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Integration of third order ordinary differential equations possessing two-parameter symmetry group by Lie's method

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Abstract The solution of a class of third order ordinary differential equations possessing two parameter Lie symmetry group is obtained by group theoretic means. It is shown that reduction to quadratures is possible according to two scenarios: (1) if upon first reduction of order the obtained second order ordinary differential equation besides the inherited point symmetry acquires at least one more new point symmetry (possibly a hidden symmetry of Type II). (2) First, reduction paths of the fourth order differential equations with four parameter symmetry group leading to the first order equation possessing one known (inherited) symmetry are constructed. Then, reduction paths along which a third order equation possessing two-parameter symmetry group appears are singled out and followed until a first order equation possessing one known (inherited) symmetry are obtained. The method uses conditions for preservation, disappearance and reappearance of point symmetries.

Keywords Ordinary differential equations · Symmetry · Lie's method

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1 Introduction

Providing a unified treatment the Lie group theory has developed into a powerful tool for solving and classifying differential equations even though the theory does not apply to all equations. In order to broaden and complement the already existing results various group theoretic approaches have been devised relying, for example, on hidden[1, 2], and nonlocal symmetries [3, 13]. Specific results on certain classes of nonlocal symmetries with numerous examples were analyzed in [7], while useful comments on hidden symmetries may be found in [8]. In this paper we use hidden and convertible symmetries in order to expand the results related to solutions of third order differential equations possessing two parameter symmetry group. We define a convertible type of symmetry of order $n-1$ as a point symmetry that disappears during the first reduction of order of an ordinary differential equation, remains hidden (non-local) during $n-1$ reductions, and reappears as a point symmetry after n reductions. Convertible symmetries may be regarded as a special class of hidden symmetries of type II [1]. Integration of third order differential equations which admit a three-dimensional solvable and non-solvable symmetry algebra has been previously discussed [6]. In general, third order ordinary differential equations possessing two parameter symmetry group are not solvable. However, due to properties of hidden or convertible symmetries in certain cases this may be possible. Since two parameter group is always solvable one symmetry generator may be used to reduce the order of the initial equation so that the other symmetry is inherited as a local symmetry of the corresponding second order equation. This property has been effectively used in order to obtain the solution of a class of second-order ordinary differential equations not possessing Lie point symmetries [5]. In this approach, the order of the initial second order equation is increased and the solution of the new third order equation is sought in the case that additional symmetries (for example hidden symmetries of type II) appear both in the new third order equation as well as in the reduced second order equation.

For third order differential equations possessing two point symmetries there are two possibilities for the complete reduction i.e. reduction to quadratures:

Case 1. If upon first reduction of order, the obtained second order ordinary differential equation besides the inherited point symmetry possesses at least one more new point symmetry which could be a type II hidden symmetry.

Case 2. Following the first reduction of order the inherited point symmetry is the only point symmetry of the obtained second order differential equation. Further reduction using that symmetry generates a first order equation with one known (inherited) point symmetry so that the initial third order equation may be solved (first order ordinary differential equations have infinite number of point symmetries however generally unknown). The idea is to use properties of hidden (convertible) symmetries that come up along the reduction paths of the fourth order differential equations with four parameter symmetry group that lead to the first order equation possessing one known symmetry. Naturally, this method does not enable the complete classification of third order equations possessing two parameter symmetry group since the source of hidden symmetries may be in equations of order higher than fourth, as well as in contact symmetries.

Although some aspects of Case 1 have been previously discussed in relation to the second order equations not possessing Lie point symmetries [5], we give a detailed account of it for completeness. A procedure similar to Case 2 was applied to the third-order ODEs [4]. Hence, a separate section of the paper is devoted to each case, and the reduction paths pertaining to the four-parameter symmetry groups are presented in the Appendix A.

Table 1 Canonical form of two-dimensional Lie algebras

Type	L_2 structure	Basis of L_2 in canonical variables	
I	$[X, Y] = 0$	$X = \frac{\partial}{\partial y}$	$Y = \frac{\partial}{\partial x}$
II	$[X, Y] = 0$	$X = \frac{\partial}{\partial y}$	$Y = x\frac{\partial}{\partial y}$
III	$[X, Y] = X$	$X = \frac{\partial}{\partial y}$	$Y = y\frac{\partial}{\partial y}$
IV	$[X, Y] = X$	$X = \frac{\partial}{\partial y}$	$Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

2 Case 1: The reduced second order equation possesses an additional, second symmetry

The starting point of the analysis is a third order differential equation

$$y''' = F(x, y, y', y''), \tag{1}$$

possessing two parameter Lie point symmetry group, which is reduced to a second order equation using one of the available symmetries. According to the Lie's classification there are four two-dimensional transitive algebras of vector fields in $\mathbb{R}^2$[9]:

In the following exposition we consider each possibility.

