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6 Topics in Nonlinear Functional Analysis

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Preface to New Edition

These lecture notes are presented here unchanged from the 1974 edition (except that the proof of Proposition 1.7.2 has been changed).

Since 1974 many books on nonlinear functional analysis have appeared. Furthermore, variational methods in nonlinear functional analysis, which are not discussed here, have seen enormous development. Here are a few more up-to-date references:

Ambrosetti, A., and Prodi, G.: *A Primer of Nonlinear Analysis.* Cambridge Studies in Advanced Mathematics, 34. Cambridge University Press, Cambridge, 1993.

Chang, K.-C.: *Infinite-Dimensional Morse Theory and Multiple Solution Problems.* Progress in Nonlinear Differential Equations and Their Applications, 6. Birkhäuser, Boston, 1993.

Deimling, K.: *Nonlinear Functional Analysis.* Springer, Berlin–New York, 1985.

Ekeland, I.: *Convexity Methods in Hamiltonian Mechanics.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 19. Springer, Berlin, 1990.

Ghoussoub, N.: *Duality and Perturbation Methods in Critical Point Theory.* Cambridge Tracts in Mathematics, 107. Cambridge University Press, Cambridge, 1993.

Ize, J.: *Bifurcation Theory for Fredholm Operators.* Mem. Amer. Math. Soc. **7** (1976), no. 174, viii + 128 pp.

Mawhin, J.: *Topological Degree Methods in Nonlinear Boundary Value Problems.* Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977. CBMS Regional Conference Series in Mathematics, 40. American Mathematical Society, Providence, R.I., 1979.

Mawhin, J., and Willem, M.: *Critical Point Theory and Hamiltonian Systems.* Applied Mathematical Sciences, 74. Springer, New York–Berlin, 1989.

Rabinowitz, P. H.: *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* CBMS Regional Conference Series in Mathematics, 65. American Mathematical Society, Providence, R.I., 1986.

Schechter, M.: *Linking Methods in Critical Point Theory.* Birkhäuser, Boston, 1999.

Struwe, M.: *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer, Berlin, 1990.

Willem, M.: *Minimax Theorems.* Progress in Nonlinear Differential Equations and Their Applications, 24. Birkhäuser, Boston, 1996.

Zeidler, E.: *Nonlinear Functional Analysis and Its Applications.* I. *Fixed-Point Theorems*. Springer, New York–Berlin, 1986.

Zeidler, E.: *Nonlinear Functional Analysis and Its Applications.* II/B. *Nonlinear Monotone Operators*. Springer, New York–Berlin, 1990.

A Russian edition of the notes was published in 1977. It contains an extra section and a very long list of further references.

Preface

In this course we shall take up a variety of topological and analytic techniques for the study of nonlinear problems, and we shall illustrate their use by applications to nonlinear differential and integral equations, primarily to rather simple nonlinear elliptic equations.

We begin with degree of mapping, first in finite dimensions and then in Banach space—the Leray-Schauder degree theory—as well as extensions of this theory. This is used in the study of existence of global solutions of nonlinear problems and also in local, perturbation problems. Concerning the latter we shall spend considerable time on bifurcation problems, i.e., problems in which various solutions may branch from a particular one.

A few topics in the calculus of variations will be treated, such as monotone operators and min-max theorems. We will also study the deep Nash-Moser extension of the implicit function theorem.

Concerning the background for the course, students should know standard linear operator theory. We also assume familiarity with basic notions of differentiable manifolds and differential forms. Almost no knowledge of topology is assumed. Occasionally some well-known results of homotopy theory will be cited without proof.

The principal reference for the course is the book:

Schwartz, J. T.: *Nonlinear Functional Analysis.* Gordon and Breach, New York, 1969, [**11**].

We now list some references to degree theory, applications, and to bifurcation theory. For general background on degree theory:

Krasnosel'skii, M. A.: *Topological Methods in the Theory of Nonlinear Integral Equations.* Macmillan, New York, 1964, [**6**].
Milnor, J. W.: *Topology from the Differentiable Viewpoint.* University Press of Virginia, Charlottesville, Va., 1965, [**8**].
Vainberg, M. M.: *Variational Methods for the Study of Nonlinear Operators.* Holden-Day, San Francisco–London–Amsterdam, 1964, [**13**].

A number of applications of degree theory may be found in the papers of

Zarantonello, E. H. (ed.): *Contributions to Nonlinear Functional Analysis.* Academic Press, New York–London, 1971, [**4**].

For recent developments and extensions of degree theory and fixed-point theory, see:

Granas, A.: *Topics in Infinite Dimensional Topology*. Sém. Collège de France, 1969–70, [**5**].

On bifurcation theory:

Aĭzengendler, P. G., and Vaĭnberg, M. M.: Methods of investigation in the theory of branching of solutions. *Mathematical Analysis 1965 (Russian)*, 7–69. Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1966. Translation in 1–72, *Progress in Math.*, vol. 2. Plenum, New York, 1968, [**1**].

Keller, J. B., and Antman, S. (eds.): *Bifurcation Theory and Nonlinear Eigenvalue Problems*. Benjamin, New York–Amsterdam, 1969, [**3**].

***Rocky Mountain J. Math.*:** Spring 1973, vol. 3, no. 2, the entire issue, [**9**].

Sattinger, D. H.: *Topics in Stability and Bifurcation Theory*. Springer Lecture Notes, No. 309. Springer, Berlin–New York, 1973, [**10**].

Stakgold, I.: Branching of solutions of nonlinear equations. *SIAM Rev.* 13: 289–332, 1971, [**12**].

Vaĭnberg, M. M., and Trenogin, V. A.: The Ljapunov and Schmidt methods in the theory of non-linear equations and their subsequent development. (Russian) *Uspehi Mat. Nauk* 17(2/104): 13–75, 1962. Translation in *Russian Math. Surveys* 17: 1–60, 1962, [**14**].

Many other interesting nonlinear problems are treated in:

Berger, M., and Berger, M.: *Perspective in Nonlinearity*. Benjamin, New York–Amsterdam, 1968, [**2**].

Lions, J. L.: *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Gauthier-Villars, Paris, 1969, [**7**].

Further references are given in the notes and are collected in the bibliography at the end.

A number of people contributed greatly to the course and the notes. The latter part of the course was conducted as a seminar, and the lectures of several participants, though not all, are included here. My warm thanks, in particular, to J. A. Ize for Sections 4.4 through 4.7 and his contributions throughout the notes, and also to E. Zehnder for his generous exposition on general implicit function theorems for Chapter 6 (which he also wrote). I also wish to thank Ralph Artino for writing the notes, and John Tavantzis for catching many errors. In addition, my thanks to Connie Engle for her cheerful and excellent typing.

CHAPTER 1

Topological Approach: Finite Dimensions

1.1. A Simple Remark

Our aim throughout the course is, speaking loosely, to solve nonlinear equations of the form

$$F(x) = 0\,. \tag{1.1}$$

We begin with a very simple result illustrating the use of topology, in particular, homotopy theory, in solving nonlinear problems.

Suppose F is a continuous map[1] of the closed unit ball $B \subset \mathbb{R}^n$ into $\mathbb{R}^k$, and suppose

$$F(x) \neq 0 \quad \text{on } \partial B\,.$$

If we let $\phi : \partial B \to \mathbb{R}^k \backslash \{0\}$ denote the restriction of F to ∂B, then the topological result expresses a condition on ϕ, which implies that for *any* extension[1] F of ϕ to B, the equation (1.1) always possesses a solution.

THEOREM 1.1.1 *Suppose ϕ maps $\partial B = \mathbb{S}^{n-1}$ into $\mathbb{R}^k \backslash \{0\}$; set*

$$\psi = \frac{\phi}{|\phi|} : \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}\,.$$

For every *extension F of ϕ inside B there exists a solution of $F(x) = 0$ if and only if the map $\psi : \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$ is homotopically nontrivial, i.e., cannot be deformed to a constant map.*

This simple result is left as an exercise.

In using the result, different cases have to be distinguished. If $n < k$, as is easily seen, every map $\psi : \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$ is homotopically

trivial, so the theorem is not useful. If $n > k$, the art of homotopy theory is still not such that one can tell whether a given map ψ is homotopically nontrivial. Many examples are known and some will be used in our applications. When $n = k$, the homotopy class of ψ is determined by the "degree" of the map ψ, ψ is homotopically trivial if and only if this degree $= 0$. The topological degree of a map is the first subject we will treat in detail. Intuitively speaking, the degree of a map at some point in the target space is the number of times, counted algebraically, the point is covered.

[1]Throughout the course all the mappings are assumed to be continuous even if not stipulated: very often they are required to be smooth.

1.2. Sard's Theorem

In defining the degree of a mapping, we will make use of a special case of Sard's theorem.

Consider a mapping $F : X \to Y$, where X and Y are open C^∞ (paracompact) manifolds of dimension n and k, respectively, and $F \in C^1 \cap C^{n-k+1}$.

DEFINITION 1.2.1 (a) A point $x_0 \in X$ is a *regular point* of F if, in terms of local coordinates, the Jacobian $\frac{\partial F}{\partial x}(x_0)$ has maximal rank (i.e., $\min(n, k)$).

(b) If x_0 is not a regular point it will be called a *critical point*.

(c) A point y_0 in Y is called a *critical value* of F if its preimage $F^{-1}(y_0)$ contains a critical point; otherwise it is called a *regular value*.

THEOREM 1.2.2 (Sard's Theorem) *If F has the properties above, then the set of its critical values has measure zero in Y.*

REMARKS. (1) Since a set of Lebesgue measure zero in $\mathbb{R}^k$ is mapped by a C^1 mapping into one of measure zero, we see that the notion of a set in Y having measure zero makes good sense. Furthermore, if $k > n$, the whole image $F(X)$ has k-dimensional measure equal to zero, since $F \subset C^1$, and thus Sard's theorem in this case is trivial.

(2) The proof of Sard's theorem in the general form as stated is not simple and will not be given here (see **[15]**).

In defining degree, we will only need Sard's theorem when $F \in C^1$ and $n = k$; we will prove it under these conditions.

PROOF: ($F \in C^1, n = k$): It suffices to consider F on a closed cube C_0 in $\mathbb{R}^n$ with side ℓ. Subdivide this cube into N^n equal pieces by dividing each edge into N pieces. For any pair of points x_0, x in one of these subcubes C, we have

$$F(x) = F(x_0) + \frac{\partial F}{\partial x}(x_0)(x - x_0) + o\left(\frac{1}{N}\right),$$

since the first derivatives of F are uniformly continuous in C_0. If x_0 is a critical point, then $\det\left(\frac{\partial F}{\partial x}(x_0)\right) = 0$, and therefore, the image of C lies in a cylinder with base in a plane of dimension $(n-1)$ and base area $\le C(\frac{\ell}{N})^{n-1}$ and height $\le o(\frac{\ell}{N})$ for some constant $C > 0$. Since there are at most N^n cubes containing critical points, by summing over all these cubes, their images under F are contained in a set of volume less than $N^n o(\frac{1}{N^n})$. Letting $N \to \infty$ the result follows. $\square$

As an illustration of the use of Sard's theorem we present a slightly curious result:

LEMMA 1.2.3 *Let Ω be an open, bounded subset of $\mathbb{R}^2$ and $f : \overline{\Omega} \to \mathbb{R}^2$, $f \in C^2(\Omega) \cap C(\overline{\Omega})$, $f = (f_1, f_2)$. Suppose that* $\det(\frac{\partial f}{\partial x}) = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2}$ *never changes sign, say ≥ 0. Let $x_0 \in \Omega$ be such that* $\det \frac{\partial f}{\partial x}(x_0) > 0$; *then f_1 takes on the value $p_0 = f_1(x_0)$ at some point on the boundary.*

PROOF: Suppose $f_1 \neq p_0$ on $\partial\Omega$. Applying Sard's theorem to $f_1 (n = 2, k = 1)$, we see that the set of regular values of f_1 is dense. Therefore, there are numbers $p_1 < p_0$, $p_2 > p_0$ arbitrarily close to p_0 which are regular values of f_1 and such that no value in $[p_1, p_2]$ is assumed by f_1 on $\partial\Omega$.

Consider

$$\tilde{\Omega} = \{x \in \Omega \mid p_1 < f_1(x) < p_2\};$$

$\tilde{\Omega} \neq \emptyset$ since $x_0 \in \tilde{\Omega}$ and $\partial\tilde{\Omega} = \{x \mid f_1(x) = p_1\} \cup \{x \mid f_1(x) = p_2\}$. Since grad $f_1 \neq 0$ at every point of $\partial\tilde{\Omega}$, it follows that $\partial\tilde{\Omega}$ consists of a finite number of simple, closed C^1 curves γ_i. Using Green's theorem

$$\iint_{\tilde{\Omega}} \det \frac{\partial f}{\partial x}\, dx_1\, dx_2 = \iint_{\tilde{\Omega}} df_1 \wedge df_2 = \int_{\partial\tilde{\Omega}} f_1\, df_2 = \sum_i \int_{\gamma_i} f_1\, df_2 = 0,$$

since f_1 is constant on each γ_i. Hence $\det \frac{\partial f}{\partial x} \equiv 0$ in $\tilde{\Omega}$ contradicting the fact that $x_0 \in \tilde{\Omega}$. □

REMARK. The function f_1 need not assume in $\partial\Omega$ every value that it takes on in Ω. For example, f_1 may be arbitrary and $f_2 \equiv 0$. However, if in addition to the hypotheses above, f satisfies $f = \operatorname{grad} u$ for some u, then it is true that every value of f_1 taken on inside Ω is attained on $\partial\Omega$.

1.3. Finite-Dimensional Degree Theory

Consider C^∞ oriented manifolds X, Y of dimension n (all manifolds are assumed to be paracompact). Before defining the degree, we recall some notions from differential geometry. The operator d of exterior differentiation maps j–forms to $j+1$ forms; in particular, if ω is a smooth $(n-1)$–form on Y

$$\omega = \sum_{j=1}^{n} (-1)^{j-1} g_j(y) dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^n$$

in local coordinates $(y^1, \ldots, y^n)$, then

$$d\omega = \sum_{j=1}^{n} \frac{\partial g_j}{\partial y^j} dy^1 \wedge \cdots \wedge dy^n .$$

For convenience, we write $dy^1 \wedge \cdots \wedge dy^n = \boxed{dy}$. Under a C^1 map $\phi : X \to Y$, forms pullback; in particular, if μ is a smooth n–form on Y,

$$\mu = f(y) \boxed{dy},$$

then its pullback is

$$(\mu \circ \phi)(x) = f(\phi(x)) J_\phi(x) \boxed{dx},$$

where J_ϕ is the Jacobian of the mapping ϕ (in local coordinates). Because of the invariance property, the integral of an n–form μ on an oriented manifold Y has invariant sense. Green's theorem takes the form: If ω is an $(n-1)$–form with compact support on Y, then

$$\int_Y d\omega = 0 .$$

We recall in addition (from advanced calculus) the effect on an integral of a one-to-one smooth change of variable $y = \phi(x)$ with nonsingular Jacobian:

$$\int_{\mathbb{R}^n} f(y)\boxed{dy} = \int_{\mathbb{R}^n} f(\phi(x))|J_\phi|\boxed{dx} .$$

Thus, for an n–form $\mu = f(y)\boxed{dy}$ on Y, we have

$$\int_Y \mu = \operatorname{sgn} J_\phi \int_X \mu \circ \phi ;$$

here X and Y are oriented, J_ϕ is nowhere singular, and ϕ is one-to-one .

We are going to define degree for maps of class C^1 and subsequently for continuous maps. We consider C^∞ oriented (paracompact) manifolds X_0 and Y of dimension n and an open subset X of X_0 with compact closure $\overline{X} = X \cup \partial X$. Let ϕ be a continuous map of $\overline{X}$ into Y, which is of class C^1 in X. Suppose $y_0 \in Y \backslash \phi(\partial X)$ is a regular value of ϕ. We shall first define the degree of the map ϕ at y_0. Since y_0 is a regular value of ϕ, it follows from the implicit function theorem that the set

$$\phi^{-1}(y_0) = \left\{x \in \overline{X} \mid \phi(x) = y_0\right\}$$

consists of isolated points in X. Since the set is compact, it is a finite set:

$$\phi^{-1}(y_0) = \{x_1, \ldots, x_k\} .$$

DEFINITION 1.3.1 If y_0 is a regular value of ϕ, then

$$d(y_0) = \sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) .$$

The following treatment of degree theory is a modification by P. Lax of that given by E. Heinz (in Schwartz' book [**11**]): We will say that a coordinate patch Ω of a point $y_0 \in Y$ is "nice" provided there are suitable coordinates, i.e., a mapping $g : \Omega \to \mathbb{R}^n$, such that $g(\Omega)$ is a cube in $\mathbb{R}^n$.

DEFINITION 1.3.2 Let $\mu = f(y)\boxed{dy}$ be a C^∞ n–form with support contained in a nice coordinate patch Ω of y_0 and lying in $Y \backslash \{\phi(\partial X)\}$ such that $\int_Y \mu = 1$; set

$$\deg(\phi, X, y_0) = \int_X \mu \circ \phi .$$

Differential forms satisfying the above conditions will be called *admissible* for y_0 and ϕ.

That $\deg(\phi, X, y_0)$ is well-defined is a consequence of the following lemma:

Lemma 1.3.3 *Suppose $\mu = f(y)\boxed{dy}$ is a C^∞ n–form on Y with $\int_Y \mu = 0$ and* supp μ *contained in a nice coordinate patch Ω, then there exists an $(n-1)$–form ω such that* supp $\omega \subset \Omega$ *and $\mu = d\omega$.*

Indeed, if ν and μ are admissible for y_0 and ψ, then $\nu - \mu$ satisfies the conditions of Lemma 1.3.3; *hence $\nu - \mu = d\omega$ with* supp $\omega \in \Omega$. *We thus have*

$$\int_X \nu \circ \phi - \int_X \mu \circ \phi = \int_X (\nu - \mu) \circ \phi = \int_X d\omega \circ \phi = \int_X d(\omega \circ \phi) = 0$$

by Green's theorem.

Proof of Lemma 1.3.3: It suffices to assume the supp μ is contained in a cube C in $\mathbb{R}^n$. Thus, given $\mu = f(y)\boxed{dy}$, $\int \mu = 0$, we must show that we can write f as $f(y) = \sum_{j=1}^n \frac{\partial g_j}{\partial y^j}(y)$ with supports of g_j in C for each j. The proof is by induction on the dimension n. When $n = 1$, $g_1(y) = \int_{-\infty}^y f(s)ds$ satisfies $f\,dy = dg_1$. Now suppose the lemma is true in n-dimensions; we wish to prove it in $n+1$ dimensions. Let $y^{n+1} = t$, $(y, t) = (y^1, \ldots, y^n, t)$, and

$$m(y) = \int_{-\infty}^{\infty} f(y,t)dt\,.$$

Since $\int m(y)\boxed{dy} = 0$, by induction

$$m(y) = \sum_{j=1}^{n} \frac{\partial g_j(y)}{\partial y^j} \quad \text{with supp}\, g_j \subset C\,.$$

Let $\tau(t)$ be a C^∞ function with support on the corresponding side of C with

$$\int_{-\infty}^{\infty} \tau(t)dt = 1\,.$$

Consider $f(y,t) - \tau(t)m(y)$; its integral with respect to t is zero, therefore,

$$g(y,t) = \int_{-\infty}^{t} (f(y,s) - \tau(s)m(y))ds$$

satisfies

$$\frac{\partial g}{\partial t} = f(y,t) - \tau(t)m(y)\,,$$

and g has support in C. Thus,

$$f(y, y^{n+1}) = \frac{\partial g}{\partial y^{n+1}}(y, y^{n+1}) + \sum_{j=1}^{n} \frac{\partial g_j(y)}{\partial y^j}\tau(y^{n+1})\,.$$

□

1.4. Properties of Degree

PROPOSITION 1.4.1 *For y_1 close to y_0,* $\deg(\phi . X, y_0) = \deg(\phi . X . y_1)$.

PROOF: If y_1 is sufficiently close to y_0, then any μ admissible for y_0 is admissible for y_1. It follows that the degree of a mapping is constant on any connected component C of $Y \backslash \{\phi(\partial X)\}$, and we shall sometimes write this as $\deg(\phi, X, C)$. □

PROPOSITION 1.4.2 *If y_0 is a regular point of ϕ, then*

$$d(y_0) = \deg(\phi, X, y_0) .$$

Consequently, we see that for $y_0 \notin \phi(\overline{X})$, so that y_0 is a regular value,

$$\deg(\phi, X, y_0) = 0 .$$

PROOF: Let $\phi^{-1}(y_0) = \{x_1, \ldots, x_k\}$. Then there exist disjoint neighborhoods N_i of x_i such that ϕ is a one-to-one mapping on each N_i. Now $N = \bigcap_{i=1}^{k} \phi(N_{x_i})$ is a neighborhood of y_0. Let μ be admissible with support in N, then

$$\deg(\phi, X, y_0) = \int_X \mu \circ \phi = \sum_{j=1}^{k} \int_{N_j} \mu \circ \phi = \sum_j \operatorname{sgn} J_\phi(x_j) \int_Y \mu$$
$$= \sum_j \operatorname{sgn} J_\phi(x_j) = d(y_0) .$$

□

It follows from Propositions 1.4.1 and 1.4.2, that $\deg(\phi, X, y_0)$ is an integer equal to $d(y)$ for any regular value y contained in the same connected component C of y_0 in $Y \backslash \{\phi(\partial Y)\}$.

PROPOSITION 1.4.3 (Homotopy Invariance) *Consider a one parameter family of maps $\phi_t(x) : \overline{X} \times [0, 1] \to Y$, continuous on $\overline{X} \times [0, 1]$ and $C^1(X)$ for each $t \in [0, 1]$. Suppose for all t, $y_0 \notin \phi_t(\partial X)$, then* $\deg(\phi_t, X, y_0)$ *is independent of t.*

PROOF: The set of points $\widetilde{Y} = \{\phi_t(x) \mid x \in \partial X, t \in [0, 1]\}$ is closed and doesn't contain y_0. Choose an admissible μ with support in a small neighborhood of y_0 disjoint from $\widetilde{Y}$; then

$$\deg(\phi_t, X, y_0) = \int_X \mu \circ \phi_t$$

and this is clearly continuous as a function of t. Since $\deg(\phi_t, X, y_0)$ is an integer, it must be constant for all t. □

The same proof yields a sharper form of the homotopy invariance.

PROPOSITION 1.4.4 (Proposition 1.4.3′) *Let X_0, Y be oriented manifolds of dimension n, and consider $X_0 \times [0, 1]$ as a subset of $X_0 \times \mathbb{R}^1$ with the induced topology. Let A be a relatively open subset of $X_0 \times [0, 1]$ with compact closure, and set*

$$A_t = \{x \in X_0 \mid (x, t) \in A\} , \qquad (\partial A)_t = \{x \in X_0 \mid (x, t) \in \partial A\} .$$

Let $y(t)$ be a continuous map of $[0, 1]$ *into* Y*, and let* ϕ *be a continuous map of* $\bar{A}$ *into* $\mathbb{R}^n$ *which is of class* C^1 *in each* A_t *and such that*

$$y(t) \notin \phi((\partial A)_t, t) \quad \textit{for each } t \textit{ in } [0, 1] .$$

Then

$$\deg(\phi(\cdot, t), A_t, y(t)) \quad \textit{is constant for } t \textit{ in } [0, 1] .$$

PROPOSITION 1.4.5 *Suppose* $X_i, i = 1, 2, \ldots,$ *is a sequence of disjoint open sets contained in the interior of* X*. Let* $y_0 \notin \phi(\overline{X} \setminus \bigcup_{i=1}^{\infty} X_i)$*; then* $\deg(\phi, X_i, y_0)$ *is zero except for finitely many* i*, and* $\deg(\phi, X, y_0) = \sum_i \deg(\phi, X_i, y_0)$.

PROOF: Since $\phi(\overline{X} \setminus \bigcup_{i=1}^{\infty} X_i)$ is closed, there exists a neighborhood N of y_0 disjoint from $\phi(\overline{X} \setminus \bigcup_{i=1}^{\infty} X_i)$, and let y be a regular value in N; then

$$\deg(\phi, X, y_0) = \deg(\phi, X, y) , \qquad \deg(\phi, X_i, y_0) = \deg(\phi, X_i, y) .$$

Since y has a finite number of preimages, $\phi^{-1}(y)$ is contained in a finite number of the X_i's and the result follows immediately from Proposition 1.4.2. □

A particular case is

PROPOSITION 1.4.6 (Excision) *If* K *is a closed set contained in* $\overline{X}$ *and* $y_0 \notin \phi(K) \cup \phi(\partial X)$*, then*

$$\deg(\phi, X, y_0) = \deg(\phi, X \setminus K, y_0) .$$

PROOF: Let $X_1 = X \setminus K$ and apply the previous proposition. □

PROPOSITION 1.4.7 *Suppose* X, Y *are manifolds of dimension* n*,* X', Y' *of dimension* m*, and*

$$\phi : X \to Y , \qquad \phi' : X' \to Y' ,$$

such that the degrees are defined at $y_0 \in Y$*,* $y_0' \in Y'$*; then*

$$\deg(\phi \times \phi', X \times X', (y_0, y_0')) = \deg(\phi, X, y_0) \cdot \deg(\phi', X', y_0') .$$

PROOF: Let μ, μ' be admissible for ϕ at y_0 and for ϕ' at y_0', respectively; then $\mu \cdot \mu'$ is an $n + m$ form admissible for $\phi \times \phi'$ at (y_0, y_0'), and

$$\int_{X \times X'} (\mu \cdot \mu') \circ \phi \times \phi' = \int_X \mu \circ \phi \cdot \int_{X'} \mu' \circ \phi' .$$

□

1.5. Further Properties and Remarks

PROPERTY 1.5.1 If ϕ is one-to-one and preserves (reverses) the orientation of X, then at any point $y_0 \in \phi(X)$, $y_0 \notin \phi(\partial X)$, $\deg(\phi, X, y_0) = 1$ (or -1). This follows directly from the definition of the degree of ϕ. In particular, if X and Y are on $\mathbb{R}^n$ and $\phi = \mathrm{Id}$ (or $-\mathrm{Id}$), then for $y_0 \in \phi(X) \cap \{Y \setminus \phi(\partial X)\}$,

$$\deg(\phi, X, y_0) = 1 \quad (\text{or } (-1)^n) .$$

PROPERTY 1.5.2 Suppose $\partial X = \emptyset$ and Y is noncompact and connected. Then $\deg(\phi, X, y)$ is defined for every y in Y, and we claim it is equal to zero. For since $\phi(X)$ is compact, there is a point $y_0 \in Y$ that is not in $\phi(X)$; but then $\deg(\phi, X, y_0) = 0$ and since $\deg(\phi, X, y)$ is independent of y, the result follows.

PROPERTY 1.5.3 (Continuous Maps) An important result of degree theory is the fact that the notion can be extended to maps ϕ which are merely continuous. One does this by approximating ϕ by C^1 maps ϕ_n tending uniformly to ϕ on $\overline{X}$. Using Property 1.4.3 one shows that for n sufficiently large and $y_0 \notin \phi(\partial X)$, $\deg(\phi_n, X, y_0)$ is independent of n, and one then defines this number as $\deg(\phi, X, y_0)$. In case Y is in $\mathbb{R}^n$, such approximations are easily constructed (using, say, mollifiers). In the general case one has to do more work. For instance, using Whitney's embedding theorem, one may suppose that Y is embedded as a regular submanifold of some $\mathbb{R}^N$. One can then use mollifiers to approximate ϕ by smooth maps ψ_n into $\mathbb{R}^N$; projecting these onto Y one obtains the ϕ_n.

We shall not carry out the details here but shall suppose that our degree theory holds for continuous maps. For such maps all the properties of this and the preceding section whose formulations make sense for such maps continue to hold and are proved by approximation by smooth maps.

PROPERTY 1.5.4 Suppose $Y = \mathbb{R}^n$ and suppose ψ is a given continuous map of ∂X into $\mathbb{R}^n \backslash y_0$. Then $\deg(\phi, X, y_0)$ is defined for any continuous extension ϕ of ψ to all of X and is independent of the extension. Indeed, if ϕ_1 is another extension form $\phi_t = t\phi_1 + (1-t)\phi$, $0 \le t \le 1$; by Proposition 1.4.3, $\deg(\phi_t, X, y_0)$ is independent of t. It makes sense then to talk of

$$\deg(\psi, X, y_0).$$

PROPERTY 1.5.5 $\deg(\psi, X, y_0)$ *depends only on the homotopy class of* $\psi : \partial X \to \mathbb{R}^n \backslash y_0$. For if ψ_t, $0 \le t \le 1$ is a homotopy deformation of $\psi = \psi_0$, then, with the aid of Tietze's extension theorem, one extends ψ_t as a map ϕ of $X \times [0, 1]$ into $\mathbb{R}^n$, and applies Proposition 1.4.3 to $\phi_t = \phi|_{X \times \{t\}}$.

We now give a generalization of the formula of Definition 1.3.2 used in defining the degree.

THEOREM 1.5.6 *Let* $\phi : X \to Y$, $\phi \in C(\overline{X})$. *Let* Ω *be a connected component of* $Y \backslash \{\phi(\partial X)\}$, *and* μ *a smooth n–form in Y with compact support in* Ω *and* $\int_Y \mu \neq 0$; *then*

$$\deg(\phi, X, \Omega) = \frac{\int_X \mu \circ \phi}{\int_Y \mu}.$$

EXAMPLE. Let X be a compact, smooth, oriented surface without boundary in $\mathbb{R}^3$, $Y = \mathbb{S}^2$. Let ϕ be the Gauss mapping (spherical map) which takes $x \in X$ into its unit normal at x. Take for μ the area element on $\mathbb{S}^2$, then $\mu \circ \phi = K(x)dA$ where A is the area element on X. K is the Gaussian curvature. According to Theorem 1.5.6,

$$\deg(\phi, X, \mathbb{S}^2) = \frac{\int_X \mu \circ \phi}{\int_{\mathbb{S}^2} \mu} = \frac{1}{4\pi} \int_X K(x)dA.$$

Thus we obtain the Gauss-Bonnet formula: The integral of the Gaussian curvature of a compact. smooth surface without boundary is $4\pi m$ where m is an integer.

PROOF OF THEOREM 1.5.6: Let $\psi_\alpha(y)$ be a partition of unity on Y such that for each α. supp ψ_α is contained in a nice coordinate patch Ω_α. Let $\mu_\alpha = \psi_\alpha \mu$. and choose $y_\alpha \in \operatorname{supp} \mu_\alpha$. From Definition 1.3.2 we have

$$\deg(\phi. X. \Omega) = \deg(\phi. X. y_\alpha) = \frac{\int \mu_\alpha \circ \phi}{\int_{\Omega_\alpha} \mu_\alpha}$$

provided $\int_{\Omega_\alpha} \mu_\alpha \neq 0$. or

$$\deg(\phi. X. \Omega) \cdot \int_{\Omega_\alpha} \mu_\alpha = \int_X \mu_\alpha \circ \phi .$$

The last formula holds in any case since if $\int_{\Omega_\alpha} \mu_\alpha = 0$. we see by Lemma 1.3.3 that $\int_X \mu_\alpha \circ \phi = 0$. Summing over α we obtain the desired result,

$$\deg(\phi. X. \Omega) = \int_Y \mu = \int_X \mu \circ \phi .$$

□

THEOREM 1.5.7 (Composition of Maps-Leray Product) *Let X be as before. and let Y and Z be oriented manifolds of dimension n. Let $\phi : \overline{X} \to Y$. $\psi : Y \to Z$ be continuous maps. If Ω_1 are the connected components of $Y \backslash \{\phi(\partial Y)\}$ having compact closure in Y. then for $z \notin \psi \circ \phi(\partial X)$.*

$$\deg(\psi \circ \phi. X. z) = \sum_i \deg(\phi. X. \Omega_i) \cdot \deg(\psi. \Omega_i. z) .$$

and the sum on the right is finite.

PROOF: We may suppose that $\phi. \psi \in C^1$ and z is a regular value of $\psi \circ \phi$ and of the map ψ. then,

$$\begin{aligned}
\deg(\psi \circ \phi. X. z) &= \sum_{\substack{x \in X \\ \psi \circ \phi(x) = z}} \operatorname{sgn} J_{\psi \circ \phi}(x) \\
&= \sum_{\substack{x \in X \\ \psi \circ \phi(x) = z}} \operatorname{sgn} J_\psi(\phi(x)) \cdot \operatorname{sgn} J_\phi(x) \\
&= \sum_{\substack{y \in Y \\ \psi(y) = z}} \operatorname{sgn} J_\psi(y) \sum_{\substack{x \in X \\ \phi(x) = y}} \operatorname{sgn} J_\phi(x) \\
&= \sum_{\substack{y \in Y \\ \psi(y) = z}} \operatorname{sgn} J_\psi(y) \cdot \deg(\phi. X. y) .
\end{aligned}$$

Now if y is contained in a component of $Y \backslash \phi(\partial X)$ whose closure is not compact. then this component contains points not in $\phi(\overline{X})$. and so $\deg(\phi. X. y) = 0$. Thus.

we may restrict ourselves to Ω_1

$$\deg(\psi \circ \phi, X, z) = \sum_i \deg(\phi, X, \Omega_i) \sum_{\substack{y \in \Omega_i \\ \psi(y)=z}} \operatorname{sgn} J_\psi(y)$$

$$= \sum_i \deg(\phi, X, \Omega_i) \cdot \deg(\psi, \Omega_i, z) .$$

□

The following important corollaries will not be used; their proofs may be found in Schwartz, [**11**, pp. 75–78]:

COROLLARY 1.5.8 (Jordan-Brouwer Theorem) *Let F be a compact set in $\mathbb{R}^n$ such that $\mathbb{R}^n \backslash F$ has a finite number k of components. Let ϕ be a homeomorphism of F into $\mathbb{R}^n$ such that $\phi(F) = G$; then $\mathbb{R}^n \backslash G$ has k components.*

COROLLARY 1.5.9 (Invariance of Domain Theorem) *The image of a continuous one-to-one mapping of an open set in $\mathbb{R}^n$ into $\mathbb{R}^n$ is open.*

REMARK 1.5.10. Let F be a continuous map of the closed unit ball in $\mathbb{R}^n$ into $\mathbb{R}^n$ with $F : \partial B \to \mathbb{R}^n \backslash \{0\}$ so that $\deg(F, B, 0)$ is defined. Consider the normalized map $\psi : \partial B = \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ defined by

$$\psi(x) = \frac{F(x)}{|F(x)|}, \quad |x| = 1 .$$

The degree of ψ is defined for every point in $\mathbb{S}^{n-1}$ and has the same value. So we can write $\deg(\psi, \mathbb{S}^{n-1}, \mathbb{S}^{n-1})$. We claim that

$$\deg(F, B, 0) = \deg\left(\psi, \mathbb{S}^{n-1}, \mathbb{S}^{n-1}\right) .$$

Indeed, we may suppose F is $C^1(B)$. Now $\deg(F, B, 0)$ depends only on $F|_{\partial B}$ and there we can deform F to ψ by

$$\psi_t(x) = \frac{F(x)}{|F(x)|^t}, \quad 0 \le t \le 1 .$$

So we may suppose $F = \psi$ on ∂B. Since $\deg(F, B, 0)$ is independent of any extension of ψ inside B, we may extend F inside B as

$$G(0) = 0 \quad \text{and} \quad G(x) = |x|^2 \psi\left(\frac{2}{\|x\|}\right) \quad \text{for } x \neq 0 .$$

Now let $y \in \mathbb{S}^{n-1}$ be a regular value of ψ, then for $\varepsilon > 0$ small, $y_0 = \varepsilon^2 y$ is a regular value of F. If $\psi^{-1}(y) = \{x_1, \ldots, x_k\}$, then $G^{-1}(\varepsilon^2 y) = \{\varepsilon x_1, \ldots, \varepsilon x_k\}$. We thus see that

$$\operatorname{sgn} J_\psi(x_j) = \operatorname{sgn} J_G(\varepsilon x_j), \quad j = 1, \ldots, k ,$$

and the result follows.

1.6. Some Applications to Nonlinear Equations

B will denote the closed unit ball in $\mathbb{R}^n$.

SPECIAL 1.6.1 *Let $\phi : B \to \mathbb{R}^n$ such that $\phi(x)$ never points opposite to x for $x \in \partial B$, i.e.,*

$$\phi(x) + \lambda x \neq 0 \quad \textit{for all } \lambda \geq 0,\ x \in \partial B\,.$$

Then $\phi(x) = 0$ has a solution inside B.

PROOF: By hypothesis $\phi(x) \neq 0$ on ∂B, so $\deg(\phi, B, 0)$ is defined. Deform ϕ on ∂B using the deformation

$$\phi_t(x) = t\phi(x) + (1-t)x\,, \quad 0 \leq t \leq 1\,.$$

By hypothesis, $\phi_t(x) \neq 0$ for $x \in \partial B$; hence

$$\deg(\phi, B, 0) = \deg(\phi_t, B, 0) = \deg(\mathrm{Id}, B, 0) = 1\,.$$

□

REMARK. The same conclusion holds if on ∂B, $\phi(x)$ never points in the same direction x, i.e.,

$$\phi(x) \neq \lambda x \quad \text{for all } \lambda \geq 0\,,$$

simply apply the preceding to $-\phi(x)$.

THEOREM 1.6.2 (Brouwer Fixed-Point Theorem) *Suppose that $F : B \to \mathbb{R}^n$, $F \in C^0(B)$ and $F(\partial B) \subset B$, then F has a fixed point.*

PROOF: Set $\phi(x) = x - F(x)$, on ∂B. Now suppose $\phi(x) \neq 0$ for $x \in \partial B$, otherwise we are through. Then $\phi(x)$ never points opposite $x \in \partial B$. Indeed, if

$$x - F(x) + \lambda x = 0 \quad \text{for some } \lambda \geq 0\,,$$

then

$$F(x) = (1+\lambda)x\,.$$

Now $\lambda > 0$ is impossible since $\|Fx\| \leq 1$. If $\lambda = 0$, $F(x) = x$ on ∂B, which we have ruled out. So by the previous result, $F(x) - x = 0$ has a solution inside B. □

REMARK. The Brouwer fixed-point theorem holds in the form: A continuous map of a closed, bounded, convex set in $\mathbb{R}^n$ into itself has a fixed point.

