*
(b) A smooth map $f : M \to N$ is called a diffeomorphism *if it is invertible and its inverse is also a smooth map.*

Example 1.2.3. The map $t \mapsto e^t$ is a diffeomorphism $(-\infty, \infty) \to (0, \infty)$. The map $t \mapsto t^3$ is a homeomorphism $\mathbb{R} \to \mathbb{R}$ but it is not a diffeomorphism! □

If M is a smooth manifold we will denote by $C^\infty(M)$ the linear space of all smooth functions $M \to \mathbb{R}$.

Remark 1.2.4. Let U be an open subset of the smooth manifold M ($\dim M = m$) and $\psi : U \to \mathbb{R}^m$ a smooth, one-to one map with open image and smooth inverse. Then ψ defines local coordinates over U compatible with the existing atlas of M. Thus (U, ψ) can be added to the original atlas and the new smooth structure is diffeomorphic with the initial one. Using Zermelo's Axiom we can produce a *maximal atlas* (no more compatible local chart can be added to it).

□

Our next result is a general recipe for producing manifolds. Historically, this is how manifolds entered mathematics.

Proposition 1.2.5. *Let M be a smooth manifold of dimension m and $f_1, \dots, f_k \in C^\infty(M)$. Define*

$$\mathcal{Z} = \mathcal{Z}(f_1, \dots, f_k) = \{p \in M \ ; \ f_1(p) = \dots = f_k(p) = 0\}.$$

Assume the functions $f_1, \ldots, f_k$ *are functionally independent along* $\mathcal{Z}$ *i.e., for each* $p \in \mathcal{Z}$ *the matrix* $\left(\frac{\partial f_i}{\partial x^j}\right)_{1\leq i\leq k,\, 1\leq j\leq m}$ *has rank* k. *Here* $x = (x^1, \ldots, x^m)$ *denotes a local chart of* M *near* p. *Then* $\mathcal{Z}$ *has a natural structure of smooth manifold of dimension* $m - k$.

Proof Step 1: Constructing the charts Let $p_0 \in \mathcal{Z}$ and denote by $(x^1, \ldots, x^m)$ local coordinates near p_0 such that $x^i(p_0) = 0$. One of the $k \times k$ minors of the matrix $\left(\frac{\partial f_i}{\partial x^j}\right)_{1\leq i\leq k,\, 1\leq j\leq m}$ is non degenerate. Assume this is determined by the last k columns (and all the k lines). We can now think of the functions $f_1, \ldots, f_k$ as defined on an open subset U of $\mathbb{R}^m$. Split $\mathbb{R}^m$ as $\mathbb{R}^{m-k} \times \mathbb{R}^k$ and set $x' = (x^1, \ldots, x^{m-k})$, $x'' = (x^{m-k+1}, \ldots, x^m)$. We are now in the setting of the implicit function theorem with $X = \mathbb{R}^{m-k}$, $Y = \mathbb{R}^k$, $Z = \mathbb{R}^k$ and $F : X \times Y \to Z$ is $x \mapsto (f_1(x), \ldots, f_k(x))$. In this case $DF_2 = \left(\frac{\partial F}{\partial x''}\right)$ is invertible since it corresponds to our non degenerate minor. Thus, in a neighborhood U of p_0, the set $\mathcal{Z}$ is the graph of some function $g : \mathbb{R}^{m-k} \to \mathbb{R}^k$, $x' \mapsto x'' = g(x')$

$$\mathcal{Z} \cap U = \{(x', g(x'))\ ;\ |x'| \text{ small}\}.$$

We now define $\psi_{p_0} : \mathcal{Z} \cap U \to \mathbb{R}^{m-k}$ by

$$p = (x', g(x')) \mapsto x' \in \mathbb{R}^{m-k}.$$

ψ_{p_0} is a local chart of $\mathcal{Z}$ near p_0.

Step2 The transition maps for the charts constructed above are smooth. The details are left to the reader.

□

Exercise 1.2.1. Complete Step 2 in the proof of Proposition1.2.5. □

Definition 1.2.6. *Let* M *be a* m*-dimensional manifold. A* codimension k submanifold *of* M *is a subspace* $N \subset M$ *locally defined as the common zero locus of functionally independent functions* $f_1, \cdots, f_k \in C^\infty(M)$.

Proposition1.2.5 shows that any submanifold has a natural smooth structure so it becomes a manifold *per se*.

1.2.2 Partitions of unity

This is a very brief technical subsection describing a trick we will extensively use in this book.

Definition 1.2.7. *Let* M *be a smooth manifold and* $(U_\alpha)_{\alpha\in\mathcal{A}}$ *an open cover of* M. *A (smooth)* partition of unity *subordinated to this cover is a family* $(f_\beta)_{\beta\in\mathcal{B}} \subset C^\infty(M)$ *satisfying the following conditions.*
(i) $0 \leq f_\beta \leq 1$
(ii) $\exists \phi : \mathcal{B} \to \mathcal{A}$ *such that* $\operatorname{supp} f_\beta \subset U_{\phi(\beta)}$.
(iii) The family $(\operatorname{supp} f_\beta)$ *is locally finite i.e. any point* $x \in M$ *admits an open neighborhood intersecting only finitely many supports* $\operatorname{supp} f_\beta$.
(iv) $\sum_\beta f_\beta(x) = 1$ *for all* $x \in M$.

We include here for the reader's convenience the basic existence result concerning partitions of unity. For a proof we refer to [73].

Proposition 1.2.8. *(a) For any open cover $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}})$ of a smooth manifold M there exists at least one smooth partition of unity $(f_\beta)_{\beta\in\mathcal{B}}$ subordinated to $\mathcal{U}$ such that* $\operatorname{supp} f_\beta$ *is compact for any β.*
(a) If we do not require compact supports then we can find a partition of unity in which $\mathcal{B} = \mathcal{A}$ and $\phi = \mathbf{1}_\mathcal{A}$.

Exercise 1.2.2. Let M be a smooth manifold and $S \subset M$ a *closed* submanifold. Prove that the restriction map

$$r : C^\infty(M) \to C^\infty(S) \quad f \mapsto f\,|_S$$

is surjective. □

1.2.3 Examples

Manifolds are everywhere and in fact, to many physical phenomena which can be modeled mathematically one can naturally associate a manifold. On the other hand, many problems in mathematics find their most natural presentation using the language of manifolds. To give the reader an idea of the scope and extent of modern geometry we present here a short list of examples of manifolds. This list will be enlarged as we enter deeper into the study of manifolds.

Example 1.2.9. (The round n-dimensional sphere) This is the codimension 1 submanifold of $\mathbb{R}^{n+1}$ given by the equation

$$|x|^2 = \sum_{i=0}^{n} (x^i)^2 = r^2, \quad x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}.$$

One checks that, along the sphere, the differential of $|x|^2$ is nowhere zero so by Proposition 1.2.5 S^n is indeed a smooth manifold. In this case one can explicitly construct an atlas (consisting of two charts) which is useful in many applications. The construction relies on *stereographic projections*. Let N and S denote the north and resp. south pole of S^n ($N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, $S = (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$). Consider the open sets $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$. They form an open cover of S^n. The stereographic projection from the north pole is the map $\sigma_N : U_N \to \mathbb{R}^n$ such that for any $P \in U_N$, $\sigma_N(P)$ is the intersection of the line NP with the hyperplane $\{x^n = 0\} \cong \mathbb{R}^n$. The stereographic projection from the south pole is defined similarly. For $P \in U_N$ denote by $(y^1(P), \cdots, y^n(P))$ the coordinates of $\sigma_N(P)$ and for $Q \in U_S$ denote by $(z^1(Q), \cdots, z^n(Q))$ the coordinates of $\sigma_S(Q)$. A simple argument shows the map $(y^1(P), \cdots, y^n(P)) \mapsto (z^1(P), \cdots, z^n(P))$ $(P \in U_N \cap U_N)$ is smooth (see the exercise below. Hence $\{(U_N, \sigma_N),\ (U_S, \sigma_S)\}$. □

Exercise 1.2.3. Show that the functions y^i, z^j constructed in the above example satisfy

$$z^i = \frac{y^i}{\left(\sum_{j=1}^n (y^j)^2\right)}, \quad \forall i = 1, \cdots, n.$$

□

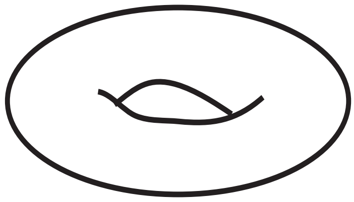

Figure 1.3: *The 2-dimensional torus*

Example 1.2.10. (The n-dimensional torus) This is the codimension n submanifold of $\mathbb{R}^{2n}(x_1, y_1; \ldots; x_n, y_n)$ defined as the zero locus

$$x_1^2 + y_1^2 = \cdots = x_n^2 + y_n^2 = 1.$$

Note that T^1 is diffeomorphic with the 1-dimensional sphere S^1 (unit circle). As a set T^n is a direct product of n circles $T^n = S^1 \times \cdots \times S^1$ (see Figure 1.3). □

The above example suggests the following general construction.

Example 1.2.11. Let M and N be smooth manifolds of dimension m and respectively n. Then their topological direct product has a natural structure of smooth manifold of dimension $m + n$. □

Example 1.2.12. (The connected sum of two manifolds) Let M_1 and M_2 be two manifolds of the same dimension m. Pick $p_i \in M_i$ $(i = 1, 2)$, choose small open neighborhoods U_i of p_i in M_i and then local charts ψ_i identifying each of these neighborhoods with the ball of radius 2 in $\mathbb{R}^m$, $B_2(0)$. Let $V_i \subset U_i$ correspond (via ψ_i) to the annulus $\{1/2 < |x| < 2\} \subset \mathbb{R}^m$. Consider

$$\phi : \{1/2 < |x| < 2\} \to \{1/2 < |x| < 2\}, \quad \phi(x) = \frac{x}{|x|^2}.$$

The action of ϕ is clear: it switches the two boundary components of $\{1/2 < |x| < 2\}$ and reverses the orientation of the radial directions. Now " glue" V_1 to V_2 using the "prescription" given by $\psi_2^{-1} \circ \phi \circ \psi_1 : V_1 \to V_2$. In this way we obtain a new topological space with a natural smooth structure induced by the smooth structures on M_i. Up to a diffeomeorphism, the new manifold thus obtained is independent of the choices of local coordinates ([16]) and it is called *the connected sum of* M_1 *and* M_2 and is denoted by $M_1 \# M_2$ (see Figure 1.4).

□

Example 1.2.13. (The real projective space $\mathbb{RP}^n$) As a topological space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$x \sim y \stackrel{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{R}^* : \quad x = \lambda y.$$

The equivalence class of $x = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1} \setminus \{0\}$ is usually denoted by $[x^0 : \ldots : x^n]$. Alternatively, $\mathbb{RP}^n$ is the set of all lines (directions) in $\mathbb{R}^{n+1}$. Traditionally one attaches to

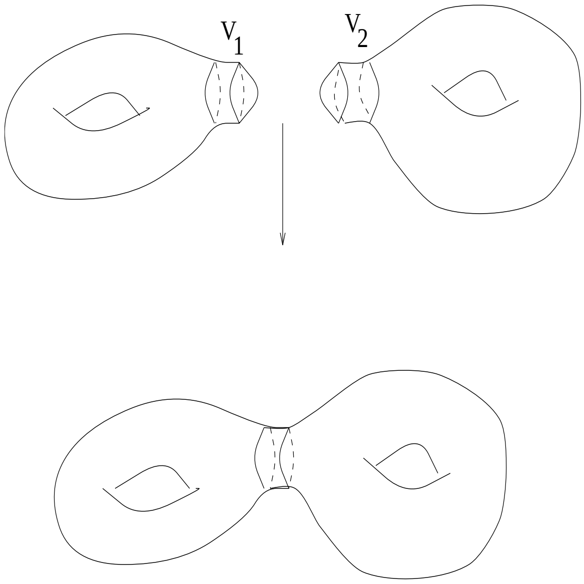

Figure 1.4: *Connected sum of tori*

each direction in $\mathbb{R}^{n+1}$ a point at infinity so that $\mathbb{RP}^n$ can be thought as the collection of all points at infinity along all directions.

$\mathbb{RP}^{n+1}$ has a natural structure of smooth manifold. To describe it consider the sets

$$U_k = \{[x^0 : \ldots : x^n] \in \mathbb{RP}^n \; ; \; x^k \neq 0\}, \quad k = 0, \ldots, n.$$

Now define

$$\psi_k : U_k \to \mathbb{R}^n \qquad [x^0 : \ldots : x^n] \mapsto (x^0/x^k, \ldots, x^{k-1}/x^k, x^{k+1}/x^k, \ldots x^n).$$

The maps ψ_k define local coordinates on the projective space. The transition map on the overlap region $U_k \cap U_m = \{[x^0 : \ldots : x^n] \; ; \; x^k x^m \neq 0\}$ can be easily described. Set

$$\psi_k([x^0 : \ldots : x^n]) = (\xi_1, \ldots, \xi_n), \;\; \psi_m([x^0 : \ldots : x^n]) = (\eta_1, \ldots, \eta_n).$$

The equality

$$[x^0 : \ldots : x^n] = [\xi_1 : \ldots : \xi_{k-1} : 1 : \xi_k : \ldots : \xi_n] = [\eta_1 : \ldots : \eta_{m-1} : 1 : \eta_m : \ldots : \eta_n]$$

immediately implies (assume $k < m$)

$$\left\{ \begin{array}{cccc} \xi_1 = \eta_1/\eta_k, & \cdots, & \xi_{k-1} = \eta_{k-1}/\eta_k & \xi_{k+1} = \eta_k \\ \xi_k = \eta_{k+1}/\eta_k, & \cdots, & \xi_{m-2} = \eta_{m-1}/\eta_k & \xi_{m-1} = 1/\eta_k \\ \xi_m = \eta_m \eta_k, & \cdots, & \xi_n = \eta_n/\eta_k & \end{array} \right. \tag{1.2.1}$$

This shows the map $\psi_k \circ \psi_m^{-1}$ is smooth and proves that $\mathbb{RP}^n$ is a smooth manifold. Note that when $n = 1$, $\mathbb{RP}^1$ is diffeomorphic with S^1. One way to see this is to observe that the projective space can be alternatively described as the quotient space of S^n modulo the equivalence relation which identifies antipodal points . $\square$

Example 1.2.14. (The complex projective space $\mathbb{CP}^n$) The definition is formally identical to that of $\mathbb{RP}^n$. $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ modulo the equivalence relation

$$x \sim y \stackrel{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{C}^* : \quad x = \lambda y.$$

The open sets U_k are defined similarly and so are the local charts $\psi_k : U_k \to \mathbb{C}^n$. They satisfy transition rules similar to (1.2.1) so that $\mathbb{CP}^n$ is a smooth manifold of dimension $2n$. $\square$

In the above example we encountered a special (and very pleasant) situation: the gluing maps not only are smooth, they are also *holomorphic* as maps $\psi_k \circ \psi_m^{-1} : U \to V$ where U and V are open sets in $\mathbb{C}^n$. This type of gluing induces a "rigidity" in the underlying manifold and it is worth distinguishing this situation.

Definition 1.2.15. (Complex manifolds) *A complex manifold is a smooth, $2n$-dimensional manifold M which admits an atlas $\{(U_i, \psi_i) : U_i \to \mathbb{C}^n\}$ such that all transition maps are holomorphic.*

The complex projective space is a complex manifold. Our next example naturally generalizes the projective spaces described above.

Example 1.2.16. (The grassmannians $G_{k,n}(\mathbb{R})$ and $G_{k,n}(\mathbb{C})$) As a set $G_{k,n}(\mathbb{R})$ consists of all k-dimensional linear subspaces of $\mathbb{R}^n$. To topologize it introduce (following [41]) the notion of *gap* between two subspaces $U, V \subset \mathbb{R}^n$ as

$$\delta(U, V) = \sup\{\operatorname{dist}(u, V)\ ;\ u \in U,\ |u| = 1\}.$$

The distance between two subspaces is then defined as

$$\hat{\delta}(U, V) = \max\{\delta(U, V),\ \delta(V, U)\}.$$

One can show easily that $(G_{k,n}(\mathbb{R}),\ \hat{\delta})$ is a metric space (exercise).

Let $\{e_1, \cdots, e_n\}$ denote the standard basis of $\mathbb{R}^n$. For any subset $I \subset \{1, \cdots, n\}$ with $\#I = k$ define $E_I = \operatorname{span}\{e_i\ ;\ i \in I\}$ so that $E_i \in G_{k,n}(\mathbb{R})$. Now define

$$\mathcal{D}_I = \{V \in G_{k,n}(\mathbb{R})\ ;\ u \cap E_I^\perp = 0\}.$$

Here $E_I^\perp$ is the orthogonal complement of E_I in $\mathbb{R}^n$ with respect to the Euclidian inner product. We leave the reader as an exercise the proofs of the following facts.

(a) $\mathcal{D}$ is a an open subset of $G_{k,n}(\mathbb{R})$.

(b) The family $(\mathcal{D}_I)$ covers $G_{k,n}(\mathbb{R})$.

(c) Any $U \in \mathcal{D}_I$ can be uniquely represented as the graph of a linear operator $T = T(U) : E_I \to E_I^\perp$, i.e.

$$U = \{(x, Tx) \in E_I \times E_I^\perp \cong \mathbb{R}^n\ ;\ x \in E_I\}.$$

We obtain continuous (*why ?*) maps $\Psi_I : \mathcal{D}_I \to L(E_I, E_I^\perp) \cong M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{k(n-k)}$. One can show that

(d) the maps Ψ_I define an atlas on $G_{k,n}(\mathbb{R})$. In particular $G_{k,n}(\mathbb{R})$ has dimension $k(n-k)$.

$G_{k,n}(\mathbb{C})$ is defined as the space of complex k-dimensional subspaces of $\mathbb{C}^n$. It can be structured as above as a smooth manifold of dimension $2k(n-k)$. Note that $G_{1,n}(\mathbb{R}) \cong \mathbb{RP}^{n-1}$ and $G_{1,n}(\mathbb{C}) \cong \mathbb{CP}^{n-1}$. The grassmannians have important applications in many classification problems. $\square$

Exercise 1.2.4. Prove the statements (a)-(d) in the above example. Show that $G_{k,n}(\mathbb{C})$ is a complex manifold (complex dimension $k(n-k)$). $\square$

Example 1.2.17. (Lie groups) A *Lie group* is a smooth manifold G together with a group structure on it such that the map

$$G \times G \to G \quad (g,h) \mapsto g \cdot h^{-1}$$

is smooth. These structures provide an excellent way to formalize the notion of symmetry .
(a) $(\mathbb{R}^n, +)$ is a commutative Lie group.

(b) The unit circle S^1 can be alternatively described as the set of complex numbers of norm one and the complex multiplication defines a Lie group structure on it. This is a commutative group. More generally the torus T^n is a Lie group as a direct product of n circles[1].

(c) The general linear group $GL(n, \mathbb{K})$ defined as the group of invertible $n \times n$ matrices with entries in the field $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ is a Lie group. Indeed, $GL(n, \mathbb{K})$ is an open subset (see Exercise 1.1.1) in the linear space of $n \times n$ matrices with entries in $\mathbb{K}$. It has dimension $d_{\mathbb{K}} n^2$ where $d_{\mathbb{K}}$ is the dimension of $\mathbb{K}$ as a linear space over $\mathbb{R}$.

(d) The orthogonal group $O(n)$ is the group of real $n \times n$ matrices satisfying

$$T \cdot T^t = \mathbb{1}.$$

To describe its smooth structure we will use the Cayley transform trick as in [65] (see also the classical [74]). Set

$$M_n(\mathbb{R})^{\#} = \{T \in M_n(\mathbb{R}) \; ; \; \det(\mathbb{1} + T) \neq 0\}.$$

The matrices in $M_n(\mathbb{R})^{\#}$ are called non exceptional. Clearly $\mathbb{1} \in O(n)^{\#} = O(n) \cap M_n(\mathbb{R})^{\#}$ so that $O(n)^{\#}$ is a *nonempty* open subset of $O(n)$. The *Cayley transform* is the map $\# : M_n(\mathbb{R})^{\#} \to M_n(\mathbb{R})$ defined by

$$A \mapsto A^{\#} = (\mathbb{1} - A)(\mathbb{1} + A)^{-1}.$$

The Cayley transform has some very nice properties.
(i) $A^{\#} \in M_n(\mathbb{R})^{\#}$ for every $A \in M_n(\mathbb{R})^{\#}$.
(ii) $\#$ is involutory i.e. $(A^{\#})^{\#} = A$ for any $A \in M_n(\mathbb{R})^{\#}$.
(iii) For every $T \in O(n)^{\#}$ the matrix $T^{\#}$ is skew-symmetric and conversely if $A \in M_n(\mathbb{R})^{\#}$ is skew-symmetric then $A^{\#} \in O(n)$.

Thus the Cayley transform is a homeomorphism from $O(n)^{\#}$ to the space of non-exceptional, skew-symmetric, matrices. The latter space is an open subset in the linear space of real $n \times n$ skew-symmetric matrices, $\underline{o}(n)$.

[1] One can show that any connected commutative Lie group has the from $T^n \times \mathbb{R}^m$.

Any $T \in O(n)$ defines a self-homeomorphism of $O(n)$ by *left translation in the group*

$$L_T : O(n) \to O(n) \quad S \mapsto L_T(S) = T \cdot S.$$

We obtain an open cover of $O(n)$:

$$O(n) = \bigcup_{T \in O(n)} T \cdot O(n)^{\#}.$$

Define $\Psi_T : T \cdot O(n)^{\#} \to \underline{o}(n)$ by $S \mapsto (T^{-1} \cdot S)^{\#}$. One can show that the collection

$$\Big(T \cdot O(n)^{\#}, \Psi_T \Big)_{T \in O(n)}$$

defines a smooth structure on $O(n)$. In particular we deduce

$$\dim O(n) = n(n-1)/2.$$

Inside $O(n)$ lies a normal subgroup (the *special orthogonal group*)

$$SO(n) = \{T \in O(n) \; ; \; \det T = 1\}.$$

$SO(n)$ is a Lie group as well and $\dim SO(n) = \dim O(n)$.
(e) The unitary group $U(n)$ is defined as

$$U(n) = \{T \in GL(n, \mathbb{C} \; ; \; T \cdot T^* = \mathbb{1}\}$$

T^* denotes the conjugate transpose (adjoint) of T. To prove that $U(n)$ is a manifold one uses again the Cayley transform trick. This time we coordinatize the group using the space $\underline{u}(n)$ of skew-adjoint (skew-Hermitian) $n \times n$ complex matrices ($A = -A^*$). Thus $U(n)$ is a smooth manifold of dimension

$$\dim U(n) = \dim \underline{u}(n) = n^2.$$

Inside $U(n)$ sits the normal subgroup $SU(n)$, the kernel of the group homomorphism $\det : U(n) \to S^1$. $SU(n)$ is also called the *special unitary group*. This a smooth manifold of dimension $n^2 - 1$. In fact the Cayley transform trick allows one to coordinatize $SU(n)$ using the space

$$\underline{su}(n) = \{A \in \underline{u}(n) \; ; \; \operatorname{tr} A = 0\}.$$

□

Exercise 1.2.5. (a) Prove the properties (i)-(iii) of the Cayley transform and then show that $\big(T \cdot O(n)^{\#}, \Psi_T\big)_{T \in O(n)}$ defines a smooth structure on $O(n)$.
(b) Prove that $U(n)$ and $SU(n)$ are manifolds.
(c) Show that $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact spaces.
(d) Prove that $SU(2)$ is diffeomorphic with S^3 (Hint: think of S^3 as the group of unit quaternions.) □

Exercise 1.2.6. Let $SL(n;\mathbb{K})$ denote the group of $n \times n$ matrices of determinant 1 with entries in the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Using the Cayley trick show that $SL(n;\mathbb{K})$ is a smooth manifold modeled on the linear space

$$\underline{sl}(n, \mathbb{K}) = \{A \in M_{n\times n}(\mathbb{K}) \; ; \; \operatorname{tr} A = 0\}.$$

In particular it has dimension $d_{\mathbb{K}}(n^2 - 1)$, where $d = \dim_{\mathbb{R}} \mathbb{K}$. □

Exercise 1.2.7. (a) Let G be a *connected* Lie group and denote by U a neighborhood of $1 \in G$. If H is the subgroup algebraically generated by U show that H is dense in G.
(b) Let G be a *compact* Lie group and $g \in G$. Show that $1 \in G$ lies in the closure of $\{g^n; \; n \in \mathbb{Z} \setminus \{0\}\}$. □

1.2.4 How many manifolds are there?

The list of examples in the previous subsection can go on for ever so one may ask whether there is any coherent way to organize the collection of all possible manifolds. This is too general a question to expect a clear cut answer. We have to be more specific. For example we can ask

Question 1: *Which are the* compact *manifolds of a given dimension d?*

For $d = 1$ the answer is very simple: the only compact 1-dimensional manifold is the circle S^1. So we can raise the stakes and try the same problem for $d = 2$. Already the situation is more elaborate. We know at least two surfaces: the sphere S^2 and the torus T^2. They clearly look different but we have not yet proved rigorously that they are indeed not diffeomorphic. This is not the end of the story. We can connect sum two tori, three tori or any number g of tori. We obtain doughnut-shaped surface as in Figure 1.5

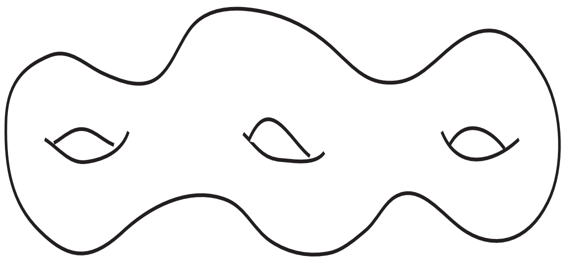

Figure 1.5: *Connected sum of 3 tori*

Again we face the same question: do we get non-diffeomorphic surfaces for different choices of g? Figure 1.5 suggests that this may be the case but this is no rigorous argument.

We know another example of compact surface, the projective plane $\mathbb{RP}^2$ and we naturally ask whether it looks like one of the surfaces constructed above. Unfortunately, we cannot visualize the real projective plane (one can prove rigorously it does not have enough room to exist inside our 3-dimensional Universe). We have to decide this question using a little more than the raw geometric intuition provided by a picture. To kill the suspense, we mention that $\mathbb{RP}^2$ does not belong to the family of donuts. The reason is that a torus for example has two faces: an inside face and an outside face (think of a car rubber tube). $\mathbb{RP}^2$ has a weird behavior: it has "no inside" and "no outside". It has only one side! One says the torus is *orientable* while the projective plane is not.

We can now connect sum any numbers of $\mathbb{RP}^2$'s to any donut an thus obtain more and more surfaces, which we cannot visualize and we have yet no idea if they are pairwise distinct. A classical result in topology says that all compact surfaces can be obtained in this way (see [52]) but in the above list some manifolds are diffeomorphic and we have to describe which. In dimensions ≥ 3 things are not settled and to make things worse in dimension ≥ 4 Question 1 is algorithmically undecidable .

We can reconsider our goals and look for all the manifolds with a given property X. In many instances one can give fairly accurate answers. Property X may refer to more than the (differential) topology of a manifold. Real life situations suggest the study of manifolds with additional structure. The following problem may give the reader a taste of the types of problems we will be concerned with in this book.

Question 2 *Can we wrap a planar piece of canvas around a metal sphere in a one-to-one fashion? (The canvas is flexible but not elastic).*

A simple do-it-yourself experiment is enough to convince anyone that this is not possible. Naturally, one asks for a rigorous explanation of what goes wrong. The best explanation of this phenomenon is contained the the celebrated Theorema Egregium (Golden Theorem) of Gauss. Canvas surfaces have additional structure (they are made of a special material) and for such objects there is a rigorous way to measure "how curved" are they. One then realizes that the problem in Question 2 is impossible since a (canvas) sphere is curved in a different way than a plane canvas.

There are many other structures Nature forced us into studying them but they may not be so easily described in elementary terms.

A word to the reader. The next two chapters are the most arid in geometry but, keep in mind that behind each construction lies a natural motivation and, even if we do not always have the time to show it to the reader, it is there, and it may take a while to reveal itself. Most of the constructions the reader will have to "endure" in the next two chapters constitute not just some difficult to "swallow" formalism but the basic language of geometry. Learning this language may not be the most pleasant thing, but surely enough, it is a very rewarding enterprise.

Chapter 2

Natural Constructions on Manifolds

The goal of this chapter is to introduce the basic terminology used in differential geometry. The key concept is that of tangent space at a point which is a first order approximation of the manifold near that point. We will be able to transport many notions in linear analysis to manifolds via the tangent space.

2.1 The tangent bundle

2.1.1 Tangent spaces

We begin with a simple example which will serve as a motivation for the abstract definitions.

Example 2.1.1. Consider the sphere

$$(S^2) : x^2 + y^2 + z^2 = 1 \quad \text{in } \mathbb{R}^3.$$

We want to find the plane passing through the north pole $N(0, 0, 1)$ which locally is "closest" to the sphere. The natural candidate for this osculator plane would be a plane given by a linear equation which best approximates the defining equation $x^2 + y^2 + z^2 = 1$ in a neighborhood of the north pole. The linear approximation of $x^2 + y^2 + z^2$ near N is

$$x^2 + y^2 + z^2 = 1 + 2(z - 1) + O(2)$$

where $O(2)$ denotes a quadratic error. Hence, the osculator plane is $z = 1$ which is the horizontal plane through the north pole. The linear subspace $\{z = 0\} \subset \mathbb{R}^3$ is called the *tangent space* to S^2 at N.

The above construction has one deficiency: it is not intrinsic since it relies on objects "outside" the manifold S^2. There is one natural way to fix this problem. Look at a smooth path $\gamma(t)$ on S^2 passing through N at $t = 0$. Hence $t \mapsto \gamma(t) \in \mathbb{R}^3$ and

$$|\gamma(t)|^2 = 1. \tag{2.1.1}$$

If we derivate (2.1.1) at $t = 0$ we get $(\dot{\gamma}(0), \gamma(0)) = 0$ i.e. $\dot{\gamma}(0) \perp \gamma(0)$ so that $\dot{\gamma}(0)$ lies in the linear subspace $z = 0$. We deduce that the tangent space consists of the tangents to the curves on S^2 passing through N.

This is apparently no major conceptual gain since we still regard the tangent space as a subspace of $\mathbb{R}^3$ and this is still an extrinsic description. However, if we use the stereographic projection from the south pole we get local coordinates (u, v) near N and any curve $\gamma(t)$ as above can be viewed as a curve $t \mapsto (u(t), v(t))$ in the (u, v) plane. If $\phi(t)$ is another curve through N given in local coordinates by $t \mapsto (\underline{u}(t), \underline{v}(t))$ then

$$\dot{\gamma}(0) = \dot{\phi}(0) \iff (\dot{u}(0), \dot{v}(0)) \;=\; (\dot{\underline{u}}(0), \dot{\underline{v}}(0)).$$

The right hand side of the above equality defines an equivalence relation $\sim$ on the set of smooth curves passing trough (0,0). Thus, there is a bijective correspondence between the tangents to the curves through N and the equivalence classes of "$\sim$". This equivalence relation is now intrinsic modulo one problem: "$\sim$" may depend on the choice of the local coordinates. Fortunately, as we are going to see, this is a non-issue. $\square$

Definition 2.1.2. *Let M^m be a smooth manifold and p_0 a point in M. Two smooth paths $\alpha,\ \beta : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = \beta(0) = p_0$ are said to have a* 1st order contact *at p_0 if there exist local coordinates $(x) = (x^1, \dots, x^m)$ near p_0 such that*

$$\dot{x}_\alpha(0) = \dot{x}_\beta(0)$$

where $\alpha(t) = (x_\alpha(t)) = (x^1_\alpha(t), \dots, x^m_\alpha(t))$ and $\beta(t) = (x_\beta(t)) = (x^1_\beta(t), \dots, x^m_\beta(t))$. We write this $\alpha \sim_1 \beta$.

Lemma 2.1.3. *$\sim_1$ is an equivalence relation.*

Sketch of proof $\sim_1$ is trivially reflexive and symmetric so we only have to check the transitivity. Let $\alpha \sim_1 \beta$ and $\beta \sim_1 \gamma$. Thus there exist local coordinates $(x) = (x^1, \dots, x^m)$ and $(y) = (y^1, \dots, y^m)$ near p_0 such that $(\dot{x}_\alpha(0)) = (\dot{x}_\beta(0))$ and $(\dot{y}_\beta(0)) = (\dot{y}_\gamma(0))$. The transitivity follows from the equality

$$\dot{y}^i_\gamma(0) = \dot{y}^i_\beta(0) = \sum_j \frac{\partial y^i}{\partial x^j} \dot{x}^j_\beta(0) = \sum_j \frac{\partial y^i}{\partial x^j} \dot{x}^j_\alpha(0) = \dot{y}^j_\alpha(0).$$

$\square$

Definition 2.1.4. *A* tangent vector *to M at p is an equivalence class of curves through p modulo the the first order contact relation. The equivalence class of a curve $\alpha(t)$ such that $\alpha(0) = p$ will be temporarily denoted by $[\dot{\alpha}(0)]$. The set of these equivalence classes is denoted by T_pM and is called the* tangent space *to M at p.*

Lemma 2.1.5. *T_pM has a natural structure of vector space.*

Proof Choose $(x^1, \dots, x^m)$ local coordinates near p such that $x^i(p) = 0$, $\forall i$ and let $\alpha,\ \beta$ two smooth curves through p. In the above local coordinates the curves $\alpha,\ \beta$ become $(x^i_\alpha(t))$, $(x^i_\beta(t))$. Construct a new curve γ through p by

$$(x^i_\gamma(t)) = (x^i_\alpha(t) + x^i_\beta(t)).$$

Set $[\dot{\alpha}(0)] + [\dot{\beta}(0)] \stackrel{def}{=} [\dot{\gamma}(0)]$. For this operation to be well defined one has to check
(a) $[\dot{\gamma}(0)]$ is independent of coordinates.
(b) If $[\dot{\alpha}_1(0)] = [\dot{\alpha}_2(0)]$ and $[\dot{\beta}_1(0)] = [\dot{\beta}_2(0)]$ then

$$[\dot{\alpha}_1(0)] + [\dot{\beta}_1(0)] = [\dot{\alpha}_2(0)] + [\dot{\beta}_2(0)].$$

We let the reader supply the routine details. □

Exercise 2.1.1. Finish the proof of the Lemma 2.1.5. □

From this point on we will omit the [,] in the notation for a tangent vector. Thus $[\dot{\alpha}(0)]$ will be written simply as $\dot{\alpha}(0)$.

As one expects, all the above notions admit a nice description using local coordinates. Let $(x^1, \dots, x^m)$ be coordinates near $p \in M$ such that $x^i(p) = 0$, $\forall i$. Consider the curves

$$e_k(t) = (t\delta_k^1, \dots, t\delta_k^m), \;\; k = 1, \dots, m$$

where δ_j^i denotes Kronecker's delta symbol. Equivalently, one can define the e_k's implicitly by $x^i = 0$, $\forall i \neq k$. We set

$$\frac{\partial}{\partial x^k}(p) \stackrel{def}{=} \dot{e}_k(0). \tag{2.1.2}$$

Note that these vectors depend on the local coordinates $(x^1, \cdots, x^m)$. Often, when the point p is clear from the context, we will omit it in the above notation.

Lemma 2.1.6. $\left(\frac{\partial}{\partial x^k}(p)\right)_{1 \leq k \leq m}$ *is a basis of* T_pM.

Proof It follows from the obvious fact that any curve through the origin in $\mathbb{R}^m$ has first order contact with a line $t \mapsto (a_1 t, \dots, a_m t)$. □

Exercise 2.1.2. Let $F : \mathbb{R}^N \to \mathbb{R}^k$ be a smooth map. Assume that
(a) $M = F^{-1}(0) \neq \emptyset$;
(b) $\operatorname{rank} D_xF = k$, for all $x \in M$.
Then M is a smooth manifold of dimension $N - k$ and

$$T_xM = (D_xF)^{-1}(0), \;\; \forall x \in M. \qquad \square$$

Example 2.1.7. We want to describe $T_{\mathbb{1}}G$, where G is one of the Lie groups discussed in Section 1.2.2.
(a) $G = O(n)$. Let $T(s)$ be a smooth path of orthogonal matrices such that $T(0) = \mathbb{1}$. Then $T^t(s) \cdot T(s) = \mathbb{1}$. Differentiating this equality at $s = 0$ we get

$$\dot{T}^t(0) + \dot{T}(0) = 0.$$

$\dot{T}(0)$ defines a vector in $T_1O(n)$ so the above equality says this tangent space lies inside the space of skew-symmetric matrices i.e. $T_1G \subset \underline{o}(n)$. On the other hand we proved in Section 1.2.2 that $\dim G = \dim \underline{o}(n)$ so that

$$T_{\mathbb{1}}O(n) = \underline{o}(n).$$

(b) $G = SL(n;\mathbb{R})$. Let $T(s)$ be a smooth path in $SL(n;\mathbb{R})$ such that $T(0) = \mathbb{1}$. Then $\det T(s) = 1$ and differentiating this equality at $s = 0$ we get (see Exercise 1.1.1) $\operatorname{tr}\dot{T}(0) = 0$. Thus the tangent space at $\mathbb{1}$ lies inside the space of traceless matrices, i.e. $T_{\mathbb{1}}SL(n;\mathbb{R}) \subset \underline{sl}(n;\mathbb{R})$. Since (according to Exercise 1.2.6) $\dim SL(n;\mathbb{R}) = \dim \underline{sl}(n;\mathbb{R})$ we deduce

$$T_{\mathbb{1}}SL(n;\mathbb{R}) = \underline{sl}(n;\mathbb{R}). \qquad \square$$

Exercise 2.1.3. Show that $T_{\mathbb{1}}U(n) = \underline{u}(n)$ and $T_1 SU(n) = \underline{su}(n)$. $\square$

2.1.2 The tangent bundle

In the previous subsection we have naturally associated to an arbitrary point p on a manifold M a vector space T_pM. It is the goal of the present subsection to coherently organize the family of tangent spaces $(T_pM)_{p\in M}$. In particular we want to give a rigorous meaning to the intuitive fact that T_pM depends smoothly upon p.

We will organize the disjoint union of all tangent spaces as a smooth manifold TM. There is a natural surjection

$$\pi : TM = \coprod_{p\in M} T_pM \to M, \quad \pi(v) = p \Longleftrightarrow v \in T_pM.$$

Any local coordinate system $x = (x^i)$ defined over an open set $U \subset M$ produces a natural basis $\left(\frac{\partial}{\partial x^i}(p)\right)$ of T_pM for any $p \in U$. Thus, an element $v \in TU = \coprod_{p\in U} T_pM$ is completely determined if we know to which tangent space it belongs (i.e. we know $p = \pi(v)$) and we also know its coordinates in the basis $\left(\frac{\partial}{\partial x^i}(p)\right)$:

$$v = \sum_i X^i(v)\left(\frac{\partial}{\partial x^i}(p)\right).$$

We thus have a bijection

$$\Psi_x : TU \to U^x \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m$$

where U^x is the image of U in $\mathbb{R}^m$ via the coordinates (x^i). We can now transfer the topology on $\mathbb{R}^m \times \mathbb{R}^m$ to TU via the map Ψ_x. Again we have to make sure this topology is independent of local coordinates.

To see this, pick a different coordinate system $y = (y^i)$ on U. The coordinate independence referred to above is equivalent to the statement that the transition map

$$\Psi_y \circ \Psi_x^{-1} : U^x \times \mathbb{R}^m \to TU \to U^y \times \mathbb{R}^m$$

is a homeomorphism. Let $A = (\overline{x}, X) \in U^x \times \mathbb{R}^m$. Then $\Psi_x^{-1}(A) = (p, \dot{\alpha}(0))$ where $x(p) = \overline{x}$ and $\alpha(t) \subset U$ is a curve through p given in the x coordinates as

$$\alpha(t) = \overline{x} + tX.$$

Denote by $F : U^x \to U^y$ the transition map $x \mapsto y$. Then

$$\Psi_y \circ \Psi_x^{-1}(A) = (y(\overline{x}); Y^1, \ldots, Y^m))$$

where $\dot{\alpha}(0) = (\dot{y}^j_\alpha(0)) = \sum Y^j \frac{\partial}{\partial y_j}(p)$. $((y_\alpha(t))$ is the defining equation for the curve $\alpha(t)$ in the coordinates y^j). Applying the chain rule we deduce

$$Y^j = \dot{y}^j_\alpha(0) = \sum_i \frac{\partial y^j}{\partial x^i}\dot{x}^i(0) = \sum_i \frac{\partial y^j}{\partial x^i} X^i. \tag{2.1.3}$$

This proves that $\Psi_y \circ \Psi_x^{-1}$ is actually smooth.

The natural topology of TM is obtained by patching together the topologies of TU_γ, where $(U_\gamma, \phi_\gamma)_\gamma$ is an atlas of M. A set $D \subset TM$ is open if its intersection with any TU_γ is open in TU_γ. The above argument shows that TM is a smooth manifold with (TU_γ, Ψ_γ) a defining atlas. Moreover, the natural projection $\pi : TM \to M$ is a smooth map.

Definition 2.1.8. *The smooth manifold TM described above is called the* tangent bundle *of M.*

Proposition 2.1.9. *A smooth map $f : M \to N$ induces a smooth map $Df : TM \to TN$ such that*
(a) $Df(T_pM) \subset T_{f(p)}N$, $\forall p \in M$
(b) The restriction to each tangent space $D_pF : T_pM \to T_{f(p)}N$ is linear. Df is called the differential *of f and one often uses the alternate notation $f_* = Df$.*

Proof Recall that T_pM is the space of tangent vectors to curves through p. If $\alpha(t)$ is such a curve $(\alpha(0) = p)$ then $\beta(t) = f(\alpha(t))$ is a smooth curve through $q = f(p)$ and define

$$Df(\dot{\alpha}(0)) = \dot{\beta}(0).$$

One checks easily that if $\alpha_1 \sim_1 \alpha_2$ then $f(\alpha_1) \sim_1 f(\alpha_2)$ so that Df is well defined. To prove the map $Df : T_pM \to T_qN$ is linear it suffices to verify it in any particular local coordinates $(x^1, \dots, x^m)$ near p and $(y^1, \dots, y^n)$ near q since any two choices differ (infinitesimally) by a linear substitution. Hence we can regard f as a collection of maps

$$(x^1, \dots, x^m) \mapsto (y^1(x^1, \dots, x^m), \dots, y^n(x^1, \dots, x^m)).$$

A basis in T_pM is given by $\{\frac{\partial}{\partial x_i}\}$ while a basis of T_qN is given by $\{\frac{\partial}{\partial y_j}\}$. Then Df is the linear operator given in these bases by the matrix $\left(\frac{\partial y^j}{\partial x^i}\right)_{1\leq j\leq n,\, 1\leq i\leq m}$. In particular, this shows that Df is smooth. □

Definition 2.1.10. *A smooth map $f : M \to N$ is called* immersion *(resp.* submersion*) if for every $p \in M$ the differential $D_pf : T_pM \to T_{f(p)}N$ is injective (resp. surjective).*

Exercise 2.1.4. Let $f : M \to N$ be a submersion. Show that for every $q \in N$ the fiber $f^{-1}(q)$ is either empty or a submanifold of M of dimension $\dim M - \dim N$. □

2.1.3 Vector bundles

The tangent bundle TM of a manifold M has some special features which makes it a very particular type of manifold. We list now the special ingredients which enter the special structure of TM since they will occur in many instances. Set for brevity $E = TM$ and $F = \mathbb{R}^m$ ($m = \dim M$).

(a) E is a smooth manifold and there exists a surjective submersion $\pi : E \to M$. For every $U \subset M$ set $E\,|_U = \pi^{-1}(U)$.
(b) From (2.1.3) we deduce that there exists a *trivializing cover* i.e. an open cover $\mathcal{U}$ of M and for every $U \in \mathcal{U}$ a diffeomorphism

$$\Psi_U : E\,|_U \to U \times F, \quad v \mapsto (p = \pi(v), \Phi_p^U(v))$$

such that

(b1) Φ_p is a diffeomorphism $E_p \to F$ for any $p \in U$.

(b2) If $U,\ V \in \mathcal{U}$ are two trivializing neighborhoods with non empty overlap $U \cap V$ then for any $p \in U \cap V$ the map $\Phi_{VU}(p) = \Phi_p^V \circ (\Phi_p^U)^{-1} : F \to F$ is a linear isomorphism and moreover, the map

$$p \mapsto \Phi_{VU}(p) \in L(F, F)$$

is smooth.

In our special case the map $\Phi_{VU}(p)$ is explicitly defined by the matrix (2.1.3)

$$A(p) = \left(\frac{\partial y^j}{\partial x^i}(p) \right)_{1 \leq i,j \leq m}.$$

In the above formula (x^i) are local coordinates on U and (y^j) are local coordinates on V.

The properties (a) and (b) make no mention of the special relationship between $E = TM$ and M. There are many quadruples (E, π, M, F) with these properties and they deserve a special name.

Definition 2.1.11. *A quadruple (E, π, M, F) (where E, M are smooth manifolds, $\pi : E \to M$ is a surjective submersion and F a vector space, real or complex) satisfying conditions (a) and (b) above is called a* smooth vector bundle *over M. E is called the* total space *and M is called the* base space*. The vector space F is called the* standard fiber *and its dimension (over its field of scalars) is called the* rank *of the bundle.*

Roughly speaking, a vector bundle is a smooth family of vector spaces. Note that the properties (b1) and (b2) imply that the fibers $\pi^{-1}(p)$ of a vector bundle have a natural structure of linear space. In particular, one can add elements in the same fiber.

The above definition has an "impurity" built in coming from the choices of the open cover (U_α) and trivializations ψ_α which are somewhat arbitrary. We can get rid of this arbitrariness by introducing an equivalence relation.

An open cover (U_α) together with trivializations

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

is said to be equivalent to the open cover (V_i) with the trivializations

$$\psi_i : \pi^{-1}(V_i) \to V_i \times F$$

if $\forall \alpha$, i there exists a smooth map $g_{\alpha i} : U_\alpha \cap V_i \to GL(F)$ such that for any $x \in U_\alpha \cap V_i$ and any $f \in F$

$$\phi_\alpha \circ \psi_i^{-1}(x, f) = (x, g_{\alpha i} f).$$

We will postulate that two equivalent trivializing covers define the same vector bundle structures.

There is an equivalent way to define vector bundles. According to Definition 2.1.11, we can find an open cover (U_α) of M such that each of the restrictions $E_\alpha = E\,|_{U_\alpha}$ is isomorphic to a product $\Psi_\alpha : E_\alpha \cong U_\alpha \times F$. Moreover, on the overlaps $U_\alpha \cap U_\beta$, *the transition maps* $g_{\alpha\beta} = \Psi_\alpha \Psi_\beta^{-1}$ can be viewed as smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(F)$$

where $GL\,(F)$ denotes the Lie group of linear automorphisms of F. They satisfy the *cocycle condition*
(a) $g_{\alpha\alpha} = \mathbb{1}_F$
(b) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathbb{1}_F$ over $U_\alpha \cap U_\beta \cap U_\gamma$.

Conversely, given a cover (U_α) of M and a collection of smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL\,(F)$ satisfying the cocycle condition we can reconstruct a vector bundle by gluing the product bundles $E_\alpha = U_\alpha \times F$ on the overlaps $U_\alpha \cap U_\beta$ according to the gluing rules

$$(x, v) \in E_\alpha \text{ is identified with } (x, g_{\beta\alpha}(x)v) \in E_\beta \ \ \forall x \in U_\alpha \cap U_\beta.$$

We may say that $g_{\beta\alpha}$ *is the transition from* α to β. In the sequel we will prefer to think of vector bundles in terms of transition maps. A cover (U_α) as above will be called a trivializing cover. The details are carried out in the exercise below.

Exercise 2.1.5. Consider a smooth manifold M, a vector space V, an open cover (U_α) and smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$$

satisfying the cocycle condition. Set

$$X = \bigcup_\alpha U_\alpha \times V \times \{\alpha\}.$$

We topologize X as the disjoint union of the topological spaces $U_\alpha \times V$. Define a relation $\sim \subset X \times X$ by

$$U_\alpha \times V \times \{\alpha\} \ni (x, u, \alpha) \sim (y, v, \beta) \in U_\beta \times V \times \{\beta\} \overset{def}{\Longleftrightarrow} x = y, \ \ v = g_{\beta\alpha}(x)u.$$

(a) Show that $\sim$ is an equivalence relation and $E = X/\sim$ equipped with the quotient topology has a natural structure of smooth manifold.
(b) Show that the projection $\pi : X \to M$, $(x, u, \alpha) \mapsto x$ descends to a submersion $E \to M$.
(c) Prove that (E, π, M, V) is naturally a smooth vector bundle. □

Definition 2.1.12. *(a) A* section *in a vector bundle $E \overset{\pi}{\to} M$ defined over the open subset $u \in M$ is a smooth map $s : U \to E$ such that $s(p) \in E_p = \pi^{-1}(p)$, $\forall p \in U$. Equivalently, this means that $\pi \circ s = \mathbb{1}_U$. The space of smooth sections of E over U will be denoted by $\Gamma(U, E)$ or $C^\infty(U, E)$. Note that $\Gamma(U, E)$ is naturally a linear space.*
(b) A section of the tangent bundle is called a vector field*. The space of vector fields over U is denoted by* Vect *(U).*

In terms of a trivializing cover (U_α) and transition maps $(g_{\alpha\beta})$ a section can be defined as a collection of smooth maps $s_\alpha : U_\alpha \to F$ satisfying a gluing condition on the overlaps

$$s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x) \ \ \forall x \in U_\alpha \cap U_\beta.$$

Definition 2.1.13. *Let $E^i \stackrel{\pi_i}{\to} M_i$ be two smooth vector bundles. A* vector bundle map *consists of a pair of smooth maps $f : M_1 \to M_2$ and $F : E^1 \to E^2$ satisfying the following properties.*
(a) The diagram below is commutative

$$\begin{array}{ccc} E^1 & \xrightarrow{F} & E^2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array},$$

i.e. $\forall p \in M_1$, $F(E^1_p) \subset E^2_{f(p)}$ (one says that F covers f).
(b) The induced map $F : E^1_p \to E^2_{f(p)}$ is linear.

The composition of bundle maps is defined in the obvious manner and so is the identity morphism so that one can define the notion of bundle isomorphism in the standard way. If E and F are two vector bundles over the same manifold then we denote by Hom (E, F) the space of bundle maps $E \to F$ which cover the identity $\mathbb{1}_M$. *Such bundle maps are called* bundle morphisms.

For example the differential Df of a smooth map $f : M \to N$ is a bundle map $Df : TM \to TN$ covering f.

Definition 2.1.14. *Let $E \stackrel{\pi}{\to} M$ be a smooth vector bundle. A* bundle endomorphism *of E is a morphism $F : E \to E$. An automorphism (or* gauge transformation*) is an invertible endomorphism.*

Given a trivializing cover (U_α) we can view a bundle map as a collection of smooth maps $h_\alpha : U_\alpha \to GL(F)$ such that

$$h_\alpha(x) = g_{\alpha\beta}(x)h_\beta(x)(g_{\alpha\beta}(x))^{-1} \ \ \forall x \in U_\alpha \cap U_\beta.$$

Exercise 2.1.6. Let V be a vector space, M a smooth manifold, $\{U_\alpha\}$ an open cover of M and $g_{\alpha\beta}$, $h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ two collections of smooth maps satisfying the cocycle conditions. Prove the two collections define isomorphic vector bundles if and only they are *cohomologous*, i.e. there exist smooth maps $\phi_\alpha : U_\alpha \to GL(V)$ such that

$$h_{\alpha\beta} = \phi_\alpha g_{\alpha\beta}\phi_\beta^{-1}. \qquad \square$$

2.1.4 Some examples of vector bundles

In this section we would like to present some important examples of vector bundles and then formulate some questions concerning the global structure of a bundle.

Example 2.1.15. The tautological line bundle over $\mathbb{RP}^n$ and $\mathbb{CP}^n$ First, let us mention that a rank 1 vector bundle is usually called a *line bundle*. We consider only the complex case. The total space of the tautological line bundle over $\mathbb{CP}^n$ is the space

$$\tau_n = \{(Z, p) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \ ; \ Z \text{ belongs to the complex line in } \mathbb{CP}^n \text{ determined by p}\}.$$

Let $\pi : \tau_n \to \mathbb{CP}^n$ denote the projection onto the second component. Note that for every $p \in \mathbb{CP}^n$ the fiber through p, $\pi^{-1}(p) = \tau_{n,p}$ coincides with the 1-dimensional subspace in $\mathbb{C}^{n+1}$ defined by p. □

Example 2.1.16. The tautological vector bundle over a grassmannian We consider here for brevity only complex grassmannian $G_{k,n}(\mathbb{C})$. The real case is completely similar. The total space of this bundle is

$$\tau_{k,n}^{\mathbb{C}} = \{(Z, p) \in \mathbb{C}^n \times G_{k,n}(\mathbb{C}) \ ; \ Z \text{ belongs to the subspace defined by } p\}.$$

If π denotes the natural projection $\pi : \tau_{k,n} \to G_{k,n}$ then for each $p \in G_{k,n}$ the fiber at p coincides with the subspace in $\mathbb{C}^n$ defined by p. Note that $\tau_n = \tau_{1,n}$. □

Exercise 2.1.7. Prove that τ_n and $\tau_{k,n}$ are indeed smooth vector bundles. Describe a collection of transition maps for $\tau_n^{\mathbb{C}}$. □

The family of vector bundles is very large. The following construction provides a very powerful method of producing vector bundles.

Definition 2.1.17. *Let $f : X \to M$ be a smooth map and E a vector bundle over M defined by an open cover (U_α) and transition maps $(g_{\alpha\beta})$. The* pullback of E by f *is the vector bundle f^*E over X defined by the open cover $f^{-1}(U_\alpha)$ and the transition maps $g_{\alpha\beta} \circ f$.*

One can check easily that the definition of the pullback is independent of the choice of open cover and transition maps. The pullback operation defines a linear map between the space of sections of E and the space of sections of f^*E. More precisely if $s \in \Gamma(E)$ is defined by the open cover (U_α) and the collection of smooth maps (s_α) then the pullback f^*s is defined by the open cover $f^{-1}(U_\alpha)$ and the smooth maps $s_\alpha \circ f$. Again there is no difficulty to check the above definition is independent of the various choices.

Exercise 2.1.8. Let $E \to X$ be a rank k (complex) smooth vector bundle over the manifold X. Assume E is *ample*, i.e. there exists a finite family $s_1, \ldots, s_N$ of smooth sections of E such that for any $x \in X$ the collection $\{s_1(x), \ldots, s_N(x)\}$ spans E_x. For each $x \in X$ set

$$S_x = \{v \in \mathbb{C}^N \ ; \ \sum_i v^i s_i(x) = 0\}$$

Note that $\dim S_x = N - k$. We have a map $F : X \to G_{k,N}(\mathbb{C})$ defined by $x \mapsto S_x^\perp$.
(a) Prove that F is smooth.
(b) Prove that E is isomorphic with the pullback $F^*\tau_{k,n}$. □

Exercise 2.1.9. Show that any vector bundle over a smooth *compact* manifold is ample. Thus any vector bundle over a compact manifold is a pullback of some tautological bundle! □

The notion of vector bundle is trickier than it may look. Its definition may suggest that a vector bundle looks like a direct product -$manifold \times$ vector space- since this happens at least locally.

Definition 2.1.18. *A rank r vector bundle $E \xrightarrow{\pi} M$ (over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$) is called* trivial *if there exists a bundle isomorphism $E \cong \mathbb{K}^r \times M$. Such an isomorphism is called a* trivialization *of E. A pair (trivial vector bundle, trivialization) is called a* trivialized *bundle.*

A trivial rank n $\mathbb{K}$-bundle over M will be denoted by $\underline{\mathbb{K}}^n_M$. One can naively ask the following question. Is every vector bundle trivial? We can even limit our search to tangent bundles. Thus we ask the following

QUESTION. *Is it true that for every smooth manifold M the tangent bundle TM is trivial (as a vector bundle)?*

Let us look at some positive examples.

Example 2.1.19. $TS^1 \cong \underline{\mathbb{R}}_{S^1}$ Let θ denote the angular coordinate on the circle. Then $\frac{\partial}{\partial\theta}$ is a globally defined, nowhere vanishing vector field on S^1. We thus get a map

$$\underline{\mathbb{R}}_{S^1} \to TS^1, \ \ (s, \theta) \mapsto (s\frac{\partial}{\partial\theta}, \theta) \in T_\theta S^1$$

which is easily seen to be a bundle isomorphism.

Let us carefully analyze this example. Think of S^1 as a Lie group (group of complex numbers of norm 1). The tangent space at $z = 1$ (i.e. $\theta, = 0$) coincides with the subspace $\mathfrak{Re} z = 0$ and $\frac{\partial}{\partial\theta} \mid_1$ is the unit vertical vector j. Denote by R_θ the counterclockwise rotation by an angle θ. Clearly R_θ is a diffeomorphism and for thus for each θ we have a linear isomorphism

$$D_\theta \mid_{\theta=0} R_\theta : T_1 S^1 \to T_\theta S^1,$$

and moreover, $\frac{\partial}{\partial\theta} = D_\theta \mid_{\theta=0} R_\theta \mathbf{j}$. The existence of the trivializing vector field $\frac{\partial}{\partial\theta}$ is due to to our ability to "move freely and coherently" inside S^1. One has a similar freedom inside a Lie group as we are going to see in the next example. □

Example 2.1.20. For any Lie group G the tangent bundle TG is trivial.

To see this let $n = \dim G$ and consider $e_1, \cdots, e_n$ a basis of the tangent space at the origin, $T_{\mathbb{1}}G$. We denote by R_g the right translation (by g) in the group defined by

$$R_g : \ x \mapsto x \cdot g, \ \ \forall x \in G.$$

R_g is a diffeomorphism with inverse $R_{g^{-1}}$ so that the differential DR_g defines a linear isomorphism $DR_g : T_{\mathbb{1}}G \to T_gG$. Set

$$E_i(g) = DR_g(e_i) \in T_gG, \ \ i = 1, \cdots, n.$$

Since the multiplication $G \times G \to G$, $(g,h) \mapsto g \cdot h$ is a smooth map we deduce that the vectors $E_i(g)$ define *smooth* vector fields over G. Moreover, for every $g \in G$, the collection $\{E_1(g), \dots, E_n(g)\}$ is a basis of T_gG so we can define without ambiguity a map

$$\Phi : \underline{\mathbb{R}}^n_G \to TG, \ \ (g; X^1, \dots X^n) \mapsto (g; \sum X^i E_i(g)).$$

One checks immediately that Φ is a vector bundle isomorphism and this proves the claim. In particular TS^3 is trivial since the sphere S^3 is a Lie group (unit quaternions). (Using the Cayley numbers one can show that TS^7 is also trivial; see [64] for details). □

We see that the tangent bundle TM of a manifold M is trivial if and only if there exist vector fields $X_1, \cdots, X_m$ ($m = \dim M$) such that for each $p \in M$, $X_1(p), \dots, X_m(p)$ span T_pM. This suggests the following more refined question.

Problem *Given a manifold M compute $v(M)$ the maximum number of pointwise linearly independent vector fields over M. Obviously $0 \leq v(M) \leq \dim M$ and TM is trivial if and only if $v(M) = \dim M$. A special instance of this problem is the celebrated vector field problem: compute $v(S^n)$ for any $n \geq 1$*

We have seen that $v(S^n) = n$ for n =1,3 and 7. Amazingly, these are the only cases when the above equality holds. This is a highly nontrivial result, first proved by J.F.Adams in [2] using very sophisticated algebraic tools. This fact is related to many other natural questions in algebra. For a nice presentation we refer to [53].

The methods we will develop in this book will not suffice to compute $v(S^n)$ for any n but we will be able to solve "half" of this problem. More precisely we will show that

$$v(S^{2n}) = 0$$

for any $n \geq 1$ so that in particular, TS^{2n} is not trivial. In odd dimensions the situation is far more elaborate (a complete answer can be found in [2]). However one can prove easily the following.

Exercise 2.1.10. $v(S^{2k-1}) \geq 1$ for any $k \geq 1$. □

The quantity $v(M)$ can be viewed as a measure of nontriviality of a tangent bundle. Unfortunately, its computation is highly nontrivial. In the second part of this book we will describe computable ways of measuring the extent of nontriviality of a vector bundle.

2.2 A linear algebra interlude

We collect in this section some classical notions of linear algebra. Most of them should be familiar to the reader but we will present them in a form suitable for applications in differential geometry. All vector spaces in this section will be assumed finite dimensional.

2.2.1 Tensor products

Let E, F be two vector spaces over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Consider the (infinite) direct sum

$$\mathcal{T}(E,F) = \bigoplus\nolimits_{(e,f) \in E \times F} \mathbb{K}.$$

Equivalently, $\mathcal{T}(E,F)$ can be identified with the space of functions $c : E \times F \to \mathbb{K}$ with finite support. $\mathcal{T}(E,F)$ has a natural basis consisting of "Dirac functions"

$$\delta_{e,f} : E \times F \to \mathbb{K}, \ \ (x,y) \mapsto \begin{cases} 1 & \text{if} \quad (x,y) = (e,f) \\ 0 & \text{if} \quad (x,y) \neq (e,f) \end{cases}$$

In particular we have an injection

$$\delta : E \times F \to \mathcal{T}(E,F), \ \ (e,f) \mapsto \delta_{e,f}.$$

(*A word of caution*: δ is not linear!) Inside $\mathcal{T}(E,F)$ sits the linear subspace $\mathcal{R}(E,F)$ spanned by

$$\lambda\delta_{e,f} - \delta_{\lambda e,f}, \ \lambda\delta_{e,f} - \delta_{e,\lambda f}, \ \delta_{e+e',f} - \delta_{e,f} - \delta_{e',f}, \ \delta_{e,f+f'} - \delta_{e,f} - \delta_{e,f'}$$

where $e, e' \in E$, $f, f' \in F$ and $\lambda \in \mathbb{K}$. Now define

$$E \otimes_{\mathbb{K}} F \stackrel{def}{=} \mathcal{T}(E,F)/\mathcal{R}(E,F).$$

and denote by π the canonical projection $\pi : \mathcal{T}(E,F) \to E \otimes F$. Set

$$e \otimes f := \pi(\delta_{e,f}).$$

We get a natural map

$$\iota : E \times F \to E \otimes F, \ \ e \times f \mapsto e \otimes f.$$

Obviously ι is bilinear. The vector space $E \otimes F$ is called the *tensor product* of E and F. Sometimes when we want to emphasize the field of scalars we write $E \otimes_{\mathbb{K}} F$. The tensor product has the following universality property.

Proposition 2.2.1. *For any bilinear map $\phi : E \times F \to G$ there exists a* unique *linear map $\Phi : E \otimes F \to G$ such that the diagram below is commutative.*

$$\begin{tikzcd} E \times F \arrow[r, "\iota"] \arrow[dr, "\phi"'] & E \otimes F \arrow[d, dashed, "\Phi"] \\ & G \end{tikzcd} .$$

The proof of this result is left to the reader as an exercise. Note that if (e_i) is a basis of $E$and (f_j) is a basis of F then $(e_i \otimes f_j)$ is a basis of $E \otimes F$ and hence

$$\dim E \otimes F = (\dim E) \cdot (\dim F).$$

Exercise 2.2.1. Using the universality property of the tensor product prove that there exists a natural isomorphism $E \otimes F \cong F \otimes E$ uniquely defined by $e \otimes f \mapsto f \otimes e$. □

The above construction can be iterated. Given three vector spaces E_1, E_2, E_3 we can construct two triple tensor products: $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$.

Exercise 2.2.2. Prove there exists a natural isomorphism $(E_1 \otimes E_2) \otimes E_3 \cong E_1 \otimes (E_2 \otimes E_3)$. □

Thus there exists a unique (up to an isomorphism) triple tensor product which we denote by $E_1 \otimes E_2 \otimes E_3$. Clearly we can now define multiple tensor products: $E_1 \otimes \cdots \otimes E_n$.

Definition 2.2.2. *The dual of a $\mathbb{K}$-linear space E is the linear space E^* defined as the space of $\mathbb{K}$-linear maps $E \to \mathbb{K}$.*

For any $e^* \in E^*$ and $e \in E$ we set

$$\langle e^*, e\rangle := e^*(e).$$

Using the universality property of the tensor product one proves easily the following result.

Proposition 2.2.3. *(a) There exists a natural isomorphism*

$$E^* \otimes F^* \cong (E \otimes F)^*$$

uniquely defined by $e^* \otimes f^* \mapsto L_{e^*\otimes f^*} \in (E \otimes F)^*$, where $\langle L_{e^*\otimes f^*}, x \otimes y\rangle = \langle e^*, x\rangle\langle f^*, y\rangle$ for all $x \in e$ and $y \in F$. In particular this shows $E^* \otimes F^*$ can be naturally identified with the space of bilinear maps $E \times F \to \mathbb{K}$.
(b) There exists a natural isomorphism

$$E^* \otimes F \cong \operatorname{Hom}(E, F)$$

uniquely determined by $e^* \otimes f \mapsto T_{e^*\otimes f} \in \operatorname{Hom}(E, F)$, where $T_{e^*\otimes f}(x) = \langle e^*, x\rangle f$ for all $x \in E$.

Exercise 2.2.3. Prove the above proposition. $\square$

The above constructions are functorial. More precisely, we have the following result.

Proposition 2.2.4. *(a) Let $T_i \in \operatorname{Hom}(E_i, F_i)$, i=1,2 be two linear operators. Then they naturally induce a linear operator $T = T_1 \otimes T_2 : E_1 \otimes E_2 \to F_1 \otimes F_2$ uniquely defined by*

$$T_1 \otimes T_2(e_1 \otimes e_2) = (T_1e_1) \otimes (T_2e_2), \ \ \forall e_i \in E_i.$$

(b) Any linear operator $S : E \to F$ induces a linear operator $S^t : F^* \to E^*$ uniquely defined by

$$\langle S^t f^*, e\rangle = \langle f^*, Se\rangle.$$

S^t is called the transpose *of S.*

Let V be a vector space. For $r, s \geq 0$ set

$$\mathcal{T}^r_s(V) := V^{\otimes r} \otimes (V^*)^{\otimes s},$$

where by definition $V^{\otimes 0} = (V^*)^{\otimes 0} = \mathbb{K}$. An element of $\mathcal{T}^r_s$ is called *tensor of type (r,s)*. Note that according to Proposition 2.2.3 a tensor of type (1,1) is a linear endomorphism of V. A tensor of type $(r, 0)$ is called *contravariant*, while a tensor of type $(0, s)$ is called *covariant*. The *tensor algebra* of V is defined to be

$$\mathcal{T}(V) := \bigoplus_{r,s} \mathcal{T}^r_s(V).$$

We use the term algebra since the tensor product induces bilinear maps

$$\otimes : \mathcal{T}^r_s \times \mathcal{T}^{r'}_{s'} \to \mathcal{T}^{r+r'}_{s+s'}.$$

The elements of $\mathcal{T}(V)$ are called *tensors*.

Exercise 2.2.4. Show that $(\mathcal{T}(V), +, \otimes)$ is an associative algebra. □

It is often useful to represent tensors using coordinates. To achieve this pick a basis (e_i) of V and let (e^i) denote the dual basis in V^* uniquely defined by

$$\langle e^i, e_j \rangle = \delta^i_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

We then obtain a basis of $\mathcal{T}^r_s(V)$

$$\{ e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} /\, 1 \leq i_\alpha,\, j_\beta \leq \dim V \}.$$

Any element $T \in \mathcal{T}^r_s(V)$ has a decomposition

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s},$$

where we use Einstein convention to sum over indices which appear twice, once as an upper index and the second time as a lower index.

On the tensor algebra there is a natural*contraction* (trace) operation

$$\mathrm{tr} \,: \mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1}$$

uniquely defined by

$$\mathrm{tr}\,(v_1 \otimes \cdots \otimes v_r \otimes u^1 \otimes \cdots \otimes u^s) = \langle u^1, v_1 \rangle v_2 \otimes \cdots v_r \otimes u^2 \otimes \cdots \otimes u^s, \;\; \forall v_i \in V, \; u^j \in V^*.$$

In the coordinates determined by a basis (e_i) of V the contraction can be described as

$$(\mathrm{tr}\, T)^{i_2 \dots i_r}_{j_2 \dots j_s} = \left(T^{i i_2 \dots i_r}_{i j_2 \dots j_s} \right),$$

where again we use Einstein's convention. In particular, we see that the contraction coincides with the usual trace on $\mathcal{T}^1_1(V) = \mathrm{End}\,(V)$.

2.2.2 Symmetric and skew-symmetric tensors

Let V be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and set $\mathcal{T}^r(V) = \mathcal{T}^r_0(V)$. The permutation group $\mathfrak{S}_r$ acts naturally on $T^r(V)$ by

$$\sigma(v_1 \otimes \,...\, \otimes v_r) = v_{\sigma(1)} \otimes \,...\, \otimes v_{\sigma(r)}, \;\; \sigma \in \mathfrak{S}_r.$$

We denote this action of $\sigma \in \mathfrak{S}_r$ on an arbitrary element $t \in \mathcal{T}^r(V)$ by σt.

In this subsection we will describe two subspaces invariant under this action. These are special instances of the so called Schur functors. (We refer to [26] for more general constructions.) Define

$$\mathcal{S}_r : \mathcal{T}^r(V) \to \mathcal{T}^r(V), \ \ \mathcal{S}_r(t) = \frac{1}{r!}\sum_{\sigma \in \mathfrak{S}_r} \sigma t$$

and

$$\mathcal{A}_r : \mathcal{T}^r(V) \to \mathcal{T}^r(V), \ \ \mathcal{A}_r(t) = \begin{cases} \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r} \epsilon(\sigma)\sigma t & \text{if} \quad r \leq \dim V \\ 0 & \text{if} \quad r > \dim V \end{cases}.$$

When $r = 0$ we set $\mathfrak{S}_0 = \{1\}$ so that $\mathcal{A}_0 = \mathcal{S}_0 = \mathbf{1}_{\mathbb{K}}$. Above $\epsilon(\sigma)$ denotes the signature of the permutation σ. The following results are immediate. Their proofs are left to the reader as exercises.

Lemma 2.2.5. *$\mathcal{A}_r$ and $\mathcal{S}_r$ are projectors of $T^r(V)$ i.e.*

$$\mathcal{S}_r^2 = \mathcal{S}_r, \ \ \mathcal{A}_r^2 = \mathcal{A}_r.$$

Moreover

$$\sigma\mathcal{S}_r(t) = \mathcal{S}_r(\sigma t) = \mathcal{S}_r(t), \ \sigma\mathcal{A}_r(t) = \mathcal{A}_r(\sigma t) = \epsilon(\sigma)\mathcal{A}_r(t).$$

Definition 2.2.6. *A tensor $T \in \mathcal{T}^r(V)$ is called* symmetric *(resp.* skew-symmetric*) if $\mathcal{S}_r(T) = T$ (resp. $\mathcal{A}_r(T) = T$). r is called the* degree *of the (skew-)symmetric tensor. The space of symmetric tensors (resp. skew-symmetric ones) will be denoted by $\mathcal{S}^r V$ (resp. $\Lambda^r V$).*

Set $\mathcal{S}^* V = \bigoplus_{r\geq 0} \mathcal{S}^r V$ and $\Lambda^* V = \bigoplus_{r\geq 0} \Lambda^r V$.

Definition 2.2.7. *The* exterior product *is the bilinear map*

$$\wedge : \Lambda^r V \times \Lambda^s V \to \Lambda^{r+s} V$$

defined by

$$\omega^r \wedge \eta^s \stackrel{def}{=} \frac{(r+s)!}{r!s!}\mathcal{A}_{r+s}(\omega \otimes \eta), \ \ \forall \omega^r \in \Lambda^r V, \ \eta^s \in \Lambda^s V.$$

Proposition 2.2.8. *The exterior product has the following properties.*
(a) Associativity: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$, $\forall \alpha, \beta \gamma \in \Lambda^* V$. In particular

$$v_1 \wedge \ldots v_k = k!\mathcal{A}_k(v_1 \otimes \ldots \otimes v_k) = \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}, \ \ \forall v_i \in V.$$

(b) Super-commutativity:

$$\omega^r \wedge \eta^s = (-1)^{rs}\eta^s \wedge \omega^r.$$

Proof We first define a new product "$\wedge_1$" by

$$\omega^r \wedge_1 \eta^s := \mathcal{A}_{r+s}(\omega \otimes \eta),$$

which will turn out to be associative and will force $\wedge$ to be associative as well.

To prove the associativity of $\wedge_1$ consider the quotient algebra $\mathcal{Q}^* = \mathcal{T}^*/\mathcal{J}^*$, where $\mathcal{T}^*$ is the algebra $(\bigoplus_{r\geq 0} \mathcal{T}^r(V), +, \otimes)$ and $\mathcal{J}^*$ is the bilateral ideal generated by the set of squares $\{v \otimes v / \ v \in V\}$. Denote the (obviously associative) multiplication in $\mathcal{Q}^*$ by $\cup$. The natural projection $\pi : \mathcal{T}^* \to \mathcal{Q}^*$ induces a linear map $\pi : \Lambda^* V \to \mathcal{Q}^*$.

Step 1. $\pi : \Lambda^* V \to \mathcal{Q}^*$ is a linear isomorphism and moreover,

$$\pi(\alpha \wedge_1 \beta) = \pi(\alpha) \cup \pi(\beta). \tag{2.2.1}$$

In particular $\wedge_1$ is an associative product.

The crucial observation is

$$\pi(T) = \pi(\mathcal{A}_r(T)), \ \ \forall t \in \mathcal{T}^r(V). \tag{2.2.2}$$

It suffices to check (2.2.2) on monomials $T = e_1 \otimes \ ... \ \otimes e_r$, $e_i \in V$. Since

$$(u+v)^{\otimes 2} \in \mathcal{J}^*, \ \ \forall u, \, v \in V$$

we deduce $u \otimes v = -v \otimes u (\mathrm{mod}\, \mathcal{J}^*)$. Hence, for any $\sigma \in \mathfrak{S}_r$

$$\pi(e_1 \otimes \ ... \ \otimes e_r) = \epsilon(\sigma)\pi(e_{\sigma(1)} \otimes \ ... \ \otimes e_{\sigma(r)}) \tag{2.2.3}$$

When we sum over $\sigma \in \mathfrak{S}_r$ in (2.2.3) we obtain (2.2.2).

To prove the injectivity of π note that $\mathcal{A}_*(\mathcal{J}^*) = 0$. If $\pi(\omega) = 0$ for some $\omega \in \Lambda^* V$ then $\omega \in \ker \pi = \mathcal{J}^* \cap \Lambda^* V$ so that

$$\omega = \mathcal{A}_*(\omega) = 0.$$

The surjectivity of π follows immediately from (2.2.2). Indeed any $\pi(T)$ can be alternatively described as $\pi(\omega)$ for some $\omega \in \Lambda^* V$. It suffices to take $\omega = \mathcal{A}_*(T)$.

To prove (2.2.1) it suffices to consider only the special cases when α and β are monomials:

$$\alpha = \mathcal{A}_r(e_1 \otimes \ ... \ \otimes e_r), \ \ \beta = \mathcal{A}_s(f_1 \otimes \ ... \ \otimes f_s).$$

We have

$$\begin{aligned}\pi(\alpha \wedge_1 \beta) &= \pi\left(\mathcal{A}_{r+s}(\mathcal{A}_r(e_1 \otimes \ ... \ \otimes e_r) \otimes \mathcal{A}_s(f_1 \otimes \ ... \ \otimes f_s))\right) \\ &\overset{(2.2.2)}{=} \pi\left(\mathcal{A}_r(e_1 \otimes \ ... \ \otimes e_r) \otimes \mathcal{A}_s(f_1 \otimes \ ... \ \otimes f_s)\right) \\ &\overset{def}{=} \pi(\mathcal{A}_r(e_1 \otimes \ ... \ \otimes e_r)) \cup \pi(\mathcal{A}_s(f_1 \otimes \ ... \ \otimes f_s)) = \pi(\alpha) \cup \pi(\beta).\end{aligned}$$

Thus $\wedge_1$ is associative.

Step 2. $\wedge$ is associative. Consider $\alpha \in \Lambda^r V$, $\beta \in \Lambda^s V$ and $\gamma \in \Lambda^t V$. We have

$$\begin{aligned}(\alpha \wedge \beta) \wedge \gamma &= \left(\frac{(r+s)!}{r!s!}\alpha \wedge_1 \beta\right) \wedge \gamma = \frac{(r+s)!}{r!s!}\frac{(r+s+t)!}{(r+s)!t!}(\alpha \wedge_1 \beta) \wedge_1 \gamma \\ &= \frac{(r+s+t)!}{r!s!t!}(\alpha \wedge_1 \beta) \wedge_1 \gamma = \frac{(r+s+t)!}{r!s!t!}\alpha \wedge_1 (\beta \wedge_1 \gamma) = \alpha \wedge (\beta \wedge \gamma).\end{aligned}$$

The associativity of $\wedge$ is proved. The computation above shows that

$$e_1 \wedge \ ... \ \wedge e_k = k! \mathcal{A}_k(e_1 \otimes \ ... \ \otimes e_k).$$

(b) The supercommutativity of $\wedge$ follows from the supercommutativity of $\wedge_1$ (or $\cup$). To prove the latter one uses (2.2.2). The details are left to the reader. $\square$

Exercise 2.2.5. Finish the proof of part (b) in the above proposition. □

$\Lambda^* V$ is called the *exterior algebra* of V. $\wedge$ is called the *exterior product* . The exterior algebra is a $\mathbb{Z}$-graded algebra i.e

$$\Lambda^* V = \bigoplus_{r \geq 0} \Lambda^r V$$

and $(\Lambda^r V) \wedge (\Lambda^s V) \subset \Lambda^{r+s} V$. In fact $\Lambda^r V = 0$ for $r > \dim V$ (pigeonhole principle).

Definition 2.2.9. *Let V be an n-dimensional $\mathbb{K}$-vector space. The one dimensional vector space $\Lambda^n V$ is called the* determinant line *of V and is denoted by* $\det V$.

There is a natural injection $\iota_V : V \hookrightarrow \Lambda^* V$ such that $(\iota_V(x))^2 = 0$ in $\Lambda^* V$ for all $x \in V$. This map enters crucially the formulation of the universality property

Proposition 2.2.10. *Let V be a vector space over $\mathbb{K}$. For any $\mathbb{K}$-algebra A and any linear map $\phi : V \to A$ such that $(\phi(x)^2 = 0$ there exists an unique morphism of $\mathbb{K}$-algebras $\Phi : \Lambda^* V \to A$ such that the diagram below is commutative*

$$\begin{tikzcd} V \arrow[r, hook, "\iota_V"] \arrow[dr, "\phi"'] & \Lambda^* V \arrow[d, dashed, "\Phi"] \\ & A \end{tikzcd} ,$$

i.e. $\Phi \circ \iota_V = \phi$.

Exercise 2.2.6. Prove Proposition 2.2.10. □

The space of symmetric tensors $\mathcal{S}^* V$ can be similarly given a structure of associative algebra with respect to the product

$$\alpha \cdot \beta := \mathcal{S}_{r+s}(\alpha \otimes \beta), \ \ \forall \alpha \in \mathcal{S}^r V, \, \beta \in \mathcal{S}^s V.$$

The symmetric product "$\cdot$" is also commutative.

Exercise 2.2.7. Formulate and prove the analogue of Proposition 2.2.10 for the algebra $\mathcal{S}^* V$. □

It is often convenient to represent (skew-)symmetric tensors in coordinates. If $e_1, \ldots, e_n$ is a basis of the vector space V then for any $1 \leq r \leq n$ the family

$$\{e_{i_1} \wedge \ldots \wedge e_{i_r} / \ 1 \leq i_1 < \ldots < i_r \leq n\}$$

is a basis for $\Lambda^r V$ so that any degree r skew-symmetric tensor ω can be uniquely represented as

$$\omega = \sum_{1 \leq i_1 < \ldots < i_r \leq n} \omega^{i_1 \ldots i_r} e_{i_1} \wedge \ldots \wedge e_{i_r}.$$

Symmetric tensors can be represented in a similar way.

The Λ^*- and $\mathcal{S}^*-$ constructions are functorial, i.e. any linear map $L : V \to W$ naturally induces morphisms of algebras

$$\Lambda^* L : \Lambda^* V \to \Lambda^* W, \;\; \mathcal{S}^* L : \mathcal{S}^* V \to \mathcal{S}^* W$$

uniquely defined by their actions on monomials:

$$\Lambda^* L(v_1 \wedge \dots \wedge v_r) = (Lv_1) \wedge \dots \wedge (Lv_r)$$

and

$$\mathcal{S}^* L(v_1 \cdot \dots \cdot v_r) = (Lv_1) \cdot \dots \cdot (Lv_r).$$

The functors Λ^* and $\mathcal{S}^*$ have an *exponential like* behavior, i.e. there exists a *natural* isomorphism

$$\Lambda^*(V \oplus W) \cong \Lambda^* V \otimes \Lambda^* W. \tag{2.2.4}$$

$$\mathcal{S}^*(V \oplus W) \cong \mathcal{S}^* V \otimes \mathcal{S}^* W. \tag{2.2.5}$$

To define the isomorphism in (2.2.4) consider the bilinear map $\phi : \Lambda^* V \times \Lambda^* W \to \Lambda^*(V \oplus W)$ uniquely determined by

$$\phi(v_1 \wedge \dots \wedge v_r \,,\, w_1 \wedge \dots \wedge w_s) = v_1 \wedge \dots \wedge v_r \wedge w_1 \wedge \dots \wedge w_s.$$

The universality property of the tensor product implies the existence of a linear map $\Phi : \Lambda^* V \otimes \Lambda^* W \to \Lambda^*(V \oplus W)$ such that $\Phi \circ \iota = \phi$ where ι is the inclusion of $\Lambda^* V \times \Lambda^* W$ in $\Lambda^* V \otimes \Lambda^* W$. To construct the inverse of Φ note that $\Lambda^* V \otimes \Lambda^* W$ is naturally a $\mathbb{K}$-algebra by

$$(\omega \otimes \eta) * (\omega' \otimes \eta') = (-1)^{\deg \eta \cdot \deg \omega'} (\omega \wedge \omega') \otimes (\eta \wedge \eta').$$

$V \oplus W$ is naturally embedded in $\Lambda^* V \otimes \Lambda^* W$ by

$$\psi(v, w) = v \otimes 1 + 1 \otimes w \in \Lambda^* V \otimes \Lambda^* W.$$

Moreover, for any $x \in V \oplus W$ we have $\psi(x) * \psi(x) = 0$. The universality property of the exterior algebra implies the existence of a unique morphism of $\mathbb{K}$-algebras $\Psi : \Lambda^*(V \oplus W) \to \Lambda^* V \otimes \Lambda^* W$ such that $\Psi \circ \iota_{V \oplus W} = \psi$. Note that Φ is also a morphism of $\mathbb{K}$-algebras and one verifies easily that $(\Phi \circ \Psi) \circ \iota_{V \oplus W} = \iota_{V \oplus W}$. The uniqueness part in the universality property of the exterior algebra implies $\Phi \circ \Psi = identity$. One proves similarly that $\Psi \circ \Phi = identity$ and this concludes the proof of (2.2.4).

We want to mention a few general facts about $\mathbb{Z}$-graded vector spaces, i.e. vector spaces equipped with a direct sum decomposition

$$V = \oplus_{n \in \mathbb{Z}} V_n.$$

(We will always assume that each V_n is finite dimensional.) The vectors in V_n are said to be *homogeneous, of degree* n. For example, the ring of polynomials $\mathbb{K}[x]$ is a $\mathbb{K}$-graded vector space. $\Lambda^* V$ and $\mathcal{S}^* V$ are $\mathbb{Z}$-graded vector spaces. The direct sum of two $\mathbb{Z}$-graded vector spaces V and W is a $\mathbb{Z}$-graded vector space with $(V \oplus W)_n \stackrel{def}{=} V_n \oplus W_n$. The tensor product of two $\mathbb{Z}$-graded vector spaces V and W is a $\mathbb{Z}$-graded vector space with

$$(V \otimes W)_n := \bigoplus_{r+s=n} V_r \otimes W_s.$$

To any $\mathbb{Z}$-graded vector space V one can naturally associate a formal series $P_V(t) \in \mathbb{Z}[[t, t^{-1}]]$ by

$$P_V(t) = \sum_{n\in\mathbb{Z}} (\dim_{\mathbb{K}} V_n) t^n.$$

$P_V(t)$ is called the *Poincaré series* of V.

Example 2.2.11. The Poincaré series of $\mathbb{K}[x]$ is

$$P_{\mathbb{K}[x]}(t) = 1 + t + t^2 + \,...\, + t^{n-1} + \,...\, = \frac{1}{1-t}. \qquad \square$$

Exercise 2.2.8. Let V and W be two $\mathbb{Z}$-graded vector spaces. Prove the following statements are true (whenever they make sense).
(a) $P_{V\oplus W}(t) = P_V(t) + P_W(t)$.
(b) $P_{V\otimes W}(t) = P_V(t) \cdot P_W(t)$.
(c) $\dim V = P_V(1)$. $\square$

Definition 2.2.12. *Let V be a $\mathbb{Z}$-graded vector space. The* Euler characteristic *of V, denoted by $\chi(V)$, is defined by (whenever it makes sense)*

$$\chi(V) \stackrel{def}{=} P_V(-1) = \sum_{n\in\mathbb{Z}} (-1)^n \dim V_n.$$

Remark 2.2.13. If we try to compute $\chi(\mathbb{K}[x])$ using the first formula in Definition 2.2.12 we get $\chi(\mathbb{K}[x]) = 1/2$ while the second formula makes no sense (divergent series). $\square$

Proposition 2.2.14. *Let V be a $\mathbb{K}$-vector space of dimension n. Then*

$$P_{\Lambda^* V}(t) = (1+t)^n \ \text{ and } \ P_{\mathbb{S}^* V}(t) = \left(\frac{1}{1-t}\right)^n = \frac{1}{(n-1)!}\left(\frac{d}{dt}\right)^{n-1}\left(\frac{1}{1-t}\right).$$

In particular, $\dim \Lambda^* V = 2^n$ *and* $\chi(\Lambda^* V) = 0$.

Proof From (2.2.4) and (2.2.5) we deduce using Exercise 2.2.8 that for any vector spaces V and W we have

$$P_{\Lambda^*(V\oplus W)}(t) = P_{\Lambda^* V}(t) \cdot P_{\Lambda^* W}(t) \ \text{ and } \ P_{\mathbb{S}^*(V\oplus W)}(t) = P_{\mathbb{S}^* V} \cdot P_{\mathbb{S}^* W}(t).$$

In particular, if V has dimension n then $V \cong \mathbb{K}^n$ so that

$$P_{\Lambda^* V}(t) = (P_{\Lambda^* \mathbb{K}}(t))^n \ \text{ and } \ P_{\mathbb{S}^* V}(t) = (P_{\mathbb{S}^* \mathbb{K}}(t))^n.$$

The proposition follows using the relations

$$P_{\Lambda^* \mathbb{K}}(t) = 1 + t; \text{ and } P_{\mathbb{S}^* \mathbb{K}}(t) = P_{\mathbb{K}[x]}(t) = \frac{1}{1-t}. \qquad \square$$

2.2.3 The "super" slang

The aim of this very brief section is to introduce the reader to the "super" terminology. We owe the "super" slang to the physicists. In the quantum world many objects have a special feature not present in the Newtonian world. They have parity (or chirality) and objects with different chiralities had to be treated differently. The "super" terminology provides an algebraic background which allows one to deal with different parities on an equal basis. From a strictly syntactic point of view the "super" slang adds the attribute super to most of the commonly used algebraic objects. In this book the prefix "s-" will abbreviate the word "super"

Definition 2.2.15. *(a) A s-space is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$.*
(b) A s-algebra over $\mathbb{K}$ is a $\mathbb{Z}_2$-graded $\mathbb{K}$-algebra, i.e. a $\mathbb{K}$-algebra $\mathcal{A}$ together with a direct sum decomposition $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ such that $\mathcal{A}^i \cdot \mathcal{A}^j \subset \mathcal{A}^{i+j\,(mod\,2)}$. The elements in $\mathcal{A}^i$ are called homogeneous of degree i. For any $a \in \mathcal{A}^i$ we denote its degree (mod 2) by $|a|$. The elements in $\mathcal{A}^0$ are said to be even *while the elements in $\mathcal{A}^1$ are said to be* odd.
(c) The supercommutator *in a s-algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ is the bilinear map*

$$[\,\cdot\,,\,\cdot\,]_s : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

defined on homogeneous elements $\omega^i \in \mathcal{A}^i$, $\eta^j \in \mathcal{A}^j$ by

$$[\omega^i, \eta^j]_s := \omega^i \eta^j - (-1)^{ij} \eta^j \omega^j.$$

A s-algebra is called s-commutative *if $[\,\cdot\,,\,\cdot\,]_s \equiv 0$. .*

Example 2.2.16. Let $E = E^0 \oplus E^1$ be a s-space. Any $T \in \mathrm{End}\,(E)$ has a block decomposition

$$T = \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix}$$

where $T_{ji} \in \mathrm{End}\,(E^i, E^j)$. $\mathrm{End}\,(E)$ is naturally a s-algebra. The even endomorphism have the form

$$\begin{bmatrix} T_{00} & 0 \\ 0 & T_{11} \end{bmatrix}$$

while the odd endomorphisms have the form

$$\begin{bmatrix} 0 & T_{01} \\ T_{10} & 0 \end{bmatrix}.$$

□

Example 2.2.17. Let V be a finite dimensional space. The exterior algebra $\Lambda^* V$ is naturally a s-algebra. The even elements are gathered in

$$\Lambda^{even} V = \bigoplus_{r \text{ even}} \Lambda^r V$$

while the odd elements are gathered in

$$\Lambda^{odd} V = \bigoplus_{r \text{ odd}} \Lambda^r V.$$

$\Lambda^* V$ is supercommutative. □

Definition 2.2.18. *Let $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ be a s-algebra. A* s-derivation *on $\mathcal{A}$ is a linear operator on $D \in \operatorname{End}(\mathcal{A})$ such that for any $x \in \mathcal{A}$*

$$[D, L_x]_s^{\operatorname{End}(\mathcal{A})} = L_{Dx} \tag{2.2.6}$$

where $[\ ,\ ]_s^{\operatorname{End}(\mathcal{A})}$ denotes the supercommutator in $\operatorname{End}(\mathcal{A})$ (with the s-structure defined in Example 2.2.16) while for any $z \in \mathcal{A}$ we denoted by L_z the left multiplication operator $a \mapsto z \cdot a$. A s-derivation is called even (resp. odd) if it is even (resp. odd) as an element of the s-algebra $\operatorname{End}(\mathcal{A})$.

Remark 2.2.19. The relation (2.2.6) is a super version of the usual Leibniz formula. Indeed, assuming D is homogeneous (as an element of the s-algebra $\operatorname{End}(\mathcal{A})$) then equality (2.2.6) becomes

$$D(xy) = (Dx)y + (-1)^{|x||D|}x(Dy)$$

for any homogeneous elements $x, y \in \mathcal{A}$. □

Example 2.2.20. Let V be a vector space. Any $u^* \in V^*$ defines an odd s-derivation of $\Lambda^* V$ denoted by i_{u^*} uniquely determined by its action on monomials.

$$i_{u^*}(v_0 \wedge v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^{r} (-1)^i \langle u^*, v_i \rangle v_0 \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \cdots \wedge v_r.$$

As usual, a hat indicates a missing entry. □

Exercise 2.2.9. Prove the statement in the above example. □

Definition 2.2.21. *Let $\mathcal{A} = (\mathcal{A}^0 \oplus \mathcal{A}^1; +; [\ ,\ ])$ be a s-algebra over $\mathbb{K}$, not necessarily associative. For any $x \in \mathcal{A}$ denote by R_x the right multiplication operator $a \mapsto [a, x]$. $\mathcal{A}$ is called* s-Lie algebra *if it is s-anticommutative, i.e.*

$$[x, y] + (-1)^{|x||y|}[y, x] = 0, \text{ for all homogeneous elements } x, y \in \mathcal{A}$$

and $\forall x \in \mathcal{A}$, R_x is a s-derivation. When $\mathcal{A}$ is purely even (i.e. $\mathcal{A}^1 = \{0\}$) then $\mathcal{A}$ is called simply a Lie algebra. The multiplication in a (s-) Lie algebra is called the (s-)bracket.

The above definition is highly condensed. In more down-to-Earth terms the fact that R_x is a s-derivation for all $x \in \mathcal{A}$ is equivalent with the *super Jacobi identity*

$$[[y, z], x] = [[y, x], z] + (-1)^{|x||y|}[y, [z, x]] \tag{2.2.7}$$

for all homogeneous elements $x, y, z \in \mathcal{A}$. When $\mathcal{A}$ is a purely even $\mathbb{K}$-algebra then $\mathcal{A}$ is a *Lie algebra* over $\mathbb{K}$ if $[\ ,\ ]$ is anticommutative and satisfies (2.2.7), which in this case is equivalent with the classical *Jacobi identity*

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \ \ \forall x, y, z \in \mathcal{A}. \tag{2.2.8}$$

Example 2.2.22. Let E be a vector space (purely even). Then $\mathcal{A} = \operatorname{End}(E)$ is a Lie algebra with bracket given by the usual commutator: $[a, b] = ab - ba$. □

Proposition 2.2.23. *Let $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ be a s-algebra and denote by $Der_s(\mathcal{A})$ the space of s-derivations of $\mathcal{A}$.*
(a) For any $D \in Der_s(\mathcal{A})$ its homogeneous components $D^0,\ D^1 \in \mathrm{End}\,(\mathcal{A})$ are also s-derivations.
(b) $\forall D,\ D' \in Der_s(\mathcal{A})$ the s-commutator $[D, D']_s^{\mathrm{End}(A)}$ is again a s-derivation.
(c) $\forall x \in \mathcal{A}$ the bracket $B^x : a \mapsto [a, x]_s$ is a s-derivation called the bracket derivative *determined by x. Moreover*

$$[B^x, B^y]_s^{\mathrm{End}\,(\mathcal{A})} = B^{[x,y]_s}, \ \ \forall x, y \in \mathcal{A}.$$

Exercise 2.2.10. Prove Proposition 2.2.23. □

Definition 2.2.24. *Let $E = E^0 \oplus E^1$ and $F = F^0 \oplus F^1$ be two s-spaces. Their s-tensor product is the s-space $E \otimes F$ with the grading $(E \otimes F)^\epsilon = \bigoplus_{i+j\equiv\epsilon\,(2)} E^i \otimes F^j$. To emphasize the super-nature of the tensor product we will use the symbol "$\hat{\otimes}$" instead of the usual "$\otimes$".*

Exercise 2.2.11. Show that there exists a natural isomorphism of s-spaces

$$V^*\hat{\otimes}\Lambda^*V \cong Der_s(\Lambda^*V),$$

uniquely determined by $v^* \times \omega \mapsto D^{v^*\otimes\omega}$ where $D^{v^*\otimes\omega}$ is s-derivation defined by

$$D^{v^*\otimes\omega}(v) = \langle v^*, v\rangle\omega, \ \ \forall v \in V.$$

Notice in particular that any s-derivation of Λ^*V is uniquely determined by its action on Λ^1V. (When $\omega = 1$, $D^{v^*\otimes 1}$ coincides with the internal derivation discussed in Example 2.2.20.) □

Let $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ be a s-algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A *supertrace* on $\mathcal{A}$ is a $\mathbb{K}$-linear map

$$\tau : \mathcal{A} \to \mathbb{K}$$

such that

$$\tau([x, y]_s) = 0 \ \ \forall x, y \in \mathcal{A}.$$

If we denote by $[\mathcal{A}, \mathcal{A}]_s$ the linear subspace of $\mathcal{A}$ spanned by the supercommutators

$$\{[x, y]_s \ ;\ x, y \in \mathcal{A}\}$$

we see that the space of s-traces is isomorphic with the dual of the quotient space $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_s$.

Proposition 2.2.25. *Let $E = E_0 \oplus E_1$ be a finite dimensional s-space and denote by $\mathcal{A}$ the s-algebra of endomorphisms of E. Then there exists a canonical s-trace tr_s on $\mathcal{A}$ uniquely defined by*

$$\mathrm{tr}_s \mathbb{1}_E = \dim E_0 - \dim E_1.$$

In fact, if $T \in \mathcal{A}$ has the block decomposition

$$T = \left[\begin{array}{cc} T_{00} & T_{01} \\ T_{10} & T_{11} \end{array}\right]$$

then

$$\mathrm{tr}_s T = \mathrm{tr}\, T_{00} - \mathrm{tr}\, T_{11}.$$

Exercise 2.2.12. Prove the above proposition. □

2.2.4 Duality

Duality is a subtle and fundamental concept which can be detected in all branches of mathematics. This section is devoted to those aspects of the atmosphere called duality which are relevant to differential geometry. In the sequel we will use Einstein's convention without mentioning it. $\mathbb{K}$ will denote one of the fields $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.2.26. *A* pairing *between two finite dimensional* $\mathbb{K}$*-vector spaces* V *and* W *is a bilinear map* $B : V \times W \to \mathbb{K}$.

Any pairing $B : V \times W \to \mathbb{K}$ defines two linear maps

$$D_V : V \to W^*, \;\; v \mapsto B(v, \bullet) \in W^*$$

and

$$D_W : W \to V^*, \;\; v \mapsto B(\bullet, w) \in V^*.$$

A pairing is called a *duality* if the the maps D_V and D_W are isomorphisms.

Example 2.2.27. The natural pairing $\langle \bullet, \bullet \rangle : V^* \times V \to \mathbb{K}$ is a duality. One sees that $D_{V^*} = \mathbb{1}_{V^*}$ and $D_V = \mathbb{1}_V$. This is called the *natural duality.* □

Example 2.2.28. Let V be a finite dimensional real vector space. Any symmetric non-degenerate quadratic form $(\bullet, \bullet) : V \times V \to \mathbb{R}$ defines a (self)duality and in particular a natural isomorphism $\mathfrak{L} : V \to V^*$. When $(\bullet, \bullet)$ is positive definite the the operator $\mathfrak{L}$ is called *metric duality.* This operator can be nicely described in coordinates as follows. Pick (e_i) a basis of V and set $g_{ij} = (e_i, e_j)$. Let (e^j) denote the dual basis of V^* (defined by $\langle e^j, e_i \rangle = \delta_i^j$). The action of $\mathfrak{L}$ is then

$$\mathfrak{L} e_i = g_{ij} e^j.$$

□

Example 2.2.29. Consider V a real vector space and $\omega : V \times V \to \mathbb{R}$ a skew-symmetric bilinear form on V. In particular ω defines a pairing. ω is said to be *symplectic* if this pairing is a duality. In this case the induced operator $V \to V^*$ is called *symplectic duality.* □

Exercise 2.2.13. Let $\omega : V \times V \to \mathbb{R}$ define a symplectic duality. Prove the following.
(a) V has even dimension.
(b) If (e_i) is a basis of V and $\omega_{ij} = \omega(e_i, e_j)$ then $\det(\omega_{ij})_{1 \leq i,j \leq \dim V} \neq 0$. □

The notion of duality is compatible with the functorial constructions introduced so far.

Proposition 2.2.30. *Let* $B_i : V_i \times W_i \to \mathbb{R}$ *(*$i = 1, 2$*) be two pairs of spaces in duality. Then there exists a natural duality*

$$B = B_1 \otimes B_2 : (V_1 \otimes V_2) \times (W_1 \otimes W_2) \to \mathbb{R}$$

uniquely determined by

$$B(v_1 \otimes v_2, w_1 \otimes w_2) = B_1(v_1, w_1) \cdot B_2(v_2, w_2).$$

Exercise 2.2.14. Prove Proposition 2.2.30. □

Proposition 2.2.30 implies that given two spaces in duality $B : V \times W \to \mathbb{K}$ there is a naturally induced duality

$$B^{\otimes n} : V^{\otimes r} \times W^{\otimes r} \to \mathbb{K}.$$

This defines by restriction a pairing

$$\Lambda^r B : \Lambda^r V \times \Lambda^r W \to \mathbb{K}$$

uniquely determined by

$$\Lambda^r B\left(v_1 \wedge \ldots \wedge v_r, w_1 \wedge \ldots \wedge w_r\right) := \det\left(B\left(v_i, w_j\right)\right)_{1 \leq i,j \leq r}.$$

Exercise 2.2.15. Prove the above pairing is a duality. □

In particular, the natural duality $\langle \bullet, \bullet \rangle : V^* \times V \to \mathbb{K}$ induces a duality

$$\langle \bullet, \bullet \rangle : \Lambda^r V^* \times \Lambda^r V \to \mathbb{R}$$

and thus defines a natural isomorphism

$$\Lambda^r V^* \cong \left(\Lambda^r V\right)^*.$$

Thus, the elements of $\Lambda^r V^*$ can be viewed as skew-symmetric r-linear forms $V^r \to \mathbb{K}$. Moreover, if $n = \dim_{\mathbb{K}}$ V then the natural duality

$$\langle \bullet, \bullet \rangle : \Lambda^n V \times \Lambda^n V^* \to \mathbb{K}$$

defines a nontrivial element in $\Lambda^n V \otimes \Lambda^n V^* \cong \mathrm{End}\left(\Lambda^n V\right)$ (the identity operator) so that $\mathrm{End}\left(\Lambda^n V\right) = \mathrm{End}\left(\det V\right)$ is naturally identified with $\mathbb{K}$. In particular any endomorphism of V induces a linear map $\det V \to \det V$ which can be identified with a $\mathbb{K}$-scalar: the classical determinant of a linear operator $\det T$.

A duality $B : V \times W \to \mathbb{K}$ naturally induces a duality $B^t : V^* \times W^* \to \mathbb{K}$ by

$$B^t(v^*, w^*) : = \langle v^*, D_V^{-1} w^* \rangle,$$

where $D_V : V \to W^*$ is the linear isomorphism induced by the duality B.

Now consider a (*real*) Euclidean vector space V. Denote its inner product by $(\bullet, \bullet)$. The self-duality defined by $(\bullet, \bullet)$ induces a self-duality

$$(\bullet, \bullet) : \Lambda^r V \times \Lambda^r V \to \mathbb{R},$$

determined by

$$\left(v_1 \wedge \ldots \wedge v_r, w_1 \wedge \ldots \wedge w_r\right) \stackrel{def}{=} \det\left(\left(v_i, w_j\right)\right)_{1 \leq i,j \leq r} \tag{2.2.9}$$

The right hand side of (2.2.9) is a Gramm determinant and in particular the quadratic form in (2.2.9) is positive definite. Thus, we have proved the following result.

Corollary 2.2.31. *An inner product on a real vector space V naturally induces an inner product on the tensor algebra $\mathcal{T}V$ and in the exterior algebra $\Lambda^* V$.*

In an Euclidean vector space V the inner product induces the metric duality $\mathcal{L} : V \to V^*$. This induces an operator $\mathcal{L} : T^r_s(V) \to T^{r-1}_{s+1}(V)$ defined by

$$\mathcal{L}(v_1 \otimes \ldots \otimes v_r \otimes u^1 \otimes \ldots \otimes u^s) = (v_2 \otimes \ldots \otimes v_r) \otimes ((\mathcal{L}v_1 \otimes u^1 \otimes \ldots \otimes u^s). \qquad (2.2.10)$$

The operation defined in (2.2.10) is classically referred to as *lowering the indices.* The reason for this nomenclature comes from the coordinate description of this operation. Thus if $T \in \mathcal{T}^r_s(V)$ is given by

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}$$

then

$$(\mathcal{L}T)^{i_2 \ldots i_r}_{j j_1 \ldots j_r} = g_{ij} T^{i i_2 \ldots i_r}_{j_1 \ldots j_s},$$

where $g_{ij} = (e_i, e_j)$. The inverse of the metric duality $\mathcal{L}^{-1} : V^* \to V$ induces a linear operation $\mathcal{T}^r_s(V) \to \mathcal{T}^{r+1}_{s-1}(V)$ called *raising the indices.*

Exercise 2.2.16. (Cartan) Let V be an Euclidean vector space. For any $v \in V$ denote by e_v (resp. i_v) the linear endomorphism of $\Lambda^* V$ defined by $e_v \omega = v \wedge \omega$ (resp. $i_v = \imath_{v^*}$ where $\imath_{v^*}$ denotes the interior derivation defined by $v^* \in V^*$-the metric dual of v; see Example 2.2.20). Show that for any $u, v \in V$

$$[e_v, i_u]_s = e_v i_u + i_u e_v = (u, v) \mathbb{1}_{\Lambda^* V}. \qquad \square$$

Definition 2.2.32. *Let V be a real vector space. A* volume form *on V is a nontrivial linear form on the determinant line of V:*

$$Det : \det V \to \mathbb{R}.$$

Equivalently, a volume form on V is an element of $\det V^*$ ($n = \dim V$). Since $\det V$ is 1-dimensional a choice of a volume form corresponds to a choice of a basis of $\det V$.

Definition 2.2.33. *Two volume forms on V, Det_1 and Det_2 are said to be equivalent if there exists a* positive *constant λ such that $Det_2 = \lambda Det_1$. (There are only two equivalence classes.) An* orientation *on a vector space is an equivalence class of volume forms. A pair (vector space + orientation) is called an* oriented vector space.

There is an equivalent way of looking at orientations. To describe it note first that any volume form Det on V uniquely determines a basis ω of $\det V$ by the requirement $Det(\omega) = 1$. We say two bases ω_1 and ω_2 of $\det V$ are equivalent if there exists $\lambda > 0$ such that $\omega_2 = \lambda \omega_1$. An orientation on V selects an equivalence class of bases of $\det V$. A basis is said to be *positively oriented* if it belongs to the equivalence class defining the orientation. Otherwise, the basis is said to be *negatively oriented.*

To any basis $\{e_1, \ldots, e_n\}$ of V one can associate a basis $e_1 \wedge \cdots \wedge e_n$ of $\det V$. Note that a permutation of the indices $1, \ldots, n$ changes the orientation of the associated basis by a factor equal to the signature of the permutation. Thus to define an orientation on a vector space it suffices to specify a total ordering of a given basis of the space. An *ordered* basis of V is said to be positively oriented if so is the associated basis of $\det V$.

Assume now V is an Euclidean space. Denote the Euclidean inner product by $g(\cdot,\cdot)$. $\det V$ has an induced Euclidean structure and in particular there exist only two length-one-vectors in $\det V$, $\pm\omega$. If we fix one of them - say ω - as a basis in $\det V$ we achieve two things. Firstly, there is a naturally associated volume form Det_g defined by $Det_g(\lambda\omega) = \lambda$. Secondly, the equivalence class of this volume form determines an orientation on V. Conversely, an orientation on V uniquely selects a length-one-vector ω in $\det V$ thus defining a volume form Det_g. Thus, we have proved the following result.

Proposition 2.2.34. *An orientation in an Euclidean vector space (V, g) canonically selects a volume form on V, henceforth denoted by Det_g*

Exercise 2.2.17. Let (V, g) be an n-dimensional, oriented, Euclidean vector space and denote by Det_g the associated volume form. Show that for any basis v_1, ..., v_n of V we have

$$Det_g(v_1 \wedge \ldots \wedge v_n) = \epsilon(v_1, \ldots, v_n)\sqrt{(\det g(v_i, v_j))}.$$

where $\epsilon(v_1, \ldots, v_n) = +1$ if the basis $v_1 \wedge \ldots \wedge v_n$ of $\det V$ is positively oriented and -1 otherwise. For $V = \mathbb{R}^2$ with its standard metric and the orientation given by $e_1 \wedge e_2$ prove that

$$|Det_g(v_1 \wedge v_2)|$$

is the area of the parallelogram spanned by v_1 and v_2. □

Definition 2.2.35. *Let (V, g) be an oriented, Euclidean space and denote by Det_g the associated volume form. The* Berezin integral *or (* berezinian*) is the linear form*

$$\widehat{\int} : \Lambda^* V \to \mathbb{R}$$

defined on homogeneous elements by

$$\widehat{\int}\omega = \begin{cases} 0 & \text{if } \deg\omega < \dim V \\ Det_g\omega & \text{if } \deg\omega = \dim V \end{cases}.$$

Definition 2.2.36. *Let $\omega \in \Lambda^2 V$, where (V, g) is an oriented, Euclidean space. We define its* pfaffian *as*

$$Pf(\omega) \stackrel{def}{=} \widehat{\int} \exp\omega = \begin{cases} 0 & \text{if } \dim V \text{ is odd} \\ \frac{1}{n!}Det_g(\omega^{\wedge n}) & \text{if } \dim V = 2n \end{cases}$$

$\exp\omega$ *denotes the exponential in the (nilpotent) algebra $\Lambda^* V$:*

$$\exp\,\omega = \sum_{k\geq 0}\frac{\omega^k}{k!}.$$

If (V, g) is as in the above definition ($\dim V = N$) and $A : V \to V$ is a skew-symmetric endomorphism of V we can form $\omega_A \in \Lambda^2 V$ by

$$\omega_A = \sum_{i<j} g(Ae_i, e_j) e_i \wedge e_j = \frac{1}{2} \sum_{i,j} g(Ae_i, e_j) e_i \wedge e_j$$

where $(e_1, \ldots, e_N)$ is a positively oriented orthonormal basis of V. The reader can check that ω_A is independent of the choice of basis as above. (Notice that $\omega_A(u, v) = g(Au, v), \ \forall u, v \in V$.)We define the pfaffian of A by

$$Pf(A) \stackrel{def}{=} Pf(\omega_A).$$

Example 2.2.37. Let $V = \mathbb{R}^2$ denote the standard Euclidean space oriented by $e_1 \wedge e_2$, where $e_1, \ e_2$ denotes the standard basis. If

$$A = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix},$$

then $\omega_A = \theta e_1 \wedge e_2$ so that $Pf(A) = \theta$. □

Exercise 2.2.18. Let $A : V \to V$ be a skew-symmetric endomorphism of an oriented Euclidean space V. Prove that $Pf(A)^2 = \det A$. □

Exercise 2.2.19. Let (V, g) be an oriented Euclidean space of dimension $2n$. Consider $A : V \to V$ a skewsymmetric endomorphism and $e_1, \ldots, e_{2n}$ a positively oriented orthonormal frame. Prove that

$$Pf(A) = \frac{(-1)^n}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}$$

$$= (-1)^n \sum_{\sigma \in \mathfrak{S}'_{2n}} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}$$

where $a_{ij} = g(e_i, Ae_j)$ is the (i, j) entry in the matrix representing A in the basis (e_i) while $\mathfrak{S}'_{2n}$ denotes the set of permutations $\sigma \in \mathfrak{S}_{2n}$ satisfying

$$\sigma(2k-1) < \min\{\sigma(2k), \sigma(2k+1)\}, \ \ \forall k.$$ □

Let (V, g) be an n-dimensional, oriented, Euclidean vector space. The metric duality $\mathcal{L} : V \to V^*$ induces both a metric and an orientation on V^*. In the sequel we will continue to use the same notation $\mathcal{L}_g$ to denote the metric duality $\mathcal{T}^r_s(V) \to \mathcal{T}^r_s(V^*) \cong \mathcal{T}^s_r(V)$.

Definition 2.2.38. *The* Hodge pairing *is defined by*

$$\Xi : \Lambda^r V \times \Lambda^{n-r} V^* \to \mathbb{R}, \quad \Xi(\omega^r, \eta^{n-r}) = Det_g(\omega^r \wedge \mathcal{L}_g \eta^{n-r}).$$

Exercise 2.2.20. Prove the Hodge pairing is a duality. □

Definition 2.2.39. *The Hodge $*$-operator is the isomorphism*

$$*: \Lambda^r V \to \Lambda^{n-r} V$$

induced by the Hodge duality.

The above definition obscures the meaning of the $*$-operator. We want to spend some time clarifying its significance.

Let $\alpha \in \Lambda^r V$ so that $*\alpha \in \Lambda^{n-r} V$. Denote by $\langle\ ,\ \rangle$ the standard duality $\Lambda^s V \times \Lambda^s V^* \to \mathbb{R}$ and by $(\ ,\ )$ the metric duality $\Lambda^s V \times \Lambda^s V \to \mathbb{R}$. Then by definition $*$ satisfies

$$Det_g(\alpha \wedge \mathfrak{L}_g \beta) = \langle *\alpha, \beta\rangle \stackrel{def}{=} (*\alpha, \mathfrak{L}_g \beta) \;\; \forall \beta \in \Lambda^{n-r} V^*. \tag{2.2.11}$$

Let ω denote the unit vector in $\det V$ defining the orientation. Then (2.2.11) can be rewritten as

$$\alpha \wedge \mathfrak{L}_g \beta = (*\alpha, \mathfrak{L}_g \beta)\omega \;\; \forall \beta \in \Lambda^{n-r} V^*.$$

Thus

$$\alpha \wedge \gamma = (*\alpha, \gamma)\omega \quad \forall \gamma \in \Lambda^{n-r} V. \tag{2.2.12}$$

Equality (2.2.12) uniquely determines the action of $*$.

Example 2.2.40. Let V be the standard Euclidean space $\mathbb{R}^3$ with standard basis $e_1,\ e_2,\ e_3$ and orientation determined by $e_1 \wedge e_2 \wedge e_3$. Then

$$*e_1 = e_2 \wedge e_3 \quad *e_2 = e_3 \wedge e_1 \quad *e_3 = e_1 \wedge e_2.$$

$$*1 = e_1 \wedge e_2 \wedge e_3 \qquad *(e_1 \wedge e_2 \wedge e_3) = 1.$$

$$*(e_2 \wedge e_3) = e_1 \quad *(e_3 \wedge e_1) = e_2 \quad *(e_1 \wedge e_2) = e_3. \qquad \square$$

The following result is left to the reader as an exercise.

Proposition 2.2.41. *The Hodge $*$-operator satisfies* ($n = \dim V$)

$$*(*\omega) = (-1)^{p(n-p)}\omega \;\; \forall \omega \in \Lambda^p V,$$

$$Det_g(*1) = 1,$$

and

$$\alpha \wedge *\beta = (\alpha, \beta) * 1, \;\; \forall \alpha \in \Lambda^k V,\ \forall \beta \in \Lambda^{n-k} V.$$

Exercise 2.2.21. Let (V, g) be an n-dimensional, oriented, Euclidean space. Denote by g_ε the rescaled metric $g_\varepsilon = \varepsilon^2 g$. If $*$ is the Hodge operator corresponding to the metric g and $*_\varepsilon$ is the Hodge operator corresponding to the metric g_ε show that

$$*_\varepsilon \omega = \varepsilon^{2p-n} * \omega \quad \forall \omega \in \Lambda^p V. \qquad \square$$

We conclude this subsection with a brief discussion of densities.

Definition 2.2.42. *Let V be a real vector space. For any $r \in \mathbb{R}$ we define a r-*density *to be a function $f : \det V \setminus \{0\} \to \mathbb{R}$ such that*

$$f(\lambda u) = |\lambda|^r f(u), \ \ \forall u \in \det V \setminus \{0\}, \ \forall \lambda \neq 0.$$

The linear space of r-densities on V will be denoted by $|\Lambda|_V^r$. When $r = 1$ we set $|\Lambda|_V \stackrel{def}{=} |\Lambda|_V^1$.

An orientation on V induces a natural isomorphism

$$\imath : \det V^* \to |\Lambda|_V \quad \omega \mapsto (\imath(\omega) : \det V \setminus \{0\} \to \mathbb{R})$$

where $\imath(\omega)(u) = \operatorname{sign}(Det\, u)\omega(u)$ and Det is an arbitrary volume form on V defining the orientation. Notice that once we fix a volume form on V, one has identifications:

$$\det V^* \cong |\Lambda|_V \cong \mathbb{R}.$$

In particular, an orientation in an Euclidean vector space canonically identifies $|\Lambda|_V$ with $\mathbb{R}$.

2.2.5 Some complex linear algebra

In this very short section we want to briefly discuss some aspects specific to linear algebra on complex vector spaces.

Thus let V be a complex vector space. Its *conjugate* is the complex vector space $\overline{V}$ which coincides with V as a *real* vector space but in which the multiplication by a scalar $\lambda \in \mathbb{C}$ is defined by

$$\lambda \cdot v := \overline{\lambda} v, \ \ \forall v \in V.$$

V has a complex dual V_c^* which can be identified with the space of complex linear maps $V \to \mathbb{C}$. If we forget the complex structure we obtain a *real* dual V_r^*consisting of all real-linear maps $V \to \mathbb{R}$.

Definition 2.2.43. *A* Hermitian metric *is a complex bilinear map*

$$(\bullet, \bullet) : V \times \overline{V} \to \mathbb{C}$$

satisfying the following properties.
(a) $(\bullet, \bullet)$ *is positive definite, i.e.*

$$(v, v) > 0, \ \ \forall v \in V \setminus \{0\}.$$

(b) $(u, v) = \overline{(v, u)}, \ \ \forall u, v \in V$.

A Hermitian metric defines a duality $V \times \overline{V} \to \mathbb{C}$ and hence it induces a complex linear isomorphism $\mathfrak{L} : \overline{V} \to V_c^*$, $v \mapsto (\cdot, v) \in V_c^*$.

If V and W are complex Hermitian vector spaces then any complex linear map $A : V \to W$ induces a complex linear map

$$A^* : \overline{W} \to V_c^* \quad A^* w := \Big(v \mapsto \langle Av, w \rangle \Big) \in V_c^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural duality between a vector space and its dual. We can rewrite the above fact as

$$\langle Av, w \rangle = \langle v, A^* w \rangle.$$

A complex linear map $\overline{W} \to V_c^*$ is the same as a complex linear map $W \to \overline{V_c}^*$. The metric duality defines a complex linear isomorphism $\overline{V_c}^* \cong V$ so we can view the *adjoint* A^* as a complex linear map $W \to V$.

Let $h = (\bullet, \bullet)$ be a Hermitian metric on the complex vector space V. If we view $(\bullet, \bullet)$ as an object over $\mathbb{R}$, i.e. as a *real* bilinear map $V \times V \to \mathbb{C}$ then the Hermitian metric decomposes as

$$h = \mathfrak{Re}\, h - \mathbf{i}\,\omega \quad \mathbf{i} = \sqrt{-1}.$$

The real part is an inner product on the *real* space V, while ω is a *real*, skew-symmetric bilinear form on V and thus can be identified with an element of $\Lambda^2_{\mathbb{R}} V^*$. ω is called the *real 2-form associated to the Hermitian metric.*

Let V be an complex vector space and $e_1, \cdots, e_n$ a basis of V (over $\mathbb{C}$). This is not a *real* basis of V since $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$. We can however complete this to a real basis. More precisely, the vectors $e_1, \mathbf{i}e_1, \cdots, e_n, \mathbf{i}e_n$ form a real basis of V.

Definition 2.2.44. *The* canonical orientation *of a complex vector space is the orientation defined by the basis* $e_1 \wedge \mathbf{i}e_1 \wedge ... \wedge e_n \wedge \mathbf{i}e_n$ *above.*

Exercise 2.2.22. Prove that the orientation defined above is indeed canonical, i.e. it is independent of the complex basis $e_1, ..., e_n$ of V. $\square$

It is convenient to have a more explicit description of the abstract objects introduced above. Let V be an n-dimensional complex vector space and h a Hermitian metric on it. Pick an unitary basis $e_1, ..., e_n$ of V i.e. $n = \dim_{\mathbb{C}} V$ and $h(e_i, e_j) = \delta_{ij}$. For each j denote by f_j the vector $\mathbf{i}e_j$. Then $e_1, f_1, \cdots, e_n, f_n$ is a $\mathbb{R}$-basis of V and $e_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n$ defines the canonical orientation of V. Denote by $e^1, f^1, \cdots, e^n, f^n$ the dual $\mathbb{R}$-basis in V^*. Then

$$\mathfrak{Re}\, h\,(e_i, e_j) = \delta_{ij} = \mathfrak{Re}\, h(f_i, f_j) \;\text{ and }\; \mathfrak{Re}\, h\,(e_i, f_j) = 0$$

i.e.

$$\mathfrak{Re}\, h = \sum_i (e^i \otimes e^i + f^i \otimes f^i).$$

Also

$$\omega(e_i, f_j) = -\mathfrak{Im}\, h(e_i, \mathbf{i}e_j) = \delta_{ij}, \;\; \omega(e_i, e_j) = \omega(f_i, f_j) = 0 \;\; \forall i, j,$$

which shows that

$$\omega = -\mathfrak{Im}\, h = \sum_i e^i \wedge f^i.$$

In particular, if we denote by Pf_h the *pfaffian* with respect to the canonical metric $\mathfrak{Re}\, h$ and orientation, we deduce

$$Pf_h(\omega) = 1.$$

Any complex space V can be also thought of as a real vector space. The multiplication by $\sqrt{-1}$ defines a (real) linear operator which we denote by J. Obviously J satisfies $J^2 = -\mathbb{1}_V$.

Conversely, if V is a real vector space then any real operator $J : V \to V$ as above defines a complex structure on V by

$$(a + b\mathbf{i})v = av + bJv, \ \ \forall v \in V, \ a + b\mathbf{i} \in \mathbb{C}.$$

We will call an operator J as above a *complex structure* .

Let V be a real vector space with a complex structure J on it. J has no eigenvectors on V. The natural extension of J to the complexification of V, $V_{\mathbb{C}} = V \otimes \mathbb{C}$ has two eigenvalues $\pm\mathbf{i}$ and thus we get a splitting of $V_{\mathbb{C}}$ as a direct sum of *complex* vector spaces (eigenspaces)

$$V_{\mathbb{C}} = \ker(J - \mathbf{i}) \oplus (\ker J + \mathbf{i}).$$

Exercise 2.2.23. Prove that we have the following isomorphisms of *complex* vector spaces

$$V \cong \ker(J - \mathbf{i}) \qquad \overline{V} \cong (\ker J + \mathbf{i}).$$

□

Set

$$V^{1,0} = \ker(J - \mathbf{i}) \cong_{\mathbb{C}} V \qquad V^{0,1} = \ker(J + \mathbf{i}) \cong_{\mathbb{C}} \overline{V}.$$

Thus $V_{\mathbb{C}} \cong V^{1,0} \oplus V^{0,1} \cong V \oplus \overline{V}$. We deduce from this an isomorphism of $\mathbb{Z}$-**graded complex** vector spaces

$$\Lambda^* V_{\mathbb{C}} \cong \Lambda^* V^{1,0} \otimes \Lambda^* V^{0,1}.$$

If we set $\Lambda^{p,q} V \stackrel{def}{=} \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$ then the above isomorphism can be reformulated as

$$\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^{p,q} V. \tag{2.2.13}$$

Note that the complex structure J on V induces by duality a complex structure J^* on V_r^* and we have an isomorphism of *complex vector spaces*

$$V_c^* = (V, J)_c^* \cong (V_r^*, J^*).$$

We can define similarly $\Lambda^{p,q} V^*$ as the $\Lambda^{p,q}$construction applied to the real vector space V_r^* equipped with the complex structure J^*. Note that

$$\Lambda^{1,0} V^* \cong (\Lambda^{1,0} V)_c^*$$

$$\Lambda^{0,1} V^* \cong (\Lambda^{0,1} V)_c^*.$$

and more generally

$$\Lambda^{p,q} V^* \cong (\Lambda^{p,q} V)_c^*.$$

If now h is a Hermitian metric on the complex vector space (V, J) then we have a natural isomorphism of complex vector spaces

$$V_c^* \cong (V_r^*, J^*) \cong_{\mathbb{C}} (V, -J) \cong_{\mathbb{C}} \overline{V}$$

so that

$$\Lambda^{p,q}V^* \cong_{\mathbb{C}} \Lambda^{q,p}V.$$

The Euclidean metric $g = \mathfrak{Re}\, h$ and the associated 2-form $\omega = -\mathfrak{Im}\, h$ are related by

$$g(u,v) = \omega(u, Jv), \;\; \omega(u,v) = g(Ju,v) \; \forall u,v \in V.$$

Moreover ω is a (1,1)-form. To see this it suffices to pick an unitary basis (e_i) of V and construct as usual the associated real orthonormal basis $\{e_1, f_1, \cdots, e_n, f_n\}$ $(f_i = Je_i)$. Denote by $\{e^i, f^i\,;\; i = 1, \cdots, n\}$ the dual orthonormal basis in V_r^*. Then $J^*e^i = -f^i$ and if we set

$$\varepsilon^i = \frac{1}{\sqrt{2}}(e^i + \mathbf{i}f^i) \quad \bar{\varepsilon}^j \frac{1}{\sqrt{2}}(e^j - \mathbf{i}f^j)$$

then

$$\Lambda^{1,0}V^* = \mathrm{span}_{\mathbb{C}}\{\varepsilon^i\} \;\; \Lambda^{0,1} = \mathrm{span}_{\mathbb{C}}\{\bar{\varepsilon}^j\}$$

and

$$\omega = \mathbf{i} \sum \varepsilon^i \wedge \bar{\varepsilon}^i.$$

2.3 Tensor fields

2.3.1 Operations with vector bundles

We now return to geometry and more specifically to vector bundles. Let $E \to M$ be a rank r vector bundle over the smooth manifold M. According to the definition of a vector bundle we can find an open cover (U_α) of M such that each restriction $E\,|_{U_\alpha}$ is trivial: $E\,|_{U_\alpha} \cong U_\alpha \times V$ where V is an r-dimensional vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The bundle E is obtained by gluing these trivial pieces on the overlaps $U_\alpha \cap U_\beta$ using a collection of transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ satisfying the cocycle condition. Conversely, a collection of gluing maps as above satisfying the cocycle condition uniquely defines a vector bundle. In the sequel we will exclusively think of vector bundles in terms of transition maps.

Let E, F be two vector bundles over the smooth manifold M given by a (common) open cover (U_α) and transition maps $(g_{\alpha\beta})$ and respectively $(h_{\alpha\beta})$. Then the collections

$$(g_{\alpha\beta} \oplus h_{\alpha\beta}), \; (g_{\alpha\beta} \otimes h_{\alpha\beta}), \;\; ((g^t_{\alpha\beta}))^{-1}, \;\; (\Lambda^* g_{\alpha\beta})$$

satisfy the cocycle condition and therefore define vector bundles which we denote by $E \oplus F$, $E \otimes F$, E^* and respectively Λ^*E. The reader can check easily that these vector bundles are independent of the choices of transition maps used to characterize E and F (use Exercise 2.1.6). The direct sum $E \oplus F$ is also called the *Whitney sum* of vector bundles. All functorial constructions discussed in the previous section have a vector bundle correspondent.

These constructions are natural in the following sense. Let E' and F' be vector bundles over the same smooth manifold M'. Any bundle maps $S : E \to E'$ and $T : F \to F'$, both covering the *same diffeomorphism* $\phi : M \to M'$, induce bundle morphisms $S \otimes T : E \otimes F \to E' \otimes F'$, covering ϕ and $S^t : (E')^* \to E^*$, covering ϕ^{-1}.

Exercise 2.3.1. Prove the assertion above.

Example 2.3.1. Let E, F, E' and F' be vector bundles over a smooth manifold M. Consider bundle isomorphisms $S : E \to E'$ and $T : F \to F'$ covering the same diffeomorphism of the base, $\phi : M \to M$. Then $(S^{-1})^t : E^* \to (E')^t$ is a bundle isomorphism covering ϕ so that we get an induced map $(S^{-1})^t \otimes T : E^* \otimes F \to (E')^* \otimes F'$. Note that we have a natural identification

$$E^* \otimes F \cong \mathrm{Hom}\,(E, F). \qquad \square$$

Definition 2.3.2. *Let $E \to M$ be a $\mathbb{K}$-vector bundle over M. A* metric *on E is a section h of $E^* \otimes_{\mathbb{K}} \overline{E^*}$ ($\overline{E} = E$ if $\mathbb{K} = \mathbb{R}$) such that for any $m \in M$, $h(m)$ defines a metric on E_m (Euclidean if $\mathbb{K} = \mathbb{R}$ or Hermitian if $\mathbb{K} = \mathbb{C}$).*

2.3.2 Tensor fields

We now specialize the previous considerations to the special situation when E is the tangent bundle of M, $E \cong TM$. The *cotangent bundle* is then

$$T^*M := (TM)^*.$$

We define the tensor bundles of M

$$\mathcal{T}^r_s(M) : = \mathcal{T}^r_s(TM) = (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}.$$

Note that $\mathcal{T}^r_s(M)$ is naturally a $C^\infty(M)$-module; multiplication by a function is multiplication by a smoothly varying scalar in each fiber of TM.

Definition 2.3.3. *(a) A* tensor field *of type (r, s) over the open set $U \subset M$ is a section of $\mathcal{T}^r_s(M)$ over U.*
(b) A degree r differential form *(r-form for brevity) is a section of $\Lambda^r(T^*M)$. The space of (smooth) r-forms over M is denoted by $\Omega^r(M)$. We set*

$$\Omega^*(M) \stackrel{def}{=} \oplus_{r \geq 0} \Omega^r(M).$$

(c) A Riemannian metric *on a manifold M is a (0,2), symmetric, (pointwise) positive definite tensor field on M.*

If we view the tangent bundle as a smooth family of vector spaces then a tensor field can be viewed as a smooth selection of a tensor in each of the tangent spaces. In particular a Riemann metric defines a smoothly varying procedure of measuring lengths of vectors in tangent spaces.

Remark 2.3.4. (a) A contravariant tensor field (i.e. a $(0, s)$-tensor field) S naturally defines a $C^\infty(M)$-multilinear map

$$S : \prod_1^s \mathrm{Vect}\,(M) \to C^\infty(M) \ \ (X_1, \dots, X_s) \mapsto (p \mapsto S_p(X_1(p), \dots, X_s(p))) \in C^\infty(M).$$

Conversely any such map uniquely defines a $(0, s)$-tensor field. In particular an r-form η can be identified with a skew-symmetric $C^\infty(M)$ multilinear map

$$\eta : \prod_1^r \mathrm{Vect}\,(M) \to C^\infty(M).$$

Notice that the wedge product in the exterior algebras induces an associative product in $\Omega^*(M)$ which we continue to denote by $\wedge$.

(b) Let $f \in C^\infty(M)$. Its Frechet derivative $Df : TM \to T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ is naturally a 1-form. Indeed, we get a smooth $C^\infty(M)$-linear map $df : \mathrm{Vect}\,(M) \to C^\infty(M)$ defined by

$$df(X)_m \stackrel{def}{=} Df(X)_{f(m)} \in T_{f(m)}\mathbb{R} \cong \mathbb{R}.$$

In the sequel we will always regard the differential of a smooth function f as a 1-form and to indicate this we will use the notation df (instead of the usual Df). □

Any diffeomorphism $f : M \to M$ induces a bundle isomorphisms $Df : TM \to TM$ and $(Df^{-1})^t : T^*M \to T^*M$ covering f. Thus a diffeomorphism f induces a $C^\infty(M)$ linear map

$$f_* : \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M) \tag{2.3.1}$$

called the *push-forward* map. Hence the group of diffeomorphisms of M acts naturally (and linearly) on the space of tensor fields.

For contravariant tensor fields a more general result is true. More precisely any smooth map $f : M \to N$ defines $C^\infty(M)$-linear map

$$f^* : \mathcal{T}^0_s(N) \to \mathcal{T}^0_s(M)$$

called the *pullback* by f. Explicitly, if S is such a tensor defined by a $C^\infty(M)$-multilinear map

$$S : (\mathrm{Vect}\,(N))^s \to C^\infty(N)$$

then f^*S is the contravariant tensor field defined by

$$(f^*S)_p(X_1(p), \dots, X_s(p)) \stackrel{def}{=} S_{f(p)}(Df\mid_p (X_1), \dots, Df\mid_p (X_s))$$

$\forall X_1, \dots, X_s \in Vect(M)$, $p \in M$. Note that when f is a *diffeomorphism* we have

$$f^* = (f_*^{-1})^t$$

where f_* is the push-forward map defined in (2.3.1).

Proposition 2.3.5. *Let $f : M \to N$ be a smooth map. The pullback by f defines a morphism of associative algebras*

$$f^* : \Omega^*(N) \to \Omega^*(M).$$

Exercise 2.3.2. Prove the above proposition. □

It is often very useful to have a local description of these objects. If $(x^1, \dots, x^n)$ are local coordinates on an open set $U \subset M$ then the vector fields $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ trivialize $TM\mid_U$. We can form a dual trivialization of $T^*M\mid_U$ using the 1-forms dx^i uniquely determined by

$$\langle dx^i, \frac{\partial}{\partial x_j} \rangle = \delta^i_j.$$

A basis in $\mathcal{T}_s^r(T_xM)$ is given by

$$\left\{\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s};\ 1\leq i_1,\ldots,i_r\leq n,\ 1\leq j_1,\ldots,j_s\leq n\right\}.$$

Hence any tensor $T\in\mathcal{T}_s^r(M)$ has a local description

$$T=T^{i_1\ldots i_r}_{j_1\ldots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s}.$$

In particular an r-form looks like

$$\omega=\sum_{1\leq i_1<\ldots<i_r\leq n}\omega_{i_1\ldots i_r}dx^{i_1}\wedge\cdots\wedge dx^{i_r}.$$

Example 2.3.6. Consider the map

$$\overline{polar}:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}^2,\quad (r,\theta)\mapsto(x=r\cos\theta,y=r\sin\theta).$$

The map $\overline{polar}$ defines the usual polar coordinates. We want to compute the pullback of the volume form $dx\wedge dy$ by $\overline{polar}$. We have

$$dx\wedge dy=d(r\cos\theta)\wedge d(r\mathrm{sin}\theta)$$

$$=(\cos\theta dr-r\mathrm{sin}\theta d\theta)\wedge(\mathrm{sin}\theta dr+r\cos\theta d\theta)$$
$$=r(\cos^2\theta+\mathrm{sin}^2\theta)dr\wedge d\theta=rdr\wedge d\theta.\qquad\square$$

All operations discussed in the previous section have natural extensions to tensor fields. There is a tensor multiplication, a Riemann metric defines a duality $\mathcal{L}:\mathrm{Vect}\,(M)\to\Omega^1(M)$ etc. In particular, there exists a contraction operator

$$\mathrm{tr}:\ \mathcal{T}^{r+1}_{s+1}(M)\to\mathcal{T}^r_s(M)$$

defined by

$$\mathrm{tr}\,(X_0\otimes\cdots\otimes X_r)\otimes(\omega_0\otimes\cdots\otimes\omega_s)=\omega_0(X_0)(X_1\otimes\cdots X_r\otimes\omega_1\otimes\cdots\otimes\omega_s).$$

($X_i\in\mathrm{Vect}\,(M)$, $\omega_j\in\Omega^1(M)$) In local coordinates the contraction has the form

$$\left\{\mathrm{tr}\left(T^{i_0\ldots i_r}_{j_0\ldots j_s}\right)\right\}=\{T^{ii_1\ldots i_r}_{ij_1\ldots j_s}\}.$$

2.3.3 Fiber bundles

We consider it is useful at this point to bring up the notion of fiber bundle. There are several reasons to do this. On the first hand, they arise naturally in geometry and they impose themselves as worth studying. On the second hand, they provide a very elegant and concise language to describe many phenomena in geometry.

We have already met examples of fiber bundles when we discussed vector bundles. These were “smooth families of vector spaces”. A fiber bundle wants to be a smooth family of copies of the same manifold. This is a very loose description but may offer a first glimpse at the notion about to be discussed.

The model situation is that of direct product $X = B \times F$ where B and F are smooth manifolds. It is convenient to regard this as a family of manifolds $(F_b)_{b \in B}$. B is called the base, F is called the standard (model) fiber and X is called the total space. This is an example of trivial fiber bundle. In general, a fiber bundle is obtained by gluing a bunch of trivial ones according to a prescribed rule. The gluing may encode a symmetry of the fiber and we would like to spend some time explaining what do we mean by symmetry.

Definition 2.3.7. *(a) Let M be a smooth manifold and G a Lie group. We say the group G acts on M from the left (resp. right) if there exists a smooth map*

$$\Phi : G \times M \to M, \;\; (g, m) \mapsto T_g m$$

such that $T_1 \equiv \mathbb{1}_M$ and

$$T_g(T_h m) = T_{gh} m \;\; (\text{resp. } T_g(T_h m) = T_{hg} m) \; \forall g, h \in G, \; m \in M.$$

In particular, we deduce that $\forall g \in G$ the map T_g is a diffeomorphism of M. For any $m \in M$ the set

$$G \cdot m = \{T_g m; \; g \in G\}$$

is called the orbit *of the action through m.*
(b) Let G act on M. The action is called free *if $\forall g \in G$ and $\forall m \in M$ $T_g m \neq m$. The action is called* effective *if $\forall g \in G$ $\exists m \in M$ such that $T_g m \neq m$.*

It is useful to think of a Lie group action as defining a symmetry.

Example 2.3.8. Let G be a Lie group. A *linear representation* of G on a vector space V is a left action of G on V such that each T_g is a linear map. One says V is a *G-module*. □

Example 2.3.9. Let G be a Lie group. For any $g \in G$ denote by L_g (resp. R_g) the left (resp right) translation by g. In this way we get the tautological left (resp. right) action of G on itself. □

Definition 2.3.10. *Let G be a Lie group. A* G-fiber bundle *is an object composed of the following:*
(a) a manifold E called the total space*;*
(b) a manifold B called the base*;*
(c) a manifold F called the standard fiber*;*
(d) a submersion $\pi : E \to B$ called the natural projection*;*
(e) A Lie group G with a fixed, effective left action on F, $g \mapsto T_g \in \mathrm{Diffeo}\,(F)$. G is called the symmetry group of the bundle*;*
(f) a collection of local trivializations *i.e. an open cover (U_α) of the base B and diffeomorphisms $\psi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that*

$$\pi \circ \psi_\alpha(b, f) = b, \;\; \forall (b, f) \in U_\alpha \times F.$$

We can form the transition (gluing) maps $\psi_{\alpha\beta} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$ *(*$U_{\alpha\beta} = U_\alpha \cap U_\beta$*) defined by* $\psi_{\alpha\beta} = \psi_\alpha^{-1} \circ \psi_\beta$*. According to (f) these maps can be written as*

$$\psi_{\alpha\beta}(b, f) = (b, T_{\alpha\beta}(b)f)$$

where $T_{\alpha\beta}(b)$ *is a diffeomorphism of* F *depending smoothly upon* $b \in U_{\alpha\beta}$*. The final condition to be imposed is* G-compatibility *i.e.*
(g) There exist smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ *satisfying the cocycle condition*

$$g_{\alpha\alpha} = 1 \in G,$$

$$g_{\gamma\alpha} = g_{\gamma\beta} \cdot g_{\beta\alpha},$$

and such that

$$T_{\alpha\beta}(b) = T_{g_{\alpha\beta}(b)}.$$

We will denote this fiber bundle by (E, π, B, F, G)*.*

The choice of an open cover (U_α) in the above definition is a source of arbitrariness since there is no natural prescription on how to perform this choice. We need to describe when two such choices are equivalent.

Two open covers (U_α) and (V_i) together with the collections of local trivializations $\phi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ and $\psi_i : V_i \times F \to \pi^{-1}(V_i)$ are said to be equivalent if for all α, i there exists a smooth map $t_{\alpha i} : U_\alpha \cap V_i \to G$ such that for any $x \in U_\alpha \cap V_i$ and any $f \in F$

$$\phi_\alpha^{-1}\psi_i(x, f) = (x, t_{\alpha i}(x)f).$$

A G-bundle structure is defined by an equivalence class of trivializing covers.

As in the case of vector bundles a collection of gluing data determines a G-fiber bundle. Indeed, if we are given a cover $(U_\alpha)_{\alpha \in A}$ of the base B and a collection of transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle condition, then we can get a bundle by gluing the trivial pieces $U_\alpha \times F$ along the overlaps. More precisely if $b \in U_\alpha \cap U_\beta$ then the element $(b, f) \in U_\alpha \times F$ is identified with the element $(b, g_{\beta\alpha} \cdot f) \in U_\beta \times F$ where $(g, f) \mapsto g \cdot f$ denotes the left action of G on F.

Definition 2.3.11. *Let* $E \xrightarrow{\pi} B$ *be a* G*-fiber bundle. A* G*-automorphism of this bundle is a diffeomorphism* $T : E \to E$ *such that* $\pi \circ T = \pi$ *(i.e.* T *maps fibers to fibers) and for any trivializing cover* (U_α) *(as in Definition 2.3.10) there exists a smooth map* $g_\alpha : U_\alpha \to G$ *such that*

$$\psi_\alpha^{-1} T \psi_\alpha(b, f) = (b, g_\alpha(b)f)), \;\; \forall b, g, f.$$

Definition 2.3.12. *(a) A* fiber bundle *is an object defined by conditions (a)-(d) and (f) in the above definition. (One can think the structure group is the group of diffeomorphisms of the standard fiber).*
(b) A section *of a fiber bundle* $E \xrightarrow{\pi} B$ *is a smooth map* $s : B \to E$ *such that* $\pi \circ s = \mathbb{1}_B$*, i.e.* $s(b) \in \pi^{-1}(b)$*,* $\forall\, b \in B$*.*

Example 2.3.13. A rank r vector bundle (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$) is a $GL(r, \mathbb{K})$-fiber bundle with standard fiber $\mathbb{K}^r$ and where the group $GL(r, \mathbb{K})$ acts on $\mathbb{K}^r$ in the natural way. □

Example 2.3.14. Let G be a Lie group. A *principal G-bundle* is a G-fiber bundle with fiber G, where G acts on itself by left translations. Equivalently, a principal G-bundle over a smooth manifold M can be described by an open cover $\mathcal{U}$ of M and a G-cocycle i.e. a collection of smooth maps

$$g_{UV} : U \cap V \to G \quad U, V \in \mathcal{U}$$

such that $\forall x \in U \cap V \cap W$ $(U, V, W \in \mathcal{U})$

$$g_{UV}(x) g_{VW}(x) g_{WU}(x) = 1 \in G.$$

□

Exercise 2.3.3. (Alternative definition of principal bundle) Let P be a fiber bundle with fiber a Lie group G. Prove the following are equivalent.
(a) P is a principal G-bundle.
(b) There exists a free, right action of G on G, $(P \times G \to P,\ (p, g) \mapsto p \cdot g$ such that its orbits coincide with the fibers of the bundle P and a trivializing cover

$$\left\{ \psi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha) \right\},$$

such that

$$\psi_\alpha(u, hg) = \psi_\alpha(u, h) \cdot g \quad \forall g, h \in G,\ u \in U_\alpha.$$

□

Exercise 2.3.4. (The frame bundle of a manifold) Let M^n be a smooth manifold. Denote by $F(M)$ the set of frames on M i.e.

$$F(M) = \{(m; X_1, \dots, X_n);\ m \in M,\ X_i \in T_m M \text{ and } \operatorname{span}(X_1, \dots, X_n) = T_m M\}.$$

(a) Prove that $F(M)$ can be naturally organized as a smooth manifold such that the natural projection $p : F(M) \to M$, $(m; X_1, \dots, X_n) \mapsto m$ is a submersion.
(b) Show $F(M)$ is a principal $GL(n, \mathbb{R})$-bundle. $F(M)$ is called the *frame bundle* of the manifold M.
Hint: A matrix $T = (T^i_j) \in GL(n, \mathbb{K})$ acts on the right on $F(M)$ by

$$(m; X_1, ..., X_n) \mapsto (m; (T^{-1})^i_1 X_i, ..., (T^{-1})^i_n X_i).$$

□

Example 2.3.15. (Associated fiber bundles) Let $\pi : P \to G$ be a principal G-bundle. Consider $(U_\alpha)_{\alpha \in A}$ a trivializing cover and denote by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ a collection of gluing maps determined by this cover. Assume G acts (on the left) on a smooth manifold F

$$\tau : G \times F \to F,\ \ (g, f) \mapsto \tau(g) f.$$

The collection $\tau_{\alpha\beta} = \tau(g_{\alpha\beta}) : U_{\alpha\beta} \to \operatorname{Diffeo}(F)$ satisfies the cocycle condition and can be used (exactly as we did for vector bundles) to define a G-fiber bundle with fiber F. This new bundle is independent of the various choices made (cover (U_α) and transition maps $g_{\alpha\beta}$). (Prove this!) It is called the bundle associated to P via τ and is denoted by $P \times_\tau F$.

□

Exercise 2.3.5. Prove that the tangent bundle of a manifold M^n is associated to $F(M)$ via the natural action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. □

Exercise 2.3.6. (The Hopf bundle) If we identify the unit odd dimensional sphere S^{2n-1} with the submanifold

$$\{ (z_1, \dots, z_n) \in \mathbb{C}^n;\ |z_0|^2 + \cdots + |z_n|^2 = 1 \}$$

then we detect an S^1-action on S^{2n-1} given by

$$e^{i\theta} \cdot (z_1, ..., z_n) = (e^{i\theta} z_1, ..., e^{i\theta} z_n).$$

The space of orbits of this action is naturally identified with the complex projective space $\mathbb{CP}^{n-1}$.
(a) Prove that $\mathbf{p} : S^{2n-1} \to \mathbb{CP}^{n-1}$ is a principal S^1 bundle called Hopf bundle. ($\mathbf{p}$ is the obvious projection). Describe one collection of transition maps.
(b) Prove that the tautological line bundle over $\mathbb{CP}^{n-1}$ is associated to the Hopf bundle via the natural action of S^1 on $\mathbb{C}^1$. □

Exercise 2.3.7. Let E be a vector bundle over the smooth manifold M. Any metric h on E (euclidian or Hermitian) defines a submanifold $S(E) \subset E$ by

$$S(E) = \{ v \in E;\ |v|_h = 1 \}.$$

Prove that $S(E)$ is a fibration over M with standard fiber a sphere of dimension rank $E - 1$. $S(E)$ is usually called the *sphere bundle* of E. □

Chapter 3

Calculus on Manifolds

This chapter describes the "kitchen" of differential geometry. We will discuss how one can operate with the various objects wehave introduced so far. In particular we will introduce several derivations of the various algebras of tensor fields an we will also present the opposite notion of integration.

3.1 The Lie derivative

3.1.1 Flows on manifolds

The notion of flow should be familiar to anyone who has had a course in ordinary differential equations. In this section we only want to describe analytic facts in a geometric light. We strongly recommend [4] for more details and excellent examples.

A neighborhood $\mathcal{N}$ of $M \times \{0\}$ in $M \times \mathbb{R}$ is called *balanced* if $\forall m \in M$

$$(\{m\} \times \mathbb{R}) \cap \mathcal{N} = \{m\} \times (-r, r), \quad \text{for some } r > 0.$$

Definition 3.1.1. *A* local flow *is a smooth map* $\Phi : \mathcal{N} \to M$, $(m, t) \mapsto \Phi^t(m)$ *(where* $\mathcal{N}$ *is a balanced neighborhood of* $M \times \{0\}$ *in* $M \times \mathbb{R}$*) such that*
(a) $\Phi^0(m) = m$, $\forall m \in M$.
(b) $\Phi^t(\Phi^s(m)) = \Phi^{t+s}(m)$ *for all* $s, t \in \mathbb{R}$, $m \in M$ *such that* (s, m), $(s + t, m)$, $(t, \Phi^s(m)) \in \mathcal{N}$.

When $\mathcal{N} = M \times \mathbb{R}$, Φ *is called a* flow *or* dynamical system.

The conditions (a) and (b) above show that a dynamical system is nothing but a left action of the additive (Lie) group $(\mathbb{R}, +)$ on M.

Example 3.1.2. Let A be a $n \times n$ real matrix. It generates a flow Φ_A^t on $\mathbb{R}^n$ by

$$\Phi_A^t x = e^{tA} x = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) x.$$

$\square$

Definition 3.1.3. *Let $\Phi : \mathcal{N} \to M$ be a local flow on M. The* infinitesimal generator *of Φ is the vector field X on M defined by*

$$X(m) = X_\Phi(m) \stackrel{def}{=} \frac{d}{dt}\,|_{t=0}\, \Phi^t(m)$$

i.e. $X(m)$ is the tangent vector to the smooth curve $t \mapsto \Phi^t(m)$ at $t = 0$. This curve is called flow line.

Exercise 3.1.1. Show X_Φ is a *smooth* vector field. □

Example 3.1.4. Consider the flow e^{tA} on $\mathbb{R}^n$ generated by an $n \times n$ matrix A. Its generator is the vector field X_A on $\mathbb{R}^n$ defined by

$$X_A(u) = \frac{d}{dt}\,|_{t=0}\, e^{tA}u = Au.$$

□

Proposition 3.1.5. *Let M be a smooth n-dimensional manifold. The map*

$$X : \{\text{Local flows on } M\} \to \text{Vect}\,(M), \quad \Phi \mapsto X_\Phi$$

is a surjection. Moreover, if $\Phi_i : \mathcal{N}_i \to N$ (i=1,2) are two local flows such that $X_{\Phi_1} = X_{\Phi_2}$ then $\Phi_1 = \Phi_2$ on $\mathcal{N}_1 \cap \mathcal{N}_2$.

Proof *Surjectivity.* Let X be a vector field on M. An *integral curve* for X is a smooth curve $\gamma : (a, b) \to M$ such that

$$\dot{\gamma}(t) = X(\gamma(t)).$$

In local coordinates (x^α) over on open subset $U \subset M$ this condition can be rewritten as

$$\dot{x}^\alpha(t) = X^\alpha\big(\,x^1(t), ..., x^n(t)\,\big), \tag{3.1.1}$$

where $\gamma(t) = (x^1(t), ..., x^n(t))$ and $X = X^\alpha \frac{\partial}{\partial x^\alpha}$. (3.1.1) is an ordinary differential equation. Classical existence results (see e.g. [4]) can be used to show that for any precompact open subset $K \subset U$ there exists $\varepsilon > 0$ such that for all $x \in K$ there exists a unique integral curve for X, $\gamma_x : (-\varepsilon, \varepsilon) \to M$ satisfying

$$\gamma_x(0) = x. \tag{3.1.2}$$

Moreover, as a consequence of the smooth dependence upon initial data we deduce that the map

$$\Phi_K : \mathcal{N}_K = K \times (-\varepsilon, \varepsilon) \to M, \;\; (x, t) \mapsto \gamma_x(t)$$

is smooth.

Now we can cover M by open, precompact local coordinate neighborhoods $(K_i)_{i \in I}$ and as above, we get smooth maps $\Phi_i : \mathcal{N}_i = K_i \times (-\varepsilon_i, \varepsilon_i) \to M$ solving the initial value problem (3.1.1-2). Moreover, by uniqueness we deduce

$$\Phi_i = \Phi_j \text{ on } \mathcal{N}_i \cap \mathcal{N}_j.$$

Define $\mathcal{N} = \cup_{i \in I} \mathcal{N}_i$ and set $\Phi : \mathcal{N} \to M$, $\Phi = \Phi_i$ on $\mathcal{N}_i$. Clearly, Φ satisfies all the conditions in the definition of a local flow. Tautologically, X is the infinitesimal generator of Φ. The second part of proposition follows from the uniqueness in initial value problems. $\square$

The family of local flows on M with the same infinitesimal generator $X \in \mathrm{Vect}(M)$ is naturally ordered according to their domains,

$$(\Phi_1 : \mathcal{N}_1 \to M) \prec (\Phi_2 : \mathcal{N}_2 \to M)$$

iff $\mathcal{N}_1 \subset \mathcal{N}_2$. This family has a *unique* maximal element which is called the *local flow generated by* X and is denoted by Φ_X.

3.1.2 The Lie derivative

Let X be a vector field on the smooth n-dimensional manifold M and denote by $\Phi = \Phi_X$ the local flow it generates. We assume Φ is actually a flow so its domain is actually $M \times \mathbb{R}$. The local flow situation is conceptually identical but unpleasantly lengthens the presentation of the facts to come.