2.1 Type I $X = \partial_y, \ Y = \partial_x$

The general form of a differential equation invariant under the action of symmetries

$$X = \partial_y, \quad Y = \partial_x. \tag{2}$$

is

$$y''' = F(y', y''). \tag{3}$$

If Equation (3) is reduced using vector field X (setting $u = x$ and $v = y'$), a second order differential equation is obtained

$$v'' = F(v, v'), \tag{4}$$

or in terms of the canonical coordinates

$$u'' = \bar{F}(v, u') = -u'^3 F\left(v, \frac{1}{u'}\right). \tag{5}$$

Equation (4) has an inherited symmetry

$$\tilde{Y}^{(1)} = \partial_u,$$

so it may be further reduced to the first order differential equation. In our notation tilde denotes restriction of the symmetry generator to corresponding local coordinates (differential invariants), while superscript index in brackets denotes the order of the prolongation. However in order to completely reduce Equation (3) to quadratures, we suppose that Equation (4) possesses an additional symmetry $\tilde{Z}$ which could be a hidden symmetry of type II. Now, there are also four possibilities for $\tilde{Y}^{(1)}$ and $\tilde{Z}$:

2.1.1 *Type IA*

Comparison of Equation (5) with the canonical form of equation corresponding to Type A shows that this case would imply the existence of three point symmetries for Equation (3) so that it does not belong to the class considered here.

2.1.2 *Type IB*

For this type the canonical form of differential equation is

$$u'' = \bar{F}(v),$$

so that

$$-u'^3 F\left(v, \frac{1}{u'}\right) = \bar{F}(v),$$

or

$$F\left(v, \frac{1}{u'}\right) = -\frac{1}{u'^3}\bar{F}(v).$$

Recalling that

$$u' = \frac{1}{v'},$$

yields

$$v'' = v'^3 F(v).$$

Consequently, the initial third order equation corresponding to Type B is

$$y''' = y''^3 f(y'). \tag{6}$$

2.1.3 *Type IC*

This case

$$-vF\left(v, \frac{1}{u'}\right) = \frac{1}{u'^3}\bar{F}(u'),$$

would imply the existence of three point symmetries, so it does not belong to the class considered here.

2.1.4 *Type ID*

The condition

$$-u'^3 F\left(v, \frac{1}{u'}\right) = u'\bar{F}(v),$$

yields

$$y''' = y''^2 f(y) \tag{7}$$

For Type IB the hidden symmetry coincides with the contact symmetry while this is not the case for ID.

2.2 Type II $X = \partial_y,\ \ Y = x\partial_y$

As for Type IA, Type IIA would imply existence of three symmetries in the initial third order equation, so this case does not belong to the class of interest. The same applies to Type IIB and Type IIC. Following essentially the same procedure as for Type I, in a straightforward manner one obtains the following third order equation for the case IID:

2.2.1 *Type IID*

$$y''' = y'' f(x). \tag{8}$$

For this type the hidden symmetry is also a contact symmetry.

2.3 Type III $X = \partial_y,\ \ Y = y\partial_y$

The form of (1) invariant under $X = \partial_y$ is

$$y''' = f(x, y', y''),$$

while the form of f under the prolongation of Y is obtained by applying the condition

$$Y^{(3)}(y''' - f)|_{(y'''-f)=0} = 0,$$

yielding

$$y''' = y'' F\left(x, \frac{y''}{y'}\right). \tag{9}$$

In terms of the fundamental differential invariants of the group generated by X and Y Equation (8) may be written as

$$u'' = u' F(v, u') - u'^2.$$

Comparison with the corresponding equations in Table 2, implies that Type IIIA would require three symmetries in the original third order differential equation, so this type does not belong to the class of interest here.