The proof is left as an exercise.

SPECIAL 1.6.3 *Suppose $\phi(x)$ is a continuous mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\frac{(\phi(x), x)}{|x|} \to +\infty \tag{1.2}$$

uniformly as $|x| \to \infty$; then ϕ is onto *$\mathbb{R}^n$, i.e., for every $y \in \mathbb{R}^n$, the equation*

$$\phi(x) = y$$

has a solution.

PROOF: We may suppose $y = 0$, since we may replace $\phi(x)$ by $\phi(x) - y$, which continues to satisfy (1.2). For some $R > 0$, we have

$$(\phi(x), x) \geq 0 \quad \text{if } |x| = R .$$

Suppose $\phi(x) \neq 0$ for $|x| = R$; otherwise we are through. Then $(\phi(x), x) \geq 0$ implies that $\phi(x)$ never points opposite to x for $|x| = R$, i.e.,

$$\phi(x) + \lambda x \neq 0 \quad \text{for } \lambda \geq 0,\ |x| = R ,$$

and the result follows from 1.6.1. □

This lemma was used in one of the early proofs of the Brouwer fixed-point theorem, and will be applied later in the course:

LEMMA 1.6.4 (Knaster, Kuratowski, and Mazurkiewicz Lemma [**16**]) *Let X be an arbitrary set in $\mathbb{R}^n$. To each $x \in X$, assign a closed set $F(x)$ in $\mathbb{R}^n$ satisfying:*

(i) *For one point $x_0 \in X$, $F(x_0)$ is compact.*
(ii) *For any finite subset $x_1, \ldots, x_k$ of X, the convex hull of $x_1, \ldots, x_k$ is contained in $\bigcup_{i=1}^k F(x_i)$; then*

$$\bigcap_{x \in X} F(x) \neq \emptyset .$$

We shall present a proof due to H. Brézis.

PROOF: Since the sets $G(x) = F(x_0) \cap F(x)$ are all compact, to show that $\bigcap_{x \in X} G(x)$ is nonempty, it suffices to show the family $\{F(x)\}_{x \in X}$ has the finite intersection property. Suppose this were false, then there would be a finite set $x_1, \ldots, x_k$ such that

$$\bigcap_{i=0}^{k} F(x_i) = \emptyset .$$

Let $U_i = \widetilde{F(x_i)} =$ the complement of $F(x_i)$; then

$$\bigcup_{i=0}^{k} U_i = \mathbb{R}^n .$$

Let ψ_i be a partition of unity in $\mathbb{R}^n$ subordinate to the cover U_i, i.e.,

$$\sum_{i=0}^{k} \psi_i(x) \equiv 1 , \quad \operatorname{supp} \psi_i \subset U_i .$$

Consider $\phi(x) = \sum_{i=0}^{k} \psi_i(x) x_i$. For any x, $\phi(x)$ is contained in the closed, convex hull of $(x_1, \ldots, x_k)$. So ϕ maps the convex hull K of $(x_1, \ldots, x_k)$ into itself. By the (extended) Brouwer fixed-point theorem, $\phi(x)$ has a fixed point $\bar{x}$ in K,

$$\bar{x} = \sum_{k=0}^{k} \psi_i(\bar{x}) x_i .$$

After reordering the indices if necessary, we may suppose that for some s (possibly k),

$$\psi_i(\bar{x}) \neq 0 \quad \text{for } i \leq s\,, \qquad \psi_i(\bar{x}) = 0 \quad \text{for } i > s\,,$$

so that $\bar{x}$ is in the convex hull of $x_1, \ldots, x_s$. By our hypothesis, $\bar{x} \in \bigcup_{i \leq s} F(x_i)$, so $\bar{x} \in F(x_i)$ for some $i \leq s$. But this implies $\bar{x} \notin U_i$ and hence $\psi_i(\bar{x}) = 0$, a contradiction. □

1.7. Borsuk's Theorem

THEOREM 1.7.1 (Borsuk's Theorem) *Let X be a bounded open subset of $\mathbb{R}^n$ symmetric about the origin such that $0 \in X$. Let $\psi : \partial X \to \mathbb{R}^n \backslash \{0\}$ be a continuous odd mapping (i.e., $\psi(-x) = -\psi(x)$); then* $\deg(\psi, X, 0)$ *is odd.*

We note by hypothesis that the $\deg(\psi, X, 0)$ is defined and independent of any extension of ψ inside X. The proof of Borsuk's theorem is based on

PROPOSITION 1.7.2 *Suppose X is an open bounded subset of $\mathbb{R}^n$ symmetric about 0 such that $0 \notin \overline{X}$, $\psi : \partial X \to \mathbb{R}^n \backslash \{0\}$ is a continuous odd mapping; then* $\deg(\psi, X, 0)$ *is even.*

PROOF OF BORSUK'S THEOREM: For $\varepsilon > 0$ sufficiently small, $B_\varepsilon = \{x : |x| \leq \varepsilon\} \cap \partial X = \emptyset$. Let ϕ be any extension of ψ that is the identity map on B_ε; then

$$\deg(\psi, X, 0) = \deg(\phi, X, 0) = \deg(\phi, X \backslash B_\varepsilon, 0) + \deg(\phi, \text{int } B_\varepsilon, 0)$$

from Proposition 1.4.5. By Proposition 1.7.2, $\deg(\phi, X \backslash B_\varepsilon, 0)$ is even while $\deg(\phi, \text{int } B_\varepsilon, 0) = \deg(\text{identity}, \int B_\varepsilon, 0) = 1$; thus $\deg(\psi, X, 0)$ is odd. □

It seems natural to try to prove Proposition 1.7.2 in the following manner: Consider

PROBLEM 1.7.3 Under the hypothesis of Proposition 1.7.2, is it true that for any $\varepsilon > 0$, there is an odd C^1 map $\phi_\varepsilon : X \to \mathbb{R}^n$, continuous in $\overline{X}$, with $|\phi_\varepsilon - \psi| < \varepsilon$ on ∂X such that 0 in $\mathbb{R}^n$ is a regular value of ϕ_ε?

If the answer to the problem is in the affirmative, then, for ε sufficiently small, $\deg(\psi, X, 0) = \deg(\phi_\varepsilon, X, 0)$. Since ϕ_ε is odd, we see that $\phi_\varepsilon(x) = 0$ implies $\phi_\varepsilon(-x) = 0$; thus $\phi_\varepsilon^{-1}(0)$ consists of an even number of regular points and consequently $\deg(\phi_\varepsilon, X, 0)$ is even.

Problem 1.7.3 suggests another one:

PROBLEM 1.7.4 If X is as in Proposition 1.7.2 and ϕ is a continuous odd map of X into $\mathbb{R}^n$ with $\phi(\partial X) \in \mathbb{R}^n \backslash \{0\}$, can ϕ be uniformly approximated by odd C^1 maps in X (continuous on $\overline{X}$) for which 0 is a regular value?

REMARK (Added in 2000). Some years after these problems were posed in these lecture notes, James Yorke provided a simple solution of Problem 1.7.4, which we now present in place of the earlier one. The solution relies on a standard form of the transversality theorem.

THEOREM 1.7.5 (Transversality Theorem) *Let X and Λ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. Let F be a smooth (C^∞) map of $X \times \Lambda$ into $\mathbb{R}^m$. Assume that 0 is a regular value of the map F, i.e., for any point $(x_0, \lambda_0) \in X \times \Lambda$ such that*

$$F(x_0, \lambda_0) = 0\,.$$

the total derivative $(\partial x, \partial\lambda) \mapsto F_x(x_0, \lambda_0)\delta_0 + F_\lambda(x_0, \lambda_0)\delta\lambda$ is surjective (from $\mathbb{R}^n \times \mathbb{R}^k$ onto $\mathbb{R}^m$). Then the set

$$\sum = \{\lambda \in \Lambda \mid 0 \textit{ is a regular map of } u \to F(u, \lambda)\}$$

is dense in Λ.

For a proof see Guillemin and Pollack **[21]**.

SOLUTION OF PROBLEM 1.7.4 AND PROOF OF PROPOSITION 1.7.2: Since we may approximate φ by smooth odd maps, we may suppose the given φ is smooth. Let $\Lambda = \mathbb{R}^{n^2}$ = the space of all $n \times n$ matrices A. We apply the transversality theorem in $X \times \Lambda$, with $Y = \mathbb{R}^n$, the mapping

$$F(x, A) = \varphi(x) + Ax\,, \quad x \in X,\ A \in \Lambda\,.$$

We claim that for some fixed, arbitrarily small matrix A, 0 is a regular value of $F(\cdot, A)$. By the transversality theorem, it suffices to show that 0 is a regular value of F. But his is trivial: If $F(x, A) = 0$, we have only to verify that for any $y \in \mathbb{R}^n$, we can solve the linear equation

$$\varphi'(x)\delta x + A\delta x + (\delta A)x = y \quad \text{for } \delta x \in \mathbb{R}^n \text{ and } \delta A \in \Lambda\,.$$

Simply take $\delta x = 0$ and, since $x \neq 0$, we can find a matrix δA so that

$$(\delta A)x = y\,.$$

Problem 1.7.4 is solved and Proposition 1.7.2 is proved. □

APPLICATIONS 1.7.6 (Applications of Borsuk's Theorem) In all of the following applications, X is a bounded, open subset of $\mathbb{R}^n$, symmetric about the origin such that $0 \in X$.

(1) Given an odd mapping $\psi : \partial X \to \mathbb{R}^k \subset \mathbb{R}^n$, $k < n$, then there exists $x \in \partial X$ for which $\psi(x) = 0$.

PROOF: Suppose $\psi(x) \neq 0$ for $x \in \partial X$; by Borsuk's theorem, $\deg(\psi, X, 0)$ is odd. Let ϕ be an extension of ψ to $\overline{X}$ as a map into $\mathbb{R}^k$; then $\deg(\phi, X, 0)$ is odd. But if $y_0 \in \mathbb{R}^n$ is a point close to the origin and not in $\mathbb{R}^k$, we have $\deg(\phi, X, y_0) = \deg(\phi, X, 0)$ while $\deg(\phi, X, y_0) = 0$, a contradiction. □

(2) Let $\psi : \partial X \to \mathbb{R}^k \subset \mathbb{R}^n$, $k < n$, be any continuous map; then there exists a point $x \in \partial X$ such that

$$\psi(x) = \psi(-x)\,.$$

PROOF: Apply (1) to $\psi(x) - \psi(-x)$. □

(3) $X \subset \mathbb{R}^n$ as above. Suppose ∂X is covered by n closed sets $A_1, A_2, A_3, \ldots, A_n$. Then one of them contains a pair of antipodal points x and $-x$.

PROOF: Suppose not. Then $\bigcap_{i=1}^{n} A_i$ is empty. For $x \in \partial X$, let

$$d_i(x) = \text{distance from } x \text{ to } A_i\,;$$

then

$$d(x) = \sum_{i=1}^{n} d_i(x) > 0\,.$$

Consider the map

$$f(x) = \left(\frac{d_1(x)}{d(x)}, \ldots, \frac{d_{n-1}(x)}{d(x)}\right)$$

of ∂X into $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. By the previous result there is a point $x_0 \in \partial X$ such that $f(x_0) = f(-x_0)$. Now x_0 belongs to some A_j, $j \le n$. Suppose x_0 belongs to some A_i with $i < n$; then $d_i(x_0) = 0$, and since $f(x_0) = f(-x_0)$, $d_i(-x_0) = 0$ so x_0 and $-x_0$ belong to A_i. Suppose now $x_0 \notin A_i$ for $i = 1, \ldots, n-1$; then for $i = 1, \ldots, n-1$, $d_i(x_0) > 0$ and hence $d_i(-x_0) > 0$. Thus x_0 and $-x_0$ belong to A_n. □

(4) *Sandwich Problem.* Let A_1, A_2, A_3 be three measurable sets in $\mathbb{R}^3$ with finite volume. Then there is a plane that simultaneously divides their volumes equally. (The sets represent bread, ham, and cheese.)

PROOF: Let $x \in \mathbb{S}^2$ be any unit vector in $\mathbb{R}^3$. If we bring up a plane perpendicular to x from the direction $-\infty \cdot x$, there is a first such plane dividing the volume of A_3 in half, and also a last such plane. Let $P(x)$ be the plane $\perp x$ lying midway between these $P(x) = \{y \mid y \cdot x = c(x)\}$. It is readily verified that $c(x)$ is continuous on $\mathbb{S}^2$. Set

$$v_i(x) = \text{meas}\{y \in A_i \mid y \cdot x > c(x)\}\,, \quad i = 1, 2\,.$$

From the definition of $P(x)$ we see that

$$v_i(x) + v_i(-x) = \text{volume of } A_i\,, \quad i = 1, 2\,.$$

The map $x \to (v_1(x), v_2(x))$ is a continuous map of $\mathbb{S}^2$ into $\mathbb{R}^2$. By (2) there exists $x_0 \in \mathbb{S}^2$ such that $v_j(x_0) = v_j(-x_0)$, $j = 1, 2$, and $P(x_0)$ is then the desired plane. □

The proof clearly yields the same result for n sets in $\mathbb{R}^n$.

1.8. Mappings in Different Dimensions

In Section 1.1 we considered a continuous mapping F of the closed unit ball B in $\mathbb{R}^n$ into $\mathbb{R}^k$ with $F : \partial B \to \mathbb{R}^k \backslash \{0\}$, together with the homotopy class of its normalization ϕ on ∂B; $\phi = F/|F|$ maps $\mathbb{S}^{n-1}$ into $\mathbb{S}^{k-1}$. Up to now, we have concentrated on the case $k = n$; we showed that if the degree of the map $\phi = \deg(F, B, 0)$ is nonzero, then ϕ is not homotopically trivial. The converse (Hopf's theorem) is also true, but we will not prove it here.

Let us consider briefly the case $n > k$. Though much is known about homotopy classes of maps of $\mathbb{S}^{n-1}$ into $\mathbb{S}^{k-1}$, their full classification is still not settled. In particular, if we are given such a map, we still do not know how to tell whether it is homotopically trivial or not—unlike the case $k = n$. where the integral formula in Theorem 1.5.6 may be used by a computer (as long as the computational error is smaller than $\frac{1}{2}$) to calculate the degree. We shall list a few facts for $n > k$ which will be used later: their proofs may be found in any book on homotopy theory. For $n > 1$. any continuous map of $\mathbb{S}^n$ to $\mathbb{S}^1$ is homotopically trivial. i.e.. $\pi_n(\mathbb{S}^1) = 0$ for $n > 1$.

The first nontrivial case is the Hopf map of $\mathbb{S}^3$ to $\mathbb{S}^2$ which we now describe:

Hopf Map. $\mathbb{S}^3$ is the boundary of the unit ball in $\mathbb{R}^4$ that we look on as $\mathbb{C}^2$. with complex coordinates $(z. w)$; $\mathbb{S}^3 = \{|z|^2 + |w|^2 = 1\}$. We may consider $\mathbb{S}^2$ as the Riemann sphere or the complex projective line CP^1, i.e., as points in $\mathbb{C}^2\backslash\{0\}$ with the equivalence relation $(z. w) \sim (\tau z. \tau w)$ for complex $\tau \neq 0$. The Hopf map $\psi : \mathbb{S}^3 \to \mathbb{S}^2$ is defined by $\psi(z. w) =$ the equivalence class of $(z. w)$. Analytically as a map into the unit vectors in $\mathbb{R}^3$,

$$\psi(z, w) = (2\,\mathrm{Re}\,\bar{w}z.\, 2\,\mathrm{Im}\,\bar{w}z,\, |z|^2 - |w|^2)\,. \tag{1.3}$$

The Hopf map is homotopically nontrivial and generates the homotopy group $\pi_3(\mathbb{S}^2)$ of map of $\mathbb{S}^3$ to $\mathbb{S}^2$. We shall make use of the following fact about $\pi_{n+1}(\mathbb{S}^n)$ = the homotopy group of maps of $\mathbb{S}^{n+1}$ to $\mathbb{S}^n$:

(1.4) For $n \geq 3$, $\pi_{n+1}(\mathbb{S}^n)$ is cyclic of order 2, and the generator is obtained from the Hopf map by $(n-2)$-times iterated suspension.

The suspension operation $\sum$ is a geometric construction on maps of $\mathbb{S}^{n-1}$ to $\mathbb{S}^{k-1}$, yielding maps of $\mathbb{S}^n$ to $\mathbb{S}^k$. It is defined as follows:

Suspension. Think of $\mathbb{S}^{n-1}$ as the equator on $\mathbb{S}^n$ and of $\mathbb{S}^{k-1}$ as the equator on $\mathbb{S}^k$. If ψ is a map of $\mathbb{S}^{n-1}$ to $\mathbb{S}^{k-1}$, the suspension construction extends the map ψ to a map $\psi_1 = \sum\psi$ of $\mathbb{S}^n$ to $\mathbb{S}^k$ in the following simple way: ψ_1 maps the north (south) pole of $\mathbb{S}^n$ to the north (south) pole of $\mathbb{S}^k$. If γ is a half great circle on $\mathbb{S}^n$ joining the poles, it hits the equator at some point x. Let γ' be the half great circle joining the poles on $\mathbb{S}^k$ and passing through $\psi(x)$. Define ψ_1 on γ mapping into γ' as a linear map (with respect to arc length). This process defines a continuous extension of ψ to a map ψ_1 of $\mathbb{S}^n$ into $\mathbb{S}^k$.

REMARK. It is clear that if ϕ and ψ are homotopically equivalent maps of $\mathbb{S}^{n-1}$ to $\mathbb{S}^{k-1}$, then their suspensions ϕ_1. ψ_1 are homotopically equivalent. However, suspension may kill homotopy; i.e., ϕ_1 and ψ_1 may turn out to be equivalent even if ϕ and ψ are not. An important fact is that after a finite number m of iterated suspensions ($n-k$ will do) one reaches the so-called stable range after which no more homotopy is killed; i.e., for $j \geq m$, $\sum^{j+1}\psi$ is homotopically nontrivial if and only if $\sum^{j}\psi$ is nontrivial.

DEFINITION A map ψ whose suspensions are all nontrivial is said to have *nontrivial stable homotopy.*

Let us write an analytic expression for the suspension of a map $\psi : \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$. Consider, more generally, a continuous map F of B_n. the closed unit ball in $\mathbb{R}^n$ into $\mathbb{R}^k$ with $F : \partial B_n \to \mathbb{R}^k\backslash\{0\}$ and $\psi = F/|F|$ on ∂B_n. Define the suspension

$F_1 = \sum F$ as a map of B_{n+1}. the closed unit ball in $\mathbb{R}^{n+1}$ into $\mathbb{R}^{k+1}$ by

(1.5) $$F_1(x, t) = (F(x), t) \in \mathbb{R}^{k+1} .$$

Here $x \in B_n$, $-1 \leq t \leq 1$; $B_{n+1} = \{(x, t) \mid |x|^2 + t^2 \leq 1\}$. One can easily see that $\psi_1 = F_1/|F_1|$ on ∂B_{n+1} is homotopic to the suspension of ψ defined above. The j^{th} iterated suspension of F is the map $F_j : B_{n+j} \to \mathbb{R}^{k+j}$

(1.6) $$F_j(x, t) = (F(x), t) \in \mathbb{R}^{k+j}$$

where $x \in B_n$, $t \in B_j$.

If $\psi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, then we note that the degree of the map ψ is unchanged under suspension. If F is an extension of ψ to B_n, then this follows immediately from the formula 1.5 and the results of Remark 1.5.10 and Property 1.4.7. We shall have need of a more general result.

PROPOSITION 1.8.1 *Let Ω be an open, bounded set in $\mathbb{R}^n$ and regard $\mathbb{R}^n$ as a direct sum of $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, so that $x \in \mathbb{R}^n$ has the unique decomposition $x = x_1 + x_2$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$. Consider a map $F : \overline{\Omega} \to \mathbb{R}^n$ of the form*

$$F(x) = x + \phi(x)$$

where $\phi : \overline{\Omega} \to \mathbb{R}^{n_1}$. Suppose $y \in \mathbb{R}^{n_1}$ and $y \notin F(\partial\Omega)$; then

$$\deg(F, \Omega, y) = \deg(F|_{\Omega_1}, \Omega_1, y)$$

where $\Omega_1 = \mathbb{R}^{n_1} \cap \Omega$.

PROOF: We may suppose that $F \in C^1$ in Ω and $y = 0$ in $\mathbb{R}^{n_1}$. For $j = 1, 2$. let $f_j(x_j)$ be C_0^∞ functions in $\mathbb{R}^{n_j}$ with supports near the origin and such that $\int_{R^{n_j}} f_j(x_j)\, \boxed{dx_j} = 1$. From our definition of degree we have

$$\deg(F, \Omega, y) = \int_{\mathbb{R}^n} (f_1 \cdot f_2) \circ F \, dx .$$

Since $\det \partial F/\partial x = \det(I + \phi_{x_1})$, the latter being an $n_1 \times n_1$ determinant, we see that

$$\deg(F, \Omega, y) =$$
$$\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} f_1(x_1 + \phi(x_1 + x_2)) f_2(x_2) |\det(I + \phi_{x_1}(x_1 + x_2))| \boxed{dx_1} \boxed{dx_2} .$$

We may replace $f_2(x_2)$ by a sequence of functions converging to the delta function without changing the degree. Thus we find

$$\deg(F, \Omega, y) = \int_{\mathbb{R}^{n_1}} f_1(x_1 + \phi(x_1)) |\det(I + \phi_{x_1}(x_1))| \boxed{dx_1} = \deg(F|_{\Omega_1}, \Omega_1, y) .$$

□

CHAPTER 2

Topological Degree in Banach Space

2.1. Schauder Fixed-Point Theorem

We wish now to extend our results to infinite-dimensional spaces, in particular, Banach spaces. However, we have to take some care. For instance, the Brouwer fixed-point theorem states that any continuous mapping taking a closed, bounded, convex set $K \subset \mathbb{R}^n$ into K has a fixed point. This is no longer true in infinite dimensions.

EXAMPLE. Let $X = \ell_2$ (i.e., the space of sequences of complex numbers $x = (x_1, x_2, \dots)$ with $\sum_{i=1}^{\infty} |x_i|^2 < +\infty$). Let B be the closed unit ball in ℓ_2, and $f : B \to B$ be defined by $f(x) = (\sqrt{1 - |x|^2}, x_1, x_2, \dots)$. The map $f(x)$ is continuous but has no fixed points. In fact, if $x = (x_1, x_2, \dots)$ were a fixed point of f, then $\|x\| = 1$ since $\|f(x)\| = 1$ for all $\|x\| \leq 1$. On the other hand, $x = (\sqrt{1 - |x|^2}, x_1, x_2, \dots)$ implies $x_1 = 0$, $x_2 = x_1$, $x_3 = x_2$, etc.; hence $x = (0, 0, \dots)$, which contradicts the fact that $\|x\| = 1$.

We see, therefore, that in infinite-dimensional spaces we must require more of f than mere continuity. We shall require compactness.

DEFINITION A continuous map f defined on a set in a Banach space X and mapping into X is called compact if, for every bounded, closed subset Ω, $\overline{f(\Omega)}$ is compact.

THEOREM 2.1.1 *Let Ω be any closed, bounded subset of X. Then $f : \Omega \to X$ is compact if and only if f is a uniform limit of finite-dimensional mappings (i.e., mappings whose ranges lie in finite-dimensional subspaces).*

PROOF: Suppose f is compact, then $\overline{f(\Omega)}$ is a compact subset of X. So given $\varepsilon > 0$, we can cover $\overline{f(\Omega)}$ by open balls $B_1, \dots, B_{j(\varepsilon)}$ with centers, $x_1, \dots, x_{j(\varepsilon)}$, in $\overline{f(\Omega)}$. Let $\psi_i(x)$ be a partition of unity on $\overline{f(\Omega)}$ subordinate to the cover $\{B_i\}_{i=1}^{j(\varepsilon)}$ i.e., $\psi_i(x) \geq 0$,

$$\sum_{i=1}^{j(\varepsilon)} \psi_i(x) = 1\,, \quad x \in \overline{f(\Omega)} \quad \text{and} \quad \psi_i = 0 \quad \text{outside } B_i\,.$$

Set

$$f_\varepsilon(x) = \sum_{i=1}^{j(\varepsilon)} \psi_i(f(x))x_i\,.$$

Then $f_\varepsilon(x)$ belongs to the convex hull of the x_i's. Also,

$$\|f(x) - f_\varepsilon(x)\| = \left\| \sum_{i=1}^{j(\varepsilon)} \psi_i(f(x))[x_i - f(x)] \right\| .$$

Now if $\psi_i(f(x)) > 0$, then $f(x) \in B_i$ and $\|x_i - f(x)\| < \varepsilon$, so $\|f - f_\varepsilon\| < \varepsilon$, uniformly in x. The argument the other way is a simple exercise. □

We can now prove the analogue of the Brouwer fixed-point theorem: the Schauder fixed-point theorem.

THEOREM 2.1.2 *Let Ω be a closed, convex, bounded subset of a Banach space X and $f : \Omega \to \Omega$ a compact map; then f has a fixed point.*

PROOF: Let $f_\varepsilon(x)$ be an ε-approximation of f as above and N_ε be the linear space spanned by $x_1, \ldots, x_{j(\varepsilon)}$. Since Ω is convex and $f_\varepsilon(\Omega)$ is contained in the convex hull of $f(\Omega)$, we have $f_\varepsilon(x) : \Omega \to \Omega \cap N_\varepsilon$. Therefore f_ε maps the closed, bounded set $N_\varepsilon \cap \Omega$, lying in N_ε, into itself. By the Brouwer fixed-point theorem, f_ε has a fixed point x_ε (i.e., $f_\varepsilon(x_\varepsilon) = x_\varepsilon$); let $\varepsilon \to 0$. By compactness $f_\varepsilon(x_\varepsilon)$ has a convergent subsequence, which we again denote by $f_\varepsilon(x_\varepsilon)$. Therefore, $x_\varepsilon = f_\varepsilon(x_\varepsilon) \to x_0$. But

$$\|x_\varepsilon - f(x_\varepsilon)\| = \|f_\varepsilon(x_\varepsilon) - f(x_\varepsilon)\| \leq \varepsilon$$

so

$$f(x_\varepsilon) \to x_0 , \quad \text{hence } f(x_0) = x_0 .$$

□

2.2. An Application

There are many interesting applications of the Schauder fixed-point theorem. We shall present a recent one to the problem of invariant subspace for a bounded, linear operator in Banach space X. It is not yet known whether every continuous linear map $A : X \to X$ has a nontrivial invariant subspace (i.e., $Y \subsetneq X$, $A(Y) \subset Y$). It was proved some years ago that if for some polynomial P, $P(A)$ is compact, then A has a nontrivial invariant subspace.

A more general result with a very simple proof was recently given by Lomonosov:

THEOREM 2.2.1 *If X is a Banach space, $K \neq 0$ is a linear, compact map $X \to X$ and $A : X \to X$ a continuous, linear map commuting with K, then A has a nontrivial invariant subspace.*

We note that if K has a nonzero eigenvalue λ, then $N = \ker(K - \lambda I)$ is finite-dimensional and is invariant under any operator A commuting with K; for if $x \in N$, then $(K - \lambda I)Ax = A(K - \lambda I)x = 0$ so $Ax \in N$. We will prove the following extension of Lomonosov's result, which we learned from Felix Browder.

THEOREM 2.2.2 *X is a Banach space over the real or complex field and $K \neq 0$ is a compact linear operator on X. B is a continuous linear operator on X which commutes with K. Suppose that B is not a multiple of the identity (if X is a real Banach space assume, in addition, that B satisfies no identity of the form $B^2 + cB + pI \equiv 0$, with $p \geq 0$ and c real constants). Then there exists a nontrivial subspace Y which is invariant for all elements in* $\text{comm}(B)$ $=$ *{set of bounded linear maps $A : X \to X$ such that $AB = BA$}.*

PROOF: Suppose not. Assume $\|K\| = 1$ and choose a vector x_0 such that $\|Kx_0\| > 1$. Let $B_1(x_0)$ be the closed unit ball about x_0, then $0 \notin \overline{KB_1(x_0)}$. For any $y \neq 0$,

$$D = \overline{\{z = Ty \mid T \in \text{comm}(B)\}}$$

is a closed invariant subspace for all $T \in \text{comm}(B)$. Hence $D = X-$ since we are assuming there are no invariant subspaces. We can thus find a linear operator A_y in $\text{comm}(B)$ such that

$$\|A_y(y) - x_0\| < 1 .$$

Let

$$N_{A_y} = \{z : \|A_y z - x_0\| < 1\},$$

then N_{A_y} is an open set containing y. Since $C = \overline{KB_1(x_0)}$ is compact, we may cover it with a finite number of open sets

$$N_{A_{y_1}}, \ldots, N_{A_{y_r}}, \quad y_1, \ldots, y_r \in C .$$

Let $\{\beta_j(x)\}_{j=1}^r$ be a partition of unity on C subordinate to the covering $\{N_{A_{y_j}}\}_{j=1}^r$. For $x \in B_1(x_0)$ define

$$\phi(x) = \sum_{j=1}^{r} \beta_j(K(x)) A_{y_j} Kx .$$

If $\beta_j(K(x)) > 0$, then $K(x) \in N_{A_{y_j}}$, and so $\|A_{y_j} K(x) - x_0\| < 1$. Hence $\phi(x)$ is a convex combination of elements in $B_1(x_0)$ and so belongs to $B_1(x_0)$. Thus $\phi : B_1(x_0) \to B_1(x_0)$; furthermore, ϕ is compact. By the Schauder fixed-point theorem, ϕ has a fixed point $\hat{x} \neq 0$ in $B_1(x_0)$. Set

$$K_0(x) = \sum_{j=1}^{r} \beta_j(K\hat{x}) Ay_j Kx .$$

Then K_0 is a compact, linear map having $\hat{x}$ as an eigenvector with eigenvalue 1. Let $M = \ker(K_0 - I)$, then $M \neq \emptyset$ since $\hat{x} \in M$. M is finite-dimensional and invariant under B because B commutes with K_0. (We have now proved Lomonosov's theorem with $B = A$.)

We have $B : M \to M$. If the field is complex, B has an eigenvector $u \neq 0$ in M with eigenvalue ζ, $Bu = \zeta u$. The subspace

$$M_1 = \ker(B - \zeta I),$$

is not all of X, since B is not a multiple of the identity. However, M_1 is invariant for $\text{comm}(B)$, a contradiction. Suppose now the field is real. If B has no eigenvector in

M then M contains a two-dimensional subspace on which the operator $B^2 + cB + pI$ vanishes, for suitable constants c and $p \geq 0$. Setting

$$M_1 = \ker(B^2 + cB + pI)$$

the proof proceeds as before. □

2.3. Leray-Schauder Degree

Let X be a Banach space, Ω a bounded, open subset of X, $\phi : \overline{\Omega} \to X$ a mapping of the form $\phi = I - K$ with K compact, and $y_0 \notin \phi(\partial\Omega)$. We wish to define $\deg(\phi, \Omega, y_0)$. We first note that if S is a closed, bounded set, then $\phi(S) = (I - K)(S)$ is closed in X. Indeed, if $x_n \in S$, $\phi(x_n) \to y$; then $x_n - K(x_n) \to y$. Since K is compact, we can take a converging subsequence again denoted by x_n such that $K(x_n) \to z$. Then $x_n \to z + y = x$ and by continuity $x - K(x) = y$. This implies that $\phi(\partial\Omega)$ is a closed set and so if $y_0 \notin \phi(\partial\Omega)$, y_0 has positive distance δ from $\partial\Omega$. Now let $\varepsilon < \delta/2$ and let K_ε be an ε-approximation of K mapping into a finite-dimensional space $N_\varepsilon = N$ containing y_0. Then $\phi_\varepsilon(x) = x - K_\varepsilon(x) \neq y_0$ on $\partial\Omega$. Consider

$$\phi_\varepsilon|_{N_\varepsilon \cap \overline{\Omega}} : N_\varepsilon \cap \overline{\Omega} \to N_\varepsilon$$

then $\deg(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0)$ is defined.

DEFINITION We set $\deg(\phi, \Omega, y_0) = \deg(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0)$. We claim that this is independent of K_ε and is then well-defined. To prove this we make use of Proposition 1.8.1. Observe first that $\deg(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0)$ is unchanged if $\dim N_\varepsilon$ is increased, i.e., if $M = N_\varepsilon \oplus W$, W finite-dimensional, then

$$\deg(\phi_\varepsilon, M \cap \Omega, y_0) = \deg(\phi_\varepsilon, N \cap \Omega, y_0)\,.$$

This follows immediately from Proposition 1.8.1.

Next suppose K_η is another approximation of K such that $K_\eta : \Omega \to N_\eta$. Let $\widehat{N}$ be a finite-dimensional space containing N_ε and N_η, then, again by Proposition 1.8.1,

$$\deg(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0) = \deg(\phi_\varepsilon, \widehat{N} \cap \Omega, y_0)$$

$$\deg(\phi_\eta, N_\eta \cap \Omega, y_0) = \deg(\phi_\eta, \widehat{N} \cap \Omega, y_0)\,.$$

Letting $\phi_t = t\phi_\varepsilon + (1 - t)\phi_\eta$, we see by homotopy invariance that

$$\deg(\phi_\varepsilon, \widehat{N} \cap \Omega, y_0) = \deg(\phi_\eta, \widehat{N} \cap \Omega, y_0)\,.$$

We also note that if $\phi = I - K : \overline{\Omega} \to X$ and $y_0 \notin \phi(\overline{\Omega})$, then $\deg(\phi, \Omega, y_0) = 0$. Indeed, $\phi(\overline{\Omega})$ is a closed set and so of positive distance from y_0. Hence we can use a finite-dimensional approximation ϕ_ε of ϕ such that $\phi_\varepsilon : N_\varepsilon \cap \Omega \to N_\varepsilon$, $y_0 \notin \phi_\varepsilon(\overline{N_\varepsilon \cap \Omega})$. But then

$$\deg(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0) = 0\,.$$

One may now extend almost all of the results concerning degree of maps in finite-dimensional space of Chapter 1 to our maps $\phi = I - K$ simply by applying these results to the finite-dimensional approximations ϕ_ε. In particular, the results of Propositions 1.4.1 and 1.4.3–1.4.6 and Theorems 1.5.7 and 1.7.1 hold. In addition, the map ϕ need only be defined on $\partial\Omega$, $\phi : \partial\Omega \to X\backslash\{y_0\}$, with $K = I - \phi$ a compact mapping of $\partial\Omega$ into X. Then we have:

> $\deg(\phi, \Omega, y_0)$ depends only on the homotopy class of $\phi : \partial\Omega \to X\{y_0\}$, where the homotopy is to consist of maps of the form
> $$\phi_t(x) = I - \widehat{K}_t(x), \quad 0 \leq t \leq 1,$$
> with $\widehat{K}$ a compact map of $\partial\Omega \times [0, 1]$ into X.

In addition, the theorem on invariance of domain of Section 1.5, as well as Borsuk's theorem of Section 1.7, hold for the maps $\phi = I - K$. We shall assume that these extensions have been carried out.

2.4. Some Compact Operators

We will give many applications of Leray-Schauder degree theory. First, let us remark that integral operators, with nice kernels, acting on function spaces are typical examples of compact operators. As an illustration, consider an integral operator K acting on $X = C[0, 1]$ or $X = L^2(0, 1)$ is

$$Ku(s) = \int_0^1 K(s, t)u(t)dt$$

where the kernel $K(s, t)$ is a continuous function on the closed square $[0, 1] \times [0, 1]$. Then K is a compact linear map of X into itself. This also maintains if $K(s, t)$ is measurable and continuous as a function of s, uniformly in t, and $|K(s,t)|$ uniformly bounded. Let us look at a nonlinear example,

(2.1) $$Ku(s) = \int_0^1 K(s, t)f(t, u(t))dt$$

where $K(s, t)$ is continuous in the closed square and f is a continuous map of $[0, 1] \times \mathbb{R} \to \mathbb{R}$ which is bounded, $|f(t, u)| \leq M$. If $X = C[0, 1]$, the map K is a compact map on any ball

$$\|u\| = \max |u(t)| \leq N.$$

The proofs are left as exercises. As an illustration of the Schauder fixed-point theorem we have

PROPOSITION 2.4.1 *The integral operator* (2.1) *above has a fixed point* $u(s)$, *i.e., a solution of*

$$Ku(s) = u(s).$$

PROOF: For any $u \in C[0, 1]$ we have

$$|Ku(s)| \leq \max_{s,t} |K(s, t)| \cdot M = C_1.$$

Thus K is a compact map of the ball $\|u\| \leq C_1$ into itself and so has a fixed point. □

When applying Leray-Schauder degree theory. we shall often make use of the classical Riesz-Schauder theory of linear compact operators: this may be found in many introductory books on functional analysis. In particular, if K is a linear compact map of a Banach space X into a Banach space Y. then $I-K$ is a Fredholm operator. i.e..

$$\ker I - K = \{x \mid Kx = x\} \quad \text{is finite-dimensional}$$

and

$$\text{range } I - K \text{ is closed in } Y \text{ and has finite codimension.}$$

In addition, the index of $I - K$,

$$\operatorname{ind} I - K \equiv \dim\ker I - K - \operatorname{codim}\operatorname{range} I - K = 0 .$$

Later we shall make use of some standard properties of Fredholm operators (see, for instance, [**24**]).

2.5. Elliptic Partial Differential Equations

Degree theory has played a fundamental role in the treatment of nonlinear elliptic boundary value problems, and we shall present some simple illustrations. First, some basic facts concerning linear elliptic operators.