For each $t \in \mathbb{R}$, Φ^t is a diffeomorphism of M and so it induces a push-forward map on the space of tensor fields. If S is a tensor field on M we define its Lie derivative along the direction given by M in a natural way

$$L_X S_m \stackrel{def}{=} -\lim_{t \to 0} \frac{1}{t}\left((\Phi^t_* S)_m - S_m \right) \ \ \forall m \in M. \tag{3.1.3}$$

Intuitively $L_X S$ measures how fast is the flow Φ changing the tensor S.

Clearly the limit in (3.1.3) exists and one sees that $L_X S$ is a tensor of the same type as S. We want provide more explicit descriptions of this operation.

Lemma 3.1.6. *For any $X \in \mathrm{Vect}\,(M)$ and $f \in C^\infty(M)$ we have*

$$Xf \stackrel{def}{=} L_X f = \langle df, X \rangle = df(X).$$

Above, $\langle \bullet, \bullet \rangle$ denotes the natural duality between T^*M and TM,

$$\langle \bullet, \bullet \rangle : C^\infty(T^*M) \times C^\infty(TM) \to C^\infty(M), \ \ C^\infty(T^*M) \times C^\infty(TM) \ni (\alpha, X) \mapsto \alpha(X).$$

In particular, L_X is a derivation of $C^\infty(M)$.

Proof Let $\Phi^t = \Phi^t_X$ be the local flow generated by X. Assume for simplicity that it is defined for all t. Φ^t acts on $C^\infty(M)$ by the pullback of its inverse i.e. $\Phi^t_*(f) = f \circ \Phi^{-t}$. Hence

$$L_X f(m) = \lim_{t \to 0} \frac{1}{t}(f(m) - f(\Phi^{-t} m)) = -\frac{d}{dt}\,|_{t=0}\, f(\Phi^{-t} m) = \langle df, X \rangle. \ \ \square$$

Exercise 3.1.2. Prove that any derivation of the algebra $C^\infty(M)$ is of the form L_X for some $X \in \mathrm{Vect}(M)$. Thus

$$Der(C^\infty(M)) \cong \mathrm{Vect}\,(M).$$

$\square$

Lemma 3.1.7. *Let $X, Y \in \mathrm{Vect}\,(M)$. Then the Lie derivative of Y along X is a new vector field $L_X Y$ which, viewed as a derivation of $C^\infty(M)$, coincides with the commutator of the two derivations of $C^\infty(M)$ defined by X and Y i.e.*

$$L_X Y f = [X, Y]f, \ \ \forall\, f \in C^\infty(M).$$

The vector field $[X, Y] = L_X Y$ is called the Lie bracket *of X and Y. In particular the Lie bracket induces a Lie algebra structure on* $\mathrm{Vect}\,(M)$.

Proof We will work in local coordinates (x^i) near a point $m \in M$ so that $X = X^i \frac{\partial}{\partial x_i}$ and $Y = Y^j \frac{\partial}{\partial x_j}$. We first describe the commutator $[X, Y]$. If $f \in C^\infty(M)$ then

$$[X, Y]f = (X^i \frac{\partial}{\partial x_i})(Y^j \frac{\partial f}{\partial x^j}) - (Y^j \frac{\partial}{\partial x_j})(X^i \frac{\partial f}{\partial x^i})$$

$$= \left(X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} \right) - \left(X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} + Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right)$$

so that the commutator of the two derivations is the derivation defined by the vector field

$$[X, Y] = \left(X^i \frac{\partial Y^k}{\partial x^i} - Y^j \frac{\partial X^k}{\partial x^j} \right) \frac{\partial}{\partial x_k}. \tag{3.1.4}$$

Note in particular that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ i.e. the basic vectors $\frac{\partial}{\partial x_i}$ commute as derivations.

So far we have not proved the vector field in (3.1.4) is independent of coordinates. We will achieve this by identifying it with the intrinsically defined vector field $L_X Y$.

Set $\gamma(t) = \Phi^t m$ so that we have a parametrization $\gamma(t) = (x^i(t))$ with $\dot{x}^i = X^i$. Then

$$\Phi^{-t} m = \gamma(-t) = \gamma(0) - \dot{\gamma}(0)t + O(t^2) = \big(x_0^i - tX^i + O(t^2)\big)$$

and

$$Y^j_{\gamma(-t)} = Y^j_m - tX^i \frac{\partial Y^j}{\partial x^i} + O(t^2). \tag{3.1.5}$$

Note that $\Phi_*^{-t} : T_{\gamma(0)}M \to T_{\gamma(-t)}M$ is the linearization of the map $(x^i) \mapsto \big(x_0^i - tX^i + O(t^2)\big)$ so it has a matrix representation

$$\Phi_*^{-t} = \mathbb{1} - t \left(\frac{\partial X^i}{\partial x^j} \right)_{i,j} + O(t^2).$$

In particular, using the Neumann's series

$$(\mathbb{1} - A)^{-1} = \mathbb{1} + A + A^2 + \cdots$$

(where A is a matrix of operator norm strictly less than 1) we deduce that $\Phi_*^t = (\Phi_*^{-t})^{-1} : T_{\gamma(-t)}M \to T_{\gamma(0)}M$ has the matrix form

$$\Phi_*^t = \mathbb{1} + t \left(\frac{\partial X^i}{\partial x^j} \right)_{i,j} + O(t^2). \tag{3.1.6}$$

Using (3.1.6) in (3.1.5) we deduce

$$Y^k_m - \left(\Phi^t_* Y_{\Phi^{-t}m}\right)^k = t\left(X^i\frac{\partial Y^k}{\partial x^i} - Y^j\frac{\partial X^k}{\partial x^j}\right) + O(t^2).$$

This concludes the proof of the lemma. □

We can now completely describe the Lie derivative on the algebra of tensor fields.

Proposition 3.1.8. *Let X be a vector field on the smooth manifold M. L_X is the unique derivation of $\mathcal{T}^*_*(M)$ with the following properties.*
(a) $L_X f = \langle df, X\rangle = Xf$, $\forall f \in C^\infty(M)$.
(b) $L_X Y = [X, Y]$, $\forall X, Y \in \mathrm{Vect}\,(M)$.
L_X commutes with the contraction $\mathrm{tr} : \mathcal{T}^{r+1}_{s+1}(M) \to \mathcal{T}^r_s(M)$. Moreover, L_X is a natural operation i.e., for any diffeomorphism $\phi : M \to N$ we have $\phi_* \circ L_X = L_{\phi_* X} \circ \phi_*$, $\forall X \in \mathrm{Vect}\,(M)$ i.e. $\phi_*(L_X) = L_{\phi_* X}$.

Proof The fact that L_X is a derivation i.e.

$$L_X(S \otimes T) = L_X S \otimes T + S \otimes L_X T$$

follows easily from the definition (3.1.3). Properties (a) and (b) were proved above. As for part (c) this is Leibniz' rule in disguise. In its simplest form when $T = Y \otimes \omega$ where $Y \in \mathrm{Vect}\,(M)$ and $\omega \in \Omega^1(M)$ we have

$$\mathrm{tr}\, T = \omega(Y)$$

and (c) is equivalent to

$$L_X\left(\omega(Y)\right) = (L_X\omega)(Y) + \omega(L_X(Y). \tag{3.1.7}$$

(3.1.7) follows immediately from (3.1.3) mimicking the proof of the usual product rule. In particular (3.1.7) uniquely determines the Lie derivative of a 1-form ω by

$$(L_X\omega)(Y) = L_X\left(\omega(Y)\right) - \omega([X, Y]). \tag{3.1.8}$$

Since L_X is a derivation of the algebra of tensor fields, its restriction to $C^\infty(M) \oplus \mathrm{Vect}\,(M) \oplus \Omega^1(M)$ uniquely determines the action on the entire algebra of tensor fields which generated by the above subspace. The reader can check easily that property (c) is satisfied in its entire generality. The naturality of L_X is another way of phrasing the coordinate independence of this operation. We leave the reader to fill in the routine details. □

Corollary 3.1.9. *For any $X, Y \in \mathrm{Vect}\,(M)$ we have*

$$[L_X, L_Y] = L_{[X,Y]}$$

as derivations of the algebra of tensor fields on M. In particular this says that $\mathrm{Vect}\,(M)$ *as a space of derivations of $\mathcal{T}^*_*(M)$ is a Lie subalgebra of the Lie algebra of derivations.*

Proof $[L_X, L_Y]$ is a derivation (as a commutator of derivations). By Lemma 3.1.7, $[L_X, L_Y] = L_{[X,Y]}$ on $C^\infty(M)$. Also, a simple computation shows that

$$[L_X, L_Y]Z = L_{[X,Y]}Z, \ \ \forall Z \in \mathrm{Vect}\,(M)$$

so that $[L_X, L_Y] = L_{[X,Y]}$ on Vect(M). Finally, since the contraction commutes with both L_X and L_Y it obviously commutes with $L_X L_Y - L_Y L_X$. The corollary is proved. □

Exercise 3.1.3. Prove that the map

$$\mathcal{D} : \mathrm{Vect}\,(M) \oplus \mathrm{End}\,(TM) \to Der(\mathcal{T}^*_*(M))$$

given by $\mathcal{D}(X, S) = L_X + S$ is well defined and is a linear isomorphism. Moreover,

$$[\mathcal{D}(X_1, S_1), \mathcal{D}(X_2, S_2)] = \mathcal{D}([X_1, X_2], [S_1, S_2]).$$ □

L_X is a derivation of $\mathcal{T}^*_*$ with the remarkable property

$$L_X(\Omega^*(M)) \subset \Omega^*(M).$$

The wedge product makes $\Omega^*(M)$ a s-algebra and it is natural to ask whether L_X is a s-derivation with respect to this product.

Proposition 3.1.10. *The Lie derivative along a vector field X is an even s-derivation of $\Omega^*(M)$ i.e.*

$$L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta), \ \ \forall \omega, \, \eta \in \Omega^*(M).$$

Proof Denote (as in Section 2.2.2) by $\mathcal{A}$ the anti-symmetrization operator $\mathcal{A} : (T^*M)^{\otimes k} \to \Omega^k(M)$. The statement in the proposition follows immediately from the straightforward observation that the Lie derivative commutes with this operator (which is a projector). We leave the reader to fill in the details. □

3.1.3 Examples

Example 3.1.11. Let $\omega = \omega_i dx^i$ be a 1-form on $\mathbb{R}^n$. If $X = X^j \frac{\partial}{\partial x^j}$ is a vector field on $\mathbb{R}^n$ then $L_X\omega = (L_X\omega)_k dx^k$ is defined by

$$(L_X\omega)_k = (L_X\omega)(\frac{\partial}{\partial x_k}) = X\omega(\frac{\partial}{\partial x_k}) - \omega(L_X\frac{\partial}{\partial x_k}) = X \cdot \omega_k + \omega\left(\frac{\partial X^i}{\partial x^k}\frac{\partial}{\partial x_i}\right).$$

Hence

$$L_X\omega = \left(X^j\frac{\partial \omega_k}{\partial x^j} + \omega_j\frac{\partial X^j}{\partial x^k}\right)dx^k.$$ □

Example 3.1.12. Consider $X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y} + H\frac{\partial}{\partial z}$ a smooth vector field on $\mathbb{R}^3$. We want to compute $L_X dv$ where dv is the volume form on $\mathbb{R}^3$, $dv = dx \wedge dy \wedge dz$. Since L_X is an even s-derivation of $\Omega^*(M)$ we deduce

$$L_X(dx \wedge dy \wedge dz) = (L_X dx) \wedge dy \wedge dz + dx \wedge (L_X dy) \wedge dz + dx \wedge dy \wedge (L_X dz).$$

Using the computation in the previous example we get

$$L_X(dx) = dF := \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz, \;\; L_X(dy) = dG, \;\; L_X(dz) = dH$$

so that

$$L_X(dv) = \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) dv = (\mathbf{div}\, X)dv.$$

In particular, we deduce that if $\mathbf{div}\, X = 0$, the local flow generated by X preserves the form dv. We will get a better understanding of this statement once we learn integration on manifolds, later in this chapter. □

Example 3.1.13. (The exponential map on Lie groups) Consider G a Lie group. Any element $g \in G$ defines two diffeomorphisms of G: the left (L_g) and the right translation (R_g)on G. A tensor field T on G is called *left* (resp. *right*) *invariant* if for any $g \in G$ $(L_g)_*T = T$ (resp. $(R_g)_*T = T$). The set of left invariant vector fields on G is denoted by $\mathfrak{L}_G$. The naturality of the Lie bracket implies

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y]$$

so that $\forall X, Y \in \mathfrak{L}_G$, $[X, Y] \in \mathfrak{L}_G$. Hence $\mathfrak{L}_G$ is a Lie subalgebra of $\mathrm{Vect}\,(G)$. $\mathfrak{L}_G$ is called the *Lie algebra* of the group G.

FACT 1. $\dim \mathfrak{L}_G = \dim G$. Indeed, the left invariance implies that the restriction map $\mathfrak{L}_G \to T_1G$, $X \mapsto X_1$ is an isomorphism (*Exercise*). We will often find it convenient to identify the Lie algebra of G with the tangent space at 1. Any $X \in \mathfrak{L}_G$ defines a local flow Φ_X^t on G.

FACT 2. Φ_X^t is defined for all $t \in \mathbb{R}$ so that it is a flow. (**Exercise**) Set $\mathbf{exp}(tX) \stackrel{def}{=} \Phi_X^t(1)$. We thus get a map

$$\mathbf{exp} : T_1G \cong \mathfrak{L}_G \to G, \;\; X \mapsto \mathbf{exp}(X)$$

called the *exponential map of the group* G.

FACT 3. $\Phi_X^t(g) = g \cdot \mathbf{exp}(tX)$ i.e.

$$\Phi_X^t = R_{\mathbf{exp}(tX)}.$$

Indeed, it suffices to check that

$$\frac{d}{dt}\, |_{t=0}\, (g\mathbf{exp}(tX)) = X_g.$$

We can write $(g\mathbf{exp}(tX)) = L_g\mathbf{exp}(tX)$ so that

$$\begin{aligned} \frac{d}{dt}\, |_{t=0}\, (L_g\mathbf{exp}(tX)) &= (L_g)_*(\frac{d}{dt}\, |_{t=0}\, \mathbf{exp}(tX)) \\ &= (L_g)_*X = X_g \;\; \text{(by left invariance)}. \end{aligned}$$

The reason for the notation $\mathbf{exp}(tX)$ is that when $G = GL(n, \mathbb{K})$ the Lie algebra of G is the Lie algebra $\underline{gl}(n, \mathbb{K})$ of all $n \times n$ matrices with the bracket given by the commutator of two matrices (**Exercise**) and for any $X \in \mathfrak{L}_G$, $\mathbf{exp}(X) = e^X = \sum_{k \geq 0} \frac{1}{k!} X^k$ (**Exercise**). □

Exercise 3.1.4. Prove the statements left as exercises in the example above. □

Exercise 3.1.5. Let G be a matrix Lie group i.e. a Lie subgroup of some general linear group $GL(N, \mathbb{K})$. This means the tangent space T_1G can be identified with a linear space of matrices. Let $X, Y \in T_1G$ and denote by $\exp(tX)$ and $\exp(tY)$ the 1-parameter groups with they generate and set

$$g(s,t) = \exp(sX)\exp(tY)\exp(-sX)\exp(-tY).$$

(a) Show that

$$g_{s,t} = 1 + [X,Y]_{alg}st + O((s^2+t^2)^{3/2}) \ \text{ as } \ s,t \to 0$$

where the bracket $[X,Y]_{alg}$ (temporarily) denotes the commutator of the two matrices X and Y.
(b) Denote (temporarily) by $[X,Y]_{geom}$ the Lie bracket of X and Y viewed as left invariant vector fields on G. Show that at $\mathbf{1} \in G$

$$[X,Y]_{alg} = [X,Y]_{geom}.$$

(c) Show that $\underline{o}(n) \subset \underline{gl}(n,\mathbb{R})$ (defined in Section 1.2.2) is a Lie subalgebra with respect to the commutator $[\cdot\,,\,\cdot]$. Similarly, show that $\underline{u}(n)$, $\underline{su}(n) \subset \underline{gl}(n,\mathbb{C})$ are *real* Lie subalgebras of $\underline{gl}(n,\mathbb{C})$, while $\underline{su}(n,\mathbb{C})$ is even a *complex* Lie subalgebra of $\underline{gl}(n,\mathbb{C})$.
(d) Prove that we have the following isomorphisms of *real* Lie algebras. $\mathfrak{L}_{O(n)} \cong \underline{o}(n)$, $\mathfrak{L}_{U(n)} \cong \underline{u}(n)$, $\mathfrak{L}_{SU(n)} \cong \underline{su}(n)$ and $\mathfrak{L}_{SL(n,\mathbb{C})} \cong \underline{sl}(n,\mathbb{C})$. □

Remark 3.1.14. In general in a non-commutative matrix Lie group G the traditional equality

$$\exp(tX)\exp(tY) = \exp(t(X+Y))$$

no longer holds. Instead, one has the *Campbell-Hausdorff formula*

$$\exp(tX)\cdot\exp(tY) = \exp\big(\, td_1(X,Y) + t^2d_2(X,Y) + t^3d_3(X,Y) + \cdots\big)$$

where d_k are homogeneous polynomials of degree k in X and Y where the multiplication between X and Y *is given by their bracket*. The d_k's are usually known as Dynkin polynomials. For example

$$d_1(X,Y) = X+Y, \ \ d_2(X,Y) = \frac{1}{2}[X,Y], \ \ d_3(X,Y) = \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) \ \text{ etc.}$$

For more details we refer to [65]. □

3.2 Derivations of $\Omega^*(M)$

3.2.1 The exterior derivative

The super-algebra of exterior forms on a smooth manifold M has additional structure and in particular its space of derivations has special features. This section is devoted precisely to these new features.

The Lie derivative along a vector field X defines an even derivation in $\Omega^*(M)$. The vector field X also defines an odd derivation i_X (called the *interior derivative along* X or the *contraction by* X) via the contraction map

$$i_X\omega \stackrel{def}{=} \mathrm{tr}\,(X \otimes \omega), \ \ \forall \omega \in \Omega^r(M).$$

More precisely $i_X\omega$ is the $(r-1)$-form determined by

$$(i_X\omega)(X_1, \dots, X_{r-1}) = \omega(X, X_1, \dots, X_{r-1}), \ \ \forall X_1, \dots, X_{r-1} \in \mathrm{Vect}\,(M).$$

The fact that i_X is an odd s-derivation is equivalent to

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^{\deg \omega}\omega \wedge (i_X\eta), \ \ \forall \omega, \eta \in \Omega^*(M).$$

Often the contraction by X is denoted by

$$X \lrcorner\, \omega := i_X\omega.$$

Exercise 3.2.1. Prove that the interior derivative along a vector field is a s-derivation. □

Proposition 3.2.1. *(a)* $[i_X, i_Y]_s = i_X i_Y + i_Y i_X = 0$.
(b) The super-commutator of L_X *and* i_Y *as s-derivations of* $\Omega^*(M)$ *is given by*

$$[L_X, i_Y]_s = L_X i_Y - i_Y L_X = i_{[X,Y]}.$$

The proof uses the fact that the Lie derivative commutes with the contraction operator and it is left to the reader as an exercise.

The above s-derivations by no means exhaust the space of s-derivations of $\Omega^*(M)$. In fact we have the following fundamental result.

Proposition 3.2.2. *There exists an universal odd s-derivation on the s-algebra of differential forms* $\Omega^*(\,\cdot\,)$ *uniquely characterized by the following conditions.*
(a) For any smooth function $f \in \Omega^0(M)$, df *coincides with the differential of* f.
(b) $d^2 = 0$.
(c) d *is natural (universal), i.e. for any smooth function* $\phi : N \to M$ *and for any form* ω *on* M *we have*

$$d\phi^*\omega = \phi^* d\omega (\Longleftrightarrow [\phi^*, d] = 0).$$

d *is called the* exterior derivative*.*

Proof We first prove the *uniqueness*. Let U be a local coordinate chart on M^n with local coordinates $(x^1, \dots, x^n)$. Then, over U, any r-form ω can be described as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \omega_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}.$$

Since d is a s-derivation and $d(dx^i) = 0$ we deduce that over U

$$d\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq n} (d\omega_{i_1 \dots i_r}) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_r})$$

$$= \sum_{1\le i_1<\cdots<i_r\le n} \left(\frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} dx^i \right) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_r}). \qquad (3.2.1)$$

Thus the form $d\omega$ is uniquely determined on any coordinate neighborhood and this completes the proof of the uniqueness of d.

Existence To prove the existence consider an r-form ω, and for each coordinate neighborhood U we define $d\omega\,|_U$ as in (3.2.1). To prove this is a well defined operation we must show that if U, V are two coordinate neighborhoods then

$$d\omega\,|_U = d\omega\,|_V \quad \text{on } U \cap V.$$

Denote by $(x^1, \dots, x^n)$ the local coordinates on U and by $(y^1, \dots, y^n)$ the local coordinates along V so that on the overlap $U \cap V$ we can describe the y's as functions of the x's. Over U we have

$$\omega = \sum_{1\le i_1<\cdots<i_r\le n} \omega_{i_1\dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$d\omega = \sum_{1\le i_1<\cdots<i_r\le n} \left(\frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} dx^i \right) \wedge (dx^{i_1} \wedge \dots \wedge dx^{i_r}),$$

while over V we have

$$\omega = \sum_{1\le j_1<\cdots<j_r\le n} \hat{\omega}_{j_1\dots j_r} dy^{j_1} \wedge \cdots \wedge dy^{j_r}$$

$$d\omega = \sum_{1\le j_1<\cdots<j_r\le n} \left(\frac{\partial \hat{\omega}_{j_1\dots j_r}}{\partial y^j} dy^j \right) (dy^{j_1} \wedge \cdots \wedge dy^{j_r}).$$

The components $\omega_{i_1\dots i_r}$ and $\hat{\omega}_{j_1\dots j_r}$ are skew-symmetric, i.e. $\forall \sigma \in \mathfrak{S}_r$

$$\omega_{i_{\sigma(1)}\dots i_{\sigma(r)}} = \epsilon(\sigma)\omega_{i_1\dots i_r},$$

and similarly for the $\hat{\omega}'$s. Since $\omega\,|_U = \omega\,|_V$ over $U \cap V$ we deduce

$$\omega_{i_1\dots i_r} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \hat{\omega}_{j_1\dots j_r}.$$

Hence

$$\frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} = \sum_{k=1}^r \left(\frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial^2 y^{j_k}}{\partial x^i \partial x^{i_k}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \hat{\omega}_{j_1\dots j_r} + \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \frac{\partial \hat{\omega}_{j_1\dots j_r}}{\partial x^i} \right),$$

where in the above equality we also sum over the indices $j_1, \dots, j_r$ according to Einstein's convention. We deduce

$$\sum_{1\le i_1<\cdots<i_r\le n} \frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$= \sum_i \sum_{k=1}^r \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial^2 x^{j_k}}{\partial x^i \partial x^{i_k}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \hat{\omega}_{j_1\dots j_r} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$+\sum_i\sum_{k=1}^r \frac{\partial y^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{j_r}}{\partial x^{i_r}}\frac{\partial\hat{\omega}_{j_1\dots j_r}}{\partial x^i}dx^i\wedge dx^{i_1}\wedge\dots\wedge dx^{i_r}. \tag{3.2.2}$$

Notice that

$$\frac{\partial^2}{\partial x^i\partial x^{i_k}}=\frac{\partial^2}{\partial x^{i_k}\partial x^i}$$

while $dx^i\wedge dx^{i_k}=-dx^{i_k}\wedge dx^i$ so that the first term in the right hand side of (3.2.2) vanishes. Consequently on $U\cap V$

$$\frac{\partial\omega_{i_1\dots i_r}}{\partial x^i}dx^i\wedge dx^{i_1}\cdots\wedge dx^{i_r}=\frac{\partial y^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{j_r}}{\partial x^{i_r}}\frac{\partial\hat{\omega}_{j_1\dots j_r}}{\partial x^i}\wedge dx^i\wedge dx^{i_1}\cdots dx^{i_r}$$

$$=\left(\frac{\partial\hat{\omega}_{j_1\dots j_r}}{\partial x^i}dx^i\right)\wedge\left(\frac{\partial y^{j_1}}{\partial x^{i_1}}dx^{i_1}\right)\wedge\cdots\wedge\left(\frac{\partial y^{j_r}}{\partial x^{i_r}}dx^{i_r}\right)$$

$$=(d\hat{\omega}_{j_1\dots j_r})\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_r}$$

$$=\frac{\partial\hat{\omega}_{j_1\dots j_r}}{\partial y^j}dy^j\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_r}.$$

This proves $d\omega\mid_U=d\omega\mid_V$ over $U\cap V$. We have thus constructed a well defined linear map

$$d:\Omega^*(M)\to\Omega^{*+1}(M).$$

To prove that d is an odd s-derivation it suffices to work in local coordinates and show the (super)product rule on monomials. Thus, let $\theta=fdx^{i_1}\wedge\cdots\wedge dx^{i_r}$ and $\omega=gdx^{j_1}\wedge\cdots\wedge dx^{j_s}$. We set for simplicity $dx^I=dx^{i_1}\wedge\cdots\wedge dx^{i_r}$ and $dx^J=dx^{j_1}\wedge\cdots\wedge dx^{j_s}$. Then

$$d(\theta\wedge\omega)=d(fgdx^I\wedge dx^J)=d(fg)\wedge dx^I\wedge dx^J$$

$$=(df\cdot g+f\cdot dg)\wedge dx^I\wedge dx^J$$

$$=df\wedge dx^I\wedge dx^J+(-1)^r(f\wedge dx^I)\wedge(dg\wedge dx^J)$$

$$=d\theta\wedge\omega+(-1)^{\deg\theta}\theta\wedge d\omega.$$

We now prove $d^2=0$. We check this on monomials fdx^I as above.

$$d^2(fdx^I)=d(df\wedge dx^I)=(d^2f)\wedge dx^I.$$

Thus it suffices to show $d^2f=0$ for all smooth functions f. We have

$$d^2f=\frac{\partial f^2}{\partial x^i\partial x^j}dx^i\wedge dx^j.$$

The desired conclusion follows from the fact that $\frac{\partial f^2}{\partial x^i\partial x^j}=\frac{\partial f^2}{\partial x^j\partial x^i}$ while $dx^i\wedge dx^j=-dx^j\wedge dx^i$.

Finally, let ϕ be a smooth map $N\to M$ and $\omega=\sum_I\omega_Idx^I$ be an r-form on M. Here I runs through all ordered multi-indices $1\le i_1<\cdots<i_r\le\dim M$. We have

$$d_N(\phi^*\omega)=\sum_I\left(d_N(\phi^*\omega_I)\wedge\phi^*(dx^I)+\phi^*\omega^I\wedge d(\phi^*dx^I)\right)$$

For functions, the usual chain rule gives $d_N(\phi^*\omega_I) = \phi^*(d_M\omega^I)$. In terms of local coordinates (x^i) the map ϕ looks like a collection of n functions $\phi^i \in C^\infty(N)$ and we get

$$\phi^*(dx^I) = d\phi^I = d_N\phi^{i_1} \wedge \cdots \wedge d_N\phi^{i_r}.$$

In particular, $d_N(d\phi^I) = 0$. We put all the above together and we deduce

$$d_N(\phi^*\omega) = \phi^*(d_M\omega^I) \wedge d\phi^I = \phi^*(d_M\omega^I) \wedge \phi^* dx^I = \phi^*(d_M\omega).$$

The proposition is proved.

□

Proposition 3.2.3. The exterior derivative satisfies the following relations ($[\ ,\ ]_s$ denotes the super-commutator in the s-algebra of real endomorphisms of $\Omega^*(M)$).
(a) $[d, d]_s = 2d^2 = 0$.
(b) **(Homotopy formula)** $[d, i_X]_s = di_X + i_X d = L_X$ $\forall X \in \mathrm{Vect}\,(M)$.
(c) $[d, L_X]_s = dL_X - L_X d = 0$, $\forall X \in \mathrm{Vect}\,(M)$.

An immediate consequence of the homotopy formula is the following invariant description of the exterior derivative:

$$(d\omega)(X_0, X_1, \ldots, X_r) = \sum_{i=0}^{r} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r))$$

$$+ \sum_{0 \le i < j \le r} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r). \tag{3.2.3}$$

Above, the hat indicates that the corresponding entry is missing.

Proof To prove the homotopy formula set $\mathcal{D} = [d, i_X]_s = di_X + i_X d$. $\mathcal{D}$ is an even s-derivation of $\Omega^*(M)$. It is a local s-derivation i.e. if $\omega \in \Omega^*(M)$ vanishes on some open set U then $\mathcal{D}\omega$ vanishes on that open set as well. The reader can check easily by direct computation that $\mathcal{D}\omega = L_X\omega$, $\forall \omega \in \Omega^0(M) \oplus \Omega^1(M)$. The homotopy formula is now a consequence of the following technical result left to the reader as an exercise.

Lemma 3.2.4. *Let $\mathcal{D}$, $\mathcal{D}'$ be two local s-derivations of $\Omega^*(M)$ which have the same parity (i.e. they are either both even or both odd). If $\mathcal{D} = \mathcal{D}'$ on $\Omega^0(M) \oplus \Omega^1(M)$ then $\mathcal{D} = \mathcal{D}'$ on $\Omega^*(M)$.*

Part (c) of the proposition is proved in a similar way. Equality (3.2.3) is a simple consequence of the homotopy formula. We prove it in two special case $r = 1$ and $r = 2$.

The case $r = 1$. Let ω be an 1-form and let $X, Y \in \mathrm{Vect}\,(M)$. We deduce from the homotopy formula

$$d\omega(X, Y) = (i_X d\omega)(Y) = (L_X\omega)(Y) - (d\omega(X))(Y).$$

On the other hand, since L_X commutes with the contraction operator, we deduce

$$X\omega(Y) = L_X(\omega(Y)) = (L_X\omega)(Y) + \omega([X, Y]).$$

Hence

$$d\omega(X,Y) = X\omega(Y) - \omega([X,Y]) - (d\omega(X))(Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

This proves (3.2.3) in the case $r = 1$.

The case $r = 2$. Consider a 2-form ω and three vector fields X, Y and Z. We deduce from the homotopy formula

$$(d\omega)(X,Y,Z) = (i_X d\omega)(Y,Z) = (L_X - di_X)\omega(Y,Z). \tag{3.2.4}$$

Since L_X commutes with contractions we deduce

$$(L_X\omega)(Y,X) = X(\omega(Y,Z)) - \omega([X,Y],Z) - \omega(Y,[X,Z]). \tag{3.2.5}$$

We substitute (3.2.5) into (3.2.4) and we get

$$(d\omega)(X,Y,Z) = X(\omega(Y,Z)) - \omega([X,Y],Z) - \omega(Y,[X,Z]) - d(i_X\omega)(Y,X). \tag{3.2.6}$$

We apply now (3.2.3) for $r = 1$ to the 1-form $i_X\omega$. We get

$$\begin{aligned} d(i_X\omega)(Y,X) &= Y(i_X\omega(Z)) - Z(i_X\omega(Y)) - (i_X\omega)([Y,Z]) \\ &= Y\omega(X,Z) - Z\omega(X,Y) - \omega(X,[Y,Z]). \end{aligned} \tag{3.2.7}$$

If we use (3.2.7) in (3.2.6) we deduce

$$\begin{aligned} (d\omega)(X,Y,Z) &= X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) \\ &-\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X). \end{aligned} \tag{3.2.8}$$

The general case in (3.2.3) can be proved by induction. The proof of the proposition is complete.

□

Exercise 3.2.2. Prove Lemma 3.2.4. □

Exercise 3.2.3. Finish the proof of (3.2.3) in the general case. □

3.2.2 Examples

Example 3.2.5. (The exterior derivative in $\mathbb{R}^3$)
(a) Let $f \in C^\infty(\mathbb{R}^3)$ then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

df looks like the gradient of f.
(b) Let $\omega \in \Omega^1(\mathbb{R}^3)$, $\omega = Pdx + Qdy + Rdz$. Then

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx$$

so that $d\omega$ looks very much like a curl.
(c) Let $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \in \Omega^2(\mathbb{R}^3)$. Then

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz.$$

This looks very much like a divergence. □

Example 3.2.6. Let G be a connected Lie group. In Example 3.1.11 we defined the Lie algebra $\mathfrak{L}_G$ of G as the space of left invariant vector fields on G. Set

$$\Omega^r_{left}(G) = \text{left invariant } r\text{-forms on } G.$$

In particular $\mathfrak{L}_G^* \cong \Omega^1_{left}(G)$. If we identify $\mathfrak{L}_G^* \cong T_1^* G$ then we get a natural isomorphism

$$\Omega^r_{left}(G) \cong \Lambda^r \mathfrak{L}_G^*.$$

The exterior derivative of a form in Ω^*_{left} can be described *only in terms of the algebraic structure of* $\mathfrak{L}_G$.

Indeed, let $\omega \in \mathfrak{L}_G^* = \Omega^1_{left}(G)$. For $X, Y \in \mathfrak{L}_G^*$ we have (see (3.2.3))

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Since ω, X and Y are left invariant $\omega(X)$ and $\omega(Y)$ are constants. Thus, the first two terms in the above equality vanish so that

$$d\omega(X, Y) = -\omega([X, Y]).$$

More generally, if $\omega \in \Omega^r_{left}$ then the same arguments applied to (3.2.3) imply that for all $X_0, ..., X_r \in \mathfrak{L}_G$ we have

$$d\omega(X_0, X_1, ..., X_r) = \sum_{0 \le i < j \le r} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_r). \tag{3.2.9}$$

□

3.3 Connections on vector bundles

3.3.1 Covariant derivatives

We learned several methods of differentiating tensor objects on manifolds. The tensor bundles are not the only vector bundles arising in geometry and very often one is interested in measuring the "oscillations" of sections of vector bundles.

Let E be a $\mathbb{K}$-vector bundle over the smooth manifold M ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). For such an arbitrary E we encounter a problem which was not present in the case of tensor bundles. Namely, the local flow generated by a vector field X on M no longer induces bundle homomorphisms. For tensor fields the transport along a flow was a method of comparing objects in different fibers which otherwise are abstract linear spaces with no natural relationship between them.

To obtain something that looks like a derivation we need to formulate clearly what properties we should expect from such an operation.
(a) It should measure how fast is a given section changing along a direction given by a vector field X. Hence it has to be an operator

$$\nabla : \mathrm{Vect}\,(M) \times C^\infty(E) \to C^\infty(E), \quad (X, u) \mapsto \nabla_X u.$$

(b) If we think of the usual directional derivative, we expect that after "rescaling" the direction X the derivative along X should only rescale by the same factor i.e.

$$\forall f \in C^\infty(M) : \quad \nabla_{fX} u = f \nabla_X u.$$

(c) Since ∇ is to be a derivation it has to satisfy a sort of (Leibniz) product rule. The only product that exists on an abstract vector bundle is the multiplication of a section with a smooth function. Hence we require

$$\nabla_X(fu) = (Xf)u + f\nabla_X u, \;\; \forall f \in C^\infty(M), \; u \in C^\infty(E).$$

Conditions (a) and (b) can be rephrased as follows: for any $u \in C^\infty(E)$ the map

$$\nabla u : \mathrm{Vect}\,(M) \to C^\infty(E), \quad X \mapsto \nabla_X u$$

is $C^\infty(M)$ linear so that it defines a bundle map

$$\nabla u \in C^\infty(\mathrm{Hom}\,(TM, E)) \cong C^\infty(T^*M \otimes E).$$

Summarizing, we can formulate the following definition.

Definition 3.3.1. *A* covariant derivative *(or* linear connection*) on E is a $\mathbb{K}$-linear map*

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$$

such that $\forall f \in C^\infty(M)$ and $\forall u \in C^\infty(E)$.

$$\nabla(fu) = df \otimes u + f\nabla u.$$

Example 3.3.2. Let $\underline{\mathbb{K}}_M \cong M \times \mathbb{K}^r$ be the rank r trivial bundle over M. The space $C^\infty(\underline{\mathbb{K}}_M)$ of smooth sections coincides with $C^\infty(M, \mathbb{K}^r)$. We can define

$$\nabla^0 : C^\infty(M, \mathbb{K}^r \to C^\infty(M, T^*M \otimes \mathbb{K}^r)$$

$$\nabla^0(f_1, ..., f_r) = (df_1, ..., df_r).$$

One checks easily that ∇ is a connection. This is called the *trivial connection.* □

Remark 3.3.3. Let ∇^0, ∇^1 be two connections on a vector bundle $E \to M$. Then for any $\alpha \in C^\infty(M)$ the map

$$\nabla = \alpha\nabla^1 + (1 - \alpha)\nabla^0 : C^\infty(E) \to C^\infty(T^* \otimes E)$$

is again a connection. □

Notation For any vector bundle F over M we set

$$\Omega^k(F) \stackrel{def}{=} C^\infty(\Lambda^k T^*M \otimes F).$$

Proposition 3.3.4. *Let E be a vector bundle. The space $\mathcal{A}(E)$ of linear connections on E is an affine space modeled on $\Omega^1(\mathrm{End}\,(E))$.*

Proof We first show $\mathcal{A}\,(E)$ is not empty. To see this, choose $\{U_\alpha\}$ an open cover of M such that $E|_{U_\alpha}$ is trivial, $\forall\alpha$. Next, pick (μ_β) a smooth partition of unity subordinated to this cover. Since $E|_{U_\alpha}$ is trivial it admits at least one connection, the trivial one, as in the above example. Denote such a connection by ∇^α. Now define

$$\nabla = \sum_{\alpha,\beta} \mu_\beta \nabla^\alpha.$$

One checks easily that ∇ is a connection so that $\mathcal{A}(E)$ is nonempty. To check that $\mathcal{A}\,(E)$ is an affine space consider two connections ∇^0 and ∇^1. Their difference $A = \nabla^1 - \nabla^0$ is an operator

$$A : C^\infty(E) \to C^\infty(T^*M \otimes E)$$

satisfying $A(fu) = fA(u)$, $\forall u \in C^\infty(E)$. Thus

$$A \in C^\infty(\mathrm{Hom}\,(E, T^*M \otimes E)) \cong C^\infty(T^*M \otimes E^* \otimes E) \cong \Omega^1(E^* \otimes E) \cong \Omega^1(\mathrm{End}E).$$

Conversely, given $\nabla^0 \in \mathcal{A}(E)$ and $A \in \Omega^1(\mathrm{End}E)$ one can verify that the operator

$$\nabla^A = \nabla^0 + A : C^\infty(E) \to \Omega^1(E).$$

is a linear connection. This concludes the proof of the proposition.

□

The tensorial operations on vector bundles extend naturally to *vector bundles with connections*. The key principle behind this fact is the Leibniz' formula. More precisely if E_i $(i = 1, 2)$ are two bundles with connections ∇^i then $E_1 \otimes E_2$ has a naturally induced connection

$$\nabla^{E_1\otimes E_2}(u_1 \otimes u_2) = (\nabla^1 u_1) \otimes u_2 + u_1 \otimes \nabla^2 u_2.$$

The dual bundle E_1^* has a natural connection ∇^* defined by the identity

$$X\langle v, u\rangle = \langle \nabla^*_X v, u\rangle + \langle v, \nabla^1_X u\rangle, \quad \forall u \in C^\infty(E_1),\ v \in C^\infty(E_1^*),\ \ X \in \mathrm{Vect}\,(M)$$

where

$$\langle \bullet, \bullet\rangle : C^\infty(E_1^*) \times C^\infty(E_1) \to C^\infty(M)$$

is the pairing induced by the natural duality between the fibers of E_1^* and E_1. In particular, any connection ∇^E on a vector bundle E induces a connection $\nabla^{End(E)}$ on $\mathrm{End}\,(E) \cong E^* \otimes E$ by

$$(\nabla^{End(E)} T)(u) = \nabla^E(Tu) - T(\nabla^E u) = [\nabla^E, T]u \tag{3.3.1}$$

$\forall T \in \mathrm{End}\,(E)\ \ u \in C^\infty(E)$.

It is often useful to have a local description of a covariant derivative. This can be obtained using Cartan's moving frame method.

Let $E \to M$ be a rank r $\mathbb{K}$-vector bundle over the smooth manifold M. Pick a coordinate neighborhood U such $E|_U$ is trivial. A moving frame is a a bundle isomorphism $\phi : U \times \mathbb{K}^r \to E|_U$. (A moving frame is what physicists call a choice of local gauge). Consider the sections $e_\alpha = \phi(\delta_\alpha)$, $\alpha = 1, ..., r$, where δ_α are the natural basic sections of $U \times \mathbb{K}^r$. As x moves in U, the collection $(e_1(x), ..., e_r(x))$ describes a basis of the moving fiber E_x and thus the terminology moving frame. A section $u \in C^\infty(E|_U)$ can be written as a linear combination

$$u = u^\alpha e_\alpha \quad u^\alpha \in C^\infty(U, \mathbb{K}).$$

Hence if ∇ is a covariant derivative in E we have

$$\nabla u = du^\alpha \otimes e_\alpha + u^\alpha \nabla e_\alpha.$$

Thus, the covariant derivative is completely described by its action on a moving frame. To get a more concrete description pick local coordinates (x^i) over U. $\nabla e_\alpha \in \Omega^1(E|_U)$ so that we can write

$$\nabla e_\alpha = \Gamma^\beta_{i\alpha} dx^i \otimes e_\beta, \;\; \Gamma^\beta_{i\alpha} \in C^\infty(U, \mathbb{K}).$$

Thus, for any section $u^\alpha e_\alpha$ of $E|_U$ we have

$$\nabla u = du^\alpha \otimes e_\alpha + \Gamma^\beta_{i\alpha} u^\alpha dx^i \otimes e_\beta. \tag{3.3.2}$$

It is convenient to view $\left(\Gamma^\beta_{i\alpha}\right)$ as an $r \times r$-matrix valued 1-form and we write this as

$$\left(\Gamma^\beta_{i\alpha}\right) = dx^i \otimes \Gamma_i.$$

The form $\Gamma = dx^i \otimes \Gamma_i$ is called the *connection 1-form* associated to the choice of local gauge. A moving frame allows one to identify sections of $E|_U$ with $\mathbb{K}^r$-valued functions on U and we can rewrite (3.3.2) as

$$\nabla u = du + \Gamma u. \tag{3.3.3}$$

A natural question arises: how does the connection 1-form changes with the change of the local gauge?

Let $\mathbf{f} = (f_\alpha)$ be another moving frame of $E|_U$. The correspondence $e_\alpha \mapsto f_\alpha$ defines an automorphism of $E|_U$. Using the local frame $\mathbf{e}$ we can identify it with a smooth map $g : U \to GL(r; \mathbb{K})$. g is called *the local gauge transformation* relating $\mathbf{e}$ to $\mathbf{f}$.

Let $\hat{\Gamma}$ denote the connection 1-form corresponding to the new moving frame i.e.

$$\nabla f_\alpha = \hat{\Gamma}^\beta_\alpha f_\beta.$$

Consider σ a section of $E|_U$. With respect to the local frame (e_α) it has the decomposition

$$\sigma = u^\alpha e_\alpha$$

while with respect to (f_β) it has a decomposition

$$\sigma = \hat{u}^\beta f_\beta.$$

The two decompositions are related by

$$u = g\hat{u}. \tag{3.3.4}$$

Now, we can identify the E-valued 1-form $\nabla\sigma$ with a $\mathbb{K}^r$-valued 1-form in two ways: either using the frame $\mathbf{e}$ or using the frame $\mathbf{f}$. In the first case $\nabla\sigma$ is identified with the $\mathbb{K}^r$-valued 1-form

$$du + \Gamma u$$

while in the second case it is identified with

$$d\hat{u} + \hat{\Gamma}\hat{u}.$$

These two identifications are related by the same rule as in (3.3.4):

$$du + \Gamma u = g(d\hat{u} + \hat{\Gamma}\hat{u}).$$

Using (3.3.4) in the above equality we get

$$(dg)\hat{u} + gd\hat{u} + \Gamma g\hat{u} = gd\hat{u} + g\hat{\Gamma}\hat{u}.$$

Hence

$$\hat{\Gamma} = g^{-1}dg + g^{-1}\Gamma g.$$

The above relation is the transition rule relating two local gauge descriptions of the same connection. The above argument can be reversed producing the following global result.

Proposition 3.3.5. *Let $E \to M$ be a rank r smooth vector bundle and (U_α) a trivializing cover with transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r;\mathbb{K})$. Then any collection of matrix valued 1-forms $\Gamma_\alpha \in \Omega^1(\mathrm{End}, \underline{\mathbb{K}}^r_{U_\alpha})$ satisfying*

$$\Gamma_\beta = (g_{\alpha\beta}^{-1}dg_{\alpha\beta}) + g_{\alpha\beta}^{-1}\Gamma^\alpha g_{\alpha\beta} = -(dg_{\beta\alpha})g_{\beta\alpha}^{-1} + g_{\beta\alpha}\Gamma_\alpha g_{\beta\alpha}^{-1} \quad \text{over } U_\alpha \cap U_\beta$$

uniquely defines a covariant derivative on E.

Exercise 3.3.1. Prove the above proposition. □

A word of warning. The identification

$$\{moving\ frames\} \cong \{local\ trivialization\}$$

should be treated carefully. These are like an object and its image in a mirror and there is a great chance of confusing the right hand with the left hand. More concretely, if $t_\alpha : E_\alpha \cong U_\alpha \times \mathbb{K}^r$ (resp. $t_\beta : E_\beta \cong U_\beta \times \mathbb{K}^r$) is a trivialization of a bundle E over an open set U_α (resp. U_β) then the transition map over $U_\alpha \cap U_\beta$ "from α to β" is $g_{\beta\alpha} = t_\beta \circ t_\alpha^{-1}$. The standard basis in $\mathbb{K}^r$, denoted by (δ_i) induces two local moving frames on E:

$$\mathbf{e}_\alpha = t_\alpha^{-1}(\delta_\cdot) \text{ and } \mathbf{e}_\beta = t_\beta^{-1}(\delta_\cdot).$$

On the overlap $U_\alpha \cap U_\beta$ these two frames are related by the local gauge transformation

$$\mathbf{e}_\beta = g_{\beta\alpha}^{-1}\mathbf{e}_\alpha.$$

This is precisely the opposite way the two trivializations are identified.

Example 3.3.6. (Complex line bundles) Let $L \to M$ be a complex line bundle over the smooth manifold M. Let $\{U_\alpha\}$ be a trivializing cover with transition maps $z_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* = GL(1, \mathbb{C})$. The bundle of endomorphisms of L, End $(L) \cong L^* \otimes L$ is trivial since it can be defined by transition maps $(z_{\alpha\beta})^{-1} \otimes z_{\alpha\beta} = 1$. Thus the space of connections on L, $\mathcal{A}(L)$ is an affine space modeled by the linear space of complex valued 1-forms. A connection on L is simply a collection of $\mathbb{C}$-valued 1-forms ω^α on U_α related on overlaps by

$$\omega^\beta = \frac{dz_{\alpha\beta}}{z_{\alpha\beta}} + \omega^\alpha = d \ln z_{\alpha\beta} + \omega^\alpha.$$

□

3.3.2 Parallel transport

As we have already pointed out, one reason we could not construct natural derivations on the space of sections of a vector bundle was the lack of a canonical procedure of identifying fibers at different points. We will see in this subsection that such a procedure is all we need to define covariant derivatives. More precisely, we will show that once a covariant derivative is chosen, it offers a simple way of identifying different fibers.

Let $E \to M$ be a rank r $\mathbb{K}$-vector bundle and ∇ a covariant derivative on E. For any smooth path $\gamma : [0, 1] \to M$ we will define a *linear* isomorphism $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ called the *parallel transport* along γ. More exactly, we will construct an entire family of linear isomorphisms

$$T_t : E_{\gamma(0)} \to E_{\gamma(t)}.$$

One should think of this T_t as identifying different fibers. In particular, if $u_0 \in E_{\gamma(0)}$ then $t \mapsto u_t = T_t u_0 \in E_{\gamma(t)}$ should be thought of as a "constant" path. The rigorous way of stating this "constancy" is via derivations: a quantity is "constant" if its derivatives are identically 0. Now, the only way we know how to derivate sections is via ∇ i.e. u_t should satisfy

$$\nabla_{\frac{d}{dt}} u_t = 0, \quad \text{where } \frac{d}{dt} = \dot{\gamma}.$$

The above equation offers a way of defining T_t. For each $u_0 \in E_{\gamma(0)}$ and $t \in [0, 1]$ define $T_t u_0$ as the value at t of the initial value problem

$$\left\{ \begin{array}{lcl} \nabla_{\frac{d}{dt}} u(t) & = & 0 \\ u(0) & = & u_0 \end{array} \right. . \tag{3.3.5}$$

The equation (3.3.5) is a linear ordinary differential equation in disguise.

To see this let us make the simplifying assumption that $\gamma(t)$ lies entirely in some coordinate neighborhood U with coordinates $(x^1, ..., x^n)$, such that $E\,|_U$ is trivial. This is always happening at least on every small portion of γ. Denote by $(e_\alpha)_{1 \le \alpha \le r}$ a local moving frame trivializing $E\,|_U$ so that $u = u^\alpha e_\alpha$. The connection 1-form corresponding to this moving frame will be denoted by $\Gamma \in \Omega^1(\text{End}\,(\underline{\mathbb{K}}^r))$. Equation (3.3.5) becomes

$$\left\{ \begin{array}{lclcl} \frac{du^\alpha}{dt} & + & \Gamma^\alpha_{t\beta} u^\beta & = & 0 \\ u^\alpha(0) & & & = & u^\alpha_0 \end{array} \right.$$

where

$$\Gamma_t = \frac{d}{dt} \lrcorner \Gamma \in \Omega^0(\mathrm{End}\,(\underline{\mathbb{K}}^r)) = \mathrm{End}\,(\underline{\mathbb{K}}^r).$$

This is obviously a linear ordinary differential equation whose solutions exist for any t. We deduce

$$\dot{u}(0) = -\Gamma_t u_0. \tag{3.3.6}$$

This gives a geometric interpretation of the connection 1-form Γ: for any vector field X the contraction $-i_X\Gamma = -\Gamma(X) \in \mathrm{End}\,(E)$ describes the infinitesimal parallel transport along the direction prescribed by the vector field X, in the non-canonical identification of nearby fibers via a local moving frame.

In more intuitive terms, if $\gamma(t)$ is an integral curve for X and T_t denotes the parallel transport along γ from $E_{\gamma(0)}$ to $E_{\gamma(t)}$ then, given a local moving frame for E in a neighborhood of $\gamma(0)$, T_t is identified with a t-dependent matrix which has a Taylor expansion of the form

$$T_t = \mathbf{id} - \Gamma_0 t + O(t^2), \quad \text{t small} \tag{3.3.7}$$

with $\Gamma_0 = (X \lrcorner \Gamma)\,|_{\gamma(0)}$.

3.3.3 The curvature of a connection

Consider $E \to M$ a rank r smooth vector bundle over the smooth manifold M and let ∇ be a covariant derivative on E:

$$\nabla : \Omega^0(E) \to \Omega^1(E).$$

Proposition 3.3.7. ∇ *has an extension to an operator*

$$d^\nabla : \Omega^r(E) \to \Omega^{r+1}(E)$$

uniquely defined by the requirements:
(a) $d^\nabla\,|_{\Omega^0(E)} = \nabla$*;*
(b)$\forall \omega \in \Omega^r(M), \;\; \eta \in \Omega^s(E)$

$$d^\nabla(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d^\nabla \eta.$$

Brief outline of the proof *Existence* For $\omega \in \Omega^r(M), \;\; u \in \Omega^0(E)$ set

$$d^\nabla(\omega \otimes u) = d\omega \otimes u + (-1)^r \omega \nabla u. \tag{3.3.8}$$

Using a partition of unity one shows that any $\eta \in \Omega^r(E)$ is a locally finite combination of monomials as above so the above definition induces an operator $\Omega^r(E) \to \Omega^{r+1}(E)$. We let the reader check that this extension satisfies conditions (a) and (b) above.
Uniqueness Any operator with the properties (a) and (b) acts on monomials as in (3.3.8) so it has to coincide with the operator described above using a given partition of unity.

□

Example 3.3.8. The trivial bundle $\underline{\mathbb{K}}_M$ has a natural connection ∇^0- the trivial connection. This coincides with the usual differential $d : \Omega^0(M) \otimes \mathbb{K} \to \Omega^1(M) \otimes \mathbb{K}$. d^{∇^0} is the usual exterior derivative. □

There is a major difference between the usual exterior derivative d and an arbitrary d^∇. In the former case we have $d^2 = 0$ which is a consequence of the commutativity $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, where (x^i) are local coordinates on M. In the second case, the equality $(d^\nabla)^2 = 0$ does not hold in general. Still, something very interesting happens.

Lemma 3.3.9. *For any smooth function $f \in C^\infty(M)$ and any $\omega \in \Omega^r(E)$ we have*

$$(d^\nabla)^2(f\omega) = f\{(d^\nabla)^2)\omega\}.$$

Hence $(d^\nabla)^2$ is a bundle morphism $\Lambda^r T^*M \otimes E \to \Lambda^{r+2}T^*M \otimes E$.

Proof We compute

$$(d^\nabla)^2(f\omega) = d^\nabla(df \wedge \omega + f d^\nabla \omega)$$
$$= -df \wedge d^\nabla\omega + df \wedge d^\nabla\omega + f(d^\nabla)^2\omega = f(d^\nabla)^2\omega.$$

□

As a map $\Omega^0(E) \to \Omega^2(E)$, $(d^\nabla)^2$ can be identified with a section of

$$\mathrm{Hom}\,(E, \Lambda^2 T^*M \otimes E) \cong E^* \otimes \Lambda^2 T^*M \otimes E \cong \Lambda^2 T^*M \otimes \mathrm{End}\,(E).$$

Thus $(d^\nabla)^2$ is an $\mathrm{End}\,(E)$-valued 2-form.

Definition 3.3.10. *For any connection ∇ on a smooth vector bundle $E \to M$ the object $(d^\nabla)^2 \in \Omega^2(\mathrm{End}\,(E))$ is called the* curvature *of ∇ and is usually denoted by $F(\nabla)$.*

Example 3.3.11. Consider the trivial bundle $\underline{\mathbb{K}}^r_M$. The sections of this bundle are smooth $\mathbb{K}^r$-valued functions on M. The exterior derivative d defines the trivial connection on $\underline{\mathbb{K}}^r_M$ and any other connection differs from d by a $M_r(\mathbb{K})$-valued 1-form on M. If A is such a form then the curvature of the connection $d + A$ denoted by $F(A)$ is

$$F(A)u = (d + A)^2 u = (dA + A \wedge A)u, \quad \forall u \in C^\infty(M, \mathbb{K}^r).$$

The $\wedge$ operation above is defined for any vector bundle E as the bilinear map

$$\Omega^r(\mathrm{End}\,(E)) \times \Omega^s(\mathrm{End}\,(E)) \to \Omega^{r+s}(\mathrm{End}\,(E))$$

uniquely defined by

$$(\omega^r \otimes A) \wedge (\eta^s \otimes B) = \omega^r \wedge \eta^s \otimes AB, \;\; A, B \in \mathrm{End}\,(E).$$

□

We conclude this subsection with an alternate description of the curvature which hopefully will shed some light on its analytical significance.

Let $E \to M$ be a smooth vector bundle on M and ∇ a connection on it. Denote its curvature by $F = F(\nabla) \in \Omega^2(\mathrm{End}\,(E))$. For any $X, Y \in \mathrm{Vect}\,(M)$ the quantity $F(X, Y)$ is an endomorphism of E. In the remaining part of this section we will give a different description of this endomorphism.

For any vector field Z denote by $i_Z : \Omega^r(E) \to \Omega^{r-1}(E)$ the $C^\infty(M) - \mathit{linear}$ operator defined by

$$i_Z(\omega \otimes u) = (i_Z\omega) \otimes u, \;\; \forall \omega \in \Omega^r(M), \; u \in \Omega^0(E).$$

The covariant derivative ∇_Z extends naturally to elements of $\Omega^r(E)$ by

$$\nabla_Z(\omega \otimes u) = (L_Z\omega) \otimes u + \omega \otimes \nabla_Z u.$$

The operators $d^\nabla, \; i_Z, \; \nabla_Z$ satisfy the usual super-commutation identities.

$$i_Z d^\nabla + d^\nabla i_Z = \nabla_Z. \tag{3.3.9}$$

$$i_X i_Y + i_Y i_X = 0. \tag{3.3.10}$$

$$\nabla_X i_Y - i_Y \nabla_X = i_{[X,Y]}. \tag{3.3.11}$$

For any $u \in \Omega^0(E)$ we compute using (3.3.9)-(3.3.11)

$$F(X,Y)u = i_Y i_X (d^\nabla)^2 u = i_Y(i_X d^\nabla)\nabla u$$

$$= i_Y(\nabla_X - d^\nabla i_X)\nabla u = (i_Y\nabla_X)\nabla u - (i_Y d^\nabla)\nabla_X u$$

$$= (\nabla_X i_Y - i_{[X,Y]})\nabla u - \nabla_Y\nabla_X u$$

$$= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})u.$$

Hence

$$F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{3.3.12}$$

If in the above formula we take $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$ where (x^i) are local coordinates on M we deduce ($\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}, \; \nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$)

$$F_{ij} = -F_{ji} = F(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = [\nabla_i, \nabla_j]. \tag{3.3.13}$$

Thus F_{ij} measures the extent to which the partial derivatives $\nabla_i, \; \nabla_j$ fail to commute. This is in sharp contrast with the classical calculus and an analytically oriented reader may object to this by saying we were careless when we picked the connection. Maybe an intelligent choice will restore the classical commutativity of partial derivatives so we should concentrate from the very beginning to covariant derivatives ∇ such that $F(\nabla) = 0$.

Definition 3.3.12. *A connection* ∇ *such that* $F(\nabla) = 0$ *is called* flat.

A natural question arises: given an arbitrary vector bundle $E \to M$ do there exist flat connections on E? If E is trivial then the answer is obviously positive. In general the answer is negative and this has to do with the *global structure* of the bundle. In the second half of this book we will understand the motivation behind this fact.

3.3.4 Holonomy

The reader may ask a very legitimate question: why have we chosen to name curvature, the deviation from commutativity of a given connection. In this subsection we describe the geometric meaning of curvature and maybe this will explain the terminology.

Let $E \to M$ be a smooth vector bundle and ∇ a connection on it. Consider $(x^1, ..., x^n)$ local coordinates on an open subset $U \subset M$ such that $E\mid_U$ is trivial. Pick $(e_1, ..., e_r)$ $(r = \operatorname{rank} E)$ a moving frame over U. The connection 1-form associated to this moving frame is

$$\Gamma = \Gamma_i dx^i = (\Gamma^\alpha_{i\beta})dx^i, \;\; 1 \le \alpha, \beta \le r.$$

It is defined by the equalities ($\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}$)

$$\nabla_i e_\beta = \Gamma^\alpha_{i\beta} e_\alpha. \tag{3.3.14}$$

Using (3.3.13) we compute

$$\begin{aligned} F_{ij}e_\beta &= (\nabla_i\nabla_j - \nabla_j\nabla_i)e_\beta \\ &= \nabla_i(\Gamma_j e_\beta) - \nabla_j(\Gamma_i e_\beta) \\ = \left(\frac{\partial\Gamma^\alpha_{j\beta}}{\partial x^i} - \frac{\partial\Gamma^\alpha_{j\beta}}{\partial x^j}\right) e_\alpha &+ \left(\Gamma^\gamma_{j\beta}\Gamma^\alpha_{i\gamma} - \Gamma^\gamma_{i\beta}\Gamma^\alpha_{j\gamma}\right) e_\alpha \\ &\left(\frac{\partial\Gamma_j}{\partial x^i} - \frac{\partial\Gamma_i}{\partial x^j} + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i\right). \end{aligned} \tag{3.3.15}$$

Though the above equation looks very complicated it will be the clue to understanding the geometric significance of curvature.

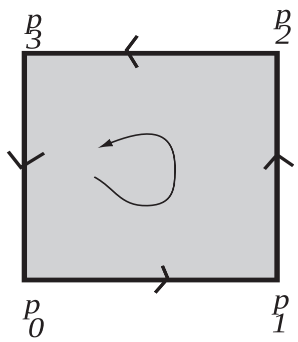

Figure 3.1: *Parallel transport along a coordinate parallelogram.*

Assume for simplicity the point of coordinates $(0, ..., 0)$ lies in U. Denote by T^s_1 the parallel transport (using the connection ∇) from $(x^1, ..., x^n)$ to $(x^1 + s, x^2, ..., x^n)$ along the curve $\tau \mapsto (x^1 + \tau, x^2, ..., x^n)$. Define T^t_2 in a similar way using the coordinate x^2 instead of x^1. Look at the parallelogram $P_{s,t}$ in the "plane" (x^1, x^2) described in Figure 3.1. We now perform the parallel transport along the boundary of $P_{s,t}$ with the counterclockwise orientation. The outcome is a linear map $\mathcal{T}_{s,t} : E_0 \to E_0$, where E_0 is the fiber of E over $p_0 = (0, ..., 0)$. Set $F_{12} = F(\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2})\mid_{(0,...,0)}$. F_{12} is an endomorphism of E_0.

Proposition 3.3.13. *for any $u \in E_0$ we have*

$$F_{12}u = -\frac{\partial^2}{\partial s \partial t} \mathfrak{T}_{s,t} u.$$

We see that the parallel transport of an element $u \in E_0$ along a closed path may not return it to itself. The curvature is an infinitesimal measure of this deviation.
Proof The parallel transport along $\partial P_{s,t}$ can be described as

$$\mathfrak{T}_{s,t} = T_2^{-t} T_1^{-s} T_2^t T_1^s.$$

Label the vertices of P_{st} in counterclockwise order starting at p_0, by $p_1,\ p_2,\ p_3$. Fix $u_0 \in E_0$. The parallel transport $T_1^s : E_0 \to E_{p_1}$ can be approximated using (3.3.6)

$$u_1 = u_1(s,t) = T_1^s u_0 = u_0 - s\Gamma_1(p_0)u_0 + C_1 s^2 + O(s^3). \tag{3.3.16}$$

C_1 is a constant vector in E_0 whose exact form is not relevant to our computations. In the sequel the letter C (eventually indexed) will denote constants.

$$\begin{aligned} u_2 = u_2(s,t) = T_2^t T_1^s u = T_2^t u_1 = u_1 - t\Gamma_2(p_1)u_1 + C_2 t^2 + O(t^3) \\ = u_0 - s\Gamma_1(p_0)u_0 - t\Gamma_2(p_1)(u_0 - s\Gamma_1(p_0)u_0) + C_1 s^2 + C_2 t^2 + O(3) \\ = \{1 - s\Gamma_1(p_0) - t\Gamma_2(p_1) + ts\Gamma_2(p_1)\Gamma_1(p_0)\}u_0 + C_1 s^2 + C_2 t^2 + O(3). \end{aligned}$$

$O(k)$ denotes an error that can be estimated from above by $C(s^2+t^2)^{k/2}$ as $s,t \to 0$. Now use

$$\Gamma_2(p_1) = \Gamma_2(p_0) + s\frac{\partial \Gamma_2}{\partial x^1}(p_0) + O(2)$$

to deduce

$$\begin{aligned} u_2 = \left\{1 - s\Gamma_1 - t\Gamma_2 - st\left(\frac{\partial \Gamma_2}{\partial x^1} - \Gamma_2\Gamma_1\right)\right\}|_{p_0}\, u_0 \\ + C_1 s^2 + C_2 t^2 + O(3). \end{aligned} \tag{3.3.17}$$

Similarly we have

$$u_3 = u_3(s,t) = T_1^{-s} T_2^t T_1^s u_0 = T_1^{-s} u_2 = u_2 + s\Gamma_1(p_2)u_2 + C_3 s^2 + O(3).$$

The Γ-term in the right-hand-side can be approximated as

$$\Gamma_1(p_2) = \Gamma_1(p_0) + s\frac{\partial \Gamma_1}{\partial x^1}(p_0) + t\frac{\partial \Gamma_1}{\partial x^2}(p_0) + O(2).$$

Using u_2 described as in (3.3.17) we get after an elementary computation

$$\begin{aligned} u_3 = u_3(s,t) = \left\{1 - t\Gamma_2 + st\left(\frac{\partial \Gamma_1}{\partial x^2} - \frac{\partial \Gamma_2}{\partial x^1} + \Gamma_2\Gamma_1 - \Gamma_1\Gamma_2\right)\right\}|_{p_0}\, u_0 \\ + C_4 s^2 + C_5 t^2 + O(3). \end{aligned} \tag{3.3.18}$$

Finally we have

$$u_4 = u_4(s,t) = T_2^{-t} = u_3 + t\Gamma_2(p_3)u_3 + C_6t^2 + O(3)$$

with

$$\Gamma_2(p_3) = \Gamma_2(p_0) + t\frac{\partial\Gamma_2}{\partial x^2}(p_0) + C_7t^2 + O(3).$$

Using (3.3.18) we get

$$u_4(s,t) = u_0 + st\left(\frac{\partial\Gamma_1}{\partial x^2} - \frac{\partial\Gamma_2}{\partial x^1} + \Gamma_2\Gamma_1 - \Gamma_1\Gamma_2\right)|_{p_0}\, u_0$$

$$+C_8s^2 + C_9t^2 + O(3)$$

$$= u_0 - stF_{12}(p_0)u_0 + C_8s^2 + C_9t^2 + O(3).$$

Clearly $\frac{\partial^2 u_4}{\partial s\partial t} = -F_{12}(p_0)u_0$ as claimed. □

Remark 3.3.14. If we had kept track of the various constants in the above computation we would have arrived at the conclusion that $C_8 = C_9 = 0$ i.e.

$$\mathfrak{T}_{s,t} = \mathbf{id} - stF_{12} + O(3).$$

Alternatively, the constant C_8 is the second order correction in the Taylor expansion of $s \mapsto \mathfrak{T}_{s,0} \equiv \mathbf{id}$ so it has to be 0. The same goes for C_9. Thus we have

$$-F_{12} = \frac{d\mathfrak{T}_{s,t}}{d\mathrm{area}P_{s,t}} = \frac{d\mathfrak{T}_{\sqrt{s},\sqrt{s}}}{ds}.$$

□

The result in the above proposition is usually formulated in terms of holonomy.

Definition 3.3.15. *Let $E \to M$ be a vector bundle with a connection ∇. The* holonomy *of ∇ along a closed path γ is the parallel transport along γ.*

We see that the curvature measures the holonomy along infinitesimal parallelograms. A connection can be viewed as an analytic way of trivializing a bundle. We can do so along paths starting at a fixed point, using the parallel transport, but using different paths ending at the same point we may wind up with trivializations which differ by a twist. The curvature provides an infinitesimal measure of that twist.

Exercise 3.3.2. Prove that any vector bundle E over the Euclidean space $\mathbb{R}^n$ is trivializable.
Hint: Use the parallel transport defined by a connection on the vector bundle E to produce a bundle isomorphism $E \to E_0 \times \mathbb{R}^n$, where E_0 is the fiber of E over the origin.

□

3.3.5 Bianchi identities

Consider $E \to M$ a smooth vector bundle with a connection $\nabla : \Omega^0(E) \to \Omega^1(E)$. We have seen that the associated exterior derivative $d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$ does not satisfy the usual $(d^\nabla)^2 = 0$ and the curvature is to blame for this. The Bianchi identity describes one remarkable algebraic feature of the curvature.

Recall that ∇ induces a connection in any tensor bundle constructed from E. In particular it induces a connection in $E^* \otimes E \cong \operatorname{End}(E)$ which we continue to denote by ∇. This extends to an "exterior derivative" $D : \Omega^p(\operatorname{End}(E)) \to \Omega^{p+1}(\operatorname{End}(E))$.

Proposition 3.3.16. (The Bianchi identity) *Let $E \to M$ and ∇ as above. Then*

$$DF(\nabla) = 0.$$

Roughly speaking, the Bianchi identity states that $(d^\nabla)^3$ is 0.

Proof We will use the identities (3.3.9) -(3.3.11). For any vector fields X, Y, Z we have

$$i_X D = \nabla_X - D i_X.$$

Hence

$$\begin{aligned}
(DF)(X,Y,Z) &= i_Z i_Y i_X DF = i_Z i_Y (\nabla_X - D i_X) F \\
&= i_Z(\nabla_X i_Y - i_{[X,Y]})F - i_Z(\nabla_Y - D i_Y) i_X F \\
&= (\nabla_X i_Z i_Y - i_{[X,Z]} i_Y - i_Z i_{[X,Y]})F - (\nabla_Y i_Z i_X - i_{[Y,Z]} i_X - \nabla_Z i_Y i_X)F \\
&= (i_{[X,Y]} i_Z + i_{[Y,Z]} i_X + i_{[Z,X]} i_Y)F \\
&\quad -(\nabla_X i_Y i_Z + \nabla_Y i_Z i_X + \nabla_Z i_X i_Y)F.
\end{aligned}$$

We compute immediately

$$i_{[X,Y]} i_Z F = F(Z,[X,Y]) = \left[\nabla_Z, \nabla_{[X,Y]}\right] - \nabla_{[Z,[X,Y]]}.$$

Also for any $u \in \Omega^0(E)$ we have

$$\begin{aligned}
(\nabla_X i_Y i_Z F)u &= \nabla_X(F(Z,Y)u) - F(Z,Y)\nabla_X u = [\nabla_X, F(Z,Y)]\, u \\
&= \left[\nabla_X, \nabla_{[Y,Z]}\right] u - [\nabla_X, [\nabla_Y, \nabla_Z]]\, u.
\end{aligned}$$

The Bianchi identity now follows from the classical Jacobi identity for commutators.

□

Example 3.3.17. Let $\underline{\mathbb{K}}$ be the trivial line bundle over a smooth manifold M. As we have seen any connection on $\underline{\mathbb{K}}$ has the form

$$\nabla^\omega = d + \omega$$

where d is the trivial connection and ω is a $\mathbb{K}$-valued 1-form on M. The curvature of this connection is

$$F(\omega) = d\omega.$$

The Bianchi identity is in this case precisely the equality $d^2\omega = 0$. □

3.3.6 Connections on tangent bundles

The tangent bundles are very special cases of vector bundles so the general theory of connections and parallel transport is applicable in this situation as well. However, the tangent bundles have some peculiar features which enrich the structure of a connection.

Recall that when looking for a local description for a connection on a vector bundle we have to first choose local coordinates and a moving frame and this is a very arbitrary decision. For tangent bundles it happens that once local coordinates (x^i) are chosen these automatically define a moving frame of the tangent bundle $\left(\frac{\partial}{\partial x_i}\right)$ and it is thus very natural to work with this frame. Hence, let ∇ be a connection on TM. With the above notations we set

$$\nabla_i \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k} \quad (\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}).$$

The coefficients Γ_{ij}^k are usually known as the *Christoffel symbols* of the connection. As usual we construct the curvature tensor

$$F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \mathrm{End}\,(TM).$$

Still, this is not the only tensor naturally associated to ∇.

Lemma 3.3.18. *For $X, Y \in \mathrm{Vect}\,(M)$ consider*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in \mathrm{Vect}\,(M).$$

Then $\forall f \in C^\infty(M)$

$$T(fX, Y) = T(X, fY) = fT(X, Y).$$

so that $T(\cdot, \cdot)$ is a tensor $T \in \Omega^2(TM)$. T is called the torsion *of the connection ∇.*

The proof of this lemma is left to the reader as an exercise. In terms of Christoffel symbols the torsion has the description

$$T(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = (\Gamma_{ij}^k - \Gamma_{ji}^k)\frac{\partial}{\partial x_k}.$$

Definition 3.3.19. *A connection on TM is said to be symmetric if $T = 0$.*

We guess by now the reader is wondering how the mathematicians came up with this object called torsion. In the remaining of this subsection we will try to sketch the geometrical meaning of torsion.

To seek such an interpretation we have to look at the finer structure of the tangent space at a point $x \in M$. It will be convenient to regard T_xM as an affine space modeled by $\mathbb{R}^n$, $n = \dim M$. Thus, we will no longer think of the elements of T_xM as vectors but instead we will treat them as points. T_xM can be coordinatized using *affine frames.* These are pairs $(p; \mathbf{e})$ where p is a point in T_xM and $\mathbf{e}$ is a basis of the underlying vector space. A frame allows one to identify T_xM with $\mathbb{R}^n$ where p is thought of as the origin.

If A, B are two affine spaces, both modelled by $\mathbb{R}^n$, and $(p; \mathbf{e})$, $(p; \mathbf{f})$ are affine frames of A and respectively B. Denote by (x^i) the coordinates in A induced by the frame $(p; \mathbf{e})$

and by (y^j) the coordinates in B induced by the frame $(q;\, \mathbf{f})$. An affine map $T : A \to B$ can then be described using these coordinates as

$$T : \mathbb{R}^n_x \to \mathbb{R}^n_y \quad x \mapsto y = Sx + v$$

where v is a vector in $\mathbb{R}^n$ and S is an invertible $n \times n$ real matrix. Thus an affine map is described by a "rotation" S followed by a translation v. This vector measures the "drift" of the origin.

If now (x^i) are local coordinates on M then they define an affine frame $\mathcal{A}_x$ at each $x \in M$: $(\mathcal{A}_x = (0; \left(\frac{\partial}{\partial x_i}\right))$. Given a connection ∇ on TM and $\gamma : I \to M$ a smooth path we will construct a family of affine isomorphisms $T_t : T_{\gamma(0)} \to T_{\gamma(t)}$ called the *affine transport of* ∇ *along* γ. In fact we will determine T_t by imposing the initial condition $T_0 = \mathbf{id}$ and then describing $\dot{T}_t$.

This is equivalent to describing the infinitesimal affine transport at a given point $x_0 \in M$ along a direction given by a vector $X = X^i \frac{\partial}{\partial x_i} \in T_{x_0}M$. The affine frame of $T_{x_0}M$ is $\mathcal{A}_{x_0} = (0; \left(\frac{\partial}{\partial x_i}\right))$.

If x_t is a point along the integral curve of X, close to x_0 then its coordinates satisfy

$$x^i_t = x^i_0 + tX^i + O(t^2).$$

This shows the origin x_0 of $\mathcal{A}_{x_0}$ "drifts" by $tX + O(t^2)$. The frame $(\frac{\partial}{\partial x_i})$ suffers a parallel transport measured as usual by $\mathbf{id} - ti_X\Gamma + O(t^2)$. The total affine transport will be

$$T_t = id + t(-i_X\Gamma + X) + O(t^2).$$

The holonomy of ∇ along a closed path will be an affine transformation and as such it has two components: a "rotation"" and a translation. As in Proposition 3.3.13 one can show the torsion measures the translation component of the holonomy along an infinitesimal parallelogram. Since we will not need this fact we will not include a proof of it.

Exercise 3.3.3. Consider the vector valued 1-form $\omega \in \Omega^1(TM)$ defined by

$$\omega(X) = X \quad \forall X \in Vect(M).$$

Show that if ∇ is a linear connection on TM then

$$d^\nabla \omega = T^\nabla$$

where T^∇ denotes the torsion of ∇. $\square$

Exercise 3.3.4. Consider a smooth vector bundle $E \to M$ over the smooth manifold M. We assume that both E and TM are equipped with connections and moreover the connection on TM is torsionless. Denote by $\hat{\nabla}$ the induced connection on $\Lambda^2 T^*M \otimes \mathrm{End}\,(E)$. Prove that $\forall X, Y, Z \in \mathrm{Vect}\,(M)$

$$\hat{\nabla}_X F(Y, Z) + \hat{\nabla}_Y F(Z, X) + \hat{\nabla}_Z F(X, Y) = 0.$$

$\square$

3.4 Integration on manifolds

3.4.1 Integration of 1-densities

We spent a lot of time learning to derivate geometrical objects but, just as in classical calculus, the story is only half complete without the reverse operation, integration.

Classically, integration requires a background measure and in this subsection we will describe the differential geometric analogue of a measure, namely the notion of 1-density on a manifold.

Let $E \to M$ be a rank k, smooth real vector bundle over a manifold M defined by an open cover (U_α) and transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{R})$ satisfying the cocycle condition. For any $r \in \mathbb{R}$ we can form the real line bundle $|\Lambda|^r(E)$ defined by the same open cover and transition maps $t_{\alpha\beta} = |\det g_{\alpha\beta}|^{-r}$. The fiber at $p \in M$ of this bundle consists of r-densities on E_p (see Section 2.2.4).

Definition 3.4.1. *Let M be a smooth manifold. The bundle of r-densities on M is*

$$|\Lambda|^r_M \stackrel{def}{=} |\Lambda|^r(TM).$$

When $r = 1$ we will use the notation $|\Lambda|_M = |\Lambda|^1_M$. We call $|\Lambda|_M$ the density bundle of M.

Denote by $C^\infty(|\Lambda|_M)$ the space of smooth sections of $|\Lambda|_M$ and by $C_0^\infty(|\Lambda|_M)$ its subspace consisting of compactly supported densities.

It helps to have local descriptions of densities. To this aim, pick an open cover of M consisting of coordinate neighborhoods, (U_α). Denote the local coordinates on U_α by (x^i_α). This choice of a cover produces a trivializing cover of TM with transition maps

$$g_{\alpha\beta} = \left(\frac{\partial x^i_\alpha}{\partial x^j_\beta} \right)_{1 \le i,j \le n},$$

where n is the dimension of M. Set $\delta_{\alpha\beta} = |\det g_{\alpha\beta}|$. A 1-density on M is then a collection of functions $\mu_\alpha \in C^\infty(U_\alpha)$ related by

$$\mu_\alpha = \delta^{-1}_{\alpha\beta} \mu_\beta.$$

It may help to think that for each point $p \in U_\alpha$ the basis $\frac{\partial}{\partial x^1_\alpha}, \dots, \frac{\partial}{\partial x^n_\alpha}$ of T_pM spans an infinitesimal parallelepiped and $\mu_\alpha(p)$ is its "volume". A change in coordinates should be thought of as a change in the measuring units. The gluing rules describe how the numerical value of the volume changes from one choice of units to another.

The densities on a manifold resemble in many respects the differential forms of maximal degree. A density can be viewed as a map

$$\mu : C^\infty(\det(TM)) \to C^\infty(M)$$

such that $\mu(f\omega) = |f|\mu(\omega)$ for all smooth functions f and all $\omega \in C^\infty(\det TM)$. In particular, any smooth map $\phi : M \to N$ between manifolds of the same dimension induces a pullback transformation

$$\phi^* : C^\infty(|\Lambda|_N) \to C^\infty(|\Lambda|_M),$$

described by

$$(\phi^*\mu)(\omega) = \mu(\det \phi\omega), \ \ \forall \omega \in C^\infty(\det TM).$$

Example 3.4.2. Consider the special case $M = \mathbb{R}^n$. Denote by $e_1, ..., e_n$ the canonical basis. This extends to a trivialization of $T\mathbb{R}^n$ and in particular the bundle of densities comes with a natural trivialization. It has a nowhere vanishing section $|dv_n|$ defined by

$$|dv_n|(e_1 \wedge ... \wedge e_n) = 1.$$

In this case any smooth density on $\mathbb{R}^n$ takes the form $\mu = f|dv_n|$ where f is some smooth function on $\mathbb{R}^n$. The reader should think of $|dv_n|$ as the standard Lebesgue measure on $\mathbb{R}^n$.

If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, viewed as a collection of n smooth functions $\phi_1 = \phi_1(x^1, ..., x^n)$,...,$\phi_n = \phi_n(x^1, ..., x^n)$ then

$$\phi^*(|dv_n|) = \left|\det\left(\frac{\partial \phi_i}{\partial x^j}\right)\right| \cdot |dv_n|.$$

$\square$

The importance of densities comes from the fact that they are precisely the objects that can be integrated. More precisely, we have the following abstract result.

Proposition 3.4.3. *There exists a natural way to associate to each smooth manifold M a linear map*

$$\int_M : C_0^\infty(|\Lambda|_M) \to \mathbb{R}$$

uniquely defined by the following conditions:
(a) $\int_M$ is invariant under diffeomorphisms, i.e. for any smooth manifolds M, N of the same dimension n, any diffeomorphism $\phi : M \to N$ and for every $\mu \in C_0^\infty(|\Lambda|_M)$ we have

$$\int_M \phi^*\mu = \int_N \mu;$$

(b) $\int_M$ is a local operation, i.e. for any open set $U \subset M$ and any $\mu \in C_0^\infty(|\Lambda|_M)$ with $\operatorname{supp}\mu \subset U$ *we have*

$$\int_M \mu = \int_U \mu.$$

(c) For any $\rho \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \rho|dv_n| = \int_{\mathbb{R}^n} \rho(x)dx,$$

where in the right-hand-side stands the Lebesgue integral of the compactly supported function ρ. $\int_M$ is called the integral *on M.*

Proof To establish the existence of an integral we associate to each manifold M a collection of data as follows.

(i) A smooth partition of unity $\mathcal{A} \subset C_0^\infty(M)$ such that $\forall \alpha \in \mathcal{A}$ the support $\operatorname{supp}\alpha$ lies entirely in some precompact coordinate neighborhood U_α and such that the cover (U_α) is locally finite.

(ii) For each U_α we pick a collection of local coordinates (x_α^i) and we denote by $|dx_\alpha^i|$ ($n = \dim M$) the density on U_α defined by

$$|dx_\alpha^i| \left(\frac{\partial}{\partial x_\alpha^1} \wedge \ldots \wedge \frac{\partial}{\partial x_\alpha^n} \right) = 1.$$

For any $\mu \in C^\infty(|\Lambda|)$ $\alpha\mu$ is a density supported in U_α and can be written as

$$\alpha\mu = \mu_\alpha |dx_\alpha^i|,$$

where μ_α is some smooth function compactly supported on U_α. The local coordinates allow us to interpret μ_α as a function on $\mathbb{R}^n$. Under this identification $|dx_\alpha^i|$ corresponds to the Lebesgue measure $|dv_n|$ on $\mathbb{R}^n$ and μ_α is a compactly supported, smooth function. We set

$$\int_{U_\alpha} \alpha\mu \stackrel{def}{=} \int_{\mathbb{R}^n} \mu_\alpha |dx_\alpha^i|.$$

Finally define

$$\int_M^{\mathcal{A}} \mu = \int_M \mu \stackrel{def}{=} \sum_{\alpha \in \mathcal{A}} \int_{U_\alpha} \alpha\mu.$$

The above sum contains only finitely many nonzero terms since $\operatorname{supp} \mu$ is compact and thus it intersects only finitely many of the $U_\alpha' s$ which form a locally finite cover.

To prove property (a) we will first prove that the integral defined as above is independent of the various choices, the partition of unity $\mathcal{A} \subset C_0^\infty(M)$ and the local coordinates $(x_\alpha^i)_{\alpha \in \mathcal{A}}$.

• **Independence of coordinates.** Fix the partition of unity $\mathcal{A}$ and consider a new collection of local coordinates (y_α^i) on each U_α. These determine two densities $|dx_\alpha^i|$ and respectively $|dy_\alpha^j|$. For each $\mu \in C_0^\infty(|\Lambda|_M)$ we have

$$\alpha\mu = \alpha\mu_\alpha^x |dx_\alpha^i| = \alpha\mu_\alpha^y |dy_\alpha^j|$$

where μ_α^x, $\mu_\alpha^y \in C_0^\infty(U_\alpha)$ are related by

$$\mu_\alpha^y = \left| \det \left(\frac{\partial x_\alpha^i}{\partial y_\alpha^j} \right) \right| \mu_\alpha^x.$$

The equality

$$\int_{\mathbb{R}^n} \mu_\alpha^x |dx_\alpha^i| = \int_{\mathbb{R}^n} \mu_\alpha^y |dy_\alpha^j|$$

is the classical change in variables formula for the Lebesgue integral.

• **Independence of the partition of unity.** Let $\mathcal{A}, \mathcal{B} \subset C_0^\infty(M)$ two partitions of unity on M. We will show that

$$\int_M^{\mathcal{A}} = \int_M^{\mathcal{B}}.$$

Form the partition of unity

$$\mathcal{A} * \mathcal{B} = \{\alpha\beta \ ; \ (\alpha, \beta) \in \mathcal{A} \times \mathcal{B}\} \subset C_0^\infty(M).$$

Note that $\operatorname{supp} \alpha\beta \subset U_{\alpha\beta} = U_\alpha \cap U_\beta$. We will prove

$$\int_M^{\mathcal{A}} = \int_M^{\mathcal{A}*\mathcal{B}} = \int_M^{\mathcal{B}}.$$

Let $\mu \in C_0^\infty(|\Lambda|_M)$. We can view $\alpha\mu$ as a compactly supported function on $\mathbb{R}^n$. We have

$$\int_{U_\alpha} \alpha\mu = \sum_\beta \int_{U_\alpha \subset \mathbb{R}^n} \beta\alpha\mu = \sum_\beta \int_{U_{\alpha\beta}} \alpha\beta\mu. \tag{3.4.1}$$

Similarly

$$\int_{U_\beta} \beta\mu = \sum_\alpha \int_{U_{\alpha\beta}} \alpha\beta\mu. \tag{3.4.2}$$

Summing (3.4.1) over α and (3.4.2) over β we get the desired conclusion.

To prove property (a) for a diffeomorphism $\phi : M \to N$ consider a partition of unity $\mathcal{A} \subset C_0^\infty(N)$. From the classical change in variables formula we deduce that for any coordinate neighborhood U_α containing the support of $\alpha \in \mathcal{A}$ and any $\mu \in C_0^\infty(|\Lambda|_N)$ we have

$$\int_{\phi^{-1}(U_\alpha)} \alpha \circ \phi \phi^* \mu = \int_{U_\alpha} \alpha\mu.$$

$\Bigl(\alpha \circ \phi\Bigr)_{\alpha \in \mathcal{A}}$ forms a partition of unity on M. Property (a) now follows by summing over α the above equality and using the independence of the integral on partitions of unity.

To prove property (b) on the local character of the integral pick $U \subset M$ and then choose a partition of unity $\mathcal{B} \subset C_0^\infty(U)$ subordinated to the open cover $(V_\beta)_{\beta\in\mathcal{B}}$. For any partition of unity $\mathcal{A} \subset C_0^\infty(M)$ with associated cover $(V_\alpha)_{\alpha\in\mathcal{A}}$ we can form a new partition of unity $\mathcal{A} * \mathcal{B}$ of U with associated cover $V_{\alpha\beta} = V_\alpha \cap V_\beta$. We use this partition of unity to compute integrals over U. For any distribution μ on M supported on U we have

$$\int_M \mu = \sum_\alpha \int_{V_\alpha} \alpha\mu = \sum_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{B}} \int_{V_{\alpha\beta}} \alpha\beta\mu = \sum_{\alpha\beta\in\mathcal{A}*\mathcal{B}} \int_{V_{\alpha\beta}} \alpha\beta\mu = \int_U \mu.$$

Property (c) is clear since for $M = \mathbb{R}^n$ we can assume that all the local coordinates chosen are cartesian. The uniqueness of the integral is immediate and we leave the reader fill in the details.

□

3.4.2 Orientability and integration of differential forms

Operating with densities on a smooth manifold is not always a very pleasant thing[1] to do. However, under some mild restrictions on the manifold, the calculus with densities can be reduced to the friendlier calculus with differential forms.

The mild restrictions referred to above have a *global nature*. We have to restrict our attention to *oriented* manifolds. Roughly speaking these are "2-sided manifolds" i.e. one

[1] This is always a matter of taste.

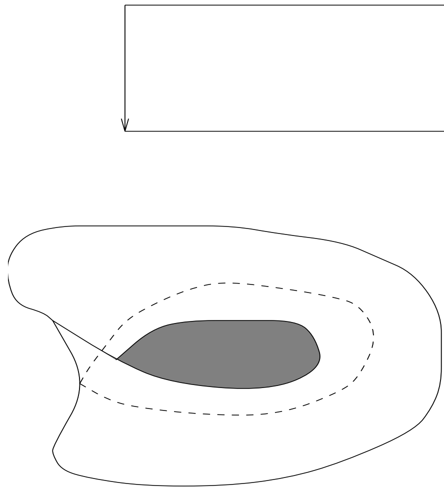

Figure 3.2: *The Mobius strip.*

can distinguish between an "inside face" and an "outside face" of the manifold. (Think of a 2-sphere in $\mathbb{R}^3$ (a soccer ball) which is naturally a "2-faced" surface.) The 2-sidedness feature is such a frequent occurrence in the real world that for many years it was taken for granted. This explains the "big surprise" produced by the famous counter-example due to Möbius in the first half of the 19th century. He produced a 1-sided surface nowadays known as the Möbius strip using paper and glue. More precisely, he glued the opposite sides of a paper rectangle attaching arrow to arrow as in Figure 3.2. The 2-sidedness can be formulated rigorously as follows.

Definition 3.4.4. *A smooth manifold M is said to be* orientable *if the determinant line bundle* $\det TM$ *(or equivalently* $\det T^*M$*) is trivial.* □

We see that $\det T^*M$ is trivial iff it admits a nowhere vanishing section. Such a section is called a *volume form* on M. We say that two volume forms ω_1 and ω_2 are equivalent if there exists $f \in C^\infty(M)$ such that

$$\omega_2 = e^f \omega_1.$$

This is indeed an equivalence relation and an equivalence class of volume forms will be called an *orientation* of the manifold. An orientable connected manifold can have two orientations. A pair (*orientable manifold, orientation*) is called an *oriented* manifold.

A natural question arises: how can one decide whether a given manifold is orientable or not. We see this is just a special instance of the more general question we addressed in Chapter 2: how can one decide whether a given vector bundle is trivial or not. The orientability question can be given a very satisfactory answer using topological techniques. However, it is often convenient to decide the orientability issue using ad-hoc arguments. In the remaining part of this section we will describe several simple ways to detect orientability.

Example 3.4.5. If the tangent bundle of a manifold M is trivial then clearly TM is orientable. In particular, all Lie groups are orientable. □

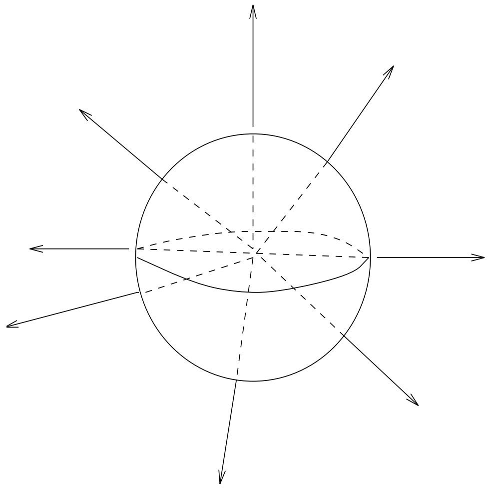

Figure 3.3: *The normal line bundle to the round sphere.*

Example 3.4.6. Suppose M is a manifold such that the Whitney sum $\underline{\mathbb{R}}^k_M \oplus TM$ is trivial. Then M is orientable. Indeed, we have

$$\det(\underline{\mathbb{R}}^k \oplus TM) = \det \underline{\mathbb{R}}^k \otimes \det TM.$$

Both $\det \underline{\mathbb{R}}^k$ and $\det(\underline{\mathbb{R}}^k \oplus TM)$ are trivial. We deduce $\det TM$ is trivial since

$$\det TM \cong \det(\underline{\mathbb{R}}^k \oplus TM) \otimes (\det \underline{\mathbb{R}}^k)^*.$$

This trick works for example when $M \cong S^n$. Indeed, let ν denote the normal line bundle. The fiber of ν at a point $p \in S^n$ is the 1-dimensional space spanned by the position vector of p as a point in $\mathbb{R}^n$; (see Figure 3.3). This is clearly a trivial line bundle since it has a tautological nowhere vanishing section $p \mapsto p \in \nu_p$. ν has a remarkable feature:

$$\nu \oplus TS^n = \underline{\mathbb{R}}^{n+1}.$$

Hence all spheres are orientable. □

Important convention The *canonical orientation* on $\mathbb{R}^n$ is the orientation defined by the volume form $dx^1 \wedge \cdots \wedge dx^n$ where $x^1, ..., x^n$ are the canonical cartesian coordinates.

The unit sphere $S^n \subset \mathbb{R}^{n+1}$ is orientable. In the sequel we will exclusively deal with its *canonical orientation*. To describe this orientation it suffices to describe a positively oriented basis of $\det T_pM$ for some $p \in S^n$. To this aim we will use the relation

$$\mathbb{R}^{n+1} \cong \nu_p \oplus T_pM.$$

An element $\omega \in \det T_pM$ defines the canonical orientation if $\vec{p} \wedge \omega \in \det \mathbb{R}^{n+1}$ defines the canonical orientation of $\mathbb{R}^{n+1}$. Above, by $\vec{p}$ we denoted the position vector of p as a point inside the euclidian space $\mathbb{R}^{n+1}$. We can think of $\vec{p}$ as the "outer" normal to the round sphere. We call this orientation *outer normal first*. When $n = 1$ it coincides with the counterclockwise orientation of the unit circle S^1.

□

Lemma 3.4.7. *A smooth manifold M is orientable if and only if there exists an open cover $(U_\alpha)_{\alpha\in\mathcal{A}}$ and local coordinates $(x_\alpha^1, ..., x_\alpha^n)$ on U_α such that*

$$\det\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right) > 0 \quad \text{on } U_\alpha \cap U_\beta. \tag{3.4.3}$$

Proof **1.** $\det T^*M$ is trivial iff there exists a volume form. Assume there exists an open cover with the properties in the lemma. Consider $\mathcal{B} \subset C_0^\infty(M)$ a partition of unity subordinated to the cover $(U_\alpha)_{\alpha\in\mathcal{A}}$ i.e. there exits a map $\varphi : \mathcal{B} \to \mathcal{A}$ such that

$$\operatorname{supp}\beta \subset U_{\varphi(\beta)} \ \ \forall \beta \in \mathcal{B}.$$

Consider

$$\omega = \sum_\beta \beta \omega_{\varphi(\beta)}$$

where for all $\alpha \in \mathcal{A}$ we define $\omega_\alpha = dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. The form ω is nowhere vanishing since condition (3.4.3) implies that on an overlap $U_{\alpha_1} \cap \cdots \cap U_{\alpha_m}$ the forms $\omega_{\alpha_1}, ..., \omega_{\alpha_m}$ differ by a *positive* multiplicative factor.

2. Conversely, let ω be a volume form on M and consider an atlas $(U_\alpha; (x_\alpha^i))$. Then $\omega\,|_{U_\alpha} = \mu_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ where the smooth functions μ_α are nowhere vanishing and on the overlaps they satisfy the gluing condition

$$\Delta_{\alpha\beta} = \det\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right) = \frac{\mu_\beta}{\mu_\alpha}.$$

A permutation φ of the variables $x_\alpha^1, ..., x_\alpha^n$ will change $dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ by a factor $\epsilon(\varphi)$ so we can always arrange these variables in such an order so that $\mu_\alpha > 0$. This will insure the positivity condition

$$\Delta_{\alpha\beta} > 0.$$

The lemma is proved. □

From the above lemma we deduce immediately the following consequence.

Proposition 3.4.8. *The connected sum of two orientable manifolds is an orientable manifold*

Exercise 3.4.1. Prove the above result. □

Exercise 3.4.2. Prove that a complex manifold is orientable. In particular, the complex grassmannians $G_{k,n}(\mathbb{C})$ are orientable. □

The reader can check immediately that the product of two orientable manifolds is again an orientable manifold. Using connected sums and products we can now produce many examples of manifolds. In particular, the connected sums of g tori is an orientable manifold.

By now the reader may ask where does orientability interact with integration. The answer lies in Section 2.2.4 where we showed that an orientation on a vector space V

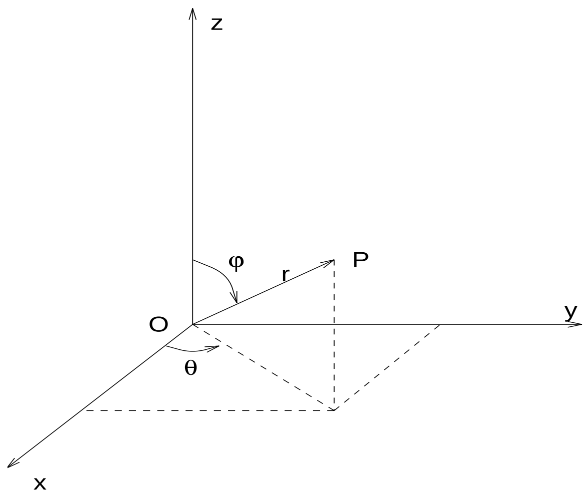

Figure 3.4: *Spherical coordinates.*

induces a *canonical* isomorphism $\imath : \det V^* \to |\Lambda|_V$. Similarly, an orientation on a smooth manifold M defines an isomorphism

$$\imath_M : C^\infty(\det T^*M) \to C^\infty(|\Lambda|_M).$$

For any compactly supported differential form ω on M of maximal degree we define its integral by

$$\int_M \omega \stackrel{def}{=} \int_M \imath_M \omega.$$

We want to emphasize that this definition *depends on the choice of orientation.*

Example 3.4.9. Consider the 2-form on $\mathbb{R}^3$

$$\omega = x dy \wedge dz$$

and let S^2 denote the unit sphere. We want to compute

$$\int_{S^2} \omega \mid_{S^2}$$

where S^2 has the canonical orientation. To compute this integral we will use spherical coordinates (r, φ, θ). These are defined by (see Figure 3.4.)

$$\left\{ \begin{array}{ccl} x & = & r \sin\varphi \cos\theta \\ y & = & r \sin\varphi \sin\theta \\ z & = & r \cos\varphi \end{array} \right.$$

At the point $p = (1, 0, 0)$ we have

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x} = \vec{p} \quad \frac{\partial}{\partial \theta} = \frac{\partial}{\partial y} \quad \frac{\partial}{\partial \varphi} = -\frac{\partial}{\partial z}$$

so that the standard orientation on S^2 is given by $d\varphi \wedge d\theta$. On S^2 we have $r \equiv 1$ and $dr \equiv 0$ so that

$$xdy \wedge dz \mid_{S^2} = \sin\varphi \cos\theta \,(\cos\theta \sin\varphi d\theta + \sin\theta \cos\varphi d\varphi)) \wedge (-\sin\varphi) d\varphi$$
$$= \sin^3\varphi \cos^2\theta d\varphi \wedge d\theta.$$

Finally, we compute

$$\int_{S^2} \omega = \int_{[0,\pi]\times[0,2\pi]} \sin^3\varphi \cos^2\theta d\varphi \wedge d\theta = \int_0^{\pi} \sin^3\varphi d\varphi \cdot \int_0^{2\pi} \cos^2\theta d\theta$$
$$= \frac{4\pi}{3} = \text{volume of the unit ball } B^3 \subset \mathbb{R}^3.$$

As we will see in the next subsection the above equality is no accident. □

Example 3.4.10. (Invariant integration on compact Lie groups.) Let G be a compact,connected Lie group. Fix once and for all an orientation on $\mathfrak{L}_G$. Consider $\omega \in \det \mathfrak{L}_G^*$ a positively oriented volume element. By left translation we can extend ω to a left-invariant volume form on G which we continue to denote by ω. This defines an orientation and in particular, by integration, we get a positive scalar

$$c = \int_G \omega.$$

Set $dg = \frac{1}{c}\omega$ so that

$$\int_G dg = 1. \tag{3.4.4}$$

dg is the unique left-invariant n-form ($n = \dim G$) on G satisfying (3.4.4) (assuming a fixed orientation on G). We claim dg is also right invariant. Assume for simplicity that G is connected.

To prove this consider the *modular function* $G \ni h \mapsto \Delta(h) \in \mathbb{R}$ defined by

$$R_h^*(dg) = \Delta(h) dg.$$

$\Delta(h)$ is a constant because $R_h^* dg$ is a left invariant form so it has to be a scalar multiple of dg. Since $(R_{h_1 h_2})^* = (R_{h_2} R_{h_1})^* = R_{h_1}^* R_{h_2}^*$ we deduce

$$\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2) \quad \forall h_1,\, h_2 \in G.$$

Hence $h \mapsto \Delta h$ is a *smooth* morphism

$$G \to (\mathbb{R} \setminus \{0\}, \cdot).$$

Since G is connected $\Delta(G) \subset \mathbb{R}_+$, and since G is compact $\Delta(G)$ is bounded. If there exists $x \in G$ such that $\Delta(x) \neq 1$ then either $\Delta(x) > 1$ or $\Delta(x^{-1}) > 1$ and in particular we would deduce the set $(\Delta(x^n))_{n\in\mathbf{Z}}$ is unbounded. Thus $\Delta \equiv 1$ which establishes the right invariance of dg.

The invariant measure dg provides a very simple way of producing invariant objects on G. More precisely, if T is tensor field on G, then for each $x \in G$ define

$$\overline{T}^{\ell}_x = \int_G ((L_g)_*T)_x dg.$$

Then $x \mapsto \overline{T}_x$ defines a smooth tensor field on G. We claim $\overline{T}$ is left invariant. Indeed, for any $h \in G$ we have

$$(L_h)_*\overline{T}^{\ell} = \int_G (L_h)_*((L_g)_*T)dg = \int_G ((L_{hg})_*T)dg$$

$$\overset{u=hg}{=} \int_G (L_u)_*Td(h^{-1}u) = \overline{T}^{\ell} \quad (\, d(h^{-1}u)) = L^*_{h^{-1}}du = du\,).$$

If we average once more on the right we get a tensor

$$x \mapsto \int_G ((R_g)_*\overline{T}^{\ell})xdg$$

which is both left and right invariant. □

Exercise 3.4.3. Let G be a Lie group. For any $X \in \mathfrak{L}_G$ denote by $ad(X)$ the linear map $\mathfrak{L}_G \to \mathfrak{L}_G$ defined by

$$\mathfrak{L}_G \ni Y \mapsto [X, Y] \in \mathfrak{L}_G.$$

(a) If ω denotes a left invariant volume form prove that $\forall X \in \mathfrak{L}_G$

$$L_X\omega = \operatorname{tr} ad(X)\omega.$$

(b) Prove that if G is a compact Lie group then $\operatorname{tr} ad(X) = 0$ for any $X \in \mathfrak{L}_G$. □

3.4.3 Stokes formula

The Stokes' formula is the higher dimensional version of the fundamental theorem of calculus (Leibniz-Newton formula)

$$\int_a^b df = f(b) - f(a)$$

where $f : [a, b] \to \mathbb{R}$ is a smooth function and $df = f'(t)dt$. In fact, the higher dimensional formula will follow from the simplest 1-dimensional situation. We will spend most of the time finding the correct formulation of the general version and this requires the concept of *manifold with boundary.* The standard example is the upper half-space

$$\mathbf{H}^n_+ = \{(x^1, ..., x^n) \in \mathbb{R}^n \ ; \ x^1 \geq 0\}.$$

Definition 3.4.11. *A smooth* manifold with boundary *is a closed subset M of a smooth manifold $\tilde{M}$ ($\dim \tilde{M} = n$) such that*
(a) $M^0 = \operatorname{int} M \neq \emptyset$.
(b) For each point $p \in \partial M = M \setminus M^0$ there exist local coordinates $(x^1, ..., x^n)$ defined on an open neighborhood $\mathcal{N}$ of p in $\tilde{M}$ such that

(b_1) $M^0 \cap \mathcal{N} = \{(x^1, ..., x^n) \; ; \; x^1 > 0\}$.
(b_2) $\partial M \cap \mathcal{N} = \{x^1 = 0\}$.
∂M is called the boundary *of M. A manifold with boundary $(M, \partial M)$ is called orientable if M^0 is orientable.*

Example 3.4.12. A closed interval $I = [a, b]$ is a smooth 1-dimensional manifold with boundary $\partial I = \{a, b\}$. □

Example 3.4.13. The closed unit ball $B^3 \subset \mathbb{R}^3$ is an orientable manifold with boundary $\partial B^3 = S^2$. □

Proposition 3.4.14. *Let $(M, \partial M)$ be a smooth manifold with boundary. Then ∂M is also a smooth manifold of dimension* $\dim \partial M = \dim M - 1$. *Moreover if M is orientable then so is its boundary.*

The proof is left to the reader as an exercise.

Important convention Let $(M, \partial M)$ be an orientable manifold with boundary. There is a (*non-canonical*) way to associate to an orientation on M^0 an orientation on the boundary. This will be the only way in which we will orient boundaries throughout this book. If we do not pay attention to this convention then our results may be off by a sign.

We now proceed to described this *induced orientation* on ∂M. For any $p \in \partial M$ choose local coordinates $(x^1, ..., x^n)$ as in Definition 3.4.11. Then the induced orientation of $T_p\partial M$ is defined by

$$\epsilon dx^2 \wedge \cdots \wedge dx^n \in \det T_p\partial M, \quad \epsilon = \pm 1$$

where ϵ is chosen so that for $x^1 > 0$ (i.e. inside M) the form

$$\epsilon(-dx^1) \wedge dx^2 \wedge \cdots \wedge dx^n$$

is positively oriented. dx^1 is usually called an *inner conormal* since x^1 increases as we go towards the interior of M. $-dx^1$ is then the *outer conormal* for analogous reasons. The rule by which we get the induced orientation on the boundary can be rephrased as

$$\{\text{outer conormal}\} \wedge \{\text{induced orientation on boundary}\} = \{\text{orientation in the interior}\}.$$

We may call this rule *"outer (co)normal first"* for obvious reasons.

□

Example 3.4.15. The canonical orientation on $S^n \subset \mathbb{R}^{n+1}$ coincides with the induced orientation of S^{n+1} as the boundary of the unit ball B^{n+1}. □

Exercise 3.4.4. Consider the hyperplane $H_i \subset \mathbb{R}^n$ defined by the equation $\{x^i = 0\}$. Prove that the induced orientation of H_i as the boundary of the half-space $\mathbf{H}_+^{n,i} = \{x^i \geq 0\}$ is given by the $(n-1)$-form $(-1)^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n$ where as usual the hat indicates a missing term. □

Theorem 3.4.16. (Stokes formula) *Let M be an oriented n-dimensional manifold with boundary ∂M and $\omega \in \Omega^{n-1}(M)$ a compactly supported form. Then*

$$\int_{M^0} d\omega = \int_{\partial M} \omega.$$

In the above formula d denotes the exterior derivative and ∂M has the induced orientation.

Proof Via partitions of unity the verification is reduced to the following two situations.
Case 1. ω is a compactly supported $(n-1)$ form in $\mathbb{R}^n$. We have to show

$$\int_{\mathbb{R}^n} d\omega = 0.$$

It suffices to consider only the special case

$$\omega = f(x) dx^2 \wedge \cdots \wedge dx^n$$

where $f(x)$ is a compactly supported smooth function. The general case is a linear combination of these special situations. We compute

$$\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^1} dx^1 \right) dx^2 \wedge \cdots \wedge dx^n = 0$$

since

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x^1} dx^1 = f(\infty, x^2, ..., x^n) - f(-\infty, x^2, ..., x^n)$$

and f has compact support.
Case 2. ω is a compactly supported $(n-1)$ form on $\mathbf{H}^n_+$. Let

$$\omega = \sum_i f_i(x) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Then

$$d\omega = \left(\sum_i (-1)^{i+1} \frac{\partial f}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n.$$

One verifies as in **Case 1** that

$$\int_{\mathbf{H}^n_+} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = 0 \quad \text{for } i \neq 1.$$

For $i = 1$ we have

$$\int_{\mathbf{H}^n_+} \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n = \int_{\mathbf{R}^{n-1}} \left(\int_0^\infty \frac{\partial f}{\partial x^1} dx^1 \right) dx^2 \wedge \cdots \wedge dx^n$$

$$= \int_{\mathbb{R}^{n-1}} \left(f(\infty, x^2, ..., x^n) - f(0, x^2, ..., x^n) \right) dx^2 \wedge \cdots \wedge dx^n$$

$$= -\int_{\mathbb{R}^{n-1}} f(0, x^2, ..., x^n) dx^2 \wedge \cdots \wedge dx^n = \int_{\partial \mathbf{H}^n_+} \omega.$$

The last equality follows from the fact that the induced orientation on $\partial \mathbf{H}^n_+$ is given by $-dx^2 \wedge \cdots \wedge dx^n$. This concludes the proof of the Stokes formula.

□

Remark 3.4.17. Stokes formula illustrates an interesting global phenomenon. It shows that the integral $\int_M d\omega$ is independent of the behavior of ω inside M. It only depends on the behavior of ω on the boundary. □

Example 3.4.18.

$$\int_{S^2} x dy \wedge dz = \int_{B^3} dx \wedge dy \wedge dz = \mathrm{vol}\,(B^3) = \frac{4\pi}{3}.$$

□

Remark 3.4.19. The above considerations extend easily to more singular situations. For example, when M is the cube $[0,1]^n$ its topological boundary is no longer a smooth manifold. However, its singularities are inessential as far as integration is concerned. The Stokes formula continues to hold

$$\int_{I^n} d\omega = \int_{\partial I} \omega \quad \forall \omega \in \Omega^{n-1}(I^n).$$

The boundary is smooth outside a set of measure zero and is given the induced orientation: " outer (co)normal first". The above equality can be used to give an explanation for the terminology "exterior derivative" we use to call d. Indeed if $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ and $I_h = [0,h]$ then we deduce

$$d\omega \mid_{x=0} = \lim_{h \to 0} h^{-n} \int_{\partial I^n_h} \omega. \tag{3.4.5}$$

When $n = 1$ this is the usual definition of the derivative. □

Example 3.4.20. We now have sufficient technical background to describe an example of vector bundle which admits no flat connections thus answering the question raised at the end of Section 3.3.3.

Let M be the unit sphere in $\mathbb{R}^3$. As in Chapter 1, we have a distinguished cover of M, $\{U_n, U_s\}$ where $U_n = S^2 \setminus \{\textit{south pole}\}$ and $U_s = S^2 \setminus \{\textit{north pole}\}$. Each of the two sets can be identified with the complex plane $\mathbb{C}$ and we get coordinates z_n on U_n and z_s on U_s. On the overlap $U_s \cap U_n$ the two sets of coordinates are related by

$$z_n = \frac{1}{z_s}.$$

Consider the complex line bundle L obtained by gluing two trivial line bundles $L_n \to U_n$ and $L_s \to U_s$ via the transition map

$$g_{sn} : U_n \cap U_s \cong \{z_n \neq 0\} \to \mathbb{C}^*, \quad g_{sn}(z_n) = z_n.$$

A connection on L is a collection of two complex valued forms $\omega_n \in \Omega^1(U_n) \otimes \mathbf{C}$, $\omega_s \in \Omega^1(U_s) \otimes \mathbb{C}$ satisfying a gluing relation on the overlap (see Example 3.3.6)

$$\omega_n(z) = \frac{dz}{z} + \omega_s(z), \ \ (z = z_n).$$

If the connection is flat then

$$d\omega_n = 0 \ \text{ on } U_n \text{ and } \ d\omega_s = 0 \ \text{ on } \omega_s.$$

Let E^+ be the equator equipped with the induced orientation as the boundary of the northern hemisphere and E^- the equator with the opposite orientation (as the boundary of the southern hemisphere). The orientation of E^+ coincides with the orientation given by the form $d\theta$ where $z_n = \exp(\mathbf{i}\theta)$. We deduce from the Stokes formula (which works for complex valued forms as well) that

$$\int_{E^+} \omega_n = 0 \quad \int_{E^-} \omega_s = -\int_{E^+} \omega_s = 0.$$

On the other hand over the equator we have

$$\omega_n - \omega_s = \frac{dz}{z} = \mathbf{i}d\theta$$

from which we deduce

$$0 = \int_{E^+} \omega_n - \omega_s = \int_{E^+} \mathbf{i}d\theta = 2\pi\mathbf{i} \ \text{ !!!}$$

Thus there exist no flat connections on the line bundle L and at fault is the gluing cocycle defining L. In a future chapter we will quantify the measure in which the gluing data obstruct the existence of flat connections. □

3.4.4 Representations and characters of compact Lie groups

The invariant integration on compact Lie groups is a very powerful tool with many uses. Undoubtedly, one of the most spectacular application is Hermann Weyl's computation of the characters of representations of compact semi-simple Lie groups. The invariant integration occupies a central place in his solution to this problem.

We devote this subsection to the description of the most elementary aspects of the representation theory of Lie groups.

Let G be a Lie group. Recall that a *(linear) representation* of G is a left action on a (finite dimensional) vector space V

$$G \times V \to V \quad (g, v) \mapsto T(g)v \in V$$

such that the map $T(g)$ is linear for any g. One also says that V has a *structure of G-module*. If V is a real (resp. complex) vector space then it is said to be a real (resp. complex) G-module.

Example 3.4.21. Let $V = \mathbb{C}^n$. Then $G = GL(n, \mathbb{C})$ acts linearly on V in the tautological manner. Moreover V^*, $V^{\otimes k}$, $\Lambda^m V$ and $S^\ell V$ are complex G-modules. □

Definition 3.4.22. *A* morphism of G-modules V_1 *and* V_2 *is a linear map* $L : V_1 \to V_2$ *such that for any* $g \in G$ *the diagram below is commutative, i.e.* $T_2(g)L = LT_1(g)$.

$$\begin{tikzcd} V_1 \arrow[r, "L"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "L"] & V_2 \end{tikzcd}$$

The space of morphisms of G-modules is denoted by $\mathrm{Hom}_G(V_1, V_2)$. *The collection of isomorphisms classes of complex G-modules is denoted by G-***Mod**.

If V is a G-module than an *invariant subspace* (or submodule) is a subspace $U \subset V$ such that $T(g)(U) \subset U$, $\forall g \in G$. A G-module is said to be irreducible if it has no invariant subspaces other than $\{0\}$ and V itself.

Proposition 3.4.23. *The direct sum "$\oplus$" and the tensor product "$\otimes$" define a structure of semi-ring with 1 on G*-Mod. *0 is represented by the null representation* $\{0\}$ *while 1 is represented by the trivial module* $G \to \mathrm{Aut}\,(\mathbb{C})$, $g \mapsto 1$.

The proof of this proposition is left to the reader.

Example 3.4.24. Let $T_i : G \to \mathrm{Aut}\,(U_i)$ $(i = 1, 2)$ be two complex G-modules. Then U_1^* is a G-module by

$$(g, u^*) \mapsto T_1^t(g^{-1})u^*.$$

Hence $\mathrm{Hom}\,(U_1, U_2)$ is also a G-module. Explicitly, the action of $g \in G$ is given by

$$(g, L) \mapsto T_2(g)LT_1(g^{-1}).$$

We see that $\mathrm{Hom}_G(U_1, U_2)$ can be identified with the linear subspace in $\mathrm{Hom}\,(U_1, U_2)$ consisting of the linear maps $U_1 \to U_2$ unchanged by the above action of G. □

Proposition 3.4.25. (Weyl's unitary trick) *Let G be a* compact *Lie group and V a complex G-module. Then there exists a Hermitian metric h on V which is G-invariant i.e.*

$$h(gv_1, gv_2) = h(v_1, v_2) \quad \forall v_1, v_2 \in V.$$

Proof Let h be an arbitrary Hermitian metric on V. Define its G-average by

$$\overline{h}(u, v) = \int_G h(gu, gv)dg$$

where dg denotes the normalized bi-invariant measure on G. One can now check easily that $\overline{h}$ is G-invariant.

□

In the sequel, G will always denote a compact Lie group.

Proposition 3.4.26. *Let V be a complex G-module and h a G-invariant Hermitian metric. If U is an invariant subspace of V then so is $U^\perp$, where "$\perp$" denotes the orthogonal complement with respect to h.*

Proof Since h is G-invariant it follows that $\forall g \in G$, $T(g)$ is an unitary operator, $T(g)T^*(g) = \mathbf{1}_V$. Hence, $T^*(g) = T^{-1}(g) = T(g^{-1})$, $\forall g \in G$.

If $x \in U^\perp$ then for all $u \in U$ and $\forall g \in G$

$$h(T(g)x, u) = h((x, T^*(g)u) = h(x, T(g^{-1})u) = 0.$$

Thus $T(g)x \in U^\perp$ so that $U^\perp$ is G-invariant.

$\square$

Corollary 3.4.27. *Every G-module V can be decomposed as a direct sum of irreducible ones.*

If we denoted by $\mathbf{Irr}$(G) the collection of isomorphism classes of irreducible G-modules then we can rephrase the above corollary by saying that $\mathbf{Irr}$(G) generates the semigroup (*G-Mod*, $\oplus$).

To gain a little more insight we need to use the following remarkable trick due to Isaac Schur.

Lemma 3.4.28. (Schur lemma) *Let V_1, V_2 be two irreducible* complex *G-modules. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V_1, V_2) = \left\{ \begin{array}{lll} 1 & \text{if} & V_1 \cong V_2 \\ 0 & \text{if} & V_1 \not\cong V_2 \end{array} \right. .$$

Proof Let $L \in \mathrm{Hom}_G(V_1, V_2)$. Then $\ker L \subset V_1$ is an invariant subspace of V_1. Similarly, $\mathrm{Range}\,(L) \subset V_2$ is an invariant subspace of V_2. Thus, either $\ker L = 0$ or $\ker L = V_1$.

The first situation forces $\mathrm{range}\, L \neq 0$ and since V_2 is irreducible $\mathrm{Range}\, L = V_2$. Hence L has to be in isomorphism of G-modules. We deduce that if V_1 and V_2 are not isomorphic as G-modules $\mathrm{Hom}_G(V_1, V_2) = \{0\}$.

Assume now that $V_1 \cong V_2$ and $S : V_1 \to V_2$ is an isomorphism of G-modules. According to the previous discussion, any other nontrivial G-morphism $L : V_1 \to V_2$ has to be an isomorphism. Consider the automorphism $T = S^{-1}L : V_1 \to V_1$. Since V_1 is a *complex* vector space T admits at least one (non-zero) eigenvalue λ.

The map $\lambda \mathbf{1}_{V_1} - T$ is an endomorphism of G-modules and $\ker(\lambda\mathbf{1}_{V_1} - T) \neq 0$. Invoking again the above discussion we deduce $T \equiv \lambda \mathbf{1}_{V_1}$, i.e. $L \equiv \lambda S$. This shows $\dim \mathrm{Hom}_G(V_1, V_2) = 1$.

$\square$

Schur's lemma is powerful enough to completely characterize S^1-**Mod**.

Example 3.4.29. The irreducible (complex)representations of S^1.) Let V be a complex irreducible S^1-module

$$(e^{i\theta}, v) \mapsto T_\theta v,$$

where

$$T_{\theta_1} \cdot T_{\theta_2} = T_{\theta_1 + \theta_2 \, mod \, 2\pi}.$$

In particular this implies that each T_θ is an S^1-automorphism since it obviously commutes with the action of this group. Hence $T_\theta = \lambda(\theta)\mathbf{1}_V$ which shows that $\dim V = 1$ since any 1-dimensional subspace of V is S^1-invariant. We have thus obtained a smooth map

$$\lambda : S^1 \to \mathbb{C}^*$$

such that

$$\lambda(e^{i\theta}\cdot e^{i\tau}) = \lambda(e^{i\theta})\lambda(e^{i\theta}).$$

Hence $\lambda : S^1 \to \mathbb{C}^*$ is a group morphism. As in the discussion of the modular function we deduce that $|\lambda| \equiv 1$. Thus λ looks like an exponential (*verify!*)

$$\lambda(e^{i\theta}) = \exp(i\alpha\theta).$$

Moreover, when $\exp(2\pi i\alpha) = 1$ so that $\alpha \in \mathbb{Z}$.

Conversely, for any integer $n \in \mathbb{Z}$ we have a representation

$$S^1 \stackrel{\rho_n}{\to} \mathrm{Aut}\,(\mathbb{C}) \quad (e^{i\theta}, z) \mapsto e^{in\theta}z.$$

The exponentials $\exp(in\theta)$ are called the *characters* of the representations ρ_n. □

Exercise 3.4.5. Describe the irreducible representations of T^n-the n-dimensional torus. □

Definition 3.4.30. *(a) Let V be a complex G-module, $g \mapsto T(g) \in \mathrm{Aut}\,(V)$. The* character *of V is the smooth function*

$$\chi_V : G \to \mathbb{C} \quad \chi_V(g) = \mathrm{tr}\,T(g).$$

(b) A class function *is a continuous function $f : G \to \mathbb{C}$ such that*

$$f(hgh^{-1}) = f(g) \quad \forall g, h \in G.$$

(The character of a representation is an example of class function).

Theorem 3.4.31. *Let G be a compact Lie group, U_1, U_2 complex G-modules and χ_{U_i} their characters. Then the following hold.*
(a)$\chi_{U_1\oplus U_2} = \chi_{U_1} + \chi_{U_2}$, $\chi_{U_1\otimes U_2} = \chi_{U_1}\cdot\chi_{U_1}$.
(b) $\chi_{U_i}(1) = \dim U_i$.
(c) $\chi_{U_i^*} = \overline{\chi}_{U_i}$*-the complex conjugate of* χ_{U_i}.
(d)

$$\int_G \chi_{U_i}(g)dg = \dim U_i^G,$$

where U_i^G denotes the space of G-invariant elements of U_i,

$$U_i^G = \{x \in U_i\ ;\ x = T_i(g)x\ \ \forall g \in G\}.$$

(e)

$$\int_G \chi_{U_1}(g)\cdot\overline{\chi}_{U_2}(g)dg = \dim \mathrm{Hom}_G(U_2, U_1).$$

Proof (a) and (b) are left to the reader. To prove (c) fix an invariant Hermitian metric on $U = U_i$. Thus each $T(g)$ is an unitary operator on U. The action of G on U^* is given by $T^t(g^{-1})$. Since T(g) is unitary we have $T^t(g^1) = \overline{T}(g)$. This proves (c).

Proof of (d). Consider

$$P : U \to U \quad Pu = \int_G T(g)udg.$$

Note that $PT(h) = T(h)P$, $\forall h \in G$ i.e. $P \in \mathrm{Hom}_G(U, U)$. We now compute

$$T(h)Pu = \int_G T(hg)udg = \int_G T(\gamma)uR^*_{h^{-1}}d\gamma$$

$$\int_G T(\gamma)ud\gamma = Pu.$$

Thus, each Pu is G-invariant. Conversely, if $x \in U$ is G-invariant then

$$Px = \int_G T(g)xdg = \int_G xdg = x$$

i.e. $U^G = \mathrm{Range}\, P$. Note also that P is a projector i.e. $P^2 = P$. Indeed

$$P^2u = \int_G T(g)Pudg = \int_G Pudg = Pu.$$

Hence P is a projection onto U^G and in particular

$$\dim_{\mathbb{C}} U^G = \mathrm{tr}\, P = \int_G \mathrm{tr}\, T(g)dg = \int_G \chi_U(g)dg.$$

Proof of (e).

$$\begin{aligned} \int_G \chi_{U_1} \cdot \overline{\chi}_{U_2} dg &= \int_G \chi_{U_1} \cdot \chi_{U_2^*} dg \\ = \int_G \chi_{U_1 \otimes U_2^*} dg &= \int_G \chi_{\mathrm{Hom}(U_2, U_1)} \\ = \dim_{\mathbb{C}} (\mathrm{Hom}\,(U_2, U_1))^G &= \dim_{\mathbb{C}} \mathrm{Hom}_G(U_2, U_1) \end{aligned}$$

since Hom_G coincides with the space of G-invariant morphisms.

□

Corollary 3.4.32. *Let U, V be irreducible G-modules. Then*

$$(\chi_U, \chi_V) = \int_G \chi_U \cdot \overline{\chi}_V dg = \delta_{UV} = \begin{cases} 1 & , \quad U \cong V \\ 0 & , \quad U \not\cong V \end{cases}.$$

Proof Follows from Theorem 3.4.31 using Schur's lemma.

□

Corollary 3.4.33. *Let U, V be two G-modules. Then $U \cong V$ if and only if $\chi_U = \chi_V$.*

Proof Decompose U and V as direct sums of irreducible G-modules

$$U = \oplus_1^m (m_i U_i) \quad V = \oplus_1^\ell (n_j V_j).$$

Hence $\chi_U = \sum m_i \chi_{V_i}$ and $\chi_V = \sum n_j \chi_{V_j}$. The equivalence "representation" $\Longleftrightarrow$ "characters" stated by this corollary now follows immediately from Schur's lemma and the previous corollary.

□

Thus, the problem of describing the representations of a compact Lie group boils down to describing the characters of its irreducible representations. This problem was completely solved by Hermann Weyl. Its solution requires a lot more work and goes beyond the scope of this book. We will spend the remaining part of this subsection analyzing the equality (d) in Theorem 3.4.31.

Describing the invariants of a group action was a very fashionable problem in the second half of the nineteenth century. Formula (d) mentioned above is a truly remarkable result. It allows (in principle) to compute the maximum number of linearly independent invariant elements.

Let V be a complex G-module and denote by χ_V its character. The complex exterior algebra $\Lambda_c^* V^*$ is a complex G-module, as the space of complex multi-linear skew-symmetric maps

$$V \times \cdots \times V \to \mathbb{C}.$$

Denote by $b_k^c(V)$ the complex dimension of the space of G-invariant elements in $\Lambda_c^k V^*$. One has the equality

$$b_k^c(V) = \int_G \chi_{\Lambda_c^k V^*} dg.$$

These facts can be presented coherently by considering the $\mathbb{Z}$-graded vector space

$$\mathcal{I}_c(V) = \oplus_k \Lambda_{inv}^k V^*.$$

Its Poincaré polynomial is

$$P_{\mathcal{I}_c(V)}(t) = \sum t^k b_k^c(V) = \int_G t^k \chi_{\Lambda_c^k V^*} dg.$$

To obtain a more concentrated formulation of the above equality we need to recall some elementary facts of linear algebra.

For each endomorphism A of V denote by $\sigma_k(A)$ the coefficient of t^k in the characteristic polynomial

$$\sigma_t(A) = \det(\mathbf{1}_V + tA).$$

Explicitly, $\sigma_k(A)$ is given by the sum

$$\sigma_k(A) = \sum_{1 \le i_1 \cdots i_k \le n} \det\left(a_{i_\alpha i_\beta}\right) \quad (n = \dim V).$$

Equivalently

$$\sigma_k(A) = \mathrm{tr}\,(\Lambda^k A)$$

where $\Lambda^k A$ is the endomorphism of $\Lambda^k V$ induced by A.

If $g \in G$ acts on V by $T(g)$ then g acts on $\Lambda^k V^*$ by $\Lambda^k T^t(g^{-1}) = \Lambda^k \bar{T}(g)$. (We implicitly assumed that each $T(g)$ is unitary with respect to some G-invariant metric on V). Hence

$$\chi_{\Lambda_c^k V^*} = \sigma_k(\bar{T}(g)). \tag{3.4.6}$$

We conclude that

$$P_{\mathcal{I}(V)}(t) = \int_G \sum t^k \sigma_k(\overline{T}(g))dg = \int_G \det(\mathbf{1}_V + t\bar{T}(g))dg. \tag{3.4.7}$$

Consider now the following situation. Let V be a *complex* V-module. Denote by $\Lambda_r^* V$ the space of $\mathbb{R}$-multi-linear, skew-symmetric maps

$$V \times \cdots \times V \to \mathbb{R}.$$

$\Lambda_r^* V^*$ is a *real* G-module. We complexify it. $\Lambda_r^* V \otimes \mathbb{C}$ is the space of $\mathbb{R}$-multi-linear, skew-symmetric maps

$$V \times \cdots \times V \to \mathbb{C}$$

and as such it is a complex G-module. The *real* dimension of the subspace $\mathcal{I}_r^k$ of G invariant elements in $\Lambda_r^k V^*$ will be denoted by $b_k^r(V)$ so that the Poincaré polynomial of $\mathcal{I}_r^*(V) = \oplus_k \mathcal{I}_r^k$ is

$$P_{\mathcal{I}_r^*(V)}(t) = \sum t^k b_k^r(V).$$

On the other hand $b_k^r(V)$ is equal to the *complex* dimension of $\Lambda_r^k V^* \otimes \mathbb{C}$. Using the results of Subsection 2.2.5 we deduce

$$\Lambda_r^* V \otimes \mathbb{C} \cong \Lambda_c^* V^* \otimes_{\mathbb{C}} \Lambda_c^* \overline{V}^* = \bigoplus_k \left(\oplus_{i+j=k} \Lambda_c^i V^* \otimes \Lambda_c^j \overline{V}^* \right). \tag{3.4.8}$$

Each of the above summands is a G-invariant subspace. Using (3.4.6) and (3.4.8) we deduce

$$P_{\mathcal{I}_r(V)} = \sum_k \int_G \sum_{i+j=k} \sigma_i(T(g)) t^i \sigma_j(\bar{T}(g)) t^j dg$$

$$\int_G \det(\mathbf{1}_V + tT(g)) \det(\mathbf{1}_{\overline{V}} + t\overline{T}(g))dg = \int_G |\det(\mathbf{1}_V + tT(g))|^2 dg. \tag{3.4.9}$$

We will have the chance to use this result in computing topological invariants of manifolds with a "high degree of symmetry" like e.g. the grassmannians.

3.4.5 Fibered calculus

In the previous section we have described the calculus associated to objects defined on a *single manifold*. The aim of this subsection is to discuss what happens when we deal with an entire family of objects parameterized by some smooth manifold. We will discuss only the fibered version of integration. The exterior derivative also has a fibered version but its true meaning can only be grasped by referring to Leray's spectral sequence of a fibration and so we will not deal with it. The interested reader can learn more about this operation from [32].

Assume now that instead of a single manifold F we have an entire (smooth) family of them $(F_b)_{b\in B}$. In more rigorous terms this means we are given a smooth fiber bundle $p : E \to B$ with standard fiber F.

On the total space E we will always work with *split coordinates* $(x^i; y^j)$ where (x^i) are local coordinates on the standard fiber F and (y^j) are local coordinates on the base B (the parameter space).

The model situation is the bundle

$$E = \mathbb{R}^k \times \mathbb{R}^m \stackrel{p}{\to} \mathbb{R}^m = B \quad (x, y) \stackrel{p}{\mapsto} y.$$

We will first define a fibered version of integration. This requires a fibered version of orientability.

Definition 3.4.34. *Let $p : E \to B$ be a smooth bundle with standard fiber F. The bundle is said to be* orientable *if*
(a) F is oriented*;*
(b) there exists an open cover (U_α) and trivializations

$$p^{-1}(U_\alpha) \stackrel{\psi_\alpha}{\to} F \times U_\alpha$$

such that the gluing maps

$$\psi_\beta \circ \psi_\alpha^{-1} : F \times U_{\alpha\beta} \to F \times U_{\alpha\beta} \quad (U_{\alpha\beta} = U_\alpha \cap U_\beta)$$

are orientation preserving in the sense that for each $y \in U_{\alpha\beta}$, the diffeomorphism

$$F \ni f \mapsto \psi_{\alpha\beta}(f, y) \in F$$

is orientation preserving.

Exercise 3.4.6. If the the base B of an orientable bundle $p : E \to B$ is orientable, then so is the total space E (as an abstract smooth manifold). □

Important convention Let $p : E \to B$ be an orientable bundle with oriented basis B. The *natural orientation* of the total space E is defined as follows.

If $E = F \times B$ then the orientation of the tangent space $T_{(f,b)}E$ is given by $\Omega_F \times \omega_B$ where $\omega_F \in \det T_f F$ (resp. $\omega_B \in \det T_b B$) defines the orientation of $T_f F$ (resp. $T_b B$).

The general case reduces to this one since any bundle is locally a product and the gluing maps are orientation preserving. This convention can be briefly described as

$$\text{orientation total space} = \text{orientation fiber} \wedge \text{orientation base}.$$

The natural orientation can thus be called the *fiber-first* orientation. In the sequel all orientable bundles will be given the fiber-first orientation.

□

Let $p : E \to B$ be an orientable fiber bundle with standard fiber F.

Proposition 3.4.35. *The* integration along fibers *is an operation*

$$p_* = \int_{E/B} : \Omega^*_{cpt}(E) \to \Omega^{*-r}_{cpt}(B \quad (r = \dim F).$$

uniquely defined by its action on forms supported on domains D of split coordinates

$$D \cong \mathbb{R}^r \times \mathbb{R}^m \xrightarrow{p} \mathbb{R}^m \quad (x; y) \mapsto y.$$

If $\omega = f dx^I \wedge dy^J$ $(f \in C_0^\infty(\mathbb{R}^{r+m}))$ then

$$\int_{E/B} = \begin{cases} 0 & , \quad |I| \neq r \\ \left(\int_{\mathbb{R}^r} f dx^I\right) dy^J & , \quad |I| = r \end{cases}.$$

The proof goes *exactly* as in the non-parametric case (i.e. when B is a point). One shows using partitions of unity that these local definitions can be patched together to produce a well defined map

$$\int_{E/B} : \Omega^*_{cpt}(E) \to \Omega^{*-r}_{cpt}(B).$$

The details are left to the reader.

Proposition 3.4.36. *Let $p : E \to B$ be an orientable bundle with an r-dimensional standard fiber F. Then for any $\omega \in \Omega^*_{cpt}(E)$ and $\eta \in \omega^*_{cpt}(B)$ such that* $\deg \omega + \deg \eta = \dim E$ *we have*

$$\int_{E/B} d_E \omega = (-1)^r d_B \int_{E/B} \omega.$$

If B is oriented and ω η ar as above then

$$\int_E \omega \wedge p^*(\eta) = \int_B \left(\int_{E/B} \omega \right) \wedge \eta. \qquad \text{(Fubini)}$$

The last equality implies immediately the *projection formula*

$$p_*(\omega \wedge p^* \eta) = p_* \omega \wedge \eta.$$

Proof It suffices to consider only the model case

$$p : E = \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m = B \quad (x; y) \xrightarrow{p} y$$

and $\omega = f dx^I \wedge dy^J$. Then

$$d_E \omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge dy^J + (-1)^{|I|} \sum_j \frac{\partial f}{\partial y^j} dx^I \wedge dy^j \wedge dy^J.$$

$$\int_{E/B} d_E\omega = \left(\int_{\mathbb{R}^r} \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I\right) (-1)^{|I|} \left(\int_{\mathbf{R}^r} \sum_j \frac{\partial f}{\partial y^j} dx^I\right) \wedge dy^j \wedge dy^J.$$

The above integrals are defined to be zero if the corresponding forms do not have degree r. The Stokes formula shows that the first integral is always zero. Hence

$$\int_{E/B} d_E\omega = (-1)^{|I|} \frac{\partial}{\partial y^j} \left(\int_{\mathbf{R}^r} \sum_j dx^I\right) \wedge dy^j \wedge dy^J = (-1)^r d_B \int_{E/B} \omega.$$

The second equality is left to the reader as a practice exercise.

$\square$

Exercise 3.4.7 (Gelfand-Leray)**.** Suppose $p : E \to B$ is an oriented fibration, ω_E is a volume form on E and ω_B is a volume form on B.

(a) Prove that for every $b \in B$ there exists a unique volume form $\omega_{E/B}$ on E_b with the property that for every $x \in E$ we have

$$\omega_E(x) = \omega_{E/B}(x) \wedge (p^*\omega_B)(x) \in \Lambda^{\dim E} T_x^* E.$$

This form is called the *Gelfand-Leray residue* of ω_E rel p.

(b) Prove that for every compactly supported smooth function $f : E \to \mathbb{R}$ we have

$$\int_E f\omega_E \int_B \left(\int_{E_b} f\omega_{E/B}\right) \omega_B$$

If we use the classical but less precise notations $dV_E = \omega_E$, $\omega_B = dV_B$, $\omega_{E/B} = \frac{dV_E}{dV_B}$ then the above equality takes the form

$$\int_E f\omega_E \int_B = \int_B \underbrace{\left(\int_{E_b} f \frac{dV_E}{dV_B}\right)}_{\frac{df}{dV_B}} dV_B.$$

The smooth function $\frac{df}{dV_B} : B \to \mathbb{R}$ is called the *Gelfand-Leray form* of f rel p.

(c) Consider the fibration $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \overset{p}{\mapsto} t = ax + by$, $a^2 + b^2 \neq 0$. Compute the Gelfand-Leray form $\frac{dx \wedge dy}{dt}$ along the fiber $p(x, y) = 0$. $\square$

Definition 3.4.37. *A ∂-bundle is a collection $(E, \partial E, p, B)$ consisting of the following.*
(i) A smooth manifold E with boundary ∂E.
(ii) A smooth map $p : E \to B$ such that the restrictions $p : \operatorname{Int} E \to B$ and $p : \partial E \to B$ are smooth bundles.

The standard fiber of $p : \operatorname{Int} E \to B$ is called the interior fiber*.*

One can think of a ∂-bundle as a smooth family of manifolds with boundary.

Example 3.4.38. The projection

$$p : [0,1] \times M \to M \quad (t; m) \mapsto m$$

defines a ∂-bundle. The interior fiber is the open interval $(0,1)$. The fiber of $p : \partial(I \times M) \to M$ is the disjoint union of two points. □

Standard Models A ∂-bundle is obtained by gluing two types of local models.

Interior models $\mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m$
Boundary models $\mathbf{H}^r_+ \times \mathbb{R}^m \to \mathbb{R}^m$ where

$$\mathbf{H}^r_+ = \big\{(x^1, \cdots, x^r) \in \mathbb{R}^r \; ; \; x^1 \geq 0\big\}.$$

Remark 3.4.39. Let $p : (E, \partial E) \to B$ be a ∂-bundle. If $p : \operatorname{Int} E \to B$ is orientable and the basis B is oriented as well, then on ∂E one can define two orientations.
(i) The fiber-first orientation as the total space of an oriented bundle $\partial E \to B$.
(ii) The induced orientation as the boundary of E. □

Exercise 3.4.8. Prove that the above orientations on ∂E coincide. □

Theorem 3.4.40. *Let $p : (E, \partial E) \to B$ be an orientable ∂-bundle with an r-dimensional interior fiber. Then for any $\omega \in \Omega^*_{cpt}(E)$ we have*

$$\int_{\partial E/B} \omega = \int_{E/B} d_E \omega - (-1)^r d_B \int_{E/B} \omega \quad \text{(Homotopy formula)}.$$

The last equality can be formulated as

$$\int_{\partial E/B} = \int_{E/B} d_E - (-1)^r d_B \int_{E/B}.$$

This is "the mother of all homotopy formulæ". It will play a crucial part in Chapter 7 when we embark on the study of DeRham cohomology.

Exercise 3.4.9. Prove the above theorem. □

Chapter 4

Riemannian Geometry

Now we can finally put to work the abstract notions discussed in the previous chapters. Loosely speaking the Riemannian geometry studies the properties of surfaces (manifolds) “made of canvas”. These are manifolds with an extra structure arising naturally in many instances. In particular we will study the problem formulated in Chapter 1: why a plane (flat) canvas disk cannot be wrapped in an one-to-one fashinon around the unit sphere in $\mathbb{R}^3$. Answering this requires the notion of Riemann curvature which will be the central theme of this chapter.

4.1 Metric properties

4.1.1 Definitions and examples

To motivate our definition we will first try to formulate rigorously what do we mean by a “canvas surface”.

A “canvas surface” can be deformed in many ways but with some limitations: it cannot be stretched as a rubber surface and this is because the fibers of the canvas are flexible but not elastic. Alternatively, this means that the only operations we can perform are those which do not change the lengths of curves on the surface. Thus one can think of “canvas surfaces” as those surfaces on which any “reasonable” curve has a well defined length.

Adapting a more constructive point of view one can say that such surfaces are endowed with a clear procedure of measuring lengths of piecewise smooth curves.

Classical vector analysis describes one method of measuring lengths of curves in $\mathbb{R}^3$. If $\gamma : [0,1] \to \mathbb{R}^3$ is such a curve then its length is given by

$$\operatorname{length}(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt$$

where $|\dot{\gamma}(t)|$ is the euclidian length of the tangent vector $\dot{\gamma}(t)$.

We want to do the same thing on an abstract manifold and we are clearly faced with one problem: how do we make sense of $|\dot{\gamma}(t)|$? Obviously, this problem can be solved if we assume that there is a procedure of measuring lengths of tangent vectors at any point on our manifold. The simplest way to do achieve this is to assume that each tangent space is endowed with an inner product (which can vary from point to point in a smooth way).

Definition 4.1.1. *(a) A* Riemann manifold *is a pair (M, g) consisting of a smooth manifold M and a metric g on the tangent bundle i.e. a smooth, symmetric positive definite (0,2) tensor field on M. g is called a* Riemann metric *on M.*
(a) Two Riemann manifolds (M_i, g_i) $(i = 1, 2)$ are said to be isometric *if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^* g_2 = g_1$.*

If (M, g) is a Riemann manifold then for any $x \in M$ the restriction $g_x : T_xM \times T_xM \to \mathbb{R}$ is an inner product on the tangent space T_xM. We will frequently use the alternative notation $\langle \cdot, \cdot \rangle_x = g_x(\cdot, \cdot)$. The length of a tangent vector $v \in T_xM$ is defined as usual as

$$|v|_x \stackrel{def}{=} g_x(v, v)^{1/2}.$$

If $\gamma : [a, b] \to M$ is a piecewise smooth curve then we define its length by

$$\mathfrak{l}(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\gamma(t)} dt.$$

If we choose local coordinates $(x^1, \cdots, x^n)$ on M then we get a local description of g as

$$g = g_{ij} dx^i dx^j, \quad g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}).$$

Proposition 4.1.2. *Let M be a smooth manifold and denote by $\mathcal{R}_M$ the set of Riemann metrics on M. Then $\mathcal{R}_M$ is a non-empty convex cone in the linear space of symmetric (0,2) tensors.*

Proof The only thing we have to prove is that $\mathcal{R}_M$ is non-empty. We will use again partitions of unity. Cover M by coordinate neighborhoods $(U_\alpha)_{\alpha \in \mathcal{A}}$. Let (x^i_α) be a collection of local coordinates on U_α. Using these local coordinates we can construct by hand the metric g_α on U_α by

$$g_\alpha = (dx^1_\alpha)^2 + \cdots + (dx^n_\alpha)^2.$$

Now pick a partition of unity $\mathcal{B} \subset C_0^\infty(M)$ subordinated to the cover $(U_\alpha)_{\alpha \in \mathcal{A}}$ i.e. there exits a map $\phi : \mathcal{B} \to \mathcal{A}$ such that $\forall \beta \in \mathcal{B}$ $\operatorname{supp} \beta \subset U_{\phi(\beta)}$. Then define

$$g = \sum_{\beta \in \mathcal{B}} \beta g_{\phi(\beta)}.$$

The reader can check easily that g is well defined and is indeed a Riemann metric on M. □

Example 4.1.3. (The Euclidean space) The space $\mathbb{R}^n$ has a natural Riemann metric

$$g_0 = (dx^1)^2 + \cdots + (dx^n)^2.$$

The geometry of $(\mathbb{R}^n, g_0)$ is the classical Euclidean geometry. □

Example 4.1.4. (Induced metrics on submanifolds) Let (M, g) be a Riemann manifold and $S \subset M$ a submanifold. If $\imath : S \to M$ denotes the natural inclusion then we obtain by pull back a metric on S

$$g_S = \imath^* g = g|_S .$$

For example, any invertible symmetric $n \times n$ matrix defines a quadratic hypersurface in $\mathbb{R}^n$ by

$$\mathcal{H}_A = \{x \in \mathbb{R}^n \; ; \; \langle Ax, x\rangle = 1\}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$. $\mathcal{H}_A$ has a natural metric induced by the Euclidean metric on $\mathbb{R}^n$. For example when $A = I_n$ then $\mathcal{H}_{I_n}$ is the unit sphere in $\mathbb{R}^n$ the induced metric is called the *round metric* of S^{n-1}. □

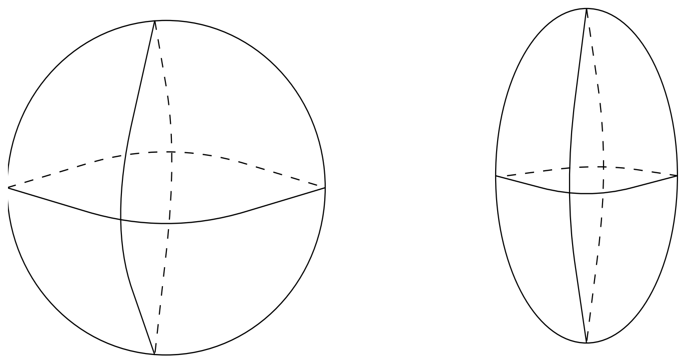

Figure 4.1: *The unit sphere and an ellipsoid look "different".*

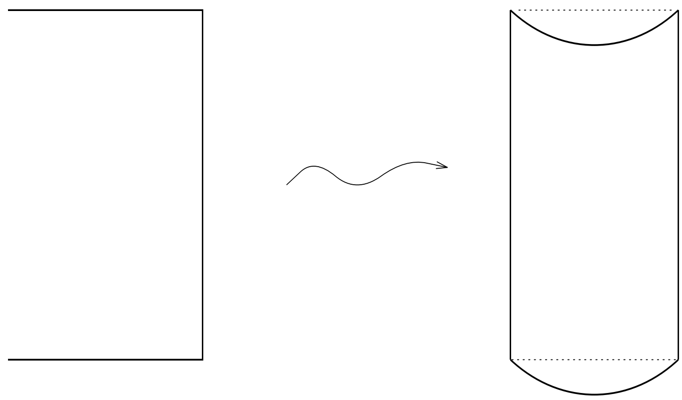

Figure 4.2: *A plane sheet and a half cylinder are "not so different".*

Remark 4.1.5. On any manifold there exist many Riemann metrics and there is no natural way of selecting one of them. One can visualize a Riemann structure as defining a "shape" of the manifold. For example the unit sphere $x^2 + y^2 + z^2 = 1$ is diffeomorphic to the ellipsoid $\frac{x^2}{1^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$ but they look "different " (see Figure 4.1). However, appearances may be deceiving. In Figure 4.2 it is illustrated the deformation of a sheet of paper to a half cylinder. They look different but the metric structures are the same since we have not changed the lengths of curves on our sheet. The conclusion to be drawn from these two examples is that we have to be very careful when we use the attribute "different". □

Example 4.1.6. (The hyperbolic plane) The Poincaré model of the hyperbolic plane is the Riemann manifold $(\mathbf{D},\ g)$ where $\mathbf{D}$ is the unit open disk in the plane $\mathbb{R}^2$ and the metric g is given by

$$g = \frac{1}{1-x^2-y^2}(dx^2+dy^2).$$

Exercise 4.1.1. Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{(u,v)\in\mathbb{R}^2\ ;\ v>0\}$$

endowed with the metric

$$h = \frac{1}{4v^2}(du^2+dv^2).$$

Show that the Cayley transform

$$z = x+\mathbf{i}y \mapsto w = -\mathbf{i}\frac{z+\mathbf{i}}{z-\mathbf{i}} = u+\mathbf{i}v$$

establishes an isometry $(\mathbf{D},g)\cong(\mathcal{H},h)$. □

Example 4.1.7. (Left invariant metrics on Lie groups.) Consider a Lie group G and denote by $\mathfrak{L}_G$ its Lie algebra. Then any inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{L}_G$ induces a Riemann metric $h=\langle\cdot,\cdot\rangle_g$ on G defined by

$$h_g(X,Y) = \langle X,Y\rangle_g = \langle (L_{g^{-1}})_*X, (L_{g^{-1}})_*Y\rangle\ \ \forall g\in G,\ X,Y\in T_gG$$

where $(L_{g^{-1}})_* : T_gG\to T_1G$ is the differential at $g\in G$ of the left translation map $L_{g^{-1}}$. One checks easily that $g\mapsto\langle\cdot,\cdot\rangle_g$ is a smooth tensor field and is left invariant i.e.

$$L_g^*h = h\quad\forall g\in G.$$

If G is also compact one then can use the averaging technique of Subsection 3.4.2 to produce metrics which are both left and right invariant. □

Exercise 4.1.2. Let $M=G_{k,n}$ be the grassmannian of complex k-planes in $\mathbb{C}^n$. For each such k-plane V denote by P_V the orthogonal projection onto V. Thus $P_V^*=P_V=P_V^2$ and $\mathrm{Range}\,(P_V)=V$ so that P_V can be viewed as a complex, selfadjoint $n\times n$ matrix. Denote by $\mathcal{S}_n$ the linear space of such matrices and by h_0 the inner product

$$h_0(A,B) = \mathrm{tr}\,(AB^*)\ \ \forall A,B\in\mathcal{S}_n.$$

(a) Prove that the map $P_k : G_{k,n}\to\mathcal{S}_n$ defined by $V\mapsto P_V$ is an embedding i.e. it is a one-to-one immersion.
(b) Let $k=1$ so that $G_{1,n}\cong\mathbb{CP}^{n-1}$. Describe the induced metric $P_1^*h_0$ on $\mathbb{CP}^{n-1}$ using the homogeneous coordinates introduced in Section 1.2.2. □

4.1.2 The Levi-Civita connection

To continue our study of Riemann manifolds we will try to follow a close parallel with classical euclidian geometry. The first question one may ask is whether there is a notion of "straight line" on a Riemann manifold. In the euclidian space $\mathbb{R}^3$ there are at least two ways to define a line segment.
(i) A line segment is the shortest path connecting two given points.
(ii) A line segment is a smooth path $\gamma : [0,1] \to \mathbb{R}^3$ satisfying

$$\ddot{\gamma}(t) = 0. \tag{4.1.1}$$

Since we have not said anything about the calculus of variations (which deals precisely with problems of type (i)) we will use the second interpretation as our starting point. We will see however that both points of view yield the same conclusion.

Let us first reformulate (4.1.1). As we know the tangent bundle of $\mathbb{R}^3$ is equipped with a natural trivialization and as such it has a natural trivial connection ∇^0 defined by

$$\nabla^0_i \partial_j = 0 \quad \forall i,j \ \ (\partial_i = \frac{\partial}{\partial x_i},\ \nabla_i = \nabla_{\partial_i}),$$

i.e. all the Christoffel symbols vanish. Moreover if g_0 denotes the Euclidean metric then

$$\left(\nabla^0_i g_0\right)(\partial_j, \partial_k) = \nabla^0_i \delta_{jk} - g_0(\nabla^0_i \partial_j, \partial_k) - g_0(\partial_j, \nabla^0_i \partial_k) = 0$$

i.e. the connection is compatible with the metric. Condition (4.1.1) can be rephrased as

$$\nabla^0_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \tag{4.1.2}$$

so that the problem of defining "lines" in a Riemann manifold reduces to choosing a "natural" connection on the tangent bundle. Of course we would like this connection to be compatible with the metric but even so, there are infinitely many connections to choose from. The following fundamental result will solve this dilemma.

Proposition 4.1.8. *Consider a Riemann manifold* (M, g)*. Then there exists a* unique symmetric *connection* ∇ *on* TM compatible *with the metric* g *i.e.*

$$T(\nabla) = 0 \quad \nabla g = 0.$$

∇ *is usually called the* Levi-Civita connection *associated to the metric* g*.*

Proof We first prove that there exists at most one connection with the required properties. We will achieve this by producing an *explicit* description of it.

Let ∇ be a connection with the desired properties, i.e.

$$\nabla g = 0 \ \text{ and } \ \nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \operatorname{Vect}(M).$$

For any $X, Y, Z \in \operatorname{Vect}(M)$ we have

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

since $\nabla g = 0$. Using the symmetry of the connection we compute

$$Zg(X,Y) - Yg(Z,X) + Xg(Y,X) = g(\nabla_Z X, Y) - g(\nabla_Y Z, X) + g(\nabla_X Y, Z)$$
$$+g(X, \nabla_Z Y) - g(Z, \nabla_Y X) + g(Y, \nabla_X Z)$$
$$= g([Z,Y], X) + g([X,Y], Z) + g([Z,X], Y) + 2g(\nabla_X Z, Y).$$

We conclude that

$$g(\nabla_X Z, Y) = \frac{1}{2}\{Xg(Y,Z) - Yg(Z,X) + Zg(X,Y)$$
$$-g([X,Y], Z) + g([Y,Z], X) - g([Z,X], Y)\}. \tag{4.1.3}$$

The above equality establishes the uniqueness of ∇.