2.3.1 Type IIIB

From

$$u' F(v, u') - u'^2 = \bar{F}(v),$$

Table 2 Canonical forms of $u'' = \tilde{F}(v, u')$ invariant under $\tilde{Y}^{(1)}$ and $\tilde{Z}$

Type	$[\tilde{Y}^{(1)}, \tilde{Z}]$	Vector field		Differential equation
A	0	$\tilde{Y}^{(1)} = \partial_u$	$\tilde{Z} = \partial_v$	$u'' = \bar{F}(u')$
B	0	$\tilde{Y}^{(1)} = \partial_u$	$\tilde{Z} = v\partial_u$	$u'' = \bar{F}(v)$
C	$\tilde{Y}^{(1)}$	$\tilde{Y}^{(1)} = \partial_u$	$\tilde{Z} = u\partial_u + v\partial_v$	$vu'' = \bar{F}(u')$
D	$\tilde{Z}$	$\tilde{Y}^{(1)} = \partial_u$	$\tilde{Z} = u\partial_u$	$u'' = u'\bar{F}(v)$

since

$$\ln y' = u, \quad \text{and} \quad u' = \frac{y''}{y'},$$

it follows in a straightforward manner that the equation we are looking for is

$$y''' = y'\left(f(x) + \left(\frac{y''}{y'}\right)^2\right). \tag{10}$$

2.3.2 Type IIIC

From

$$u' F(v, u') - u'^2 = \frac{1}{v}\bar{F}(u'),$$

and Equation (8) in a straightforward manner one obtains

$$y''' = y'\left(\frac{1}{x}F\left(\frac{y''}{y'}\right) + \left(\frac{y''}{y'}\right)^2\right). \tag{11}$$

2.3.3 Type IIID

From

$$u' F(v, u') - u'^2 = u'\bar{F}(v),$$

the following equation is obtained

$$y''' = y'\left(f(x)\frac{y''}{y'} + \left(\frac{y''}{y'}\right)^2\right). \tag{12}$$

For Type III all hidden symmetries are also contact symmetries of the obtained equations.

2.4 Type IV $X = \partial_y,\ \ Y = x\partial_x + y\partial_y$

In a manner similar to the one applied to Type III, the form of Equation (1) invariant under the canonical representation of Type IV algebra is

$$y'' = \frac{1}{x^2} f(y', xy''). \tag{13}$$

In terms of the fundamental differential invariants of the group generated by X and Y Equation (11) becomes

$$u'' = -u'^3 F\left(v, \frac{1}{u'}\right) - u'^2.$$

As in all previous cases Type IA would imply existence of the third symmetry in the initial third order equation so this case does not belong to the class in focus here.

2.4.1 Type IVB

Equation

$$u'' = -u'^3 F\left(v, \frac{1}{u'}\right) - u'^2 = \bar{F}(v),$$

leads to

$$F\left(v, \frac{1}{u'}\right) = \frac{1}{u'} + \frac{1}{u'^3}\bar{F}(v),$$

and after straightforward calculation to the equation

$$y''' = \frac{y''}{x} + xy''^3 f(y'). \tag{14}$$

2.4.2 Type IVC

From

$$u'' = -u'^3 F\left(v, \frac{1}{u'}\right) - u'^2 = \frac{1}{v}\bar{F}(u'),$$

and

$$F\left(v, \frac{1}{u'}\right) = -\frac{1}{u'^3}\left(\frac{1}{v}\bar{F}(u') + u'^2\right),$$

the following equation is obtained

$$y''' = -\frac{y''}{x} - \frac{xy''^3}{y'} F\left(\frac{1}{xy''}\right). \tag{15}$$

2.4.3 Type IVD

The equation in Table 2, corresponding to Type IV yields

$$u'' = -u'^3 F\left(v, \frac{1}{u'}\right) - u'^2 = u'\bar{F}(v),$$

so that the equation of interest is

$$y''' = -\frac{y''}{x}\Big(1 - y'' F(y')\Big). \tag{16}$$

For Type IVD hidden symmetry is not the contact symmetry of the obtained equation while for Types B and C these symmetries coincide.

Therefore, the class of third order equations that reduce to second order ODEs which posses at least one more new point symmetry (possibly a Type II hidden symmetry) includes the equations whose general form is given by expressions (6) and (7) (Type I), expression (8) (Type II), (9), (10) and (11) (Type III) and (14), (15) and (16) (Type IV). The results are summarized in Table 3.