2.5.1. We shall consider real-valued functions $u(x)$ defined in a bounded region G in $\mathbb{R}^n$ having smooth C^∞ boundary, though everything can be carried over to vector bundles on manifolds. Set

$$\frac{\partial}{\partial x_j} = \partial_j , \quad \partial = (\partial_1, \dots, \partial_n) . \quad \partial^\alpha = \partial^{\alpha_1} \dots \partial^{\alpha_n} .$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i = a nonnegative integer. ∂^∂ is a differential operator of order

$$\sum \alpha_j = |\alpha| .$$

and any linear partial differential operator with real C^∞ coefficients $a_\alpha(x)$ in $\overline{G}$ has the form

$$P = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha .$$

Consider the polynomial associated with the operator P,

$$P(x, \xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

DEFINITION (Ellipticity) The operator P is called *elliptic* if the leading homogeneous part of P does not vanish for $\xi \ne 0$, i.e.,

$$p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ne 0 \quad \text{for } x \in \overline{G}.\ \xi \in \mathbb{R}^n\backslash\{0\} .$$

The most familiar example is the Laplace operator

$$\Delta = \sum_{1}^{n} \partial_j^2 , \qquad p_2(\xi) = \sum \xi_j^2 = |\xi|^2 .$$

In connection with elliptic operators one studies
Boundary Value Problems.

$$Pu = f \text{ in } G , \quad B_j u = g_j \text{ on } \partial G , \quad j = 1, \dots, k , \tag{2.2}$$

where f and g_j are given functions in $\overline{G}$ and ∂G, respectively, and B_j are certain partial differential operators defined at boundary points. We shall suppose that the order of each B_j is less than m. The boundary operators $B = \{B_j\}$ are to be chosen so that the problem (2.2) is, in some sense, well-posed. In the best case this means that there is existence and uniqueness of the solution. This may not hold and one understands the notion of well-posedness in a more general sense. Let us restrict ourselves to functions u satisfying homogeneous boundary conditions $Bu = 0$. For P acting on such functions, we say the boundary value problem is well-posed provided

(1) $\ker P$ belongs to C^∞ and $\dim \ker P = \nu < \infty$
(2) In suitable function spaces X, Y, the operator $P : X \to Y$ is continuous and has closed range in Y, of finite codimension ν^*, i.e., P is Fredholm.

Then

$$\text{index } P = \text{ind } P = \nu - \nu^* .$$

For example, the following are well-posed:

$$\Delta u = f \text{ in } G , \quad u = 0 \text{ on } \partial G ,$$

$$\Delta u = f \text{ in } G , \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial G \quad \left(\frac{\partial}{\partial n} \text{ is normal derivative} \right)$$

$$\Delta u = f \text{ in } G , \quad a(x)\frac{\partial u}{\partial n} + b(x)u = 0 \text{ on } \partial G , \quad a(x) > 0 ,$$

and each has index zero.

Much of the theory of linear elliptic boundary value problems is taken up with the problem of characterizing those boundary conditions $Bu = 0$ leading to well posed problems, and also with investigating which function spaces may serve for X and Y in condition 2. It is a fact of life (unfortunate or not) that the spaces $X = C^{k+m}(\overline{G})$, $Y = C^k(\overline{G})$, are not suitable candidates, though they are the first to spring to mind. (Here $k \geq 0$ is an integer and $C^k(\overline{G})$ represents the space of functions having continuous derivatives up to order k in $\overline{G}$.) For any nonnegative integer k the operator

$$\Delta : C^{k+2} \to C^k , \qquad u = 0 \text{ on } \partial G ,$$

is continuous but does not have closed range in C^k. In fact, for $f \in C^0$ the solution of $\Delta u = f$, $u = 0$ on ∂G, is in general not in C^2.

2.5.2. Hölder Spaces. A suitable choice of function spaces are the *Hölder spaces* $C^{k+\mu}$, $k \geq 0$ an integer and $0 < \mu < 1$; i.e., those functions u in $C^k(\overline{G})$ with finite norm

$$|u|_{k+\mu} = |u|_k + \sum_{|\alpha|=k} \sup_{x \neq y \in G} \frac{|\partial^\alpha u(x) - \partial - \partial^\alpha u(y)|}{|x-y|^\mu}, \tag{2.3}$$

where

$$|u|_k = \sum_{|\alpha| \leq k} \max_{\overline{G}} |\partial k^\alpha u(x)|.$$

It is a fact that for any integer $k \geq 0$ and $0 < \mu < 1$,

$$\Delta : C_0^{k+2+\mu}(\overline{G}) \to C^{k+\mu}(\overline{G})$$

is an isomorphism onto; here the subscript 0 denotes the functions vanishing on the boundary.

For general elliptic operators P the corresponding "nice" boundary operators B have been characterized. These are the so-called coercive, or complementing, or Lopatinsky-Shapira boundary conditions. (See, for example, **[17]** or section 19 in **[20]**.) For these one has the following basic results, assuming, as we always shall, that the orders of the operators B_j are all less than m:

Basic Results. Let $k \geq 0$ be an integer and $0 < \mu < 1$.

(1) The map

$$P : \left\{u \in C^{k+m+\mu}(\overline{G}) \mid Bu = 0 \text{ on } \partial G\right\} \to C^{k+\mu}(\overline{G})$$

is Fredholm and its index I is independent of k.

(2) If u is a suitably generalized solution of

$$Pu = f \in C^{k+\mu}(\overline{G}), \qquad Bu = 0 \quad \text{on } \partial G,$$

then $u \in C^{k+m+\mu}(\overline{G})$. Thus all functions in $\ker P$ belong to C^∞.

(3) For any u in $C^{k+\mu}$ satisfying the nice boundary conditions $Bu = 0$,

$$|u|_{k+m+\mu} \leq C|Pu|_{k+\mu} + C|u|_0 \tag{2.4}$$

where C is a constant depending on the operators and on k, μ but not on u.

It follows that if $\ker P = 0$, then there is a constant C' independent of u such that the stronger inequality

$$|u|_{k+m+\mu} \leq C'|Pu|_{k+\mu} \tag{2.4'}$$

holds.

2.5.3. Sobolev Spaces. Another class of spaces that are suitable for elliptic equations are the spaces $H_{k,p}$, $k \geq 0$ an integer, and $1 < p < \infty$, with norm

$$\|u\|_{k,p} = \left|\int_G \sum_{|\alpha| \leq k} |\partial^\alpha u|^p \, dx\right|^{1/p}$$

the space $H_{k,p}$ is the completion of $C^\infty(G)$ in this norm.

Consider P acting on the functions in $H_{k+m,p}$ satisfying $Bu = 0$ (i.e., $u \in H_{k+m,p}$ and u is the limit in $H_{m,p}$ of C^∞ functions in $\overline{G}$ satisfying $Bu = 0$ in ∂G). Then one has the corresponding results:

Basic Results in $H_{k,p}$.

(1) The map

$$P : \{u \in H_{k+m,p} \mid Bu = 0 \text{ on } \partial G\} \to H_{k,p}$$

is Fredholm and its index is i (as above).

(2) If u is a suitably generalized solution of

$$Pu = f \in H_{k,p}, \qquad Bu = 0 \text{ on } \partial G, \tag{2.5}$$

then $u \in H_{k+m,p}$.

(3) A solution of (2.5) satisfies

$$\|u\|_{k+m,p} \le C\|Pu\|_{k,p} + C\|u\|_{0,1} \tag{2.6}$$

with a constant C independent of u. Furthermore, if $\ker P = 0$, then

$$\|u\|_{k+m,p} \le C'\|Pu\|_{k,p} \tag{2.6$'$}$$

for a constant C' independent of u.

With the aid of the theorem of Arzelà-Ascoli one may prove that for any $M > 0$:

(2.7)
If $k + \mu > k' + \mu'$, the bounded set $|u|_{k+\mu} \le M$ is compact in $C^{k'+\mu'}$.
If $k > k'$, the bounded set $\|u\|_{k,p} \le M$ is compact in $H_{k',p}$.

It follows, therefore, from the basic results that if

$$i = \operatorname{ind} P = 0 \quad \text{and} \quad \ker P = 0,$$

so that P^{-1} exists, then

the maps $P^{-1} : C^{k+\mu} \to C^{k+\mu}$ and $P^{-1} : H_{k,p} \to H_{k,p}$ are compact.

It is naturally very useful to understand the relationships between the spaces $C^{k+\mu}$ and $H_{m,p}$. Some of these are described by the

THEOREM 2.5.1 (Sobolev Embedding Theorem) *Consider functions $u \in H_{m,p}(G)$, G is a bounded domain in $\mathbb{R}^n$ with smooth boundary, m is a positive integer, and $1 \le p < \infty$.*

(i) *If j is an integer, $0 \le j < m$, such that*

$$0 < \frac{1}{q} \equiv \frac{1}{p} - \frac{m-j}{n} \le 1,$$

then $\partial^j u \in L_q(G)$ and $u \in H_{j,q}$, i.e., the inclusion map $H_{m,p} \hookrightarrow H_{j,q}$ is continuous. Furthermore, for $q' < q$ the inclusion map $H_{m,p} \hookrightarrow H_{j,q'}$ is compact.

(ii) *If j is an integer, $0 \le j < m$ such that*

$$0 < \mu \equiv m - \frac{n}{p} - j < 1 ,$$

then $u \in C^{j+\mu}(\overline{G})$.

Proofs may be found in sections 8–11 of the book by A. Friedman cited above.

2.5.4. A Nonlinear Elliptic Equation. We shall present here a simple application of degree theory to a mildly nonlinear elliptic boundary value problem. Let P be an elliptic operator of order m with "nice" associated boundary conditions $Bu = 0$ on ∂G. We wish to solve

$$Pu = g(x, u, \partial^\beta u), \qquad Bu = 0 \quad \text{on } \partial G . \tag{2.8}$$

Here g is a C^∞ function of x in $\overline{G}$ and of u and its derivatives up to order $m-1$ growing less than linearly in these arguments; i.e., for some positive constants $\gamma < 1$ and M,

$$\left|g(x, u, \partial^\beta u)\right| \le M \left(1 + \sum_{|\beta| \le m-1} |\partial^\beta u|\right)^\gamma . \tag{2.9}$$

We shall consider only the simplest case, that is,

$$i = \operatorname{ind} P = 0 \quad \text{and} \quad \ker P = 0 ,$$

so that P^{-1} exists. Set $g(x, u, \partial^\beta u) = G[u]$ and rewrite the equation as

$$u - P^{-1}G[u] = 0 . \tag{2.8'}$$

THEOREM 2.5.2 *Under the hypotheses above,* (2.8) *has a solution in $C^\infty(\overline{G})$.*

To solve (2.8′) we will apply degree theory in the Banach space $X = \{u \in C^{m-1}(\overline{G}) \mid Bu = 0 \text{ on } \partial G\}$. First we seek

A Priori Estimates for the Solution. Fix $p > n$ and suppose there is a solution u in $H_{m,p}$. Then, according to (2.6′) for $k = 0$, we have

$$\|u\|_{m,p} \le C'\|G[u]\|_{0,p} \le C'M \left[\int_G \left(1 + \sum_{|\beta|<m} |\partial^\beta u|\right)^{p\gamma} dx\right]^{1/p} \quad \text{by (2.9).}$$

Since $\gamma < 1$ it follows easily that there is a constant C_1 such that the solution u satisfies

$$\|u\|_{m,p} \le C_1 .$$

If we apply the Sobolev embedding theorem (ii) (noting that $p > n$) we obtain the a priori bound

$$|u|_{m-1} \le C_2 \tag{2.10}$$

for some constant C_2.

PROOF OF THE THEOREM: In the space X as defined above let Ω be the ball in X

$$|u|_{m-1} \le C_2 + 1 .$$

In Ω we define the map

$$\phi(u) = u - P^{-1}G[u] .$$

In view of the a priori estimate (2.10) there is no solution of $\phi(u) = 0$ on $\partial\Omega$. Furthermore, for $u \in \overline{\Omega}$ there is a constant C_3 such that $|G[u](x)| \le C_3$; i.e., fixing $p > n$ as before,

$$\|G[u]\|_{0,p} \le C_4 .$$

Hence, by (2.6′),

$$\|P^{-1}G[u]\|_{m,p} \le C_5 \tag{2.11}$$

and, by the Sobolev embedding theorems,

$$|P^{-1}G[u]|_{m-1+\mu} \le C_6 \quad \text{for } \mu = 1 - \frac{n}{p} .$$

It follows that $P^{-1}G[\cdot]$ is a compact map of $\overline{\Omega}$ into X, and it is easy to verify that this map is continuous. Consequently,

$$\deg(\phi, \Omega, 0) \text{ is defined.}$$

From our estimates it follows that for $\phi_t(u) = u - tP^{-1}G[u]$, $0 \le t \le 1$,

$$\deg(\phi_t, \Omega, 0) \text{ is independent of } t,$$

and hence equals the degree for $t = 0$, namely, one. Thus (2.8′) has a solution in Ω.

To complete our proof we observe that from (2.11) it follows that the solution u is in $H_{m,p}$. Since $u \in \Omega$, it follows easily that $G[u]$ is in $H_{1,p}$. Applying the basic results in $H_{k,p}$ we find that $u \in H_{m+1,p}$ and hence, since $p > n$, $u \in C^m(\overline{G})$. Continuing in this way we find that u is a $C^\infty(\overline{G})$ solution of (2.8). □

EXERCISE Prove the existence of a solution using the Schauder fixed-point theorem in place of degree theory.

2.6. Mildly Nonlinear Perturbations of Linear Operators

We know that for a compact linear map T of a Banach space X into itself, $I - T$ is Fredholm with index zero, and that its adjoint $T^* : X^* \to X^*$ is also compact with $\dim\ker(I - T)^* = \dim\ker(I - T)$. Furthermore, for given y,

$$(I - T)x = y$$

has a solution $x \to x^*(y) = 0$ for all $x^* \in \ker(I - T^*)$. In this section we will present some simple extensions to nonlinear operators.

First we study the effect on a linear operator due to a compact perturbation which is suitably small at infinity.

THEOREM 2.6.1 *Let X and Y be real Banach spaces and $A : X \to Y$ a bounded linear map such that*

(i) $\text{range}(A)$ *is closed.*
(ii) $X_1 = \ker A$ *has a complementing closed subspace* X_2.

Let $K : X \to Y$ *be a nonlinear compact map such that*

(iii) $K(X) \subseteq \text{range}(A)$.
(iv) $K(x) = o(\|x\|)$ *as* $\|x\| \to +\infty$ *uniformly.*

Then

$$\text{range}(A + K) = \text{range}(A),$$

PROOF: Decompose $X = X_1 \oplus X_2$; then $A : X_2 \to \text{range}(A)$ is an isomorphism with a bounded inverse A^{-1} by the closed graph theorem. Write $x = x_1 + x_2$ with $x_1 \in \ker A$, $x_2 \in X_2$. We shall prove more than is claimed, namely, that for each y in $\text{range}(A)$ and each $x_1 \in X_1$ there is a solution x_2 of

$$Ax_2 + K(x_1 + x_2) = y\,.$$

Set $Ax_2 = z \in \text{range}(A)$ and write $x_2 = A^{-1}z$; the last equation takes the form

$$z + K\big(x_1 + A^{-1}z\big) = y\,. \tag{2.12}$$

We will find a solution $z = z(x_1)$ of (2.12) with the aid of degree theory in $\text{range}(A)$ for the map

$$(I + T)z\,, \qquad T(z) = K\big(x_1 + A^{-1}z\big)\,.$$

Keep x_1 and y fixed. In the ball $\|z\| \le R = R(x_1, y)$, for R sufficiently large, $\deg(I + T, \|z\| \le R, y)$ is defined, since there are no solutions for $\|z\| = R$; this follows from hypothesis (iv). By homotopy invariance we see that for $0 \le t \le 1$,

$$\deg(I + T, \|z\| \le R, y) = \deg(I + tT, \|z\| \le R, y) = 1\,.$$

Hence $z + T(z) = y$ has a solution. □

COROLLARY 2.6.2 [22] *If* T *is a nonlinear compact mapping* $X \to X$ *such that*

$$T(x) = T_\infty(x) - K(x)$$

with $T_\infty(x)$ *a linear operator* $X \to X$ *and* $K(x) = o(\|x\|)$ *as* $\|x\| \to \infty$ *uniformly, then*:

(i) $T_\infty(x)$ *is compact.*
(ii) *If* $\text{range}(K) \subset \text{range}(I - T_\infty)$, *the equation*

$$(I - T)x = y$$

has a solution if and only if $y \in \text{range}(I - T_\infty)$.

PROOF: (i) If T_∞ is not compact, there is a sequence of unit vectors x_j such that $\|T_\infty x_i - T_\infty x_j\| \ge \delta > 0$ for all $i \ne j$. Now for R large this implies

$$\|T(Rx_i) - T(Rx_j)\| \ge \delta R - o(R) > 1\,,$$

which contradicts the compactness of T.

(ii) Write $(I - T)x = y$ as $(1 - T_\infty)x + K(x) = y$, then apply the Theorem 2.6.1 with $A = I - T_\infty$.

□

The condition $K(X) \subset \text{range}(A)$ in the previous theorem is rather restrictive and sometimes difficult to verify. We present a variation on this result in which this restriction is dropped. However, we impose other conditions enabling us to use degree theory. It should be remarked that these conditions are not necessarily the best or most natural; many alternative conditions can be invented.

We shall formulate a rather general result, but we shall only prove a special case of it here using degree theory. The general case may be proved following the proof of Theorem 4.1.4 in Section 4.1.

Let X, Y be Banach spaces and $A : X \to Y$ a continuous linear map which is Fredholm of index $i \geq 0$, i.e.,

(i) $\ker A = X_1$ has dimension $d < \infty$ and
(ii) range $A = Y_1$ is closed in Y with codimension $d^* = d - i$.

Decompose as direct sums

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2 = QY \oplus (I - Q)Y,$$

where Q is a projection operator in Y onto Y_1.

THEOREM 2.6.3 *Let $K : X \to Y$ be a nonlinear compact map for which there exist positive constants R_0, ε such that*

(iii)$_1$ $\|QK(x)\| = o(\|x\|)$ *uniformly as* $\|x\| \to \infty$,
(iii)$_2$ $(I - Q)K(x_1 + x_2) \neq 0$ *for* $x_1 \in X_1$, $x_2 \in X_2$ *and* $\|x_1\| \geq R_0$, $\|x_2\| \leq \varepsilon\|x_1\|$, *and*
(iv) *the mapping* $(I - Q)K(x_1)$ *for* $\|x_1\| = R_0$ *into* $Y_2 \backslash \{0\}$ *has nontrivial stable homotopy; i.e., all its suspensions are nontrivial. (In case* $d^* = d$ *this means that the degree of this map at the origin is not zero.)*

Then

(a) *for any* $y_0 \in Y_1$, *there is a solution of*

$$Ax + K(x) = y_0. \tag{2.13}$$

(b) *The same is true for any* $y_0 \in Y$ *if* K *also satisfies (here* $x = x_1 + x_2$*)*
(iii)$_3$ $\|(I - Q)K(x)\| \to \infty$ *uniformly as* $\|x\| \to \infty$ *provided* $\|x_2\| \leq \varepsilon\|x_1\|$.

PROOF IN CASE $d^* = d$: We note first that (b) follows from (a) and that in either case it suffices to consider $y_0 = 0$, for it is easy to see that $K_0(x) = K(x) - y_0$ satisfies the conditions (iii)$_1$, (iii)$_2$, and (iv) with different constants R_0, ε, and hence $A(x) + K_0(x) = 0$ has a solution. So we need only prove (a) with $y_0 = 0$.

Applying Q and $(I - Q)$ to the equation $Ax + K(x) = 0$, we see that it is equivalent to the system

$$Ax_2 + QK(x_1 + x_2) = 0, \qquad (I - Q)K(x_1 + x_2) = 0.$$

Writing $z = Ax_2 \in \text{range}(A)$ we obtain as before

$$z + QK(x_1 + A^{-1}z) = 0, \qquad (I - Q) \cdot K(x_1 + A^{-1}z) = 0.$$

Since X_1 and $(I - Q)Y = Y_2$ have the same dimension d, there is a linear isomorphism $B : X_1 \to Y_2$. Hence, setting $Bx_1 = y_2$, we may rewrite the system as

$$(2.14)\qquad z + QK(B^{-1}y_2 + A^{-1}z) = 0, \qquad (I-Q)K(B^{-1}y_2 + A^{-1}z) = 0.$$

The left-hand sides of these equations may be viewed as an operator of the form $I + C$, C compact, mapping $y = z + y_2 \in Y$ into Y. Here

$$C(z + y_2) = K(B^{-1}y_2 + A^{-1}z) - y_2.$$

We claim that the degree of the map in a large ball $\|y\| \leq R$ is defined. For suppose (2.14) has a solution y on $\|y\| = R$, R large. Then from the first equation in (2.14) and from $(\text{iii})_1$, we see that

$$\|z\| \leq o(\|y_2\| + \|z\|) \quad \text{and hence} \quad \|z\| = o(\|y_2\|).$$

For R sufficiently large we see easily from $(\text{iii})_2$ that the second equation in (2.14) cannot hold.

The preceding argument also shows that for the deformation, $0 \leq t \leq 1$,

$$F_t(y): \begin{array}{l} z + t\, QK(B^{-1}y_2 + A^{-1}z) \\ (I-Q)\cdot K(B^{-1}y_2 + tA^{-1}z), \end{array}$$

we have $F_t(y) \neq 0$ for $\|y\| = R$ large. Thus

$$\deg(F_t, \|y\| \leq R, 0)$$

is independent of t. The map F_0 is simply

$$z + y_2 \hookrightarrow z + (I-Q)K(B^{-1}y_2).$$

This is a product map (in fact suspension) and therefore has the same degree as the finite-dimensional map $(I-Q)K(B^{-1}y_2)$ at the origin. Since B is an isomorphism, this degree $= \pm\deg((I-Q)K(x_1), \|x_1\| \leq R_0, 0) \neq 0$ by (iv). Hence (2.14), and so (2.13), has a solution. □

REMARK. If $(I-Q)Y = Y_2$ has a scalar product $\langle\ ,\ \rangle$, then condition (iv) automatically holds if K satisfies

$$\langle (I-Q)K(x_1), Bx_1\rangle \neq 0 \quad \text{for } \|x_1\| = R_0.$$

The degree of $(I-Q)K(x_1)$ is then ± 1.

EXERCISE Prove the remark.

PROBLEM Using Theorem 2.6.3, formulate and prove an existence theorem for an elliptic boundary value problem of the form (2.8) in which index $P = 0$ but $\ker P \neq 0$.

2.7. Calculus in Banach Space

In this section we will present several forms of the classical implicit function theorem. This is based on the material in [**19, 23**].

Let X and Y be Banach spaces, and $f: X \to Y$ a continuous mapping defined on an open subset of X. Let $B(X, Y)$ denote the set of bounded linear maps $X \to Y$.

DEFINITION f is (Frechet) *differentiable* at $x_0 \in X$ if there exists a bounded linear mapping $A \in B(X, Y)$ such that

$$\|f(x_0 + u) - f(x_0) - Au\| = o(r)$$

for $\|u\| \leq r$ as $r \to 0$.

We list several properties of the Frechet derivative A:

(1) If A exists it is unique; it is sometimes denoted by $f_x(x_0)$, $Df(x_0)$, or $f'(x_0)$.

(2) If $f_x(x_0)$ is a continuous linear map $x_0 \mapsto B(X, Y)$, then f is said to be of class C^1. We can define, inductively, $f \in C^p$, $p = 1, 2, \ldots$, i.e., if $D(D^{p-1} f)(x_0)$ is in

$$B(X, B(X, B(X, \ldots, B(X, Y \underbrace{) \cdots)))}_{p} .$$

(3) Compositions of C^p maps are C^p.

(4) If X and Y are complex Banach spaces, U open in X, and $f : U \to X$ is differentiable at each point in U (with $f_x(x)$ linear over the complex field), then f is said to be *holomorphic* on U. One can show that if S is any finite-dimensional subspace of X and f is continuous, then f is holomorphic iff for any continuous linear functional y^* on Y, $y^* \circ f$ is holomorphic on $U \cap S$.

If X and Y are real Banach spaces, then f is real analytic in U if f is the restriction to U of a holomorphic map of a neighborhood of U in complexified-X into complexified-Y.

LEMMA 2.7.1 *If $f : X \to Y$ is of class C^1 and is compact in a neighborhood of x_0, then $Df(x_1)$ is a compact linear map $X \to Y$.*

PROOF: If $Df(x_0)$ were not compact, there would exist a sequence $\{x_i\}$, $\|x_i\| \leq 1$, and $\varepsilon > 0$ such that

$$\|Ax_i - Ax_j\| \geq \varepsilon > 0 \quad \text{for all } i \text{ and } j .$$

Choose $\delta > 0$ small enough so that

$$\|f(x_0 + \delta x_i) - f(x_0) - \delta A x_i\| \leq \frac{\varepsilon\delta}{4} ;$$

then (setting $x_0 = 0$)

$$\begin{aligned} \frac{\varepsilon\delta}{2} &\geq \|f(\delta x_i) - f(\delta x_j) - \delta A x_i + \delta A x_j\| \\ &\geq \|\delta A x_i - \delta A x_j\| - \|f(\delta x_i) - f(\delta x_j)\| \\ &\geq \delta\varepsilon - \|f(\delta x_i) - f(\delta x_j)\| \end{aligned}$$

or

$$\|f(\delta x_i) - f(\delta x_j)\| \geq \frac{\varepsilon\delta}{2} ,$$

contradicting the fact that f is compact.

We shall often make use of the *integral theorem of the mean*: If $f \in C^1$ on a convex open set U, then for any $x, x' \in U$

$$f(x') - f(x) = \int_0^1 \frac{d}{dt} f(tx' + (1-t)x)dt = \int_0^1 f_x(tx' + (1-t)x)dt(x' - x).$$

We will prove the implicit function theorem with the aid of the strict contraction mapping principle, which asserts that if (X, d) is a complete metric space and $f : X \to X$ is a continuous map which is contracting, i.e.,

$$d(f(x), f(y)) \le \theta d(x, y), \ 0 \le \theta < 1 \quad \text{for all } x, y \text{ in } X,$$

then $f(x)$ has a unique fixed point. □

THEOREM 2.7.2 (Implicit Function Theorem[1]) *Let X, Y, and Z be Banach spaces and f a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that f has a Frechet derivative with respect to x, $f_x(x, y)$, which is continuous in U. Let $(x_0, y_0) \in U$ and $f(x_0, y_0) = 0$. If $A = f_x(x_0, y_0)$ is an isomorphism of X onto Z then*:

(i) *There is a ball $\{y : \|y - y_0\| < r\} = B_r(y_0)$ and a unique continuous map $u : B_r(y_0) \to X$ such that $u(y_0) = x_0$ and $f(u(y), y) \equiv 0$.*

(ii) *If f is of class C^1, then $u(y)$ is of class C^1 and*

$$u_y(y) = -[f_x(u(y), y)]^{-1} \circ f_y(u(y), y).$$

(iii) *$u_y(y)$ belongs to C^p if f is in C^p, $p > 1$.*

PROOF: We may suppose $x_0 = 0$, $y_0 = 0$. The equation $f(x, y) = 0$ may be written in the form $Ax = Ax - f(x, y) \equiv R(x, y)$ or

$$x = x - A^{-1} f(x, y) \equiv A^{-1} R(x, y) \equiv g(x, y).$$

We will show that for suitable $r, \delta > 0$, and each fixed $y \in B_r(0)$ the map $g(x, y) : B_\delta(0) \to B_\delta(0)$ is a strict contraction. So there is for each fixed y a unique $x = u(y)$ in $B_\delta(0)$ such that $g(u(y), y) = u(y)$ or

$$f(u(y), y) = 0.$$

Choose $\varepsilon > 0$ with $\varepsilon \|A^{-1}\| \le \frac{1}{2}$. (Note that as a consequence of the closed graph theorem A^{-1} is bounded.) We first show that $\|R(x_1, y) - R(x_2, y)\| \le \varepsilon \|x_1 - x_2\|$ when x_i belongs to some ball $B_\delta(0)$ and $y \in B_r(0)$.

$$\begin{aligned} R(x_1, y) - R(x_2, y) &= Ax_1 - Ax_2 - (f(x_1, y) - f(x_2, y)) \\ &= A(x_1 - x_2) - \left[\int_0^1 f_x(tx_1 + (1-t)x_2), y)dt\right](x_1 - x_2) \\ &= \left[A - \int_0^1 f_x(tx_1 + (1-t)x_2, y)dt\right](x_1 - x_2) \\ &= \int_0^1 [f_x(0, 0) - f_x(tx_1 + (1-t)x_2, y)]dt(x_1 - x_2). \end{aligned}$$

[1] The formulation is that of [18] (see p. 339).

Since f_x is continuous, we can choose $r, \delta > 0$ such that

$$\|f_x(0,0) - f_x(x,y)\| \le \varepsilon$$

when $\|x\| \le \delta$, $\|y\| \le r$. Then

$$\|R(x_1, y) - R(x_2, y)\| \le \varepsilon \|x_1 - x_2\|$$

and

$$\|g(x_1, y) - g(x_2, y)\| \le \varepsilon \|A^{-1}\| \, \|x_1 - x_2\| \le \tfrac{1}{2}\|x_1 - x_2\| . \tag{2.15}$$

This shows that $g(x, y)$ is contracting for $x \in B_\delta(0)$ for each $y \in B_r(0)$.

Next we show $g(x, y)$ maps $B_\delta(0) \to B_\delta(0)$ when y is restricted to a suitable ball. By continuity of $g(x, y)$ at (0,0), restrict $r > 0$ small enough so that

$$\|g(0, y) \le \tfrac{1}{2}\delta$$

and (2.15) holds. Then

$$\|g(x, y)\| \le \|g(0, y)\| + \tfrac{1}{2}\|x\| \le \delta .$$

Hence by the strict contraction mapping principle, there exists for each $y \in B_r(0)$ a unique x, denoted by $u(y)$, such that

$$\|x\| \le \delta \quad \text{and} \quad f(u(y), y) = 0 .$$

Continuity of $u(y)$. Let $y_1, y_2 \in B_r(0)$; then

$$\begin{aligned}
&\|u(y_1) - u(y_2)\| \\
&\quad = \|g(u(y_1), y_1) - g(u(y_2), y_2)\| \\
&\quad \le \|g(u(y_1), y_1) - g(u(y_2), y_1)\| + \|g(u(y_2), y_1) - g(u(y_2), y_2)\| \\
&\quad \le \tfrac{1}{2}\|u(y_1) - u(y_2)\| + \|g(u(y_2), y_1) - g(u(y_2), y_2)\| ,
\end{aligned}$$

or

$$\|u(y_1) - u(y_2)\| \le 2\|g(u(y_2), y_1) - g(u(y_2), y_2)\| .$$

Since the right side of this equation approaches zero as $y_1 \to y_2$, in view of the continuity of g in the y variable, the result follows. Thus (i) is proved.

To prove (ii), consider $y + \delta y$ such that $\|y + \delta y\| \le r$ and set $\delta u = u(y + \delta y) - u(y)$. Since $u(y)$ is continuous in $\|y\| \le r$, $\delta u \to 0$ as $\delta y \to 0$. Now by the differentiability of f,

$$\begin{aligned}
&\|f(u(y + \delta y), y + \delta y) - f(u(y), y) - f_x(u(y), y)\delta u - f_y(u(y), y)\delta y\| \\
&\quad \le \varepsilon(\|\delta y\| + \|\delta u\|) \quad \text{for any } \varepsilon > 0
\end{aligned}$$

provided $\|\delta y\|$ is small enough, i.e.,

$$\|f_x(u(y), y)\delta u + f_y(u(y), y)\delta y\| \le \varepsilon(\|\delta y\| + \|\delta u\|) . \tag{2.16}$$

Since $f_x(u(y), y) \to f_x(0, 0)$ and $[f_x(0, 0)]^{-1}$ is bounded, we see that $[f_x(u(y), y)]^{-1}$ exists and is bounded for $\|y\|$ sufficiently small. From (2.16) it follows that

$$\|\delta u + [f_x(u(y), y)]^{-1} f_y(u(y), y)\delta y\| \le C\varepsilon(\|\delta y\| + \|\delta u\|)$$

for some constant C. Let $v = [f_x(u(y), y]^{-1} f_y(u(y), y)\delta y$; then

$$\|\delta u + v\| \le C\varepsilon(\|\delta y\| + \|\delta u + v\| + \|v\|) .$$

Choosing $\varepsilon > 0$ small enough so that $C\varepsilon < \frac{1}{2}$, we find for some constants $C_1, C_2 > 0$,

$$\|\delta u + v\| \le \varepsilon C_1(\|\delta y\| + \|v\|) \le \varepsilon C_2\|\delta y\| .$$

This shows that u has a Frechet derivative at y,

$$u_y(y) = -[f_x(u(y), y)]^{-1} f_y(u(y), y) . \tag{2.17}$$

Clearly, if f is C^1, then the right-hand side of the last equation is continuous in y; we see that $u(y) \in C^1$.

Finally, if $f \in C^2$, the right-hand side of (2.17) is in C^1, and so $u \in C^2$; by induction it follows that $u \in C^p$ if $f \in C^p$. The theorem is proved. □

COROLLARY 2.7.3 *If f is a C^p map, $p \ge 1$, of a neighborhood of $x_0 \in X$ into Y with $y_0 = f(x_0)$ and $f_x(x_0)$ an isomorphism onto Y, then there is a ball $\{y \mid \|y - y_2\| \le r\} = B_r(y_0)$ for which there is a unique C^p solution*

$$x = u(y) \quad \textit{of } f(u(y)) = y , \quad x_0 = u(y_0) .$$

PROOF: Let $F(x, y) = f(x) - y = 0$ and $Z = Y$ in the previous theorem. □

There is a useful global extension of this result due to Hadamard:

THEOREM 2.7.4 (Monodromy Type) *Let f be a C^1 map of a Banach space X to a Banach space Y. Assume that for each $x \in X$, $f_x(x)^{-1}$ exists and has norm bounded by a fixed constant. Then f is a homeomorphism of X* onto *Y.*

This is theorem 1.22 in Schwartz **[11]** and the proof will be omitted.

REMARK. If X, Y, and Z are complex Banach spaces and if f is holomorphic from $X \times Y \to Z$, then so is the solution $f(u(y), y) = 0$. This is because the solution is obtained, via the strict contraction mapping principle, as a unique limit of iterates, all of which are holomorphic. Hence the limit is holomorphic. The same is true for real analytic functions when X, Y, and Z are real Banach spaces, by extending to the complexified spaces.

The following form of the implicit function theorem is often used:

THEOREM 2.7.5 *Let $f(x, y)$ be a C^p map, $p \ge 1$, of a neighborhood of $(0, 0)$ in $X \times Y$ into a Banach space Z such that*

(i) $f(0, 0) = 0$,
(ii) range $f_x(0, 0) \equiv Rf_x(0, 0) = Z$,
(iii) ker $f_x(0, 0) = X_1$ *has a closed complementing subspace X_2 in X; i.e., X is a direct sum $X = X_1 \oplus X_2$.*

Then for each $x_1 \in X_1$, $\|x_1\| \le \delta$, and $y \in Y$, $\|y\| \le r$, for suitably small $\delta, r > 0$, there is a unique C^p solution $x_2 = u(x_1, y)$ of

$$f(x_1 + u(x_1, y), y) \equiv 0$$

with $u(0, 0) = 0$.

PROOF: Set $\widetilde{Y} = X_1 \times Y$, i.e., $\tilde{y} = (x_1, y)$, and apply the implicit function theorem to $G(x_2, \tilde{y}) \equiv f(x_1 + x_2, y)$, mapping a neighborhood of the origin in $X_2 \times \widetilde{Y}$ into Z. □

2.7.6 (**Lyapunov-Schmidt Procedure**) We will apply the preceding in a framework that will occur often in bifurcation theory.

Let X, Λ, Y be Banach spaces (we think of Λ as the parameter space) and $f(x, \lambda)$ a C^p map, $p \geq 1$, of a neighborhood of (x_0, λ_0) in $X \times \Lambda$ into Y, with $f(x_0, \lambda_0) = 0$. We wish to study the set of solutions near (x_0, λ_0) of

$$f(x, \lambda) = 0 .$$

Assuming $f_x(x_0, \lambda_0)$ is Fredholm, the Lyapunov-Schmidt procedure reduces this problem to one of solving a finite number of equations, i.e., we make the

HYPOTHESES (a) $\ker f_x(x_0, \lambda_0) = X_1$ is finite-dimensional and

(b) range $f_x(x_0, \lambda_0) = Y_1$ is a closed linear subspace of Y of finite codimension.

We may suppose $(x_0, \lambda_0) = (0, 0)$. Decompose $Y = Y_1 \oplus Y_2$ as a direct sum with $\dim Y_2 < \infty$, and let Q be the associated projection operator onto Y_1. We also decompose $X = X_1 \oplus X_2$ as a direct sum. Applying Q and $(I - Q)$ to the equation $f(x, \lambda) = 0$, we see that it is equivalent to the equations

$$Q(f(x, \lambda)) = 0 , \qquad (1 - Q)(f(x, \lambda)) = 0 .$$

Applying Theorem 2.7.5 to

$$Qf(x_1 + x_2, \lambda) : X_2 \times (X_1 \times \Lambda) \to Y_1 ,$$

we see that there exists a unique solution $x_2 = u(x_1, \lambda)$ near 0 of

$$Qf(x_1 + u(x_1, \lambda), \lambda) = 0 .$$

Hence $x_1 + u(x_1, \lambda)$ *is a solution of* $f(x, \lambda) = 0$ *if and only if*

$$(1 - Q) f(x_1 + u(x_1, \lambda), \lambda) = 0 . \tag{2.18}$$

Since the range of $(I - Q)$ is finite-dimensional, (2.18), called the *bifurcation equation*, is a finite set of equations. If the parameter space Λ is also finite-dimensional, then the local study of the equation $f(x, \lambda) = 0$ is reduced to a finite number of equations for a finite number of unknowns.