Using local coordinates $(x^1, \cdots, x^n)$ on M we deduce from (4.1.3) (with $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_k}$, $Z = \frac{\partial}{\partial x_j}$) that

$$g(\nabla_i \partial_j, \partial_k) = g_{k\ell}\Gamma^\ell_{ij} = \frac{1}{2}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right).$$

If $(g^{i\ell})$ denotes the inverse of $(g_{i\ell})$ we deduce

$$\Gamma^\ell_{ij} = \frac{1}{2} g^{k\ell}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right). \tag{4.1.4}$$

Existence It boils down to showing that (4.1.3) indeed defines a connection with the required properties. The routine details are left to the reader.

□

We can now define the notion of "straight line" on a Riemann manifold.

Definition 4.1.9. *A* geodesic *on a Riemann manifold (M,g) is a smooth curve $\gamma : (a,b) \to M$ satisfying*

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0, \tag{4.1.5}$$

where ∇ is the Levi-Civita connection.

Using local coordinates $(x^1, ..., x^n)$ with respect to which the Christoffel symbols are (Γ^k_{ij}) and $\gamma(t) = (x^1(t), ..., x^n(t))$ we can reformulate the geodesic equation as a second order nonlinear system of ordinary differential equations. Set $\frac{d}{dt} = \dot\gamma(t) = \dot x^i \partial_i$. Then

$$\nabla_{\frac{d}{dt}}\dot\gamma(t) = \ddot x^i \partial_i + \dot x^i \nabla_{\frac{d}{dt}}\partial_i$$
$$= \ddot x^i \partial_i + \dot x^i \dot x^j \nabla_j \partial_i$$
$$= \ddot x^k \partial_k + \Gamma^k_{ji}\dot x^i \dot x^j \partial_k \quad (\Gamma^k_{ij} = \Gamma^k_{ji})$$

so that the geodesic equation is equivalent to

$$\ddot x^k + \Gamma^k_{ij}\dot x^i \dot x^j = 0 \quad \forall k = 1, ..., n. \tag{4.1.6}$$

Since the coefficients $\Gamma^k_{ij} = \Gamma^k_{ij}(x)$ depend smoothly upon x we can use the classical Banach-Picard theorem on existence in initial value problems (see e.g. [4]). We deduce the following local existence result.

Proposition 4.1.10. *Let (M, g) be a Riemann manifold. For any compact subset $K \subset TM$ there exists $\varepsilon > 0$ such that for any $(x, X) \in K$ there exists a unique geodesic $\gamma = \gamma_{x,X} : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = X$.*

One can think of a geodesic as defining a path in the tangent bundle $t \mapsto (\gamma(t), \dot{\gamma}(t))$. The above proposition shows that the geodesics define a local flow Φ on TM by

$$\Phi_t(x, X) = (\gamma(t), \dot{\gamma}(t)) \quad \gamma = \gamma_{x,X}.$$

Definition 4.1.11. *The local flow defined above is called the* geodesic flow *of the Riemann manifold (M, g). When the geodesic glow is a global flow i.e. any $\gamma_{x,X}$ is defined at each moment of time t for any $(x, X) \in TM$ then the Riemann manifold is called* geodesically complete.

The geodesic flow has some remarkable properties.

Conservation of energy. If $\gamma(t)$ is a geodesic then the length of $\dot{\gamma}(t)$ is independent of time.

□

Indeed

$$\frac{d}{dt}|\dot{\gamma}(t)|^2 = \frac{d}{dt}g(\dot{\gamma}(t), \dot{\gamma}(t)) = 2g(\nabla_{\dot{\gamma}(t)}, \dot{\gamma}(t)) = 0.$$

Thus if we consider the sphere bundles

$$S_r(M) = \{X \in TM \; ; \; |X| = r\}$$

we deduce that $S_r(M)$ are invariant subsets of the geodesic flow.

Exercise 4.1.3. Describe the infinitesimal generator of the geodesic flow. □

Example 4.1.12. Let G be a connected Lie group and $\mathfrak{L}_G$ its Lie algebra. Any $X \in \mathfrak{L}_G$ defines an endomorphism $ad\,(X)$ of $\mathfrak{L}_G$ by

$$ad(X)Y = [X, Y].$$

The Jacobi identity implies that

$$ad([X, Y]) = [ad(X), ad(Y)],$$

where the bracket in the right hand side is the usual commutator of two endomorphisms.

Assume there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{L}_G$ such that for any $X \in \mathfrak{L}_G$, $ad(X)$ is skew-adjoint i.e.

$$\langle [X, Y], Z \rangle = -\langle Y, [X, Z] \rangle \tag{4.1.7}$$

We can now extend this inner product to a left invariant metric h on G. We want to describe its geodesics. First, we have to determine the associated Levi-Civita connection. Using (4.1.3) we get

$$h(\nabla_X Z, Y) = \frac{1}{2}\{Xh(Y, Z) - Y(Z, X) + Zh(X, Y)$$

$$-h([X,Y],Z)+h([Y,Z],X)-h([Z,X],Y)\}.$$

If we take $X,Y,Z\in\mathfrak{L}_G$, i.e. these vector fields are left invariant, then $h(Y,Z)=constant$, $h(Z,X)=const.$, $h(X,Y)=const.$ so that the first three terms in the above formula vanish. We obtain the following equality (at $1\in G$)

$$\langle\nabla_X Z,Y\rangle=\frac{1}{2}\{-\langle[X,Y],Z\rangle+\langle[Y,Z],X\rangle-\langle[Z,X],Y\rangle\}.$$

Using the skew-symmetry of $ad(X)$ and $ad(Z)$ we deduce

$$\langle\nabla_X Z,Y\rangle=\frac{1}{2}\langle[X,Z],Y\rangle$$

so that at $1\in G$

$$\nabla_X Z=\frac{1}{2}[X,Z]\quad\forall X,Z\in\mathfrak{L}_G. \tag{4.1.8}$$

This formula correctly defines a connection since any $X\in\operatorname{Vect}(G)$ can be written as a linear combination

$$X=\sum\alpha_i X_i\quad\alpha_i\in C^\infty(G)\quad X_i\in\mathfrak{L}_G.$$

Now, if $\gamma(t)$ is a geodesic we can write $\dot\gamma(t)=\sum\gamma_i X_i$ so that

$$0=\nabla_{\dot\gamma(t)}\dot\gamma(t)=\sum_i\dot\gamma_i X_i+\frac{1}{2}\sum_{i,j}\gamma_i\gamma_j[X_i,X_j].$$

Since $[X_i,X_j]=-[X_j,X_i]$ we deduce $\dot\gamma_i=0$ i.e.

$$\dot\gamma(t)=\sum\gamma_i(0)X_i=X.$$

This means γ is an integral curve of the left invariant vector field X so that the geodesics through the origin with initial direction $X\in T_1G$ are

$$\gamma_X(t)=\exp(tX).$$

□

Exercise 4.1.4. Let G be a Lie group and h a bi-invariant metric on G. Prove that its restriction to $\mathfrak{L}_G$ satisfies (4.1.7). In particular, on any compact Lie groups there exist metrics satisfying (4.1.7). □

Definition 4.1.13. *Let $\mathfrak{L}$ be a finite dimensional real Lie algebra. The* Killing pairing *or* form *is the bilinear map*

$$\kappa:\mathfrak{L}\times\mathfrak{L}\to\mathbb{R},\ \kappa(x,y)=-\operatorname{tr}(ad(x)ad(y))\quad x,y\in\mathfrak{L}.$$

The Lie algebra $\mathfrak{L}$ is said to be semisimple *if the Killing pairing is a duality. A Lie group G is called semisimple if its Lie algebra is semisimple.*

Exercise 4.1.5. Prove that $SO(n)$ and $SU(n)$ and $SL(n,\mathbb{R})$ are semisimple Lie groups but $U(n)$ is not. □

Exercise 4.1.6. Let G be a *compact* Lie group. Prove that the Killing form is positive semi-definite[1] and satisfies (4.1.7).
Hint Use Exercise 4.1.4 □

Exercise 4.1.7. Show that the parallel transport of X along $\exp(tY)$ is

$$(L_{\exp(\frac{t}{2}Y)})_*(R_{\exp(\frac{t}{2}Y)})_*X.$$

□

Example 4.1.14. (Geodesics on flat tori and on $SU(2)$) The n-dimensional torus $T^n \cong S^1 \times \cdots \times S^1$ is an Abelian, compact Lie group. If $(\theta^1, ..., \theta^n)$ are the natural angular coordinates on T^n then the flat metric is defined by

$$g = (d\theta^1)^2 + \cdots (d\theta^n)^2.$$

g is a bi-invariant metric on T^n and obviously its restriction to the origin satisfies the skew-symmetry condition (4.1.7) since the bracket is 0. The geodesics through 1 will be the exponentials

$$\gamma_{\alpha_1,\dots,\alpha_n}(t) \quad t \mapsto (e^{\mathbf{i}\alpha_1 t}, ..., e^{\mathbf{i}\alpha_n t}) \quad \alpha_k \in \mathbb{R}.$$

If the numbers α_k are linearly dependent over $\mathbb{Q}$ then obviously $\gamma_{\alpha_1,\dots,\alpha_n}(t)$ is a closed curve. On the contrary, when the α's are linearly independent over $\mathbb{Q}$ then a classical result of Kronecker (see e.g. [33]) states that the image of $\gamma_{\alpha_1,\dots,\alpha_n}$ is dense in T^n!!! (see also Section 7.4 to come)

The special unitary group $SU(2)$ can also be identified with the group of unit quaternions

$$\{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}\ ;\ a^2 + b^2 + c^2 + d^2 = 1\}$$

so that $SU(2)$ is diffeomorphic with the unit sphere $S^3 \subset \mathbb{R}^4$. The round metric on S^3 is bi-invariant with respect to left and right (unit) quaternionic multiplication (verify this) and its restriction to (1,0,0,0) satisfies (4.1.7). The geodesics of this metric are the 1-parameter subgroups of S^3 and we let the reader verify that these are in fact the great circles of S^3 i.e. the circles of maximal diameter on S^3. Thus, all geodesics on S^3 are closed. □

4.1.3 The exponential map and normal coordinates

We have already seen that there are many differences between the classical Euclidean geometry and the the general riemannian geometry in the large. In particular we have seen examples in which one of the basic axioms of Euclidean geometry no longer holds: two distinct geodesic (read lines) may intersect in more than one point. The global topology of the manifold is responsible for this "failure".

[1]The converse of the above exercise is also true, i.e. any semisimple Lie group with positive definite Killing form is compact. This is known as Weyl's theorem. Its proof, which will be given later in the book, requires substantially more work.

Locally however, things are not "as bad". Local Riemannian geometry is similar in many respects with the Euclidean geometry. For example, locally, all of the classical incidence axioms hold.

In this section we will define using the metric some special collections of local coordinates in which things are very close to being Euclidean.

Let (M, g) be a Riemann manifold and U an open coordinate neighborhood with coordinates $(x^1, ..., x^n)$. We will try to find a local change in coordinates $(x^i) \mapsto (y^j)$ in which the expression of the metric is as close as possible to the Euclidean metric $g_0 = \delta_{ij} dy^i dy^j$.

Let $q \in U$ be the point with coordinates (0,...,0). Via a linear change in coordinates we may as well assume that

$$g_{ij}(q) = \delta_{ij}.$$

We can formulate this by saying that (g_{ij}) is Euclidean up to order zero.

We would like to "spread" the above equality to an entire neighborhood of q. To achieve this we try to find local coordinates (y^j) near q such that in these new coordinates the metric is Euclidean up to order one i.e.

$$g_{ij}(q) = \delta_{ij} \quad \frac{\partial g_{ij}}{\partial y^k}(q) = \frac{\partial \delta_{ij}}{\partial y^k}(q) = 0. \quad \forall i, j, k.$$

We now describe a geometric way of producing such coordinates using the geodesic flow.

Denote as usual the geodesic from q with initial direction $X \in T_qM$ by $\gamma_{q,X}(t)$. Note the following simple fact

$$\forall s > 0 \quad \gamma_{q,sX}(t) = \gamma_{q,X}(st).$$

Hence, there exists a small neighborhood V of $0 \in T_qM$ such that for any $X \in V$ the geodesic $\gamma_{q,X}(t)$ is defined for all $|t| \leq 1$. We define the *exponential map* at q by

$$\exp_q : V \subset T_qM \to M \quad X \mapsto \gamma_{q,X}(1).$$

T_qM is an Euclidean space and we can define $\mathbf{D}_q(r) \subset T_qM$ - the open "disk" of radius r centered at the origin. We have the following result.

Proposition 4.1.15. *Let (M, g) and $q \in M$ as above. Then there exists $r > 0$ such that the exponential map*

$$\exp_q : \mathbf{D}_q(r) \to M$$

is a diffeomorphism onto. The supremum of all radii r with this property is denoted by $\rho_M(q)$.

Definition 4.1.16. *$\rho_M(q)$ is called the* injectivity radius *of M at q. The infimum*

$$\rho_M = \inf_q \rho_M(q)$$

is called the injectivity radius of M.

The proof relies on the following key fact.

Lemma 4.1.17. *The Frechet differential at $0 \in T_qM$ of the exponential map*

$$D_0 \exp_q : T_qM \to T_{\exp_q(0)}M = T_qM$$

is the identity $T_qM \to T_qM$.

Proof of the lemma. Consider $X \in T_qM$. It defines a line in T_qM by $t \mapsto tX$ which is mapped via the exponential map to the geodesic $\gamma_{q,X}(t)$. By definition

$$(D_0 \exp_q)X = \dot{\gamma}_{q,X}(0) = X.$$

□

Proposition 4.1.15 follows immediately from the above lemma using the inverse function theorem.

□

Now choose an orthonormal frame $(\mathbf{e}_1, ..., \mathbf{e}_n)$ of T_qM and denote by $(\mathbf{x}^1, ..., \mathbf{x}^n)$ the resulting cartesian coordinates. For $0 < r < \rho_M(q)$ any point $p \in \exp_q(\mathbf{D}_q(r))$ can be uniquely written as

$$p = \exp_q(\mathbf{x}^i \mathbf{e}_i)$$

so the collection $(\mathbf{x}^1, ..., \mathbf{x}^n)$ provides a coordinatization of the open set $\exp_q(\mathbf{D}_q(r)) \subset M$. The coordinates thus obtained are called *normal coordinates* at q , the open set $\exp_q(\mathbf{D}_q(r))$ is called a *normal neighborhood* and will be denoted by $\mathbf{B}_r(q)$ for reasons that will become apparent a little later.

Proposition 4.1.18. *Let $(\mathbf{x}^i)$ be normal coordinates at $q \in M$ and denote by $\mathbf{g}_{ij}$ the expression of the metric tensor in these coordinates. Then we have*

$$\mathbf{g}_{ij}(q) = \delta_{ij} \ \text{ and } \ \frac{\partial \mathbf{g}_{ij}}{\partial \mathbf{x}^k}(q) = 0 \quad \forall i, j, k.$$

Thus, the normal coordinates provide a first order contact between g and the Euclidean metric.

Proof By construction, the vectors $\mathbf{e_i} = \frac{\partial}{\partial \mathbf{x}^i}$ form an orthonormal basis of T_qM and this proves the first equality. To prove the second equality we need the following auxiliary result.

Lemma 4.1.19. *In normal coordinates $(\mathbf{x}_i)$ (at q) the Christoffel symbols Γ^i_{jk} vanish at q.*

Proof of the lemma. For any $(m^1, ..., m^n) \in \mathbb{R}^n$ the curve $\mathbf{x}^i = m^i t$ is the geodesic $t \mapsto \exp_q\left(\sum m^i \frac{\partial}{\partial \mathbf{x}^i}\right)$ so that

$$\Gamma^i_{jk}(\mathbf{x}(t)) m^j m^k = 0.$$

In particular

$$\Gamma^i_{jk}(0) m^j m^k = 0 \quad \forall m^j \in \mathbb{R}^n$$

from which we deduce the lemma.

□

The result in the above lemma can be formulated as

$$g\left(\nabla_{\frac{\partial}{\partial \mathbf{x}^j}} \frac{\partial}{\partial \mathbf{x}^i}, \frac{\partial}{\partial \mathbf{x}^k}\right) = 0 \quad \forall i, j, k$$

so that

$$\nabla_{\frac{\partial}{\partial \mathbf{x}^j}} \frac{\partial}{\partial \mathbf{x}^i} = 0 \quad \text{at } q \ \ \forall i, j. \tag{4.1.9}$$

Using $\nabla g = 0$ we deduce

$$\frac{\partial \mathbf{g}_{ij}}{\partial \mathbf{x}^k}(q) = (\frac{\partial}{\partial \mathbf{x}^k} \mathbf{g}_{ij}) \mid_q = 0.$$

□

The reader may ask whether we can go one step further and find local coordinates which produce a second order contact with the Euclidean metric. At this step we are in for a big surprise. This thing is in general not possible and in fact there is a geometric way of measuring the "second order distance" between an arbitrary metric and the Euclidean metric. This is where the curvature of the Levi-Civita connection comes in, and we will devote an entire section to this subject.

4.1.4 The minimizing property of geodesics

We defined geodesics via a 2nd order equation imitating the 2nd order equation defining lines in an Euclidean space. As we have already mentioned, this is not the unique way of extending the notion of Euclidian straight line to arbitrary Riemann manifolds. One may try to look for curves of minimal length joining two given points. We will prove that the geodesics defined as in the previous subsection do just that, at least locally. We begin with a technical result which is interesting in its own. Let (M, g) be a Riemann manifold.

Lemma 4.1.20. *For each $q \in M$ there exists $0 < r < \rho_M(q)$ and $\varepsilon > 0$ such that $\forall m \in \mathbf{B}_r(q)$ we have $\varepsilon < \rho_M(m)$ and $\mathbf{B}_\varepsilon(m) \supset \mathbf{B}_r(q)$. In particular, any two points of $\mathbf{B}_r(q)$ can be joined by a* unique *geodesic of length $< \varepsilon$.*

We must warn the reader that the above result does not guarantee that the postulated connecting geodesic lies entirely in $\mathbf{B}_r(q)$. This is a different ball game.

Proof of the lemma. Using the smooth dependence upon initial data in ordinary differential equations we deduce that there exists an open neighborhood V of $(q, 0) \in TM$ such that $\exp_m X$ is well defined for all $(m, X) \in V$. We get a smooth map

$$F : V \to M \times M \quad (m, X) \mapsto (m, \exp_m X).$$

We compute the differential of F at $(q, 0)$. First, using normal coordinates $(\mathbf{x}^i)$ near q we get coordinates $(\mathbf{x}^i; \mathbf{X}^j)$ near $(q, 0) \in TM$. The partial derivatives of F at $(q, 0)$ are

$$D_{(q,0)}F(\frac{\partial}{\partial \mathbf{x}^i}) = \frac{\partial}{\partial \mathbf{x}^i} + \frac{\partial}{\partial \mathbf{X}^i}$$

$$D_{(q,0)}F(\frac{\partial}{\partial \mathbf{X}^i}) = \frac{\partial}{\partial \mathbf{X}^i}.$$

Thus the matrix defining $D_{(q,0)}F$ has the form

$$\begin{bmatrix} \mathbb{1} & 0 \\ * & \mathbb{1} \end{bmatrix}$$

and in particular it is nonsingular.

It follows from the implicit function theorem that F maps some neighborhood V of $(q,0) \in TM$ diffeomorphically onto some neighborhood U of $(q,q) \in M \times M$. We can choose V to have the form $\{(m,X)\ ;\ |X|_m < \varepsilon,\ \ m \in \mathbf{B}_\delta(q)\}$ for some sufficiently small ε and δ. Choose $0 < r < \min(\varepsilon, \rho_M(q))$ such that

$$\forall\, m_1, m_2 \in \mathbf{B}_r(q) \quad (m_1, m_2) \in U.$$

In particular, it follows that for any $m \in \mathbf{B}_r(q)$

$$\exp_m : \mathbf{D}_\varepsilon(m) \subset T_mM \to M$$

is a diffeomorphism onto, and

$$\mathbf{B}_\varepsilon(m) = \exp_m(\mathbf{D}_\varepsilon(m)) \supset \mathbf{B}_r(q).$$

Clearly, for any m the curve $t \mapsto \exp_m(tX)$ is a geodesic of length $< \varepsilon$ joining m to $\exp_m(X)$. It is the unique geodesic with this property since $F : V \to U$ is injective.

□

We can now formulate the main result of this subsection.

Theorem 4.1.21. *Let q, r and ε as in the previous lemma and consider $\gamma : [0,1] \to M$ the unique geodesic of length $< \varepsilon$ joining two points of $\mathbf{B}_r(q)$. If $\omega : [0,1] \to M$ is a piecewise smooth path with the same endpoints as γ then*

$$\int_0^1 |\dot\gamma(t)|dt \le \int_0^1 |\dot\omega(t)|dt$$

with equality if and only if $\omega([0,1]) = \gamma([0,1])$. Thus γ is the shortest path joining its endpoints.

The proof relies on two lemmata. Let $m \in M$ be an arbitrary point and assume $0 < R < \rho_M(m)$.

Lemma 4.1.22. (Gauss) *In $\mathbf{B}_R(m)$ the geodesics through m are orthogonal to the hypersurfaces*

$$\Sigma_\delta = \exp_q(S_\delta(0)) = \{\exp_m(X)\ ;\ |X| = \delta\} \quad 0 < \delta < R.$$

Proof Let $t \mapsto X(t)$, $0 \le t \le 1$ denote an arbitrary smooth curve in T_mM such that $|X(t)|_m = 1$ i.e. $X(t)$ is a curve on the unit sphere $S_1(0) \subset T_mM$. We have to prove that the curves $t \mapsto \exp_m(\delta X(t))$ are orthogonal to the radial geodesic $s \mapsto \exp_m(sX(0))$, $0 \le s \le R$.

Consider the parameterized surface

$$f(s,t) = \exp_m(sX(t)) \quad (s,t) \in [0,R) \times [0,1].$$

Set

$$\frac{\partial f}{\partial s} \stackrel{def}{=} f_*\left(\frac{\partial}{\partial s}\right) \in T_{f(s,t)}M.$$

Define $\frac{\partial f}{\partial t}$ similarly. $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ are not vector fields in the usual sense since the surface f may have double points and at such point f_* may associate two tangent vectors. One way around this problem is to pullback everything (metric, connection etc.) to the rectangle $[0,R] \times [0,1]$ via f and everything will be correctly defined. We will denote the pulled-back objects by the same symbols as the originals. It is convenient to view these "vector fields" as the tangent vectors to the "coordinate lattice" $f(t = const.)$, $f(s = const.)$. We have to show

$$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = \langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle = 0 \quad \forall (s,t)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product defined by g. The first equality is tautological. Using the metric compatibility of the (pulled-back) Levi-Civita connection we compute

$$\frac{\partial}{\partial s}\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle = \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle + \langle \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} \rangle.$$

Since the curves $f(t = const.)$ are geodesics we deduce

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0.$$

On the other hand, since $\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0$ we deduce (using the symmetry of the Levi-Civita connection)

$$\langle \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} \rangle = \langle \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} \rangle = \frac{1}{2}\frac{\partial}{\partial t}\left|\frac{\partial}{\partial s}\right|^2 = 0$$

since $\left|\frac{\partial f}{\partial s}\right| = |X(t)| = 1$. We conclude that the quantity $\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle$ is independent of s. For $s = 0$ we have $f(0,t) = \exp_m(0)$ so that $\frac{\partial f}{\partial t} = 0$, and therefore

$$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = 0 \quad \forall (s,t)$$

as needed.

□

Now consider any continuous, piecewise smooth curve

$$\omega : [a,b] \to \mathbf{B}_R(m) \setminus \{m\}.$$

Each $\omega(t)$ can be uniquely expressed in the form

$$\omega(t) = \exp_m(\rho(t)X(t)) \quad |X(t)| = 1 \quad 0 < |\rho(t)| < R.$$

Lemma 4.1.23. *The length of the curve $\omega(t)$ is $\geq |\rho(b) - \rho(a)|$. The equality holds iff $X(t) = const$ and $\dot{\rho}(t) \geq 0$. In other words, the shortest path joining two concentrical shells Σ_δ is a radial geodesic.*

Proof Let $f(\rho, t) = \exp_m(\rho X(t))$ so that

$$\omega(t) = f(\rho(t), t).$$

Then

$$\dot{\omega} = \frac{\partial f}{\partial \rho}\dot{\rho} + \frac{\partial f}{\partial t}.$$

Since the vectors $\frac{\partial f}{\partial \rho}$ and $\frac{\partial f}{\partial t}$ are mutually orthogonal and since

$$\left|\frac{\partial f}{\partial \rho}\right| = |X(t)| = 1$$

we get

$$|\dot{\omega}|^2 = |\dot{\rho}|^2 + \left|\frac{\partial f}{\partial \rho}\right|^2 \geq |\dot{\rho}|^2.$$

The equality holds if and only if $\frac{\partial f}{\partial \rho} = 0$ i.e. $\dot{X} = 0$. Thus

$$\int_a^b |\dot{\omega}| dt \geq \int_a^b |\dot{\rho}| dt \geq |\rho(b) - \rho(a)|.$$

Equality holds if and only if $\rho(t)$ is monotone and $X(t)$ is constant. This completes the proof of the lemma.

□

The proof of Theorem 4.1.21 is now immediate. Let $m_0, m_1 \in \mathbf{B}_r(q)$ and $\gamma : [0,1] \to M$ a geodesic of length $< \varepsilon$ such that $\gamma(i) = m_i$, $i = 0, 1$. We can write

$$\gamma(t) = \exp_{m_0}(tX) \quad X \in \mathbf{D}_\varepsilon(m_0).$$

Set $R = |X|$. Consider any other piecewise smooth path $\omega : [a,b] \to M$ joining m_0 to m_1. For any $\delta > 0$ this path must contain a portion joining the shell $\Sigma_\delta(m_0)$ to the shell $\Sigma_R(m_0)$ and lying between them. By the previous lemma the length of this segment will be $\geq R - \delta$. Letting $\delta \to 0$ we deduce

$$\mathfrak{l}(\omega) \geq R = \mathfrak{l}(\gamma).$$

If $\omega([a,b])$ does not coincide with $\gamma([0,1])$ then we obtain a strict inequality.

□

Any Riemann manifold has a natural structure of metric space. More precisely we set

$$d(p,q) = \inf\Big\{\mathfrak{l}(\omega) \;\; \omega : [0,1] \to M \;\text{ piecewise smooth path joining } p \text{ to } q\Big\}$$

A piecewise smooth path ω connecting two points p, q such that $\mathfrak{l}(\omega) = d(p,q)$ is said to be *minimal*. From Theorem 4.1.21 we deduce immediately the following consequence.

Corollary 4.1.24. *The image of any minimal path coincides with the image of a geodesic. In other words, any minimal path can be reparameterized such that it satisfies the geodesic equation.*

Exercise 4.1.8. Prove the above corollary. □

Theorem 4.1.21 also implies that any two nearby points can be joined by a unique minimal geodesic. In particular we have the following consequence.

Corollary 4.1.25. *Let $q \in M$. Then for all $r > 0$ sufficiently small*

$$\exp_q(\mathbf{D}_r(0))(= \mathbf{B}_r(q)) = \{p \in M \; ; \; d(p, q) < r\} \tag{4.1.10}$$

Corollary 4.1.26. *For any $q \in M$ we have the equality*

$$\rho_M(q) = \sup\{r \; ; \; r \text{ satisfies } (4.1.10)\}.$$

Proof The same argument as in the proof of Theorem 4.1.21 shows that $\forall r < \rho_M(q)$ the radial geodesics $\exp_q(tX)$ are minimal.

□

Definition 4.1.27. *A subset $U \subset M$ is said to be* convex *if any two points in U can be joined by a* unique minimal *geodesic which lies entirely* inside *U.*

Proposition 4.1.28. *For any $q \in M$ there exists $0 < R < \iota_M(q)$ such that for any $r < R$ the ball $\mathbf{B}_r(q)$ is convex.*

Proof Choose a small $\varepsilon > 0$ ($0 < 2\varepsilon < \rho_M(q)$) and $0 < R < \varepsilon$ such any two points m_0, m_1 in $B_R(q)$ can be joined by a unique minimal geodesic $\gamma_{m_0,m_1}(t)$ ($0 \leq t \leq 1$) of length $< \varepsilon$ (not necessarily contained in $\mathbf{B}_R(q)$). We will prove that $\forall m_0, m_1 \in \mathbf{B}_R(q)$ the map $t \mapsto d(q, \gamma_{m_0,m_1}(t))$ is convex and thus it achieves its maxima at the endpoints $t = 0, 1$. Note that

$$d(q, \gamma(t)) < R + \varepsilon < \rho_M(q).$$

The geodesic $\gamma_{m_0,m_1}(t)$ can be uniquely expressed as

$$\gamma_{m_0,m_1}(t) = \exp_q(r(t)X(t)) \quad X(t) \in T_qM \quad \text{with } r(t) = d(q, \gamma_{m_0,m_1}(t)).$$

It suffices to show $\frac{d^2}{dt^2}(r^2) \geq 0$ for $t \in [0, 1]$ if $d(q, m_0)$ and $d(q, m_1)$ are sufficiently small.

At this moment it is convenient to use normal coordinates $(\mathbf{x}^i)$ near q. The geodesic γ_{m_0,m_1} takes the form $(\mathbf{x}^i(t))$ and we have

$$r^2 = (\mathbf{x}^1)^2 + \cdots + (\mathbf{x}^n)^2.$$

We compute easily

$$\frac{d^2}{dt^2}(r^2) = 2r^2(\ddot{\mathbf{x}}^1 + \cdots + \ddot{\mathbf{x}}^n) + |\dot{\mathbf{x}}|^2 \tag{4.1.11}$$

where $\dot{\mathbf{x}}(t) = \sum \dot{\mathbf{x}}^i \mathbf{e}_i \in T_qM$. γ satisfies the equation

$$\ddot{\mathbf{x}}^i + \Gamma^i_{jk}(\mathbf{x})\dot{\mathbf{x}}^j\dot{\mathbf{x}}^k = 0$$

Since $\Gamma^i_{jk}(0) = 0$ (normal coordinates) we deduce that there exists a constant $C > 0$ (depending only on the magnitude of the second derivatives of the metric at q) such that

$$|\Gamma^i_{jk}(\mathbf{x})| \leq C|x|.$$

Using the geodesic equation we obtain

$$\ddot{\mathbf{x}}^i \geq -C|\mathbf{x}||\dot{\mathbf{x}}|^2.$$

We substitute the above inequality in (4.1.11) to get

$$\frac{d^2}{dt^2}(r^2) \geq 2|\dot{\mathbf{x}}|^2 \left(1 - nC|\mathbf{x}|^3\right). \tag{4.1.12}$$

If we choose from the very beginning

$$R + \varepsilon \leq \left(\frac{1}{nC}\right)^{1/3}$$

then because along the geodesic $|x| \leq R + \varepsilon$ the right-hand-side of (4.1.12) is nonnegative. This establishes the convexity of $t \mapsto r^2(t)$ and concludes the proof of the proposition.

□

In the last result of this subsection we return to the concept of geodesic completeness. We will see that this can be described in terms of the metric space structure alone.

Theorem 4.1.29. (Hopf-Rinow) *Let M be a Riemann manifold and $q \in M$. The following assertions are equivalent:*
(a) $\exp_q$ is defined on all of T_qM.
(b) The closed and bounded (with respect to the metric structure) sets of M are compact.
(c) M is complete as a metric space.
(d) M is geodesically complete.
(e) There exists a sequence of compact sets $K_n \subset M$, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$ such that if $p_n \notin K_n$ then $d(q, p_n) \to \infty$.

Moreover, on a (geodesically) complete manifold any two points can be joined by a minimal geodesic.

Remark 4.1.30. On a complete manifold there could exist points (sufficiently far apart) which can be joined by more than one minimal geodesic. Think for example of a manifold where there exist closed geodesic (e.g the tori T^n). □

Exercise 4.1.9. Prove the Hopf-Rinow theorem. □

Exercise 4.1.10. Let (M, g) be a Riemann manifold and let (U_α) an open cover consisting of *bounded geodesically convex open sets*. Set $d_\alpha = (\text{diameter}\,(U_\alpha))^2$. Denote by g_α the metric on U_α defined by $g_\alpha = d_\alpha^{-1} g$ so that the diameter of U_α in the new metric is 1. Using a partition of unity (φ_i) subordinated to this cover we can form a new metric

$$\tilde{g} = \sum_i \varphi_i g_{\alpha(i)} \quad (\operatorname{supp} \varphi_i \subset U_{\alpha(i)}).$$

Prove that $\tilde{g}$ is a complete Riemann metric. Hence, on any manifold there exist complete Riemann metrics. □

4.1.5 Calculus on Riemann manifolds

The classical vector analysis extends nicely to Riemann manifolds. We devote this subsection to describing this more general "vector analysis".

Let (M, g) be an oriented Riemann manifold. We now have two structures at our disposal: a Riemann metric and an orientation and we will use both of them to construct a plethora of operations on tensors.

First, using the *metric* we can construct by duality an isomorphism $\mathfrak{L} : \mathrm{Vect}\,(M) \to \Omega^1(M)$ (called *lowering the indices*).

Example 4.1.31. Let $M = \mathbb{R}^3$ with the Euclidean metric. A vector field V on M has the form

$$V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

Then

$$W = \mathfrak{L}V = Pdx + Qdy + Rdz.$$

If we think of V as a field of forces in the space then W is the infinitesimal work of V. $\square$

On a Riemann manifold there is an equivalent way of describing the exterior derivative.

Proposition 4.1.32. *Let*

$$\varepsilon : C^\infty(T^*M \otimes \Lambda^k T^*M) \to C^\infty(\Lambda^{k+1}T^*M)$$

denote the exterior multiplication operator

$$\varepsilon(\alpha \otimes \beta) = \alpha \wedge \beta, \quad \forall \alpha \in \Omega^1(M),\ \beta \in \Omega^k(M).$$

Then

$$d = \varepsilon \circ \nabla$$

where d denotes the exterior derivative and ∇ is the connection induced on $\Lambda^k T^*M$ by the Levi-Civita connection. $\square$

Proof We will use a strategy useful in many other situations. Our discussion about normal coordinates will payoff. Denote temporarily by D the operator $\varepsilon \circ \nabla$. The equality $d = D$ is a local statement and it suffices to prove it in any coordinate neighborhood. Choose $(\mathbf{x}^i)$ normal coordinates at an arbitrary point $p \in M$ and set $\partial_i = \frac{\partial}{\partial \mathbf{x}^i}$. Note that

$$D = \sum_i dx^i \wedge \nabla_i \quad \nabla_i = \nabla_{\partial_i}.$$

Let $\omega \in \Omega^k(M)$. Near p it can be written as

$$\omega = \sum_I \omega_I d\mathbf{x}^I$$

where as usual, for any ordered multi-index I: $(1 \leq i_1 < \cdots < i_k \leq n)$ we set $d\mathbf{x}^I = d\mathbf{x}^{i_1} \wedge \cdots \wedge d\mathbf{x}^{i_k}$. In normal coordinates *at* p we have $(\nabla_i \partial_i)\,|_p = 0$ from which we get the equalities

$$(\nabla_i d\mathbf{x}^j)\,|_p\,(\partial_k) = -(d\mathbf{x}^j(\nabla_i \partial_k))\,|_p = 0.$$

Thus *at* p

$$D\omega = \sum_I d\mathbf{x}^i \wedge \nabla_i(\omega_I d\mathbf{x}^I)$$

$$= \sum_I d\mathbf{x}^i \wedge (\partial_j \omega_I d\mathbf{x}^I + \omega_I \nabla_i(d\mathbf{x}^I)) = \sum_I d\mathbf{x}^i \wedge \partial_j \omega_I = d\omega.$$

Since the point p was chosen arbitrarily this completes the proof of Proposition 4.1.32.

□

Exercise 4.1.11. Show that for any k-form ω on the Riemann manifold (M,g) the exterior derivative $d\omega$ can be expressed by

$$d\omega(X_0, X_1, \cdots, , X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \cdots, \hat{X}_i, \cdots, X_k)$$

for all $X_0, \cdots, X_k \in \mathrm{Vect}\,(M)$. ($\nabla$ denotes the Levi-Civita connection.) □

The Riemann metric defines by duality a metric in any tensor bundle $\mathcal{T}^r_s(M)$ which we continue to denote by g. Thus, given two tensor fields T_1, T_2 of the same type (r,s) we can form their pointwise scalar product

$$M \ni p \mapsto g(T,S)_p = g_p(T_1(p), T_2(p)).$$

In particular, any such tensor has a pointwise norm

$$M \ni p \mapsto |T|_{g,p} = (T,T)_p^{1/2}.$$

Using the *orientation* we can construct (using the results in subsection 2.2.4) a natural *volume form* on M which we denote by dv_g and we call it the *metric volume* . This is the positively oriented volume form of pointwise norm $\equiv 1$. If $(x^1, ..., x^n)$ are local coordinates such that $dx^1 \wedge \cdots \wedge dx^n$ is positively oriented then

$$dv_g = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

where $|g| = \det(g_{ij})$. In particular we can integrate (compactly supported) functions on M by

$$\int_{(M,g)} f \stackrel{def}{=} \int_M f dv_g \quad \forall f \in C_0^\infty(M).$$

We have the following (not so surprising) result.

Proposition 4.1.33.

$$\nabla_X dv_g = 0 \quad \forall X \in \mathrm{Vect}\,(M).$$

Proof We have to show that for any $p \in M$

$$(\nabla_X dv_g)(e_1, ..., e_n) = 0 \tag{4.1.13}$$

where $e_1, ..., e_p$ is a basis of T_pM. Choose $(\mathbf{x}^i)$ normal coordinates near p. Set $\partial_i = \frac{\partial}{\partial \mathbf{x}^i}$, $\mathbf{g}_{ij} = g(\partial_i, \partial_k)$ and $e_i = \partial_i |_p$. Since the expression in (4.1.13) is linear in X we may as well assume $X = \partial_k$ for some $k = 1, ..., n$. We compute

$$(\nabla_X dv_g)(e_1, ..., e_n) = X(dv_g(\partial_1, ..., \partial_n))|_p - \sum_i dv_g(e_1, ..., (\nabla_X \partial_i)|_p, ..., \partial_n). \qquad (4.1.14)$$

We consider each term separately. Note first that $dv_g(\partial_1, ..., \partial_n) = (\det(\mathbf{g}_{ij}))^{1/2}$ so that $X(\det(\mathbf{g}_{ij}))^{1/2} |_p = \partial_k(\det(\mathbf{g}_{ij}))^{1/2} |_p$ is a linear combination of products in which each product has a factor of the form $\partial_k \mathbf{g}_{ij} |_p$. Such a factor is zero since we are working in normal coordinates. Thus the first term in(4.1.14) is zero. The other terms are zero as well since in normal coordinates p

$$\nabla_X \partial_i = \nabla_{\partial_k} \partial_i = 0.$$

Proposition 4.1.33 is proved.

□

Once we have an orientation we also have the *Hodge $*$-operator*

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

uniquely determined by

$$\alpha \wedge *\beta = (\alpha, \beta) dv_g \quad \forall \alpha\ \beta \in \Omega^k(M). \qquad (4.1.15)$$

In particular

$$*1 = dv_g.$$

Example 4.1.34. To any vector field $F = P\partial_x + Q\partial_y + R\partial_z$ on $\mathbb{R}^3$ we associated its *infinitesimal work*

$$W_F = \mathcal{L}(F) = Pdx + Qdy + Rdz.$$

The *infinitesimal energy flux* of F is the 2-form

$$\Phi_F = *W_F = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.$$

The exterior derivative of W_F is the infinitesimal flux of the vector field $\mathbf{curl}\, F$

$$dW_F = (\partial_y R - \partial_z Q) dy \wedge dz + (\partial_z P - \partial_x R) dz \wedge dx + (\partial_x Q - \partial_y P) dx \wedge dy$$

$$= \Phi_{\mathbf{curl}\, F} = *W_{\mathbf{curl}\, F}.$$

The divergence of F is the scalar defined as

$$\mathbf{div}\, F = *d * W_F = *d\Phi_F = *\{(\partial_x P + \partial_y Q + \partial_z R) dx \wedge dy \wedge dz\} = \partial_x P + \partial_y Q + \partial_z R.$$

If f is a function on $\mathbb{R}^3$ then we compute easily

$$*d * df = \partial_x^2 f + \partial_y^2 f + \partial_x^2 f = \Delta f.$$

□

Definition 4.1.35. *(a) For any smooth function f on M we denote by* $\mathbf{grad}\, f$ *the vector field dual to the 1-form df. In other words*

$$(\mathbf{grad}\, f, X) = df(X) = X \cdot f \quad \forall X \in \mathrm{Vect}\,(M).$$

(b) For any $X \in \mathrm{Vect}\,(M)$ we denote by $\mathbf{div}\, X$ *the smooth function defined by the equality*

$$L_X dv_g = (\mathbf{div}\, X) dv_g.$$

Proposition 4.1.36. *Let X be a vector field on M and denote by α the 1-form dual to X. Then*
(a) $\mathbf{div}\, X = \mathrm{tr}\,(\nabla X)$ *where we view ∇X as an element of $C^\infty(\mathrm{End}\,(TM))$ via the identifications*

$$\nabla X \in \Omega^1(TM) \cong C^\infty(T^*M \otimes TM) \cong C^\infty(\mathrm{End}\,(TM)).$$

(b) $\mathbf{div}\, X = *d * \alpha$.
(c) If $(x^1, ..., x^n)$ are local coordinates such that $dx^1 \wedge \cdots \wedge dx^n$ is positively oriented then

$$\mathbf{div}\, X = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} X^i)$$

where $X = X^i \partial_i$.

The proof will rely on the following technical result which is interesting in its own.

Lemma 4.1.37. *Denote by δ the operator*

$$\delta = *d* : \Omega^k(M) \to \Omega^{k-1}(M).$$

Let α be a $(k-1)$-form and β a k-form such that at least one of them is compactly supported. Then

$$\int_M (d\alpha, \beta) dv_g = (-1)^{\nu_{n,k}} \int_M (\alpha, \delta\beta) dv_g$$

where $\nu_{n,k} = nk + n + 1$.

Definition 4.1.38. *Define $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ by*

$$d^* = (-1)^{\nu_{n,k}} \delta = (-1)^{\nu_{n,k}} * d * .$$

Proof of the lemma. We have

$$\int_M (d\alpha, \beta) dv_g = \int_M d\alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) + (-1)^k \int_M \alpha \wedge d * \beta.$$

The first integral in the right-hand-side vanishes by the Stokes formula since $\alpha \wedge *\beta$ has compact support. Since

$$d * \beta \in \Omega^{n-k+1}(M) \text{ and } *^2 = (-1)^{(n-k+1)(k-1)} \text{ on } \Omega^{n-k+1}(M)$$

we deduce

$$\int_M (d\alpha, \beta) dv_g = (-1)^{k+(n-k+1)(n-k)} \int_M \alpha \wedge *\delta\beta.$$

This establishes the assertion in the lemma since

$$(n-k+1)(k-1)+k \equiv \nu_{n,k} \,(\mathrm{mod}\, 2).$$

$\square$

Proof of the proposition. Set $\Omega = dv_g$ and let $(X_1, ..., X_n)$ be a local moving frame of TM in a neighborhood of some point. Then

$$(L_X\Omega)(X_1, ..., X_n) = X(\Omega(X_1, ..., X_n)) - \sum_i \Omega(X_1, ..., [X, X_i], ..., X_n). \tag{4.1.16}$$

Since $\nabla\Omega = 0$ we get

$$X \cdot (\Omega(X_1, ..., X_n)) = \sum_i \Omega(X_1, ..., \nabla_X X_i, ..., X_n).$$

Using the above equality in (4.1.16) we deduce from $\nabla_X Y - [X, Y] = \nabla_Y X$ that

$$(L_X\Omega)(X_1, ..., X_n) = \sum_i \Omega(X_1, ..., \nabla_{X_i} X, ..., X_n). \tag{4.1.17}$$

Over the neighborhood where the local moving frame is defined we can find smooth functions f_i^j such that

$$\nabla_{X_i} X = f_i^j X_j \Rightarrow \mathrm{tr}\,(\nabla X) = f_i^i.$$

Part (a) of the proposition follows after we substitute the above equality in (4.1.17).
Proof of (b) From (a) we deduce that for any $f \in C_0^\infty(M)$ we have

$$L_X(f\omega) = (Xf)\Omega + f\mathrm{tr}\,(\nabla X)\Omega.$$

On the other hand

$$L_X(f\Omega) = (i_X d + d\, i_X)(f\Omega) = d\, i_X(f\Omega).$$

Hence

$$\{(Xf) + f\mathrm{tr}\,(\nabla X)\}\, dv_g = d(i_X f\Omega).$$

Since the form $f\Omega$ is *compactly supported* we deduce from Stokes formula

$$\int_M d(i_X f\Omega) = 0.$$

We have thus proved that for any compactly supported function f we have the equality

$$-\int_M f\mathrm{tr}\,(\nabla X) dv_g = \int_M Xf dv_g = \int_M df(X) dv_g$$

$$= \int_M (\mathbf{grad}\, f, X) dv_g = \int_M (df, \alpha) dv_g.$$

Using Lemma 4.1.37 we deduce

$$-\int_M f\mathrm{tr}\,(\nabla X) dv_g = -\int_M f\delta\alpha dv_g \quad \forall f \in C_0^\infty(M).$$

This completes the proof of (b).
Proof of (c) We use the equality

$$L_X(\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n) = \mathbf{div}\,(X)(\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n).$$

The desired formula follows derivating in the left-hand-side. One uses the fact that L_X is an even s-derivation and the equalities

$$L_X dx^i = \partial_i X^i dx^i \quad \text{(no summation)}$$

proved in Subsection 3.1.3.

□

Exercise 4.1.12. Let (M, g) be a Riemann manifold and $X \in \mathrm{Vect}\,(M)$. Show that the following conditions on X are equivalent.
(a) $L_X g = 0$.
(b) $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathrm{Vect}\,(M)$.
(A vector field X satisfying the above equivalent conditions is called a *Killing* vector field).
□

Exercise 4.1.13. Consider a Killing vector field X on the Riemann manifold (M, g) and denote by η the 1-form dual to X. Show that $\delta\eta = 0$ i.e. in other words $\mathbf{div}\,(X) = 0$. □

Definition 4.1.39. *Let (M, g) be an oriented Riemann manifold (possibly with boundary). For any k-forms α, β define*

$$\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle_M = \int_M (\alpha, \beta) dv_g = \int_M \alpha \wedge *\beta$$

whenever the integrals in the right-hand-side are finite.

Let (M, g) be an oriented Riemann manifold with boundary ∂M. By definition, M is a closed subset of a boundary-less manifold $\tilde{M}$ of the same dimension. Along ∂M we have a vector bundle decomposition

$$(T\tilde{M}) \mid_{\partial M} = T(\partial M) \oplus \mathfrak{n}$$

where $\mathfrak{n} = (T\partial M)^{\perp}$ is the orthogonal complement of $T\partial M$ in $(TM) \mid_{\partial M}$. Since both M and ∂M are oriented manifolds it follows that $\mathfrak{n}$ is a trivial line bundle. Indeed, over the boundary

$$\det TM = \det(T\partial M) \otimes \mathfrak{n}$$

so that

$$\mathfrak{n} \cong \det TM \otimes \det(T\partial M)^*.$$

In particular $\mathfrak{n}$ admits nowhere vanishing sections and each such section defines an orientation in the fibers of $\mathfrak{n}$. An *outer normal* is a nowhere vanishing section σ of $\mathfrak{n}$ such that for each $x \in \partial M$ and any positively oriented $\omega_x \in \det T_x\partial M$ $\sigma_x \wedge \omega_x$ is a positively oriented element of $\det T_x M$. Since $\mathfrak{n}$ carries a fiber metric we can select a unique outer normal of pointwise length $\equiv 1$. This will be called the *unit outer normal* and will be denoted by $\vec{\nu}$. Using partitions of unity we can extend $\vec{\nu}$ to a vector field defined on M.

Proposition 4.1.40. (Integration by parts) *Let (M, g) be a compact, oriented Riemann manifold with boundary, $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$. Then*

$$\int_M (d\alpha, \beta) dv_g = \int_{\partial M} (\alpha \wedge *\beta) \mid_{\partial M} + \int_M (\alpha, d^*\beta) dv_g$$

$$= \int_{\partial M} \alpha \mid_{\partial M} \wedge \hat{*}(i_{\vec{\nu}}\beta) \mid_{\partial M} + \int_M (\alpha, d^*\beta) dv_g$$

where $\hat{*}$ denotes the Hodge $*$-operator on ∂M with the induced metric $\hat{g}$ and orientation.

Using the $\langle \cdot, \cdot \rangle$ notation of Definition 4.1.39 we can rephrase the above equality as

$$\langle d\alpha, \beta \rangle_M = \langle \alpha, i_{\vec{\nu}}\beta \rangle_{\partial M} + \langle \alpha, d^*\beta \rangle_M .$$

Proof of the proposition. As in the proof of Lemma 4.1.37 we have

$$(d\alpha, \beta) dv_g = d\alpha \wedge *\beta = d(\alpha \wedge *\beta) + (-1)^k \alpha \wedge d * \beta.$$

The first part of the proposition follows from Stokes formula arguing precisely as in Lemma 4.1.37. To prove the second part we have to check that

$$(\alpha \wedge *\beta) \mid_{\partial M} = \alpha \mid_{\partial M} \wedge \hat{*}(i_{\vec{\nu}}\beta) \mid_{\partial M} .$$

This is a local (even a pointwise) assertion so we may as well assume $M = \mathbf{H}^n_+ = \{(x^1, ..., x^n) \in \mathbb{R}^n \ ; \ x^1 \geq 0\}$ and the metric is the Euclidean metric. Note that $\vec{\nu} = -\partial_1$. Let I be an ordered $(k-1)$-index and J an ordered k- index. Denote by J^c the ordered $(n-k)$-index complementary to J so that (as sets) $J \cup J^c = \{1, ..., n\}$. By linearity, it suffices to consider only the cases $\alpha = dx^I$, $\beta = dx^J$. We have

$$*dx^J = \epsilon_J dx^{J^c} \quad (\epsilon_J = \pm 1) \tag{4.1.18}$$

and

$$i_{\vec{\nu}} dx^J = \begin{cases} 0 & , \quad 1 \notin J \\ -dx^{J'} & , \quad 1 \in J \end{cases}$$

where $J' = J \setminus \{1\}$. Note that if $1 \notin J$ then $1 \in J^c$ so that

$$(\alpha \wedge *\beta) \mid_{\partial M} = 0 = \alpha \mid_{\partial M} \wedge \hat{*}(i_{\vec{\nu}}\beta) \mid_{\partial M}$$

so the only nontrivial situation left to be discussed is $1 \in J$. On the boundary

$$\hat{*}(i_{\vec{\nu}} dx^J) = -\hat{*}(dx^{J'}) = -\epsilon'_J dx^{J^c} \quad (\epsilon'_J = \pm 1). \tag{4.1.19}$$

We have to compare the two signs ϵ_J and ϵ'_J. in (4.1.18) and (4.1.19). ϵ_J is the signature of the permutation $J \vec{\cup} J^c$ of $\{1, ..., n\}$ obtained by writing the two increasing multi-indices one after the other, first J and then J^c. Similarly, since the boundary ∂M has the orientation $-dx^2 \wedge \cdots \wedge dx^n$, ϵ'_J is $(-1) \times$(the signature of the permutation $J' \vec{\cup} J^c$ of $\{2, ..., n\}$). Obviously

$$\text{sign}\,(J \vec{\cup} J^c) = \text{sign}\,(J' \vec{\cup} J^c)$$

so that $\epsilon_J = -\epsilon'_J$. The proposition now follows from (4.1.18) and (4.1.19).

$\square$

Corollary 4.1.41. (Gauss) *Let (M,g) be a compact, oriented Riemann manifold with boundary and X a vector field on M. Then*

$$\int_M \mathbf{div}\,(X)dv_g = \int_{\partial M}(X,\vec{\nu})dv_{g_\partial}$$

where $g_\partial = g\,|_{\partial M}$.

Proof Denote by α the 1-form dual to X. We have

$$\int_M \mathbf{div}\,(X)dv_g = \int_M 1\wedge *d*\alpha dv_g$$

$$= \int_M (1, *d*\alpha)dv_g = -\int_M (1, d^*\alpha)dv_g$$

$$= \int_{\partial M}\alpha(\vec{\nu})dv_{g_\partial} = \int_{\partial M}(X,\vec{\nu})dv_{g_\partial}.$$

Remark 4.1.42. The compactness assumption on M can be replaced with an integrability condition on the forms α, β so that the previous results hold for noncompact manifolds as well provided all the integrals are finite. □

Definition 4.1.43. *Let (M,g) be an oriented Riemann manifold. The* geometric Laplacian *is the linear operator $\Delta_M : C^\infty(M) \to C^\infty(M)$ defined by*

$$\Delta_M = d^*df = -*d*df = -\mathbf{div}\,(\mathbf{grad}\,f).$$

A smooth function f on M satisfying the equation $\Delta_M f = 0$ is called harmonic.

Using Proposition 4.1.36 we deduce that in local coordinates $(x^1, ..., x^n)$ the geometric laplacian takes the form

$$\Delta_M = -\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial j\right)$$

where (g^{ij}) denotes as usual the matrix inverse to (g_{ij}). Note that when g is the Euclidean metric the geometric Laplacian is

$$\Delta_0 = -(\partial_i^2 + \cdots + \partial_n^2)$$

which differs from the physicists Laplacian by a sign.

Corollary 4.1.44. (Green) *Let (M,g) as in Proposition 4.1.40 and f, $g \in C^\infty(M)$. Then*

$$\langle f, \Delta_M g\rangle_M = \langle df, dg\rangle_M - \langle f, \frac{\partial g}{\partial\vec{\nu}}\rangle_{\partial M}.$$

$$\langle f, \Delta_M g\rangle_M - \langle \Delta_M f, g\rangle_M = \langle \frac{\partial f}{\partial\vec{\nu}}, g\rangle_{\partial M} - \langle f, \frac{\partial g}{\partial\vec{\nu}}\rangle_{\partial M}.$$

Proof The first equality follows immediately from the integration by parts formula (Proposition 4.1.40) with $\alpha = f$ and $\beta = dg$. The second identity is now obvious.

□

Exercise 4.1.14. (a) Prove that the only harmonic functions on a compact oriented Riemann manifold M are the constant ones.
(b) If $u, f \in C^\infty(M)$ are such that $\Delta_M u = f$ show that

$$\int_M f = 0.$$

$\square$

Exercise 4.1.15. Denote by $(u^1, \dots, u^n)$ the coordinates on the round sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$ obtained via the stereographic projection from the south pole.
(a) Show that the round metric g_0 on S^n is given in these coordinates by

$$g_0 = \frac{4}{1+r^2}\{(du^1)^2 + \cdots + (du^n)^2\}$$

where $r^2 = (u^1)^2 + \cdots + (u^n)^2$.
(b) Show that the n-dimensional "area" of S^n is

$$\sigma_n = \int_{S^n} dv_{g_0} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$$

where Γ is Euler's gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

$\square$

Hint: Need to know the "doubling formula"

$$\pi^{1/2}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)$$

and (see [31])

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}dr = \frac{(\Gamma(n/2))^2}{2\Gamma(n)}.$$

Exercise 4.1.16. Consider the Killing form on $\underline{su}(2)$ (the Lie algebra of $SU(2)$) defined by

$$\langle X, Y\rangle = -\mathrm{tr}\, X \cdot Y.$$

(a) Show that the Killing form defines a bi-invariant metric on $SU(2)$ and then compute the volume of the group with respect to this metric. $SU(2)$ is given the orientation defined by $e_1 \wedge e_2 \wedge e_3 \in \Lambda^3\underline{su}(2)$ where $e_i \in \underline{su}(2)$ are the *Pauli matrices*

$$e_1 = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}$$

(b) Show that the trilinear form on $\underline{su}(2)$ defined by

$$B(X, Y, Z) = \langle [X, Y], Z\rangle$$

is skew-symmetric. In particular $B \in \Lambda^3 \underline{su}(2)^*$.
(c) B has a unique extension as a left-invariant 3-form on $SU(2)$ known as the *Cartan form*on $SU(2)$) which we continue to denote by B. Compute $\int_{SU(2)} B$.
Hint: Use the natural diffeomorphism $SU(2) \cong S^3$ and the computations in the previous exercise.

□

4.2 The Riemann curvature

A Riemann metric on a manifold has roughly speaking the effect of "giving a shape" to the manifold. Thus a very short (in diameter) manifold is different from a very long one. A large (in volume) manifold is different from a small one. However there is a lot more information encoded in the Riemann manifold than just its size. To recover it we need to look deeper in the structure and go beyond the first order approximations we have used so far. The Riemann curvature tensor achieves just that. It is an object which is very rich in information about the "shape" of a manifold and loosely speaking provides a second order approximation to the geometry of the manifold. As Riemann himself observed we do not need to go beyond this order of approximation to recover all the informations. In this section we introduce the reader to the Riemann curvature tensor and its associates. We will describe some special examples and we will conclude with the Gauss-Bonnet theorem which shows that the local object which is the Riemann curvature has global effects.

Unless otherwise indicated, we will use Einstein's summation convention.

4.2.1 Definitions and properties

Let (M, g) be a Riemann manifold and denote by ∇ the Levi-Civita connection.

Definition 4.2.1. *The* Riemann curvature *denoted by $R = R(g)$ is defined as*

$$R(g) = F(\nabla)$$

where $F(\nabla)$ is the curvature of the Levi-Civita connection.

The Riemann curvature is a tensor $R \in \Omega^2(\mathrm{End}\,(TM))$ explicitly defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

In local coordinates $(x^1, \cdots, x^n)$ we have the description

$$R^\ell_{ijk}\partial_\ell = R(\partial_j, \partial_k)\partial_i.$$

In terms of the Christoffel symbols we have

$$R^\ell_{ijk} = \partial_j\Gamma^\ell_{ik} - \partial_k\Gamma^\ell_{ij} + \Gamma^\ell_{mj}\Gamma^m_{ik} - \Gamma^\ell_{mk}\Gamma^m_{ij}.$$

Lowering the indices we get a new tensor

$$R_{ijk\ell} := g_{im}R^m_{jk\ell} = (R(\partial_k, \partial_\ell)\partial_j, \partial_i).$$

Theorem 4.2.2. (The symmetries of the curvature tensor.) *The Riemann curvature tensor R satisfies the following identities ($X, Y, Z, U, V \in \mathrm{Vect}\,(M)$).*
(a) $g(R(X,Y)U,V) = -g(R((Y,X),U,V)$.
(b) $g(R(X,Y)U,V) = -g(R(X,Y)V,U)$.
(c) (The 1st Bianchi identity)

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

(d) $g(R(X,Y)U,V) = g(R(U,V)X,Y)$.
(e) (The 2nd Bianchi identity)

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$$

In local coordinates the above identities have the form

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k},$$

$$R_{ijk\ell} = R_{k\ell ij},$$

$$R^i_{jk\ell} + R^i_{\ell jk} + R^i_{k\ell j} = 0,$$

$$(\nabla_i R)^j_{mk\ell} + (\nabla_\ell R)^j_{mik} + (\nabla_k R)^j_{m\ell i} = 0.$$

Proof (a) It follows immediately from the definition of R as an $\mathrm{End}(TM)$-valued skew-symmetric bilinear map $(X,Y) \mapsto R(X,Y)$.
(b) We have to show the symmetric bilinear form

$$Q(U,V) = g(R(X,Y)U,V) + g(R(X,Y)V,U)$$

is trivial. Thus it suffices to check $Q(U,U) = 0$. We may as well assume that $[X,Y] = 0$ since (locally) X, Y can be written as linear combinations (over $C^\infty(M)$) of commuting vector fields. (E.g. $X = X^i\partial_i$). Then

$$Q(U,U) = g((\nabla_X\nabla_Y - \nabla_Y\nabla_X)U,U).$$

We compute

$$\begin{aligned} Y(Xg(U,U)) &= 2Yg(\nabla_X U,U) \\ &= 2g(\nabla_Y\nabla_X U,U) + 2g(\nabla_X U,\nabla_Y U) \end{aligned}$$

and similarly

$$X(Yg(U,U)) = 2g(\nabla_X\nabla_Y U,U) + 2g(\nabla_X U,\nabla_Y U).$$

Subtracting the two equalities we deduce (b).
(c) As before, we can assume the vector fields X, Y, Z pairwise commute. The 1st Bianchi identity is then equivalent to

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X = 0.$$

The identity now follows from the symmetry of the connection: $\nabla_X Y = \nabla_Y X$ etc.
(d) We will use the following algebraic lemma ([44], Chapter 5).

Lemma 4.2.3. *Let $R : E \times E \times E \times E \to \mathbb{R}$ be a quadrilinear map on a real vector space E. Define*

$$S(X_1, X_2, X_3, X_4) = R(X_1, X_2, X_3, X_4) + R(X_2, X_3, X_1, X_4) + R(X_3, X_1, X_2, X_4).$$

If R satisfies the symmetry conditions

$$R(X_1, X_2, X_3, X_4) = -R(X_2, X_1, X_3, X_4)$$

$$R(X_1, X_2, X_3, X_4) = -R(X_1, X_2, X_4, X_3)$$

then

$$R(X_1, X_2, X_3, X_4) - R(X_3, X_4, X_1, X_2)$$
$$= \frac{1}{2}\left\{S(X_1, X_2, X_3, X_4) - S(X_2, X_3, X_4, X_1) - S(X_3, X_4, X_1, X_2) + S(X_4, X_3, X_1, X_2)\right\}.$$

The proof of the lemma is a straightforward (but tedious) computation which is left to the reader. The Riemann curvature $R = g(R(X_1, X_2)X_3, X_4)$ satisfies the symmetries required in the lemma and moreover, the 1st Bianchi identity shows the associated form S is identically zero. This concludes the proof of (d).
(e) This is the Bianchi identity we established for any linear connection (see Exercise 3.3.4).
□

The Riemann curvature tensor is the source of many important invariants associated to a Riemann manifold. We begin by presenting the simplest ones.

Definition 4.2.4. *Let (M, g) be a Riemann manifold with curvature tensor R. Any two vector fields X, Y on M define an endomorphism of TM by*

$$U \mapsto R(U, X)Y.$$

The Ricci curvature *is the trace of this endomorphism i.e.*

$$\mathrm{Ric}\,(X, Y) = \mathrm{tr}\,(U \mapsto R(U, X)Y).$$

We view it as a (0,2)-tensor $(X, Y) \mapsto \mathrm{Ric}\,(X, Y) \in C^\infty(M)$.

If $(x^1, \cdots, x^n)$ are local coordinates on M and the curvature R has the local expression $R = (R^\ell_{kij})$ then the Ricci curvature has the local description

$$\mathrm{Ric} = (R_{ij}) = \sum_\ell R^\ell_{j\ell i}.$$

The symmetries of the Riemann curvature imply that Ric is a *symmetric* (0,2)-tensor (as the metric).

Definition 4.2.5. *The* scalar curvature *s of a Riemann manifold is the trace of the Ricci tensor, i.e. in local coordinates*

$$s = g^{ij} R_{ij} = g^{ij} R^\ell_{i\ell j}$$

where as usual (g^{ij}) is the inverse matrix of (g_{ij}).

Let (M, g) be a Riemann manifold and $p \in M$. For any linearly independent $X, Y \in T_pM$ set

$$K_p(X, Y) = \frac{(R(X, Y)Y, X)}{|X \wedge Y|},$$

where $|X \wedge Y|$ denotes the Gramm determinant

$$|X \wedge Y| = \begin{vmatrix} (X, X) & (X, Y) \\ (Y, X) & (Y, Y) \end{vmatrix}$$

which is non-zero since X and Y are linearly independent. ($|X \wedge Y|^{1/2}$ measures the area of the parallelogram in T_pM spanned by X and Y.)

Exercise 4.2.1. Let $X, Y, Z, W \in T_pM$ such that span(X,Y)= span(Z,W) is a 2-dimensional subspace of T_pM prove that $K_p(X, Y) = K_p(Z, W)$. □

According to the above exercise the quantity $K_p(X, Y)$ depends only upon the 2-plane in T_pM generated by X and Y. Thus K_p is in fact a function on $Gr_2(p)$ the grassmannian of 2-dimensional subspaces of T_pM.

Definition 4.2.6. *The function $K_p : Gr_2(p) \to \mathbb{R}$ defined above is called the* sectional curvature *of M at p.* □

Exercise 4.2.2. Prove that

$$Gr_2(M) = \text{disjoint union of } Gr_2(p) \quad p \in M$$

can be organized as a smooth fiber bundle over M with standard fiber $Gr_{2,n}(\mathbb{R})$, $n = \dim M$ such that if M is a Riemann manifold $Gr_2(M) \ni (p; \pi) \mapsto K_p(\pi)$ is a smooth map. □

4.2.2 Examples

Example 4.2.7. Consider again the situation discussed in Example 4.1.12. Thus G is a Lie group and $\langle \cdot, \cdot \rangle$ is a metric on $\mathfrak{L}_G$ satisfying

$$\langle ad(X)Y, Z \rangle = -\langle Y, ad(X)Z \rangle$$

In other words, $\langle \cdot, \cdot \rangle$ is the restriction of a bi-invariant metric $\mathfrak{m}$ on G. We have shown that the Levi-Civita connection of this metric is

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \forall X, Y \in \mathfrak{L}_G.$$

We can now easily compute the curvature

$$R(X, Y)Z = \frac{1}{4}\{[X, [Y, Z]] - [Y, [X, Z]]\} - \frac{1}{2}[[X, Y], Z]$$

$$\text{(Jacobi identity)} = \frac{1}{4}[[X, Y], Z] + \frac{1}{4}[Y, [X, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z] = -\frac{1}{4}[[X, Y], Z].$$

We deduce

$$\langle R(X,Y)Z,W\rangle = -\frac{1}{4}\langle [[X,Y],Z],W\rangle = \frac{1}{4}\langle ad(Z)[X,Y],W\rangle$$
$$= -\frac{1}{4}\langle [X,Y], ad(Z)W\rangle = -\frac{1}{4}\langle [X,Y],[Z,W]\rangle.$$

Now let $\pi \in Gr_2(g)$ for some $g \in G$. If (X,Y) is an orthonormal basis of π (viewed as left invariant vector fields on G) then the sectional curvature along π is

$$K_g(\pi) = \frac{1}{4}\langle [X,Y],[X,Y]\rangle \geq 0$$

Denote the Killing form by $\kappa(X,Y) = -\mathrm{tr}\,(ad(X)ad(Y))$. To compute the Ricci curvature we pick an orthonormal basis $E_1, \cdots, E_n$ of $\mathfrak{L}_G$. For any $X = X^iE_i, Y = Y^jE_j \in \mathfrak{L}_G$ we have

$$\mathrm{Ric}\,(X,Y) = \frac{1}{4}\mathrm{tr}\,(Z \mapsto [[X,Z],Y])$$
$$= \frac{1}{4}\sum_i \langle [[X,E_i],Y],E_i)\rangle = -\frac{1}{4}\sum_i \langle ad(Y)[X,E_i],E_i\rangle$$
$$= \frac{1}{4}\sum_i \langle [X,E_i],[Y,E_i]\rangle = \frac{1}{4}\sum_i \langle ad(X)E_i, ad(Y)E_i\rangle$$
$$= -\frac{1}{4}\sum_i \langle ad(Y)ad(X)E_i,E_i\rangle = -\frac{1}{4}\mathrm{tr}\,(ad(Y)ad(X)) = \frac{1}{4}\kappa(X,Y).$$

In particular, on a compact semisimple Lie group the Ricci curvature is a symmetric positive definite (0,2)-tensor (in fact it is a scalar multiple of the Killing metric.)

We can now easily compute the scalar curvature. Using the same notations as above we get

$$s = \frac{1}{4}\sum_i \mathrm{Ric}\,(E_i,E_i) = \frac{1}{4}\sum \kappa(E_i,E_i).$$

In particular, if G is a compact semisimple group and the metric is given by the Killing form then the scalar curvature is

$$s(\kappa) = \frac{1}{4}\dim G.$$

□

Remark 4.2.8. Many problems in topology lead to a slightly more general situation than the one discussed in the above example namely to metrics on Lie groups which are only left invariant. Although the results are not as "crisp" as in the bi-invariant case many nice things do happen. For details we refer to [57]. □

Example 4.2.9. Let M be a 2-dimensional Riemann manifold (surface) and consider local coordinates on M, (x^1,x^2). Due to the symmetries of R

$$R_{ijkl} = -R_{ijlk} = R_{klij}$$

we deduce that the only nontrivial component of the Riemann tensor is $R = R_{1212}$. The sectional curvature is simply a function on M

$$K = \frac{1}{|g|}R_{1212} = \frac{1}{2}s(g) \quad \text{where } |g| = \det(g_{ij}).$$

In this case K is known as the *total curvature* or the *Gauss curvature* of the surface. If in particular M is oriented and the form $dx^1 \wedge dx^2$ defines the orientation, we can construct a 2-form

$$\varepsilon(g) = \frac{1}{2\pi}K dv_g = \frac{1}{4\pi}s(g)dv_g = \frac{1}{2\pi\sqrt{|g|}}R_{1212}dx^1 \wedge dx^2.$$

$\varepsilon(g)$ is called the *Euler form associated to the metric g.* We want to emphasize that this form is defined *only* when M is *oriented*.

We can rewrite this using the pfaffian construction of Subsection 2.2.4. The curvature R is a 2-form with coefficients in the bundle of skew-symmetric endomorphisms of TM so we can write

$$R = A \otimes dv_g \quad A = \frac{1}{\sqrt{|g|}}\begin{bmatrix} 0 & R_{1212} \\ R_{2112} & 0 \end{bmatrix}$$

Assume for simplicity that (x^1, x^2) are normal coordinates at a point $q \in M$. Thus at q, $|g| = 1$ since ∂_1, ∂_2 is an orthonormal basis of T_qM. Hence at q, $dv_g = dx^1 \wedge dx^2$ and

$$\varepsilon(g) = \frac{1}{2\pi}g(R(\partial_1, \partial_2)\partial_2, \partial_1)dx^1 \wedge dx^2.$$

Hence we can write

$$\varepsilon(g) = \frac{1}{2\pi}Pf_g(-A)dv_g \stackrel{def}{=} \frac{1}{2\pi}Pf_g(-R).$$

The Euler form has a very nice interpretation in terms of holonomy. Assume as before that (x^1, x^2) are normal coordinates at q and consider the square $S_t = [0, \sqrt{t}] \times [0, \sqrt{t}]$ in the (x^1, x^2) plane. Denote the (counterclockwise) holonomy along ∂S_t by $\mathcal{T}_t$. This is an orthogonal transformation of T_qM and with respect to the orthogonal basis (∂_1, ∂_2) of T_qM it has a matrix description as

$$\mathcal{T}_t = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}.$$

The result in Subsection 3.3.4 can be rephrased as

$$\sin\theta(t) = -tg(R(\partial_1, \partial_2)\partial_2, \partial_1) + O(t^2)$$

so that

$$R_{1212} = \dot{\theta}(0).$$

Hence R_{1212} is simply the infinitesimal angle measuring the infinitesimal rotation suffered by ∂_1 along S_t. We can think of the Euler form as a "density" of holonomy since it measures the holonomy per elementary parallelogram. □

4.2.3 Cartan's moving frame method

This method was introduced by Elie Cartan at the beginning of this century. Cartan's insight was that the local properties of a manifold equipped with a geometric structure can be very well understood if one knows how the frames of the tangent bundle (compatible with the geometric structure) vary from one point of the manifold to another. We will begin our discussion with the model case of $\mathbb{R}^n$.

Example 4.2.10. Consider $X_\alpha = X_\alpha^i \frac{\partial}{\partial x_i}$, $\alpha = 1, ..., n$ an *orthonormal moving frame* on $\mathbb{R}^n$ where $(x^1, \cdots, x^n)$ are the usual cartesian coordinates. Denote by (θ^α) the *dual coframe* i.e. the moving frame of $T^*\mathbb{R}^n$ defined by

$$\theta^\alpha(X_\beta) = \delta^\alpha_\beta.$$

The 1-forms measure the infinitesimal displacement of a point P with respect to the frame (X_α). Note that the TM-valued 1-form $\theta = \theta^\alpha X_\alpha$ is the differential of the identity map $\mathbf{id} : \mathbb{R}^n \to \mathbb{R}^n$ expressed using the given moving frame.

Introduce the 1-forms ω^α_β defined by

$$dX_\beta = \omega^\alpha_\beta X_\alpha \tag{4.2.1}$$

where we set

$$dX_\alpha = (\frac{\partial X^i_\alpha}{\partial x^j} dx^j) \otimes \frac{\partial}{\partial x_i}.$$

We can form the matrix valued 1-form (ω^α_β) which measures the infinitesimal rotation suffered by the moving frame (X_α) following the infinitesimal displacement $x \mapsto x + dx$. In particular $\omega = (\omega^\alpha_\beta)$ is a skew-symmetric matrix since

$$0 = d\langle X_\alpha, X_\beta\rangle = \langle \omega \cdot X_\alpha, X_\beta\rangle + \langle X_\alpha, \omega \cdot X_\beta\rangle.$$

Since $\theta = d\mathbf{id}$ we deduce $d\theta = 0$ and we can rewrite this as

$$d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta$$

or

$$d\theta = -\omega \wedge \theta. \tag{4.2.2}$$

Using $d^2 X_\beta = 0$ in (4.2.1) we deduce

$$d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega^\gamma_\beta$$

or equivalently

$$d\omega = -\omega \wedge \omega. \tag{4.2.3}$$

The equations (4.2.2)-(4.2.3) are called *the structural equations* of the Euclidean space. The significance of these structural equations will become evident in a little while. □

We now try to perform the same computations on an arbitrary Riemann manifold M. We choose a local orthonormal moving frame (X_α) and construct similarly its dual coframe (θ^α). Unfortunately, there is no natural way to define dX_α to produce the forms ω^α_β entering the structural equations. We will find them using a different (dual) search strategy.

Proposition 4.2.11. (E. Cartan) *There exists a collection of 1-forms* ω^α_β uniquely defined *by the requirements*
(a)$\omega^\alpha_\beta = -\omega^\beta_\alpha$.
(b)$d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta$, $\forall \alpha$.

Proof **Uniqueness** Suppose ω^α_β satisfy the conditions (a)&(b) above. Then there exist functions $f^\alpha_{\beta\gamma}$ and $g^\alpha_{\beta\gamma}$ such that

$$\omega^\alpha_\beta = f^\alpha_{\beta\gamma}\theta^\gamma$$

$$d\theta^\alpha = \frac{1}{2} g^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \quad (g^\alpha_{\beta\gamma} = -g^\alpha_{\gamma\beta}).$$

Then the condition (a) is equivalent to
(a1) $f^\alpha_{\beta\gamma} = -f^\beta_{\alpha\gamma}$
while (b) gives
(b1) $f^\alpha_{\beta\gamma} - f^\alpha_{\gamma\beta} = g^\alpha_{\beta\gamma}$

The above two relations uniquely determine the f's in terms of the g's via a cyclic permutation of the indices α, β, γ

$$f^\alpha_{\beta\gamma} = \frac{1}{2}(g^\alpha_{\beta\gamma} + g^\beta_{\gamma\alpha} - g^\gamma_{\alpha\beta}) \tag{4.2.4}$$

Existence Consider the functions $g^\alpha_{\beta\gamma}$ defined by

$$d\theta^\alpha = \frac{1}{2} g^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \quad (g^\alpha_{\beta\gamma} = -g^\alpha_{\gamma\beta}).$$

Next define $\omega^\alpha_\beta = f^\alpha_{\beta\gamma}\theta^\gamma$ where the f's are given by (4.2.4). We let the reader check that the forms ω^α_β satisfy both (a) and (b).

□

The reader may now ask why go through all this trouble. What have we gained by constructing the forms ω, and after all, what is their significance?

To answer these questions consider the Levi-Civita connection ∇. Define $\hat{\omega}^\alpha_\beta$ by

$$\nabla X_\beta = \hat{\omega}^\alpha_\beta X_\alpha.$$

Hence

$$\nabla_{X_\gamma} X_\beta = \hat{\omega}^\alpha_\beta(X_\gamma) X_\alpha.$$

Since ∇ is compatible with the Riemann metric we deduce in standard manner that $\hat{\omega}^\alpha_\beta = -\hat{\omega}^\beta_\alpha$.

The differential of θ^α can be computed in terms of the Levi-Civita connection (see Subsection 4.1.5) and we have

$$d\theta^\alpha(X_\beta, X_\gamma) = X_\beta\theta^\alpha(X_\gamma) - X_\gamma\theta^\alpha(X_\beta) - \theta^\alpha(\nabla_{X_\beta}X_\gamma) + \theta^\alpha(\nabla_{X_\gamma}X_\beta)$$

$$(\text{use } \theta^\alpha(\mathrm{X}_\beta) = \delta^\alpha_\beta = \text{ const}) = -\theta^\alpha(\hat{\omega}^\delta_\gamma(X_\beta)X_\delta) + \theta^\alpha(\hat{\omega}^\delta_\beta(X_\gamma)X_\delta)$$

$$= \hat{\omega}^\alpha_\beta(X_\gamma) - \hat{\omega}^\alpha_\gamma(X_\beta) = (\theta^\beta \wedge \hat{\omega}^\alpha_\beta)(X_\beta, X_\gamma).$$

Thus the $\hat{\omega}$'s satisfy both conditions (a) and (b) of Proposition 4.2.11 so that we must have

$$\hat{\omega}^{\alpha}_{\beta} = \omega^{\alpha}_{\beta}.$$

In other words, the matrix valued 1-form $(\omega^{\alpha}_{\beta})$ is the 1-form associated to the Levi-Civita connection in this local moving frame. In particular, the 2-form

$$\Omega = (d\omega + \omega \wedge \omega)$$

is the Riemannian curvature (see the computation in Example 3.3.11). The structural equations of a Riemann manifold take the form

$$d\theta = -\omega \wedge \theta$$

$$d\omega + \omega \wedge \omega = \Omega.$$

Comparing these with the Euclidean structural equations we deduce another interpretation of the Riemann curvature: it measures "the distance" between the given Riemann metric and the Euclidean one". We refer to [70] for more details on this aspect of the Riemann tensor.

The technique of orthonormal frames is extremely versatile in concrete computations.

Example 4.2.12. We will use the moving frame method to compute the curvature of the hyperbolic plane i.e. the upper half space

$$\mathbf{H}_+ = \{(x, y)\,;\ y > 0\}$$

endowed with the metric

$$g = \frac{1}{y^2}(dx^2 + dy^2).$$

$(y\partial_x, y\partial_y)$ is an orthonormal moving frame and $(\theta^x = \frac{1}{y}dx, \theta^y = \frac{1}{y}dy)$ is its dual coframe. We compute easily

$$d\theta^x = d(\frac{1}{y}dx) = \frac{1}{y^2}dx \wedge dy = (\frac{1}{y}dx) \wedge \theta^y$$

$$d\theta^y = d(\frac{1}{y}dy) = 0 = (-\frac{1}{y}dx) \wedge \theta^x.$$

Thus the connection 1-form in this local moving frame is

$$\omega = \begin{bmatrix} 0 & -\frac{1}{y} \\ \frac{1}{y} & 0 \end{bmatrix} dx.$$

The Riemann curvature is

$$\Omega = d\omega = \begin{bmatrix} 0 & \frac{1}{y^2} \\ -\frac{1}{y^2} & 0 \end{bmatrix} dy \wedge dx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \theta^x \wedge \theta^y.$$

The Gauss curvature is

$$K = \frac{1}{|g|} g(\Omega(\partial_x, \partial_y)\partial_y, \partial_x) = y^4(-\frac{1}{y^4}) = -1.$$

□

4.2.4 The geometry of submanifolds

We now want to apply Cartan's method of moving frames to discuss the the local geometry of submanifolds of a Riemann manifold.

Let (M, g) be a Riemann manifold of dimension m and S a k-dimensional submanifold in M. The restriction of g to S induces a Riemann metric on S. We want to analyze the relationship between the Riemann geometry of M (assumed to be known) and the geometry of S with the induced metric.

Denote by ∇^M (resp. ∇^S) the Levi-Civita connection on (M, g) (resp on (S,$g\,|_S$). The metric g produces an orthogonal splitting of vector bundles

$$(TM)\,|_S \cong TS \oplus N_S.$$

N_S is called the normal bundle of $S \hookrightarrow M$. Thus, a section of $(TM)\,|_S$, that is a vector field X of M along S, splits into two components: a tangential component X^τ and a normal component X^ν.

Now choose a local orthonormal moving frame $(X_1, ..., X_k; X_{k+1}, ..., X_m)$ such that the first k vectors $(X_1, ..., X_k)$ are tangent to S. Denote the dual coframe by $(\theta^\alpha)_{1\leq\alpha\leq m}$. Note that

$$\theta^\alpha\,|_S = 0 \quad \text{for } \alpha > k.$$

Denote by (μ^α_β), $(1 \leq \alpha, \beta \leq m)$ the connection 1-forms associated to ∇^M in this frame and let σ^α_β, $(1 \leq \alpha, \beta \leq k)$ be the connection 1-forms of ∇^S. We will analyze the structural equations of M restricted to $S \hookrightarrow M$.

$$d\theta^\alpha = \theta^\beta \wedge \mu^\alpha_\beta \quad 1 \leq \alpha, \beta \leq m. \tag{4.2.5}$$

We distinguish two situations.
A. $1 \leq \alpha \leq k$. Since $\theta^\beta\,|_S = 0$ for $\beta > k$ the equality (4.2.5) yields

$$d\theta^\alpha = \sum_{\beta=1}^{k} \theta^\beta \wedge \mu^\alpha_\beta, \quad \mu^\alpha_\beta = -\mu^\beta_\alpha \quad 1 \leq \alpha, \beta \leq k.$$

The uniqueness part of Proposition 4.2.11 implies that along S

$$\sigma^\alpha_\beta = \mu^\alpha_\beta \quad 1 \leq \alpha, \beta \leq k.$$

This can be equivalently rephrased as

$$\nabla^S_X Y = (\nabla^M_X Y)^\tau \quad \forall X, Y \in \operatorname{Vect}(S). \tag{4.2.6}$$

B. $k < \alpha \leq m$. We deduce

$$0 = \sum_{\beta=1}^{k} \theta^\beta \wedge \mu^\alpha_\beta.$$

At this point we want to use the following elementary result.

Exercise 4.2.3. (Cartan Lemma) Let V be a d-dimensional real vector space and consider p linearly independent elements $\omega_1, \dots, \omega_p \in \Lambda^1 V$, $p \leq d$. If $\theta_1, \dots, \theta_p \in \Lambda^1 V$ are such that

$$\sum_{i=1}^{p} \theta_i \wedge \omega_i = 0,$$

then there exist scalars A_{ij}, $1 \leq i, j \leq p$ such that $A_{ij} = A_{ji}$ and

$$\theta_i = \sum_{j=1}^{p} A_{ij}\omega_j.$$

□

Using Cartan Lemma we can find smooth functions $f^{\lambda}_{\beta\gamma}$, $\lambda > k$, $1 \leq \beta, \gamma \leq k$ satisfying

$$f^{\lambda}_{\beta\gamma} = f^{\lambda}_{\gamma\beta}.$$

$$\mu^{\lambda}_{\beta} = f^{\lambda}_{\beta\gamma}\theta^{\gamma}.$$

Now form

$$\mathcal{N} = f^{\lambda}_{\beta\gamma}\theta^{\beta} \otimes \theta^{\gamma} \otimes X_{\lambda}.$$

We can view $\mathcal{N}$ as a symmetric bilinear map

$$\mathrm{Vect}\,(S) \times \mathrm{Vect}\,(S) \to C^{\infty}(N_S).$$

If $U, V \in \mathrm{Vect}\,(S)$

$$U = U^{\beta}X_{\beta} = \theta^{\beta}(U)X_{\beta} \quad 1 \leq \beta \leq k$$

$$V = V^{\gamma}X_{\gamma} = \theta^{\gamma}(V)X_{\gamma} \quad 1 \leq \gamma \leq k$$

then

$$\mathcal{N}(U, V) = \sum_{\lambda > k}\left\{\sum_{\beta}\left(\sum_{\gamma} f^{\lambda}_{\beta\gamma}\theta^{\gamma}(V)\right)\theta^{\beta}(U)\right\}X^{\lambda}$$

$$= \sum_{\lambda > k}\left(\sum_{\beta}\mu^{\lambda}_{\beta}(V)U^{\beta}\right)X_{\lambda}.$$

The last term is precisely the normal component of $\nabla^M_V U$ so that we have established

$$\left(\nabla^M_V U\right)^{\nu} = \mathcal{N}(U, V) = \mathcal{N}(V, U) = \left(\nabla^M_U V\right)^{\nu}. \tag{4.2.7}$$

$\mathcal{N}$ is called the *2nd fundamental form*[2] of $S \hookrightarrow M$.

There is an alternative way of looking at $\mathcal{N}$. Choose $U, V \in \mathrm{Vect}\,(S)$, $N \in C^{\infty}(N_S)$. We have $(g(\cdot, \cdot) = \langle \cdot, \cdot \rangle)$

$$\langle \mathcal{N}(U, V), N \rangle = \langle \left(\nabla^M_U V\right)^{\nu}, N \rangle = \langle \nabla^M_U V, N \rangle$$

[2]The *first* fundamental form is the induced metric.

$$= \nabla^M_U \langle V, N\rangle - \langle V, \nabla^M_U N\rangle$$
$$= -\langle V, \big(\nabla^M_U N\big)^\tau\rangle.$$

We have thus established

$$-\langle V, \big(\nabla^M_U N\big)^\tau\rangle = \langle \mathcal{N}(U,V), N\rangle = \langle \mathcal{N}(V,U), N\rangle = -\langle U, \big(\nabla^M_V N\big)^\tau\rangle. \tag{4.2.8}$$

The 2nd fundamental form can be used to determine a relationship between the curvature of M and that of S. More precisely we have the following celebrated result.

THEOREMA EGREGIUM (Gauss) Let R^M (resp. R^S) denote the Riemann curvature of (M,g) (resp. $(S,\, g|_S)$. Then for any $X,Y,Z,T \in \mathrm{Vect}\,(S)$ we have

$$\langle R^M(X,Y)Z,T\rangle = \langle R^S(X,Y)Z,T\rangle$$
$$+\Big\langle \mathcal{N}(X,Z)\,,\,\mathcal{N}(Y,T)\Big\rangle - \Big\langle \mathcal{N}(X,T)\,,\,\mathcal{N}(Y,Z)\Big\rangle. \tag{4.2.9}$$

Proof Note that

$$\nabla^M_X Y = \nabla^S_X Y + \mathcal{N}(X,Y).$$

We have

$$R^M(X,Y)X = [\nabla^M_X, \nabla^M_Y]Z - \nabla^M_{[X,Y]}Z$$
$$= \nabla^M_X(\nabla^S_Y Z + \mathcal{N}(Y,Z)) - \nabla^M_Y(\nabla^S_X Z + \mathcal{N}(X,Z)) - \nabla^S_{[X,Y]}Z - \mathcal{N}([X,Y],Z).$$

Take the inner product with T of both sides above. Since $\mathcal{N}(\cdot,\cdot)$ is N_S-valued we deduce using (4.2.6)-(4.2.8)

$$\langle R^M(X,Y)Z,T\rangle = \langle \nabla^M_X\nabla^S_Y Z, T\rangle + \langle \nabla^M_X \mathcal{N}(Y,Z), T\rangle$$
$$-\langle \nabla^M_Y\nabla^S_X Z, T\rangle - \langle \nabla^M_Y \mathcal{N}(X,Z), T\rangle - \langle \nabla^S_{[X,Y]}Z, T\rangle$$
$$= \langle [\nabla^S_X, \nabla^S_Y]Z, T\rangle - \langle \mathcal{N}(Y,Z), \mathcal{N}(X,T)\rangle$$
$$+\langle \mathcal{N}(X,Z), \mathcal{N}(Y,T)\rangle - \langle \nabla^S_{[X,Y]}Z, T\rangle.$$

This is precisely the equality (4.2.9). □

The above result is especially interesting when S is a transversally oriented hypersurface, i.e. a codimension 1 submanifold such that the normal bundle N_S is trivial[3]. Pick an orthonormal frame $\vec{n}$ of N_S, i.e. a length 1 section of N_S, and choose an orthonormal moving frame $(X_1, ..., X_{m-1})$ of TS. Then $(X_1, ..., X_{m-1}, \vec{n})$ is an orthonormal moving frame of $(TM)|_S$ and the second fundamental form is completely described by

$$\mathcal{N}_{\vec{n}}(X,Y) = \langle \mathcal{N}(X,Y), \vec{n}\rangle.$$

$\mathcal{N}_{\vec{n}}$ is a bona-fide symmetric bilinear form and moreover according to (4.2.8) we have

$$\mathcal{N}_{\vec{n}}(X,Y) = -\langle \nabla^M_X \vec{n}, Y\rangle = -\langle \nabla^M_Y \vec{n}, X\rangle.$$

[3]Locally, all hypersurfaces are transversally oriented since N_S is locally trivial by definition.

Gauss formula becomes in this case

$$\langle R^S(X,Y)Z,T\rangle = \langle R^M(X,Y)Z,T\rangle - \begin{vmatrix} \mathcal{N}_{\vec{n}}(X,Z) & \mathcal{N}_{\vec{n}}(X,T) \\ \mathcal{N}_{\vec{n}}(Y,Z) & \mathcal{N}_{\vec{n}}(Y,T) \end{vmatrix}.$$

Let us further specialize and assume $M = \mathbb{R}^m$. Then

$$\langle R^S(X,Y)Z,T\rangle = \begin{vmatrix} \mathcal{N}_{\vec{n}}(X,T) & \mathcal{N}_{\vec{n}}(X,Z) \\ \mathcal{N}_{\vec{n}}(Y,T) & \mathcal{N}_{\vec{n}}(Y,Z) \end{vmatrix}. \tag{4.2.10}$$

In particular, the sectional curvature along the plane spanned by X, Y is

$$\langle R^S(X,Y)Y,X\rangle = \mathcal{N}_{\vec{n}}(X,X)\cdot\mathcal{N}_{\vec{n}}(Y,Y) - |\mathcal{N}_{\vec{n}}(X,Y)|^2.$$

This is a truly remarkable result. On the right-hand-side we have an extrinsic term (it depends on the "space surrounding S") while in the left-hand-side we have a purely intrinsic term (which is defined entirely in terms of the internal geometry of S). Historically, the extrinsic term was discovered first (by Gauss) and very much to Gauss surprise (?!?) one does not need to look outside S to compute it. This marked the beginning of a new era in geometry. It changed dramatically the way people looked at manifolds and thus it fully deserves the name of The Golden (egregium) Theorem of Geometry.

Example 4.2.13. (Quadrics in $\mathbb{R}^3$.) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a selfadjoint, invertible linear operator with at least one positive eigenvalue. This implies the quadric

$$Q_A = \{u \in \mathbb{R}^3 \ ; \ \langle Au, u\rangle = 1\}$$

is nonempty and smooth (use implicit function theorem to check this). Let $u_0 \in Q_A$. Then

$$T_{u_0}Q_A = \{x \in \mathbb{R}^3 \ ; \ \langle Au_0, x\rangle = 0\} = (Au_0)^\perp.$$

Q_A is a transversally oriented hypersurface in $\mathbb{R}^3$ since the map $Q_A \ni u \mapsto Au$ defines a nowhere vanishing section of the normal bundle. Set $\vec{n} = \frac{1}{|Au|}Au$. Consider (e_0, e_1, e_2) an orthonormal frame of $\mathbb{R}^3$ such that $e_0 = \vec{n}(u_0)$. Denote the cartesian coordinates in $\mathbb{R}^3$ with respect to this frame by (x^0, x^1, x^2) and set $\partial_i = \frac{\partial}{\partial x_i}$. Extend (e_1, e_2) to a local moving frame of TQ_A near u_0.

The second fundamental form of Q_A at u_0 is

$$\mathcal{N}_{\vec{n}}(\partial_i, \partial_j) = \langle \partial_i\vec{n}, \partial_j\rangle\,|_{u_0}.$$

We compute

$$\partial_i\vec{n} = \partial_i\left(\frac{Au}{|Au|}\right) = \partial_i(\langle Au, Au\rangle^{-1/2})Au + \frac{1}{|Au|}A\partial_i u$$

$$= -\frac{\langle \partial_i Au, Au\rangle}{|Au|^{3/2}}Au + \frac{1}{|Au|}\partial_i Au.$$

Hence

$$\mathcal{N}_{\vec{n}}(\partial_i, \partial_j)\,|_{u_0} = \frac{1}{|Au_0|}\langle A\partial_i u, e_j\rangle\,|_{u_0}$$

$$= \frac{1}{|Au_0|}\langle \partial_i u, Ae_j\rangle|_{u_0} = \frac{1}{|Au_0|}\langle e_i, Ae_j\rangle. \qquad (4.2.11)$$

We can now compute the Gaussian curvature at u_0.

$$K_{u_0} = \frac{1}{|Au_0|^2}\begin{vmatrix} \langle Ae_1, e_1\rangle & \langle Ae_1, e_2\rangle \\ \langle Ae_2, e_1\rangle & \langle Ae_2, e_2\rangle \end{vmatrix}.$$

In particular, when $A = r^{-2}I$ so that Q_A is the round sphere of radius r we deduce

$$K_u = \frac{1}{r^2} \quad \forall |u| = r.$$

Thus, the round sphere has constant positive curvature.

We can now explain rigorously why we cannot wrap a plane canvas around the sphere. Notice that when we deform a plane canvas the only thing that changes is the *extrinsic geometry* while the *intrinsic geometry* is not changed since the lengths of the"fibers" stays the same. Thus, any intrinsic quantity is invariant under "bending". In particular, no matter how we deform the plane canvas we will always get a surface with Gauss curvature 0 which cannot be wrapped on a surface of constant *positive* curvature! Gauss himself called the total curvature a"bending invariant".

Example 4.2.14. (Gauss) Let Σ be a transversally oriented, compact surface in $\mathbb{R}^3$. (E.g. a connected sum of g tori). Note that the Whitney sum $N_\Sigma \oplus T\Sigma$ is the trivial bundle $\underline{\mathbb{R}}^3_\Sigma$. We orient N_Σ such that

$$\text{orientation}\, N_\Sigma \wedge \text{orientation}\;\; T\Sigma = \text{orientation}\,\mathbb{R}^3.$$

Let $\vec{n}$ be the unit section of N_Σ defining the above orientation. We obtain in this way a map

$$\mathcal{G} : \Sigma \to S^2 = \{u \in \mathbb{R}^3 \; ; \; |u| = 1\}, \quad \Sigma \ni x \mapsto \vec{n}(x) \in S^2.$$

$\mathcal{G}$ is called the *Gauss map* of $\Sigma \hookrightarrow S^2$. It really depends on how Σ is embedded in $\mathbb{R}^3$ so it is an *extrinsic object*. Denote by $\mathcal{N}_{\vec{n}}$ the second fundamental form of $\Sigma \hookrightarrow \mathbb{R}^3$, and let (x^1, x^2) be normal coordinates at $q \in \Sigma$ such that

$$\text{orientation}\, T_q\Sigma = \partial_1 \wedge \partial_2.$$

Consider ε_Σ the Euler form on Σ with the metric induced by the Euclidean metric in $\mathbb{R}^3$. Then, taking into account our orientation conventions, we have

$$2\pi\varepsilon_\Sigma(\partial_1, \partial_2) = R_{1212}$$

$$= \begin{vmatrix} \mathcal{N}_{\vec{n}}(\partial_1, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\vec{n}}(\partial_2, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_2, \partial_2) \end{vmatrix}. \qquad (4.2.12)$$

Now notice that

$$\partial_i \vec{n} = -\mathcal{N}_{\vec{n}}(\partial_i, \partial_1)\partial_1 - \mathcal{N}_{\vec{n}}(\partial_i, \partial_2)\partial_2.$$

We can think of $\vec{n}, \partial_1 |_q$ and $\partial_2 |_q$ as defining a (positively oriented) $\mathbb{R}^3$. The last equality can be rephrased by saying that the derivative of the Gauss map

$$\mathcal{G}_* : T_q\Sigma \to T_{\vec{n}(q)}S^2$$

acts according to

$$\partial_i \mapsto -\mathcal{N}_{\vec{n}}(\partial_i, \partial_1)\partial_1 - \mathcal{N}_{\vec{n}}(\partial_i, \partial_2)\partial_2.$$

In particular, we deduce

$$\mathcal{G}_* \text{ preserves (reverses) orientations} \Longleftrightarrow R_{1212} > 0 \ (< 0) \tag{4.2.13}$$

because the orientability issue is decided by the sign of the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -\mathcal{N}_{\vec{n}}(\partial_1, \partial_1) & -\mathcal{N}_{\vec{n}}(\partial_1, \partial_2) \\ 0 & -\mathcal{N}_{\vec{n}}(\partial_2, \partial_1) & -\mathcal{N}_{\vec{n}}(\partial_2, \partial_2) \end{vmatrix}.$$

At q, $\partial_1 \perp \partial_2$ so that

$$\langle \partial_i \vec{n}, \partial_j \vec{n} \rangle = \mathcal{N}_{\vec{n}}(\partial_i, \partial_1)\mathcal{N}_{\vec{n}}(\partial_j, \partial_1) + \mathcal{N}_{\vec{n}}(\partial_i, \partial_2)\mathcal{N}_{\vec{n}}(\partial_j, \partial_2).$$

We can rephrase this coherently as an equality of matrices

$$\begin{bmatrix} \langle \partial_1 \vec{n}, \partial_1 \vec{n} \rangle & \langle \partial_1 \vec{n}, \partial_2 \vec{n} \rangle \\ \langle \partial_2 \vec{n}, \partial_1 \vec{n} \rangle & \langle \partial_2 \vec{n}, \partial_2 \vec{n} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{N}_{\vec{n}}(\partial_1, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\vec{n}}(\partial_2, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_2, \partial_2) \end{bmatrix} \times \begin{bmatrix} \mathcal{N}_{\vec{n}}(\partial_1, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\vec{n}}(\partial_2, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_2, \partial_2) \end{bmatrix}^t.$$

Hence

$$\begin{vmatrix} \mathcal{N}_{\vec{n}}(\partial_1, \partial_1) & \mathcal{N}_{\vec{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\vec{n}}(\partial_1, \partial_2) & \mathcal{N}_{\vec{n}}(\partial_2, \partial_2) \end{vmatrix}^2 = \begin{vmatrix} \langle \partial_1 \vec{n}, \partial_1 \vec{n} \rangle & \langle \partial_1 \vec{n}, \partial_2 \vec{n} \rangle \\ \langle \partial_1 \vec{n}, \partial_2 \vec{n} \rangle & \langle \partial_2 \vec{n}, \partial_2 \vec{n} \rangle \end{vmatrix}. \tag{4.2.14}$$

If we denote by dv_0 the metric volume form on S^2 induced by the restriction of the Euclidean metric on $\mathbb{R}^3$ we see that (4.2.12) and (4.2.14) put together yield

$$2\pi |\varepsilon_\Sigma(\partial_1, \partial_2)| = |dv_0(\partial_1 \vec{n}, \partial_2 \vec{n})| = |dv_0(\mathcal{G}_*(\partial_1), \mathcal{G}_*(\partial_2))|.$$

Using (4.2.13) we get

$$\varepsilon_\Sigma = \frac{1}{2\pi} \mathcal{G}_\Sigma^* dv_0 = \frac{1}{2\pi} \mathcal{G}_\Sigma^* \varepsilon_{S^2}. \tag{4.2.15}$$

This is one form of the celebrated Gauss-Bonnet theorem . We will have more to say about it in the next subsection.

Note that the last equality offers yet another interpretation of the Gauss curvature. From this point of view the curvature is a *"distortion factor"*. The Gauss map "stretches" an infinitesimal parallelogram to some infinitesimal region on the unit sphere. The Gauss curvature describes by what factor the area of this parallelogram was changed. □

4.2.5 The Gauss-Bonnet theorem

We conclude this chapter with one of the most beautiful results in geometry. Its meaning reaches deep inside the structure of a manifold and can be viewed as the origin of many fertile ideas.

Recall one of the questions we formulated at the beginning of our study: explain unambiguously why a sphere is "different" from a torus. This may sound like forcing our way in through an open door since everybody can "see" they are different. Unfortunately this is not a conclusive explanation since we can see only 3-dimensional things and possibly there are many ways to deform a surface outside our tight 3D Universe.

The elements of Riemann geometry we discussed so far will allow us to produce an invariant powerful enough to distinguish a sphere from a torus. But it will do more than that.

Theorem 4.2.15. (Gauss-Bonnet Theorem. Preliminary version.) *Let S be a compact oriented surface without boundary. If g_0 and g_1 are two Riemann metrics on S and $\varepsilon_{g_i}(S)$ $(i = 0, 1)$ are the corresponding Euler forms then*

$$\int_S \varepsilon_{g_0}(S) = \int_S \varepsilon_{g_1}(S).$$

Hence the quantity $\int_S \varepsilon_g(S)$ is independent of the Riemann metric g so that it really depends only on the topology of S!!!

The idea behind the proof is very natural. Denote by g_t the metric $g_t = g_0 + t(g_1 - g_0)$. We will show

$$\frac{d}{dt}\int_S \varepsilon_{g_t} = 0 \quad \forall t \in [0, 1].$$

It is convenient to consider a more general problem.

Definition 4.2.16. *Let M be a compact oriented manifold. For any Riemann metric g on E define*

$$\mathfrak{E}(g) = \int_M s(g)dv_g,$$

where $s(g)$ denotes the scalar curvature of (M, g). $\mathfrak{E}(g)$ is called the Hilbert-Einstein functional.

We have the following remarkable result.

Lemma 4.2.17. *Let M be a compact oriented manifold without boundary and $g^t = (g^t_{ij})$ be a 1-parameter family of Riemann metrics on M depending smoothly upon $t \in \mathbb{R}$. Then*

$$\frac{d}{dt}\mathfrak{E}(g^t) = -\int_M \langle \mathrm{Ric} - \frac{1}{2}s(g^t)g^t, \dot{g}^t\rangle_t dv_{g^t} \quad \forall t.$$

In the above formula $\langle\cdot,\cdot\rangle_t$ denotes the inner product induced by g^t on the space of symmetric (0,2)-tensors while the dot denotes the t-derivative.

The $(0,2)$-tensor

$$\mathcal{E}_{ij} = R_{ij}(x) - \frac{1}{2}s(x)g_{ij}(x)$$

is called the *Einstein tensor* of (M, g).

Definition 4.2.18. *A Riemann manifold (M, g) is said to be Einstein if the metric g satisfies* Einstein's equation

$$\mathrm{Ric}_g = \frac{1}{2}s(g)g.$$

Remark 4.2.19. The attribute Einstein usually refers to a larger class of Riemann manifolds satisfying the condition

$$\mathrm{Ric}_g(x) = \lambda(x)g(x)$$

for some smooth function $\lambda \in C^\infty(M)$. We refer to [12] for a comprehensive presentation of this subject. In this book we reserve the name Einstein to the special case described in the above theorem. □

Exercise 4.2.4. Consider a 3-dimensional Riemann manifold (M, g). Show that

$$R_{ijk\ell} = \mathcal{E}_{ik}g_{j\ell} - \mathcal{E}_{i\ell}g_{jk} + \mathcal{E}_{j\ell}g_{ik} - \mathcal{E}_{jk}g_{i\ell} + \frac{s}{2}(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}).$$

In particular, this shows that on a Riemann 3-manifold the full Riemann tensor is completely determined by the Einstein tensor. □

Exercise 4.2.5. (Schouten-Struik, [67]) Prove that the scalar curvature of an Einstein manifold of dimension ≥ 3 is constant.
Hint Use the 2nd Bianchi identity. □

Notice that when (S, g) is a compact oriented Riemann surface two very nice things happen.
(i) (S, g) is Einstein. (Recall that only R_{1212} is nontrivial).
(ii) $\mathfrak{E}(g) = 2\int_S \varepsilon_g$.

Theorem 4.2.15 is thus an immediate consequence of Lemma 4.2.17.

Proof of the lemma We will produce a very explicit description of the integrand

$$\frac{d}{dt}(s(g^t)dv_{g^t} = (\frac{d}{dt}(s(g^t))dv_{g^t} + s(g^t)(\frac{d}{dt}dv_{g^t}) \tag{4.2.16}$$

of $\frac{d}{dt}\mathfrak{E}(g^t)$. We will adopt a "get down in the mud and just do it" strategy reminiscent to the good old days of the tensor calculus frenzy. By this we mean that we will work in a nicely chosen collection of local coordinates and keep track of the zillion indices we will encounter. As we will see, "life in the mud" isn't as bad as it may seem to be.

We will study the integrand (4.2.16) at $t = 0$. The general case is entirely analogous. For typographical reasons we will be forced to introduce new notations. Thus, $\hat{g}$ will denote (g^t) for $t = 0$, while g^t will be denoted simply by g. A hat over a quantity means we think of that quantity at $t = 0$ while a dot means differentiation at $t = 0$.

Let q be an arbitrary point on S and denote by $(x^1, \cdots, x^n)$ a collection of $\hat{g}$-normal coordinates at q. Denote by ∇ the Levi-Civita connection of g and let Γ^i_{jk} denote its Christoffel symbols in the coordinates (x^i).

Many nice things happen *at q* and we list a few of them which will be used later.

$$\hat{g}_{ij} = \hat{g}^{ij} = \delta_{ij} \quad \partial_k \hat{g}_{ij} = 0. \tag{4.2.17}$$

$$\hat{\nabla}_i \partial_j = 0 \quad \hat{\Gamma}^i_{jk} = 0. \tag{4.2.18}$$

If $\alpha = \alpha_i dx^i$ is a 1-form then *at q*

$$\delta_{\hat{g}} \alpha = \sum_i \partial_i \alpha_i. \quad (\delta := *d*) \tag{4.2.19}$$

In particular, for any smooth function u

$$(\Delta_{M,\hat{g}} u)(q) = -\sum_i \partial_i{}^2 u. \tag{4.2.20}$$

Set

$$h = (h_{ij}) = (\dot{g}) = (\dot{g}_{ij}).$$

h is a symmetric (0,2)-tensor. Its $\hat{g}$-trace is the scalar

$$\mathrm{tr}_{\hat{g}} h = \hat{g}^{ij} h_{ij} = \mathrm{tr}\, \mathcal{L}^{-1}(h)$$

where $\mathcal{L}$ is the lowering the indices operator defined by $\hat{g}$. In particular *at q*

$$\mathrm{tr}_{\hat{g}}\, h = \sum_i h_{ii}. \tag{4.2.21}$$

The curvature of g is given by

$$R^\ell_{ikj} = -R^\ell_{ijk} = \partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^\ell_{mk} \Gamma^m_{ij} - \Gamma^\ell_{mj} \Gamma^m_{ik}.$$

The Ricci tensor is

$$R_{ij} = R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{mk} \Gamma^m_{ij} - \Gamma^k_{mj} \Gamma^m_{ik}.$$

Finally the scalar curvature is

$$\begin{aligned} s &= \mathrm{tr}_g\, R_{ij} = g^{ij} R_{ij} \\ &= g^{ij} \left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{mk} \Gamma^m_{ij} - \Gamma^k_{mj} \Gamma^m_{ik} \right). \end{aligned}$$

Derivating s at $t = 0$ and then evaluating *at q* we obtain

$$\begin{aligned} \dot{s} &= \dot{g}^{ij} \left(\partial_k \hat{\Gamma}^k_{ij} - \partial_j \hat{\Gamma}^k_{ik} \right) + \delta^{ij} \left(\partial_k \dot{\Gamma}^k_{ij} - \partial_j \dot{\Gamma}^k_{ik} \right) \\ &= \dot{g}^{ij} \hat{R}_{ij} + \sum_i \left(\partial_k \dot{\Gamma}^k_{ii} - \partial_i \dot{\Gamma}^k_{ik} \right). \end{aligned} \tag{4.2.22}$$

The term $\dot{g}^{ij}$ can be computed by derivating the equality $g^{ik}g_{jk} = \delta^i_k$ at $t = 0$. We get

$$\dot{g}^{ik}\hat{g}_{jk} + \hat{g}^{ik}h_{jk} = 0$$

so that

$$\dot{g}^{ij} = -h_{ij}. \tag{4.2.23}$$

To evaluate the derivatives $\dot{\Gamma}$'s we use the known formulæ

$$\Gamma^m_{ij} = \frac{1}{2}g^{km}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right)$$

which upon derivation at $t = 0$ yield

$$\begin{aligned}\dot{\Gamma}^m_{ij} &= \frac{1}{2}\left(\partial_i \hat{g}_{jk} - \partial_k \hat{g}_{ij} + \partial_j \hat{g}_{ik}\right) + \frac{1}{2}\hat{g}^{km}\left(\partial_i h_{jk} - \partial_k h_{ij} + \partial_j h_{ik}\right)\\ &= \frac{1}{2}\left(\partial_i h_{jm} - \partial_m h_{ij} + \partial_j h_{im}\right).\end{aligned} \tag{4.2.24}$$

We substitute (4.2.23) -(4.2.24) in (4.2.22), and we get *at* q

$$\begin{aligned}\dot{s} &= -\sum_{i,j} h_{ij}\hat{R}_{ij} + \frac{1}{2}\sum_{i,k}(\partial_k\partial_i h_{ik} - \partial_k{}^2 h_{ii} + \partial_k\partial_i h_{ik}) - \frac{1}{2}\sum_{i,k}(\partial_i{}^2 h_{kk} - \partial_i\partial_k h_{ik})\\ &= -\sum_{i,j} h_{ij}\hat{R}_{ij} - \sum_{i,k}\partial_i{}^2 h_{kk} + \sum_{i,k}\partial_i\partial_k h_{ik}\\ &= -\langle\widehat{\mathrm{Ric}}, \dot{g}\rangle_{\hat{g}} + \Delta_{M,\hat{g}}\mathrm{tr}_{\hat{g}}\,\dot{g} + \sum_{i,k}\partial_i\partial_k h_{ik}.\end{aligned} \tag{4.2.25}$$

To get a coordinate free description of the last term note that *at* q

$$(\hat{\nabla}_k h)(\partial_i, \partial_m) = \partial_k h_{im}.$$

The total covariant derivative $\hat{\nabla}h$ is a (0,3)-tensor. Using the $\hat{g}$-trace we can construct a (0,1)-tensor

$$\mathrm{tr}_{\hat{g}}(\hat{\nabla}h) = \mathrm{tr}(\mathcal{L}_{\hat{g}}^{-1}\hat{\nabla}h)$$

where $\mathcal{L}_{\hat{g}}^{-1}$ is the raising the indices operator defined by $\hat{g}$. In the local coordinates (x^i) we have

$$\mathrm{tr}_{\hat{g}}(\hat{\nabla}h) = \hat{g}^{ij}(\hat{\nabla}_i h)_{jk}dx^k.$$

Using (4.2.17) and (4.2.19) we deduce that the last term in (4.2.25) can be rewritten (*at* q) as

$$\delta\mathrm{tr}_{\hat{g}}\,(\hat{\nabla}h) = \delta\mathrm{tr}_{\hat{g}}(\hat{\nabla}\dot{g}).$$

We have thus established that

$$\dot{s} = -\langle\widehat{\mathrm{Ric}}, \dot{g}\rangle_{\hat{g}} + \Delta_{M,\hat{g}}\mathrm{tr}_{\hat{g}}\,\dot{g} + \delta\mathrm{tr}_{\hat{g}}(\hat{\nabla}\dot{g}). \tag{4.2.26}$$

The second term of the integrand (4.2.16) is a lot easier to compute.

$$dv_g = \pm\sqrt{|g|}dx^1\cdots dx^n$$

so that

$$d\dot{v}_g = \pm\frac{1}{2}|\hat{g}|^{-1/2}\frac{d}{dt}|g|dx^1\cdots dx^n.$$

At q the metric is Euclidian, $\hat{g}_{ij} = \delta_{ij}$ and

$$\frac{d}{dt}|g| = \sum_i \dot{g}_{ii} = |\hat{g}|\cdot \mathrm{tr}_{\hat{g}}(\dot{g}) = |\hat{g}|\langle \hat{g}, \dot{g}\rangle_{\hat{g}}|\hat{g}|.$$

Hence

$$\dot{\mathfrak{E}}(g) = \int_M \langle\frac{1}{2}s(\hat{g})\hat{g} - \mathrm{Ric}(\hat{g}), \dot{g}\rangle_{\hat{g}} dv_{\hat{g}} + \int_M \Big(\Delta_{M,\hat{g}}\mathrm{tr}_{\hat{g}}\,\dot{g} + \delta\mathrm{tr}_{\hat{g}}(\hat{\nabla}\dot{g})\Big)dv_{\hat{g}}.$$

Green's formula shows the last two terms vanish and the proof of the Lemma is concluded. □

Definition 4.2.20. *Let* S *be a compact, oriented surface without boundary. We define its* Euler characteristic *as the number*

$$\chi(S) = \frac{1}{2\pi}\int_S \varepsilon(g),$$

where g *is an arbitrary Riemann metric on* S*. The number*

$$g(S) = \frac{1}{2}(2 - \chi(S))$$

is called the genus *of the surface.*

Remark 4.2.21. According to the theorem we have just proved the Euler characteristic is independent of the metric used to define it. Hence, the Euler characteristic is a *topological invariant* of the surface. The reason for this terminology will become apparent when we discuss DeRham cohomology, a $\mathbb{Z}$-graded vector space naturally associated to a surface whose Euler characteristic coincides with the number defined above. So far we have no idea whether $\chi(S)$ is even an integer. □

Proposition 4.2.22.

$$\chi(S^2) = 2 \ \text{ and } \ \chi(T^2) = 0.$$

Proof To compute $\chi(S^2)$ we use the round metric g_0 for which $K = 1$ so that

$$\chi(S^2) = \frac{1}{2\pi}\int_{S^2} dv_{g_0} = \frac{1}{2\pi}\mathrm{area}_{g_0}(S^2) = 2.$$

To compute the Euler characteristic of the torus we think of it as an Abelian Lie group with a bi-invariant metric. Since the Lie bracket is trivial we deduce from the computations in Subsection 4.2.2 that its curvature is zero. This concludes the proof of the proposition. □

Proposition 4.2.23. *If* S_i *(i=1,2) are two compact oriented surfaces without boundary then*

$$\chi(S_1\# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

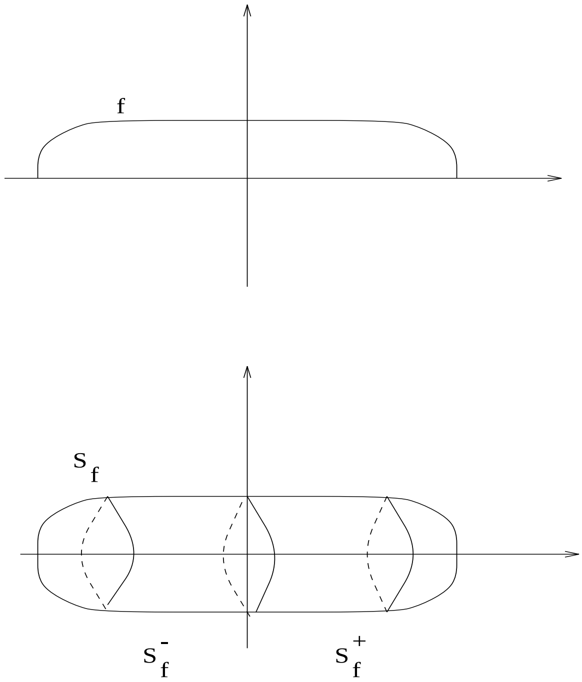

Figure 4.3: *Generating a hot-dog-shaped surface*

Thus upon iteration we get

$$\chi(S_1\#\cdots\#S_k) = \sum_{i=1}^{k}\chi(S_i) - 2(k-1)$$

for any compact oriented surfaces $S_1, \dots, S_k$*. In terms of genera, the last equality can be rephrased as*

$$g(S_1\#\cdots\#S_k) = \sum_{i=1}^{k} g(S_i).$$

In the proof of this proposition we will use special metrics on connected sums of surfaces which require a preliminary analytical discussion.

Consider $f : (-4, 4) \to (0, \infty)$ a smooth, even function such that
(i) $f(x) = 1$ for $|x| \le 2$.
(ii) $f(x) = \sqrt{1-(x+3)^2}$ for $x \in [-4, -3.5]$.
(iii) $f(x) = \sqrt{1-(x-3)^2}$ for $x \in [3.5, 4]$.
(iv) f is non-decreasing on [-4,0].
One such function is graphed in Figure 4.3

Denote by S_f the surface inside $\mathbb{R}^3$ obtained by rotating the graph of f about the x-axis. Because of properties (i)-(iv) S_f is a smooth surface diffeomorphic to S^2. (One such diffeomorphism can be explicitly constructed projecting along radii starting at the origin). We denote by g the metric on S_f induced by the euclidian metric in $\mathbb{R}^3$. Since S_f

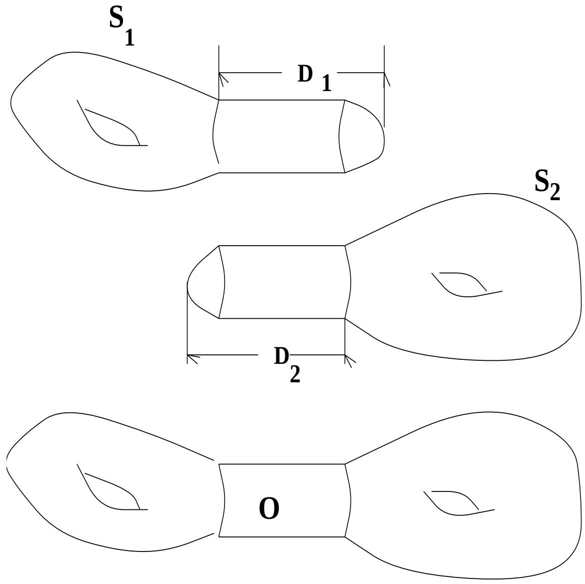

Figure 4.4: *Special metric on a connected sum*

is diffeomorphic to a sphere

$$\int_{S_f} K_g dv_g = 2\pi\chi(S^2) = 4\pi.$$

Set

$$S_f^{\pm} = S_f \cap \{\pm x > 0\} \quad S_f^{\pm 1} = S_f \cap \{\pm x > 1\}$$

Since f is even we deduce

$$\int_{S_f^{\pm}} K_g dv_g = \frac{1}{2}\int_{S_f} K_g dv_g = 2\pi. \tag{4.2.27}$$

On the other hand, on the neck $C = \{|x| \leq 2\}$ the metric g is locally euclidian

$$g = dx^2 + d\theta^2$$

so that over this region $K_g = 0$. Hence

$$\int_C K_g dv_g = 0. \tag{4.2.28}$$

Proof of the Proposition 4.2.23 Let $D_i \subset S_i$ $(i = 1, 2)$ be a local coordinate neighborhood diffeomorphic with a disk in the plane. Pick a metric g_i on S_i such that (D_1, g_1) is *isometric* with S_f^+ and (D_2, g_2) is isometric to S_f^-. The connected sum $S_1\#S_2$ is obtained by chopping off the regions S_f^1 from D_1 and S_f^{-1} from D_2 and (isometrically) identifying the remaining cylinders $S_f^{\pm} \cap \{|x| \leq 1\} = C$ and call O the overlap created by gluing (see Figure 4.4). Denote the metric thus obtained on $S_1\#S_2$ by $\hat{g}$. We can now compute

$$\chi(S_1\#S_2) = \frac{1}{2\pi}\int_{S_1\#S_2} K_{\hat{g}} dv_{\hat{g}}$$

$$= \frac{1}{2\pi}\int_{S_1\setminus D_1} K_{g_1} dv_{g_1} + \frac{1}{2\pi}\int_{S_2\setminus D_2} K_{g_2} dv_{g_2} + \frac{1}{2\pi}\int_O K_g dv_g$$

$$\overset{(4.2.28)}{=} \frac{1}{2\pi}\int_{S_1} K_{g_1} dv_{g_1} - \frac{1}{2\pi}\int_{D_1} K_g dv_g$$

$$+\frac{1}{2\pi}\int_{S_2} K_{g_2} dv_{g_2} - \frac{1}{2\pi}\int_{D_2} K_g dv_g$$

$$\overset{(4.2.27)}{=} \chi(S_1) + \chi(S_2) - 2.$$

The proposition is proved.

□

Corollary 4.2.24. (Gauss-Bonnet) *Let Σ_g denote the connected sum of g-tori. (By definition $\Sigma_0 = S^2$. Then*

$$\chi(\Sigma_g) = 2 - 2g \text{ and } g(\Sigma_g) = g.$$

In particular a sphere is not diffeomorphic to a torus.

Remark 4.2.25. It is a classical result that the only compact oriented surfaces are the connected sums of g-tori (see [52]) so that the genus of a compact oriented surface is a complete topological invariant. □

Chapter 5

Elements of the calculus of variations

This is a very exciting subject lieing at the frontier between mathematics and physics. The limited space we will devote to this subject will hardly do it justice and we will barely touch its physical significance. We recommend to anyone looking for an intellectual feast the Chapter 16 in vol.2 of [27] which in our opinion is the most eloquent argument for the raison d'être of the calculus of variations.

5.1 The least action principle

5.1.1 1-dimensional Euler-Lagrange equations

From a very "dry" point of view the fundamental problem of the calculus of variations can be easily formulated as follows.

Consider a smooth manifold M and denote $L : \mathbb{R} \times TM \to \mathbb{R}$ be a smooth function called the *lagrangian*. Fix two points p_0, $p_1 \in M$. For any piecewise smooth path $\gamma : [0,1] \to M$ connecting these points we define its *action* by

$$S(\gamma) = S_L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t))dt.$$

In the calculus of variations one is interested in those paths as above with minimal action.

Example 5.1.1. Given $U : \mathbb{R}^3 \to \mathbb{R}$ a smooth function called the *potential* we can form the lagrangian

$$L(q, \dot{q}) : \mathbb{R}^3 \times \mathbb{R}^3 \cong T\mathbb{R}^3 \to \mathbb{R}$$

as

$$L = Q - U = \text{kinetic energy} - \text{potential energy} = \frac{1}{2}m|\dot{q}|^2 - U(q).$$

(m is the mass). The action of a path (trajectory) $\gamma : [0,1] \to \mathbb{R}^3$ is a quantity called the newtonian action. □

Example 5.1.2. To any Riemann manifold (M, g) one can naturally associate two lagrangians $L_1,\ L_2 : TM \to \mathbb{R}$ defined by

$$L_1(q, v) = g_q(v, v)^{1/2} \quad (v \in T_qM)$$

$$L_2(q, v) = \frac{1}{2} g(v, v).$$

We see that the action defined by L_1 coincides with the length of a path. The action defined by L_2 is called the *energy* of a path. □

Before we present the main result of this subsection we need to introduce a bit of notation.

Tangent bundles are very peculiar manifolds. Any collection $(q^1, \dots, q^n)$ of local coordinates on a smooth manifold M automatically induces local coordinates on TM. Any point in TM can be described by a pair (q, v) where $q \in M$, $v \in T_qM$. Furthermore, v has a decomposition

$$v = v^i \partial_i \quad (\partial_i = \frac{\partial}{\partial q^i}).$$

We set $\dot{q}^i = v^i$ so that

$$v = \dot{q}^i \partial_i.$$

The collection $(q^1, \dots, q^n; \dot{q}^1, \dots, \dot{q}^n)$ defines local coordinates on TM. These are said to be *holonomic* local coordinates on TM. This will be the only type of local coordinates we will ever use.

Theorem 5.1.3. (The least action principle) *Let $L : \mathbb{R} \times TM \to \mathbb{R}$ be a lagrangian and $p_0,\ p_1 \in M$ two fixed points. If $\gamma : [0, 1] \to M$ is a* smooth *path such that*
(i)$\gamma(i) = p_i$, $i = 0, 1$.
(ii) $S_L(\gamma) \leq S_L(\tilde{\gamma})$ for any smooth path $\tilde{\gamma} : [0, 1] \to M$ joining p_0 to p_1.
Then γ satisfies the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\gamma}} L(t, \gamma, \dot{\gamma}) = \frac{\partial L}{\partial \gamma}.$$

More precisely, if $(q^i; \dot{q}^j)$ are holonomic local coordinates on TM such that $\gamma(t) = (q^i(t))$ and $\dot{\gamma} = (\dot{q}^j(t))$ then γ is a solution of the system of nonlinear ordinary differential equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^k}(t, q^i; \dot{q}^j) = \frac{\partial L}{\partial q^k}(t, q^i; \dot{q}^j) \quad k = 1, \dots, n = \dim M.$$

Definition 5.1.4. *A path $\gamma : [0, 1] \to M$ satisfying the Euler-Lagrange equations with respect to some lagrangian L is said to be an* extremal *of L.*

To get a better feeling of these equations consider the special case discussed in Example 5.1.1

$$L = \frac{1}{2} m |\dot{q}|^2 - U(q).$$

The Euler-Lagrange equations become

$$m\ddot{q} = -\nabla U(q). \tag{5.1.1}$$

These are precisely Newton's equation of the motion of a particle of mass m in the force field $-\nabla U$ generated by the potential U.

In the proof of the least action principle we will use the notion of *variation of a path.*

Definition 5.1.5. *Let $\gamma : [0,1] \to M$ be a smooth path. A* variation *of γ is a smooth map*

$$\alpha = \alpha_s(t) : (-\varepsilon, \varepsilon) \times [0,1] \to M$$

such that $\alpha_0(t) = \gamma(t)$. If moreover, $\alpha_s(i) = p_i$ $\forall s$ (i=0,1) then we say that α is a variation rel endpoints.

Proof of Theorem 5.1.3. Let α_s be a variation of γ rel endpoints. Then

$$S_L(\alpha_0) \leq S_L(\alpha_s) \quad \forall s$$

so that

$$\frac{d}{ds}|_{s=0}\, S_L(\alpha_s) = 0.$$

Assume for simplicity that the image of γ is entirely contained in some open coordinate neighborhood U with coordinates $(q^1, \cdots, q^n)$. Then for very small $|s|$ we can write

$$\alpha_s(t) = (q^i(s,t)) \text{ and } \frac{d\alpha_s}{dt} = (\dot{q}^i(s,t)).$$

Following the tradition, we set

$$\delta\alpha = \frac{\partial\alpha}{\partial s}|_{s=0} = \frac{\partial q^i}{\partial s}\partial_i \quad \delta\dot{\alpha} = \frac{\partial}{\partial s}|_{s=0}\, \frac{d\alpha_s}{dt} = \frac{\partial \dot{q}^j}{\partial s}\frac{\partial}{\partial \dot{q}^j}.$$

$\delta\alpha$ is a vector field along γ called *infinitesimal variation* (see Figure 5.1). In fact, the pair $(\delta\alpha; \delta\dot{\alpha}) \in T(TM)$ is a vector field along $t \mapsto (\gamma(t), \dot{\gamma}(t)) \in TM$. Note that $\delta\dot{\alpha} = \frac{d}{dt}\delta\alpha$ and at endpoints $\delta\alpha = 0$.

Exercise 5.1.1. Prove that if $t \mapsto X(t) \in T_{\gamma(t)}M$ is a smooth vector field along γ such that $X(t) = 0$ for $t = 0, 1$ then there exists at lest one variation rel endpoints α such that $\delta\alpha = X$.

(Hint: Use the exponential map of some Riemann metric on M.)

We compute (at $s = 0$)

$$0 = \frac{d}{ds}S_L(\alpha_s) = \frac{d}{ds}\int_0^1 L(t, \alpha_s, \dot{\alpha_s})$$

$$= \int_0^1 \frac{\partial L}{\partial q^i}\delta\alpha^i dt + \int_0^1 \frac{\partial L}{\partial \dot{q}^j}\delta\dot{\alpha}_s^j dt.$$

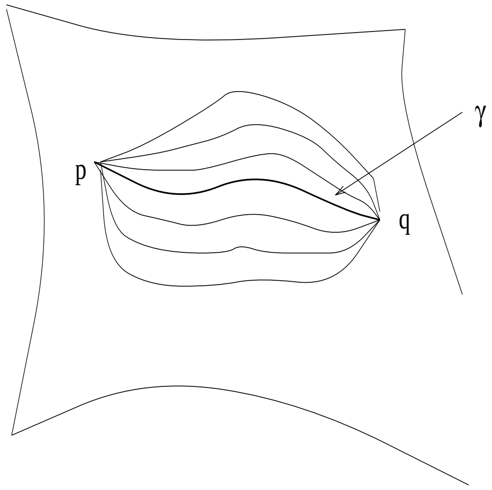

Figure 5.1: *Deforming a path rel endpoints*

Integrating by parts in the second term we deduce

$$\int_0^1 \left\{ \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \right\} \delta\alpha^i dt. \tag{5.1.2}$$

The last equality holds *for any variation* α. From Exercise 5.1.1 deduce it holds for any vector field $\delta\alpha^i \partial_i$ along γ. At this point we use the following classical result of analysis.

" If $f(t)$ is a continuous function on [0,1] such that

$$\int_0^1 f(t)g(t)dt = 0 \quad \forall g \in C_0^\infty(0,1)$$

then f is identically zero."

Using this result in (5.1.2) we deduce the desired conclusion.

□

Remark 5.1.6. (a) In the proof of the least action principle we used a simplifying assumption i.e. the image of γ lies in a coordinate neighborhood. This is true locally and for the above arguments to work it suffices to choose only a special type of variations, localized on small intervals of [0,1]. In terms of infinitesimal variations this means we need to look only at vector fields along γ supported in local coordinate neighborhoods. We leave the reader fill in the details.

(b) The Euler-Lagrange equations were described using holonomic local coordinates. The minimizers of the action (if any) are objects independent of any choice of local coordinates so that the Euler-Lagrange equations have to be independent of such choices. We check this directly. If (x^i) is another collection of local coordinates on M and $(x^i; \dot{x}^j)$ are the coordinates induced on TM then we have the transition rules

$$x^i = x^i(q^1, \cdots, q^n), \quad \dot{x}^j = \frac{\partial x^j}{\partial q^k}\dot{q}^k$$

so that

$$\frac{\partial}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\frac{\partial}{\partial x^j} + \frac{\partial^2 x^j}{\partial q^k \partial q^i}\dot{q}^k \frac{\partial}{\partial \dot{x}^j}$$

$$\frac{\partial}{\partial \dot{q}^j} = \frac{\partial \dot{x}^i}{\partial \dot{q}^j}\frac{\partial}{\partial \dot{x}^j} = \frac{\partial x^j}{\partial q^i}\frac{\partial}{\partial \dot{x}^j}$$

Then

$$\frac{\partial L}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\frac{\partial L}{\partial x^j} + \frac{\partial^2 x^j}{\partial q^k \partial q^i}\dot{q}^k \frac{\partial L}{\partial \dot{x}^j}.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^i}\frac{\partial L}{\partial \dot{x}^j}\right)$$

$$= \frac{\partial^2 x^j}{\partial q^k \partial q^i}\dot{q}^k \frac{\partial L}{\partial \dot{x}^j} + \frac{\partial x^j}{\partial q^i}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^j}\right).$$

We now see that the Euler-Lagrange equations in the q-variables imply the Euler-Lagrange in the x-variable i.e. these equations are *independent of coordinates*.

The æsthetically conscious reader may object to the way we chose to present the Euler-Lagrange equations. These are intrinsic equations we formulated in a coordinate dependent fashion. Is there any way of writing these equation so that the intrinsic nature is visible "on the nose"?

If the lagrangian L satisfies certain nondegeneracy conditions there are two ways of achieving this goal. One method is to consider a natural nonlinear connection ∇^L on TM as in [59] . The Euler-Lagrange equations for an extremal $\gamma(t)$ can then be rewritten as a "geodesics equation"

$$\nabla^L_{\dot{\gamma}}\dot{\gamma}.$$

The example below will illustrate this approach on a very special case when L is the lagrangian L_2 defined in Example 5.1.2 in which the extremals are precisely the geodesics on a Riemann manifold.

Another far reaching method of globalizing the formulation of the Euler-Lagrange equation is through the Legendre transform which again requires a nondegeneracy condition on the lagrangian. Via the Legendre transform the Euler-Lagrange equations become a system of *first order equations* on the cotangent bundle T^*M known as the Hamilton equations. These equations have the advantage that can be formulated on manifolds more general than the cotangent bundles namely on *symplectic manifolds*. These are manifolds carrying a closed 2-form whose restriction to each tangent space defines a symplectic duality (see §2.2.4.) Much like the geodesics equations on a Riemann manifold the Hamilton equations carry many informations about the structure of symplectic manifolds and are currently the focus of very intense research. For more details and examples we refer to the monographs [5] or [21]. □

Example 5.1.7. Let (M, g) be a Riemann manifold. We will compute the Euler-Lagrange equations for the lagrangians L_1, L_2 in Example 5.1.2.

$$L_2(q, \dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$$

so that

$$\frac{\partial L_2}{\partial \dot{q}^k} = g_{jk}\dot{q}^j \quad \frac{\partial L_2}{\partial q^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j.$$

The Euler-Lagrange equations are

$$\ddot{q}^j g_{jk} + \frac{\partial g_{jk}}{\partial q^i}\dot{q}^i\dot{q}^j = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j. \tag{5.1.3}$$

Since $g^{km}g_{jm} = \delta^m_j$ we get

$$\ddot{q}^m + g^{km}\left(\frac{\partial g_{jk}}{\partial q^i} - \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\right)\dot{q}^i\dot{q}^j = 0. \tag{5.1.4}$$

When we derivate with respect to t the equality

$$g_{ik}\dot{q}^i = g_{jk}\dot{q}^j$$

we deduce

$$g^{km}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^i\dot{q}^j = \frac{1}{2}g^{km}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^i\dot{q}^j.$$

We substitute this equality in (5.1.4) and we get

$$\ddot{q}^m + \frac{1}{2}g^{km}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)\dot{q}^i\dot{q}^j = 0. \tag{5.1.5}$$

The coefficient of $\dot{q}^i\dot{q}^j$ in (5.1.5) is none other than the Christoffel symbol Γ^m_{ij} so this equation is precisely the geodesic equation. □

Consider now the lagrangian $L_1(q, \dot{q}) = (g_{ij}\dot{q}^i\dot{q}^j)^{1/2}$. Note that the action

$$\int_{p_0}^{p_1} L(q, \dot{q})dt$$

is independent of the parameterization $t \mapsto q(t)$ since it computes the length of the path. Thus, when we express the Euler-Lagrange equations for a minimizer γ_0 of this action we may as well assume it is parameterized by arclength i.e.

$$|\dot{\gamma}_0| = 1.$$

The Euler-Lagrange equations for L_1 are

$$\frac{d}{dt}\frac{g_{kj}\dot{q}^j}{\sqrt{g_{ij}\dot{q}^i\dot{q}^j}} = \frac{\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j}{2\sqrt{g_{ij}\dot{q}^i\dot{q}^j}}.$$

Along the extremal we have $g_{ij}\dot{q}^i\dot{q}^j = 1$ (arclength parameterization) so that the previous equations can be rewritten as

$$\frac{d}{dt}\left(g_{kj}\dot{q}^j\right) = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j.$$

We recognize here the equation (5.1.3) which, as we have seen, is the geodesic equation in disguise. This fact almost explains why the geodesics are the shortest paths between two nearby points.

5.1.2 Noether's conservation principle

This subsection is intended to offer the reader a glimpse at a fascinating subject touching both physics and geometry. We first need to introduce a bit of traditional terminology commonly used by physicists.

Consider a smooth manifold M. The tangent bundle TM is usually referred to as the *space of states* or the *lagrangian phase space*. A point in TM is said to be a *state*. A lagrangian $L : \mathbb{R} \times TM \to \mathbb{R}$ associates to each state several meaningful quantities.
(a) The *generalized momenta*: $p_i = \frac{\partial L}{\partial \dot{q}^i}$.
(b) The *energy*: $H = p_i \dot{q}^i - L$.
(c) The *generalized force*: $F = \frac{\partial L}{\partial q^i}$.

This terminology can be justified by looking at the lagrangian of a classical particle in a potential force field $F = -\nabla U$

$$L = \frac{1}{2} m |\dot{q}|^2 - U(q).$$

The momenta associated to this lagrangian are the usual kinetic momenta of the Newtonian mechanics

$$p_i = m \dot{q}^i$$

while H is simply the total energy

$$H = \frac{1}{2} m |\dot{q}|^2 + U(q).$$

It will be convenient to think of an extremal for an arbitrary lagrangian $L(t, q, \dot{q})$ as describing the motion of a particle under the influence of the generalized force.

Proposition 5.1.8. (Conservation of energy) *Let $\gamma(t)$ be an extremal of a* time independent *lagrangian $L = L(q, \dot{q})$. Then the energy is conserved along γ i.e.*

$$\frac{d}{dt} H(\gamma, \dot{\gamma}) = 0.$$

Proof By direct computation we get

$$\frac{d}{dt} H(\gamma, \dot{\gamma}) = \frac{d}{dt}(p_i \dot{q}^i - L)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \dot{q}^i + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i - \frac{\partial L}{\partial q^i} \dot{q}^i - \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i$$

$$= \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \right\} \dot{q}^i = 0 \quad (\text{by Euler} - \text{Lagrange}).$$

□

At the beginning of this century (1918) Emmy Noether discovered that many of the conservation laws of the classical mechanics had a geometric origin: they were, most of them, reflecting a symmetry of the lagrangian!!! This became a driving principle in the search for conservation laws and in fact, conservation became synonymous with symmetry.

It eased the leap from classical to quantum mechanics, and one can say it is a very important building block of quantum physics in general. In the few instances of conservation laws where the symmetry was not apparent the conservation was always "blamed" on a "hidden symmetry". What is then this Noether principle?

To answer this question we need to make some simple observations.

Let X be a vector field on a smooth manifold M defining a global flow Φ^t. This flow induces a flow Ψ^t on the tangent bundle TM defined by

$$\Psi^t(x, v) = (\Phi^t(x), \Phi^t_*(v)).$$

One can think of Ψ^t as defining an action of the additive group $\mathbb{R}$ on TM. Alternatively, the physicists say that X is an *infinitesimal symmetry* of the given mechanical system described by the lagrangian L.

Example 5.1.9. Let M be the unit round sphere $S^2 \subset \mathbb{R}^3$. The rotations about the z-axis define a 1-parameter group of isometries of S^2 generated by $\frac{\partial}{\partial\theta}$ (θ is the longitude on S^2).

Definition 5.1.10. *Let L be a lagrangian on TM and X a vector field on M. The lagrangian L is said to be X-* invariant *if*

$$L \circ \Psi^t = L \quad \forall t.$$

Denote by $\mathcal{X} \in \mathrm{Vect}\,(TM)$ the infinitesimal generator of Ψ^t and by $\mathcal{L}_{\mathcal{X}}$ the Lie derivative on TM along $\mathcal{X}$. We see that L is X-invariant iff

$$\mathcal{L}_{\mathcal{X}} L = 0.$$

We describe this derivative using the local coordinates $(q^i, \dot{q}^j)$. Set $(q^i(t), \dot{q}^j(t)) = \Psi^t(q^i, \dot{q}^j)$.

$$\frac{d}{dt}|_{t=0}\, q^i(t) = X^k \delta^i_k.$$

To compute $\frac{d}{dt}|_{t=0}\, \dot{q}^j(t)\frac{\partial}{\partial q^j}$ we use the definition of the Lie derivative on M

$$-\frac{d}{dt}\dot{q}^j\frac{\partial}{\partial q^j} = L_X(\dot{q}^i\frac{\partial}{\partial q^i})$$

$$= \left(X^k\frac{\partial \dot{q}^j}{\partial q^k} - \dot{q}^k\frac{\partial X^j}{\partial q^k}\right)\frac{\partial}{\partial q^j} = -\dot{q}^i\frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial q^j}$$

since $\partial\dot{q}^j/\partial q^i = 0$ on TM. Hence

$$\mathcal{X} = X^i\frac{\partial}{\partial q^i} + \dot{q}^k\frac{\partial X^j}{\partial q^k}\frac{\partial}{\partial \dot{q}^j}.$$

Corollary 5.1.11. *L is X-invariant iff*

$$X^i\frac{\partial L}{\partial q^i} + \dot{q}^k\frac{\partial X^j}{\partial q^k}\frac{\partial L}{\partial \dot{q}^j} = 0.$$

Theorem 5.1.12. (E. Noether) *If the lagrangian L is X-invariant then the quantity*

$$P_X = X^i \frac{\partial L}{\partial \dot{q}^i} = X^i p_i$$

is conserved along the extremals of L.

Proof Consider an extremal $\gamma = \gamma(q^i(t))$ of L. We compute

$$\frac{d}{dt} P_X(\gamma, \dot{\gamma}) = \frac{d}{dt} \left\{ X^i(\gamma(t)) \frac{\partial L}{\partial \dot{q}^i} \right\} = \frac{\partial X^i}{\partial q^k} \dot{q}^k \frac{\partial L}{\partial \dot{q}^i} + X^i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right)$$

$$= (\text{Euler} - \text{Lagrange}) \; \frac{\partial X^i}{\partial q^k} \dot{q}^k \frac{\partial L}{\partial \dot{q}^i} + X^i \frac{\partial L}{\partial q^i} = 0 \;\; \text{by Corollary 5.1.11.}$$

□

The classical conservation of momentum law is a special consequence of Noether's theorem.

Corollary 5.1.13. *Consider a lagrangian $L = L(t, q, \dot{q})$ on $\mathbb{R}^n$. If $\frac{\partial L}{\partial q^i} = 0$ (the i-th component of the force is zero) then $\frac{dp_i}{dt} = 0$ along any extremal (the i-th component of the momentum is conserved).*

To prove this it suffices to take $X = \frac{\partial}{\partial q^i}$ in Noether's conservation law..

The conservation of momentum has an interesting application in the study of geodesics.

Example 5.1.14. (Geodesics on surfaces of revolution.) Consider a surface of revolution S in $\mathbb{R}^3$ obtained by rotating about the z-axis the curve $y = f(z)$ situated in the yz plane. If we use cylindrical coordinates (r, θ, z) we can describe S as

$$r = f(z).$$

In these coordinates the Euclidean metric in $\mathbb{R}^3$ has the form

$$ds^2 = dr^2 + dz^2 + r^2 d\theta^2.$$

We can choose (z, θ) as local coordinates on S then the induced metric has the form

$$g_S = \{1 + (f'(z))^2\} dz^2 + f^2(z) d\theta^2 = A(z) dz^2 + r^2 d\theta^2. \;\; (r = f(z))$$

The lagrangian defining the geodesics on S is

$$L = \frac{1}{2} \left(A\dot{z}^2 + r^2 \dot{\theta}^2 \right).$$

We see that L is independent of θ: $\frac{\partial L}{\partial \theta} = 0$ so that the generalized momentum

$$\frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta}$$

is conserved along the geodesics.

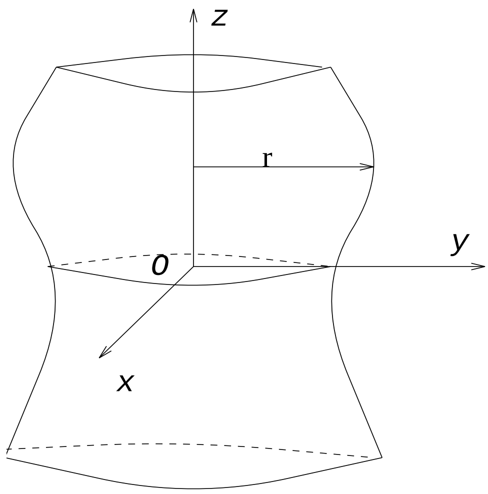

Figure 5.2: *A surface of revolution*

This fact can be given a nice geometric interpretation. Consider a geodesic $\gamma = (z(t), \theta(t))$ and compute the angle ϕ between $\dot{\gamma}$ and $\frac{\partial}{\partial\theta}$. We get

$$\cos\phi = \frac{\langle\dot{\gamma}, \partial/\partial\theta\rangle}{|\dot{\gamma}|\cdot|\partial/\partial\theta|} = \frac{r^2\dot{\theta}}{r|\dot{\gamma}|}$$

i.e. $r\cos\phi = r^2\dot{\theta}|\dot{\gamma}|^{-1}$. The conservation of energy implies that $|\dot{\gamma}|^2 = 2L = H$ is constant along the geodesics. We deduce the following classical result.

Theorem 5.1.15. (Clairaut) *On a a surface of revolution the quantity $r\cos\phi$ is constant along any geodesic. $\phi \in (-\pi, \pi)$ is the oriented angle the geodesic makes with the parallels $z = const$.*

Exercise 5.1.2. Describe the geodesics on the round sphere S^2 and on the cylinder $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$.

5.2 The variational theory of geodesics

We have seen that the paths of minimal length between two points on a Riemann manifold are necessarily geodesics. Conversely, given a geodesic joining two points q_0, q_1 it may happen it is not a minimal path. This should be compared with the situation in calculus when a critical point of a function f may not be a minimum or a maximum. To decide this issue one has to look at the second derivative. This is precisely what we intend to do in the case of geodesics. This situation is a bit more complicated since the action functional

$$S = \frac{1}{2}\int |\dot{\gamma}|^2 dt$$

is not defined on a finite dimensional manifold. It is a function defined on the "space of all paths" joining the two given points. With some extra effort this space can be organized as an infinite dimensional manifold. We will not attempt to formalize these prescriptions but rather follow the ad-hoc, intuitive approach of [55].

5.2.1 Variational formulæ

Let M be a connected Riemann manifold and consider $p, q \in M$. Denote by $\Omega_{p,q} = \Omega_{p,q}(M)$ the space of all *continuous*, *piecewise smooth* paths $\gamma : [0,1] \to M$ connecting p to q.

An *infinitesimal variation* of a path $\gamma \in \Omega_{p,q}$ is a *continuous*, *piecewise smooth* vector field V along γ such that $V(0) = 0$ and $V(1) = 0$. The space of infinitesimal variations of γ is an infinite dimensional linear space denoted by $T_\gamma = T_\gamma \Omega_{p,q}$.

Definition 5.2.1. *Let $\gamma \in \Omega_{p,q}$. A* variation *of γ is a continuous map*

$$\alpha = \alpha_s(t) : (-\varepsilon, \varepsilon) \times [0,1] \to M$$

such that
(i) $\forall s \in (-\varepsilon, \varepsilon)$, $\alpha_s \in \Omega_{p,q}$.
(ii) There exists a partition $0 = t_0 < t_1 \cdots < t_{k-1} < t_k = 1$ of [0,1] such that the restriction of α to each $(-\varepsilon, \varepsilon) \times (t_{i-1}, t_i)$ is a smooth map.

Every variation α of γ defines an *infinitesimal variation*

$$\delta\alpha \stackrel{def}{=} \frac{\partial \alpha_s}{\partial s}|_{s=0} \ .$$

Exercise 5.2.1. Given $V \in T_\gamma$ construct a variation α such that $\delta\alpha = V$. □

Consider now the *energy functional*

$$E : \Omega_{p,q} \to \mathbb{R} \quad E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt.$$

Fix $\gamma \in \Omega_{p,q}$ and let α be a variation of γ. The velocity $\dot\gamma(t)$ has a finite number of discontinuities so that the quantity

$$\Delta_t \dot\gamma = \lim_{h \to 0^+} (\dot\gamma(t+h) - \dot\gamma(t-h))$$

is nonzero only for finitely many t's.

Theorem 5.2.2. (The first variation formula)

$$E_*(\delta\alpha) \stackrel{def}{=} \frac{d}{ds}|_{s=0} E(\alpha_s) = -\sum_t \langle (\delta\alpha)(t), \Delta_t\dot\gamma \rangle - \int_0^1 \langle \delta\alpha, \nabla_{\frac{d}{dt}} \dot\gamma \rangle dt. \qquad (5.2.1)$$

∇ denotes the Levi-Civita connection. (Note that the right-hand-side depends on α only through $\delta\alpha$ so it is really a linear function on T_γ.)

Proof Set $\dot\alpha_s = \frac{\partial \alpha_s}{\partial t}$. We derivate under the integral sign using the equality

$$\frac{\partial}{\partial s}|\dot\alpha_s|^2 = 2\langle \nabla_{\frac{\partial}{\partial s}} \dot\alpha_s, \dot\alpha_s \rangle$$

and we get

$$\frac{d}{ds}|_{s=0} E(\alpha_s) = \int_0^1 \langle \nabla_{\frac{\partial}{\partial s}} \dot\alpha_s, \dot\alpha_s \rangle |_{s=0} \, dt.$$

Since the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ commute we have $\nabla_{\frac{\partial}{\partial s}}\frac{\partial \alpha}{\partial t} = \nabla_{\frac{\partial}{\partial t}}\frac{\partial \alpha}{\partial s}$. Let $0 = t_0 < t_2 < \cdots < t_k = 1$ be a partition of [0,1] as in Definition 5.2.1. Since $\alpha_s = \gamma$ for $s = 0$ we conclude

$$E_*(\delta\alpha) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \nabla_{\frac{\partial}{\partial t}} \delta\alpha, \dot{\gamma}\rangle.$$

We use the equality

$$\frac{\partial}{\partial t}\langle \delta\alpha, \dot{\gamma}\rangle = \langle \nabla_{\frac{\partial}{\partial t}} \delta\alpha, \dot{\gamma}\rangle + \langle \delta\alpha, \nabla_{\frac{\partial}{\partial t}} \dot{\gamma}\rangle$$

to integrate by parts and we obtain

$$E_*(\delta\alpha) = \sum_{i=1}^{k} \langle \delta\alpha, \dot{\gamma}\rangle \Big|_{t_{i-1}}^{t_i} - \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \delta\alpha, \nabla_{\frac{\partial}{\partial t}} \dot{\gamma}\rangle dt.$$

This is precisely equality (5.2.1).

□

Definition 5.2.3. *A path $\gamma \in \Omega_{p,q}$ is critical if*

$$E_*(V) = 0 \quad \forall V \in T_\gamma.$$

Corollary 5.2.4. *A path $\gamma \in \Omega_{p,q}$ is critical if and only if it is a geodesic.*

Exercise 5.2.2. Prove the above corollary. □

Note that a priori a critical path may have a discontinuous first derivative. The above corollary shows that this is not the case; the criticality also implies smoothness. This is a manifestation of a more general analytical phenomenon called *elliptic regularity*. We will have more to say about it in Chapter 9.

The map $E_* : T_\gamma \to \mathbb{R}$, $\delta\alpha \mapsto E_*(\delta\alpha)$ is called the first derivative of E at $\gamma \in \Omega_{p,q}$. We want to define a second derivative of E in order to address the issue raised at the beginning of this section. We will imitate the finite dimensional case which we now briefly analyze.

Let $f : X \to \mathbb{R}$ be a smooth function on the finite dimensional smooth manifold X. If x_0 is a critical point of f, i.e. $df(x_0) = 0$, then we can define the *hessian* at x_0

$$f_{**} : T_{x_0}X \times T_{x_0}X \to \mathbb{R}$$

as follows. Given $V_1, V_2 \in T_{x_0}X$ consider a smooth map $(s_1, s_2) \mapsto \alpha(s_1, s_2) \in X$ such that

$$\alpha(0,0) = x_0 \text{ and } \frac{\partial \alpha}{\partial s_i}(0,0) = V_i, \quad i = 1, 2. \tag{5.2.2}$$

Now set

$$f_{**}(V_1, V_2) = \frac{\partial^2 f(\alpha(s_1, s_2))}{\partial s_1 \partial s_2}\Big|_{(0,0)}.$$

Note that since x_0 is a critical point of f the hessian $f_{**}(V_1, V_2)$ is independent of the function α satisfying (5.2.2).

We now return to our energy functional $E : \Omega_{p,q} \to \mathbb{R}$. Let $\gamma \in \Omega_{p,q}$ be a critical path. Consider a 2-parameter variation of γ

$$\alpha_{s_1,s_2}(t) : U \times [0,1] \to M.$$

U is the tiny square $(-\varepsilon,\varepsilon) \times (-\varepsilon,\varepsilon) \subset \mathbb{R}^2$ and $\alpha_{0,0} = \gamma$. α is continuous and has second derivatives everywhere except maybe on finitely many "coordinate planes $s_i = const$ or $t = const$. Set $\delta_i\alpha = \frac{\partial\alpha}{\partial s_i}|_{(0,0)}$, $i = 1, 2$. Note that $\delta_i\alpha \in T_\gamma$.