3 Case 2: Reduction to a first order equation possessing one known (inherited) point symmetry

This approach uses properties of convertible (hidden) symmetries that arise along reduction paths of the

Table 3 Third order equations of Case1 acquiring a second point symmetry upon first reduction of order

Type	I	II	III	IV
A				
B	$y''' = y''^3 f(y')$		$y''' = y'\left(f(x) + \left(\frac{y''}{y'}\right)^2\right)$	$y''' = \frac{y''}{x} + xy''^3 f(y')$
C			$y''' = y'\left(\frac{1}{x}F\left(\frac{y''}{y'}\right) + \left(\frac{y''}{y'}\right)^2\right)$	$y''' = -\frac{y''}{x} - \frac{xy''^3}{y'}F\left(\frac{1}{xy''}\right)$
D	$y''' = y''^2 f(y)$	$y''' = y'' f(x)$	$y''' = y'\left(f(x)\frac{y''}{y'} + \left(\frac{y''}{y'}\right)^2\right)$	$y''' = -\frac{y''}{x}\Big(1 - y''F(y')\Big)$

fourth order differential equations with four parameter symmetry group that lead to the first order equation possessing one known symmetry, hence reducible to quadratures. In our approach the method includes classification of all reduction paths of a fourth order ODE possessing certain four dimensional algebra of local symmetries to a one dimensional ODE, and choosing ones that have third order ODE possessing two parameter symmetry group along one of the paths. Again we stress the fact that complete classification is not attempted here since the source of hidden symmetries may be in the equations of order higher than four as well as in contact symmetries. In the following exposition we use the notation of reference [10], so that $A_{n,m}$ denotes a Lie algebra of dimension n and isomorphism type m. We illustrate the method using algebras $A_{4,1}$ and $A_{4,4}$.

3.1 Example 1: Algebra $A_{4,1}$

The algebra $A_{4,1}$ has a basis

$$X = \partial_y, \quad Y = x\partial_y, \quad Z = x^2\partial_y, \quad U = \partial_x, \tag{17}$$

with nonzero commutator relations

$$[Y, U] = X, \quad [Z, U] = Y. \tag{18}$$

In reference [12] the vector field U is expressed as

$$U = -\partial_x + f(x)\partial_y. \tag{19}$$

However it may be noticed that since canonical coordinates are not uniquely defined and satisfy transformations

$$\begin{aligned} \bar{x} &= F(x) \\ \bar{y} &= y + G(x), \end{aligned} \tag{20}$$

for arbitrary smooth functions F and G (with additional constraint $F^{'}(x) \neq 0$), expression (19) may be reduced to the form in (17). Consequently, the most general invariant fourth order equation admitting the Lie symmetry algebra $A_{4,1}$ has a simpler form

$$\triangle^4\colon\ y^{iv} = F(y'''), \tag{21}$$

where $\triangle^n$ denotes a differential equation of order n. Although Equation (21) may be easily integrated without the use of symmetry methods, it is instructive to analyze the reduction paths for this equation. The reduction scheme of $A_{4,1}$ algebra is presented in Table A.1 of Appendix A. In the notation used superscript N denotes nonlocal symmetry and tilde denotes restriction of the inherited symmetry generator to corresponding fundamental differential invariants, while the superscript in the parenthesis denotes the order of the prolongation.

In the construction of the reduction paths, well known conditions for disappearance, preservation and reappearance of point symmetries have been used. The particular reduction path chosen for clarification of the method is presented below in a schematic manner.

$\triangle^4\colon \mathrm{A}_{4,1}(X, Y, Z, U)$

$\Downarrow Z$

$\triangle^3\colon \mathrm{A}_{2,1}(\tilde{X}^{(1)}; \tilde{Y}^{(1)}); \tilde{U}^{(1),N}$

$\swarrow$ $\qquad\qquad\qquad\qquad$ $\searrow$

$\Downarrow \tilde{X}^{(1)}$ $\qquad\qquad\qquad\qquad$ $\tilde{Y}^{(1)} \Downarrow$

$\triangle^2\colon\ \mathrm{A}_1(\tilde{Y}^{(2)}); \tilde{U}^{(2),N}$ $\qquad\qquad$ $\triangle^2\colon\ \mathrm{A}_1(\tilde{X}^{(2)}); \tilde{U}^{(2),N}$

$\Downarrow \tilde{Y}^{(2)}$ $\qquad\qquad\qquad\qquad$ $\Downarrow \tilde{X}^{(2)}$

$\triangle^1\colon \mathrm{A}_1(\tilde{U}^{(3)})$ $\qquad\qquad\qquad\qquad$ $\triangle^1\colon\ \mathrm{A}_1(\tilde{U}^{(3)})$