2.8. The Leray-Schauder Degree for Isolated Solutions, the Index

Suppose X is a Banach space $\Omega \subset X$ a bounded open set; let $\phi : \overline{\Omega} \to X$, $\phi \neq 0$ on $\partial\Omega$, $\phi \in C^1(\Omega)$ with $K = I - \phi$ compact. Assume that $x_0 \in \Omega$ is an isolated solution of $\phi(x_0) = 0$ and that $A = \phi_x(x_0) = I - K_x(x_0)$ is invertible. Let $B_\varepsilon(x_0)$ be a ball with radius $\varepsilon > 0$ and center x, chosen so that $B_\varepsilon(x_0)$ contains no other solution of $\phi(x) = 0$. The existence of such a ball is ensured by the implicit function theorem. By Lemma 2.7.1, $T = K_x(x_0)$ is compact. It is possible to compute $\deg(\phi, B_\varepsilon(x_0), 0)$. For $0 < \varepsilon \leq \varepsilon_0$ this is independent of ε and is called the *index of the map* ϕ *at* x_0.

Consider the set $\{\lambda\}$ of real eigenvalues of T bigger than 1. Clearly 1 is not an eigenvalue since, by assumption, $(1 - T)$ is invertible. For such an eigenvalue λ, let n_λ be its multiplicity:

$$n_\lambda = \dim\left(\bigcup_{p=1}^{\infty} \ker(\lambda I - T)^p\right).$$

That n_λ is finite is part of the Riesz-Schauder theory for linear compact operators.

THEOREM 2.8.1 (Leray-Schauder) *Under the preceding assumptions,*

$$\deg(\phi, B_\varepsilon(x_0), 0) = (-1)^\beta, \qquad \beta = \sum_{\lambda>1} n_\lambda .$$

The theorem is based on the corresponding result in finite dimensions:

REMARKS. (1) If A is a real nonsingular matrix in $\mathbb{R}^n$, and $T = I - A$, then

$$\operatorname{sgn}\det A = (-1)^\beta, \qquad \beta = \sum_{\lambda>1} n_\lambda(T),$$

where the sum is taken over the real eigenvalues of T that are greater than one. This, in turn, is based on:

(2) If λ_0 is a real eigenvalue of $T : \mathbb{R}^n \to \mathbb{R}^n$ of multiplicity m, then for small $\varepsilon > 0$

$$\operatorname{sgn}\det(\lambda_0 + \varepsilon - T) = (-1)^m \operatorname{sgn}\det(\lambda_0 - \varepsilon - T).$$

Remark (2) is easily proved since

$$\det(\lambda I - t) = \Pi(\lambda - \lambda_j)^{m_j},$$

where λ_j are all the eigenvalues of T (including complex) and m_j are the respective multiplicities. Remark (1) follows from Remark (2) by looking at the sgn $\det(\lambda I - T)$ for λ large, and then letting $\lambda \to 1$.

PROOF OF THEOREM: We may assume $x_0 = 0$. By the deformation, $\frac{1}{t}K(tx)$, $0 < t \le 1$ of K to T, one sees easily that $\deg(I - K, B_\varepsilon, 0) = \deg(I - T, B_\varepsilon, 0)$. We may decompose $X = X_1 \oplus X_2$ where X_1 is spanned by all the generalized eigenvectors of T (i.e., $\bigcup_p \ker(\lambda I - T)^p, \lambda > 1$), and X_2 is invariant under T. Then, by the product property

$$\deg(I - T, B_\varepsilon, 0) = \deg((I - T)|_{X_1}, B_\varepsilon \cap X_1, 0) \cdot \deg((I - T)|_{X_2}, B_\varepsilon \cap X_2, 0).$$

Now in $B_\varepsilon \cap X_2$ the mapping $I - T$ admits the deformation $I - tT, 0 \le t \le 1$, to the identity, since $(I - tT)x_2 = 0$ for $0 \le t \le 1$, $x_2 \in X_2$ implies $x_2 = 0$. Thus,

$$\deg(I - T, B_\varepsilon, 0) = (\deg(I - T)|_{X_1}, B_\varepsilon \cap X_1, 0) = (-1)^\beta \quad \text{by Remark (1).}$$

□

EXERCISE Let Ω and ϕ be as above. Let G be a connected open set in $X \backslash \phi(\partial\Omega)$ consisting only of regular values of ϕ; i.e., for any point $y \in G$, $\phi_x(x)$ is invertible at each point x in $\phi^{-1}(y)$. Prove that $n(y) =$ the number of points in $\phi^{-1}(y)$ is constant on G.

CHAPTER 3

Bifurcation Theory

Let f be a mapping of a neighborhood of a point x_0 in a Banach space X into a Banach space Y, with $f(x_0) = 0$. We wish to study the set of solutions of

$$f(x) = 0.$$

In this degree of generality we cannot hope to say much. Even in finite dimensions the problem is extremely complicated; classical algebraic geometry is concerned with the case that f is a polynomial.

The equation

$$f(x, \lambda) = 0,$$

where f depends, in addition, on one or more parameters λ, occurs often. It sometimes happens that, as λ varies, there is a nice family of solutions $x(\lambda)$, but that at some critical value of λ this family may disappear, or may split into several branches, hence the name bifurcation. A familiar example is the problem in elasticity of a straight rod lying on a table which is being compressed by forces at the ends. For small forces the rod maintains its shape, i.e., the only (local) solution of the equations of elasticity is the trivial one. But as the forces increase they reach a first critical value beyond which the rod may buckle.

In this chapter, with the aid of the tools developed earlier, we will study local solutions under a variety of assumptions. In Section 3.4, we also present a global result. We will usually suppose that $f(x)$ is of class C^p, $p \geq 1$, and that the Banach spaces are over the real field (if they are over the complex field we do not assume that f is holomorphic). We will also usually assume that f is Fredholm, i.e.,

$$\begin{aligned} &\ker f_x(x_0) = X_1 \text{ has dimension } d < \infty, \\ &\text{range } f_x(x_0) = Y_1 \text{ is a closed subspace of } Y \text{ of finite codimension.} \end{aligned} \tag{3.1}$$

If the range is not closed, then very little is known.

3.1. The Morse Lemma

Consider first the simplest case: $Y_1 = Y$. The implicit function Theorem 2.7.5 tells us that in a neighborhood of x_0, the set $f^{-1}(0)$ consists of a d-dimensional submanifold of class C^p through x_0. If $Y_1 \neq Y$, the problem is then called, speaking loosely, a *bifurcation problem.*

The next simplest case is

$$\text{codim } Y_1 = 1;$$

i.e., for some continuous, linear functional $y^* \neq 0$, $y^* \in Y^*$, $Y_1 = \{y \in Y \mid y^*(y) = 0\}$. We may suppose $x_0 = 0$. In Section 2.7.6, we have seen that the local study of $f(x) = 0$ reduces to the single bifurcation equation

$$y^* f(x_1 + u(x_1)) = 0\,, \quad x_1 \in X_1\,. \tag{3.2}$$

Here X is decomposed as $X = X_1 \oplus X_2$, and $u(x_1) \in X_2$ is a function of class C^p. The bifurcation equation is thus one equation for d unknowns.

Even for a single equation, however, the solution set may be very complicated. Consider a single equation

$$F(x) = 0\,,$$

where F is a C^p function, $p \geq 2$, defined in a neighborhood of the origin in $\mathbb{R}^d$ with $F(0) = 0$. If $F_x(0) \neq 0$, then, as we saw, the set of solutions of $F(x) = 0$ near the origin is a C^p hypersurface (i.e., of dimension $d - 1$).

The next generic case is

(3.3) $F(0) = 0\,, \quad F_x(0) = 0\,,$ and the matrix of second derivatives $F_{xx}(0)$ is nonsingular,

i.e., the origin is a nondegenerate stationary point of F. This is just the situation in which one has the

THEOREM 3.1.1 (Morse Lemma) *If $F \in C^p$, $p \geq 2$, and satisfies (3.3), there exists a local C^{p-2} coordinate change $y(x)$ defined in a neighborhood of the origin with $y(0) = 0$, $y_x(0) = I$ such that*

$$F(x) \equiv \frac{1}{2}\big(F_{xx}(0)y(x), y(x)\big)$$

near the origin.

In this case the solution set of $F(x) = 0$ is very easy to analyze. In particular, we have

COROLLARY 3.1.2 *Under the conditions of the lemma, if $d = 2$ and the quadratic form $(F_{xx}(0)y, y)$ is indefinite, the set of solutions of $F(x) = 0$ near the origin consists of two C^{p-2} curves intersecting only at the origin (transversally in case $p > 2$).*

In general for $d > 2$, if $F_{xx}(0)$ is indefinite, the set of solutions of $F(x) = 0$ looks like a deformed cone (see Figure 3.1).

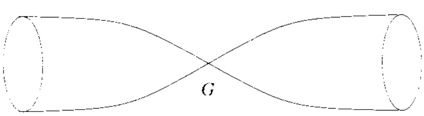

FIGURE 3.1. Example for $d = 3$.

In the next section we will apply the Morse lemma to a bifurcation problem. The idea of applying it in such problems was suggested by J. Duistermaat. Indeed, it's clear that whenever we can find a suitable change of variable reducing some nonlinear equation, or finite system of equations $F(x) = 0$, in $\mathbb{R}^d$ to normal forms whose solution set can be easily analyzed, we can describe the structure of the solutions of the original system. Thus, in particular, the Thom-Mather theory of the stability of maps should play a very useful role in bifurcation theory.

We shall prove a generalized form of the Morse lemma which, together with the proof, is taken from [**29**, lemma 3.2.3].

LEMMA 3.1.3 (Generalized Morse Lemma) *Let $F(x, y)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$, be a C^p real function, $p \geq 2$, in a neighborhood of $(0, 0)$, with $F_x(0, 0) = 0$ and $Q = F_{xx}(0, 0)$ nonsingular. In a neighborhood of the origin there is a C^p function $x(y)$, with $x(0) = 0$, satisfying*

$$F_x(x(y), y) \equiv 0 \tag{3.4}$$

and also a C^{p-2} function $\xi(x, y)$ with values in $\mathbb{R}^d$ of the form

$$\xi = x - x(y) + 0(|x - x(y)|^2)$$

such that

$$F(x, y) \equiv F(x(y), y) + \frac{1}{2}(Q(y)\xi, \xi) \tag{3.5}$$

where $Q(y) = F_{xx}|_{x=x(y)}$.

COROLLARY 3.1.4 *If $F(0, 0) = 0$ and $Q(0)$ is indefinite, then for every y near the origin, the equation $F(x, y) = 0$ has as solution set a $(d - 1)$ dimensional surface in C^{p-2} except for a possible conical singularity at $x = x(y)$, in case $F(x(y), y) = 0$.*

PROOF: The implicit function theorem yields the solution $x(y)$ of (3.4). If we replace the variable x by $x - x(y)$, we may suppose that $F_x(0, y) \equiv 0$ for y near the origin.

We shall seek ξ of the form

$$\xi = R(x, y)x$$

where R is a $d \times d$ matrix to be determined, with $R(0, y) = I$, so that (3.5) holds: i.e., if R^* is the adjoint of R,

$$\frac{1}{2}(R^* Q(y) Rx, x) = F(x, y) - F(0, y) .$$

Writing $F(x, y) - F(0, y) = \int_0^1 \frac{d}{dt} F(tx, y)dt$ and integrating by parts, we have

$$F(x, y) - F(0, y) = \int_0^1 (1 - t)(F_{xx}(tx, y)x, x)dt = \frac{1}{2}(B(x, y)x, x) .$$

where $B(x, y) = 2\int_0^1 (1-t)F_{xx}(tx, y)dt$; note that B is a symmetric matrix. Thus, we wish to find R so that

$$R^* Q(y) R = B(x, y) \tag{3.6}$$

and $R(0, y) = I$.

We solve (3.6) with the aid of the implicit function theorem. At $x = 0$ we have $B(0, y) = Q(y)$, and $R = I$ satisfies (3.6) there. The Frechet derivative of the map $R^* Q(y) R$ at this point is the linear map

$$R \mapsto R^* Q(y) + Q(y) R .$$

This map is *onto* the space of symmetric matrices, for if S is a symmetric matrix then $R = \frac{1}{2} Q^{-1} S$ satisfies $R^* Q + Q R = S$. It follows from the implicit function theorem, Theorem 2.7.5, that (3.6) has a solution in C^{p-2} in a neighborhood of (0,0). □

REMARK. It is clear from the proof that the regularity assumptions, in particular, with respect to y, may be weakened. It is also clear that if F is C^∞ or analytic, so are $x(y)$ and $\xi(x, y)$.

3.2. Application of the Morse Lemma

Let us consider the case discussed above, in which f satisfies (3.1), with $x_0 = 0$, and codim $Y_1 = 1$; i.e., there is $y^* \in Y^*$, $y^* \neq 0$, such that

$$Y_1 = \{y \in Y \mid y^*(y) = 0\} .$$

THEOREM 3.2.1 *Assume that f is as above, in C^p with $p \geq 2$, and that its restriction to X_1 satisfies*

(3.7) *the $d \times d$ symmetric matrix $y^* f_{x_1 x_1}(0)$ is nondegenerate and indefinite.*

Then in a neighborhood of the origin, the set of solutions of $f(x) = 0$ consists of a deformed cone of dimension $d - 1$ with vertex at the origin. In particular, if $d = 2$, then it consists of two C^{p-2} curves crossing only at the origin (transversally if $p > 2$).

It is clear that if $y^* f_{x_1 x_1}(0)$ is definite, then $x = 0$ is the only local solution of $f(x) = 0$.

PROOF: As we have remarked, the equation $f(x) = 0$ is equivalent to the bifurcation equation (3.2):

$$F(x_1) \equiv y^* f(x_1 + x_2(x_1)) = 0 .$$

The Morse lemma applied to $F(x_1)$ yields the desired result. We have only to check that the hypotheses of the lemma hold; namely, we show

(i) $F_{x_1}(0) = 0$,
(ii) $F_{x_1 x_1}(0) = y^* f_{x_1 x_1}(0)$.

To check these we recall that $x_2(x_1)$ was obtained as the solution of

$$Q f(x_1 + x_2(x_1)) = 0 , \quad x_2(0) = 0 .$$

Differentiating this, we find

$$Q f_x(0)(x_1 + x_{2_{x_1}}(0) x_1) = 0 .$$

Since $f_x(0) x_1 = 0$ and Q is projection into $R(f_x(0))$, we see that

$$f_x(0) x_{2_{x_1}}(0) x_1 = 0 ,$$

and since $f_x(0)$ is an isomorphism on X_2 and $x_{2_{x_1}}(0)x_1 \in X_2$, it follows that $x_{2_{x_1}}(0)x_1 = 0$, i.e.,

$$x_{2_{x_1}}(0) = 0\,. \tag{3.8}$$

Consequently, since $y^* f_{x_1}(0) = 0$, we have

$$F_{x_1}(0) = y^* f_{x_1}(0) = 0\,, \qquad F_{x_1x_1}(0) = y^* f_{x_1x_1}(0)\,.$$

□

REMARK. 3.8′ In case $d = 2$ and $p \geq 3$ in Theorem 3.2.1, it follows from the Morse lemma and (3.8) that at the origin, the tangents of the two solution curves lie in the plane X_1 and thus have the directions v satisfying $y^*(f_{x_1x_1}(0)(v, v)) = 0$.

Let us apply Theorem 3.2.1 in a situation occurring frequently in bifurcation theory. Consider $f(x, \lambda) \in C^p$, $p \geq 1$, mapping a neighborhood of $(0, \lambda_0)$ in $X \times \mathbb{R}$ into a Banach space Y, with

$$f(0, \lambda_0) = 0\,. \tag{3.9}$$

DEFINITION The point $(0, \lambda_0)$ is called a *bifurcation point* of f if every neighborhood of $(0, \lambda_0)$ in $X \times \mathbb{R}$ contains a solution (x, λ), $x \neq 0$ of

$$f(x, \lambda) = 0\,.$$

In many problems f satisfies

$$f(0, \lambda) \equiv 0\,. \tag{3.9′}$$

In this case it follows from the implicit function theorem that if $f_x(0, \lambda_0)$ is an isomorphism of X onto Y, then $(0, \lambda_0)$ is *not* a bifurcation point.

THEOREM 3.2.2 *Let $f(x, \lambda)$ be a C^p map, $p \geq 2$, of a neighborhood of $(0, \lambda_0)$ in $X \times \mathbb{R}$ into Y with $f(0, \lambda_0) = 0$. Suppose*

(i) $f_\lambda(0, \lambda_0) = 0$,
(ii) ker $f_x(0, \lambda_0)$ *is one-dimensional, spanned by* x_0,
(iii) range $f_x(0, \lambda_0) = Y_1$ *has codimension* 1, *and*
(iv) $f_{\lambda\lambda}(0, \lambda_0) \in Y_1$ *and* $f_{\lambda x}(0, \lambda_0)x_0 \notin Y_1$.

Then $(0, \lambda_0)$ is a bifurcation point of f. In fact, the set of solutions of $f(x, \lambda)$ near the origin consists of two C^{p-2} curves Γ_1, Γ_2 intersecting only at $(0, \lambda_0)$. Furthermore, if $p > 2$,

Γ_1 is tangent to the λ-axis at $(0, \lambda_0)$ and so may be parametrized by λ:

$$(x(\lambda), \lambda), \qquad |\lambda| \leq \varepsilon\,,$$

Γ_2 may be parametrized by a variable s, $|s| \leq \varepsilon$, as

$$(sx_0 + x_2(s), \lambda(s))\,,$$

with $x_2(0) = x_{2_s}(0) = 0$, $\lambda(0) = \lambda_0$.

REMARK. In case f satisfies (3.9′) the curve Γ_1 is the λ-axis. This theorem has been observed by several authors using the implicit function theorem. See, for example, theorem 1 in Crandall and Rabinowitz [**18**].

PROOF: We may suppose $\lambda_0 = 0$. Let $\widehat{X} = X \times \mathbb{R}$ and $f(x, \lambda) = f(\hat{x})$. Then $f_{\hat{x}}(0) = f_x(0, 0) \oplus f_\lambda(0, 0)$. From (i) and (ii) it follows that ker $f_{\hat{x}}(0)$ is spanned by $(x_0, 0)$ and $(0, 1)$, and so is two-dimensional. Let $y^* \neq 0$ be a linear functional annihilating Y_1. We claim that $f(\hat{x})$ satisfies the hypotheses of Theorem 3.2.1. We have only to verify (3.7). The 2×2 matrix in question has the form, in rather obvious notation,

$$Q = \begin{pmatrix} y^* f_{x_0 x_0}(0, 0) & y^* f_{x_0 \lambda}(0, 0) \\ y^* f_{x_0 \lambda}(0, 0) & y^* f_{\lambda\lambda}(0, 0) \end{pmatrix}$$

and from the hypotheses (iv) it follows that lower diagonal term is zero while the off-diagonal terms are not zero. Hence det $Q < 0$ and therefore Q, is nonsingular and indefinite.

Applying Theorem 3.2.1, we infer the existence of the two curves Γ_1, Γ_2. Suppose now $p \geq 3$. These curves are then of class C^{p-2} and intersect transversally at the origin. To complete the proof of the theorem we have only to prove that one of them is tangent to the λ-axis there. This follows from Remark 3.8′ and the fact that $y^* f_{\lambda\lambda}(0, 0) = 0$. □

Here is a very simple application to a nonlinear elliptic boundary value problem: Let G be a bounded region in $\mathbb{R}^n$ with smooth boundary. Consider the boundary value problem for a real function u

$$f(u, \lambda) = \Delta u - \lambda g(u) = 0 \quad \text{in } G, \qquad \text{normal derivative } \frac{\partial u}{\partial n} = \alpha u \quad \text{on } \partial G;$$

here α is a constant and $g(0) = 0$. Assume $g'(0) \neq 0$. The linearized problem at $u = 0$ is

$$f_u(0, \lambda)u = \Delta u - \lambda g'(0)u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = \alpha u \quad \text{on } \partial\Omega.$$

Suppose λ_0 is an eigenvalue of this problem with null space spanned by ϕ (i.e., one-dimensional). The linearized problem is self-adjoint so that range $f_u(0, \lambda_0)$ is the set of elements $\psi \perp_{L_2} \phi$. Since $(f_{u\lambda}(0, \lambda_0)\phi, \phi) = -(g'(0)\phi, \phi)$ and $g'(0) \neq 0$, we find

$$f_{u\lambda}(0, \lambda_0)\phi \notin Y_1.$$

By the preceding theorem we have therefore:

Conclusion. $(0, \lambda_0)$ is a bifurcation point. In fact, there is a one-parameter family of nontrivial solutions $(s\phi + u_2(s), \lambda(s))$, $|s| \leq \varepsilon$.

The same result holds for the Dirichlet boundary condition $u = 0$ on ∂G in place of the one above.

As an application of the generalized Morse lemma, Lemma 3.1.3, we may derive the following:

THEOREM 3.2.3 *Let $f(x, \lambda)$ be a C^p map, $p \geq 2$, of a neighborhood of $(0, 0)$ in $X \times \mathbb{R}$ into Y with $f(0, 0) = 0$. Suppose*

(i) $f(0, 0) = 0$,
(ii) $X_1 = \ker f_x(0, 0)$ *is d-dimensional,* $d > 1$,
(iii) range $f_x(0, 0) = Y_1$ *has codimension* 1,
(iv) $f_{\lambda\lambda}(0, 0) \in Y_1$, *and*

(v) *for some* $x_0 \in X_1$, $f_{\lambda x}(0,0)x_0 \notin Y_1$.

Then $(0,0)$ *is a bifurcation point of* f. *Furthermore, if we decompose* $X_1 = \{ax_0\} \oplus X_1'$ *and* $X = \{ax_0\} \oplus X_1' \oplus X_2 = P_1X \oplus P_1'X \oplus P_2X$, *where* P_1, P_1', P_2 *are the associated projections, then for each (small) element* $x_1' \in X_1'$, *the set of solutions of* $f(x,\lambda)$ *near the origin with*

$$P_1'x = x_1'$$

consists of two C^{p-2} *curves. For* $p \geq 3$, *and each* x_1' *fixed, these two curves either intersect transversally or else they look like two branches of a hyperbola.*

EXERCISE Prove Theorem 3.2.3.

3.3. Krasnoselski's Theorem

In chapter 4 of his book [**6**], Krasnoselski has given a general sufficient condition for a point to be a bifurcation point within the category of compact operators. Though we will present a more general result later, we first present his result.

Let X be a Banach space and $f(x,\lambda)$ a map with domain $D \subset X \times \mathbb{R}$ into X of the form: $f(x,\lambda) = x - (\mu_0 + \lambda)Tx + g(x,\lambda)$.

We will assume:

(1) $\mu_0 \neq 0$ and $(0, \mu_0) \in D$,
(2) T is a linear compact map $X \to X$,
(3) $g(x,\lambda)$ is a nonlinear compact map D into X, and
(4) $g(0,\lambda) \equiv 0$ and $g(x,\lambda) = o(\|x\|)$ uniformly for $|\lambda| < \varepsilon$.

We wish to determine when (0,0) is a bifurcation point of $f(x,\lambda) = 0$.

We see immediately that a necessary condition is that $I - \mu_0 T$ not be invertible. Indeed, if $I - \mu_0 T$ had a bounded inverse, the implicit function theorem would give a unique local solution $x(\lambda)$, and this is $x(\lambda) \equiv 0$. (This is not quite correct, since we have not assumed any regularity of g. However, it is easily verified that $(0,\lambda)$ is the only small solution of $f(x,\lambda) = 0$ for $|\lambda|$ small by writing the equation in the form $x = (I - \mu_0 T)^{-1}[\lambda Tx - g(x,\lambda)]$ and estimating the right-hand side.) So (0,0) is not a bifurcation point.

Thus, a necessary condition for (0,0) to be a bifurcation point is that μ_0^{-1} is an eigenvalue of T.

THEOREM 3.3.1 (Krasnoselski) *Under assumptions* 1–4 *above, suppose* $1/\mu_0$ *is an eigenvalue of* T *with odd multiplicity; then* $(0,0)$ *is a bifurcation point of* $f(x,\lambda)$.

Recall that multiple $\mu_0^{-1} = \dim \bigcup_1^\infty \ker(\mu_0^{-1} I - T)^p$.

PROOF: Suppose (0,0) is not a bifurcation point then for $\varepsilon > 0$ sufficiently small and λ fixed and also sufficiently small. $\deg(f(x,\lambda), \|x\| \leq \varepsilon, 0)$ is defined and independent of λ. By Theorem 2.8.1, for $\lambda_1 > 0$.

$$\deg(f(x,\lambda_1), \|x\| \leq \varepsilon, 0) = (-1)^{\beta(\lambda_1)}$$

where $\beta(\lambda) = \sum$ multiplicities of eigenvalues of T which are $> \frac{1}{\mu_0+\lambda}$, and for $\lambda_2 < 0$,

$$\deg(f(x,\lambda_2), \|x\| \le \varepsilon, 0) = (-1)^{\beta(\lambda_2)}$$

and $\beta(\lambda_2) - \beta(\lambda_1) =$ multiplicity of the eigenvalue μ_0^{-1}. Since the multiplicity of μ_0^{-1} is odd,

$$\deg(f(x,\lambda_2), \|x\| \le \varepsilon, 0) = -\deg(f(x,\lambda_1), \|x\| \le \varepsilon, 0),$$

contradicting the fact that $\deg(f(x,\lambda), \|x\| \le \varepsilon, 0)$ is independent of λ. □

EXAMPLES. (1) If $1/\mu_0$ has even multiplicity, the conclusion of the theorem need not hold. Let $X = \mathbb{R}^2$, $x = \binom{x_1}{x_2}$. Consider the equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (\mu_0+\lambda)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_2^3 \\ x_1^3 \end{pmatrix} = 0$$

with $\mu_0 = 1$. Then $T = I$. Multiplying the equations by x_2 and x_1, respectively, and subtracting, we find $x_2^4 + x_1^4 = 0$ or $x_1 = x_2 = 0$, as the only solution. Hence (0,0) is not a bifurcation point. In this case $\ker(1-T) = \mathbb{R}^2$, so the multiplicity of $\mu_0 = 1$ is 2.

(2) A similar example in $\mathbb{R}^2$ in which $\ker(I-T) = \mathbb{R}^1$, $\ker(I-T)^2 = \mathbb{R}^2$ is

$$x_2 - \lambda x_1 + x_2^3 = 0, \qquad -\lambda x_2 - x_1^3 = 0.$$

Here $I - T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. As above, one finds that any solution (x_1, x_2) satisfies $x_2^2 + x_1^4 + x_2^4 = 0$, and so is trivial.

(3) In the case that g is smooth and $m = 1$, Theorem 3.3.1 is a special case of Theorem 3.2.2. In this case, in addition to the trivial line of solutions $(0,\lambda)$, we also have another smooth curve of solutions cutting this transversally. If $m > 1$ this need not be the case. Consider the example in $X = \mathbb{R}^3 : T = I$, $\mu_0 = 1$ and g independent of λ:

$$g(x) = v\left(\frac{x}{|x|}\right) e^{-\frac{1}{|x|^2}},$$

where v is a map of $\mathbb{S}^2$ into $\mathbb{R}^3$ with $v(y) \perp y$ for every y, and v vanishes at only one point, the north pole. If $x \neq 0$ is a solution of

$$-\lambda x + g(x) = 0$$

then, since $g(x) \perp x$, we see that $\lambda = 0$ and $x = (0,0,x_3)$, $x_3 > 0$. Thus, there is a nontrivial segment of solutions of the form $(0,0,x_3)$, $x_3 > 0$ and $\lambda = 0$.

3.4. A Theorem of Rabinowitz

P. Rabinowitz has proved the following global extension of Krasnoselski's theorem [**31**]:

THEOREM 3.4.1 *Let X be a Banach space and $f(x, \mu)$ a continuous mapping of a domain G in $X \times R$ into X of the form*

$$f(x, \mu) = (I - \mu T)x - g(x, \mu)$$

satisfying

(i) *T is a linear compact map of X into X,*
(ii) *g is a nonlinear compact map of G into X with $g(x, \mu) = o(\|x\|)$ uniformly on bounded μ intervals, and*
(iii) *$(0, \mu_0) \in G$ where $\mu_0 \neq 0$ and μ_0^{-1} is an eigenvalue of T odd multiplicity.*

Let S denote the closure of the nontrivial (i.e., $x \neq 0$) solutions (x, μ) of $f(x, \mu) = 0$ in G, and C be the connected component of S containing $(0, \mu_0)$.

Then either

(1) C is not compact in G (in case $G = X \times R$ this means C is not bounded), or
(2) C contains a finite number of points $(0, \mu_j)$ with $1/\mu_j$ eigenvalues of T. Furthermore, the number of such points having odd multiplicity, including $(0, \mu_0)$, is even.

The proof that we present is due to J. Ize. It makes use of

LEMMA 3.4.2 (Ize) *Consider $f(x, \mu)$ as in the theorem. For $\mu = \mu_0 + \lambda$ with $\lambda \neq 0$, $|\lambda|$ small, μ^{-1} is not an eigenvalue of T and hence*

$$i_- = \text{index of } 0 \text{ for } (I - \mu T) = \deg(I - \mu T, \|x\| \leq r, 0)\,, \quad \lambda < 0\,,$$

for $r = r(\lambda)$ sufficiently small is defined and independent of r and λ.

So is

$$i_+ = \deg(I - \mu T, \|x\| \leq r, 0) \quad \text{for } \lambda > 0\,.$$

For fixed small $r > 0$, consider the following map in a neighborhood of the origin in $X \times R \to X \times R$ defined by $H_r(x, \lambda) = (y, \tau)$ where

$$(I - (\mu_0 + \lambda)T)x - g(x, \mu_0 + \lambda) = y\,, \qquad \|x\|^2 - r^2 = \tau\,.$$

CLAIM For suitably small $\lambda_0, r > 0$,

$$\deg\left(H_r, \|x\|^2 + \lambda^2 \leq r^2 + \lambda_0^2, (0, 0)\right) = i_- - i_+\,.$$

PROOF: Let $\lambda_0 > 0$ be so small that the only inverse of an eigenvalue of T in the interval $[\mu_0 - \lambda_0, \mu_0 + \lambda_0]$ is μ_0. (Recall that nonzero eigenvalues of a compact map are isolated.) As in the preceding section, $[I - (\mu \pm \lambda_0)T]^{-1}$ exists and is bounded, and the only solution x, with $\|x\|$ sufficiently small, of

$$[I - (\mu \pm \lambda_0)T]x - g(x, \mu_0 \pm \lambda_0) = 0 \tag{3.10}$$

is $x = 0$.

We claim that for r small, $H_r(x, \lambda) = (0, 0)$ has no solutions satisfying $\|x\|^2 + \lambda^2 = r^2 + \lambda_0^2$. Indeed, if (x, λ) is such a solution, then $\lambda = \pm\lambda_0$ and, for r small, the only solution of (3.10) is $x = 0$.

Consider the deformation, $0 \le t \le 1$,

$$H_r^t(x, \lambda) = (y^t, \tau^t),$$
$$y^t = (I - (\mu_0 + \lambda)T)x - tg(x, \mu_0 + \lambda),$$
$$\tau^t = t(\|x\|^2 - \|r^2\|) + (1 - t)(\lambda_0^2 - \lambda^2).$$

As before, $\deg(H_r^t(x, \lambda), \|x\|^2 + |\lambda|^2 \le r^2 + \lambda_0^2, (0, 0))$ is well-defined (i.e., there are no solutions on the boundary). Hence the degree is independent of t. For $t = 0$

$$H_r^0(x, \lambda) = \left(I - (\mu_0 + \lambda)T, \lambda_0^2 - \lambda^2\right).$$

If $H_r^0(x, \lambda) = (0, 0)$, then $\lambda = \pm\lambda_0$ and $x = 0$. So the only solutions are $(0, \lambda_0)$, $(0, -\lambda_0)$. However, the Frechet derivative of $H_r^0(x, \lambda)$ at $(0, \lambda)$ is

$$DH_r^0(0, \lambda)(x, \lambda') = ((I - (\mu_0 + \lambda)T)x, -2\lambda\lambda').$$

This is a product map, and so the degree at $\lambda = \lambda_0$ is $-i_+$ and the degree at $\lambda = -\lambda_0$ is i_-. Hence the total degree is $i_- - i_+$. □

PROOF OF RABINOWITZ' THEOREM: Suppose C is compact in G. Recall that the only possible accumulation point of the eigenvalues of a compact map is zero, and so, in any finite interval in R there are a finite number of inverses of eigenvalues. Consequently, C contains at most a finite number of $(0, \mu_j)$, $j = 0, \ldots, k$, such that μ_j^{-1} is an eigenvalue of T. Let Ω be any open set in $X \times R$ containing C such that there are no nontrivial solutions (x, μ), $x \ne 0$, of the equation $f(x, \mu) = 0$ on $\partial\Omega$, and so that Ω contains no other point $(0, \mu)$ such that μ^{-1} is an eigenvalue of T, as in Figure 3.2.

In Ω for $r > 0$ consider the map $f_r(x, \mu) : \overline{\Omega} \to X \times R$,

$$f_r(x, \mu) = \left(f(x, \mu), \|x\|^2 - r^2\right).$$

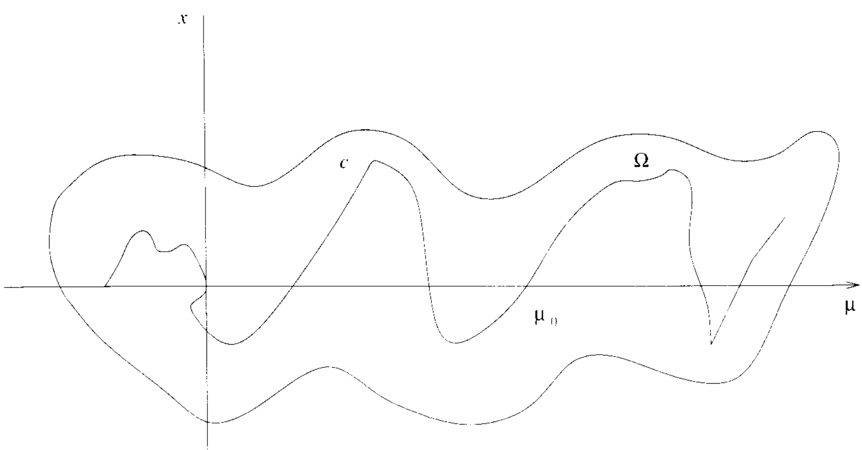

FIGURE 3.2

Now $\deg(f_r(x, \mu), \Omega, (0, 0))$ is defined, since on $\partial\Omega$ there are no nonzero solutions of $f(x, \mu) = 0$, and hence $0 = \|x\| < r$ for such a solution. Furthermore, the degree is independent of r. For r large, $f_r(x, \mu) = 0$ has no solutions in Ω, and hence has zero degree. On the other hand, for r small, if (x, μ) is a solution of $f_r(x, \mu) = 0$, then $\|x\| = r$, and hence, as before, μ is close to one of the μ_j, $j = 0, 1, \dots, k$. (Namely, if this is not the case, then $(I - \mu T)^{-1}$ is bounded and $x = 0$ is the only solution of $f(x, \mu) = 0$, contradicting $\|x\| = r > 0$.) But then the sum of the local degrees of f_r in the neighborhoods of each of the μ_j is equal to zero. By Lemma 3.4.2,

$$0 = \sum_{j=0}^{k} \left(i_-(j) - i_+(j)\right).$$

Since $i_+(j) = (-1)^{m_j} i_-(j)$, where m_j is the multiplicity of μ_j, the nonzero terms involve only the μ_j with odd multiplicity, and since these terms add up to zero, there must be an even number of them. □

3.5. Extension of Krasnoselski's Theorem

A number of people have observed that with the aid of the Lyapunov-Schmidt procedure, Krasnoselski's theorem may be considerably generalized. In particular, compactness may be dropped; in place of Leray-Schauder degree, one uses degree theory in finite dimensions.

The material in this section is taken from the doctoral dissertation **[30]** of J. Ize (Courant Institute, 1974). A related reference is **[28]**.

Let X be a Banach space over the real or complex field Λ with norm $\| \; \|$. Suppose X_0 is a linear subspace of X complete under the norm $\| \; \|_0$ with $\| \; \|_0 \geq \| \; \|$. Then $X_0 \hookrightarrow X$ is a continuous injection. (Typical example: X is a space of functions $H_{m,p}$, and X_0 is a subspace of more regular functions.)

We wish to study the equation

$$(A - \lambda)x - G(x, \lambda) = 0 \tag{3.11}$$

near the origin. Here A is a continuous linear operator taking X_0 into X, and $G(x, \lambda)$ a C^1 function[1] in a neighborhood of the origin of $X_0 \times \Lambda$ into X satisfying:

(1) $\|G(x, \lambda)\| = O(\|x\|_0^2 + \|\lambda\|^p)$ for some power p (to be specified in (3.17′)).

(2) A is a Fredholm operator of index zero, i.e.,
   (a) $\dim \ker A = q < \infty$
   (b) range A is closed and has finite codimension equal to q.

(3) Zero is an eigenvalue of A with finite multiplicity, i.e.,

$$\dim \bigcup_{j=1}^{\infty} \ker A^j = m < +\infty.$$

[1] By this it is not meant that G is holomorphic in case $\Lambda = \mathbb{C}$.

Here the domain of $A^j = \mathcal{D}(A^j) = \{x \in X_0 \mid A^k x \in X_0,\ k = 1, \dots, j-1\}$.

THEOREM 3.5.1 (Ize) *Assume* (1)–(3). *Then in each of the following cases, the origin* (0,0) *is a bifurcation point; i.e., there are nontrivial solutions* (x, λ), $x \neq 0$ *near* (0,0), *of*

$$(A - \lambda)x - G(x, \lambda) = 0 .$$

The cases are:

(i) *m is odd.*
(ii) *Λ is complex and* $q = \dim \ker A = 1$.
(iii) *m is even, λ is real or complex, and G satisfies some special (reasonable) conditions.*

The proof is somewhat technical and takes up the remainder of Section 3.5.