Exercise 5.2.3. Given $V_1, V_2 \in T_\gamma$ construct a 2-parameter variation α such that $V_i = \delta_i\alpha$. □

We can now define the hessian of E at γ by

$$E_{**}(\delta_1\alpha, \delta_2\alpha) \stackrel{def}{=} \frac{\partial^2 E(\alpha_{s_1,s_2})}{\partial s_1 \partial s_2}|_{(0,0)} .$$

Theorem 5.2.5. (The second variation formula)

$$E_{**}(\delta_1\alpha, \delta_2\alpha) = -\sum_t \langle \delta_2\alpha, \Delta_t\delta_1\alpha\rangle - \int_0^1 \langle \delta_2\alpha, \nabla^2 \frac{\partial}{\partial t}\delta_1\alpha - R(\dot\gamma, \delta_1\alpha)\dot\gamma\rangle dt, \tag{5.2.3}$$

where R denotes the Riemann curvature. In particular, E_{**} is a bilinear functional on T_γ.

Proof According to the first variation formula we have

$$\frac{\partial E}{\partial s_2} = -\sum_t \langle \delta_2\alpha, \Delta_t \frac{\partial\alpha}{\partial t}\rangle - \int_0^1 \langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\rangle dt.$$

Hence

$$\frac{\partial^2 E}{\partial s_1 \partial s_2} = -\sum_t \langle \nabla_{\frac{\partial}{\partial s_1}} \delta_2\alpha, \Delta_1\dot\gamma\rangle - \sum_t \langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial s_1}} \left(\Delta_t \frac{\partial\alpha}{\partial t}\right)\rangle$$
$$- \int_0^1 \langle \nabla_{\frac{\partial}{\partial s_1}} \delta_2\alpha, \nabla_{\frac{\partial}{\partial t}}\dot\gamma\rangle dt - \int_0^1 \langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial s_1}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\rangle dt. \tag{5.2.4}$$

Since γ is a geodesic $\Delta_t\dot\gamma = 0$ and $\nabla_{\frac{\partial}{\partial t}}\dot\gamma = 0$. Using the commutativity of $\frac{\partial}{\partial t}$ with $\frac{\partial}{\partial s_1}$ we deduce

$$\nabla_{\frac{\partial}{\partial s_1}} \left(\Delta_t \frac{\partial\alpha}{\partial t}\right) = \Delta_t \left(\nabla_{\frac{\partial}{\partial s_1}} \frac{\partial\alpha}{\partial t}\right) = \Delta_t \left(\nabla_{\frac{\partial}{\partial t}} \delta_1\alpha\right).$$

Finally, the definition of the curvature implies

$$\nabla_{\frac{\partial}{\partial s_1}} \nabla_{\frac{\partial}{\partial t}} = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s_1}} + R(\delta_1\alpha, \dot\gamma).$$

Putting all the above together we deduce immediately the equality (5.2.3).

□

Corollary 5.2.6.

$$E_{**}(V_1, V_2) = E_{**}(V_2, V_1) \quad \forall V_1, V_2 \in T_\gamma.$$

5.2.2 Jacobi fields

In this subsection we will put to work the elements of calculus of variations presented so far. Let (M, g) be a Riemann manifold and $p, q \in M$.

Definition 5.2.7. *Let $\gamma \in \Omega_{p,q}$ be a geodesic. A* geodesic variation *of γ is a smooth map $\alpha_s(t) : (-\varepsilon, \varepsilon) \times [0,1] \to M$ such that $\alpha_0 = \gamma$ and $t \mapsto \alpha_s(t)$ is a geodesic for all s. We set as usual $\delta\alpha = \frac{\partial \alpha}{\partial s}|_{s=0}$.*

Proposition 5.2.8. *Let $\gamma \in \Omega_{p,q}$ be a geodesic and (α_s) a geodesic variation of γ. Then the infinitesimal variation $\delta\alpha$ satisfies the* Jacobi equation

$$\nabla_t^2 \delta\alpha = R(\dot\gamma, \delta\alpha)\dot\gamma \quad (\nabla_t = \nabla_{\frac{\partial}{\partial t}}).$$

Proof

$$\nabla_t^2 \delta\alpha = \nabla_t\left(\nabla_t \frac{\partial\alpha}{\partial s}\right) = \nabla_t\left(\nabla_s \frac{\partial\alpha}{\partial t}\right)$$

$$= \nabla_s\left(\nabla_t \frac{\partial\alpha}{\partial t}\right) + R(\dot\gamma, \delta\alpha)\frac{\partial\alpha}{\partial t} = R(\dot\gamma, \delta\alpha)\frac{\partial\alpha}{\partial t}.$$

□

Definition 5.2.9. *A smooth vector field J along a geodesic γ is called a* Jacobi field *if it satisfies the Jacobi equation*

$$\nabla_t^2 J = R(\dot\gamma, J)\dot\gamma.$$

Exercise 5.2.4. Show that if J is a Jacobi field along a geodesic γ then there exists a geodesic variation α_s of γ such that $J = \delta\alpha$. □

Exercise 5.2.5. Let $\gamma \in \Omega_{p,q}$ and J a vector field along γ.
(a) Prove that J is a Jacobi field if and only if

$$E_{**}(J, V) = 0 \;\; \forall V \in T_\gamma.$$

(b) Show that $J \in T_\gamma$ (i.e. J is a vector field along γ *vanishing at endpoints*) is a Jacobi field if any only if $E_{**}(J, W) = 0$ for *all* vector fields W along γ. (It is important to emphasize the point that *not all* vector fields W along γ belong to T_γ.) □

Exercise 5.2.6. Let $\gamma \in \Omega_{p,q}$ be a geodesic. Define $\mathfrak{J}_p$ to be the space of Jacobi fields V along γ such that $V(p) = 0$. Show that $\dim \mathfrak{J}_p = \dim M$ and moreover, the evaluation map

$$\mathbf{ev}_q : \mathfrak{J}_p \to T_pM \quad V \mapsto \nabla_t V(p)$$

is a linear isomorphism. □

Definition 5.2.10. *Let $\gamma(t)$ be a geodesic. Two points $\gamma(t_1)$ and $\gamma(t_2)$ on γ are said to be* conjugate *along γ if there exists a nontrivial Jacobi field J along γ such that $J(t_i) = 0$, $i = 1, 2$.*

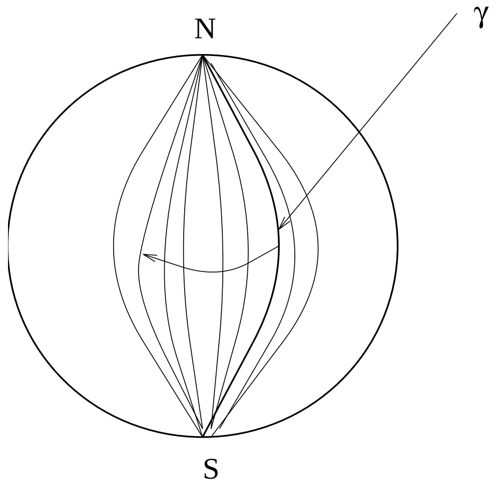

Figure 5.3: *The poles are conjugate along meridians.*

Example 5.2.11. Consider $\gamma : [0, 2\pi] \to S^2$ a meridian on the round sphere connection the poles. One can verify easily (using Clairaut's theorem) that γ is a geodesic. The counterclockwise rotation by an angle θ about the z-axis will produce a new meridian, hence a new geodesic γ_θ. Thus (γ_θ) is a geodesic variation of γ with fixed endpoints. $\delta\gamma$ is a Jacobi field vanishing at the poles. We conclude that the poles are conjugate along any meridian (see Figure 5.3). □

Definition 5.2.12. *Let $\gamma \in \Omega_{p,q}$ be a geodesic. γ is said to be* nondegenerate *if q is not conjugated to p along γ.*

The following result (partially) explains the geometric significance of conjugate points.

Theorem 5.2.13. *Let $\gamma \in \Omega_{p,q}$ be a* nondegenerate, minimal geodesic*. Then p is conjugate with no point on γ other than itself. In particular, a geodesic segment containing conjugate points* cannot be minimal !

Proof We argue by contradiction. Let $p_1 = \gamma(t_1)$ be a point on γ conjugate with p. Denote by J_t a Jacobi field along $\gamma|_{[0,t_1]}$ such that $J_0 = 0$ and $J_{t_1} = 0$. Define $V \in T_\gamma$ by

$$V_t = \begin{cases} J_t & , \quad t \in [0, t_1] \\ 0 & , \quad t \geq t_1 \end{cases} .$$

We will prove that V_t is a Jacobi field along γ which contradicts the nondegeneracy of γ. In the sequel $\mathfrak{l}$ denotes the length.

Step 1

$$E_{**}(U, U) \geq 0 \quad \forall U \in T_\gamma. \tag{5.2.5}$$

Indeed, let α_s denote a variation of γ such that $\delta\alpha = U$. One computes easily that

$$\frac{d^2}{ds^2} E(\alpha_{s^2}) = 2E_{**}(U, U).$$

Since γ is *minimal* for any small s we have $\mathfrak{l}(\alpha_{s^2}) \geq \mathfrak{l}(\alpha_0)$ so that

$$E(\alpha_{s^2}) \geq \frac{1}{2}\left(\int_0^1 |\dot{\alpha}_{s^2}|dt\right)^2 = \frac{1}{2}\mathfrak{l}(\alpha_{s^2})^2 \geq \frac{1}{2}\mathfrak{l}(\alpha_0)^2$$

$$= \frac{1}{2}\mathfrak{l}(\gamma)^2 = E(\alpha_0).$$

Hence

$$\frac{d^2}{ds^2}|_{s=0}\, E(\alpha_{s^2}) \geq 0.$$

This proves (5.2.5).

Step 2 $E_{**}(V, V) = 0$. This follows immediately from the second variation formula and the fact that the nontrivial portion of V is a Jacobi field.

Step 3

$$E_{**}(U, V) = 0 \quad \forall U \in T_\gamma.$$

From (5.2.5) and Step 2 we deduce

$$0 = E_{**}(V, V) \leq E_{**}(V + \tau U, V + \tau U) = f_U(\tau) \quad \forall \tau.$$

Thus, $\tau = 0$ is a global minimum of $f_U(\tau)$ so that

$$f'_U(0) = 0.$$

Step 3 follows from the above equality using the bilinearity and the symmetry of E_{**}. The final conclusion (that V is a Jacobi field) follows from Exercise 5.2.5.

□

Exercise 5.2.7. Let $\gamma : \mathbb{R} \to M$ be a geodesic. Prove that the set

$$\{t \in \mathbb{R}\ ;\ \gamma(t) \text{ is conjugate to } \gamma(0)\}$$

is discrete. □

Definition 5.2.14. *Let $\gamma \in \Omega_{p,q}$ be a geodesic. We define its* index*, denoted by* ind$\,(\gamma)$*, as the cardinality of the set*

$$C_\gamma = \{t \in (0, 1)\ ;\ \text{is conjugate to } \gamma(0)\}$$

which by Exercise 5.2.7 is finite.

Theorem 5.2.13 can be reformulated as follows: the index of a nondegenerate minimal geodesic is zero.

The index of a geodesic obviously depends on the curvature of the manifold. Often, this dependence is very powerful.

Theorem 5.2.15. *Let M be a Riemann manifold with non-positive* sectional curvature*, i.e.*

$$\langle R(X, Y)Y, X\rangle \leq 0 \ \ \forall X, Y \in T_xM \ \ \forall x \in M. \tag{5.2.6}$$

Then for any $p, q \in M$ and any geodesic $\gamma \in \Omega_{p,q}$, ind$\,(\gamma) = 0$.

Proof It suffices to show that for any geodesic $\gamma : [0,1] \to M$ the point $\gamma(1)$ is not conjugated to $\gamma(0)$.

Let J_t be a Jacobi field along γ vanishing at the endpoints. Thus

$$\nabla_t^2 J = R(\dot{\gamma}, J)\dot{\gamma}$$

so that

$$\int_0^1 \langle \nabla_t^2 J, J\rangle dt = \int_0^1 \langle R(\dot{\gamma}, J)\dot{\gamma}, J\rangle dt = -\int_0^1 \langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle dt.$$

We integrate by parts the left-hand-side of the above equality and we deduce

$$\langle \nabla_t J, J\rangle|_0^1 - \int_0^1 |\nabla_t J|^2 dt = -\int_0^1 \langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle dt.$$

Since $J(\tau) = 0$ for $\tau = 0, 1$ we deduce using (5.2.6)

$$\int_0^1 |\nabla_t J|^2 dt \leq 0.$$

This implies $\nabla_t J = 0$ which coupled with the condition $J(0) = 0$ implies $J \equiv 0$. The proof is complete.

□

The notion of conjugacy is intimately related to the behavior of the exponential map.

Definition 5.2.16. *Let $f : X \to Y$ be a smooth map between the smooth manifolds x and Y.*
(a) A point $x \in X$ is said to be critical *if*

$$\operatorname{rank}(Df_x : T_x X \to T_{f(x)} Y) < \min\{\dim T_x X,\ \dim T_{f(x)} Y\}.$$

(b) A point $y \in Y$ is called a critical value *for f if $f^{-1}(y)$ contains at least one critical point of f.*
(c) $y \in Y$ is a regular value *if it is not a critical value.*

Theorem 5.2.17. *Let (M, g) be a connected, complete, Riemann manifold and $q_0 \in M$. A point $q \in M$ is conjugated to q_0 along some geodesic if and only if it is a critical value for the exponential map*

$$\exp_{q_0} : T_{q_0} M \to M.$$

Proof Let $q = \exp_{q_0} v$ ($v \in T_{q_0} M$). Assume first that q is a critical value for $\exp_{q_0}$ and v is a critical point. Then $D_v \exp_{q_0}(X) = 0$ for some $X \in T_v(T_{q_0} M)$. Let $v(s)$ be a path in $T_{q_0} M$ such that $v(0) = v$ and $\dot{v}(0) = X$. The map $(s, t) \mapsto \exp_{q_0}(tv(s))$ is a geodesic variation of the radial geodesic $\gamma_v :\ t \mapsto \exp_{q_0}(tv)$. Hence the vector field

$$W = \frac{\partial}{\partial s}|_{s=0} \exp_{q_0}(tv(s))$$

is a Jacobi field along γ_v. Obviously $W(0) = 0$ and moreover

$$W(1) = \frac{\partial}{\partial s}|_{s=0} \exp_{q_0}(v(s)) = D_v \exp_{q_0}(X) = 0.$$

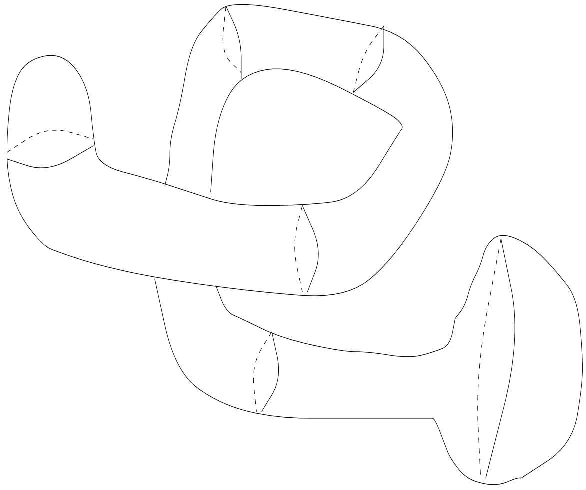

Figure 5.4: *Lengthening a sphere.*

On the other hand this is a nontrivial field since

$$\nabla_t W = \nabla_s |_{s=0} \frac{\partial}{\partial t} \exp_{q_0}(tv(s)) = \nabla_s v(s) |_{s=0} \neq 0.$$

This proves q_0 and q are conjugated along γ_v.

Conversely, assume v is not a critical point for $\exp_{q_0}$. For any $X \in T_v(T_{q_0}M)$ denote by J_X the Jacobi field along γ_v such that

$$J_X(q_0) = 0. \tag{5.2.7}$$

The existence of such a Jacobi field follows from Exercise 5.2.6. As in that exercise denote by $\mathfrak{J}_{q_0}$ the space of Jacobi fields J along γ_v such that $J(q_0) = 0$. The map

$$T_v(T_{q_0}M) \to \mathfrak{J}_{q_0} \quad X \mapsto J_X$$

is a linear isomorphism. Thus, a Jacobi field along γ_v vanishing at both q_0 and q must have the form J_X where $X \in T_v(T_{q_0}M)$ satisfies $D_v \exp_{q_0}(X) = 0$. Since v is not a critical point this means $X = 0$ so that $J_X \equiv 0$.

$\square$

Corollary 5.2.18. *On a complete Riemann manifold M with non-positive sectional curvature the exponential map $\exp_q$ has no critical values for any $q \in M$.*

We will see in the next chapter that this corollary has a lot to say about the topology of M.

Consider now the following experiment. Stretch the round two-dimensional sphere of radius 1 until it becomes "very long". A possible shape one can obtain may look like in Figure 5.4. The long tube is very similar to a piece of cylinder so that the total (= scalar) curvature is very close to zero, in other words is very small. The lesson to learn from this intuitive experiment is that the price we have to pay for lengthening the sphere is decreasing the curvature. Equivalently, a highly curved surface cannot have a large diameter. Our next result offers a more quantitative description of this phenomenon.

Theorem 5.2.19. (Myers) *Let M be an n-dimensional complete Riemann manifold. If for all $X \in \mathrm{Vect}\,(M)$*

$$\mathrm{Ric}\,(X, X) \geq \frac{(n-1)}{r^2}|X|^2$$

then every geodesic of length $\geq \pi r$ has conjugate points and thus is not minimal. Hence

$$\mathrm{diam}\,(M) = \sup\{\mathrm{dist}\,(p, q)\ ;\ p, q \in M\} \leq \pi r$$

and in particular the Hopf-Rinow theorem implies that M must be compact.

Proof Fix a minimal geodesic $\gamma : [0, \ell] \to M$ of length ℓ and let $e_i(t)$ be an orthonormal basis of vector fields along γ such that $e_n(t) = \dot{\gamma}(t)$. Set $q_0 = \gamma(0)$ and $q_1 = \gamma(\ell)$. Since γ is minimal we deduce

$$E_{**}(V, V) \geq 0 \quad \forall V \in T_\gamma.$$

Set $W_i = \sin(\pi t/\ell)e_i$. Then

$$E_{**}(W_i, W_i) = -\int_0^\ell \langle W_i, \nabla_t W_i + R(W_i, \dot{\gamma})\dot{\gamma}\rangle dt$$

$$= \int_0^\ell \sin^2(\pi t/\ell)\left(\pi^2/\ell^2 - \langle R(e_i, \dot{\gamma})\dot{\gamma}, e_i\rangle\right) dt.$$

We sum over $i = 1, \ldots, n-1$ and we obtain

$$\sum_{i=1}^{n-1} E_{**}(W_i, W_i) = \int_0^\ell \sin^2 \pi t/\ell \left((n-1)\pi^2/\ell^2 - \mathrm{Ric}\,(\dot{\gamma}, \dot{\gamma})\right) dt \geq 0.$$

If $\ell > \pi r$ then

$$(n-1)\pi^2/\ell^2 - \mathrm{Ric}\,(\dot{\gamma}, \dot{\gamma}) < 0$$

so that

$$\sum_{i=1}^{n-1} E_{**}(W_i, W_i) < 0.$$

Hence for at least for some W_i we have $E_{**}(W_i, W_i) < 0$. This contradicts the minimality of γ. The proof is complete.

□

Corollary 5.2.20. *A semisimple Lie group G with* positive definite *Killing pairing is compact.*

Proof The Killing form defines in this case a bi-invariant Riemann metric on G. Its geodesics through the origin $1 \in G$ are the 1-parameter subgroups $\exp(tX)$ which are defined for all $t \in \mathbb{R}$. Hence, by Hopf-Rinow theorem G has to be complete.

On the other hand we have computed the Ricci curvature of the Killing metric and we found

$$\mathrm{Ric}\,(X, Y) = \frac{1}{4}\kappa(X, Y) \quad \forall X, Y \in \mathfrak{L}_G.$$

The corollary now follows from Myers' theorem.

□

Exercise 5.2.8. Let M be a Riemann manifold and $q \in M$. For the unitary vectors $X, Y \in T_qM$ consider the family of geodesics

$$\gamma_s(t) = \exp_q t(X + sY).$$

Denote by $W_t = \delta\gamma_s$ the associated Jacobi field along $\gamma_0(t)$. Form $f(t) = |W_t|^2$. Prove the following.
(a) $W_t = D_{tX} \exp_q(Y)$ = Frechet derivative of $v \mapsto \exp_q(v)$
(b) $f(t) = t^2 - \frac{1}{3}\langle R(Y, X)X, Y\rangle_q t^4 + O(t^5)$.
(c) Denote by $\mathbf{x}^i$ a collection of normal coordinates at q. Prove that

$$g_{k\ell}(\mathbf{x}) = \delta_{k\ell} - \frac{1}{3}R_{kij\ell}\mathbf{x}^i\mathbf{x}^j + O(3).$$

$$\det g_{ij}(\mathbf{x}) = 1 - \frac{1}{3}R_{ij}\mathbf{x}^i\mathbf{x}^j + O(3).$$

(e) Let

$$\mathbf{D}_r(q) = \{x \in T_qM \ ; \ |x| \leq r\}.$$

Prove that if the Ricci curvature is negative definite at q then

$$\mathrm{vol}_0\left(\mathbf{D}_r(q)\right) \leq \mathrm{vol}_g\left(\exp_q(\mathbf{D}_r(q)\right)$$

for all r sufficiently small. vol_0 denotes the Euclidean volume in T_qM while vol_g denotes the volume on the Riemann manifold M. □

Remark 5.2.21. The interdependence "curvature-topology" on a Riemann manifold has deep reaching ramifications which stimulate the curiosity of many researchers. We refer to [23] or [55] and the extensive references therein for a presentation of some of the most attractive results in this direction. □

Chapter 6

The fundamental group and covering spaces

In the previous chapters we almost exclusively studied local properties of manifolds. This study is interesting only if some additional structure is present since otherwise all manifolds are locally alike.

We noticed an interesting phenomenon: the global "shape" (topology) of a manifold restricts the types of structures that can exist on a manifold. For example, the Gauss-Bonnet theorem implies that on a connected sum of two tori there cannot exist metrics with curvature everywhere positive because the integral of the curvature is a negative universal constant.

We used Gauss-Bonnet theorem in the opposite direction and we deduced-the intuitively obvious fact- that a sphere is not diffeomorphic to a torus because they have distinct genera. The Gauss-Bonnet theorem involves a heavy analytical machinery which obscures the intuition. Notice that S^2 has a remarkable property which distinguishes it from T^2: on the sphere any closed curve can be shrunk to a point while on the torus there exist at least two "independent" unshrinkable curves (see Figure 6.1). In particular this means the sphere is not diffeomorphic to a torus.

This chapter will set the above observations on a rigorous foundation.

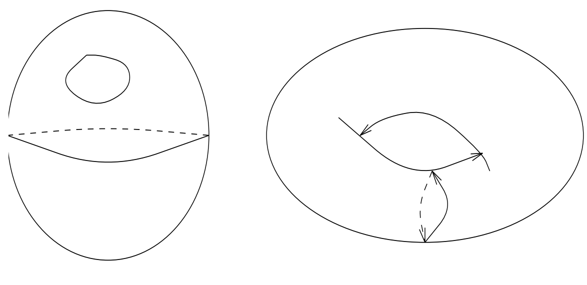

Figure 6.1: *Looking for unshrinkable loops.*

6.1 The fundamental group

6.1.1 Basic notions

In the sequel all topological spaces will be locally path connected spaces.

Definition 6.1.1. *(a) Let X and Y be two topological spaces and $f_0, f_1 : X \to Y$ continuous maps. f_0 and f_1 are said to be* homotopic *if there exists a continuous map*

$$F : [0,1] \times X \to Y \quad (t, x) \mapsto F_t(x)$$

such that $F_i \equiv f_i$ for $i = 0, 1$. We write this as $f_0 \simeq_h f_1$.
(b) Two topological spaces X, Y are said to be homotopically equivalent *if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_h \mathbf{1}_Y$ and $g \circ f \simeq_h \mathbf{1}_X$. We write this $X \simeq_h Y$.*
(c) A topological space is said to be contractible if it is homotopically equivalent to a point.

Example 6.1.2. The unit disk in the plane is contractible. The annulus $\{1 \leq |z| \leq 2\}$ is homotopically equivalent with the unit circle. □

Definition 6.1.3. *(a) Let X be a topological space and $x_0 \in X$. A* loop based at x_0 *is a continuous map*

$$\gamma : [0,1] \to X \text{ such that } \gamma(0) = \gamma(1) = x_0.$$

The space of loops in X based at x_0 is denoted by $\Omega(X, x_0)$.
(b) Two loops $\gamma_0, \gamma_1 : I \to X$ based at x_0 are said to be homotopic rel x_0 *if there exists a continuous map*

$$\Gamma : [0,1] \times I \to X \quad (t, s) \mapsto \Gamma_t(s)$$

such that

$$\Gamma_i(s) = \gamma_i(s) \quad i = 0, 1$$

and

$$(s \mapsto \Gamma_t(s)) \in \Omega(X, x_0) \quad \forall t \in [0,1].$$

We write this as $\gamma_0 \simeq_{x_0} \gamma_1$.

Note that a loop is more than a closed curve; it is a closed curve + a description of a motion of a point around the closed curve.

Example 6.1.4. The two loops $\gamma_1, \gamma_2 : I \to \mathbb{C}$, $\gamma_k(t) = \exp(2k\pi t)$, $k = 1, 2$ are different though they have the same image. □

Definition 6.1.5. *(a) Let γ_1, γ_2 be two loops based at $x_0 \in X$. The* product *of γ_1 and γ_2 is the loop*

$$\gamma_1 * \gamma_2(s) = \begin{cases} \gamma_1(2s) & , \quad 0 \leq s1/2 \\ \gamma_2(2s-1) & , \quad 1/2 \leq s \leq 1 \end{cases}.$$

The inverse *of a based loop γ is the based loop γ^- defined by*

$$\gamma^-(s) = \gamma(1-s).$$

(c) The identity *loop is the constant loop $\mathfrak{e}_{x_0}(s) \equiv x_0$.*

Intuitively, the product of two loops γ_1 and γ_2 is the loop obtained by first following γ_1 (twice faster) and then γ_2 (twice faster).

The following result is left to the reader as an exercise.

Lemma 6.1.6. *Let $\alpha_0 \simeq_{x_0} \alpha_1$, $\beta_0 \simeq_{x_0} \beta_1$ and $\gamma_0 \simeq_{x_0} \gamma_1$ be three pairs of homotopic based loops.. Then*
(a) $\alpha_0 * \beta_0 \simeq_{x_0} \alpha_1 * \beta_1$.
(b) $\alpha_0 * \alpha_0^- \simeq_{x_0} \mathfrak{e}_{x_0}$.
(c) $\alpha_0 * \mathfrak{e}_{x_0} \simeq_{x_0} \alpha_0$.
(d) $(\alpha_0 * \beta_0) * \gamma_0 \simeq_{x_0} \alpha_0 * (\beta_0 * \gamma_0)$.

Hence the product operation descends to an operation "$\cdot$" on $\Omega(X, x_0)/\simeq_{x_0}$- the set of homotopy classes of based loops. Moreover the induced operation is associative, it has a unit and each element has an inverse. Hence $(\Omega(X, x_0)/\simeq_{x_0}, \cdot)$ is a group.

Definition 6.1.7. *The group $(\Omega(X, x_0)/\simeq_{x_0}, \cdot)$ is called the* fundamental group *(or the Poincaré group) of the topological space X and is denoted by $\pi_1(X, x_0)$. The image of a based loop γ in $\pi_1(X, x_0)$ is denoted by $[\gamma]$.*

The elements of $\pi_1(X, x_0)$ are the "unshrinkable loops" discussed at the beginning of this chapter.

The fundamental group $\pi_1(X, x_0)$ "sees" only the connected component of X which contains x_0. To get more information about X one should study all the groups $\{\pi_1(X, x)\}_{x \in X}$.

Proposition 6.1.8. *Let X and Y be two topological spaces $x_0 \in X$ and $y_0 \in Y$. Then any continuous map $f : X \to Y$ such that $f(x_0) = y_0$ induces a morphism of groups*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

satisfying the following functoriality properties.
(a) $(\mathbf{1}_X)_* = \mathbf{1}_{\pi_1(X, x_0)}$.
(b) If

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

are continuous maps (such that $f(x_0) = y_0$ and $g(y_0) = z_0$) then $(g \circ f)_* = g_* \circ f_*$.
(c) Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ be two base-point-preserving continuous maps. Assume f_0 is homotopic to f_1 rel x_0 i.e. there exists a continuous map $F : I \times X \to Y$, $(t, x) \mapsto F_t(x)$ such that $F_i(x) \equiv f_i(x)$ for $i = 0, 1$ and $F_t(x_0) \equiv y_0$. Then $(f_0)_* = (f_1)_*$.

Proof Let $\gamma \in \Omega(X, x_0)$ Then $f(\gamma) \in \Omega(Y, y_0)$ and one can check immediately that

$$\gamma \simeq_{x_0} \gamma' \Rightarrow f(\gamma) \simeq_{y_0} f(\gamma').$$

Hence the correspondence

$$\Omega(X, x_0) \ni \gamma \mapsto f(\gamma) \in \Omega(Y, y_0)$$

descends to a map $f : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. This is clearly a group morphism. (a) and (b) are now obvious. We prove (c).

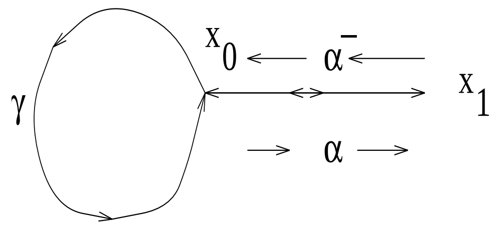

Figure 6.2: *Connecting base points.*

Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ be two continuous maps and F_t a homotopy rel x_0 connecting them. For any $\gamma \in \Omega(X, x_0)$

$$\beta_0 = f_0(\gamma) \simeq_{y_0} f_1(\gamma) = \beta_1.$$

The above homotopy is realized by $\mathcal{B}_t = F_t(\gamma)$.

□

A priori, the fundamental group of a topological space X may change as the base point varies and it almost certainly does if X has several connected components. However, if X is connected (and thus path connected since it is locally so) all fundamental groups $\pi_1(X, x)$, $x \in X$ are isomorphic.

Proposition 6.1.9. *Let X be a connected topological space. Any continuous path $\alpha : [0, 1] \to X$ joining x_0 to x_1 induces an isomorphism*

$$\alpha_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

defined by

$$\alpha_*([\gamma]) = [\alpha^- * \gamma * \alpha]$$

(see Figure 6.2).

Exercise 6.1.1. Prove Proposition 6.1.9.

Thus, the fundamental group of a connected space X is independent of the base point modulo some isomorphism. We will write $\pi_1(X, pt)$ to underscore this weak dependence on the base point.

Corollary 6.1.10. *Two homotopically equivalent connected spaces have isomorphic fundamental groups.*

Example 6.1.11. (a) $\pi_1(\mathbb{R}^n, pt) \sim \pi_1(pt, pt) = \{1\}$.
(b) $\pi_1(\text{annulus}) \sim \pi_1(S^1)$. □

Definition 6.1.12. *A connected space X such that $\pi_1(X, pt) = \{1\}$ is said to be* simply connected.

Exercise 6.1.2. Prove that the spheres of dimension ≥ 2 are simply connected. □

Exercise 6.1.3. Let G be a connected Lie group. Define a new operation "$\star$ " on $\Omega(G, 1)$ by

$$(\alpha \star \beta)(s) = \alpha(s) \cdot \beta(s)$$

where $\cdot$ denotes the group multiplication.
(a) Prove that $\alpha \star \beta \simeq_1 \alpha * \beta$.
(b) Prove that $\pi_1(G, 1)$ is abelian. $\square$

Exercise 6.1.4. Let $E \to X$ be a rank r complex vector bundle over the smooth manifold X and let ∇ be a flat connection on E i.e. $F(\nabla) = 0$. Pick $x_0 \in X$ and identify the fiber E_{x_0} with $\mathbb{C}^r$. For any continuous, piecewise smooth $\gamma \in \Omega(X, x_0)$ denote by $T_\gamma = T_\gamma(\nabla)$ the parallel transport along γ so that $T_\gamma \in GL\,(r, \mathbb{C})$.
(a) Prove that $\alpha \simeq_{x_0} \beta \Rightarrow T_\alpha = T_\beta$.
(b) $T_{\beta * \alpha} = T_\alpha \circ T_\beta$.

Thus, any flat connection induces a group morphism

$$T : \pi_1(X, x_0) \to GL\,(r, \mathbb{C}) \quad \gamma \mapsto T_\gamma^{-1}.$$

This morphism (representation) is called the *monodromy* of the connection. $\square$

Example 6.1.13. We want to compute the fundamental group of the complex projective space $\mathbb{CP}^n$. More precisely, we want to show it is simply connected. We will establish this by induction.

For $n = 1$, $\mathbb{CP}^1 \cong S^2$ and by Exercise 6.1.2 the sphere S^2 is simply connected. We next assume $\mathbb{CP}^k$ is simply connected for $k < n$ and prove the same is true for n.

Notice first that the natural embedding $\mathbb{C}^{k+1} \hookrightarrow \mathbb{C}^{n+1}$ induces an embedding $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^n$. More precisely, in terms of homogeneous coordinates this embedding is given by

$$[z_0 : \ldots : z_k] \mapsto [z_0 : \ldots : z_k : 0 : \ldots : 0] \in \mathbb{CP}^n.$$

Choose as base point $\mathbf{pt} = [1 : 0 : \ldots : 0] \in \mathbb{CP}^n$ and let $\gamma \in \Omega(\mathbb{CP}^n, pt)$. We may assume γ avoids the point $P = [0 : \ldots : 0 : 1]$ since we can homotop it out of any neighborhood of P.

We now use a classical construction of projective geometry. We project γ from P to the "hyperplane" $\mathcal{H} = \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$. More precisely, if $\zeta = [z_0 : \ldots : z_n] \in \mathbb{CP}^n$ we denote by $\pi(\zeta)$ the intersection of the line $P\zeta$ with the hyperplane $\mathcal{H}$. In homogeneous coordinates

$$\pi(\zeta) = [z_0(1 - z_n) : \ldots : z_{n-1}(1 - z_n) : 0] \; (= [\mathrm{z_0 : \ldots : z_{n-1}} : 0] \text{ when } z_n \neq 1).$$

Clearly π is continuous. For $t \in [0, 1]$ define

$$\pi_t(\zeta) = [z_0(1 - tz_n) : \ldots : z_{n-1}(1 - tz_n) : (1 - t)z_n].$$

Geometrically, π_t flows the point ζ along the line $P\zeta$ until it reaches the hyperplane $\mathcal{H}$. Note that $\pi_t(\zeta) = \zeta \; \forall t$ and $\forall \zeta \in \mathcal{H}$. Clearly π_t is a homotopy rel pt connecting $\gamma = \pi_0(\gamma)$ to a loop γ_1 in $\mathcal{H} \cong \mathbb{CP}^{n-1}$ based at pt. Our induction hypothesis shows that γ_1 can be shrunk to pt inside $\mathcal{H}$. This proves $\mathbb{CP}^n$ is simply connected. $\square$

6.1.2 Of categories and functors

The considerations in the previous subsection can be very elegantly presented using the language of categories and functors. This brief subsection is a minimal introduction to this language. The interested reader can learn more about it from the monograph [51].

A *category* is a triple $\mathcal{C}$=(**Ob**, **Hom**, $\circ$) where

(i) **Ob** is a collection of elements called the *objects* of the category.

(ii) **Hom** is a family of sets $\mathrm{Hom}\,(X,Y)$, one for each pair of objects X and Y. The elements of $\mathrm{Hom}\,(X,Y)$ are called the *morphisms* (or arrows) from X to Y.

(iii) $\circ$ is a collection of maps

$$\circ : \mathrm{Hom}\,(X,Y) \times \mathrm{Hom}\,(Y,Z) \to \mathrm{Hom}\,(X,Z)$$

$$(f,g) \mapsto g \circ f.$$

which satisfies the following conditions.

(C1) For any object X there exists a unique element $\mathbf{1}_X \in \mathrm{Hom}\,(X,X)$ such that

$$f \circ \mathbf{1}_X = f \quad g \circ \mathbf{1}_X = g \;\; \forall f \in \mathrm{Hom}\,(X,Y)\; \forall g \in \mathrm{Hom}\,(Z,X).$$

(C2) $\forall f \in \mathrm{Hom}\,(X,Y)$ $g \in \mathrm{Hom}\,(Y,Z)$ $h \in \mathrm{Hom}\,(Z,W)$

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Remark 6.1.14. We note in passing that we were deliberately sloppy when we used the term "collection" when we introduced **Ob**. It may happen **Ob** or **Hom** is not a set and one has to deal with thorny foundational issues which are beyond the scope of this book □

Example 6.1.15. • **Top** is the category of topological spaces. The objects are topological spaces and the morphisms are the continuous maps.

• (**Top**, $*$) is the category of marked topological spaces. the objects are pairs (X, $*$) where X is a topological space and $*$ is a distinguished point of X. The morphisms

$$(X,*) \xrightarrow{f} (Y,\diamond)$$

are the continuous maps $f : X \to Y$ such that $f(*) = \diamond$.

• ${}_{\mathbf{F}}$**Vect** is the category of vector spaces over the field **F**. The morphisms are the **F**-linear maps.

• **Gr** is the category of groups, while **Ab** denotes the category of abelian groups. The morphisms are the obvious ones.

• ${}_R$**Mod** denotes the category of left R-modules, where R is some ring. □

Definition 6.1.16. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. A* covariant *(resp.* contravariant*) functor is a map*

$$\mathcal{F} : \mathbf{Ob}\,(\mathcal{C}_1) \times \mathbf{Hom}\,(\mathcal{C}_1) \to \mathbf{Ob}\,(\mathcal{C}_2) \times \mathbf{Hom}\,(\mathcal{C}_2)$$

$$(X,f) \mapsto (\mathcal{F}(X), \mathcal{F}(f))$$

such that if $X \xrightarrow{f} Y$ *then* $\mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y)$ *(resp.*$\mathcal{F}(X) \xleftarrow{\mathcal{F}(f)} \mathcal{F}(Y)$*) and*
(i) $\mathcal{F}(\mathbf{1}_X) = \mathbf{1}_{\mathcal{F}(X)}$
(ii) $\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f)$ *(resp.* $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathcal{F}(g \circ f)$*).*

Example 6.1.17. Let V be a real vector space. Then right tensoring with V defines a covariant functor

$$\otimes V : {}_{\mathbb{R}}\mathbf{Vect} \to {}_{\mathbb{R}}\mathbf{Vect}$$

defined by

$$U \mapsto U \otimes V \quad (U_1 \xrightarrow{L} U_2) \mapsto (U_1 \otimes V \xrightarrow{L \otimes \mathbf{1}_V} U_2 \otimes V).$$

On the other hand, taking the dual

$$^* : {}_{\mathbb{R}}\mathbf{Vect} \to {}_{\mathbb{R}}\mathbf{Vect}$$

$$V \mapsto V^* \quad (U \xrightarrow{L} V) \mapsto (V^* \xrightarrow{L^t} U^*)$$

defines a contravariant functor.

The fundamental group construction of the previous is a covariant functor

$$\pi_1 : (\mathbf{Top}, *) \to \mathbf{Gr}.$$

□

In Chapter 7 we will introduce other functors very important in geometry.

6.2 Covering spaces

6.2.1 Definitions and examples

As in the previous section we will assume that all topological spaces are locally path connected.

Definition 6.2.1. *(a) A continuous map* $\pi : X \to Y$ *is said to be a* covering map *if for any* $y \in Y$ *there exists an open neighborhood* U *such that* $\pi^{-1}(U)$ *is a disjoint union of open sets* $V_i \subset X$ *each of which is mapped homeomorphically onto* U *by* π*. Such an* U *is said to be an* evenly covered *neighborhood. The sets* V_i *are called the* sheets *over* U*.*
(b) Let Y *be a topological space. A* covering space *of* Y *is a topological space* X *together with a covering map* $\pi : X \to Y$*.*

If $\pi : X \to Y$ *is a covering map then for any* $y \in Y$ *the set* $\pi^{-1}(y)$ *is called the* fiber *over* y*.*

Example 6.2.2. Let D be a discrete set. Then for any topological space X the product $X \times D$ is a covering of X with covering projection $\pi(x, d) = x$. This type of covering space is said to be trivial. □

Exercise 6.2.1. Show that a fibration with standard fiber a discrete space is a covering. □

Example 6.2.3. The exponential map $\exp : \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi \mathbf{i} t)$ is a covering map. However its restriction to $(0, \infty)$ is no longer a a covering map. (Prove this!). □

Exercise 6.2.2. Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Riemann manifolds of the same dimension such that $(\tilde{M}, \tilde{g})$ is complete. Let $\phi : \tilde{M} \to M$ a surjective local isometry i.e. Φ is smooth and

$$|v|_g = |D\phi(v)|_{\tilde{g}} \quad \forall v \in T\tilde{M}.$$

Prove that ϕ is a covering map. □

The above exercise has a particularly nice consequence.

Theorem 6.2.4. (Cartan-Hadamard) *Let (M, g) be a* complete *Riemann manifold with non-positive sectional curvature. Then for every point $q \in M$ the exponential map*

$$\exp_q : T_qM \to M$$

is a covering map.

Proof Consider the pull-back $h = \exp_q^*(g)$. h is a symmetric non-negative definite (0,2)-tensor field on T_qM. It is in fact a metric (i.e. it is *positive definite*) since the map $\exp_q$ has no critical points due to the non-positivity of the sectional curvature.

The lines $t \mapsto tv$ through the origin of T_qM are geodesics of h and they are defined for all $t \in \mathbb{R}$. By Hopf-Rinow theorem we deduce that (T_qM, h) is complete. The theorem now follows from Exercise 6.2.2.

□

Exercise 6.2.3. Let $\tilde{G}$ and G two Lie groups of the same dimension and $\phi : \tilde{G} \to G$ a smooth, surjective group morphism. Prove that ϕ is a covering map. In particular, this explains why $\exp : \mathbb{R} \to S^1$ is a covering map. □

Exercise 6.2.4. Identify $S^3 \subset \mathbb{R}^4$ with the group of unit quaternions

$$S^3 = \{q \in \mathbb{H}\ ;\ |q| = 1\}.$$

The linear space $\mathbb{R}^3$ can be identified with the space of purely imaginary quaternions

$$\mathbb{R}^3 = \mathfrak{Im}\,\mathbb{H} = \{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\}.$$

(a) Prove that $\forall q \in S^3$ $qxq^{-1} \in \mathfrak{Im}\,\mathbb{H}$.
(b) Prove that for any $q \in S^3$ the linear map

$$T_q : \mathfrak{Im}\,\mathbb{H} \to \mathfrak{Im}\mathbb{H} \quad x \mapsto qxq^{-1}$$

is an isometry so that $T_q \in$SO(3). Moreover the map

$$S^3 \ni q \mapsto T_q \in SO(3)$$

is a group morphism.
(c) Prove the above group morphism is a covering map. □

Example 6.2.5. Let M be a smooth manifold. A Riemann metric on M induces a metric on the determinant line bundle $\det TM$. The sphere bundle of $\det TM$ (with respect to this metric) is a covering space of M called the *orientation cover* of M. □

Definition 6.2.6. *Let $X_1 \stackrel{\pi_1}{\to} Y$ and $X_2 \stackrel{\pi_2}{\to} Y$ be two covering spaces of Y. A* morphism of covering spaces *is a continuous map $F : X_1 \to X_2$ such that $\pi_2 \circ F = \pi_1$ i.e. the diagram below is commutative.*

$$\begin{tikzcd} X_1 \arrow[rr, "F"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

If F is also a homeomorphism we say F is an isomorphism of covering spaces.

Finally, if $X \stackrel{\pi}{\to} Y$ is a covering space then its automorphisms are called deck transformations. *The deck transformations form a group denoted by* Deck (X, π).

Exercise 6.2.5. Show that Deck $(\mathbb{R} \stackrel{\exp}{\to} S^1) \cong \mathbb{Z}$. □

Exercise 6.2.6. (a) Prove that the natural projection $S^n \to \mathbb{RP}^n$ is a covering map.
(b) Denote by $\tau_{\mathbb{R}}^1$ the tautological (real) line bundle over $\mathbb{RP}^n$. Using a metric on this line bundle form the associated sphere bundle $S(\tau_{\mathbb{R}}^1) \to \mathbb{RP}^n$. Prove that this defines a covering space isomorphic with the one described at part (a). □

6.2.2 Unique lifting property

Definition 6.2.7. *Let $X \stackrel{\pi}{\to} Y$ be a covering space and $F : Z \to Y$ a continuous map. A* lift *of f is a continuous map $F : Z \to X$ such that $\pi \circ F = f$, i.e. the diagram below is commutative.*

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Z \arrow[ur, dashed, "F"] \arrow[r, "f"] & Y \end{tikzcd}$$

Proposition 6.2.8. (Unique Path Lifting) *Let $X \stackrel{\pi}{\to} Y$ be a covering map, $\gamma : [0,1] \to Y$ a path in Y and x_0 a point in the fiber over $y_0 = \gamma(0)$, $x_0 \in \pi^{-1}(y_0)$. Then there exists at most one lift of γ, $\Gamma : [0,1] \to Y$ such that $\Gamma(0) = x_0$.*

Proof We argue by contradiction. Assume there exist two such lifts, $\Gamma_1, \Gamma_2 : [0,1] \to X$. Set

$$S = \{t \in [0,1] \; ; \; \Gamma_1(t) = \Gamma_2(t)\}.$$

$S \neq \emptyset$ since $0 \in S$. Obviously S is closed so it suffices to prove that it is also open. We will prove that there exists $r_0 > 0$ such that $[0, r_0] \subset S$. The general situation is entirely similar.

Pick a small open neighborhood U of x_0 such that π restricts to a homeomorphism onto $\pi(U)$. There exists $r_0 > 0$ such that $\gamma_i([0, r_0] \subset U$, $i = 1, 2$. Since $\pi \circ \Gamma_1 = \pi \circ \Gamma_2$ we deduce $\Gamma_1 |_{[0,r_0]} = \Gamma_2 |_{[0,r_0]}$. The proposition is proved.

□

Theorem 6.2.9. *Let $X \xrightarrow{\pi} Y$ be a covering space and $f : Z \to Y$ a continuous map, where Z is a connected space. Fix $z_0 \in Z$ and x_0 a point in $\pi^{-1}(y_0)$, where $y_0 = f(z_0)$. Then there exists at most one lift $F : Z \to X$ of f such that $F(z_0) = x_0$.*

Proof For each $z \in Z$ let α_z be a continuous path connecting z_0 to z. If F_1, F_2 are two lifts of f such that $F_1(z_0) = F_2(z_0) = x_0$ then for any $z \in Z$ the paths $\Gamma_1 = F_i(\alpha_z)$ and $\Gamma_2 = F_2(\alpha_z)$ are two lifts of $\gamma = f(\alpha_z)$ starting at the same point. From Proposition 6.2.8 we deduce that $\Gamma_1 \equiv \Gamma_2$, i.e. $F_1(z) = F_2(z)$ for any $z \in Z$.

□

6.2.3 Homotopy lifting property

Theorem 6.2.10. (Homotopy lifting property) *Let $X \xrightarrow{\pi} Y$ be a covering space, $f : Z \to Y$ a continuous map and $F : Z \to X$ a lift of f. If*

$$h : [0,1] \times Z \to Y \quad (t, z) \mapsto h_t(z)$$

is a homotopy of f ($h_0(z) \equiv f(z)$) then there exists a unique lift of h

$$H : [0,1] \times Z \to X \quad (t, z) \mapsto H_t(z)$$

such that $H_0(z) \equiv F(z)$.

Proof For each $z \in Z$ we can find an open neighborhood U_z of $z \in Z$ and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ (depending on z) such that h maps $[t_{i-1}, t_i] \times U_z$ into an evenly covered neighborhood of $h_{t_{i-1}}(z)$. Following this partition can now succesively lift $h\,|_{I \times U_z}$ to a continuous map $H = H^z : I \times U_z \to X$ such that $H_0(\zeta) = F(\zeta)$, $\forall \zeta \in U_z$. By unique lifting property the liftings on $I \times U_{z_1}$ and $I \times U_{z_2}$ agree on $I \times (U_{z_1} \cap U_{z_2})$ for any $z_1, z_2 \in Z$ and hence we can glue all these local lifts together to obtain the desired lift H on $I \times Z$.

□

Corollary 6.2.11. (Path lifting property) *Let $X \xrightarrow{\pi} Y$ be a covering space and $\gamma : [0,1] \to Y$ a continuous path starting at $y_0 \in Y$. Then for every $x_0 \in \pi^{-1}(y_0)$* there exists *a unique lift $\Gamma : [0,1] \to X$ of γ starting at x_0.*

Proof Use the previous theorem with $f : \{pt\} \to Y$, $f(pt) = \gamma(0)$ and $h_t(pt) = \gamma(t)$.

□

Corollary 6.2.12. *Let $X \xrightarrow{\pi} Y$ be a covering space, $y_0 \in Y$ and $\gamma_0, \gamma_1 \in \Omega(Y, y_0)$ If γ_0 and γ_1 are homotopic rel y_0 then any lifts Γ_0, Γ_1 which start at the same point also end at the same point, i.e. $\Gamma_0(1) = \Gamma_1(1)$.*

Proof Lift the homotopy connecting γ_0 to γ_1 to a homotopy in X. By unique lifting property this lift connects Γ_0 to Γ_1. We thus get a continuous path $\Gamma_t(1)$ inside the fiber $\pi^{-1}(y_0)$ which connects $\Gamma_0(1)$ to $\Gamma_1(1)$. Since the fibers are discrete this path must be constant.

□

Let $X \stackrel{\pi}{\to} Y$ be a covering space and $y_0 \in Y$. Then for every $x \in \pi^{-1}(y_0)$ and any $\gamma(s) \in \Omega^(Y, y_0)$ denote by $\Gamma_x(s)$ the unique lift of γ starting at x. Set

$$x \cdot \gamma \stackrel{def}{=\!\!\to} \Gamma_x(1).$$

By Corollary 6.2.12 if $\omega \in \Omega(Y, y_0)$ is homotopic to γ rel y_0 then

$$x \cdot \gamma = x \cdot \omega.$$

Hence $x \cdot \gamma$ depends only upon the equivalence class $[\gamma] \in \pi_1(Y, y_0)$. Clearly

$$x \cdot ([\gamma] \cdot [\omega]) = (x \cdot [\gamma]) \cdot [\omega]$$

and

$$x \cdot \mathfrak{e}_{y_0} = x$$

so that the correspondence

$$\pi^{-1}(y_0) \times \pi_1(Y, y_0) \ni (x, \gamma) \mapsto x \cdot \gamma \in x \cdot \gamma \in \pi^{-1}(y_0)$$

defines a right action of $\pi_1(Y, y_0)$ on the fiber $\pi^{-1}(y_0)$. This action is called the *monodromy of the covering*. The map $x \mapsto x \cdot \gamma$ is called the monodromy along γ. Note that when Y is simply connected the monodromy is trivial. The map π induces a group morphism

$$\pi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \quad x_0 \in \pi^{-1}(y_0).$$

Proposition 6.2.13. π_* *is injective.*

Proof Indeed, let $\gamma \in \Omega(X, x_0)$ such that $\pi(\gamma)$ is trivial in $\pi_1(Y, y_0)$. The homotopy connecting $\pi(\gamma)$ to $\mathfrak{e}_{y_0}$ lifts to a homotopy connecting γ to the unique lift of $\mathfrak{e}_{y_0}$ at x_0 which is $\mathfrak{e}_{x_0}$.

□

6.2.4 On the existence of lifts

Theorem 6.2.14. *Let $X \stackrel{\pi}{\to} Y$ be a covering space, $x_0 \in X$, $y_0 = \pi(x_0) \in Y$, $f : Z \to Y$ a continuous map and $z_0 \in Z$ such that $f(z_0) = y_0$. Assume the spaces Y and Z are connected (and thus path connected). f admits a lift $F : Z \to X$ such that $F(z_0) = x_0$ if and only if*

$$f_*\left(\pi_1(Z, z_0)\right) \subset \pi_*\left(\pi_1(X, x_0)\right). \tag{6.2.1}$$

Proof **Necessity**. If F is such a lift then using the functoriality of the fundamental group construction we deduce $f_* = \pi_* \circ F_*$. This implies the inclusion (6.2.1).

Sufficiency For any $z \in Z$ choose a path γ_z from z_0 to z. Then $\alpha_z = f(\gamma_z)$ is a path from y_0 to $y = f(z)$. Denote by A_z the unique lift of α_z starting at x_0 and set $F(z) = A_z(1)$. We claim that F is a well defined map.

Indeed, let ω_z be another path in Z connecting z_0 to z. Set $\lambda_z = f(\omega_z)$ and denote by Λ_z its unique lift in X starting at x_0. We have to show that $\Lambda_z(1) = A_z(1)$. Construct the loop based at z_0

$$\beta_z = \omega_z * \gamma_z^- .$$

$f(\beta_z)$ is a loop in Y based at y_0. From (6.2.1) we deduce that the lift B_z of $f(\beta_z)$ at $x_0 \in X$ is a closed path (i.e. the monodromy along $f(\beta_z)$ is trivial). We now have

$$\Lambda_z(1) = B_z(1/2) = A_z(0) = A_z(1).$$

This proves F is a well defined map. We still have to show this map is also continuous.

Pick $z \in Z$. Since f is continuous, for every arbitrarily small, evenly covered neighborhood U of $f(z) \in Y$ there exists a path connected neighborhood V of $z \in Z$ such that $f(V) \subset U$. For any $\zeta \in V$ pick a path $\sigma = \sigma_\zeta$ in V connecting z to ζ. Let ω denote the path $\omega = \gamma_z * \sigma_\zeta$ (go from z_0 to z along γ_z and then from z to ζ along σ_ζ). Then $F(\zeta) = \Omega(1)$ where Ω is the unique lift of $f(\omega)$ starting at x_0. Since $(f(\zeta) \in U$ we deduce that $\Omega(1)$ belongs to the local sheet Σ, containing $F(z)$, which homeomorphically covers U. We have thus proved $z \in V \subset F^{-1}(\Sigma)$. The proof is complete since the local sheets Σ form a basis of neighborhoods of $F(z)$.

□

Definition 6.2.15. *Let Y be a connected space. A covering space $X \xrightarrow{\pi} Y$ is said to be* universal *if X is simply connected.*

Corollary 6.2.16. *Let $X_1 \xrightarrow{p_i} Y$ (i=0,1) be two covering spaces of Y. Fix $x_i \in X_i$ such that $p_0(x_0) = p(x_1) = y_0 \in Y$. If X_0 is universal there exists a* unique *covering morphism $F : X_0 \to X_1$ such that $F(x_0) = x_1$.*

Proof A bundle morphism $F : X_0 \to X_1$ can be viewed as a lift of the map $p_0 : X_0 \to Y$ to the total space of the covering space defined by p_1. The corollary follows immediately from Theorem 6.2.14 and the unique lifting property.

□

Corollary 6.2.17. *Every space admits at most one universal covering space (up to isomorphism).*

Theorem 6.2.18. *Let Y be a connected, locally path connected space such that each of its points admits a simply connected neighborhood. Then Y admits an (essentially unique) universal covering space.*

Sketch of proof Assume for simplicity that Y is a metric space. Fix $y_0 \in Y$. Let $\mathcal{P}_{y_0}$ denote the collection of continuous paths in Y starting at y_0. It can be topologized using the metric of uniform convergence in Y. Two paths in $\mathcal{P}_{y_0}$ are said to be homotopic rel endpoints if they can be deformed from one to another keeping the endpoints fixed. This defines an equivalence relation on $\mathcal{P}_{y_0}$. We denote the space of equivalence classes by $\tilde{Y}$ and we endow it with the quotient topology. Define $p : \tilde{Y} \to Y$ by

$$p([\gamma]) = \gamma(1) \quad \forall \gamma \in \mathcal{P}_{y_0}.$$

Then $(\tilde{Y}, p)$ is an universal covering space of Y.

□

Exercise 6.2.7. Finish the proof of the above theorem. □

Example 6.2.19. $\mathbb{R} \stackrel{\exp}{\to} S^1$ is the universal cover of S^1. More generally, $\mathrm{Exp} : \mathbb{R}^n \to T^n$

$$(t_1, \ldots, t_n) \mapsto (\exp(2\pi i t_1), \ldots, \exp(2\pi i t_n))$$

is the universal cover of T^n. The natural projection

$$p : S^n \to \mathbb{RP}^n$$

is the universal cover of $\mathbb{RP}^n$. □

Example 6.2.20. Let (M, g) be a complete Riemann manifold with non-positive sectional curvature. By Cartan-Hadamard theorem the exponential map $\exp_q : T_qM \to M$ is a covering map. Thus the universal cover of such a manifold is a linear space of the same dimension. In particular the universal covering space is *contractible*!!!

We now have another explanation why $\mathrm{Exp} : \mathbb{R}^n \to T^n$ is a universal covering space of the torus: the sectional curvature of the (flat) torus is zero. □

Exercise 6.2.8. Let (M, g) be a complete Riemann manifold and $p : \tilde{M} \to M$ its universal covering space.
(a) Prove that $\tilde{M}$ has a natural structure of smooth manifold such that p is a local diffeomorphism.
(b) Prove that the pullback p^*g defines a complete Riemann metric on $\tilde{M}$ locally isometric with g. □

Example 6.2.21. Let (M, g) be a complete Riemann manifold such that

$$\mathrm{Ric}\,(X, X) \geq const.|X|_g^2, \tag{6.2.2}$$

where *const* denotes a strictly positive constant. By Myers theorem M is compact. Using the previous exercise we deduce that the universal cover $\tilde{M}$ is a complete Riemann manifold locally isometric with (M, g). Hence the inequality (6.2.2) continues to hold on the covering $\tilde{M}$. Myers theorem implies again that the universal cover $\tilde{M}$ is *compact*!! In particular, the universal cover of a semisimple, compact Lie group is compact!!! □

6.2.5 The universal cover and the fundamental group

Theorem 6.2.22. *Let $\tilde{X} \stackrel{p}{\to} X$ be the universal cover of a space X. Then*

$$\pi_1(X, pt) \cong \mathrm{Deck}\,(\tilde{X} \to X).$$

Proof Fix $\xi_0 \in \tilde{X}$ and set $x_0 = p(\xi_0)$. There exists a bijection

$$\mathbf{Ev} : \mathrm{Deck}\,(\tilde{X}) \to p^{-1}(x_0)$$

given by the **ev**aluation

$$\mathbf{Ev}\,(F) = F(\xi_0).$$

For any $\xi \in \pi^{-1}(x_0)$ let γ_ξ be a path connecting ξ_0 to ξ. Any two such paths are homotopic rel endpoints since $\tilde{X}$ is simply connected (check this). Their projections on the base X determine identical elements in $\pi_1(X, x_0)$. We thus have a natural map

$$\Psi : \mathrm{Deck}\,(\tilde{X}) \to \pi_1(X, x_0) \quad F \mapsto p(\gamma_{F(\xi_0)}).$$

Ψ is clearly a group morphism. (Think monodromy!). The injectivity and the surjectivity of Ψ are consequences of the lifting properties of the universal cover.

□

Corollary 6.2.23. *If the space X has a compact universal cover then $\pi_1(X, pt)$ is finite.*

Proof Indeed the fibers of the universal cover have to be both discrete and compact. Hence they must be finite. The map **Ev** in the above proof maps the fibers bijectively onto Deck $(\tilde{X})$. □

Corollary 6.2.24. (H. Weyl) *The fundamental group of a compact semisimple group is finite.*

Indeed, we deduce from Example 6.2.21 that the universal cover of such a group is compact.

Example 6.2.25. From Example 6.2.5 we deduce that $\pi_1(S^1) \cong (\mathbb{Z}, +)$. □

Exercise 6.2.9. (a)Prove that $\pi_1(\mathbb{RP}^n, pt) \cong \mathbb{Z}_2$, $\forall n \geq 2$.
(b) Prove that $\pi_1(T^n) \cong \mathbb{Z}^n$. □

Exercise 6.2.10. Show that the natural inclusion $U(n-1) \hookrightarrow U(n)$ induces an isomorphism between the fundamental groups. Conclude that the map

$$\det : U(n) \to S^1$$

induces an isomorphism

$$\pi_1(U(n)) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

□

Chapter 7

Cohomology

7.1 DeRham cohomology

7.1.1 Speculations around the Poincaré lemma

To start off, consider the following partial differential equation in the plane: given two smooth functions P and Q find a smooth function u such that

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q. \tag{7.1.1}$$

As is, the formulation is still fuzzy since we have not specified the domains of the functions u, P and Q. As it will turn out, this aspect has an incredible relevance in geometry.

Equation (7.1.1) has another interesting feature: it is overdetermined ,i.e. it imposes too many conditions on too few unknowns. It is therefore quite natural to impose some additional restrictions on the data P, Q just like the zero determinant condition when solving overdetermined linear systems.

To see what restrictions one should add it is convenient to introduce the 1-form $\alpha = Pdx + Qdy$. (7.1.1) can be rewritten as

$$du = \alpha. \tag{7.1.2}$$

If (7.1.2) has at least one solution u then

$$0 = d^2u = d\alpha$$

so that a necessary condition for existence is

$$d\alpha = 0, \tag{7.1.3}$$

i.e.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

A form satisfying (7.1.3) is said to be *closed*. Thus, if the equation $du = \alpha$ has a solution then α is necessarily closed. Is the converse also true?

Let us introduce a bit more terminology. A form α such that the equation (7.1.2) has a solution is said to be *exact*. The motivation for this terminology comes from the fact that sometimes du is called the exact differential of u. We thus have an inclusion of vector spaces

$$\{\text{exact forms}\} \subset \{\text{closed forms}\}.$$

Is it true the opposite inclusion also holds?

Amazingly, the answer to this question *depends on the domain* on which we study (7.1.2). The Poincaré lemma comes to raise our hopes. It says that this is always true at least *locally*.

Lemma 7.1.1. (Poincaré lemma) *Let C be an open convex set in $\mathbb{R}^n$ and $\alpha \in \Omega^k(C)$. Then the equation*

$$du = \alpha \tag{7.1.4}$$

has a solution $u \in \Omega^{k-1}(C)$ if and only if *α is closed,* $d\alpha = 0$.

Proof The necessity is clear. We prove the sufficiency. We may as well assume $0 \in C$. Consider the radial vector field on C

$$\vec{r} = x^i \frac{\partial}{\partial x_i}$$

and denote by Φ_t the flow it generates. Φ_t is the linear flow

$$\Phi_t(x) = e^t x, \quad x \in \mathbb{R}^n.$$

Note that since C is *convex* the flow lines of Φ_t, which are the straight lines through the origin, intersect C along *connected* segments originating at 0.

We begin with an a priori study of (7.1.4). Let u satisfy $du = \alpha$. Using the homotopy formula

$$L_{\vec{r}} = di_{\vec{r}} + i_{\vec{r}}d$$

we get

$$dL_{\vec{r}}u = d(di_{\vec{r}} + i_{\vec{r}}d)u = di_{\vec{r}}\alpha,$$

i.e.

$$d(L_{\vec{r}}u - i_{\vec{r}}\alpha) = 0.$$

This suggests looking for solutions of

$$L_{\vec{r}}u = i_{\vec{r}}\alpha. \tag{7.1.5}$$

If u is a solution of this equation then

$$L_{\vec{r}}du = dL_{\vec{r}}u = di_{\vec{r}}\alpha = L_{\vec{r}}\alpha - i_{\vec{R}}d\alpha = L_{\vec{r}}\alpha.$$

Hence u also satisfies

$$L_{\vec{r}}(du - \alpha) = 0.$$

Set $\omega = du - \alpha = \sum_I \omega_I dx^I$. Using the computations in Subsection 3.1.3 we deduce

$$L_{\vec{r}}dx^i = dx^i$$

so that

$$L_{\vec{r}}\omega = \sum_I (L_{\vec{r}}\omega_I) dx^I = 0.$$

We deduce that $L_{\vec{r}}\omega_I = 0$ and consequently the coefficients ω_I are constants along the the flow lines of Φ_t which all converge at 0. Thus

$$\omega_I = c_I = constant.$$

Each monomial $c_I dx^I$ is exact i.e. there exits $\eta_I \in \Omega^{k-1}(C)$ such that $d\eta_I = c_I dx^I$. For example, when $I = 1 < 2 < \cdots < k$

$$dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k = d(x^1 dx^2 \wedge \cdots \wedge dx^k).$$

Thus, the equality $L_{\vec{r}}\omega = 0$ implies ω is exact. Hence there exists $\eta \in \Omega^{k-1}(C)$ such that

$$d(u - \eta) = \alpha$$

i.e. $u - \eta$ solves (7.1.4). Conclusion: any solution of (7.1.5) produces a solution of (7.1.4).

We now proceed to solve (7.1.5). At this point the flow Φ_t enters crucially. Define

$$u = \int_{-\infty}^{0} \Phi_t^*(i_{\vec{r}}\alpha) dt. \tag{7.1.6}$$

Here the convexity assumption on C enters essentially since it implies that

$$\Phi_t(C) \subset C \quad \forall t \le 0$$

so that if the above integral is convergent then u is a form on C. If we write

$$\Phi_t^*(i_{\vec{r}}\alpha) = \sum_{|I|=k-1} \eta_I^t(x) dx^I$$

then

$$u(x) = \sum_I \left(\int_{-\infty}^{0} \eta_I^t(x) dt \right) dx^I. \tag{7.1.7}$$

We have to check two things.

A. The integral in (7.1.7) is well defined. To see this we first write

$$\alpha = \sum_{|J|=k} \alpha_J dx^J$$

and set

$$A(x) = \max_{J; 0 \le \tau \le 1} |\alpha_J(\tau x)|.$$

Then

$$\Phi_t^*(i_{\vec{r}}\alpha) = e^t i_{\vec{r}} \left(\sum_J \alpha_J(e^t x) dx^J \right)$$

so that

$$|\eta_I^t(x)| \leq Ce^t|x|A(x) \quad \forall t \leq 0.$$

This proves the integral in (7.1.7) converges.

B. u defined by (7.1.6) is a solution of (7.1.5). Indeed, on differential forms

$$L_{\vec{r}}u = \lim_{s\to 0}\frac{1}{s}\left(\Phi_s^*u - u\right) = \lim_{s\to 0}\frac{1}{s}\left(\Phi_s^*\int_{-\infty}^{0}\Phi_t^*(i_{\vec{r}}\alpha)dt - \int_{-\infty}^{0}\Phi_t^*(i_{\vec{r}}\alpha)dt\right)$$

$$= \lim_{s\to 0}\left(\int_{-\infty}^{s}\Phi_t^*(i_{\vec{r}}\alpha)dt - \int_{-\infty}^{0}\Phi_t^*(i_{\vec{r}}\alpha)dt\right) = \lim_{s\to 0}\int_{0}^{s}\Phi_t^*(i_{\vec{r}}\alpha)dt = \Phi_0^*(i_{\vec{r}}\alpha) = i_{\vec{r}}\alpha.$$

The Poincaré lemma is proved.

□

The local solvability does not in any way implies global solvability. Something happens when one tries to go from local to global.

Example 7.1.2. Consider the form $d\theta$ on $\mathbb{R}^2\setminus\{0\}$ where (r,θ) denote the polar coordinates in the punctured plane. To write it in cartesian coordinates (x,y) we use the equality

$$\tan\theta = \frac{y}{x}$$

so that

$$(1+\tan^2\theta)d\theta = -\frac{y}{x^2}dx + \frac{dy}{x}$$

$$(1+\frac{y^2}{x^2})d\theta = \frac{-ydx+xdy}{x^2},$$

i.e.

$$d\theta = \frac{-ydx+xdy}{x^2+y^2} = \alpha.$$

Obviously, $d\alpha = d^2\theta = 0$ on $\mathbb{R}^2\setminus\{0\}$ so that α is closed on the punctured plane. Can we find a smooth function u on $\mathbb{R}^2\setminus\{0\}$ such that $du=\alpha$?

We know that we can always do this locally. However, we cannot achieve this globally. Indeed, if this was possible then

$$\int_{S^1} du = \int_{S^1}\alpha = \int_{S^1} d\theta = 2\pi.$$

On the other hand, using polar coordinates $u = u(r,\theta)$ we get

$$\int_{S^1} du = \int_{S^1}\frac{\partial u}{\partial\theta}d\theta$$

$$= \int_0^{2\pi}\frac{\partial u}{\partial\theta}d\theta = u(1,2\pi) - u(1,0) = 0.$$

Hence on $\mathbb{R}^2\setminus\{0\}$

$$\{\text{exact forms}\} \neq \{\text{closed forms}\}.$$

We see what a dramatic difference a point can make: $\mathbb{R}^2\setminus\{\text{point}\}$ is structurally very different from $\mathbb{R}^2$. □

The artifice in the previous example simply increases the mystery. It is still not clear what makes it impossible to patch-up local solutions. The next subsection describes two ways to deal with this issue.

7.1.2 Čech vs. DeRham

Let us try to analyze what prevents the "spreading" of local solvability of (7.1.4) to global solvability. We will stay in the low degree range.

The Čech approach Consider ω a closed 1-form on a smooth manifold. To solve the equation $du = \omega$, $u \in C^\infty(M)$ we first cover M by geodesically convex open sets (U_α) (with respect to some fixed Riemann metric). By Poincaré lemma we can solve $du = \omega$ on each open set U_α so that we can find a smooth function $f_\alpha \in C^\infty(U_\alpha)$ such that $df_\alpha = \omega$. We get a global solution if and only if

$$f_{\alpha\beta} = f_\alpha - f_\beta = 0 \quad \text{on each } U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset.$$

For fixed α the solutions of the equation $du = \omega$ on U_α differ only by additive constants (i.e. closed 0-forms).

The quantities $f_{\alpha\beta}$ satisfy $df_{\alpha\beta} = 0$ on the (*connected*) overlaps $U_{\alpha\beta}$ so they are constants. Clearly they satisfy the conditions

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \quad \text{on } U_{\alpha\beta\gamma} \neq \emptyset. \tag{7.1.8}$$

On each U_α we have as we have seen several choices of solution. Altering a choice is tantamount to adding a constant $f_\alpha \to f_\alpha + c_\alpha$. The quantities $f_{\alpha\beta}$ change according to

$$f_{\alpha\beta} \to f_{\alpha\beta} + c_\alpha - c_\beta.$$

Thus the global solvability issue leads to the following situation. Pick a collection of local solutions f_α. The equation $du = \omega$ is globally solvable if we can alter each f_α by a constant c_α such that

$$f_{\alpha\beta} = (c_\beta - c_\alpha) \quad \forall \alpha, \beta \text{ such that } U_{\alpha\beta} \neq \emptyset. \tag{7.1.9}$$

We can start the alteration at some open set U_α and work our way up from one such open set to its neighbors, always trying to implement (7.1.9). It may happen that in the process we might have to return to an open set whose solution was already altered. Now we are in trouble. (Try this on S^1 and $\omega = d\theta$.) After several attempts one can point the finger to the culprit: the global topology of the manifold may force us to always return to some already altered local solution.

Notice that we replaced the partial differential equation $du = \omega$ with a system of linear equations (7.1.9), where the constants $f_{\alpha\beta}$ are subject to the constraints (7.1.8). This is no computational progress since the combinatorics of this system makes it impossible solve in most cases.

The above considerations extend to higher degree and one can imagine the complexity increases considerably. This is however the approach Čech adopted in order to study the topology of manifolds and although it may seem computationally hopeless, its theoretical insights are invaluable.

The DeRham Approach This time we postpone asking why the global solvability is not always possible. Instead, for each smooth manifold M one considers the $\mathbb{Z}$-graded vector spaces

$$B^*(M) = \oplus_{k\geq 0} B^k(M),\ (B^0(M) \stackrel{def}{=} \{0\})\quad Z^*(M) = \oplus_{k\geq 0} Z^k(M)$$

where

$$B^k(M) = \{d\omega \in \Omega^k(M)\ ;\ \omega \in \Omega^{k-1}(M)\} = \text{exact } k-\text{forms}$$

and

$$Z^k(M = \{\eta \in \Omega^k(M)\ ;\ d\eta = 0\} = \text{closed } k-\text{forms}.$$

Clearly $B^k \subset Z^k$. Form the quotients

$$H^k(M) = Z^k(M)/B^k(M).$$

Intuitively, this space consists of those closed k-forms ω for which the equation

$$du = \omega$$

has no global solution $u \in \Omega^{k-1}(M)$. Thus if we can somehow describe these spaces we may get an idea " who" is responsible for the global nonsolvability.

Definition 7.1.3. *For any smooth manifold M the vector space $H^k(M)$ is called the k-th DeRham cohomology group.*

Clearly $H^k(M) = 0$ for $k > \dim M$.

Example 7.1.4. The Poincaré lemma shows that $H^k(\mathbb{R}^n) = 0$ for $k > 0$. The discussion in Example 7.1.2 shows that $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$. □

Proposition 7.1.5. *For any smooth manifold M*

$$\dim H^0(M) = \text{number of connected components of } M.$$

Proof Indeed

$$H^0(M) = Z^0(M) = \{f \in C^\infty(M)\ ;\ df = 0\}$$

Thus $H^0(M)$ coincides with the linear space of locally constant functions. These are constant on the connected components of M.

□

Already $H^0(M)$ contains an important topological information. Obviously the groups H^k are diffeomorphism invariants and its is reasonably to suspect the higher cohomology groups may contain more topological information.

Thus, to any manifold we can now associate the graded vector space

$$H^*(M) \stackrel{def}{=} \bigoplus_{k\geq 0} H^k(M).$$

A priori, the spaces H^k may be infinite dimensional. The Poincaré polynomial, denoted by $P_M(t)$ is defined by

$$P_M(t) = \sum_{k\geq 0} t^k \dim H^k(M)$$

every time the right-hand-side expression makes sense. The number $\dim H^k(M)$ is usually denoted by $b_k(M)$ and is called the *k-th Betti number* of M. Hence

$$P_M(t) = \sum_k b_k(M)t^k.$$

The alternating sum

$$\chi(M) := \sum_k (-1)^k b_k(M)$$

is called the *Euler characteristic* of M.

Exercise 7.1.1. Show that $P_{S^1}(t) = 1 + t$. □

We will spend the remaining of this chapter trying to understand what is that these groups do and which (if any) is the connection between the two approaches outlined above.

7.1.3 Very little homological algebra

At this point it is important to isolate the common algebraic skeleton on which both DeRham and Čech approaches are built. This requires a little terminology from homological algebra. In the sequel all rings will be assumed commutative with 1.

Definition 7.1.6. *(a) Let R be a ring, $C = \oplus_{n\in\mathbb{Z}} C_n$ and $D = \oplus_{n\in\mathbb{Z}} D_n$ two $\mathbb{Z}$-graded left R-modules. A degree k-morphism $\phi : C \to D$ is a R-module morphism such that*

$$\phi(C_n) \subset D_{n+k} \quad \forall n \in \mathbb{Z}.$$

(b) Let $C = \oplus_{n\in\mathbb{Z}}$ be a $\mathbb{Z}$-graded R-module. A boundary *(resp.* coboundary*) operator is a degree -1 (resp. a degree 1) endomorphism $d : C \to C$ such that $d^2 = 0$.*

A chain *(resp.* cochain*) complex over R is a pair (C, d) where C is a $\mathbb{Z}$-graded R-module and d is a boundary operator (resp. a coboundary operator).*

In this book we will be interested mainly in cochain complexes so in the remaining part of this subsection we will stick to this situation. In this case cochain complexes are usually described as $(C = \oplus_{n\in\mathbb{Z}} C^n, d)$. Moreover we will consider only the case $C^n = 0$ for $n < 0$.

Traditionally, a cochain complex is represented as a long sequence of R-modules and morphisms of R-modules

$$(C, d) : \quad \cdots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \to \cdots$$

such that $\operatorname{range}(d_{n-1}) \subset \ker(d_n)$, i.e. $d_n d_{n-1} = 0$.

Definition 7.1.7. *Let*

$$\cdots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \to \cdots$$

be a cochain complex. Set

$$Z^n(C) = \ker d_n \quad B^n(C) = \operatorname{range}(d_{n-1}).$$

The elements of $Z^n(C)$ are called cocycles*, while the elements of $B^n(C)$ are called coboundaries. Two cocycles $c, c' \in Z^n(C)$ are said to be* cohomologous *if $c - c' \in B^n(C)$. The quotient module*

$$H^n(C) \stackrel{def}{=} Z^n(C)/B^n(C)$$

is called the n-th cohomology group *(module) of C. It can be identified with the set of equivalence classes of cohomologous cocycles. A cochain complex complex C is said to be* acyclic *if $H^n(C) = 0$ for all $n > 0$.* □

For a cochain complex C one usually writes

$$H^*(C) = \bigoplus_{n \geq 0} H^n(C).$$

Example 7.1.8. (The DeRham complex) Let M be an m-dimensional smooth manifold. Then the sequence

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \to 0$$

(where d is the exterior derivative) is a cochain complex called the DeRham complex. Its cohomology is the DeRham cohomology of the manifold. □

Example 7.1.9. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a real Lie algebra. Define

$$d : \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^*$$

by

$$(d\omega)(X_0, X_1, \ldots, X_k) = \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),$$

where as usual, the hat indicates a missing argument. According to the computations in Example 3.2.6 d is a coboundary operator so that $(\Lambda^* \mathfrak{g}^*, d)$ is a cochain complex. Its cohomology is called the *Lie algebra cohomology* and is denoted by $H^*(\mathfrak{g})$. □

Exercise 7.1.2. (a) Let $\mathfrak{g}$ be a real Lie algebra. Show that

$$H^1(\mathfrak{g}) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$$

where $[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{[X, Y]\ ;\ X, Y \in \mathfrak{g}\}$.
(b) Compute $H^1(\underline{gl}(n, \mathbb{R}))$ where $\underline{gl}(n, \mathbb{R})$ denotes the Lie algebra of $n \times n$ real matrices with the bracket given by the commutator.
(c) (Whitehead) Let $\mathfrak{g}$ be a semisimple Lie algebra, i.e. its Killing pairing is nondegenerate. Prove that $H^1(\mathfrak{g}) = \{0\}$. (**Hint:** Prove that $[\mathfrak{g}, \mathfrak{g}]^\perp = 0$ where $\perp$ denotes the orthogonal complement with respect to the Killing pairing.) □

Proposition 7.1.10. *Let*

$$(C,d): \ \cdots \to C^{n-1} \stackrel{d_{n-1}}{\longrightarrow} C^n \stackrel{d_n}{\longrightarrow} C^{n+1} \to \cdots$$

be a cochain complex of R-modules. Assume moreover that C is also a $\mathbb{Z}$-graded R-algebra, i.e. there exists an associative multiplication such that

$$C^n \cdot C^m \subset C^{m+n} \quad \forall m, n.$$

If d is a quasi-derivation, i.e.

$$d(x \cdot y) = \pm(dx) \cdot y \pm x \cdot (dy) \quad \forall x, y \in C$$

then $H^*(C)$ inherits a structure of $\mathbb{Z}$-graded R-algebra.

A cochain complex as in the above proposition is called a *differential graded algebra* or *DGA*.

Proof It suffices to show $Z^*(C) \cdot Z^*(C) \subset Z^*(C)$ and $B^*(C) \cdot B^*(C) \subset B^*(C)$.

If $dx = dy = 0$ then $d(xy) = \pm(dx)y \pm x(dy) = 0$. If $x = dx'$ and $y = dy'$ then since $d^2 = 0$ we deduce $xy = \pm(dx'dy')$.

□

Corollary 7.1.11. *The DeRham cohomology of a smooth manifold has an $\mathbb{R}$-algebra structure induced by the exterior multiplication of differential forms.*

Definition 7.1.12. *Let (A, d) and (B, δ) be two cochain complexes.*
(a) A cochain map *is a degree 0 morphism $\phi : A \to B$ such that $\phi \circ d = \delta \circ \phi$ i.e. the diagram below is commutative for any n.*

$$\begin{array}{ccc} A^n & \xrightarrow{d_n} & A^{n+1} \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle \phi_{n+1}} \\ B^n & \xrightarrow{\delta_n} & B^{n+1} \end{array}$$

(b) Two cochain maps $\phi, \psi : A \to B$ are said to be cochain homotopic *and we write this $\phi \simeq_h \psi$ if there exists a degree -1 morphism $\chi : \{A^n \to B^{n-1}\}$ such that*

$$\phi(a) - \psi(a) = \pm\delta \circ \chi(a) \pm \chi \circ d(a).$$

(c) Two cochain complexes (A, d) and (B, δ) are said to be homotopic *if there exist cochain maps*

$$\phi : A \to B \ \text{ and } \ \psi : B \to A$$

such that $\psi \circ \phi \simeq_h \mathbb{1}_A$ and $\phi \circ \psi \simeq_h \mathbb{1}_B$.

Example 7.1.13. The commutation rules in Subsection 3.2.1, namely $[L_X, d] = 0$ and $[i_X, d]_s = L_X$ show that for each vector field X on a smooth manifold M the Lie derivative along X, $L_X : \Omega^*(M) \to \Omega^*(M)$ is a cochain map homotopic with the trivial map ($\equiv 0$). The interior derivative i_X is the cochain homotopy achieving this. □

Proposition 7.1.14. *(a) Any cochain map $\phi : (A, d) \to (B, \delta)$ induces a degree zero morphism in cohomology*

$$\phi_* : H^*(A) \to H^*(B).$$

(b) If the cochain maps $\phi, \psi : A \to B$ are cochain homotopic then they induce identical *morphisms in cohomology, $\phi_* = \psi_*$.*
(c) $(\mathbb{1}_A)_* = \mathbb{1}_{H^*(A)}$ and if

$$(A, d) \xrightarrow{\phi} (A', d') \xrightarrow{\psi} (A'', d'')$$

are cochain maps then $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Proof (a) It boils down to checking the inclusions

$$\phi(Z^n(A)) \subset Z^n(B) \text{ and } \phi(B^n(A)) \subset B^n(B).$$

These follow immediately from the definition of a cochain map.
(b) We have to show that

$$\phi(\text{cocycle}) - \psi(\text{cocycle}) = \text{coboundary}.$$

Let $da = 0$. Then

$$\phi(a) - \psi(a) = \pm\delta(\chi(a) \pm \chi(da) = \delta(\pm\chi(a)) = \text{coboundary in } B.$$

(c) Obvious.

□

Corollary 7.1.15. *If two cochain complexes (A, d) and $(B,\ \delta)$ are cochain homotopic then their cohomology modules are isomorphic.*

Proposition 7.1.16. *Let*

$$0 \to (A, d^A) \xrightarrow{\phi} (B, d^B) \xrightarrow{\psi} (C, d^C) \to 0$$

be a short exact sequence of cochain complexes. This means we have a commutative diagram

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^{n+1} \arrow[u, "d^A"] \arrow[r, "\phi_{n+1}"] & B^{n+1} \arrow[u, "d^B"] \arrow[r, "\psi_{n+1}"] & C^{n+1} \arrow[u, "d^C"] \arrow[r] & 0 \\
0 \arrow[r] & A^{n} \arrow[u, "d^A"] \arrow[r, "\phi_{n}"] & B^{n} \arrow[u, "d^B"] \arrow[r, "\psi_{n}"] & C^{n} \arrow[u, "d^C"] \arrow[r] & 0 \\
0 \arrow[r] & A^{n-1} \arrow[u, "d^A"] \arrow[r, "\psi_{n-1}"] & B^{n-1} \arrow[u, "d^B"] \arrow[r, "\psi_{n-1}"] & C^{n-1} \arrow[u, "d^C"] \arrow[r] & 0 \\
 & \vdots \arrow[u, "d^A"] & \vdots \arrow[u, "d^B"] & \vdots \arrow[u, "d^C"] &
\end{tikzcd}
\tag{7.1.10}
$$

in which the rows are exact. Then there exists a long exact sequence

$$\cdots \to H^{n-1}(C) \stackrel{\partial_{n-1}}{\to} H^n(A) \stackrel{\phi_*}{\to} H^n(B) \stackrel{\psi_*}{\to} H^n(C) \stackrel{\partial_n}{\to} H^{n+1}(A) \to \cdots . \tag{7.1.11}$$

We will not include a proof of this proposition. We believe this is one proof in homological algebra the reader should try it on his/her own. We will just indicate the construction of the connecting maps ∂_n.

This construction and in fact the entire proof relies on a simple technique called diagram chasing. Start with $x \in H^n(C)$. x can be represented by some cocycle $c \in Z^n(C)$. Since ψ_n is surjective there exists $b \in B^n$ such that $c = \psi_n(b)$. From the commutativity of the diagram (7.1.10) we deduce $0 = d^C\psi_n(b) = \psi_{n+1}d^B(b)$ i.e. $d^B(b) \in \ker\psi_{n+1} = \operatorname{range}\phi_{n+1}$. In other words, there exists $a \in A^{n+1}$ such that $\phi_{n+1}(a) = d_n^B b$. We claim a is a cocycle. Indeed

$$\phi_{n+2}d^A_{n+1}a = d^B_{n+1}\phi_{n+1}a = d^B_{n+1}d^B_n b = 0.$$

Since ϕ_{n+2} is injective we deduce $d^A_{n+1}a = 0$ i.e. a is a cocycle.

If we trace back the path which lead us from $c \in Z^n(C)$ to $a \in Z^{n+1}(A)$ we can write

$$a = \phi^{-1}_{n+1} \circ d^B \circ \psi^{-1}_n(c) = \phi^{-1}_{n+1} \circ d^B b.$$

This is not entirely rigorous since a depends on various choices. We let the reader check that the correspondence $Z^n(C) \ni c \mapsto a \in Z^{n+1}(A)$ above induces a well defined map in cohomology, $\partial_n : H^n(C) \to H^{n+1}(A)$ and moreover, the sequence (7.1.11) is exact.

Exercise 7.1.3. [1] Suppose R is a commutative ring with 1. For any cochain complex $(K^\bullet, d_K)$ of R-modules, and any integer n we denote by $K[n]^\bullet$ the complex defined by $K[n]^m = K^{n+m}$, $d_{K[n]} = (-1)^n d_K$. We associate to any cochain map $f : K^\bullet \to L^\bullet$ two new cochain complexes:

(a) *The cone* $\Big(C(f)^\bullet, d_{C(f)}\Big)$ where

$$C(f)^\bullet = K[1]^\bullet \oplus L^\bullet, \quad d_{C(f)} \begin{bmatrix} k^{i+1} \\ \ell^i \end{bmatrix} = \begin{bmatrix} -d_K & 0 \\ f & d_L \end{bmatrix} \cdot \begin{bmatrix} k^{i+1} \\ \ell^i \end{bmatrix}.$$

(b) *The cylinder* $\Big(Cyl(f), d_{Cyl(f)}\Big)$

$$Cyl(f)^\bullet \cong K^\bullet \oplus C(f)^\bullet, \quad d_{Cyl(f)} \begin{bmatrix} k^i \\ k^{i+1} \\ \ell^i \end{bmatrix} = \begin{bmatrix} d_K & -\mathbb{1}_{K^{i+1}} & 0 \\ 0 & -d_K & 0 \\ 0 & f & d_L \end{bmatrix} \cdot \begin{bmatrix} k^i \\ k^{i+1} \\ \ell^i \end{bmatrix}.$$

We have canonical inclusions $\alpha : L^\bullet \to Cyl(f)$, $\bar{f} : K^\bullet \to Cyl(f)^\bullet$, a canonical projections $\beta : Cyl(f)^\bullet \to L^\bullet$, $\delta = \delta(f) : C(f)^\bullet \to K[1]^\bullet$, and $\pi : Cyl(f) \to C(f)$.

(i) Prove that $\alpha, \beta\bar{f}, \delta(f)$ are cochain maps, $\beta \circ \alpha = \mathbb{1}_L$ and $\alpha \circ \beta$ is cochain homotopic to $\mathbb{1}_{Cyl(f)}$.
(ii) Show that we have the following commutative diagram of cochain complexes, where the rows are exact.

$$\begin{tikzcd}
 & 0 \arrow[r] & L^\bullet \arrow[r, "\bar{\pi}"] \arrow[d, "\alpha"] & C(f)^\bullet \arrow[r, "\delta(f)"] \arrow[d, "\mathbb{1}_{C(f)}"] & K[1]^\bullet \arrow[r] & 0 \\
0 \arrow[r] & K^\bullet \arrow[r, "\bar{f}"] \arrow[d, "\mathbb{1}_K"] & Cyl(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\
 & K^\bullet \arrow[r, "f"] & L^\bullet & & &
\end{tikzcd}$$

(iii) Show that the connecting morphism in the long exact sequence corresponding to the shor exact sequence

$$0 \to K^\bullet \xrightarrow{\bar{f}} Cyl(f) \xrightarrow{\pi} C(f) \to 0$$

coincides with the morphism induced in cohomology by $\delta(f) : C(f) \to K[1]^\bullet$.
(iv) Prove that f induces an isomorphism in cohomology if and only if the cone of f is acyclic. □

[1] This exercise describes additional features of the long exact sequence in cohomology. They are particularly useful in the study of derived categories.

Exercise 7.1.4. (Abstract Morse inequalities) Let $C = \bigoplus_{n\geq 0} C^n$ be a cochain complex over the field $\mathbf{F}$. Assume each of the vector spaces C^n is finite dimensional. Form the Poincaré series

$$P_C(t) = \sum_{n\geq 0} t^n \dim_{\mathbf{F}} C^n$$

and

$$P_{H^*(C)}(t) = \sum_{n\geq 0} t^n \dim_{\mathbf{F}} H^n(C).$$

Prove that there exists a formal series $R(t) \in \mathbb{Z}[[t]]$ with *non-negative* coefficients such that

$$P_C(t) = P_{H^*(C)}(t) + (1+t)R(t).$$

In particular, whenever it makes sense, the graded spaces C^* and H^* have identical Euler characteristics

$$\chi(C^*) = P_C(-1) = P_{H^*(C)}(-1) = \chi(H^*(C)).$$

□

Exercise 7.1.5. (Additivity of Euler characteristic) Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of cochain complexes over the field $\mathbf{F}$. Prove that if at least two of the cohomology modules $H^*(A)$, $H^*(B)$ and $H^*(C)$ have finite dimension over $\mathbf{F}$ then the same is true about the third one and moreover

$$\chi(H^*(B)) = \chi(H^*(A)) + \chi(H^*(C)).$$

□

Exercise 7.1.6. (Finite dimensional Hodge theory) Let $(\oplus_{n\geq 0} V^n, d_n)$ be a cochain complex over the reals such that $\dim V^n < \infty$ for all n. Assume each V^n is an Euclidean space and denote by $d_n^* : V^{n+1} \to V^n$ the adjoint of d_n. We can now form the Laplacians

$$\Delta_n : V^n \to V^n \quad \Delta_n = d_n^* d_n + d_{n-1} d_{n-1}^*.$$

(a) Prove that $\Delta_n c = 0$ iff $d_n c = 0$ and $d_{n-1}^* c = 0$.
(b) Let $c \in Z^n(C)$. Prove that there exists a unique $\bar{c} \in Z^n(C)$ cohomologous to c such that

$$|\bar{c}| = \min\{|c'| \; ; \; c - c' \in B^n(C)\}$$

where $|\cdot|$ denotes the Euclidean norm in V^n.
(c) Prove that $\bar{c}$ determined in part (b) satisfies $\Delta_n \bar{c} = 0$. Deduce from all the above the Hodge theorem

$$H^n(V) \cong \ker \Delta_n.$$

□

Exercise 7.1.7. Let V be a real vector space and $v_0 \in V$. Define $d_k : \Lambda^k V \to \Lambda^{k+1} V$ by $\omega \mapsto v_0 \wedge \omega$.
(a) Prove that

$$\cdots \stackrel{d_{k-1}}{\to} \Lambda^k V \stackrel{d_k}{\to} \Lambda^{k+1} V \stackrel{d_{k+1}}{\to} \cdots$$

is a cochain complex (known as Koszul complex).
(b) Use the finite dimensional Hodge theory described in previous exercise to prove that the Koszul complex is acyclic if $v_0 \neq 0$ i.e.

$$H^k(\Lambda^* V, d) = 0 \quad \forall k \geq 0. \qquad \square$$

7.1.4 Functorial properties of DeRham cohomology

Let M and N be two smooth manifolds and $\phi : M \to N$ a smooth map. The pullback

$$\phi^* : \Omega^*(N) \to \Omega^*(M)$$

is a cochain map, i.e.

$$\phi^* d^N = d^M \phi^*.$$

Thus ϕ^* induces a morphism in cohomology which we continue to denote by ϕ^*;

$$\phi^* : H^*(N) \to H^*(M).$$

In fact we have a more precise statement.

Proposition 7.1.17. *The DeRham cohomology construction is a contravariant functor from the category of smooth manifolds and smooth maps to the category of $\mathbb{Z}$-graded vector spaces with degree zero morphisms.*

Note that the pull-back is an algebra morphism $\phi^* : \Omega^*(N) \to \Omega^*(M)$ and the exterior differentiation is a quasi-derivation so that the map it induces in cohomology will also be a ring morphism.

Definition 7.1.18. *(a) Two smooth maps $\phi_0, \phi_1 : M \to N$ are said to be (smoothly) homotopic (and we write this $\phi_0 \simeq_{sh} \phi_1$) if there exists a smooth map*

$$\Phi : I \times M \to N \quad (t, m) \mapsto \Phi_t(m)$$

such that $\Phi_i = \phi_i$ for $i = 0, 1$.
(b) A smooth map $\phi : M \to N$ is said to be a (smooth) homotopy equivalence if there exists a smooth map $\psi : N \to M$ such that $\phi \circ \psi \simeq_{sh} \mathbf{1}_N$ and $\psi \circ \phi \simeq_{sh} \mathbf{1}_M$.
(c) Two smooth manifolds M and N are said to be homotopically equivalent if there exists a homotopy equivalence $\phi : M \to N$.

Proposition 7.1.19. *Let $\phi_0, \phi_1 : M \to N$ be two homotopic smooth maps. Then they induce* identical *maps in cohomology*

$$\phi_0^* = \Phi_1^* : H^*(N) \to H^*(M).$$

Proof According to the general results in homological algebra it suffices to show the pullbacks

$$\phi_0^*, \phi_1^* : \Omega^*(N) \to \Omega^*(M)$$

are cochain homotopic. Thus, we have to produce a map

$$\chi : \Omega^*(N) \to \Omega^{*-1}(M)$$

such that

$$\phi_1^*(\omega) - \phi_0^*(\omega) = \pm\chi(d\omega) \pm d\chi\omega.$$

At this point, our discussion on the fibered calculus of Subsection 3.4.5 will pay off.

The projection $\Phi : I \times M \to M$ defines an oriented ∂-bundle (with standard fiber I). For any $\omega \in \Omega^*(N)$ we have the equality

$$\begin{aligned}\phi_1^*(\omega) - \phi_0^*(\omega) &= \Phi^*(\omega)\,|_{1\times M} - \Phi^*(\omega)\,|_{0\times M} \\ &= \int_{(\partial I\times M)/M} \Phi^*(\omega).\end{aligned}$$

We now use the homotopy formula of Theorem 3.4.40 of Subsection 3.4.5. We get

$$\begin{aligned}\int_{(\partial I\times M)/M} \Phi^*(\omega) &= \int_{(I\times M)/M} d_{I\times M}\Phi^*(\omega) - d_M \int_{(I\times M)/M} \Phi^*(\omega) \\ &= \int_{(I\times M)/M} \Phi^*(d_N\omega) - d_M \int_{(I\times M)/M} \Phi^*(\omega).\end{aligned}$$

Thus

$$\chi(\omega) = \int_{(I\times M)/M} \Phi^*(\omega)$$

is the sought for cochain homotopy. □

Corollary 7.1.20. *Two homotopically equivalent spaces have isomorphic cohomology rings.*

Consider now a smooth manifold and U, V two open subsets such that $M = U \cup V$. Denote by $\imath_U$ (resp. $\imath_V$) the inclusions $U \hookrightarrow M$ (resp. $V \hookrightarrow M$). These induce the restriction maps

$$\imath_U^* : \Omega^*(M) \to \Omega^*(U) \quad \omega \mapsto \omega\,|_U$$

and

$$\imath_V^* : \Omega^*(M) \to \Omega^*(V) \quad \omega \mapsto \omega\,|_V\ .$$

We get a cochain map

$$\begin{aligned}r : \Omega^*(M) &\to \Omega^*(U) \oplus \Omega^*(V) \\ \omega &\mapsto (\imath_U^*\omega, \imath_V^*\omega).\end{aligned}$$

There exists another cochain map

$$\delta : \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V)$$

$$(\omega, \eta) \mapsto -\omega\,|_{U\cap V} + \eta\,|_{U\cap V}\ .$$

Lemma 7.1.21. *The* short Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \xrightarrow{r} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\delta} \Omega^*(U \cap V) \to 0$$

is exact.

Proof Obviously r is injective. The proof of the equality $\operatorname{Range} r = \ker \delta$ can be safely left to the reader. The surjectivity of δ requires a little more effort.

$\{U, V\}$ is an open cover of M so we can find a partition of unity

$$\{\varphi_U, \varphi_V\} \subset C^\infty(M)$$

subordinated to this cover, i.e.

$$\operatorname{supp} \varphi_U \subset U, \quad \operatorname{supp} \varphi_V \subset V$$

$$0 \leq \varphi_U,\ \varphi_V \leq 1, \quad \varphi_U + \varphi_V = 1.$$

Note that for any $\omega \in \Omega^*(U \cap V)$

$$\operatorname{supp} \varphi_V \omega \subset \operatorname{supp} \varphi_V \subset V$$

and thus, upon extending $\varphi_V \omega$ by 0 outside V we can view it as a form on U. Similarly, $\varphi_U \omega \in \Omega^*(V)$. Note that

$$\delta(-\varphi_V \omega, \varphi_U \omega) = (\varphi_V + \varphi_U)\omega = \omega.$$

This establishes the surjectivity of δ.

□

Using the abstract results in homological algebra we deduce from the above lemma the following fundamental result.

Theorem 7.1.22. (Mayer-Vietoris) *Let $M = U \cup V$ be an open cover of the smooth manifold M. Then there exists a long exact sequence*

$$\cdots \to H^k(M) \xrightarrow{r} H^k(U) \oplus H^k(V) \xrightarrow{\delta} H^k(U \cap V) \xrightarrow{\partial} H^{k+1}(M) \to \cdots$$

called the long Mayer-Vietoris sequence*.*

The *connecting morphisms* ∂ can be explicitly described using the prescriptions following Proposition 7.1.16 in the previous subsection. Start with $\omega \in \Omega^k(U \cap V)$ such that $d\omega = 0$. Writing as before

$$\omega = \varphi_V \omega + \varphi_U \omega$$

we deduce

$$d(\varphi_V \omega) = d(-\varphi_U \omega) \quad \text{on } U \cap V.$$

Thus we can find $\eta \in \Omega^{k+1}(M)$ such that

$$\eta\,|_U = d(\varphi_V \omega) \quad \eta\,|_V = d(-\varphi_U \omega).$$

Then

$$\partial \omega = \eta.$$

The reader can prove directly that the above definition is independent of the various choices.

The Mayer-Vietoris sequence has the following functorial property.

Proposition 7.1.23. *Let $\phi : M \to N$ be a smooth map and $\{U, V\}$ an open cover of N. Then $U' = \phi^{-1}(U), V' = \phi^{-1}(V)$ form an open cover of M and moreover, the diagram below is commutative.*

$$\begin{array}{ccccccc}
H^k(N) & \xrightarrow{r} & H^k(U) \oplus H^k(V) & \xrightarrow{\delta} & H^k(U \cap V) & \xrightarrow{\partial} & H^{k+1}(N) \\
\downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\
H^k(M) & \xrightarrow{r'} & H^k(U') \oplus H^k(V') & \xrightarrow{\delta'} & H^k(U' \cap V') & \xrightarrow{\partial'} & H^{k+1}(M)
\end{array}$$

Exercise 7.1.8. Prove the above proposition. □

7.1.5 Some simple examples

The Mayer-Vietoris theorem established in the previous subsection is a very powerful tool for computing the cohomology of manifolds. In principle, it allows one to recover the cohomology of a manifold decomposed into simpler parts, knowing the cohomologies of its constituents. In this subsection we will illustrate this principle on some simple examples.

Example 7.1.24. (The cohomology of spheres.) The cohomology of S^1 can be easily computed using the definition of DeRham cohomology. We have $H^0(S^1) = \mathbb{R}$ since S^1 is connected. The closed 1-forms on S^1 have the form *const.$d\theta$* so that $H^1(S^1) \cong \mathbb{R}$. Thus

$$P_{S^1}(t) = 1 + t.$$

To compute the cohomology of higher dimensional spheres we use Mayer-Vietoris theorem.

The $(n+1)$-dimensional sphere S^{n+1} can be covered by two open sets

$$U_{south} = S^{n+1} \setminus \{\text{north pole}\} \text{ and } U_{north} = S^{n+1} \setminus \{\text{south pole}\}.$$

Each is diffeomorphic to $\mathbb{R}^{n+1}$. Note that the overlap $U_{north} \cap U_{south}$ is homotopically equivalent with the "Equator" S^n. The Poincaré lemma implies that

$$H^{k+1}(U_{north}) \oplus H^{k+1}(U_{south}) \cong 0$$

for $k \geq 0$. The Mayer-Vietoris sequence gives

$$H^k(U_{north}) \oplus H^k(U_{south}) \to H^k(U_{north} \cap U_{south}) \to H^{k+1}(S^{n+1}) \to 0.$$

For $k > 0$ the group on the left is also trivial so that we have the isomorphisms

$$H^k(S^n) \cong H^k(U_{north} \cap U_{south}) \cong H^{k+1}(S^{n+1}) \quad k > 0.$$

Denote by $P_n(t)$ the Poincaré polynomial of S^n and set $Q_n(t) = P_n(t) - P_n(0) = P_n(t) - 1$. We can rewrite the above equality as

$$Q_{n+1}(t) = tQ_n(t) \quad n > 0.$$

Since $Q_1(t) = t$ we deduce $Q_n(t) = t^n$, i.e.

$$P_{S^n}(t) = 1 + t^n.$$

□

Example 7.1.25. Let $\{U, V\}$ be an open cover of the smooth manifold M. We assume that all the Betti numbers of U, V and $U \cap V$ are finite. Using the Mayer-Vietoris short exact sequence and Exercise 7.1.5 in Subsection 7.1.3 we deduce that all the Betti numbers of M are finite and moreover we have

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V). \tag{7.1.12}$$

This resembles very much the classical inclusion-exclusion principle in combinatorics. We will use this simple observation to prove that the Betti numbers of a connected sum of g tori is finite and then compute its Euler characteristic.

Let Σ be a surface with finite Betti numbers. From the decomposition $\Sigma = (\Sigma \setminus \text{disk}) \cup \text{disk}$ we deduce (using again Exercise 7.1.5)

$$\chi(\Sigma) = \chi(\Sigma \setminus \text{disk}) + \chi(\text{disk}) - \chi\left((\Sigma \setminus \text{disk}) \cap (\text{disk})\right).$$

Since $(\Sigma \setminus \text{disk}) \cap \text{disk}$ is homotopically a circle and $\chi(\text{disk}) = 1$ we deduce

$$\chi(\Sigma) = \chi(\Sigma \setminus \text{disk}) + 1 - \chi(S^1) = \chi(\Sigma \setminus \text{disk}) + 1.$$

If now Σ_1 and Σ_2 are two surfaces with finite Betti numbers then

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 \setminus \text{disk}) \cup (\Sigma_2 \setminus \text{disk})$$

where the two holed surfaces intersect over an entire annulus, which is homotopically a circle. Thus

$$\begin{aligned} \chi(\Sigma_1 \# \Sigma_2) &= \chi(\Sigma_1 \setminus \text{disk}) + \chi(\Sigma_2 \setminus \text{disk}) - \chi(S^1) \\ &= \chi(\Sigma_1) + \chi(\Sigma_2) - 2. \end{aligned}$$

This equality is identical with the one proved in Proposition 4.2.23 of Subsection 4.2.5.

We can decompose a torus as a union of two cylinders. The intersection of these cylinders is the disjoint union of two annuli so homotopically, this overlap is a disjoint union of two circles. In particular, the Euler characteristic of the intersection is zero. Hence

$$\chi(\text{torus}) = 2\chi(\text{cylinder}) = 2\chi(\text{circle}) = 0.$$

We conclude as in Proposition 4.2.23 that

$$\chi(\text{connected sum of } g \text{ tori}) = 2 - 2g.$$

This is a pleasant, surprising connection with the Gauss-Bonnet theorem. And the story is not over. □

7.1.6 The Mayer-Vietoris principle

We describe in this subsection a "patching" technique which is extremely versatile in establishing general homological results about arbitrary manifolds building up from elementary ones.

Definition 7.1.26. *A smooth manifold M is said to be of* finite type *if it can be covered by finitely many open sets $U_1, \dots, U_m$ such that any nonempty intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ $(k \geq 1)$ is diffeomorphic to $\mathbb{R}^{\dim M}$. Such a cover is said to be a* good cover.

Example 7.1.27. All compact manifolds are of finite type. To see this it suffices to cover such a manifold by finitely many open sets which are geodesically convex with respect to some Riemann metric.

If M is a finite type manifold and $U \subset M$ is a closed subset homeomorphic with the closed unit ball in $\mathbb{R}^{\dim M}$ them $M \setminus U$ is a finite type non-compact manifold. (It suffices to see that $\mathbb{R}^n \setminus$ closed ball is of finite type).

The connected sums and the direct products of finite type manifolds are finite type manifolds. □

Proposition 7.1.28. *Let $p : E \to B$ be a smooth vector bundle. If the base B is of finite type then so is the total space E.*

In the proof of this proposition we will use the following fundamental result.

Lemma 7.1.29. *Let $p : E \to B$ be a smooth vector bundle such that B is diffeomorphic to $\mathbb{R}^n$. Then $p : E \to B$ is a* trivial *bundle !!!*

Proof of Proposition 7.1.28. Denote by F the standard fiber of E. F is a vector space. Let (U_i) be a MV-cover of B. For each ordered multi-index $I : \{i_1 < \cdots < i_k\}$ denote by U_I the multiple overlap $U_{i_1} \cap \cdots \cap U_{i_k}$. Using the previous lemma we deduce that each $E_I = E|_{U_I}$ is a product $F \times U_i$ and thus it is diffeomorphic with some vector space. Hence (E_i) is a MV-cover.

□

Exercise 7.1.9. Prove Lemma 7.1.29.
Hint Assume E is a vector bundle over the unit open ball $B \subset \mathbb{R}^n$. Use ∇-parallel transport along $\vec{r} = -x^i \frac{\partial}{\partial x_i}$ where ∇ is a connection on E.

□

We organize the family of n-dimensional, finite type smooth manifolds as a category $\mathfrak{M}_n$. The morphisms of this category are the embeddings (i.e. 1-1 immersions) $M_1 \hookrightarrow M_2$ $(M_i \in \mathfrak{M}_n)$.

Definition 7.1.30. *Let R be a commutative ring with 1. A contravariant Mayer-Vietoris functor (or MV-functor for brevity) is a contravariant functor from the category $\mathfrak{M}_n$ to the category of $\mathbb{Z}$-graded R-modules*

$$\mathcal{F} = \oplus_{n \in \mathbb{Z}} \mathcal{F}^n \to \mathbf{Grad}\ {}_R\mathbf{Mod} \quad M \mapsto \oplus \mathcal{F}^n(M)$$

with the following property. If $\{U, V\}$ is a MV-cover *of $M \in \mathfrak{M}_n$, i.e. U, V, $U \cap V \in \mathfrak{M}_n$, then there exist morphisms of R-modules*

$$\partial_n : \mathcal{F}^n(U \cap V) \to \mathcal{F}^{n+1}(M)$$

such that the sequence below is exact.

$$\cdots \to \mathcal{F}^n(M) \xrightarrow{r^*} \mathcal{F}^n(U) \oplus \mathcal{F}^n(V) \xrightarrow{\delta} \mathcal{F}^n(U \cap V) \xrightarrow{\partial_n} \mathcal{F}^{n+1}(M) \to \cdots$$

where r^* is defined by

$$r^* = \mathcal{F}(\imath_U) \oplus \mathcal{F}(\imath_V)$$

while δ is defined by

$$\delta(x \oplus y) = \mathcal{F}(\imath_{U\cap V})(y) - \mathcal{F}(\imath_{U\cap V})(x).$$

(The maps $\imath_\bullet$ denote natural embeddings.) Moreover, if $N \in \mathfrak{M}_n$ is an open submanifold of N and $\{U, V\}$ is a MV cover of M such that $\{U \cap N, V \cap N\}$ is a MV-cover of N then the diagram below is commutative.

$$\begin{tikzcd} \mathcal{F}^n(U\cap V) \arrow[r, "\partial_n"] \arrow[d] & \mathcal{F}^{n+1}(M) \arrow[d] \\ \mathcal{F}^n(U\cap V\cap N) \arrow[r, "\partial_n"] & \mathcal{F}^{n+1}(N) \end{tikzcd} .$$

The vertical arrows are the morphisms $\mathcal{F}(\imath_\bullet)$ induced by inclusions.

The covariant MV-functors are defined in the dual way, by reversing the orientation of all the arrows in the above definition.

Definition 7.1.31. *Let $\mathcal{F}$, $\mathcal{G}$ be two contravariant MV-functors $\mathfrak{M}_n \to$ **Grad** $_R$**Mod**. A* correspondence *between these functors is a collection of R-module morphisms*

$$\phi_M = \bigoplus_{n\in\mathbb{Z}} \phi_M^n : \oplus\mathcal{F}^n(M) \to \oplus\mathcal{G}^n(M)$$

(one collection for each $M \in \mathfrak{M}_n$) such that for any embedding $M_1 \overset{\varphi}{\hookrightarrow} M_2$ the diagram bellow is commutative

$$\begin{tikzcd} \mathcal{F}^n(M_2) \arrow[r, "\mathcal{F}(\varphi)"] \arrow[d, "\phi_{M_2}"'] & \mathcal{F}^n(M_1) \arrow[d, "\phi_{M_1}"'] \\ \mathcal{G}^n(M_2) \arrow[r, "\mathcal{G}(\varphi)"] & \mathcal{G}^n(M_1) \end{tikzcd}$$

and for any $M \in \mathfrak{M}_n$ and any MV cover $\{U, V\}$ of M the diagram below is commutative.

$$\begin{tikzcd} \mathcal{F}^n(U\cap V) \arrow[r, "\partial_n"] \arrow[d, "\phi_{U\cap V}"'] & \mathcal{F}^{n+1}(M) \arrow[d, "\phi_M"'] \\ \mathcal{G}^n(U\cap V) \arrow[r, "\partial_n"] & \mathcal{G}^{n+1}(M) \end{tikzcd}$$

The correspondence is said to be a natural equivalence *if all the morphisms ϕ_M are isomorphisms.*

Theorem 7.1.32. (Mayer-Vietoris principle) *Let $\mathcal{F}$, $\mathcal{G}$ be two (contravariant) Mayer-Vietoris functors on $\mathfrak{M}_n$ and $\phi : \mathcal{F} \to \mathcal{G}$ a correspondence. If*

$$\phi^k_{\mathbb{R}^n} : \mathcal{F}^k(\mathbb{R}^n) \to \mathcal{G}^k(\mathbb{R}^n)$$

is an isomorphism for any $k \in \mathbb{Z}$ then ϕ is a natural equivalence.

Proof The family of finite type manifolds $\mathfrak{M}_n$ has a natural filtration

$$\mathfrak{M}^1_n \subset \mathfrak{M}^2_n \subset \cdots \subset \mathfrak{M}^r_n \subset \cdots$$

where $\mathfrak{M}^r_n$ is the collection of all smooth manifolds which admit a good cover consisting of at most r open sets. We will prove the theorem using an induction over r.

The theorem is clearly true for $r = 1$ by hypothesis. Assume ϕ^k_M is an isomorphism for all $M \in \mathfrak{M}^{r-1}_n$. Let $M \in \mathfrak{M}^r_n$ and consider a good cover $\{U_1, \ldots, U_r\}$ of M. Then

$$\{U = U_1 \cup \cdots \cup U_{r-1}, U_r\}$$

is a MV-cover of M. We thus get a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F}^n(U) \oplus \mathcal{F}^n(U_r) & \longrightarrow & \mathcal{F}^n(U \cap U_r) & \xrightarrow{\partial} & \mathcal{F}^{n+1}(M) & \longrightarrow & \mathcal{F}^{n+1}(U) \oplus \mathcal{G}^{n+1}(U_r) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{G}^n(U) \oplus \mathcal{G}^n(U_r) & \longrightarrow & \mathcal{G}^n(U \cap U_r) & \xrightarrow{\partial} & \mathcal{G}^{n+1}(M) & \longrightarrow & \mathcal{F}^{n+1}(U) \oplus \mathcal{G}^{n+1}(U_r)
\end{array}$$

The vertical arrows are defined by the correspondence ϕ. Note the inductive assumption implies that in the above infinite sequence only the morphisms ϕ_M may not be isomorphisms. At this point we invoke the following technical result.

Lemma 7.1.33. (The five lemma) *Consider the following commutative diagram of R-modules.*

$$\begin{array}{ccccccccc}
A_{-2} & \longrightarrow & A_{-1} & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 \\
{\scriptstyle f_{-2}}\downarrow & & {\scriptstyle f_{-1}}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\
B_{-2} & \longrightarrow & B_{-1} & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & B_2
\end{array}$$

If f_i is an isomorphism for any $i \neq 0$ then so is f_0.

Exercise 7.1.10. Prove the five lemma. □

The five lemma applied to our situation shows that the morphisms ϕ_M must be isomorphisms.

□

Remark 7.1.34. (a) The Mayer-Vietoris principle is true for covariant MV-functors as well. The proof is obtained by reversing the orientation of the horizontal arrows in the above proof.

(b) The Mayer-Vietoris principle can be refined a little bit. Assume that $\mathcal{F}$ and $\mathcal{G}$ are functors from $\mathfrak{M}_n$ to the category of $\mathbb{Z}$-graded R-algebras and $\phi : \mathcal{F} \to \mathcal{G}$ is a correspondence compatible with the multiplicative structures, i.e. each of the R-module morphisms ϕ_M are in fact morphisms of R-algebras. Then if $\phi_{\mathbb{R}^n}$ are isomorphisms of $\mathbb{Z}$-graded R-algebras then so are the ϕ_M's for any $M \in \mathfrak{M}_n$.
(c) Assume R is a field. The proof of the Mayer-Vietoris principle shows that if $\mathcal{F}$ is a MV-functor and $\dim_R \mathcal{F}^*(\mathbb{R}^n) < \infty$ then $\dim \mathcal{F}^*(M) < \infty$ for all $M \in \mathfrak{M}_n$. □

Corollary 7.1.35. *Any finite type manifold has finite Betti numbers.*

7.1.7 The Künneth formula

We learned in principle how to compute the cohomology of an"union of manifolds". We will now use the Mayer-Vietoris principle to compute the cohomology of products of manifolds.

Theorem 7.1.36. (Künneth formula) *Let $M \in \mathfrak{M}_m$ and $N \in \mathfrak{M}_n$. Then there exists a natural isomorphism of graded $\mathbb{R}$-algebras*

$$H^*(M \times N) \cong H^*(M) \otimes H^*(N) = \bigoplus_{n \geq 0} \left(\oplus_{p+q=n} H^p(M) \otimes H^q(N)\right).$$

In particular, we deduce

$$P_{M\times N}(t) = P_M(t) \cdot P_N(t).$$

Proof We construct two functors

$$\mathcal{F}, \mathcal{G} : \mathfrak{M}_m \to \textbf{Grad}\,_{\mathbb{R}}\textbf{Alg}$$

$$\mathcal{F} : M \mapsto \bigoplus_{r\geq 0} \mathcal{F}^r(M) = \bigoplus_{r \geq 0} \{\oplus_{p+q=r} H^p(M) \otimes H^q(N)\}$$

$$\mathcal{G} : M \mapsto \oplus_{r\geq 0}\mathcal{G}(M) = \oplus_{r\geq 0}\mathcal{G}^r(M \times N).$$

$$\mathcal{F}(f) = \bigoplus_{r\geq 0} \left(\oplus_{p+q=r} f^*|_{H^p(M_2)} \otimes \mathbf{1}_{H^q(N)}\right) \quad \forall f : M_1 \hookrightarrow M_2$$

$$\mathcal{G}(f) = \bigoplus_{r\geq 0} (f \times \mathbf{1}_N)^*|_{H^r(M_2\times N)} \quad \forall f : M_1 \hookrightarrow M_2.$$

We let the reader check the following elementary fact.

Exercise 7.1.11. $\mathcal{F}$ and $\mathcal{G}$ are contravariant MV-functors. □

For $M \in \mathfrak{M}_M$ define $\phi_M : \mathcal{F}(M) \to \mathcal{G}(N)$ by

$$\phi_M(\omega \otimes \eta) = \omega \times \eta \stackrel{def}{=} \pi_M^*\omega \wedge \pi_N^*\eta \quad (\omega \in H^*(M), \eta \in H^*(M))$$

where π_M (resp. π_N) are the canonical projections $M \times N \to M$ (resp. $M \times N \to N$). The operation

$$\times : H^*(M) \otimes H^*(N) \to H^*(M \times N) \quad (\omega \otimes \eta) \mapsto \omega \times \eta$$

is called the *cross product.* The Künneth formula is a consequence of the following lemma.

Lemma 7.1.37. *(a) ϕ is a correspondence of MV-functors.*
(b) $\phi_{\mathbb{R}^m}$ is an isomorphism.

Proof of the lemma The only nontrivial thing to prove is that for any MV-cover $\{U, V\}$ of $M \in \mathfrak{M}_m$ the diagram below is commutative.

$$\begin{array}{ccc}
\oplus_{p+q=r} H^p(U\cap V)\otimes H^q(N) & \stackrel{\partial}{\longrightarrow} & \oplus_{p+q=r} H^{p+1}(M)\otimes H^q(N) \\
\Big\downarrow{\scriptstyle \phi_{U\cap V}} & & \Big\downarrow{\scriptstyle \phi_M} \\
H^r((U\times N)\cap(V\times N)) & \stackrel{\partial'}{\longrightarrow} & H^{r+1}(M\times N)
\end{array}$$

We briefly recall the construction of the connecting morphisms ∂ and ∂'. One considers a partition of unity $\{\varphi_U, \varphi_V\}$ subordinated to the cover $\{U, V\}$. Then $\psi_U = \pi_M^*\varphi_U$ and $\psi_V = \pi_M^*\varphi_V$ form a partition of unity subordinated to the cover $\{U\times N, V\times N\}$ of $M\times N$. If $\omega\otimes\eta \in H^*(U\cap V)\otimes H^*(N)$ then

$$\partial(\omega\otimes\eta) = \hat{\omega}\otimes\eta$$

where

$$\hat{\omega}\,|_U = -d(\varphi_V\omega) \quad \hat{\omega} = d(\varphi_U\omega).$$

On the other hand

$$\phi_{U\cap V}(\omega\otimes\eta) = \omega\times\eta$$

and

$$\partial'(\omega\times\eta) = \hat{\omega}\times\eta.$$

This proves (a).

To establish (b) note that the inclusion

$$\jmath : N \hookrightarrow \mathbb{R}^m\times N \quad x\mapsto (0, x)$$

is a homotopy equivalence with π_N a homotopy inverse. Hence, by the homotopy invariance of the DeRham cohomology we deduce

$$\mathcal{G}(\mathbb{R}^m) \cong H^*(N).$$

Using the Poincaré lemma and the above isomorphism we can identify the morphism $\phi_{\mathbb{R}^m}$ with $\mathbf{1}_{\mathbb{R}^m}$.

□

Example 7.1.38. Consider the n-dimensional torus, T^n. By writing it as a direct product of n circles we deduce from Künneth formula that

$$P_{T^n}(t) = \{P_{S^1}(t)\}^n = (1+t)^n.$$

Thus

$$b_k(T^n) = \binom{n}{k}, \quad \dim H^*(T^n) = 2^n,$$

and

$$\chi(T^n) = 0.$$

One can easily describe a basis of $H^*(T^n)$. Choose angular coordinates $(\theta^1, \dots, \theta^n)$ on T^n. For each ordered multi-index $I: \ 1 \leq i_1 < \cdots < i_k \leq n$ we have a closed, non-exact form $d\theta^I$. These monomials are linearly independent (over $\mathbb{R}$) and there are 2^n of them. Thus, they form a basis of $H^*(T^n)$. In fact, one can read the multiplicative structure using this basis. We have an isomorphism of $\mathbb{R}$-algebras

$$H^*(T^n) \cong \Lambda^* \mathbb{R}^n.$$

$\square$

Exercise 7.1.12. Let $M \in \mathfrak{M}_m$ and $N \in \mathfrak{M}_n$. Show that for any $\omega_i \in H^*(M)$, $\eta_j \in H^*(N)$ (i,j=0,1) the following equality holds.

$$(\omega_0 \times \eta_0) \wedge (\omega_1 \times \eta_1) = (-1)^{\deg \eta_0 \deg \omega_1} (\omega_0 \wedge \omega_1) \times (\eta_0 \wedge \eta_1).$$

$\square$

Exercise 7.1.13. (Leray-Hirsch) Let $p: E \to M$ be smooth bundle with standard fiber F. We assume the following:
(a) Both M and F are of finite type.
(b) There exists cohomology classes $e_1, \dots, e_r \in H^*(E)$ such that their restrictions to any fiber generate the cohomology algebra of that fiber.

The projection p induces a $H^*(M)$-module structure on $H^*(E)$ by

$$\omega \cdot \eta = p^*\omega \wedge \eta \quad \forall \omega \in H^*(M) \eta \in H^*(E).$$

Show that $H^*(E)$ is a free $H^*(M)$-module with generators $e_1, \dots, e_r$. $\square$

7.2 The Poincaré duality

7.2.1 Cohomology with compact supports

Let M be a smooth n-dimensional manifold. Denote by $\Omega^k_{cpt}(M)$ the space of smooth compactly supported k-forms. Then

$$0 \to \Omega^0_{cpt}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{cpt}(M) \to 0$$

is a cochain complex. Its cohomology is denoted by $H^*_{cpt}(M)$ and is called the DeRham cohomology with compact supports. Note that when M is compact this cohomology coincides with the usual DeRham cohomology.

Although it looks very similar to the usual DeRham cohomology, there are many important differences. The most visible one is that if $\phi: M \to N$ and $\omega \in \Omega^*_{cpt}(N)$ is a smooth map then the pull-back $\phi^*\omega$ may not have compact support so this new construction is no longer a contravariant functor from the category of smooth manifolds and smooth maps to the category of graded vector spaces.

On the other hand if $\dim M = \dim N$ and ϕ is an embedding we can identify M with an open subset of N and then any $\eta \in \Omega^*_{cpt}(M)$ can be extended with 0 outside $M \subset N$. This extension by zero defines a *push-forward* map

$$\phi_* : \Omega^*_{cpt}(M) \to \Omega^*_{cpt}(N).$$

One can verify easily that ϕ_* is a cochain map so that it induces a morphism

$$\phi_* : H^*_{cpt}(M) \to H^*_{cpt}(N).$$

In terms of our category $\mathfrak{M}_n$ we see that H^*_{cpt} is a *covariant functor* from the category $\mathfrak{M}_n$ to the category of graded real vector spaces. As we will see, it is a rather nice functor.

Theorem 7.2.1. *H^*_{cpt} is a covariant MV-functor and moreover*

$$H^k_{cpt}(\mathbb{R}^n) = \begin{cases} 0 & , \quad k < n \\ \mathbb{R} & , \quad k = n \end{cases}.$$

The last assertion of this theorem is usually called the Poincaré lemma for compact supports.

We first prove the Poincaré lemma for compact supports. The crucial step is the following technical result we borrowed from [10].

Lemma 7.2.2. *Let $E \xrightarrow{p} B$ be a rank r real vector bundle, orientable in the sense described in Subsection 3.4.5. Denote by p_* the integration-along-fibers map*

$$p_* : \Omega^*_{cpt}(E) \to \Omega^{*-r}_{cpt}(B).$$

Then there exists a smooth bilinear map

$$\mathfrak{m} : \Omega^i_{cpt}(E) \times \Omega^j_{cpt}(E) \to \Omega^{i+j-r-1}_{cpt}(E)$$

such that

$$p^*p_*\alpha \wedge \beta - \alpha \wedge p^*p_*\beta = (-1)^r d(\mathfrak{m}(\alpha, \beta)) - \mathfrak{m}(d\alpha, \beta) + (-1)^{\deg \alpha}\mathfrak{m}(\alpha, d\beta).$$

Proof Consider the ∂-bundle

$$\pi : \mathcal{E} = I \times (E \oplus E) \to E$$

$$\pi : (t; v_0, v_1) \mapsto (t; v_0 + t(v_1 - v_0)).$$

Note that

$$\partial\mathcal{E} = (\{0\} \times (E \oplus E)) \sqcup (\{1\} \times (E \oplus E)).$$

Define $\pi^t : E \oplus E \to E$ as the composition

$$E \oplus E \cong \{t\} \times E \oplus E \hookrightarrow I \times (E \oplus E) \xrightarrow{\pi} E.$$

Observe that

$$\partial\pi = \pi\,|_{\partial\mathcal{E}} = (-\pi^0) \sqcup \pi^1.$$

For $(\alpha, \beta) \in \Omega^*_{cpt}(E) \times \Omega^*_{cpt}(E)$ define $\alpha \odot \beta \in \Omega^*_{cpt}(E \oplus E)$ by

$$\alpha \odot \beta := (\pi^0)^*\alpha \wedge (\pi^1)^*\beta.$$

(Verify that $\alpha \odot \beta$ has indeed compact support.) For $\alpha \in \Omega^i_{cpt}(E)$ and $\beta \in \Omega^j_{cpt}(E)$ we have the equalities

$$p^*p_*\alpha \wedge \beta = \pi^1_*(\alpha \odot \beta) \in \Omega^{i+j-r}_{cpt}(E)$$

$$\alpha \wedge p^*p_*\beta = \pi^0_*(\alpha \odot \beta) \in \Omega^{i+j-r}_{cpt}(E).$$

Hence

$$D(\alpha, \beta) = p^*p_*\alpha \wedge \beta - \alpha \wedge p^*p_*\beta = \pi^1_*(\alpha \odot \beta) - \pi^0_*(\alpha \odot \beta)$$

$$= \int_{\partial\mathcal{E}/E} \alpha \odot \beta.$$

We now use the fibered Stokes formula to get

$$D(\alpha, \beta) = \int_{\mathcal{E}/E} d_{\mathcal{E}}\mathcal{T}^*(\alpha \odot \beta) + (-1)^r d_E \int_{\mathcal{E}/E} \mathcal{T}^*(\alpha \odot \beta).$$

$\mathcal{T}$ is the natural projection $\mathcal{E} = I \times (E \oplus E) \to E \oplus E$. The lemma holds with

$$\mathfrak{m}(\alpha, \beta) = \int_{\mathcal{E}/E} \mathcal{T}^*(\alpha \odot \beta). \qquad \square$$

Proof of the Poincaré lemma for compact supports Consider $\delta \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \delta \leq 1 \quad \int_{\mathbb{R}^n} \delta(x)dx = 1.$$

Define the (closed) compactly supported n-form

$$\tau = \delta(x)dx^1 \wedge \cdots \wedge dx^n.$$

We want to use Lemma 7.2.2 in which E is the rank n bundle over a point i.e. $E = \{pt\} \times \mathbb{R}^n \xrightarrow{p} \{pt\}$. The integration along fibers is simply the integration map.

$$p_* : \Omega^*_{cpt}(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega \mapsto p_*\omega = \begin{cases} 0 & , \ \deg\omega < n \\ \int_{\mathbb{R}^n} \omega & , \ \deg\omega = n \end{cases}$$

If now ω is a closed, compactly supported form on $\mathbb{R}^n$ we have

$$\omega = (p^*p_*\tau) \wedge \omega.$$

Using Lemma 7.2.2 we deduce

$$\omega - \tau \wedge p^*p_*\omega = (-1)^n d\mathfrak{m}(\tau, \omega).$$

Thus any closed, compactly supported form ω on $\mathbb{R}^n$ is cohomologous to $\tau \wedge p^* p_* \omega$. The latter is always zero if $\deg \omega < n$. When $\deg \omega = n$ we deduce that ω is cohomologous to $(\int \omega)\tau$. This completes the proof of the Poincaré lemma. □

To finish the proof of Theorem 7.2.1 we must construct a Mayer-Vietoris sequence. Let M be a smooth manifold decomposed as an union of two open sets

$$M = U \cup V.$$

The sequence of inclusions

$$U \cap V \hookrightarrow U, V \hookrightarrow M$$

induces a short sequence

$$0 \to \Omega^*_{cpt}(U \cap V) \xrightarrow{i} \Omega^*_{cpt}(U) \oplus \Omega^*_{cpt}(V) \xrightarrow{j} \Omega^*_{cpt}(M) \to 0$$

where

$$i(\omega) = (\hat{\omega}, \hat{\omega}) \quad j(\omega, \eta) = \hat{\eta} - \hat{\omega}.$$

The hat denotes the extension by zero outside the support. This sequence is called the Mayer-Vietoris short sequence for compact supports.

Lemma 7.2.3. *The above Mayer-Vietoris sequence is exact.*

Proof i is obviously injective. Clearly, Range $(i) = \ker(j)$. We have to prove that j is surjective. Let (φ_U, φ_V) a partition of unity subordinated to the cover $\{U, V\}$. Then for any $\eta \in \Omega^*_{cpt}(M)$

$$\varphi_U \eta \in \Omega^*_{cpt}(U) \quad \varphi_V \eta \in \Omega^*_{cpt}(V).$$

In particular we have

$$\eta = j(-\varphi_U \eta, \varphi_V \eta).$$

Hence j is surjective. □

We get a long exact sequence called the long Mayer-Vietoris sequence for compact supports.

$$\cdots \to H^k_{cpt}(U \cap V) \to H^k_{cpt}(U) \oplus H^k_{cpt}(V) \to H^k_{cpt}(M) \xrightarrow{\delta} H^{k+1}_{cpt} \to \cdots$$

The connecting homomorphism can be explicitly described as follows. If $\omega \in \Omega^k_{cpt}(M)$ is a closed form then

$$d(\varphi_U \omega) = d(-\varphi_V \omega) \quad \text{on } U \cap V.$$

We set $\delta\omega = d(\varphi_U \omega)$. The reader can check immediately that the cohomology class of $\delta\omega$ is independent of all the choices made.

If $\phi : N \hookrightarrow M$ is a morphism of $\mathfrak{M}_n$ then for any MV-cover of $\{U, V\}$ of M $\{\phi^{-1}(U), \phi^{-1}(V)\}$ is an MV-cover of N. Moreover, we almost tautologically get a commutative diagram

$$\begin{array}{ccc}
H^k_{cpt}(N) & \xrightarrow{\delta} & H^{k+1}_{cpt}(\phi^{-1}(U \cap V)) \\
\phi_* \downarrow & & \phi_* \downarrow \\
H^k_{cpt}(M) & \xrightarrow{\delta} & H^{k+1}_{cpt}(U \cap V)
\end{array} .$$

This proves H^*_{cpt} is a covariant Mayer-Vietoris sequence. □

Remark 7.2.4. To be perfectly honest (from a categorial point of view) we should have considered the *chain complex*

$$\bigoplus_{k\leq 0}\tilde{\Omega}^k = \bigoplus_{k\leq 0}\Omega^{-k}_{cpt}$$

and correspondingly the associated *homology*

$$\tilde{H}_* := H^{-*}_{cpt}.$$

This makes the sequence

$$0 \leftarrow \tilde{\Omega}^{-n} \leftarrow \cdots \leftarrow \tilde{\Omega}^{-1} \leftarrow \tilde{\Omega}^0 \leftarrow 0$$

a *chain* complex and its *homology* $\tilde{H}_*$ is a bona-fide *covariant* Mayer-Vietoris functor since the connecting morphism δ goes in the right direction $\tilde{H}^* \to \tilde{H}^{*-1}$. However, the simplicity of the original notation is worth the small formal ambiguity so we stick to our upper indices. □

From the proof of the Mayer-Vietoris principle we deduce the following.

Corollary 7.2.5. *For any $M \in \mathfrak{M}_n$ and any $k \leq n$ we have* $\dim H^k_{cpt}(M) < \infty$.

7.2.2 The Poincaré duality

Definition 7.2.6. *Denote by $\mathfrak{M}^+_n$ the category of n-dimensional, finite type, oriented manifolds. The morphisms are the embeddings of such manifold. The MV functors on $\mathfrak{M}^+_n$ are defined exactly as for $\mathfrak{M}_n$.*

Given $M \in \mathfrak{M}^+_n$ there is a natural pairing

$$\langle \bullet, \bullet \rangle_\kappa : \Omega^k(M) \times \Omega^{n-k}_{cpt}(M) \to \mathbb{R}$$

defined by

$$\langle \omega, \eta \rangle_\kappa := \int_M \omega \wedge \eta.$$

which is called the *Kronecker pairing* . We can extend this pairing to any $(\omega, \eta) \in \Omega^* \times \Omega^*_{cpt}$ as

$$\langle \omega, \eta \rangle_\kappa = \begin{cases} 0 & , \ \deg\omega + \deg\eta \neq n \\ \int_M \omega \wedge \eta & , \ \deg\omega + \deg\eta = n \end{cases}.$$

This pairing induces maps

$$\mathcal{D} = \mathcal{D}^k : \Omega^k(M) \to (\Omega^{n-k}_{cpt}(M))^*$$

$$\langle \mathcal{D}(\omega), \eta \rangle = \langle \omega, \eta \rangle_\kappa.$$

$\langle \bullet,$
bullet$\rangle$ denotes the natural duality between a vector space and its dual, $V^* \times V \to \mathbb{R}$. We

continue to denote by $\mathcal{D}(\omega)$ the restriction to the space of closed, compactly supported $(n-k)$-forms, Z^{n-k}_{cpt}. If moreover ω is closed, this functional on Z^{n-k}_{cpt} vanishes on the subspace B^{n-k}_{cpt} of exact, compactly supported $(n-k)$-forms. Indeed, if $\eta = d\eta'$, $\eta' \in \Omega^{n-k-1}_{cpt}(M)$ then

$$\langle \mathcal{D}(\omega), \eta\rangle = \int_M \omega \wedge d\eta' \stackrel{Stokes}{=} \pm \int_M d\omega \wedge \eta' = 0.$$

Thus, if ω is closed $\mathcal{D}(\omega)$ defines an element of $(H^{n-k}_{cpt}(M))^*$. If moreover ω is exact, a computation as above shows that $\mathcal{D}(\omega) = 0 \in (H^{n-k}_{cpt}(M))^*$. Hence $\mathcal{D}$ descends to a map in cohomology

$$\mathcal{D} : H^k(M) \to (H^{n-k}_{cpt}(M))^*$$

which is the same as saying that the Kronecker pairing descends to a pairing in cohomology.

Theorem 7.2.7. (Poincaré duality) *The Kronecker pairing in cohomology is a duality for all $M \in \mathfrak{M}^+_n$.*

Proof The functor

$$\mathfrak{M}^+_n \to \textbf{Graded Vector Spaces}$$

defined by

$$M \to \bigoplus_k \tilde{H}^k(M) = \bigoplus_k (H^{n-k}_{cpt}(M))^*$$

is a contravariant MV-functor. (The exactness of the Mayer-Vietoris sequence is preserved by transposition. This is where the fact that all the cohomology groups are finite dimensional vector spaces plays a very important role).

For purely formal reasons (which will become apparent in a little while) we define the connecting morphism of the functor $\tilde{H}^k$ $(\tilde{H}^k(U \cap V) \stackrel{\tilde{d}}{\to} \tilde{H}^{k+1}(U \cup V))$ to be $(-1)^k \delta^t$, where $H^{n-k-1}_{cpt}(U \cap V) \stackrel{\delta}{\to} H^{n-k}_{cpt}(U \cup V)$ denotes the connecting morphism in the DeRham cohomology with compact supports and δ^t denotes its transpose.

The Poincaré lemma for compact supports can be rephrased

$$\tilde{H}^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & , \quad k = 0 \\ 0 & , \quad k > 0 \end{cases}.$$

The Kronecker pairing induces linear maps

$$\mathcal{D}_M : H^k(M) \to \tilde{H}^k(M).$$

Lemma 7.2.8. $\bigoplus_k \mathcal{D}^k$ *is a correspondence of MV functors.*

Proof We have to check two facts.

FACT A. Let $M \stackrel{\phi}{\hookrightarrow} N$ be a morphism in $\mathfrak{M}^+_n$. Then the diagram below is commutative.

$$\begin{array}{ccc} H^k(N) & \xrightarrow{\phi^*} & H^k(M) \\ {\scriptstyle \mathcal{D}_N}\downarrow & & {\scriptstyle \mathcal{D}_M}\downarrow \\ \tilde{H}^k(N) & \xrightarrow{\tilde{\phi}^*} & \tilde{H}^k(M) \end{array}.$$

FACT B. If $\{U, V\}$ is a MV-cover of $M \in \mathfrak{M}_n^+$ then the diagram bellow is commutative

$$\begin{array}{ccc} H^k(U\cap V) & \stackrel{\partial}{\longrightarrow} & H^{k+1}(M) \\ {\scriptstyle \mathcal{D}_{U\cap V}}\Big\downarrow & & {\scriptstyle \mathcal{D}_M}\Big\downarrow \\ \tilde{H}^k(U\cap V) & \stackrel{(-1)^k\delta^t}{\longrightarrow} & \tilde{H}^{k+1}(M) \end{array}.$$

Proof of FACT A. Let $\omega \in H^k(N)$. Denoting by $\langle \bullet, \bullet\rangle$ the natural duality between a vector space and its dual we deduce that for any $\eta \in H^{n-k}_{cpt}(M)$ we have

$$\begin{aligned} \big\langle \tilde{\phi}^* \circ \mathcal{D}_N(\omega), \eta\big\rangle &= \big\langle (\phi_*)^t \mathcal{D}_N(\omega), \eta\big\rangle = \big\langle \mathcal{D}_N(\omega), \phi_*\eta\big\rangle \\ &= \int_N \omega\wedge\phi_*\eta = \int_{M\hookrightarrow N} \omega|_M \wedge\eta = \int_M \phi^*\omega\wedge\eta \\ &= \langle \mathcal{D}_M(\phi^*\omega), \eta\rangle. \end{aligned}$$

Hence $\tilde{\phi}^*\circ\mathcal{D}_N = \mathcal{D}_M\circ\phi^*$.

Proof of FACT B. Let φ_U, φ_V be a partition of unity subordinated to the MV-cover $\{U, V\}$ of $M \in \mathfrak{M}_n^+$. Consider a closed k-form $\omega \in \Omega^k(U\cap V)$. Then the connecting morphism in usual DeRham cohomology acts as

$$\partial\omega = \begin{cases} d(-\varphi_V\omega) & \text{on } U \\ d(\varphi_U\omega) & \text{on } V \end{cases}.$$

Choose $\eta \in \Omega^{n-k-1}_{cpt}(M)$ such that $d\eta = 0$. We have

$$\langle \mathcal{D}_M\partial\omega, \eta\rangle = \int_M \partial\omega\wedge\eta$$

$$= \int_U \partial\omega\wedge\eta + \int_V \partial\omega\wedge\eta - \int_{U\cap V}\partial\omega\wedge\eta$$

$$= -\int_U d(\varphi_V\omega)\wedge\eta + \int_V d\varphi_U\omega)\wedge\eta + \int_{U\cap V} d(\varphi_V\omega)\wedge\eta.$$

Note that the first two integrals vanish. Indeed, over U we have the equality

$$(d(\varphi_V\omega)\wedge\eta = d\left(\varphi_V\omega\wedge\eta\right)$$

and the vanishing now follows from Stokes formula. The second term is dealt with in a similar fashion. As for the last term we have

$$\begin{aligned} \int_{U\cap V}(d\varphi_V\omega)\wedge\eta &= \int_{U\cap V} d\varphi_V\wedge\omega\wedge\eta = (-1)^{\deg\omega}\int_{U\cap V}\omega\wedge(d\varphi_V\wedge\eta) \\ &= (-1)^k\int_{U\cap V}\omega\wedge\delta\eta = (-1)^k\langle\mathcal{D}_{U\cap V}\omega, \delta\eta\rangle \\ &= \langle(-1)^k\delta^t\mathcal{D}_{U\cap V}\omega, \eta\rangle. \end{aligned}$$

This concludes the proof of Fact B.

The Poincaré duality now follows from the Mayer-Vietoris principle. □

Remark 7.2.9. Using the Poincaré duality we can associate to any smooth map $f : M \to N$ between compact oriented manifolds of dimensions m and respectively n a natural *push-forward morphism*

$$f_* : H^*(M) \to H^{*+q}(N) \quad (q = \dim N - \dim M = n - m)$$

defined by the composition

$$H^*(M) \stackrel{\mathcal{D}_M}{\to} (H^{m-*}(M))^* \stackrel{(f^*)^t}{\to} (H^{m-*}(N))^* \stackrel{\mathcal{D}_N^{-1}}{\to} H^{*+n-m}(N)$$

where $(f^*)^t$ denotes the transpose of the pullback morphism. $\square$

Corollary 7.2.10. *If $M \in \mathfrak{M}_n^+$ then*

$$H^k_{cpt}(M) \cong (H^{n-k}(M))^*.$$

Proof Since $H^k_{cpt}(M)$ is finite dimensional the transpose

$$\mathcal{D}_M^t : (H^{n-k}_{cpt}(M))^{**} \to (H^k(M))^*$$

is an isomorphism. On the other hand, for any finite dimensional vector space there exists a natural isomorphism

$$V^{**} \cong V.$$ $\square$

Corollary 7.2.11. *Let M be a compact oriented n-dimensional manifold. Then the pairing*

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R} \quad (\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is a duality. In particular $b_k(M) = b_{n-k}(M)$, $\forall k$.

If M is connected $H^0(M) \cong H^n(M) \cong \mathbb{R}$ so that $H^n(M)$ is generated by any volume form defining the orientation.

The symmetry of Betti numbers can be translated in the language of Poincaré polynomials as

$$t^n P_M(\frac{1}{t}) = P_M(t). \tag{7.2.1}$$

Example 7.2.12. Let Σ_g denote the connected sum of g tori. We have shown that

$$\chi(\Sigma_g) = b_0 - b_1 + b_2 = 2 - 2g.$$

Since Σ_g is connected, the Poincaré duality implies $b_2 = b_0 = 1$. Hence $b_1 = 2g$ i.e.

$$P_{\Sigma_g}(t) = 1 + 2gt + t^2.$$ $\square$

Consider now a compact oriented smooth manifold such that $\dim M = 2k$. The Kronecker pairing induces a *non-degenerate* bilinear form

$$\mathcal{I} : H^k(M) \times H^k(M) \to \mathbb{R} \quad \mathcal{I}(\omega, \eta) = \int_M \omega \wedge \eta.$$

$\mathcal{I}$ is called the *cohomological intersection form* of M. When k is even (so that n is divisible by 4) $\mathcal{I}$ is a symmetric form. Its signature is called the *signature* of M and is denoted by $\sigma(M)$. When k is odd $\mathcal{I}$ is skew-symmetric, i.e. it is a symplectic form. In particular, we deduce the following result.

Corollary 7.2.13. *For any compact manifold $M \in \mathfrak{M}^+_{4k+2}$ the middle Betti number $b_{2k+1}(M)$ is even.*

Exercise 7.2.1. (a) Let $P \in \mathbb{Z}[t]$ be an odd degree polynomial with non-negative integer coefficients such that $P(0) = 1$. Show that if P satisfies the symmetry condition (7.2.1) there exists a compact, connected, oriented manifold M such that $P_M(t) = P(t)$.
(b) Let $P \in \mathbb{Z}[t]$ be a polynomial of degree $2k$ with non-negative integer coefficients. Assume $P(0) = 1$ and P satisfies (7.2.1). If the coefficient of t^k is even then there exists a compact connected manifold $M \in \mathfrak{M}^+_{2k}$ such that $P_M(t) = P(t)$.
Hint: Describe the Poincaré polynomial of a connect sum in terms of the polynomials of its constituents. Combine this fact with the Künneth formula. □

Remark 7.2.14. The result in the above exercise is sharp. Using his intersection theorem F. Hirzebruch showed that there exist no *smooth* manifolds M of dimension 12 or 20 with Poincaré polynomials $1 + t^6 + t^{12}$ and respectively $1 + t^{10} + t^{20}$. Note that in each of these cases both middle Betti numbers are odd. For details we refer to J. P. Serre, "*Travaux de Hirzebruch sur la topologie des variétés*", Seminaire Bourbaki 1953/54,n° **88**. □

7.3 Intersection theory

7.3.1 Cycles and their duals

Definition 7.3.1. *Let $M \in \mathfrak{M}^+_n$. A k-dimensional* cycle *in M is a pair (S, ϕ) where S is a compact, oriented k-dimensional manifold and $\phi : S \to M$ is a smooth map.*

We denote by $\mathcal{C}_k(M)$ the set of k-dimensional cycles in M.

Definition 7.3.2. *(a)Two cycles $(S_0, \phi_0), (S_1, \phi_1) \in \mathcal{C}_k(M)$ are said to be* cobordant *(we write this $(S_0, \phi_0) \sim_c (S_1, \phi_1)$) if there exists a compact, oriented manifold with boundary Σ and a smooth map $\Phi : \Sigma \to M$ such that*
(a1) $\partial\Sigma = (-S_0) \sqcup S_1$ where $-S_0$ denotes the oriented manifold S_0 with the opposite orientation and $\sqcup$ denotes the disjoint union.
(a2) $\Phi|_{S_i} = \phi_i$, $i = 0, 1$.
(b) A cycle $(S, \phi) \in \mathcal{C}_k(M)$ is said to be degenerate *if ϕ is homotopic to the constant map $S \to M$. We write $(S, \phi) \sim_c 0$.*

We denote by $\mathfrak{D}_k(M)$ the set of degenerate cycles.

Exercise 7.3.1. Let $(S_0, \phi_0) \sim_c (S_1, \phi_1)$. Prove that $(-S_0 \sqcup S_1, \phi_0 \sqcup \phi_1) \sim_c 0$. □

We denote by $\mathcal{Z}_k(M)$ the free abelian group generated by $\mathcal{C}_k(M)$ and by $\mathcal{B}_k(M) \subset \mathcal{Z}_k(M)$ the subgroup generated by cycles cobordant to degenerated ones. We can form the quotient group $\mathcal{H}_k(M)$. For any cycle $(S, \phi) \in \mathcal{C}_k(M)$ we denote by $[S, \phi]$ its image in $\mathcal{H}_k(M)$.

Exercise 7.3.2. (a) Prove that

$$[S_0, \phi_0] + [S_1, \phi_1] = [S_0 \sqcup S_1, \phi_0 \sqcup \phi_1] \quad \forall (S_i, \phi_i) \in gC_k(M)$$

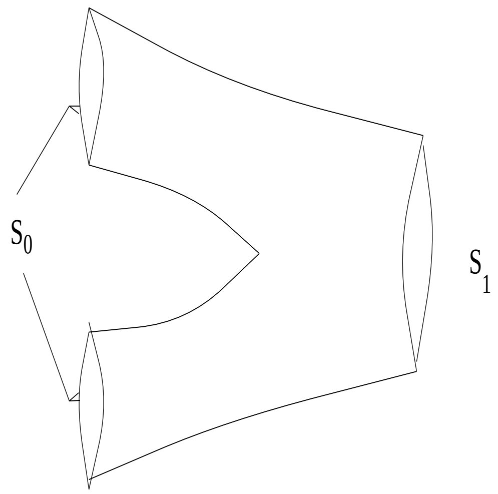

Figure 7.1: *A cobordism in* $\mathbb{R}^3$

and

$$-[S, \phi] = [-S, \phi] \in \mathcal{H}_k(M).$$

(b) Prove that$[S, \phi] = 0$ in $\mathcal{H}_k(M)$ if and only if there exists a degenerate cycle $(S'\phi')$ such that $(S, \phi) \sim_c (S', \phi')$. □

Any k-cycle (S, ϕ) defines a linear map $H^k(M) \to \mathbb{R}$ by

$$H^k(M) \ni \omega \mapsto \int_S \phi^*\omega.$$

Stokes formula shows that this map is well defined i.e. it is independent of the closed form representing a cohomology class. Indeed, if ω is exact, i.e. $\omega = d\omega'$ then

$$\int_S \phi^* d\omega' = \int_S d\phi^*\omega' = 0.$$

In other words, each cycle defines an element in $(H^k(M))^*$ which can be identified via the Poincaré duality with $H^{n-k}_{cpt}(M)$. Thus there exists $\delta_S \in H^{n-k}_{cpt}(M)$ such that

$$\int_M \omega \wedge \delta_S = \int_S \phi^*\omega \quad \forall \omega \in H^k(M).$$

δ_S is called the *Poincaré dual* of (S, ϕ).

There exist many closed forms $\eta \in \Omega^{n-k}_{cpt}(M)$ representing δ_S. When there is no risk of confusion, we continue denote any such representative by δ_S.

Example 7.3.3. Let $M = \mathbb{R}^n$ and S is a point $\{pt\} \subset \mathbb{R}^n$. pt is canonically a 0-cycle. Its Poincaré dual is a compactly supported n-form ω such that for any constant λ (i.e. closed 0-form)

$$\int_{\mathbb{R}^n} \lambda\omega = \int_{pt} \lambda = \lambda,$$

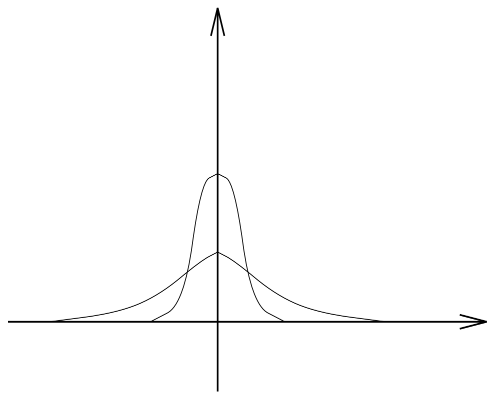

Figure 7.2: *The dual of a point is Dirac's distribution*

i.e.

$$\int_{\mathbb{R}^n} \omega = 1.$$

Thus δ_{pt} can be represented by any compactly supported n-form with integral 1. In particular we can choose representatives with arbitrarily small supports. Their "profiles" look like in Figure 7.2. "At limit" they approach Dirac's delta distribution.

□

Example 7.3.4. Consider an n-dimensional, compact, oriented manifold M. We denote by $[M]$ the cycle $(M, \mathbb{1}_M)$. Then $\delta_{[M]} = 1 \in H^0(M)$. □

For any differential form ω we set (for typographical reasons) $|\omega| = \deg \omega$.

Example 7.3.5. Consider the compact manifolds $M \in \mathfrak{M}_m^+$ and $N \in \mathfrak{M}_n^+$. To any cycle $(S, \phi) \in \mathcal{C}_p(M)$ and $(T, \psi) \in \mathcal{C}_q(N)$ we can associate the cycle $(S \times T, \phi \times \psi) \in \mathcal{C}_{p+q}(M \times N)$. We denote by π_M (resp. π_N) the natural projection $M \times N \to M$ (resp. $M \times N \to N$). We want to prove the equality

$$\delta_{S\times T} = (-1)^{(m-p)q} \delta_S \times \delta_T \tag{7.3.1}$$

where

$$\omega \times \eta \stackrel{def}{=} \pi_M^* \omega \wedge \pi_N^* \eta \quad \forall (\omega, \eta) \in \Omega^*(M) \times \Omega^*(N).$$

By Künneth formula we have $H^*(M \times N) \cong H^*(M) \otimes H^*(N)$ and the cohomology of $M \times N$ is generated (as a linear space) by the cross products $\omega \times \eta$.