In terms of the fundamental differential invariants r and v of the group generated by Z, Equation (21) becomes

$$r^2 v^{'''} + 8rv'' + 12v' = F(r^2 v'' + 6rv' + 6v). \tag{22}$$

Restrictions of the remaining vector fields in terms of r and v are:

$$\begin{aligned} \tilde{X}^{(1)} &= -\frac{2}{r^3}\partial_v, \\ \tilde{Y}^{(1)} &= -\frac{1}{r^2}\partial_v, \\ \tilde{U}^{(1),N} &= \partial_r + \left(r^2 \int v dr - \frac{2v}{r}\right)\partial_v. \end{aligned}$$

In terms of the fundamental differential invariants of symmetry $\tilde{X}^{(1)}$ which we denote as x and y,

$$\begin{aligned} x &= r, \\ y &= -\frac{1}{2} r^3 v. \end{aligned}$$

Table 4 Generators of four-dimensional Lie algebras whose most general ODEs reduce to the third order ODEs possessing two point symmetries

Lie algebra	Generators X, Y, Z, U
$2A_2$	$-x\partial_x,\ \partial_y,\ -y\partial_y,\ \partial_y$
$A_{3,2}\oplus A_1$	$\partial_y,\ -x\partial_x,\ \partial_x + (y + f(x))\partial_y,\ e^x\partial_y$
$A_{3,6} \oplus A_1$	$\partial_y,\ (x^2-1)^{1/2}\partial_y,\ -x(x^2-1)^{1/2}\partial_x + (f(x) - y(x^2-1)^{1/2})\partial_y,\ x\partial_y$
$A^a_{3,7} \oplus A_1 (a > 0)$	$\partial_y,\ x\partial_y,\ -(1+x^2)\partial_x + ((a-x)y + f(x))\partial_y,\ (1+x^2)^{1/2}e^{\arctan(x)}\partial_y$
$A_{3,8} \oplus A_1$	$\partial_y,\ y\partial_y,\ -y^2\partial_y,\ f(x)\partial_x$
$A_{4,1}$	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ -\partial_x + f(x)\partial_y$
$A^a_{4,2}\ (a \neq 0, 1)$	$e^{(1-a)x}\partial_y,\ -\partial_y,\ x\partial_y,\ \partial_x + y\partial_y$
$A_{4,3}$	$\partial_y,\ x\partial_y,\ -x\log(x)\partial_y,\ x\partial_x + (y + f(x))\partial_y$
$A_{4,4}$	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ -\partial_x + (y + f(x))\partial_y$
$A^{a,b}_{4,6}\ (a \neq 0, b \geq 0)$	$(1+x^2)^{1/2}e^{(b-a)\arctan(x)}\partial_y,\ x\partial_y,\ \partial_y,\ (1+x^2)\partial_x + (xy + by)\partial_y$
$A_{4,8}$	$\partial_y,\ \partial_x,\ x\partial_y,\ x\partial_x$
$A_{4,12}$	$\partial_y,\ x\partial_y,\ y\partial_y,\ -(1+x^2)\partial_x - xy\partial_y$

A straightforward calculation turns Equation (21) into

$$y''' = \frac{y''}{x} + xF\left(\frac{y''}{x}\right). \tag{23}$$

This equation has two point symmetries $X = \partial_y$, $Y = x\partial_y$ and reduces to quadrature as shown in the diagram above. Naturally, as presented in the schematic form above, reducing the order of a third order equation in the reduction path using symmetry $\tilde{Y}^{(1)}$ first also leads to a first order equation solvable by quadrature. In terms of the fundamental differential invariants of symmetry $\tilde{Y}^{(1)} = -\frac{1}{r^2}\partial_v$, denoted by ρ and θ, an equation also possessing two parameter symmetry group is obtained however it is of slightly more complicated form. A straightforward calculation for this case yields

$$\rho = r,$$
$$\theta = -r^2 v,$$

so that with $w = \theta'$ Equation (21) becomes

$$w'' + 2\frac{w'}{\rho} - 2\frac{w}{\rho^2} = F\left(w' + 2\frac{w}{\rho}\right). \tag{24}$$

The restriction of the inherited symmetry to the fundamental differential invariants is

$$\tilde{X}^{(2)} = -\frac{2}{\rho^2}\partial_w,$$

while the nonlocal symmetry is

$$\tilde{U}^{(2),N} = \partial_\rho + \frac{2}{\rho^2}\int w d\rho\ \partial_w.$$

Further reduction using $\tilde{X}^{(2)}$ yields the following first order equation

$$y' - 2\frac{y}{x} = x^2 F\left(\frac{y}{x^2}\right), \tag{25}$$

whose symmetry is

$$\tilde{U}^{(3)} = \partial_x + 2\frac{y}{x}\partial_y.$$

Equation (25) is of Riccati type and easily solvable by quadrature.