Reduction to Finite Dimensions. As in the Lyapunov-Schmidt procedure, the first step will be the reduction of the problem to a finite-dimensional one. However, the finite-dimensional space which we will consider is not $X_1 = \ker A$ but

$$X^1 = \bigcup_{j=1}^{\infty} \ker A^j .$$

Let n be the first number such that

$$X^1 = \bigcup_{j=1}^{n} \ker A^j ;$$

thus $\ker A^{n+k} = \ker A^n$ for $k > 0$. Decompose

$$X_0 = X^1 \oplus X_3 , \quad X_3 \text{ closed.} \tag{3.12}$$

LEMMA 3.5.2 *X admits the direct sum decomposition*

$$X = X^1 \oplus AX_3 . \tag{3.13}$$

PROOF: With $X_1 = \ker A$, decompose

$$X^1 = X_1 \oplus X_2, \quad \text{so } X_0 = X_1 \oplus X_2 \oplus X_3 ; \tag{3.14}$$

$\dim X_1 = q$, $\dim X_2 = m - q$. However, range A is spanned by AX_2 and AX_3, and we claim that, in fact, it has the direct sum decomposition

$$\text{range } A = AX_2 \oplus AX_3 . \tag{3.15}$$

To verify this we have only to show that $Ax_2 \cap AX_3 = 0$. Suppose $x_2 \in X_2$, $x_3 \in X_3$, and $Ax_2 = Ax_3$; then $x_2 - x_3 \in X_1$. By our direct sum decomposition (3.14) it follows that $x_2 = 0$, $x_3 = 0$. Thus (3.15) is verified.

The map $A : X_2 \to AX_2$ is one-to-one and so has $\dim AX_2 = m - q$. Since range A has codimension q in X, it follows from (3.15) that AX_3 has codimension m in X. However, X^1 has dimension m. Thus to prove (3.13), we have only to show that

$$X^1 \cap AX_3 = 0 .$$

Suppose, then, $Ax_3 \in X^1$ for some $x_3 \in X_3$. By the definition of X^1 we have $A^{n+1}x_3 = 0$, and so by the property of n, $A^n x_3 = 0$, i.e., $x_3 \in X_3 \cap X^1$. Thus, $x_3 = 0$ and the lemma is proved.

The lemma furnishes a splitting of the operator A

$$A : X^1 \to X^1, \qquad A : X_3 \to AX_3,$$

with the latter mapping being one-to-one . By the closed graph theorem this map $A : X_3 \to AX_3$ has a bounded inverse. We can now reduce the problem to a finite-dimensional one. Write $x = x^1 + x_3$, $x^1 \in X^1$, $x_3 \in X_3$, and let Q be the projection in X onto AX_3 associated with the splitting (3.13). Then, since $(A - \lambda)x^1 \in X^1$, equation (3.11) is equivalent to the system

$$Q(A - \lambda)x_3 = QG(x^1 + x_3, \lambda),$$
$$(I - Q)(A - \lambda)x^1 = (I - Q)G(x^1 + x_3, \lambda) + \lambda(I - Q)x_3 .$$

Using the fact that $A : X_3 \to AX_3$ has a bounded inverse, there is a unique solution near the origin of the first of these equations: $x_3 = x_3(x^1, \lambda)$, $x_3(0, 0) = 0$, by the implicit function theorem. Thus equation (3.11) is reduced to the finite-dimensional problem:

$$(3.16) \quad (A - \lambda)x^1 = (I - Q)G\big(x^1 + x_3(x^1, \lambda), \lambda\big) + \lambda(I - Q)x_3 \equiv \widetilde{G}(x^1, \lambda) .$$

One verifies easily that

$$(3.17) \qquad \widetilde{G}(x^1, \lambda) = O(\|x^1\|_0^2 + |\lambda|^p) .$$

The Value of p. We will now specify the value of p in Theorem 3.5.1. On X^1 = the generalized null space of A, the operator A is nilpotent, $A^n \equiv 0$. Thus, by introducing a suitable basis in X^1, we may put $A|_{X^1}$ into Jordan canonical form with $q = \dim\ker A$ Jordan blocks of size $k_1, \ldots, k_q$. Setting

$$k = \max k_1 ,$$

we choose

$$(3.17') \qquad p = 2k + 1 .$$

□

REMARK. This value of p is optimal. Consider the following finite-dimensional example in $\mathbb{R}^m$ with A consisting of one Jordan block:

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}$$

Here $q = 1$, $k = m$. The system, for $x = (x_1 \dots x_n)^{\mathsf{T}}$ is

$$(A - \lambda I) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ |x_1|^2 + |\lambda|^{2m} \end{pmatrix} .$$

i.e.,

$$\begin{aligned} x_2 - \lambda x_1 &= 0 , \\ x_3 - \lambda x_2 &= 0 , \\ x_m - \lambda x_{m-1} &= 0 , \\ -\lambda x_m &= |x_1|^2 + |\lambda|^{2m} . \end{aligned}$$

Thus $p = 2k$. Solving for $x_2, \dots, x_m$ in terms of x_1, we find $x_2 = \lambda x_1$, $x_3 = \lambda^2 x_1, \dots, x_m = \lambda^{m-1} x_1$; hence the last equation yields

$$-\lambda^m x_1 = |x_1|^2 + |\lambda|^{2m} ,$$

which has $x_1 = 0$, $\lambda = 0$, as the only solution.

PROOF OF THEOREM 3.5.1: We have reduced the problem to the finite-dimensional one (3.16) with $\widetilde{G}$ satisfying (3.17); from now on we shall work only in finite dimensions. Thus, we may assume that $X = X^1$ has dimension m, and that A is nilpotent on X, with n the first integer such that $A^n \equiv 0$.

By choosing suitable coordinates in X, we may suppose that A is in Jordan normal form with q blocks of size $k_1, \dots, k_q$. We shall denote these coordinates in a special way

$$AX = \left(\begin{array}{cccc|cccc} 0 & 1 & & 0 & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ 0 & & & 0 & & & & \\ \hline & & & & 0 & 1 & & 0 \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & 0 & & & 0 \\ & & & & & & & & \ddots \end{array} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_2 \\ \vdots \\ x_3 \\ \vdots \end{pmatrix}$$

where x is a column vector, which for convenience we note here as a row vector:

$$\begin{aligned} x &= (x_1, \dots, x_2, \dots, x_q \dots) \\ &= \left(x_1, x_1^1, \dots, x_1^{k_1 - 1}, x_2, \dots, x_2^{k_2 - 1}, \dots, x_q, x_q^1, \dots, x_q^{k_q - 1}\right) . \end{aligned}$$

Thus, the vectors with all components zero except possibly for $x_1, x_2, \dots, x_q$, span $\ker A$. Equation (3.11) now consists of q blocks; the j^{th} block has the form

$$(3.11_j)\qquad \begin{aligned} x_j^1 - \lambda x_j &= g_j^1(x,\lambda)\,,\\ x_j^2 - \lambda x_j^1 &= g_j^2(x,\lambda)\,,\\ &\ \ \vdots\\ x_j^{k_j-1} - \lambda x_j^{k_j-2} &= g_j^{k_j-1}(x,\lambda)\,,\\ -\lambda x_j^{k_j-1} &= g_j^{k_j}(x,\lambda)\,, \end{aligned}$$

By (3.17′) and condition (1), each g_j^i satisfies

$$(3.17'')\qquad g_j^i(x,\lambda) = O\big(|x|^2 + |\lambda|^{2k+1}\big)$$

where $k = \max k_i$.

In finite dimensions we are going to prove a sharper form of Theorem 3.5.1 in which the conditions depend more on the specific Jordan structure. Namely, in place of (3.17″) we shall assume

$$(3.18)\qquad g_j^i(x,\lambda) = O\big(|x|^2 + |\lambda|^{k+i+1}\big)\,, \qquad i = 1,\dots,k_j,\ j = 1,\dots,q\,.$$

To prove this result we make a further reduction of the problem in $X \times \Lambda$ to one in $X_1 \times \Lambda$ where $X_1 = \ker A$. Namely, with the aid of the implicit function theorem we may solve the reduced system of $q(j-1)$ equations, consisting of the first $j-1$ equations in the j^{th} block, $j = 1,\dots,q$, for the coordinates x_j^i, $j = 1,\dots,k_{j-1}$, $j = 1,\dots,q$, in terms of $x_1,\dots,x_q$. It is clear from the form of these equations that for $j = 1,\dots,q$

$$\begin{aligned} x_j^1 &= \lambda x_j + O\big(|x|^2 + |\lambda|^{k+2}\big)\,,\\ x_j^2 &= \lambda^2 x_j + O\big(|x|^2 + |\lambda|^{k+3}\big)\,,\\ &\ \ \vdots\\ x_j^{k_j-1} &= \lambda^{k_j-1} x_j + O\big(|x|^2 + |\lambda|^{k+k_j}\big)\,. \end{aligned}$$

Thus, for $j = 1,\dots,q$, the last equation in the j^{th} block takes the form

$$(3.19)\qquad \lambda^{k_j} x_j = g_j(x_1,\dots,x_q,\lambda)$$

for functions g_j satisfying

$$(3.18')\qquad g_j = O\left(\sum_1^q |x_i|^2 + |\lambda|^{k+k_j+1}\right).$$

We have reduced our system to the system (3.19) of q equations for $(x_1,\dots,x_q,\lambda)$. Theorem 3.5.1 then follows from the following result, in which we rename $(x_1,\dots,x_q) = x$. □

THEOREM 3.5.3 *Let X be $\mathbb{R}^q$ or $\mathbb{C}^q$, let Λ be $\mathbb{R}$ or $\mathbb{C}$, and let $k_1,\dots,k_q$ be nonnegative integers, with $k = \max k_i > 0$. In a neighborhood of the origin in $X \times \Lambda$,* *for $j = 1, \ldots, q$, let $g_j(x, \lambda)$ be a C^1 function with values in Λ satisfying (3.18′). For $r > 0$ consider the system*

$$\begin{aligned} \lambda^{k_j} x_j - g_j(x, \lambda) &= 0, \quad j = 1, \ldots, q, \\ |x|^2 - r^2 &= 0. \end{aligned} \tag{3.20}$$

Then for $r > 0$ sufficiently small, there is a solution of the system near the origin in any one of the following cases:

(i) *$m = \sum_1^q k_j$ is odd,*
(ii) *$\Lambda = \mathbb{C}$ and $q = 1$, and*
(iii) *$m = \sum_1^q k_j$ is even and, for some j, say $j = q : k_q > 0$, and g_q is of the form*

$$g_q(x, \lambda) = \lambda h_q(x, \lambda), \qquad h_q \in C^1 \quad \text{and} \quad h_q = O\big(|x|^2 + |\lambda|^{k+k_q}\big). \tag{3.21}$$

REMARK. Case (iii) follows from case (i) by replacing the equation

$$\lambda^{k_q} x_q - g_q = 0 \quad \text{by} \quad \lambda^{k_q - 1} x_q - h_q = 0$$

and replacing k_q by $k_q' = k_q - 1$. Then we are exactly in case (i). Returning to our larger system $(3.5.1)_j$, $j = 1, \ldots, q$, we see that if $g_q^{k_q}$ is of the form

$$g_q^{k_q}(x, \lambda) = \lambda h_q^{k_q}(x, \lambda) \quad \text{with} \quad h_q^{k_q} = O\big(|x|^2 + |\lambda|^{k+k_q}\big), \tag{3.21′}$$

$h_q^{k_q} \in C^1$, then the reduced system (3.20) satisfies (3.21). *This is the class of terms G referred to in* Theorem 3.5.1(iii).

PROOF OF THEOREM 3.5.3: We consider two cases: $\Lambda = \mathbb{R}$ and $\Lambda = \mathbb{C}$.

If $\Lambda = \mathbb{R}$, let $F(x, \lambda) = (y, \tau)$ be the map defined in a neighborhood of the origin in $\mathbb{R}^q \times \mathbb{R}$ into $\mathbb{R}^q \times \mathbb{R}$:

$$\begin{aligned} y_j &= \lambda^{k_j} x_j - g_j(x, \lambda), \quad j = 1, \ldots, q, \\ \tau &= |x|^2 - r^2. \end{aligned} \tag{3.22}$$

We claim that for M a sufficiently large constant and all $r > 0$ sufficiently small,

$$\deg\big(F, |x|^2 + |\lambda|^2 \le r^2 + M^2 r^{2/k}, 0\big)$$

is defined; i.e., there are no solutions of $F(x, \lambda) = 0$ on the boundary. For suppose (x, λ) is such a solution; then $\lambda = \pm M r^{1/k}$. Furthermore,

$$x_j = \lambda^{-k_j} g_j = O\big(\lambda^{-k_j} r^2 + |\lambda|^{k+1}\big), \qquad j = 1, \ldots, q.$$

Squaring and adding we find

$$r^2 = O\left(r^4 \sum \lambda^{-2k_j} + |\lambda|^{2k+2}\right),$$

or, since $\lambda = \pm M r^{1/k}$,

$$1 \le \left[C r^2 \sum_j (M r^{1/k})^{-2k_j} \right] + C M^{2k+2} r^{2/k}.$$

We may choose M so large (and independent of r) that the first term on the right is less than $\frac{1}{2}$. Fixing M, we then require r to be so small that the second term is less than $\frac{1}{2}$, and we have a contradiction.

Now consider the deformation, for $0 \le t \le 1$,

$$\begin{cases} \lambda^{k_j} x_j - t g_j(x, \lambda), & 1 \le j \le q, \\ t(|x|^2 - r^2) + (1-t)(M^2 r^{2/k} - |\lambda|^2). \end{cases}$$

Just as above, zero is never attained on the boundary. Thus, the degree is independent of t and for $t = 0$ the map is

$$\widetilde{F}(x, \lambda) = \left(\lambda^{k_1} x_1, \dots, \lambda^{k_q} x_q, M^2 r^{2/k} - \lambda^2\right).$$

The preimages of zero are $(0, \pm M r^{1/k})$.

The Jacobian of $\widetilde{F}(x, \lambda)$ is

$$J\widetilde{F}(x, \lambda) = \begin{pmatrix} \lambda^{k_1} & & & & 0 \\ & \lambda^{k_2} & & & \\ & & \ddots & & \\ & & & \lambda^{k_q} & \\ 0 & & & & -2\lambda \end{pmatrix}$$

$$\det J\widetilde{F}(\lambda, \lambda) = -2\lambda^{\sum_{j=1}^{\xi} k_j + 1} = -2\lambda^{m+1}.$$

Thus the points $(0, \pm M r^{1/k})$ are regular points of the map, and the determinant of the Jacobians at these points is $-2(\pm\lambda_0)^{m+1}$. Hence if m is odd, the degree is -2 and we have a nontrivial solution. For $\Lambda = \mathbb{R}$ the theorem is proved. Note that the degree is zero if m is even.

Suppose $\Lambda = \mathbb{C}$. In the complex case we consider the system

$$\lambda^{k_j} x_j = \tilde{g}_j = O\left(\sum |x_i|^2 + |\lambda|^{k_j + k + 1}\right), \quad j = 1, \dots, q,$$
$$|x|^2 - r^2 = 0,$$

as $2q + 1$ real equations for $2q + 2$ real unknowns.

As above, we see that there are no solutions of the system on the boundary of

$$D : |x|^2 + |\lambda|^2 \le r^2 + M^2 r^{2/k}$$

for suitably large M and all $r > 0$ sufficiently small.

Consequently, the map $F(x, \lambda) = (y, \tau)$ of a neighborhood of the origin in $\mathbb{C}^q \times \mathbb{C}$ into $\mathbb{C}^q \times \mathbb{R}$ given by (3.22) maps ∂D into $\mathbb{C}^q \times \mathbb{R}\backslash\{0\}$. We may regard it as a map

$$F : \mathbb{S}^{2q+1} \to \mathbb{R}^{2q+1}\backslash\{0\}.$$

We cannot use degree theory, but we wish to show that the homotopy class of this map into $\mathbb{R}^{2q+1}\backslash\{0\}$ is nontrivial.

As before we may deform this map on ∂D to

$$(x, \lambda) \mapsto \left(\lambda^{k_1} x_1, \dots, \lambda^{k_q} x_q, M^2 r^{2/k} - |\lambda|^2\right). \tag{3.23}$$

By a series of deformations, which we postpone for the moment, we deform this map on ∂D to

$$(3.24)\qquad (x,\lambda)\mapsto \left(\lambda^m x_1, x_2, \ldots, x_q, |x_1|^2-|\lambda|^2\right), m=\sum k_j\,.$$

We now use the results stated in Section 1.8. If $m = 1$ and $q = 1$, this is just minus the Hopf map $\psi(z,\lambda)$ of (1.3), mapping $\mathbb{S}^3 \to \mathbb{S}^2$, and has nontrivial homotopy type.

For $m > 0$, $q > 1$, this is a $2(q-1)$-fold suspension of $-m$ times the Hopf map

$$\mathbb{S}^{2q+1} \to \mathbb{R}^{2q+1}\backslash\{0\}$$

and is nontrivial if and only if m is odd. In case $q = 1$, there is no suspension; hence the map is always nontrivial whether m is even or odd, proving case (ii) of Theorem 3.5.3. (In this case, $q = 1$, Ize's thesis contains a stronger result.)

To complete the proof we have to show that in ∂D the map (3.23) may be deformed to (3.24) via maps into $\mathbb{C}^q \times \mathbb{R}\backslash\{0\}$. We shall construct a permissible deformation of (3.23) on ∂D to the map

$$(3.25)\qquad (x,\lambda)\mapsto \left(\lambda^{k_1}x_1,\ldots,\lambda^{k_{q-2}}x_{q-2}, \lambda^{k_{q-1}+k_q}x_{q-1}, x_q, M^2r^{2/k}-|\lambda|^2\right).$$

By repeating this, one obtains a deformation to

$$(x,\lambda)\mapsto \left(\lambda^m x_1, x_2,\ldots,x_q, M^2r^{2/k}-|\lambda|^2\right).$$

Finally, via the deformation

$$(x,\lambda)_t\mapsto \left(\lambda^m x_1, x_2,\ldots,x_q, t|x_1|^2+(1-t)M^2r^{2/k}-|\lambda|^2\right),\quad 0\le x\le 1,$$

we obtain for $t = 1$ the desired map (3.24). Note that if for some t and some (x,λ) on ∂D, the point $(x,\lambda)_t = 0$, then $x_1 = \cdots = x_q = 0$ and either λ or x_1 is zero. If $\lambda = 0$, then $|x_1|^2 = r^2 + M^2r^{2/k}$, and so the last component of $(x,\lambda)_t$ cannot vanish. Likewise, if $x_1 = 0$, we have $|\lambda|^2 = r^2 + M^2r^{2/k}$, and again the last component $\neq 0$.

The deformation of (3.23) to (3.25) is obtained in two steps. First we construct the deformation, $0 \le t \le 1$, in which only the $(q-1)^{\text{st}}$ and q^{th} components change:

$$(x,\lambda)_t\mapsto \left(\ldots,(1-t)\lambda^{k_{q-1}}x_{q-1}-tx_q, t\lambda^{k_{q-1}+k_q}x_{q-1}+(1-t)\lambda^{k_q}x_q,\ldots\right).$$

For $t = 1$ this gives the map

$$(x,\lambda)\mapsto \left(\ldots,-x_q,\lambda^{k_{q-1}+k_q}x_{q-1},\ldots\right).$$

If we now perform the deformation, $0 \le t \le 1$ where, again, only the $(q-1)^{\text{st}}$ and q^{th} components change:

$$(x,\lambda)_t\mapsto \left(\ldots,t\lambda^{k_{q-1}+k_q}x_{q-1}-(1-t)x_q, (1-t)\lambda^{k_{q-1}+k_q}x_{q-1}+tx_q,\ldots\right).$$

We obtain the desired map (3.25) for $t = 1$. □

EXERCISE Prove that these deformations are admissible.

It is clear that in special circumstances the argument used here may apply under weaker conditions than (1) or (3.18). Furthermore, the theorem holds under considerable modification of these conditions:

EXERCISE Prove Theorem 3.5.1 assuming, in place of (1),

$$\|G(x,\lambda)\| = O\left(\|x\|_0^a + |\lambda|^{b+k}\right) \tag{1'}$$

provided

$$a > 1 \quad \text{and} \quad b > \frac{k}{a-1}.$$

For a and b as above, prove the sharper form in finite dimensions with (3.18) replaced by

$$g_j^i(x,\lambda) = O\left(|x|^a + |\lambda|^{b+i}\right), \qquad i = 1, \dots, k_q;\ j = 1 \dots q.$$

Prove Theorem 3.5.3 with (3.18′) replaced by

$$g_j = O\left(\left(\sum_1^q |x_i|^2\right)^{a/2} + |\lambda|^{b+k_j}\right).$$

3.6. Stability of Solutions

A solution x_0 of a nonlinear problem $f(x) = 0$ may correspond to a steady state solution of a time-dependent problem

$$\frac{dx}{dt} = \dot{x} = f(x).$$

It is then of interest to know whether it is stable or not. If we perturb x_0 slightly to $x_0 + \delta x_0$ and solve the initial value problem $\dot{x} = f(x)$, $x(0) = x_0 + \delta x_0$, assuming this is well-posed, it is of interest to know if the resulting solution $x(t)$ is close to x_0 for all t or tends to x_0 as $t \to \infty$. Some information may be obtained by considering the linearized problem

$$\delta\dot{x} = f_x(x_0)\delta x.$$

If the spectrum of $f_x(x_0)$ lies to the left of the imaginary axis, the solution δx decays exponentially as $t \to \infty$. We then say that the solution x_0 of $f(x_0)$ is (linearly) stable. If the spectrum contains points in the right half-plane, the solution is called (linearly) unstable. In this section we wish to study the (linear) stability or instability of solutions of certain bifurcation problems.

3.6.1. Some Examples of Bifurcation.

EXAMPLE 1. Consider G a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the Dirichlet problem for $u(x)$ in G

$$(\Delta - \mu)u = u^2 \quad \text{in } G, \qquad u = 0 \quad \text{on } \partial G.$$

The eigenvalues of Δ are $\cdots \le \mu_2 \le \mu_1 < \mu_0 < 0$ with μ_0 simple and having a positive eigenfunction u_0. By the result that we proved as an application of Theorem 3.2.2, we know that $(0, \mu_0)$ is a bifurcation point for this problem. In fact, the set of solutions (u, μ) near $(0, \mu_0)$ consists of the trivial curve $(0, \mu)$ and an analytic curve

$$(u(s), \mu(s)) = (su_0 + u_2(s), \mu(s)), \qquad (u(0), \mu(0)) = (0, \mu_0),$$

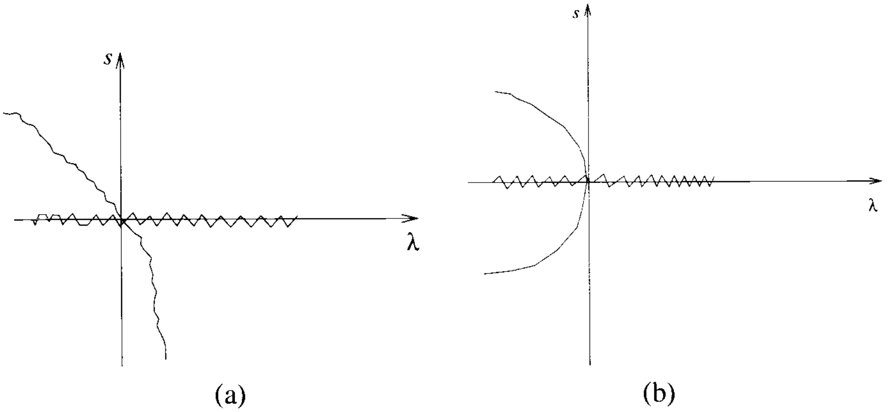

FIGURE 3.3

where $u_2 \in X_2$ = space of functions in $C^\infty(\overline{G})$ that are L_2 orthogonal to u_0.

We wish to investigate the behavior of $(u(s), \mu(s))$ near $s = 0$. If the equation with $\mu - \mu_0 = \lambda(s)$

$$Au = (\Delta - \mu_0)u = \lambda u + u^2$$

is differentiated with respect to s, we find, since $\dot{u} = u_0 + \dot{u}_2$,

$$A\dot{u}_2(0) = 0\,.$$

Hence $\dot{u}_2(0) = 0$ and $\dot{u}(0) = u_0$, as we already know from Theorem 3.2.2. Differentiating again, we have $\ddot{u}(0) = \ddot{u}_2(0)$ and, at $s = 0$,

$$A\ddot{u}_2 = 2\dot{\lambda}\dot{u} + 2\dot{u}^2 = 2\dot{\lambda}u_0 + 2u_0^2\,.$$

If we take the L_2 scalar product with u_0, we find, assuming $(u_0, u_0) = 1$,

$$2\dot{\lambda}(0) + 2\int_G u_0^3\,dx = (A\ddot{u}_2, u_0) = 0$$

since A is self-adjoint, $u_0 \perp$ range A. Thus $\dot{\lambda}(0) < 0$, and we therefore have Figure 3.3(a).

EXAMPLE 2. Consider G as in the previous example, and the Dirichlet problem

$$(\Delta - \mu)u = u^3 \text{ in } G\,, \quad u = 0 \text{ on } \partial G\,.$$

We again have an analytic family of solutions

$$(u(s), \mu(s)) = (su_0 + u_2(s), \mu(s))\,, \quad \mu = \mu_0 + \lambda(s)\,.$$

By a similar analysis to the above, we find $\lambda(0) = \dot{\lambda}(0) = 0$, $\ddot{\lambda}(0) = -2\int_G u_0^4\,dx$ and the corresponding Figure 3.3(b).

We wish to investigate whether these solution branches are (linearly) stable or not. In doing this we shall study, in general, the stability of the branches of the solution curves obtained in Theorem 3.2.2 (with $p \geq 3$).[2] In that theorem we obtained two C^{p-2} solution curves $(x(s) = sx_0 + x_2(s), \lambda(s))$ and $(x = \phi(\sigma), \lambda = \sigma)$ with $\phi(0) = \dot{\phi}(0) = 0$. Zero is an eigenvalue of $f_x(0,0)$, and we are interested in the spectrum of $f_x(x(s), \lambda(s))$ and $f_x(\phi(\sigma), \sigma)$. We shall assume that aside from an eigenvalue near zero, the remainder of the spectra of these operators lies in the left half-plane.

DEFINITION (Crandall-Rabinowitz) Let T_0, K be bounded linear maps $X \to Y$ of Banach spaces X, Y; μ_0 is a *K-simple eigenvalue* of T_0 if

(i) $\ker(T_0 - \mu_0 K)$ is one-dimensional spanned by some x_0,
(ii) $\text{range}(T_0 - \mu_0 K)$ is closed and of codimension 1, and
(iii) $Kx_0 \notin \text{range}(T_0 - \mu_0 K)$.

REMARK. If $X = Y$, $K = I$, T_0 compact, then μ_0 being an I-simple eigenvalue of T_0 is equivalent to μ_0 being a simple eigenvalue.

LEMMA 3.6.1 (Crandall-Rabinowitz) *Suppose that μ_0 is a K-simple eigenvalue of T_0. There is a number $\delta > 0$ such that for $\|T - T_0\| < \delta$, there is a unique $\mu(T)$ with $\mu(T)_0 = \mu_0$, which is a K-simple eigenvalue of T and* $\ker(T - \mu(T)K)$ *is spanned by $x(T) = x_0 + x_2(T)$, where $X = \text{span}\{x_0\} \oplus X_2$, $x_2 \in X_2$. In addition, $\mu(T), x(T)$ are unique and analytic in T.*

PROOF: We may suppose $\mu_0 = 0$; we wish to solve

$$(T - \mu(T)K)x(T) = 0$$

with $x(T) = x_0 + x_2(T)$. Consider the mapping (here $r \in \mathbb{R}$)

$$(T, r, x_2) \mapsto (T - rK)(x_0 + x_2)\,.$$

For each T close to T_0, we will find a zero in the map with the aid of the implicit function theorem. The Frechet derivative with respect to r and x_2 is

$$-\delta r K x_0 + T_0 \delta x_2\,.$$

Since 0 is a K-simple eigenvalue of T_0, it follows that this map is one-to-one onto Y. By the implicit function theorem there is a unique analytic solution for T near T_0, $r(T)$: $x_2(T)$ with $r(T_0) = 0$, $x_2(T_0) = 0$.

Next we have to check the uniqueness in the lemma. For δ and r small, $T - rK$ is Fredholm of index zero, and since the dimension of the null space is uppersemicontinuous, we see that

$$\dim\ker(T - rK) \leq 1\,.$$

We know that $(T - r(T)K)(x_0 + x_2(T)) = 0$. Suppose for some T and r, $T - rK$ annihilates some vector $(\beta x_0 + x_2) \neq 0$, $x_2 \in X_2$. Then

$$T_0 x_2 - r\beta K x_0 = (T_0 - T)(\beta x_0 + x_2) + rKx_2\,.$$

[2]This material is based on **[27]**. See also **[32]**. In his notes, Sattinger **[10]** has studied the stability of a variety of problems.

As a linear map of $X_2 \otimes R_1$, the left-hand operator is an isomorphism onto Y. Thus, for some constant C

$$|\beta r| + \|x_2\| \leq C\big(\|T - T_0\|(|\beta| + \|x_2\|) + |r|\|x_2\|\big) .$$

For δ, r sufficiently small, it follows that $\beta \neq 0$; otherwise $\beta = 0$ and $x_2 = 0$. So we may suppose $\beta = 1$. But then it follows from the preceding inequality that $\|x_2\|$ is small. Thus, $x_0 + x_2$ is close to x_0. From the uniqueness of the small solution $r(T), x_2(T)$ obtained from the implicit function theorem, we conclude that $r = r(T)$ and $x_2 = x_2(T)$.

Finally, we leave as an exercise to show that $K(x_0 + x_2(T)) \notin \text{range}(T - \mu(T)K)$.

Returning to our solution curves obtained in Theorem 3.2.2, we shall assume that X is a linear subspace of Y and that the inclusion map i is continuous. We shall apply the lemma to $T_0 = f_x(0,0)$, and $K = i =$ conclusion. Then 0 is an i-simple eigenvalue of $f_x(0,0)$. By the lemma there exist unique $\mu(s)$ and $\omega(s) = x_0 + x_2(s)$ in C^{p-2} such that

$$f_x(x(s), \lambda(s))\omega(s) = \mu(s)\omega(s) .$$

Similarly, along the other branch $(\phi(\sigma), \sigma)$,

$$f_x(\phi(\sigma), \sigma)u(\sigma) = \gamma(\sigma)u(\sigma) ,$$

where $u(\sigma) = x_0 + \tilde{x}_2(\sigma)$ and $\gamma(\sigma)$ are in C^{p-2}. □

THEOREM 3.6.2 (Crandall-Rabinowitz) *$\gamma'(0) \neq 0$; $s\dot{\lambda}(s)\gamma'(0)$ and $\mu(s)$ vanish together and have opposite sign. Furthermore, if $\mu(s) \neq 0$ for $s \neq 0$, then*

$$\frac{s\dot{\lambda}(s)\gamma'(0)}{\mu(s)} \to -1 \quad \text{as } s \to 0 .$$

Before proving the theorem, let us apply it to the examples. In both cases $\sigma = \lambda$, $\phi(\sigma) \equiv 0$, $f_x(0,\lambda)u_0 = (\Delta - (\mu_0 + \lambda))u_0 = -\lambda u_0$, i.e., $\gamma = -\lambda$, and we therefore have stability of the solution $(0, \lambda)$ when $\lambda > 0$, instability when $\lambda < 0$. We also note that $\gamma'(0) < 0$. In example 1 for s small, $s > 0$, $\dot{\lambda}(s) < 0$, $\gamma'(0) < 0$. Then, by the theorem, $\mu(s) < 0$, which means stability.

For $s < 0$, we have $\dot{\lambda}(s) < 0$, $\gamma'(0) < 0$, and so $\mu(s) > 0$, which means instability; we therefore have Figure 3.4(a). Similarly, in example 2 we obtain Figure 3.4(b).

PROOF OF THEOREM 3.6.2: On the branch $(\lambda = \sigma, x = \phi(\sigma)$, $\phi(0) = \phi'(0) = 0$, we have $f_x u(\sigma) = \gamma(\sigma)u(\sigma)$. Differentiating with respect to σ and evaluating at $\sigma = 0$, we find $f_{x\lambda}(0,0)u(0) + f_x(0,0)u'(0) = \gamma'(0)u(0)$. Suppose $y^* \neq 0$ is a continuous linear functional on Y that vanishes on range $f_x(0,0)$; applying y^* we find

$$\langle y^*, f_{x\lambda}(0,0)x_0\rangle = \gamma'(0)\langle y^*, x_0\rangle , \tag{3.26}$$

the left-hand side is not equal to zero since $f_{x\lambda}(0,0)x_0$ is not in range $f_x(0,0)$. Therefore we have $\gamma'(0) \neq 0$ and $(y^*, x_0) \neq 0$. On the other branch

$$f_x(x(s), \lambda(s))\omega(s) = \mu(s)\omega(s) .$$

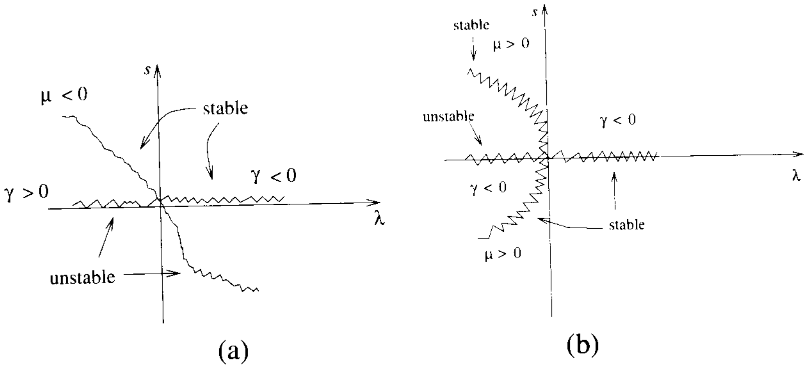

FIGURE 3.4

Differentiating $f(x(s), \lambda(s)) = 0$, we have

$$f_x(x(s), \lambda(s))\dot{x}(s) + f_\lambda(x(s), \lambda(s))\dot{\lambda}(s) = 0 .$$

We also have $x(s) = sx_0 + x_2(s)$, $\dot{x}(0) = x_0$. Subtracting one equation from the other gives

$$f_x(x(s), \lambda(s))(x(s) - \omega(s)) + f_\lambda(x(s), \lambda(s))\dot{\lambda}(s) + \mu(s)\omega(s) = 0 .$$

Expand $f_x(x(s), \lambda(s))$, $f_\lambda(x(s), \lambda(s))$ in Taylor series about $s = 0$ and substitute in above:

$$\begin{aligned}(3.27)\quad & f_x(0,0)(\dot{x}(s) - \omega(s)) + O(1)(\dot{x}(s) - \omega(s)) + f_{\lambda x}(0,0)\dot{x}(0)s\dot{\lambda}(s) \\ & + f_{\lambda\lambda}(0,0)s\dot{\lambda}(s)^2 + o(s)\dot{\lambda}(s) + \mu(s)x_0 + \mu(s)O(1) = 0 .\end{aligned}$$

Recall that $\dot{x}(s) - \omega(s) = x_0 + \dot{x}_2(s) - (x_0 + \tilde{x}_2(s)) = \dot{x}_2(s) - \tilde{x}_2(s) \in X_2$. Since $f_x(0,0)$ is invertible in X_2, with bounded inverse, we see that

$$(3.28)\qquad \|\dot{x}(s) - \omega(s)\| \le C(|s\dot{\lambda}(s)| + |\mu(s)|) .$$

Next, apply y^* to (3.27); using (3.28) we find

$$s\dot{\lambda}(s)\langle y^*, f_{\lambda x}(0,0)x_0\rangle + \mu(s)\langle y^*, x_0\rangle = O(1)(|s\dot{\lambda}(s)| + |\mu(s)|) .$$

In view of (3.26) we have

$$(s\dot{\lambda}(s)\gamma'(0) + \mu(s))\langle y^*, x_0\rangle = O(1)(|s\dot{\lambda}(s)| + |\mu(s)|) .$$

The assertion in the theorem follows easily from this. □

3.6.2. Another Application. Consider the ordinary differential equation

$$f(u, \lambda) - \ddot{u} = h(u^2 + \dot{u}^2)u + (1 - \lambda)u = 0$$

on the interval $(0, \pi)$ with Dirichlet conditions $u(0) = u(\pi) = 0$. Let X be the space of C^2 functions on $[0, \pi]$ satisfying the boundary conditions and $Y = C[0, \pi]$. Here $h(r)$ is a real C^2 function defined for $r \ge 0$, with $h(0) = 0$.

We see that $\ker f_u(0,0)$ is spanned by $u_0 = \sin t$, and $f_{u\lambda}(0,0)\delta u = -\delta u$ so $f_{\lambda u}(0,0)u_0$ is not in range $f_u(0,0)$. Consequently, we have a bifurcating curve $(u(s), \lambda(s))$; in fact, we observe that

$$u(s) = s \sin t, \quad \lambda = h(s^2).$$

The other curve of solutions is the trivial one $(0, \lambda)$ so that $\gamma(\lambda) = -\lambda$. From the theorem we conclude that for $s \neq 0$, $\mu(s)$ and $s\lambda'(s)$ have the same sign; i.e., $\mu(s)$ has the same sign as $\dot{h}(s^2)$ for $s \neq 0$.

3.7. The Number of Global Solutions of a Nonlinear Problem

Lecture of M. Kalka

In this section we take up papers [**25, 26**]. These treat an elliptic boundary value problem

$$\Delta u + f(u(x)) = y(x) \quad \text{in } G, \qquad u = 0 \quad \text{on } \partial G, \tag{3.29}$$

where G is a bounded domain in $\mathbb{R}^n$ with smooth boundary and f is a given convex function $\mathbb{R} \to \mathbb{R}$ satisfying some additional conditions. The problem is to find for which functions $y(x)$ in G there exist solutions, and how many there are.

First we prove some general functional analytic results taken from [**11**] (with minor changes).

DEFINITION A set M in a real Banach space X is said to be a C^k *manifold of codimension* 1 if for every point $u_0 \in M$ there is an open neighborhood U of u_0 and a weak C^k function Γ defined on U with

(i) $\Gamma'(u_0) \neq 0$ and
(ii) $M \cap U = \{u \in U \mid \Gamma(u) = 0\}$.