Pick $(\omega, \eta) \in \Omega^*(M) \times \Omega^*(N)$ such that $\deg \omega + \deg \eta = (m+n) - (p+q)$. Then, using Exercise 7.1.12

$$\int_{M\times N} (\omega \times \eta) \wedge (\delta_S \times \delta_T) = (-1)^{(m-p)|\eta|} \int_{M\times N} (\omega \wedge \delta_S) \times (\eta \wedge \delta_T).$$

The above integral should be understood in the generalized sense of Kronecker pairing. The only time when this pairing does not vanish is when $|\omega| = p$ and $|\eta| = q$. In this case the last term equals

$$(-1)^{q(m-p)}\left(\int_S \omega\wedge\delta_S\right)\left(\int_T \eta\wedge\delta_T\right) = (-1)^{q(m-p)}\left(\int_S \phi^*\omega\right)\left(\int_T \psi^*\eta\right)$$

$$= (-1)^{q(m-p)}\int_{S\times T}(\phi\times\psi)^*(\omega\wedge\eta).$$

This establishes equality (7.3.1). □

Example 7.3.6. Consider a compact manifold $M\in\mathfrak{M}_n^+$. Fix a basis (ω_i) of $H^*(M)$ such that each ω_i is homogeneous of degree $|\omega_i| = d_i$ and denote by ω^i the basis of $H^*(M)$ dual to (ω_i) with respect to the Kronecker pairing i.e.

$$\langle\omega^i,\omega_j\rangle_\kappa = (-1)^{|\omega^i|\cdot|\omega_j|}\langle\omega_j,\omega^i\rangle_\kappa = \delta^i_j.$$

In $M\times M$ there exists a remarkable cycle, the *diagonal*

$$\Delta : M\to M\times M \quad x\mapsto (x,x).$$

We claim that the Poincaré dual of this cycle is

$$\delta_\Delta = \mathfrak{d}_M \stackrel{def}{=} \sum_i (-1)^{|\omega^i|}\omega^i\times\omega_i. \tag{7.3.2}$$

Indeed, for any homogeneous forms $\alpha,\beta\in\Omega^*(M)$ such that $|\alpha|+|\beta| = n$ we have

$$\int_{M\times M}(\omega\times\eta)\wedge\mathfrak{d}_M = \sum_i(-1)^{|\omega^i|}\int_{M\times M}(\alpha\times\beta)\wedge(\omega^i\times\omega_i)$$

$$= \sum_i(-1)^{|\omega^i|}(-1)^{|\beta|\cdot|\omega^i|}\int_{M\times M}(\alpha\wedge\omega^i)\times(\beta\wedge\omega_i)$$

$$= \sum_i(-1)^{|\omega^i|}(-1)^{|\beta|\cdot|\omega^i|}\left(\int_M\alpha\wedge\omega^i\right)\left(\int_M\beta\wedge\omega_i\right).$$

The i-th summand is nontrivial only when $|\beta| = |\omega^i|$ and $|\alpha| = |\omega_i|$. Using the equality $|\omega^i| + |\omega^i|^2 \equiv 0 (\bmod\ 2)$ we deduce

$$\int_{M\times M}(\alpha\times\beta)\wedge\mathfrak{d}_M = \sum_i\left(\int_M\alpha\wedge\omega^i\right)\left(\int_M\beta\wedge\omega_i\right)$$

$$= \sum_i\langle\alpha,\omega^i\rangle_\kappa\langle\beta,\omega_i\rangle_\kappa.$$

From the equalities

$$\alpha = \sum_i\omega_i\langle\omega^i,\alpha\rangle_\kappa \quad \beta = \sum_j\langle\beta,\omega_j,\rangle_\kappa\omega^j$$

we conclude

$$\int_M \Delta^*(\alpha \times \beta) = \int_M \alpha \wedge \beta = \int_M (\sum_i \omega_i \langle \omega^i, \alpha \rangle_\kappa) \wedge (\sum_j \langle \beta, \omega_j \rangle_\kappa \omega^j)$$

$$= \int_M \sum_{i,j} \langle \omega^i, \alpha \rangle_\kappa \langle \beta, \omega_j \rangle_\kappa \omega_i \wedge \omega^j = \sum_{i,j} \langle \omega^i, \alpha \rangle_\kappa \langle \beta, \omega_j \rangle_\kappa \langle \omega_i, \omega^j \rangle_\kappa$$

$$= \sum_i (-1)^{|\omega_i|(n-|\omega_i|)} \langle \alpha, \omega^i \rangle_\kappa (-1)^{|\omega_i|(n-|\omega_i|)} \langle \beta, \omega_i \rangle_\kappa = \sum_i \langle \alpha, \omega^i \rangle_\kappa \langle \beta, \omega_i \rangle_\kappa.$$

Equality (7.3.2) is proved. □

Proposition 7.3.7. *Let $M \in \mathfrak{M}_n^+$ and $(S_i, \phi_i) \in \mathcal{C}_k(M)$ $(i = 0, 1)$ - two cycles in M.*
(a) If $(S_0, \phi_0) \sim_c (S_1, \phi_1)$ then $\delta_{S_0} = \delta_{S_1}$ in $H_{cpt}^{n-k}(M)$
(b) If (S_0, ϕ_0) is degenerate then $\delta_{S_0} = 0$ in $H_{cpt}^{n-k}(M)$.
(c) $\delta_{S_0 \sqcup S_1} = \delta_{S_0} + \delta_{S_1}$ in $H_{cpt}^{n-k}(M)$.
(d) $\delta_{-S_0} = -\delta_{S_0}$ in $H_{cpt}^{n-k}(M)$.

Proof (a) Consider a compact manifold Σ with boundary $\partial\Sigma = S_0 \sqcup S_1$ and a smooth map $\Phi : \Sigma \to M$ such that $\Phi|_{\partial\Sigma} = \phi_0 \sqcup \phi_1$. For any closed k-form $\omega \in \Omega^k(M)$ we have

$$0 = \int_\Sigma \Phi^*(d\omega) = \int_\Sigma d\Phi^*\omega$$

$$\stackrel{Stokes}{=} \int_{\partial\Sigma} \omega = \int_{S_1} \phi_1^*\omega - \int_{S_0} \phi_0^*\omega.$$

Part (b) is left to the reader. Part (c) is obvious. To prove (d) consider $\Sigma = [0, 1] \times S_0$ and $\Phi : [0, 1] \times S_0 \to M$, $\Phi(t, x) = \phi_0(x)$ $\forall (t, x) \in \Sigma$. Note that $\partial\Sigma = (-S_0) \sqcup S_0$ so that

$$\delta_{-S_0} + \delta_{S_0} = \delta_{-S_0 \sqcup S_0} = \delta_{\partial\Sigma} = 0.$$ □

The above proposition shows that the correspondence

$$\mathcal{C}_k(M) \ni (S, \phi) \mapsto \delta_S$$

descends to a map

$$\delta : \mathcal{H}_k(M) \to H_{cpt}^{n-k}(M).$$

This is usually called the *homological Poincaré duality.*

7.3.2 Intersection theory

Consider $M \in \mathfrak{M}_n^+$ and S a k-dimensional compact oriented submanifold of M. We denote by $\imath : S \hookrightarrow M$ inclusion map so that $(S, \imath)$ is a k-cycle.

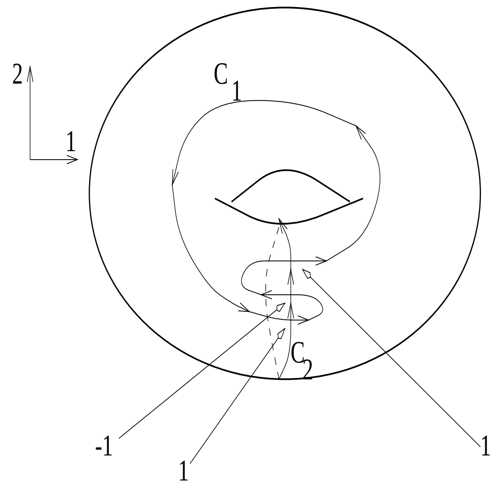

Figure 7.3: *The intersection number of the two cycles on T^2 is 1*

Definition 7.3.8. *A smooth map $\phi : T \to M$ from an $(n-k)$-dimensional, oriented manifold T is said to be* transversal *to S (and we write this $S \pitchfork \phi$) if*
(a) $\phi^{-1}(S)$ is a finite subset of T;
(b) for every $x \in \phi^{-1}(S)$ we have

$$\phi_*(T_xT) + T_{\phi(x)}S = T_{\phi(x)}M \quad \text{(direct sum)}.$$

In this case, for each $x \in \phi^{-1}(S)$ we define the local intersection number *at x to be (* or= *orientation)*

$$i_x(S,T) = \begin{cases} 1 & , \quad \mathbf{or}\,(T_{\phi(x)}S) \wedge \mathbf{or}(\phi_* T_xT) = \mathbf{or}(T_{\phi(x)}M) \\ -1 & , \quad \mathbf{or}(T_{\phi(x)}S) \wedge \mathbf{or}(\phi_* T_xT) = -\mathbf{or}(T_{\phi(x)}M) \end{cases}.$$

Finally, we define the intersection number *of S with T to be*

$$S \cdot T = \sum_{x \in \phi^{-1}(S)} i_x(S,T).$$

Our next result offers a different description of the intersection number indicating how one can drop the transversality assumption from the original definition.

Proposition 7.3.9. *Let $M \in \mathfrak{M}_n^+$. Consider $S \hookrightarrow M$ a compact, oriented, k-dimensional submanifold and $(T,\phi) \in \mathcal{C}_{n-k}(M)$ a $(n-k)$-dimensional cycle intersecting S transversally, i.e. $S \pitchfork \phi$. Then*

$$S \cdot T = \int_M \delta_S \wedge \delta_T, \tag{7.3.3}$$

where $\delta_\bullet$ denotes the Poincaré dual of $\bullet$.

The proof of the proposition relies on a couple of technical lemmata of independent interest.

Lemma 7.3.10. (Localization lemma) *Let $M \in \mathfrak{M}_n^+$ and $(S, \phi) \in \mathcal{C}_k(M)$. Then for any neighborhood $\mathcal{N}$ of $\phi(S)$ in M there exists $\delta_S^{\mathcal{N}} \in \Omega_{cpt}^{n-k}(M)$ such that*
(a) $\delta_S^{\mathcal{N}}$ represents the Poincaré dual $\delta_S \in H_{cpt}^{n-k}(M)$;
(b) $\operatorname{supp} \delta_S^{\mathcal{N}} \subset \mathcal{N}$.

Proof Fix a Riemann metric on M. Each point $p \in \phi(S)$ has a geodesically convex neighborhood entirely contained in $\mathcal{N}$. Cover $\phi(S)$ by finitely many such neighborhoods and denote by N their union. Then $N \in \mathfrak{M}_n^+$ and $(S, \phi) \in \mathfrak{M}_n^+(N)$. Denote by δ_S^N the Poincaré dual of S in N. It can be represented by a closed form in $\Omega_{cpt}^{n-k}(N)$ which we continue to denote by δ_S^N. If now we pick a closed form $\omega \in \Omega^k(M)$ then $\omega|_N$ is also closed and

$$\int_M \delta_S^N \wedge \omega = \int_N \delta_S^N \wedge \omega = \int_S \phi^* \omega.$$

Hence, δ_S^N represents the Poincaré dual of S in $H_{cpt}^{n-k}(M)$ and moreover, $\operatorname{supp} \delta_S^N \subset \mathcal{N}$. □

Definition 7.3.11. *Let $M \in \mathfrak{M}_n^+$ and $S \hookrightarrow M$ a compact, k-dimensional, oriented submanifold of M. A* local transversal *at $p \in S$ is an embedding*

$$\phi : B \subset \mathbb{R}^{n-k} \to M \quad (B = \text{open ball centered at } 0)$$

such that $S \pitchfork \phi$ and $\phi^{-1}(S) = \{0\}$.

Lemma 7.3.12. *Let $M \in \mathfrak{M}_n^+$, $S \hookrightarrow M$ a compact, k-dimensional, oriented submanifold of M and (B, ϕ) a local transversal at $p \in S$. Then for any sufficiently "thin" closed neighborhood $\mathcal{N}$ of $S \subset M$ we have*

$$S \cdot (B, \phi) = \int_B \phi^* \delta_S^{\mathcal{N}}.$$

Proof Using the transversality $S \pitchfork \phi$, the implicit function theorem and eventually restricting ϕ to a smaller ball, we deduce that (for some sufficiently "thin" neighborhood $\mathcal{N}$ of S) there exist local coordinates $(x^1, \ldots, x^n)$ defined on some neighborhood U of $p \in M$ diffeomorphic with the cube

$$\{|x^i| < 1, \ \forall i\}$$

such that
(i)$S \cap U = \{x^{k+1} = \cdots = x^n = 0\}$, $p = (0, \ldots, 0)$
(ii) The orientation of $S \cap U$ is defined by $dx^1 \wedge \cdots \wedge dx^k$.
(iii) The map $\phi : B \subset \mathbb{R}^{n-k}_{(y^1, \ldots, y^{n-k})} \to M$ is expressed in these coordinates as

$$x^1 = 0, \ldots, x^k = 0, x^{k+1} = y^1, \ldots, x^n = y^{n-k}.$$

(iv) $\mathcal{N} \cap U = \{|x^j| \leq 1/2\ ;\ j = 1, \ldots, n\}$.

Let $\epsilon = \pm 1$ such that $\epsilon dx^1 \wedge \ldots \wedge dx^n$ defines the orientation of TM. In other words

$$\epsilon = S \cdot (B, \phi).$$

For each $\xi = (x^1, \dots, x^k) \in S \cap U$ denote by P_ξ the $(n-k)$-"plane"

$$P_\xi = \{(\xi; x^{k+1}, \dots, x^n)\ ;\ |x^j| < 1\ \ j > k\}.$$

We orient each P_ξ using the $(n-k)$-form $dx^{k+1} \wedge \cdots \wedge dx^n$ and set

$$v(\xi) = \int_{P_\xi} \delta_S^{\mathcal{N}}.$$

Equivalently,

$$v(\xi) = \int_B \phi_\xi^* \delta_S^{\mathcal{N}},$$

where $\phi_\xi : B \to M$ is defined by

$$\phi_\xi(y^1, \dots, y^{n-k}) = (\xi; y^1, \dots, y^{n-k}).$$

To any function $\varphi = \varphi(\xi) \in C^\infty(S \cap U)$ such that

$$\operatorname{supp} \varphi \subset \{|x^i| \leq 1/2\ ;\ i \leq k\}$$

we associate the k-form

$$\omega_\varphi = \varphi dx^1 \wedge \cdots \wedge dx^k = \varphi d\xi \in \Omega^k_{cpt}(S \cap U).$$

Extend the functions $x^1, \dots, x^k \in C^\infty(U \cap \mathcal{N})$ to smooth compactly supported functions

$$\tilde{x}^i \in C_0^\infty(M) \to [0, 1]$$

The form ω_φ is then the restriction to $U \cap S$ of the closed compactly supported form

$$\tilde{\omega}_\varphi = \varphi(\tilde{x}^1, \dots, \tilde{x}^k) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^k.$$

We have

$$\int_M \tilde{\omega}_\varphi \wedge \delta_S^{\mathcal{N}} = \int_U \omega_\varphi \wedge \delta_S^{\mathcal{N}} = \int_S \omega_\varphi = \int_{\mathbb{R}^k} \varphi(\xi) d\xi. \tag{7.3.4}$$

The integral over U can be evaluated using the Fubini theorem. Write

$$\delta_S^{\mathcal{N}} = f dx^{k+1} \wedge \cdots \wedge dx^n + \varrho$$

where ϱ is an $(n-k)$-form not containing the monomial $dx^{k+1} \wedge \cdots \wedge dx^n$. Then

$$\int_U \omega_\varphi \wedge \delta_S^{\mathcal{N}} = \int_U f\varphi dx^1 \wedge \cdots \wedge dx^n$$

$$= \epsilon \int_U f\varphi |dx^1 \wedge \cdots \wedge dx^n| \quad (|dx^1 \wedge \cdots \wedge dx^n| = \text{Lebesgue density})$$

$$\stackrel{Fubini}{=} \epsilon \int_{S \cap U} \varphi(\xi) \left(\int_{P_\xi} f |dx^{k+1} \wedge \cdots \wedge dx^n| \right) |d\xi|$$

$$= \epsilon \int_S \varphi(\xi) \left(\int_{P_\xi} \delta_S^{\mathcal{N}} \right) |d\xi| = \epsilon \int_S \varphi(\xi) v(\xi) |d\xi|.$$

Comparing with (7.3.4) and taking into account that φ was chosen arbitrarily we deduce

$$v(0) = \epsilon.$$

If in the above equality we use the assumption (iii) we recognize in the left-hand-side the integral

$$\int_B \phi^* \delta_S^{\mathcal{N}}.$$

The local transversal lemma is proved. □

Proof of Proposition 7.3.9 Let

$$\phi^{-1}(S) = \{p_1, \ldots, p_m\}.$$

The transversality assumption implies that each p_i has an open neighborhood B_i diffeomorphic to an open ball such that $\phi_i = \phi|_{B_i}$ is a local transversal at $y_i = \phi(x_i)$. Moreover, we can choose the neighborhoods B_i to be mutually disjoint. Then

$$S \cdot T = \sum_i S \cdot (B_i, \phi_i).$$

The compact set

$$K = \phi(T \setminus \cup D_i)$$

does not intersect S so that we can find a "thin", closed neighborhood" $\mathcal{N}$ of $S \hookrightarrow M$ such that $K \cap \mathcal{N} = \emptyset$. Then $\phi^* \delta_S^{\mathcal{N}}$ is compactly supported in $\cup D_i$

$$\int_M \delta_S^{\mathcal{N}} \wedge \delta_T = \int_T \phi^* \delta_S^{\mathcal{N}}$$

$$= \sum_i \int_{D_i} \phi_i^* \delta_S^{\mathcal{N}}.$$

From the local transversal lemma we get

$$\int_{D_i} \phi^* \delta_S^{\mathcal{N}} = (-1)^{k(n-k)} S \cdot (B_i, \phi) = i_{p_i}(S, T).$$ □

Equality (7.3.3) has a remarkable feature. Its right-hand-side is an integer but is defined only for cycles S, T such that S is embedded and $S \pitchfork T$, while the left-hand-side makes sense for any cycles of complementary dimensions but a priori it may not be an integer. In any event, we have a remarkable consequence.

Corollary 7.3.13. *Let $(S_i, \phi_i) \in \mathcal{C}_k(M)$ and $((T_i, \psi_i) \in \mathcal{C}_{n-k}(M)$ where $M \in \mathfrak{M}_n^+$, $i = 0, 1$. If*
(a) $S_0 \sim_c S_1$, $T_0 \sim_c T_1$,
(b) the cycles S_i are embedded and
(c) $S_i \pitchfork T_i$
Then

$$S_0 \cdot T_0 = S_1 \cdot T_1.$$

Definition 7.3.14. *The* homological intersection pairing *is the $\mathbb{Z}$-bilinear map*

$$\mathcal{I} = \mathcal{I}_M : \mathcal{H}_k(M) \times \mathcal{H}_{n-k}(M) \to \mathbb{R}$$

($M \in \mathfrak{M}_n^+$) defined by

$$\mathcal{I}(S,T) = \int_M \delta_S \wedge \delta_T.$$

We have proved that in some special instances $\mathcal{I}(S,T) \in \mathbb{Z}$. We want to prove that when M is *compact* this is *always* the case.

Theorem 7.3.15. *Let $M \in \mathfrak{M}_n^+$ be* compact *manifold. Then for any $(S,T) \in \mathcal{H}_k(M) \times \mathcal{H}_{n-k}(M)$ the intersection number $\mathcal{I}(S,T)$ is always an integer.*

The theorem will follow from two lemmata. The first one will show that it suffices to consider only the situation when one of the two cycles is embedded. The second one will show that if one of the cycles is embedded then the second cycle can be deformed so that it intersects the former transversally. (This is called a *general position result.*)

Lemma 7.3.16. (Diagonal trick) *Let M, S and T as in Theorem 7.3.15. Then*

$$\mathcal{I}(S,T) = (-1)^{n-k} \int_{M \times M} \delta_{S \times T} \wedge \delta_\Delta$$

where Δ is the diagonal cycle $\Delta : M \to M \times M$, $x \mapsto (x,x)$. (It is here where the compactness of M is essential, since otherwise Δ would not be a cycle).

Proof We will use the equality 7.3.1

$$\delta_{S \times T} = (-1)^{n-k} \delta_S \times \delta_T.$$

Then

$$(-1)^{n-k} = \int_{M \times M} \delta_{S \times T} \wedge \delta_\Delta = \int_{M \times M} (\delta_S \times \delta_T) \wedge \delta_\Delta$$

$$= \int_M \Delta^*(\delta_S \times \delta_T) = \int_M \delta_S \wedge \delta_T. \qquad \square$$

Lemma 7.3.17. (Moving lemma) *Let $S \in \mathcal{C}_k(M)$ and $T \in \mathcal{C}_{n-k}(M)$ be two cycles in $M \in \mathfrak{M}_n^+$. If S is embedded then T is cobordant to a cycle $\tilde{T}$ such that $S \pitchfork \tilde{T}$.*

The proof of this result relies on Sard's theorem. For details we refer to [35, Chap.3].

Proof of Theorem 7.3.15 Let $(S,T) \in \mathcal{C}_k(M) \times \mathcal{C}_{n-k}(M)$. Then

$$\mathcal{I}(S,T) = (-1)^{n-k} \mathcal{I}(S \times T, \Delta).$$

Since Δ is embedded we may assume by the moving lemma that $(S \times T) \pitchfork \Delta$ so that $\mathcal{I}(S \times T, \Delta) \in \mathbb{Z}$. $\square$

7.3.3 The topological degree

Consider two compact, connected, oriented smooth manifolds M, N having the same dimension n. Any smooth map $F : M \to N$ canonically defines an n-dimensional cycle in $M \times N$

$$\Gamma_F : M \to M \times N \quad x \mapsto (x, F(x)).$$

Γ_F is called the *graph of F*.

Any point $y \in N$ defines an n-dimensional cycle $M \times \{y\}$. Since N is connected all these cycles are cobordant so that the integer $\Gamma_F \cdot (M \times \{y\})$ is independent of y.

Definition 7.3.18. *The* topological degree *of the map F is defined by*

$$\deg F \stackrel{def}{=} (M \times \{y\}) \cdot \Gamma_F.$$

Note that the intersections of Γ_F with $M \times \{y\}$ correspond to the solutions of the equation $F(x) = y$. Thus the topological degree counts these solutions (with sign).

Proposition 7.3.19. *Let $F : M \to N$ as above. Then for any n-form $\omega \in \Omega^n(N)$*

$$\int_M F^*\omega = \deg F \int_N \omega.$$

Remark 7.3.20. The map F induces a morphism

$$\mathbb{R} \cong H^n(N) \stackrel{F^*}{\to} H^n(M) \cong \mathbb{R}$$

which can be identified with a real number. The above proposition guarantees this number is an integer. □

Proof of the proposition Note that if $\omega \in \Omega^n(N)$ is exact then

$$\int_N \omega = \int_M F^*\omega = 0.$$

Thus, to prove the proposition it suffices to check it for any particular form which generates $H^n(N)$. Our candidate will be the Poincaré dual δ_y of a point $y \in N$. We have

$$\int_N \delta_y = 1$$

while equality (7.3.1) gives

$$\delta_{M \times \{y\}} = \delta_M \times \delta_y = 1 \times \delta_y.$$

We can then compute the degree of F using Theorem 7.3.15

$$\deg F = \deg F \int_N \delta_y = \int_{M \times N} (1 \times \delta_y) \wedge \delta_{\Gamma_F}$$

$$= \int_M \Gamma_F^*(1 \times \delta_y) = \int_M F^*\delta_y.$$ □

Corollary 7.3.21. (Gauss-Bonnet) *Consider a connected sum of g-tori $\Sigma = \Sigma_g$ embedded in $\mathbb{R}^3$ and let $\mathcal{G}_\Sigma : \Sigma \to S^2$ be its Gauss map. Then*

$$\deg \mathcal{G}_\Sigma = \chi(\Sigma) = 2 - 2g.$$

This corollary follows immediately from the considerations at the end of Subsection 4.2.4.

Exercise 7.3.3. Consider the compact, connected manifolds $M_0, M_1, N \in \mathfrak{M}_n^+$ and the smooth maps $F_i : M_i \to N$, $i = 0, 1$. Show that if F_0 is cobordant to F_1 then $\deg F_0 = \deg F_1$. In particular, homotopic maps have the same degree. □

Exercise 7.3.4. Let A be a nonsingular $n \times n$ real matrix. It defines a smooth map

$$F_A : S^{n-1} \to S^{n-1} \quad x \mapsto \frac{Ax}{|Ax|}.$$

Prove that $\deg F_A = \text{sign}\det A$.
Hint: Use the polar decomposition $A = P \cdot O$ (where P is a positive symmetric matrix and O is an orthogonal one) to deform A inside $GL(n, \mathbb{R})$ to a diagonal matrix. □

Exercise 7.3.5. Let $M \stackrel{F}{\to} N$ be a smooth map (M, N are smooth, compact oriented of dimension n). Assume $y \in N$ is a regular value of F i.e. for all $x \in F^{-1}(y)$ the derivative

$$D_x F : T_x M \to T_y N$$

is invertible. For $x \in F^{-1}(y)$ define

$$\deg(F, x) = \begin{cases} 1 & , \ D_x F \text{ preserves orientations} \\ -1 & , \ \text{otherwise} \end{cases}$$

Prove that

$$\deg F = \sum_{F(x)=y} \deg(F, x).$$

□

Exercise 7.3.6. Let M denote a compact oriented manifold and consider a smooth map

$$F : M \to M.$$

Regard $H^*(M)$ as a superspace with the obvious $\mathbb{Z}_2$-grading

$$H^*(M) = H^{even}(M) \oplus H^{odd}(M)$$

and define the *Lefschetz number* $\lambda(F)$ of F as the supertrace of the pull back $F^* : H^*(M) \to H^*(M)$. Prove that

$$\lambda(F) = \Delta \cdot \Gamma_F.$$

and deduce from this the Lefschetz fixed point theorem: $\lambda(F) \neq 0 \Rightarrow F$ has a fixed point. □

7.3.4 Thom isomorphism theorem

Let $p : E \to B$ be an orientable fiber bundle with standard fiber F and compact, oriented basis B. Let $\dim B = m$ and $\dim F = r$.

In this subsection we will extensively use the techniques of fibered calculus described in Subsection 3.4.5. The integration along fibers

$$\int_{E/B} = p_* : \Omega^*_{cpt}(E) \to \Omega^{*-r}(B)$$

satisfies

$$p_* d_E = (-1)^r d_B p_*$$

so that it induces a map in cohomology

$$p_* : H^*_{cpt}(E) \to H^{*-r}(B).$$

This induced map in cohomology is sometimes called the *Gysin map*.

Exercise 7.3.7. Consider a smooth map $f : M \to N$ between compact, oriented manifolds M, N of dimensions m and respectively n. Denote by i_f the embedding of M in $M \times N$ as the graph of f

$$M \ni x \mapsto (x, f(x)) \in M \times N.$$

The natural projection $M \times N \to N$ allows us to regard $M \times N$ as a trivial fiber bundle over N. Show that the push-forward map $f_* : H^*(M) \to H^{*+n-m}(N)$ defined in Remark 7.2.9 can be equivalently defined by

$$f_* = \pi_* \circ (i_f)_*$$

where π_* denotes the integration along fibers while

$$(i_f)_* : H^*_{cpt}(M) \to H^*_{cpt}(N)$$

is the natural morphism defined by the *embedding* i_f. □

Let us return to the fiber bundle $p : E \to B$. Any smooth section

$$\sigma : B \to E$$

defines an embedded cycle in E of dimension $m = \dim B$. Denote by δ_σ its Poincaré dual in $H^r_{cpt}(E)$. Using the properties of the integration along fibers we deduce that for any $\omega \in \Omega^m(B)$ we have

$$\int_E \delta_\sigma \wedge p^*\omega = \int_B \left(\int_{E/B} \delta_\sigma \right) \omega.$$

On the other hand by Poincaré duality we get

$$\int_E \delta_\sigma \wedge p^*\omega = (-1)^{rm} \int_E p^*\omega \wedge \delta_\sigma$$

$$= (-1)^{rm} \int_B \sigma^* p^* \omega = (-1)^{rm} \int_B (p\sigma)^* \omega = (-1)^{rm} \int_B \omega.$$

Hence

$$p_* \delta_\sigma = \int_{E/B} \delta_\sigma = (-1)^{rm} \in \Omega^0(B).$$

Proposition 7.3.22. *Let $p : E \to B$ a bundle as above. If it admits at least one section then the Gysin map*

$$p_* : H^*_{cpt}(E) \to H^{*-r}(B)$$

is surjective.

Proof Denote by τ_σ the map

$$\tau_\sigma : H^*(B) \to H^{*+r}_{cpt}(E) \quad \omega \mapsto (-1)^{rm} \delta_\sigma \wedge p^* \omega = p^* \omega \wedge \delta_\sigma.$$

Then τ_σ is a right inverse for p_*. Indeed

$$\omega = (-1)^{rm} p_* \delta_\sigma \wedge \omega = (-1)^{rm} p_* (\delta_\sigma \wedge p^* \omega) = p_*(\tau_\sigma \omega). \qquad \square$$

The map p_* is not injective in general. For example, if (S, ϕ) is a k-cycle in F, then it defines a cycle in any fiber $\pi^{-1}(b)$ an consequently in E. Denote by δ_S its Poincaré dual in $H^{m+r-k}_{cpt}(E)$. Then for any $\omega \in \Omega^{m-k}(B)$ we have

$$\int_B (p_* \delta_S) \wedge \omega = \int_E \delta_S \wedge p^* \omega$$

$$= \pm \int_D \phi^* p^* \omega = \int_S (p \circ \phi)^* \omega = 0$$

since $p \circ \phi$ is constant. Hence $p_* \delta_S = 0$. Hence if F carries nontrivial cycles $\ker p_*$ may not be trivial.

The simplest example of standard fiber with only trivial cycles is a vector space.

Definition 7.3.23. *Let $p : E \to B$ be an orientable vector bundle over the compact oriented manifold B ($\dim B = m$, $\operatorname{rank}(E) = r$). The* Thom class *of E, denoted by τ_E is the Poincaré dual of the cycle defined by the zero section $\zeta_0 : B \to E$, $b \mapsto 0 \in E_b$. Note that $\tau_E \in H^r_{cpt}(E)$.*

Theorem 7.3.24. (Thom isomorphism) *Let $p : E \to B$ as in the above definition. Then the map*

$$\tau : H^*(B) \to H^{*+r}_{cpt}(E) \quad \omega \mapsto \tau_E \wedge p^* \omega$$

is an isomorphism called the Thom isomorphism. *Its inverse is the Gysin map*

$$(-1)^{rm} p_* : H^*_{cpt}(E) \to H^{*-r}(B).$$

Proof We have already established that

$$p_* \tau = (-1)^{rm}$$

To prove the reverse equality $\tau p_* = (-1)^{rm}$ we will use Lemma 7.2.2 of Subsection 7.2.1. For $\beta \in \Omega^*_{cpt}(E)$ we have

$$(p^* p_* \tau_E) \wedge \beta - \tau_E \wedge (p^* p_* \beta) = (-1)^r d(\mathfrak{m}(\tau_E, \beta))$$

where $\mathfrak{m}(\tau_E, \beta) \in \Omega^*_{cpt}(E)$. Since $p^* p_* \tau_E = (-1)^{rm}$ we deduce

$$(-1)^{rm} \beta = (\tau_E \wedge p^*(p_* \beta) + \text{exact form},$$

i.e.

$$(-1)^{rm} \beta = \tau_E \circ p_*(\beta) \ \text{ in } H^*_{cpt}(E). \qquad \square$$

Exercise 7.3.8. Show that $\tau_E = \zeta_* 1$ where $\zeta_* : H^*(M) \to H^{*+\dim M}_{cpt}(E)$ is the push-forward map defined by a section $\zeta : M \to E$. $\square$

7.3.5 Gauss-Bonnet revisited

We now examine a very special type of vector bundle: the tangent bundle of a compact, oriented, smooth manifold M. Note first the following fact.

Exercise 7.3.9. Prove that M is orientable if and only if TM is orientable as a bundle. $\square$

Definition 7.3.25. *Let $E \to M$ be a real orientable vector bundle over the compact, oriented n-dimensional smooth manifold. Denote by $\tau_E \in H^n_{cpt}(E)$ the Thom class of E. The* Euler class *of E is defined by*

$$\mathbf{e}(E) = \zeta_0^* \tau_E \in H^n(M),$$

where $\zeta_0 : M \to E$ denotes the zero section. $\mathbf{e}(TM)$ is called the Euler! class *of M and is denoted by $\mathbf{e}(M)$.*

Note that the sections of TM are precisely the vector fields on M. Moreover, any such section $\sigma : M \to TM$ tautologically defines an n-dimensional cycle in TM and in fact any two such cycles are homotopic - try an affine homotopy along the fibers of TM. Any two sections $\sigma_0, \sigma_1 : M \to TM$ determine cycles of complementary dimension and thus the intersection number $\sigma_0 \cdot \sigma_1$ is a well defined integer, independent of the two sections. It is a number reflecting the topological structure of the manifold.

Proposition 7.3.26. *Let $\sigma_0, \sigma_1 : M \to TM$ be two sections of TM. Then*

$$\int_M \mathbf{e}(M) = \sigma_0 \cdot \sigma_1.$$

In particular, if $\dim M$ *is odd then*

$$\int_M \mathbf{e}(M) = 0.$$

Proof The section σ_0, σ_1 are cobordant and their Poincaré dual in $H^n_{cpt}(TM)$ is the Thom class τ_M. Hence

$$\sigma_0 \cdot \sigma_1 = \int_{TM} \delta_{\sigma_0} \wedge \delta_{\sigma_1} = \int_{TM} \tau_M \wedge \tau_M$$
$$= \int_{TM} \tau_M \wedge \delta_{\zeta_0} = \int_M \zeta_0^* \tau_M = \int_M e(M).$$

If $\dim M$ is odd then

$$\int_M e(M) = \sigma_0 \cdot \sigma_1 = -\sigma_1 \cdot \sigma_0 = -\int_M e(M). \qquad \square$$

Theorem 7.3.27. *Let M be a compact oriented n-dimensional manifold and denote by $\mathbf{e}(M)$ its Euler class. Then the integral of $\mathbf{e}(M)$ over M is equal to the Euler characteristic of M,*

$$\int_M \mathbf{e}(M) = \chi(M) = \sum_{k=0}^{n} (-1)^k b_k(M).$$

In the proof we will use an equivalent description of $\chi(M)$.

Lemma 7.3.28. *Denote by Δ the diagonal cycle in $M \times M$. Then*

$$\chi(M) = \Delta \cdot \Delta.$$

Proof of the lemma Consider a basis (ω_j) of $H^*(M)$ consisting of homogeneous elements. We denote by (ω^i) the dual basis i.e.

$$\langle \omega^i, \omega_j \rangle_\kappa = \delta^i_j.$$

According to (7.3.2) we have

$$\delta_\Delta = \sum_i (-1)^{|\omega^i|} \omega^i \times \omega_i.$$

Similarly, if we start first with the basis (ω^i) then its dual basis is

$$(-1)^{|\omega^i| \cdot |\omega_i|} \omega_i$$

so we also have (taking into account that $|\omega_j| + |\omega^j| \cdot |\omega_j| \equiv n|\omega_j| \pmod 2$)

$$\delta_\Delta = \sum_i (-1)^{n \cdot |\omega_j|} \omega_j \times \omega^j.$$

Using Exercise 7.1.12 we deduce

$$\Delta \cdot \Delta = \int_{M \times M} \left(\sum_i (-1)^{|\omega^i|} \omega^i \times \omega_i \right) \wedge \left(\sum_j (-1)^{n|\omega_j|} \omega_j \times \omega^j \right)$$

$$= \int_{M\times M} \left(\sum_{i,j} (-1)^{|\omega^i|}(-1)n \cdot |\omega_j|(-1)^{|\omega_i|\cdot|\omega_j|}\omega^i \wedge \omega_j \right) \times \left(\omega_i \times \omega^j\right)$$

$$= \sum_{i,j} (-1)^{|\omega^i|}(-1)n \cdot |\omega_j|(-1)^{|\omega_i|\cdot|\omega_j|}\langle \omega_i, \omega_j\rangle_\kappa \, \langle \omega_i, \omega^j\rangle_\kappa .$$

In the last expression we now use the duality equations

$$\langle \omega_i, \omega^j\rangle_\kappa = (-1)^{|\omega_i|\cdot|\omega^j|}\delta_i^j$$

and the congruence

$$|\omega^i| + n|\omega_i| + |\omega_i|^2 + |\omega_i| \cdot ||\omega^i| \equiv |\omega^i| + n|\omega_i| + |\omega_i|^2 + |\omega_i|(n - |\omega_i|) \equiv |\omega^i| \pmod 2$$

to conclude that

$$\Delta \cdot \Delta = \sum_{\omega^i} (-1)^{|\omega^i|} = \chi(M). \qquad \square$$

Proof of theorem 7.3.27 The tangent bundle of $M \times M$ restricts to the diagonal Δ as a rank $2n$ vector bundle. If we choose a Riemann metric on $M \times M$ then we get an orthogonal splitting

$$T(M \times M)\,|_\Delta = N_\Delta \oplus T\Delta.$$

The diagonal map $M \to M \times M$ identifies M with Δ so that $T\Delta \cong TM$. We now have the following remarkable result.

Lemma 7.3.29. *$N_\Delta \cong TM$.*

Proof Use the isomorphisms

$$T(M \times M)\,|_\Delta \cong T\Delta \oplus N_\Delta \cong TM \oplus N_\Delta$$

and

$$T(M \times M)\,|_\Delta = TM \oplus TM. \qquad \square$$

From this lemma we immediately deduce the equality of Thom classes

$$\tau_{N_\Delta} = \tau_M. \tag{7.3.5}$$

At this point we want to invoke a technical result whose proof is left to the reader as an exercise in Riemann geometry.

Lemma 7.3.30. *Denote by* exp *the exponential map of a Riemann metric g on $M \times M$. Regard Δ as a submanifold in N_Δ via the embedding given by the zero section. Then there exists an open neighborhood $\mathcal{N}$ of $\Delta \subset N_\Delta \subset T(M \times M)$ such that*

$$\exp|_{\mathcal{N}} : \mathcal{N} \to M \times M$$

is an embedding.

Let $\mathcal{N}$ be a neighborhood of $\Delta \subset N_\Delta$ as in the above lemma and set $\mathcal{N} = \exp(\mathcal{N})$. We can view Δ as a submanifold of both $\mathcal{N}$ and $\mathcal{N}$. Denote by $\delta_\delta^{\mathcal{N}}$ (resp. $\delta_\Delta^{\mathcal{N}}$) the Poincaré dual of Δ in $\mathcal{N}$ (resp. $\mathcal{N}$). $\delta_\Delta^{\mathcal{N}}$ is the Thom class of $N_\Delta \to \Delta$ which in view of (7.3.5) means

$$\delta_\Delta^{\mathcal{N}} = \tau_{N_\Delta} = \tau_\Delta = \tau_M.$$

We get

$$\Delta \cdot \Delta = \int_\Delta \delta_\Delta^{\mathcal{N}} = \int_\Delta \delta_\Delta^{\mathcal{N}} = \int_\Delta \zeta_0^* \tau_\Delta$$
$$= \int_M \zeta_0^* \tau_M = \int_M e(M).$$

Hence

$$\int_M e(M) = \chi(M). \qquad \square$$

If M is a connected sum of g tori then we can rephrase the Gauss-Bonnet theorem as follows.

Corollary 7.3.31. *For any Riemann metric h on a connected sum of g-tori Σ_g we have*

$$\frac{1}{2\pi}\varepsilon(h) = \frac{1}{4\pi} s_h dv_h = \mathbf{e}(\Sigma_g) \text{ in } H^*(\Sigma_g).$$

The remarkable feature of the Gauss-Bonnet theorem is that once we choose a metric we can *explicitly* describe a representative of the Euler class in terms of the Riemann curvature. The same is true for any compact oriented even dimensional Riemann manifold. In this generality the result is known as Gauss-Bonnet-Chern and we will have more to say about it in the next chapter.

We now have a new interpretation of the Euler characteristic of a compact oriented manifold M.

Given a smooth vector field X on M, its "graph" in TM

$$\Gamma_X = \{(x, X(x)) \in T_xM \;;\; x \in M\}$$

is an n-dimensional submanifold of TM. The Euler characteristic is then the intersection number

$$\chi(M) = \Gamma_X \cdot M$$

where we regard M as a submanifold in TM via the embedding given by the zero section. In other words, the Euler characteristic counts (with sign) the zeroes of the vector fields on M. For example if $\chi(M) \neq 0$ this means that any vector field on M *must have a zero !* We have thus proved the following result.

Corollary 7.3.32. *If $\chi(M) \neq 0$ then the tangent bundle TM is nontrivial.*

The equality $\chi(S^{2n}) = 2$ is particularly relevant in the vector field problem discussed in Subsection 2.1.4. Using the notations of that subsection we can write

$$v(S^{2n}) = 0.$$

We have thus solved "half" the vector field problem.

Exercise 7.3.10. Let X be a vector field over the compact oriented manifold M. A point $x \in M$ is said to be a *non-degenerate* zero of X if $X(0) = 0$ and

$$\det\left(\frac{\partial X_i}{\partial x^j}\right)|_{x=x_0} \neq 0.$$

for some local coordinates (x^i) near x_0 such that the orientation of $T^*_{x_0}M$ is given by $dx^1 \wedge \cdots \wedge dx^n$. Prove that the local intersection number of Γ_X with M at x_0 is given by

$$i_{x_0}(\Gamma_X, M) = \text{sign}\det\left(\frac{\partial X_i}{\partial x^j}\right)|_{x=x_0}.$$

(This is sometimes called the *local index* of X at x_0 and is denoted by $i(X, x_0)$. □

From the above exercise we deduce the following celebrated result.

Corollary 7.3.33. (Poincaré-Hopf) *If X is a vector field along a compact, oriented manifold M with only non-degenerated zeros $x_1, \ldots, x_k$ then*

$$\chi(M) = \sum_j i(X, x_j).$$

Exercise 7.3.11. Let X be a vector field on $\mathbb{R}^n$ and having a non-degenerate zero at the origin.
(a) prove that for all $r > 0$ sufficiently small X has no zeros on $S_r = \{|x| = r\}$.
(b) Consider $F_r : S_r \to S^{n-1}$ defined by

$$F_r(x) = \frac{1}{|X(x)|}X(x).$$

Prove that $i(X, 0) = \deg F_r$ for all $r > 0$ sufficiently small.
Hint: Deform X to a linear vector field. □

7.4 Symmetry and topology

The symmetry properties of a manifold have a great impact on its global (topological) structure. We devote this section to presenting some aspects of this avenue.

7.4.1 Symmetric spaces

Definition 7.4.1. *A* homogeneous space *is a smooth manifold M acted transitively by a Lie group G called the* symmetry group.

Recall that a left action

$$G \times M \to M \quad (g, m) \mapsto g \cdot m$$

is called transitive if for any $m \in M$ the map

$$\Psi_m : G \ni g \mapsto g \cdot m \in M$$

is surjective. For any point x of a homogeneous space M we define the *isotropy group* at x by

$$\mathcal{I}_x = \{g \in G \,;\, g \cdot x = x\}.$$

Lemma 7.4.2. *Let M be a homogeneous space with symmetry group G and $x, y \in M$. Then*
(a) $\mathcal{I}_x$ is a closed subgroup of G;
(b) $\mathcal{I}_x \cong \mathcal{I}_y$.

Proof (a) is immediate. To prove (b), choose $g \in G$ such that $y = g \cdot x$. Then note that

$$\mathcal{I}_y = g \mathcal{I}_x g^{-1}.$$

□

Remark 7.4.3. It is worth mentioning some fundamental results in the theory of Lie groups which will shed a new light on the considerations of this section. Their proofs can be found in the monograph [73].

FACT 1. Any closed subgroup of a Lie group is also a Lie group. In particular, the isotropy groups $\mathcal{I}_x$ of a homogeneous space are all Lie groups. They are smooth submanifolds of the symmetry group.

FACT 2. Let G be a closed group and H a closed subgroup. Then the space of left cosets

$$G/H = \{g \cdot H \,;\, g \in G\}$$

can be given a smooth structure such that the map

$$G \times (G/H) \to G/H \quad (g_1, g_2H) \mapsto (g_1 g_2) \cdot H$$

is smooth. G/H becomes a homogeneous space with symmetry group G. All isotropy groups are isomorphic to H.

FACT 3. If M is a homogeneous space with symmetry group G and $x \in M$ then M is equivariantly diffeomorphic to $G/\mathcal{I}_x$, i.e. there exists a diffeomorphism

$$\phi : M \to G/\mathcal{I}_x$$

such that $\phi(g \cdot y) = g \cdot \phi(y)$. □

We will be mainly interested in a very special class of homogeneous spaces.

Definition 7.4.4. *A* symmetric space *is a collection of data $(M, h, G, \sigma, \mathfrak{i})$ satisfying the following conditions.*
(a) (M, h) is a Riemann manifold.
(b) G is a connected Lie group acting isometrically and transitively on M

$$G \times M \ni (g, m) \mapsto g \cdot m \in M.$$

(c) $\sigma : M \times M \to M$ *is a smooth map* $(m_1, m_2) \mapsto \sigma_{m_1}(m_2)$ *such that*
(c1) $\forall m \in M$ $\sigma_m : M \to M$ *is an isometry and* $\sigma_m(m) = m$.
(c2) $\sigma_m \circ \sigma_m = \mathbf{1}_M$.
(c3) $D\sigma_m|_{T_mM} = -\mathbf{1}_{T_mM}$.
(c4) $\sigma_{gm} = g\sigma_m g^{-1}$

(d) $\mathfrak{i} : M \times G \to G$, $(m, g) \mapsto \mathfrak{i}_m g$ *is a smooth map such that*
(d1) $\forall m \in M$, $\mathfrak{i}_m : G \to G$ *is a homomorphism of* G.
(d2) $\mathfrak{i}_m \circ \mathfrak{i}_m = \mathbf{1}_G$.
(d3) $\mathfrak{i}_{gm} = g\mathfrak{i}_m g^{-1}$, $\forall (m, g) \in M \times G$.

(e) $\sigma_m g \sigma_m^{-1}(x) = \sigma_m g \sigma_m(x) = \mathfrak{i}_m(g) \cdot x$, $\forall m, x \in M$, $g \in G$.

Remark 7.4.5. This may not be the most elegant definition of a symmetric space and certainly it is not the minimal one. Optimizing it will require a substantial amount of work and we refer to [34] for an extensive presentation of this subject. This definition has one advantage. It lists all the properties we need to establish the topological results of this section. □

The next exercise offers the reader a feeling of what symmetric spaces are all about. In particular, it describes the geometric significance of the family of involutions σ_m.

Exercise 7.4.1. Let (M, h) be a symmetric space. Denote by ∇ the Levi-Civita connection and by R the Riemann curvature tensor.
(a) Prove that $\nabla R = 0$.
(b) Fix $m \in M$ and let $\gamma(t)$ be a geodesic of M such that $\gamma(0) = m$. Show that $\sigma_m\gamma(t) = \gamma(-t)$. □

Example 7.4.6. Perhaps the most popular example of symmetric space is the round sphere $S^n \subset \mathbb{R}^{n+1}$. The symmetry group is $SO(n+1)$ - the group of orientation preserving "rotations" of $\mathbb{R}^{n+1}$. For each $m \in S^{n+1}$ we denote by σ_m the orthogonal reflection through the 1-dimensional space determined by the radius Om. We then set $\mathfrak{i}_m(T) = \sigma_m T \sigma_m^{-1}$, $\forall T \in SO(n+1)$. We let the reader check that σ and $\mathfrak{i}$ satisfy all the required axioms. □

Example 7.4.7. Let G be a connected Lie group and $\mathfrak{m}$ a bi-invariant Riemann metric on G. The direct product $G \times G$ acts on G by

$$(g_1, g_2) \cdot h = g_1 h g_2^{-1}.$$

This action is clearly transitive and since $\mathfrak{m}$ is bi-invariant its is also isometric. Define

$$\sigma : G \times G \to G \quad \sigma_g h = gh^{-1}g^{-1}$$

and

$$\mathfrak{i} : G \times (G \times G) \to G \times G \quad \mathfrak{i}_g(g_1, g_2) = (gg_1g^{-1}, gg_2g^{-1}).$$

We leave the reader check that these data define a symmetric space structure on $(G, \mathfrak{m})$. The symmetry group is $G \times G$. □

Example 7.4.8. Consider the complex grassmannian $M = G_k(n, \mathbb{C})$. Any element $S \in M$ can be identified with a rank k, selfadjoint idempotent

$$P : \mathbb{C}^n \to \mathbb{C}^n \quad P^* = P = P^2.$$

$P = P_S$ is the orthogonal projection onto the k-dimensional subspace $S \subset \mathbb{C}^n$. Denote by Sym_n the linear space of selfadjoint $n \times n$ complex matrices. The map

$$M \ni S \mapsto P_S \in Sym_n$$

is an embedding of M in Sym_n. The linear space Sym_n has a natural metric

$$g_0(A, B) = \mathfrak{Re}\, \mathrm{tr}\,(AB^*).$$

It defines by restriction a Riemann metric g_0 on M.

The unitary group $U(n)$ acts on Sym_n by

$$A \mapsto T \star A \stackrel{de}{=} TAT^* \quad T \in U(n)\ A \in Sym_n.$$

Note that $U(n) \star M = M$ and g_0 is $U(n)$-invariant. Thus $U(n)$ acts transitively and isometrically on M.

For each subspace $S \in M$ define

$$R_S = P_S - P_{S^\perp} = 2P_S - 1.$$

R_S is the orthogonal reflection through $S^\perp$. Note that $R_S \in U(n)$ and $R_S^2 = 2$. The map

$$A \mapsto R_S \star A$$

is an involution of Sym_n. It descends to an involution of M. We thus get an entire family of involutions

$$\sigma : M \times M \to M, \quad (P_{S_1}, P_{S_2}) \mapsto R_{S_1} \star P_{S_2}.$$

define

$$\mathfrak{i} : M \times U(n) \to U(n), \quad \mathfrak{i}_S T = R_S T R_S.$$

We leave the reader check that the above collection of data defines a symmetric space structure on $G_k(n, \mathbb{C})$. □

Exercise 7.4.2. Fill in the details left out in the above example. □

7.4.2 Symmetry and cohomology

Definition 7.4.9. *Let M be a homogeneous space with symmetry group G. A differential form $\omega \in \Omega^*(M)$ is said to be (left)* invariant *if $\ell_g^*\omega = \omega\ \forall g \in G$, where we denoted by*

$$\ell_g^* : \Omega^*(M) \to \Omega^*(M)$$

the pullback defined by the left action by g: $m \mapsto g \cdot m$.

Proposition 7.4.10. *Let M be a compact homogeneous space with compact, connected symmetry group G. Then any cohomology class of M can be represented by a (not necessarily unique) invariant form.*

Proof Denote by $d\mu_g$ the normalized bi-invariant volume form on G. For any form $\omega \in \Omega^*(M)$ we define its G-average by

$$\overline{\omega} = \int_G \ell_g^* \omega d\mu_g.$$

$\overline{\omega}$ is an invariant form on M. The proposition is a consequence of the following result.

Lemma 7.4.11. *If ω is a closed form on M then $\overline{\omega}$ is closed and cohomologous to ω.*

Proof of the lemma The form $\overline{\omega}$ is obviously closed so we only need to prove it is cohomologous to ω. Consider a bi-invariant Riemann metric $\mathfrak{m}$ on G. Since G is connected the exponential map

$$\exp : \mathfrak{L}_G \to G \quad X \mapsto \exp(tX)$$

is surjective. Choose $r > 0$ sufficiently small such that

$$\exp : D_r = \{|X|_{\mathfrak{m}} = r \; ; \; X \in \mathfrak{L}_G\} \to G$$

is an embedding. Set $B_r = \exp D_r$. We can select finitely many $g_1, \dots, g_m \in G$ such that

$$G = \bigcup_{j=1}^m B_j \quad (B_j = g_j B_r).$$

Now pick a partition of unity $(\alpha_j) \subset C^\infty(G)$ subordinated to the cover (B_j), i.e.

$$0 \leq \alpha_j \leq 1, \quad \operatorname{supp} \alpha_j \subset B_j, \quad \sum_j \alpha_j = 1.$$

Set

$$a_j = \int_G \alpha_j d\mu_g.$$

Since the volume of G is normalized to 1 and $\sum_j \alpha_j = 1$ we deduce $\sum_j a_j = 1$. For any $j = 1, \dots, m$ define $T_j : \Omega^*(M) \to \Omega^*(M)$ by

$$T_j \omega = \int_G \alpha_j(g) \ell_g^* \omega d\mu_g.$$

Note that

$$\overline{\omega} = \sum_j T_j \omega \;\text{ and }\; dT_j\omega = T_j d\omega.$$

Each T_j is thus a cochain map. It induces a morphism in cohomology which we continue to denote by T_j. The proof of the lemma will be completed in several steps.

Step 1. $\ell_g^* = \mathbf{id}$ on $H^*(M)$ for all $g \in G$. Let $X \in \mathfrak{L}_G$ such that $g = \exp X$. Define

$$f : I \times M \to M \quad f_t(m) = \exp(tX) \cdot m = \ell_{\exp(tX)} m.$$

f is a homotopy connecting $\mathbf{1}_M$ with ℓ_g^*. This concludes Step 1.

Step 2

$$T_j = a_j \mathbf{1}_{H^*(M)}.$$

For $t \in [0,1]$ consider $\phi_{j,t} : B_j \to G$ defined as the composition

$$B_j \stackrel{g_j^{-1}}{\to} B_r \stackrel{\exp^{-1}}{\to} D_r \stackrel{t \cdot X}{\to} D_{tr} \stackrel{\exp}{\to} B_{tr} \stackrel{g_j}{\hookrightarrow} G.$$

Define $T_{j,t} : \Omega^*(M) \to \Omega^*(M)$ by

$$T_{j,t}\omega = \int_G \alpha_j(g) \ell^*_{\phi_{j,t}(t)} \omega d\mu_g = T_j \phi^*_{j,t}\omega. \tag{7.4.1}$$

We claim that $T_{j,0}$ is cochain homotopic to $T_{j,1}$.

To verify this claim set $t = e^s$, $-\infty < s \leq 0$ and

$$g_s = \exp(e^s \exp^{-1}(g)) \quad \forall g \in B_r.$$

Then

$$U_s\omega \stackrel{def}{=} T_{j,e^s}\omega = \int_{B_r} \alpha_j(g_j g) \ell^*_{g_j g_s}\omega = \int_{B_r} \alpha(g_j g) \ell^*_{g_s} \ell^*_{g_j} \omega d\mu_g.$$

For each $g \in B_r$ the map

$$\Psi_s(m) = g_s(m)$$

defines a local flow on M. We denote by X_g its infinitesimal generator. Then

$$\frac{d}{ds}(U_s\omega) = \int_{B_r} \alpha_j(g_j g) L_{X_g} \ell^*_{g_s} \ell^*_{g_j} \omega d\mu_g$$

$$= \int_{B_r} \alpha_j(g_j g)(d_M i_{X_g} + i_{X_g} d_M)(\ell^*_{g_s} \ell^*_{g_j}\omega) d\mu_g.$$

Consequently

$$T_{j,0}\omega - T_{j,1}\omega = U_{-\infty}\omega - U_0\omega = -\int_{-\infty}^0 \left(\int_{B_r} \alpha_j(g_j g)(d i_{X_g} + i_{X_g} d)(\ell^*_{g_s}\ell^*_{g_j}\omega) d\mu_g \right) ds.$$

(An argument entirely similar to the one we used in the proof of the Poincaré lemma shows that the above improper integral is pointwise convergent). From the above formula we immediately read a cochain homotopy $\chi : \Omega^*(M) \to \Omega^{*-1}(M)$ connecting $U_{-\infty}$ to U_0. More precisely

$$\{\chi(\omega)\}\,|_{x \in M} = -\int_{-\infty}^0 \left(\int_{B_r} \alpha_j(g_j g) \left\{ i_{X_g} \ell^*_{g_s} \ell^*_{g_j} \omega \right\} |_x \, d\mu_g \right) ds.$$

Now notice that

$$T_{j,0}\omega = \left(\int_{B_j} \alpha_j(g) d\mu_g \right) = a_j \ell^*_{g_j}\omega$$

while

$$T_{j,1}\omega = T_j\omega.$$

Taking into account Step 1 we deduce $T_{j,0} = a_j\mathbf{id}$. Step 2 is completed.

The lemma and hence the proposition follow from

$$\mathbf{1}_{H^*(M)} = \sum_j a_j \mathbf{1}_{H^*(M)} = \sum_j T_j = G - \text{average}.$$

□

The proposition we have just proved has a greater impact when M is a symmetric space.

Proposition 7.4.12. *Let (M, h) be an, oriented symmetric space with symmetry group G. Then the following are true.*
(a) Every invariant form on M is closed.
(b) If moreover M is compact then the only invariant form cohomologous to zero is the trivial one.

Proof (a) Consider an invariant k-form ω. Fix $m \in M$ and set $\hat{\omega} = \sigma_m^*\omega$. We claim $\hat{\omega}$ is invariant. Indeed, $\forall g \in G$

$$\ell_g^*\hat{\omega} = \ell_g^*\sigma_m^*\omega = (\sigma_m g)^*\omega$$
$$= (gi_g\sigma_m g)^*\omega = \sigma_m^*\ell_{\mathrm{i}(g)}^*\omega = \sigma_m^*\omega = \hat{\omega}.$$

Since $D\sigma_m|_{T_mM} = -\mathbf{1}_{T_mM}$ we deduce that at $m \in M$

$$\hat{\omega} = (-1)^k\omega.$$

Because both ω and $\hat{\omega}$ are G-invariant we deduce that the above equality holds for any $g\cdot m$. Invoking the transitivity of the G-action we conclude that

$$\hat{\omega} = (-1)^k\omega \ \text{ on } M.$$

In particular

$$d\hat{\omega} = (-1)^k d\omega \ \text{ on } M.$$

The $(k+1)$-forms $d\hat{\omega} = \sigma_m^* d\omega$ and $d\omega$ are both invariant and as above we deduce

$$d\hat{\omega} = \hat{d\omega} = (-1)^{k+1}d\omega.$$

The last two inequalities imply $d\omega = 0$.
(b) Let ω be an invariant form cohomologous to zero, i.e. $\omega = d\alpha$. Denote by $*$ the Hodge $*$-operator corresponding to the invariant metric h. Since G acts by isometries $\eta = *\omega$ is also invariant so that $d\eta = 0$. We can now integrate (M is compact) and use Stokes theorem to get

$$\int_M \omega \wedge *\omega = \int_M d\alpha\wedge = \pm\int_M \alpha \wedge d\eta = 0.$$

This forces $\omega \equiv 0$.

□

Form Proposition 7.4.10 and the above theorem we deduce the following celebrated result of Élie Cartan ([17])

Corollary 7.4.13. (Cartan) *Let (M, h) be a compact, oriented symmetric space with compact, connected symmetry group G. Then the cohomology algebra $H^*(M)$ of M is isomorphic with the graded algebra $\Omega^*_{inv}(M)$ of invariant forms on M.*

In the coming subsections we will apply this result to the symmetric spaces discussed in the previous subsection: the Lie groups and the complex grassmannians.

7.4.3 The cohomology of compact Lie groups

Consider a compact, connected Lie group G and denote by $\mathfrak{L}_G$ its Lie algebra. According to Proposition 7.4.10, in computing its cohomology it suffices to restrict our considerations to the subcomplex consisting of left invariant forms. This can be identified with the exterior algebra $\Lambda^*\mathfrak{L}_G^*$. We deduce

Corollary 7.4.14. *$H^*(G) \cong H^*(\mathfrak{L}_G) \cong \Lambda^*_{inv}\mathfrak{L}_G$ where $\Lambda^*_{inv}\mathfrak{L}_G$ denote the algebra of bi-invariant forms on G, while $H^*(\mathfrak{L}_G)$ denotes the Lie algebra cohomology introduced in Example 7.1.9.*

Using the Exercise 7.1.2 we deduce the following consequence.

Corollary 7.4.15. *If G is a compact semisimple Lie group then $H^1(G) = 0$.*

Proposition 7.4.16. *Let G be a compact semisimple Lie group. Then $H^2(G) = 0$.*

Proof A closed bi-invariant 2-form ω on G is uniquely defined by its restriction to $\mathfrak{L}_G$ and satisfies the following conditions.

$$d\omega = 0 \Longleftrightarrow \omega([X_0, X_1], X_2) - \omega([X_0, X_2], X_1) + \omega([X_1, X_2], X_0) = 0$$

and (right-invariance)

$$(L_{X_0}\omega)(X_1, X_2) = 0 \;\; \forall X_0 \in \mathfrak{L}_G \Longleftrightarrow \omega([X_0, X_1], X_2) - \omega([X_0, X_2], X_1) = 0.$$

Thus

$$\omega([X_0, X_1], X_2) = 0 \quad \forall X_0, X_1, X_2 \in \mathfrak{L}_G.$$

On the other hand since $H^1(\mathfrak{L}_G) = 0$ we deduce (see Exercise 7.1.2) $\mathfrak{L}_G = [\mathfrak{L}_G, \mathfrak{L}_G]$ so that the last equality can be rephrased as

$$\omega(X, Y) = 0 \quad \forall X, Y \in \mathfrak{L}_G. \quad \square$$

Definition 7.4.17. *A Lie algebra is called* simple *if it has no nontrivial ideals. A Lie group is called simple if its Lie algebra is simple.*

Exercise 7.4.3. Prove that $SU(n)$ and $SO(m)$ are simple. $\square$

Proposition 7.4.18. *Let G be a compact, simple Lie group. Then $H^3(G) \cong \mathbb{R}$. Moreover, $H^3(G)$ is generated by the* Cartan form

$$\alpha(X, Y, Z) = \kappa([X, Y], Z)$$

where κ denotes the Killing pairing.

The proof of the proposition is contained in the following sequence of exercises.

Exercise 7.4.4. Prove that a simple Lie algebra is necessarily semi-simple. □

Exercise 7.4.5. Let ω be a closed, bi-invariant 3-form on a Lie group G. Then

$$\omega(X, Y, [Z, T]) = \omega([X, Y], Z, T) \quad \forall X, Y, Z, T \in \mathfrak{L}_G.$$

□

Exercise 7.4.6. Let ω be a closed, bi-invariant 3-form on a compact, semisimple Lie group.
(a) Prove that for any $X \in \mathfrak{L}_G$ there exists a unique left invariant $\eta_X \in \Omega^1(G)$ such that

$$(i_X\omega)(Y, Z) = \eta_X([Y, Z]).$$

Moreover, the correspondence $X \mapsto \eta_X$ is linear.
Hint: Use $H^1(G) = H^2(G) = 0$.
(b) Denote by A the linear operator $\mathfrak{L}_G \to \mathfrak{L}_G$ defined by

$$\kappa(AX, Y) = \eta_X(Y).$$

Prove that A is selfadjoint with respect to the Killing metric.
(c) Prove that the eigenspaces of A are ideals of $\mathfrak{L}_G$. Use this to prove Proposition 7.4.18.
□

Exercise 7.4.7. Compute

$$\int_{SU(2)} \alpha \quad \text{and} \int_{SO(3)} \alpha$$

where α denotes the Cartan form. (These groups are oriented by their Cartan forms.)
Hint: Use the computation in the Exercise 4.1.16 and the double cover $SU(2) \to SO(3)$ described in the Subsection 6.2.1. Pay very much attention to the various constants.

□

7.4.4 Invariant forms on grassmannians and Weyl's integral formula

We will use the results of Subsection 7.4.2 to compute the Poincaré polynomial of the complex Grassmannian $G_k(n, \mathbb{C})$. Set $\ell = n - k$.

$G_k(n, \mathbb{C})$ is a symmetric space with symmetry group $U(n)$. It is a complex manifold so that it is orientable (cf. Exercise 3.4.2). Alternatively, the orientability of $G_k(n, \mathbb{C})$ is a consequence of the following fact.

Exercise 7.4.8. If M is a homogeneous space with connected isotropy groups then M is orientable.

We have to describe the $U(n)$-invariant forms on $G_k(n, \mathbb{C})$. These forms are completely determined by their values at a particular point in the Grassmannian. We choose this point to correspond to the subspace S_0 determined by the canonical inclusion $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$.

The isotropy of S_0 is the group $H = U(k) \times U(\ell)$. H acts linearly on the tangent space $V_0 = T_{S_0} G_k(n, \mathbb{C})$. If ω is an $U(n)$-invariant form then its restriction to V_0 is an H-invariant skew-symmetric, multilinear map

$$V_0 \times \cdots \times V_0 \to \mathbb{R}.$$

Conversely, any H-invariant element of $\Lambda^* V_0^*$ extends via the transitive action of $U(n)$ to an invariant form on $G_k(n, \mathbb{C})$. Denote by Λ^*_{inv} the space of H-invariant elements of $\Lambda^* V_0^*$. We have thus established the following result.

Proposition 7.4.19. *There exists an isomorphism of graded* $\mathbb{R}$-*algebras:*

$$H^*(G_k(n, \mathbb{C})) \cong \Lambda^*_{inv}.$$

We want to determine the Poincaré polynomial of the complexified graded space, $\Lambda^*_{inv} \otimes \mathbb{C}$

$$P_{k,\ell}(t) = \sum_j t^j \dim_{\mathbb{C}} \Lambda^j_{inv} \otimes \mathbb{C} = P_{G_k(n,\mathbb{C})}(t).$$

Denote the action of H on V_0 by

$$H \ni h \mapsto T_h \in \operatorname{Aut}(V_0).$$

Using the equality (3.4.9) of Subsection 3.4.4 we deduce

$$P_{k,\ell}(t) = \int_H |\det(\mathbb{1}_{V_0} + tT_h)|^2 dh \tag{7.4.2}$$

where dh denotes the normalized bi-invariant volume form on H.

At this point the above formula may look hopelessly complicated. Fortunately, it can be dramatically simplified using a truly remarkable idea of H.Weyl.

Note first that the function

$$H \ni h \mapsto \varphi(h) = |\det(\mathbb{1}_{V_0} + tT_h)|^2$$

is a class function, i.e. $\varphi(ghg^{-1}) = \varphi(h)$, $\forall g, h \in H$.

Inside H sits the *maximal torus*

$$\mathbb{T} = \mathbb{T}^k \times \mathbb{T}^\ell,$$

where

$$\mathbb{T}^k = \{\operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_k}) \in U(k)\},$$

and similarly

$$\mathbb{T}^\ell = \{\operatorname{diag}(e^{i\phi_1}, \dots, e^{i\phi_\ell}) \in U(\ell)\}.$$

Each $h \in U(k) \times U(\ell)$ is conjugate to diagonal unitary matrix, i.e. there exists $g \in H$ such that $ghg^{-1} \in \mathbb{T}$.

We can rephrase this fact in terms of the adjoint action of H on itself

$$\operatorname{Ad} : H \times H \to H \quad (g, h) \mapsto \operatorname{Ad}_g(h) = ghg^{-1}.$$

The class functions are constant along the orbits of the adjoint action and each such orbit intersects the maximal torus $\mathbb{T}$. In other words, a class function is completely determined by its restriction to the maximal torus. Hence, at least theoretically, we should be able to describe the integral in (7.4.2) as an integral over $\mathbb{T}$. This is achieved in a very explicit manner by the next result.

Define $\Delta_n : \mathbb{T}^n \to \mathbb{C}$ by

$$\Delta_n(\theta_1, \dots, \theta_n) = \prod_{1 \leq i < j \leq n} (e^{\mathbf{i}\theta_i} - e^{\mathbf{i}\theta_j}), \quad \mathbf{i} = \sqrt{-1}.$$

Proposition 7.4.20. (Weyl's integration formula) *Consider a class function φ on the group $G = U(k_1) \times \cdots \times U(k_s)$. Then*

$$\int_G \varphi(g) dg = \frac{1}{k_1! \cdots k_s!} \int_{\mathbb{T}} \varphi(t_1, \dots, t_s) |\Delta_{k_1}(t_1)|^2 \cdots |\Delta_{k_s}(t_s)|^2 dt_1 \wedge \cdots \wedge dt_s.$$

Above, dg denotes the normalized *bi-invariant volume form on G. For each $j = 1, \dots, s$ we denoted by $t = t_j$ the collection of angular coordinates on $\mathbb{T}^k = \mathbb{T}^{k_j}$ while dt denotes the normalized bi-invariant volume on $\mathbb{T}^k$*

$$dt = \frac{1}{(2\pi)^k} d\theta_1 \wedge \cdots \wedge d\theta_k.$$

The remainder of this subsection is devoted to the proof of this proposition. The reader may skip this part at the first lecture and go directly to Subsection 7.4.5 where this formula is used to produce an explicit description of the Poincaré polynomial of a complex Grassmannian.

Proof of Proposition 7.4.20 We will consider only the case $s = 1$. The general situation is entirely similar. Thus $G = U(k)$ and $\mathbb{T} = \mathbb{T}^k$. Denote the angular coordinates on $\mathbb{T}$ by $(\theta^1, \dots, \theta^k)$. Given a class function $\varphi : G \to \mathbb{C}$ form the complex valued form $\omega_\varphi = \varphi(g) dg$.

Consider the homogeneous space $G/\mathbb{T}$ and the smooth map

$$q : \mathbb{T} \times G/\mathbb{T} \to G \quad (t, gT) = gtg^{-1}.$$

Note that if $g_1\mathbb{T} = g_2\mathbb{T}$ then $g_1 t g_1^{-1} = g_1 t g_2^{-1}$ so q is well defined. Pick a real metric $\mathfrak{m}$ on $\mathfrak{L}_G$ which is Ad-invariant i.e.

$$\mathrm{Ad}_g^* \mathfrak{m} = \mathfrak{m} \quad \forall g \in G.$$

The natural choice

$$\mathfrak{m}(X, Y) = -\mathfrak{Re}\, \mathrm{tr}\,(XY^*), \quad \underline{u}(k) = \mathfrak{L}_G$$

will do the trick. The Lie algebra $\mathfrak{L}_G$ splits orthogonally as

$$\mathfrak{L}_G = \mathfrak{L}_{\mathbb{T}} \oplus \mathfrak{L}_{G/\mathbb{T}}. \quad (\mathfrak{L}_{G/\mathbb{T}} \stackrel{def}{=} \mathfrak{L}_{\mathbb{T}}^{\perp})$$

The tangent space to $1 \cdot \mathbb{T} \in G/\mathbb{T}$ can be identified with $\mathfrak{L}_{G/\mathbb{T}}$.

Fix $x \in G/\mathbb{T}$. Any $g \in G$ defines a linear map $L_g : T_x G/\mathbb{T} \to T_{gx} G/\mathbb{T}$. Moreover, if $gx = hx = y$ then L_g and L_h differ by an element in the stabilizer of $x \in G/\mathbb{T}$. This stabilizer is isomorphic to $\mathbb{T}$ and in particular it is connected. Hence, if $\omega \in \det T_x G/\mathbb{T}$ then $L_g\omega \in \det T_y G/\mathbb{T}$ and $L_h\omega \in \det T_y G/\mathbb{T}$ define the same orientation of $T_y G/\mathbb{T}$. In other words, an orientation in one of the tangent spaces of $G/\mathbb{T}$ "spreads" via the action of G to an orientation of the entire manifold. Thus, we can orient $G/\mathbb{T}$ by fixing an orientation on $\mathfrak{L}_{G/\mathbb{T}}$. We fix an orientation on $\mathfrak{L}_G$ and orient $\mathfrak{L}_{\mathbb{T}}$ using the form $d\theta^1 \wedge \cdots \wedge d\theta^k$. The orientation on $\mathfrak{L}_{G/\mathbb{T}}$ will be determined by the condition (*or*= orientation)

$$\mathbf{or}(\mathfrak{L}_G) = \mathbf{or}(\mathfrak{L}_{\mathbb{T}}) \wedge \mathbf{or}(\mathfrak{L}_{G/\mathbb{T}}).$$

The proof of Weyl's integration formula will be carried out in two steps.

Step 1

$$\int_G \omega = \frac{1}{k!}\int_{\mathbb{T}\times G/\mathbb{T}} q^*\omega \quad \forall \omega.$$

Step 2 For any class function φ on G we have

$$\int_{\mathbb{T}\times G/\mathbb{T}} q^*\omega_\varphi = \int_{\mathbb{T}} \varphi(t)|\Delta_k(t)|^2 dt.$$

Step 1. We use the equality

$$\int_{\mathbb{T}\times G/\mathbb{T}} q^*\omega = \deg q \int_G \omega$$

so it suffices to compute the degree of q.

Denote by $N(\mathbb{T})$ the normalizer of $\mathbb{T}$ in G, i.e.

$$N(\mathbb{T}) = \{g \in G\ ;\ g\mathbb{T}g^{-1} \subset \mathbb{T}\}$$

and then form the *Weyl group*

$$\mathcal{W} = N(\mathbb{T})/\mathbb{T}.$$

Lemma 7.4.21. *$\mathcal{W} \cong \mathfrak{S}_k$-the group of permutations of k symbols.*

Proof This is a pompous rephrasing of the classical statement in linear algebra that two unitary matrices are similar iff they have the same spectrum (multiplicities included). The adjoint action of $N(\mathbb{T})$ on $\mathbb{T}$= diagonal unitary matrices simply permutes the entries of a diagonal unitary matrix. This action descends to an action on the quotient $\mathcal{W}$ so that $\mathcal{W} \subset \mathfrak{S}_k$.

Conversely, any permutation of the entries of a diagonal matrix can be achieved by a conjugation. Geometrically this corresponds to a reordering of an orthonormal basis.

□

Lemma 7.4.22. *Let $\alpha^1, \ldots, \alpha^k) \in \mathbb{R}^k$ such that $1, \frac{\alpha^1}{2\pi}, \ldots, \frac{\alpha^k}{2\pi}$ are linearly independent over $\mathbb{Q}$. Set $\tau = (\exp(\mathbf{i}\alpha^1), \ldots, \exp(\mathbf{i}\alpha^k)) \in \mathbb{T}^k$. Then the sequence $(\tau^n)_{n\in\mathbb{Z}}$ is dense in $\mathbb{T}^k$. (τ is said to be a generator of $\mathbb{T}^k$.)*

For the sake of clarity, we defer the proof of this lemma to the end of this subsection.

Lemma 7.4.23. *Let $\tau \in \mathbb{T}^k \subset G$ be a generator of $\mathbb{T}^k$. Then $q^{-1}(\tau) \subset \mathbb{T} \times G/\mathbb{T}$ consists of $|\mathcal{W}| = k!$ points.*

Proof

$$q(s, gT) = \tau \Longleftrightarrow gsg^{-1} = \tau \Longleftrightarrow g\tau g^{-1} = s \in \mathbb{T}.$$

In particular, $g\tau^n g^{-1} = s^n \in \mathbb{T}$, $\forall n \in \mathbb{Z}$. Since (τ^n) is dense in $\mathbb{T}$ we deduce

$$gTg^{-1} \subset \mathbb{T} \Rightarrow g \in N(\mathbb{T}).$$

Hence

$$q^{-1}(\tau) = \{(g^{-1}\tau g, gT) \in \mathbb{T} \times G/\mathbb{T}\ ;\ g \in N(\mathbb{T})\}$$

and thus $q^{-1}(\tau)$ has the same cardinality as the Weyl group $\mathcal{W}$. □

The metric $\mathfrak{m}$ on $\mathfrak{L}_{G/\mathbb{T}}$ extends to a G-invariant metric on $G/\mathbb{T}$. It defines a left-invariant volume form on $G/\mathbb{T}$, $dv_{\mathfrak{m}}$. Let

$$v_0 = \int_{G/\mathbb{T}} dv_{\mathfrak{m}}$$

and set $d\mu = \frac{1}{v_0} dv_{\mathfrak{m}}$.

Lemma 7.4.24. *$q^*dg = |\Delta_k(t)|^2 dt \wedge d\mu$. In particular, any generator τ of $\mathbb{T}^k \subset G$ is a regular value of q since $\Delta_k(\sigma) \neq 0$, $\forall \sigma \in q^{-1}(\tau)$.*

Proof Fix $x_0 = (t_0, g_0\mathbb{T}) \in \mathbb{T} \times G/\mathbb{T}$ and set $h_0 = g_0 t_0 g_0^{-1}$. Via the action of $\mathbb{T} \times G$ on $\mathbb{T} \times G/\mathbb{T}$ we can identify $T_{x_0}(\mathbb{T} \times G/\mathbb{T})$ with $\mathfrak{L}_{\mathbb{T}} \oplus \mathfrak{L}_{G/\mathbb{T}}$. Fix $X \in \mathfrak{L}_{\mathbb{T}}$ and $Y \in \mathfrak{L}_G$ and consider

$$h_s = q(t_0 \exp(sX), g_0 \exp(sY)\mathbb{T}) = g_0 \exp(sY) t_0 \exp(sX) \exp(-sY) g_0^{-1} \in G.$$

We want to describe

$$\frac{d}{ds} h_0^{-1} h_s \in T_1 G = \mathfrak{L}_G.$$

Using the Taylor expansions

$$\exp(sX) = 1 + sX + O(s^2) \ \text{ and } \ \exp(sY) = 1 + sY + O(s^2)$$

we deduce

$$\begin{aligned} h_0^{-1} h_s &= g_0 t_0^{-1}(1 + sY) t_0 (1 + sX)(1 - sY) g_0^{-1} + O(s^2) \\ &= 1 + s\left(g_0 t_0^{-1} Y t_0 g_0^{-1} + g_0 X g_0^{-1} - g_0 Y g_0^{-1}\right) + O(s^2). \end{aligned}$$

Hence

$$D_{x_0} : T_{x_0}(\mathbb{T} \times G/\mathbb{T}) \cong \mathfrak{L}_{\mathbb{T}} \oplus \mathfrak{L}_{G/\mathbb{T}} \to \mathfrak{L}_{\mathbb{T}} \oplus \mathfrak{L}_{G/\mathbb{T}} \cong \mathfrak{L}_G$$

can be written as

$$D_{x_0} q(X \oplus Y) = \mathrm{Ad}_{g_0}(\mathrm{Ad}_{t_0^{-1}} - \mathbf{id})Y + \mathrm{Ad}_{g_0} X$$

or in block form

$$D_{x_0} q = \mathrm{Ad}_{g_0} \left[\begin{array}{cc} \mathbb{1}_{\mathfrak{L}_{\mathbb{T}}} & 0 \\ 0 & \mathrm{Ad}_{t_0^{-1}} - \mathbb{1}_{\mathfrak{L}_{G/\mathbb{T}}} \end{array} \right]$$

Ad_g is an $\mathfrak{m}$-orthogonal endomorphism of $\mathfrak{L}_G$ so that $\det \mathrm{Ad}_g = \pm 1$. On the other hand, since $G = U(k)$ is connected $\det \mathrm{Ad}_g = \det \mathrm{Ad}_1 = 1$. Hence

$$\det D_{x_0} q = \det(\mathrm{Ad}_{t^{-1}} - \mathbb{1}_{\mathfrak{L}_{G/\mathbb{T}}}).$$

We can identify $\mathfrak{L}_{G/\mathbb{T}}$ with

$$\{X \in \underline{u}(k) \; ; \; X_{jj} = 0 \;\; j = 1, \ldots, k\}.$$

Given $t = \mathrm{diag}\,(\exp(\mathbf{i}\theta^1), \ldots, \exp(\mathbf{i}\theta^k)) \in \mathbb{T}^k \subset U(k)$ we can explicitly compute the eigenvalues of $\mathrm{Ad}_{t^{-1}}$ acting on $\mathfrak{L}_{G/\mathbb{T}}$. They are

$$\{\exp(-\mathbf{i}(\theta_i - \theta_j)) \; ; \; 1 \le i \ne j \le k\}.$$

Consequently

$$\det D_{x_0} q = \det(\mathrm{Ad}_{t^{-1}} - 1) = |\Delta_k(t)|^2.$$

□

Lemma 7.4.24 shows that q is an orientation preserving map. Using Lemma 7.4.23 and Exercise 7.3.5 we deduce $\deg q = |\mathcal{W}| = k!$. Step 1 is completed.

Step 2 follows immediately from Lemma 7.4.24. Weyl's integration formula is proved.

□

Proof of Lemma 7.4.22 We follow Weyl's original approach ([75, 76]) in a modern presentation.

Let $X = C(\mathbb{T}, \mathbb{C})$ denote the Banach space of continuous complex valued functions on $\mathbb{T}$. We will prove that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} f(\tau^j) = \int_T f dt, \quad \forall f \in X. \tag{7.4.3}$$

If $U \subset \mathbb{T}$ is an open subset and f is a continuous, non-negative function supported in U ($f \not\equiv 0$) then for very large n

$$\frac{1}{n+1} \sum_{j=0}^{n} f(\tau^j) \approx \int_T f dt \ne 0.$$

This means that $f(\tau^j) \ne 0$, i.e. $\tau^j \in U$ for some j.

To prove the equality (7.4.3) consider the continuous linear functionals $L_n, L : X \to \mathbb{C}$

$$L_n(f) = \frac{1}{n+1} \sum_{j=0}^{n} f(\tau^j),$$

and

$$L(f) = \int_T f dt.$$

We have to prove that

$$\lim_{n\to\infty} L_n(f) = L(f) \quad f \in X. \tag{7.4.4}$$

It suffices to establish (7.4.4) for any $f \in \mathcal{S}$, where $\mathcal{S}$ is a subset of X spanning a dense subspace. We let $\mathcal{S}$ be the subset consisting of the trigonometric monomials

$$e_\zeta(\theta^1, \dots, \theta^k) = \exp(\mathbf{i}\zeta_1\theta^1)\cdots\exp(\mathbf{i}\zeta_k\theta^k) \quad \zeta = (\zeta_1, \dots, \zeta_k) \in \mathbb{Z}^k.$$

The Weierstrass approximation theorem guarantees that this $\mathcal{S}$ spans a dense subspace. We compute easily

$$L_n(e_\zeta) = \frac{1}{n+1}\sum_{j=0}^{n} e_{j\zeta}(\alpha) = \frac{1}{n+1}\frac{e_\zeta(\alpha)^{n+1}-1}{e_\zeta(\alpha)-1}.$$

Since $1, \frac{1}{2\pi}\alpha_1, \dots, \frac{1}{2\pi}\alpha_k$ are linearly independent over $\mathbb{Q}$ we deduce that $e_\zeta(\alpha) \neq 1$ for all $\zeta \in \mathbf{Z}^k$. Hence

$$\lim_{n\to\infty} L_n(e_\zeta) = 0 = \int_T e_\zeta dt = L(e_\zeta).$$

Lemma 7.4.22 is proved.

□

7.4.5 The Poincaré polynomial of a complex grassmannian

After this rather long detour we can continue our search for the Poincaré polynomial of $G_k(n, \mathbb{C})$.

Let S_0 denote the canonical subspace $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$. The tangent space of $G_k(n, \mathbb{C})$ at S_0 can be identified with the linear space $\mathfrak{E}$ of complex linear maps $\mathbb{C}^k \to \mathbb{C}^\ell$, $\ell = n-k$. The isotropy group at S_0 is $H = U(k) \times U(\ell)$.

Exercise 7.4.9. Prove that the isotropy group H acts on $\mathfrak{E} = \{L : \mathbb{C}^k \to \mathbb{C}^\ell\}$ by

$$(T, S) \cdot L = SLT^* \quad \forall L \in \mathfrak{E}, T \in U(k), S \in U(\ell).$$ □

Consider the maximal torus $T^k \times T^\ell \subset H$ formed by the diagonal unitary matrices. We will denote the elements of T^k by $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$, $\varepsilon_\alpha = \exp(\mathbf{i}\tau_\alpha)$ and the elements of T^ℓ by $\underline{e} = (e_1, \dots, e_\ell)$, $e_j = \exp(\mathbf{i}\theta_j)$. The normalized measure on T^k is denoted by $d\tau$ while the normalized measure on T^ℓ is denoted by $d\theta$.

The element $(\underline{\varepsilon}, \underline{e}) \in T^k \times T^\ell$ viewed as a linear operator on $\mathfrak{E}$ has eigenvalues

$$\{\bar{\varepsilon}_\alpha e_j \,;\, 1 \le \alpha \le k \;\; 1 \le j \le \ell\}.$$

Using the Weyl integration formula we deduce that the Poincaré polynomial of $G_k(n, \mathbb{C})$ is

$$P_{k,\ell}(t) = \frac{1}{k!\ell!}\int_{T^k\times T^\ell}\prod_{\alpha,j}|1 + t\bar{\varepsilon}_\alpha e_j|^2 |\Delta_k(\underline{\varepsilon}(\tau))|^2 |\Delta_\ell(\underline{e}(\theta))|^2 d\tau \wedge d\theta$$

$$= \frac{1}{k!\ell!}\int_{T^k\times T^\ell}\prod_{\alpha,j}|\varepsilon_\alpha + te_j|^2 |\Delta_k(\underline{\varepsilon}(\tau))|^2 |\Delta_\ell(\underline{e}(\theta))|^2 d\tau \wedge d\theta.$$

We definitely need to analyze the integrand in the above formula. Set

$$I_{k,\ell}(t) = \prod_{\alpha,j}(\varepsilon_\alpha + te_j)\Delta_k(\underline{\varepsilon})\Delta_\ell(\underline{e})$$

so that

$$P_{k,\ell}(t) = \frac{1}{k!\ell!}\int_{T^k\times T^\ell} I_{k,\ell}(t)\overline{I_{k,\ell}(t)}d\tau\wedge d\theta. \tag{7.4.5}$$

We will study in great detail the formal expression

$$J_{k,\ell}(t;x;y) = \prod_{\alpha,j}(x_\alpha + ty_j).$$

The Weyl group $\mathcal{W} = \mathfrak{S}_k \times \mathfrak{S}_\ell$ acts on the variables $(x;y)$ by separately permuting the x-components and the y-components. If $(\sigma,\varphi)\in\mathcal{W}$ then

$$J_{k,\ell}(t;\sigma(x);\varphi(y)) = J_{k,\ell}(t;x;y).$$

Thus we can write $J_{k,\ell}$ as a sum

$$J_{k,\ell}(t) = \sum_{d\geq 0} t^d Q_d(x)R_d(y)$$

where $Q_d(x)$ and $R_d(y)$ are symmetric polynomials in x and respectively y.

To understand the nature of these polynomials we need to introduce a very useful class of symmetric polynomials, namely the *Schur polynomials*. This will require a short trip in the beautiful subject of symmetric polynomials. An extensive presentation of this topic is contained in the monograph [50].

A *partition* is a compactly supported, decreasing function

$$\lambda : \{1,2,\dots\}\to\{0,1,2,\dots\}.$$

We will describe a partition by an ordered finite collection $(\lambda_1,\lambda_2,\dots,\lambda_n)$ where $\lambda_1\geq\cdots\geq\lambda_n\geq\lambda_{n+1}=0$. The *length* of a partition λ is the number

$$L(\lambda) = \max\{n\ ;\ \lambda_n\neq 0\}.$$

The *weight* of a partition λ is the number

$$|\lambda| = \sum_{n\geq 1}\lambda_n.$$

Traditionally, one can visualize a partition using *Young diagrams.* A Young diagram is an array of boxes arranged in left justified rows (see Figure 7.4). Given a partition $(\lambda_1\geq\cdots\geq\lambda_n)$ its Young diagram will have λ_1 boxes on the first row λ_2 boxes on the second row etc.

Any partition λ has a *conjugate* $\hat{\lambda}$ defined by

$$\hat{\lambda}_n = \#\{j\geq 0\ ;\ \lambda_j\geq n\}.$$

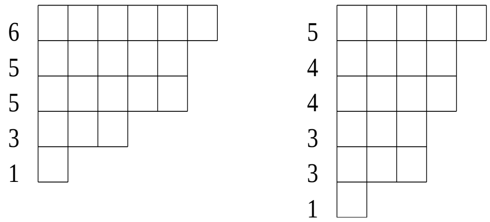

Figure 7.4: *The conjugate of (6,5,5,3,1) is (5,4,4,3,3,1)*

The Young diagram of $\hat{\lambda}$ is the transpose of the Young diagram of λ (see Figure 7.4).

A *strict partition* is a partition which is *strictly decreasing* on its support. Denote by $\mathcal{P}_n$ the set of partitions of length $\leq n$ and by $\mathcal{P}_n^*$ the set of *strict* partitions λ of length $n-1 \leq L(\lambda) \leq n$. Clearly $\mathcal{P}_n^* \subset \mathcal{P}_n$. Denote by $\delta = \delta_n \in \mathcal{P}_n^*$ the partition $(n-1, n-2, \ldots, 1, 0, \ldots)$.

Remark 7.4.25. The correspondence

$$\mathcal{P}_n \ni \lambda \mapsto \lambda + \delta_n \in \mathcal{P}_n^*$$

is a bijection. □

To any $\lambda \in \mathcal{P}_n^*$ we can associate a skew-symmetric polynomial

$$a_\lambda(x_1, \ldots, x_n) = \det(x_j^{\lambda_i}) = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) x_{\sigma(i)}^{\lambda_i}.$$

Note that a_{δ_n} is the Vandermonde determinant

$$a_\delta(x_1, \ldots, x_n) = \det(x_i^{n-1-i}) = \prod_{i<j}(x_i - x_j) = \Delta_n(x).$$

For each $\lambda \in \mathcal{P}_n$ we have $\lambda + \delta \in \mathcal{P}_n^*$ so that $a_{\lambda+\delta}$ is well defined and nontrivial.

Note that $a_{\lambda+\delta}$ vanishes when $x_i = x_j$ so that the polynomial $a_{\lambda+\delta}(x)$ is divisible by each of the differences $(x_i - x_j)$ and consequently is divisible by a_δ. Hence

$$S_\lambda(x) := \frac{a_{\lambda+\delta}(x)}{a_\delta(x)}$$

is a well defined polynomial. It is a symmetric polynomial since each of the quantities $a_{\lambda+\delta}$ and a_δ is skew-symmetric in its arguments. $S_\lambda(x)$ is called the *Schur polynomial* corresponding to the partition λ. Note that each Schur polynomial S_λ is homogeneous of degree $|\lambda|$. We have the following remarkable result.

Lemma 7.4.26.

$$J_{k,\ell}(t) = \sum_{\lambda \in \mathcal{P}_{k,\ell}} t^{|\overline{\lambda}|} S_{\hat{\lambda}}(x) S_{\overline{\lambda}}(y)$$

where

$$\mathcal{P}_{k,\ell} = \{\lambda \,;\, \lambda_1 \leq k \;\; L(\lambda) \leq \ell\}.$$

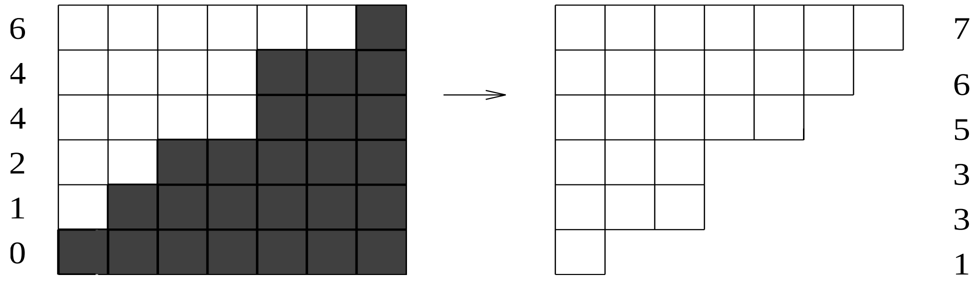

Figure 7.5: *The complementary of* $(6,4,4,2,1,0) \in \mathcal{P}_{7,6}$ *is* $(7,6,5,3,3,1) \in \mathcal{P}_{6,7}$

For each $\lambda \in \mathcal{P}_{k,\ell}$ *we denoted by* $\overline{\lambda}$ *the complementary partition*

$$\overline{\lambda} = (k-\lambda_\ell, k-\lambda_{\ell-1}, \dots, k-\lambda_1).$$

Geometrically, the partitions in $\mathcal{P}_{k,\ell}$ are precisely those partitions whose Young diagrams fit inside a $\ell \times k$ rectangle. If λ is such a partition then the Young diagram of the complementary of $\overline{\lambda}$ is (up to a 180° rotation) the complementary of the diagram of λ in the $\ell \times k$ rectangle (see Figure 7.5).

For a proof of Lemma 7.4.26 we refer to [50, Section I.4, Example 5]. The true essence of the Schur polynomials is however representation theoretic and a reader with a little more representation theoretic background may want to consult the classical reference [49, Ch.VI, Sec. 6.4,Thm. V] for a very exciting presentation of the Schur polynomials and the various identities they satisfy, including the one in Lemma 7.4.26.

Using (7.4.5), Lemma 7.4.26 and the definition of the Schur polynomial we can describe the Poincaré polynomial of $G_k(n,\mathbb{C})$ as

$$P_{k,\ell}(t) = \frac{1}{k!\ell!}\int_{T^k\times T^\ell}\left|\sum_{\lambda\in\mathcal{P}_{k,\ell}} t^{|\overline{\lambda}|} a_{\hat{\lambda}+\delta_k}(\underline{\varepsilon}) a_{\overline{\lambda}+\delta_\ell}(\underline{e})\right|^2 d\tau\wedge d\theta. \tag{7.4.6}$$

The integrand in (7.4.6) is a linear combination of trigonometric monomials $\varepsilon_1^{r_1}\cdots\varepsilon_k^{r_k}\cdot e_1^{s_1}\cdots e_\ell^{s_\ell}$ where the r's and s' are nonnegative integers.

Note that if $\lambda,\mu\in\mathcal{P}_{k,\ell}$ are distinct partitions then the terms $a_{\hat{\lambda}+\delta}(\underline{\varepsilon})$ and $a_{\hat{\mu}+\delta}(\underline{\varepsilon})$ have no monomials in common. Hence

$$\int_{T^k} a_{\hat{\lambda}+\delta}(\underline{\varepsilon})\overline{a_{\hat{\mu}+\delta}(\underline{\varepsilon})}\,d\tau = 0.$$

Similarly

$$\int_{T^\ell} a_{\overline{\lambda}+\delta}(\underline{e})\overline{a_{\overline{\mu}+\delta}(\underline{e})}d\theta = 0 \quad \text{if } \lambda\neq\mu.$$

On the other hand a simple computation shows that

$$\int_{T^k} |a_{\hat{\lambda}+\delta}(\underline{\varepsilon})|^2 d\tau = k!$$

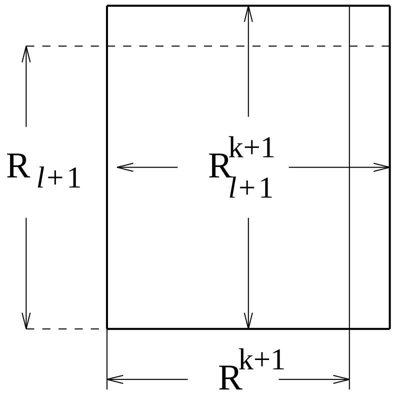

Figure 7.6: *The rectangles $R_{\ell+1}$ and R^{k+1} are "framed" inside $R^{k+1}_{\ell+1}$*

and

$$\int_{T^\ell} |a_{\overline{\lambda}+\delta}(\underline{e})|^2 d\theta = \ell!.$$

In other words, the terms

$$\left(\frac{1}{k!\ell!}\right)^{1/2} a_{\hat{\lambda}+\delta}(\underline{\varepsilon}) a_{\overline{\lambda}+\delta}(\underline{e})$$

form an *orthonormal* system in the space of trigonometric (Fourier) polynomials endowed with the L^2 inner product. We deduce immediately from (7.4.6) that

$$P_{k,\ell}(t) = \sum_{\lambda \in \mathcal{P}_{k,\ell}} t^{2|\overline{\lambda}|}. \tag{7.4.7}$$

The map

$$\mathcal{P}_{k,\ell} \ni \lambda \mapsto \overline{\lambda} \in \mathcal{P}_{\ell,k}$$

is a bijection so that

$$P_{k,\ell}(t) = \sum_{\lambda \in \mathcal{P}_{\ell,k}} t^{2|\lambda|} = P_{\ell,k}(t). \tag{7.4.8}$$

Computing the Betti numbers, i.e. the number of partitions in $\mathcal{P}_{k,\ell}$ with a given weight is a very complicated combinatorial problem and currently there are no exact general formulæ. We will achieve the next best thing and rewrite the Poincaré polynomial as a "fake" rational function.

Denote by $b_{k,\ell}(w)$ the number of partitions $\lambda \in \mathcal{P}_{\ell,k}$ with weight $|\lambda| = w$. Hence

$$P_{k,\ell}(t) = \sum_{w=1}^{k\ell} b_{k,\ell}(w) t^{2w}.$$

Alternatively, $b_{k,\ell}(w)$ is the number of Young diagrams of weight w which fit inside a $k \times \ell$ rectangle.

Lemma 7.4.27.

$$b_{k+1,\ell+1}(w) = b_{k,\ell+1}(w) + b_{k+1,\ell}(w - \ell - 1).$$

Proof Look at the $(k+1)\times\ell$ rectangle R^{k+1} inside the $(k+1)\times(l+1)$-rectangle $R^{k+1}_{\ell+1}$ (see Figure 7.6). Then

$$\begin{aligned} b_{k+1,\ell+1}(w) &= \#\,\{\text{diagrams of weight } w \text{ which fit inside } R^{k+1}_{\ell+1}\} \\ &= \#\{\text{diagrams which fit inside } R^{k+1}\} \\ &\quad +\#\{\text{diagrams which do not fit inside } R^{k+1}\}. \end{aligned}$$

On the other hand

$$b_{k+1,\ell} = \#\{\text{diagrams which fit inside } R^{k+1}\}.$$

If a diagram does not fit inside R^{k+1} this means that its first line consists of $\ell+1$ boxes. When we drop this line we get a diagram of weight $w-\ell-1$ which fits inside the $k\times(\ell+1)$ rectangle $R_{\ell+1}$ of Figure 7.6. Thus, the second contribution to $b_{k+1,\ell+1}(w)$ is $b_{k,\ell+1}(w-\ell-1)$. □

The result in the above lemma can be reformulated as

$$P_{k+1,\ell+1}(t) = P_{k+1,\ell}(t) + t^{2(\ell+1)}P_{k,\ell+1}(t).$$

Because the roles of k and ℓ are symmetric (cf. (7.4.8)) we also have

$$P_{k+1,\ell+1}(t) = P_{k,\ell+1}(t) + t^{2(k+1)}P_{k+1,\ell}(t).$$

These two equality together yield

$$P_{k,\ell+1}(1-t^{2(\ell+1)}) = P_{k+1,\ell}(1-t^{2(k+1)}).$$

Let $m = k+\ell+1$ and set $Q_{d,m}(t) = P_{d,m-d}(t) = P_{G_d(m,\mathbb{C})}(t)$. The last equality can be rephrased as

$$Q_{k+1,m}(t) = Q_{k,m}\cdot\frac{1-t^{m-k}}{1-t^{2(k+1)}}$$

so that

$$Q_{k+1,m}(t) = \frac{1-t^{2(m-k)}}{1-t^{2(k+1)}}\cdot\frac{1-t^{2(m-k+1)}}{1-t^{2k}}\cdots\frac{1-t^{2(n-1)}}{1-t^4}.$$

Now we can check easily that $b_{1,m-1}(w)=1$ i.e.

$$Q_{1,m}(t) = 1+t^2+t^4+\cdots+t^{2(m-1)} = \frac{1-t^{2m}}{1-t^2}.$$

Hence

$$\begin{aligned} P_{G_k(m,\mathbb{C})}(t) = Q_{k,m}(t) &= \frac{(1-t^{2(m-k+1)})\cdots(1-t^{2m})}{(1-t^2)\cdots(1-t^{2k})} \\ &= \frac{(1-t^2)\cdots(1-t^{2m})}{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(m-k)})}. \end{aligned}$$

Remark 7.4.28. (a) The invariant theoretic approach in computing the cohomology of $G_k(n, \mathbb{C})$ was used successfully for the first time by C. Ehresmann[24]. His method was then extended to arbitrary compact, oriented symmetric spaces by H. Iwamoto [38]. However, we followed a different avenue which did not require Cartan's maximal weight theory.
(b) We borrowed the idea of using the Weyl's integration formula from Weyl's classical monograph [74]. In turn, Weyl attributes this line of attack to R. Brauer. Our strategy is however quite different from Weyl's. Weyl uses an equality similar to (7.4.5) to produce an upper estimate for the Betti numbers (of $U(n)$ in his case) and then produces by hand sufficiently many invariant forms. The upper estimate is then used to established that these are the only ones. We refer also to [70] for an explicit description of the invariant forms on Grassmannians. □

Exercise 7.4.10. Show that the cohomology *algebra* of $\mathbb{CP}^n$ is isomorphic to the truncated ring of polynomials

$$\mathbb{R}[x]/(x^{n+1})$$

where x is a formal variable of degree 2 while (x^{n+1}) denotes the ideal generated by x^{n+1}.
Hint: Describe $\Lambda^*_{inv}\mathbb{CP}^n$ explicitly.

□

7.5 Čech cohomology

In this last section we return to the problem formulated in the beginning of this chapter: what is the relationship between the Čech and the DeRham approach. We will see that these are essentially two equivalent facets of the same phenomenon. Understanding this equivalence requires the introduction of a new and very versatile concept, namely that of a sheaf. This is done in the first part of the section. The second part is a fast paced introduction to Čech cohomology. A concise yet very clear presentation of these topics can be found in [36]. For a very detailed presentation of this subject we refer to [29].

7.5.1 Sheaves and presheaves

Consider a topological space X. The collection $\mathcal{O}_X$ of its open sets can be organized as a category. The morphisms are the inclusions $U \hookrightarrow V$. A *presheaf* of Abelian groups on X is a contravariant functor $\mathcal{S} : \mathcal{O}_X \to \mathbf{Ab}$.

In other words, $\mathcal{S}$ associates to each open set an Abelian group $\mathcal{S}_U$ and to each inclusion $U \hookrightarrow V$ a group morphism $r^U_V : \mathcal{S}_V \to \mathcal{S}_U$ such that if $U \hookrightarrow V \hookrightarrow W$ then $r^U_W = r^U_V \circ r^V_W$. If $x \in \mathcal{S}_V$ then for any $U \hookrightarrow V$ we set

$$x\,|_U \stackrel{def}{=} r^U_V(x) \in \mathcal{S}_U.$$

If $f \in \mathcal{S}_U$ then we define $\operatorname{dom} f \stackrel{def}{=} U$.

The presheaves of rings, modules, vector spaces are defined in an obvious fashion.

Example 7.5.1. Let X be a topological space. For each open set $U \subset X$ denote by $C(U)$ the space of continuous functions $U \to \mathbb{R}$. The assignment $U \mapsto C(U)$ defines a presheaf of $\mathbb{R}$-algebras on X. The maps r_V^U are determined by the restrictions $|_U: C(V) \to C(U)$. If X is a smooth manifold we get another presheaf $U \mapsto C^\infty(U)$, of smooth function. More generally the differential forms of degree k can be organized in a presheaf $\Omega^k(\bullet)$. If E is a smooth vector bundle then the E-valued differential forms of degree k can be organized as a presheaf of vector spaces

$$U \mapsto \Omega_E^k(U) = \Omega^k(E|_U).$$

If G is an Abelian group equipped with the discrete topology then the G-valued continuous functions $C(U, G)$ determine a presheaf called the constant G-presheaf which is denoted by $\underline{G}_X$. □

Definition 7.5.2. *A presheaf $\mathcal{S}$ on a topological space X is said to be a* sheaf *if the following hold.*
(a) If (U_α) is an open cover of the open set U and $f, g \in \mathcal{S}_U$ satisfy $f|_{U_\alpha} = g|_{U_\alpha}$ $\forall \alpha$ then $f = g$.
(b) If (U_α) is an open cover of the open set U and $f_\alpha \in \mathcal{S}_{U_\alpha}$ satisfy

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta} \quad \forall U_\alpha \cap U_\beta \neq \emptyset$$

then there exists $f \in \mathcal{S}_U$ such that $f|_{U_\alpha} = f_\alpha$, $\forall \alpha$.

Example 7.5.3. All the presheaves discussed in Example 7.5.1 are sheaves. □

Example 7.5.4. Consider the presheaf $\mathcal{S}$ over $\mathbb{R}$ defined by

$$\mathcal{S}(U) = \text{continuous, bounded functions } f : U \to \mathbb{R}.$$

We let the reader verify this is not a sheaf since the condition (b) is violated. The reason behind this violation is that in the definition of this presheaf we included a *global* condition namely the boundedness assumption. □

Definition 7.5.5. *Let X be a topological space and R a commutative ring with 1.*
(a) A space of germs *over X (*“espace étalé ” *in the french literature) is a topological space $\mathcal{E}$ together with a continuous map $\pi : \mathcal{E} \to X$ such that*

(a1) π is a local homeomorphism (that is each point $e \in \mathcal{E}$ has a neighborhood U such that $\pi|_U$ is a homeomorphism onto the open subset $\pi(U) \subset X$).

(a2)The stalk *$\mathcal{E}_x = \pi^{-1}(x)$ is an R-module $\forall x \in X$.*

(a3) The module operations $(u, v) \mapsto (ru + sv)$ ($r, s \in R$, $u, v \in \pi^{-1}(x)$) depend continuously upon u, v.

(b) A section *of a space of germs $\pi : \mathcal{E} \to X$ defined over $U \subset X$ is a continuous function $s : U \to \mathcal{E}$ such that $s(x) \in \mathcal{E}_x$ $\forall x \in X$. The spaces of sections defined over U will be denoted by $\mathcal{E}(U)$.*

Example 7.5.6. (The space of germs associated to a presheaf) Let $\mathcal{S}$ be a presheaf of Abelian groups over a topological space X. For each $x \in X$ define an equivalence relation $\sim_x$ on

$$\bigsqcup_{U \ni x} S_U$$

by

$$f \sim_x g \Longleftrightarrow \exists \text{ open } U \ni x \text{ such that} f|_U = g|_U .$$

The equivalence class of $f \in \bigsqcup_{U \ni x} S_U$ is denoted by $[f]_x$ and is called the *germ* of f at x. Set

$$\mathcal{S}_x = \{[f]_x \; ; \; \operatorname{dom} f \ni x\}.$$

and

$$\hat{\mathcal{S}} = \bigsqcup_{x \in X} \mathcal{S}_x.$$

There exists a natural projection $\pi : \hat{\mathcal{S}} \to X$ which maps $[f]_x$ to x. The "fibers" of this map are $\pi^{-1}(x) = \mathcal{S}_x$ - the germs at $x \in X$. Any $f \in \mathcal{S}_U$ defines a subset

$$f(U) = \{[f]_u \; ; \; u \in U\} \subset \hat{\mathcal{S}}$$

We can define a topology in $\hat{\mathcal{S}}$ by indicating a basis of neighborhoods. A basis of *open* neighborhoods of $[f]_x \in \hat{\mathcal{S}}$ is given by the collection

$$\{g(U) \; ; \; U \ni x, \; g \in \mathcal{S}_U \; [g]_x = [f]_x\}.$$

(We leave the reader check that this collection of sets satisfies the axioms of a basis of neighborhoods as discussed e.g. in [43].) With this choice of topology each $f \in \mathcal{S}_U$ defines a *continuous* section of π over U

$$[f] : \; U \ni u \mapsto [f]_u \in \mathcal{S}_u.$$

Note that each fiber $\mathcal{S}_x$ has a well defined structure of Abelian group

$$[f]_x + [g]_x = [(f+g)|_U]_x \;\; U \ni x \text{ is open and } U \subset \operatorname{dom} f \cap \operatorname{dom} g.$$

(Check this addition is independent of the various choices.) Since $\pi : f(U) \to U$ is a homeomorphism it follows that $\pi : \hat{\mathcal{S}} \to X$ is a space of germs. It is called the *space of germs associated to the presheaf* $\mathcal{S}$. □

If the space of germs associated to a sheaf $\mathcal{S}$ is a covering space we say that $\mathcal{S}$ is a sheaf of locally constant functions (valued in some discrete Abelian group). When the covering is trivial i.e. it is isomorphic to a product $X \times \{\text{discrete set}\}$ then the sheaf is really the constant sheaf associated to a discrete Abelian group.

Example 7.5.7. (The sheaf associated to a space of germs) Consider a space of germs $\mathcal{E} \xrightarrow{\pi} X$ over the topological space X. For each open subset $U \subset X$ denote by $\bar{\mathcal{E}}(U)$ the space of continuous sections $U \to \mathcal{E}$. The correspondence $U \mapsto \bar{\mathcal{E}}(U)$ clearly a sheaf. $\bar{\mathcal{E}}$ is called the sheaf associated to the space of germs. □

Proposition 7.5.8. *(a) Let* $\mathcal{E} \xrightarrow{\pi} X$ *be a space of germs. Then* $\hat{\bar{\mathcal{E}}} = \mathcal{E}$.
(b) Let $\mathcal{S}$ *be a presheaf over the topological space* X. $\mathcal{S}$ *is a sheaf if and only if* $\bar{\hat{\mathcal{S}}} = \mathcal{S}$.

Exercise 7.5.1. Prove the above proposition. □

Definition 7.5.9. *If* $\mathcal{S}$ *is a presheaf over the topological space* X *then* $\bar{\hat{\mathcal{S}}}$ *is called the* sheaf associated to $\mathcal{S}$.

Definition 7.5.10. *(a) Let A* morphism *between the (pre)sheaves of Abelian groups (modules etc.)* $\mathcal{S}$ *and* $\tilde{\mathcal{S}}$ *over the topological space* X *is a collection of morphisms of Abelian groups (modules etc.)* $h_U : \mathcal{S}_U \to \tilde{\mathcal{S}}_U$, *one for each open set* $U \subset X$, *such that when* $V \subset U$ $h_V \circ r_U^V = \tilde{r}_U^V \circ h_U$. *Above,* r_U^V *denotes the restriction morphisms of* $\mathcal{S}$ *while the* $\tilde{r}_U^V$ *denotes the restriction morphisms of* $\tilde{\mathcal{S}}$. *A morphism h is said to be* injective *if each* h_U *is injective.*

(b) Let $\mathcal{S}$ *be a presheaf over the topological space* X. *A* sub-presheaf *of* $\mathcal{S}$ *is a pair* $(\mathcal{T}, \imath)$ *where* $\mathcal{T}$ *is a presheaf over* X *and* $\imath : \mathcal{T} \to \mathcal{S}$ *is an injective morphism.* $\imath$ *is called the* canonical inclusion *of the sub-presheaf.*

Let $h : \mathcal{S} \to \mathcal{T}$ be a morphism of presheaves. The correspondence $U \mapsto \ker h_U \subset \mathcal{S}_U$ defines a presheaf called the *kernel* of the morphism h. It is a sub-presheaf of $\mathcal{S}$.

Proposition 7.5.11. *Let* $h : \mathcal{S} \to \mathcal{T}$ *be a morphism of presheaves. If both* $\mathcal{S}$ *and* $\mathcal{T}$ *are sheaves then so is the kernel of* h.

The proof of this proposition is left to the reader as an exercise.

Definition 7.5.12. *(a)Let* $\mathcal{E}_i \xrightarrow{\pi_i} X$ $(i = 0, 1)$ *be two spaces of germs over the same topological space* X. *A morphism of spaces of germs is a continuous map* $h : \mathcal{E}_0 \to \mathcal{E}_1$ *such that*

(a1) $\pi_1 \circ h = \pi_0$, *i.e.* $h(\pi_0^{-1}(x)) \subset \pi_1^{-1}(x)$ $\forall x \in X$.

(a2) For any $x \in X$ *the induced map* $h_x : \pi_0^{-1}(x) \to \pi_1^{-1}(x)$ *is a morphism of Abelian groups (modules etc.).*

The morphism is called injective if each h_x *is injective.*

(b) Let $\mathcal{E} \xrightarrow{\pi} X$ *be a space of germs. A subspace of germs is a pair* $(\mathcal{F}, \jmath)$ *where* $\mathcal{F}$ *is a space of germs over* X *and* $\jmath : \mathcal{F} \to \mathcal{E}$ *is an injective morphism.*

Proposition 7.5.13. *(a) Let* $h : \mathcal{E}_0 \to \mathcal{E}_1$ *be a morphism between two spaces of germs over* X. *Then* $h(\mathcal{E}_0) \xrightarrow{\pi_1} X$ *is a space of germs over* X *called the image of* h *and denoted by* $\operatorname{Im} h$. *It is a subspace of* $\mathcal{E}_1$.

Exercise 7.5.2. Prove the above proposition. □

Lemma 7.5.14. *Consider two sheaves* $\mathcal{S}$ *and* $\mathcal{T}$ *and let* $h : \mathcal{S} \to \mathcal{T}$ *be a morphism. Then* h *induces a morphism between the associated spaces of germs* $\hat{h} : \hat{\mathcal{S}} \to \hat{\mathcal{T}}$.

The definition of $\hat{h}$ should be obvious. If $f \in \mathcal{S}_U$ and $x \in U$ then $\hat{h}([f]_x) = [h(f)]_x$ where $h(f)$ is now an element of $\mathcal{T}(U)$. We leave the reader check that $\hat{h}$ is independent of the various choices and it is a *continuous* map $\hat{\mathcal{S}} \to \hat{\mathcal{T}}$ with respect to the topologies described in Example 7.5.6.

The sheaf associated to the space of germs $\operatorname{Im} \hat{h}$ is a subsheaf of $\mathcal{T}$ called the *image* of h and is denoted by $\operatorname{Im} h$.

Exercise 7.5.3. Consider a morphism of sheaves (over X) $h : \mathcal{S} \to \mathcal{T}$. Let $U \subset X$ be an open set. Show that a section $g \in \mathcal{T}_U$ belongs to $(\operatorname{Im} h)_U$ if an only if for every $x \in X$ there exists an open neighborhood $V_x \subset U$ such that $g|_{V_x} = h(f)$ for some $f \in \mathcal{S}_{V_x}$. □

Definition 7.5.15. *(a) A sequence of sheaves and morphisms of sheaves*

$$\cdots \longrightarrow \mathcal{S}_n \stackrel{h_n}{\rightarrow} \mathcal{S}_{n+1} \stackrel{h_{n+1}}{\rightarrow} \mathcal{S}_{n+2} \longrightarrow \cdots$$

is said to be exact if $\operatorname{Im} h_n = \ker h_{n+1}$, $\forall n$.
(b) Consider a sheaf $\mathcal{S}$ *over the space* X. *A* resolution *of* $\mathcal{S}$ *is a long exact sequence*

$$0 \hookrightarrow \mathcal{S} \stackrel{\imath}{\hookrightarrow} \mathcal{S}_0 \stackrel{d_0}{\rightarrow} \mathcal{S}_1 \stackrel{d_1}{\rightarrow} \cdots \longrightarrow \mathcal{S}_n \stackrel{d_n}{\rightarrow} \mathcal{S}_{n+1} \rightarrow \cdots.$$

Exercise 7.5.4. Consider a short exact sequence of sheaves

$$0 \rightarrow \mathcal{S}_{-1} \rightarrow \mathcal{S}_0 \rightarrow \mathcal{S}_1 \rightarrow 0.$$

For each open set U define $\mathcal{S}(U) = \mathcal{S}_0(U)/\mathcal{S}_{-1}(U)$.
(a) Prove that $U \mapsto \mathcal{S}(U)$ is a presheaf.
(b) Prove that $\mathcal{S}_1 \cong \bar{\bar{\mathcal{S}}}$ = the sheaf associated to the presheaf $\mathcal{S}$. □

Example 7.5.16. (The DeRham resolution) Let M be a smooth n-dimensional manifold. Using the Poincaré lemma and the Exercise 7.5.3 we deduce immediately that the sequence

$$0 \hookrightarrow \underline{\mathbb{R}}_M \hookrightarrow \Omega^0_M \stackrel{d}{\rightarrow} \Omega^1_M \stackrel{d}{\rightarrow} \cdots \stackrel{d}{\rightarrow} \Omega^n_M \rightarrow 0$$

is a resolution of the constant sheaf $\underline{\mathbb{R}}_M$. Ω^k_M denotes the sheaf of k-forms on M while d denotes the exterior differentiation. □

7.5.2 Čech cohomology

Let $U \mapsto \mathcal{S}(U)$ be pre-sheaf of Abelian groups over a topological space X. Consider an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{A}}$ of X. A *q-simplex* ($q \in \mathbb{Z}_+$) is an ordered $(q+1)$-uple

$$\sigma = (\alpha_0, \alpha_1, \ldots, \sigma_q) \in \mathcal{A}^{q+1}$$

such that

$$U_\sigma \stackrel{def}{=} \bigcap_0^q U_{\alpha_i} \neq \emptyset.$$

The set of all q-simplices is denoted by $\mathcal{U}_{(q)}$. Their union

$$\bigcup_q \mathcal{U}_{(q)}$$

is denoted by $\mathcal{N}(\mathcal{U})$ and is called the *nerve* of the cover. Define

$$C^q(\mathcal{S}, \mathcal{U}) = \prod_{\sigma \in \mathcal{U}_{(q)}} \mathcal{S}_\sigma \quad (\mathcal{S}_\sigma = \mathcal{S}(U_\sigma)).$$

The elements of $C^q(\mathcal{S}, \mathcal{U})$ are called Čech q-cochains (subordinated to the cover $\mathcal{U}$). In other words, a q-cochain c associates to each q-simplex σ an element $\langle c, \sigma \rangle \in \mathcal{S}_\sigma$.

For each q-simplex $\sigma = (\alpha_0, \ldots, \alpha_q)$ we define its j-th boundary as the $(q-1)$-simplex

$$\partial^j \sigma = \partial^j_q \sigma = (\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_q) \in \mathcal{U}_{(q-1)},$$

where as usual a hat indicates a missing entry.

Exercise 7.5.5. Prove that $\partial^i_{q-1}\partial^j_q = \partial^{j-1}_{q-1}\partial^i_q$ for $j > i$. □

We can now define an operator

$$\delta : C^{q-1}(\mathcal{S}, \mathcal{U}) \to C^q(\mathcal{S}, \mathcal{U})$$

which assigns to each $(q-1)$-cochain c a q-cochain δc whose value on a q-simplex σ is given by

$$\langle \delta c, \sigma \rangle = \sum_{j=0}^{q} (-1)^j \langle c, \partial^j \sigma \rangle \,|_{U_\sigma} \; .$$

Using the Exercise 7.5.5 above one deduces immediately the following result.

Lemma 7.5.17. $\delta^2 = 0$ *so that*

$$0 \hookrightarrow C^0(\mathcal{S}, \mathcal{U}) \xrightarrow{\delta} C^1(\mathcal{S}, \mathcal{U}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^q(\mathcal{S}, \mathcal{U}) \xrightarrow{\delta} \cdots$$

is a cochain complex.

The cohomology of this cochain is called the *Čech cohomology* of the cover $\mathcal{U}$ with coefficients in the pre-sheaf $\mathcal{S}$.

Example 7.5.18. Let $\mathcal{U}$ and $\mathcal{S}$ as above. A 0-cochain is is a correspondence which associates to each open set $U_\alpha \in \mathcal{U}$ an element $c_\alpha \in \mathcal{S}(U_\alpha)$. It is a cocycle if for any 1-simplex (α, β) of the nerve we have

$$c_\beta - c_\alpha = 0.$$

A 1-cochain associates to each 1-simplex (α, β) an element

$$c_{\alpha\beta} \in \mathcal{S}(U_{\alpha\beta}).$$

This correspondence is a cocycle if for any 2-simplex (α, β, γ) we have

$$c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta} = 0.$$

For example if X is a smooth manifold and $\mathcal{U}$ is a good cover then we can associate to each closed 1-form $\omega \in \Omega^1(M)$ a Čech 1-cocycle valued in $\underline{\mathbb{R}}_X$ as follows.

First, select for each U_α a solution $f_\alpha \in C^\infty(U_\alpha)$ of

$$df_\alpha = \omega.$$

Since $d(f_\alpha - f_\beta) \equiv 0$ on $U_{\alpha\beta}$ we deduce there exist constants $c_{\alpha\beta}$ such that $f_\alpha - f_\beta = c_{\alpha\beta}$. Obviously this is a cocycle and it is easy to see that its cohomology class is independent of the initial selection of local solutions f_α. Moreover, if ω is exact this cocycle is a coboundary. In other words we have a natural map

$$H^1(X) \to H^1(\mathcal{N}(\mathcal{U}), \underline{\mathbb{R}}_X).$$

We will see later this is an isomorphism. □

Definition 7.5.19. *Consider two open covers* $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ *and* $\mathcal{V} = (V_\beta)_{\beta\in\mathcal{B}}$ *of the same topological space* X*. We say* $\mathcal{V}$ *is finer than* $\mathcal{U}$ *and we write this* $\mathcal{U} \prec \mathcal{V}$ *if there exists a map* $\varrho : \mathcal{B} \to \mathcal{A}$ *such that*

$$V_\beta \subset U_{\varrho(\beta)} \quad \forall \beta \in \mathcal{B}.$$

The map ϱ *is said to be a* refinement map.

Proposition 7.5.20. *Consider two open covers* $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ *and* $\mathcal{V} = (V_\beta)_{\beta\in\mathcal{B}}$ *of the same topological space* X *such that* $\mathcal{U} \prec \mathcal{V}$*. Fix a sheaf of Abelian groups* $\mathcal{S}$*. Then the following are true.*
(a) Any refinement map ϱ *induces a cochain map*

$$\varrho_* : \bigoplus C^q(\mathcal{U}, \mathcal{S}) \to \bigoplus C^q(\mathcal{V}, \mathcal{S}).$$

(b) If $r : \mathcal{A} \to \mathcal{B}$ *is another refinement map then* ϱ_* *is cochain homotopic to* r_**. In particular, any relation* $\mathcal{U} \prec \mathcal{V}$ *defines a unique morphism in cohomology*

$$\imath_{\mathcal{U}}^{\mathcal{V}} : H^*(\mathcal{U}, \mathcal{S}) \to H^*(\mathcal{V}, \mathcal{S}).$$

(c) If $\mathcal{U} \prec \mathcal{V} \prec \mathcal{W}$ *then*

$$\imath_{\mathcal{U}}^{\mathcal{W}} = \imath_{\mathcal{V}}^{\mathcal{W}} \circ \imath_{\mathcal{U}}^{\mathcal{V}}.$$

Proof (a) We define $\varrho_* : C^q(\mathcal{U}) \to C^q(\mathcal{V})$ by

$$\mathcal{S}(V_\sigma) \ni \langle \varrho_*(c), \sigma\rangle := \langle c, \varrho(\sigma)\rangle|_{V_\sigma} \quad \forall c \in C^q(\mathcal{U})\ \sigma \in \mathcal{V}_{(q))}$$

where by definition $\varrho(\sigma) \in \mathcal{A}^q$ is the q-simplex $(\varrho(\beta_0), \dots, \varrho(\beta_q))$. The fact that ϱ_* is a cochain map follows immediately from the obvious equality

$$\varrho \circ \partial_q^j = \partial_q^j \circ \varrho.$$

(b) We define $h_j : \mathcal{V}_{(q-1)} \to \mathcal{U}_{(q)}$ by

$$h_j(\beta_0, \dots, \beta_{q-1}) = (\varrho(\beta_0), \dots, \varrho(\beta_j), r(\beta_j), \cdots, r(\beta_{q-1})).$$

The reader should check that $h_j(\sigma)$ is indeed a simplex of $\mathcal{U}$ for any simplex σ of $\mathcal{V}$. Note that $V_\sigma \subset U_{h_j(\sigma)}$ $\forall j$. Now define

$$\chi = \chi_q : C^q(\mathcal{U}) \to C^{q-1}(\mathcal{V})$$

by

$$\langle \chi_q(c), \sigma\rangle := \sum_{j=0}^{q-1} (-1)^j \langle c, h_j(\sigma)\rangle|_{V_\sigma} \quad \forall c \in C^q(\mathcal{U})\ \forall \sigma \in \mathcal{V}_{(q-1)}.$$

We will show that

$$\delta \circ \chi_q(c) + \chi_{q+1} \circ \delta(\sigma) = \varrho_*(c) - r_*(c) \quad \forall c \in C^q(\mathcal{U}).$$

Let $\sigma = (\beta_0, \dots, \beta_q) \in \mathcal{V}_{(q)}$ and set

$$\varrho(\sigma) = (\lambda_0, \dots, \lambda_q),\ r(\sigma) = (\mu_0, \dots, \mu_q) \in \mathcal{U}_{(q)}$$

so that

$$h_j(\sigma) = (\lambda_0, \dots, \lambda_j, \mu_j, \dots, \mu_q).$$

Then

$$\begin{aligned}
\langle \chi \circ \delta(c), \sigma \rangle &= \sum (-1)^j \langle \delta c, h_j(\sigma) \rangle \,|_{V_\sigma} \\
&= \sum_{j=0}^{q} (-1)^j (\sum_{k=0}^{q+1} (-1)^k \langle c, \partial^k_{q+1} h_j(\sigma) \rangle \,|_{V_\sigma})
\end{aligned}$$

$$= \sum_{j=0}^{q} (-1)^j \left(\sum_{k=0}^{j} (-1)^k \langle c, (\lambda_0, \dots, \hat{\lambda}_j, \dots, \lambda_j, \mu_j, \dots, \mu_q) \rangle \,|_{V_\sigma} \right.$$
$$\left. + \sum_{\ell=j}^{q} (-1)^{\ell+1} \langle c, (\lambda_0, \dots, \lambda_j, \mu_j, \dots, \hat{\mu}_k, \dots, \mu_q) \rangle \,|_{V_\sigma} \right)$$
$$= \sum_{j=0}^{q} (-1)^j \left(\sum_{k=0}^{j-1} (-1)^k \langle c, (\lambda_0, \dots, \hat{\lambda}_j, \dots, \lambda_j, \mu_j, \dots, \mu_q) \rangle \,|_{V_\sigma} \right.$$
$$\left. + \sum_{\ell=j+1}^{q} (-1)^{\ell+1} \langle c, (\lambda_0, \dots, \lambda_j, \mu_j, \dots, \hat{\mu}_k, \dots, \mu_q) \rangle \,|_{V_\sigma} \right)$$
$$+ \sum_{j=0}^{q} (-1)^j \left\{ \langle c, (\lambda_0, \dots, \lambda_{j-1}, \mu_j, \dots, \mu_q) \rangle \,|_{V_\sigma} + \langle c, (\lambda_0, \dots, \lambda_j, \mu_{j+1}, \dots, \mu_q) \rangle \,|_{V_\sigma} \right\}.$$

The last term is a telescopic sum which is equal to

$$\langle c, (\lambda_0, \dots, \lambda_q \rangle \,|_{V_\sigma} - \langle c, (\mu_0, \dots, \mu_q) \rangle \,|_{V_\sigma} = \langle \varrho_* c, \sigma \rangle - \langle r_* c, \sigma \rangle.$$

If we change the order of summation in the first two term we recover the term $\langle -\delta \chi c, \sigma \rangle$.
(c) is left to the reader as an exercise.

□

We now have a collection of groups

$$\{ H^*(\mathcal{U}, \mathcal{S}) \ ; \ \mathcal{U} - \text{open cover of } X \}$$

and maps

$$\left\{ \imath^{\mathcal{V}}_{\mathcal{U}} : H^*(\mathcal{U}, \mathcal{S}) \to H^*(\mathcal{V}, \mathcal{S}); \ \ \mathcal{U} \prec \mathcal{V} \right\}$$

such that

$$\imath^{\mathcal{U}}_{\mathcal{U}} = \mathbf{id}$$

and

$$\imath^{\mathcal{W}}_{\mathcal{U}} = \imath^{\mathcal{W}}_{\mathcal{V}} \circ \imath^{\mathcal{V}}_{\mathcal{U}}$$

whenever $\mathcal{U} \prec \mathcal{V} \prec \mathcal{W}$. We can thus define the inductive limit

$$H^*(X, \mathcal{S}) \stackrel{def}{=} \lim_{\mathcal{U}} H^*(\mathcal{U}, \mathcal{S}).$$

$H^*(X, \mathcal{S})$ is called the *Čech cohomology* of the space X with coefficients in the pre-sheaf $\mathcal{S}$. Let us briefly recall the definition of the direct limit. One defines an equivalence relation on the disjoint union

$$\coprod_{\mathcal{U}} H^*(\mathcal{U}, \mathcal{S})$$

by

$$H^*(\mathcal{U}) \ni f \sim g \in H^*(\mathcal{V}) \Longleftrightarrow \exists \mathcal{W} \succ \mathcal{U},\ \mathcal{V}:\ \imath_{\mathcal{U}}^{\mathcal{W}} f = \imath_{\mathcal{V}}^{\mathcal{W}} g.$$

We denote the equivalence class of f by $\overline{f}$. Then

$$\lim_{\mathcal{U}} H^*(\mathcal{U}) = \left(\coprod_{\mathcal{U}} H^*(\mathcal{U}, \mathcal{S}) \right) / \sim .$$

At limit we get maps

$$\imath_{\mathcal{U}} : H^*(\mathcal{U}, \mathcal{S}) \to H^*(X, \mathcal{S}).$$

Example 7.5.21. Let $\mathcal{S}$ be a *sheaf* over the space X. For any open cover $\mathcal{U} = (U_\alpha)$ a 0-cycle subordinated to $\mathcal{U}$ is a collection of sections $f_\alpha \in \mathcal{S}(U_\alpha)$ such that every time $U_\alpha \cap U_\beta \neq \emptyset$

$$f_\alpha |_{U_{\alpha\beta}} = f_\beta |_{U_{\alpha\beta}} .$$

According to the properties of a sheaf, such a collection defines a unique *global* section $f \in \mathcal{S}(X)$. Hence $H^0(X, \mathcal{S}) = \mathcal{S}(X)$. □

Proposition 7.5.22. *Any morphism of pre-sheaves $h : \mathcal{S}_0 \to \mathcal{S}_1$ over X induces a morphism in cohomology*

$$h_* : H^*(X, \mathcal{S}_0) \to H^*(X, \mathcal{S}_1).$$

Sketch of proof Let $\mathcal{U}$ be an open cover of X. Define

$$h_* : C^q(\mathcal{U}, \mathcal{S}_0) \to C^q(\mathcal{U}, \mathcal{S}_1)$$

by

$$\langle h_* c, \sigma \rangle = h_U(\langle c, \sigma \rangle) \quad \forall c \in C^q(\mathcal{U}, \mathcal{S}_0)\ \sigma \in \mathcal{U}_(q).$$

The reader can check easily that h_* is a cochain map so it induces a map in cohomology

$$h_*^{\mathcal{U}} : H^*(\mathcal{U}, \mathcal{S}_0) \to H^*(\mathcal{U}, \mathcal{S}_1)$$

which commutes with the refinements $\imath_{\mathcal{U}}^{\mathcal{V}}$. The proposition follows by passing to direct limits.

□

Theorem 7.5.23. *Let*

$$0 \to \mathcal{S}_{-1} \xrightarrow{\jmath} \mathcal{S}_0 \xrightarrow{p} \mathcal{S}_1 \to 0$$

be an exact sequence of sheaves over a paracompact *space X. Then there exists a natural long exact sequence*

$$\cdots \to H^q(X, \mathcal{S}_{-1}) \xrightarrow{\jmath_*} H^q(X, \mathcal{S}_0) \xrightarrow{p_*} H^q(X, \mathcal{S}_1) \xrightarrow{\delta_*} H^{q+1}(X, \mathcal{S}_{-1}) \to \cdots .$$

Sketch of proof For each open set $U \subset X$ define $\mathcal{S}(U) = \mathcal{S}_0(U)/\mathcal{S}_{-1}(U)$. Then the correspondence $U \mapsto \mathcal{S}(U)$ defines a pre-sheaf on X. Its associated sheaf is isomorphic with $\mathcal{S}_1$ (see Exercise 7.5.4). Thus for each open cover $\mathcal{U}$ we have a short exact sequence

$$0 \to C^q(\mathcal{U}, \mathcal{S}_{-1}) \xrightarrow{\jmath} C^q(\mathcal{U}, \mathcal{S}_0) \xrightarrow{\pi} C^q(\mathcal{U}, \mathcal{S}) \to 0.$$

We thus get a long exact sequence in cohomology

$$\cdots \to H^q(\mathcal{U}, \mathcal{S}_{-1}) \to H^q(\mathcal{U}, \mathcal{S}_0) \to H^q(\mathcal{U}, \mathcal{S}) \to H^{q+1}(\mathcal{U}, X) \to \cdots.$$

Passing to direct limits we get a long exact sequence

$$\cdots \to H^q(X, \mathcal{S}_{-1}) \to H^q(X, \mathcal{S}_0) \to H^q(X, \mathcal{S}) \to H^{q+1}(X, \mathcal{S}_{-1}) \to \cdots.$$

To conclude the proof of the proposition we invoke the following technical result. Its proof can be found in [68].

Lemma 7.5.24. *If two pre-sheaves $\mathcal{S}$, $\mathcal{S}'$ over a* paracompact *topological space X have isomorphic associated sheaves then*

$$H^*(X, \mathcal{S}) \cong H^*(X, \mathcal{S}').$$

□

Definition 7.5.25. *A sheaf $\mathcal{S}$ is said to be* fine *if for any locally finite open cover $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{A}}$ there exist morphisms $h_\alpha : \mathcal{S} \to \mathcal{S}$ with the following properties.*
(a) For any $\alpha \in \mathcal{A}$ there exists a closed set C_α such that $C_\alpha \subset U_\alpha$ and $h_\alpha(\mathcal{S}_x) = 0$ for $x \notin C_\alpha$ where $\mathcal{S}_x$ denotes the stalk of $\mathcal{S}$ at $x \in X$. The set C_α is called a support of h_α and and we write this $\operatorname{supp} h_\alpha \subset C_\alpha$.
(b)$\sum_\alpha h_\alpha = \mathbf{1}_{\mathcal{S}}$. This sum is well defined since the cover $\mathcal{U}$ is locally finite.

Example 7.5.26. Let X be a smooth manifold. Using partitions of unity we deduce that the sheaf Ω^k_X of smooth k-forms is fine. More generally, if E is a smooth vector bundle over X then the space Ω^k_E of E-valued k-forms is fine. □

Proposition 7.5.27. *Let $\mathcal{S}$ be a fine sheaf over a paracompact space X. Then $H^q(X, \mathcal{S}) \cong 0$ for $q \geq 1$.*

Proof Because X is paracompact any open cover admits a locally finite refinement. Thus it suffices to show that for each locally finite open cover $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{A}}$ the cohomology groups $H^q(\mathcal{U}, \mathcal{S})$ are trivial for $q \geq 1$. We will achieve this by showing that the identity map $C^q(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S})$ is cochain homotopic with the trivial map. We thus need to produce a map

$$\chi^q : C^q(\mathcal{U}, \mathcal{S}) \to C^{q-1}(\mathcal{U}, \mathcal{S})$$

such that

$$\chi^{q+1}\delta^q + \delta^{q-1}\chi^q = \mathbf{id}. \tag{7.5.1}$$

Consider the morphisms $h_\alpha : \mathcal{S} \to \mathcal{S}$ associated to the cover $\mathcal{U}$ postulated in the definition of a fine sheaf. For each $\alpha \in \mathcal{A}$, $\sigma \in \mathcal{U}_{(q-1)}$ and $f \in C^q(\mathcal{U}, \mathcal{S})$ we construct $\langle t_\alpha(f), \sigma\rangle \in \mathcal{S}(U_\sigma)$ as follows. Consider the open cover of U_σ

$$\{V = U_\alpha \cap U_\sigma, W = U_\sigma \setminus C_\alpha\} \quad (\operatorname{supp} h_\alpha \subset C_\alpha).$$

Note that $h_\alpha f|_{V\cap W} = 0$ and according to the axioms of a sheaf $h_\alpha(f|_V)$ can be extended by zero to a section $\langle t_\alpha(f), \sigma\rangle \in \mathcal{S}(U_\sigma)$. Now, for every $f \in C^q(\mathcal{U}, \mathcal{S})$ define $\chi^q f \in C^{q-1}(\mathcal{U}, \mathcal{S})$ by

$$\langle \chi^q(f), \sigma\rangle = \sum_\alpha \langle t_\alpha(f), \sigma\rangle.$$

The above sum is well defined since the cover $\mathcal{U}$ is locally finite. We leave the reader check that χ^q satisfies (7.5.1).

□

Definition 7.5.28. *Let $\mathcal{S}$ be a sheaf over a space X. A* fine resolution *is a resolution*

$$0 \to \mathcal{S} \hookrightarrow \mathcal{S}_0 \xrightarrow{d} \mathcal{S}_1 \xrightarrow{d} \cdots$$

such that each of the sheaves $\mathcal{S}_j$ is fine.

Theorem 7.5.29. (Abstract DeRham theorem) *Let*

$$0 \hookrightarrow \mathcal{S} \to \mathcal{S}_0 \xrightarrow{d^0} \mathcal{S}_1 \xrightarrow{d^1} \cdots$$

be a fine resolution of the sheaf $\mathcal{S}$ over the paracompact *space X. Then*

$$0 \to \mathcal{S}_0(X) \xrightarrow{d_0} \mathcal{S}_1(X) \xrightarrow{d_1} \cdots$$

is a cochain complex and there exists a natural isomorphism

$$H^q(X, \mathcal{S}) \cong H^q(\mathcal{S}_q(X)).$$

Proof The first statement in the theorem can be safely left to the reader. For $q \geq 1$ denote by $\mathcal{Z}_q$ the kernel of the sheaf morphism d_q. We set for uniformity $\mathcal{Z}_0 = \mathcal{S}$. We get a short exact sequence of sheaves

$$0 \to \mathcal{Z}_q \to \mathcal{S}_q \to \mathcal{Z}_{q+1} \to 0 \quad q \geq 0. \tag{7.5.2}$$

We use the associated long exact sequence in which $H^k(X, \mathcal{S}_q) = 0$ for $k \geq 1$ since $\mathcal{S}_q$ is a fine sheaf. This yields the isomorphisms

$$H^{k-1}(X, \mathcal{Z}_{q+1}) \cong H^k(X, \mathcal{Z}_q) \quad k \geq 2.$$

We deduce inductively that

$$H^m(X, \mathcal{Z}_0) \cong H^1(X, \mathcal{Z}_{m-1}) \quad m \geq 1. \tag{7.5.3}$$

Using again the long sequence associated to (7.5.2) we get an exact sequence

$$H^0(X, \mathcal{Z}_{m-1}) \stackrel{d_*^{m-1}}{\to} H^0(X, \mathcal{Z}_m) \to H^1(X, \mathcal{Z}_{m-1}) \to 0.$$

We apply the computation in Example 7.5.21 we get

$$H^1(X, \mathcal{Z}_{m-1}) \cong \mathcal{Z}_m(X)/d_*^{m-1}\left(\mathcal{S}_{m-1}(X)\right).$$

This is precisely the content of the theorem.

□

Corollary 7.5.30. *Let M be a smooth manifold. Then*

$$H^*(M, \underline{\mathbb{R}}_M) \cong H^*(M).$$

Proof M is paracompact. We conclude using the fine resolution

$$0 \to \underline{\mathbb{R}}_M \hookrightarrow \Omega_M^0 \stackrel{d}{\to} \Omega_M^1 \to \cdots .$$

□

Exercise 7.5.6. Describe explicitly the isomorphisms

$$H^1(M) \cong H^1(M, \underline{\mathbb{R}}_M)$$

and

$$H^2(M) \cong H^2(M, \underline{\mathbb{R}}_M).$$

□

Remark 7.5.31. The above corollary has a surprising implication. Obviously the Čech cohomology is a *topological* invariant and thus, so must be the DeRham cohomology which is defined in terms of a smooth structure. Hence if two *smooth* manifolds are *homeomorphic* they must have isomorphic DeRham groups! Such exotic situations do exist. In a celebrated paper [54], John Milnor has constructed a family of nondiffeomorphic manifolds all homeomorphic to the sphere S^7. More recently, the work of Simon Donaldson in gauge theory was used by Michael Freedman to construct a smooth manifold homeomorphic to $\mathbb{R}^4$ but not diffeomorphic with $\mathbb{R}^4$ equipped with the natural smooth structure. (This is possible only for 4-dimensional vector spaces!) These three mathematicians, J. Milnor, S. Donaldson and M. Freedman were awarded Fields medals for their contributions. □

Theorem 7.5.32. *Let M be a smooth manifold and $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{A}}$ a good cover of M i.e.*

$$U_\sigma \cong \mathbb{R}^{\dim M}.$$

Then

$$H^*(\mathcal{U}, \underline{\mathbb{R}}_M) \cong H^*(M).$$

Proof Let $\mathcal{Z}^k$ denote the sheaf of closed k-forms on M. Using the Poincaré lemma we deduce

$$\mathcal{Z}^k(U_\sigma) = d\Omega^{k-1}(U_\sigma).$$

We thus have a short exact sequence

$$0 \to C^q(\mathcal{U}, \mathcal{Z}^k) \to C^q(\mathcal{U}, \Omega^{k-1}) \stackrel{d_*}{\to} C^q(\mathcal{U}, \mathcal{Z}^k) \to 0.$$

Using the associated long exact sequence and the fact that Ω^{k-1} is a fine sheaf we deduce as in the proof of the abstract DeRham theorem that

$$H^q(\mathcal{U}, \underline{\mathbb{R}}_M) \cong H^{q-1}(\mathcal{U}, \mathcal{Z}^1) \cong \cdots$$
$$\cong H^1(\mathcal{U}, \mathcal{Z}^{q-1}) \cong H^0(\mathcal{U}, \mathcal{Z}^q)/d_* H^0(\mathcal{U}, \Omega^{q-1}) \cong \mathcal{Z}^q(M)/d\Omega^{q-1}(M) \cong H^*(M).$$

□

Remark 7.5.33. The above result is a special case of a theorem of Leray: if $\mathcal{S}$ is a sheaf on a *paracompact* space X and $\mathcal{U}$ is an open cover such that

$$H^q(U_\sigma, \mathcal{S}) = 0 \quad \forall q \geq 1 \; \sigma \in \mathcal{U}_{(k)}$$

then $H^*(\mathcal{U}, \mathcal{S}) = H^*(X, \mathcal{S})$. For a proof we refer to [29].

When $\underline{G}_M$ is a constant sheaf (where G is an arbitrary Abelian group) we have a Poincaré lemma (see [25, Chapter IX, Thm. 5.1])

$$H^q(\mathbb{R}^n, \underline{G}) = 0 \quad q \geq 1.$$

Hence for any good cover $\mathcal{U}$

$$H^*(M, \underline{G}_M) = H^*(\mathcal{U}, \underline{G}_M).$$

□

Example 7.5.34. Let M be a smooth manifold and $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{A}}$ a good cover of M. A 1-cocycle of $\underline{\mathbb{R}}_M$ is a collection of real numbers $f_{\alpha\beta}$ - one for each pair $(\alpha, \beta) \in \mathcal{A}^2$ such that $U_{\alpha\beta} \neq \emptyset$ satisfying

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$$

whenever $U_{\alpha\beta\gamma} \neq \emptyset$. The collection is a coboundary if there exist the constants f_α such that $f_{\alpha\beta} = f_\beta - f_\alpha$. This is precisely the situation encountered in Subsection 7.1.2. The abstract DeRham theorem explains why the Čech approach is equivalent with the DeRham approach. □

Remark 7.5.35. Often in concrete applications one finds it is convenient to work with skew-symmetric Čech cochains. A cochain $c \in C^q(\mathcal{S}, \mathcal{U})$ is skew-symmetric if for any q simplex $\sigma = (\alpha_0, \ldots, \alpha_q)$ and for all $\varphi \in \mathfrak{S}_{q+1}$ we have

$$\langle c, (\alpha_0, \ldots, \alpha_q) \rangle = \epsilon(\varphi) \langle c, (\alpha_{\varphi(0)}, \ldots, \alpha_{\varphi(q)}) \rangle.$$

One can then define a "skew-symmetric" Čech cohomology following the same strategy. (The various intervening formulæ will be more elaborate.) The resulting cohomology coincides with the cohomology described in this subsection. For a proof of this fact we refer to [68]. □

Chapter 8

Characteristic classes

We now have sufficient background to approach a problem formulated in Chapter 2: find a way to measure the "extent of nontriviality" of a given vector bundle. This is essentially a topological issue but, as we will see, in the context of smooth manifolds there are powerful differential geometric methods which will solve a large part of this problem. Ultimately, only topological techniques yield the best results.

8.1 Chern-Weil theory

8.1.1 Connections in principal G-bundles

In this subsection we will describe how to take into account the possible symmetries of a vector bundle when describing a connection.

All the Lie groups we will consider will be assumed to be *matrix Lie groups* i.e. Lie subgroups of a general linear group $GL(n, \mathbb{K})$.

This restriction is neither severe, nor necessary. It is not severe since, according to a nontrivial result (Peter-Weyl theorem), any compact Lie group is isomorphic with a matrix Lie group and these groups are sufficient for the applications in geometry. It is not necessary since all the results of this subsection are true for any Lie group. We stick with this assumption since most proofs are easier to "swallow" in this context.

For a matrix Lie group G, its Lie algebra $\mathfrak{g}$ is a Lie algebra of matrices in which the bracket is the usual commutator.

Let M be a smooth manifold. Recall that a principal G-bundle P over M can be defined by an open cover (U_α) of M and a gluing cocycle

$$g_{\alpha\beta} : U_{\alpha\beta} \to G.$$

The Lie group G operates on its Lie algebra $\mathfrak{g}$ via the *adjoint action*

$$Ad : G \to GL(\mathfrak{g}), \quad g \mapsto Ad(g) \in GL(\mathfrak{g})$$

where

$$Ad(g)X = gXg^{-1} \quad \forall X \in \mathfrak{g}.$$

We denote by $Ad(P)$ the vector bundle with standard fiber $\mathfrak{g}$ associated to P via the adjoint representation. In other words, $Ad(P)$ is the vector bundle defined by the open cover (U_α) and gluing cocycle

$$Ad(g_{\alpha\beta}) : U_{\alpha\beta} \to GL(\mathfrak{g}).$$

The bracket operation in the fibers of $Ad(P)$ induces a bilinear map

$$[\cdot, \cdot] : \Omega^k(Ad(P)) \times \Omega^\ell(Ad(P)) \to \Omega^{k+\ell}(Ad(P))$$

defined by

$$[\omega^k \otimes X, \eta^\ell \otimes Y] = (\omega^k \wedge \eta^\ell) \otimes [X, Y] \tag{8.1.1}$$

for all $\omega^k \in \Omega^k(M)$, $\eta^\ell \in \Omega^\ell(M)$ and $X, Y \in \Omega^0(Ad(P))$.

Exercise 8.1.1. Prove that for any $\omega, \eta, \phi \in \Omega^*(Ad(P))$ the following hold.

$$[\omega, \eta] = -(-1)^{|\omega|\cdot|\eta|}[\eta, \omega]. \tag{8.1.2}$$

$$[[\omega, \eta], \phi] = [[\omega, \phi], \eta] + (-1)^{|\omega|\cdot|\phi|}[\omega, [\eta, \phi]]. \tag{8.1.3}$$

In other words, $\Omega^*(Ad(P)), [\ ,\ ])$ is a super Lie algebra. □

Definition 8.1.1. *(a) A* connection *on the principal bundle P defined by the open cover U_α and the gluing cocycle $g_{\beta\alpha} : U_{\alpha\beta} \to G$is a collection*

$$A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$$

satisfying the transition rules

$$A_\beta(x) = g_{\alpha\beta}^{-1}(x) dg_{\alpha\beta}(x) + g_{\alpha\beta}^{-1}(x) A_\alpha(x) g_{\alpha\beta}(x) \quad \forall x \in U_{\alpha\beta}.$$

(b) The curvature *of the above connection is defined by the collection $F_\alpha \in \Omega^2(U_\alpha) \otimes \mathfrak{g}$ where*

$$F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha].$$

Proposition 8.1.2. *(a) The set $\mathcal{A}(P)$ of connections on P is an affine space modeled by $\Omega^1(Ad(P))$.*
(b) The collection (F_α) defines a global $Ad(P)$-valued 2-form.
(c) (The Bianchi identity)

$$dF_\alpha + [A_\alpha, F_\alpha] = 0 \quad \forall \alpha. \tag{8.1.4}$$

Proof (a) If (A_α), $(B_\alpha) \in \mathcal{A}(P)$ then their difference $C_\alpha = A_\alpha - B_\alpha$ satisfies the gluing rules

$$C_\beta = g_{\beta\alpha} C_\alpha g_{\beta\alpha}^{-1}$$

so that it defines an element of $\Omega^1(Ad(P))$.
(b) We need to check that the forms F_α satisfy the gluing rules

$$F_\beta = g^{-1} F_\alpha g$$

where $g = g_{\alpha\beta} = g_{\beta\alpha}^{-1}$. We have

$$F_\beta = dA_\beta + \frac{1}{2}[A_\beta, A_\beta]$$

$$= d(g^{-1}dg + g^{-1}A_\alpha g) + \frac{1}{2}[g^{-1}dg + g^{-1}A_\alpha g, g^{-1}dg + g^{-1}A_\alpha g].$$

Set $\varpi = g^{-1}dg$. Using (8.1.2) we get

$$F_\beta = d\varpi + \frac{1}{2}[\varpi, \varpi]$$

$$+d(g^{-1}A_\alpha g) + [\varpi, g^{-1}A_\alpha g] + \frac{1}{2}[g^{-1}A_\alpha g, g^{-1}A_\alpha g]. \tag{8.1.5}$$

We will check two things.
A. The *Maurer-Cartan* structural equations.

$$d\varpi + \frac{1}{2}[\varpi, \varpi] = 0.$$

B.

$$d(g^{-1}A_\alpha g) + [\varpi, g^{-1}A_\alpha g] = g^{-1}(dA_\alpha)g.$$

Proof of A. Let us first introduce a new operation. Let $\underline{gl}(n, \mathbb{K})$ denote the associative algebra of $\mathbb{K}$-valued $n \times n$ matrices. There exists a natural operation

$$\wedge : \Omega^k(U_\alpha) \otimes \underline{gl}(n, \mathbb{K}) \times \Omega^\ell(U_\alpha) \otimes \underline{gl}(n, \mathbb{K}) \to \Omega^{k+\ell}(U_\alpha) \otimes \underline{gl}(n, \mathbb{K})$$

uniquely defined by

$$(\omega^k \otimes A) \wedge (\eta^\ell \otimes B) = (\omega^k \wedge \eta^\ell) \otimes (A \cdot B) \tag{8.1.6}$$

where $\omega^k \in \Omega^k(U_\alpha)$, $\eta^\ell \in \Omega^\ell(U_\alpha)$ and $A, B \in \underline{gl}(n, \mathbb{K})$ (see also Example 3.3.11). The space $\underline{gl}(n, \mathbb{K})$ is naturally a Lie algebra with respect to the commutator of two matrices. This structure induces a bracket

$$[\ ,\ ] : \Omega^k(U_\alpha) \otimes \underline{gl}(n, \mathbb{K}) \times \Omega^\ell(U_\alpha) \otimes \underline{gl}(n, \mathbb{K}) \to \Omega^{k+\ell}(U_\alpha) \otimes \underline{gl}(n, \mathbb{K})$$

defined as in (8.1.1). A very simple computation yields the following identity.

$$\omega \wedge \eta = \frac{1}{2}[\omega, \eta] \quad \forall \omega, \eta \in \Omega^1(U_\alpha) \otimes \underline{gl}(n, \mathbb{K}). \tag{8.1.7}$$

Assume the Lie group lies inside $GL(n, \mathbb{K})$ so that its Lie algebra $\mathfrak{g}$ lies inside $\underline{gl}(n, \mathbb{K})$. We can think of the map $g_{\alpha\beta}$ as a matrix valued map so that we have

$$d\varpi = d(g^{-1}dg) = (dg^{-1}) \wedge dg = -(g^{-1} \cdot dg \cdot g^{-1})dg = -(g^{-1}dg) \wedge (g^{-1}dg)$$

$$= -\varpi \wedge \varpi \stackrel{(8.1.7)}{=} -\frac{1}{2}[\varpi, \varpi].$$

Proof of B. We compute

$$d(g^{-1}A_\alpha g) = (dg^{-1}A_\alpha) \cdot g + g^{-1}(dA_\alpha)g + g^{-1}A_\alpha dg$$

$$= -g^{-1} \cdot dg \cdot g^{-1} \wedge A_\alpha \cdot g + g^{-1}(dA_\alpha)g + (g^{-1}A_\alpha g) \wedge g^{-1}dg$$
$$= -\varpi \wedge g^{-1}A_\alpha g + g^{-1}A_\alpha g \wedge \varpi + g^{-1}(dA_\alpha)g$$
$$\stackrel{(8.1.7)}{=} -\frac{1}{2}[\varpi, g^{-1}A_\alpha g] + \frac{1}{2}[\varpi, g^{1}A_\alpha g] + g^{-1}(dA_\alpha)g \stackrel{(8.1.2)}{=} -[\varpi, g^{-1}A_\alpha g] + g^{-1}(dA_\alpha)g.$$

Part (b) of the proposition now follows from **A**, **B** and (8.1.5).
(c) First, we let the reader check the following identity

$$d[\omega, \eta] = [d\omega, \eta] + (-1)^{|\omega|}[\omega, d\eta] \tag{8.1.8}$$

where $\omega, \eta \in \Omega^*(U_\alpha) \otimes \mathfrak{g}$. Using the above equality we get

$$d(F_\alpha) = \frac{1}{2}\{[dA_\alpha, A_\alpha] - [A_\alpha, dA_\alpha]\} \stackrel{(8.1.2)}{=} [dA_\alpha, A_\alpha]$$
$$= [F_\alpha, A_\alpha] - \frac{1}{2}[[A_\alpha, A_\alpha], A_\alpha] \stackrel{(8.1.3)}{=} [F_\alpha, A_\alpha].$$

The proposition is proved.