3.2 Example 2: Algebra $A_{4,4}$

The algebra $A_{4,4}$ has a basis

$$X = \partial_y,\ \ Y = x\partial_y,\ \ Z = x^2\partial_y,\ \ U = -\partial_x + y\partial_y, \tag{26}$$

with nonzero commutator relations

$$[X, U] = X,\ \ [Y, U] = X + U,\ \ [Z, U] = Y + Z. \tag{27}$$

Table 5 Four dimensional real Lie algebras, the corresponding fourth order equations, the operator employed in the first reduction and the obtained third order equation possessing only two point symmetries

Algebra	Δ^4: Fourth-order ODE	1st	Δ^3: Third-order ODE
$2A_2$	$y^{iv}=\frac{y'''}{y'^2}F(\frac{y'y'''}{y''^2})$	X	$y'''=x^4(F(\xi)-6)y'^4-x^2(3F(\xi)-11)y'^3+$ $(3F(\xi)-18)y'^2-(6y''-\frac{(15-(F(\xi))}{x^2})y'+10\frac{y''}{x}+3\frac{y''^2}{y'}$, $\xi=\frac{3y'^2-3y^2y'+2y^4-y''y}{(y^2-y')^2}$
$A_{3,2}\oplus A_1$	$y^{iv}=(y'''-y'')F((y'''-y'')e^{-x})+y''$	Y	$y'''=\frac{2(1-F(\xi))}{x}y+\frac{x=(3-x)F(\xi)}{x}y'+\frac{2(1-F(\xi))}{x}y''$, $\xi=e^{-x}(2y+y'(x-3)-xy'')$
$A_{3,6}\oplus A_1$	$y^{iv}=-\frac{(8x^4y'''+12x^3y''-10x^2y''-9xy''+2y''')x^2}{}$ $-\frac{F(\xi)}{(x^2-1)^2x^3}$ $\xi=(x^2-1)^{1/2}(x^2y'''+3xy''-y''')x^2$	X (or Y)	$y'''=\frac{F(\xi)-(8x^4y''+12x^3y'-10x^2y''-9xy'+2y'')x^2}{(x^2-1)^2x^3}$, $\xi=(x^2-1)^{1/2}(x^2y''+3xy'-y'')x^2$
$A^a_{3,7}\oplus A_1$	$y^{iv}=-\frac{8x^3y'''+12x^2y''+8xy'''+3y''-a^2y''}{(x^2-1)^2x^3}$ $+e^{-\arctan(x)a}(x^2+1)^{-7/2}F(\xi)$, $\xi=e^{\arctan(x)a}(x^2+1)^{3/2}(x^2y'''+(3x+a)y''+y''')$,	X (or Y)	$[y'''((2+x^2)(x^3+ax^2)+x)+$ $y''(8x(x^3+12ax^2+4xa^2+8x+4a)]\frac{e^{\arctan(x)a}}{(1+x^2)^{5/2}}-e^{\arctan(x)a}(1+x^2)^{-7/2}F(\xi)=0$, $\xi=e^{\arctan(x)a}(x^2+1)[y''x^2(a+x)(x^2+1)+y'(3x^2+2a(2x+a)+1)+1)]$
$A_{3,8}\oplus A_1$	$y^{iv}=\frac{-3y''^3+4y'y''y'''+y'^3F(\xi)}{(x^2-1)^2x^3}$, $\xi=\frac{y'''}{y'}-\frac{3y''^2}{2y'^2}$	X	$y'''=y'y''+F(y''-\frac{1}{2}y'^2)$
$A_{4,1}$	$y^{iv}=F(y''')$	Y (or Z)	$y'''=\frac{y''}{x}+xF(\frac{y''}{x})$
$A^a_{4,2}$	$y^{iv}=(a-1)^2y''+e^xF(\xi)$, $\xi=\frac{(a-1)y''+y'''}{e^x}$	Z	$y'''=-\frac{4y''}{x}-y'(a-1)^2+\frac{2(a-1)^2}{x}y+\frac{e^xF(\xi)}{x}$, $\xi=e^{-x}[y'(x(a-1)+3)+y''x+2y(1-a)]$
$A_{4,3}$	$y^{iv}=\frac{2xy''+F(xy''+x^2y''')}{x^3}$	Z	$y'''=\frac{y''}{x}(\frac{7}{\eta}-1)-\frac{y'}{\eta x^2}(3+\frac{14}{\eta})$ $+\frac{y}{x^3}(\frac{14}{\eta^4}+\frac{10}{\eta^2}-\frac{5}{\eta}-4)+\frac{F(\xi)}{x^3}$, $\eta=\log(x)$, $\xi=y\frac{1-\eta}{x\eta^2}-y'\frac{2\eta+1}{\eta}-y''x$
$A_{4,4}$	$y^{iv}=e^{-x}F(y'''e^x)$	Y (or Z)	$y'''=\frac{y''}{x}+xe^{-x}F(e^x\frac{y''}{x})$