PROPOSITION 3.7.1 *Let M be a closed connected C^k manifold, $k \geq 1$, of codimension 1 in the Banach space X. Then $X - M$ has at most two components.*

PROOF: Suppose A_1, A_2, A_3 open in $X - M$ (hence in X since M is closed) such that $\bigcup_{i=1}^{3} A_i = X - M$, $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $B_i = \partial A_i$, $B_i \neq \emptyset$ for if so A_i is open and closed, implying $A_i = X$. Also $B_i \subset M$.

Now any $u_0 \in M$ has a neighborhood U such that $U \cap (X - M)$ has exactly twp components. Thus only two of the A_i can have a nonempty intersection with U. Hence $U \cap M$ can be contained in at most two of the B_i.

Also, if $u_0 \in B_i$ and one of the two components of $U \cap (X - M)$ is contained in A_i, then every boundary point on $U \cap M$ is a boundary point of A_i (i.e., $\in B_i$). Hence the B_i are open and closed in $M - M = B_i$. But every point $u_0 \in M$ belongs to at most two B_i. □

Let X and Y be real Banach spaces and ϕ a C^k map, $k \geq 1$, of an open set Ω in X into Y.

DEFINITION $x_0 \in \Omega$ is a *singular point* of ϕ if $\phi'(x_0)$ is not an isomorphism of X onto Y. The set of singular points of ϕ is called the *singular set* W; $\phi(W)$ is called the *set of singular values* of ϕ.

THEOREM 3.7.2 *Let $\phi : \Omega \to Y$ be of class C^k, $k \geq 2$, and assume that at $x_0 \in \Omega$*

(i) *$\phi'(x_0)$ has kernel X_1 spanned by a vector v.*

(ii) *range $\phi'(x_0) = Y_1$ is a closed linear subspace of Y that is annihilated by a linear functional $y^* \neq 0$.*

(iii) *The linear functional on X : $F(x) = y^*(\phi''(x_0)(v, x))$ is not identically zero.*

Then, in a neighborhood of x_0, the singular set W of ϕ is a C^{k-1} manifold of codimension 1. If, in addition, we require

(iii)′. *$y^*(\phi''(x_0)(v, v)) \neq 0$,*

then for some neighborhood U of x_0, $\phi(W \cap U)$ is a C^{k-1} manifold of codimension 1 in Y.

A point x_0 satisfying (i), (ii), and (iii)′ is called an *ordinary singular point* of ϕ.

PROOF: We may suppose $x_0 = 0$ and remark first that for x near 0, the linear operator $\phi'(x)$ is Fredholm of index zero and hence, since $\dim \ker \phi'(x)$ is upper-semicontinuous, $\ker \phi'(x)$ is either zero or one-dimensional. In addition, we know from the theory of Fredholm operators that if $\ker \phi'(x)$ is nonempty, then it is spanned by a vector close to v.

With this in mind, consider the following problem: Decompose $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$, where Y_2 is spanned by some vector $y_2 \neq 0$; we may suppose $y^* y_2 = 1$. For x near 0 in X, w near 0 in X_2, and t near 0 in $\mathbb{R}$, solve the equation

$$\phi'(x)(v + w) + t y_2 = 0$$

for $w(x), t(x)$ as functions of x. It is immediately verified that the conditions of the implicit function theorem are satisfied, so that one has a unique solution $w(x)$, $t(x)$.

Near 0 the critical set W is simply the set

$$\{x \text{ near } 0 \mid t(x) = 0\}.$$

If we differentiate the equation with respect to x at $x = 0$ and apply y^*, we find, since $y^*(\phi'(0)x) = 0$ and $y^* y_2 = 1$,

$$y^*(\phi''(0)(v, x)) + t_x(0)x \equiv 0.$$

Therefore, by condition (iii), the linear functional $t_x(0)$ is not identically zero, and hence, near the origin W is a C^{k-1} manifold of codimension 1. The first assertion of the theorem is proved.

To prove the second part of the theorem, consider the mapping defined in a neighborhood U of the origin

$$\psi(x) = \phi(x) + y^*(\phi'(x)(v + w(x)) y_2.$$

By our choice of $w(x)$, $\psi(x) = \phi(x)$ on $W \cap U$. Differentiating, we have

$$\psi'(0)x = \phi'(0)x + y^*(\phi''(0)(v, x)) y_2,$$

and from condition (iii)′, we easily see that $\psi'(0)$ is an isomorphism of X onto Y. Thus, by the implicit function theorem, ψ is a C^{k-1} diffeomorphism of a neighborhood of the origin in X to a neighborhood of $\phi(0)$ in Y, and hence near $\phi(0)$, $\phi(W) \equiv \psi(W)$ is a C^{k-1} manifold of codimension 1. □

COROLLARY 3.7.3 *Let $\Omega \subset X$ open $\phi : \Omega \to Y$ a map of class C^k, $k \geq 2$, and $x_0 \in \Omega$ an ordinary singular point of ϕ, with $y^*(\phi''(x_0)(v, v)) > 0$ say. If $y_2 \in Y$ is a vector transversal to the set $\phi(W)$ at $y_0 = \phi(x_0)$, where W is the singular set of ϕ, then there exist a neighborhood U of x_0 and $\varepsilon \in \mathbb{R}$ such that*

(i) *$\forall y \in (y_0, y_0 + \varepsilon y_2]$ the equation $\phi(x) = y$ has exactly two solutions in U and*

(ii) *$\forall y \in (y_0, y_0 - \varepsilon y_2]$ the equation $\phi(x) = y$ has no solutions in U.*

PROOF: By the theorem, x_0 has a neighborhood U such that $\phi(W \cap U)$ is a C^{k-1} manifold of codimension 1. For small real η, set $y = y_0 + \eta y_2$. Using the notation of the theorem we may assume that $y^* y_2 = 1$ and $x_0 = 0$, $y_0 = \phi(x_0) = 0$.

Using the by-now familiar Lyapunov-Schmidt procedure of Section 2.7.6, we may reduce the equation for $x = x_1 + x_2$, $x_1 = av \in X_1$, $x_2 \in X_2$,

$$\phi(x_1 + x_2) = \eta y_2\,,$$

by first solving for $x_2(x_1)$ and then obtaining the bifurcation equation

$$F(a) \equiv y^* \phi(av + x_2(av)) = \eta$$

to be solved for a. As in (3.8) we have

$$x_{2_{x_1}}(0) = 0\,,$$

and therefore $F(0) = F'(0) = 0$, $F''(0) = y^*(\phi''(0)(v, v)) > 0$. Consequently, $F(a) = \eta$ for $0 < |\eta|$ small has two solutions for $\eta > 0$ and none for $\eta < 0$. □

Now for a global form of these results.

THEOREM 3.7.4 *Consider a C^k map, $k \geq 2$, $\phi : X \to Y$ satisfying*

(i) *ϕ is proper, i.e., the preimage of every compact set is compact,*

(ii) *the singular set W of ϕ is not empty, closed, and connected, and consists entirely of ordinary singular points, and*

(iii) *the preimage of every point $y \in \phi(W)$ consists of one point.*

Then $M = \phi(W)$ is a closed, connected C^{k-1} manifold of codimension 1, and $Y \backslash M$ contains exactly two connected components A_1, A_2 such that

(a) *if $y \in A_1$ then $\phi^{-1}(y)$ is empty and*

(b) *if $y \in A_2$ then $\phi^{-1}(y)$ consists of two points.*

PROOF: Since ϕ is proper and W is closed and connected, it follows that $M = \phi(W)$ is closed and connected. Furthermore, from (ii) and Theorem 3.7.2, we see that M is a C^{k-1} manifold of codimension 1 and, by (iii), ϕ is a homeomorphism of W onto M. By Proposition 3.7.1, $Y \backslash M$ has at most two connected components.

The points in $Y \backslash M$ are all regular values of ϕ, and since ϕ is proper, it follows that any point $y \in Y \backslash M$ has a finite number $N(y)$ of preimages. Furthermore, it is easily seen that $N(y)$ is locally constant and hence N is constant on each component of $Y \backslash M$. To determine N we note that for every neighborhood U of $x_0 \in W$, there is a neighborhood V of $y_0 = \phi(x_0)$ such that $\phi^{-1}(V) \subset U$. Otherwise there would exist a sequence of points x_n bounded away from x_0 with $\phi(x_n) \to y_0$. Since ϕ is proper, a subsequence would converge to some point $u \neq x_0$ with $\phi(u) = y_0$, contradicting (iii). Since x_0 is an ordinary singular point, we may apply Corollary 3.7.3 to find, locally, the number of solutions of $\phi(x) = y$ when y lies on a line segment transversal to M at y_0. The number of solutions is zero or two depending on which side of M the point y lies, and the theorem is proved. □

We turn now to the problem (3.29). Assume that $f(u)$ satisfies

(1) $f \in C^3$ and is real increasing,
(2) $f''(t) \geq 0$ and $f''(0) > 0$,
(3) $\lim_{t \to -\infty} f'(t) = \ell_1, 0 < \ell_1 < \lambda_1$, and
(4) $\lim_{t \to +\infty} f'(t) = \ell_2, \lambda_1 < \ell_2 < \lambda_2$,

where λ_1, λ_2 are, respectively, the first and second eigenvalues for the equation $\Delta u + \lambda u = 0$ in $G, u|_{\partial G} = 0$. In **[25, 26]** it is supposed that $f''(t) > 0$ and $f(0) = 0$; however, the proof uses only (2).

Using the notation of Section 2.5, we consider $y \in Y = C^{\mu}(G), 0 < \mu < 1$, and look for solutions u in

$$X = \left\{u \in C^{2+\mu}(G) \mid u = 0 \text{ on } \partial G\right\}.$$

THEOREM 3.7.5 *There exists in Y a closed connected C^1 manifold M of codimension 1 such that $Y \backslash M$ consists exactly of two connected components A_0, A_2, and*

(i) *if $y \in A_0$,* (3.29) *has no solution*;
(ii) *if $y \in A_2$,* (3.29) *has exactly two solutions*; *and*
(iii) *if $y \in M$,* (3.29) *has exactly one solution.*

Ambrosetti and Prodi prove the theorem by showing that the map

$$\phi(u) = \Delta u + f(u) \quad \text{of } X \text{ into } Y$$

satisfies conditions (i), (ii), and (iii) of Theorem 3.7.4.

In **[26]**, Berger and Podolak give a somewhat different proof; they show, furthermore, that M has a Cartesian representation. Following their paper, with some difference in the details, we will also establish this stronger result.

Let $u_0(x)$ be an eigenfunction of Δ corresponding to the first eigenvalue λ_1; we shall suppose that its L_2-norm is one, $(u_0, u_0) = 1$. It is well-known that $u_0(x) \neq 0$ in G, and we shall suppose $u_0(x) > 0$. Also, $(\Delta + \lambda_1)X = Y_1$ consists of those functions in Y that are L_2-orthogonal to u_0. For s real and $g \in Y_1$, we shall first solve the following problem for $v \in X_2 = \{v \in X \mid v \perp u_0\}$:

$$(3.30) \qquad \Delta v + Pf(su_0 + v) = g(x) \quad \text{in } G, \quad v = 0 \quad \text{on } \partial G.$$

Here P is the L_2-orthogonal projection in Y on Y_1:

LEMMA 3.7.6 *There exists a unique solution* $v(x) = v(x, s, g)$ *in* X_2 *of* (3.30) *which is of class* C^2 *in* s *and* g. *For fixed* s *the correspondence* $v \leftrightarrow g$ *is a diffeomorphism of* X_2 *onto* Y_1.

PROOF: We shall use degree theory to solve (3.30).

(1) First we derive an a priori estimate for the solution v and prove uniqueness. This is done with the aid of the inequalities

$$|f(u)| \le C + \ell_2|u|, \quad |f(u) - f(u')| \le \ell_2|u - u'|,$$

for some constant C. Suppose v is a solution; multiplying (3.30) by v and integrating by parts we find (here $v_j = \partial v/\partial x_j$),

$$\sum_j \|v_j\|^2 = (v, f(su_0 + v)) - (v, g) < \|v\|(C + \ell_2|s| + \ell_2\|v\|)$$

for some different constant C depending also on g. Since $(v, u_0) = 0$, we infer that

$$\sum_j \|v_j\|^2 \ge \lambda_2\|v\|^2 .$$

Thus we obtain the a priori bound

$$\|v\| \le \frac{1}{\lambda_2 - \ell_2}(C + |s|\ell_2) . \tag{3.31}$$

Before proceeding with more bounds, let us demonstrate uniqueness. Let v' be a solution for g', and set $w = v - v'$. The same analysis shows that

$$\begin{aligned}\lambda_2\|w\|^2 &\le (w, f(su_0 + v) - f(su_0 + v')) + (w, g' - g)\\ &\le \ell_2\|w\|^2 + \|g' - g\|\ \|w\|\end{aligned}$$

so that

$$\|w\| = \|v - v'\| \le \frac{1}{\lambda_2 - \ell_2}\|g - g'\| . \tag{3.31'}$$

Having a bound for $\|v\|$ we see from (3.30) that since f grows at most linearly, we also have an a priori bound for

$$\|\Delta v\| .$$

Applying the results of Section 2.5.3, there is an a priori bound for

$$\sum \|v_{x_i x_j}\| .$$

We may now apply the Sobolev embedding theorem of that section and derive a bound (depending on the dimension n) for $\|v\|_{L^q}$, with some $q > 2$, or for $|v|_\delta$, $0 < \delta < 1$, i.e., some Hölder norm of v. But then it follows, say in the first case, that $\|f(v)\|_{L^q} \le$ some constant, and we infer again via the results of Section 2.5.3 that $\|v\|_{L^r} \le$ some constant for $r > q$, and so on. By repeating this argument a finite number of times, each time obtaining improved estimates for v, we may finally conclude that

$$|v|_{2+\mu} \leq C_1(1+|s|) \tag{3.31''}$$

for some constant C_1 depending of course on g.

(2) In the ball $|v|_{2+\mu} \leq C_1(1+|s|)+1$ in X_2, we will solve (3.30), which we may write in the following form:

$$T(v) \equiv v + Kv \equiv v + \Delta^{-1}Pf(su_0+v) - \Delta^{-1}g = 0\,.$$

We note that if $g \perp u_0$, then $\Delta^{-1}g \perp u_0$. For if u is the solution of

$$\Delta u = g \quad \text{in } G\,, \qquad u = 0 \quad \text{on } \partial G\,,$$

then, taking the scalar product with u_0, we find by Green's theorem,

$$0 = (g, u_0) = (\Delta u, u_0) = (u, \Delta u_0) = -\lambda_1(u, u_0)\,.$$

The operator $T(v)$ thus maps the ball in X_2 into X_2, and by the results of Section 2.5.3, K is a compact operator. Since there are no solutions on the boundary of $Tv = 0$, $\deg(T, \text{ball}, 0) = \deg T$ is defined. Our derivation of the a priori estimate also works if f and g are replaced by tf and tg, $0 \leq t \leq 1$, and we conclude by deformation that

$$\deg T = \deg I = 1\,.$$

Thus, the existence and uniqueness of solutions of (3.30) are proved.

To prove the regularity of the solution in its dependence on s and g is simply a somewhat tedious exercise, using the results of Section 2.5.3, and is left to the reader. The fact that for fixed s the correspondence $v \leftrightarrow g$ is a diffeomorphism follows then from (3.31′). □

Returning to the equation

$$\phi(u) = \Delta u + f(u) = y(x)\,, \qquad u = 0 \quad \text{on } \partial G\,,$$

let us write

$$u = su_0 + u_2\,, \qquad u_2 \perp u_0\,,$$

and

$$y(x) = tu_0 + g(x)\,, \qquad g \perp u_0, s, t \text{ real.}$$

Applying P and $(I-P)$, we see that the equation may be written as a pair

$$\Delta u_2 + Pf(su_0+u_2) = g(x)\,, \qquad -s\lambda_1 + (f(su_0+u_2), u_0) = t\,.$$

By Lemma 3.7.6, this is equivalent to the pair

$$u_2 = v(s,g)\,, \qquad F(s) \equiv F(s,g) \equiv -s\lambda_1 + \big(f(su_0+v(s,g)), u_0\big) = t\,.$$

For fixed s, $v(s,g)$ is a diffeomorphism of g to u_2, and this implies the first part of

LEMMA 3.7.7 *Suppose* $u = su_0 + u_2 \in X$ *and* $y(x) = tu_0 + g(x)$, $g \perp u_0$. *Then* u *is a singular point of* ϕ *if and only if*

$$F_s(s,g) \equiv -\lambda_1 + \int f'(su_0 + v(s,g))(u_0 + v_s(s,g))u_0\,dx = 0\,. \tag{3.32}$$

Furthermore, u *is then an ordinary singular point.*

PROOF: It is clear from (3.32) that $\ker\phi'(u)$ is one-dimensional, spanned by some function $z(x)$. It is well-known that range $\phi'(u)$ is closed in Y and consists of those functions in Y which are L_2-orthogonal to z. To check that u is a regular singular point, we have to verify $3'$ of Theorem 3.7.2. Now, in that condition,

$$y^*(\phi''(u)(z,z)) = \int f''(u(x))z^3(x)dx\,.$$

To complete the proof we show that $z(x) \neq 0$ in G, say $z(x) > 0$; then the last expression is positive. The function z satisfies

$$\Delta z + f'(u)z = 0 \quad \text{in } G\,, \qquad z = 0 \quad \text{on } \partial G\,, \tag{3.33}$$

i.e., $\mu = 1$ is an eigenvalue of the problem $\Delta z + \mu\rho(x)z = 0$ in G, $z = 0$ in ∂G, where $\rho(x) = f'(u(x))$.

It is well-known that if $\rho(x) > 0$, then the eigenvalues of such a problem are $0 < \mu_1 < \mu_2 \leq \ldots$; the first eigenvalue μ_1 is simple and the corresponding eigenfunction does not vanish in G. Furthermore, the r^{th} eigenvalue μ_r is a decreasing functional of the coefficient $\rho(x)$.

Since $f'(\mu) < \lambda_2$, we claim that the first eigenvalue μ_1 of the problem (3.33) is $\mu_1 = 1$. Indeed, $\mu = 1$ is the second eigenvalue of the problem $\Delta w + \mu\lambda_2 w = 0$, $w = 0$ on ∂G, and by the preceding remarks it follows that the second eigenvalue of $\Delta z + \mu f'(u)z = 0$, $z = 0$ on ∂G exceeds 1. Thus, $\ker \Delta + f'(u)$ is one-dimensional, spanned by a positive function z, and so, as we observed,

$$y^*(\phi''(u)(z,z)) = \int_G f''(u(x))z^3(x)\,dx > 0\,.$$

□

For each $g \in Y_1$, we are going to show that there is exactly one value of $s = s_0(g) = s_0$ for which (3.32) holds and that $s_0(g)$ is the unique minimum point of the function $F(s)$. It then follows from Theorem 3.7.2 that $s_0(g)$ is a C^2 function of g. Thus, the set M has the following Cartesian representation: M consists of points

$$y(x) = g(x) - s_0(g)\lambda_1 u_0 + \big(f(s_0(g)u_0 + v(s_0(g),g)), u_0\big)u_0\,, \quad g \in Y_1\,.$$

Furthermore, for

$$y(x) = g(x) + tu_0\,,$$

we have, for $t_0(g) = -s_0(g)\lambda_1 + \big(f(s_0(g))u_0 + v(s_0(g),g), u_0\big)$,

- if $t < t_0(g)$, the equation $\phi(u) = y$ has no solutions,
- if $t > t_0(g)$ the equation $\phi(u) = y$ has exactly two solutions, and
- if $t = t_0(g)$, the equation $\phi(u) = y$ has exactly one solution.

The set W of singular points also has a Cartesian representation; W consists of

$$s_0(g)u_0 + v(s_0(g), g) \quad \text{for } g \in Y_1 .$$

In particular, Theorem 3.7.5 is proved.

The existence of a unique minimum point $s_0(g)$ of $F(s, g)$ follows easily from the following:

(1) At any point s where $F_s(s, g) = 0$, we have $F_{ss}(s, g) > 0$.
(2) For fixed g, $F(s, g) \to +\infty$ as $s \to \pm\infty$.

PROOF OF (1): Keeping g fixed, set $v(s, g) = v(s)$ and $su_0 + v(s, g) = u(s)$. Thus,

$$\Delta v(s) + Pf(u(s)) = g , \qquad F(s) = -s\lambda_1 + (f(u(s)), u_0) .$$

Differentiating with respect to s, we find, since $u_{ss} = v_{ss}$,

$$\Delta v_s + Pf'(u(s))u_s = 0 , \quad \Delta v_{ss} + Pf'v_{ss} + Pf''u_s^2 = 0 . \tag{3.34}$$

If we take scalar products of the first equation with v_{ss} and the second with v_s and subtract, we find by Green's theorem,

$$(f'u_s, v_{ss}) - (f'v_{ss}, v_s) - (f''u_s^2, v_s) = 0$$

or, since $u_s = u_0 + v_s$,

$$(f'u_0, v_{ss})(f''u_s^2, v_s) .$$

Next, we have

$$F_{ss} = (f''u_s^2, u_0) + (f'v_{ss}, u_0) = (f''u_s^2, u_0) + (f''u_s^2, v_s)$$

by the preceding,

$$= (f''u_s^2, u_s) = \int f''(su_0 + v(s))u_s^3 \, dx .$$

Now we suppose that for some point s, $F_s = 0$, i.e.,

$$-\lambda_1 + (f'(u(s))u_s, u_0) = 0 .$$

Combining this with (3.34), we see that for this value of s, u_s satisfies

$$\Delta u_s + f'(u(s))u_s = 0 .$$

As in the proof of Lemma 3.7.7, we may show that $u_s = u_0 + v_s$ does not vanish in G. Since $v_s \perp u_0$ and $u_0 > 0$ in G, $v_s \geq 0$ at some point in G, and hence $u_s > 0$ in G. Consequently, for this value of s,

$$F_{ss} = \int_G f''(u(s))u_s^3 \, dx > 0 .$$

□

PROOF OF (2): Up to now we have not used the hypothesis that $\ell_1 < \lambda_1$; it is here that we use it. We will show that $\lim_{s\to-\infty} F(s, g) = +\infty$. For $s \to +\infty$ the argument is similar. In view of (1), it suffices to show that for some sequence $s_k \to -\infty$, $F(s_k, g) \to +\infty$.

Recalling the estimate (3.31″) for $v(s, g)$,

$$|v|_{2+\mu} \le C_1(1 + |s|)\,.$$

We see that as $s \to -\infty$, $v(s, g)/s$ is uniformly bounded in $C^{2+\mu}(G)$. Hence we can choose a sequence $s_k \to -\infty$ for which

$$\frac{v(s_k)}{s_k} \quad \text{converges uniformly to } w \in C(\overline{G})\,, \qquad w \perp u_0\,.$$

Divide G into three sets G_+, G_-, G_0 according as $u_0(x) + w(x)$ is positive, negative, or zero. Conditions (3) and (4) (see p. 65) on f imply that

$$\lim_{s\to-\infty} \frac{f(s)}{s} = \ell_1\,, \qquad \lim_{s\to+\infty} \frac{f(s)}{s} = \ell_2\,.$$

By the Lebesgue dominated convergence theorem we find

$$\lim_{s_k\to-\infty} \frac{F(s_k, g)}{s_k} = -\lambda_1 + \ell_1 \int_{G_+} (u_0 + w)u_0\,dx + \ell_2 \int_{G_-} (u_0 + w)u_0\,dx\,.$$

Since $w \perp u_0$, we have

$$\int_{G_+} (u_0 + w)u_0\,dx + \int_{G_-} (u_0 + w)u_0\,dx = \int_G u_0^2\,dx = 1\,, \tag{3.35}$$

and hence

$$\begin{aligned}\lim_{s_k\to-\infty} \frac{F(s_k, g)}{s_k} &= -\lambda_1 + \ell_2 + (\ell_1 - \ell_2) \int_{G_+} (u_0 + w)u_0\,dx \\ &\le -\lambda_1 + \ell_2 + \ell_1 - \ell_2 \quad \text{by (3.35)} \\ &= \ell_1 - \lambda_1 < 0\,.\end{aligned}$$

□

EXERCISE Under the conditions of Theorem 3.7.5, with $f(0) = 0$, consider the equation $\Delta u + f(u) = 0$. Show that it has exactly one nonzero solution if and only if $f'(0) \ne \lambda_1$.

CHAPTER 4

Further Topological Methods

In Section 1.8 we considered nonlinear mappings from $\mathbb{R}^n$ into $\mathbb{R}^k$ with $k < n$. We wish now to extend some of the results of that section to Banach space. According to Theorem 1.1.1, if B is a closed unit ball in $\mathbb{R}^n$, ϕ a map $\partial B \to \mathbb{R}^k \backslash \{0\}$, $k \leq n$, then, for every continuous extension F of ϕ inside B, the equation $F(x) = 0$ is solvable if and only if the map

$$\psi(x) = \frac{\phi(x)}{\|\phi(x)\|} : \partial B \to \mathbb{S}^{k-1}$$

is homotopically nontrivial.

DEFINITION Any map $\phi : \partial B \to \mathbb{R}^k \backslash \{0\}$ with the property that $F(x) = 0$ is solvable for every continuous extension of F inside B is called *essential.*

A special case occurs when

$$\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, \quad x = x_1 + x_2, \quad \text{and} \quad \phi(x) = \phi(x_1 + x_2) = x_1 + \Phi(x_2),$$

where ϕ maps the unit ball in $\mathbb{R}^{n_2}$ into a subspace $\mathbb{R}^{n_3}$ of $\mathbb{R}^{n_2}$ and

$$\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_3} \subset \mathbb{R}^k .$$

Note that $\phi(\partial B) \neq 0$ implies $\phi(x_2) \neq 0$ for $\|x_2\| = 1$. So the homotopy class of $\phi : \mathbb{S}^{n-1} \to \mathbb{R}^k \backslash \{0\}$ is obtained by n_1 suspensions of the map $\Phi : \mathbb{S}^{n_2 - 1} \to \mathbb{R}^{n_3} \backslash \{0\}$.

4.1. Extension of Leray-Schauder Degree

Suppose X is a real Banach space, $B = B_\rho$ the closed ball $\|x\| \leq \rho$, and T a continuous map: $B_\rho \to X$ of the form $I - K$, where K is a compact map. We will assume T maps B into a closed subspace Y of X with finite codimension $= i$ and $T(\partial B) \neq 0$. Then, as we have seen earlier, there is an $\varepsilon > 0$ such that $\|T(x)\| \geq \varepsilon$ for $x \in \partial B$. Let

$$T_0 = T|_{\partial B}, \quad T_0 : \partial B \to Y \backslash \{0\} .$$

DEFINITION We say T_0 is *essential* (relative to Y) if the equation $T(x) = 0$ is solvable for every (permissible) extension T of T_0 inside B_ρ; i.e., $K = I - T$ is compact, and T maps into Y.

We remark here that the Leray-Schauder $\deg(T_0, B, 0)$ is defined and is zero. Indeed, $\deg(T_0, B, 0) = \deg(T, B, x)$ for $\|x\|$ small and $x \notin Y$. Since x is not an image point of T, $\deg(T, B, x) = 0$.

DEFINITION A deformation $T_t = I - K_t$, with K_t a compact map of $\partial B_\rho \times [0,1] \to X$ and $T_t(x) \in Y\backslash\{0\}$, is called a *permissible deformation*.

REMARK. Consider a map $T_0 = I - K : \partial B \to Y\backslash\{0\}$. Then whether T_0 is essential or not depends only on its homotopy class, defined by permissible deformations. In fact, suppose T_0 is not essential and $T_t = I - K_t$, $K_0 = K$, is a permissible deformation. Then T_1 is not essential. Indeed, the map $\widehat{T}_1$ defined by

$$\widehat{T}_1 = \begin{cases} \frac{1}{2}(2x - K(2x)) & \text{for } \|x\| \le \frac{1}{2} \\ \|x\|\{\frac{x}{\|x\|} - K_{2\|x\|-1}(\frac{x}{\|x\|})\} & \text{for } \|x\| \ge \frac{1}{2} \end{cases}$$

is a continuous extension of T_1 inside $\|x\| \le 1$ of the admissible form such that $\widehat{T}_1(x) = 0$ has no solution.

We shall determine necessary and sufficient conditions for a mapping T_0 to be essential. First,

PROPOSITION 4.1.1 *Suppose* $X = X_0 \oplus W$, $\dim W = d$, *and* V *is a linear subspace of* W *of dimension* d^*. *Let* F_0 *be a map defined on* ∂B_ρ *of the following form: For* $x \in X$, $x = x_0 + w$,

$$F_0(x) = x_0 + \Phi(w)$$

where Φ *maps* $\{w \in W \mid \|w\| \le \rho\}$ *into* V, *and suppose* $\Phi(w) \ne 0$ *if* $\|w\| = \rho$. *Then* F_0 *is essential (for* $Y = X_0 \oplus V$*) if and only if all suspensions of the map* $\psi(\tau) = \frac{\Phi(\rho\tau)}{\|\Phi(\rho\tau)\|}$ *for* $|\tau| = 1$, *i.e.,* $\psi(\tau) : \mathbb{S}^{d-1} \to \mathbb{S}^{d^*-1}$, *are nontrivial (i.e.,* ψ *has nontrivial, stable homotopy).*

In order to characterize essential maps we show now that we can deform any T_0 to a map F_0, as in the proposition (via a permissible deformation). Write X as a direct sum

$$X = Y \oplus Z \quad \text{with } \dim Z = i$$

so that any $x \in X$ has the unique decomposition $x = y + z$, $y \in Y$, $z \in Z$. We may suppose T_0 has the form

$$T_0(x) = T_0(y+z) = y + z - K(y+z), \qquad K \text{ compact.}$$

Then $T_0(x) = y - K_1(x)$ with K_1 compact and $K_1(x) \in Y$. For convenience, we shall suppose $\rho = 1$.

We know that $\|T_0(x)\| \ge \varepsilon > 0$ when $\|x\| = 1$. There exists a finite-dimensional map K_2 such that

$$\|K_1(x) - K_2(x)\| \le \frac{\varepsilon}{2} \quad \text{with } K_2(x) \subset V \subset Y$$

with $\dim V < +\infty$. Using the deformation

$$H_t(x) = y - (1-t)K_1 - tK_2(x), \quad 0 \le t \le 1,$$

deform $y - K_1(x)$ to $T_1 = y - K_2(x)$. For $\|x\| = 1$,

$$\|H_t(x)\| = \|y - K_1(x) + t(K_1(x) - K_2(x))\| \ge \|T_0(x)\| - t\|K_1(x) - K_2(x)\| \ge \frac{\varepsilon}{2} > 0,$$

so that $H_t(x)$ is a permissible deformation.

Decompose Y as a direct sum,

$$Y = V \oplus X_0\,, \qquad y = v + x_0\,,$$

with X_0 a closed linear subspace of Y. Then

$$T_1 = y - K_2(x) = v + x_0 - K_2(x) = x_0 - K_3(x) = x_0 - K_3(v + x_0 + z)$$

with $K_3(x) \in V$ and K_3 compact.

The deformation

$$G_t(x) = x_0 - K_3(v + tx_0 + z)$$

is permissible. Indeed, if $G_t(x) = 0$ for some v, z, t, x_0 with $\|x\| = 1$, then $x_0 = 0$, which implies $K_3(v+z) = 0$. But $T_1(x) \neq 0$ implies $K_3(x) \neq 0$, which gives a contradiction.

Writing

$$X = Y \oplus Z = X_0 \oplus V \oplus Z = X_0 \oplus W\,, \quad \text{with } W = V \oplus Z\,,$$

we have deformed T_0 to the form

$$G_0(x) = x_0 + \Phi(w)\,, \tag{4.1}$$

where Φ is a continuous map of $\|w\| \leq 1$ into a closed linear subspace V of W; $\Phi(w) \neq 0$ for $\|w\| = 1$. If $\dim W = d$ and $\dim V = d^*$, then $d - d^* = i = \dim Z$.

The proposition thus implies:

THEOREM 4.1.2 *Given T_0 as above. There is a permissible deformation of T_0 to a map G_0 of the form* (4.1). *For $\|w\| = \rho$, set $\tau = w/\rho$ and*

$$\psi(\tau) = \frac{\Phi(\rho\tau)}{\|\Phi(\rho\tau)\|}\,.$$

Then T_0 is essential if and only if $\psi(\tau) : \mathbb{S}^{d-1} \to \mathbb{S}^{d^*-1}$ has nontrivial stable homotopy.

PROOF OF PROPOSITION: We first prove sufficiency.

(1) Suppose X is finite-dimensional, $\dim X = j + d$. Then our hypothesis means that F_0, being the j-fold suspension of Φ, has nontrivial homotopy as a map into $Y \backslash \{0\}$. (Here $Y = X_0 \oplus V$.) Thus by Theorem 1.1.1, F_0 is essential.

(2) The next step is the standard reduction to finite dimensions. Let $T = I - K$ be a permissible extension of F_0 inside B, $T : B \to Y$. Using the decomposition $Y = X_0 \oplus V$, write $T(x) = T_0(x) + T'(x)$, $T_0(x) \in X_0$, $T'(x) \in V$, and using the decomposition $X = X_0 \oplus W$, write $x = x_0 + w$. Then

$$x_0 - T_0(x) = K(x) + T'(x) - w$$

and is a compact map of B into X_0. For any $\varepsilon > 0$ we may approximate $x_0 - T_0(x)$ within ε by a mapping $K_{0\varepsilon}(x)$ of B into a finite-dimensional subspace X_1 of X_0. Then the operator

$$K_\varepsilon(x) = K_{0\varepsilon}(x) + w - T'(x)$$

is compact and satisfies $\|K(x) - K_\varepsilon(x)\| = \|x_0 - T_0(x) - K_{0\varepsilon}(x)\| \le \varepsilon$.

It suffices to show that there is an x_ε in $B \cap (X_1 \oplus W)$ satisfying $x_\varepsilon = K_\varepsilon(x_\varepsilon)$. Letting $\varepsilon \to 0$ through a sequence, and choosing a subsequence for which $K(x_\varepsilon)$ converges to some x_0 in B, we have

$$\lim(x_\varepsilon - x_0) = \lim(K_\varepsilon(x_\varepsilon) - K(x_\varepsilon)) + \lim(K(x_\varepsilon) - x_0) = 0 .$$

Hence, by continuity,

$$x_0 = \lim_{\varepsilon \to 0} K(x_\varepsilon) = K(x_0) .$$

To prove the existence of x_ε satisfying

$$T_\varepsilon(x_\varepsilon) \equiv x_\varepsilon - K_\varepsilon(x_\varepsilon) = 0 , \quad x_\varepsilon \in B \cap (X_1 \oplus W) = B' ,$$

we have only to show that the homotopy class of $T_\varepsilon : \partial B' \to X_1 \oplus V \backslash \{0\}$ is nontrivial. On $\partial B'$ we may deform T_ε to T via

$$T_t = (1-t)T_\varepsilon + tT , \qquad 0 \le t \le 1 ,$$

and then use the finite-dimensional result of (1).

□

We omit the proof of necessity; it may be found in [**41**]; see also [**36, 38, 46**].

REMARK 4.1.3 (P. Rabinowitz). Suppose X is a B-space and Y is a closed subspace of X, with finite codimension i. Let $T = I - K$, with K compact; map the unit ball B in X into Y and suppose $T : \partial B \to Y \backslash \{0\}$. Then, as we have seen, $\deg(T, B, 0)$ is defined and equal to zero. Suppose, now, that T is odd. Then according to Borsuk's theorem, $\deg(T, B, 0)$ is odd. Hence, the hypothesis that $T(\partial B) \ne 0$ cannot hold for odd maps.

In fact, if $T = I - K$ maps $B \to Y$, Y a closed subspace of X, $Y \ne X$, then the equation $T(x) = 0$ has a solution with $\|x\| = r \le 1$ for any $r \le 1$.

We shall present an application of Theorem 4.1.2 to elliptic boundary value problems as in [**41**]. In [**34**], J. Cronin has put that argument in a more abstract setting, and we shall begin with that result.

X and Y are real Banach spaces. Consider a map

$$Ax - Gx : X \to Y$$

where G is compact and A is a continuous Fredholm map with index $i \ge 0$. This means $\dim \ker A = d < \infty$, codim range $A = d^* = d - i$.

With $X_1 = \ker A$, $Y_1 = $ range A, we can decompose X and Y into the direct sums

$$X = X_1 \oplus X_2 , \quad x = x_1 + x_2 , \quad Y = Y_1 \oplus Y_2 , \quad \dim Y_2 = d^* .$$

Let P be the associated projection in Y onto Y_1. By the closed graph theorem, $A : X_2 \to Y_1$ has a bounded inverse with bound C.

THEOREM 4.1.4 *Assume that for some positive constant M,*

(i) $\|PG(x)\| \le M$ *for all* x,

(ii) *there exists a constant* $N > 0$ *such that* $(I - P)G(x_1 + x_2) \neq 0$ *for* $\|x_2\| \leq CM$ *and* $\|x_1\| \geq N$, *and*

(iii) *the map* $(I - P)G : X_1 \to Y_2 \backslash \{0\}$ *for* $\|x_1\| = N$ *has nontrivial stable homotopy.*

Then $A(x) - G(x) = 0$ *has a solution.*

REMARKS. (1) (ii) and (iii) automatically hold if

(iv) For $x_1 \in X_1$, $\|x_1\| = 1$ and $\|x_2\| \leq CM$, $\phi(x_1) = \lim_{r\to\infty}(I - P)G(rx_1 + x_2)$ exists (uniformly in x_2), is independent of x_2, and satisfies

$$\phi|_{\text{unit sphere in } X_1} \to Y_2 - \{0\} \text{ has nontrivial stable homotopy.}$$

(2) If $d = d^*$, then $(I - P) : X_1 \to Y_2 \backslash \{0\}$ having nontrivial stable homotopy means

$$\deg((1 - P)G(x_1), \|x_1\| - N, 0) \neq 0$$

and (iv) means $\deg(\phi, \|x_1\| = N, 0) \neq 0$.