□

Exercise 8.1.2. Let $\omega_\alpha \in \Omega^k(U_\alpha) \otimes \mathfrak{g}$ satisfy the gluing rules

$$\omega_\beta = g_{\beta\alpha}\omega_\alpha g_{\beta\alpha}^{-1} \quad \text{on } U_{\alpha\beta}.$$

In other words, the collection ω_α defines a global k-form $\omega \in \Omega^k(Ad(P))$. Prove that the collection

$$d\omega_\alpha + [A_\alpha, \omega_\alpha]$$

defines a global $Ad(P)$-valued $(k+1)$-form on M which we denote by $d_A\omega$. Thus, the Bianchi identity can be rewritten as

$$d_A F(A) = 0$$

for any $A \in \mathcal{A}(P)$. □

8.1.2 G-vector bundles

Definition 8.1.3. *Let G be a Lie group and $E \to M$ a vector bundle with standard fiber a vector space V. A* G structure *on E is defined by the following collection of data.*
(a) A representation $\rho : G \to GL(V)$.
(b) A principal G-bundle P over M such that E is associated to P via ρ. In other words, there exists an open cover (U_α) of M and a gluing cocycle

$$g_{\alpha\beta} : U_{\alpha\beta} \to G$$

such that the vector bundle E can be defined by the cocycle

$$\rho(g_{\alpha\beta}) : U_{\alpha\beta} \to GL(V).$$

We denote a G-structure by the pair (P, ρ).

Two G-structures (P_i, ρ_i) on E $(i = 1, 2)$ are said to be isomorphic if $\rho_1 = \rho_2$ and the principal G-bundles P_i are isomorphic.

Example 8.1.4. Let $E \to M$ be a rank r real vector bundle over a smooth manifold M. A metric on E allows us to talk about orthonormal moving frames. They are easily produced from arbitrary ones via the Gramm-Schimdt orthonormalization technique. In particular, two different orthonormal local trivializations are related by a transition map valued in the orthogonal group $O(r)$ so that a metric on a bundle allows one to replace an arbitrary collection of gluing data by an equivalent (cohomologous) one with transitions in $O(r)$. In other words, a metric on a bundle induces an $O(r)$ structure. The representation ρ is in this case the natural injection $O(r) \hookrightarrow GL(r,\mathbb{R})$.

Conversely, an $O(r)$ structure on a rank r real vector bundle is tantamount to choosing a metric on that bundle.

Similarly, a Hermitian metric on a rank k complex vector bundle defines an $U(k)$-structure on that bundle. □

Let $E = (P, \rho, V)$ be a G-vector bundle. Assume P is defined by an open cover (U_α) and gluing maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to G.$$

If the collection $\{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}$ defines a connection on the principal bundle P then the collection $\rho_*(A_\alpha)$ defines a connection on E. Above, $\rho_* : \mathfrak{g} \to \mathrm{End}\,(V)$ denotes the derivative of ρ at $\mathbf{1} \in G$. A connection of E obtained in this manner is said to be compatible with the G-structure. Note that if $F(A_\alpha)$ is the curvature of the connection on P then the collection $\rho_*(F(A_\alpha))$ coincides with the curvature $F(\rho_*(A_\alpha))$ of the connection $\rho_*(A_\alpha)$.

For example, a connection compatible with some metric on a vector bundle is compatible with the orthogonal/unitary structure of that bundle. The curvature of such a connection is skew-symmetric which shows the infinitesimal holonomy is an infinitesimal orthogonal/unitary transformation of a given fiber.

8.1.3 Invariant polynomials

Let V be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Consider the symmetric power

$$S^k(V^*) \subset (V^*)^{\otimes k}$$

which consists of symmetric, multilinear maps

$$\varphi : V \times \cdots \times V \to \mathbb{K}.$$

Note that any $\varphi \in S^k(V^*)$ is completely determined by

$$P_\varphi(v) = \varphi(v, \ldots, v).$$

This follows immediately via the *polarization formula*

$$\varphi(v_1, \ldots, v_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1 \cdots \partial t_k} P_\varphi(t_1 v_1 + \cdots + t_k v_k).$$

If $\dim V = n$, then fixing a basis of V we can identify $S^k(V^*)$ with the space of degree k homogeneous polynomials in n variables.

Assume now that $\mathcal{A}$ is a $\mathbb{K}$-algebra with 1. Starting with $\varphi \in S^k(V^*)$ we can produce a $\mathbb{K}$-multilinear map

$$\varphi = \varphi_{\mathcal{A}} : (\mathcal{A} \otimes V) \times \cdots \times (\mathcal{A} \otimes V) \to \mathcal{A}$$

uniquely determined by

$$\varphi(a_1 \otimes v_1, \ldots, a_k \otimes v_k) = \varphi(v_1, \ldots, v_k) a_1 a_2 \cdots a_k \in \mathcal{A}.$$

If moreover the algebra $\mathcal{A}$ is *commutative* then $\varphi_{\mathcal{A}}$ is uniquely determined by the polynomial

$$P_\varphi(x) = \varphi_{\mathcal{A}}(x, \ldots, x) \quad x \in \mathcal{A} \otimes V.$$

Let us emphasize that when $\mathcal{A}$ is not commutative the above function *is not* symmetric in its variables. For example if $a_1 a_2 = -a_2 a_1$ then

$$P(a_1 X_1, a_2 X_2, \cdots) = -P(a_2 X_2, a_1 X_1, \cdots).$$

It will be so if $\mathcal{A}$ is commutative. For applications to geometry $\mathcal{A}$ will be the algebra $\Omega^*(M)$ of complex valued differential forms on a smooth manifold M. When restricted to the commutative subalgebra

$$\Omega^{even}(M) = \bigoplus_{k \geq 0} \Omega^{2k}(M) \otimes \mathbb{C}.$$

we do get a symmetric function.

Let us point out a useful identity. If $P \in I_k(\mathfrak{g})$, U is an open subset of $\mathbb{R}^n$,

$$F_i = \omega_i \otimes X_i \in \Omega^{d_i}(U) \otimes \mathfrak{g}, \quad A = \omega \otimes X \in \Omega^d(U) \otimes \mathfrak{g}$$

then

$$\begin{gathered} P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1} \cdots, F_k) \\ = (-1)^{d(d_1 + \cdots d_{i-1})} \omega \omega_1 \cdots \omega_k P(X_1, \cdots, [X, X_i], \cdots X_k). \end{gathered}$$

In particular, if $F_1, \cdots, F_{k-1}$ have even degree we deduce that for every $i = 1, \cdots, k$ we have

$$P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = \omega \omega_1 \cdots \omega_k P(X_1, \cdots, [X, X_i], \cdots X_k)$$

Summing over i and using the Ad-invariance of P we deduce

$$\sum_{i=1}^{k} P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = 0, \tag{8.1.9}$$

$\forall F_1 \cdots, F_{k-1} \in \Omega^{even}(U) \otimes \mathfrak{g}, \quad F_k, A \in \Omega^*(U) \otimes \mathfrak{g}$.

Example 8.1.5. Let $V = \underline{gl}(n, \mathbb{C})$. For each matrix $T \in V$ we denote by $c_k(T)$ the coefficient of λ^k in the characteristic polynomial

$$c_\lambda(T) = \det\left(\mathbb{1} - \frac{\lambda}{2\pi\mathbf{i}} T\right) = \sum_{k \geq 0} c_k(T) \lambda^k, \quad (\mathbf{i} = \sqrt{-1}).$$

$c_k(T)$ is a degree k homogeneous polynomial in the entries of T. For example

$$c_1(T) = -\frac{1}{2\pi\mathbf{i}}\operatorname{tr} T, \quad c_n(T) = \left(-\frac{1}{2\pi\mathbf{i}}\right)^n \det T.$$

Via polarization, $c_k(T)$ defines an element of $S^k(\underline{gl}(n,\mathbb{C})^*)$.

If $\mathcal{A}$ is a commutative $\mathbb{C}$-algebra with 1 then $\mathcal{A}\otimes\underline{gl}(n,\mathbb{C})$ can be identified with the space $\underline{gl}(n,\mathcal{A})$ of $n\times n$ matrices with entries in $\mathcal{A}$. For each $T\in\underline{gl}(n,\mathcal{A})$

$$\det\left(\mathbb{1} - \frac{\lambda}{2\pi\mathbf{i}}T\right) \in \mathcal{A}[\lambda]$$

and $c_k(T)$ continues to be the coefficient of λ^k in the above polynomial. □

Consider now a matrix Lie group G. The adjoint action of G on its Lie algebra $\mathfrak{g}$ induces an action on $S^k(\mathfrak{g}^*)$ still denoted by *Ad*. We denote by $I^k(G)$ the *Ad*-invariant elements of $S^k(\mathfrak{g}^*)$. It consists of those $\varphi\in S^k(\mathfrak{g}^*)$ such that

$$\varphi(gX_1g^{-1},\dots,gX_kg^{-1}) = \varphi(X_1,\dots,X_k)$$

for all $X_1,\dots,X_k\in\mathfrak{g}$. Set

$$I^*(G) = \bigoplus_{k\geq 0} I^k(G)$$

and

$$I^{**}(G) = \prod_{k\geq 0} I^k(G).$$

The elements of $I^*(G)$ are usually called *invariant polynomials*. $I^{**}(G)$ can be identified (as vector space) with the space of *Ad*-invariant formal power series with variables from $\mathfrak{g}^*$.

Example 8.1.6. Let $G = GL(n,\mathbb{C})$ so that $\mathfrak{g} = \underline{gl}(n,\mathbb{C})$. The map

$$\underline{gl}(n,\mathbb{C}) \ni X \mapsto \operatorname{tr}\exp(X)$$

defines an element of $I^{**}(GL(n,\mathbb{C}))$. To see this we use the "Taylor expansion"

$$\exp(X) = \sum_{k\geq 0}\frac{1}{k!}X^k$$

which yields

$$\operatorname{tr}\exp(X) = \sum_{k\geq 0}\frac{1}{k!}\operatorname{tr} X^k.$$

For each k, $\operatorname{tr} X^k \in I^k(GL(n,\mathbb{C}))$ since

$$\operatorname{tr}(gXg^{-1})^k = \operatorname{tr} gX^kg^{-1} = \operatorname{tr} X^k.$$

□

Proposition 8.1.7. *Let $\varphi\in I^k(G)$. Then for any $X, X_1,\dots,X_k\in\mathfrak{g}$*

$$\varphi([X,X_1],X_2,\dots,X_k) + \cdots + \varphi(X_1,X_2,\dots,[X,X_k]) = 0. \tag{8.1.10}$$

Proof The proposition follows immediately from the equality

$$\frac{d}{dt}|_{t=0}\ \varphi(e^{tX}X_1e^{-tX},\dots,e^{tX}X_ke^{-tX}) = 0.$$ □

8.1.4 The Chern-Weil theory

Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and $P \to M$ a principal G-bundle over the smooth manifold M.

Assume P is defined by an open cover (U_α) and a gluing cocycle

$$g_{\alpha\beta} : U_{\alpha\beta} \to G.$$

Pick $A \in \mathcal{A}(P)$ defined by the collection $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$. Its curvature is then defined by the collection

$$F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha].$$

Given $\phi \in I^k(G)$ we can define as in the previous section (with $\mathcal{A} = \Omega^{even}(U_\alpha)$, $V = \mathfrak{g}$)

$$P_\phi(F_\alpha) = \phi(F_\alpha, \dots, F_\alpha) \in \Omega^{2k}(U_\alpha).$$

Because ϕ is Ad-invariant and $F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1}$ we deduce

$$P_\phi(F_\alpha) = P_\phi(F_\beta) \text{ on } U_{\alpha\beta}$$

so that the locally defined forms $P_\phi(F_\alpha)$ patch-up to a global $2k$-form on M which we denote by $\phi(F(A))$.

Theorem 8.1.8. (Chern-Weil) *(a) The form $\phi(F(A))$ is closed $\forall A \in \mathcal{A}(P)$.*
(b) If A^0, $A^1 \in \mathcal{A}(P)$ then the forms $\phi(F(A^0))$ and $\phi(F(A^1))$ are cohomologous. In other words, $\phi(F(A))$ defines a cohomology class in $H^{2k}(M)$ which is independent *of the connection $A \in \mathcal{A}(P)$.*

Proof We use the Bianchi identity $dF_\alpha = -[A_\alpha, F_\alpha]$. The Leibniz' rule yields

$$d\phi(F_\alpha, \dots, F_\alpha) = \phi(dF_\alpha, F_\alpha, \dots, F_\alpha) + \cdots + \phi(F_\alpha, \dots, F_\alpha, dF_\alpha)$$

$$= -\phi([A_\alpha, F_\alpha], F_\alpha, \dots, F_\alpha) - \cdots - \phi(F_\alpha, \dots, F_\alpha, [A_\alpha, F_\alpha]) \stackrel{(8.1.10)}{=} 0.$$

(b) Let $A^i \in \mathcal{A}(P)$ $(i = 0, 1)$ be defined by the collections

$$A_\alpha^i \in \Omega^1(U_\alpha) \otimes \mathfrak{g}.$$

Set $C_\alpha = A_\alpha^1 - A_\alpha^0$ and for $0 \le t \le 1$ we define $A_\alpha^t \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ by $A_\alpha^t = A_\alpha^0 + tC_\alpha$. The collection (A_α^t) defines a connection $A^t \in \mathcal{A}(P)$ and $t \mapsto A^t \in \mathcal{A}(P)$ is an (affine) path connecting A^0 to A^1. Note that

$$C = (C_\alpha) = \dot{A}^t.$$

We denote by $F^t = (F_\alpha^t)$ the curvature of A^t. A simple computation yields

$$F_\alpha^t = F_\alpha^0 + t(dC_\alpha + [A_\alpha^0, C_\alpha]) + \frac{t^2}{2}[C_\alpha, C_\alpha]. \tag{8.1.11}$$

Hence

$$\dot{F}_\alpha^t = dC_\alpha + [A_\alpha^0, C_\alpha] + t[C_\alpha, C_\alpha] = dC_\alpha + [A_\alpha^t, C_\alpha].$$

Consequently

$$\phi(F_\alpha^1) - \phi(F_\alpha^0) = \int_0^1 \Big\{ \phi(\dot F_\alpha^t, F_\alpha^t, \dots, F_\alpha^t) + \cdots + \phi(F_\alpha^t, \dots, F_\alpha^t, \dot F_\alpha^t) \Big\}\, dt$$

$$= \int_0^1 \left\{ \phi(dC_\alpha, F_\alpha^t, \dots, F_\alpha^t) + \cdots + \phi(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha) \right\} dt$$

$$+ \int_0^1 \left\{ \phi([A_\alpha^t, C_\alpha], F_\alpha^t, \dots, F_\alpha^t) + \cdots + \phi(F_\alpha^t, \dots, F_\alpha^t, [A_\alpha^t, C_\alpha]) \right\} dt.$$

Because the algebra $\Omega^{even}(U_\alpha)$ is commutative we deduce

$$\phi(\omega_{\sigma(1)}, \dots, \omega_{\sigma(k)}) = \phi(\omega_1, \dots, \omega_k)$$

for all $\sigma \in \mathfrak{S}_k$ and any $\omega_1, \dots, \omega_k \in \Omega^{even}(U_\alpha) \otimes \mathfrak{g}$. Hence

$$\phi(F_\alpha^1) - \phi(F_\alpha^0) = k \int_0^1 \phi(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t C_\alpha])dt.$$

We claim that

$$\phi(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]) = d\phi(F_\alpha^t, \dots, F_\alpha^t, C_\alpha).$$

Using the Bianchi identity we get

$$d\phi(F_\alpha^t, \dots, F_\alpha^t, C_\alpha)$$

$$= \phi(F_\alpha^t, \cdots, F_\alpha^t, dC_\alpha) + \phi(dF_\alpha^t, \cdots, F_\alpha^t, C_\alpha) + \cdots + \phi(F_\alpha^t, \cdots, dF_\alpha^t, C_\alpha)$$

$$= \phi(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha) - \phi(C_\alpha, [A_\alpha^t, F_\alpha^t], F_\alpha^t, \dots, F_\alpha^t) - \cdots - \phi(C_\alpha, F_\alpha^t, \dots, F_\alpha^t, [A_\alpha^t, F_\alpha^t])$$

$$= \phi(C_\alpha], F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t)$$

$$-\phi(F_\alpha^t, \dots, F_\alpha^t, [A_\alpha^t, C_\alpha]) - \phi([A_\alpha^t, F_\alpha^t], F_\alpha^t, \dots, F_\alpha^t, C_\alpha) - \cdots - \phi(F_\alpha^t, \dots, F_\alpha^t, [A_\alpha^t, F_\alpha^t], C_\alpha)$$

$$\stackrel{(8.1.9)}{=} \phi(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]) = \phi(dC_\alpha + [A_\alpha^t, C_\alpha], F_\alpha^t, \dots, F_\alpha^t).$$

Hence

$$\phi(F_\alpha^1) - \phi(F_\alpha^0) = d \int_0^1 k\phi(\dot A_\alpha^t, F_\alpha^t, \dots, F_\alpha^t)dt \stackrel{def}{=} dT_\phi(A_\alpha^1, A_\alpha^0). \tag{8.1.12}$$

Since $C_\beta = g_{\beta\alpha} C_\alpha g_{\beta\alpha}^{-1}$ and $F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1}$ on $U_{\alpha\beta}$ we conclude from the Ad-invariance of ϕ that the collection $T_\phi(A_\alpha^1, A_\alpha^0)$ defines a global $(2k-1)$-form on M which we denote by $T(A^1, A^0)$ and we name it the ϕ-transgression from A^0 to A^1. We have thus established the *transgression formula*

$$\phi(F(A^1)) - \phi(F(A^0)) = d\, T_\phi(A^1, A^0). \tag{8.1.13}$$

The Chern-Weil theorem is proved.

□

Example 8.1.9. Consider a matrix Lie group G with Lie algebra $\mathfrak{g}$ and denote by P_0 the trivial principal G-bundle over G, $P_0 = G \times G$. Denote by ϖ the tautological 1-form $\varpi = g^{-1}dg \in \Omega^1(G) \otimes \mathfrak{g}$. Note that for every left invariant vector field $X \in \mathfrak{g}$ we have

$$\varpi(X) = X.$$

Denote by d the trivial connection on P_0. Clearly d is a flat connection. Moreover, the Maurer-Cartan equation implies that $d+\varpi$ is also a flat connection. Thus for any $\phi \in I^k(G)$

$$\phi(F(d)) = \phi(F(d+\varpi)) = 0.$$

The transgression formula implies that the form

$$\tau_\phi = T_\phi(d+\varpi, d) = k\int_0^1 \phi(\varpi, F(d+t\varpi), \cdots, F(d+t\varpi))dt \in \Omega^{2k-1}(G)$$

is closed. Clearly τ_ϕ is closed. A simple computation using the Maurer-Cartan equations shows that

$$\begin{aligned} \tau_\phi &= \frac{k}{2^{k-1}}\left(\int_0^1 (t^2-1)^{k-1}dt\right)\cdot \phi(\varpi, [\varpi,\varpi], \cdots, [\varpi,\varpi]) \\ &= (-1)^{k-1}\frac{k}{2^{k-1}}\frac{2^{2k-1}k!(k-1)!}{(2k)!}\phi(\varpi, [\varpi,\varpi], \cdots, [\varpi,\varpi]) \\ &= (-1)^{k-1}\frac{2^k}{\binom{2k}{k}}\cdot \phi(\varpi, [\varpi,\varpi], \cdots, [\varpi,\varpi]). \end{aligned}$$

We thus have a natural map $\tau : I^*(G) \to H^{odd}(G)$ called *transgression*. The elements in the range of τ are called *transgressive*. When G is compact and connected then a nontrivial result due to the combined efforts of H. Hopf, C. Chevalley, H. Cartan, A. Weil and L. Koszul states that the cohomology of G is generated as an $\mathbb{R}$-algebra by the transgressive elements. We refer to [18] for a beautiful survey of this subject. □

Exercise 8.1.3. Let $G = SU(2)$. The Killing form κ is a degree 2 *Ad*-invariant polynomial on $\underline{su}(2)$. Describe $\tau_\kappa \in \Omega^3(G)$ and then compute

$$\int_G \tau_\kappa.$$

Compare this result with the similar computations in Subsection 7.4.3. □

Let us now analyze the essentials of the Chern-Weil construction.

Input: (a) A principal G-bundle P over a smooth manifold M (defined by an open cover (U_α) and gluing cocycle $g_{\alpha\beta} : U_{\alpha\beta} \to G$).
(b) A connection $A \in \mathcal{A}(P)$ defined by the collection

$$A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$$

satisfying the transition rules

$$A_\beta = g_{\alpha\beta}^{-1}dg_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta} \quad \text{on } U_{\alpha\beta}.$$

(c) $\phi \in I^k(G)$.
Output: A closed form $\phi(F(A)) \in \Omega^{2k}(M)$ whose cohomology class is independent of the connection A. We denote this cohomology class by $\phi(P)$.

Thus, the principal bundle P defines a map, called the *Chern-Weil correspondence*

$$\mathfrak{cw}_P : I^*(G) \to H^*(M) \quad \phi \mapsto \phi(P).$$

One can check easily the map $\mathfrak{cw}_P$ is a morphism of $\mathbb{R}$-algebras.

Definition 8.1.10. *Let M and N be two smooth manifolds and $F : M \to N$ a smooth map. If P is a principal G-bundle over N defined by an open cover (U_α) and gluing cocycle $g_{\alpha\beta} : U_{\alpha\beta} \to G$ then the* pullback *of P by F is the principal bundle $F^*(P)$ over M defined by the open cover $F^{-1}(U_\alpha)$ and gluing cocycle*

$$F^{-1}(U_{\alpha\beta}) \stackrel{F}{\to} U_{\alpha\beta} \stackrel{g_{\alpha\beta}}{\to} G.$$

The pullback of a connection on P is defined similarly.
The following result should be obvious.

Proposition 8.1.11. *(a) If P is a trivial G-bundle over the smooth manifold M then $\phi(P) = 0 \in H^*(M)$ for any $\phi \in I^*(G)$.*
(b) Let $M \stackrel{F}{\to} N$ be a smooth map between the smooth manifolds M and N. Then for every principal G-bundle over N and any $\phi \in I^*(G)$ we have

$$\phi(F^*(P)) = F^*(\phi(P)).$$

Equivalently, this means the diagram below is commutative.

$$\begin{tikzcd} I^*(G) \arrow[r, "\mathfrak{cw}_P"] \arrow[dr, "\mathfrak{cw}_{F^*(P)}"'] & H^*(N) \arrow[d, "F^*"] \\ & H^*(M) \end{tikzcd}$$

Denote by $\mathfrak{P}_G$ the collection of smooth principal G-bundles (over smooth manifolds). For each $P \in \mathfrak{P}_G$ we denote by $\mathcal{B}_P$ the base of P. Finally, we denote by $\mathcal{F}$ a *contravariant* functor from the category of smooth manifolds (and smooth maps) to the category of abelian groups.

Definition 8.1.12. *An $\mathcal{F}$-valued G-* characteristic class *is a correspondence*

$$\mathfrak{P}_G \ni P \mapsto c(P) \in \mathcal{F}(\mathcal{B}_P)$$

such that
(a) $c(P) = 0$ if P is trivial and
(b) $\mathcal{F}(F)(c(P)) = c(F^*(P))$ for any smooth map $F : M \to N$ and any principal G-bundle $P \to N$.

Hence, the Chern-Weil construction is just a method of producing G-characteristic classes valued in the DeRham cohomology.

Remark 8.1.13. (a) We see that each characteristic class provides a way of measuring the nontriviality of a principal G-bundle.
(b) A very legitimate question arises. Do there exist characteristic classes (in the DeRham cohomology) not obtainable via the Chern-Weil construction?

The answer is negative but the proof requires an elaborate topological technology which is beyond the reach of this course. The interested reader can find the details in the monograph [58] which is the ultimate reference on the subject of characteristic classes.
(b) There exist characteristic classes valued in contravariant functors other then the DeRham cohomology. E.g., for each abelian group A the Čech cohomology with coefficients in the constant sheaf $\underline{A}$ defines a contravariant functor $H^*(-, A)$ and using topological techniques one can produce $H^*(-, A)$-valued characteristic classes. For details we refer to [58] or the classical [71]. □

8.2 Important examples

We devote this section to the description of some of the most important examples of characteristic classes. In the process we will describe the invariants of some commonly encountered Lie groups.

8.2.1 The invariants of the torus T^n

The n-dimensional torus $T^n = U(1) \times \cdots \times U(1)$ is an Abelian group so that the adjoint action on its Lie algebra $\mathfrak{t}^n$ is trivial. Hence

$$I^*(T^n) = S^*((\mathfrak{t}^n)^*).$$

In practice one uses a more explicit description. This is obtained as follows. Pick angular coordinates $0 \leq \theta^i \leq 2\pi$ $(1 \leq i \leq n)$ and set

$$x_j = -\frac{1}{2\pi\mathbf{i}} d\theta^j.$$

The x_j's form a basis of $(\mathfrak{t}^n)^*$ and now we can identify

$$I^*(T^n) \cong \mathbb{R}[x_1, \ldots, x_n].$$

8.2.2 Chern classes

Let E be a rank r complex vector bundle over the smooth manifold M. We have seen that a Hermitian metric on E induces an $U(r)$-structure (P, ρ) where ρ is the tautological representation

$$\rho : U(r) \hookrightarrow GL(r, \mathbb{C}).$$

Exercise 8.2.1. Prove that different Hermitian metrics on E define isomorphic $U(r)$-structures. □

Thus, we can identify such a bundle with a principal $U(r)$ bundle in a tautological way. A connection on the principal bundle is then equivalent with a linear connection ∇ on E compatible with a Hermitian metric $\langle \bullet, \bullet \rangle$, i.e.

$$\nabla_X \langle \lambda u, v \rangle = \lambda \{ \langle \nabla_X u, v \rangle + \langle u, \nabla_X v \rangle \}$$

$\forall \lambda \in \mathbb{C},\ u, v \in C^\infty(E),\ X \in \mathrm{Vect}\,(M)$.

The characteristic classes of E are by definition the characteristic classes of the tautological principal $U(r)$-bundle. To describe these characteristic classes we need to elucidate the structure of the ring of invariants $I^*(U(r))$.

$I^*(U(r))$ consists of symmetric, r-linear maps

$$\phi : \underline{u}(r) \times \cdots \times \underline{u}(r) \to \mathbb{R}$$

invariant under the adjoint action

$$\underline{u}(r) \ni X \mapsto TXT^{-1} \in \underline{u}(r), \ \ T \in U(r).$$

It is convenient to identify such a map with its polynomial form

$$P_\phi(X) = \phi(X, \dots, X).$$

The Lie algebra $\underline{u}(r)$ consists of $r \times r$ complex skew-adjoint matrices. Classical results of linear algebra show that for any $X \in \underline{u}(r)$ there exists $T \in U(r)$ such that TXT^{-1} is diagonal

$$TXT^{-1} = \mathbf{i}diag(\lambda_1, \dots, \lambda_r).$$

The set of diagonal matrices in $\underline{u}(r)$ is called the *Cartan algebra* of $\underline{u}(r)$ and is denoted by $\mathcal{C}_{\underline{u}(r)}$. It is a (maximal) Abelian Lie subalgebra of $\underline{u}(r)$. Consider the stabilizer

$$\mathfrak{F}_{U(r)} = \{ T \in U(r) \ ; \ TXT^{-1} = X \ \ \forall X \in \mathcal{C}_{\underline{u}(r)} \}$$

and the normalizer

$$\mathcal{N}_{U(r)} = \{ T \in U(r) \ ; \ T\mathcal{C}_{\underline{u}(r)}T^{-1} \subset \mathcal{C}_{\underline{u}(r)} \}.$$

$\mathfrak{F}_{U(r)}$ is a normal subgroup of $\mathcal{N}_{U(r)}$ so we can form the quotient

$$\mathcal{W}(U(r)) \stackrel{def}{=} \mathcal{N}_{U(r)}/\mathfrak{F}_{U(r)}$$

called the *Weyl group* of $U(r)$. As in §7.4.4 we see that it is isomorphic with the symmetric group $\mathfrak{S}_r$ because two diagonal skew-Hermitian matrices are unitarily equivalent if and only if they have the same eigenvalues (including multiplicities).We see that P_ϕ is Ad-invariant if and only if its restriction to the Cartan algebra is invariant under the action of the Weyl group.

The Cartan algebra is the Lie algebra of the (maximal) torus T^n (consisting of diagonal unitary matrices) and as in the previous subsection we introduce the variables

$$x_j = -\frac{1}{2\pi\mathbf{i}} d\theta_j.$$

The restriction of P_ϕ to $\mathcal{C}_{\underline{u}(r)}$ is a polynomial in the variables $x_1, \ldots, x_r$. The Weyl group $\mathfrak{S}_r$ permutes these variables so that P_ϕ is Ad-invariant if and only if $P_\phi(x_1, \ldots, x_r)$ is a symmetric polynomial in its variables. According to the fundamental theorem of symmetric polynomials the ring of these polynomials is generated (as an $\mathbb{R}$-algebra) by the elementary ones

$$\begin{array}{ccc} c_1 & = & \sum_j x_j \\ c_2 & = & \sum_{i<j} x_i x_j \\ \vdots & \vdots & \vdots \\ c_r & = & x_1 \cdots x_r \end{array}$$

Thus

$$I^*(U(r)) = \mathbb{R}[c_1, c_2, \ldots, c_r].$$

In terms of matrices $X \in \underline{u}(r)$ we have

$$\sum_k c_k(X) t^k = \det\left(\mathbf{1} - \frac{t}{2\pi\mathbf{i}} X\right) \in I^*(U(r))[t].$$

The above polynomial is known as the *universal Chern polynomial* and its coefficients are called the *universal Chern classes*.

Returning to our rank r vector bundle E we get the *Chern classes*

$$c_k(E) = c_k(F(\nabla)) \in H^{2k}(M)$$

and the *Chern polynomial*

$$c_t(E) = \det\left(\mathbf{1} - \frac{t}{2\pi\mathbf{i}} F(\nabla)\right).$$

∇ denotes a connection compatible with a Hermitian metric $\langle\ ,\ \rangle$ on E while $F(\nabla)$ denotes its curvature.

Remark 8.2.1. The Chern classes produced via the Chern-Weil method capture only a part of what topologists usually refer to characteristic classes of complex bundles. To give the reader a feeling of what the Chern-Weil construction is unable to capture we will sketch a different definition of the 1st Chern class of a complex line bundle. The following facts are essentially due to Kodaira and Spencer [45]; see also [32] for a nice presentation.

Let $L \to M$ be a smooth complex Hermitian line bundle over the smooth manifold M. Upon choosing a good open cover (U_α) of M we can describe L by a collection of smooth maps $z_{\alpha\beta} : U_{\alpha\beta} \to U(1) \cong S^1$ satisfying the cocycle condition

$$z_{\alpha\beta} z_{\beta\gamma} z_{\gamma\alpha} = \mathbf{1} \quad \forall \alpha, \beta, \gamma. \tag{8.2.1}$$

If we denote by $C^\infty(\cdot, S^1)$ the sheaf of multiplicative groups of smooth S^1-valued functions we see that the family of complex line bundles on M can be identified with the Čech group $H^1(M, C^\infty(\cdot, S^1))$. This group is called the *smooth Picard group* of M. The group multiplication is precisely the tensor product of two line bundles. We will denote it by $\mathrm{Pic}^\infty(M)$.

If we write $z_{\alpha\beta} = \exp(2\pi\mathbf{i}\theta_{\alpha\beta})$ ($\theta_{\beta\alpha} = -\theta_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{R})$) we deduce from (8.2.1) we deduce that $\forall U_{\alpha\beta\gamma} \neq \emptyset$

$$\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = n_{\alpha\beta\gamma} \in \mathbb{Z}.$$

It is not difficult to see that $\forall U_{\alpha\beta\gamma\delta} \neq \emptyset$

$$n_{\beta\gamma\delta} - n_{\alpha\gamma\delta} + n_{\alpha\beta\delta} - n_{\alpha\beta\gamma} = 0.$$

In other words $n_{\alpha\beta\gamma}$ defines a Čech 2-cocycle of the constant sheaf $\mathbb{Z}$.

On a more formal level we can capture the above cocycle starting from the exact sequence of sheaves

$$0 \to \mathbb{Z} \hookrightarrow C^\infty(\cdot, \mathbb{R}) \stackrel{\exp(2\pi\mathbf{i}\cdot)}{\to} C^\infty(\cdot, S^1) \to 0.$$

The middle sheaf is a fine sheaf so its cohomology vanishes in positive dimensions. The long exact sequence in cohomology then gives

$$0 \to \operatorname{Pic}^\infty(M) \stackrel{\delta}{\to} H^2(M, \mathbb{Z}) \to 0.$$

The cocycle $(n_{\alpha\beta\gamma})$ represents precisely the class $\delta(L)$.

The class $\delta(L)$, $L \in \operatorname{Pic}^\infty(M)$ is called the topological 1st Chern class and is denoted by $c_1^{top}(L)$. This terminology is motivated by the following result of Kodaira and Spencer, [45]:

"The image of $c_1^{top}(L)$ in the DeRham cohomology via the natural morphism

$$H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{R}) \cong H^*_{DR}(M)$$

coincides with the 1st Chern class obtained via the Chern-Weil procedure.".

The Chern-Weil construction misses precisely the torsion elements in $H^2(M, \mathbb{Z})$. For example if a line bundle admits a flat connection then its Chern class is trivial. This may not be the case with the topological one. The line bundle may not be topologically trivial. □

8.2.3 Pontryagin classes

Let E be a rank r real vector bundle over the smooth manifold M. An Euclidean metric on E induces an $O(r)$ structure (P, ρ). The representation ρ is the tautological one

$$\rho : O(r) \hookrightarrow GL(r, \mathbb{R}).$$

Exercise 8.2.2. Prove that two metrics on E induce isomorphic $O(r)$-structures. □

Hence, as in the complex case, we can naturally identify the rank r real vector bundles with principal $O(r)$-bundles. A connection on the principal bundle can be viewed as a metric compatible connection in the associated vector bundle. To describe the various characteristic classes we need to understand the ring of invariants $I^*(O(r))$.

As usual we will identify the elements of $I^k(O(r))$ with the degree k, *Ad*-invariant polynomials on the Lie algebra $\underline{o}(r)$ consisting of skewsymmetric $r \times r$ real matrices. Fix $P \in I^k(O(r))$. Set $m = [r/2]$ and denote by J the 2×2 matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Consider the *Cartan algebra*

$$\mathcal{C}_{\underline{o}(r)} = \begin{cases} \{\lambda_1 J \oplus \cdots \oplus \lambda_m J \in \underline{o}(r)\ ;\ \lambda_j \in \mathbb{R}\} & , \quad r = 2m \\ \{\lambda_1 J \oplus \cdots \oplus \lambda_m J \oplus 0 \in \underline{o}(r)\ ;\ \lambda_j \in \mathbb{R}\} & , \quad r = 2m+1 \end{cases}$$

$\mathcal{C}_{\underline{o}(r)}$ is the Lie algebra of the (maximal) torus

$$T^m = \begin{cases} R_{\theta_1} \oplus \cdots \oplus R_{\theta_m} \in O(r) & , \quad r = 2m \\ R_{\theta_1} \oplus \cdots \oplus R_{\theta_m} \oplus \mathbf{1}_{\mathbb{R}} \in O(r) & , \quad r = 2m+1 \end{cases}$$

where for each $\theta \in [0, 2\pi]$ we denoted by R_θ the 2×2 rotation

$$R_\theta = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right].$$

As in Subsection 8.2.1 we introduce the variables

$$x_j = -\frac{1}{2\pi} d\theta_j.$$

Using standard results concerning the normal Jordan form of a skew-symmetric matrix we deduce that for every $X \in \underline{o}(r)$ there exists $T \in O(r)$ such that $TXT^{-1} \in \mathcal{C}_{\underline{o}(r)}$. Consequently, any Ad-invariant polynomial on $\underline{o}(r)$ is uniquely defined by its restriction to the Cartan algebra.

Following the approach in the complex case, we consider

$$\mathfrak{F}_{O(r)} = \{T \in O(r)\ ;\ TXT^{-1} = X \ \ \forall X \in \mathcal{C}_{\underline{o}(r)}\}$$

$$\mathcal{N}_{O(r)} = \{T \in O(r)\ ;\ T\mathcal{C}_{\underline{o}(r)}T^{-1} \subset \mathcal{C}_{\underline{o}(r)}\}.$$

$\mathfrak{F}_{O(r)}$ is a normal subgroup in $\mathcal{N}(O(r))$ so we can form the *Weyl group*

$$\mathcal{W}(O(r)) = \mathcal{N}_{O(r)}/\mathfrak{F}_{O(r)}.$$

Exercise 8.2.3. Prove that $\mathcal{W}(O(r))$ is the subgroup of $GL(m, \mathbb{R})$ generated by the involutions

$$\sigma_{ij} :\ (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m) \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_m)$$

$$\varepsilon_j :\ (x_1, \ldots, x_j, \ldots, x_m) \mapsto (x_1, \ldots, -x_j, \ldots, x_m).$$

□

The restriction of $P \in I^k(O(r))$ to $\mathcal{C}_{\underline{o}(r)}$ is a degree k homogeneous polynomial in the variables $x_1, \ldots, x_m$ invariant under the action of the Weyl group. Using the above exercise we deduce that P must be a symmetric polynomial $P = P(x_1, \ldots, x_m)$ separately even in each variable. Invoking once again the fundamental theorem of symmetric polynomials we conclude that P must be a polynomial in the elementary symmetric ones

$$\begin{array}{ccc} p_1 & = & \sum_j x_j^2 \\ p_2 & = & \sum_{i<j} x_i^2 x_j^2 \\ \vdots & \vdots & \vdots \\ p_m & = & x_1^2 \cdots x_m^2 \end{array}.$$

Hence

$$I^*(O(r)) = \mathbb{R}[p_1, \ldots, p_{[r/2]}].$$

In terms of $X \in \underline{o}(r)$ we have

$$p_t(X) = \sum_j p_j(X)t^{2j} = \det\left(\mathbf{1} - \frac{t}{2\pi}X\right) \in I^*(O(r))[t].$$

The above polynomial is called the *universal Pontryagin polynomial* while its coefficients $p_j(X)$ are called the *universal Pontryagin classes.*

The *Pontryagin classes* $p_1(E), \ldots, p_m(E)$ of our real vector bundle E are then defined by the equality

$$p_t(E) = \sum_j p_j(E)t^{2j} = \det\left(\mathbf{1} - \frac{t}{2\pi}F(\nabla)\right) \in H^*(M)[t]$$

where ∇ denotes a connection compatible with some (real) metric on E, while $F(\nabla)$ denotes its curvature. Note that $p_j(E) \in H^{4j}(M)$.

8.2.4 The Euler class

Let E be a rank r, real *oriented* vector bundle. A metric on E induces an $O(r)$-structure but the existence of an orientation implies the existence of a finer structure, namely an $SO(r)$-symmetry.

The groups $O(r)$ and $SO(r)$ share the same Lie algebra $\underline{so}(r) = \underline{o}(r)$. The inclusion

$$\imath : SO(r) \hookrightarrow O(r)$$

induces a morphism of $\mathbb{R}$-algebras

$$\imath^* : I^*(O(r)) \to I^*(SO(r)).$$

Because $\underline{so}(r) = \underline{o}(r)$ one deduces immediately that $\imath^*$ is injective.

Lemma 8.2.2. *When r is odd then $\imath^* : I^*(O(r)) \to I^*(SO(r))$ is an isomorphism.*

Exercise 8.2.4. Prove the above lemma. □

The situation is different when r is even, $r = 2m$. To describe the ring of invariants $I^*(SO(2m))$ we need to study in greater detail the Cartan algebra

$$\mathcal{C}_{\underline{o}(2m)} = \{\lambda_1 J \oplus \cdots \oplus \lambda_m J \in \underline{o}(2m)\}$$

and the corresponding Weyl group action. The Weyl group $\mathcal{W}(SO(2m))$, defined as usual as the quotient

$$\mathcal{W}(SO(2m) = \mathcal{N}_{SO(2m)}/\mathfrak{F}_{SO(2m)}$$

is the subgroup of $GL(\mathcal{C}_{\underline{o}(2m)})$ generated by the involutions

$$\sigma_{ij} : \ (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_m) \mapsto (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_m)$$

and

$$\varepsilon : \ (\lambda_1, \ldots, \lambda_m) \mapsto (\varepsilon_1 \lambda_1, \ldots, \varepsilon_m \lambda_m),$$

where $\varepsilon_1, \ldots, \varepsilon_m = \pm 1$ and $\varepsilon_1 \cdots \varepsilon_m = 1$. (*Check this!*)

Set as usual $x_i = -\lambda_i / 2\pi$. The Pontryagin invariants

$$p_j(x_1, \ldots, x_m) = \sum_{1 \leq i_1 < \cdots < i_j \leq m} (x_{i_1} \cdots x_{i_j})^2$$

continue to be $\mathcal{W}(SO(2m))$ invariants. There is however a new invariant

$$\Delta(x_1, \ldots, x_m) = \prod_j x_j.$$

In terms of

$$X = \lambda_1 J \oplus \cdots \oplus \lambda_m J \in \mathcal{C}_{SO(2m)}$$

we can write

$$\Delta(X) = \left(\frac{-1}{2\pi}\right)^m Pf(X),$$

where Pf denotes the pfaffian of the skewsymmetric matrix X viewed as a linear map $\mathbb{R}^{2m} \to \mathbb{R}^{2m}$, when $\mathbb{R}^{2m}$ is endowed with the canonical orientation. Note that $p_m = \Delta^2$.

Proposition 8.2.3.

$$I^*(SO(2m)) \cong \mathbb{R}[Z_1, Z_2, \ldots, Z_m; Y]/(Y^2 - Z_m) \quad (Z_j = p_j, \ Y = \Delta)$$

where $(Y^2 - Z_m)$ *denotes the ideal generated by the polynomial* $Y^2 - Z_m$.

Sketch of proof We follow an approach used by H. Weyl in describing the invariants of the alternate group ([74], Sec. II.2). The isomorphism will be established in two steps.
Step 1 $I^*(SO(2m))$ is generated (as an $\mathbb{R}$-algebra) by $p_1, \cdots, p_m, \Delta$.
Step 2 The kernel of the morphism

$$\mathbb{R}[Z_1, \ldots, Z_m, ; Y] \stackrel{\psi}{\to} I^*(SO(2m))$$

defined by $Z_j \mapsto p_j$, $Y \mapsto \Delta$ is the ideal $(Y^2 - Z_m)$.

Proof of Step 1 Note that $\mathcal{W}(SO(2m))$ has index 2 as a subgroup in $\mathcal{W}(O(2m))$. Thus $\mathcal{W}(SO(2m))$ is a normal subgroup and

$$\mathcal{G} = \mathcal{W}(O(2m))/\mathcal{W}(SO(2m)) \cong \mathbb{Z}_2.$$

$\mathcal{G} = \{\mathbb{1}, \mathfrak{e}\}$ acts on $I^*(SO(2m))$ by

$$(\mathfrak{e}F)(x_1, x_2, \ldots, x_m) = F(-x_1, x_2, \ldots, x_m) = \cdots = F(x_1, x_2, \ldots, -x_m),$$

and moreover

$$I^*(O(2m)) = \ker(\mathbb{1} - \mathfrak{e}).$$

For each $F \in I^*(SO(2m))$ we have

$$F^+ \stackrel{def}{=} (\mathbb{1} + \mathfrak{e})F \in \ker(1 - \mathfrak{e})$$

so that

$$F^+ = P(p_1, \ldots, p_m).$$

On the other hand,

$$F^- \stackrel{def}{=} (\mathbb{1} - \mathfrak{e})F$$

is separately odd in each of its variables. Indeed,

$$F^-(-x_1, x_2, \ldots, x_m) = \mathfrak{e}F^-(x_1, \ldots, x_m)$$

$$= \mathfrak{e}(\mathbb{1} - \mathfrak{e})F(x_1, \ldots, x_m) = -(\mathbb{1} - \mathfrak{e})F(x_1, \ldots, x_m) = -F^-(x_1, \ldots, x_m).$$

Hence, F^- vanishes when any of its variables vanishes so that F^- is divisible by their product $\Delta = x_1 \cdots x_m$

$$F^- = \Delta \cdot G.$$

Since $\mathfrak{e}F^- = -F^-$ and $\mathfrak{e}\Delta = -\Delta$ we deduce $\mathfrak{e}G = G$ i.e. $G \in I^*(O(2m))$. Consequently, G can be written as

$$G = Q(p_1, \ldots, p_m)$$

so that

$$F^- = \Delta \cdot Q(p_1, \ldots, p_m).$$

Step 1 follows from

$$F = \frac{1}{2}(F^+ + F^-) = \frac{1}{2}(P(p_1, \ldots, p_m) + \Delta \cdot Q(p_1, \ldots, p_m)).$$

Proof of Step 2. From the equality

$$\det X = Pf(X)^2 \quad \forall X \in \underline{so}(2m)$$

we deduce

$$(Y^2 - Z_m) \subset \ker \psi$$

so that we only need to establish the opposite inclusion.

Let $P = P(Z_1, Z_2, \ldots, Z_m; Y) \in \ker \psi$. Consider P as a polynomial in Y with coefficients in $\mathbb{R}[Z_1, \ldots, Z_m]$. Divide P by the quadratic polynomial (in Y) $Y^2 - Z_m$. The remainder is linear

$$R = A(Z_1, \ldots, Z_m)Y + B(Z_1, \ldots, Z_m).$$

Since $Y^2 - Z_m$, $P \in \ker \psi$ we deduce $R \in \ker \psi$. Thus

$$A(p_1, \ldots, p_m)\Delta + B(p_1, \ldots, p_m) = 0.$$

Applying the morphism $\mathfrak{e}$ we get

$$-A(p_1, \ldots, p_m)\Delta + B(p_1, \ldots, p_m) = 0.$$

Hence $A \equiv B \equiv 0$ so that P is divisible by $Y^2 - Z_m$. □

Let E be a rank $2m$, real, oriented vector bundle over the smooth manifold M. As in the previous subsection we deduce that we can use a metric to naturally identify E with a principal $SO(2m)$-bundle and in fact, this principal bundle is independent of the metric. Finally, choose a connection ∇ compatible with some metric on E.

Definition 8.2.4. *(a) The* universal Euler class *is defined by*

$$\mathbf{e} = \mathbf{e}(X) = \frac{1}{(2\pi)^m} Pf(-X) \in I^m(SO(2m)).$$

(b) The Euler class *of E, denoted by $\mathbf{e}(E) \in H^{2m}(M)$ is the cohomology class represented by the* Euler form

$$\mathbf{e}(\nabla) = \frac{1}{(2\pi)^m} Pf(-F(\nabla)) \in \Omega^{2m}(M).$$

(According to the Chern-Weil theorem this cohomology class is independent of the metric and the connection.) □

Example 8.2.5. Let (Σ, g) be a compact, oriented, Riemann surface and denote by ∇^g the Levi-Civita connection. The the Euler form

$$\varepsilon(g) = \frac{1}{4\pi} s(g) dv_g$$

coincides with the Euler form $\mathbf{e}(\nabla^g)$ obtained via the Chern-Weil construction. □

Remark 8.2.6. Let E be a rank $2m$, real, oriented vector bundle over the smooth, compact, oriented manifold M. We now have two apparently conflicting notions of Euler classes.
A *topological Euler class* $\mathbf{e}_{top}(E) \in H^{2m}(M)$ defined as the pullback of the Thom class via an arbitrary section of E.
A *geometric Euler class* $\mathbf{e}_{geom}(E) \in H^{2m}(M)$ defined via the Chern-Weil construction.

The most general version of the Gauss-Bonnet theorem, which will be established later in this chapter, will show that these two notions coincide! □

8.2.5 Universal classes

In each of the situations discussed so far we discussed characteristic classes for vector bundles with a given rank. In this section we show how one can coherently present these facts all at once, irrespective of rank. The algebraic machinery which will achieve this end is called *inverse limit.* We begin by first describing a special example of inverse limit.

A *projective sequence* of rings is a a sequence of rings $\{R_n\}_{n\geq 0}$ together with a sequence of ring morphisms $R_n \stackrel{\phi_n}{\leftarrow} R_{n+1}$. The inverse limit of a projective system (R_n, ϕ_n) is the subring

$$\lim_{\leftarrow} R_n \subset \prod_{n\geq 0} R_n$$

consisting of the sequences $(x_1, x_2, \ldots)$ such that $\phi_n(x_{n+1}) = x_n$, $\forall n \geq 0$.

Example 8.2.7. Let $R_n = \mathcal{R}[[X_1, \dots, X_n]]$ be the ring of formal power series in n variables with coefficients in the commutative ring with unit $\mathcal{R}$. ($R_0 = \mathcal{R}$.) Denote by $\phi_n : R_{n+1} \to R_n$ the natural morphism defined by setting $X_{n+1} = 0$. The inverse limit of this projective system is denoted by $\mathcal{R}[[X_1, X_2, \dots]]$.

Given a sequence $F_n \in \mathcal{R}[[X]]$ $(n \geq 1)$ such that $F_n(0) = 1$ we can form the sequence of products

$$(1, F_1(X_1), F_1(X_1)F_2(X_2), \dots, F_1(X_1)\cdots F_n(X_n), \dots)$$

which defines an element in $\mathcal{R}[[X_1, X_2, \dots]]$ denoted by $F_1(X_1)F_2(X_2)\cdots$. When $F_1 = F_2 = \cdots F_n = \cdots = F$ the corresponding elements is denoted by $(F)^\infty$.

Exercise 8.2.5. (a) Let $\mathcal{R}[[x]]^\flat$ denote the set of formal power series $F \in \mathcal{R}[[x]]$ such that $F(0) = 1$. Prove that $(\mathcal{R}[[x]]^\flat, \cdot)$ is an abelian group.
(b) Prove that $\forall F, G \in \mathcal{R}[[x]]^\flat$

$$(F \cdot G)^\infty = (F)^\infty \cdot (G)^\infty$$

Similarly, given $G_n \in \mathcal{R}[[x]]$, $(n \geq 1)$ such that $F_n(0) = 0$ we can from the sequence of sums

$$(0, G_1(X_1), G_1(X_1) + G_2(X_2), \dots, G_1(X_1) + \cdots \oplus G_n(X_n), \dots)$$

which defines an element in $\mathcal{R}[[X_1, X_2, \dots]]$ denoted by $G_1(X_1) + G_2(X_2) + \cdots$. When $G_1 = G_2 = \cdots = F_n = \cdots = F$ we denote the corresponding element $(G)_\infty$.

In dealing with characteristic classes of vector bundles one naturally encounters the increasing sequences

$$U(1) \hookrightarrow U(2) \hookrightarrow \cdots \tag{8.2.2}$$

(in the complex case) and (in the real case)

$$O(1) \hookrightarrow O(2) \hookrightarrow \cdots . \tag{8.2.3}$$

We will discuss these two situations separately.
The complex case The sequence in (8.2.2) induces a projective sequence of rings

$$\mathbb{R} \leftarrow I^{**}(U(1)) \leftarrow I^{**}(U(2)) \leftarrow \cdots . \tag{8.2.4}$$

We know that $I^{**}(U(n)) = \mathfrak{S}^n[[x_j]]$= the ring of symmetric formal power series in n variables with coefficients in $\mathbb{R}$. Set

$$\mathfrak{S}^\infty[[x_j]] = \lim_{\leftarrow} \mathfrak{S}^n.$$

Given a rank n vector bundle E over a smooth manifold M and $\phi \in I^{**}(U(n))$ the characteristic class $\phi(E)$ is well defined since $u^{\dim M + 1} = 0$ for any $u \in \Omega^*(M)$. Thus we can work with the ring $I^{**}(U(n))$ rather than $I^*(U(n))$ as we have done so far. An element

$$\phi = (\phi_1, \phi_2, \dots) \in I^{**}(U(\infty)) \stackrel{def}{=} \lim_{\leftarrow} I^{**}(U(n)) = \mathfrak{S}^\infty[[x_j]]$$

is called *universal characteristic class*. If E is a complex vector bundle we set $\phi(E) = \phi_r(E)$ where $r = \operatorname{rank} E$. More precisely, to define $\phi_r(E)$ we need to pick a connection ∇ compatible with some Hermitian metric on E and then set

$$\phi(E) = \phi_r(F(\nabla)).$$

Example 8.2.8. We denote by $c_k^{(n)}$ $(n \geq k)$ the elementary symmetric polynomial in n variables

$$c_k^{(n)} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

Then the sequence

$$(0, \ldots, 0, c_k^{(k)}, c_k^{(k+1)}, \ldots)$$

defines an element in $\mathfrak{S}^\infty[[x_j]]$ denoted by c_k which we call the universal k-th Chern class. Formally, we can write

$$c_k = \sum_{1 \leq i_1 < \cdots < i_k < \infty} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

We can present the above arguments in a more concise form as follows. Consider the function $F(x) = (1 + tx) \in \mathcal{R}[[x]]$ where $\mathcal{R} = \mathbb{R}[t]$. Then $(F)^\infty$ defines an element in $\mathcal{R}[[X_1, X_2, \ldots]]$. One sees immediately that in fact $(F)^\infty \in \mathfrak{S}^\infty[[x_j]]\,[[t]]$ and moreover

$$(F)^\infty(x_1, x_2, \ldots) = (1 + tx_1)(1 + tx_2) \cdots = 1 + c_1 t + c_2 t^2 + \cdots .$$

We can perform the above trick with any $F \in \mathbb{R}[[x]]$ such that $F(0) = 1$. We get a semigroup morphism

$$(\mathbb{R}[[x]]^\flat, \cdot) \ni F \mapsto (F)^\infty \in (\mathfrak{S}^\infty[[x_j]], \cdot).$$

(see Exercise 8.2.5) One very important example is

$$F(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k} \in \mathbb{R}[[x]].$$

The coefficients B_k are known as the *Bernoulli numbers*. The product

$$(F)^\infty = \left(\frac{x_1}{1 - e^{-x_1}} \right) \cdot \left(\frac{x_2}{1 - e^{-x_2}} \right) \cdots$$

defines an element in $\mathfrak{S}^\infty[[x_j]]$ called the *universal Todd class* and denoted by $\mathbf{Td}$. Using the fundamental theorem of symmetric polynomials we can write

$$\mathbf{Td} = 1 + \mathbf{Td}_1 + \mathbf{Td}_2 + \cdots$$

where $\mathbf{Td}_n \in \mathfrak{S}^\infty[[x_j]]$ is an universal symmetric, homogeneous "polynomial" of degree n, hence expressible as a combination of the elementary symmetric "polynomials" $c_1, c_2, \ldots$. By an universal "polynomial" we understand element in the inverse limit

$$\mathbb{R}[x_1, x_2, \ldots] = \varprojlim \mathbb{R}[x_1, x_2, \ldots].$$

An universal "polynomial" P is said to be homogeneous of degree d if it can be represented as a sequence

$$P = (P_1, P_2, \dots)$$

where P_m is a homogeneous polynomial of degree d in m variables and

$$P_{m+1}(x_1, x_2, \dots, x_m, 0) = P_m(x_1, \dots x_m).$$

For example

$$\mathbf{Td}_1 = \frac{1}{2}c_1, \ \ \mathbf{Td}_2 = \frac{1}{12}(c_1^2 + c_2), \ \ \mathbf{Td}_3 = \frac{1}{24}c_1c_2 \text{ etc.}$$

Analogously, any function $G \in \mathbb{R}[[x]]$ such that $G(0) = 0$ defines an element

$$(G)_\infty = G(x_1) + G(x_2) + \cdots \in \mathfrak{S}^\infty[[x_j]]$$

We have two examples in mind. First, consider $G(x) = x^k$. We get the symmetric function

$$s_k = (x^k)_\infty = (G)_\infty = x_1^k + x_2^k + \cdots$$

called the *universal k-th power sum.* We will denote these by s_k.

Next, consider

$$G(x) = e^x - 1 = \sum_{m \geq 1} \frac{x^m}{m!}.$$

We have

$$(e^x - 1)_\infty = \sum_{m \geq 1} \frac{1}{m!}(x^m)_\infty = \sum_{m=1} \frac{1}{m!} s_m \in \mathfrak{S}^\infty[[x_j]]$$

Given a complex vector bundle E we define

$$\mathbf{ch}(E) = \operatorname{rank} E + (e^x - 1)_\infty(E).$$

$\mathbf{ch}(E)$ is called the *Chern character* of the bundle E. If ∇ is a connection on E compatible with some Hermitian metric then we can express the Chern character of E as

$$\begin{aligned} \mathbf{ch}(E) &= \operatorname{tr}\left(e^{F(\nabla)}\right) \\ &= \operatorname{rank}(E) + \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{tr}\left(F(\nabla)^{\wedge k}\right) \end{aligned}$$

where $\wedge$ is the bilinear map

$$\Omega^i(\operatorname{End}(E)) \times \Omega^j(\operatorname{End}(E)) \to \Omega^{i+j}(\operatorname{End}(E))$$

defined in Example 3.3.11.

Proposition 8.2.9. *Consider two complex vector bundles E_1, E_2 over the same manifold M. Then*

$$\mathbf{ch}(E_1 \oplus E_2) = \mathbf{ch}(E_1) + \mathbf{ch}(E_2)$$

and

$$\mathbf{ch}(E_1 \otimes E_2) = \mathbf{ch}(E_1) \cdot \mathbf{ch}(E_2) \in H^*(M).$$

Proof Consider a connection ∇^i on E_i compatible with some Hermitian metric h_i, $i = 1, 2$. Then $\nabla^1 \oplus \nabla^2$ is a connection on $E_1 \oplus E_2$ compatible with the metric $h_1 \oplus h_2$ and moreover

$$F(\nabla^1 \oplus \nabla^2) = F(\nabla_1) \oplus F(\nabla^2).$$

Hence

$$\exp(F(\nabla^1 \oplus \nabla^2)) = \exp(F(\nabla^1)) \oplus \exp(F(\nabla^2))$$

from which we deduce the first equality. As for the second equality consider the connection ∇ on $E_1 \otimes E_2$ uniquely defined by the Leibniz rule

$$\nabla(s_1 \otimes s_2) = (\nabla^1 s_1) \otimes s_2 + s_1 \otimes (\nabla^2 s_2), \quad s_i \in C^\infty(E_i)$$

where the operation

$$\otimes : \Omega^k(E_1) \times \Omega^\ell(E_2) \to \Omega^{k+\ell}(E_1 \otimes E_2)$$

is defined by

$$(\omega^k \otimes s_1) \otimes (\eta^\ell s_2) = (\omega^k \wedge \eta^\ell) \otimes (s_1 \otimes s_2) \tag{8.2.5}$$

$s_i \in C^\infty(E_i)$, $\omega^k \in \Omega^k(M)$ and $\eta^\ell \in \Omega^\ell(M)$.

We compute the curvature of ∇ using the equality $F(\nabla) = (d^\nabla)^2$. If $s_i \in C^\infty(E_i)$ then

$$F(\nabla)(s_1 \otimes s_2) = d^\nabla\{(\nabla^1 s_1) \otimes s_2 + s_1 \otimes (\nabla^2 s_2)\}$$

$$= \left\{(F(\nabla^1)s_1) \otimes s_2 - (\nabla^1 s_1) \otimes (\nabla^2 s_2) + (\nabla^1 s_1) \otimes (\nabla^2 s_2) + s_1 \otimes (F(\nabla^2))\right\}$$

$$= F(\nabla^1) \otimes \mathbb{1}_{E_2} + \mathbb{1}_{E_1} \otimes F(\nabla^2).$$

The second equality in the proposition is a consequence the following technical lemma.

Lemma 8.2.10. *Let A (resp. B) be a skew-adjoint, $n \times n$ (resp. $m \times m$) complex matrix. Then*

$$\operatorname{tr}(\exp(A \otimes \mathbb{1}_{\mathbb{C}^m} + \mathbb{1}_{\mathbb{C}^n} \otimes B)) = \operatorname{tr}(\exp(A)) \cdot \operatorname{tr}(\exp(B)).$$

Proof of the lemma Pick an orthonormal basis (e_i) of $\mathbb{C}^n$ (resp. an orthonormal basis (f_j) of $\mathbb{C}^m$) such that with respect to this basis $A = \mathit{diag}(\lambda_1, \ldots, \lambda_n)$ (resp. $B = \mathit{diag}(\mu_1, \ldots, \mu_m)$). Then with respect to the basis $(e_i \otimes f_j)$ of $\mathbb{C}^n \otimes \mathbb{C}^m$ we have

$$(A \otimes \mathbb{1}_{\mathbb{C}^m} + \mathbb{1}_{\mathbb{C}^n} \otimes B) = \mathit{diag}(\lambda_i + \mu_j).$$

Hence

$$\exp(A \otimes \mathbb{1}_{\mathbb{C}^m} + \mathbb{1}_{\mathbb{C}^n} \otimes B) = \mathit{diag}(e^{\lambda_i} e^{\mu_j})$$

so that

$$\operatorname{tr}(A \otimes \mathbb{1}_{\mathbb{C}^m} + \mathbb{1}_{\mathbb{C}^n} \otimes B) = \sum e^{\lambda_i} e^{\mu_j} = \operatorname{tr}(e^A) \cdot \operatorname{tr}(e^B). \ \Box$$

The proposition is proved. $\Box$

Exercise 8.2.6. (Newton's formulæ) Consider the symmetric polynomials

$$c_k = \sum_{1\leq i_1 \leq \cdots i_k \leq n} x_{i_1}\cdots x_{i_k} \in \mathbb{R}[x_1,\dots,x_n]$$

and $(r \geq 0)$

$$s_r = \sum_j x_j^r \in \mathbb{R}[x_1,\dots,x_n].$$

Set

$$f(t) = \prod_{j=1}^n (1 - x_j t).$$

(a) Show that

$$\frac{f'(t)}{f(t)} = -\sum_r s_r t^{r-1}.$$

(b) Prove the Newton formulæ

$$\sum_{j=1}^r (-1)^j s_{r-j} c_j = 0 \quad \forall 1 \leq r \leq n.$$

(c) Deduce from the above formulae the following identities between universal symmetric polynomials.

$$s_1 = c_1,\ \ s_2 = c_1^2 - 2c_2,\ \ s_3 = c_1^3 - 3c_1c_2 + 3c_3.$$

The real case The sequence (8.2.3) induces a projective system

$$I^{**}(O(1)) \leftarrow I^{**}(O(2)) \leftarrow \cdots .$$

We have proved that $I^{**}(O(n)) = \mathfrak{S}^{[n/2]}[[x_j^2]]$= the ring of even, symmetric power series in $[n/2]$ variables. The inverse limit of this system is

$$I^* * (O(\infty) = \lim_{\leftarrow} I^{**}(O(r)) = \mathfrak{S}^{\infty}[[x_j^2]] \stackrel{def}{=} \lim_{\leftarrow} \mathfrak{S}^m[[x_j^2]].$$

As in the complex case, any element of this ring is called a *universal characteristic class.* In fact, for any real vector bundle E and any $\phi \in \mathfrak{S}^{\infty}[[x_j^2]]$ there is a well defined characteristic class $\phi(E)$ which can be expressed exactly as in the complex case, using metric compatible connections. If $F \in \mathbb{R}[[x]]^{\flat}$ then $(F(x^2))^{\infty}$ defines an element of $\mathfrak{S}^{\infty}[[x_j^2]]$.

In topology, the most commonly encountered situations are the following.

A.

$$F(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 + \sum_{k\geq 1} (-1)^k \frac{2^{2k-1}-1}{2^{2k-1}(2k)!} B_k x^k.$$

The universal characteristic class $(F(x))^{\infty}$ is denoted by $\hat{\mathbf{A}}$ and is called the $\hat{\mathbf{A}}$ − **genus**. We can write

$$\hat{\mathbf{A}} = 1 + \hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_2 + \cdots$$

where A_k are universal, symmetric, even, homogeneous "polynomials" and as such they can be described using universal Pontryagin classes

$$p_m = \sum_{1 \leq j_1 < \cdots j_m} x_{j_1}^2 \cdots x_{j_m}^2.$$

The first couple of terms are

$$\hat{\mathbf{A}}_1 = -\frac{p_1}{24}, \quad \hat{\mathbf{A}}_2 = \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2) \text{ etc.}$$

B. Consider

$$F(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}} = 1 + \frac{1}{3}x + \frac{1}{45}x^2 + \cdots = 1 + \sum_{k \geq 1} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k x^k.$$

The universal class $(F(x^2))^\infty$ is denoted by $\mathbf{L}$ and is called the $\mathbf{L}$-genus. As before we can write

$$\mathbf{L} = 1 + \mathbf{L}_1 + \mathbf{L}_2 + \cdots$$

where the $\mathbf{L}_j$'s are universal, symmetric, even, homogeneous "polynomials". They can be expressed in terms of the universal Pontryagin classes. The first few terms are

$$\mathbf{L}_1 = \frac{1}{3}p_1, \ \ \mathbf{L}_2 = \frac{1}{45}(7p_2 - p_1^2) \text{ etc.}$$

8.3 Computing characteristic classes

The theory of characteristic classes is as useful as one's ability to compute them. In this section we will describe some methods of doing this.

Most concrete applications require the aplication of a combination of techniques from topology, differential and algebraic geometry and Lie group theory that go beyond the scope of this book. We will discuss in some detail a few invariant theoretic methods and we will present one topological result more precisely the Gauss-Bonnet-Chern theorem.

8.3.1 Reductions

In applications, the symmetries of a vector bundle are implicitly described through topological properties.

For example, if a rank r complex vector bundle E splits as a Whitney sum $E = E_1 \oplus E_2$ with rank $E_i = r_i$ then E, which has a natural $U(r)$- symmetry, can be given a finer structure of $U(r_1) \times U(r_2)$ vector bundle.

More generally assume that a given rank r complex vector bundle E admits a G-structure (P, ρ), where P is a principal G bundle and $\rho : G \to U(r)$ is a representation of G. Then we can perform two types of Chern-Weil constructions: using the $U(r)$ structure and using the G structure and in particular we obtain two collections of characteristic classes associated to E. One natural question is whether there is any relationship between them.

In terms of the Whitney splitting $E = E_1 \oplus E_2$ above, the problem takes a more concrete form: compute the Chern classes of E in terms of the Chern classes of E_1 and E_2. Our next definition formalizes the above situations.

Definition 8.3.1. Let $\varphi : H \to G$ be a smooth morphism of (matrix) Lie groups.
(a) If P is a principal H-bundle over the smooth manifold M defined by the open cover (U_α) and gluing cocycle

$$h_{\alpha\beta} : U_{\alpha\beta} \to H$$

then the principal G-bundle defined by the gluing cocycle

$$g_{\alpha\beta} = \varphi \circ h_{\alpha\beta} : U_{\alpha\beta} \to G$$

is said to be the φ-associate of P and is denoted by $\varphi(P)$.
(b) A principal G-bundle Q over M is said to be φ-reducible if there exists a principal H-bundle $P \to M$ such that $Q = \varphi(P)$.

The morphism $\varphi : H \to G$ in the above definition induces a $\mathbb{R}$-algebra morphism

$$\varphi^* : I^*(G) \to I^*(H).$$

The elements of $\ker \varphi^*$ are called *universal identities.*

The following result is immediate.

Proposition 8.3.2. Let P be a principal G-bundle which can be reduced to a principal H-bundle Q. Then for every $\eta \in \ker \varphi^*$ we have

$$\eta(P) = 0 \ \text{ in } H^*(M).$$

Proof Denote by φ_* the differential of φ at $1 \in H$

$$\varphi_* : \mathfrak{L}_H \to \mathfrak{L}_G.$$

Pick a connection (A_α) on Q and denote by (F_α) its curvature. Then the collection $\varphi_*(A_\alpha)$ defines a connection on P with curvature $\varphi_*(F_\alpha)$. Now

$$\eta(\varphi_*(F_\alpha)) = (\varphi^*\eta)(F_\alpha) = 0. \quad \square$$

The above result should be seen as a guiding principle in proving identities between characteristic classes rather than a rigid result. What is important about this result is the simple argument used to prove it.

We conclude this subsection with some simple but very important applications of the above principle.

Example 8.3.3. Let E and F be two complex vector bundles over the same smooth manifold M of ranks r and respectively s. Then

$$c_t(E \oplus F) = c_t(E) \cdot c_t(F) \tag{8.3.1}$$

where the "$\cdot$" denotes the $\wedge$-multiplication in $H^{even}(M)$. Equivalently, this means

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \cdot c_j(F).$$

To check this, pick a Hermitian metric g on E and a Hermitian metric h on F. $g \oplus h$ is a Hermitian metric on $E \oplus F$. Hence, $E \oplus F$ has an $U(r+s)$ structure reducible to an $U(r) \times U(s)$ structure.

The Lie algebra of $U(r) \times U(s)$ is the direct sum $\underline{u}(r) \oplus \underline{u}(s)$. Any element X in this algebra has a block decomposition X_E

$$X = \begin{bmatrix} X_r & 0 \\ 0 & X_s \end{bmatrix} = X_r \oplus X_s$$

where X_r (respectively X_s) is an $r \times r$ (resp. $s \times s$) complex, skew-adjoint matrix. Let $\imath$ denote the natural inclusion $\underline{u}(r) \oplus \underline{u}(s) \hookrightarrow \underline{u}(r+s)$ and denote by $c_t^{(\nu)} \in I^*(U(\nu))[t]$ the Chern polynomial.

We have

$$\begin{aligned} \imath^*(c_t^{(r+s)})(X_r \oplus X_s) &= \det\left(\mathbf{1}_{r+s} - \frac{t}{2\pi\mathbf{i}} X_r \oplus X_s\right) \\ &= \det\left(\mathbf{1}_r - \frac{t}{2\pi\mathbf{i}} X_r\right) \cdot \det\left(\mathbf{1}_s - \frac{t}{2\pi\mathbf{i}} X_s\right) \\ &= c_t^{(r)} \cdot c_t^{(s)}(s). \end{aligned}$$

The equality (8.3.1) now follows using the argument in the proof of Proposition 8.3.2.

Exercise 8.3.1. Let E and F be two complex vector bundles over the same manifold M. Show that

$$\mathbf{Td}(E \oplus F) = \mathbf{Td}(E) \cdot \mathbf{Td}(F).$$

Exercise 8.3.2. Let E and F be two real vector bundles over the same manifold M. Prove that

$$p_t(E \oplus F) = p_t(E) \cdot p_t(F) \tag{8.3.2}$$

where p_t denotes the Pontryagin polynomials.

Exercise 8.3.3. Let E and F be two real vector bundles over the same smooth manifold M. Show that

$$\mathbf{L}(E \oplus F) = \mathbf{L}(E) \cdot \mathbf{L}(F)$$

$$\hat{\mathbf{A}}(E \oplus F) = \hat{\mathbf{A}}(E) \cdot \hat{\mathbf{A}}(F).$$

Example 8.3.4. The natural inclusion $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$ induces an embedding $\imath : O(n) \hookrightarrow U(n)$. (An orthogonal map $T : \mathbb{R}^n \to \mathbb{R}^n$ extends by complexification to an unitary map $T_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$). This is mirrored at the Lie algebra level by an inclusion

$$\underline{o}(n) \hookrightarrow \underline{u}(n)$$

and we obtain a morphism

$$\imath^* : I^*(U(n)) \to I^*(O(n))$$

We claim that

$$i^*(c_{2k+1}) = 0$$

and

$$\imath^*(c_{2k}) = (-1)^k p_k.$$

Indeed, for $X \in \underline{o}(n)$ we have

$$\imath^*(c_{2k+1})(X) = \left(-\frac{1}{2\pi\mathbf{i}}\right)^{2k+1} \sum_{1\leq i_1<\cdots<i_{2k+1}\leq n} \lambda_{i_1}(X)\cdots\lambda_{i_{2k+1}}(X)$$

where $\lambda_j(X)$ are the eigenvalues of X over $\mathbb{C}$. Since X is in effect a real skew-symmetric matrix we have

$$\lambda_j(\overline{X}) = \overline{\lambda_j(X)} = -\lambda_j(X).$$

Consequently

$$\imath^*(c_{2k+1})(X) = \imath^*(c_{2k+1}(\overline{X})) = \left(-\frac{1}{2\pi\mathbf{i}}\right)^{2k+1} \sum_{1\leq i_1<\cdots<i_{2k+1}\leq n} \overline{\lambda_{i_1}(X)\cdots\lambda_{i_{2k+1}}(X)}$$

$$= (-1)^{2k+1}\left(-\frac{1}{2\pi\mathbf{i}}\right)^{2k+1} \sum_{1\leq i_1<\cdots<i_{2k+1}\leq n} \lambda_{i_1}(X)\cdots\lambda_{i_{2k+1}}(X) = -\imath^*(c_{2k+1})(X).$$

The equality $\imath^*(c_{2k}) = (-1)^k p_k$ is proved similarly.

From the above example we deduce immediately the following consequence.

Proposition 8.3.5. If $E \to M$ is a real vector bundle and $E \otimes \mathbb{C}$ is its complexification then

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}), \quad k = 1, 2, \ldots. \tag{8.3.3}$$

In a more concentrated form this means

$$p_t(E) = p_{-t}(E) = c_{\mathbf{i}t}(E \otimes \mathbb{C}).$$

Exercise 8.3.4. Let $E \to M$ be a complex vector bundle of rank r.
(a) Show that $c_k(E^*) = (-1)^k c_k(E)$ i. e.

$$c_t(E^*) = c_{-t}(E).$$

(b) One can also regard E as a **real, oriented** vector bundle $E_{\mathbb{R}}$. Prove that

$$\sum_k (-1)^k t^{2k} p_k(E_{\mathbb{R}}) = c_t(E) \cdot c_{-t}(E)$$

i.e.

$$p_{\mathbf{i}t}(E_{\mathbb{R}}) = c_t(E) \cdot c_{-t}(E).$$

and moreover

$$c_r(E) = \mathbf{e}(E_{\mathbb{R}}).$$

Exercise 8.3.5. (a) The natural morphisms $I^{**}(U(r)) \to I^{**}(O(r))$ described above induce a morphism

$$\Phi_\infty : I^{**}(U(\infty)) \to I^{**}(O(\infty)).$$

As we already know, for any $F \in \mathbb{R}[[x]]^\flat$, $(F)^\infty$ is an element of $I^{**}(U(\infty))$ and if moreover F is **even** then $(F)^\infty$ can be regarded as an element of $I^{**}(O(\infty))$. Show that for every $F \in \mathbb{R}[[x]]^\flat$

$$\Phi_\infty((F)^\infty) = (F \cdot F^-)^\infty \in I^{**}(O(\infty))$$

where $F^-(x) = F(-x)$.
(b) Let E be a real vector bundle. Deduce from part (a) that

$$\mathbf{Td}(E \otimes \mathbb{C}) = \hat{\mathbf{A}}(E)^2.$$

Example 8.3.6. Consider the inclusion

$$\imath : SO(2k) \times SO(2\ell)) \hookrightarrow SO(2k + 2\ell).$$

This induces a ring morphism

$$\imath^* : I^*(SO(2k + 2\ell) \to I^*(SO(2k) \times SO(2\ell)).$$

Note that

$$I^*(SO(2k) \times SO(2\ell)) \cong I^*(SO(2k)) \otimes_{\mathbb{R}} I^*(SO(2\ell).$$

Denote by $\mathbf{e}^{(\nu)}$ the Euler class in $I^*(SO(2\nu))$. We want to prove that

$$\imath^*(\mathbf{e}^{(k+\ell)}) = \mathbf{e}^{(k)} \otimes \mathbf{e}^{(\ell)}.$$

Let $X = X_k \oplus X_\ell \in \underline{so}(2k) \oplus \underline{so}(2\ell)$. Modulo a conjugation by $(S, T) \in SO(2k) \times SO(2\ell)$ we may assume that

$$X_k = \lambda_1 J \oplus \cdots \oplus \lambda_k J$$

and

$$X_\ell = \mu_1 J \oplus \cdots \oplus \mu_\ell J$$

where as usual J denotes the 2×2 matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We have

$$\imath^*(\mathbf{e}^{(k+\ell)})(X) = \left(-\frac{1}{2\pi}\right)^{k+\ell} \lambda_1 \cdots \lambda_k \cdot \mu_1 \cdots \mu_\ell = \mathbf{e}^{(k)}(X_k) \cdot \mathbf{e}^{(\ell)}(X_\ell)$$

$$= \mathbf{e}^{(k)} \otimes \mathbf{e}^{(\ell)}(X_k \oplus X_\ell).$$

The above example has an interesting consequence.

Proposition 8.3.7. Let E and F be two real, oriented vector bundle over the same manifold M and having even ranks. Then

$$\mathbf{e}(E \oplus F) = \mathbf{e}(E) \cdot \mathbf{e}(F)$$

where as usual the $\cdot$ denotes the $\wedge$-multiplication in $H^{even}(M)$.

Example 8.3.8. Let E be a rank $2k$ real, oriented vector bundle over the smooth manifold M. We claim that if E admits a nowhere vanishing section ξ then $\mathbf{e}(E) = 0$.

To see this, fix an Euclidean metric on E so that E is now endowed with an $SO(2k)$-structure. Denote by L the real line subbundle of E generated by the section ξ. Clearly, L is a trivial line bundle and E splits as an orthogonal sum

$$E = L \oplus L^{\perp}.$$

The orientation on E and the orientation on L defined by ξ induce an orientation on $L^{\perp}$ so that $L^{\perp}$ has an $SO(2k-1)$-structure.

What we have just said shows that the $SO(2k)$ structure of E can be reduced to an $SO(1) \times SO(2k-1) \cong SO(2k-1)$-structure. Denote by $\imath^*$ the inclusion induced morphism

$$I^*(SO(2k)) \to I^*(SO(2k-1)).$$

Since $\imath^*(\mathbf{e}^{(k)}) = 0$ we deduce from the Proposition 8.3.2 that $\mathbf{e}(E) = 0$.

The result proved in the above example can be reformulated more suggestively as follows.

Corollary 8.3.9. Let E be a real oriented vector bundle of even rank over the smooth manifold M. If $\mathbf{e}(E) \neq 0$ then any section of E must vanish somewhere on M!

8.3.2 The Gauss-Bonnet-Chern theorem

If $E \to M$ is a real oriented vector bundle over a smooth, compact, oriented manifold M then there are two apparently conflicting notions of Euler class naturally associated to E.
The topological Euler class

$$\mathbf{e}_{top}(E) = \zeta_0^* \tau_E$$

where τ_E is the Thom class of E and $\zeta_0 : M \to E$ is the zero section.
The geometric Euler class

$$\mathbf{e}_{geom}(E) = \begin{cases} \frac{1}{(2\pi)^r} Pf(-F(\nabla)) & \text{if rank}\,(E) \text{ is even} \\ 0 & \text{if rank}\,(E) \text{ is odd} \end{cases}$$

where ∇ is a connection on E compatible with some metric and $2r = \text{rank}\,(E)$. The next result, which generalizes the Gauss-Bonnet theorem, will show that these two notions of Euler class coincide.

Theorem 8.3.10. (Gauss-Bonnet-Chern) Let $E \xrightarrow{\pi} M$ be a real, oriented vector bundle over the compact oriented manifold M. Then

$$\mathbf{e}_{top}(E) = \mathbf{e}_{geom}(E).$$

Proof We will distinguish two cases.
A rank (E) is odd. Consider the automorphism of E

$$\mathfrak{i} : E \to E \quad u \mapsto -u \;\; \forall u \in E.$$

Since the fibers of E are odd dimensional we deduce that $\mathfrak{i}$ reverses the orientation in the fibers. In particular, this implies

$$\pi_* \mathfrak{i}^* \tau_E = -\pi_* \tau_E = \pi_*(-\tau_E)$$

where π_* denotes the integration along fibers. Since π_* is an isomorphism (Thom isomorphism theorem) we deduce

$$\mathfrak{i}^* \tau_E = -\tau_E.$$

Hence

$$\mathbf{e}_{top}(E) = -\zeta_0^* \mathfrak{i}^* \tau_E. \tag{8.3.4}$$

On the other hand notice that

$$\zeta_0^* \mathfrak{i}^* = \zeta_0.$$

Indeed

$$\zeta_0^* \mathfrak{i}^* = (\mathfrak{i}\zeta_0)^* = (-\zeta_0)^* = (\zeta)^* \;\; (\zeta_0 = -\zeta_0).$$

The equality $\mathbf{e}_{top} = \mathbf{e}_{geom}$ now follows from (8.3.4).
B. rank $(E) = 2k$. We will use a variation of the original argument due to Chern ([19]). Let ∇ denote a connection on E compatible with a metric g. The strategy of proof is very simple. We will explicitly construct a closed form $\omega \in \Omega^{2k}_{cpt}(E)$ such that
(i) $\pi_* \omega = 1 \in \Omega^0(M)$.
(ii) $\zeta_0^* \omega = \mathbf{e}(\nabla) = (2\pi)^{-k} Pf(-F(\nabla))$.
The Thom isomorphism theorem coupled with (i) implies that ω represents the Thom class in $H^{2k}_{cpt}(E)$. (ii) simply states the sought for equality $\mathbf{e}_{top} = \mathbf{e}_{geom}$.

Denote by $S(E)$ the unit sphere bundle of E

$$S(E) = \{u \in E \; ; \; |u|_g = 1\}.$$

$S(E)$ is a compact manifold and

$$\dim S(E) = \dim M + 2k - 1.$$

Denote by π_0 the natural projection $S(E) \to M$ and by $\pi_0^*(E) \to S(E)$ the pullback of E to $S(E)$ via the map π_0. $\pi_0^*(E)$ has an $SO(2k)$-structure and moreover it admits a tautological, nowhere vanishing section

$$\Upsilon : S(E_x) \ni e \mapsto e \in E_x \equiv (\pi_0^*(E)_x)_e \;\; (x \in M).$$

Thus according to Example 8.3.8 we must have

$$\mathbf{e}_{geom}(\pi_0^*(E)) = 0 \in H^{2k}(S(E))$$

where $\mathbf{e}_{geom}(\pi_0^* E)$ denotes the differential form

$$\mathbf{e}_{geom}(\pi_0^*\nabla) = \frac{1}{(2\pi)^k} Pf(-F(\pi_0^*\nabla)).$$

Hence there must exist $\psi \in \Omega^{2k-1}(S(E))$ such that

$$d\psi = \mathbf{e}_{geom}(\pi_0^* E).$$

The decisive step in the proof of Gauss-Bonnet-Chern theorem is contained in the following lemma.

Lemma 8.3.11. There exists $\Psi = \Psi(\nabla) \in \Omega^{2k-1}(S(E))$ such that

$$d\Psi(\nabla) = \mathbf{e}_{geom}(\pi_0^*(E)) \tag{8.3.5}$$

and

$$\int_{S(E)/M} \Psi(\nabla) = -1 \in \Omega^0(M). \tag{8.3.6}$$

The form $\Psi(\nabla)$ is sometimes referred to as the **global angular form** of the pair (E, ∇). For the clarity of the exposition we will conclude the proof of the Gauss-Bonnet-Chern theorem assuming Lemma 8.3.11 which will be proved later on.

Denote by $r : E \to \mathbb{R}_+$ the norm function

$$E \ni e \mapsto |e|_g$$

If we set $E^0 = E \setminus \{\text{zero section}\}$ then we can identify

$$E^0 \cong (0, \infty) \times S(E) \quad e \mapsto (|e|, \frac{1}{|e|}e).$$

Consider the smooth cutoff function

$$\rho = \rho(r) : [0, \infty) \to \mathbb{R}$$

such that $\rho(r) = -1$ for $r \in [0, 1/4]$ and $\rho(r) = 0$ for $r \geq 3/4$. Finally define

$$\omega = \omega(\nabla) = -\rho'(r) dr \wedge \Psi(\nabla) - \rho(r)\pi^*(\mathbf{e}(\nabla)).$$

ω is well defined since $\rho'(r) \equiv 0$ near the zero section. Obviously ω has compact support on E and satisfies the condition (ii) since

$$\zeta_0^*\omega = -\rho(0)\zeta_0^*\pi^*\mathbf{e}(\nabla) = \mathbf{e}(\nabla).$$

From the equality

$$\int_{E/M} \rho(r)\pi^*\mathbf{e}(\nabla) = 0$$

we deduce

$$\begin{aligned}\int_{E/M}\omega &= -\int_{E/M}\rho'(r)dr\wedge\Psi(\nabla)\\ &= -\int_0^\infty \rho'(r)dr\cdot\int_{S(E)/M}\Psi(\nabla)\\ &= -(\rho(1)-\rho(0))\int_{S(E)/M}\Psi(\nabla)\stackrel{(8.3.6)}{=}1.\end{aligned}$$

To complete the program outlined at the beginning of the proof we need to show that ω is closed.

$$\begin{aligned}d\omega &= \rho'(r)dr\wedge d\Psi(\nabla)-\rho'(r)\wedge\pi^*\mathbf{e}(\nabla)\\ &\stackrel{(8.3.5)}{=}\rho'(r)dr\wedge\{\pi_0^*\mathbf{e}(\nabla)-\pi^*\mathbf{e}(\nabla)\}.\end{aligned}$$

The above form is identically zero since $\pi_0^*\mathbf{e}(\nabla)=\pi^*\mathbf{e}(\nabla)$ on the support of ρ'. Thus ω is closed and the theorem is proved. □

Proof of Lemma 8.3.11 We denote by $\overline{\nabla}$ the pullback of ∇ to π_0^*E. The tautological section $\Upsilon : S(E)\to\pi_0^*E$ can be used to produce an orthogonal splitting

$$\pi_0^*E = L\oplus L^\perp$$

where L is the line bundle spanned by Υ while $L^\perp$ is its orthogonal complement in π_0^*E with respect to the pullback metric g. Denote by

$$P:\pi_0^*E\to\pi_0^*E$$

the orthogonal projection onto $L^\perp$. Using P we can produce a new metric compatible connection $\hat{\nabla}$ on π_0^*E by

$$\hat{\nabla} = (\text{trivial connection on } L)\oplus P\overline{\nabla}P.$$

We have an equality of differential forms

$$\pi_0^*\mathbf{e}(\nabla)=\mathbf{e}(\overline{\nabla})=\frac{1}{(2\pi)^k}Pf(-F(\overline{\nabla})).$$

Since the curvature of $\hat{\nabla}$ splits as a direct sum

$$F(\hat{\nabla}) = 0\oplus F'(\hat{\nabla})$$

where $F'(\hat{\nabla})$ denotes the curvature of $\hat{\nabla}|_{L^\perp}$ we deduce

$$Pf(F(\hat{\nabla}))=0.$$

We denote by ∇^t the connection $\hat{\nabla}+t(\overline{\nabla}-\hat{\nabla})$ so that $\nabla^0=\hat{\nabla}$ and $\nabla^1=\overline{\nabla}$. If F^t is the curvature of ∇^t we deduce from the transgression formula (8.1.12) that

$$\pi_0^*\mathbf{e}(\nabla)=\mathbf{e}(\overline{\nabla})-\mathbf{e}(\hat{\nabla})$$

$$= d\left\{ \left(\frac{-1}{2\pi}\right)^k k \int_0^1 Pf(\overline{\nabla} - \hat{\nabla}, F^t, \dots, F^t)dt \right\}.$$

We claim that the form

$$\Psi(\nabla) = \left(\frac{-1}{2\pi}\right)^k k \int_0^1 Pf(\overline{\nabla} - \hat{\nabla}, F^t, \dots, F^t)dt$$

satisfies all the conditions in Lemma 8.3.11. By construction

$$d\Psi(\nabla) = \pi_0^* \mathbf{e}(\nabla)$$

so all that we need to prove is

$$\int_{S(E)/M} \Psi(\nabla) = -1 \in \Omega^0(M).$$

It suffices to show that for each fiber E_x of E we have

$$\int_{E_x} \Psi(\nabla) = -1.$$

Along this fiber $\pi_0^* E$ is naturally isomorphic with a trivial bundle

$$\pi_0^* E\,|_{E_x} \cong (E_x \times E_x \to E_x).$$

Moreover $\overline{\nabla}$ restricts as the trivial connection. By choosing an orthonormal basis of E_x we can identify $\pi_0^* E\,|_{E_x}$ with the trivial bundle $\underline{\mathbb{R}}^{2k}$ over $\mathbb{R}^{2k}$. The unit sphere $S(E_x)$ is identified with the unit sphere $S^{2k-1} \subset \mathbb{R}^{2k}$. The splitting $L \oplus L^\perp$ over $S(E)$ restricts over $S(E_x)$ as the splitting

$$\underline{\mathbb{R}}^{2k} = \nu \oplus TS^{2k-1}$$

where ν denotes the normal bundle of $S^{2k-1} \hookrightarrow \mathbb{R}^{2k}$. The connection $\hat{\nabla}$ is then the direct sum between the trivial connection on ν and the Levi-Civita connection on TS^{2k-1}.

Fix a point $p \in S^{2k-1}$ and denote by $(x^1, \dots, x^{2k-1})$ a collection of normal coordinates near p such that the basis $(\frac{\partial}{\partial x_i}|_p)$ is positively oriented. Set $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \dots, 2k-1$. Denote the unit outer normal vector field by ∂_0. For $\alpha = 0, 1, \dots, 2k-1$ set $\mathbf{f}_\alpha = \partial_\alpha|_p$. The vectors $\mathbf{f}_\alpha$ form a positively oriented orthonormal basis of $\mathbb{R}^{2k}$.

We will use Latin letters to denote indices running from 1 to $2k-1$ and Greek letters indices running from 0 to $2k-1$.

$$\overline{\nabla}_i \partial_\alpha = (\overline{\nabla}_i \partial_\alpha)^\nu + (\overline{\nabla}_i \partial_\alpha)^\tau$$

where the superscript ν indicates the normal component while the superscript τ indicates the tangential component. Since at p

$$0 = \hat{\nabla}_i \partial_j = (\overline{\nabla}_i \partial_j)^\tau$$

we deduce

$$\overline{\nabla}_i \partial_j = (\overline{\nabla}_i \partial_j)^\nu \text{ at } p.$$

Hence

$$\overline{\nabla}_i\partial_j = (\overline{\nabla}_i\partial_j, \partial_0)\partial_0 = -(\partial_j, \overline{\nabla}_i\partial_0)\partial_0.$$

Recall that $\overline{\nabla}$ is the trivial connection in $\underline{\mathbb{R}}^{2k}$ and we have

$$\overline{\nabla}_i\partial_0\,|_p = \left(\frac{\partial}{\partial \mathbf{f}_i}\partial_0\right)|_p = \mathbf{f}_i = \partial_i\,|_p\ .$$

Consequently

$$\overline{\nabla}_i\partial_j = -\delta_{ji}\partial_0 \quad \text{at } p.$$

If we denote by θ^i the local frame of TS^{2k-1} to (∂_i) then we can rephrase the above equality as

$$\overline{\nabla}\partial_j = -(\theta^1 + \cdots + \theta^{2k-1}) \otimes \partial_0.$$

On the other hand

$$\overline{\nabla}_i\partial_0 = \partial_i$$

i.e.

$$\overline{\nabla}\partial_0 = \theta^1 \otimes \partial_1 + \cdots + \theta^{2k-1} \otimes \partial_{2k-1}.$$

Since $(x^1, \cdots, x^{2k-1})$ are normal coordinates with respect to the Levi-Civita connection $\hat{\nabla}$ we deduce that $\hat{\nabla}\partial_\alpha = 0$, $\forall\alpha$ so that

$$A = (\overline{\nabla} - \hat{\nabla})\,|_p = \left[\begin{array}{cccc} 0 & -\theta^1 & \cdots & -\theta^{2k-1} \\ \theta^1 & 0 & \cdots & 0 \\ \theta^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \theta^{2k-1} & 0 & \cdots & 0 \end{array}\right].$$

Denote by F^0 the curvature of $\nabla^0 = \hat{\nabla}$ at p. Then

$$F^0 = 0 \oplus R$$

where R denotes the Riemann curvature of $\hat{\nabla}$ at p. The computations in Example 4.2.13 show that the second fundamental form of the embedding

$$S^{2k-1} \hookrightarrow \mathbb{R}^{2k}$$

coincides with the induced Riemann metric (which is the first fundamental form). Using Teorema Egregium we get

$$\langle R(\partial_i, \partial_j)\partial_k, \partial_\ell\rangle = \delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}.$$

In matrix format we have

$$F^0 = 0 \oplus (\Omega_{ij})$$

where $\Omega_{ij} = \theta^i \wedge \theta^j$. The curvature F^t at p of $\nabla^t = \hat{\nabla} + tA$ can be computed using the equation (8.1.11) of subsection 8.1.4 and we get

$$F^t = F^0 + t^2 A \wedge A = 0 \oplus (1 - t^2)F^0.$$

We can now proceed to evaluate $\Psi(\nabla)$.

$$\Psi(\nabla)\,|_p = \left(\frac{-1}{2\pi}\right)^k k\int_0^1 Pf(A,(1-t^2)F^0,\cdots,(1-t^2)F^0)dt$$

$$= \left(\frac{-1}{2\pi}\right)^k k\left(\int_0^1 (1-t^2)^{k-1}dt\right) Pf(A,F^0,F^0,\cdots,F^0). \tag{8.3.7}$$

We need to evaluate the pfaffian in the right-hand-side of the above formula. Set for simplicity $F=F^0$. Using the polarization formula and Exercise 2.2.19 in Subsection 2.2.4 we get

$$Pf(A,F,F,\cdots,F) = \frac{(-1)^k}{2^k k!}\sum_{\sigma\in\mathfrak{S}_{2k}} \epsilon(\sigma)A_{\sigma_0\sigma_1}F_{\sigma_2\sigma_3}\cdots F_{\sigma_{2k-2}\sigma_{2k-1}}.$$

For $i=0,1$ define

$$\mathfrak{S}^i = \{\sigma\in\mathfrak{S}_{2k}\ ;\ \sigma_i=0\}.$$

We deduce

$$2^k k! Pf(A,F,\cdots,F) = (-1)^k\sum_{\sigma\in\mathfrak{S}^0}\epsilon(\sigma)(-\theta^{\sigma_1})\wedge\theta^{\sigma_2}\wedge\theta^{\sigma_3}\wedge\cdots\wedge\theta^{\sigma_{2k-2}}\wedge\theta^{\sigma_{2k-1}}$$

$$+(-1)^k\sum_{\sigma\in\mathfrak{S}^1}\epsilon(\sigma)\theta^{\sigma_0}\wedge\theta^{\sigma_2}\wedge\theta^{\sigma_3}\wedge\cdots\wedge\theta^{\sigma_{2k-2}}\wedge\theta^{\sigma_{2k-1}}.$$

For each $\sigma\in\mathfrak{S}^0$ we get a permutation

$$\phi:\ (\sigma_1,\sigma_2,\cdots,\sigma_{2k-1})\in\mathfrak{S}_{2k-1}$$

such that $\epsilon(\sigma)=\epsilon(\phi)$. Similarly, for $\sigma\in\mathfrak{S}^1$ we get a permutation

$$\phi = (\sigma_0,\sigma_2,\cdots,\sigma_{2k-1})\in\mathfrak{S}_{2k-1}$$

such that $\epsilon(\sigma)=\epsilon(\phi)$. Hence

$$2^k k! Pf(A,F,\cdots,F) = 2(-1)^{k+1}\sum_{\phi\in\mathfrak{S}_{2k-1}}\epsilon(\phi)\theta^{\phi_1}\wedge\cdots\cdots\theta^{\phi_{2k-1}}$$

$$= 2(-1)^{k+1}(2k-1)!\theta^1\wedge\cdots\wedge\theta^{2k-1}$$

$$= 2(-1)^{k+1}dvol_{S^{2k-1}}.$$

Using the last equality in (8.3.7) we get

$$\Psi(\nabla)\,|_p = \left(\frac{-1}{2\pi}\right)^k k\left(\int_0^1(1-t^2)^{k-1}dt\right)\cdot 2(-1)^{k+1}(2k-1)!dvol_{S^{2k-1}}$$

$$= -\frac{(2k)!}{(4\pi)^k k!}\left(\int_0^1(1-t^2)^{k-1}dt\right)dvol_{S^{2k-1}}.$$

Using the Exercise 4.1.15 we get that

$$V_{2k-1} = \int_{S^{2k-1}} dvol_{S^{2k-1}} = \frac{2\pi^k}{(k-1)!}$$

and consequently

$$\int_{S^{2k-1}} \Psi(\nabla) = -V_{2k-1}\frac{(2k)!}{(4\pi)^k k!}\left(\int_0^1 (1-t^2)^{k-1}dt\right)$$

$$= -\frac{(2k)!}{2^{2k-1}k!(k-1)!}\left(\int_0^1 (1-t^2)^{k-1}dt\right).$$

The above integral can be evaluated inductively using the substitution $t = \cos\varphi$

$$I_k = \left(\int_0^1 (1-t^2)^{k-1}dt\right) = \int_0^{\pi/2} (\cos\varphi)^{2k-1}d\varphi$$

$$= (\cos\varphi)^{2k-2}\sin\varphi|_0^{\pi/2} + (2k-2)\int_0^{\pi/2}(\cos\varphi)^{2k-3}(\sin\varphi)^2 d\varphi$$

$$= (2k-1)I_{k-1} - (2k-2)I_k$$

so that

$$I_k = \frac{2k-3}{2k-2}I_{k-1}.$$

One now sees immediately that

$$\int_{S^{2k-1}} \Psi(\nabla) = -1.$$

Lemma 8.3.11 is proved. □.

Corollary 8.3.12. (Chern [Ch]) Let (M,g) be a compact, oriented Riemann Manifold of dimension $2n$. If R denotes the Riemann curvature then

$$\chi(M) = \frac{1}{(2\pi)^n}\int_M Pf(-R).$$

The next exercises provide another description of the Euler class of a real oriented vector bundle $E \to M$ over the compact oriented manifold M in terms of the homological Poincaré duality. Set $r = \operatorname{rank} E$ and let τ_E be a compactly supported form representing the Thom class.

Exercise 8.3.6. Let $\Phi : N \to M$ be a smooth map, where N is compact and oriented. We denote by $\Phi^\#$ the bundle map $\Phi^*E \to E$ induced by the pullback operation. Show that $\Phi^*\tau_E \stackrel{def}{=} (\Phi^\#)^*\tau_E \in \Omega^r(\Phi^*E)$ is compactly supported and represents the Thom class of Φ^*E.

In the next exercise we will also assume M is endowed with a Riemann structure.

Exercise 8.3.7. Consider a nondegenerate smooth section s of E i.e.
(i) $\mathcal{Z} = s^{-1}(0)$ is a codimension r smooth submanifold of M.
(ii) There exists a connection ∇^0 on E such that

$$\nabla^0_X s \neq 0 \quad \text{along } \mathcal{Z}$$

for any vector field X normal to $\mathcal{Z}$ (along $\mathcal{Z}$).
(a) **(Adjunction formula)** Let ∇ be an **arbitrary** connection on E and denote by $N_{\mathcal{Z}}$ the normal bundle of $\mathcal{Z} \hookrightarrow M$. Show that the **adjunction map**

$$\mathfrak{a}_\nabla : N_{\mathcal{Z}} \to E|_{\mathcal{Z}}$$

defined by

$$X \mapsto \nabla_X s \quad X \in C^\infty(N_{\mathcal{Z}})$$

is a bundle **isomorphism**. Conclude that $\mathcal{Z}$ is endowed with a natural orientation.
(b) Show that there exists an open neighborhood $\mathcal{N}$ of $\mathcal{Z} \hookrightarrow N_{\mathcal{Z}} \hookrightarrow TM$ such that

$$\exp : \mathcal{N} \to \exp(\mathcal{N})$$

is a diffeomorphism. (Compare with Lemma 7.3.30.) Deduce that the Poincaré dual of $\mathcal{Z}$ in M can be identified (via the above diffeomorphism) with the Thom class of $N_{\mathcal{Z}}$.
(c) Prove that the Euler class of E coincides with the Poincaré dual of $\mathcal{Z}$.

The part (c) of the above exercise generalizes the Poincare-Hopf theorem (see Corollary 7.3.33). In that case the section was a nondegenerate vector field and its zero set was a finite collection of points. The local index of each zero measures the difference between two orientations of the normal bundle of this finite collection of points: one is the orientation obtained if one tautologically identifies this normal bundle with the restriction of TM to this finite collection of points while and the other one is obtained via the adjunction map.

In many instances one can explicitly describe a nondegenerate section and its zero set and thus one gets a description of the Euler class which is satisfactory for most topological applications.

Example 8.3.13. Let τ_n denote the tautological line bundle over the complex projective space $\mathbb{CP}^n$. According to Exercise 8.3.4

$$c_1(\tau_n) = \mathbf{e}(\tau_n)$$

when we view τ_n as a rank 2 oriented real vector bundle. Denote by $[H]$ the $(2n-2)$-cycle defined by the natural inclusion

$$\imath : \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n, ;\ [z_0 : \dots z_{n-1}] \mapsto [z_0 : \dots : z_{n-1} : 0] \in \mathbb{CP}^n.$$

We claim that the (homological) Poincaré dual of $c_1(\tau_n)$ is $-[H]$. We will achieve this by showing that $c_1(\tau_n^*) = -c_1(\tau_n)$ is the Poincaré dual of $[H]$.

Let P be a degree 1 homogeneous polynomial $P \in \mathbb{C}[z_0, \dots, z_n]$. For each complex line $L \hookrightarrow \mathbb{C}^{n+1}$ the polynomial P defines a complex linear map $L \to \mathbb{C}$ and thus an element of L^* which we denote by $P|_L$. We thus have a well defined map

$$\mathbb{CP}^n \ni L \mapsto P|_L \in L^* = \tau_n^*|_L$$

and the reader can check easily that this is a **smooth** section of τ_n^* which we denote by $[P]$. Consider the special case $P_0 = z_n$. The zero set of the section $[P_0]$ is precisely the image of $[H]$. We let the reader keep track of all the orientation conventions in Exercise 8.3.7 and conclude that $c_1(\tau_n^*)$ is indeed the Poincaré dual of $[H]$.

Exercise 8.3.8. Let $U_S = S^2 \setminus \{\text{north pole}\}$ and $U_N = S^2 \setminus \{\text{south pole}\} \cong \mathbb{C}$. The overlap $U_N \cap U_S$ is diffeomorphic with the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For each map $g : \mathbb{C}^* \to U(1) \cong S^1$ denote by L_g the line bundle over S^2 defined by the gluing map

$$g_{NS} : U_N \cap U_S \to U(1), \;\; g_{NS}(z) = g(z).$$

Show that

$$\int_{S^2} c_1(L_g) = \deg g$$

where $\deg g$ denotes the degree of the smooth map $g\,|_{S^1 \subset \mathbb{C}^*} \to S^1$.

Chapter 9

Elliptic equations on manifolds

Almost all the objects in differential geometry are defined by expressions involving partial derivatives. The curvature of a connection is the most eloquent example.

One is often led to studying such objects with specific properties. For example, we inquired whether on a given vector bundle there exist flat connections. This situation can be dealt with topologically, using the Chern-Weil theory of characteristic classes.

Very often, topological considerations alone are not sufficient, and one has look into the microstructure of the problem. This is where analysis comes in and more specifically, one is led to the study of partial differential equations. Among them, the elliptic ones play a crucial role in modern geometry.

This chapter is an introduction to this vast and dynamic subject which has numerous penetrating applications in geometry and topology.

9.1 Partial differential operators: algebraic aspects

9.1.1 Basic notions

We first need to introduce the concept of partial differential operator (p.d.o. for brevity) on a smooth manifold M. To understand what we are looking for we begin with the simplest of the situations, $M = \mathbb{R}^N$.

Perhaps the best known partial differential operator is the Laplacian

$$\Delta : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N), \quad \Delta u = -\sum_i {\partial_i}^2 u,$$

where as usual $\partial_i = \frac{\partial}{\partial x_i}$. This is a scalar operator in the sense that it acts on scalar valued functions. Note that our definition of the Laplacian differs from the usual one by a sign. The Laplacian defined as above is sometimes called *the geometric Laplacian.*

Next in line is the exterior derivative

$$d : \Omega^k(\mathbb{R}^N) \to \Omega^{k+1}(\mathbb{R}^N).$$

This is a vectorial operator in the sense it acts on vector valued functions. A degree k form ω on $\mathbb{R}^N$ can be viewed as a collection of $\binom{N}{k}$ smooth functions or equivalently, as a smooth function $\omega : \mathbb{R}^N \to \mathbb{R}^{\binom{N}{k}}$. Thus d can be viewed as an operator

$$d : C^\infty\Big(\mathbb{R}^N, \mathbb{R}^{\binom{N}{k}}\Big) \to C^\infty\Big(\mathbb{R}^N, \mathbb{R}^{\binom{N}{k+1}}\Big).$$

It is convenient to think of $C^\infty(\mathbb{R}^N, \mathbb{R}^\nu)$ as the space of smooth sections of the trivial bundle $\underline{\mathbb{R}}^\nu$ over $\mathbb{R}^N$

For any smooth $\mathbb{K} = \mathbb{R}, \mathbb{C}$-vector bundles E, F over a smooth manifold M we denote by $\mathfrak{O}p(E, F)$ the space of $\mathbb{K}$-linear operators

$$C^\infty(E) \to C^\infty(F).$$

$\mathfrak{O}p(E, E)$ is an associative $\mathbb{K}$-algebra.

The spaces $C^\infty(E)$, $C^\infty(F)$ are more than just $\mathbb{K}$-vector spaces. *They are modules over the ring of smooth functions* $C^\infty(M)$. The partial differential operators are elements of $\mathfrak{O}p$ which interact in a special way with the above $C^\infty(M)$-module structures. First define

$$\mathbf{PDO}^0(E, F) = \mathrm{Hom}\,(E, F).$$

Given $T \in \mathbf{PDO}^0$, $u \in C^\infty(E)$ and $f \in C^\infty(M)$ we have

$$T(fu) - f(Tu) = 0$$

or, in terms of commutators

$$[T, f]u = T(fu) - f(Tu) = 0. \tag{9.1.1}$$

Each $f \in C^\infty(M)$ defines a map

$$ad(f) : \mathfrak{O}p\,(E, F) \to \mathfrak{O}p\,(E, F)$$

by

$$ad(f)T = T \circ f - f \circ T = [T, f] \;\; \forall T \in \mathfrak{O}p\,(E, F).$$

Above, f denotes the $C^\infty(M)$-module multiplication by f. We can rephrase the equality (9.1.1) as

$$\mathbf{PDO}^0(E, F) = \{T \in \mathfrak{O}p\,(E, F)\,;\; ad(f)T = 0\; \forall f \in C^\infty(M)\} \stackrel{def}{=} \ker ad.$$

Define

$$\mathbf{PDO}^{(m)}(E, F) = \ker ad^{m+1} \stackrel{def}{=} \{T \in \ker ad(f_0)ad(f_1)\cdots ad(f_m)\,;\; \forall f_i \in C^\infty(M)\}.$$

The elements of $\mathbf{PDO}^{(m)}$ are called *partial differential operators* of order $\leq m$. We set

$$\mathbf{PDO}\,(E, F) = \bigcup_{m \geq 0} \mathbf{PDO}^{(m)}(E, F).$$

Remark 9.1.1. Note that we could have defined $\mathbf{PDO}^{(m)}$ inductively as

$$\mathbf{PDO}^{(m)} = \{T \in \mathfrak{O}p\,;\; [T, f] \in \mathbf{PDO}^{(m-1)} \;\; \forall f \in C^\infty(M)\}.$$

This point of view is especially useful in induction proofs. □

Example 9.1.2. Denote by $\underline{\mathbb{R}}$ the trivial line bundle over $\mathbb{R}^N$. The sections of $\underline{\mathbb{R}}$ are precisely the real functions on $\mathbb{R}^N$. We want to analyze $\mathbf{PDO}^{(1)} = \mathbf{PDO}^{(1)}(\underline{\mathbb{R}}, \underline{\mathbb{R}})$.

Let $L \in \mathbf{PDO}^{(1)}$, $u, f \in C^\infty(\mathbb{R}^N)$. Then

$$[L, f]u = \sigma(f) \cdot u$$

where $\sigma(f) \in C^\infty(\mathbb{R}^N)$. On the other hand, for any $f, g \in C^\infty(\mathbb{R}^N)$

$$\sigma(fg)u = [L, fg]u = [L, f](gu) + f([L, g]u) = \sigma(f)g \cdot u + f\sigma(g) \cdot u.$$

Hence $\sigma(fg) = \sigma(f)g + f\sigma(g)$. In other words the map $f \mapsto \sigma(f)$ is a derivation of $C^\infty(\mathbb{R}^N)$ and consequently (see Exercise 3.1.2) there exists a smooth vector field X on $\mathbb{R}^N$ such that

$$\sigma(f) = X \cdot f, \quad \forall f \in C^\infty(\mathbb{R}^N).$$

Let $\mu = L(1) \in C^\infty(\mathbb{R}^N)$. Then for all $u \in C^\infty(\mathbb{R}^N)$ we have

$$Lu = L(u \cdot 1) = [L, u] \cdot 1 + u \cdot L(1) = X \cdot u + \mu \cdot u.$$

$\square$

Lemma 9.1.3. *Any $L \in \mathbf{PDO}^{(m)}(E, F)$ is a* local operator *i.e.* $\forall u \in C^\infty(E)$

$$\operatorname{supp} Lu \subset \operatorname{supp} u.$$

Proof We argue by induction over m. For $m = 0$ the result is obvious. Let $L \in \mathbf{PDO}^{(m+1)}$ and $u \in C^\infty(E)$. For every $f \in C^\infty(M)$ we have

$$L(fu) = [L, f]u + fLu.$$

Since $[L, f] \in \mathbf{PDO}^{(m)}$ we deduce by induction

$$\operatorname{supp} L(fu) \subset \operatorname{supp} u \cup \operatorname{supp} f \forall f \in C^\infty(M).$$

For any open set $\mathcal{O}$ such that $\mathcal{O} \supset \operatorname{supp} u$ we can find $f \equiv 1$ on $\operatorname{supp} u$ and $f \equiv 0$ outside $\mathcal{O}$ so that $fu \equiv u$). This concludes the proof of the lemma.

$\square$

The above lemma shows that in order to analyze the action of a p.d.o. one can work in *local coordinates*. Thus understanding the structure of an arbitrary p.d.o. boils down to understanding the action of a p.d.o. in $\mathbf{PDO}^{(m)}(\underline{\mathbb{K}}^p, \underline{\mathbb{K}}^q)$, where $\underline{\mathbb{K}}^p$ and $\underline{\mathbb{K}}^q$ are trivial $\mathbb{K}$-vector bundles over $\mathbb{R}^N$. This is done in the exercises at the end of this subsection.

Proposition 9.1.4. *Let E, F, G be smooth $\mathbb{K}$-vector bundles over the same manifold M. If $P \in \mathbf{PDO}^{(m)}(F, G)$ and $Q \in \mathbf{PDO}^{(n)}(E, F)$ then $P \circ Q \in \mathbf{PDO}^{(m+n)}(E, G)$.*

Proof We argue by induction over $m+n$. For $m+n=0$ the result is obvious. In general, if $f \in C^\infty(M)$

$$[P \circ Q, f] = [P, f] \circ Q + P \circ [Q, f].$$

By induction, the operators in the right-hand-side have orders $\leq m+n-1$. The proposition is proved.

□

Corollary 9.1.5. *The operator*

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N) \quad (\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N})$$

($\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, $|\alpha| = \sum \alpha_i$) is a p.d.o. of order $\leq m$.

Proof According to the computation in Example 9.1.2 each partial derivative ∂_i is a 1st order p.d.o. According to the above proposition multiple compositions of such operators are again p.d.o.'s.

□

Lemma 9.1.6. *Let $E, F \to M$ be two smooth vector bundles over the smooth manifold M. Then for any $P \in \mathbf{PDO}\,(E, F)$ and any $f, g \in C^\infty(M)$*

$$ad(f) \cdot (ad(g)P) = ad(g) \cdot (ad(f)P).$$

Proof

$$ad(f) \cdot (ad(g)P) = [[P, g], f] = [[P, f], g] + [P, [f, g]$$

$$= [[P, f], g] = ad(g) \cdot (ad(f)P).$$

□

From the above lemma we deduce that if $P \in \mathbf{PDO}^{(m)}$ then for any $f_1, \ldots, f_m \in C^\infty(M)$ the *bundle morphism*

$$ad(f_1) ad(f_2) \cdots ad(f_m) P$$

does not change if we permute the f's.

Proposition 9.1.7. *Let $P \in \mathbf{PDO}^{(m)}(E, F)$, $f_i, g_i \in C^\infty(M)$ ($i = 1, \ldots, m$) such that at a point $x_0 \in M$*

$$df_i(x_0) = dg_i(x_0) \in T^*_{x_0} M \quad \forall i = 1, \ldots, m.$$

Then

$$\{ad(f_1) ad(f_2) \cdots ad(f_m) P\} \mid_{x_0} = \{ad(g_1) ad(g_2) \cdots ad(g_m) P\} \mid_{x_0}.$$

In the proof we will use the following technical result which we leave to the reader as an exercise.

Lemma 9.1.8. *For each $x_0 \in M$ consider the ideals of $C^\infty(M)$*

$$\mathfrak{m}_{x_0} = \{f \in C^\infty(M)\ ;\ f(x_0) = 0\}$$

$$\mathfrak{J}_{x_0} = \{f \in C^\infty(M)\ ;\ f(x_0) = 0,\ df(x_0) = 0\}$$

Then

$$\mathfrak{J}_{x_0} = \mathfrak{m}_{x_0}^2,$$

i.e. any function f which vanishes at x_0 together with its derivatives can be written as

$$f = \sum_j g_j h_j \quad g_j, h_j \in \mathfrak{m}_{x_0}.$$

Exercise 9.1.1. Prove the above lemma. □

Proof Let $P \in \mathbf{PDO}^{(m)}$ and $f_i, g_i \in C^\infty(M)$ such that

$$df_i(x_0) = dg_i(x_0) \quad \forall i = 1, \dots, m.$$

Since

$$ad(const.) = 0$$

we may assume (eventually altering the f's and the g's by additive constants) that

$$f_i(x_0) = g_i(x_0) \quad \forall i.$$

We will show that

$$\{ad(f_1)ad(f_2)\cdots ad(f_m)P\}\mid_{x_0} = \{ad(g_1)ad(f_2)\cdots ad(f_m)P\}\mid_{x_0}.$$

Iterating we get the desired conclusion.

Let $\phi = f_1 - g_1$ and set $Q = ad(f_2)\cdots ad(f_m)P \in \mathbf{PDO}^{(1)}$. We have to show that

$$\{ad(\phi)Q\}\mid_{x_0} = 0. \tag{9.1.2}$$

Note that $\phi \in \mathfrak{J}_{x_0}$ so according to the above lemma we can write

$$\phi = \sum_j \alpha_j \beta_j \quad \alpha_j,\ \beta_j \in \mathfrak{m}_{x_0}.$$

We have

$$\{ad(\phi)Q\}\mid_{x_0} = \left\{\sum_j ad(\alpha_j\beta_j)Q\right\}\mid_{x_0}$$

$$= \sum_j \{[Q, \alpha_j]\beta_j\}\mid_{x_0} + \sum_j \{\alpha_j[Q, \beta_j]\}\mid_{x_0} = 0.$$

This proves the equality (9.1.2) and hence the proposition.

□

The proposition we have just proved has an interesting consequence. Given $P \in \mathbf{PDO}^{(m)}(E,F)$, $x_0 \in M$ and $f_i \in C^\infty(M)$ $(i = 1, \dots, m)$ the linear map

$$\{\frac{1}{m!} ad(f_1) \cdots ad(f_m)P\}|_{x_0}: E_{x_0} \to F_{x_0}$$

depends only on the quantities $\xi_i = df_i(x_0) \in T^*_{x_0}M$. Hence for any $\xi_i \in T^*_{x_0}M$ $(i = 1, \dots, m)$ the above expression unambiguously induces a linear map

$$\sigma(P)(\xi_1, \dots, \xi_m)(P) : E_{x_0} \to F_{x_0}$$

which moreover is symmetric in the variables ξ_i. Using the polarization trick of Chapter 8 we see this map is uniquely determined by the polynomial

$$\sigma(P)(\xi) = \sigma_m(P)(\xi) = \sigma(P)(\underbrace{\xi, \cdots, \xi}_{m}).$$

If we denote by $\pi : T^*M \to M$ the natural projection then for each $P \in \mathbf{PDO}^{(m)}(E,F)$ we have a well defined map

$$\sigma_m(P)(\cdot) \in \mathrm{Hom}\,(\pi^*E, \pi^*F)$$

where π^*E and π^*F denote the pullbacks via π. Along the fibers of T^*M $\sigma_m(P)(\xi)$ looks like a degree m homogeneous "polynomial" with coefficients in $\mathrm{Hom}\,(E_{x_0}, F_{x_0})$.

Proposition 9.1.9. *Let $P \in \mathbf{PDO}^{(m)}(E,F)$ and $Q \in \mathbf{PDO}^{(n)}(F,G)$. Then*

$$\sigma_{m+n}(Q \circ P) = \sigma_n(Q) \circ \sigma_m(P).$$

Exercise 9.1.2. Prove the above proposition. □

Definition 9.1.10. *A p.d.o. $P \in \mathbf{PDO}^{(m)}$ is said to have order m if $\sigma_m(P) \not\equiv 0$. In this case $\sigma_m(P)$ is called the* (principal) symbol *of P.*

The set of p.d.o.'s of order m will be denoted by $\mathbf{PDO}^m$.

Definition 9.1.11. *Let $P \in \mathbf{PDO}^m(E,F)$. P is said to be* elliptic *if for any $x \in M$ and any $\xi \in T^*_xM \setminus \{0\}$*

$$\sigma_m(P)(\xi) : E_x \to F_x$$

is a linear isomorphism.

The following exercises provide a complete explicit description of p.d.o.'s on $\mathbb{R}^N$.

Exercise 9.1.3. Consider the scalar p.d.o. on $\mathbb{R}^N$ described in Corollary 9.1.5

$$L = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha.$$

Show that

$$\sigma_m(P)(\xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha = \sum_{|\alpha| \le m} a_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N}.$$

□

Exercise 9.1.4. Let $C : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ be a p.d.o. of order m. Its principal symbol has the form

$$\sigma_m(L)(\xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha.$$

Show that

$$L - \sum_{|\alpha|=m} a_\alpha(x)\partial^\alpha$$

is an p.d.o. of order $\le m-1$. Conclude that the only scalar p.d.o. on $\mathbb{R}^N$ are those indicated in Corollary 9.1.5. □

Exercise 9.1.5. Let $L \in \mathbf{PDO}^{(m)}(E, F)$ and $u \in C^\infty(E)$. Show the operator

$$C^\infty(M) \ni f \mapsto [L, f]u \in C^\infty(F)$$

belongs to $\mathbf{PDO}^{(m)}(\underline{\mathbb{R}}_M, F)$. □

Exercise 9.1.6. Let $\underline{\mathbb{K}}^p$ and $\underline{\mathbb{K}}^q$ denote the trivial $\mathbb{K}$-vector bundles over $\mathbb{R}^N$ of rank p and respectively q. Show that any $L \in \mathbf{PDO}^{(m)}(\underline{\mathbb{R}}^p, \underline{\mathbb{R}}^q)$ has the form

$$L = \sum_{|\alpha| \le m} A_\alpha(x)\partial^\alpha$$

where for any α $A_\alpha \in C^\infty(\mathbb{R}^N, \mathrm{Hom}\,(\mathbb{K}^p, \mathbb{K}^q))$.
Hint: Use the previous exercise to reduce the problem to the case $p = 1$.

□

9.1.2 Examples

At a first glance, the notions introduced so far may look too difficult to "swallow". To help the reader get a friendlier feeling towards them we included in this subsection a couple of classical examples which hopefully will ease this process. More specifically, we will compute the principal symbols of some p.d.o.'s we have been extensively using in this book.

In the sequel $\cdot$ will denote the multiplication by a smooth scalar function.

Example 9.1.12. (The Euclidean Laplacian) This is the second order p.d.o.

$$\Delta : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N) \quad \Delta = -\sum_i {\partial_i}^2$$

Let $f \in C^\infty(\mathbb{R}^N)$. Then ($\partial_i = \frac{\partial}{\partial x_i}$)

$$ad(f)(\Delta) = -\sum_i ad(f)\,(\partial_i)^2 = -\sum_i \{ad(f)(\partial_i) \circ \partial_i + \partial_i \circ ad(f)(\partial_i)\}$$

$$= -\sum_i \{f_{x^i} \cdot \partial_i + \partial_i(f_{x^i} \cdot)\}$$

$$= -\sum_i \{ f_{x_i} \cdot \partial_i + f_{x^i x^i} \cdot + f_{x^i} \cdot \partial_i \}$$

$$= (\Delta f) \cdot -2 \sum_i f_{x^i} \cdot \partial_i.$$

Hence

$$ad(f)^2(\Delta) = ad(f)(\Delta f \cdot) - 2 \sum_i f_{x^i} \cdot ad(f)\,(\partial_i)$$

$$= -2 \sum_i (f_{x^i})^2 \cdot = -2|df|^2 \cdot .$$

If we set $\xi = df$ in the above equality we deduce

$$\sigma_2(\Delta)(\xi) = -|\xi|^2 \cdot .$$

In particular this shows Δ is an elliptic operator. □

Example 9.1.13. (Covariant derivatives) Consider a vector bundle $E \to M$ over the smooth manifold M and ∇ a connection on E. We can view ∇ as a p.d.o.

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E).$$

Its symbol can be read from $ad(f)\nabla$, $f \in C^\infty(M)$. For any $u \in C^\infty(E)$

$$(ad(f)\nabla)u = \nabla(fu) - f(\nabla u) = df \otimes u.$$

By setting $\xi = df$ we deduce

$$\sigma_1(\nabla)(\xi) = \xi \otimes,$$

i.e. the symbol is the tensor multiplication by ξ. □

Example 9.1.14. (The exterior derivative) Let M be a smooth manifold. The exterior derivative

$$d : \Omega^*(M) \to \Omega^*(M)$$

is a first order p.d.o. To compute its symbol consider $\omega \in \Omega^k(M)$ and $f \in C^\infty(M)$. Then

$$(ad(f)d)\omega = d(f\omega) - fd\omega = df \wedge \omega.$$

If we set $\xi = df$ we deduce $\sigma_1(d) = e(\xi) =$ the left exterior multiplication by ξ. □

Example 9.1.15. Consider an oriented, n-dimensional Riemann manifold (M, g). As in Chapter 4 we can produce an operator

$$\delta = *d* : \Omega^*(M) \to \Omega^*(M)$$

where $*$ is the Hodge $*$-operator

$$* : \Omega^*(M) \to \Omega^{n-*}(M).$$

δ is a first order p.d.o. and moreover

$$\sigma_1(\delta) = *\sigma_1(d)* = *e(\xi) * .$$

This description can be further simplified.

Fix $\xi \in T_x^*M$ and denote by $\xi^* \in T_xM$ its metric dual. For simplicity we assume $|\xi| = |\xi^*| = 1$. Extend $\xi^1 = \xi$ to an oriented orthonormal basis $(\xi^1, \dots, \xi^n)$ of T_x^*M and denote by ξ_i the dual basis of T_xM.

Consider $\omega \in \Lambda^k T_x^*M$ a monomial $d\xi^I$ where $I = (i_1, \dots, i_k)$ denotes as usual an ordered multi-index. Note that if $1 \notin I$ then

$$\sigma_1(\xi^1)\omega = 0. \tag{9.1.3}$$

If $1 \in I$ e.g. $I = (1, \dots, k)$ then

$$*e(\xi)(*\omega) = *(\xi^1 \wedge \xi^{k+1} \wedge \dots \wedge \xi^n) = (-1)^{(n-k)(k-1)}\xi^2 \wedge \dots \wedge \xi^k$$

$$= -(-1)^{\nu_{n,k}} i(\xi^*)\omega \tag{9.1.4}$$

where $\nu_{n,k} = nk + n + 1$ is the exponent introduced in Subsection 4.1.5 while $i(\xi^*)$ denotes the interior derivative along ξ^*. Putting together (9.1.3) and (9.1.4) we deduce

$$\sigma_1(\delta)(\xi) = -(-1)^{\nu_{n,k}} i(\xi^*). \tag{9.1.5}$$

□

Example 9.1.16. (The Hodge-DeRham operator) Let (M, g) be as in the above example. The Hodge-DeRham operator is

$$d + d^* : \Omega^*(M) \to \Omega^*(M)$$

where $d^* = (-1)^{\nu_{n,k}}\delta$. Hence $d + d*$ is a first order p.d.o. and moreover

$$\sigma(d + d^*)(\xi) = \sigma(d)(\xi) + \sigma(d^*)(\xi) = e(\xi) - i(\xi^*).$$

The Hodge Laplacian is the operator $(d + d^*)^2$. We call it Laplacian since

$$\sigma((d + d^*)^2)(\xi) = \{\sigma(d + d^*)(\xi)\}^2 = (e(\xi) - i(\xi^*))^2$$

while Exercise 2.2.16 shows

$$(e(\xi) - i(\xi^*))^2 = -(e(\xi)i(\xi^*) + i(\xi^*)e(\xi)) = -|\xi|_g^2 \cdot .$$

Notice that $d + d^*$ is elliptic (since the square of its symbol is invertible). □

Definition 9.1.17. *Let $E \to M$ be a smooth vector bundle over the Riemann manifold (M, g). A second order p.d.o.*

$$L : C^\infty(E) \to C^\infty(E)$$

is called a generalized Laplacian *if $\sigma_2(L)(\xi) = -|\xi|_g^2$.*

Notice that all generalized Laplacians are elliptic operators.

9.1.3 Formal adjoints

For simplicity, all vector bundles in this subsection will be assumed complex, unless otherwise indicated.

Let $E_1, E_2 \to M$ be vector bundles over a smooth *oriented* manifold. Fix a Riemann metric g on M and Hermitian metrics $\langle \cdot, \cdot \rangle_i$ on E_i, $i = 1, 2$. We denote by $dv_g = *1$ the volume form on M defined by the metric g. Finally, $C_0^\infty(E_i)$ denotes the space of smooth, *compactly supported* sections of E_i.

Definition 9.1.18. *Let $P \in \mathbf{PDO}(E_1, E_2)$. The operator $Q \in \mathbf{PDO}(E_2, E_1)$ is said to be a* formal adjoint *of P if $\forall u \in C_0^\infty(E_1)$ and $\forall v \in C_0^\infty(E_2)$*

$$\int_M \langle Pu, v\rangle_2 dv_g = \int_M \langle u, Qv\rangle_1 dv_g.$$

Lemma 9.1.19. *Any $P \in \mathbf{PDO}(E_1, E_2)$ admits at most one formal adjoint.*

Proof Let Q_1, Q_2 be two formal adjoints of P. Then $\forall v \in C_0^\infty(E_2)$

$$\int_M \langle u, (Q_1 - Q_2)v\rangle_1 dv_g = 0 \quad \forall u \in C_0^\infty(E_1).$$

This implies $(Q_1 - Q_2)v = 0$ $\forall v \in C_0^\infty(E_2)$. If now $v \in C^\infty(E_2)$ is not necessarily compactly supported then , choosing $(\alpha) \subset C_0^\infty(M)$ a partition of unity, we conclude using the locality of $Q = Q_1 - Q_2$ that

$$Qv = \sum \alpha Q(\alpha v) = 0.$$

□

The formal adjoint of a p.d.o. $P \in \mathbf{PDO}(E_1, E_2)$ (whose existence is not yet guaranteed) is denoted by P^*. It is worth emphasizing that P^* depends on the choices of g and $\langle \cdot, \cdot \rangle_i$.

Proposition 9.1.20. *(a) Let $L_0 \in \mathbf{PDO}(E_0, E_1)$ and $L_1 \in \mathbf{PDO}(E_1, E_2)$ admit formal adjoints $L_i^* \in \mathbf{PDO}(E_{i+1}, E_i)$ ($i = 0, 1$) (with respect to a metric g on the base and metric $\langle \cdot, \cdot \rangle_j$ on E_j, $j = 0, 1, 2$. Then $L_1 L_0$ admits a formal adjoint and*

$$(L_1 L_0)^* = L_0^* L_1^*.$$

(b) If $L \in \mathbf{PDO}^{(m)}(E_0, E_1)$ then $L^* \in \mathbf{PDO}^{(m)}(E_1, E_0)$.

Proof (a) For any $u_i \in C_0^\infty(E_i)$ we have

$$\int_M \langle L_1 L_0 u_0, u_2\rangle_2 dv_g = \int_M \langle L_0 u_0, L_1^* u_0\rangle_1 dv_g = \int_M \langle u_0, L_0^* L_1^* u_2\rangle_0 dv_g.$$

(b) Let $f \in C^\infty(M)$. Then

$$(ad(f)L)^* = (L \circ f - f \circ L)^* = -[L^*, f] = -ad(f)L^*.$$

Thus

$$ad(f_0)ad(f_1)\cdots ad(f_m)L^* = (-1)^{m+1}(ad(f_0)ad(f_1)\cdots ad(f_m)L)^* = 0.$$

□

The above computation yields the following result.

Corollary 9.1.21. *If $P \in \mathbf{PDO}^{(m)}$ admits a formal adjoint then*

$$\sigma_m(P^*) = (-1)^m \sigma_m(P)^*$$

where the $*$ in the right-hand-side denotes the conjugate transpose of a linear map.

Let E be a Hermitian vector bundle over the oriented Riemann manifold (M, g).

Definition 9.1.22. *A p.d.o. $L \in \mathbf{PDO}\,(E, E)$ is said to be* formally selfadjoint *if $L = L^*$.*

The above notion depends clearly on the various metrics

Example 9.1.23. Using the integration by parts formula of Subsection 4.1.5 we deduce that the Hodge-DeRham operator

$$d + d^* : \Omega^*(M) \to \Omega^*(M)$$

on an oriented Riemann manifold (M, g) is formally selfadjoint with respect with the metrics induced by g in the various intervening bundles. In fact d^* is the formal adjoint of d. □

Proposition 9.1.24. *Let $(E_i, \langle\cdot,\cdot\rangle_i)$ $(i = 1, 2)$ be two arbitrary Hermitian vector bundle over the oriented Riemann manifold (M, g). Then any $L \in \mathbf{PDO}\,(E_1, E_2)$ admits at least (and hence exactly) one formal adjoint L^*.*

Sketch of proof We prove this result only in the case when E_1 and E_2 are trivial vector bundles over $\mathbb{R}^N$. However *we do not assume the Riemann metric over $\mathbb{R}^N$ is the Euclidean one.* The general case can be reduced to this via partitions of unity and we leave the reader check it for him/her-self.

Let $E_1 = \underline{\mathbb{C}}^p$ and $E_2 = \underline{\mathbb{C}}^q$. By choosing orthonormal moving frames we can assume the metrics on E_i are the Euclidean ones. According to the exercises at the end of Subsection 9.1.1 any $L \in \mathbf{PDO}^{(m)}(E_1, E_2)$ has the form

$$L = \sum_{|\alpha| \le m} A_\alpha(x) \partial^\alpha,$$

where $A_\alpha \in C^\infty(\mathbb{R}^N, M_{q\times p}(\mathbb{C}))$. Clearly, the formal adjoint of A_α is the conjugate transpose

$$A_\alpha^* = \overline{A}_\alpha^t.$$

To prove the proposition it suffices to show that each

$$\frac{\partial}{\partial x_i} \in \mathbf{PDO}^1(E_1, E_1)$$

admits a formal adjoint. It is convenient to consider the slightly more general situation.

Lemma 9.1.25. *Let $X = X^i \frac{\partial}{\partial x_i} \in \mathrm{Vect}\,(\mathbb{R}^N)$ and denote by ∇_X the first order p.d.o.*

$$\nabla_X u = X^i \frac{\partial}{\partial x_i} u \quad u \in C_0^\infty(\underline{\mathbb{C}}^p).$$

Then

$$\nabla_X = -\nabla_X - \mathbf{div}_g(X)$$

where $\mathbf{div}_g(X)$ denotes the divergence of X with respect to the metric g.

Proof of the lemma Let $u, v \in C_0^\infty(\mathbb{R}^N, \mathbb{C}^p)$. Choose $R \gg 0$ such that the Euclidean ball B_R of radius R centered at the origin contains the supports of both u and v. From the equality

$$X \cdot \langle u, v\rangle = \langle \nabla_X u, v\rangle + \langle u, \nabla_X v\rangle$$

we deduce

$$\int_{\mathbb{R}^N} \langle \nabla_X u, v\rangle dv_g = \int_{B_R} \langle \nabla_X u, v\rangle dv_g$$

$$= \int_{B_R} X \cdot \langle u, v\rangle dv_g - \int_{B_R} \langle u, \nabla_X v\rangle dv_g. \tag{9.1.6}$$

Set $f = \langle u, v\rangle \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$ and denote by $\alpha \in \Omega^1(\mathbb{R}^N)$ the 1-form dual to X with respect to the Riemann metric g i.e.

$$(\alpha, \beta)_g = \beta(X) \quad \forall \beta \in \Omega^1(\mathbb{R}^N).$$

Equivalently

$$\alpha = g_{ij} X^i dx^j.$$

The equality (9.1.6) can be rewritten

$$\int_{B_R} \langle \nabla_X u, v\rangle dv_g = \int_{B_R} df(X) dv_g - \int_{\mathbb{R}^N} \langle u, \nabla_X v\rangle dv_g$$

$$= \int_{B_R} (df, \alpha)_g dv_g - \int_{\mathbb{R}^N} \langle u, \nabla_X v\rangle dv_g.$$

The integration by parts formula of Subsection 4.1.5 yields

$$\int_{B_R} (df, \alpha)_g dv_g = \int_{\partial B_R} (df \wedge *_g \alpha)\,|_{\partial B_R} + \int_{B_R} f d^* \alpha dv_g.$$

Since $f \equiv 0$ on a neighborhood of of ∂B_R we get

$$\int_{B_R} (df, \alpha)_g dv_g = \int_{B_R} \langle u, v\rangle d^* \alpha \, dv_g.$$

Since $d^*\alpha = -\mathbf{div}_g(X)$ (see Subsection 4.1.5) we deduce

$$\int_{B_R} X \cdot \langle u, v\rangle dv_g = -\int_{B_R} \langle u, v\rangle \mathbf{div}_g(X) dv_g = \int_{B_R} \langle u, -\mathbf{div}_g(X) v\rangle dv_g.$$

Putting all the above together we get

$$\int_{\mathbb{R}^N} \langle \nabla_X u, v\rangle dv_g = \int_{\mathbb{R}^N} \langle u, (-\nabla_X - \mathbf{div}_g(X)) v\rangle dv_g,$$

i.e.

$$\nabla_X^* = -\nabla_X - \mathbf{div}_g(X) = -\nabla_X - \frac{1}{\sqrt{|g|}} \sum_i \partial_i(\sqrt{|g|} X^i). \tag{9.1.7}$$

The lemma and consequently the proposition is proved.

$\square$

Example 9.1.26. Let E be a rank r smooth vector bundle over the *oriented* Riemann manifold (M,g), $\dim M = m$. Let $\langle\cdot,\cdot\rangle$ denote a Hermitian metric on E and consider a connection ∇ on E compatible with this metric. ∇ defines a first order p.d.o.

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E).$$

The metrics g and $\langle\cdot,\cdot\rangle$ induce a metric on $T^*M \otimes E$. We want to describe the formal adjoint of ∇ with respect to these choices of metrics.

As we have mentioned in the proof of the previous proposition this is an entirely local issue. So we fix $x_0 \in M$ and denote by $(x^1,\dots,x^m)$ a collection of g-normal coordinates on a neighborhood U of x. Next, we pick a local *synchronous* frame of E near x_0 i.e. a local orthonormal frame (e_α) such that *at* x_0

$$\nabla_i e_\alpha = 0 \quad \forall i = 1,\dots,m.$$

The adjoint of the operator

$$dx^k\otimes : C^\infty(E\,|_U) \to C^\infty(T^*U \otimes E\,|_U)$$

is the interior derivative (contraction) along the vector field g-dual to the 1-form dx^k i.e.

$$(dx^k\otimes)^* = C^k \stackrel{def}{=} g^{jk}\cdot i(\partial_j).$$

Since ∇ is a metric connection we deduce as in the proof of Lemma 9.1.25 that

$$\nabla_k{}^* = -\nabla_k - \mathbf{div}_g(\partial_k).$$

Hence

$$\begin{aligned}\nabla^* &= \sum_k \nabla_k{}^* \circ C^k = \sum_k (-\nabla_k - \mathbf{div}_g(\partial_k)) \circ C^k \\ &= -\sum_k (\nabla_k + \partial_k(\log(\sqrt{|g|})) \circ C^k. \end{aligned} \tag{9.1.8}$$

In particular, since $\partial_k g = 0$ at x_0 we get

$$\nabla^*\,|_{x_0} = -\sum_k \nabla_k \circ C^k.$$

The *covariant Laplacian* is the second order p.d.o.

$$\Delta = \Delta_\nabla : C^\infty(E) \to C^\infty(E), \quad \Delta = \nabla^*\nabla.$$

To justify the attribute *Laplacian* we will show that Δ is indeed a generalized Laplacian. Using (9.1.8) we deduce that over U (chosen as above) we have

$$\Delta = -\left\{\sum_k (\nabla_k + \partial_k \log\sqrt{|g|}) \circ C^k\right\} \circ \left\{\sum_k dx^j \otimes \nabla_j\right\}$$

$$= -\left\{\sum_k (\nabla_k + \partial_k \log \sqrt{|g|})\right\} \circ \left(g^{kj} \cdot \nabla_j\right)$$

$$= -\sum_{k,j} \{g^{kj}\nabla_k + \partial_k g^{kj} + g^{kj}\partial_k \log \sqrt{|g|}\} \circ \nabla_j$$

$$= -\sum_{k,j} \{g^{kj}\nabla_k\nabla_j + \frac{1}{\sqrt{|g|}}\partial_k(\sqrt{|g|}g^{kj}) \cdot \nabla_j\}.$$

The symbol of Δ can be read easily from the last equality. More precisely

$$\sigma_2(\Delta)(\xi) = -g^{jk}\xi_j\xi_k = -|\xi|_g^2.$$

Hence Δ is indeed a generalized Laplacian. □

In the following exercise we use the notations in the previous example.

Exercise 9.1.7. (a) Show that

$$g^{k\ell}\Gamma_{k\ell}^i = -\frac{1}{\sqrt{|g|}}\partial_k(\sqrt{|g|}g^{ik})$$

where $\Gamma_{k\ell}^i$ denote the Christoffel symbols of the Levi-Civita connection associated to the metric g.
(b) Show that

$$\Delta_\nabla = -\mathrm{Tr}_g(\nabla^{T^*M\otimes E}\nabla^E)$$

where $\nabla^{T^*M\otimes E}$ is the connection on $T^*M \otimes E$ obtained by tensoring the Levi-Civita connection on T^*M and the connection ∇^E on E while

$$\mathrm{Tr}_g : C^\infty(T^*M^{\otimes 2} \otimes E) \to C^\infty(E)$$

denotes the double contraction by g

$$\mathrm{Tr}_g(S_{ij} \otimes u) = g^{ij}S_{ij}u.$$

□

Let E and M as above.

Proposition 9.1.27. *Let $L \in \mathbf{PDO}^2(E)$ be a generalized Laplacian. Then there* exists *a* unique *metric connection ∇ on E and $\mathcal{R} = \mathcal{R}(L) \in \mathrm{End}\,(E)$ such that*

$$L = \nabla^*\nabla + \mathcal{R}.$$

The endomorphism $\mathcal{R}$ is known as the Weitzenböck remainder *of the Laplacian L.*

Exercise 9.1.8. Prove the above proposition.
Hint Try ∇ defined by

$$\nabla_{f\mathbf{grad}(h)}u = \frac{f}{2}\{(\Delta_g h)u - (ad(h)L)u\} \quad f, h \in C^\infty(M),\ u \in C^\infty(E).$$

□

Exercise 9.1.9. (General Green formula) Consider a compact Riemannian manifold (M, g) with boundary ∂M. Denote by $\vec{n}$ the unit outer normal along ∂M. Let $E, F \to M$ be Hermitian vector bundles over M and suppose $L \in \mathbf{PDO}^k(E, F)$. Set $g_0 = g\,|_{\partial M}$, $E_0 = E\,|_{\partial M}$ and $F_0 = F\,|_{\partial M}$. The Green formula states that there exists a sesquilinear map

$$B_L : C^\infty(E) \times C^\infty(F) \to C^\infty(\partial M)$$

such that

$$\int_M \langle Lu, v\rangle dv(g) = \int_{\partial M} B_L(u,v) dv(g_0) + \int_M \langle u, L^* v\rangle dv(g).$$

Prove the following.
(a) If L is a zeroth order operator (i.e. L is a bundle morphism) then $B_L = 0$.
(b) If $L_1 \in \mathbf{PDO}(F, G)$ and $L_2 \in \mathbf{PDO}(E, F)$ then

$$B_{L_1 L_2}(u, v) = B_{L_1}(L_2 u, v) + B_{L_2}(u, L_1^* v).$$

(c)

$$B_{L^*}(v, u) = -\overline{B_L(u, v)}.$$

(d) Suppose ∇ is a Hermitian connection on E and $X \in \mathrm{Vect}\,(M)$. Set $L = \nabla_X : C^\infty(E) \to C^\infty(E)$. Then

$$B_L(u, v) = \langle u, v\rangle g(X, \vec{n}).$$

(e) Let $L = \nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$. Then

$$B_L(u, v) = \langle u, i_{\vec{n}} v\rangle_E.$$

where $i_{\vec{n}}$ denotes the contraction by $\vec{n}$.
(f) Denote by $\vec{\nu}$ the section of $T^*M\,|_{\partial M}$ g-dual to $\vec{n}$. Suppose L is a first order p.d.o. and set $J := \sigma_L(\vec{\nu})$. Then

$$B_L(u, v) = \langle Ju, v\rangle_F.$$

(g) Using (a)-(f) show that for all $u \in C^\infty(E)$, $v \in C^\infty(F)$ and any $x_0 \in \partial M$ the quantity $B_L(u, v)(x_0)$ depends only on the jets of u, v at x_0 of order at most $k - 1$.

$\square$

9.2 Functional framework

The partial differential operators are linear operators in infinite dimensional spaces and this feature requires special care in dealing with them. Linear algebra alone is not sufficient. This is where functional analysis comes in.

In this section we introduce a whole range of functional spaces which represent the suitable enviroment for p.d.o.'s to live in.

The presentation assumes the reader is familiar with some basic principles of functional analysis. As a reference for these facts we recommend the excellent monograph [15] or the very comprehensive [79].

9.2.1 Sobolev spaces in $\mathbb{R}^N$

Let D denote an open subset of $\mathbb{R}^N$. We denote by $L^1_{loc}(D)$ the space of locally integrable real functions on D, i.e. Lebesgue measurable functions $f : \mathbb{R}^N \to \mathbb{K}$ such that $\forall \alpha \in C_0^\infty(\mathbb{R}^N)$ $\alpha f \in L^1(\mathbb{R}^N)$.

Definition 9.2.1. *Let $f \in L^1_{loc}(D)$. A function $g \in L^1_{loc}(D)$ is said to be the* weak k-th *partial derivative of f, and we write this $g = \partial_k f$ weakly, if*

$$\int_D g\varphi dx = -\int_D f\partial_k\varphi dx \quad \forall \varphi \in C_0^\infty(D).$$

Lemma 9.2.2. *Any $f \in L^1_{loc}(D)$ admits* at most *one weak partial derivative.*

The proof of this lemma is left to the reader.
Warning: Not all locally integrable functions admit weak derivatives.

Exercise 9.2.1. Let $f \in C^\infty(D)$. Prove that the classical partial derivative $\partial_k f$ is also its weak k-th derivative. □

Exercise 9.2.2. Let $H \in L^1_{loc}(\mathbb{R})$ denote the *Heaviside function*, $H(t) \equiv 1$ for $t \geq 0$, $H(t) \equiv 0$ for $t < 0$. Prove that H is not weakly differentiable. □

Exercise 9.2.3. Let $f_1, f_2 \in L^1_{loc}(D)$. If $\partial_k f_i = g_i \in L^1_{loc}$ weakly then $\partial_k(f_1 + f_2) = g_1 + g_2$ weakly. □

The definition of weak derivative can be generalized to higher order derivatives as follows. Consider $L : C^\infty(D) \to C^\infty(D)$ a scalar p.d.o. and $f, g \in L^1_{loc}(D)$. Then we say that $Lf = g$ **weakly** if

$$\int_D g\varphi dx = \int_D fL^*\varphi dx \quad \forall \varphi \in C_0^\infty(D).$$

Above, L^* denotes the formal adjoint of L with respect to the Euclidean metric on $\mathbb{R}^N$.

Exercise 9.2.4. Let $f, g \in C^\infty(D)$. Prove that

$$Lf = g \ \text{classically} \iff Lf = g \ \text{weakly}.$$

□

Definition 9.2.3. *Let $k \in \mathbb{Z}_+$ and $p \in [1, \infty]$. The* Sobolev space *$L^{k,p}(D)$ consists of all the functions $f \in L^p(D)$ such that for any multi-index α satisfying $|\alpha| \leq k$ the mixed partial derivative $\partial^\alpha f$ exists weakly and moreover $\partial^\alpha f \in L^p(D)$. For every $f \in L^{k,p}(D)$ we set*

$$\|f\|_{k,p} = \|f\|_{k,p,D} = \left(\sum_{|\alpha| \leq k} \int_D |\partial^\alpha f|^p dx \right)^{1/p}$$

if $p < \infty$ while if $p = \infty$

$$\|f\|_{k,\infty} = \sum_{|\alpha| \leq k} \operatorname{ess\,sup} |\partial^\alpha f|.$$

When $k = 0$ we write $\|f\|_p$ instead of $\|f\|_{0,p}$.

Definition 9.2.4. *Let $k \in \mathbb{Z}_+$ and $p \in [1, \infty]$. Set*

$$L^{k,p}_{loc}(D) = \{f \in L^1_{loc}(D) \; ; \; \varphi f \in L^{k,p}(D) \;\; \forall \varphi \in C_0^\infty(D)\}.$$

Exercise 9.2.5. Let $f(t) = |t|^\alpha$, $t \in \mathbb{R}$, $\alpha > 0$. Show that for every $p > 1$ such that $\alpha > (p-1)/p$

$$f(t) \in L^{1,p}_{loc}(\mathbb{R}).$$

□

Theorem 9.2.5. *Let $k \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$. Then*
(a) $(L^{k,p}(\mathbb{R}^N), \|\cdot\|_{k,p})$ is a Banach space.
(b) If $1 \leq p < \infty$ the subspace $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{k,p}(\mathbb{R}^N)$.
(c) If $1 < p < \infty$ the Sobolev space $L^{k,p}(\mathbb{R}^N)$ is reflexive.

The proof of this theorem relies on a collection of basic techniques frequently used in the study of partial differential equations. This is why we choose to cover the proof of this theorem in some detail. We will consider only the case $p < \infty$ leaving the $p = \infty$ situation to the reader.

Proof Using the exercise 9.2.3 we deduce that $L^{k,p}$ is a vector space. From the classical Minkowski inequality

$$\left(\sum_{i=1}^{\nu} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\nu} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\nu} |y_i|^p\right)^{1/p}.$$

we deduce that $\|\cdot\|_{k.p}$ is a norm. To prove that $L^{k,p}$ is complete we will use the well established fact that L^p is complete.

Convention: *To simplify the notations, throughout this chapter all the extracted subsequences will be denoted by the same symbols as the sequences they originate from.*

Let $(f_n) \subset L^{k,p}(\mathbb{R}^N)$ be a Cauchy sequence i.e.

$$\lim_{m,n \to \infty} \|f_m - f_n\|_{k,p} = 0.$$

In particular, for each multi-index $|\alpha| \leq k$ the sequence $(\partial^\alpha f_n)$ is Cauchy in $L^p(\mathbb{R}^N)$ and thus

$$\partial^\alpha f_n \stackrel{L^p}{\to} g_\alpha \quad \forall |\alpha| \leq k.$$

Set $f = \lim_n f_n$. We claim that $\partial^\alpha f = g_\alpha$ weakly.

Indeed, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \partial^\alpha f_n \varphi dx = (-1)^{|\alpha|} \int_{\mathbb{R}^N} f_n \partial^\alpha \varphi.$$

Since $\partial^\alpha f_n \to g_\alpha$ and $f_n \to f$ in L^p and $\varphi \in L^q(\mathbb{R}^N)$ where $1/q = 1 - 1/p)$ we conclude

$$\int_{\mathbb{R}^N} g_\alpha \cdot \varphi dx = \lim_n \int_{\mathbb{R}^N} \partial^\alpha f_n \cdot \varphi dx$$

$$= \lim_n (-1)^{|\alpha|} \int_{\mathbb{R}^N} f_n \cdot \partial^\alpha \varphi dx = (-1)^{|\alpha|} \int_{\mathbb{R}^N} f \cdot \partial^\alpha \varphi dx.$$

Part (a) is proved. To prove that $C_0^\infty(\mathbb{R}^N)$ is dense we will use *mollifiers*. Their definition uses the operation of *convolution*. Given $f, g \in L^1(\mathbb{R}^N)$ define

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy.$$

We leave the reader check that $f * g$ is well defined (i.e. $y \mapsto f(x-y)g(y) \in L^1$ for almost all x).

Exercise 9.2.6. If $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ then $f * g$ is well defined. Moreover $f * g \in L^p(\mathbb{R}^N)$ and

$$\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p. \tag{9.2.1}$$

□

To define the mollifiers one usually starts with a function $\rho \in C_0^\infty(\mathbb{R}^N)$ such that

$$\rho \geq 0, \quad \operatorname{supp} \rho \subset \{|x| < 1\}$$

and

$$\int_{\mathbb{R}^N} \rho dx = 1.$$

Next, for each $\delta > 0$ we define

$$\rho_\delta(x) = \delta^{-N} \rho(x/\delta).$$

Note that $\operatorname{supp} \rho_\delta \subset \{|x| < \delta\}$ and

$$\int_{\mathbb{R}^N} \rho_\delta dx = 1.$$

The sequence (ρ_δ) is called a mollifying sequence. The next result describes the main use of this construction.