(*Continued on next page*)

Table 5 (*Continued*)

Algebra	Δ^4 : Fourth-order ODE	1st	Δ^3: Third-order ODE
$A_{4,6}^{a,b}$	$y^{iv} = \frac{(a-b)^2 y'' - (8x^3 y''' + 12x^2 y'' + 8xy''' + 3y'')}{(x^2+1)^2}, + e^{\arctan(x)b}(x^2+1)^{-7/2}F(\xi)$, $\xi = e^{-\arctan(x)b}(x^2+1)^{3/2}((a-b)y'' + x^2 y''' + 3xy'' + y''')$	Y (or Z)	$y''' = \frac{(a-b)^2 y' - (8x^3 y'' + (12x^2+1)y' + 8xy'')}{(x^2+1)^2} + e^{\arctan(x)b}(x^2+1)^{-7/2}F(\xi)$, $\xi = (x^2+1)^{3/2}[(a-b)y' + (x^2+1)y'' + 3xy']e^{-\arctan(x)b}$
$A_{4,8}$	$y^{iv} = y''^2 F(\frac{y'''^2}{y''^3})$	Y (or Z)	$y'' = 7yy' - 6y^3 - y^2 F(4y - 4\frac{y'}{y} - \frac{y'^2}{y^3})$
$A_{4,12}$	$y^{iv} = \frac{y''F(\xi) - 8xy'''(1+x^2) - 12x^2 y''}{(1+x^2)^2}$, $\xi = \frac{y'''(1+x^2)}{y''} + 3x$	X	$y''' = y'\frac{F(\xi) - 12x^2}{(x^2+1)^2} - \frac{8x(y'' + y'^2)}{1+x^2} + y'(3y'' + y'^2)$, $\xi = (\frac{y''}{y'} + y')(1+x^2) + 3x$

The most general invariant fourth order equation admitting the Lie symmetry algebra $A_{4,4}$ is

$$\triangle^4 : y^{iv} = e^{-x}\ F(e^x y'''). \tag{28}$$

From the reduction paths presented in the Appendix, the only reduction path of interest is presented below in a schematic manner.

$$\triangle^4\text{: } A_{4,4}(X, Y, Z, U)$$
$$\Downarrow Z$$
$$\swarrow \qquad \triangle^3\text{: } A_{2,1}(\tilde{X}^{(1)}; \tilde{Y}^{(1)}); \tilde{U}^{(1),N} \qquad \searrow$$
$$\Downarrow \tilde{X}^{(1)} \qquad\qquad \tilde{Y}^{(1)} \Downarrow$$
$$\triangle^2\text{: } A_1(\tilde{Y}^{(2)}); \tilde{U}^{(2),N} \qquad\qquad \triangle^2\text{: } A_1(\tilde{X}^{(2)}); \tilde{U}^{(2),N}$$
$$\Downarrow \tilde{Y}^{(2)} \qquad\qquad \Downarrow \tilde{X}^{(2)}$$
$$\triangle^1\text{: } A_1(\tilde{U}^{(3)}) \qquad\qquad \triangle^1\text{: } A_1(\tilde{U}^{(3)})$$

Calculations for this algebra are very similar to the ones for algebra $A_{4,1}$ so that the third order equation possessing two parameter symmetry group reducible to quadratures is

$$y''' = \frac{y''}{x} + xe^{-x}F\left(e^x \frac{y''}{x}\right).$$

Following a straightforward calculation the first order equation corresponding to the Equation (25) is

$$y' - 2\frac{y}{x} = x^2 e^{-x}F\left(e^x \frac{y}{x^2}\right).$$

4 Tables

Tables 4 and 5 are presented in order to complement the results of studies on fourth-order ODEs, such as [12], while Table 6 contains reduction paths pertaining to the Case 2 of this work.

In Table 4, we list the generators of the four-dimensional real Lie algebras whose most general ODEs of order four reduce to the third order ODEs possessing only two point symmetries. In this table $f(x)$ denotes an arbitrary function.