PROOF OF THEOREM: $Ax - Gx = 0$ is equivalent to the system

$$Ax_2 - PG(x) = 0, \qquad (I - P)G(x_1 + x_2) = 0,$$

or

$$x_2 - A^{-1}PG(x) = 0, \qquad (I - P)G(x_1 + x_2) = 0.$$

Suppose X_1 is spanned by $w_1, \ldots, w_d$; then we can write $x_1 = \sum_{j=1}^{d} a_j w_j$. Suppose that Y_1 is the subspace of Y on which the continuous linear functionals (on Y) $\ell_1, \ldots, \ell_{d^*}$ vanish. Then the system may be written

$$\begin{cases} x_2 - A^{-1}PG\left(\sum a_j w_j + x_w\right) = 0 \\ \left\langle \ell_\alpha, G\left(\sum a_j w_j + x_2\right)\right\rangle = 0, \qquad \alpha = 1, \ldots, d^*. \end{cases}$$

From the first equation and condition (i), we see that a solution satisfies $\|x_2\| \leq CM$. By (ii), it therefore satisfies

$$\|x_2\| \leq CM, \quad \|x_1\| < N. \tag{4.2}$$

Let us give a different description of X. Write $x_1 + x_2$ as $[x_2, a]$ with $a = (a_1, \ldots, a_d)$, and define the norm of $[x_2, a]$ as

$$\|[x_2, a]\| = \left\| \sum a_j w_j + x_2 \right\|.$$

Thus, we may regard X as $X = X_2 \times \mathbb{R}^d$. Consider the map $X_2 \times \mathbb{R}^d \to X_2 \times \mathbb{R}^{d^*}$ given by

$$\begin{aligned} &x_2 - A^{-1}PG\left(\sum a_j w_j + x_2\right), \\ &\left\langle \ell_\alpha, G\left(\sum a_j w_j + x_2\right)\right\rangle, \qquad \alpha = 1, \ldots d^*. \end{aligned} \tag{4.3}$$

To prove the theorem, we apply Theorem 4.1.2 in the ball

$$\|[x_2, a]\| \leq CM + N + 1. \tag{4.4}$$

Under the deformation,

$$x_2 - tA^{-1}PG\Big(\sum a_j w_j + x_2\Big), \quad 0 \le t \le 1,$$
$$\Big\langle \ell_\alpha, G\Big(\sum a_j w_j + tx_2\Big)\Big\rangle, \qquad \alpha = 1, \dots, d^*.$$

(4.3) deforms to

$$x_2$$
$$\langle \ell_\alpha, G(x_1)\rangle, \quad \alpha = 1, \dots, d^*.$$

In view of condition (iii), we may apply Theorem 4.1.2. □

Theorem 4.1.4 is related to Theorem 2.6.3.

EXERCISE Prove Theorem 2.6.3 for $d^* < d$.

In [**33**] M. S. Berger and E. Podolak have observed that in some cases it suffices to assume a weaker form of (iii): that only a finite number of suspensions of $(I - P)G(x_1)$ are nontrivial. We present a form of their result.

THEOREM 4.1.5 *Consider A and G as in Theorem* 4.1.4, *satisfying* (i) *and* (ii). *The conclusion of the theorem holds if condition* (iii) *is replaced by* (ii)′ *and* (iii)′ *below*:

(ii)′ *There is a decomposition of X_2 as a sum of closed subspaces*

$$X_2 = X_2' \oplus X_2'', \quad \dim X_2' = m,$$

such that if we decompose

$$Y_1 = AX_2' \oplus AX_2''$$

and consider the associated projections P', P'', $I - P' - P''$ in Y,

$$Y = AX_2' \oplus AX_2'' \oplus Y_2 = P'Y \oplus P''Y + (I - P' - P'')Y.$$

Then

$A^{-1}P''G$ satisfies a Lipschitz condition with respect to x_2''

with Lipschitz constant $C < 1$.

(iii)′ *The m-fold suspension of the map $(I-P)G : X_1 \to Y_2\backslash\{0\}$ for $\|x_1\| = N$ is nontrivial. Here $P = P' + P''$.*

REMARK. The condition (ii)′ may seem artificial, but in fact, it occurs in practice for elliptic operators A with discrete spectrum going to infinity. In such cases, for any $\varepsilon > 0$, one can usually find such a decomposition, with X_2' spanned by a finite number of eigenvectors of A, such that $\varepsilon\|Ax_2''\| \ge \|x_2''\|$. In this case if $P''G$ satisfies some Lipschitz condition, then condition (ii)′ may be realized.

PROOF OF THEOREM 4.1.5: Write $x = x_1 + x_2' + x_2''$. We have to solve

$$Ax_2' + Ax_2'' = G(x_1 + x_2' + x_2''). \tag{4.5}$$

Applying the projection P'':

$$Ax_2'' = P''G(x), \quad x_2'' = A^{-1}P''G(x_1 + x_2' + x_2'').$$

Since the right-hand side satisfies a Lipschitz condition in x_2'' with constant $C < 1$, the equation has a unique fixed point $x_2''(x_1 + x_2')$, which one verifies is continuous in $x_1 + x_2'$. Inserting this in (4.5), we obtain the finite-dimensional system

$$Ax_2' = P'G\big(x_1 + x_2' + x_2''(x_1 + x_2')\big), \quad (I - P)G\big(x_1 + x_2' + x_2''(x_1 + x_2')\big) = 0.$$

The argument used in proving Theorem 4.1.4 may now be employed in the finite-dimensional space $X_1 \oplus X_2'$. First we obtain a priori bounds for a solution $x_1 + x_2'$ analogous to (4.2). In fact, since $x_1 + x_2' + x_2''(x_1 + x_2')$ is a solution of the original problem, it follows from (4.2) that for some constant M',

$$\|x_2' + x_2''(x_1 + x_2')\| \leq CM, \quad \|x_2'\| \leq M', \quad \|x_1\| \leq N.$$

We may now carry out the rest of the proof of Theorem 4.1.4, working in the ball $\|[x_2', a]\| \leq M' + N + 1$ in place of (4.4). □

4.2. Applications to Partial Differential Equations

We shall generalize the following results due to Landesman and Lazer **[40]**:

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let L be a formally self-adjoint, elliptic, second-order operator in $\overline{\Omega}$, with real coefficients C^∞ in $\overline{\Omega}$. We will assume all functions are real. Consider the Dirichlet problem

$$Lu = f(x) - g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $f(x)$ is a given smooth function and $g(u)$ is a continuous function having limits as $u \to \pm\infty$:

$$\lim_{u \to \pm\infty} g(u) = g(\pm\infty)$$

with

(4.6) $$g(-\infty) < g(u) < g(\infty).$$

Suppose $\ker L$ is one-dimensional, spanned by $w(x)$. If we take the L_2 scalar product $(\,,\,)$, we find a necessary condition for u to be a solution is that

$$(f - g, w) = (Lu, w) = 0,$$

since L is formally self-adjoint and $Lw = 0$. Applying (4.6), we obtain the necessary condition for solvability of the Dirichlet problem above:

(4.7)
$$(f, w) < g(\infty)\int_{w>0} w\,dx + g(-\infty)\int_{w<0} w\,dx.$$
$$(f, w) > g(\infty)\int_{w<0} w\,dx + g(-\infty)\int_{w>0} w\,dx.$$

Landesman and Lazer proved that the above condition is also sufficient for solvability.

Note that if L had an inverse, then we could apply the Schauder fixed-point theorem to solve

$$u = L^{-1}(f(x) - g(u)).$$

We shall present a generalization of this result (see reference [**41**, p. 133]). Using the Leray-Schauder degree and the main theorem in the last section, we can derive similar results for arbitrary, elliptic operators L with null space of any dimension, and which are not necessarily self-adjoint.

Let L be a linear elliptic, partial differential operator with coefficients in $C^\infty(\overline{\Omega})$, of order m acting on (for convenience) scalar functions u, satisfying "nice" boundary conditions as in Section 2.5:

$$Bu = 0 \quad \text{on } \partial\Omega$$

expressed in terms of differential operators of order $< m$. Consider the problem

$$Lu = g(x, u, \dots, D^{m-1}u) \quad \text{in } G, \qquad Bu = 0 \quad \text{on } \partial\Omega. \tag{4.8}$$

The conditions $Bu = 0$ guarantee that for L acting on functions satisfying $Bu = 0$, we have:

(1) $\ker L$ is finite-dimensional, spanned by $w_1, \dots, w_d$,
(2) range L has finite codimension d^*. There are $C^\infty(\Omega)$ functions $w'_1, \dots, w'_{d^*}$, such that range $L \perp_{L_2} w'_j$ for $j = 1, \dots, d^*$.

Assume that

$$d^* \le d.$$

Concerning g we assume: Writing $\eta = (u, \dots, D^{m-1}u)$:

(a) There is a constant $M > 0$ such that $|g(x, \eta)| \le M$ for all x in Ω and all η, and g is C^∞ for $x \in \overline{\Omega}$ and all η.
(b) $h(x, \eta) = \lim_{r\to\infty} g(x, r\eta)$ for $|\eta| = 1$ exists uniformly for x in $\overline{\Omega}$ and $|\eta| = 1$.

Furthermore, we make the following technical hypotheses:

(c) The only solution w of

$$Lw = 0 \quad \text{in } \Omega, \qquad Bw = 0 \quad \text{on } \partial\Omega,$$

which vanishes on a set of positive measure in Ω is $w = 0$.
(d) Define a map $\phi : \mathbb{S}^{d-1} \to \mathbb{R}^{d^*}$, $\phi = (\phi_1, \dots, \phi_{d^*})$ by

$$\phi_\beta(a_1, \dots, a_d) = \left(h\left(x, D^\alpha \sum_{j=1}^{d} a_j w_j(x)\right), w'_\beta\right), \qquad \beta = 1, \dots, d^*,$$

$$\phi_\beta(a) = \phi_\beta(a_1, \dots, a_d) = (h(x, w, \dots, D^{m-1}w), w'_\beta), \qquad \beta = 1, \dots, d^*,$$

where $w = \sum_{j=1}^{d} a_j w_j$.

THEOREM 4.2.1 *If $\phi : \mathbb{S}^{d-1} \to \mathbb{R}^{d^*}\backslash\{0\}$ and has nontrivial stable homotopy, then*

$$Lu = g \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega \text{ has a solution.}$$

REMARKS. (1) If $d = d^*$, our condition means that the degree of ϕ at the origin is nonzero; for $d = d^* = 1$ and $g = g(x, u)$, this means simply that if

$$h_\pm(x) = \lim_{u\to\pm\infty} g(x, u),$$

then

$$A_1 = \int_{w>0} h_+ w'\,dx + \int_{w<0} h_- w'\,dx \quad \text{and} \quad A_2 = \int_{w<0} h_+ w'\,dx + \int_{w>0} h_- w'\,dx$$

has opposite sign. This agrees with (4.7) in that particular problem.

(2) If L is self-adjoint, under the boundary conditions $Bu = 0$, then $d = d^*$. Then one may take $w'_\alpha = w_\alpha$, $\alpha = 1, \dots, d$. Suppose the following condition is satisfied:

$$\int_{w>0} h_+ w\,dx + \int_{w<0} h_- w\,dx > 0$$

for every $w \not\equiv 0$ in $\ker L$. Then necessarily, the mapping ϕ has degree 1 and our problem is solvable.

SKETCH OF THE PROOF OF THE THEOREM: Set $g(x, u, \dots, D^{m-1}u) = G[u]$. Using the notation of Section 2.5, for some fixed δ, $0 < \delta < 1$, let $X = \{u \in C^{m+\delta}(\overline{\Omega}) \mid Bu = 0 \text{ on } \partial\Omega\}$, and let $Y = C^\delta\{\overline{\Omega}\}$. Then $G : X \to Y$ is a compact map. Let $X_1 = \ker L$, $X_2 = \ker L^\perp$. We apply Theorem 4.1.4, with $A = L$; (a) implies (i). Using the technical hypothesis (c), one proves (after some work) that

$$\lim_{r\to\infty} \left(w'_\beta, G\left(r \sum a_j w_j + x_2\right)\right)$$

exists and, for at least one $\beta = 1, \dots, d^*$, is not zero. The main condition in the theorem then yields condition (iv). (See [**41**] for details.) □

We now present a result of P. Rabinowitz [**43**] related to Remark 4.1.3:

THEOREM 4.2.2 *Consider the nonlinear elliptic problem* (4.8) *with L as in Theorem* 4.2.1 *and* $d* < d$. *Assume that* $G[u] = g(x, u, \dots, D^{m-1}u)$ *is odd in* u, *i.e.,* $G[-u] = -G[u]$, *and that* g *is* C^∞ *for* x *in* $\overline{\Omega}$ *and all values of the other arguments. Then, for any* $r > 0$, *there exists a* C^∞ *solution* u *in* $C^{m+\delta}$ *with* $\|u\|_{m-1+\delta} = r$. *(Here* $0 < \delta < 1$.*)*

PROOF: Let $X = \{u \in C^{m+\delta}(\overline{\Omega}) \mid Bu = 0 \text{ on } \partial\Omega\}$, $Y = C^\delta(\Omega)$. Decompose

$$X = X_1 \oplus X_2\,, \quad X_1 = \ker L\,, \quad X_2 = \perp_{L_2} X_1\,,$$
$$Y = Y_1 \oplus Y_2 = PY \oplus (I - P)Y \quad \text{where } Y_1 = \text{range}\, L,\ Y_2 \perp_{L_2} Y_1\,.$$

Write $u = u_1 + u_2$, $u_1 = \sum_1^d a_j w_j \in X_1$, $u_2 \in X_2$. Then problem (4.8) is equivalent in the usual way to the system

$$u_2 - L^{-1} PG\left[u_2 + \sum a_j w_j\right] = 0\,, \qquad (I - P)G\left[u_2 + \sum a_j w_j\right] = 0;$$

as before, write this as a map of $[X_2, \mathbb{R}^d] \to [X_2, \mathbb{R}^{d^*}]$, $d^* < d$.

If we now work in the space of functions $[u_2 \in C^{m-1+\delta} \mid Bu = 0 \text{ on } \partial\Omega\}$ and apply Borsuk's theorem, we obtain the desired solution in $C^{m-1+\delta}(\overline{\Omega})$. Then we can prove its regularity as in Section 2.5. □

REMARK. In this theorem, G need not be a differential operator. For instance, we could take

$$G[u] = f(x) \int_{\Omega} \left(u^5 + \sum_{i=1}^{n} (\partial_{x_i} u)^3 \right) dx .$$

If $f(x) \notin$ range L then it follows that there is a nontrivial function $u \in C^\infty(\overline{\Omega})$ with $Bu = 0$ on $\partial\Omega$ such that $Lu = 0$ and

$$\int_{\Omega} \left(u^5 + \sum (\partial_i u)^3 \right) dx = 0 .$$

4.3. Framed Cobordism

For mappings $\phi : X \to Y$ between oriented manifolds of the same dimension, we have defined $\deg(\phi, X, y_0)$ and used it to solve equations of the form $\phi(x) = 0$ in case $Y = \mathbb{R}^k$. In Section 1.8, for $X =$ a closed ball B in $\mathbb{R}^d$, $d > k$, we were led to consider the homotopy class of $\phi|_{\partial B} \to \mathbb{R}^k \backslash 0$. In the preceding sections of this chapter these results were extended to infinite-dimensional spaces with the aid of suspension and stable homotopy.

If X is not a ball, however, the methods we used are no longer applicable. To treat a general manifold X, Pontrjagin introduced the notion of framed cobordism to replace degree. A very elegant description of this is contained in [**8**, sec. 7]. In this section we will give a brief description of this concept.

Let X_0^n and Y^k be two oriented manifolds of dimension n and k, respectively, with $n \geq k$. Let X be an open subset of X_0 whose closure $\overline{X}$ is compact in X_0. If $\phi : \overline{X} \to Y$ is a smooth map and $y \notin \phi(\partial X)$ is a regular value of ϕ (i.e., $\partial\phi(x)/\partial x$ has rank $= k$ for each $x \in \phi^{-1}(y)$), then it follows from the implicit function theorem that $\phi^{-1}(y)$ is a compact submanifold N of X of dimension $n - k$ without boundary. Recall that when $n = k$, $\phi^{-1}(y)$ consists of a finite number of points and

$$\deg(\phi, \Omega, y) = \sum_{x \in \phi^{-1}(y)} (-1)^{\operatorname{sgn} \det(\frac{\partial u(x)}{\partial x})} .$$

This is the algebraic count of the number of times y is covered. Heuristically in the case $n \neq k$, we want a way to count the number of connected components of $\phi^{-1}(y)$.

DEFINITION Let X be an oriented n-dimensional manifold, and N_1, N_2 be two oriented, compact submanifolds of dimension $(n - k)$ in X without boundary. N_1 is *cobordant* to N_2 within X if, for $\varepsilon > 0$ small,

$$(N_1 \times [0, \varepsilon)) \cup (N_2 \times (1 - \varepsilon, 1])$$

can be extended to a compact manifold M in $X \times [0, 1]$ with

$$\partial M = (N_1 \otimes \{0\}) \cup (N_2 \times \{1\}) , \quad M \cap [(X \times \{0\}) \cup (X \times \{1\})] = \partial M .$$

The orientation of ∂M is then consistent with that of N_1 and N_2. The manifold M in the definition is said to be a *cobordism* between N_1 and N_2.

It is a straightforward exercise to show that the cobordism is an equivalence relation on the oriented submanifolds of X of dimension $(n-k)$ without boundary.

Suppose now that X is a Riemannian manifold (i.e., on $T_x(X)$, the tangent space of X at x, there is a positive definite scalar product defined $\langle u, v\rangle_x$, $u, v \in T_x(X)$, such that $\langle u, v\rangle_x$ is smooth in x).

DEFINITION A *framing* of a submanifold $N \subset X$ of dimension $n-k$ is the assignment ν of k linearly independent vectors

$$(\nu^1(x), \ldots, \nu^k(x)) \quad \text{in } T_x(X)$$

that are normal to N. The pair (N, ν) is called a *Pontrjagin framed manifold.*

DEFINITION Let (N_1, ν_1), (N_2, ν_2) be two framed manifolds. (N_1, ν_1) is said to be *framed cobordant* to (N_2, ν_2) if there is a cobordism $M \subset X \times [0, 1]$ between N_1 and N_2 and a framing u of M such that

$$\begin{aligned} u^i(x, t) &= \nu_1^i(x) \quad \text{for } (x, t) \in N_1 \times [0, \varepsilon) \\ &= \nu_2^i(x) \quad \text{for } (x, t) \in N_2 \times (1-\varepsilon, 1]. \end{aligned}$$

Again it is an exercise to check that this defines an equivalence relation.

EXAMPLE. Consider a smooth map $\phi : X \to Y$; for $y \in Y \backslash \phi(\partial X)$, a regular value of ϕ, set $N = \phi^{-1}(y)$. Let $v^1, \ldots, v^k$ be a positively-oriented basis for $T_y(Y)$. Let ϕ_* be the natural map

$$\phi_* : T_x(X) \to T_{\phi(x)}(X) \quad \text{induced by } \phi .$$

If $x \in \phi^{-1}(y)$, then $\frac{\partial\phi}{\partial x}(x)$ has rank k and ϕ_* restricted to $T_x(N)^{\perp}$, the subspace of $T_x(X)$ orthogonal to $T_x(N)$ is one-to-one and onto $T_y(Y)$. The inverse gives a framing for N, called a Pontrjagin framed manifold associated with the map ϕ.

LEMMA 4.3.1 *With $v^1, \ldots, v^k$ a given basis of $T_y(Y)$, let $(N, \nu) = (\phi^{-1}(y), \nu)$ be the resulting framing of $\phi^{-1}(y)$; then the framed cobordism class of $\phi^{-1}(y)$ is independent of the choice of $v^1, \ldots, v^k$.*

PROOF: If $u^1, \ldots, u^k$ is another similarly oriented basis for $T_y(Y)$, the pair may be connected by $v_t^1, \ldots, v_t^k$, where for each $t \in [0, 1]$, $v_t^1, \ldots, v_t^k$ is a basis for $T_y(Y)$.

$$(v_0^1, \ldots, v_0^k) = (v^1, \ldots, v^k), \qquad (v_1^1, \ldots, v_1^k) = (u^1, \ldots, u^k).$$

This is simply because the set of $k \times k$ real matrices with positive determinant is connected. Letting $M = N \times [0, 1]$, we see that (M, u) with $u(x, t) = (v_t^1(x), \ldots, v_t^k(x))$ is a framed cobordism between N and N. □

PROPOSITION 4.3.2 *Suppose $\phi, \psi : X \to Y$ are smooth mappings where X and Y are two Riemannian manifolds, $\overline{X}$ compact as before. Suppose $y \in Y$ is a regular value of both maps and $y \notin \phi(\partial X)$, $y \notin \psi(\partial X)$. If the distance $d(\psi, \phi) = \sup_{x \in X} d(\phi(x), \psi(x))$ is sufficiently small, then, with their framings, $\phi^{-1}(y)$ and $\psi^{-1}(y)$ are cobordant.*

PROOF: To construct the cobordism M, let us first connect ϕ and ψ, using the deformation $\phi_t : X \to Y$ defined by moving along the shortest geodesic in Y joining $\phi(x)$ to $\psi(x)$. By a suitable choice of parametrization, we may suppose ϕ_t independent of t in $(0, \varepsilon)$ and $(1 - \varepsilon, 1)$. Consider

$$\Phi : \overline{X} \times [0, 1] \to Y \quad \text{such that } \Phi = \phi_t(x) .$$

If y were a regular value of Φ, then $\Phi^{-1}(y)$, together with its framing, would give the desired framed cobordism between $\phi^{-1}(y)$ and $\psi^{-1}(y)$. However, if y is not a regular value of Φ, then $\Phi^{-1}(y)$ may not be a smooth manifold. In order to get around this difficulty, we use the following theorem:

THEOREM 4.3.3 (Transversality Theorem) [**15**, chap. 4] *Let $\phi : Z \to Y$ with ϕ a smooth mapping. Let y be a fixed point in Y such that outside of a relatively compact open set $U \subset Z$, $\phi^{-1}(y)$ consists of regular points. Then we can deform Φ slightly to $\tilde{\Phi}$ so that y is a regular value of $\tilde{\Phi}$ and, in the complement of U, $\Phi = \tilde{\Phi}$.*

Returning to the proof, we may perturb Φ slightly so that y is a regular value of the perturbed map $\tilde{\Phi}$ and Φ is unchanged in $X \times (0, \varepsilon)$ and $X \times (1 - \varepsilon, 1)$. Then $\tilde{\Phi}^{-1}(y)$ is a cobordism of $\phi^{-1}(y)$ and $\psi^{-1}(y)$, and with its framing $\tilde{\Phi}^{-1}(y)$ is a framed cobordism of $\phi^{-1}(Y)$ and $\psi^{-1}(y)$. □

LEMMA 4.3.4 *Suppose $\phi : \overline{X} \to Y$ is smooth and $y_0 \notin Y \backslash \phi(\partial X)$. Then there is a neighborhood U of y_0, such that for y_1 and y_2 in U, with y_1, y_2 regular values of ϕ, $\phi^{-1}(y_1)$ and $\phi^{-1}(y_2)$ are framed cobordant.*

COROLLARY 4.3.5 *If y_1 and y_2 are regular values of ϕ in the same component C of $Y \backslash \phi(\partial X)$, then $\phi^{-1}(y_1)$ and $\phi^{-1}(y_2)$ are framed cobordant.*

The proof is a simple exercise.

PROOF OF LEMMA: If U is a small neighborhood of y_0, let g be a C^∞ diffeomorphism of Y, which is the identity outside a neighborhood of U, and maps U onto U and $g(y_2) = y_1$.

Consider $\phi(x)$ and $\psi(x) = g \circ \phi$. Since y_2 is a regular value of ϕ, it follows that y_1 is a regular value of ψ, and ϕ and ψ are close if we choose the diameter of U small. By Proposition 4.3.2, $\phi^{-1}(y_1)$ is framed cobordant to $\psi^{-1}(y_1) = \phi^{-1}(y_2)$. □

Extension to Continuous Maps. Let ϕ be a continuous map $\phi : X \to Y$ and C a component of $Y \backslash \phi(\partial X)$. Approximating ϕ in the C^0 topology by a C^∞ mapping $\tilde{\phi}$, we may associate with ϕ and C a well-defined framed cobordism class of X.

The following theorem connects the theory of framed cobordism classes with homotopy classes, and its proof may be found in section 7 of Milnor's book:

THEOREM 4.3.6 (Pontrjagin) *If X is an n-dimensional compact manifold without boundary and $Y = \mathbb{S}^k$, then there is a one-to-one correspondence between framed cobordism classes $(\mathbb{N}^{n-k}, \nu)$ in X and homotopy classes of maps $X \to \mathbb{S}^k$.*

Framed cobordism theory has been extended to infinite-dimensional spaces. Some references which are also related to the material of the preceding two sections are [**36, 38, 42, 44**].

In the next two sections we will take up a related theory of Gęba and Granas.

4.4. Stable Cohomotopy Theorem

Lecture of J. Ize

Recall some facts: Let Ω be an open, bounded set in $\mathbb{R}^n$ and $F : \overline{\Omega} \to \mathbb{R}^n$ a continuous map such that $F \mid \partial\Omega \neq \{0\}$. Then $\deg F = \text{degree}(F, \Omega, 0)$ is well-defined and depends only on the homotopy type of F restricted to $\partial\Omega$. Furthermore, if $\deg F \neq 0$, then every continuous extension of $F \mid \partial\Omega$ to Ω has a zero.

Now let X be a *closed bounded* set in $\mathbb{R}^d$ and $F : X \to \mathbb{R}^{d^*}\backslash\{0\}$ a continuous map.

DEFINITION F is *inessential* iff for all Y closed bounded subsets of $\mathbb{R}^d$ containing X, F extends to $\widetilde{F} : Y \to \mathbb{R}^{d^*}\backslash\{0\}$.

EXAMPLES. (1) $X = \mathbb{S}^{d-1}$, F is inessential iff the homotopy class of $F/|F|$ in $\pi_{d-1}(\mathbb{S}^{d^*-1})$ is trivial.

(2) X is a compact manifold without boundary and $F : X \to \mathbb{S}^{d^*-1}$. As a consequence of the Pontrjagin-Hopf theorem, F is inessential iff the framed cobordism class of F is trivial; i.e., given a regular value p of F, in case F is smooth, then $(F^{-1}(p), F^*TS_p^{d^*-1})$ is the framed boundary of a manifold M in $X \times [0, 1]$.

PROPOSITION 4.4.1 $F_0, F_1 : X \subset \mathbb{R}^d \to \mathbb{R}^{d^*}\backslash\{0\}$. *Suppose F_1 is homotopic to F_0 and F_0 is inessential. Then F_1 is inessential.*

PROOF: Let Y be closed and bounded, $X \subset Y \subset \mathbb{R}^d$, and $\widetilde{F}_0$ an extension of F_0; $\widetilde{F}_0 : Y \to \mathbb{R}^{d^*}\backslash\{0\}$. Suppose $F(t, s) : X \times [0, 1] \to \mathbb{R}^{d^*}\backslash\{0\}$ with $F(0, x) = F_0(x)$, $F(1, x) = F_1(x)$. Let $Z = \{X \times [0, 1]\} \cup \{Y \times \{0\}\}$. Define $\widetilde{F} : Z \to \mathbb{R}^{d^*}\backslash\{0\}$ as

$$\widetilde{F}(x, t) = F(x, t), \quad x \in X,$$
$$\widetilde{F}(y, 0) = \widetilde{F}_0(y), \quad y \in Y,$$

Z being closed, we may extend $\widetilde{F}$ to $G : Y \times [0, 1] \to \mathbb{R}^{d^*}$ by Tietze's extension theorem. Let $A = \{y \in Y \mid G(y, t) = 0 \text{ for some } t\}$; then $A \cap X = \emptyset$ and A and X are closed in Y. There is a continuous separating function $\lambda : Y \to [0, 1]$ such that $\lambda(A) = 0$, $\lambda(X) = 1$. Set $\widetilde{G}(y, t) = G(y, \lambda(y)t)$. Clearly $\widetilde{G} : Y \times [0, 1] \to \mathbb{R}^{d^*}\backslash\{0\}$ for if $\widetilde{G}(y, t) = 0$, then $y \in A$ and $\lambda(y) = 0$. But $G(y, 0) = \widetilde{F}(y, 0) = \widetilde{F}_0(y) \neq \{0\}$. Also for $t = 1$ and y in X, $G(y, \lambda(y)) = G(y, 1) = F(x, 1) = F_1(x)$, so $\widetilde{G}_1(y, 1)$ extends F_1. □

COROLLARY 4.4.2 $F : X \to \mathbb{R}^{d^*}\backslash\{0\}$ *is inessential* $\to$ *F is homotopic to $G : X \to pt$ in $\mathbb{R}^{d^*}\backslash\{0\}$.*

PROOF: $\Rightarrow$ F inessential and X bounded; then X is contained in a large ball B, centered at $\{0\}$, and F admits an extension $\widetilde{F} : B \to \mathbb{R}^{d^*}\backslash\{0\}$. Define $F(x,t) : X \times [0,1] \to \mathbb{R}^{d^*}\backslash\{0\}$ by $F(x,t) = \widetilde{F}((1-t)x)$. Thus F is homotopic to $G : X \to \widetilde{F}(0)$.

The converse follows from Proposition 4.4.1. □

In these lectures, we plan to study:

- *Cohomotopy Groups.* We shall investigate homotopy classes of maps $X \to \mathbb{S}^n$.
- *Extension to Banach Spaces E.* If X is a closed, bounded set in E, we will consider compact vector fields into subspaces of finite codimension: $X \xrightarrow{f} E^{\infty-n}\backslash\{0\}$.

  Here $E^{\infty-n}$ is a closed subspace of codimension n, and $I - f$ is compact. We study stable cohomotopy.
- *Application.* We will look at an extension of Rabinowitz's theorem in Section 3.4 on existence of solutions in the large for $(I - \lambda T)x = g(x, \lambda)$ to the complex case.

4.5. Cohomotopy Groups

Lecture of J. Ize

(See [**45**].) Let X be a compact topological space of dim $\leq 2m - 2$, and A closed in X. (*Topological dimension n* of X is defined as Inf$n$ for which for any finite open covering of X, one has an open refinement such that the intersection of any $n + 2$ sets in it is empty.)

DEFINITION Let $\Pi^m(X, A)$ be the set of homotopy classes of continuous maps $f : (X, A) \to (\mathbb{S}^m, pt)$.

THEOREM 4.5.1 *$\Pi^m(X, A)$ is an abelian group.*

Idea of Proof When X Is a Complex. Let $f, g : (X, A) \to (\mathbb{S}^m, pt)$; approximate f and g by simplicial maps. Consider

$$(X, A) \xrightarrow{d=\text{diagonal map}} (X, A) \times (X, A) \xrightarrow{f \times g} (\mathbb{S}^m, pt) \times (\mathbb{S}^m, pt)\,.$$

So $(f \times g) \circ d$ is a simplicial map from a complex of dimension $\leq 2m - 2$ to a $2m$ complex. Consequently, $(f \times g) \cdot d(X)$ lies in the $2m - 2$ skeleton of $\mathbb{S}^m \times \mathbb{S}^m$, and furthermore, one can deform this to lie in $\mathbb{S}^m \vee_{pt} \mathbb{S}^m$ = two copies of $\mathbb{S}^m$ wedged at the point pt.

EXAMPLE. A closed curve on a torus T can be deformed to the generators of $\Pi_1(T)$.

Let $\omega : \mathbb{S}^m v_{pt} \mathbb{S}^m \longrightarrow (\mathbb{S}^m, pt)$ be the map $\omega(x, pt) = \omega(pt, x) = x$. Define $[f] + [g] = [\omega(f \times g) \cdot d]$ via this deformation. It is clear that this operation is commutative. The extra free dimension is needed to prove that the "addition" is independent of the representatives of the homotopy classes, since $X \times [0, 1]$ has dim $\leq 2m - 1$, so one can move freely.

REMARKS. (1) *Natural element*: $(X, A) \to (pt, pt)$.

(2) *Inverse of* $[f]$: $(X, A) \xrightarrow{f} (\mathbb{S}^m, pt) \xrightarrow{i} (\mathbb{S}^m, pt)$, i is an orientation reversing map of degree -1.

(3) $f : Y \to X$ induces $f^* : \Pi^m(X) \to \Pi^m(Y)$ by $Y \xrightarrow{f} X \xrightarrow{g} \mathbb{S}^m$, $f^*[g] = [g \circ f]$.

(4) Finally, we introduce the *coboundary operator* $\delta : \Pi^m(A) \longrightarrow \Pi^{m+1}(X, A)$. Suppose $f : A \to \mathbb{S}^m$, E_+^{m+1} = upper hemisphere of $\mathbb{S}^{m+1}$ with $\mathbb{S}^m$ the equator of $\mathbb{S}^{m+1}$, $A \subset X$. Since E_+^{m+1} is contractible we may extend f to $\tilde{f} : X \to E_+^{m+1}$. Let h be a deformation of $\mathbb{S}^m$ to the south pole p of $\mathbb{S}^{m+1}$, and stretch E_+^{m+1} over $\mathbb{S}^{m+1} - \{p\}$. Then define $\delta[f] = [h \cdot \tilde{f}] : (X, A) \to (\mathbb{S}^{m+1}, pt)$.

This cohomotopy theory satisfies the Eilenberg-Steenrod axioms for cohomology theory.

4.6. Stable Cohomotopy Theory

Lecture of J. Ize

Let E be a real Banach space. Give an orientation to E via a sequence of subspaces E_n, with $\dim E_n = n$, and complementary closed subspaces $E^{\infty-n}$ satisfying $E_n \subset E_{n+1}$, $E = E_n \oplus E^{\infty-n}$, $E_{n+1} = E_n \oplus R$, $E^{\infty-n} = E^{\infty-n-1} \oplus \mathbb{R}$. Let

$$P^{\infty-n} = E^{\infty-n} \backslash \overline{\mathbb{R}}^+ = \{x \in E^{\infty-n} \mid x = x_1 + r, x_1 \in E^{\infty-n-1}, r \in \mathbb{R}, r < 0\}.$$

DEFINITION $L \cdot S(E)$: *Leray-Schauder category* of E.

> Objects: $X \subset E$, closed and bounded.
> Morphisms: $I-$ compact.
> Homotopies: $x - F(x, t)$, F compact: $X \times I \to E$, $I = [0, 1]$.

DEFINITION $(X, A) \in L \cdot S(E)$, $A \subset X$. $\Pi^{\infty-n}(X, A)$ = set of homotopy classes of maps $f = I - F$, in the category, with

$$\begin{aligned} I - F : X &\to E^{\infty-n}\backslash\{0\} \\ : A &\to P^{\infty-n}. \end{aligned}$$

Define $\Pi^{\infty-n}(X) = \Pi^{\infty-n}(X, \emptyset)$.

DEFINITION $f : X \to E^{\infty-n}\backslash\{0\}$ is *inessential* iff for all Y in $L \cdot S(E)$, $Y \supset X$, f extends to $\tilde{f} : Y \to E^{\infty-n}\backslash\{0\}$ in the category.

REMARK 4.6.1. As before (replacing Tietze's extension theorem with that of Dugundji), if f is compactly homotopic to g and g is inessential, then f is inessential.

THEOREM 4.6.2 (Gęba-Granas) *$\Pi^{\infty-n}(X, A)$ is an abelian group.*

Idea of Proof.

(a) *A Finite-Dimensional Approximation.* Let $f = I - F$ map X into $E^{\infty-n}\setminus\{0\}$ and A into $P^{\infty-n}$. F compact implies that $f(X)$ and $f(A)$ are closed in $E^{\infty-n}$. Set $\varepsilon < \min\{\text{dist}(f(X), E_n), \text{dist}(f(A), E_n \oplus \overline{R^+})\}$. Approximate F within ε by G, mapping X into a finite-dimensional subspace L (assume $E_{n+1} \subset L$ and $\dim L = n+m+1$). Then $g = I - G$ maps X into an ε-neighborhood of $E^{\infty-n}$. Let P_0 be the projection of E onto $E^{\infty-n}$ parallel to E_n, and set $\tilde{f}(x) = P_0 g(x) = x - (I - P_0)x - P_0 G(x) = x - \widetilde{F}(x)$ with $\widetilde{F}(x)$ in L, since $P_0 G = G - (I - P_0)G$, $(I - P_0)G$ lies in E_n, which is contained in L. Then f can be deformed to $\tilde{f}$ via $P_0((1-t)f(x) + tg(x))$ by our choice of ε. Set $\tilde{f}_L : (X \cap L, A \cap L) \xrightarrow{\tilde{f}} (E^{\infty-n} \cap L - \{0\}, P^{\infty-n} \cap L) = (E^{m+1}\setminus\{0\}, P^{m+1})$, since $\dim E^{\infty-n} \cap L = m + 1$. P^{m+1} being contractible, the last pair is homotopy equivalent to $(\mathbb{S}^m, pt)$.

So we obtain an element in $\Pi^m(X \cap L, A \cap L)$ which is a group if $\dim L = n + m + 1 \leq 2m - 2$, i.e., $m \geq n + 3$.

Also one can prove that f and g are homotopic in $L \cdot S(E)$ iff $\tilde{f}_L$ and $\tilde{g}_L$ are homotopic (the converse uses the homotopy extension theorem).

(b) *Maps with Finite Range.* Let L be a fixed $(n + m + 1)$–dimensional subspace containing E_{n+1}. Define $\Pi_L^{\infty-n}(X, A)$ as the set of homotopy classes of maps $f : X \to E^{\infty-n}\setminus\{0\}$, $A \to P^{\infty-n}$, $f = I - F$ with $F(X) \subset L$, and with homotopies $x - H(x, t)$, $H(x, t) \subset L$. Let τ be the map: $\Pi_L^{\infty-n}(X, A) \xrightarrow{\tau} \Pi^m(X \cap L, A \cap L)$ induced by the above restriction to $X \cap L : f \to \tilde{f}_L$. Then τ is one-to-one and onto.