Lemma 9.2.6. *(a) For any $f \in L^1_{loc}(\mathbb{R}^N)$ the convolution $\rho_\delta * f$ is a* smooth[1] *function!*
(b) If $f \in L^p(\mathbb{R}^N)$ $(1 \leq p < \infty)$ then

$$\rho_\delta * f \stackrel{L^p}{\to} f \ \text{ as } \delta \to 0.$$

Proof of the lemma Part (a) is left to the reader as an exercise in the differentiability of integrals with parameters.

To establish part (b) we will use the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$. Fix $\varepsilon > 0$ and choose $g \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|f - g\|_p \leq \varepsilon/3.$$

[1]This explains the term *mollifier*: ρ_δ smoothes out the asperities.

We have

$$\|\rho_\delta * f - f\|_p \leq \|\rho_\delta * (f-g)\|_p + \|\rho_\delta * g - g\|_p + \|g - f\|_p.$$

Using the inequality (9.2.1) we deduce

$$\|\rho_\delta * f - f\|_p \leq 2\|f-g\|_p + \|\rho_\delta * g - g\|_p. \tag{9.2.2}$$

We need to estimate $\|\rho_\delta * g - g\|_p$. Note that

$$\rho_\delta * g\,(x) = \int_{\mathbb{R}^N} \rho(z) g(x - \delta z) dz$$

and

$$g(x) = \int_{\mathbb{R}^N} \rho(z) g(x) dz.$$

Hence

$$|\rho_\delta * g\,(x) - g(x)| \leq \int_{\mathbb{R}^N} \rho(z)|g(x-\delta z) - g(x)| dz \leq \delta \sup |dg|.$$

Since

$$\operatorname{supp} \rho_\delta * g \subset \{x + z \,;\; x \in \operatorname{supp} g\,; z \in \operatorname{supp} \rho_\delta\}$$

there exists a compact set $K \subset \mathbb{R}^N$ such that

$$\operatorname{supp}(\rho_\delta * g - g) \subset K \quad \forall \delta \in (0,1).$$

We conclude that

$$\|\rho_\delta * g - g\|_p \leq \left(\int_K \delta^p (\sup |dg|)^p dx\right)^{1/p} = \operatorname{vol}(K)^{1/p} \delta \sup |dg|.$$

If now we pick δ such that

$$\operatorname{vol}(K)^{1/p} \delta \sup |dg| \leq \varepsilon/3$$

we conclude from (9.2.2) that

$$\|\rho_\delta * f - f\|_p \leq \varepsilon.$$

The lemma is proved.

□

The next auxiliary result describes another useful feature of the mollification technique especially versatile in as far as the study of partial differential equations is concerned.

Lemma 9.2.7. *Let $f, g \in L^1_{loc}(\mathbb{R}^N)$ such that $\partial_k f = g$ weakly. Then $\partial_k(\rho_\delta * f) = \rho_\delta * g$. More generally, if*

$$L = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$$

is a p.d.o with constant *coefficients $a_\alpha \in \mathbb{R}$ and $Lf = g$ weakly then*

$$L(\rho_\delta * f) = \rho_\delta * g \;\text{ classically.}$$

Remark 9.2.8. The above lemma is a commutativity result. It shows that if L is a p.d.o. with constant coefficients then

$$[L, \rho_\delta *]f = L(\rho_\delta * f) - \rho_\delta * (Lf) = 0.$$

This fact has a fundamental importance in establishing regularity results for elliptic operators. □

Proof of the lemma It suffices to prove only the second part. We will write L_x to emphasize that L acts via derivatives with respect to the variables $x = (x^1, \ldots, x^N)$. Note that

$$L(\rho_\delta * f) = \int_{\mathbb{R}^N} (L_x \rho_\delta(x-y)) f(y) dy. \tag{9.2.3}$$

Since

$$\frac{\partial}{\partial x_i} \rho_\delta(x-y) = -\frac{\partial}{\partial y^i} \rho_\delta(x-y)$$

and

$$\partial_i{}^* = -\partial_i$$

we deduce from (9.2.3) that

$$L(\rho_\delta * f) = \int_{\mathbb{R}^N} (L_y^* \rho_\delta(x-y)) f(y) dy = \int_{\mathbb{R}^N} \rho_\delta(x-y) g(y) dy = \rho_\delta * g$$

since $Lf = g$ weakly and $y \mapsto \rho_\delta(x-y) \in C_0^\infty(\mathbb{R}^N)$ $\forall x$. The lemma is proved.

□

After this rather long detour we return to the proof of Theorem 9.2.5. Let $f \in L^{k,p}(\mathbb{R}^N)$. We will construct $f_n \in C_0^\infty(\mathbb{R}^N)$ such that $f_n \to f$ in $L^{k,p}$ using two basic techniques: *truncation* and *mollification*.

Step 1: Truncation The essentials of this technique are contained in the following result.

Lemma 9.2.9. *Let $f \in L^{k,p}(\mathbb{R}^N)$. Consider for each $R > 0$ a smooth function $\eta_R \in C_0^\infty(\mathbb{R}^N)$ such that $\eta(x) \equiv 1$ for $|x| \le R$, $\eta_R(x) \equiv 0$ for $|x| \ge R+1$ and $|d\eta_R(x)| \le 2$ $\forall x$. Then $\eta_R \cdot f \in L^{k,p}(\mathbb{R}^N)$ $\forall R \ge 0$ and moreover*

$$\eta_R \cdot f \stackrel{L^{k,p}}{\to} f \ \text{ as } R \to \infty.$$

Proof We consider only the case $k = 1$. The general situation can be proved by induction. We first prove that $\partial_i(\eta_R f)$ exists weakly and as expected

$$\partial_i(\eta_R \cdot f) = (\partial_i \eta_R) \cdot f + \eta_R \cdot \partial_i f.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $\eta_R \varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (\partial_i \eta_R)\varphi + \eta_R \cdot \partial_i \varphi dx = \int_{\mathbb{R}^N} \partial_i(\eta_R \varphi) f = -\int_{\mathbb{R}^N} \eta_R \varphi \partial_i f dx.$$

This confirms our claim. Clearly $(\partial_i \eta_R) f + \eta_R \partial_i f \in L^p(\mathbb{R}^N)$ so that $\eta_R f \in L^{1.p}(\mathbb{R}^N)$.

Note that $\partial_i \eta_R \equiv 0$ for $|x| \le R$ and $|x| \ge R+1$. In particular we deduce

$$(\partial_i \eta_R) f \to 0 \ \text{ a.e.}$$

Clearly $|(\partial_i \eta_R) f| \le 2|f(x)|$ so that by the dominated convergence theorem we conclude

$$(\partial_i \eta_R) \cdot f \stackrel{L^p}{\to} 0 \ \text{ as } R \to \infty.$$

Similarly

$$\eta_R \partial_i f \to \partial_i f \ \text{ a.e.}$$

and $|\eta_R \partial_i f| \le |\partial_i f|$ which implies $\eta_R \partial_i f \to \partial_i f$ in L^p. The lemma is proved.

□

According to the above lemma the space of compactly supported $L^{k,p}$-functions is dense in $L^{k,p}(\mathbb{R}^N)$. Hence it suffices to show that any such function can be arbitrarily well approximated in the $L^{k,p}$-norm by smooth, compactly supported functions.

Let $f \in L^{k,p}(\mathbb{R}^N)$ and assume

$$\operatorname{ess\,supp} f \subset \{|x| \le R\}.$$

Step 2: Mollification $\rho_\delta * f \stackrel{L^{k,p}}{\to} f$ as $\delta \to 0$.

Note that each $\rho_\delta * f$ is a smooth, function supported in $\{|x| \le R+\delta\}$. According to Lemma 9.2.7

$$\partial^\alpha(\rho_\delta * f) = \rho_\delta * (\partial^\alpha f) \ \ \forall |\alpha| \le k.$$

The desired conclusion now follows using Lemma 9.2.6.

To conclude the proof of Theorem 9.2.5 we need to show that $L^{k,p}$ is reflexive if $1 < p < \infty$. We will use the fact that L^p is reflexive for p in this range.

Note first that $L^{k,p}(\mathbb{R}^N)$ can be viewed as a closed subspace of the direct product

$$\prod_{|\alpha| \le k} L^p(\mathbb{R}^N).$$

via the map

$$T : L^{k,p}(\mathbb{R}^N) \to \prod_{|\alpha| \le k} L^p(\mathbb{R}^N) \ \ f \mapsto (\partial^\alpha f)_{|\alpha| \le k}$$

which is continuous 1-1 and and has closed range. Indeed, if

$$\partial^\alpha f_n \stackrel{L^p}{\to} f_\alpha$$

then arguing as in the proof of completeness we deduce that

$$f_\alpha = \partial^\alpha f_0 \ \text{ weakly}$$

where $f_0 = \lim f_n$. Hence $(f_\alpha) = T f_0$. We now conclude that $L^{k,p}(\mathbb{R}^N)$ is reflexive as a closed subspace of a reflexive space. Theorem 9.2.5 is proved.

□

Remark 9.2.10. (a) For $p = 2$ the spaces $L^{k,2}(\mathbb{R}^N)$ are in fact Hilbert spaces. The inner product is given by

$$\langle u, v\rangle_k = \int_{\mathbb{R}^N} \left(\sum_{|\alpha|\leq k} \partial^\alpha u \cdot \partial^\alpha v dx \right) dx.$$

(b) If $D \subset \mathbb{R}^N$ is open then $L^{k,p}(D)$ is a Banach space, reflexive if $1 < p < \infty$. However $C_0^\infty(D)$ is no longer dense in $L^{k,p}(D)$. The closure of $C_0^\infty(D)$ in $L^{k,p}(D)$ is denoted by $L_0^{k,p}(D)$. Intuitively, $L_0^{k,p}(D)$ consists of the functions $u \in L^{k,p}(D)$ such that

$$\frac{\partial^j u}{\partial \nu^j} = 0 \ \text{ on } \partial D \ \ \forall j = 0, 1, \dots, k-1$$

where $\partial/\partial\nu$ denotes the normal derivative along the boundary. The above statement should be taken with a grain of salt since at this point it is not clear how one can define $u|_{\partial D}$ when u is defined only almost everywhere. We refer to [Adm] for a way around this issue.

The larger space $C^\infty(D) \cap L^{k,p}(D)$ is dense in $L^{k,p}(D)$ provided the boundary of D is sufficiently regular. We refer again to [3] for details. □

Exercise 9.2.7. Prove that the following statements are equivalent.
(a) $u \in L^{1,p}(\mathbb{R}^N)$.
(b) There exists a constant $C > 0$ such that for all $\varphi \in C^\infty(\mathbb{R}^N)$

$$\left| \int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial x^i} \right| \leq C\|\varphi\|_{L^{p'}} \quad \forall i = 1, \cdots, N$$

where $p' = p/(p-1)$.
(c) There exists $C > 0$ such that for all $h \in \mathbb{R}^N$

$$\|\Delta_h u\|_{L^p} \leq C|h|$$

where $\Delta_h u(x) = u(x+h) - u(x)$. □

Exercise 9.2.8. Let $f \in L^{1,p}(\mathbb{R}^N)$ and $\phi \in C^\infty(\mathbb{R})$ such that

$$|d\phi| \leq const.$$

and $\phi(f) \in L^p$. Then $\phi(f) \in L^{1,p}(\mathbb{R}^N)$ and

$$\partial_i \phi(f) = \phi'(f) \cdot \partial_i f.$$

□

Exercise 9.2.9. Let $f \in L^{1,p}(\mathbb{R}^N)$. Show that $|f| \in L^{1,p}(\mathbb{R}^N)$ and

$$\partial_i |f| = \begin{cases} \partial_i f & \text{a.e. on } \{f \geq 0\} \\ -\partial_i f & \text{a.e. on } \{f < 0\} \end{cases}$$

Hint: Show that $f_\varepsilon = (\varepsilon^2 + f^2)^{1/2}$ converges to f in $L^{1,p}$ as $\varepsilon \to 0$.

□

9.2.2 Embedding theorems: integrability properties

The embedding theorems describe various inclusions between the Sobolev spaces $L^{k,p}(\mathbb{R}^N)$.

Define the "strength" of the Sobolev space $L^{k,p}(\mathbb{R}^N)$ as the quantity

$$\sigma(k,p) = \sigma_N(k,p) = k - N/p.$$

The "strength" is a measure of the size of a Sobolev size. Loosely speaking, the bigger the strength the more regular are the functions in that space and thus it consists of "fewer" functions.

Remark 9.2.11. The origin of the quantities $\sigma_N(k,p)$ can be explained by using the notion of *conformal weight*. A function $u : \mathbb{R}^N \to \mathbb{R}$ can be thought as a dimensionless physical quantity. Its conformal weight is 0. Its partial derivatives $\partial_i u$ are physical quantities measured in $meter^{-1}$= variation per unit of distance and they have conformal weight -1. More generally, a mixed partial $\partial^\alpha u$ has conformal weight $-|\alpha|$. The quantities $|\partial^\alpha|^p$ have conformal weight $-p|\alpha|$. The volume form dx is assigned conformal weight N: the volume is measured in $meter^N$. The integral of a quantity of conformal weight w is a quantity of conformal weight $w + N$. For example

$$\int_{\mathbb{R}^N} |\partial^\alpha u|^p dx$$

has conformal weight $N - p|\alpha|$. In particular the quantity

$$\left(\int \{ \sum_{|\alpha|=k} |\partial^\alpha u|^p \} dx \right)^{1/p}$$

has conformal weight $(N - kp)/p = -\sigma_N(k,p)$. Geometrically, the conformal weight is captured by the behavior under the rescalings $x = \lambda y$. If $u_\lambda(y) = u(\lambda y)$ then

$$\partial_i u_\lambda = \lambda (\partial_i u)_\lambda$$

$$\partial^\alpha u_\lambda = \lambda^{|\alpha|} (\partial^\alpha u)_\lambda.$$

On an abstract manifold M of dimension N the quantities of conformal weight w are the sections of the bundle of w/N-densities $|\Lambda|_M^{w/N}$. For example a 1-density (measure) can be integrated and has weight N. □

Theorem 9.2.12. (Sobolev) *If*

$$\sigma_N(k,p) = \sigma_N(m,q) < 0 \text{ and } k > m$$

then

$$L^{k,p}(\mathbb{R}^N) \hookrightarrow L^{m,p}(\mathbb{R}^N)$$

and the natural inclusion is continuous, i.e. there exists $C = C(N,k,m,p,q) > 0$ *such that*

$$\|f\|_{m,q} \le C \|f\|_{k,p} \quad \forall f \in L^{k,p}(\mathbb{R}^N).$$

Proof We follow the approach of [63] which relies on the following elementary but ingenious lemma.

Lemma 9.2.13. *Let $N \geq 2$ and $f_1, \dots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. for each $x \in \mathbb{R}^N$ and $1 \leq i \leq N$ define*

$$\xi_i = (x^1, \dots, \hat{x}^i, \dots, , x^N) \in \mathbb{R}^{N-1}.$$

Then

$$f(x) = f_1(\xi_1) f_2(\xi_2) \cdots f_N(\xi_N) \in L^1(\mathbb{R}^N)$$

and moreover

$$\|f\|_1 \leq \prod_{i=1}^{N} \|f_i\|_{N-1}.$$

Exercise 9.2.10. Prove Lemma 9.2.13. □

We first prove the theorem in the case $k = 1$, $p = 1$ which means $m = 0$ and $q = N/(N-1)$. We will show that

$$\exists C > 0 : \ \|u\|_{N/(N-1)} \leq C \| \, |du| \, \|_1 \quad \forall u \in C_0^\infty(\mathbb{R}^N)$$

where $|du|^2 = |\partial_1 u|^2 + \cdots + |\partial_N u|^2$. This result then extends by density to any $u \in L^{1,1}(\mathbb{R}^N)$.

We have

$$|u(x^1, \dots, x^N)| \leq \int_{-\infty}^{x^1} |\partial_i u(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^N)| dt \stackrel{def}{=} g_i(\xi_i).$$

Note that $g_i \in L^1(\mathbb{R}^{N-1})$ so that

$$f_i(\xi_i) = g_i(\xi_i)^{1/(N-1)} \in L^{N-1}(\mathbb{R}^{N-1}).$$

Since

$$|u(x)|^{N/(N-1)} \leq f_1(\xi_1) \cdots f_N(\xi_N)$$

we conclude from Lemma 9.2.13 that $u(x) \in L^{N/(N-1)}(\mathbb{R}^N)$ and

$$\|u\|_{N/(N-1)} \leq \left(\prod_1^N \|g_i(\xi_i)\|_1 \right)^{1/N} = (\left(\prod_1^N \|\partial_i u\|_1 \right)^{1/N}.$$

Using the classical inequality

$$\text{geometric mean } \leq \text{ arithmetic mean}$$

we conclude

$$\|u\|_{N/(N-1)} \leq \frac{1}{N} \sum_1^N \|\partial_i u\|_1 \leq \frac{const.}{N} \sum_1^N \| \, |du| \, \|_1.$$

We have thus proved that $L^{1,1}(\mathbb{R}^N)$ embeds continuously in $L^{N/(N-1)}(\mathbb{R}^N)$.

Now let $1 < p < \infty$ such that $\sigma_N(1,p) = 1 - N/p < 0$ (i.e. $p < N$). We have to show that $L^{1,p}(\mathbb{R}^N)$ embeds continuously in $L^{p^*}(\mathbb{R}^N)$ where

$$p^* = \frac{Np}{N-p}.$$

Let $u \in C_0^\infty(\mathbb{R}^N)$. Set $v = |v|^{r-1}v$ where $r > 1$ will be specified later. The inequality

$$\|v\|_{N/(N-1)} \leq \left(\prod_1^N \|\partial_i v\|_1\right)^{1/N}$$

implies

$$\|u\|_{rN/(N-1)}^r \leq r\left(\prod_1^N \| \, |u|^{r-1}\partial_i u\|_1\right)^{1/N}.$$

If $q = p/(p-1)$ is the conjugate exponent of p then using the Hölder inequality we get

$$\| \, |u|^{r-1}\partial_i u\| \leq \|u\|_{q(r-1)}^{r-1}\|\partial_i u\|_p.$$

Consequently

$$\|u\|_{rN/(N-1)}^r \leq r\|u\|_{q(r-1)}^{r-1}\left(\prod_1^N \|\partial_i u\|_p\right)^{1/N}.$$

Now choose r such that $rN/(N-1) = q(r-1)$. This gives

$$r = p^*\frac{N-1}{N}$$

and we get

$$\|u\|_{p^*} \leq r\left(\prod_1^N \|\partial_i u\|_p\right)^{1/N} \leq C(N,p)\| \, |du| \, \|_p.$$

This shows $L^{1,p} \hookrightarrow L^{p^*}$ if $1 \leq p < N$. The general case

$$L^{k,p} \hookrightarrow L^{m,q} \ \text{ if } \sigma_N(k,p) = \sigma_N(m,q) < 0 \ \ k > m$$

follows easily by induction over k. We leave the reader fill in the details.

□

Theorem 9.2.14. (Rellich-Kondratchov) *Let $(k,p), (m,q) \in \mathbb{Z}_+ \times [1,\infty)$ such that*

$$k > m \ \text{ and } \ 0 > \sigma_N(k,p) > \sigma_N(m,q).$$

Then any bounded sequence $(u_n) \subset L^{k,p}(\mathbb{R}^N)$ supported in a ball $B_R(0)$, $R > 0$ has a subsequence strongly convergent in $L^{m,q}(\mathbb{R}^N)$.

Proof We discuss only the case $k = 1$ so that the condition $\sigma_N(1, p) < 0$ imposes $1 \le p < N$. Let (u_n) be a bounded sequence in $L^{1,p}(\mathbb{R}^N)$ such that

$$\operatorname{ess\,supp} u_n \subset \{|x| \le R\} \ \ \forall n.$$

We have to show that for every $1 \le q < p^* = Np/(N-p)$ the sequence (u_n) contains a subsequence convergent in L^q. The proof will be carried out in several steps.

Step 1 We will prove that for every $0 < \delta < 1$ the mollified sequence $u_{n,\delta} = \rho_\delta * u_n$ admits a subsequence *uniformly convergent on* $\{|x| \le R+1\}$.

To prove this we will use the Arzela-Ascoli theorem and we will show there exists $C = C(\delta) > 0$ such that

$$|u_{n,\delta}(x)| < C \ \ \forall n, \ \forall |x| \le R+1$$

$$|u_{n,\delta}(x_1) - u_{n,\delta}(x_2)| \le C|x_1 - x_2| \ \ \forall n, \ \forall |x_1|, \, |x_2| \le R+1.$$

Indeed

$$\begin{aligned} |\rho_\delta * u\,(x)| &\le \delta^{-N} \int_{|y-x|\le\delta} \rho\left(\frac{x-y}{\delta}\right) |u_n(y)| dy \\ &\le \delta^{-N} \int_{B_\delta(x)} |u_n(y)| dy \\ &\le C(N,p)\delta^{-N} \|u_n\|_{p^*} \cdot \operatorname{vol}(B_\delta)^{(p^*-1)/p^*} \\ &\le C(\delta)\|u_n\|_{1,p} \ \ \text{(by Sobolev embedding theorem).} \end{aligned}$$

Similarly

$$\begin{aligned} |u_{n,\delta}(x_1) - u_{n,\delta}(x_2)| &\le \int_{B_{R+1}} |\rho_\delta(x_1 - y) - \rho_\delta(x_2 - y)| \cdot |u_n(y)| dy \\ &\le C(\delta) \cdot |x_1 - x_2| \int_{B_{R+1}} |u_n(y)| dy \\ &\le C(\delta) \cdot |x_1 - x_2| \cdot \|u_n\|_{1,p}. \end{aligned}$$

Step 1 is completed.

Step 2: Conclusion Using the diagonal procedure we can extract a subsequence of (u_n) (still denoted by (u_n)) and a subsequence $\delta_n \searrow 0$ such that $(g_n = u_{n,\delta_n})$
(i) g_n is uniformly convergent on B_R.
(ii) $\lim_n \|g_n - u_n\|_{1,p,\mathbb{R}^N} = 0$.

We claim the subsequence (u_n) as above is convergent in $L^q(B_R)$ for all $1 \le q < p$. Indeed for all n, m

$$\|u_n - u_m\|_{q,B_R} \le \|u_n - g_n\|_{q,B_R} + \|g_n - g_m\|_{q,B_R} + \|g_m - u_m\|_{q,B_R}.$$

We now examine separately each of the three terms in the right-hand-side.
A. The first and the third term.

$$\|u_n - g_n\|_{q,B_R}^q = \int_{B_R} |u_n(x) - g_n(x)| dx$$

$$\text{(Hölder inequality)} \leq \left(\int_{B_R} |u_n(x) - g_n(x)|^q dx \right)^{1/r} \cdot \operatorname{vol}(B_R)^{(r-1)/r} \quad (r = p^*/q)$$

$$\leq C(R) \|u_n - g_n\|^q_{p^*, \mathbb{R}^N}$$

$$\text{(Sobolev)} \leq C(R) \|u_n - g_n\|_{1,p,\mathbb{R}^N} \to 0.$$

B. The middle term.

$$\|g_n - g_m\|^q_{q,B_R} \leq \{\sup_{B_R} |g_n(x) - g_m(x)|\}^q \cdot \operatorname{vol}(B_R) \to 0.$$

Hence the (sub)sequence (u_n) is Cauchy in $L^q(B_R)$ and thus it converges. The compactness theorem is proved.

□

9.2.3 Embedding theorems: differentiability properties

A priori, the functions in the Sobolev spaces $L^{k,p}$ are only measurable and are defined only almost everywhere. However, if the strength $\sigma_N(k,p)$ is sufficiently large then the functions of $L^{k,p}$ have a built-in regularity: each can be modified on a negligible set to become continuous and even differentiable.

To formulate our next results we must introduce another important family of Banach spaces, namely the spaces of Hölder continuous functions.

Let $\alpha \in (0,1)$. A function $u : D \subset \mathbb{R}^N \to \mathbb{R}$ is said to be *α-Hölder continuous* if

$$[u]_\alpha \stackrel{def}{=} \sup_{0<R<1, z\in D} R^{-\alpha} \operatorname{osc}(u;\ B_R(z) \cap D) < \infty$$

where for any set $S \subset D$ we denoted by $\operatorname{osc}(u;\ S)$ the oscillation of u on S i.e.

$$\operatorname{osc}(u;\ S) = \sup\{|u(x) - u(y)|\ ;\ x, y \in S\}.$$

Set

$$\|u\|_{\infty,D} = \sup_{x\in D} |u(x)|$$

and define

$$C^{0,\alpha}(D) = \{u : D \to \mathbb{R}\ ;\ \|u\|_{0,\alpha,D} \stackrel{def}{=} \|u\|_\infty + [u]_\alpha < \infty\}.$$

More generally, for every integer $k \geq 0$ define

$$C^{k,\alpha}(D) = \{u \in C^m(D)\ ;\ \partial^\beta u \in C^{0,\alpha}(D),\ \forall |\beta| \leq k\}.$$

$C^{k,\alpha}(D)$ is a Banach space with respect to the norm

$$\|u\|_{k,\alpha} = \sum_{|\beta|\leq k} \|\partial^\beta u\|_{\infty,D} + \sum_{|\beta|=k} [\partial^\beta u]_{\alpha,D}.$$

Define the strength of the Hölder space $C^{k,\alpha}$ as the quantity

$$\sigma(k,\alpha) = k + \alpha.$$

Theorem 9.2.15. (Morrey) *Consider* $(m,p) \in \mathbb{Z}_+ \times [1,\infty]$ *and* $(k,\alpha) \in \mathbb{Z}_+ \times (0,1)$ *such that* $m > k$ *and* $\sigma_N(m,p) = \sigma(k,\alpha) > 0$. *Then* $L^{m,p}(\mathbb{R}^N)$ *embeds continuously in* $C^{k,\alpha}(\mathbb{R}^N)$.

Proof We consider only the case $k = 1$ and (necessarily) $m = 0$. The proof relies on the following elementary observation.

Lemma 9.2.16. *Let* $u \in C^\infty(B_R) \cap L^{1,1}(B_R)$ *and set*

$$\overline{u} = \frac{1}{\mathrm{vol}\,(B_R)} \int_{B_R} u(x)dx.$$

Then

$$|u(x) - \overline{u}| \leq \frac{2^N}{\sigma_{N-1}} \int_{B_R} \frac{|du(y)|}{|x-y|^{N-1}} dy. \tag{9.2.4}$$

In the above inequality σ_{N-1} *denotes the "area" of the* $(N-1)$*-dimensional round sphere* $S^{N-1} \subset \mathbb{R}^N$.

Proof of the lemma

$$u(x) - u(y) = -\int_0^{|x-y|} \frac{\partial}{\partial r} u(x + r\omega)dr \quad (\omega = -\frac{x-y}{|x-y|}).$$

Integrating the above equality with respect to y we get

$$\mathrm{vol}\,(B_R)(u(x) - \overline{u}) = -\int_{B_R} dy \int_0^{|x-y|} \frac{\partial}{\partial r} u(x + r\omega)dr.$$

If we set $|\partial_r u(x + r\omega)| = 0$ for $|x + r\omega| > R$ then

$$\mathrm{vol}\,(B_R)|u(x) - \overline{u}| \leq \int_{|x-y| \leq 2R} dy \int_0^\infty |\partial_r u(x + r\omega)|dr.$$

If we use polar coordinates (ρ, ω) centered at x then $dy = \rho^{N-1} d\rho d\omega$ where $\rho = |x-y|$ and $d\omega$ denotes the euclidian "area" form on the unit round sphere. We deduce

$$\begin{aligned}
\mathrm{vol}\,(B_R)|u(x) - \overline{u}| &\leq \int_0^\infty dr \int_{S^{N-1}} d\omega \int_0^{2R} |\partial_r u(x + r\omega)|\rho^{N-1} d\rho \\
&= \frac{(2R)^N}{N} \int_0^\infty dr \int_{S^{N-1}} |\partial_r(x + r\omega)|d\omega dr \\
&= \frac{(2R)^N}{N} \int_0^\infty r^{N-1} dr \int_{S^{N-1}} \frac{1}{r^{N-1}} |\partial_r u(x + r\omega)|d\omega \\
(z = x + r\omega) \quad &= \frac{(2R)^N}{N} \int_{B_R} \frac{|\partial_r u(z)|}{|x-z|^{N-1}} \leq \frac{(2R)^N}{N} \int_{B_R} \frac{|du(z)|}{|x-z|^{N-1}} dy.
\end{aligned}$$

The lemma is proved.

□

We want to make two simple observations.

1. In the above lemma we can replace the round ball B_R centered at origin by any other ball centered at any other point. In the sequel for any $R > 0$ and $x_0 \in \mathbb{R}^N$ we set

$$\overline{u}_{x_0,R} = \frac{1}{\operatorname{vol}(B_R(x_0))} \int_{B_R(x_0)} u(y)dy.$$

2. The inequality (9.2.4) can be extended by density to any $u \in L^{1,1}(B_R)$.

We will complete the proof of Morrey's theorem in three steps.
Step 1: *L^∞-estimates.* We will show there exists $C > 0$ such that $\forall u \in L^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$

$$\|u\|_\infty \leq C\|u\|_{1,p}.$$

For each $x \in \mathbb{R}^N$ denote by $B(x)$ the unit ball centered at x and set $\overline{u}_x = \overline{u}_{x,1}$. Using (9.2.4) we deduce

$$|u(x)| \leq |\overline{u}_x| + C_N \int_{B(x)} \frac{|du(y)|}{|x-y|^{N-1}} dy$$

$$C\left(\|u\|_p + \int_{B(x)} \frac{|du(y)|}{|x-y|^{N-1}} dy\right). \tag{9.2.5}$$

Since $\sigma_N(1,p) > 0$ we deduce that $p > N$ so that its conjugate exponent q satisfies

$$q = \frac{p}{p-1} < \frac{N}{N-1}.$$

In particular, the function $y \mapsto |x-y|^{-(N-1)}$ lies in $L^q(B(x))$ and

$$\int_{B(x)} \frac{1}{|x-y|^{q(N-1)}} dy \leq C(N,q)$$

where $C(N,q)$ is an universal constant depending only on N and q. Using the Hölder inequality in (9.2.5) we conclude

$$|u(x)| \leq C\|u\|_{1,p} \quad \forall x.$$

Step 2: *Oscillation estimates.* We will show there exists $C > 0$ such that for all $u \in L^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$

$$[u]_\alpha \leq C\|u\|_{1,p}.$$

Indeed, from the inequality (9.2.4) we deduce that for any ball $B_R(x_0)$ and any $x \in B_R(x_0)$

$$|u(x) - \overline{u}_{x_0,R}| \leq C \int_{B_R(x_0)} \frac{|du(y)|}{|x-y|^{N-1}}$$

$$\text{(Hölder inequality)} \;\leq\; C\|\,|du|\,\|_p \cdot \left(\int_{B_R(x_0)} \frac{1}{|x-y|^{q(N-1)}} dy\right)^{1/q} \quad (q = p/(p-1))$$

$$\leq C\|\,|du|\,\|_p R^\nu$$

where

$$\nu = \frac{1-(q-1)N-1)}{q} = 1-\frac{1}{p}-\frac{N}{p}+\frac{1}{p} = 1-\frac{N}{p} = \alpha.$$

Hence

$$|u(x)-\overline{u}_{x_0,R}| \leq CR^{\alpha} \quad \forall x \in B_R(x_0)$$

and consequently

$$\operatorname{osc}(u; B_R(x_0)) \leq CR^{\alpha}.$$

Step 2 is completed.
Step 3: *Conclusion.* Given $u \in L^{1,p}(\mathbb{R}^N)$ we can find $(u_n) \in C_0^{\infty}(\mathbb{R}^N)$ such that

$$u_n \to u \ \text{ in } L^{1,p} \text{ and almost everywhere.}$$

The estimates established at Step 1 and 2 are preserved as $n \to \infty$. In fact these estimates actually show the sequence (u_n) converges in the $C^{0,\alpha}$-norm to function $v \in C^{0,\alpha}(\mathbb{R}^N)$ which agrees almost everywhere with u.

□

Exercise 9.2.11. Let $u \in L^{1,1}(\mathbb{R}^N)$ satisfy a (q,ν)-*energy estimate* i.e.

$$\exists C > 0 : \ \frac{1}{r^N}\int_{B_r(x)} |du(y)|^q dy \leq C_1 r^{-\nu} \ \ \forall x \in \mathbb{R}^N,\ 0 < r < 2$$

where $0 \leq \nu < q$ and $q > 1$. Show that (up to a change on a negligible set) u is α-Hölder continuous ($\alpha = 1-\nu/q$) and moreover

$$[u]_{\alpha} \leq C_2$$

where the constant C_2 depends only on N, q, ν and C_1.
Hint: Prove that

$$\int_{B_r(x)} \frac{|du(y)|}{|x-y|^{N-1}} \leq Cr^{\alpha}$$

and then use the inequality (9.2.4).

□

Remark 9.2.17. The result in the above exercise has a suggestive interpretation. If u satisfies the (q,ν)-energy estimate then although $|du|$ may not be bounded, on average, it "explodes" no worse that $r^{\alpha-1}$ as $r \to 0$. Thus

$$|u(x)-u(0)| \approx C\int_0^{|x|} t^{\alpha-1}dt \approx C|x|^{\alpha}.$$

The energy estimate is a very useful tool in the study of nonlinear elliptic equations. □

The Morrey embedding theorem can be complemented by a compactness result. Let $(k,p) \in \mathbb{Z}_+ \times [1,\infty]$ and $(m,\alpha) \in \mathbb{Z}_+ \times (0,1)$ such that

$$\sigma_N(k,p) > \sigma(m,\alpha) \quad k > m.$$

Then a simple application of the Arzela-Ascoli theorem shows that any bounded sequence in $L^{k,p}(\mathbb{R}^N)$ admits a subsequence which converges in the $C^{m,\alpha}$ norm on any bounded open subset of $\mathbb{R}^N$.

The last results we want to discuss are the *interpolation inequalities*. They play an important part in applications but we chose not to include their proofs long but elementary proofs since they do not use any concept we will need later. The interested reader may consult [3] or [11] for details.

Theorem 9.2.18. (Interpolation inequalities) *For each $R > 0$ choose a smooth, cutoff function $\eta_R \in C_0^\infty(\mathbb{R}^N)$ such that*

$$\eta_R \equiv 1 \text{ if } |x| \le R$$

$$\eta_R \equiv 0 \text{ if } |x| \ge R+1$$

$$|d\eta_R(x)| \le 2 \;\; \forall x \in \mathbb{R}^N.$$

Fix $(m,p) \in \mathbb{Z}_+ \times [1,\infty)$ and $(k,\alpha) \in \mathbb{Z}_+ \times (0,1)$.
(a) For every $0 < r \le R+1$ there exists $C = C(r,R,m,p)$ such that for every $0 \le j < m$, $\varepsilon > 0$ and for all $u \in L^{m,p}(\mathbb{R}^N)$

$$\|\eta_R u\|_{j,p,\mathbb{R}^N} \le C\varepsilon \|\eta_R u\|_{m,p,\mathbb{R}^N} + C\varepsilon^{-j(m-j)}\|\eta_R u\|_{p,B_r}.$$

(b) For every $0 < r \le R+1$ there exists $C = C(r,R,k,\alpha)$ such that for every $0 \le j < k$, $\varepsilon > 0$ and for all $u \in C^{k,\alpha}(\mathbb{R}^N)$

$$\|\eta_R u\|_{j,\alpha,\mathbb{R}^N} \le C\varepsilon \|\eta_R u\|_{k,\alpha,\mathbb{R}^N} + C\varepsilon^{-j(m-j)}\|\eta_R u\|_{0,\alpha,B_r}.$$

The results in this and the previous section extend verbatim to slightly more general situations namely to functions

$$f : \mathbb{R}^N \to H$$

where H is a finite dimensional Hilbert space.

9.2.4 Functional spaces on manifolds

The Sobolev and the Hölder spaces can be defined over manifolds as well. To define these spaces we need two things: an oriented Riemann manifold (M,g) and a $\mathbb{K}$-vector bundle $\pi : E \to M$ endowed with a metric $h = \langle \bullet, \bullet \rangle$ and a connection $\nabla = \nabla^E$ compatible with h. The metric $g = (\bullet, \bullet)$ defines two important objects:
(i) the Levi-Civita connection ∇^g and
(ii) a volume form $dv_g = *1$. In particular, dv_g defines a Borel measure on M. We denote by $L^p(M,\mathbb{K})$ the space of $\mathbb{K}$-valued p-integrable functions on (M, dv_g) (modulo the equivalence relation of equality almost everywhere).

Definition 9.2.19. *Let $p \in [1,\infty]$. An L^p-section of E is a Lebesgue measurable map $\psi : M \to E$ (i.e. $\psi^{-1}(U)$ is Lebesgue measurable for any open subset $U \subset E$) such that:*
(i) $\pi \circ \psi(x) = x$ for almost all $x \in M$ except possibly a negligible set.
(ii) The function $x \mapsto |\psi(x)|_h$ belongs to $L^p(M,\mathbb{R})$.

The space of L^p-sections of E (modulo equality almost everywhere) is denoted by $L^p(E)$. We leave the reader check the following fact.

Proposition 9.2.20. *$L^p(E)$ is a Banach space with respect to the norm*

$$\|\psi\|_{p,E} = \begin{cases} \left(\int_M |\psi(x)|^p dv_g(x)\right)^{1/p} & \text{if } p < \infty \\ \operatorname{ess\,sup}_x |\psi(x)| & \text{if } p = \infty \end{cases}.$$

Note that if $p, q \in [1, \infty]$ are conjugate, $1/p + 1/q = 1$, then the metric $h : E \times E \to \underline{\mathbb{K}}_M$ defines a continuous pairing

$$\langle \bullet, \bullet \rangle : L^p(E) \times L^q(E) \to L^1(M, \mathbb{K}),$$

i.e.

$$\left| \int_M \langle \psi, \phi \rangle dv_g \right| \leq \|\psi\|_{p,E} \cdot \|\phi\|_{q,E}$$

This follows immediately from the Cauchy inequality

$$|h(\psi(x), \phi(x)| \leq |\psi(x)| \cdot |\phi(x)| \ \text{ a.e. on } M$$

and the usual Hölder inequality.

Exercise 9.2.12. Let $E_i \to M$ $(i = 1, \dots, k)$ be vector bundles with metrics and consider a multilinear bundle map

$$\chi : E_1 \times \cdots \times E_k \to \underline{\mathbb{K}}_M.$$

We regard χ as a section of $E_1^* \otimes \cdots \otimes E_k^*$. If

$$\chi \in L^{p_0}(E_1^* \otimes \cdots \otimes E_k^*)$$

then for every $p_1, \dots, p_k \in [1, \infty]$ such that

$$1 - 1/p_0 = 1/p_1 + \cdots + 1/p_k$$

and $\forall \psi_j \in L^{p_j}(E_j)$, $j = 1, \dots, k$

$$\left| \int_M \chi(\psi_1, \dots, \psi_k) dv_g \right| \leq \|\chi\|_{p_0} \cdot \|\psi_1\|_{p_1} \cdots \|\psi_k\|_{p_k}.$$

$\square$

For each $m = 1, 2, \dots$ define ∇^m as the composition

$$\nabla^m : \ C^\infty(E) \xrightarrow{\nabla^E} C^\infty(T^*M \otimes E) \xrightarrow{\nabla^{T^*M \otimes E}} C^\infty(T^*M^{\otimes 2} \otimes E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} C^\infty(T^*M^{\otimes m} \otimes E)$$

where we used the symbol ∇ to generically denote the connections in the tensor products $T^*M^{\otimes j} \otimes$E induced by ∇^g and ∇^E

The metrics g and h induce metrics in each of the tensor bundles $T^*M^{\otimes m} \otimes E$ and in particular, we can define the spaces $L^p(T^*M^{\otimes m} \otimes E)$.

Definition 9.2.21. *(a) Let* $u \in L^1_{loc}(E)$ *and* $v \in L^1_{loc}(T^*M^{\otimes m} \otimes E)$. *We say that* $\nabla^m u = v$ *weakly if*

$$\int_M \langle v, \phi \rangle dv_g = \int \langle u, (\nabla^m)^* \phi \rangle dv_g \quad \forall u \in C_0^\infty(T^*M^{\otimes m} \otimes E).$$

(b) Define $L^{m,p}(E)$ *as the space of sections* $u \in L^p(E)$ *such that* $\forall j = 1, \ldots, m$ *there exist* $v_j \in L^p(T^*M^{\otimes j} \otimes E)$ *such that* $\nabla^j u = v_j$ *weakly. We set*

$$\|u\|_{m,p} = \|u\|_{m,p,E} = \sum_{j=1}^{p} \|\nabla^j u\|_p.$$

A word of warning The Sobolev space $L^{m,p}(E)$ introduced above depends on several choices: the metrics on M and E and the connection on E. When M is non-compact this dependence is very dramatic and has to be seriously taken into consideration.

Example 9.2.22. Let (M, g) be the space $\mathbb{R}^N$ endowed with the euclidian metric. The trivial line bundle $E = \underline{\mathbb{R}}_M$ is naturally equipped with the trivial metric and connection. Then $L^p(\underline{\mathbb{R}}_M) = L^p(M, \mathbb{R})$. Denote by D the Levi-Civita connection. Then for every $u \in C^\infty(M)$ and $m \in \mathbb{Z}_+$ we have

$$D^m u = \sum_{|\alpha|=m} dx^{\otimes\alpha} \otimes \partial^\alpha u$$

where for every multi-index α we denoted by $dx^{\otimes\alpha}$ the monomial

$$dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_N}.$$

The length of $D^m u(x)$ is

$$\left(\sum_{|\alpha|=m} |\partial^\alpha u(x)|^2 \right)^{1/2}.$$

The space $L^{m,p}(\underline{\mathbb{R}}_M)$ coincides as a set with the Sobolev space $L^{k,p}(\mathbb{R}^N)$. The norm $\| \bullet \|_{m,p,\underline{\mathbb{R}}_M}$ is equivalent with the norm $\| \bullet \|_{m,p,\mathbb{R}^N}$ introduced in the previous sections. □

Proposition 9.2.23. $(L^{k,p}(E), \| \cdot \|_{k,p,E})$ *is a Banach space which is reflexive if* $1 < p < \infty$.

The proof of this result is left to the reader as an exercise.

The Hölder spaces can be defined on manifolds as well. If (M, g) is a Riemann manifold then g canonically defines a metric space structure on M (see Chapter 4) and in particular we can talk about the oscillation of a function $u : M \to \mathbb{K}$. On the other hand, defining the oscillation of a section of some bundle over M requires a little more work.

Let (E, h, ∇) as before. We assume the *injectivity radius* ρ_M *of* M *is positive*. Set $\rho_0 = \min\{1, \rho_M\}$. If x, $y \in M$ are two points such that $\mathrm{dist}_g(x, y) \leq \rho_0$ then they can be joined by a unique minimal geodesic $\gamma_{x,y}$ starting at x and ending at y. We denote by $T_{x,y} : E_y \to E_x$ the ∇^E-parallel transport along $\gamma_{x,y}$. For each $\xi \in E_x$ and $\eta \in E_y$ we set by definition

$$|\xi - \eta| = |\xi - T_{x,y}\eta|_x = |\eta - T_{y,x}\xi|_y.$$

If $u : M \to E$ is a section of E and $S \subset M$ has the diameter $< \rho_0$ we define

$$\operatorname{osc}(u\ ;\ S) = \sup\{|u(x) - u(y)|\ ;\ x,\, y \in S\}.$$

Finally set

$$[u]_{\alpha,E} = \sup\{r^{-\alpha}\operatorname{osc}(u\ ;\ B_r(x))\ ;\ 0 < r < \rho_0,\ x \in M\}.$$

For any $k \geq 0$ define

$$\|u\|_{k,\alpha,E} = \sum_{j=0}^{k} \|\nabla^j u\|_{\infty,E} + [\nabla^m u]_{\alpha,T^*M^{\otimes m}\otimes E}$$

and set

$$C^{k,\alpha}(E) = \{u \in C^k(E)\ ;\ \|u\|_{k,\alpha} < \infty\}.$$

Theorem 9.2.24. *Let (M, g) be a* compact*, N-dimensional, oriented Riemann manifold and E a vector bundle over M equipped with a metric h and compatible connection ∇. Then the following are true.*
(a) The Sobolev space $L^{m,p}(E)$ and the Hölder spaces $C^{k,\alpha}(E)$ do not depend on the metrics g, h and on the connection ∇. More precisely, if g_1 is a different metric on M and ∇^1 is another connection on E compatible with some metric h_1 then

$$L^{m,p}(E, g, h, \nabla) = L^{m,p}(E, g_1, h_1, \nabla^1) \quad \text{as sets of sections}$$

and the identity map between these two spaces is a Banach space isomorphism. A similar statement is true for the Hölder spaces.
(b) If $1 \leq p < \infty$ then $C^\infty(E)$ is dense in $L^{k,p}(E)$.
(c) If $(k_i, p_i) \in \mathbb{Z}_+ \times [1, \infty)$ $(i = 0, 1)$ are such that

$$k_0 \geq k_1 \ \text{ and } \ \sigma_N(k_0, p_0) = k_0 - N/p_0 \geq k_1 - N/p_1 = \sigma_N(k_1, p_1)$$

then $L^{k_0,p_0}(E)$ embeds continuously in $L^{k_1,p_1}(E)$. If moreover

$$k_0 > k_1 \ \text{ and } \ k_0 - N/p_0 > k_1 - N/p_1$$

then the embedding $L^{k_0,p_0}(E) \hookrightarrow L^{k_1,p_1}(E)$ is compact i.e. any bounded sequence of $L^{k_0,p_0}(E)$ admits a subsequence convergent in the L^{k_1,p_1}-norm.
(d) If $(m, p) \in \mathbb{Z}_+ \times [1, \infty)$ and $(k, \alpha) \in \mathbb{Z}_+ \times (0, 1)$ and

$$m - N/p \geq k + \alpha$$

then $L^{m,p}(E)$ embeds continuously in $C^{k,\alpha}(E)$. If moreover

$$m - N/p > k + \alpha$$

then the embedding is also compact.

We developed all the tools needed to prove this theorem and we leave this task to the reader. The method can be briefly characterized by two phrases: partition of unity and interpolation inequalities. We will see them at work in the next section.

9.3 Elliptic partial differential operators: analytic aspects

This section represents the analytical heart of this chapter. We discuss two notions which play a pivotal role in the study of elliptic partial differential equations. More precisely we will introduce the notion of *weak solution* and *a priori estimate*.

Consider the following simple example. Suppose we want to solve the partial differential equation

$$\Delta u + u = f \in L^2(S^2, \mathbb{R}) \tag{9.3.1}$$

where Δ denotes the Laplace-Beltrami operator on the round sphere. Riemann suggested one should consider the energy functional

$$E(u) = \int_{S^2} \Big\{ \frac{1}{2}(|du|^2 + u^2) - fu \Big\} dv_g.$$

If u_0 is a minimum of E i.e.

$$E(u_0) \leq E(u) \;\; \forall u$$

then

$$0 = \frac{d}{dt}|_{t=0}\, E(u_0 + tv) = \int_{S^2} (du_0, dv) + u_0 \cdot v - f \cdot v dv_g \quad \forall v. \tag{9.3.2}$$

Integrating by parts we get

$$\int_{S^2} (d^* du_0 + u_0 - f) \cdot v \; dv_g = 0 \quad \forall v$$

so that necessarily

$$\Delta u_0 + u_0 = f.$$

There are a few grey areas in this approach and Weierstrass was quick to point them out: what is the domain of E, u_0 may not exist and if it does it may not be C^2 so the integration by parts is ilegal etc. This avenue was abandoned until the dawns of this century when Hilbert reintroduced them into the spotlight and emphasized the need to deal with these issues. His important new point of view was that the approach suggested by Riemann does indeed produce a solution of (9.3.1) *" provided if need be that the notion of solution be suitable extended "*. The suitable notion of solution is precisely described in (9.3.2). Naturally, one asks when this extended notion of solution coincides with the classical one. Clearly, it suffices that u of (9.3.2) be at least C^2 so that everything boils down to a question of *regularity*.

Riemann's idea was first rehabilitated in Weyl's faimous treatise [77] on Riemann surfaces. It took the effort of many talented people to materialize Hilbert's program formulated as his 19th and 20th problem in the famous list of 27 problems he presented at the Paris conference at the beginning of this century. We refer the reader to [1] for more details.

This section takes up the issues raised in the above simple example. The key fact which will allow us to legitimize Riemann's argument is the ellipticity of the partial differential operator involved in this equation.

9.3.1 Elliptic estimates in $\mathbb{R}^N$

Let $E = \underline{\mathbb{C}}^r$ denote the trivial vector bundle over $\mathbb{R}^N$. Denote by $\langle \bullet, \bullet \rangle$ the natural Hermitian metric on E and by ∂ the trivial connection. The norm of $L^{k,p}(E)$ (defined using the Euclidean volume) will be denoted by $\| \bullet \|_{k,p}$. Consider an elliptic operator of order m

$$L = \sum_{|\alpha| \leq m} A_\alpha(x) \partial_x^\alpha : C^\infty(E) \to C^\infty(E).$$

For each $k \in \mathbb{Z}_+$ and $R > 0$ define

$$\|L\|_{k,R} = \sum_{|\alpha| \leq m, |\beta| \leq k+m-|\alpha|} \sup_{B_R(0)} \|\partial_x^\beta (A_\alpha(x))\|.$$

In this subsection we will establish the following fundamental result.

Theorem 9.3.1. *(a) Let $(k,p) \in \mathbb{Z}_+ \times (1,\infty)$ and $R > 0$. Then there exists $C = C(\|L\|_{k+1,R}, k, p, N, R) > 0$ such that $\forall u \in C_0^\infty(E|_{B_R(0)})$*

$$\|u\|_{k+m,p} \leq C(\|Lu\|_{k,p} + \|u\|_p). \tag{9.3.3}$$

(b) Let $(k,\alpha) \in \mathbb{Z}_+ \times (0,1)$ and $R > 0$. Then there exists $C = C(\|L\|_{k+1,R}, k, \alpha, N, R) > 0$ such that $\forall u \in C_0^\infty(E|_{B_R(0)})$

$$\|u\|_{k+m,\alpha} \leq C(\|Lu\|_{k,\alpha} + \|u\|_{0,\alpha}). \tag{9.3.4}$$

The proof consists of two conceptually distinct parts. In the first part we establish the result under the supplementary assumption that L has *constant coefficients*. In the second part, the general result is deduced from the special case using perturbation techniques in which the interpolation inequalities play an important role. Throughout the proof we will use the same letter C to denote various constants $C = C(\|L\|_{k+1,R}, k, p, N, R) > 0$

Step 1. We assume L has the form

$$L = \sum_{|\alpha|=m} A_\alpha \partial_x^\alpha$$

where A_α are $r \times r$ complex matrices , independent of $x \in \mathbb{R}^N$. We set

$$\|L\| = \sum \|A_\alpha\|.$$

We will prove the conclusions of the theorem hold in this special case. We will rely on a very deep analytical result whose proof goes beyond the scope of this book.

For each $f \in L^1(\mathbb{R}^N, \mathbb{C})$ denote by $\hat{f}(\xi)$ its Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(-\mathbf{i} x \cdot \xi) dx.$$

Theorem 9.3.2. (Calderon-Zygmund) *Let $\overline{m} : S^{N-1} \to \mathbb{C}$ be a smooth function and define*

$$m(\xi) : \mathbb{R}^N \setminus \{0\} \to \mathbb{C}$$

by

$$m(\xi) = \overline{m}\left(\frac{\xi}{|\xi|}\right).$$

Then the following hold.
(a) There exists $\Omega \in C^\infty(S^{N-1}, \mathbb{C})$ and $c \in \mathbb{C}$ such that
(a_1) $\int_{S^{N-1}} \Omega dv_{S^{N-1}} = 0$.
(a_2) For any $u \in C_0^\infty(\mathbb{R}^N)$ the limit

$$(Tu)(x) = \lim_{\varepsilon \searrow 0} \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^N} u(x-y) dy$$

exists for almost every $x \in \mathbb{R}^N$ and moreover

$$cu(x) + Tu(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(\mathbf{i}x \cdot \xi) m(\xi) \hat{u}(\xi) d\xi.$$

(b) For every $1 < p < \infty$ there exists $C = C(p, \|\overline{m}\|_\infty) > 0$ such that

$$\|Tu\|_p \leq C\|u\|_p \quad \forall u \in C_0^\infty(\mathbb{R}^N).$$

*(c)***(Korn-Lichtenstein)** *For every $0 < \alpha < 1$ and any $R > 0$ there exists $C = C(\alpha, \|\overline{m}\|_{0,\alpha}, R) > 0$ such that $\forall u \in C_0^{0,\alpha}(B_R)$*

$$[Tu]_\alpha \leq C\|u\|_{0,\alpha,B_R}.$$

For a proof of part (a) and (b) we refer to [72], Chap II §4.4. Part (c) is "elementary" and we suggest the reader to try and prove it. In any case a proof of this inequality can be found in [11], Part II.5.

Let us now return to our problem. Assuming L has the above special form we will prove (9.3.3). The proof of (9.3.4) is entirely similar and is left to the reader. We discuss first the case $k = 0$.

Let $u \in C_0^\infty(E\,|_{B_R})$. u can be viewed as a collection

$$u(x) = (u^1(x), \ldots, u^r(x))$$

of smooth functions compactly supported in B_R. Define

$$\hat{u}(\xi) = (\hat{u}^1(\xi), \ldots, \hat{u}^r(\xi)).$$

If we set $v = Lu$ then for any multi-index β such that $|\beta| = m$ we have

$$L\partial^\beta u = \partial^\beta Lu = \partial^\beta v$$

because L has constant coefficients. We Fourier transform the above equality and we get

$$(-\mathbf{i})^m \sum_{|\alpha|=m} A_\alpha \xi^\alpha \widehat{\partial^\beta u}(\xi) = (-\mathbf{i})^m \xi^\beta \hat{v}(\xi). \tag{9.3.5}$$

Note that since L is *elliptic* the operator

$$\sigma(L)(\xi) = A(\xi) = \sum_{|\alpha|=m} A_\alpha \xi^\alpha : \mathbb{C}^r \to \mathbb{C}^r$$

is invertible for any $\xi \neq 0$. From (9.3.5) we deduce

$$\widehat{\partial^\beta u(\xi)} = \xi^\beta B(\xi) v\hat{(}\xi) \;\; \forall \xi \neq 0$$

where $B(\xi) = A(\xi)^{-1}$. Note that $B(\xi)$ is homogeneous of degree $-m$ so that $M(\xi) = \xi^\beta B(\xi)$ is homogeneous of degree 0. Thus we can find functions $m_{ij}(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ which are homogeneous of degree 0 such that

$$\widehat{\partial^\beta u^i}(\xi) = \sum_j m_{ij}(\xi) \hat{v}^j(\xi).$$

Using Theorem 9.3.2 (a) and (b) we deduce

$$\|\partial^\beta u\|_p \leq C\|v\|_p = C\|Lu\|_p.$$

This proves (9.3.3) when L has this special form.

Step 2 *The general case.* L is now an arbitrary elliptic operator of order m. Let $r > 0$ sufficiently small (to be specified later). Cover B_R by finitely many balls $B_r(x_\nu)$ and consider $\eta_\nu \in C_0^\infty(B_r(x_\nu))$ such that each point in B_R is covered by at most 10^N of these balls and

$$\eta_\nu \geq 0 \quad \sum_\nu \eta_\nu = 1$$

$$\|\partial^\beta \eta_\nu\| \leq C r^{-|\beta|} \;\; \forall |\beta| \leq m.$$

If $u \in C_0^\infty(B_R)$ then

$$v = Lu = L\left(\sum_\nu \eta_\nu u\right) = \sum_\nu L(\eta_\nu u).$$

Set $u_\nu = \eta_\nu u$ and

$$L_\nu = \sum_{|\alpha|=m} A_\alpha(x_\nu)\partial^\alpha.$$

We rewrite the equality $v_\nu \stackrel{def}{=} Lu_\nu$ as

$$L_\nu u_\nu = (L_\nu - L)u_\nu + v_\nu = \sum_{|\alpha|=m} \varepsilon_{\alpha,\nu}(x)\partial^\alpha u_\nu - \sum_{|\beta|<m} A_\beta(x)\partial^\beta u_\nu + v_\nu,$$

where $\varepsilon_{\alpha,\nu}(x) = A_\alpha(x_\nu) - A_\alpha(x)$. Using (9.3.3) we deduce

$$\|u_\nu\|_{m,p} \leq C(\|u_\nu\|_{p,B_r(x_\nu)} + \|v_\nu\|_{p,B_r(x_\nu)} + \sum_{|\alpha|=m} \|\varepsilon_{\alpha,\nu}(x)\partial^\alpha u\|_{p,B_r(x_\nu)} + \|u\|_{m-1,p,B_r(x_\nu)}).$$

Since $|\varepsilon_{\alpha,\nu}(x)| \leq Cr$ on $B_r(x_\nu)$, where $C = C(\|L\|_{1,R})$ we deduce

$$\|u_\nu\|_{m,p,B_r(x_\nu)} \leq C(r\|u_\nu\|_{p,B_r(x_\nu)} + \|u_\nu\|_{p,B_r(x_\nu)} + \|v_\nu\|_{p,B_r(x_\nu)}).$$

We can now specify $r > 0$ such that $Cr < 1/2$ in the above inequality. Hence

$$\|u_\nu\|_{m,p,B_r(x_\nu)} \leq C(\|u_\nu\|_{p,B_r(x_\nu)} + \|v_\nu\|_{p,B_r(x_\nu)}).$$

We need to estimate $\|v_\nu\|_p$. We use the equality

$$v_\nu = L(\eta_\nu u) = \eta_\nu Lu + [L, \eta_\nu]u$$

in which $[L, \eta_\nu] = ad(\eta_\nu)L$ is a p.d.o. of order $m-1$ so that

$$\|[L, \eta_\nu]u\|_{p,B_r(x_\nu)} \leq Cr^{-(m-1)}\|u\|_{m-1,p,B_r(x_\nu)}.$$

Hence

$$\|v_\nu\|_{p,B_r(x_\nu)} \leq C(\|\eta_\nu Lu\|_{p,B_r(x_\nu)} + r^{-(m-1)}\|u\|_{m-1,p,B_r(x_\nu)})$$

so that

$$\|u_\nu\|_{m,p,B_r(x_\nu)} \leq C(\|Lu\|_{p,B_R} + r^{-(m-1)}\|u\|_{m-1,p,B_R}).$$

We sum over ν taking into account that the number of spheres $B_r(x_\nu)$ is $O((R/r)^N)$ we deduce

$$\|u\|_{m,p,B_R} \leq \sum_\nu \|u_\nu\|_{m,p} \leq CR^N(r^{-N}\|Lu\|_{p,B_R} + r^{-(m+N-1)}\|u\|_{m-1,p,B_R}).$$

Note that r depends only on R, p, $\|L\|_{1,R}$ so that

$$\|u\|_{m,p,B_R} \leq C(\|Lu\|_{p,B_R} + \|u\|_{m-1,p,B_R})$$

where C is as in the statement of Theorem 9.3.1.

We still need to deal with the term $\|u\|_{m-1,p,B_R}$ in the above inequality. It is precisely at this point where the interpolation inequalities enter crucially.

View u as a section of $C_0^\infty(E|_{B_{2R}})$. If we pick $\eta \in C_0^\infty(B_{2R})$ such that $\eta \equiv 1$ on B_R we deduce from the interpolation inequalities that there exists $C > 0$ such that

$$\|u\|_{m-1,B_R} \leq \varepsilon\|u\|_{m,p,R} + C\varepsilon^{-(m-1)}\|u\|_{p,B_R}.$$

Hence

$$\|u\|_{m,p,B_R} \leq C(\|Lu\|_{p,B_R} + \varepsilon\|u\|_{m,p,B_R} + \varepsilon^{-(m-1)}\|u\|_{p,B_R}).$$

If now we choose $\varepsilon > 0$ sufficiently small we deduce (9.3.3) with $k = 0$.

To establish it for arbitrary k we argue by induction. Consider a multi-index $|\beta| = k$. If $u \in C_0^\infty(E|_{B_R})$ and $Lu = v$ then

$$L(\partial^\beta u) = \partial^\beta Lu + [L, \partial^\beta]u = \partial^\beta v + [L, \partial^\beta]u.$$

The crucial observation is that $[L, \partial^\beta]$ is a p.d.o. of order $\leq m + k - 1$. Indeed

$$\sigma_{m+k}([L, \partial^\beta]) = [\sigma_m(L), \sigma_k(\partial^\beta)] = 0.$$

Using (9.3.3) with $k = 0$ we deduce

$$\begin{aligned} \|\partial^\beta u\|_{m,p,B_R} &\leq C(\|\partial^\beta v\|_{p,B_R} + \|[L, \partial^\beta]u\|_{p,B_R} + \|\partial^\beta u\|_{p,B_R}) \\ &\leq C(\|v\|_{k,p,B_R} + \|u\|_{m+k-1,p,B_R} + \|u\|_{k,p,B_R}). \end{aligned}$$

The term $\|u\|_{m+k-1,p,B_R} + \|u\|_{k,p,B_R}$ can be estimated from above by

$$\varepsilon\|u\|_{m+k,p,B_R} + C\|u\|_{p,B_R}$$

using the interpolation inequalities as before. The inequality (9.3.3) is completely proved. The Hölder case is entirely similar. It is left to the reader as an exercise. Theorem 9.3.1 is proved.

□

Using the truncation technique and the interpolation inequalities we deduce the following consequence.

Corollary 9.3.3. *Let L as in Theorem 9.3.1 and fix $0 < r < R$. Then for every $k \in \mathbb{Z}_+$, $1 < p < \infty$ and $\alpha \in (0, 1)$ there exists $C = C(k, p, \alpha, N, \|L\|_{k+1,R}, R, r) > 0$ such that $\forall u \in C^\infty(E)$*

$$\|u\|_{k+m,p,B_r} \leq C(\|Lu\|_{k,p,B_R} + \|u\|_{p,B_R})$$

and

$$\|u\|_{k+m,\alpha,B_r} \leq C(\|Lu\|_{k,\alpha,B_R} + \|u\|_{0,\alpha,B_R}).$$

Exercise 9.3.1. Prove the above corollary. □

9.3.2 Elliptic regularity

In this subsection we continue to use the notations of the previous subsection.

Definition 9.3.4. *Let u, $v : \mathbb{R}^N \to \mathbb{C}^r$ be measurable functions.*
(a) u is a classical solution *of the partial differential equation*

$$Lu = \sum_{|\beta| \leq m} A_\alpha(x)\partial^\alpha u\,(x) = v(x) \tag{9.3.6}$$

if there exists $\alpha \in (0, 1)$ such that $v \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$, $v \in C^{m,\alpha}_{loc}(\mathbb{R}^N)$ and (9.3.6) holds everywhere.

(b) u is said to be an L^p strong solution *of (9.3.6) if $u \in L^{m,p}_{loc}(\mathbb{R}^N)$ and $v \in L^p_{loc}(\mathbb{R}^N)$ and (9.3.6) hold almost everywhere. (The partial derivatives of u should be understood in generalized sense.)*
(c) u is said to be an L^p weak solution *if $u, v \in L^p_{loc}(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} \langle u, L^*\phi\rangle dx = \int_{\mathbb{R}^N} \langle v, \phi\rangle dx \quad \forall \phi \in C_0^\infty(E).$$

Note the following obvious inclusion

$$\{L^p \text{ weak solutions}\} \supset \{L^p \text{ strong solutions}\}.$$

The principal result of this subsection will show that when L is elliptic then the above inclusion is an equality.

Theorem 9.3.5. *Let $1 < p < \infty$ and $L : C^\infty(E) \to C^\infty(E)$ an elliptic operator of order m. Then any L^p-weak solution u of*

$$Lu = v \in L^p_{loc}(E)$$

is an L^p strong solution ,i.e. $u \in L^{m,p}_{loc}(E)$.

Remark 9.3.6. Loosely speaking the above theorem says that if a "clever" (i.e. elliptic) combination of mixed partial derivatives can be defined weakly then any mixed partial derivative (up to a certain order) can be weakly defined as well. □

The essential ingredient in the proof is the technique of mollification. For each $\delta > 0$ set

$$u_\delta = \rho_\delta * u \in C^\infty(E) \quad v_\delta = \rho_\delta * v \in C^\infty(E).$$

The decisive result in establishing the regularity of u is the following.

Lemma 9.3.7. *Let $w_\delta = Lu_\delta - v_\delta \in C^\infty(E)$. Then for every $\phi \in C_0^\infty(E)$*

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N} \langle w_\delta, \phi\rangle dx = 0,$$

i.e. w_δ converges weakly to 0 in L^p_{loc}.

Roughly speaking this lemma says that

$$[L, \rho_\delta *] \to 0 \text{ as } \delta \to 0.$$

We first show how one can use Lemma 9.3.7 to prove $u \in L^{m,p}_{loc}(E)$. Fix $0 < r < R$. Note that u_δ is a classical solution of

$$Lu_\delta = v_\delta + w_\delta.$$

Using the elliptic estimates of Corollary 9.3.3 we deduce

$$\|u_\delta\|_{m,p,B_{2r}} \le C(\|u_\delta\|_{p,B_{3r}} + \|v_\delta\|_{p,B_{3r}} + \|w_\delta\|_{p,B_{3r}}).$$

Since $u_\delta \to u$ and $v_\delta \to v$ in $L^p(E\,|_{B_{3r}})$ we deduce

$$\|u_\delta\|_{p,B_{3r}},\ \|v_\delta\|_{p,B_{3r}} \leq C.$$

On the other hand, since w_δ is weakly convergent in $L^p(E\,|_{B_{3r}})$ we deduce it must be bounded in this norm. Hence

$$\|u\|_{m,p,B_r} \leq C.$$

In particular, because $L^{m,p}(E\,|_{B_{2r}})$ is *reflexive* we deduce that a subsequence of (u_δ) converges weakly to some $\overline{u} \in L^{m,p}(E\,|_{B_{2r}})$. Moreover, using Rellich-Kondratchov compactness theorem we deduce that on a subsequence

$$u_\delta \to \overline{u} \text{ strongly in } L^p(E\,|_{B_r}).$$

Since $u_\delta \to u$ in L^p_{loc} (as mollifiers) we deduce $u\,|_{B_r} = \overline{u} \in L^{m,p}(E\,|_{B_r})$. This shows $u \in L^{m,p}_{loc}$ because r is arbitrary. Theorem 9.3.5 is proved.

□

Proof of Lemma 9.3.7 Pick $\phi \in C_0^\infty(E)$. Assume $\operatorname{supp}\phi \subset B = B_R$. We have to show

$$\lim_{\delta\to 0}\left(\int_B \langle Lu_\delta, \phi\rangle dx - \int_B \langle v_\delta, \phi\rangle dx\right) = 0.$$

We analyze each of the above terms separately. Assume the formal adjoint of L has the form

$$L^* = \sum_{|\beta|\leq m} B_\beta(x)\partial_x^\beta.$$

We have

$$\int_B \langle Lu_\delta, \phi\rangle dx = \int_B \langle u_\delta(x), L_x^*\phi\,(x)\rangle dx$$

$$= \int_B \left(\int_{\mathbb{R}^N} \langle \rho_\delta(x-y)u(y), L_x^*\phi\,(x)\rangle dy\right) dx$$

$$= \sum_\beta \int_B \left(\int_{\mathbb{R}^N} \langle \rho_\delta(x-y)u(y), B_\beta(x)\partial_x^\beta\phi\,(x)\rangle dy\right) dx$$

$$= \sum_\beta \int_B \left(\int_{\mathbb{R}^N} \langle u(y), \rho_\delta(x-y)B_\beta(x)\partial_x^\beta\phi\,(x)\rangle dy\right) dx.$$

Similarly

$$\int_B \langle v_\delta(x), \phi(x)\rangle dx = \int_B \int_{\mathbb{R}^N} \langle v(y), \rho_\delta(x-y)\phi(x)\rangle dydx$$

$$= \int_B \int_{\mathbb{R}^N} \langle u(y), L_y^*\rho_\delta(x-y)\phi(x)\rangle dydx$$

$$= \sum_\beta \int_B \int_{\mathbb{R}^N} \langle u(y), B_\beta(y)\partial_y^\beta(\rho_\delta(x-y)\phi(x))\rangle dydx$$

(switch the order of integration)

$$= \sum_\beta \int_{\mathbb{R}^N} \int_B \langle u(y), B_\beta(y) \partial_y^\beta (\rho_\delta(x-y)\phi(x)) \rangle dx dy$$

$(\partial_x \rho_\delta(x-y) = -\partial_y \rho_\delta(x-y))$

$$= \sum_\beta (-1)^\beta \left(\int_{\mathbb{R}^N} \int_B \langle u(y), B_\beta(y)(\partial_x^\beta \rho_\delta(x-y))\phi(x) \rangle dx dy \right)$$

(integrate by parts in the interior integral)

$$= \sum_\beta \int_{\mathbb{R}^N} \int_B \langle u(y), B_\beta(y) \rho_\delta(x-y) \partial_x^\beta \phi(x) \rangle dx dy$$

(switch back the order of integration)

$$= \sum_\beta \int_B \int_{\mathbb{R}^N} \langle u(y), B_\beta(y) \rho_\delta(x-y) \partial_x^\beta \phi(x) \rangle dy dx.$$

Hence

$$\int_B \langle L u_\delta, \phi \rangle dx - \int_B \langle v_\delta, \phi \rangle dx =$$

$$= \sum_\beta \int_B \int_{\mathbb{R}^N} \langle u(y), \rho_\delta(x-y) \left(B_\beta(x) - B_\beta(y) \right) \partial_x^\beta \phi(x) \rangle dy dx.$$

We will examine separately each term in the above sum.

$$\begin{aligned} &\int_B \int_{\mathbb{R}^N} \langle u(y), \rho_\delta(x-y) \left(B_\beta(x) - B_\beta(y) \right) \partial_x^\beta \phi(x) \rangle dy dx \\ &= \int_B \int_{\mathbb{R}^N} \rho_\delta(x-y) \langle u(y), (B_\beta(x) - B_\beta(y)) \partial_x^\beta \phi(x) \rangle dy dx \\ &= \int_B \langle \left(\int_{\mathbb{R}^N} \rho_\delta(x-y) u(y) dy \right), B_\beta(x) \partial_x^\beta \phi(x) \rangle dx \\ &\quad - \int_B \langle \left(\int_{\mathbb{R}^N} \rho_\delta(x-y) B_\beta^*(y) u(y) dy \right), \partial_x^\beta \phi(x) \rangle dx \\ &= \int_B \langle u_\delta(x), B_\beta(x) \partial_x^\beta \phi(x) \rangle dx - \int_B \langle (B_\beta^* u)_\delta(x), \partial_x^\beta \phi(x) \rangle dx \end{aligned}$$

where $(B_\beta^* u)_\delta$ denotes the mollification of $B_\beta^* u$. As $\delta \to 0$

$$u_\delta \to u \text{ and } (B_\beta^* u)_\delta \to B_\beta^* u \text{ in } L^p_{loc}.$$

Hence

$$\lim_{\delta \to 0} \int_B \int_{\mathbb{R}^N} \langle u(y), \rho_\delta(x-y) \left(B_\beta(x) - B_\beta(y) \right) \partial_x^\beta \phi(x) \rangle dy dx$$

$$= \int_B \langle u(x), B_\beta(x) \partial_x^\beta \phi(x) \rangle dx - \int_B \langle B_\beta^*(x) u(x), \partial_x^\beta \phi(x) \rangle dx = 0.$$

Lemma 9.3.7 is proved.

$\square$

Corollary 9.3.8. *If $u \in L^p_{loc}(E)$ is weak L^p-solution of*

$$Lu = v$$

($1 < p < \infty$) and $v \in L^{k,p}_{loc}(E)$ then $u \in L^{k+m,p}_{loc}(E)$ and for every $0 < r < R$

$$\|u\|_{m+k,p,B_r} \leq C(\|v\|_{k,p,B_R} + \|u\|_{p,B_R})$$

where as usual $C = C(\|L\|_{k+m+1,R}, R, r, \ldots)$.

Proof We already know that $u \in L^{k+m,p}_{loc}(E)$. Pick a sequence $u_n \in C_0^\infty(E)$ such that

$$u_n \to u \ \text{ strongly in } L^{k+m,p}_{loc}(E).$$

Then

$$Lu_n \to Lu \ \text{ strongly in } L^{k+m,p}_{loc}(E)$$

and

$$\|u_n\|_{m+k,p,B_r} \leq C(\|u_n\|_{0,p,B_R} + \|Lu_n\|_{k,p,B_R}).$$

The desired estimate is obtained by letting $n \to \infty$ in the above inequality.

□

Corollary 9.3.9. (Weyl Lemma) *If $u \in L^p_{loc}(E)$ is a weak L^p-solution of $Lu = v$ and v is smooth then u must be smooth.*

Proof Since v is smooth we deduce $v \in L^{k,p}_{loc}$ $\forall k \in \mathbb{Z}_+$. Hence $u \in L^{k+m,p}_{loc}$ $\forall k$. Using Morrey embedding theorem we deduce that $u \in C^{m,\alpha}_{loc}$ $\forall m \geq 0$.

□

The results in this and the previous section are local and so extend to the more general case of p.d.o. on manifolds. They take a particularly nice form for operators on compact manifolds.

Let (M, g) be a compact, oriented Riemann manifold and E, $F \to M$ two metric vector bundles with compatible connections. Denote by $L \in \mathbf{PDO}^m(E, F)$ an elliptic operator of order m.

Theorem 9.3.10. *(a) Let $u \in L^p(E)$ and $v \in L^{k,p}(F)$ ($1 < p < \infty$) such that*

$$\int_M \langle v, \phi\rangle_F dv_g = \int_M \langle u, L^*\phi\rangle_E dv_g \quad \forall \phi \in C_0^\infty(F).$$

then $u \in L^{k+m,p}(E)$ and

$$\|u\|_{k+mp,E} \leq C(\|v\|_{k,p,F} + \|u\|_{p,E})$$

where $C = C(L, k, p)$.
(b) If $u \in C^{m,\alpha}(E)$ and $v \in C^{k,\alpha}(F)$ ($0 < \alpha < 1$) are such that

$$Lu = v$$

then $u \in C^{m+k,\alpha}(E)$ and

$$\|u\|_{k+m,\alpha,E} \leq C(\|u\|_{0,\alpha,E} + \|v\|_{k,\alpha,F})$$

where $C = C(L, k, \alpha)$.

Remark 9.3.11. The regularity results and the a priori estimates we have established so far represent only the minimal information one needs to become an user of the elliptic theory. Regrettably we have mentioned nothing about two important topics: equations with non smooth coefficients and boundary value problems. For generalized Laplacians these topics are discussed in great detail in [28] and [61]. The boundary value problems for first order elliptic operators require a more delicate treatment. We refer to [13] for a very nice presentation of this subject. □

Exercise 9.3.2. (Kato's inequalities) Let (M, g) denote a compact oriented Riemann manifold without boundary. Consider a metric vector bundle $E \to M$ equipped with a compatible connection ∇.
(a) Show that for every $u \in L^{1,2}(E)$ the function $x \mapsto |u(x)|$ is in $L^{1,2}(M)$ and moreover

$$|d\,|u(x)|\,| \leq |\nabla u(x)|$$

for almost all $x \in M$.
(b) Set $\Delta_E = \nabla^*\nabla$ and denote by Δ_M the scalar Laplacian. Show that for all $u \in L^{2,2}(E)$ we have

$$\Delta_M(|u|^2) = 2\langle \Delta_E u, u\rangle_E - 2|\nabla u|^2.$$

Conclude that $\forall \phi \in C^\infty(M)$ such that $\phi \geq 0$ we have

$$\int_M (d|u|, d(\phi|u|)\,)_g \, d\,v_g \leq \int_M \langle \Delta_E u, u\rangle_E \phi \, d\,v_g,$$

i.e.

$$|u(x)|\Delta_M(|u(x)|) \leq \langle \Delta_E u(x), u(x)\rangle_E \ \text{ weakly.}$$

□

9.3.3 An application: prescribing the curvature of surfaces

In this subsection we will illustrate the power of the results we proved so far by showing how they can be successfully used to prove an important part of the celebrated uniformization theorem. In the process we will have the occasion to introduce the reader to some tricks frequently used in the study of nonlinear elliptic equations. We will consider a slightly more general situation than the one required by the uniformization theorem.

Let (M, g) be a compact, connected, oriented Riemann manifold of dimension N. Denote by $\Delta = d^*d : C^\infty(M) \to C^\infty(M)$ the scalar Laplacian. We assume for simplicity that

$$\mathrm{vol}_g(M) = \int_M dv_g = 1$$

so that the average of any integrable function φ is defined by

$$\overline{\varphi} = \int_M \varphi(x) dv_g(x).$$

We will study the following partial differential equation.

$$\Delta u + f(u) = s(x) \tag{9.3.7}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $s \in C^\infty(M)$ satisfy the following conditions.
(C_1) f is smooth and strictly increasing.
(C_2) There exist $a > 0$ and $b \in \mathbb{R}$ such that

$$f(t) \geq at + b \quad \forall t \in \mathbb{R}.$$

Set

$$F(u) = \int_0^u f(t)dt.$$

We assume
(C_3) $\lim_{|t|\to\infty}(F(t) - \overline{s}t) = \infty$ where $\overline{s}$ denotes the average of s

$$\overline{s} = \int_M s(x)dv_g.$$

Theorem 9.3.12. *Let f and $s(x)$ satisfy the conditions $(C_1 - C_3)$ Then there exists a unique $u \in C^\infty(M)$ such that*

$$\Delta u\,(x) + f(u(x)) = s(x) \;\; \forall x \in M.$$

The proof of this theorem will be carried out in two steps.

Step 1: Existence of a weak solution A weak solution of (9.3.7) is a function $u \in L^{1,2}(M)$ such that $f(u(x)) \in L^2(M)$ and

$$\int_M \{(du, d\phi) + f(u)\phi\}dv_g = \int_M s(x)\phi(x)dv_g(x) \;\; \forall \phi \in L^{1,2}(M).$$

Step 2: Regularity We show that a weak solution is in fact a classical solution.

Proof of Step 1 We will use the direct method of the calculus of variations (outlined at the beginning of the current section). Consider the energy functional

$$I : L^{1,2}(M) \to \mathbb{R}$$

$$I(u) = \int_M \{\frac{1}{2}|du|^2 + F(u) - g(x)u(x)\}dv_g(x).$$

This functional is not quite well defined since there is no guarantee that $F(u) \in L^1(M)$ for all $u \in L^{1,2}(M)$.

Leaving this issue aside for a moment we can perform a formal computation *à la* Riemann. Assume u is a minimizer of I, i.e.

$$I(u) \leq I(v) \quad \forall v \in L^{1,2}(M).$$

Thus for all $\phi \in L^{1,2}(M)$

$$I(u) \leq I(u + t\phi) \quad \forall t \in \mathbb{R}.$$

Hence $t = 0$ is a minimum of $h_\phi(t) = I(u+t\phi)$ so that $h'_\phi(0) = 0 \; \forall \phi$. A simple computation shows that

$$h'_\phi(0) = \int_M \{(du, d\phi) + f(u)\phi - s(x)\phi(x)\}dv_g = 0$$

so that a minimizer of I is a weak solution provided we deal with the integrability issue raised at the beginning of this discussion. Any way, the lesson we learn from this formal computation is that minimizers of I are strong candidates for solutions of (9.3.7).

We will circumvent the trouble with the possible non-integrability of $F(u)$ by using a famous trick in elliptic partial differential equations called *the maximum principle*

Lemma 9.3.13. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous,* strictly increasing *function and $u, v \in L^{1,2}(M)$ such that*
(i) $h(u)$, $h(v) \in L^2(M)$.
(ii) $\Delta u + h(u) \geq \Delta v + h(v)$ weakly i.e.

$$\int_M \{(du, d\phi) + h(u)\phi\} dv_g \geq \int_M \{dv, d\phi) + h(v)\phi\} dv_g \tag{9.3.8}$$

$\forall \phi \in L^{1,2}(M)$ such that $\phi \geq 0$ a.e. M. Then $u \geq v$ a.e. on M.

Proof of the lemma Let

$$(u-v)^- = \min\{u - v, 0) = \frac{1}{2}\{(u-v) - |u-v|\}.$$

According to the Exercise 9.2.9 we have $(u-v)^- \in L^{1,2}(M)$ and

$$d(u-v)^- = \begin{cases} d(u-v) & \text{a.e. on } \{u < v\} \\ 0 & \text{a.e. on } \{u \geq v\} \end{cases}$$

Using $\phi = -(u-v)$ in (9.3.8) we deduce

$$-\int_M \{(d(u-v), d(u-v)^-) + (h(u) - h(v))(u-v)^-\} dv_g \geq 0.$$

Clearly

$$(d(u-v), d(u-v)^-) = |d(u-v)^-|^2$$

and since h is nondecreasing

$$(h(u) - h(v))(u-v)^- \geq 0.$$

Hence

$$\int_M |d(u-v)^-|^2 dv_g \leq -\int_M (h(u) - h(v))(u-v)^- dv_g \leq 0$$

so that

$$|d(u-v)^-| \equiv 0.$$

Since M is connected this means $(u-v)^- \equiv c \leq 0$. If $c < 0$ then

$$(u-v) \equiv (u-v)^- \equiv c$$

so that $u = v + c < v$. Since h is strictly increasing we conclude

$$\int_M h(u) dv_g < \int_M h(v) dv_g.$$

On the other hand, using $\phi \equiv 1$ in 9.3.8 we deduce

$$\int_M h(u)dv_g \geq \int_M h(v)dv_g!$$

Hence c cannot be negative so that $(u-v)^- \equiv 0$ which is another way of saying $u \geq v$. The maximum principle is proved.

□

Exercise 9.3.3. Assume h in the above lemma is only non-decreasing but u and v satisfy the supplementary condition

$$\overline{u} \geq \overline{v}.$$

Show the conclusion of Lemma 9.3.13 continues to hold. □

We now return to the equation (9.3.7). Note first that if u and v are two weak solutions of this equation then

$$\Delta u + f(u) \geq \ (\leq)\, \Delta v + h(v) \ \text{ weakly}$$

so that by the maximum principle $u \geq \ (\leq)\, v$. This shows the equation (9.3.7) has *at most one weak solution.*

To proceed further we need the following a priori estimate.

Lemma 9.3.14. *Let u be a weak solution of (9.3.7). If C is a positive constant such that*

$$f(C) \geq \sup_M s(x)$$

then $u(x) \leq C$ a.e. on M.

Proof The equality $f(C) \geq \sup s(x)$ implies

$$\Delta C + f(C) \geq s(x) = \Delta u + f(u) \ \text{ weakly}$$

so the conclusion follows from the maximum principle.

□

Fix $C_0 > 0$ such that $f(C_0) \geq \sup s(x)$. Consider a strictly increasing C^2-function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(u) = f(u) \quad \text{for } u \leq C_0$$

$$\tilde{f}(u) \ \text{ is linear for } u \geq C_0 + 1.$$

The condition (C_2) implies there exist A, $B > 0$ such that

$$|\tilde{f}(u)| \leq A|u| + B. \tag{9.3.9}$$

Lemma 9.3.15. *If u is a weak solution of*

$$\Delta u + \tilde{f}(u) = s(x) \tag{9.3.10}$$

then u is also a weak solution of (9.3.7).

Proof We deduce as in the proof of Lemma 9.3.14 that $u \leq C_0$ which is precisely the range where f coincides with $\tilde{f}$.

□

The above lemma shows that instead of looking for a weak solution of (9.3.7) we should try to find a weak solution of (9.3.10). We will use the direct method of the calculus of variations on a new functional

$$\tilde{I}(u) = \int_M \{\frac{1}{2}|du(x)|^2 + \tilde{F}(u(x)) - s(x)u(x)\}dv_g(x)$$

where

$$\tilde{F}(u) = \int_0^u \tilde{f}(t)dt.$$

The advantage we gain by using this new functional is clear. The inequality (9.3.9) shows $\tilde{F}$ has at most quadratic growth so that $\tilde{F}(u) \in L^1(M)$ for all $u \in L^2(M)$. The existence of a minimizer is a consequence of the following fundamental principle of the calculus of variations.

Proposition 9.3.16. *Let X be a* reflexive *Banach space and $J : X \to \mathbb{R}$ a convex, weakly lower semi-continuous, coercive functional i.e. the level sets*

$$J^c = \{x\ ;\ J(x) \leq c\}$$

are respectively convex, weakly closed and bounded in X. Then J admits a minimizer i.e. there exists $x_0 \in X$ such that

$$J(x_0) \leq J(x) \quad \forall x \in X.$$

Proof Note that

$$\inf_X J(x) = \inf_{J^c} J(x)$$

Consider $x_n \in J^c$ such that

$$\lim_n J(x_n) = \inf J.$$

Since X is reflexive and J^c is convex, weakly closed and bounded in the norm of X we deduce that J^c is weakly compact. Hence a (generalized) subsequence (x_ν) of x_n converges weakly to some $x_0 \in J^c$. Using the lower semi-continuity of J we deduce

$$J(x_0) \leq \liminf_\nu J(x_\nu) = \inf J.$$

Hence x_0 is a minimizer of J. The proposition is proved.

□

The next result will conclude the proof of Step 1.

Lemma 9.3.17. *$\tilde{I}$ is convex, weakly lower semi-continuous and coercive (with respect to the $L^{1,2}$-norm).*

Proof The convexity is clear since $\tilde{F}$ is convex on account that $\tilde{f}$ is strictly increasing. The terms

$$u \mapsto \frac{1}{2}\int_M |du|^2 dv_g \quad u \mapsto -\int_M s(x)u(x)dv_g(x)$$

are clearly weakly lower semi-continuous. We need to show the functional

$$u \mapsto \int_M \tilde{F}(u)dv_g$$

is also weakly lower semi-continuous.

Since $\tilde{F}$ is convex we can find $\alpha > 0, \beta \in \mathbb{R}$ such that

$$\tilde{F}(u) - \alpha u - \beta \geq 0.$$

If $u_n \to u$ strongly in $L^{1,2}(M)$ we deduce using the Fatou lemma that

$$\int_M \{\tilde{F}(u) - \alpha u - \beta\}dv_g \leq \liminf_{n\to\infty} \int_M \{\tilde{F}(u_n) - \alpha u_n - \beta\}dv_g.$$

On the other hand

$$\lim_n \int_M \alpha u_n + \beta dv_g = \int_M \alpha u + \beta dv_g$$

which shows that

$$\int_M \tilde{F}(u)dv_g \leq \liminf_n \int \tilde{F}(u_n)dv_g.$$

This means the functional

$$L^{1,2}(M) \ni u \mapsto \int_M \tilde{F}(u)dv_g$$

is strongly lower semi-continuous. Thus the level sets

$$\{u \; ; \; \int_M \tilde{F}(u)dv_g \leq c\}$$

are both convex and *strongly closed.* Hahn-Banach separation principle can now be invoked to conclude these level sets are also *weakly* closed. We have thus established that $\tilde{I}$ is convex and weakly lower semi-continuous.

Remark 9.3.18. We see that the lower semi-continuity and the convexity conditions are very closely related. In some sense they are almost equivalent. We refer to [20] for a presentation of the direct method of the calculus of variations were the lower semi-continuity issue is studied in great detail. □

The coercivity will require a little more work. The key ingredient will be a *Poincaré inequality.* We first need to introduce some more terminology.

For any $u \in L^2(M)$ we denoted by $\overline{u}$ its average. Now set

$$u^\perp(x) = u(x) - \overline{u}.$$

Note the average of $u^\perp$ is 0. This choice of notation is motivated by the fact that $u^\perp$ is perpendicular (with respect to the $L^2(M)$-inner product) to the kernel of Δ which is the 1-dimensional space spanned by the constant functions.

Lemma 9.3.19. (Poincaré inequality) *There exists $C > 0$ such that*

$$\int_M |du|^2 dv_g \geq C \int_M |u^\perp|^2 dv_g \ \ \forall u \in L^{1,2}(M).$$

Proof We argue by contradiction. Assume that for any $\varepsilon > 0$ there exists $u_\varepsilon \in L^{1,2}(M)$ such that

$$\int_M |u_\varepsilon^\perp|^2 dv_g = 1$$

and

$$\int_M |du_\varepsilon|^2 dv_g \leq \varepsilon.$$

The above two conditions imply the family $(u_\varepsilon^\perp)$ is bounded in $L^{1,2}(M)$. Since $L^{1,2}(M)$ is reflexive we deduce that on a subsequence

$$u_\varepsilon^\perp \rightharpoonup v \ \text{ weakly in } L^{1,2}(M).$$

The inclusion $L^{1,2}(M) \hookrightarrow L^2(M)$ is compact (Rellich-Kondratchov) so that on a subsequence

$$u_\varepsilon^\perp \to v \ \text{ strongly in } L^2(M).$$

This implies $v \neq 0$ since

$$\int_M |v|^2 dv_g = \lim_\varepsilon \int_M |u_\varepsilon^\perp|^2 dv_g = 1.$$

On the other hand

$$du_\varepsilon = du_\varepsilon^\perp \to 0 \ \text{ strongly in } L^2.$$

We conclude

$$\int_M (dv, d\phi) dv_g = \lim_\varepsilon \int_M (du_\varepsilon, d\phi) dv_g = 0 \ \ \forall \phi \in L^{1,2}(M).$$

In particular

$$\int_M (dv, dv) dv_g = 0$$

so that $dv \equiv 0$. Since M is connected we deduce $v \equiv c = const.$ and moreover

$$c = \int_M v(x) dv_g(s) = \lim_\varepsilon \int_M u_\varepsilon^\perp dv_g = 0.$$

This contradicts the fact that $v \neq 0$. The Poincaré inequality is proved.

□

We can now establish the coercivity of $\tilde{I}(u)$. Let $\kappa > 0$ and $u \in L^{1,2}(M)$ such that

$$\int_M \frac{1}{2} |du|^2 + \tilde{F}(u) - s(x) u dv_g \leq \kappa. \tag{9.3.11}$$

Since

$$\int_M |du|^2 dv_g = \int_M |du^\perp|^2 dv_g$$

and
$$\int_M |u|^2 = |\overline{u}|^2 + \int_M |u^\perp|^2 dv_g$$
it suffices to show the quantities
$$\overline{u},\ \int_M |u^\perp|^2 dv_g,\ \int_M |du^\perp| dv_g$$
are bounded. In view of the Poincaré inequality the boundedness of
$$\int_M |du^\perp|^2 dv_g$$
implies the boundedness of $\|u^\perp\|_{2,M}$ so that we should concentrate only on $\overline{u}$ and $\|du^\perp\|_2$.

The inequality (9.3.11) can be rewritten as
$$\int_M \{\frac{1}{2}|du^\perp|^2 + \tilde{F}(u) - s^\perp u^\perp\} dv_g - \overline{s}\cdot\overline{u} \leq c.$$
The Poincaré and Cauchy inequalities imply
$$C\|u^\perp\|_2^2 - \|s^\perp\|_2 \cdot \|u^\perp\|_2 + \int_M \tilde{F}(u) dv_g - \overline{s}\cdot\overline{u} \leq \kappa.$$
Since $\mathrm{vol}_g(M) = 1$ and $\tilde{F}$ is convex we have a Jensen inequality
$$\tilde{F}\left(\int_M u dv_g\right) \leq \int_M \tilde{F}(u) dv_g$$
so that
$$C\|u^\perp\|_2^2 - \|s^\perp\|_2 \cdot \|u^\perp\|_2 + \tilde{F}(\overline{u}) - \overline{s}\cdot\overline{u} \leq \kappa. \tag{9.3.12}$$
Set $P(t) = Ct^2 - \|s^\perp\|_2 t$ and let $m = \inf P(t)$. From the inequality (9.3.12) we deduce
$$\tilde{F}(\overline{u}) - \overline{s}\cdot\overline{u} \leq \kappa - m.$$
Using condition (C_3) we deduce that $|\overline{u}|$ must be bounded. Feed this information back in (9.3.12). We conclude that $P(\|u^\perp\|_2)$ must be bounded. This forces $\|u^\perp\|_2$ to be bounded. Thus $\tilde{I}$ is coercive and Lemma 9.3.17 is proved.

□

Step 2: The regularity of the minimizer. We will use a technique called *bootstrapping* which blends the elliptic regularity theory and the Sobolev embedding theorems to gradually improve the regularity of the weak solution.

Let u be the weak solution of (9.3.10). Then $u(x)$ is a weak L^2 solution of
$$\Delta u = h(x) \text{ on } M$$
where $h(x) = -\tilde{f}(u(x)) - s(x)$. Note that since the growth of $\tilde{f}$ is at most linear $\tilde{f}(u(x)) \in L^2(M)$. The elliptic regularity theory implies that $u \in L^{2,2}(M)$. Using Sobolev (or Morrey) embedding theorem we can considerably improve the integrability of u. We deduce that

(i) either $u \in L^q(M)$ if $-N/q \leq 2 - N/2 \leq 0$ ($\dim M = N$)
(ii) or u is Hölder continuous if $2 - N/2 > 0$

In any case this shows $u(x) \in L^{q_1}(M)$ for some $q_1 > 2$ which implies $h(x) \in L^{q_1}(M)$. Using again elliptic regularity we deduce $u \in L^{2,q_1}$ and Sobolev inequality implies that $u \in L^{q_2}(M)$ for some $q_2 > q_1$. After a finite number of steps we conclude that $h(x) \in L^q(M)$ for all $q > 1$. Elliptic regularity implies $u \in L^{2,q}(M)$ for all $q > 1$. This implies $h(x) \in L^{2,q}(M)$ for all $q > 1$. Invoking elliptic regularity again we deduce that $u \in L^{4,q}(M)$ for any $q > 1$. (At this point it is convenient to work with f rather than with $\tilde{f}$ which was only C^2). Feed this back in $h(x)$ and regularity theory improves the regularity of u two orders at a time. In view of Morrey embedding theorem the conclusion is clear: $u \in C^\infty(M)$. The proof of Theorem 9.3.12 is complete.

□

From the theorem we have just proved we deduce immediately the following consequence.

Corollary 9.3.20. *Let (M, g) be a compact, connected, oriented Riemann manifold and $s(x) \in C^\infty(M)$. Assume* $\mathrm{vol}_g(M) = 1$. *Then the following two conditions are equivalent.*
(a) $\overline{s} = \int_M s(x) dv_g > 0$

(b) For every $\lambda > 0$ there exists a unique $u = u_\lambda \in C^\infty(M)$ such that

$$\Delta u + \lambda e^u = s(x). \tag{9.3.13}$$

Proof (a) $\Rightarrow$ (b) follows from Theorem 9.3.12.
(b) $\Rightarrow$ (a) follows by multiplying (9.3.13) with $v(x) \equiv 1$ and then integrating by parts so that

$$\bar{s} = \lambda \int_M e^{u(x)} dv_g(x) > 0.$$

□

Although the above corollary may look like a purely academic result it has a very nice geometrical application. We will use it to prove a special case of the celebrated *uniformization theorem.*

Definition 9.3.21. *Let M be a smooth manifold. Two Riemann metrics g_1 and g_2 are said to be* conformal *if there exists $f \in C^\infty(M)$ such that $g_2 = e^f g_1$.*

Exercise 9.3.4. Let (M, g) be an oriented Riemann manifold of dimension N and $f \in C^\infty(M)$. Denote by $\tilde{g}$ the conformal metric $\tilde{g} = e^f g$. If $s(x)$ is the scalar curvature of g and $\tilde{s}$ is the scalar curvature of $\tilde{g}$ show that

$$\tilde{s}(x) = e^{-f}\{s(x) + (N-1)\Delta_g f - \frac{(N-1)(N-2)}{4}|df(x)|_g^2\}$$

where Δ_g denotes the scalar Laplacian of the metric g while $|\cdot|_g$ denotes the length measured in the metric g. □

Remark 9.3.22. A long standing problem in differential geometry which was only relatively recently solved is the *Yamabe problem*:

"If (M, g) is a compact oriented Riemann manifold does there exist a metric conformal to g whose scalar curvature is constant?"

In dimension 2 this problem is related to the uniformization problem of complex analysis.

The Yamabe problem was solved in its complete generality due to the combined efforts of T. Aubin, [7, 8] and R. Schoen [66]. For a very beautiful account of its proof we recommend the excellent survey of J. Lee and T.Parker, [47].

One can formulate a more general question than the Yamabe problem. Given a compact oriented Riemann manifold (M, g) decide whether a smooth function $s(x)$ on M is the scalar curvature of some metric on M conformal to g. This problem is known as the Kazdan-Warner problem. The case $\dim M = 2$ is completely solved in [42]. The higher dimensional situation $\dim M > 2$ is far more complicated both topologically and analytically. □

Theorem 9.3.23. (Uniformization Theorem) *Let (Σ, g) be a compact, oriented Riemann manifold of dimension 2. Assume $\mathrm{vol}_g(\Sigma) = 1$. If $\chi(\Sigma) < 0$ (or equivalently if its genus is ≥ 2) then there exists a unique metric $\tilde{g}$ conformal to g such that*

$$s(\tilde{g}) \equiv -1.$$

Proof We look for $\tilde{g}$ of the form $\tilde{g} = e^u g$. Using Exercise 9.3.4 we deduce that u should satisfy

$$-1 = e^{-u}\{s(x) + \Delta u\},$$

i.e.

$$\Delta u + e^u = -s(x)$$

where $s(x)$ is the scalar curvature of the metric g. The Gauss-Bonnet theorem implies that

$$\overline{s} = 4\pi\chi(\Sigma) < 0$$

so that the existence of u is guaranteed by Corollary 9.3.20. The uniformization theorem is proved.

□

On a manifold of dimension 2 the scalar curvature coincides up to a positive factor with the sectional curvature. The uniformization theorem implies that the compact oriented surfaces of negative Euler characteristic admit metrics of constant negative sectional curvature. Now, using the Cartan-Hadamard theorem we deduce the following topological consequence.

Corollary 9.3.24. *The universal cover of a compact, oriented surface of negative Euler characteristic is diffeomorphic to $\mathbb{R}^2$.*

In the following exercises (M, g) denotes a compact, oriented Riemann manifold without boundary.

Exercise 9.3.5. Fix $c > 0$. Show that for every $f \in L^2(M)$ the equation

$$\Delta u + cu = f$$

has an unique solution $u \in L^{2,2}(M)$. $\square$

Exercise 9.3.6. Consider a smooth function $f : M \times \mathbb{R} \to \mathbb{R}$ such that for every $x \in M$ the function $u \mapsto f(x, u)$ is increasing. Assume the equation

$$\Delta_g u = f(x, u) \tag{9.3.14}$$

admits a pair of *comparable* sub/super-solutions i.e. there exist $u_0, U_0 \in L^{1,2}(M) \cap L^\infty(M)$ such that

$$U_0(x) \geq u_0(x) \ \text{ a.e. on } M$$

and

$$\Delta_g U_0 \geq f(x, U_0(x)) \geq f(x, u_0(x)) \geq \Delta_g u_0 \quad \text{weakly in } L^{1,2}(M).$$

Fix $c > 0$ and define $(u_n)_{n \geq 1} \subset L^{2,2}(M)$ inductively as the unique solution of the equation

$$\Delta u_n(x) + cu_n(x) = cu_{n-1}(x) + f(x, u_{n-1}(x))$$

(a) Show that

$$u_0(x) \leq u_1(x) \leq u_2(x) \leq \cdots \leq u_n(x) \leq \cdots \leq U_0(x) \quad \forall x \in M.$$

(b) Show that u_n converges uniformly on M to a solution $u \in C^\infty(M)$ of (9.3.14) satisfying

$$u_0 \leq u \leq U_0.$$

(c) Prove that the above conclusions continue to hold even if the monotonicity assumption on f is dropped. $\square$

9.4 Elliptic operators on compact manifolds

The elliptic operators on compact manifolds behave in many respects as finite dimensional operators. It is the goal of this last section to present the reader some fundamental analytic facts which will transform the manipulation with such p.d.o. into a less painful task. What makes these operators so "friendly" is the existence of a priori estimates. These coupled with the Rellich-Kondratchov compactness theorem are the keys which will open many doors.

9.4.1 The Fredholm theory

Throughout this section we assume the reader is familiar with some fundamental facts about unbounded linear operators. We refer to [15] Ch.II for a very concise presentation of these notions. An exhaustive presentation of this subject can be found in [41]. For the reader's convenience we describe the fundamental notions related to unbounded operators.

Definition 9.4.1. *(a) Let X, Y be two Hilbert spaces over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and*

$$T : D(T) \subset X \to Y$$

*a linear operator (*not necessarily continuous*) defined on the linear subspace $D(T) \subset X$. T is said to be densely defined if $D(T)$ is dense in X. T is said to be* closed *if its graph*

$$\Gamma_T = \{(x, Tx) \in D(T) \times Y \subset X \times Y\}$$

is a closed subspace in $X \times Y$.
(b) Let $T : D(T) \subset X \to Y$ be a closed, densely defined linear operator. The adjoint *of T is the operator $T^* : D(T^*) \subset Y \to X$ defined by its graph*

$$\Gamma_{T^*} = \{(y^*, x^*) \in Y \times X \; ; \; \langle x^*, x\rangle = \langle y^*, Tx\rangle \;\; \forall x \in D(T).$$

where $\langle \cdot, \cdot \rangle : Z \times Z \to \mathbb{K}$ denotes the inner product on a generic Hilbert space Z.
(c) A closed, densely defined operator $T : D(T) \subset X \to X$ is said to be selfadjoint *if $T = T^*$.*

Remark 9.4.2. (a) In more concrete terms $T : D(T) \subset X \to Y$ is closed if for any sequence $(x_n) \subset D(T)$ such that $(x_n, Tx_n) \to (x, y)$ it follows that (i) $x \in D(T)$ and (ii) $y = Tx$.
(b) If $T : X \to Y$ is a closed operator then T is bounded (closed graph theorem). Also note that if $T : D(T) \subset X \to Y$ is a closed, densely defined operator then $\ker T$ is a *closed* subspace of X
(c) One can show that the adjoint of any closed, densely defined operator is a closed, *densely defined* operator.
(d) The closed, densely defined operator $T : D(T) \subset X \to X$ is selfadjoint if the following two conditions hold.
(i) $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in D(T)$ and
(ii)

$$D(T) = \{y \in X \; ; \; \exists C > 0 : \; |\langle Tx, y\rangle| \leq C|x| \quad \forall x \in D(T)\}.$$

If only the condition (i) is satisfied the operator T is called *symmetric.* □

Let (M, g) be a compact, oriented Riemann manifold, E, F two metric vector bundles with compatible connections and $L \in \mathbf{PDO}^k = \mathbf{PDO}^k(E, F)$ a k-th order elliptic p.d.o. We will denote the various L^2 norms by $\| \cdot \|$ and the $L^{k,2}$-norms by $\| \cdot \|_k$.

Definition 9.4.3. *The* analytical realization *of L is the linear operator*

$$L_a : D(L_a) \subset L^2(E) \to L^2(E)$$

which acts by $u \mapsto Lu$ for all $u \in D(L_a) = L^{k,2}(E)$.

Proposition 9.4.4. *(a) The analytical realization L_a of L is a closed, densely defined linear operator.*
(b) If $L^* : C^\infty(F) \to C^\infty(E)$ is the formal adjoint of L then

$$(L^*)_a = (L_a)^*.$$

Proof (a) Since $C^\infty(E) \subset D(L_a) = L^{k,2}(E)$ is dense in $L^2(E)$ we deduce that L_a is densely defined. To prove L_a is also closed consider a sequence $(u_n) \subset L^{k,2}(E)$ such that

$$u_n \to u \text{ strongly in } L^2(E) \text{ and } Lu_n \to v \text{ strongly in } L^2(F).$$

From the elliptic estimates we deduce

$$\|u_n - u_m\|_k \leq C(\|Lu_n - Lu_m\| + \|u_n - u_m\|) \to 0 \ \text{ as } m, n \to \infty.$$

Hence (u_n) is a Cauchy sequence in $L^{k,2}(E)$ so that $u_n \to u$ in $L^{k,2}(E)$. Now it is clear that $v = Lu$.
(b) From the equality

$$\int_M \langle Lu, v\rangle dv_g = \int_M \langle u, L^* v\rangle dv_g \ \ \forall u \in L^{k,2}(E),\ v \in L^{k,2}(F)$$

we deduce

$$D((L_a)^*) \supset D((L^*)_a) = L^{k,2}(F)$$

and $(L_a)^* = (L^*)_a$ on $L^{k,2}(F)$. To prove that

$$D((L_a)^*) \subset D((L^*)_a) = L^{k,2}(F)$$

we need to show that if $v \in L^2(F)$ is such that $\exists C > 0$

$$\left|\int_M \langle Lu, v\rangle dv_g\right| \leq C\|u\| \ \ \forall u \in L^{k,2}(E)$$

then $v \in L^{k,2}(F)$. Indeed, the above inequality shows that the functional

$$u \mapsto \int_M \langle Lu, v\rangle dv_g$$

extends to a continuous linear functional on $L^2(E)$. Hence there exists $\phi \in L^2(E)$ such that

$$\int_M \langle (L^*)^* u, v\rangle dv_g = \int_M \langle u, \phi\rangle dv_g \ \ \forall u \in L^{k,2}(E).$$

In other words, v is a weak L^2-solution of the elliptic equation $L^* v = \phi$. Using elliptic regularity theory we deduce $v \in L^{k,2}(M)$. The proposition is proved.

□

Following the above result we will not make any notational distinction between an elliptic operator (on a compact manifold) and its analytical realization.

Definition 9.4.5. *(a) Let X and Y be two Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $T : D(T) \subset X \to Y$ a closed, densely defined linear operator. T is said to be* semi-Fredholm *if*
(i) $\dim\ker T < \infty$ *and*
(ii) The range $\mathrm{R}\,(T)$ *of T is closed.*
(b) The operator T is called Fredholm *if both T and T^* are semi-Fredholm. In this case the integer*

$$\operatorname{ind} T \stackrel{def}{=} \dim_{\mathbb{K}} \ker T - \dim_{\mathbb{K}} \ker T^*$$

is called the Fredholm index *of T.*

Remark 9.4.6. The above terminology has its origin in the work of Ivar Fredholm at the beginning of this century. His result (later considerably generalized by F. Riesz) states that if $K : H \to H$ is a compact operator from a Hilbert space to itself then $\mathbf{1}_H + K$ is a Fredholm operator of index 0. □

Consider again the elliptic operator L of Proposition 9.4.4.

Theorem 9.4.7. *The operator $L_a : D(L_a) \subset L^2(E) \to L^2(F)$ is Fredholm.*

Proof The Fredholm property is a consequence of the following compactness result.

Lemma 9.4.8. *Any sequence $(u_n) \subset L^{k,2}(E)$ such that $\{\|u_n\| + \|Lu_n\|\}$ is bounded contains a subsequence strongly convergent in $L^2(E)$.*

Proof of the lemma Using elliptic estimates we deduce that

$$\|u_n\|_k \leq C(\|Lu_n\| + \|u_n\|) \leq const.$$

Hence (u_n) is also bounded in $L^{k,2}(E)$. On the other hand, since M is compact $L^{k,2}(E)$ embeds compactly in $L^2(E)$. The lemma is proved.

□

We will first show that $\dim \ker L < \infty$. In the proof we will rely on the classical result of F. Riesz which states that a Banach space is finite dimensional if and only if its bounded subsets are precompact (see [15], Chap. VI).

Note first that according to Weyl's lemma $\ker L \subset C^\infty(E)$. Next, notice that $\ker L$ is a Banach space with respect to the L^2-norm since according to Remark 9.4.2 (a) $\ker L$ is closed in $L^2(E)$. We will show that any sequence $(u_n) \subset \ker L$ which is also bounded in the L^2-norm contains a subsequence convergent in L^2. This follows immediately from Lemma 9.4.8 since $\|u_n\| + \|Lu_n\| = \|u_n\|$ is bounded.

To prove that the range $\mathrm{R}\,(T)$ is closed we will rely on the following very useful inequality, a special case of which we have seen at work in Subsection 9.3.3.

Lemma 9.4.9. (Poincaré inequality) *There exists $C > 0$ such that*

$$\|u\| \leq C\|Lu\|$$

for all $u \in L^{k,2}(E)$ which are L^2-orthogonal to $\ker L$ i.e.

$$\int_M \langle u, \phi\rangle dv_g = 0 \;\; \forall \phi \in \ker L.$$

Proof We will argue by contradiction. Denote by $X \subset L^{k,2}(E)$ the subspace consisting of sections L^2-orthogonal to $\ker L$. Assume that for any $n > 0$ there exists $u_n \in X$ such that

$$\|u_n\| = 1 \;\text{ and }\; \|Lu_n\| \leq 1/n.$$

Thus $\|Lu_n\| \to 0$ and in particular $\|u_n\| + \|Lu_n\|$ is bounded. Using Lemma 9.4.8 we deduce that a subsequence of (u_n) is convergent in $L^2(E)$ to some u. Note that $\|u\| = 1$. It is not difficult to see that in fact $u \in X$. We get a sequence

$$(u_n, Lu_n) \subset \Gamma_L = \text{ the graph of } L$$

such that

$$(u_n, Lu_n) \to (u, 0).$$

Since L is closed we deduce $u \in D(L)$ and $Lu = 0$. Hence $u \in \ker L \cap X = \{0\}$. This contradicts the condition $\|u\| = 1$.

□

We can now conclude the proof of Theorem 9.4.7. Consider a sequence $(v_n) \subset \mathrm{R}\,(L)$ such that $v_n \to v$ in $L^2(F)$. We want to show $v \in \mathrm{R}\,(L)$.

For each v_n we can find an unique $u_n \in X = (\ker L)^{\perp}$ such that

$$Lu_n = v_n.$$

Using the Poincaré inequality we deduce

$$\|u_n - u_m\| \leq C\|v_n - v_m\|.$$

When we couple this inequality with the elliptic estimates we get

$$\|u_n - u_m\|_k \leq C(\|Lu_n - Lu_m\| + \|u_n - u_m\|) \leq C\|v_n - v_m\| \to 0 \text{ as } m, n \to \infty.$$

Hence (u_n) is a Cauchy sequence in $L^{k,2}(E)$ so that $u_n \to u$ in $L^{k,2}(E)$. Clearly $Lu = v$ so that $v \in \mathrm{R}\,(L)$.

We have so far proved that $\ker L$ is finite dimensional and $\mathrm{R}\,(L)$ is closed i.e. L_a is semi-Fredholm. Since $(L_a)^* = (L^*)_a$ and L^* is also an elliptic operator we deduce $(L_a)^*$ is also semi-Fredholm. This completes the proof of Theorem 9.4.7.

□

Using the closed range theorem of functional analysis we deduce the following important consequence.

Corollary 9.4.10. (Abstract Hodge decomposition) *Any k-th order elliptic operator $L : C^\infty(E) \to C^\infty(F)$ over the compact manifold M defines natural orthogonal decompositions of $L^2(E)$ and $L^2(F)$. More precisely we have*

$$L^2(E) = \ker L \oplus \mathrm{R}\,(L^*) \text{ and } L^2(F) = \ker L^* \oplus \mathrm{R}\,(L).$$

Corollary 9.4.11. *If* $\ker L^* = 0$ *then for every* $v \in L^2(F)$ *the partial differential equation*

$$Lu = v$$

admits at least one weak L^2-solution $u \in L^2(E)$.

The last corollary is really unusual. It states the equation $Lu = v$ has a solution provided the dual equation $L^*v = 0$ has no nontrivial solution. A nonexistence hypothesis implies an existence result! This partially explains the importance of the vanishing results in geometry, i.e. the results to the effect that $\ker L^* = 0$. With an existence result in our hands presumably we are more capable of producing geometric objects. In the next chapter we will describe one powerful technique of producing vanishing theorems based on the so called *Weitzenböck identities*.

Corollary 9.4.12. *Over a* compact *manifold*

$$\ker L = \ker L^*L \quad \ker L^* = \ker LL^*.$$

Proof Clearly $\ker L \subset \ker L^*L$. Conversely, let $\psi \in C^\infty(E)$ such that $L^*L\psi = 0$. Then

$$\|L\psi\|^2 = \int_M \langle L\psi L\psi\rangle dv_g = \int_M \langle L^*L\psi, \psi\rangle dv_g = 0.$$

□

The Fredholm property of an elliptic operator has very deep topological ramifications culminating with one of the most beautiful results in mathematics: the Atiyah-Singer index theorems. Unfortunately this would require a lot more extra work to include it here. However, in the remaining part of this subsection we will try to unveil some of the natural beauty of elliptic operators. We will show that the index of an elliptic operator has many of the attributes of a topological invariant.

We stick to the notations used so far. Denote by $\mathbf{Ell}_k(E, F)$ the space of elliptic operators $C^\infty(E) \to C^\infty(F)$ of order k. By using the attribute space when referring to $\mathbf{Ell}_k$ we implicitly suggested it carries some structure. It is not a vector space, it is not an affine space it is not even a convex set. It is only a cone in the linear space $\mathbf{PDO}^{(m)}$. But it carries a natural structure of metric space which we now proceed to describe.

Let $L_1,\ L_2 \in \mathbf{Ell}_k(E, F)$. We set

$$\delta(L_1, L_2) = \sup\{\|L_1u - L_2u\|\ ;\ \|u\|_k = 1\}.$$

Define

$$d(L_1, L_2) = \max\{\delta(L_1, L_2), \delta(L_1^*, L_2^*)\}.$$

We let the reader check that $(\mathbf{Ell}_k, d)$ is indeed a metric space. A continuous family of elliptic operators $(L_\lambda)_{\lambda\in\Lambda}$ (where Λ is a topological space) is then a continuous map

$$\Lambda \ni \lambda \mapsto L_\lambda \in \mathbf{Ell}_k.$$

In more intuitive terms this means that the coefficients of L_λ and their derivatives up to order k depend continuously upon λ.

Theorem 9.4.13. *The index map*

$$\mathrm{ind} : \mathbf{Ell}_k(E, F) \to \mathbb{Z}, \quad L \mapsto \mathrm{ind}\,(L)$$

is continuous.

The proof relies on a very simple algebraic trick which however requires some analytical foundation.

Let X, Y be two Hilbert spaces. For any Fredholm operator $L : D(L) \subset X \to Y$ denote by $\imath_L : \ker L \to X$ (resp. by $P_L : X \to \ker L$) the natural inclusion $\ker L \hookrightarrow X$ (resp. the orthogonal projection $X \to \ker L$). If $L_i : D(L_i) \subset X \to Y$ $(i = 0, 1)$ are two Fredholm operators define

$$\mathcal{R}_{L_0}(L_1) : D(L_1) \oplus \ker L_0^* \subset X \oplus \ker L_0^* \to Y \oplus \ker L_0$$

by

$$\mathcal{R}_{L_0}(L_1)(u,\phi) = (L_1 u + \imath_{L_0^*}\phi, P_{L_0}u), \;\; u \in D(L_1), \; \phi \in \ker L_0^*.$$

In other words $\mathcal{R}_{L_0}(L_1)$ is given by the block decomposition

$$\mathcal{R}_{L_0}(L_1) = \begin{bmatrix} L_1 & \imath_{L_0^*} \\ P_{L_0} & 0 \end{bmatrix}.$$

We will call $\mathcal{R}_{L_0}(L_1)$ is the *regularization of* L_1 *at* L_0. The operator L_0 is called the *pivot* of the regularization. For simplicity, when $L_0 = L_1 = L$, we write

$$\mathcal{R}_L = \mathcal{R}_L(L).$$

The result below lists the main properties of the regularization.

Lemma 9.4.14. *(a)* $\mathcal{R}_{L_0}(L_1)$ *is a Fredholm operator.*
(b) $\mathcal{R}^*_{L_0}(L_1) = \mathcal{R}_{L_0^*}(L_1^*)$.
(c) $\mathcal{R}_{L_0}$ *is invertible (with bounded inverse).*

Exercise 9.4.1. Prove the above lemma. □

We strongly recommend the reader who feels less comfortable with basic arguments of functional analysis to try to provide the no-surprise proof of the above result. It is a very good “routine booster”.

Proof of Theorem 9.4.13 Let $L_0 \in \mathbf{Ell}_k(E,F)$. We have to find $r > 0$ such that $\forall L \in \mathbf{Ell}_k(E,F)$ satisfying $d(L_0, L) \leq r$ we have

$$\operatorname{ind}(L) = \operatorname{ind}(L_0).$$

We will achieve this end in two steps.

Step 1 We will find $r > 0$ such that $\forall L$ satisfying $d(L_0, L) < r$ the regularization of L at L_0 is invertible (with bounded inverse).

Step 2 We will conclude that if $d(L, L_0) < r$ where $r > 0$ is determined at Step 1 then $\operatorname{ind}(L) = \operatorname{ind}(L_0)$.

Step 1 Since $\mathcal{R}_{L_0}(L)$ is Fredholm it suffices to show that both $\mathcal{R}_{L_0}(L)$ and $\mathcal{R}_{L_0^*}(L^*)$ are injective if L is sufficiently close to L_0. We will do this only for $\mathcal{R}_{L_0}(L)$ since the remaining case is entirely similar.

We argue by contradiction. Assume there exists a sequence $(u_n, \phi_n) \subset L^{k,2}(E) \times \ker L_0^*$ and a sequence $(L_n) \subset \mathbf{Ell}_k(E,F)$ such that

$$\|u_n\|_k + \|\phi_n\| = 1 \tag{9.4.1}$$

$$\mathcal{R}_{L_0}(L_n)(u_n, \phi_n) = (0, 0) \tag{9.4.2}$$

and

$$d(L_0, L_n) \leq 1/n. \tag{9.4.3}$$

From (9.4.1) we deduce that (ϕ_n) is a bounded sequence in the *finite dimensional space* $\ker L_0^*$. Hence it contains a subsequence *strongly convergent* in L^2 and in fact in any Sobolev norm. Set $\phi = \lim \phi_n$. Note that $\|\phi\| = \lim_n \|\phi_n\|$. Using (9.4.2) we deduce

$$L_n u_n = -\phi_n,$$

i.e. the sequence $(L_n u_n)$ is strongly convergent to $-\phi$ in $L^2(F)$. The condition (9.4.3) now gives

$$\|L_0 u_n - L_n u_n\| \leq 1/n,$$

i.e.

$$\lim_n L_n u_n = \lim_n L_0 u_n = -\phi \in L^2(F).$$

Since $u_n \perp \ker L_0$ (by (9.4.2)) we deduce from the Poincaré inequality combined with the elliptic estimates that

$$\|u_n - u_m\|_k \leq C\|L_0 u_n - L_0 u_m\| \to 0 \;\text{ as } m, n \to \infty.$$

Hence the sequence u_n strongly converges in $L^{k,2}$ to some u. Moreover

$$\lim_n \|u_n\|_k = \|u\|_k \;\text{ and }\; \|u\|_k + \|\phi\| = 1.$$

Putting all the above together we conclude that there exists a pair $(u, \phi) \in L^{k,2}(E) \times \ker L_0^*$ such that

$$\|u\|_k + \|\phi\| = 1$$

$$L_0 u = -\phi \;\text{ and }\; u \perp \ker L. \tag{9.4.4}$$

This contradicts the abstract Hodge decomposition which coupled with (9.4.4) implies $u = 0$ and $\phi = 0$. Step 1 is completed.

Step 2 Let $r > 0$ as determined at Step 1 and $L \in \mathbf{Ell}_k(E, F)$. Hence

$$\mathcal{R}_{L_0}(L) = \begin{bmatrix} L & \imath_{L_0^*} \\ P_{L_0} & 0 \end{bmatrix}$$

is invertible. We will use the invertibility of this operator to produce an injective operator

$$\ker L^* \oplus \ker L_0 \hookrightarrow \ker L \oplus \ker L_0^*.$$

This implies $\dim \ker L^* + \dim \ker L_0 \leq \dim \ker L + \dim \ker L_0^*$ i.e.

$$\mathrm{ind}\,(L_0) \leq \mathrm{ind}(L).$$

A dual argument with L replaced by L^* and L_0 replaced by L_0^* will produce the opposite inequality and thus finish the proof of Theorem 9.4.13. Now let us provide the details.

First, we orthogonally decompose

$$L^2(E) = (\ker L)^{\perp} \oplus \ker L$$

and

$$L^2(F) = (\ker L^*)^\perp \oplus \ker L^*.$$

Set $U = \ker L \oplus \ker L_0^*$ and $V = \ker L^* \oplus \ker L_0$. We will regard $\mathcal{R}_{L_0}(L)$ as an operator

$$\mathcal{R}_{L_0}(L) : (\ker L)^\perp \oplus U \to (\ker L^*)^\perp \oplus V.$$

As such it has a block decomposition

$$\mathcal{R}_{L_0}(L) = \begin{bmatrix} T & A \\ B & C \end{bmatrix},$$

where

$$T : L^{k,2}(E) \cap (\ker L)^\perp \subset (\ker L)^\perp \to (\ker L^*)^\perp = \mathrm{Range}\,(L)$$

denotes the restriction of L to $(\ker L)^\perp$. T *is an invertible operator with bounded inverse.*

Since $\mathcal{R}_{L_0}(L)$ is invertible, $\forall v \in V$ there exists a *unique* pair $(\phi, u) \in (\ker L)^\perp \oplus U$ such that

$$\mathcal{R}_{L_0}(L) \begin{bmatrix} \phi \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

This means

$$T\phi + Au = 0 \;\text{ and }\; B\phi + Cu = v.$$

Thus we can view both ϕ and u as (linear) functions of v, $\phi = \phi(v)$ and $u = u(v)$. We claim the map $v \mapsto u = u(v)$ is injective. Indeed if $u(v) = 0$ for some v then $T\phi = 0$ and since T is injective ϕ must be zero. From the equality $v = B\phi + Cu$ we deduce $v = 0$. We have thus produced the promised injective map $V \hookrightarrow U$. Theorem 9.4.13 is proved.

□

The theorem we have just proved has many topological consequences. We mention only one of them.

Corollary 9.4.15. *Let* L_0, $L_1 \in \mathbf{Ell}_k(E, F)$ *if* $\sigma_k(L_0) = \sigma_k(L_1)$ *then* $\mathrm{ind}\,(L_0) = \mathrm{ind}\,(L_1)$.

Proof For every $t \in [0, 1]$ $L_t = (1 - t)L_0 + tL_1$ is a k-th order elliptic operator depending continuously on t. (Look at the symbols). Thus $\mathrm{ind}\,(L_t)$ is an integer depending continuously on t so it must be independent of t.

□

This corollary allows us to interpret the index as as a continuous map from the elliptic symbols to the integers. The analysis has vanished ! This is (almost) a purely algebraic-topologic object. There is one (major) difficulty. These symbols are "polynomials with coefficients in some spaces of endomorphism". The deformation invariance of the index provides a very powerful method for computing it by deforming a"complicated" situation to a "simpler" one. Unfortunately our deformation freedom is severely limited by the "polynomial" character of the symbols. There aren't many polynomials around. Two polynomial-like elliptic symbols may be homotopic in a larger classes of symbols (e.g. symbols which are only positively homogeneous along the fibers of the cotangent bundle). At this point one should return to analysis and try to conceive some operators that behave very much like elliptic p.d.o. and have more general symbols. Such objects exist and are called *pseudo-differential operators*. We refer to [46] for a very efficient presentation of this subject. We will not follow this long but very rewarding path but we believe the reader who reached this point can complete this journey alone.

Exercise 9.4.2. Let $L \in \mathbf{Ell}_k(E, F)$. A finite dimensional subspace $V \subset L^2(F)$ is called a *stabilizer* of L if the operator

$$S_{L,V} : L^{k,2}(E) \oplus V \to L^2(F) \quad S_{L,V}(u \oplus v) = Lu + v$$

is surjective.
(a) Show that any subspace $V \subset L^2(F)$ containing $\ker L^*$ is a stabilizer of L. More generally, any finite dimensional subspace of $L^2(F)$ containing a stabilizer is itself a stabilizer. Conclude that if V is a stabilizer then

$$\operatorname{ind} L = \dim \ker S_{L,V} - \dim V.$$

$\square$

Exercise 9.4.3. Consider a compact manifold Λ and $L : \Lambda \to \mathbf{Ell}_k(E, F)$ a continuous family of elliptic operators.
(a) Show that the family L admits an *uniform stabilizer* i.e. there exists a finite dimensional subspace $V \subset L^2(F)$ such that V is a stabilizer of each operator L_λ in the family L.
(b)Show that if V is an uniform stabilizer of the family L then the family of subspaces $\ker S_{L_\lambda,V}$ defines a vector bundle over Λ.
(c) Show that if V_1 and V_2 are two uniform stabilizers of the family L then we have a natural isomorphism vector bundles

$$\ker S_{L,V_1} \oplus \underline{V_2} \cong \ker S_{L,V_2} \oplus \underline{V_1}.$$

In particular, we have an isomorphism of line bundles

$$\det \ker S_{L,V_1} \otimes \det V_1^* \cong \det \ker S_{L,V_2} \otimes \det V_2^*.$$

Thus the line bundle $\det \ker S_{L,V} \otimes \det V^* \to \Lambda$ is independent of the uniform stabilizer V. It is called the *determinant line bundle of the family* L and is denoted by $\det \mathbf{ind}(L)$. $\square$

9.4.2 Spectral theory

We mentioned at the beginning of this section that the elliptic operators on compact manifold behave very much like matrices. Perhaps nothing illustrates this feature better than their remarkable spectral properties. This is the subject we want to address in this subsection.

Consider as usual a compact, oriented Riemann manifold (M, g) and a complex vector bundle $E \to M$ endowed with a Hermitian metric $\langle \bullet, \bullet \rangle$ and compatible connection. Throughout this subsection L will denote a k-th order, formally selfadjoint elliptic operator $L : C^\infty(E) \to C^\infty(E)$. Its analytical realization

$$L_a : L^{k,2}(E) \subset L^2(E) \to L^2(E)$$

is a selfadjoint, elliptic operator so its spectrum is an unbounded closed subset of $\mathbb{R}$. Note that for any $\lambda \in \mathbb{R}$ the operator $\lambda - L_a$ is the analytical realization of the elliptic p.d.o. $\lambda \mathbf{1}_E - L$ and in particular $\lambda - L_a$ is a Fredholm operator so that

$$\lambda \in \sigma(L) \Longleftrightarrow \ker(\lambda - L) \neq 0.$$

Thus the spectrum of L consists only of eigenvalues of finite multiplicities. The main result of this subsection states that one can find an orthonormal basis of $L^2(E)$ which diagonalizes L_a.

Theorem 9.4.16. *Let $L \in \mathbf{Ell}_k(E)$ be a formally selfadjoint elliptic operator. Then the following are true.*
(a) The spectrum $\sigma(L)$ is real $\sigma(L) \subset \mathbb{R}$ and for each $\lambda \in \sigma(L)$ the subspace $\ker(\lambda - L)$ *is finite dimensional and consists of* smooth *sections.*
(b) $\sigma(L)$ is a closed, countable, discrete, unbounded set.
(c) There exists an orthogonal decomposition

$$L^2(E) = \bigoplus_{\lambda \in \sigma(L)} \ker(\lambda - L).$$

(d) Denote by P_λ the orthogonal projection onto $\ker(\lambda - L)$*. Then*

$$L^{k,2}(E) = D(L_a) = \{\psi \in L^2(E)\ ;\ \sum_\lambda \lambda^2 \|P_\lambda \psi\|^2 < \infty\}.$$

Part (c) of this theorem allows one to write

$$1 = \sum_\lambda P_\lambda$$

and

$$L = \sum_\lambda \lambda P_\lambda.$$

The first identity is true over the entire $L^2(E)$ while part (d) of the theorem shows the domain of validity of the second equality is precisely the domain of L.

Proof (a) We only need to show that $\ker(\lambda - L)$ consists of smooth sections. In view of Weyl's lemma this is certainly the case since $\lambda - L$ is an elliptic operator.
(b)&(c) We first show $\sigma(L)$ is discrete. More precisely given $\lambda_0 \in \sigma(L)$ we will find $\varepsilon > 0$ such that $\ker(\lambda - L) = 0$, $\forall |\lambda - \lambda_0|, \varepsilon$, $\lambda \neq \lambda_0$.

Assume for simplicity $\lambda_0 = 0$. We will argue by contradiction. Thus there exist $\lambda_n \to 0$ and $u_n \in C^\infty(E)$ such that

$$Lu_n = \lambda_n u_n,\ \ \|u_n\| = 1.$$

Clearly $u_n \in \mathrm{R}\,(L) = (\ker L^*)^\perp = (\ker L)^\perp$ so that the Poincaré inequality implies

$$1 = \|u_n\| \le C\|Lu_n\| = C\lambda_n \to 0.$$

Thus $\sigma(L)$ must be a discrete set.

Now consider $t_0 \in \mathbb{R} \setminus \sigma(L)$. Thus $t_0 - L$ has a bounded inverse

$$T = (t_0 - L)^{-1}.$$

Obviously T is a selfadjoint operator. We claim that T is also a compact operator.

Assume (v_n) is a bounded sequence in $L^2(E)$. We have to show $u_n = Tv_n$ admits a subsequence which convergent in $L^2(E)$. Note that u_n is a solution of the partial differential equation

$$(t_0 - L)u_n = t_0 u_n - Lu_n = v_n$$

so that using elliptic estimates we deduce

$$\|u_n\|_k \leq C(\|u_n\| + \|v_n\|).$$

Obviously $u_n = Tv_n$ is bounded in $L^2(E)$ so the above inequality implies $\|u_n\|_k$ is also bounded. The desired conclusion follows from the compactness of the embedding $L^{k,2}(E) \to L^2(E)$.

Thus T is a compact, selfadjoint operator. We can now use the spectral theory of such well behaved operators as described for example in [15], Chap. 6. The spectrum of T is a closed, bounded, countable set with one accumulation point, $\mu = 0$. Any $\mu \in \sigma(T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity and since $\ker T = 0$

$$L^2(E) = \bigoplus_{\mu \in \sigma(T) \setminus \{0\}} \ker(\mu - T).$$

Using the equality

$$L = t_0 - T^{-1}$$

we deduce

$$\sigma(L) = \{t_0 - \mu^{-1}\ ;\ \mu \in \sigma(T) \setminus \{0\}\}.$$

This proves (b)&(c).

To prove (d) note that if $\psi \in L^{k,2}(E)$ then $L\psi \in L^2(E)$ i.e.

$$\| \sum_\lambda \lambda P_\lambda \psi\|^2 = \sum_\lambda \lambda^2 \|P_\lambda \psi\|^2 < \infty.$$

Conversely, if

$$\sum_\lambda \lambda^2 \|P_\lambda \psi\|^2 < \infty$$

consider the sequence of smooth sections

$$\phi_n = \sum_{|\lambda| \leq n} \lambda P_\lambda \psi$$

which converges in $L^2(E)$ to

$$\phi = \sum_\lambda \lambda P_\lambda \psi.$$

On the other hand

$$\phi_n = L\psi_n$$

where

$$\psi_n = \sum_{|\lambda|\leq n} P_\lambda \psi$$

converges in $L^2(E)$ to ψ. Using the elliptic estimates we deduce

$$\|\psi_n - \psi_m\|_k \leq C(\|\psi_n - \psi_m\| + \|\phi_n - \phi_m\|) \to 0 \ \text{ as } n, m \to \infty.$$

Hence $\psi \in L^{k,2}(E)$ as a $L^{k,2}$-limit of smooth sections. The theorem is proved.

□

Example 9.4.17. Let $M = S^1$, $E = \underline{\mathbb{C}}_M$ and

$$L = -\mathbf{i}\frac{\partial}{\partial\theta} : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C}).$$

L is clearly a formally selfadjoint elliptic p.d.o. The eigenvalues and the eigenvectors of L are determined from the periodic boundary value problem

$$-\mathbf{i}\frac{\partial u}{\partial\theta} = \lambda u, \ \ u(0) = u(2\pi)$$

which implies

$$u(\theta) = C\exp(\mathbf{i}\lambda\theta) \ \text{ and } \ \exp(2\pi\lambda\mathbf{i}) = 1.$$

Hence

$$\sigma(L) = \mathbb{Z} \ \text{ and } \ \ker(n - L) = \operatorname{span}_{\mathbb{C}}\{\exp(\mathbf{i}n\theta)\}.$$

The orthogonal decomposition

$$L^2(S^1) = \bigoplus_n \ker(n - L)$$

is the usual Fourier decomposition of periodic functions. Note that

$$u(\theta) = \sum_n u_n \exp(\mathbf{i}n\theta) \in L^{1,2}(S^1)$$

if and only if

$$\sum_{n\in\mathbb{Z}}(1 + n^2)|u_n|^2 < \infty.$$

□

The following exercises provide a variational description of the eigenvalues of formally selfadjoint elliptic operator $L \in \mathbf{Ell}_k(E)$ which is bounded from below i.e.

$$\inf\{\int_M \langle Lu, u\rangle dv_g \ ; \ u \in L^{k,2}(E), \ \ \|u\| = 1\} > -\infty.$$

Exercise 9.4.4. Let $V \in L^{k,2}(E)$ be a finite dimensional invariant subspace of L. Show that
(a) V consists only of smooth sections.
(b) The quantity

$$\lambda(V^{\perp}) = \inf\{\int_M \langle Lu, u\rangle dv_g \ ; \ u \in L^{k,2}(E) \cap V^{\perp}, \ \ \|u\| = 1\}$$

is an eigenvalue of L. $V^{\perp}$ denotes the orthogonal complement of V in $L^2(E)$. □

Exercise 9.4.5. Set $V_0 = 0$ and denote $\lambda_0 = \lambda(V_0^{\perp})$ According to the previous exercise λ_0 is an eigenvalue of L. Pick ϕ_0 an eigenvector corresponding to λ_0 such that $\|\phi_0\| = 1$ and form $V_1 = V_0 \oplus \mathrm{span}\,\{\phi_0\}$. Set $\lambda_1 = \lambda(V_1^{\perp})$ and iterate the procedure. After m steps we have produced $m+1$ vectors $\phi_0, \phi_1, \ldots, \phi_m$ corresponding to $m+1$ eigenvalues $\lambda_0, \lambda_1 \leq \cdots \lambda_m$ of L. Set $V_{m+1} = \mathrm{span}_{\mathbb{C}}\{\phi_0, \phi_1, \ldots, \phi_m\}$ and $\lambda_{m+1} = \lambda(V_{m+1}^{\perp})$ etc.
(a) Prove that

$$\{\phi_1, \ldots, \phi_m, \ldots\}$$

is a Hilbert basis of $L^2(E)$ and

$$\sigma(L) = \{\lambda_1 \leq \cdots \leq \lambda_m \cdots\}.$$

(b) Denote by G_m the grassmannian of m-dimensional subspaces of $L^{k,2}(E)$. Show that

$$\lambda_m = \inf_{V \in G_m} \max\{\int_M \langle Lu, u\rangle dv_g \ ; \ u \in V \ \|u\| = 1\}.$$

□

Exercise 9.4.6. Use the results in the above exercises to show that if L is a bounded from below, k-th order formally selfadjoint elliptic p.do. over an N-dimensional manifold then

$$\lambda_m(L) = O(m^{k/N}) \ \text{ as } m \to \infty$$

and

$$d(\Lambda) = \dim \oplus_{\lambda \leq \Lambda} \ker(\lambda - L) = O(\Lambda^{N/k}) \ \text{ as } \Lambda \to \infty.$$

□

Remark 9.4.18. (a)When L is a formally selfadjoint generalized Laplacian then the result in the above exercise can be considerably sharpened. More precisely H.Weyl showed that

$$\lim_{\Lambda \to \infty} \Lambda^{-N/2} d(\Lambda) = \frac{\mathrm{rank}\,(E) \cdot \mathrm{vol}_g(M)}{(4\pi)^{N/2}\Gamma(N/2+1)}.$$

The very ingenious proof of this result relies on another famous p.d.o. namely the heat operator $\partial_t + L$. For details we refer to [10].

(b) Assume L is the scalar Laplacian Δ on a compact Riemann manifold (M, g) of dimension M. Weyl's formula shows that the asymptotic behavior of the spectrum of Δ contains several geometric informations about M: we can read the dimension and the volume of M from it. If we think of M as the elastic membrane of a drum then the eigenvalues of Δ describe all the frequencies of the sounds the "drum" M can produce. Thus "we can hear" the dimension and the volume of a drum. This is a special case of a famous question raised by V.Kac in [39]: can one hear the shape of a drum? In more rigorous terms this question asks how much of the geometry of a Riemann manifold can be recovered from the spectrum of its Laplacian. This is what spectral geometry is all about.

It has been established recently that the answer to Kac's original question is negative. We refer to [30] and the references therein for more details. □

Exercise 9.4.7. Compute the spectrum of the scalar Laplacian on the torus T^2 equipped with the flat metric and then use this information to prove the above Weyl asymptotic formula in this special case. □

9.4.3 Hodge theory

We now have enough theoretical background to discuss the celebrated Hodge theorem. It is convenient to work in a slightly more general context than Hodge's original theorem.

Definition 9.4.19. *Let (M, g) be an oriented Riemann manifold. An* elliptic complex *is a sequence of first order p.d.o.'s*

$$0 \to C^\infty(E_0) \stackrel{D_0}{\to} C^\infty(E_1) \stackrel{D_1}{\to} \cdots \stackrel{D_{m-1}}{\to} C^\infty(E_m) \to 0$$

satisfying the following conditions.
(i) $(C^\infty(E_i), D_i)$ is a cochain complex, i.e. $D_i D_{i-1} = 0$, $\forall 1 \leq i \leq m$.
(ii) For each $(x, \xi) \in T^*M \setminus \{0\}$ the sequence of principal symbols

$$0 \to (E_0)_x \stackrel{\sigma(D_0)(x,\xi)}{\longrightarrow} (E_1)_x \to \cdots \stackrel{\sigma(D_{m-1})(x,\xi)}{\longrightarrow} (E_m)_x \to 0$$

is exact.

Example 9.4.20. The DeRham complex $(\Omega^*(M), d)$ is an elliptic complex. In this case the associated sequence of principal symbols is ($e(\xi)$ = exterior multiplication by ξ)

$$0 \to \mathbb{R} \stackrel{e(\xi)}{\to} T_x^*M \stackrel{e(\xi)}{\to} \cdots \stackrel{e(\xi)}{\to} \det(T_x^*M) \to 0$$

is the Koszul complex of Exercise 7.1.7 of Subsection 7.1.3 where it is shown to be exact. Hence the DeRham complex is elliptic. We will have the occasion to discuss another famous elliptic complex in the next chapter. □

Consider an elliptic complex $(C^\infty(E_\cdot, D_\cdot))$ over a *compact* oriented Riemann manifold (M, g). Denote its cohomology by $H^*(E_\cdot, D_\cdot)$. A priori these may be infinite dimensional spaces. We will see that the combination ellipticity + compactness prevents this from happening. Endow each E_i with a metric and compatible connection. We can now talk about Sobolev spaces and formal adjoints D_i^*. Form the operators

$$\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^* : C^\infty(E_i) \to C^\infty(E_i).$$

We can now state and prove the celebrated Hodge theorem.

Theorem 9.4.21. (Hodge) *Assume M is* compact. *Then the following are true.*
(a) $H^i(E., D.) \cong \ker \Delta_i \subset C^\infty(E_i)$.
(b) $\dim H^i(E., D.) < \infty \ \forall i$.
(c) **(Hodge decomposition)** *There exists an orthogonal decomposition*

$$L^2(E_i) = \ker \Delta_i \oplus \mathrm{R}\,(D_{i-1}) \oplus \mathrm{R}\,(D_i^*)$$

where we view both D_{i-1} and D_i as bounded operators $L^{1,2} \to L^2$.

Proof Set

$$E = \oplus E_i, \quad D = \oplus D_i, \quad D^* = \oplus D_i^*$$
$$\Delta = \oplus \Delta_i, \quad \hat{D} = D + D^*.$$

Thus D, D^* and Δ are p.d.o.'s $C^\infty(E) \to C^\infty(E)$. Since $D_i D_{i-1} = 0$ we deduce $D^2 = (D^*)^2 = 0$. We deduce

$$\Delta = D^*D + DD^* = (D + D^*)^2 = \hat{D}^2.$$

We now invoke the following elementary algebraic fact which is a consequence of the exactness of the symbol sequence. (Look back to the finite dimensional Hodge theory in §7.1.3.)

Exercise 9.4.8. The operators $\hat{D}$ and Δ are *elliptic* formally selfadjoint p.d.o. □

Note that according to Corollary 9.4.12 $\ker \Delta = \ker \hat{D}$ so that we have an orthogonal decomposition

$$L^2(E) = \ker \Delta \oplus \mathrm{R}\,(\hat{D}). \tag{9.4.5}$$

This is precisely part (c) of Hodge's theorem.

For each i denote by P_i the orthogonal projection $L^2(E_i) \to \ker \Delta_i$. Set

$$Z^i = \{u \in C^\infty(E_i)\ ;\ D_i u = 0\}$$

and

$$B^i = D_{i-1}(C^\infty(E_{i-1}))$$

so that

$$H^i(E., D.) = Z^i / B^i.$$

We claim that the map $P_i : Z^i \to \ker \Delta_i$ descends to an isomorphism

$$H^i(E., D.) \to \ker \Delta_i.$$

This will complete the proof of Hodge theorem. The above claim is a consequence of several simple facts.
Fact 1 $\ker \Delta_i \subset Z^i$. This follows from the equality $\ker \Delta = \ker \hat{D}$.

Fact 2 If $u \in Z_i$ then $u - P_i u \in B_i$. Indeed, using the decomposition (9.4.5) we have

$$u = P_i u + \hat{D}\psi \quad \psi \in L^{1,2}(E).$$

Since $u - P_i u \in C^\infty(E_i)$ we deduce from Weyl's lemma that $\psi \in C^\infty(E)$. Thus there exist $v \in C^\infty(E_{i-1})$ and $w \in C^\infty(E_i)$ such that $\psi = v \oplus w$ and

$$u = P_i u + D_{i-1} v + D_i^* w.$$

Applying D_i on both sides of this equality we get

$$0 = D_i u = D_i P_i u + D_i D_{i-1} u + D_i D_i^* w = D_i D_i^* w.$$

Since $\ker D_i^* = \ker D_i D_i^*$ the above equalities imply

$$u - P_i u = D_{i-1} v \in B^i.$$

We conclude that P_i descends to a linear map $\ker \Delta_i \to H^i(E., D.)$. Thus $B^i \subset \mathrm{R}\,(\hat{D}) = (\ker \Delta)^\perp$ and we deduce that no two distinct elements in $\ker \Delta_i$ are cohomologous since otherwise their difference would have been orthogonal to $\ker \Delta_i$. Hence the induced linear map $P_i : \ker \Delta_i \to H^i(E., D.)$ is injective. Fact 1 shows it is also surjective so that

$$\ker \Delta_i \cong H^i(E., D.).$$

Hodge theorem is proved.

$\square$

Let us apply Hodge theorem to the DeRham complex on a compact oriented Riemann manifold

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0 \ \ (n = \dim M).$$

We know the formal adjoint of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is

$$d^* = (-1)^{\nu_{n,k}} * d *$$

where $\nu_{n,k} = nk + n + 1$ and $*$ denoted the Hodge $*$-operator defined by the Riemann metric g and the fixed orientation on M. Set

$$\Delta = dd^* + d^* d.$$

Corollary 9.4.22. (Hodge) *Any smooth k-form $\omega \in \Omega^k(M)$ decomposes uniquely as*

$$\omega = \omega_0 + d\eta + d^*\zeta \ \ \eta \in \Omega^{k-1}(M),\ \zeta \in \Omega^{k+1}(M)$$

and $\omega_0 \in \Omega^k(M)$ is g-harmonic i.e.

$$\Delta\omega_0 = 0 \iff d\omega = 0 \ \text{ and } \ d^*\omega = 0.$$

If moreover ω is closed then $\zeta = 0$ and this means any cohomology class $[z] \in H^k(M)$ is uniquely represented by a harmonic k-form.

Denote by $\mathbf{H}^k(M, g)$ the space of g-harmonic k-forms on M. The above corollary shows

$$\mathbf{H}^k(M, g) \cong H^k(M)$$

for any metric g.

Corollary 9.4.23. *The Hodge $*$-operator defines a bijection*

$$* : \mathbf{H}^k(M,g) \to \mathbf{H}^{n-k}(M,g).$$

Proof If ω is g-harmonic then so is $*\omega$ since

$$d * \omega = \pm * d * \omega = 0$$

and

$$*d * (*\omega) = \pm * (d\omega) = 0.$$

$*$ is bijective since $*^2 = (-1)^{k(n-k)}$.

$\square$

Using the L^2-inner product on $\Omega^*(M)$ we can identify $\mathbf{H}^{n-k}(M,g)$ with its dual and thus we can view $*$ as an isomorphism

$$\mathbf{H}^k \xrightarrow{*} (\mathbf{H}^{n-k})^*.$$

On the other hand the Poincaré duality described in Chapter 7 induces another isomorphism

$$\mathbf{H}^k \xrightarrow{PD} (\mathbf{H}^{n-k})^*$$

defined by

$$\langle PD(\omega), \eta \rangle_0 = \int_M \omega \wedge \eta$$

where $\langle \cdot, \cdot \rangle_0$ denotes the natural pairing between a vector space and its dual.

Proposition 9.4.24. $PD = *$, *i.e.*

$$\int_M \langle *\omega, \eta \rangle_g dv_g = \int_M \omega \wedge \eta \;\; \forall \omega \in \mathbf{H}^k \;\; \eta \in \mathbf{H}^{n-k}.$$

$\square$

Proof We have

$$\langle *\omega, \eta \rangle_g dv_g = \langle \eta, *\omega \rangle_g dv_g = \eta \wedge *^2\omega = (-1)^{k(n-k)} \eta \wedge \omega = \omega \wedge \eta.$$

$\square$

Exercise 9.4.9. Let $\omega_0 \in \Omega^k(M)$ be a harmonic k form and denote by $\mathcal{C}_{\omega_0}$ its cohomology class. Show that

$$\int_M |\omega_0|_g^2 dv_g \leq \int_M |\omega|_g^2 dv_g \quad \forall \omega \in \mathcal{C}_{\omega_0}$$

with equality if and only if $\omega = \omega_0$. $\square$

Exercise 9.4.10. Let G denote a compact connected Lie group equipped with a bi-invariant Riemann metric h. Prove that a differential form on G is h-harmonic if and only if it is bi-invariant. $\square$

Chapter 10

Dirac operators

We devote this last chapter to a presentation of a very important class of first order elliptic operators which have numerous applications in modern geometry. We will first describe their general features and then we will spend the remaining part discussing some frequently encountered examples.

10.1 The structure of Dirac operators

10.1.1 Basic definitions and examples

Consider a Riemann manifold (M, g) and a smooth vector bundle $E \to M$.

Definition 10.1.1. *A* Dirac operator *is a first order p.d.o.*

$$D : C^\infty(E) \to C^\infty(E)$$

such that D^2 is a generalized Laplacian, i.e.

$$\sigma(D^2)(x, \xi) = -|\xi|_g^2 \mathbf{1}_{E_x} \quad \forall (x, \xi) \in T^*M.$$

The Dirac operator is said to be graded *if E splits as $E = E_0 \oplus E_1$ and $D(C^\infty(E_i)) \subset C^\infty(E_{(i+1) \bmod 2})$. In other words, D has a block decomposition*

$$D = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

Note that the Dirac operators are 1st order elliptic p.d.o.

Example 10.1.2. (Hamilton-Floer) Denote by E the trivial vector bundle $\underline{\mathbb{R}}^{2n}$ over the circle S^1. Thus $C^\infty(E)$ can be identified with the space of smooth functions

$$u : S^1 \to \mathbb{R}^{2n}.$$

Let $J : C^\infty(E) \to C^\infty(E)$ denote the endomorphism of E which has the block decomposition

$$J = \begin{bmatrix} 0 & -\mathbf{1}_{\underline{\mathbb{R}}^n} \\ \mathbf{1}_{\underline{\mathbb{R}}^n} & 0 \end{bmatrix}$$

with respect to the natural splitting $\underline{\mathbb{R}}^{2n} = \underline{\mathbb{R}}^n \oplus \underline{\mathbb{R}}^n$. We define the Hamilton-Floer operator

$$\mathfrak{F} : C^\infty(E) \to C^\infty(E)$$

by

$$\mathfrak{F}u = J\frac{du}{d\theta} \quad \forall u \in C^\infty(E).$$

Clearly, $\mathfrak{F}^2 = -\frac{d^2}{d\theta^2}$ is a generalized Laplacian. □

Example 10.1.3. (Hodge-DeRham) Let (M, g) be an oriented Riemann manifold. Then, according to the computations in the previous chapter, the Hodge-DeRham operator

$$d + d^* : \Omega^*(M) \to \Omega^*(M).$$

is a Dirac operator. □

Let $D : C^\infty(E) \to C^\infty(E)$ be a Dirac operator over the oriented Riemann manifold (M, g). Its symbol is an endomorphism

$$\sigma(D) : \pi^*E \to \pi^*E,$$

where $\pi : T^*M \to M$ denotes the natural projection. Thus, for any $x \in M$ and any $\xi \in T_x^*M$, $c(\xi) = \sigma(D)(x, \xi)$ is an endomorphism of E_x depending linearly upon ξ. Since D^2 is a generalized Laplacian

$$c(\xi)^2 = \sigma(D^2)(x, \xi) = -|\xi|_g^2 \mathbf{1}_{E_x}.$$

To summarize, we see that each Dirac operator induces a bundle morphism

$$c : T^*M \otimes E \to E \quad (\xi, e) \mapsto c(\xi)e$$

such that $c(\xi)^2 = -|\xi|^2$. From the equality

$$c(\xi + \eta) = -|\xi + \eta|^2 \quad \forall \xi, \eta \in T_x^*M,\ x \in M$$

we conclude that

$$\{c(\xi), c(\eta)\} = -2g(\xi, \eta)\mathbf{1}_{E_x},$$

where for any linear operators A, B we denoted by $\{A, B\}$ their anticommutator

$$\{A, B\} \stackrel{def}{=} AB + BA.$$

Definition 10.1.4. *(a) A* Clifford structure *on a vector bundle E over a Riemann manifold (M, g) is a bundle morphism*

$$c : T^*M \otimes E \to E$$

such that

$$\{c(\xi), c(\eta)\} = -2g(\xi, \eta)\mathbf{1}_E.$$

c is usually called the Clifford multiplication *of the (Clifford) structure. A pair (vector bundle, Clifford structure) is called a* Clifford bundle.
(b) A $\mathbb{Z}_2$-grading on a Clifford bundle $E \to M$ is a splitting $E = E_0 \oplus E_1$ such that $\forall \alpha \in \Omega^1(M)$ the Clifford multiplication by α is an odd endomorphism of the superspace $C^\infty(E_0) \oplus C^\infty(E_1)$.

Proposition 10.1.5. *Let $E \to M$ be a smooth vector bundle over the Riemann manifold (M, g). Then the following conditions are equivalent.*
(a) There exists a Dirac operator $D : C^\infty(E) \to C^\infty(E)$.
(b) The bundle E admits a Clifford structure.

Proof We have just seen that (a)⇒(b). To prove the reverse implication let

$$c : T^*M \otimes E \to E$$

be a Clifford multiplication. Then for every connection

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$$

the composition

$$D = c \circ \nabla : C^\infty(E) \xrightarrow{\nabla} C^\infty(T^*M \otimes E) \xrightarrow{c} C^\infty(E)$$

is a first order p.d.o. with symbol c. Clearly D is a Dirac operator.