In Table 5 we list four-dimensional real Lie algebras, the corresponding fourth-order equations, the operator employed in the first reduction and the obtained third order equation possessing only two point symmetries.

In Table 6 we list paths for complete integration of third order equations possessing two point symmetries.

Table 6 Paths for complete integration of third order equations possessing two point symmetries

$\triangle^4$	1st	$\triangle^3$	2nd	$\triangle^2$	3rd	$\triangle^1$
$2\,A_2$	$\to X$	$A_{2,2}(\tilde{Z}^{(1)}, \tilde{U}^{(1)}); \tilde{Y}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{Z}^{(2)}); \tilde{Y}^{(2),N}$	$\to \tilde{Z}^{(2)}$	$A_1(\tilde{Y}^{(3)})$
$A_{3,2}\oplus\ A_1$	$\to Y$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{X}^{(2)}), \tilde{Z}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
$A_{3,6}\oplus\ A_1$	$\to X$	$A_{2,1}(\tilde{Y}^{(1)}, \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{Y}^{(2)}); \tilde{Z}^{(2),N}$	$\to \tilde{Y}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
	$\to Y$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{Z}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
$A^a_{3,7}\oplus\ A_1$	$\to X$	$A_{2,1}(\tilde{Y}^{(1)}, \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{Y}^{(2)}); \tilde{Z}^{(2),N}$	$\to \tilde{Y}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
	$\to Y$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{Z}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
$A_{3,8}\oplus\ A_1$	$\to X$	$A_{2,1}(\tilde{Y}^{(1)}; \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{Y}^{(1)}$	$A_1(\tilde{U}^{(2)}); \tilde{Z}^{(2),N}$	$\to \tilde{U}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
$A_{4,1}$	$\to Y$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Z}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{Z}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{U}^{(3)})$
	$\to Z$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Y}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{X}^{(1)}$	$A_1(\tilde{Y}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{Y}^{(2)}$	$A_1(\tilde{U}^{(3)})$
			$\to \tilde{Y}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{U}^{(3)})$
$A^a_{4,2}$	$\to Z$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Y}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{X}^{(1)}$	$A_1(\tilde{Y}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{Y}^{(2)}$	$A_1(\tilde{U}^{(3)})$
$A_{4,3}$	$\to Z$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Y}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{X}^{(1)}$	$A_1(\tilde{Y}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{Y}^{(2)}$	$A_1(\tilde{U}^{(3)})$
$A_{4,4}$	$\to Z$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Y}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{Y}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{U}^{(3)})$
$A^{a,b}_{4,6}$	$\to Y$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Z}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{X}^{(1)}$	$A_1(\tilde{Z}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{Z}^{(2)}$	$A_1(\tilde{U}^{(3)})$
	$\to Z$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{Y}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{X}^{(1)}$	$A_1(\tilde{Y}^{(2)}); \tilde{U}^{(2),N}$	$\to \tilde{Y}^{(2)}$	$A_1(\tilde{U}^{(3)})$
$A_{4,8}$	$\to Y$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{U}^{(1)}); \tilde{Z}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{Z}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{Z}^{(3)})$
	$\to Z$	$A_{2,1}(\tilde{X}^{(1)}, \tilde{U}^{(1)}); \tilde{Y}^{(1),N}$	$\to \tilde{U}^{(1)}$	$A_1(\tilde{X}^{(2)}); \tilde{Y}^{(2),N}$	$\to \tilde{X}^{(2)}$	$A_1(\tilde{U}^{(3)})$
$A_{4,12}$	$\to X$	$A^a_{2,2}(\tilde{Y}^{(1)}, \tilde{Z}^{(1)}); \tilde{U}^{(1),N}$	$\to \tilde{Z}^{(1)}$	$A_1(\tilde{U}^{(2)}); \tilde{Y}^{(2),N}$	$\to \tilde{U}^{(2)}$	$A_1(\tilde{Y}^{(3)})$

In the notation used superscript N denotes nonlocal symmetry and tilde denotes restriction of the inherited symmetry generator to the corresponding fundamental differential invariants, while the superscript in the parenthesis denotes the order of the prolongation. Column labels 1st, 2nd and 3rd denote the operators used in the first, second and third reductions. Known conditions for disappearance, preservation and reappearance of point symmetries have been used in the construction of reduction paths along with conditions derived and discussed in [11]. A complete set of paths for each four-dimensional Lie algebra is presented in an extended version of this work [14].

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