□

Example 10.1.6. Let (M, g) be a Riemann manifold. For each $x \in M$ and $\xi \in T^*M$ define

$$c(\xi) : \Lambda^*T^*M \to \Lambda^*T^*M$$

by

$$c(\xi)\omega = (e_\xi - i_\xi)\omega$$

where e_ξ denotes the (left) exterior multiplication by ξ while i_ξ denotes the interior differentiation along $\xi^* \in T_xM$ - the metric dual of ξ. The Exercise 2.2.16 of Section 2.2.4 shows that c defines a Clifford multiplication on Λ^*T^*M. If ∇ denotes the Levi-Civita connection on Λ^*T^*M then the Dirac operator $c \circ \nabla$ is none other than the Hodge-DeRham operator. □

Exercise 10.1.1. Prove the last assertion in the above example. □

The above proposition reduces the problem of describing which vector bundles admit Dirac operators to an algebraic-topological one: find the bundles admitting a Clifford structure. In the following subsections we will address precisely this issue.

10.1.2 Clifford algebras

The first thing we want to understand is the object called Clifford multiplication.

Consider (V, g) a (real) finite dimensional, Euclidean space. A Clifford multiplication is then a pair (E, ρ) where E is a $\mathbb{K}$-vector space and $\rho : V \to \mathrm{End}(E)$ is an $\mathbb{R}$-linear map such that

$$\{\rho(u), \rho(v)\} = -2g(u, v)\mathbf{1}_E \quad \forall u, v \in V.$$

If (e_i) is an orthonormal basis of V then ρ is completely determined by the linear operators $\rho_i = \rho(e_i)$ which satisfy the anticommutation rules

$$\{\rho_i, \rho_j\} = -2\delta_{ij}\mathbf{1}_E.$$

The collection (ρ_i) generates an associative subalgebra in $\mathrm{End}(E)$ and it is natural to try to understand its structure. We will look at the following universal situation.

Definition 10.1.7. *Let V be a real, finite dimensional vector space and*

$$q : V \times V \to \mathbb{R}$$

a symmetric bilinear form. The Clifford algebra $Cl(V, q)$ is the associative $\mathbb{R}$–algebra with unit, generated by V subject to the relations

$$\{u, v\} = uv + vu = -2q(u, v) \cdot \mathbf{1} \quad \forall u, v \in V.$$

Proposition 10.1.8. *The Clifford algebra $Cl(V, q)$ exists and is uniquely defined by its universality property: for every linear map $\jmath : V \to \mathcal{A}$ such that $\mathcal{A}$ is an associative $\mathbb{R}$-algebra with unit and $\{\jmath(u), \jmath(v)\} = -2q(u, v)\cdot\mathbf{1}$ there exists an unique morphism of algebras $\Phi : Cl(V, q) \to \mathcal{A}$ such that the diagram below is commutative.*

$$\begin{tikzcd} V \arrow[r, "\imath"] \arrow[dr, "\jmath"'] & Cl(V,q) \arrow[d, "\Phi"] \\ & \mathcal{A} \end{tikzcd}$$

$\imath$ denotes the natural inclusion $V \hookrightarrow Cl(V, q)$.

Sketch of proof Let $\mathcal{A} = \oplus_{k\geq 0} V^{\otimes k}$ ($V^{\otimes 0} = \mathbb{R}$) denote the free associative $\mathbb{R}$–algebra with unit generated by V. Set

$$Cl(V, q) = \mathcal{A}/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by

$$\{\ u \otimes v + v \otimes u + 2q(u, v) \otimes 1\ ;\ u, v \in V\}.$$

$\imath$ is the composition $V \hookrightarrow \mathcal{A} \to Cl(V, q)$ where the second arrow is the natural projection. We let the reader check the universality property.

□

Exercise 10.1.2. Prove the universality property.

Remark 10.1.9. (a)When $q \equiv 0$ then $Cl(V, 0)$ is the exterior algebra $\Lambda^* V$.
(b) In the sequel the inclusion $V \hookrightarrow Cl(V, q)$ will be thought of as being part of the definition of a Clifford algebra. This makes a Clifford algebra a structure richer than merely an abstract $\mathbb{R}$–algebra: it is an algebra with a distinguished real subspace. Thus when thinking of automorphisms of this structure one should really concentrate only on those automorphisms of $\mathbb{R}$–algebras preserving the distinguished subspace. □

Corollary 10.1.10. *Let (V_i, q_i) ($i = 1, 2$) be two real, finite dimensional vector spaces endowed with quadratic forms $q_i : V \to \mathbb{R}$. Then any linear map $T : V_1 \to V_2$ such that $q_2(Tv) = q_1(v)$, $\forall v \in V_1$ induces an unique morphism of algebras $T_\# : Cl(V_1, q_1) \to Cl(V_2, q_2)$ such that $T_\#(V_1) \subset V_2$, where we view V_i as a linear subspace in $Cl(V_i, q_i)$. The correspondence $T \mapsto T_\#$ constructed above is functorial i.e. $(\mathbf{1}_{V_i})_\# = \mathbf{1}_{Cl(V_i,q_i)}$ and $(S \circ T)_\# = S_\# \circ T_\#$ for all admissible S and T.*

The above corollary shows the algebra $Cl(V,q)$ depends only on the isomorphism class of the pair (V,q)= vector space + quadratic form. It is known from linear algebra that the isomorphism classes of such pairs are classified by some simple invariants:

$$(\dim V, \operatorname{rank} q, \operatorname{sign} q).$$

We will be interested in the special case when $\dim V = \operatorname{rank} q = sign\, q = n$, i.e. when q is an Euclidean metric on the n-dimensional space V. In this case the Clifford algebra $CL(V,q)$ is usually denoted by $Cl(V)$ or Cl_n. If (e_i) is an orthonormal basis of V then we can alternatively describe Cl_n as the associative $\mathbb{R}$–algebra with $\mathbf{1}$ generated by (e_i) subject to the relations

$$e_ie_j + e_je_i = -2\delta_{ij}.$$

Using the universality property of Cl_n we deduce that the map

$$V \to Cl(V) \quad v \mapsto -v \in Cl(V)$$

extends to an automorphism of algebras $\alpha : Cl(V) \to Cl(V)$. Note that α is involutive i.e. $\alpha^2 = \mathbf{1}$. Set

$$Cl^0(V) = \ker(\alpha - \mathbf{1}), \;\; Cl^1(V) = \ker(\alpha + \mathbf{1}).$$

Note that $Cl(V) = Cl^0(V) \oplus Cl^1(V)$ and moreover

$$Cl^{\varepsilon}(V) \cdot Cl^{\eta}(V) \subset Cl^{(\varepsilon+\eta) \bmod 2}(V),$$

i.e. the automorphism α naturally defines a $\mathbb{Z}_2$-grading of $Cl(V)$. In other words, the Clifford algebra $Cl(V)$ is naturally a super-algebra.

Let $(\tilde{Cl}(V), +, *)$ denote the opposite algebra of $Cl(V)$. $\tilde{Cl}(V)$ coincides with $Cl(V)$ as a vector space but its multiplication $*$ is defined by

$$x * y \stackrel{def}{=} y \cdot x \quad \forall x, y \in \tilde{Cl}(V),$$

where "$\cdot$" denotes the usual multiplication in $Cl(V)$. Note that for any $u, v \in V$

$$u \cdot v + v \cdot u = u * v + v * u$$

so that using the universality property of Clifford algebras we conclude that the natural injection $V \hookrightarrow \tilde{Cl}(V)$ extends to a morphism of algebras $Cl(V) \to \tilde{Cl}(V)$. This may as well be regarded as an antimorphism $Cl(V) \to Cl(V)$ which we call the *transposition* map, $x \mapsto x^{\flat}$. Note that

$$(u_1 \cdot u_2 \cdots u_r)^{\flat} = u_r^{\flat} \cdots u_1^{\flat} \quad \forall u_i \in V.$$

For $x \in Cl(V)$ we set $x^* = (\alpha(x))^{\flat} = \alpha(x^{\flat})$. x^* is called the *adjoint* of x.

For each $v \in V$ define $\mathbf{c}(v) \in \operatorname{End}(\Lambda^* V)$ by

$$\mathbf{c}(v)\omega = (e_v - i_v)\omega \quad \forall \omega \in \Lambda^* V$$

where as usual e_v denotes the (left) exterior multiplication by v while i_v denotes the interior derivative along the metric dual of v. Invoking again the Exercise 2.2.16 we deduce

$$\mathbf{c}(v)^2 = -|v|_g^2$$

so that by the universality property of the Clifford algebras the map c extends to a morphism of algebras $c : Cl(V) \to \operatorname{End}(\Lambda^* V)$.

Exercise 10.1.3. Prove that $\forall x \in Cl(V)$ we have

$$\mathbf{c}(x^*) = \mathbf{c}(x)^*,$$

where the * in the right-hand-side denotes the adjoint of $\mathbf{c}(x)$ viewed as a linear operator on the linear space $\Lambda^* V$ endowed with the metric induced by the metric on V. □

For each $x \in Cl(V)$ $\mathbf{c}(x)1$ is an element of $\Lambda^* V$ called the *symbol* of x. The linear map $Cl(V) \ni x \mapsto \sigma(x) \in \Lambda^* V$ is called the *symbol* map. If (e_i) is an orthonormal basis then

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k} \quad \forall e_{i_j}.$$

This shows the symbol map is bijective since the ordered monomials

$$\{e_{i_1} \cdots e_{i_k} \ ; \ 1 \leq i_1, \cdots i_k \leq \dim V\}$$

form a basis of $Cl(V)$. The inverse of the symbol map is called the *quantization* map and is denoted by $\mathfrak{q} : \Lambda^* V \to Cl(V)$.

Exercise 10.1.4. Show that $\mathfrak{q}(\Lambda^{even/odd} V) = Cl^{even/odd}(V)$. □

Definition 10.1.11. *(a) A $\mathbb{K}$(=$\mathbb{R}$-,$\mathbb{C}$)-vector space E is said to be a $\mathbb{K}$-* Clifford module *if there exists a morphism of $\mathbb{R}$–algebras*

$$\rho : Cl(V) \to \mathrm{End}_{\mathbb{K}}(E).$$

(b) A $\mathbb{K}$-superspace E is said to be a $\mathbb{K}$-Clifford s-module if there exists a morphism of s-algebras

$$\rho : Cl(V) \to \widehat{\mathrm{End}}_{\mathbb{K}}(E).$$

(c) Let E be a $\mathbb{K}$-Clifford module, $\rho : Cl(V) \to \mathrm{End}_{\mathbb{K}}(E)$. E (or ρ) is said to be selfadjoint if there exists a metric on E (Euclidean if $\mathbb{K} = \mathbb{R}$, Hermitian if $\mathbb{K} = \mathbb{C}$) such that

$$\rho(x^*) = \rho(x)^* \quad \forall x \in Cl(V).$$

We now see that what we originally called a Clifford structure is precisely a Clifford module.

Example 10.1.12. $\Lambda^* V$ is a selfadjoint, real $Cl(V)$ super-module. □

In the following two subsections we intend to describe the complex Clifford modules. The real theory is far more elaborate. For more information we refer the reader to the excellent monograph [46].

10.1.3 Clifford modules: the even case

In studying complex Clifford modules it is convenient to work with the complexified Clifford algebras

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}.$$

The (complex) representation theory of $\mathbb{C}l_n$ depends on the parity of n so that we will discuss each case separately. The reader may want to refresh his/her memory of the considerations in Subsection 2.2.5.

Let $n = 2k$ and consider an n-dimensional Euclidean space (V, g) . The decisive step in describing the complex $Cl(V)$ modules is the following.

Proposition 10.1.13. *There exists a complex $Cl(V)$-module $\mathbb{S} = \mathbb{S}(V)$ such that*

$$\mathbb{C}l(V) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}) \ \text{ as } \mathbb{C}-\text{algebras}.$$

(The above isomorphism is not natural; it depends on several auxiliary choices.)

$\mathbb{S}(V)$ is known as the (even) *complex spinor* module.

The reader familiar with the representation theory of associative algebras can immediately grasp the relevance of this proposition since Weddeburn's theorem completely describes the modules over the algebra of endomorphisms of a vector space. We will have to say more about that a little later.

Proof Consider a complex structure on V i.e. a skew-symmetric operator $J : V \to V$ such that $J^2 = -\mathbf{1}_V$. Such a J exists since V is even dimensional. Let $\{e_1, f_1; \dots ; e_k, f_k\}$ be an orthonormal basis of V such that $Je_i = f_i \ \forall i$.

Extend J by complex linearity to $V \otimes_{\mathbb{R}} \mathbb{C}$. We can now decompose $V \otimes \mathbb{C}$ into the eigenspaces of J

$$V = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0} = \ker(\mathbf{i} - J)$ and $V^{0,1} = \ker(\mathbf{i} + J)$. Alternatively,

$$V^{1,0} = \mathrm{span}_{\mathbb{C}}(e_j - \mathbf{i}f_j), \quad V^{0,1} = \mathrm{span}_{\mathbb{C}}(e_j + \mathbf{i}f_j).$$

The metric on V defines a Hermitian metric on the *complex* vector space (V, J)

$$h(u, v) = g(u, v) + \mathbf{i}g(u, Jv)$$

which allows us to identify (see Subsection 2.2.5)

$$V^{0,1} \cong_{\mathbb{C}} \overline{(V, J)} \cong_{\mathbb{C}} V_c^* \cong_{\mathbb{C}} (V^{1,0})^*.$$

(V_c^* denotes the complex dual of the complex space (V, J)). With respect to this Hermitian metric the collection

$$\{\varepsilon_j = \frac{1}{\sqrt{2}}(e_j - \mathbf{i}f_j) \ ; \ 1 \le j \le k\}$$

is an orthonormal basis of $V^{1,0}$ while

$$\{\bar{\varepsilon}_j = \frac{1}{\sqrt{2}}(e_j + \mathbf{i}f_j) \ ; \ 1 \le j \le k\}$$

is an orthonormal basis of $V^{0,1}$. Set

$$\mathbb{S}_{2k} = \Lambda^* V^{1,0} = \Lambda^{*,0} V.$$

Any morphism

$$\rho : \mathbb{C}l(V) \to \mathrm{End}\,(\mathbb{S}_{2k})$$

is uniquely defined by its restriction to

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

We thus have to specify the action of each of the components $V^{1,0}$ and $V^{0,1}$. The elements $w \in V^{1,0}$ will act by exterior multiplication

$$\mathbf{c}(w)\omega = \sqrt{2}e(w)\omega = \sqrt{2}w \wedge \omega \quad \forall \omega \in \Lambda^{*,0}V.$$

The elements $\overline{w} \in V^{0,1}$ can be identified with complex linear functionals on $V^{1,0}$ and as such they will act by interior differentiation

$$\mathbf{c}(\overline{w})w_1 \wedge \cdots \wedge w_\ell = -\sqrt{2}i(\overline{w})(w_1 \wedge \cdots \wedge w_\ell)$$

$$= \sqrt{2}\sum_{j=1}^{\ell}(-1)^j g_{\mathbb{C}}(w_j, \overline{w})w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_\ell,$$

where $g_{\mathbb{C}}$ denotes the extension of g to $(V \otimes \mathbb{C}) \times (V \otimes \mathbb{C})$ by complex linearity.

To check that the above constructions do indeed define an action of $\mathbb{C}l(V)$ we need to check that $\forall v \in V$

$$\mathbf{c}(v)^2 = -\mathbf{1}_{\mathbb{S}_{2k}}.$$

This boils down to verifying the anticommutation rules

$$\{\mathbf{c}(e_i), \mathbf{c}(f_j)\} = 0, \;\; \{\mathbf{c}(e_i), \mathbf{c}(e_j)\} = -2\delta_{ij} = \{\mathbf{c}(f_i), \mathbf{c}(f_j)\}.$$

We have

$$e_i = \frac{1}{\sqrt{2}}(\varepsilon_i + \overline{\varepsilon}_i), \;\; f_j = \frac{\mathbf{i}}{\sqrt{2}}(\varepsilon_j - \overline{\varepsilon}_j)$$

so that

$$\mathbf{c}(e_i) = e(\varepsilon_i) - i(\overline{\varepsilon}_i), \quad \mathbf{c}(f_j) = \mathbf{i}(e(\varepsilon_j) + i(\overline{\varepsilon}_j)).$$

The anticommutation rules follow as in the Exercise 2.2.16 using the equalities

$$g_{\mathbb{C}}(\varepsilon_i, \overline{\varepsilon}_j) = \delta_{ij}.$$

This shows $\mathbb{S}_{2k}$ is naturally a $\mathbb{C}l(V)$-module. Note that $\dim_{\mathbb{C}} \mathbb{S}_{2k} = 2^k$ so that

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{S}) = 2^n = \dim_{\mathbb{C}} \mathbb{C}l(V).$$

A little work (left to the reader) shows the Clifford multiplication map $c : \mathbb{C}l(V) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_{2k})$ is injective. This completes the proof of the proposition.

□

Using basic algebraic results about the representation theory of the algebra of endomorphisms of a vector space we can draw several useful consequences. (See [69] for a very nice presentation of these facts.)

Corollary 10.1.14. *There exists an unique (up to isomorphism) irreducible complex $\mathbb{C}l_{2k}$-module and this is the complex spinor module $\mathbb{S}_{2k}$.*

Corollary 10.1.15. *Any complex $\mathbb{C}l_{2k}$-module has the form $\mathbb{S}_{2k} \otimes W$, where W is an arbitrary complex vector space. The action of $\mathbb{C}l_{2k}$ on $\mathbb{S}_{2k} \otimes W$ is defined by*

$$v \cdot (s \otimes w) = \mathbf{c}(v)s \otimes w.$$

W is called the twisting *space of the given Clifford module.*

Remark 10.1.16. Given a complex $\mathbb{C}l_{2k}$-module E its twisting space can be recovered as the space of morphisms of Clifford modules

$$W = \mathrm{Hom}_{\mathbb{C}l_{2k}}(\mathbb{S}_{2k}, E).$$

$\square$

Assume now that (V, g) is an *oriented*, $2k$-dimensional Euclidean space. For any positively oriented orthonormal basis $e_1, \ldots, e_{2k}$ we can form the element

$$\Gamma = \mathbf{i}^k e_1 \cdots e_{2k} \in \mathbb{C}l(V).$$

One can check easily this element is independent of the *oriented* basis and thus it is an element intrinsically induced by the orientation. It is called the *chirality operator* defined by the orientation. Note that

$$\Gamma^2 = 1 \text{ and } \Gamma x = (-1)^{\deg x}\Gamma \quad \forall x \in \mathbb{C}l^0(V) \cup \mathbb{C}l^1(V).$$

Let $\mathbb{S} = \mathbb{S}(V)$ denote the spinor module of $\mathbb{C}l(V)$. The chirality operator defines an involutive endomorphism of $\mathbb{S}$ and thus defines a $\mathbb{Z}_2$-grading on $\mathbb{S}$

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- \quad (\mathbb{S}^\pm = \ker(\pm \mathbf{1} - \Gamma))$$

and hence a $\mathbb{Z}_2$-grading of $\mathrm{End}(\mathbb{S})$. Since $\{v, \Gamma\} = 0 \quad \forall v \in V$ we deduce the Clifford multiplication by v is an odd endomorphism of $\mathbb{S}$. This means any isomorphism $\mathbb{C}l(V) \cong \mathrm{End}\,(\mathbb{S}(V))$ is an isomorphism of $\mathbb{Z}_2$-graded algebras.

Exercise 10.1.5. Let J be a complex structure on V. This produces two things: it defines an orientation on V and identifies $\mathbb{S} = \mathbb{S}(V) \cong \Lambda^{*,0}V$. Prove that with respect to these data the chiral grading of $\mathbb{S}$ is

$$\mathbb{S}^{+/-} \cong \Lambda^{even/odd,0}V.$$

$\square$

The above considerations extend to arbitrary Clifford modules. The chirality operator introduces a $\mathbb{Z}_2$-grading in any complex Clifford module which we call the *chiral grading*. However this does not exhaust the family of Clifford s-modules. The family of s-modules can be completely described as

$$\{\mathbb{S}\hat{\otimes} W \; ; \; W \text{ complex s} - \text{space}\}$$

where $\hat{\otimes}$ denotes the s-tensor product. The modules endowed with the chiral grading form the subfamily in which the twisting s-space W is purely even.

Example 10.1.17. Let (V, g) be a $2k$-dimensional, *oriented, Euclidean space.* Then $\Lambda^*_{\mathbb{R}} V \otimes \mathbb{C}$ is naturally a Clifford module. Thus it has the form

$$\Lambda^*_{\mathbb{R}} V \otimes \mathbb{C} \cong \mathbb{S} \otimes W.$$

To find the twisting space we pick a complex structure on V whose induced orientation agrees with the given orientation of V. This complex structure produces an isomorphism

$$\Lambda^*_{\mathbb{R}} V \otimes \mathbb{C} \cong \Lambda^{*,0} V \otimes \Lambda^{0,*} V \cong \mathbb{S} \otimes (\Lambda^{*,0} V)^* \cong \mathbb{S} \otimes \mathbb{S}^*.$$

This shows the twisting space is $\mathbb{S}^*$.

On the other hand the chirality operator defines $\mathbb{Z}_2$ gradings on both $\mathbb{S}$ and $\mathbb{S}^*$ so that $\Lambda^*_{\mathbb{R}} V \otimes \mathbb{C}$ can be given two different s-structures: the chiral superstructure (in which the grading of $\mathbb{S}^*$ is forgotten) and the grading as s-tensor product $\mathbb{S} \hat{\otimes} \mathbb{S}^*$. Using Exercise 10.1.5 we deduce that the second grading is precisely the degree grading

$$\Lambda^*_{\mathbb{R}} V \otimes \mathbb{C} = \Lambda^{even} V \otimes \mathbb{C} \oplus \Lambda^{odd} V \otimes \mathbb{C}.$$

To understand the chiral grading we need to describe the action of the chiral operator on $\Lambda^*_{\mathbb{R}} V \otimes \mathbb{C}$. This can be done via the Hodge $*$-operator. More precisely we have

$$\Gamma \cdot \omega = \mathbf{i}^{k+p(p-1)} * \omega \quad \forall \omega \in \Lambda^p V \otimes \mathbb{C}. \tag{10.1.1}$$

□

Exercise 10.1.6. Prove the equality 10.1.1. □

In order to formulate the final result of this subsection we need to extend the automorphism α and the anti-automorphism $^\flat$ to the complexified Clifford algebra $\mathbb{C}l(V)$. α can be extended by complex linearity

$$\alpha(x \otimes z) = \alpha(x) \otimes z \quad \forall x \in Cl(V)$$

while $^\flat$ is extends according to

$$(x \otimes z)^\flat = x^\flat \otimes \overline{z}.$$

As in the real case set $y^* = \alpha(y^\flat) = \alpha(y)^\flat$ $\forall y \in \mathbb{C}l(V)$.

Proposition 10.1.18. *Let $\mathbb{S}(V)$ denote the spinor module of the $2k$-dimensional Euclidean space (V, g). Then for every morphism of algebras*

$$\rho : \mathbb{C}l(V) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}(V))$$

there exists a Hermitian metric on $\mathbb{S}(V)$ such that

$$\rho(y^*) = \rho(y)^* \quad \forall y \in \mathbb{C}l(V).$$

Moreover, this metric is unique up to a multiplicative constant.

Sketch of proof Choose an orthonormal basis $\{e_1, \ldots, e_{2k}\}$ of V and denote by G the group generated by these elements. G is finite and it consists of the monomials

$$\{e_{i_1} \cdots e_{i_\ell}\ ;\ 1 \leq i_1, \ldots, i_\ell \leq 2k\}.$$

As a set, G generates $\mathbb{C}l(V)$ as a complex vector space.

Pick a Hermitian metric h on $\mathbb{S}(V)$ and for each $g \in G$ denote by h_g the pulled-back metric

$$h_g(s_1, s_2) = h(\rho(g)s_1, \rho(g)s_2) \quad \forall s_1, s_2 \in \mathbb{S}.$$

We can now form the averaged metric

$$h_G = \frac{1}{|G|} \sum_{g \in G} h_g.$$

Each $\rho(g)$ is an unitary operator with respect to this metric. We leave the reader to check that this is the metric we are after. The uniqueness follows from the irreducibility of $\mathbb{S}(V)$ using Schur's lemma.

$\square$

Corollary 10.1.19. *Let (V, g) as above and $\rho : Cl(V) \to \mathrm{End}_{\mathbb{C}}(E)$ be a complex Clifford module. Then E admits at least one Hermitian metric with respect to which ρ is selfadjoint.*

Proof Decompose E as $\mathbb{S} \otimes W$ and ρ as $\Delta \otimes \mathbf{id}_W$ for some isomorphism of algebras $\Delta : \mathbb{C}l(V) \to \mathrm{End}(\mathbb{S})$. The sought for metric is a tensor product of the canonical metric on $\mathbb{S}$ and some metric on W.

$\square$

10.1.4 Clifford modules: the odd case

The odd dimensional situation can be deduced using the facts we have just established concerning the algebras $\mathbb{C}l_{2k}$. The bridge between these two situations is provided by the following general result.

Lemma 10.1.20. $\mathbb{C}l_m \cong \mathbb{C}l_{m+1}^{even}$.

Proof Pick an orthonormal basis $\{e_0, e_1, \ldots, e_m\}$ in the standard Euclidean space $\mathbb{R}^{m+1}$. These generate the algebra Cl_{m+1}. We view Cl_m as the Clifford algebra generated by $\{e_1, \ldots, e_m\}$. Now define

$$\Psi : Cl_m \to Cl_{m+1}^{even}$$

by

$$\Psi(x^0 + x^1) = x^0 + e_0 \cdot x^1$$

where $x^0 \in Cl_m^{even}$ and $x^1 \in Cl_m^{odd}$. We leave the reader to check that this is indeed an isomorphism of algebras.

$\square$

Proposition 10.1.21. *Let (V, g) be a $(2k+1)$-dimensional Euclidean space. Then there exist two complex, irreducible $\mathbb{C}l(V)$ modules $\mathbb{S}^+(V)$ and $\mathbb{S}^-(V)$ such that*

$$\mathbb{C}l(V) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}^+) \oplus \mathrm{End}_{\mathbb{C}}(\mathbb{S}^-) \quad \text{as ungraded algebras.}$$

The direct sum $\mathbb{S}(V) = \mathbb{S}_V^+ \oplus \mathbb{S}_V^-$ is called the (odd) spinor module.

Proof Fix an orientation on V and a positively oriented orthonormal basis $e_1, e_2, \dots, e_{2k+1}$. Denote by $\mathbb{S}_{2k+2}$ the spinor module of $\mathbb{C}l(V \oplus \mathbb{R})$ where $V \oplus \mathbb{R}$ is given the direct sum Euclidean metric and the orientation

$$\mathbf{or}(V \oplus \mathbb{R}) = \mathbf{or}(V) \wedge \mathbf{or}(\mathbb{R}).$$

Choose an isomorphism

$$\rho : \mathbb{C}l(V \oplus \mathbb{R}) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_{2k+2}).$$

Then $\mathbb{S}_{2k+2}$ becomes naturally a supermodule

$$\mathbb{S}_{2k+2} = \mathbb{S}_{2k+1}^+ \oplus \mathbb{S}_{2k+1}^-$$

and we thus we get the isomorphisms of algebras

$$\mathbb{C}l(V) \cong \mathbb{C}l(V \oplus \mathbb{R})^{even} \cong \mathrm{End}^{even}(\mathbb{S}_{2k+2}) \cong \mathrm{End}\,(\mathbb{S}_{2k+2}^+) \oplus \mathrm{End}\,(\mathbb{S}_{2k+2}^-).$$

$\square$

The above characterization can be used to describe the complex (super)modules of $\mathbb{C}l_{2k+1}$. We will not present the details since the applications we have in mind do not require these facts. For more details we refer to [46].

10.1.5 A look ahead

In this heuristic section we interrupt a little bit the flow of arguments to provide the reader a sense of direction. The next step in our story is to glue all the pointwise data presented so far into smooth families (i.e. bundles). To produce a Dirac operator on an n-dimensional Riemann manifold (M, g) one needs several things.
(a) A bundle of Clifford algebras $\mathcal{C} \to M$ such that $\mathcal{C}_x \cong Cl_n$ or $\mathbb{C}l_n$, $\forall x \in M$
(b) A fiberwise injective morphism of vector bundles $\imath : T^*M \hookrightarrow \mathcal{C}$ such that $\forall x \in M$

$$\{\imath(u), \imath(v)\}_{\mathcal{C}} = -2g(u, v) \quad \forall u, v \in T_x^*M.$$

(c) A bundle of Clifford modules i.e. a vector bundle $\mathcal{E} \to M$ together with a morphism $c : \mathcal{C} \to \mathrm{End}\,(\mathcal{E})$ whose restrictions to the fibers are morphisms of algebras.
(d) A connection on $\mathcal{E}$.

The above collection of data can be constructed from bundles associated to a common principal bundle. The symmetry group of this principal bundle has to be a Lie group with several additional features which we now proceed to describe.

Let (V, g) denote the standard fiber of T^*M and denote by Aut_V the group of automorphisms ϕ of $Cl(V)$ such that $\phi(V) \subset V$. The group Aut_V^c is defined similarly, using the complexified algebra $\mathbb{C}l(V)$ instead of $Cl(V)$. For brevity, we discuss only the real case.

We need a Lie group G which admits a smooth morphism $\rho : G \to \mathrm{Aut}_V$. Tautologically, ρ defines a representation $\rho : G \to GL(V)$ which we assume is orthogonal.

We also need a Clifford module $c : Cl(V) \to \mathrm{End}\,(E)$ and a representation $\mu : G \to GL(E)$ such that for every $v \in V$ and any $g \in G$ the diagram below is commutative.

$$\begin{array}{ccc} E & \xrightarrow{\mathbf{c}(v)} & E \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ E & \xrightarrow{\mathbf{c}(g\cdot v)} & E \end{array} \tag{10.1.2}$$

This commutativity can be given an invariant theoretic interpretation as follows. View the Clifford multiplication $c : V \to \mathrm{End}\,(E)$ as an element $c \in V^* \otimes E^* \otimes E$. The group G acts on this tensor product and the above commutativity simply means that c is invariant under this action.

In concrete applications E comes with a metric and we need to require that μ is an orthogonal/unitary representation.

To produce all the data (a)-(d) all we now need is a principal G-bundle $P \to M$ such that the associated bundle $P \times_\rho V$ is isomorphic with T^*M. (This may not be always feasible due to possible topological obstructions). Any connection ∇ on P induces by association metric connections ∇^M on[1] T^*M and ∇^E on the bundle of Clifford modules $\mathcal{E} = P \times_\mu E$. With respect to these connections the Clifford multiplication is covariant constant i.e.

$$\nabla^E(\mathbf{c}(\alpha)u) = \mathbf{c}(\nabla^M\alpha) + \mathbf{c}(\alpha)\nabla^E u \quad \forall \alpha \in \Omega^1(M)\ u \in C^\infty(E).$$

This follows from the following elementary invariant theoretic result.

Lemma 10.1.22. *Let G be a Lie group and $\rho : G \to \mathrm{Aut}\,(E)$ a linear representation of G. Assume there exists $e_0 \in E$ such that $\rho(g)e_0 = e_0$ $\forall g \in G$. Consider an arbitrary principal G-bundle $P \to X$ and an arbitrary connection ∇ on P. Then e_0 canonically determines a section u_0 on $P \times_\rho E$ which is covariant constant with respect to the induced connection $\nabla^E = \rho_*(\nabla)$ i.e.*

$$\nabla^E u_0 = 0.$$

Exercise 10.1.7. Prove the above lemma. □

Apparently the chances that a Lie group G with the above properties exists are very slim. The very pleasant surprise is that all these (and even more) happen in many geometrically interesting situations.

Example 10.1.23. Let (V, g) be an oriented Euclidean space. Using the universality property of Clifford algebras we deduce that each $g \in SO(V)$ induces an automorphism of $Cl(V)$ preserving $V \hookrightarrow Cl(V)$. Moreover it defines an orthogonal representation on the canonical Clifford module

$$c : Cl(V) \to \mathrm{End}\,(\Lambda^* V)$$

[1] In practice one requires a little more namely that ∇^M is precisely the Levi-Civita connection on T^*M. This leads to significant simplifications in many instances.

such that

$$\mathbf{c}(g \cdot v)(\omega) = g \cdot (\mathbf{c}(v)(g^{-1} \cdot \omega)) \quad \forall g \in SO(V), v \in V, \omega \in \Lambda^* V$$

i.e. $SO(V)$ satisfies the equivariance property (10.1.2).

If (M, g) is an oriented Riemann manifold we can now build our bundle of Clifford modules starting from the principal SO bundle of its oriented orthonormal coframes. As connections we can now pick the Levi-Civita connection and its associates. The corresponding Dirac operator is the Hodge-DeRham operator. □

The next two sections discuss two important examples of Lie groups with the above properties. These are the spin groups $Spin(n)$ and its "complexification" $Spin^{\mathbf{c}}(n)$. It turns out that all the groups one needs to build Dirac operators are these three classes: SO, $Spin$ and $Spin^c$.

10.1.6 *Spin*

Let (V, g) be a finite dimensional Euclidean space. The group of automorphisms of the Clifford algebra $Cl(V)$ contains a very rich subgroup consisting of the interior ones. These have the form

$$\varphi_x : Cl(V) \to Cl(V) \quad u \mapsto \varphi_x(u) = x \cdot u \cdot x^{-1} \;\; \forall u \in Cl(V),$$

where x is some invertible element in $Cl(V)$. The candidates for the Lie groups with the properties outlined in the previous section will be sought for amongst subgroups of interior automorphisms. It is thus natural to determine the subgroup

$$\{x \in Cl(V)^\star \; ; \; x \cdot V \cdot x^{-1} \subset V\},$$

where $Cl(V)^\star$ denotes the group of invertible elements. We will instead try to understand the *Clifford group*

$$\Gamma(V) = \{x \in Cl^\star(V) \; ; \; \alpha(x) \cdot V \cdot x^{-1} \subset V\}$$

where $\alpha : Cl(V) \to Cl(V)$ denotes the involutive automorphism of $Cl(V)$ defining its $\mathbb{Z}_2$-grading. In general, the map $\rho_x = \{Cl(V) \ni u \mapsto \alpha(x)ux^{-1} \in Cl(V)\}$ is not an automorphism of algebras but, as we will see by the end of this subsection, if $x \in \Gamma(V)$ then $\rho_x = \pm\varphi_x$ and hence *a posteriori* this alteration has no impact. Its impact is mainly on the æsthetics of the presentation which we borrowed from the elegant paper [6].

By construction $\Gamma(V)$ comes equipped with a tautological representation

$$\rho : \Gamma(V) \to GL(V) \quad \rho(x) : v \mapsto \alpha(x) \cdot v \cdot x^{-1}.$$

Proposition 10.1.24. $\ker \rho = (\mathbb{R}^*, \cdot) \subset Cl(V)^\star$.

Proof Clearly $\mathbb{R}^* \subset \ker \rho$. To establish the opposite inclusion choose an orthonormal basis (e_i) of V and let $x \in \ker \rho$. x decomposes into even/odd components

$$x = x_0 + x_1,$$

and the condition

$$\alpha(x) e_i x^{-1} = e_i$$

translates into

$$(x_0 - x_1)e_i = e_i(x_0 + x_1) \ \ \forall i.$$

This is equivalent to the following two conditions

$$[x_0, e_i] = x_0 e_i - e_i x_0 = x_1 e_i + e_i x_1 = \{x_1, e_i\} = 0 \ \ \forall i.$$

In terms of the s-commutator the above two equalities can be written as one

$$[e_i, x]_s = 0 \ \ \forall i.$$

Since $[\cdot, x]$ is a superderivation of $Cl(V)$ we conclude that

$$[y, x]_s = 0 \quad \forall y \in Cl(V).$$

In particular, x_0 lies in the center of $Cl(V)$. We let the reader check the following elementary fact.

Lemma 10.1.25. *The s-center*[2] *of the Clifford algebra is the field of scalars* $\mathbb{R} \subset Cl(V)$.

Note that since $\{x_1, e_i\} = 0$ then x_1 should be a linear combination of elementary monomials $e_{j_1} \cdots e_{j_s}$ none of which containing e_i as a factor. Since this should happen for every i this means $x_1 = 0$ and this concludes the proof of the proposition.

□

Definition 10.1.26. *The* spinorial norm *is the map*

$$N : Cl(V) \to Cl(V) \quad N(x) = x^\flat x.$$

Proposition 10.1.27. *(a)* $N(\Gamma(V)) \subset \mathbb{R}^*$.
(b) The map $N : \Gamma(V) \to \mathbb{R}^*$ *is a group morphism.*

Proof Let $x \in \Gamma(V)$. We first prove that $x^\flat \in \Gamma(V)$. Since $\alpha(x)vx^{-1} \in V, \ \ \forall v \in V$ we deduce that

$$\alpha(\{\alpha(x) \cdot v \cdot x^{-1}\}^\flat) = -\alpha(x) \cdot v \cdot x^{-1} \in V.$$

Using the fact that $x \mapsto x^\flat$ is an anti-automorphism we deduce

$$\alpha(\,(x^\flat)^{-1} \cdot v \cdot \alpha(x^\flat)\,) \in V$$

so that

$$\alpha((x^\flat)^{-1}) \cdot v \cdot x^\flat \in V,$$

that is $(x^\flat)^{-1} \in \Gamma(V)$. Hence $x^* \in \Gamma(V) \ \forall x \in \Gamma(V)$. In particular, since $\alpha(\Gamma(V)) \subset \Gamma(V)$, we deduce $N(\Gamma(V)) \subset \Gamma(V)$. For any $v \in V$ we have

$$\alpha(N(x)) \cdot v \cdot (N(x))^{-1} = \alpha(x^\flat x) \cdot v \cdot (x^\flat x)^{-1} = \alpha(x^\flat) \cdot \{\alpha(x) \cdot v \cdot x^{-1}\} \cdot (x^\flat)^{-1}.$$

[2]The super-center consists of those elements super-commuting with every element in the s-algebra.

On the other hand $y \stackrel{def}{=} \alpha(x)\cdot v\cdot x^{-1}$ is an element in V which implies $y^* = \alpha(y^\flat) = -y$. Hence

$$\alpha(N(x))\cdot v\cdot (N(x))^{-1} = -\alpha(x^\flat)\cdot y^*\cdot (x^\flat)^{-1}$$
$$= -\alpha(x^\flat)\cdot \alpha(x^\flat)^{-1}\cdot v^*\cdot x^\flat\cdot (x^\flat)^{-1} = -v^* = v.$$

This means $N(x)\in \ker\rho = \mathbb{R}^*$.
(b) If $x, y\in\Gamma(V)$ then

$$N(x\cdot y) = (xy)^\flat(xy) = y^\flat x^\flat xy = y^\flat N(x) y = N(x) y^\flat y = N(x)N(y).$$

□

Theorem 10.1.28. *(a) For every $x\in\Gamma(V)$ the transformation $\rho(x)$ of V is orthogonal.*
(b) There exists a short exact sequence of groups

$$1\rightarrow \mathbb{R}^*\hookrightarrow \Gamma(V)\xrightarrow{\rho} O(V)\rightarrow 1.$$

(c) Every $x\in\Gamma(V)$ can be written (in a non-unique way) as a product $x = v_1\cdots v_k$, $v_j\in V$. In particular, every element of $\Gamma(V)$ is $\mathbb{Z}_2$-homogeneous i.e. it is either purely even or purely odd.

Proof (a) Note that $\forall v\in V$ we have $N(v) = -|v|_g^2$. For every $x\in\Gamma(V)$ we get

$$N(\rho(x)(v)) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = N(\alpha(x))N(x)^{-1}N(v).$$

On the other hand $x^2 = \alpha(x^2) = \alpha(x)^2$ we deduce

$$N(x)^2 = N(\alpha(x))^2$$

so that $N(\rho(x)(v)) = \pm N(v)$. Since both $N(v)$ and $N(\rho_x(v))$ are *negative numbers* we deduce that the only logical choice of signs in the above equality is $+$. Hence $\rho(x)$ is an orthogonal transformation.
(b) & (c) We only need to show $\rho(\Gamma(V)) = O(V)$. For $x\in V$ with $|x|_g = 1$ we have

$$\alpha(x) = -x = x^{-1}.$$

If we decompose $v\in V$ as $\lambda x + u$ where $\lambda\in\mathbb{R}$ and $u\perp x$ then we deduce

$$\rho(x)v = -\lambda x + u.$$

In other words, $\rho(x)$ is the orthogonal reflection in the hyperplane through origin which is perpendicular to x. Since any orthogonal transformation of V is a composition of such reflections we deduce that for each $T\in O(V)$ we can find $v_1,\dots,v_k\in V$ such that

$$T = \rho(v_1)\cdots\rho(v_k).$$

Incidentally this also establishes (c).

□

Set $\Gamma^0(V) = \Gamma(V) \cap Cl^{even}$. Note that

$$\rho(\Gamma^0(V)) \subset SO(V) = \{T \in O(V)\ ;\ \det T = 1\}.$$

Hence we have a short exact sequence

$$1 \to \mathbb{R}^* \hookrightarrow \Gamma^0(V) \xrightarrow{\rho} 1.$$

Definition 10.1.29. *Set*

$$Pin(V) := \{x \in \Gamma(V)\ ;\ |N(x)| = 1\},$$

and

$$Spin(V) := \{x \in \Gamma(V)\ ;\ N(x) = 1\} = Pin(V) \cap \Gamma^0(V).$$

The results we proved so far show that $Spin(V)$ can be alternatively described by the following "friendlier" equality

$$Spin(V) = \{v_1 \cdots v_{2k}\ ;\ k \geq 0, v_i \in V,\ \ |v_i| = 1,\ \forall i = 1. \dots 2k\}.$$

Proposition 10.1.30. *There exist short exact sequences*

$$1 \to \mathbb{Z}_2 \to Pin(V) \to O(V) \to 1$$

$$1 \to \mathbb{Z}_2 \to Spin(V) \to SO(V) \to 1.$$

$Spin(V)$ is a Lie group and $\rho : Spin(V) \to SO(V)$ is a covering map. It is connected if $\dim V \geq 2$ *and simply connected if* $\dim V \geq 3$*. In particular, $Spin(V)$ is the universal cover of $SO(V)$ when* $\dim V \geq 3$.

Proof The exactness of the two sequences is left to the reader. It is also fairly easy to prove that $\rho : Spin(V) \to SO(V)$ is a covering map using the following simple observations:
(i) ρ is a group morphism ;
(ii) ρ is continuous;
(iii) $\ker \rho$ is discrete.
This shows that $Spin(V)$ can be naturally endowed with a smooth structure pulled back from $SO(V)$ via ρ. Since $SO(V)$ is connected if $\dim V \geq 2$ the fact that $Spin(V)$ is connected would follow if we showed that any points in the same fiber of ρ can be connected by arcs. It suffices to look at the fiber $\rho^{-1}(1) = \{-1, 1\}$.

Using Theorem 10.1.28(c) we see that

$$Spin(V) = \{v_1 \cdots v_{2k}\ ;\ k \geq 1, v_j \in V,\ |v_j| = 1\}.$$

Thus if $u, v \in V$ are such that $|u| = |v| = 1,\ u \perp v$ the path

$$\gamma(t) = (u \cos t + v \sin t)(u \cos t - v \sin t), \quad 0 \leq t \leq \pi/2$$

lies inside $Spin(V)$ and moreover

$$\gamma(0) = -1,\ \ \gamma(\pi/2) = 1.$$

To prove that $Spin(V)$ is simply connected if $\dim V \geq 3$ it suffices to observe that each closed path in $Spin(V)$ is homotopic to a "monomial loop"

$$\gamma(t) = v_1(t) \cdots v_{2k}(t),$$

where $v_j(t)$ are closed,piecewise smooth paths in the unit sphere $S^{\dim V - 1} \subset (V, g)$. If $\dim V \geq 3$ then the unit sphere $S^{\dim V - 1}$ is simply connected which forces $Spin(V)$ to be simply connected.

$\square$

Corollary 10.1.31. $\pi_1(SO(n)) = \mathbb{Z}_2$ if $n \geq 3$.

Let $Spin(n) = Spin(\mathbb{R}^n)$ where $\mathbb{R}^n$ denotes the standard Euclidean space. Since the natural map $\rho : Spin(n) \rightarrow SO(n)$ is a cover we deduce that its derivative at the "origin" induces an isomorphism of Lie algebras

$$\tau = \rho_* : \underline{spin}(n) \stackrel{\cong}{\rightarrow} \underline{so}(n).$$

We would like to spend some time discussing some often confusing aspects of this isomorphism.

We can view $Spin(V)$ as a submanifold of $Cl(V)$ and as such we can identify its Lie algebra $\underline{spin}(V)$ with a linear subspace of $Cl(V)$. The next result offers a more precise description.

Proposition 10.1.32. *Consider the quantization map* $\mathfrak{q} : \Lambda^* V \rightarrow Cl(V)$. *Then*

$$\underline{spin}(V) = \mathfrak{q}(\Lambda^2 V).$$

The Lie bracket is given by the commutator in $Cl(V)$.

Proof The group $\Gamma(V)$ is a Lie group as a closed subgroup of the group of linear transformations of $Cl(V)$. Since the elements of $\Gamma(V)$ are either purely even or purely odd we deduce that the tangent space at $1 \in \Gamma(V)$ can be identified with the subspace

$$E = \{x \in Cl^{even}(V) \; ; \; xv - vx \in V, \quad \forall v \in V\}.$$

Fix $x \in E$ and let $e_1, \ldots, e_n$ be an orthonormal basis of V. We can decompose x as

$$x = x_0 + e_1 x_1,$$

where $x_0 \in Cl(V)^{even}$ and $x_1 \in Cl(V)^{odd}$ are linear combinations of monomials involving only the vectors $e_2, \ldots e_n$. Since $[x_0, e_1] = 0$ and $\{x_1, e_1\} = 0$ we deduce

$$e_1 x_1 = \frac{1}{2}[e_1, x_1] = \frac{1}{2}[e_1, x] \in V.$$

In particular this means $x_1 \in \mathbb{R} \oplus V \in Cl(V)$. Repeating the same argument with every vector e_i we deduce that

$$x \in \mathbb{R} \oplus \operatorname{span}\{e_i \cdot e_j \; ; \; 1 \leq i < j \leq \dim V\} = \mathbb{R} \oplus \mathfrak{q}(\Lambda^2 V).$$

Thus

$$T_1\Gamma^0(V) \subset \mathbb{R} \oplus \mathfrak{q}(\Lambda^2 V).$$

The tangent space to $Spin(V)$ satisfies a further restriction obtained by differentiating the condition $N(x) = 1$. This gives

$$\underline{spin}(V) \subset \{x \in \mathbb{R} \oplus \mathfrak{q}(\Lambda^2 V)\ ;\ x^\flat + x = 0\} = \mathfrak{q}(\Lambda^2 V).$$

Since $\dim \underline{spin}(V) = \dim \underline{so}(V) = \dim \Lambda^2 V$ we conclude the above inclusion is in fact an equality of vector spaces.

Now consider two smooth paths $x, y : (-\varepsilon, \varepsilon) \to Spin(V)$ such that $x(0) = y(0) = 1$. The Lie bracket of $\dot{x}(0)$ and $\dot{y}(0)$ is then found (using the Exercise 3.1.5) from the equality

$$x(t)y(t)x(t)^{-1}y(t)^{-1} = 1 + [\dot{x}(0), \dot{y}(0)]t^2 + O(t^3) \ \ (\text{as } t \to 0)$$

where the above bracket is the commutator of $\dot{x}(0)$ and $\dot{y}(0)$ viewed as elements in the associative algebra $Cl(V)$.

□

To get a more explicit picture of the isomorphism

$$\rho_* : \underline{spin}(V) \to \underline{so}(V)$$

we fix an orientation on V and then choose a positively oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of V, $(n = \dim V)$. For every $x \in \underline{spin}(V)$ the element $\rho_*(x) \in \underline{spin}(V)$ acts on V according to

$$\rho_*(x)v = x \cdot v - v \cdot x.$$

If

$$x = \sum_{i<j} x_{ij} e_i e_j$$

then

$$\rho_*(x)e_j = -2\sum_i x_{ij} e_i, \quad (x_{ij} = -x_{ji}).$$

Note the following often confusing fact. If we identify as usual $\underline{so}(V) \cong \Lambda^2 V$ by

$$\underline{so}(n) \ni A \mapsto \omega_A = \sum_{i<j} g(Ae_i, e_j) e_i \wedge e_j = -\sum_{i<j} g(e_i, Ae_j) e_i \wedge e_j$$

then the Lie algebra isomorphism ρ_* takes the form

$$\omega_{\rho_*(x)} = -\sum_{i<j} g(e_i, \rho_*(x)e_j) e_i \wedge e_j$$

$$= 2\sum_{i<j} x_{ij} e_i \wedge e_j = 2\sigma(x) \in \Lambda^2 V.$$

where $\sigma : Cl(V) \to \Lambda^* V$ is the symbol map, $e_i e_j \mapsto e_i \wedge e_j$.

A word of warning If $A \in \underline{so}(V)$ has the matrix description

$$Ae_j = \sum_i a_{ij} e_i$$

with respect to an oriented orthonormal basis $\{e_1, \cdots, e_n\}$, $(n = \dim V)$ then the 2-form associated to A has the form

$$\omega_A = -\sum_{i<j} a_{ij} e_i \wedge e_j$$

so that

$$\rho_*^{-1}(A) = -\frac{1}{2}\sum_{i<j} a_{ij} e_i e_j = -\frac{1}{4}\sum_{i,j} a_{ij} e_i e_j!$$

The above negative sign is essential, and in many concrete problems it makes a world of difference.

Any Clifford module $\phi : Cl(V) \to \mathrm{End}_{\mathbb{K}}(E)$ defines by restriction a representation

$$\phi : Spin(V) \to GL_{\mathbb{K}}(E).$$

The (complex) representation theory described in the previous sections can be used to determine the representations of $Spin(V)$.

Example 10.1.33. (The complex spinor representations) Consider a finite dimensional oriented Euclidean space (V, g). Assume first that $\dim V$ is even. The orientation on V induces a $\mathbb{Z}_2$-grading on the spinor module $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Since $Spin(V) \subset Cl^{even}(V)$ we deduce that each of the spinor spaces $\mathbb{S}^\pm$ is a representation space for $Spin(V)$. They are in fact irreducible, nonisomorphic complex $Spin(V)$-modules. They are called the positive/negative complex spin representations.

Assume next that $\dim V$ is odd. The spinor module $\mathbb{S}(V)$

$$\mathbb{S}(V) = \mathbb{S}^+(V) \oplus \mathbb{S}^-(V).$$

is not irreducible as a $\mathbb{C}l(V)$ modules. Each of the modules $\mathbb{S}^\pm(V)$ is a representation space for $Spin(V)$. They are irreducible but also *isomorphic* as $Spin(V)$-modules. If we pick an oriented orthonormal basis $e_1, \cdots, e_{2n+1}$ of V then the Clifford multiplication by $\omega = e_1 \cdots e_{2n+1}$ intertwines the $\pm$ components. This is a $Spin(V)$ isomorphism since ω lies in the center of $Cl(V)$. □

Convention For each positive integer n we will denote by $\mathbb{S}_n$ the $Spin(n)$ module defined by

$$\mathbb{S}_{2k} \cong \mathbb{S}^+(\mathbb{R}^{2k}) \oplus \mathbb{S}^-(\mathbb{R}^{2k}) \ \ (n = 2k),$$

and

$$\mathbb{S}_{2k+1} \cong \mathbb{S}^+(\mathbb{R}^{2k+1}) \cong_{Spin(2k+1)} \mathbb{S}^-(\mathbb{R}^{2k+1}) \ \ (n = 2k+1).$$

$\mathbb{S}_n$ will be called the *fundamental complex spinor module of* $Spin(n)$.

Exercise 10.1.8. Let (V, g) be a $2k$-dimensional Euclidean space and $J : V \to V$ a complex structure compatible with the metric g. Thus we have an explicit isomorphism

$$\Delta : \mathbb{C}l(V) \to \mathrm{End}\,(\Lambda^{*,0}V).$$

Choose $u \in V$ such that $|u|_g = 1$ and then for each $t \in \mathbb{R}$ set

$$q = q(t) = \cos t + u \cdot v \sin t, \quad v = Jv.$$

Note that $q \in Spin(V)$ so that $\Delta(q)$ preserves the parities when acting on $\Lambda^{*,0}(V)$. Hence $\Delta(q) = \Delta_+(q) \oplus \Delta_-(q)$ where $\Delta_{+/-}(q)$ acts on $\Lambda^{even/odd,0}V$. Compute $\mathrm{tr}\,(\Delta_{+/-}(q))$ and then conclude that $\Lambda^{even,/odd,0}V$ are non-isomorphic $Spin(V)$-modules. $\square$

Proposition 10.1.34. *Let $\phi : Cl(V) \to \mathrm{End}\,(E)$ be a selfadjoint Clifford module. Then the induced representation of $Spin(V)$ is orthogonal (unitary).*

Exercise 10.1.9. Prove the above proposition. $\square$

Exercise 10.1.10. Prove that the group $Spin(V)$ satisfies all the conditions discussed in Subsection 10.1.5. $\square$

10.1.7 $Spin^c$

The considerations in the previous case have a natural extension to the complexified Clifford algebra $\mathbb{C}l_n$. The canonical involutive automorphism $\alpha : Cl_n \to Cl_n$ extends by complex linearity to an automorphism of $\mathbb{C}l_n$ while the adjoint anti-automorphism $^\flat : Cl_n \to Cl_n$ extend to $\mathbb{C}l_n$ according to the rule

$$(v \otimes z)^\flat = v \otimes \overline{z}.$$

As in the real case set $x^* = \alpha(x)^\flat$ and $N(x) = x^\flat \cdot x$.

Let (V, g) be an Euclidean space. The *complex Clifford group* $\Gamma^c(V)$ is defined by

$$\Gamma^c(V) = \{x \in \mathbb{C}l(V)^\star; \alpha(x) \cdot v \cdot x^{-1} \in V \ \ \forall v \in V\}.$$

We denote by ρ^c the tautological representation

$$\rho^c : \Gamma^c(V) \to GL(V, \mathbb{R}).$$

As in the real case one can check that

$$\rho^c(\Gamma^c(V)) = O(V)$$

and

$$\ker \rho^c = \mathbb{C}^*.$$

The spinorial norm $N(x)$ defines a homomorphism

$$N : \Gamma^c(V) \to \mathbb{C}^*.$$

Define

$$Pin^c(V) = \{x \in \Gamma^c(V) \ ; \ |N(x)| = 1\}.$$

We leave the reader to check the following result.

Proposition 10.1.35. *There exists a short exact sequence*

$$1 \to S^1 \to Pin^c(V) \to O(V) \to 1.$$

Corollary 10.1.36. *There exists a natural isomorphism*

$$Pin^c(V) \cong (Pin(V) \times S^1)/\sim$$

where "$\sim$" is the equivalence relation

$$(x, z) \sim (-x, -z) \quad \forall (x, z) \in Pin(V) \times S^1.$$

Proof The inclusions $Pin(V) \subset Cl(V)$, $S^1 \subset \mathbb{C}$ induce an inclusion

$$(Pin(V) \times S^1)/\sim \to \mathbb{C}l(V).$$

The image of this morphism lies obviously in $\Gamma^c(V) \cap \{|N| = 1\}$ so that $(Pin(V) \times S^1)/\sim$ can be viewed as a subgroup of $Pin^c(V)$. The sought for isomorphism now follows from the exact sequence

$$1 \to S^1 \to (Pin(V) \times S^1)/\sim \to O(V) \to 1.$$

□

We define $Spin^c(V)$ as the inverse image of $SO(V)$ via the morphism

$$\rho^c : Pin^c(V) \to O(V).$$

Arguing as in the above corollary we deduce

$$Spin^c(V) \cong (Spin(V) \times S^1)/\sim \cong (Spin(V) \times S^1)/\mathbb{Z}_2.$$

Exercise 10.1.11. Prove $Spin^c(V)$ satisfies all the conditions outlined in Subsection 10.1.5.

□

Assume now $\dim V$ is even. Then any any isomorphism

$$\mathbb{C}l(V) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}(V))$$

induces a complex unitary representation

$$Spin^c(V) \to \mathrm{Aut}\,(\mathbb{S}(V))$$

called the complex spinorial representation of $Spin^c$. It is not irreducible since (once we fix an orientation on V), $\mathrm{End}\,(\mathbb{S}(V))$ has a natural superstructure and by definition $Spin^c$ acts through even automorphism. As in the real case, $\mathbb{S}(V)$ splits into a direct sum of irreducible representations $\mathbb{S}^{\pm}(V)$.

Any complex structure J on V defines two things:
(i) a canonical orientation on V and
(ii) a natural subgroup

$$U(V, J) = \{T \in SO(V)\ ;\ [T, J] = 0\} \subset SO(V).$$

Denote by $\imath_J : U(V, J) \to SO(V)$ the inclusion map.

Proposition 10.1.37. *There exists a natural group morphism*

$$\xi_J : U(V,J) \to Spin^c(V)$$

such that the diagram below is commutative.

$$\begin{tikzcd} U(V,J) \arrow[r, "\xi_J"] \arrow[dr, "\imath_J"'] & Spin^c(V) \arrow[d, "\rho^c"] \\ & SO(V) \end{tikzcd}$$

Proof Let $\omega \in U(V)$ and consider a path $\gamma : [0,1] \to U(V)$ connecting $\mathbf{1}$ to ω. Via the inclusion $U(V) \hookrightarrow SO(V)$ we may regard γ as a path in $SO(V)$. As such, it admits a unique lift $\tilde{\gamma} : [0,1] \to Spin(V)$ such that $\tilde{\gamma}(0) = \mathbf{1}$.

Using the double cover $S^1 \to S^1$ $z \mapsto z^2$ we can find a path $\delta(t)$ in S^1 such that

$$\delta(0) = 1 \text{ and } \delta^2(t) = \det\gamma(t).$$

Define $\xi(\omega)$ to be the image of $(\tilde{\gamma}(1), \delta(1))$ in $Spin^c(V)$. We have to check that
(i) ξ is well defined and
(ii) σ is a smooth group morphism.

To prove (i) we need to show that if $\eta : [0,1] \to U(V)$ is a different path connecting $\mathbf{1}$ to ω and $\lambda : [0,1] \to S^1$ is such that $\lambda(0) = 1$ and $\lambda(t) = \det\eta(t)^2$ then

$$(\tilde{\eta}(1), \lambda(1)) = (\tilde{\gamma}(1), \delta(1)) \text{ in } Spin^c(V).$$

The elements $\tilde{\gamma}(1)$ and $\tilde{\eta}(1)$ lie in the same fiber of the covering $Spin(V) \xrightarrow{\rho} SO(V)$ so that they differ by an element in $\ker\rho$. Hence

$$\tilde{\gamma}(1) = \epsilon\tilde{\eta}(1) \quad \epsilon = \pm\mathbf{1}.$$

We can identify ϵ as the holonomy of the covering $Spin(V) \to SO(V)$ along the loop $\gamma * \eta^-$ which goes from $\mathbf{1}$ to ω along γ and back to $\mathbf{1}$ along $\eta^-(t) = \eta(1-t)$.

The map $\det : U(V) \to S^1$ induces an isomorphism between the fundamental groups (see Exercise 6.2.10 of Subsection 6.2.5). Hence in describing the holonomy ϵ it suffices to replace the loop $\gamma * \eta^- \subset U(V)$ by any loop $\nu(t)$ such that

$$\det\nu(t) = \det(\gamma * \eta^-) = \Delta(t) \in S^1.$$

Such a loop will be homotopic to $\gamma * \eta^{-1}$ in $U(V)$ and thus in $SO(V)$ as well. Select $\nu(t)$ of the form

$$\nu(t)e_1 = \Delta(t)e_1, \;\; \nu(t)e_i = e_i \;\forall i \geq 2$$

where (e_i) is a complex, orthonormal basis of (V,J). Set $f_i = Je_i$. With respect to the real basis $(e_1, f_1; e_2, f_2; \cdots)$ $\nu(t)$ (viewed as an element of $SO(V)$) has the matrix description

$$\left[\begin{array}{ccc} \cos\theta(t) & -\sin\theta(t) & \cdots \\ \sin\theta(t) & \cos\theta(t) & \cdots \\ \vdots & \vdots & \mathbf{Id} \end{array}\right],$$

where $\theta : [0,1] \to \mathbb{R}$ is a continuous map such that $\Delta(t) = e^{\mathbf{i}\theta(t)}$. The lift of $\nu(t)$ to $Spin(V)$ has the form

$$\tilde{\nu}(t) = (\cos\frac{\theta(t)}{2} - e_1 f_1 \sin\frac{\theta(t)}{2}).$$

We see that the holonomy defined by $\tilde{\nu}(t)$ is nontrivial if and only if the holonomy of the loop $t \mapsto \delta(t)$ in the double cover $S^1 \xrightarrow{z^2} S^1$ is nontrivial. This means that $\delta(1)$ and $\lambda(1)$ differ by the same element of $\mathbb{Z}_2$ as $\tilde{\gamma}(1)$ and $\tilde{\eta}(1)$. This proves ξ is well defined. We leave the reader to check that ξ is indeed a smooth morphism of groups.

□

10.1.8 Low dimensional examples

In low dimensions the objects discussed in the previous subsections can be given more suggestive interpretations. In this subsection we will describe some of these interpretations.

$\boxed{n=1}$ The Clifford algebra Cl_1 is isomorphic with the field of complex numbers $\mathbb{C}$. The $\mathbb{Z}_2$-grading is $\mathfrak{Re}\,\mathbb{C} \oplus \mathfrak{Im}\,\mathbb{C}$. The group $Spin(1)$ is isomorphic with $\mathbb{Z}_2$.

$\boxed{n=2}$ The Clifford algebra Cl_2 is isomorphic with the algebra of quaternions $\mathbb{H}$. This can be seen by choosing an orthonormal basis $\{e_1, e_2\}$ in $\mathbb{R}^2$. The isomorphism is given by

$$1 \mapsto 1,\;\; e_1 \mapsto \mathbf{i},\;\; e_2 \mapsto \mathbf{j},\;\; e_1e_2 \mapsto \mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the imaginary units in $\mathbb{H}$. Note that

$$Spin(2) = \{a + b\mathbf{k}\ ;\ a, b \in \mathbb{R}\ a^2 + b^2 = 1\} \cong S^1.$$

The natural map $Spin(1) \to SO(2) \cong S^1$ takes the form $e^{\mathbf{i}\theta} \mapsto e^{2\mathbf{i}\theta}$.

$\boxed{n=3}$ As an (ungraded) algebra Cl_3 is isomorphic to the direct sum $\mathbb{H}\oplus\mathbb{H}$. More relevant is the isomorphism $Cl_3^{even} \cong Cl_2 \cong \mathbb{H}$ given by

$$1 \mapsto 1,\;\; e_1e_2 \mapsto \mathbf{i},\;\; e_2e_3 \mapsto \mathbf{j},\;\; e_3e_1 \mapsto \mathbf{k}$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{R}^3$. Under this identification the operation $x \mapsto x^\flat$ coincides with the conjugation in $\mathbb{H}$

$$x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto \overline{x} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

In particular the spinorial norm coincides with the usual norm on $\mathbb{H}$

$$N(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = a^2 + b^2 + c^2 + d^2.$$

Thus any $x \in Cl_3^{even} \setminus \{0\}$ is invertible and

$$x^{-1} = \frac{1}{N(x)}x^\flat.$$

Moreover, a simple computation shows that $x\mathbb{R}^3x^{-1} \subset \mathbb{R}^3 \; \forall x \in Cl_3^{even} \setminus \{0\}$ so that

$$\Gamma^0(\mathbb{R}^3) \cong \mathbb{H} \setminus \{0\}.$$

Hence

$$Spin(3) \cong \{x \in \mathbb{H} \, ; \, |x| = 1\} \cong SU(2).$$

The natural map $Spin(3) \to SO(3)$ is precisely the map described in the Exercise 6.2.4 of Subsection 6.2.1.

The isomorphism $Spin(3) \cong SU(2)$ can be visualized by writing each $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ as

$$q = u + \mathbf{j}v \;\; u = a + b\mathbf{i}, \; v = (c - d\mathbf{i}) \in \mathbb{C}.$$

To a quaternion $q = u + \mathbf{j}v$ one associates the 2×2 complex matrix

$$S_q = \begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix} \in SU(2).$$

Note that $S_{\bar{q}} = S_q^*$, $\forall q \in \mathbb{H}$. For each quaternion $q \in \mathbb{H}$ we denote by L_q (resp. R_q) the left (resp. right) multiplication. The right multiplication by $\mathbf{i}$ defines a complex structure on $\mathbb{H}$. Define $T : \mathbb{H} \to \mathbb{C}^2$ by

$$q = u + \mathbf{j}v \mapsto Tq = \begin{bmatrix} u \\ v \end{bmatrix}.$$

A simple computation shows that

$$T(R_{\mathbf{i}}q) = \mathbf{i}Tq$$

i.e. T is a complex linear map. Moreover $\forall q \in Spin(3) \cong S^3$ the matrix S_q is in $SU(2)$ and the diagram below is commutative.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{T} & \mathbb{C}^2 \\ {\scriptstyle L_q}\downarrow & & {\scriptstyle S_q}\downarrow \\ \mathbb{H} & \xrightarrow{T} & \mathbb{C}^2 \end{array}$$

In other words, the representation

$$Spin(3) \ni q \mapsto L_q \in GL_{\mathbb{C}}(\mathbb{H})$$

of $Spin(3)$ is isomorphic with the tautological representation of $SU(2)$ on $\mathbb{C}^2$. On the other hand the correspondences

$$e_1e_2 \mapsto S_{\mathbf{i}} \oplus S_{\mathbf{i}} \in \mathrm{End}\,(\mathbb{C}^2) \oplus \mathrm{End}\,(\mathbb{C}^2)$$

$$e_2e_3 \mapsto S_{\mathbf{j}} \oplus S_{\mathbf{j}} \in \mathrm{End}\,(\mathbb{C}^2) \oplus \mathrm{End}\,(\mathbb{C}^2)$$

$$e_3e_1 \mapsto S_{\mathbf{k}} \oplus S_{\mathbf{k}} \in \mathrm{End}\,(\mathbb{C}^2) \oplus \mathrm{End}\,(\mathbb{C}^2)$$

$$e_1e_2e_3 \mapsto R = \mathbf{1}_{\mathbb{C}^2} \oplus (-\mathbf{1}_{\mathbb{C}^2}) \in \mathrm{End}\,(\mathbb{C}^2) \oplus \mathrm{End}\,(\mathbb{C}^2)$$

extend to an isomorphism of algebras $\mathbb{C}l_3 \to \mathrm{End}\,(\mathbb{C}^2) \oplus \mathrm{End}\,(\mathbb{C}^2)$. This proves that the tautological representation of $SU(2)$ is precisely the complex spinorial representation $\mathbb{S}_3$

From the equalities

$$[R_{\mathbf{j}}, L_q] = \{R_{\mathbf{j}}, R_{\mathbf{i}}\} = 0$$

we deduce that $R_{\mathbf{j}}$ defines an isomorphism of $Spin(3)$-modules

$$R_{\mathbf{j}} : \mathbb{S}_3 \to \overline{\mathbb{S}}_3.$$

This implies there exists a $Spin(3)$-invariant bilinear map

$$\beta : \mathbb{S}_3 \times \mathbb{S}_3 \to \mathbb{C}.$$

This plays an important part in the formulation of the recently introduced Seiberg-Witten equations (see [78]).

Exercise 10.1.12. The left multiplication by $\mathbf{i}$ introduces a different complex structure on $\mathbb{H}$. Prove the representation $Spin(3) \ni q \mapsto (R_{\overline{q}} : \mathbb{H} \to \mathbb{H})$ is
(i) complex with respect to the above introduced complex structure on $\mathbb{H}$ and
(ii) it is isomorphic with complex spinorial representation described by the left multiplication. □

$\boxed{n = 4}$ Cl_4 can be realized as the algebra of 2×2 matrices with entries in $\mathbb{H}$. To describe this isomorphism we have to start from a natural embedding $\mathbb{R}^4 \hookrightarrow M_2(\mathbb{H})$ given by the correspondence

$$\mathbb{H} \cong \mathbb{R}^4 \ni x \mapsto \left[\begin{array}{cc} 0 & -x \\ \overline{x} & 0 \end{array}\right]$$

A simple computation shows that the conditions in the universality property of a Clifford algebra are satisfied and this correspondence extends to a bona-fide morphism of algebras $Cl_4 \to M_2(\mathbb{H})$. We let the reader check this morphism is also injective and a dimension count concludes it must also be surjective.

Proposition 10.1.38. $Spin(4) \cong SU(2) \times SU(2)$.

Proof We will use the description of $Spin(4)$ as the universal (double-cover) of $SO(4)$ so we will explicitly produce a smooth $2-1$ group morphism $SU(2) \times SU(2) \to SO(4)$.

Again we think of $SU(2)$ as the group of unit quaternions. Thus each pair $(q_1, q_2) \in SU(2) \times SU(2)$ defines a real linear map

$$T_{q_1,q_2} : \mathbb{H} \to \mathbb{H} \quad x \mapsto T_{q_1,q_2}x = q_1 x \overline{q}_2.$$

Clearly $|x| = |q_1| \cdot |x| \cdot |\overline{q_2}| = |T_{q_1,q_2}x|$ $\forall x \in \mathbb{H}$ so that each T_{q_1,q_2} is an orthogonal transformation of $\mathbb{H}$. Since $SU(2) \times SU(2)$ is connected all the operators T_{q_1,q_2} belong to the component of $O(4)$ containing $\mathbf{1}$ i.e. T defines an (obviously smooth) group morphism

$$T : SU(2) \times SU(2) \to SO(4).$$

Note that $\ker T = \{1, -1\}$ so that T is $2-1$. In order to prove T is a double cover it suffices to show it is onto. This follows easily by noticing T is an immersion (*verify this* !) so that its range must contain an entire neighborhood of $\mathbf{1} \in SO(4)$. Since the range of T is closed (*verify this*!) we conclude that T must be onto because the closure of the subgroup (algebraically) generated by an open set in a connected Lie group coincides with the group itself (see Subsection 1.2.3).

□

The above result shows that

$$\underline{so}(4) \cong \underline{spin}(4) \cong \underline{su}(2) \oplus \underline{su}(2) \cong \underline{so}(3) \oplus \underline{so}(3).$$

Exercise 10.1.13. Using the identification $Cl_4 \cong M_2(\mathbb{H})$ show that $Spin(4)$ corresponds to the subgroup

$$\{diag\,(p,q)\ ;\ p,q \in \mathbb{H}, |p| = |q| = 1\} \subset M_2(\mathbb{H}).$$

□

Exercise 10.1.14. Let $\{e_1, e_2, e_3, e_4\}$ be an oriented orthonormal basis of $\mathbb{R}^4$. Let $*$ denote the Hodge operator defined by the canonical metric and the above chosen orientation. Note that

$$* : \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$$

is involutive $*^2 = \mathbf{id}$ so that we can split Λ^2 into the ± 1 eigenspaces of $*$

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4.$$

(a) Show that

$$\Lambda^2_\pm = \mathrm{span}_{\mathbb{R}}\{\eta_1^\pm,\ \eta_2^\pm,\ \eta_3^\pm\}$$

where

$$\eta_1^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$$

$$\eta_2^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2)$$

$$\eta_3^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

(b) Show that the above splitting of $\Lambda^2\mathbb{R}^4$ corresponds to the splitting $\underline{so}(4) = \underline{so}(3) \oplus \underline{so}(3)$ under the natural identification $\Lambda^2\mathbb{R}^4 \cong \underline{so}(4)$. □

To obtain an explicit realization of the complex spinorial representations $\mathbb{S}_4^\pm$ we need to describe a concrete realization of the complexification $\mathbb{C}l_4$. We start from the morphism of $\mathbb{R}$-algebras

$$\mathbb{H} \ni x = u + \mathbf{j}v \mapsto S_x = \begin{bmatrix} u & -\bar{v} \\ v & \overline{u} \end{bmatrix}$$

This extends by complexification to an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C}).$$

(*Verify this*!) We now use this isomorphism to achieve the identification.

$$\mathrm{End}_{\mathbb{H}}(\mathbb{H} \oplus \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2 \oplus \mathbb{C}^2).$$

The embedding $\mathbb{R}^4 \to \mathbb{C}l_4$ now takes the form

$$\mathbb{H} \cong \mathbb{R}^4 \ni x \mapsto T_x = \begin{bmatrix} 0 & -S_{\bar{x}} \\ S_x & 0 \end{bmatrix} \in \mathrm{End}\,(\mathbb{C}^2 \oplus \mathbb{C}^2).$$

Note that the chirality operator $\Gamma = -e_1e_2e_3e_4$ is represented by the canonical involution

$$\Gamma \mapsto \mathbf{1}_{\mathbb{C}^2} \oplus (-\mathbf{1}_{\mathbb{C}^2}).$$

We deduce the $\mathbb{S}_4^{\pm}$ representations of $Spin(4) = SU(2) \times SU(2)$ are given by

$$\mathbb{S}_4^+ : SU(2) \times SU(2) \ni (p, q) \mapsto p \in GL(2; \mathbb{C})$$

$$\mathbb{S}_4^- : SU(2) \times SU(2) \ni (p, q) \mapsto q \in GL(2; \mathbb{C}).$$

As for $Spin(3)$ these representations can be given quaternionic descriptions.

Exercise 10.1.15. The space $\mathbb{R}^4 \cong \mathbb{H}$ has a canonical complex structure defined by $R_{\mathbf{i}}$ which defines (following the prescriptions in §10.1.3) an isomorphism $\mathbf{c} : \mathbb{C}l_4 \to \mathrm{End}\,(\Lambda^*\mathbb{C}^2)$. Identify $\mathbb{C}^2$ in the obvious way with $\Lambda^1\mathbb{C}^2$ and with $\Lambda^{even}\mathbb{C}^2$ via $e_1 \mapsto 1 \in \Lambda^0\mathbb{C}^2,\ e_2 \mapsto e_1 \wedge e_2$. Show that under this identification we have

$$\mathbf{c}(x) = T_x \ \ \forall x \in \mathbb{R}^4 \cong (\mathbb{H}, R_{\mathbf{i}}) \cong \mathbb{C}^2$$

where T_x is the odd endomorphism of $\mathbb{C}^2 \oplus \mathbb{C}^2$ defined above. □

Exercise 10.1.16. Let V be a 4-dimensional oriented Euclidean space. Denote by $\mathfrak{q} : \Lambda^*V \to Cl(V)$ the quantization map and fix an isomorphism $\Delta : \mathbb{C}l(V) \to \mathrm{End}\,(\mathbb{S}(V))$ of $\mathbb{Z}_2$-graded algebras. Show that for any $\eta \in \Lambda^2_+(V)$ the image $\Delta \circ \mathfrak{q}(\eta) \in \mathrm{End}\,(\mathbb{S}(V))$ is an endomorphism of the form $T \oplus 0 \in \mathrm{End}\,(\mathbb{S}^+(V)) \oplus \mathrm{End}\,(\mathbb{S}^-(V))$. □

Exercise 10.1.17. Denote by V a 4-dimensional oriented Euclidean space.
(a) Show that the representation $\mathbb{S}_4^+ \otimes \mathbb{S}_4^+$of $Spin(4)$ descends to a representation of $SO(4)$ and moreover

$$\mathbb{S}^+(V) \otimes_{\mathbb{C}} \mathbb{S}^+(V) \cong (\Lambda^0(V) \oplus \Lambda^2_+(V)) \otimes_{\mathbb{R}} \mathbb{C}$$

as $SO(4)$ representations.
(b) Show that $\mathbb{S}^+(V) \cong \bar{\mathbb{S}}^+(V)$ as $Spin(4)$ modules.
(c)The above isomorphism defines an element $\phi \in \mathbb{S}^+(V) \otimes_{\mathbb{C}} \mathbb{S}^+(V)$. Show that via the correspondence at (a) the isomorphism ϕ spans $\Lambda^0(V)$. □

10.1.9 Dirac bundles

In this subsection we discuss a distinguished type of Clifford bundle which is both frequently encountered in applications and is rich in geometric informations. We will touch only the general aspects. The special characteristics of the most important concrete examples are studied in some detail in the following section. In the sequel all Clifford bundles will be assumed to be complex.

Definition 10.1.39. *Let $E \to M$ be a Clifford bundle over the oriented Riemann manifold (M, g). A* Dirac structure *on E is a pair (h, ∇) consisting of a Hermitian metric h on E and a* Clifford connection *i.e. a connection ∇ compatible with h such that*
(i) $\forall \alpha \in \Omega^1(M)$ the Clifford multiplication by α is a skew-Hermitian endomorphism of E.
(ii) $\forall \alpha \in \Omega^1(M)$, $X \in \mathrm{Vect}\,(M)$, $u \in C^\infty(E)$

$$\nabla_X(\mathbf{c}(\alpha)u) = \mathbf{c}(\nabla^M_X \alpha)u + \mathbf{c}(\alpha)(\nabla_X u)$$

where ∇^M denotes the Levi-Civita connection on T^*M. (This condition means the Clifford multiplication is covariant constant). A pair (Clifford bundle, Dirac structure) will be called a Dirac bundle. *A $\mathbb{Z}_2$-grading on a Dirac bundle (E, h, ∇) is a $\mathbb{Z}_2$ grading of the underlying Clifford structure $E = E_0 \oplus E_1$ such that $h = h_0 \oplus h_1$ and $\nabla = \nabla^0 \oplus \nabla^1$ where h_i and resp. ∇^i is a metric (resp. a metric connection) on E_i.*

The next result addresses the fundamental consistency question: do there exist Dirac bundles?

Proposition 10.1.40. *Let $E \to M$ be a Clifford bundle over the oriented Riemann manifold (M, g). Then there exist Dirac structures on E.*

Proof Denote by $\mathcal{D}$ the (possible empty) family of Dirac structure on E. Note that if $(h_i, \nabla^i) \in \mathcal{D}$ $(i = 1, 2)$ and $f \in C^\infty(M)$ then

$$(fh_1 + (1-f)h_2, f\nabla^1 + (1-f)\nabla^2) \in \mathcal{D}.$$

This elementary fact shows the existence of Dirac structures is essentially a local issue: local Dirac structures can be patched-up via partitions of unity.

Thus it suffices to consider only the case when M is an open subset of $\mathbb{R}^n$ and E is a trivial vector bundle. On the other hand we cannot assume the metric g is also trivial (i.e. Euclidean) since the local obstructions given by the Riemann curvature cannot be removed. We will distinguish two cases.
A. $n = \dim M$ is even. The proof will be completed in three steps.

Step 1. *A special example.* Fix a selfadjoint endomorphism

$$\mathbf{c} : \mathbb{C}l_n \to \mathrm{End}\,(\mathbb{S}_n),$$

where $\mathbb{S}_n$ denotes the fundamental spinor representation. We will continue to denote by $\mathbf{c}$ the restriction to $Spin(n) \hookrightarrow \mathbb{C}l_n$.

Fix a global, oriented, orthonormal frame (e_i) of TM and denote by (e^j) its dual coframe. Denote by $\omega = (\omega_{ij})$ the connection 1-form of the Levi-Civita connection on T^*M with respect to this moving frame i.e.

$$\nabla e^j = \omega e^j = \sum_i \omega_{ij} \otimes e^i, \;\; \omega \in \Omega^(M) \otimes \underline{so}(n).$$

Using the canonical isomorphism $\rho_* : \underline{spin}(n) \to \underline{so}(n)$ we define

$$\tilde{\omega} = \rho_*^{-1}(\omega) = -\frac{1}{2}\sum_{i<j} \omega^{ij} e^i \cdot e^j \in \Omega^1(M) \otimes \underline{spin}(n).$$

This defines a connection $\nabla^{\mathbb{S}}$ on the trivial vector bundle $\underline{\mathbb{S}}_M$ by

$$\nabla^{\mathbb{S}} u = du - \frac{1}{2}\sum_{i<j} \omega_{ij} \otimes \mathbf{c}(e^i)\cdot \mathbf{c}(e^j)u \quad \forall u \in C^\infty(\underline{\mathbb{S}}_M).$$

The considerations in §10.1.5 show that ∇^S is indeed a Clifford connection so that $\underline{\mathbb{S}}_M$ is a Dirac bundle.

Step 2. *Constructing general Dirac bundles* Fix a Dirac bundle (E, h_E, ∇^E) over M. For any Hermitian vector bundle (W, h_W) equipped with a Hermitian connection ∇^W we can construct the tensor product $E \otimes W$ equipped with the metric $h_E \otimes h_W$ and the product connection $\nabla^{E\otimes W}$. These two data define a Dirac structure on $E \otimes W$ (Exercise 10.1.18 at the end of this subsection).

The representation theory of the Clifford algebra $\mathbb{C}l_n$ with n even shows that any Clifford bundle over M must be a twisting $\mathbb{S} \otimes W$ of the spinor bundle $\mathbb{S}$. This completes the proof of the proposition when $\dim M$ is even.

Step 3 *Conclusion.* The odd case is dealt with similarly using the different representation theory of the $\mathbb{C}_{2k+1}$. Now instead of one generating model $\mathbb{S}$ there are two but the proof is conceptually identical. The straightforward details are left to the reader.

□

Denote by (E, h, ∇) a Dirac bundle over the oriented Riemann manifold (M, g). The exists a Dirac operator on E canonically associated to this structure

$$D = c \circ \nabla : C^\infty(E) \xrightarrow{\nabla} C^\infty(T^*M \otimes E) \xrightarrow{c} C^\infty(E).$$

A Dirac operator associated to a Dirac structure is said to be a *geometric Dirac operator.*

Proposition 10.1.41. *Any geometric Dirac operator is formally selfadjoint.*

Proof The assertion in the above proposition is local so we can work with local orthonormal moving frames. Fix $x_0 \in M$ and denote by (x^i) a collection of normal coordinates near x_0. Set $e_i = \frac{\partial}{\partial x_i}$. Denote by (e^i) the dual coframe of (e_i). If (h, ∇) is a Dirac structure on the Clifford bundle E then *at* x_0 the associated Dirac operator can be described as

$$D = \sum_i \mathbf{c}(e^i)\nabla_i \quad (\nabla_i = \nabla_{e_i}).$$

We deduce

$$D^* = (\nabla_i)^* \mathbf{c}(e^i)^*.$$

Since the connection ∇ is compatible with h and $\mathbf{div}(e_i)\,|_{x_0}= 0$ we deduce

$$(\nabla_i)^* = -\nabla_i$$

while

$$\mathbf{c}(e^i)^* = -\mathbf{c}(e^i)$$

since the Clifford multiplication is skew-Hermitian. Hence, *at* x_0

$$D^* = \sum_i \nabla_i \circ \mathbf{c}(e^i) = \sum_i [\nabla_i, \mathbf{c}(e^i)] + D.$$

Since the Clifford multiplication is covariant constant and $(\nabla_i^M e^i)\,|_{x_0} = 0$ we conclude

$$[\nabla_i, \mathbf{c}(e^i)]\,|_{x_0} = 0 \quad \forall i.$$

This concludes the proof of the proposition.

□

Let D be a geometric operator associated to the Dirac bundle (E, h, ∇). By definition D^2 is a generalized Laplacian and consequently

$$D^2 = \tilde{\nabla}^* \tilde{\nabla} + \mathcal{R},$$

where $\tilde{\nabla}$ is a connection on E and $\mathcal{R}$ is the Weitzenböck remainder $-$ an endomorphism of E. For geometric Dirac operators, $\tilde{\nabla} = \nabla$ (!!!) and this remainder can be given a very explicit description with remarkable geometric consequences. To formulate it we need a little foundational work.

Let (E, h, ∇) be a Dirac bundle over the oriented Riemann manifold (M, g). Denote by $Cliff(M) \to M$ the bundle of Clifford algebras generated by (T^*M, g).

The curvature $F(\nabla)$ of ∇ is a section of $\Lambda^2 T^*M \otimes \mathrm{End}\,(E)$. Using the quantization map $\mathfrak{q} : \Lambda^* T^*M \to Cliff(M)$ we get a section

$$\mathfrak{q}(F) \in Cliff(M) \otimes \mathrm{End}\,(E).$$

On the other hand, the Clifford multiplication $c : Cliff(M) \to \mathrm{End}\,(E)$ defines a linear map

$$Cliff(M) \otimes \mathrm{End}\,(E) \to \mathrm{End}(E) \quad \omega \otimes T \mapsto \mathbf{c}(\omega) \circ T.$$

This map associates to the element $\mathfrak{q}(F)$ an endomorphism of E which we denote by $\mathfrak{c}(F)$. If (e^i) is a local, oriented, orthonormal moving frame for T^*M then we can write

$$F(\nabla) = \sum_{i<j} e^i \wedge e^j \otimes F_{ij}$$

and

$$\mathfrak{c}(F) = \sum_{i<j} \mathbf{c}(e^i)\mathbf{c}(e^j) F_{ij} = \frac{1}{2} \sum_{i,j} \mathbf{c}(e^i)\mathbf{c}(e^j) F_{ij}.$$

Theorem 10.1.42. (Bochner-Weitzenböck) *Let D be the geometric Dirac operator associated to the Dirac bundle (E, h, ∇) over the oriented Riemann manifold (M, g). Then*

$$D^2 = \nabla^* \nabla + \mathfrak{c}(F(\nabla)).$$

Proof Fix $x \in M$ and then choose an oriented, local orthonormal moving frame (e_i) of TM near x such that

$$[e_i, e_j]\,|_x = (\nabla^M e_i)\,|_x = 0,$$

where ∇^M denotes the Levi-Civita connection. Such a choice is always possible because the torsion of the Levi-Civita connection is zero. Finally denote by (e^i) the dual coframe of (e_i). Then

$$D^2\,|_x = \sum_i \mathbf{c}(e^i)\nabla_i \left(\sum_j \mathbf{c}(e^j)\nabla_j \right).$$

Since $[\nabla_i, \mathbf{c}(e_j)]\,|_x = 0$ we deduce

$$D^2\,|_x = \sum_{i,j} \mathbf{c}(e^i)\mathbf{c}(e^j)\nabla_i\nabla_j = -\sum_i \nabla_i^2 + \sum_{i\neq j} \mathbf{c}(e^i)\mathbf{c}(e^j)\nabla_i\nabla_j$$

$$= -\sum_i \nabla_i^2 + \sum_{i<j} \mathbf{c}(e^i)\mathbf{c}(e^j)[\nabla_i, \nabla_j]$$

$$= -\sum_i \nabla_i^2 + \sum_{i<j} \mathbf{c}(e^i)\mathbf{c}(e^j)F_{ij}(\nabla).$$

We want to emphasize again *the above equalities hold only at x*. The theorem now follows by observing that

$$(\nabla^*\nabla)\,|_x = -\left(\sum_i \nabla_i^2 \right)|_x\,.$$

□

Exercise 10.1.18. Let (E, h, ∇) be a Dirac bundle over the oriented Riemann manifold (M, g) with associated Dirac operator D. Consider a Hermitian bundle $W \to M$ and a connection ∇^W compatible with the Hermitian metric h_W.
(a) Show that $(E \otimes W, h \otimes h_W, \hat{\nabla} = \nabla \otimes \mathbf{1}_W + \mathbf{1}_E \otimes \nabla^W)$ defines a Dirac structure on $E \otimes W$ in which the Clifford multiplication by $\alpha \in \Omega^1(M)$ is defined by

$$\mathbf{c}(\alpha)(e \otimes w) = (\mathbf{c}(\alpha)e) \otimes w, \quad e \in C^\infty(E),\ w \in C^\infty(W).$$

We denote by D_W the corresponding geometric Dirac operator.
(b) Denote by $\mathfrak{c}_W(F(\nabla^W))$ the endomorphism of $E \otimes W$ defined by the sequence

$$F(\nabla^W) \in C^\infty(\Lambda^2 T^*M \otimes \mathrm{End}\,(W)) \overset{\mathfrak{q}}{\mapsto} C^\infty(Cl(T^*M) \otimes \mathrm{End}\,(W)) \overset{c}{\mapsto} C^\infty(\mathrm{End}\,(E \otimes W)).$$

Show that

$$D_W^2 = \hat{\nabla}^*\hat{\nabla} + \mathfrak{c}(F(\nabla)) + \mathfrak{c}_W(F(\nabla^W)).$$

□

10.2 Fundamental examples

This section is entirely devoted to the presentation of some fundamental examples of Dirac operators. More specifically we will discuss the Hodge-DeRham operator, the Dolbeault operator the *spin* and *spin*^{c} Dirac. We will provide more concrete descriptions of the Weitzenböck remainder presented in Subsection 10.1.9 and show some of its uses in establishing vanishing theorems.

10.2.1 The Hodge-DeRham operator

Let (M, g) be an oriented Riemann manifold and set

$$\Lambda^*_{\mathbb{C}} T^*M = \Lambda^* T^*M \otimes \mathbb{C}.$$

For simplicity we continue to denote by $\Omega^*(M)$ the space of smooth differential forms on M with complex coefficients. We have already seen that the Hodge-DeRham operator

$$d + d^* : \Omega^*(M) \to \Omega^*(M)$$

is a Dirac operator. In fact, we will prove this operator is a geometric Dirac operator. Continue to denote by g the Hermitian metric induced by the metric g on the complexification $\Lambda^*_{\mathbb{C}} T^*M$. ∇^g will denote the Levi-Civita and its associates. When we want to be more specific about which Levi-Civita connection we are using at a given moment we will indicate the bundle it acts on as a superscript. E.g., ∇^{T^*M} is the Levi-Civita connection on T^*M.

Proposition 10.2.1. *The pair (g, ∇^g) defines a Dirac structure on the Clifford bundle $\Lambda_{\mathbb{C}} T^*M$ and $d + d^*$ is the associated Dirac operator.*

Proof In Subsection 4.1.5 we have proved that d can be alternatively described as the composition

$$C^\infty(\Lambda^* T^*M) \xrightarrow{\nabla} C^\infty(T^*M \otimes \Lambda^* T^*M) \xrightarrow{\varepsilon} C^\infty(\Lambda^* T^*M),$$

where ε denotes the exterior multiplication map. Thus

$$d + d^* = \varepsilon \circ \nabla + \nabla^* \circ \varepsilon^*.$$

If $X_1, \cdots, X_n$ is a local orthonormal frame of TM and $\theta^1, \cdots, \theta^n$ is its dual coframe then for any ordered multi-index I we have

$$\varepsilon^*(\theta^I) = \sum_j i_{X_j} \theta^I.$$

Thus, for any $\omega \in C^\infty(\Lambda^* T^*M)$

$$\nabla^* \circ \varepsilon^* = -\sum_{,} \nabla_{X_k} i_{X_k} + \mathbf{div}\,(X_k) i_{X_k} \omega.$$

Fortunately, we have the freedom to choose the frame (X_k) in any manner we find convenient. Fix an *arbitrary* point $x_0 \in M$ and choose (X_k) such that at x_0 we have $X_k =$

$\frac{\partial}{\partial x_k}$, where (x^k) denotes a collection of normal coordinates near x_0. With such a choice $\mathbf{div}\,(X_k) = 0$ at x_0 and thus

$$d^*\omega\,|_{x_0} = -\sum_k i_{\partial_k} \nabla_{\partial_k} \omega.$$

This shows $d + d^*$ can be written as the composition

$$C^\infty(\Lambda^* T^* M) \stackrel{\nabla}{\to} C^\infty(T^* M \otimes \Lambda^* T^* M) \stackrel{c}{\to} C^\infty(\Lambda^* T^* M),$$

where $\mathbf{c}$ denotes the usual Clifford multiplication on an exterior algebra. We leave the reader to verify that the Levi-Civita connection on $\Lambda^* T^* M$ is indeed a Clifford connection i.e. the Clifford multiplication is covariant constant. This shows that $d + d^*$ is a geometric Dirac operator.

□

We want to spend some time elucidating the structure of the Weitzenböck remainder. First of all, we need a better description of the curvature of ∇^g viewed as a connection on $\Lambda^* T^* M$.

Denote by R the Riemann curvature tensor, i.e. the curvature of the Levi-Civita connection

$$\nabla^g : C^\infty(TM) \to C^\infty(T^* M \otimes TM).$$

Thus R is a bundle morphism

$$R : C^\infty(TM) \to C^\infty(\Lambda^2 T^* M \otimes TM).$$

We have a dual morphism

$$\tilde{R} : C^\infty(T^* M) \to C^\infty(\Lambda^2 T^* M \otimes T^* M)$$

uniquely determined by the equality

$$(\tilde{R}(Y, Z)\alpha)(X) = -\alpha(R(Y, Z)X) \quad \forall \alpha \in \Omega^1(M)\ X,\ Y,\ Z \in \mathrm{Vect}\,(M).$$

Lemma 10.2.2. *$\tilde{R}$ is the curvature of the Levi-Civita connection ∇^{T^*M}.*

Proof The Levi-Civita connection on $T^* M$ is determined by the equalities

$$(\nabla_Z^{T^*M} \alpha)(X) = Z \cdot \alpha(X) - \alpha(\nabla_Z^{TM} X) \quad \forall \alpha \in \Omega^1(M),\ X,\ Z \in \mathrm{Vect}\,(M).$$

Derivating along $Y \in \mathrm{Vect}\,(M)$ we get

$$(\nabla_Y \nabla_Z \alpha)(X) = Y \cdot (\nabla_Z \alpha)(X) - (\nabla_Z \alpha)(\nabla_Y X)$$
$$= Y \cdot Z \cdot \alpha(X) - Y \cdot \alpha(\nabla_Z X) - Z \cdot \alpha(\nabla_Y X) - \alpha(\nabla_Z \nabla_Y X).$$

Similar computations give $\nabla_Z \nabla_Y \alpha$ and $\nabla_{[Y,Z]}$ and we get

$$(R^{T^*M}(Y, Z)\alpha)(X) = -\alpha(R^{TM}(Y, Z)X).$$

□

The Levi-Civita connection $\nabla^g = \nabla^{T^*M}$ extends as an even derivation to a connection on $\Lambda^* T^* M$. More precisely, for every $X \in \mathrm{Vect}\,(M)$ and any $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$ we define

$$\nabla^g_X(\alpha_1 \wedge \cdots \wedge \alpha_k) = (\nabla^g_X \alpha_1) \wedge \cdots \wedge \alpha_k + \cdots + \alpha_1 \wedge \cdots \wedge (\nabla^g_X \alpha_k).$$

Lemma 10.2.3. *The curvature of the Levi-Civita connection on Λ^*T^*M is defined by*

$$R^{\Lambda^*T^*M}(X,Y)(\alpha_1 \wedge \cdots \wedge \alpha_k) = (\tilde{R}(X,Y)\alpha_1) \wedge \cdots \wedge \alpha_k + \cdots + \alpha_1 \wedge \cdots \wedge (\tilde{R}(X,Y)\alpha_k)$$

for any vector fields X, Y and any 1-forms $\alpha_1, \ldots, \alpha_k$.

Exercise 10.2.1. Prove the above lemma. □

The Weitzenböck remainder of the Dirac operator $d + d^*$ is $\mathfrak{c}(R^{\Lambda^*T^*M})$. To better understand its action we need to pick a local, oriented, orthonormal moving frame (e_i) of TM. We denote by (e^i) its dual coframe. The Riemann curvature tensor can be expressed as

$$R = \sum_{i<j} e^i \wedge e^j R_{ij},$$

where R_{ij} is the skew-symmetric endomorphism

$$R_{ij} = R(e_i, e_j) : TM \to TM.$$

Thus

$$\mathfrak{c}(R^{\Lambda^*T^*M}) = \sum_{i<j} \mathbf{c}(e^i)\mathbf{c}(e^j) R^{\Lambda^*T^*M}(e_i, e_j) = \frac{1}{2}\sum_{i,j} \mathbf{c}(e^i)\mathbf{c}(e^j) R^{\Lambda^*T^*M}(e_i, e_j),$$

where $\mathbf{c}(e^j) = e(e^j) - i(e_j)$.

Note that since both $\nabla^*\nabla$ and $(d+d^*)^2$ preserve the $\mathbb{Z}$-grading of $\Lambda^*_{\mathbb{C}}T^*M = \oplus_k \Lambda^k_{\mathbb{C}}T^*M$ so does the Weitzenböck remainder and consequently it must split as

$$\mathfrak{c}(R^{\Lambda^*T^*M}) = \oplus_{k\geq 0} \mathcal{R}^k.$$

Since $\mathcal{R}^0 \equiv 0$ so the first interesting case is $\mathcal{R}^1$. To understand its form pick normal coordinates (x^i) near $\mathbf{x}_0 \in M$. Set $e_i = \frac{\partial}{\partial x_i}|_{\mathbf{x}_0} \in T_{\mathbf{x}_0}M$ and $e^i = dx^i|_{\mathbf{x}_0} \in T^*_{\mathbf{x}_0}M$.

At $\mathbf{x}_0$ the Riemann curvature tensor R has the form

$$R = \sum_{k<\ell} e^k \wedge e^\ell R(e_k, e_\ell)$$

where $R(e_k, e_\ell)e_j = R^i_{jk\ell} e_i = R_{ijk\ell} e_i$. Using Lemma 10.2.2 we get

$$\tilde{R}(e_k, e_\ell)e^j = R_{ijk\ell} e^i.$$

Using this in the expression of $\mathcal{R}^1$ at $\mathbf{x}_0$ we get

$$\mathcal{R}^1(\sum_j \alpha_j e^j) = \frac{1}{2}\sum_{k,\ell}\sum_{i,j} \alpha_j R_{ijk\ell} \mathbf{c}(e^k)\mathbf{c}(e^\ell) e^i.$$

We need to evaluate the Clifford actions in the above equality.

$$\mathbf{c}(e^\ell)e^i = e^\ell \wedge e^i - \delta_{i\ell}$$

and

$$\mathbf{c}(e^k)\mathbf{c}(e^\ell)e^i = e^k \wedge e^\ell \wedge e^i - \delta_{i\ell}e^k - \delta_{k\ell}e^i + \delta_{ik}e^\ell.$$

Hence

$$\mathcal{R}^1(\sum_j \alpha_j e^j) = \frac{1}{2}\sum_{i,j,k,\ell} \alpha_j R_{ijk\ell}(e^k \wedge e^\ell \wedge e^i - \delta_{i\ell}e^k - \delta_{k\ell}e^i + \delta_{ik}e^\ell).$$

Using the first Bianchi identity we deduce that

$$\sum_{i,k,\ell} R_{ijk\ell}e^k \wedge e^\ell \wedge e^i = -\sum_{i,k,\ell} R_{jik\ell}e^i \wedge e^k \wedge e^\ell = 0 \;\; \forall j.$$

Because of the skew-symmetry $R_{ijk\ell} = -R_{ij\ell k}$ we conclude that

$$\sum_{ijk\ell} \alpha_j R_{ijk\ell}\delta_{k\ell}e^i = 0.$$

Hence

$$\mathcal{R}^1(\sum_j \alpha_j e^j) = \frac{1}{2}\sum_{i,j,k,\ell} \alpha_j R_{ijk\ell}(\delta_{ik}e^\ell - \delta_{i\ell}e^k)$$

$$= \frac{1}{2}\sum_{i,j,\ell} \alpha_j R_{iji\ell}e^\ell - \frac{1}{2}\sum_{i,j,k} \alpha_j R_{ijki}e^k = \sum_{i,j,k} \alpha_j R_{ijik}e^k = \sum_{jk} \alpha_j R_{jk}e^k,$$

where

$$R_{jk} = \sum_i R_{ijik}$$

denotes the Ricci tensor at $\mathbf{x}_0$. Hence

$$\mathcal{R}^1 = \mathrm{Ric}. \tag{10.2.1}$$

In the above equality Ric is regarded (via the metric duality) as a selfadjoint endomorphism of T^*M.

The identity (10.2.1) has a beautiful consequence.

Theorem 10.2.4. (Bochner) *Let (M, g) be a compact, connected, oriented Riemann manifold.*
(a) If the Ricci tensor is non-negative definite then $b_1(M) \leq \dim M$.
(b) If Ricci tensor is non-negative definite but is somewhere strictly positive definite then $b_1(M) = 0$.
(Recall that $b_1(M)$ denotes the first Betti number of M).

The above result is truly remarkable. The condition on the Ricci tensor is purely local but with global consequences. We have proved a similar result using geodesics (see Myers Theorem, Subsection 5.2.2 , 6.2.4 and 6.2.5). Under more restrictive assumptions on the Ricci tensor (uniformly positive definite) one deduces a stronger conclusion namely that the fundamental group is finite. If the uniformity assumption is dropped then the conclusion of the Myers theorem no longer holds (think of the flat torus). In particular this result gives yet another explanation for the equality

$$H^1(G) = 0$$

where G is a compact semisimple Lie group. Recall that in this case the Ricci curvature is $\frac{1}{4} \times \{\text{the Killing metric}\}$.

Proof (a) Let Δ_1 denote the metric Laplacian

$$\Delta_1 = dd^* + d^*d : \Omega^1(M) \to \Omega^1(M).$$

Hodge theory asserts that

$$b_1(M) = \dim \ker \Delta_1$$

so that in order to estimate the first Betti number we need to estimate the "number" of solutions of the elliptic equation

$$\Delta_1 \eta = 0 \quad \eta \in \Omega^1.$$

Using the Bochner-Weitzenböck theorem and the equality (10.2.1) we deduce

$$\Delta_1 \eta = \nabla^* \nabla \eta + \mathrm{Ric}\, \eta = 0 \quad \text{on } M.$$

Taking the L^2-inner product by η and then integrating by parts we get

$$\int_M |\nabla \eta|^2 dv_g + \int_M (\mathrm{Ric}\, \eta, \eta) dv_g = 0. \tag{10.2.2}$$

Since Ric is non-negative definite we deduce

$$\nabla \eta = 0.$$

Hence any harmonic 1-form must be covariant constant. In particular, since M is connected, the number of linearly independent harmonic 1-forms is no greater than the rank of $T^*M = \dim M$.

(b) Using the equality (10.2.2) we deduce that any harmonic 1-form η must satisfy

$$(\mathrm{Ric}(x)\eta_x, \eta_x)_x = 0 \quad \forall x \in M.$$

If the Ricci tensor is positive at some $x_0 \in M$ then $\eta(x_0) = 0$. Since η is also covariant constant and M is connected, we conclude that $\eta \equiv 0$.

□

Remark 10.2.5. For a very nice survey of some beautiful applications of this technique we refer to [9]. □

10.2.2 The Dolbeault operator

This subsection introduces the reader to the Dolbeault operator which plays a central role in complex geometry. Since we had almost no contact with this beautiful branch of geometry we will present only those aspects concerning the "Dirac nature" of these operators. To define this operator we need a little more differential geometric background.

Definition 10.2.6. *(a) Let $E \to M$ be a smooth real vector bundle over the smooth manifold M. An* almost complex *structure on E is an endomorphism $J : E \to E$ such that $J^2 = -\mathbf{1}_E$.*
(b) An almost complex structure on a smooth manifold M is an almost complex structure J on the tangent bundle. An almost complex manifold is a pair (manifold, almost complex structure).

Note that any almost complex manifold is necessarily even dimensional and orientable so from the start we know that not any manifold admits almost complex structure. In fact the existence of such a structure is determined by topological invariants finer than the dimension and orientability.

Example 10.2.7. (a) Any complex manifold is almost complex. Indeed any such manifold is locally modeled by $\mathbb{C}^n$ and the transition maps are holomorphic maps $\mathbb{C}^n \to \mathbb{C}^n$. The multiplication by $\mathbf{i}$ defines a real endomorphism on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ which induces the almost complex structure on TM.
(b) For any manifold M the total space of its tangent bundle TM is an almost complex manifold. If (x^i) are local coordinates on M and (X^j) are the coordinates introduced in the fibers of TM ($TM \ni X = X^i \frac{\partial}{\partial x_i}$) then the almost complex structure on TM is determined by

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial X^i} \quad \frac{\partial}{\partial X^i} \mapsto -\frac{\partial}{\partial x_i}.$$

We let the reader check this is a well defined operator (independent of the choice of local coordinates). □

Let (M, J) be an almost complex manifold. Using the results of Subsection 2.2.5 we deduce that the complexified tangent bundle $TM \otimes \mathbb{C}$ splits as

$$TM \otimes \mathbb{C} = (TM)^{1,0} \otimes (TM)^{0,1}.$$

The complex bundle $TM^{1,0}$ is isomorphic (over $\mathbb{C}$) with (TM, J).

By duality J induces an almost complex structure in the cotangent bundle T^*M and similarly we get a decomposition

$$T^*M \otimes \mathbb{C} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}.$$

In turn this defines a decomposition

$$\Lambda^*_{\mathbb{C}} T^*M = \bigoplus_{p,q} \Lambda^{p,q} T^*M.$$

We set $\Omega^{p,q}(M) = C^\infty(\Lambda^{p,q} T^*M)$.

Example 10.2.8. Let M be a complex manifold. If $(z^j = x^j + \mathbf{i} y^j)$ are local holomorphic coordinates on M then $(TM)^{1,0}$ is generated (locally) by the complex tangent vectors

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \mathbf{i}\frac{\partial}{\partial y^j}\right)$$

while $(T^*M)^{1,0}$ is locally generated by the complex 1-forms

$$dz^j = dx^j + \mathbf{i}y^j.$$

$(TM)^{1,0}$ is also known as the *holomorphic tangent space* while $(T^*M)^{1,0}$ is called the *holomorphic cotangent space.* The space $(T^*M)^{0,1}$ is called the anti-holomorphic cotangent space and is locally generated by the complex 1-forms

$$d\bar{z}^j = dx^j - \mathbf{i}y^j.$$

Note that any (p,q)-form $\eta \in \Omega^{p,q}(M)$ can be locally described as

$$\eta = \sum_{|A|=p,|B|=q} \eta_{A,B} dz^A \wedge d\bar{z}^B \quad (\eta_{A,B} \in C^\infty(M,\mathbb{C}))$$

where we use capital Latin letters $A, B, C \ldots$ to denote ordered multi-indices and for each such index A

$$dz^A = dz^{A_1} \wedge \cdots \wedge dz^{A_p}.$$

$d\bar{z}^B$ is defined similarly. □

Exercise 10.2.2. Let (M, J) be an arbitrary smooth almost complex manifold.
(a) Prove that

$$d\Omega^{p,q}(M) \subset \Omega^{p+2,q-1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)\Omega^{p-1,q+2}(M).$$

(b) Show that if M is a *complex* manifold then

$$d\Omega^{p,q}(M) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M). \tag{10.2.3}$$

(The converse is also true and is known as the Newlander-Nirenberg theorem. Its proof is far from trivial. For details we refer to the original paper [62]). □

Definition 10.2.9. *An almost complex structure on a smooth manifold is called* integrable *if it can derived from a holomorphic atlas, i.e. an atlas in which the transition maps are holomorphic.*

Thus, the Newlander-Nirenberg theorem mentioned above states that the condition (10.2.3) is necessary and sufficient for an almost complex structure to be integrable.

Exercise 10.2.3. Assuming the Newlander-Nirenberg theorem prove that an almost complex structure J on the smooth manifold M is integrable if and only if the *Nijenhuis tensor* $N \in C^\infty(T^*M^{\otimes 2} \otimes TM)$ defined by

$$N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y] \;\; \forall X,Y \in \mathrm{Vect}\,(M)$$

vanishes identically. □

In the sequel we will exclusively consider only complex manifolds. Let M be such a manifold. The above exercise shows that the exterior derivative

$$d : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$$

splits as a direct sum

$$d = \oplus_{p+q=k} d^{p,q},$$

where

$$d^{p,q} = \{d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)\}.$$

The component $\Omega^{p,q} \to \Omega^{p+1,q}$ is denoted by $\partial = \partial^{p,q}$ while the other component $\Omega^{p,q} \to \Omega^{p,q+1}$ is denoted by $\overline{\partial} = \overline{\partial}^{p,q}$.

Example 10.2.10. In local holomorphic coordinates (z^i) the action of the operator $\overline{\partial}$ is described by

$$\overline{\partial}\left(\sum_{A,B} \eta_{AB} dz^A \wedge d\overline{z}^B\right) = \sum_{j,A,B} (-1)^{|A|} \frac{\partial \eta_{AB}}{\partial \overline{z}^j} dz^A \wedge d\overline{z}^j \wedge d\overline{z}^B.$$

□

It is not difficult to see that

$$\overline{\partial}^{p,q+1} \circ \overline{\partial}^{p,q} = 0 \quad \forall p,\, q.$$

In other words for any $0 \leq p \leq \dim_{\mathbb{C}} M$ the sequence

$$0 \to \Omega^{p,0}(M) \stackrel{\overline{\partial}}{\to} \Omega^{p,1}(M) \stackrel{\overline{\partial}}{\to} \cdots$$

is a cochain complex known as the p-th *Dolbeault complex* of the complex manifold M. Its cohomology groups are denoted by $H^{p,q}_{\overline{\partial}}(M)$.

Lemma 10.2.11. *The Dolbeault complex is an elliptic complex.*

Proof The symbol of $\overline{\partial}^{p,q}$ is very similar to the symbol of the exterior derivative. For any $x \in M$ and any $\xi \in T^*X$

$$\sigma(\overline{\partial}^{p,q})(\xi) : \Lambda^{p,q} T^*_x M \to \Lambda^{p,q+1} T^*_x M$$

is (up to a multiplicative constant) the (left) exterior multiplication by $\xi^{0,1}$, where $\xi^{0,1}$ denotes the $(0,1)$ component of ξ viewed as an element of the complexified tangent space. More precisely

$$\xi^{0,1} = \frac{1}{2}(\xi + \mathbf{i} J_0 \xi),$$

where $J_0 : T^*_x M \to T^*_x M$ denotes the canonical complex structure induced on $T^*_x M$ by the holomorphic charts. The sequence of symbols is the cochain complex

$$0 \to \Lambda^{p,0} \otimes \Lambda^{0,0} \stackrel{\mathbf{id} \otimes (-1)^p \xi^{0,1} \wedge}{\longrightarrow} \Lambda^{p,0} \otimes \Lambda^{p,0} \otimes \Lambda^{0,1} \stackrel{\mathbf{id} \otimes (-1)^p \xi^{0,1} \wedge}{\longrightarrow} \cdots .$$

This complex is the ($\mathbb{Z}$-graded) tensor product of the trivial complex

$$0 \to \Lambda^{p,0} \stackrel{\mathbf{id}}{\to} \Lambda^{p,0} \to 0$$

with the Koszul complex

$$0 \to \Lambda^{0,0} \stackrel{(-1)^p\xi^{0,1}\wedge}{\longrightarrow} \Lambda^{0,1} \stackrel{(-1)^p\xi^{0,1}\wedge}{\longrightarrow} \cdots .$$

Since $\xi^{0,1} \neq 0$ for any $\xi \neq 0$ the Koszul complex is exact (see Subsection 7.1.3) . This proves the Dolbeault complex is elliptic.

□

To study the Dirac nature of this complex we need to introduce a Hermitian metric h on TM. Its real part is a Riemann metric g on M and the canonical almost complex structure on TM is a skew-symmetric endomorphism with respect to this real metric. The associated 2-form $\Omega_h = -\mathfrak{Im}\, h$ is nondegenerate in the sense that Ω^n ($n = \dim_{\mathbb{C}} M$) is a volume form on M. According to the results of Subsection 2.2.5 the orientation of M defined by Ω^n coincides with the orientation induced by the complex structure.

We form the *Dolbeault operator*

$$\overline{\partial} + \overline{\partial}^* : \Omega^{p,*}(M) \to \Omega^{p,*}(M).$$

Proposition 10.2.12. *$\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ is a Dirac operator.*

Proof We need to show that

$$\big(\sigma(\overline{\partial})(\xi) - \sigma(\overline{\partial})(\xi)^*\big)^2 = -\frac{1}{2}|\xi|^2\mathbf{id} \quad \forall \xi \in T^*M.$$

Denote by J the canonical complex structure on T^*M and set $\eta = J\xi$. Note that $\xi \perp \eta$ and $|\xi| = |\eta|$. Then

$$\sigma(\overline{\partial})(\xi) = (-1)^p\frac{1}{2}e(\xi + \mathbf{i}\eta) = \frac{1}{2}(e(\xi) + \mathbf{i}e(\eta)),$$

where as usual $e(\cdot)$ denotes the (left) exterior multiplication. The adjoint of $\sigma(\overline{\partial})(\xi)$ is

$$(-1)^p\sigma(\overline{\partial})(\xi)^* = \frac{1}{2}(i(\xi^*) - \mathbf{i}i(\eta^*)),$$

where ξ^* (resp. η^*) denotes the metric dual of ξ (resp. η). We deduce

$$\big(\sigma(\overline{\partial})(\xi) - \sigma(\overline{\partial})(\xi)^*\big)^2 = \frac{1}{4}\left\{e(\xi) + \mathbf{i}e(\eta) - i(\xi^*) + \mathbf{i}i(\eta^*)\right\}^2$$

$$= \frac{1}{4}\left\{(e(\xi) - i(\xi^*)) + \mathbf{i}(e(\eta) + i(\eta^*))\right\}^2 \stackrel{def}{=} \frac{1}{4}\{\mathbf{c}(\xi) + \mathbf{i}\tilde{c}(\eta)\}^2$$

$$= \frac{1}{4}\left\{\mathbf{c}(\xi)^2 - \tilde{c}(\eta)^2 + \mathbf{i}(\mathbf{c}(\xi)\tilde{c}(\eta) + \tilde{c}(\eta)\mathbf{c}(\xi))\right\}.$$

Note that

$$\mathbf{c}(\xi)^2 = -(e(\xi)i(\xi^*) + i(\xi^*)e(\xi)) = -|\xi|^2$$

and

$$\tilde{c}(\eta)^2 = e(\eta)i(\eta^*) + i(\eta^*)e(\eta) = |\eta|^2.$$

On the other hand since $\xi \perp \eta$ we deduce as above that

$$\mathbf{c}(\xi)\tilde{c}(\eta) + \tilde{c}(\eta)\mathbf{c}(\xi) = 0.$$

(Verify this!) Hence

$$\left\{ \sigma(\overline{\partial})(\xi) - \sigma(\overline{\partial})(\xi)^* \right\}^2 = -\frac{1}{4}(|\xi|^2 + |\eta|^2) = -\frac{1}{2}|\eta|^2.$$

□

A natural question arises as to when the above operator is a *geometric* Dirac operator. Note first that the Clifford multiplication is certainly skew-adjoint since it is the symbol of a formally selfadjoint operator. Thus all we need to inquire is when the Clifford multiplication is covariant constant. Since

$$\mathbf{c}(\xi) = \frac{(-1)^p}{\sqrt{2}}(e(\xi) + \mathbf{i}e(\eta) - i(\xi^*) - \mathbf{i}i(\eta^*))$$

where $\eta = J\xi$ we deduce the Clifford multiplication is covariant constant if $\nabla^g J = 0$.

Definition 10.2.13. *Let M be a complex manifold and h a Hermitian metric on TM (viewed as a complex bundle). Then h is said to be a* Kähler metric *if $\nabla J = 0$ where ∇ denotes the Levi-Civita connection associated to the Riemann metric $\mathfrak{Re}\, h$ and J is the canonical almost complex structure on TM. A pair (complex manifold, Kähler metric) is called a Kähler manifold.*

Exercise 10.2.4. Let (M, J) be an almost complex manifold and h a Hermitian metric on TM. Let $\Omega = -\mathfrak{Im}\, h$. Using the exercise 10.2.3 show that if

$$d\Omega_h = 0$$

then the almost complex is integrable and the metric h is Kähler. Conversely, assuming that J is integrable and h is Kähler show that

$$d\Omega = 0.$$

□

We see that on a Kähler manifold the above Clifford multiplication is covariant constant. In fact, a more precise statement is true.

Proposition 10.2.14. *Let M be a complex manifold and h a Kähler metric on TM. Then the Levi-Civita induced connection on $\Lambda^{0,*}T^*M$ is a Clifford connection with respect to the above Clifford multiplication and moreover the Dolbeault operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ is the geometric Dirac operator associated to this connection.*

Exercise 10.2.5. Prove the above proposition. □

Example 10.2.15. Let (M, g) be an oriented 2-dimensional Riemann manifold (surface). The Hodge $*$ operation defines an endomorphism

$$* : TM \to TM$$

satisfying $*^2 = -\mathbf{1}_{TM}$ i.e. an almost complex structure on M. Using the Exercise 10.2.4 we deduce this almost complex structure is integrable since, by dimensionality

$$d\Omega = 0,$$

where Ω is the natural 2-form $\Omega(X, Y) = g(*X, Y)$ $X, Y \in \text{Vect}\,(M)$. This complex structure is said to be canonically associated to the metric. □

Example 10.2.16. Perhaps the favorite example of Kähler manifold is the complex projective space $\mathbb{C}^n$. To describe this structure consider the tautological line bundle $L_1 \to \mathbb{CP}^n$. It can be naturally viewed as a subbundle of the trivial bundle $\underline{\mathbb{C}}^{n+1} \to \mathbb{CP}^n$. Denote by h_0 the canonical Hermitian metric on $\underline{\mathbb{C}}^{n+1}$ and by ∇^0 the trivial connection. If we denote by $P : \underline{\mathbb{C}}^{n+1} \to L_1$ the orthogonal projection then $\nabla = P \circ \nabla^0 |_{L_1}$ defines a connection on L_1 compatible with $h_1 = h|_{L_1}$. Denote by ω the 1st Chern form associated to this connection

$$\omega = \frac{\mathbf{i}}{2\pi} F(\nabla)$$

and set

$$h_{FS}(X, Y)_x = -\omega_x(X, JY) + \mathbf{i}\omega(X, Y) \quad \forall x \in M,\ X, Y \in T_xM.$$

Then h_{FS} is a Hermitian metric on $\mathbb{CP}^n$ (verify!) called the *Fubini-Study* metric. It is clearly a Kähler metric since (see the Exercise 10.2.4) $d\Omega_h = -d\omega = -dc_1(\nabla) = 0$. □

Exercise 10.2.6. Describe h_{FS} in projective coordinates and then prove that h is indeed a Hermitian metric, i.e. it is positive definite). □

Remark 10.2.17. Any complex submanifold of a Kähler manifold is obviously Kähler. In particular, any complex submanifold of $\mathbb{CP}^n$ is automatically Kähler. A celebrated result of Chow states that any complex submanifold of $\mathbb{CP}^n$ is automatically algebraic i.e. it can be defined as the zero set of a family of homogeneous polynomials. Thus all complex nonsingular algebraic varieties admit a natural Kähler structure. It is thus natural to ask whether there exist Kähler manifolds which are not algebraic. The answer is positive and a very thorough resolution of this problem is contained in the famous Kodaira embedding theorem which provides a simple necessary and sufficient condition for a compact complex manifold to be algebraic. For this result (and many more others) Kodaira was awarded the Fields medal in 1954. His proofs rely essentially on some vanishing results deduced from the Weitzenböck formulæ for the Dolbeault operator $\overline{\partial} + \overline{\partial}^*$ and its twisted versions. A very clear presentation of this subject can be found in the beautiful monograph [32]. □

10.2.3 The *spin* Dirac operator

Like the Dolbeault operator, the *spin* Dirac operator exists only on manifolds with a bit of extra structure. We will first describe this new structure.

Let (M^n, g) be an n-dimensional, oriented Riemann manifold. In other words, the tangent bundle TM admits an $SO(n)$ structure so that it can be defined by an open cover (U_α) and transition maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)$$

satisfying the cocycle condition.

The manifold is said to posses a *spin structure* if there exist smooth maps

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$$

satisfying the cocycle condition and such that

$$\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta} \;\; \forall \alpha,\, \beta,$$

where $\rho : Spin(n) \to SO(n)$ denotes the canonical double cover. The collection $\tilde{g}_{\alpha\beta}$ as above is also called a lifting of the $SO(n)$ structure. A pair (manifold, spin structure) is called a spin manifold.

Not all manifolds admit spin structure. To understand what can go wrong let us start with a trivializing cover $\mathcal{U} = (U_\alpha)$ for TM with transition maps $g_{\alpha\beta}$ and such that all multiple intersection $U_{\alpha\beta\cdots\gamma}$ are contractible. In other words, $\mathcal{U}$ is a good cover. Since each of the overlaps $U_{\alpha\beta}$ is contractible each map $g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)$ admits at least one lift

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n).$$

From the equality $\rho(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathbf{1}$ we deduce

$$\epsilon_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} \in \ker \rho = \mathbb{Z}_2.$$

Thus any lift of the gluing data $g_{\alpha\beta}$ to $Spin(n)$ produces a degree 2 Čech cochain of the trivial sheaf $\mathbb{Z}_2$ namely the 2-cochain

$$(\epsilon_\bullet) : \quad U_{\alpha\beta\gamma} \mapsto \epsilon_{\alpha\beta\gamma}.$$

Note that for any $\alpha, \beta, \gamma, \delta$ such that $U_{\alpha\beta\gamma\delta} \neq \emptyset$ we have

$$\epsilon_{\beta\gamma\delta} - \epsilon_{\alpha\gamma\delta} + \epsilon_{\alpha\beta\delta} - \epsilon_{\alpha\beta\gamma} = 0 \in \mathbb{Z}_2.$$

In other words, $\epsilon_\bullet$ defines a Čech 2-cocycle and thus defines an element in the Čech cohomology group $H^2(M, \mathbb{Z}_2)$. It is not difficult to see this element is independent of the various choices: the cover $\mathcal{U}$, the gluing data $g_{\alpha\beta}$ and the lifts $\tilde{g}_{\alpha\beta}$. This element is intrinsic to the tangent bundle TM. It is called the *second Stiefel-Whitney class* of M and it is denoted by $w_2(M)$. We see that if $w_2(M) \neq 0$ then M cannot admit a spin structure. In fact, the converse is also true.

Proposition 10.2.18. *An oriented Riemann manifold M admits a spin structure if and only if $w_2(M) = 0$.*

Exercise 10.2.7. Prove the above result. $\square$

Remark 10.2.19. The usefulness of the above proposition depends strongly on the ability of computing w_2. This is a good news/bad news situation. The good news is that algebraic topology has produced very efficient tools for doing this. The bad news is that we will not mention them since it would lead us far astray. See [46] and [58] for more details. □

Remark 10.2.20. The definition of isomorphism of *spin*-structures is rather subtle (see [56]). More precisely, two *spin* structures defined by the cocycles $\tilde{g}_{\bullet\bullet}$ and $\tilde{h}_{\bullet\bullet}$ are isomorphic if there exists a collection $\varepsilon_\alpha \in \mathbb{Z}_2 \subset Spin(n)$ such that the diagram below is commutative for all $x \in U_{\alpha\beta}$

$$\begin{array}{ccc} Spin(n) & \xrightarrow{\varepsilon_\alpha} & Spin(n) \\ \Big\downarrow{\scriptstyle \tilde{g}_{\beta\alpha}(x)} & & \Big\downarrow{\scriptstyle \tilde{h}_{\beta\alpha}(x)} \\ Spin(n) & \xrightarrow{\varepsilon_\beta} & Spin(n) \end{array}$$

The group $H^1(M, \mathbb{Z}_2)$ acts on $Spin(M)$ as follows. Take an element $\varepsilon \in H^1(M, \mathbb{Z}_2)$ represented by a Čech cocycle, i.e. a collection of *continuous maps* $\varepsilon_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2 \subset Spin(n)$ satisfying the cocycle condition

$$\varepsilon_{\alpha\beta} \cdot \varepsilon_{\beta\gamma} \cdot \varepsilon_{\gamma\alpha} = 1.$$

Then the collection $\varepsilon_{\bullet\bullet} \cdot \tilde{g}_{\bullet\bullet}$ is a $Spin(n)$ gluing cocycle defining a *spin* structure we denote by $\varepsilon \cdot \sigma$. It is easy to check that the isomorphism class of $\varepsilon \cdot \sigma$ is independent of the various choice, i.e the Čech representatives for ε and σ. Clearly the correspondence

$$H^1(M, \mathbb{Z}_2) \times Spin(M) \ni (\varepsilon, \sigma) \mapsto \varepsilon \cdot \sigma \in Spin(M)$$

defines a left action of $H^1(M, \mathbb{Z}_2)$ on $Spin(M)$. This action is transitive and free. □

Exercise 10.2.8. Prove the above proposition and the statement in the above remark. □

Exercise 10.2.9. Describe the only 2 spin structures on S^1. □

Example 10.2.21. (a) A simply connected Riemann manifold M of dimension ≥ 5 admits spin structures if and only if every compact orientable surface embedded in M has trivial normal bundle.
(b) A simply connected four-manifold M admits spin structures if and only if the normal bundle N_Σ of any embedded compact, orientable surface Σ has even Euler class i.e.

$$\int_\Sigma \mathbf{e}(N_\Sigma)$$

is an even integer.
(c) Any compact oriented surface admits spin structures. Any sphere S^n admits a unique spin structure. The product of two spin manifolds is canonically a spin manifold.
(d) $w_2(\mathbb{RP}^n) = 0$ iff $n \equiv 3 \pmod 4$ while $\mathbb{CP}^n$ admits spin structures iff n is odd. □

Let (M^n, g) be a spin manifold. Assume the tangent bundle TM is defined by the open cover (U_α) and transition maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to SO(n).$$

Moreover assume the spin structure is given by the lifts

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n).$$

As usual, we regard the collection $g_{\alpha\beta}$ as defining the principal $SO(n)$ bundle of oriented frames of TM. We call this bundle $P_{SO(M)}$. The collection $\tilde{g}_{\alpha\beta}$ defines a principal $Spin(n)$ bundle which we denote by $P_{Spin(M)}$. We can regard $P_{SO(M)}$ as a bundle associated to $P_{Spin(M)}$ via $\rho : Spin(n) \to SO(n)$. Using the unitary spinorial representation

$$\Delta_n : Spin(n) \to \mathrm{Aut}\,(\mathbb{S}_n)$$

we get a Hermitian vector bundle

$$\mathbb{S}(M) = P_{Spin(M)} \times_{\Delta_n} \mathbb{S}_n$$

called the *complex spinor bundle*.

From Lemma 10.1.22 we deduce that $\mathbb{S}(M)$ is naturally a bundle of $Cl(TM)$ modules. Then, the natural isomorphism $Cl(TM) \cong Cl(T^*M)$ (via the metric) induces on $\mathbb{S}(M)$ a structure of $Cl(T^*M)$-module. As it turns out $\mathbb{S}(M)$ has a natural Dirac structure whose associated Dirac operator is the *spin* Dirac operator on M. We will denote it by $\mathfrak{D}$.

Since Δ_n is a unitary representation the spinorial bundle $\mathbb{S}(M)$ comes with a natural metric with respect to which the Clifford multiplication is self-adjoint. All we now need is to describe a natural connection on $\mathbb{S}(M)$ with respect to which the Clifford multiplication is covariant constant.

We start with the Levi-Civita connection ∇^g which we can regard as a connection on the principal bundle $P_{SO(M)}$. Alternatively, ∇^g can be defined by a collection of $\underline{so}(n)$-valued 1-forms $\omega_\alpha \in \Omega^1(U_\alpha) \otimes \underline{so}(n)$ such that

$$\omega_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta} \quad \text{on } U_{\alpha\beta}.$$

Denote by τ the canonical isomorphism of Lie algebras

$$\tau : \underline{spin}(n) \to \underline{so}(n).$$

Then the collection $\tilde{\omega}_\alpha = \tau^{-1}(\omega_\alpha)$ defines a connection $\hat{\nabla}$ on the principal bundle $P_{Spin(M)}$ and thus via the representation Δ_n it defines a connection $\nabla = \nabla^{\mathbb{S}}$ on the spinor bundle $\mathbb{S}(M)$.

The above construction can be better visualized if we work in local coordinates. Choose a local, oriented orthonormal frame (e_i) of $TM\,|_{U_\alpha}$ and denote by (e^j) is dual coframe. The Levi-Civita connection has the form

$$\nabla e_j = e^k \otimes \omega_{kj}^i e_i$$

so that $\omega_\alpha = e^k \otimes (\omega_{ki}^j)$ where for each k the collection $(\omega_{ki}^j)_{i,j}$ is a skew-symmetric matrix. Using the concrete description of τ given in Subsection 10.1.6 we deduce

$$\tilde{\omega}_\alpha = -\sum_k e^k \otimes \Big(\frac{1}{2}\sum_{i<j} \omega_{kj}^i e_i e_j\Big) = -\frac{1}{4}\sum_{i,j,k} \omega_{kj}^i e^k \otimes e_i e_j.$$

Lemma 10.2.22. *$\hat{\nabla}$ is a Clifford connection on $\mathbb{S}(M)$ so that $(\mathbb{S}(M), \hat{\nabla})$ is a Dirac bundle called the* bundle of pure spinors.

Proof Use Lemma 10.1.22.

□

We now want to understand the structure of the Weitzenböck remainder of the geometric Dirac operator $\mathfrak{D}$ associated to the bundle of pure spinors. If

$$R = \sum_{k<\ell} e^k \wedge e^\ell R_{k\ell} \quad R_{k\ell} = (R^i_{jk\ell}) = (R_{ijk\ell})$$

is the Riemann curvature tensor (in the above trivializations over U_α) we deduce that the curvature of $\hat{\nabla}$ is

$$\tilde{R} = \sum_{k<\ell} e^k \wedge e^\ell \otimes \tau^{-1}(R_{k\ell}) = -\frac{1}{4}\sum_{k<\ell}\sum_{ij} e^k \wedge e^\ell \otimes R_{ijk\ell} e_i e_j.$$

From this we obtain

$$\mathfrak{c}(F(\hat{\nabla})) = -\frac{1}{8}\sum_{ijk\ell} R_{ijk\ell}\mathbf{c}(e^i)\mathbf{c}(e^j)\mathbf{c}(e^k)\mathbf{c}(e^\ell).$$

In the above sum the terms corresponding to indices (i, j, k, ℓ) such that $i = j$ or $k = \ell$ vanish due to the corresponding skew-symmetry of the Riemann tensor. Thus we can write

$$\mathfrak{c}(F(\hat{\nabla})) = -\frac{1}{8}\sum_{i\neq j}\sum_{k\neq\ell} R_{ijk\ell}\mathbf{c}(e^i)\mathbf{c}(e^j)\mathbf{c}(e^k)\mathbf{c}(e^\ell)$$

Using the equalities $\mathbf{c}(e^i)\mathbf{c}(e^j) + \mathbf{c}(e^j)\mathbf{c}(e^i) = -2\delta_{ij}$ we deduce that the monomial $\mathbf{c}(e^i)\mathbf{c}(e^j)$ anti-commutes with $\mathbf{c}(e^k)\mathbf{c}(e^\ell)$ if the two sets$\{i, j\}$ and $\{k, \ell\}$ have a unique element in common. Such pairs of monomials will have no contributions in the above sum due to the curvature symmetry

$$R_{ijk\ell} = R_{k\ell ij}.$$

Thus we can split the above sum into two parts

$$\mathfrak{c}(F(\hat{\nabla})) = -\frac{1}{4}\sum_{i,j} R_{ijij}\mathbf{c}(e^i)\mathbf{c}(e^j)\mathbf{c}(e^i)\mathbf{c}(e^j) + \sum_{i,j,k,\ell\,\text{distinct}} R_{ijkl}\mathbf{c}(e^i)\mathbf{c}(e^j)\mathbf{c}(e^k)\mathbf{c}(e^\ell).$$

Using the first Bianchi identity we deduce that the second sum vanishes. The first sum is equal to

$$-\frac{1}{4}\sum_{i,j} R_{ijij}(\mathbf{c}(e^i)\mathbf{c}(e^j))^2 = \frac{1}{4}\sum_{i,j} R_{ijij} = \frac{s}{4},$$

where s denotes the scalar curvature of M. We have thus proved the following result.

Theorem 10.2.23. (Lichnerowicz) *([48])*

$$\mathfrak{D}^2 = \nabla^*\nabla + \frac{1}{4}s.$$

A section $\psi \in C^\infty(\mathbb{S}(M))$ such that $\mathfrak{D}^2\psi = 0$ is called a *h*armonic spinor. Lichnerowicz theorem shows that a compact spin manifold with positive scalar curvature admits no harmonic spinors.

Exercise 10.2.10. Consider an oriented 4-dimensional Riemann spin manifold (M, g) and $W \to M$ a Hermitian vector bundle equipped with a Hermitian connection ∇. Form the twisted Dirac operator

$$\mathfrak{D}_W : C^\infty\left((\mathbb{S}^+(M) \oplus \mathbb{S}^-(M)) \otimes W\right) \to C^\infty\left((\mathbb{S}^+(M) \oplus \mathbb{S}^-(M)) \otimes W\right)$$

defined in Exercise 10.1.18. $\mathfrak{D}_W$ is $\mathbb{Z}_2$-graded and hence it has a block decomposition

$$\mathfrak{D}_W = \left[\begin{array}{cc} 0 & \mathfrak{D}_{W,+}^* \\ \mathfrak{D}_{W,+} & 0 \end{array}\right],$$

where $\mathfrak{D}_{W,+} : C^\infty(\mathbb{S}^+(M) \otimes W) \to C^\infty(\mathbb{S}^-(M) \otimes W)$. Show that

$$\mathfrak{D}_{W,+}^*\mathfrak{D}_{W,+} = \tilde{\nabla}^*\tilde{\nabla} + \frac{s}{4} + \mathfrak{c}(F_+(\nabla)),$$

where $\tilde{\nabla}$ denotes the product connection of $\mathbb{S}(M) \otimes W$ while

$$F_+(\nabla) = \frac{1}{2}(F(\nabla) + *F(\nabla))$$

denotes the self-dual part of the curvature of the bundle (W, ∇). □

10.2.4 The *spin*^{c} Dirac operator

Our last example of Dirac operator generalizes both the *spin* Dirac operator and the Dolbeault operator. The common ingredient behind both these examples is the notion of *spin*^{c} structure. We begin by introducing it to the reader.

Let (M^n, g) be an oriented, n-dimensional Riemann manifold. As in the previous section we can regard the tangent bundle as associated to the principal bundle $P_{SO(M)}$ of oriented orthonormal frames. Assume $P_{SO(M)}$ is defined by a good open cover $\mathcal{U} = (U_\alpha)$ and transition maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to SO(n).$$

The manifold M is said to posses a *spin*^{c} structure if there exists a principal $Spin^c(n)$ -bundle P_{Spin^c} such that $P_{SO(M)}$ is associated to P_{Spin^c} via the natural morphism $\rho^c : Spin^c(n) \to SO(n)$:

$$P_{SO(M)} = P_{Spin^c} \times_{\rho^c} SO(n).$$

Equivalently, this means there exist smooth maps $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin^c(n)$, satisfying the cocycle condition, such that

$$\rho^c(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}.$$

As for *spin* structures, there are obstructions to *spin*^{c} structures as well which clearly are less restrictive. Let us try to understand what can go wrong. We stick to the assumption that all the overlaps $U_{\alpha\beta\cdots\gamma}$ are contractible.

Since $Spin^c(n) = (Spin(n) \times S^1)/\mathbb{Z}_2$, lifting the $SO(n)$ structure $(g_{\alpha\beta})$ reduces to finding smooth maps

$$h_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$$

and

$$z_{\alpha\beta} : U_{\alpha\beta} \to S^1$$

such that

$$\rho(h_{\alpha\beta}) = g_{\alpha\beta}$$

and

$$(\epsilon_{\alpha\beta\gamma}, \zeta_{\alpha\beta\gamma}) \stackrel{def}{=} (h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}\,,\, z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}) \in \{(-1,-1),(1,1)\}. \tag{10.2.4}$$

If we set $\lambda_{\alpha\beta} = z_{\alpha\beta}^2 : U_{\alpha\beta} \to S^1$ we deduce from (10.2.4) that the collection $(\lambda_{\alpha\beta})$ should satisfy the cocycle condition. In particular, it defines a principal S^1 bundle over M, or equivalently, a complex line bundle $\mathcal{L}$. This line bundle should be considered as part of the data defining a *spin*^{c} structure. The collection $(\epsilon_{\alpha\beta\gamma})$ is an old acquaintance: it is a Čech 2-cocycle representing the 2nd Stiefel-Whitney class.

As in Subsection 8.2.2 we can represent the cocycle $\lambda_{\alpha\beta}$ as

$$\lambda_{\alpha\beta} = \exp(\mathbf{i}\theta_{\alpha\beta}).$$

The collection

$$n_{\alpha\beta\gamma} = \frac{1}{2\pi}(\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha})$$

defines a 2-cocycle of the constant sheaf $\mathbb{Z}$ representing the the topological 1st Chern class of $\mathcal{L}$. The condition (10.2.4) shows that

$$n_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} \,(\mathrm{mod}\ 2).$$

To summarize, we see that the existence of a *spin*^{c} structure implies the existence of a complex line bundle $\mathcal{L}$ such that

$$c_1^{top}(\mathcal{L}) = w_2(M)\ (\mathrm{mod}\ 2).$$

It is not difficult to prove the above condition is also sufficient. In fact one can be more precise.

Denote by $Spin^c(M)$ the collection of isomorphism classes of *spin*^{c} structures on the manifold M. Any $\sigma \in Spin^c(M)$ is defined by a a lift $(h_{\alpha\beta}, z_{\alpha\beta})$ as above. We denote by $\mathcal{L}_\sigma$ the complex line bundle defined by the gluing data $(z_{\alpha\beta})$. We have seen that

$$c_1^{top}(\mathcal{L}_\sigma) \equiv w_2(M)\ (\mathrm{mod}\ 2).$$

Denote by $\mathcal{L}_M \subset H^2(M, \underline{\mathbb{Z}})$ the "affine " subspace consisting of those cohomology classes satisfying the above congruence modulo 2. We thus have a map

$$Spin^c(M) \to \mathcal{L}_M, \quad \sigma \mapsto c_1^{top}(\mathcal{L}_\sigma).$$

Proposition 10.2.24. *The above map is a surjection.*

Exercise 10.2.11. Complete the proof of the above proposition. □

The smooth Picard group $\mathrm{Pic}^{\infty}(M)$ acts on $Spin^c(M)$ by

$$Spin^c(M) \times \mathrm{Pic}^{\infty}(M) \ni (\sigma, L) \mapsto \sigma \otimes L.$$

More precisely if $\sigma \in Spin^c(M)$ is given by the cocycle

$$\sigma = [h_{\alpha\beta}, z_{\alpha\beta}] : U_{\alpha\beta} \to Spin\,(n) \times S^1/\sim$$

and L is given by the S^1 cocycle

$$\zeta_{\alpha\beta} : U_{\alpha\beta} \to S^1$$

then $\sigma \otimes L$ is given by the cocycle

$$[h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}].$$

Note that

$$\mathfrak{L}_{\sigma\otimes L} = \mathfrak{L}_{\sigma} \otimes L^2$$

so that

$$c_1^{top}(\sigma \otimes L) = c_1^{top}(\sigma) + 2c_1^{top}(L).$$

Proposition 10.2.25. *The above action of* $\mathrm{Pic}^{\infty}(M)$ *on* $Spin^c(M)$ *is is free and transitive.*

Proof Consider two *spin*^{c} structures σ^1 and σ^2 defined by the good cover (U_α) and the gluing coycles

$$[h_{\alpha\beta}^{(i)}, z_{\alpha\beta}^{(i)}], \;\; i = 1, 2.$$

Since $\rho^c(h_{\alpha\beta}^{(1)}) = \rho^c(h_{\alpha\beta}^{(2)}) = g_{\alpha\beta}$ we can assume (eventually modifying the maps $h_{\alpha\beta}^{(2)}$ by a sign) that

$$h_{\alpha\beta}^{(1)} = h_{\alpha\beta}^{(2)}$$

This implies that

$$\zeta_{\alpha\beta} = z_{\alpha\beta}^{(2)}/z_{\alpha\beta}^{(1)}$$

is an S^1-cocycle defining a complex line bundle L. Obviously $\sigma^2 = \sigma^1 \otimes L$. This shows the action of $\mathrm{Pic}^{\infty}(M)$ is transitive. We leave the reader verify this action is indeed free. The proposition is proved.

□

Given two *spin*^{c} structures σ_1 and σ_2 we can define their "difference" σ_2/σ_1 as the unique line bundle L such that $\sigma_2 = \sigma_1 \otimes L$. This shows that the collection of *spin*^{c} structures is (non-canonically) isomorphic with $H^2(X, \mathbb{Z}) \cong \mathrm{Pic}^{\infty}$. It is a sort of affine space modelled on $H^2(X, \mathbb{Z})$ in the sense that the "difference " between two *spin*^{c} structures is an element in $H^2(X, \mathbb{Z})$ but there is no distinguished origin of this space. A structure as above is usually called a $H^2(M, \mathbb{Z})$-**torsor**.

Without a sufficient background in algebraic topology the above results may look of very little help in detecting *spin*^{c} structures. This is not the case and to convince the reader we will list below (without proofs) some examples of *spin*^{c} manifolds.

Example 10.2.26. (a) Any *spin* manifold admits a *spin*^{c} structure.
(b) Any almost complex manifold has a natural *spin*^{c} structure.
(c) (Hirzebruch-Hopf, [37]; see also [61]) Any oriented manifold of dimension ≤ 4 admits a *spin*^{c} structure. □

Let us analyze the first two example above. If M is a spin manifold then the lift

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$$

of the SO structure to a *spin* structure canonically defines a *spin*^{c} structure via the trivial morphism

$$Spin(n) \to Spin^c(n) \times_{\mathbb{Z}_2} S^1, \quad g \mapsto (g,1) \bmod \text{the } \mathbb{Z}_2 - \text{action}.$$

We see that in this case the associated complex line bundle is the trivial bundle. This is called the *canonical spin*^{c} structure of a *spin* manifold. Thus on a *spin* manifold the torsor of *spin*^{c}-structures does in fact possess a "canonical origin" so in this case there is a canonical identification

$$Spin^c(M) \cong \mathrm{Pic}^\infty \cong H^2(M, \mathbb{Z}).$$

To any complex line bundle L defined by the S^1-cocycle $(z_{\alpha\beta})$ we can associate the *spin*^{c} structure defined by the gluing data

$$\{(\tilde{g}_{\alpha\beta}, z_{\alpha\beta})\}.$$

Clearly, the line bundle associated to this structure is $L^2 = L^{\otimes 2}$. In particular this shows that on a spin manifold M for any $\sigma \in Spin^c(M)$ there exists a square root $\mathcal{L}_\sigma^{1/2}$ of $\mathcal{L}_\sigma$.

To understand why an almost complex manifold (necessarily of even dimension $n = 2k$) admits a canonical *spin*^{c} structure it suffices to recall the natural morphism $U(k) \to SO(2k)$ factors through a morphism

$$\xi : U(k) \to Spin^c(2k).$$

The $U(k)$-structure of TM, defined by the gluing data

$$h_{\alpha\beta} : U_{\alpha\beta} \to U(k)$$

induces a *spin*^{c} structure defined by the gluing data $\xi(h_{\alpha\beta})$. Its associated line bundle is given by the S^1-cocycle

$$\det{}_{\mathbb{C}}(h_{\alpha\beta}) : U_{\alpha\beta} \to S^1$$

and it is precisely the determinant line bundle

$$\det{}_{\mathbb{C}} T^{1,0}M = \Lambda^{k,0}TM.$$

The dual of this line bundle, $\det_{\mathbb{C}}(T^*M)^{1,0} = \Lambda^{k,0}T^*M$ plays a special role in algebraic geometry. It usually denoted by K_M and it is called the *canonical line bundle.* Thus the line bundle associated to this spin^c structure is $K_M^{-1} \stackrel{def}{=} K_M^*$.

From the considerations in Subsection 10.1.5 and 10.1.7 we see that many (complex) vector bundles associated to the principal *Spin*^{c} bundle of a *spin*^{c} manifold carry natural

Clifford structures and in particular one can speak of Dirac operators. We want to discuss in some detail a very important special case.

Assume that (M, g) is an oriented, n-dimensional Riemann manifold. Fix $\sigma \in Spin^c(M)$ (assuming there exist spin^c structures). Denote by $(g_{\alpha\beta})$ a collection of gluing data defining the SO structure $P_{SO(M)}$ on M with respect to some good open cover (U_α). Moreover, we assume σ is defined by the data

$$h_{\alpha\beta} : U_{\alpha\beta} \to Spin^c(n).$$

Denote by Δ_n^c the fundamental complex spinorial representation (defined in §10.1.6).

$$\Delta_n : Spin^c(n) \to \operatorname{Aut}(\mathbb{S}_n).$$

We obtain a complex bundle

$$\mathbb{S}_\sigma(M) = P_{Spin^c} \times_{\Delta_n} \mathbb{S}_n$$

which has a natural Clifford structure. This is called the bundle of complex spinors associated to σ.

Example 10.2.27. (a) Assume M is a spin manifold. We denote by σ_0 the *spin^c* structure corresponding to the fixed spin structure. The corresponding bundle of spinors $\mathbb{S}_0(M)$ coincides with the bundle of pure spinors defined in the previous section. Moreover for any complex line bundle L we have

$$\mathbb{S}_L \stackrel{def}{=} \mathbb{S}_\sigma \cong \mathbb{S}_0 \otimes L.$$

where $\sigma = \sigma_0 \otimes L$. Note that in this case $L^2 = \mathfrak{L}_\sigma$ so one can write

$$\mathbb{S}_\sigma \cong \mathbb{S}_0 \otimes \mathfrak{L}_\sigma^{1/2}.$$

(b) Assume M is an almost complex manifold. The bundle of complex spinors associated to the canonical *spin^c* structure σ (such that $\mathfrak{L}_\sigma = K_M^{-1}$) is denoted by $\mathbb{S}_{\mathbb{C}}(M)$. Note that

$$\mathbb{S}_{\mathbb{C}}(M) \cong \Lambda^{0,*}T^*M.$$

□

We will construct a natural family of Dirac structures on the bundle of complex spinors associated to a *spin^c* structure. Consider for warm-up the special case when TM is trivial. Then we can assume $g_{\alpha\beta} \equiv \mathbf{1}$ and

$$h_{\alpha\beta} = (\mathbf{1}, z_{\alpha\beta}) : U_{\alpha\beta} \to Spin(n) \times S^1 \to Spin^c(n).$$

The S^1 cocycle $(z_{\alpha\beta}^2)$ defines the line bundle $\mathfrak{L}_\sigma$. It this case something more happens. The collection $(z_{\alpha\beta})$ is also an S^1-cocycle defining a complex Hermitian line bundle $\hat{L}$ such that $\hat{L}^2 \cong \mathfrak{L}_\sigma$. Traditionally, $\hat{L}$ is denoted by $\mathfrak{L}_\sigma^{1/2}$ though the square root may not be uniquely defined.

We can now regard $\mathbb{S}_\sigma(M)$ as a bundle associated to the trivial $Spin(n)$-bundle P_{Spin} and as such there exists an isomorphism of complex $Spin(n)$ vector bundles

$$\mathbb{S}_\sigma(M) \cong \mathbb{S}(M) \otimes \mathfrak{L}_\sigma^{1/2}.$$

As in the exercise 10.1.18 of Subsection 10.1.9 we deduce that twisting the canonical connection on the bundle of pure spinors $\mathbb{S}_0(M)$ with any Hermitian connection on $\mathfrak{L}_\sigma^{1/2}$ we obtain a Clifford connection on $\mathbb{S}_\sigma(M)$. Notice that if the collection

$$\{\omega_\alpha \in \underline{u}(1) \otimes \Omega^1(U_\alpha)\}$$

defines a connection on $\mathfrak{L}_\sigma$, i.e.

$$\omega_\beta = \frac{dz_{\alpha\beta}^2}{z_{\alpha\beta}^2} + \omega_\alpha \text{ over } U_{\alpha\beta}$$

then the collection

$$\hat{\omega}_\alpha = \frac{1}{2}\omega_\alpha$$

defines a Hermitian connection on $\hat{L} = \mathfrak{L}_\sigma^{1/2}$. Moreover if F denotes the curvature of $(\omega.)$ then the curvature of $(\hat{\omega}.)$ is given by

$$\hat{F} = \frac{1}{2}F. \tag{10.2.5}$$

Hence any connection on $\mathfrak{L}_\sigma$ defines in an unique way a Clifford connection on $\mathbb{S}_\sigma(M)$.

Assume now that TM is not necessarily trivial. We can however cover M by open sets (U_α) such that each TU_α is trivial. If we pick from the start a connection on $\mathfrak{L}_\sigma$ this induces a Clifford connection on each $\mathbb{S}_\sigma(U_\alpha)$. These can be glued back to a Clifford connection on $\mathbb{S}_\sigma(M)$ using partitions of unity. We let the reader check the connection obtained in this way is independent of the various choices.

Example 10.2.28. Assume (M, g) is both complex and spin. Then a choice of a spin structure canonically selects a square root ${K_M}^{-1/2}$ of the line bundle K_M^{-1} because K_M^{-1} is the line bundle associated to the *spin*^{c} structure determined by the complex structure on M. Then

$$\mathbb{S}_{\mathbb{C}} \cong \mathbb{S}_0 \otimes {K_M}^{-1/2}.$$

Any Hermitian connection on K_M induces a connection of $\mathbb{S}_{\mathbb{C}}$. If M happens to be Kähler then the Levi-Civita connection induces a complex Hermitian connection K_M and thus a Clifford connection on $\mathbb{S}_{\mathbb{C}}(M) \cong \Lambda^{0,*}T^*M$. □

Let ω be a connection on $\mathfrak{L}_\sigma$. Denote by ∇^ω the Clifford connection it induces on $\mathbb{S}_\sigma(M)$ and by $\mathfrak{D}_\omega$ the associated geometric Dirac operator. Since the Weitzenböck remainder of this Dirac operator is a *local object* so to determine its form we may as well assume $\mathbb{S}_\sigma = \mathbb{S} \otimes \mathfrak{L}_\sigma^{1/2}$. Using the computation of Exercise 10.1.18 and the form of the Weitzenböck remainder for the *spin* operator we deduce

$$\mathfrak{D}_{\sigma,\omega}^2 = (\nabla^\omega)^* \nabla^\omega + \frac{1}{4}s + \frac{1}{2}\mathfrak{c}_{\mathfrak{L}_\sigma}(F(\omega)),$$

where F denotes the curvature of the connection ω on $\mathfrak{L}_\sigma$. Since $F(\omega)$ has the form

$$F = \mathbf{i} \times \Omega, \quad \Omega \in \Omega^2(M)$$

we deduce

$$\mathfrak{c}_{\mathcal{L}_\sigma}(F) = \mathbf{i}\mathfrak{q}(\Omega) \in Cl(T^*M) \otimes \mathbb{C}.$$

Hence

$$\mathfrak{D}^2_{\sigma,\omega} = (\nabla^\omega)^*\nabla^\omega + \frac{1}{4}s + \frac{\mathbf{i}}{2}\mathfrak{q}(\Omega). \tag{10.2.6}$$

Exercise 10.2.12. Consider the canonical *spin*^{c} structure σ^c on a compact Kähler manifold (M, g, J). The Levi-Civita connection induces a Clifford connection on K_M^{-1} and thus a connection on the associated bundle of spinors

$$\mathbb{S}_{\mathbb{C}}(M) = \Lambda^{0,*}T^*M.$$

(a) Show that the associated *spin*^{c} Dirac operator coincides with the Dolbeault operator.
(b) Assume $L \to M$ is a Hermitian line bundle over M equipped with a Hermitian connection and denote by $\mathfrak{D}_L$ the corresponding *spin*^{c} Dirac operator on

$$\mathbb{S}_{\sigma^c \otimes L} \cong \mathbb{S}_{\mathbb{C}} \otimes L \cong \Lambda^{0,*}T^*M \otimes L.$$

Prove that

$$\Box_L : C^\infty(\Lambda^{0,q}T^*M) \subset C^\infty(\Lambda^{*,q}T^*M),$$

where $\Box_L$ is the generalized Laplacian $\mathfrak{D}^2_L$.
(c)(Kodaira) Assume L is a *positive* line bundle, i.e.

$$h(X, Y) := c_1(\nabla)(X, JY) = \frac{\mathbf{i}}{2\pi}F_L(\nabla)(X, JY) \quad \forall X, Y \in \mathrm{Vect}\,(M).$$

defines a Riemann metric on M. Denote by $H^{0,q}(L)$ the kernel of the restriction of $\Box_L$ to $\Lambda^{0,q}T^*M$. Show that there exists $n_0 \geq 0$ such that

$$H^{0,q}(L^{\otimes n}) = 0 \;\; \forall n \geq n_0, \;\; \forall q \geq 1.$$

$\Box$

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Index