PROOF THAT τ IS ONTO: $\tilde{f}_L(x_1) = x_1 - \widetilde{F}_L(x_1)$ represents an element of $\Pi^m(X \cap L, A \cap L)$ with x_1 in L. Writing any element x in E as $x = x_1 \oplus x_2$, set $f(x) = x_2 \oplus x_1 - \widetilde{F}_L(x_1)$; then $f(x)$ is an appropriate extension of $\tilde{f}_L$ if one chooses the complementing subspace of L in E to be contained in $E^{\infty-n}$ (possible since $E_{n+1} \subset L$). □

PROOF THAT τ IS ONE-TO-ONE: Suppose f and g have $\tilde{f}_L$ and $\tilde{g}_L$ homotopic via $\tilde{h}_L(x, t)$. Set $Z = \{X \times \{0\}\} \cup \{X \cap L \times [0, 1]\} \cup \{X \times \{1\}\}$ closed in $X \times [0, 1]$. Construct $H(x, t)$ a map from Z and L (hence compact), defining H on $X \times \{0\}$ as $x - f(x)$, on $X \cap L \times [0, 1]$ as $x - \tilde{h}_L(x, t)$, on $X \times \{1\}$ as $x - g(x)$. Extend H to $\overline{H} : X \times [0, 1] \to L$ (Dugundji's theorem) and define $h(x, t) = P_0(x - \overline{H}(x, t)) = x - (I - P_0)x - P_0\overline{H}(x, t)$, P_0 the projection of E onto $E^{\infty-n}$. (If $h(x, t) = 0$, then $x - \overline{H}(x, t) \in E_n \subset L$, so x lies in L and $h(x, t) = \tilde{h}_L(x, t)$.)

Thus, we can give to $\Pi_L^{\infty-n}(X, A)$ the group structure of $\Pi^m(X \cap L, A \cap L)$. □

(c) *The Limit Process.* If M is a finite-dimensional subspace containing L, then there exists a Mayer-Vietoris homomorphism $\tilde{\Delta} : \Pi^m(X \cap L,$

$A \cap L) \to \Pi^{\dim M-1-n}(X \cap M, A \cap M)$, which, via τ, induces a homomorphism $\Delta : \Pi_L^{\infty-n}(X, A) \to \Pi_M^{\infty-n}(X, A)$, well behaved with respect to induced maps and coboundaries. So one obtains an *inductive* family of abelian groups and

$$\varinjlim_{L+\infty} \Pi^{\dim L-1-n}(X \cap L, A \cap L) \equiv \sum^{\infty-n}(X, A)$$

is called the *stable cohomotopy group of* (X, A). The limit of τ gives an isomorphism between $\sum^{\infty-n}(X, A)$ and $\Pi^{\infty-n}(X, A)$, which inherits all the functorial properties of the stable group.

Note that from the corollary to Proposition 4.4.1 the neutral element in $\Pi^{\infty-n}$ (X) is represented by the set of inessential maps.

THEOREM 4.6.3 $\Pi^{\infty-n}$ *defines a generalized cohomology functor for* $L \cdot S(E)$. $f = I - F : (X, A) \to (Y, B)$ *induces* $f^* : \Pi^{\infty-n}(Y, B) \to \Pi^{\infty-n}(X, A)$ *by* $f^*[g] = [g \circ f]$.

(i) $(\mathrm{Id})^* = \mathrm{Id}$.
(ii) $(fg)^* = g^* f^*$.
(iii) *There exists a map* $\delta : \Pi^{\infty-n}(A) \to \Pi^{\infty-n+1}(X, A)$ *such that if* $f_0 = f|_A$, *then* $f^*\delta = \delta f_0^*$.
(iv) $\xrightarrow{\delta} \Pi^{\infty-n}(X, A) \to \Pi^{\infty-n}(X) \to \Pi^{\infty-n}(A) \xrightarrow{\delta} \Pi^{\infty-n+1}(X, A)$ *is exact, the other maps being induced by inclusions.*
(v) *If* f *is homotopic to* g, *then* $f^* = g^*$.
(vi) *Strong excision:* $X = A \cup B : \Pi^{\infty-n}(X, A) \cong \Pi^{\infty-n}(B, A \cap B)$.
(vii) $\Pi^{\infty-n}(pt) = 0$.

Complete details may be found in **[5, 37, 39]**.

EXAMPLES. (1) $X = S$ *unit sphere in* E. Here there is no need of a Mayer-Vietoris sequence; we use only suspension: $\Pi^{\infty-n}(S) =$ [suspension of maps $S \cap L = \mathbb{S}^{n+m} \to \mathbb{S}^m] = \Pi^m(\mathbb{S}^{n+m}) = \Pi_{n+m}(\mathbb{S}^m) = \Pi_n$. So $\Pi^{\infty-0}(S) = Z$, $\Pi^{\infty-1}(S) = Z_2$ generated by the suspension of the Hopf map $\eta : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^3$, $\eta(\lambda, z) = (\bar{\lambda} z, |z|^2 - |\lambda|^2)$. $\Pi^{\infty-2}(S) = Z_2$, $\Pi^{\infty-3}(S) = Z_{24}, \ldots$.

(2) $X = \overline{B}$ *closed ball of radius* R, then $\Pi^{\infty-n}(\overline{B}) = 0$ for all n, for if $f(x) = x - F(x)$ is a map from $\overline{B}$ to $E^{\infty-n}\backslash\{0\}$, then assuming that $\overline{B}$ is centered at the origin, $\tilde{f}(x) = x - \frac{\|x\|}{R} F(\frac{xR}{\|x\|})$ extends f to E for $\|x\| \geq R$.

Connection with the Leray-Schauder Degree Theory. The Alexander-Pontrjagin duality theorem between $\Pi^{\infty-n}(X)$ and $\sum_n(E\backslash X)$, the stable homotopy group of $E\backslash X$, shows that $\Pi^{\infty-0}(X) = \bigoplus Z$, as many copies as there are bounded components of $E\backslash X$. Thus, if f is a map from X into $E\backslash\{0\}$, then $[f] = \sum m_i \alpha_i$, where α_i is represented by $x - x_i$, x_i any point in the i^{th} bounded component of $E\backslash X$. If $X = \partial\Omega$, Ω an open, bounded subset of E, define degree $(f, \Omega, 0) = \sum m_i$.

EXERCISE Show that this degree has the usual properties of the Leray-Schauder degree.

For theories in other directions, see the references at the end of Section 4.3. See also the survey article [**35**].

4.7. Application to Existence of Global Solutions

Lecture of J. Ize

We shall extend Rabinowitz's result on global solutions to complex Banach spaces. (This material is taken from the doctoral dissertation of J. Ize, Courant Institute, 1974.)

Recall the bifurcation result of Section 3.5. E is a complex Banach space, T a compact linear map from E into E, $g(x, \lambda)$ a compact map from its domain D in $E \times \mathbb{C}$ into E, with $g(0, \lambda) \equiv 0$ and $\|g(x, \lambda)\| = o(\|x\|)$ as $\|x\| \to 0$ in D. Consider the equation $x - \lambda Tx - g(x, \lambda) = 0$. Then $(0, \lambda_0)$ is a bifurcation point if λ_0 is a characteristic value of T (i.e., λ_0^{-1} is an eigenvalue of T) of odd multiplicity. Note that bifurcation occurs only at characteristic values of T.

Let S be the closure in D of the nontrivial solutions (x, λ), $x \neq 0$. Let λ_0 be a characteristic value of T at which bifurcation takes place, and C be the connected component of S containing $(0, \lambda_0)$. Thus, if $(0, \lambda)$ belongs to C, then λ is a characteristic value of T.

THEOREM 4.7.1 *C is either*

(i) *not compact in D (and if $D = E \times \mathbb{C}$, C is unbounded), or*
(ii) *C is bounded in D and contains a finite number of points $(0, \lambda_i)$, $i = 0, \dots, p$, λ_i characteristic value of T, of multiplicity m_i, and $\sum_0^p m_i$ is even.*

This implies that the number of characteristic values of T of odd multiplicity, in C, is even.

LEMMA 4.7.2 *Let λ_0 be a characteristic value of T of multiplicity m. For $r > 0$, let*

$$H_r(x, \lambda) = \left\{(I - \lambda T)x - g(x, \lambda), \|x\|^2 - r^2\right\} : D \to E \times \mathbb{C}$$

and

$$S = \left\{(x, \lambda) \mid \|x\|^2 + |\lambda - \lambda_0|^2 = r^2 + \rho^2\right\}.$$

Then there are two positive constants r and ρ such that the stable homotopy class of $H_r(x, \lambda)$ with respect to S is defined and equal to $\Sigma(m\eta)$. (Σ suspension, η Hopf map.)

PROOF: Since λ_0 is isolated, we may choose $\rho > 0$, so that for some constant $M > 0$, $\|(I - (\lambda_0 + \rho e^{i\theta})T)^{-1}\| \le M$ for all θ. Using the smallness condition on g, choose r so small that $(I - \lambda T)x - g(x, \lambda) \neq 0$ for $\lambda = \lambda_0 + \rho e^{i\theta}$, all θ, and $0 < \|x\| \le r$. Thus on the sphere S, one can deform H_r to $((I - \lambda T)x, \|x\|^2 - r^2)$. Let $f(x) : [0, 1] \to \mathbb{C}$ be a path such that $f(0) = 0$, $f(1) = 1$, and $f(t)\lambda_0$ is a path from 0 to λ_0 avoiding all other characteristic values of T. Since these points are discrete, there is a δ-neighborhood of the path which also avoids such points. Choose ρ with $\rho < \delta / \max |f(t)|$. We then have for a suitable α,

$$E = \ker(I - \lambda_0 T)^\alpha \oplus R(I - \lambda_0 T)^\alpha , \quad m = \dim \ker(I - \lambda_0 T)^\alpha ,$$
$$x = x_1 \oplus x_2 , \quad x_1 \text{ in } \ker(I - \lambda_0 T)^\alpha , \quad \text{and}$$
$$H_r(x, \lambda) = \left((I - \lambda T)x_1 \oplus (I - \lambda T)x_2, \ \|x\|^2 - r^2 = \rho^2 - |\lambda - \lambda_0|^2\right) \text{ on } S .$$

Deform this map to $((I - \lambda T)x_1 \oplus x_2, \ \rho^2 - |\lambda - \lambda_0|^2)$ via $((I - \lambda T)x_1 \oplus (I - \lambda f(t)T)x_2, \ \rho^2 - |\lambda - \lambda_0|^2)$ on S. (If this is zero, then $\lambda = \lambda_0 + \rho e^{i\theta}$, and so $x_1 = 0$, and from the choice of ρ, $\lambda f(t)$ belongs to the δ-neighborhood of the path, so $x_2 = 0$ or $\lambda f(t) = \lambda_0$, which implies that x_2 is in $\ker(I - \lambda_0 T)$ and so $x_2 = 0$. Then $\|x\|$ cannot be equal to r.) Thus, $[H_r(x, \lambda)] =$ [suspension of $((I - \lambda T)x_1$, $\rho^2 - |\lambda - \lambda_0|^2)$ on $\|x_1\|^2 + |\lambda - \lambda_0|^2 = r^2 + \rho^2$]. Using a Jordan form for $I - \lambda_0 T$, we may deform this map to $((\lambda_0 - \lambda)x_1, \ \rho^2 - |\lambda - \lambda_0|^2)$, and as in the proof of Theorem 3.5.3 with $\Lambda = \mathbb{C}$, a further deformation leads to $\sum m\eta$. □

PROOF OF THE THEOREM: Suppose C is a compact set in D. As in the real case, let Ω be an open, bounded subset of D such that C is contained in Ω, Ω contains no points $(0, \lambda)$ of S but $(0, \lambda_i)$, $i = 0, \ldots, p$, as stated in the theorem, and $(I - \lambda T)x - g(x, \lambda) = 0$ on $\partial\Omega$ has no solution but $x = 0$.

Set $E^{\infty - 0} = E \times \mathbb{C}$, $E^{\infty - 2} = E$, $E^{\infty - 1} = E \times (\mathrm{Re}\, \mathbb{C})$; then for all $r > 0$, $H_r(x, \lambda)$ represents the same element in $\Pi^{\infty-1}(\partial\Omega)$: $H_r(x, \lambda)$ avoids $\{0\} \times (\mathrm{Im}\, \mathbb{C})$. Note that by identifying $\mathbb{R}^{2n}$ and $\mathbb{C}^n$, we may use the stable cohomotopy theory of Section 4.4.

(a) *Global Class*. For $r = R$ large, $\|x\|^2 - R^2 < 0$ on Ω, so $H_R(x, \lambda)$ defines an element in $\Pi^{\infty-1}(\overline{\Omega}, \partial\Omega)$ so that in the exact sequence,

$$\begin{array}{ccccc} \Pi^{\infty-1}(\overline{\Omega}, \partial\Omega) & \longrightarrow & \Pi^{\infty-1}(\overline{\Omega}) & \longrightarrow & \Pi^{\infty-1}(\partial\Omega) \\ [H_R(x, \lambda)] & \longrightarrow & [H_R(x, \lambda) & \longrightarrow & [H_R(x, \lambda)] = [H_r(x, \lambda)] . \end{array}$$

It follows that $[H_r(x, \lambda)|_{\partial\Omega}] = 0$, i.e., $H_r(x, \lambda)|_{\partial\Omega}$ is inessential for all $r > 0$.

Choose r so small that any solution of $H_r(x, \lambda) = 0$ in Ω must lie inside a ball $\overline{B}_j = \{(x, \lambda) \mid \|x\|^2 + |\lambda - \lambda_j|^2 \leq r^2 + \rho^2\}$, $j = 0, \ldots, p$. Then Lemma 4.7.2 is applicable.

(b) *Replacement of* $\Pi^{\infty-1}(\partial\Omega)$ *by a Group Easier to Compute*. Since $H_r(x, \lambda)$ is inessential on $\partial\Omega$, let B be a ball containing Ω and extend H_r to $\overline{B}$. Set $\widetilde{H}_r(x, \lambda) =$ this extension on $\overline{B} \backslash \Omega$, $H_r(x, \lambda)$ on Ω. It is clear that $\widetilde{H}_r$ is inessential on $\partial B = S$. By construction, $\widetilde{H}_r(x, \lambda)$ maps $\overline{B} \backslash \bigcup_0^p B_j$ into $E^{\infty-1} \backslash \{0\}$.

(c) *Local Classes*. If $S_j = \partial B_j$, then $\widetilde{H}_r(x, \lambda) | \bigcup_0^p S_j$ represents an element in $\Pi^{\infty-1}(\bigcup_0^p S_j) = \bigoplus_0^p \Pi^{\infty-1}(S_j) = \bigoplus_0^p Z_2$, as noted in Section 4.4, and the S_j are disjoint. From the lemma, this element is $\bigoplus_0^p [\sum m_j \eta] = \bigoplus_0^p m_j \alpha_j$ [**8**].

The situation is now the following:

$$\begin{array}{ccccc}\Pi^{\infty-1}(S) & \xleftarrow{k^*} & \Pi^{\infty-1}\left(\overline{B}\setminus\bigcup_0^p B_j\right) & \xrightarrow{i^*} & \Pi^{\infty-1}\left(\bigcup_0^p S_j\right)\\ 0=[\widetilde{H}_r|_S] & \longleftarrow & \left[\widetilde{H}_r|_{\overline{B}-\bigcup_0^p B_j}\right] & \longrightarrow & \left[H_r|_{\bigcup_0^p S_j}\right]=\bigoplus_0^p m_j\alpha_j\end{array}$$

where i^* and k^* are induced by inclusions.

(d) i^* *an Isomorphism*. By excision of $\bigcup_0^p B_j$ we have

$$\Pi^{\infty-n}\left(\overline{B}-\bigcup_0^p B_j,\bigcup_0^p S_j\right)\cong\Pi^{\infty-n}\left(\overline{B},\bigcup_0^p\widetilde{B}_j\right).$$

Moreover, in the exact sequence

$$\Pi^{\infty-n-1}\left(\bigcup_0^p\overline{B}_j\right)\to\Pi^{\infty-n}\left(\overline{B},\bigcup_0^p\overline{B}_j\right)\to\Pi^{\infty-n}(\overline{B})$$

the extreme groups are zero; hence so is $\Pi^{\infty-n}(\overline{B},\bigcup_0^p\overline{B}_j)$. So we have

$$\Pi^{\infty-1}\left(\overline{B}-\bigcup_0^p B_j,\bigcup_0^p S_j\right)\longrightarrow\Pi^{\infty-1}\left(\overline{B}-\bigcup B_j\right)\xrightarrow{i^*}\Pi^{\infty-1}\left(\bigcup_0^p S_j\right)$$
$$\longrightarrow\Pi^{\infty-0}\left(\overline{B}-\bigcup_0^p B_j,\bigcup_0^p S_j\right),$$

the groups on both ends vanishing. This implies that i^* is an isomorphism, and $\bigoplus_0^p\alpha_j$ generate $\Pi^{\infty-1}(\overline{B}-\bigcup_0^p B_j)$.

(e) k^* *Onto*. Choose any element x_0 of E, and decompose E as $Y_2\oplus\mathbb{C}x_0$, so that any x in E can be written as $x=y_2\oplus z$, $z\in\mathbb{C}$. Then $(y_2,(\lambda-\lambda_j)z,\|x\|^2-r^2)$, restricted to S_j, represents α_j and is inessential on S_i, $i\neq j$ (the map is nonzero on $\overline{B}_i$). Moreover, it also represents the generator α of $\Pi^{\infty-1}(S)$, for it is just the suspension of the Hopf map. Hence $k^*(\alpha_j)=\alpha$ for all j. k^* being a homomorphism, $k^*(\bigoplus_0^p m_j\alpha_j)=(\sum_0^p m_j)\alpha$ [**8**]. But $k^*[\widetilde{H}_r|_{\overline{B}-\bigcup_0^p B_j}]=[\widetilde{H}_r|_S]=0$. Hence $\sum_0^p m_j$ is even.

□

Generalization to the Case $(I-T(\lambda))x-g(x,\lambda)=0$. Here $T(\lambda)$ is a compact operator from $E\times K$ into E, $K=\mathbb{R}$ or $\mathbb{C}$, and $T(\lambda)$ is analytic in λ. We may obtain exactly the same results as in the case where $T(\lambda)=\lambda T$, but with a different notion of multiplicity. Namely, if $\ker(I-T(\lambda_0))\neq\emptyset$, then, setting $A=I-T(\lambda_0)$, $C(\lambda)=T(\lambda)-T(\lambda_0)$, the above equation can be written

(4.9) $$Ax-C(\lambda)x-g(x,\lambda)=0.$$

Here A is a Fredholm operator of index 0. Decompose E as $E=\ker A\oplus X_2$; then $A|_{X_2}$ has an inverse $K:R(A)\to X_2$. Write any element x of E as $x=x_1+x_2$, x_2 in X_2, and let Q be the projection on $R(A)$. Apply Q to (4.9): $Ax_2-QC(\lambda)x_2-$

$QC(\lambda)x_1 - Qg(x, \lambda) = 0$. Apply K; then $x_2 - KQC(\lambda)x_2 - KQC(\lambda)x_1 - KQg(x, \lambda) = 0$. Since $C(\lambda_0) = 0$, one has for λ near λ_0:

$$x_1 - (I - KQC(\lambda))^{-1}(KQC(\lambda)x_1 + KQg(x, \lambda)) = 0.$$

By the principle of contracting mappings, we may solve this for x_2 in terms of x_1 and λ. The other equation $(I - Q)C(\lambda)(x_1 + x_2) + (I - Q)g(x, \lambda) = 0$ is equivalent to

$$(I - Q)C(\lambda)(I + (I - KQC(\lambda))^{-1}KQC(\lambda))x_1$$
$$+ (I - Q)(I + C(\lambda)(I - KQC(\lambda))^{-1}KQ)g(x, \lambda) = 0,$$

i.e.,

$$(I - Q)C(\lambda)(I - KQC(\lambda))^{-1}x_1 + (I - Q)(I - C(\lambda)KQ)^{-1}g(x_1 + x_2(x_1, \lambda)) = 0.$$

Suppose there exists $\tilde{\lambda}$ such that $I - T(\tilde{\lambda})$ is invertible; then the set of points λ with $\ker(I - T(\lambda) = \emptyset$ is discrete. So if $B(\lambda)$ denotes $(I - Q)C(\lambda)(I - KQC(\lambda))^{-1}|_{\ker A}$, then $B(\lambda)$ is a matrix with entries analytic in λ, $B(\lambda_0) = 0$, and $\det(B(\lambda)) = (\lambda - \lambda_0)^m a(\lambda)$ with $a(\lambda_0) \neq 0$. (For $g = 0$, $B(\lambda)$ invertible means that λ_0 is isolated in the spectrum of $I - T(\lambda)$.) Then, if one defines the *algebraic multiplicity* of λ_0 to be m, the theorems on global solutions, in the real and the complex cases, remain valid.

CHAPTER 5

Monotone Operators and the Min-Max Theorem

In this chapter we give a very brief introduction to the theory of monotone operators and some related results. An excellent source of material is [**47**].

5.1. Monotone Operators in Hilbert Space

DEFINITION A mapping $f : X \to X$ of a Banach space is *nonexpansive* if

$$\|f(x) - f(y)\| \leq \|x - y\| .$$

DEFINITION Let H be a Hilbert space; a mapping $A : H \to H$ is *monotone* if

$$(Ax - Ay), x - y) \geq 0, \qquad \forall x, y \in H .$$

REMARK. A is monotone if and only if $I + \lambda A$ is expanding for all $\lambda > 0$. In fact, $\|(x + \lambda Ax) - (y + \lambda Ay)\| \geq \|x - y\|$ because $(x + \lambda Ax - y - \lambda Ay, x - y) \geq \|x - y\|^2$. The converse is similarly easy to verify.

We first state and prove an extension of the contraction mapping principle for nonexpansive maps in a Hilbert space.

THEOREM 5.1.1 *Let H be a Hilbert space and B a bounded, closed, convex subset of H. Let $f : B \to B$ be nonexpansive. Then f has a fixed point in B, and the set of fixed points is convex.*

Note that this theorem is not true in general Banach spaces. Indeed, let X be the space of bounded sequences of real numbers $x = (a_1, a_2, \dots)$ such that $|a_i| \to 0$ as $i \to \infty$. Let $\|x\| = \max_i |a_i|$; then X is a Banach space. Define a mapping f on the unit ball B in X by $f(x) = (1, a_1, a_2, \dots)$. If $y = (b_1, b_2, \dots) \in B$, then

$$\|f(x) - f(y)\| = \|(0, a_1 - b_1, \dots)\| = \|x - y\| ,$$

so $f : B \to B$ is nonexpansive. Now if $(a_1, \dots, a_n, \dots)$ is a fixed point in B, then $(a_1, a_2, a_3, \dots) = (1, a_1, a_2, \dots)$, but this implies $a_i = 1$ for all i; hence $(a_1, a_2, \dots) \notin X$ since $|a_i| \neq 0$.

The theorem *is* true in any uniformly convex Banach space (see [**47**]). It is an open problem whether it holds in any reflexive Banach space. The proof is based on the following lemma, which is a useful trick in many arguments dealing with monotone operators:

LEMMA 5.1.2 (Minty) *Let Ω be a convex subset of H and $A : \Omega \to H$ monotone and continuous on finite-dimensional subspaces. The following are equivalent for fixed $u \in \Omega$ and z in H:*

(i) $(Au - z, v - u) \geq 0$ *for all* $v \in \Omega$, *and*
(ii) $(Av - z, v - u) \geq 0$ *for all* $v \in \Omega$.

REMARK. Observe that if u is an interior point of Ω, then condition (i) means that $Au = z$.

PROOF OF LEMMA: It follows from the monotonicity of A that

$$(Au - z, v - u) - (Av - z, v - u) \leq 0,$$

so (i) $\Rightarrow$ (ii). Now for any $w \in \Omega$ and $0 \leq t \leq 1$, set $v = tu + (1 - t)w$ so that $v - u = (1 - t)(w - u)$. Suppose $(Av - z, w - u) \geq 0$; letting $t \to 1$ and using the continuity of A on line segments, we find $(Au - z, w - u) \geq 0$. □

PROOF OF THEOREM: We may suppose $0 \in B$. For $0 < \lambda < 1$, consider $\lambda f(x)$. By the contraction mapping principle the equation

$$\lambda f(x) = x \quad \text{has a unique solution; } x_\lambda \text{ in } B.$$

Let $A = I - f : B \to H$, then A is a monotone map; so is $A_\lambda = I - \lambda f$, $0 < \lambda < 1$, and $A_\lambda x_\lambda = 0$. Let $\lambda \to 1$ through a sequence and choose a weakly convergent subsequence, again denoted by x_λ, such that $x_\lambda - u \in B$. For any $v \in B$,

$$(A_\lambda v, v - x_\lambda) \geq (A_\lambda x_\lambda, v - x_\lambda) = 0.$$

Hence $(Av, v - u) \geq 0$. Using the lemma with $z = 0$, we find

$$(Au, v - u) \geq 0 \quad \text{for all } v \in B.$$

So

$$(u - f(u), v - u) \geq 0 \quad \text{for every } v \in B.$$

Setting $v = f(u)$, we have

$$(u - f(u), f(u) - u) \geq 0 \Rightarrow u = f(u).$$

So u is a fixed point of f.

We have just proved that for $u \in B$,

$$Au = 0 \to (Av, v - u) \geq 0 \quad \text{for all } v \in B.$$

It follows that the set of solutions of $Au = 0$ is convex. □

EXERCISE Prove the convexity in the above theorem directly.

EXERCISE Suppose $f(x)$ is a real smooth function defined in some open set in a real Hilbert space. Since $f_x(x)$ is a continuous linear functional, there is a $z(x)$ in H such that $f_x(x)y = (y, z(x))$. Show that $z(x)$ is monotone if and only if f is a convex functional.

In what follows, we want to study solutions of $Ax = 0$ in a Hilbert space when A is a monotone operator. In view of the last exercise, if A is the gradient of a convex functional, this is related to the variational problem of minimizing convex functionals. In this connection we recall the following well-known result:

THEOREM 5.1.3 *Let X be a reflexive Banach space and K a closed convex subset of X. Suppose f is a real convex functional on K, lower-semicontinuous and bounded below on K. Suppose $f(x) \to \infty$ as $\|x\| \to \infty$ uniformly; then f achieves its minimum on K. (f is lower-semicontinuous at x_0 means that if $x_i \to x_0$, then $\underline{\lim}\, f(x_i) \geq f(x_0)$, or, equivalently, for any constant c, the set $\{x \mid f(x) > c\}$ is open.)*

The proof of this theorem is based on two results, the first of which shows how compactness can be used:

PROPOSITION 5.1.4 (Eberlein-Smulyan) *A Banach space X is reflexive if and only if every closed, bounded convex set K is compact in the weak topology.*

PROPOSITION 5.1.5 (Mazur) *If $x_n - x_0$ weakly, then there is a sequence of convex combinations*

$$y_n = \sum_{j=1}^{n} \alpha_{nj} x_j \text{ of } x_j' s \quad \text{with} \sum_{j=1}^{n} \alpha_{nj} = 1, \alpha_j \geq 0$$

such that $y_n \to x_0$ strongly.

The proofs of both these theorems may be found in almost any book on functional analysis.

PROOF OF THEOREM 5.1.3: Suppose $d = \inf_{x \in K} f(x)$. Let x_i be a minimizing sequence, i.e., $f(x_i) \to d$, $f(x_i) \geq d$ for each i. The norms $\|x_i\|$ are bounded, so x_i has a weakly convergent subsequence, again denoted by x_i, such that $x_i - x$ weakly, $x \in K$. We must prove that $f(x) = d$; clearly, $f(x) \geq d$. Now for any $\varepsilon > 0$, $f(x_i) \leq d + \varepsilon$ for i sufficiently large.

Let y_i be a sequence of convex combinations of x_i such that $y_i \to x$ strongly. Since f is convex,

$$f(y_i) \leq d + \varepsilon\,.$$

By the lower-semicontinuity of f,

$$f(x) \leq d + \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, $f(x) \leq d$. Thus $f(x) = d$. □

EXERCISE If x_0 is a minimum point of a smooth convex functional $f(x)$ on a closed, bounded convex set C in a Hilbert space, show that $(A(x_0), y - x_0) \geq 0$ for all y in C. Here $A(x) = f_x(x)$.

We shall derive a similar result for any monotone operator $A(x)$.

THEOREM 5.1.6 *Suppose B is the closed unit ball in a real Hilbert space H and $A : B \to H$ is a monotone operator that is continuous on finite-dimensional subspaces. Then*

(i) *There is a point x_0 in B satisfying*

(5.1) $$(Ax_0, y - x_0) \geq 0 \quad \textit{for all } y \textit{ in } B\,.$$

Furthermore, the set of such points is convex.

(ii) *If, in addition, for every $x \in \partial B$, Ax never points opposite to x, i.e.,*

$$x + \lambda Ax \neq 0 \quad \text{for all } \lambda \geq 0, \; \|x\| = 1,$$

then $Ax_0 = 0$.

PROOF: (1) We note first that (ii) follows easily from (i), for if x_0 is an interior point of B, then clearly $A(x_0) = 0$, while if $x_0 \in \partial B$, then, if $A(x_0) \neq 0$, $A(x_0)$ points opposite to x_0.

(2) Next we show that it suffices to prove (i) in finite dimensions. Suppose we know the result in that case. For any $y \in B$, let $S(y)$ be the closed convex set

$$S(y) = \{x \in B \mid (Ay, y - x) \geq 0\}.$$

We claim that the sets $S(y)$ for $y \in B$ have the finite intersection property. Indeed, if $y_1, \ldots, y_k \in B$, let E be a finite-dimensional subspace containing these points. By the finite-dimensional result there exists $x \in E \cap B$ such that

$$(Ax, y - x) \geq 0 \quad \text{for all } y \in E \cap B.$$

Since A is monotone, it follows that

$$(Ay, y - x) \geq (Ax, y - x) \geq 0 \quad \text{for all } y \in E \cap B;$$

i.e., the claim holds. Now the sets $S(y)$ are compact in the weak topology and it follows that they have nonempty intersection; i.e., there exists $x_0 \in B$ such that

$$(Ay, y - x_0) \geq 0 \quad \text{for all } y \in B.$$

By Lemma 5.1.2 it follows that $(Ax_0, y - x_0) \geq 0$ for all $y \in B$.

That the set of all solutions of (5.1) is convex follows from the fact that the set is also the set of solutions x_0 of $(Ay, y - x_0) \geq 0$ for all $y \in B$.

(3) We now prove (i) in case E is finite-dimensional. If (i) is not true, then for every $x \in \partial B$, Ax does not point opposite to x. Then, according to the result in 1.6.1, $Ax = 0$ has a solution inside B.

□

REMARK. The proof of the theorem, and hence the theorem itself, holds if the assumption that A is monotone is replaced by a weaker assumption:

(5.2) For every pair $x, y \in B$, if $(Ax, x - y) \leq 0$, then $(Ay, y - x) \geq 0$.

No application of this more general result is known. In case $H = R$ and $A : R \to \mathbb{R}$, the condition (5.2) means simply that if $A(x_0) = 0$ for some x_0, then $A(x) \leq 0$ for $x \leq x_0$ and $A(x) \geq 0$ for $x \geq x_0$.

COROLLARY 5.1.7 *Suppose H is a real Hilbert space and $A : H \to H$ satisfies*

(i) *A is monotone,*

(ii) *A is continuous on finite-dimensional subspaces, and*

(iii) $\frac{(Ax, x)}{\|x\|} \to \infty$ *as* $\|x\| \to \infty$ *uniformly.*

Then A maps H onto H.

PROOF: Since $Ax - y$ is also monotone and satisfies (iii), it suffices to solve $Ax = 0$. Applying Theorem 5.1.6(ii) in a big ball about the origin, we obtain a solution of $Ax = 0$. □

A stronger form of the above corollary is the following:

COROLLARY 5.1.8 *Suppose $A : H \to H$ is*

(i) *monotone,*
(ii) *continuous on finite-dimensional subspaces, and*
(iii)′ $\|Ax\| \to \infty$ *as* $\|x\| \to \infty$ *uniformly.*

Then A maps H onto H.

PROOF: For $\varepsilon > 0$, the operator $A_\varepsilon = A + \varepsilon I$ is monotone and

$$\frac{(A_\varepsilon x, x)}{\|x\|} = \varepsilon\|x\| = \frac{(Ax, x)}{\|x\|} \geq \varepsilon\|x\| + \frac{(A(0), x)}{\|x\|}\,.$$

Since $(A(0), x)/\|x\|$ is bounded, the right-hand side tends to ∞ as $\|x\| \to \infty$. By the corollary above, A_ε maps H onto H. Let x_ε be a solution of $A_\varepsilon x = y \in H$.

We claim $\|x_\varepsilon\|$ is bounded uniformly for all $\varepsilon > 0$. Indeed,

$$\frac{(y, x_\varepsilon)}{\|x_\varepsilon\|} = \frac{(A_\varepsilon x_\varepsilon, x_\varepsilon)}{\|x_\varepsilon\|} \geq \varepsilon\|x_\varepsilon\| - \|A(0)\|\,,$$

so $\varepsilon\|x_\varepsilon\| \leq \|A(0)\| + \|y\| = K$, and K is independent of ε. Since $Ax_\varepsilon + \varepsilon x_\varepsilon = y$, we have $\|Ax_\varepsilon\| \leq \|\varepsilon x_\varepsilon\| + \|y\| =$ constant independent of ε. By (iii)′, this implies $\|x_\varepsilon\| \leq$ constant independent of ε.

Now let $\varepsilon \to 0$ through a sequence; then x_ε has a weakly convergent subsequence, again denoted by x_ε, which converges weakly to $x \in H$. Since

$$Ax_\varepsilon + \varepsilon x_\varepsilon = y\,, \quad Ax_\varepsilon \to y\,,$$

by monotonicity,

$$(Ax_\varepsilon - Av, x_\varepsilon - v) \geq 0 \quad \text{for all } v \in H\,.$$

Letting $\varepsilon \to 0$, we find

$$(y - Av, x - v) \geq 0 \quad \text{for all } v \in H\,.$$

By Lemma 5.1.2, we infer that

$$(y - Ax, x - v) \geq 0 \quad \text{for all } v\,.$$

As before, since v can take any direction, this implies

$$y - Ax = 0\,.$$

□

Some Open Problems.

(1) Suppose T is a continuous map $H \to H$ which is expanding, i.e., $\|Tx - Ty\| \geq \|x - y\|$ and $T(0) = 0$. Suppose T maps a neighborhood of the origin onto a neighborhood of the origin. Does T map H onto H?

(2) (R. Bott) Suppose B is the closed unit ball in a Hilbert space H and $A : B \to H$ such that for some positive number $\theta < 1$,

$$(Ax, x) \geq -\theta \|Ax\| \, \|x\| \quad \text{when } \|x\| = 1$$

and that, instead of monotonicity, we have

$$(Ax - Ay, x - y) \geq -\theta \|Ax - Ay\| \, \|x - y\|$$

for all $x, y \in H$. Assuming A continuous on finite-dimensional subspaces, can we solve

$$Ax = 0 \quad \text{on } B?$$

5.2. Min-Max Theorem

Theorem 5.1.6, as well as the stronger form given in the subsequent remark, are very special cases of a rather general result related to the min-max theorem. We shall describe this result, which is taken from [**48**].

First we recall

THEOREM 5.2.1 (Von Neumann Min-Max Theorem) *(In form given by M. Shiffman.) For $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, let $E \subset X$, $F \subset Y$ be convex compact sets and $K : E \times F \to \mathbb{R}$ a function satisfying the following:*

(i) *For each y, $K(x, y)$ is a continuous convex function x.*

(ii) *For each x, $K(x, y)$ is a continuous concave function of y. Then there exists an $(x_0, y_0) \in E \times F$ such that $K(x_0, y_0)$ is minimum with respect to $x \in E$ and maximum with respect to y in F, i.e.,*

$$K(x_0, y) \leq K(x_0, y_0) \leq K(x, y_0)\,.$$

REMARKS. (1) This conclusion of the above theorem is equivalent to

$$\max_y(\min_x K(x, y)) = \min_x(\max_y K(x, y))\,.$$

(2) Convexity and concavity can be replaced by quasi convexity and quasi concavity, respectively.

DEFINITION A real function $\phi(x)$ defined on a convex set is *quasi-convex* if for every real constant c, the set $\{x \mid \phi(x) < c\}$ is convex. $\phi(x)$ is *quasi-concave* if $-\phi(x)$ is quasi-convex.

Remark 1 is easily verified; suppose

$$\max_y \min_x K(x, y) = \min_x \max_y K(x, y)\,.$$

Let x_0, y_0 be points in E, F such that

$$\max_y K(x_0, y) = \min_x \max_y K(x, y) = \alpha = \max_y \min_x K(x, y) = \min_x K(x, y_0)\,.$$

Then

$$K(x_0, y) \le \alpha \le K(x, y_0),$$

and consequently $\alpha = K(x_0, y_0)$.

EXERCISE Prove the other part of Remark 1.

We now describe a much more general form.

THEOREM 5.2.2 (Generalized Ky Fan Min-Max Theorem) *Let F be a Hausdorff topological vector space and G a vector space. Let $A \subset F$, $B \subset G$ be convex sets and $K(u, v)$ a real function defined on $A \times B$ satisfying the following:*

(i) *For each $v \in B$, $K(u, v)$ is quasi-convex in u and lower-semicontinuous in A.*
(ii) *For each $u \in A$, $-K(u, v)$ is quasi-convex in v and lower-semicontinuous on the intersection of B with any finite-dimensional space.*
(iii) *For some $\tilde{v} \in B$ and some*

$$\lambda > \sup_{v \in B} \inf_{u \in A} K(u, v) \equiv \alpha,$$

the set $\{u \in A \mid K(u, \tilde{v}) \le \lambda\}$ is compact. Then

$$\alpha \equiv \sup_v \inf_u K(u, v) = \inf_u \sup_v (K(u, v)) \equiv \beta.$$

Question. Can (i) be replaced by the following: For each v, $K(u, v)$ is lower-semicontinuous in u on finite-dimensional subspaces?

Theorem 5.2.2 is proved with the aid of a result which is again a slight extension of a theorem of Ky Fan.

THEOREM 5.2.3 *Let E be a Hausdorff topological vector space*[1] *and C a convex set in E. Let $f(x, y)$ be a real function defined on $C \times C$ satisfying the following:*

(i) $f(x, x) \le 0$.
(ii) *For every $x \in C$, the set*

$$\{y \in C \mid f(x, y) > 0\} \quad \text{is convex}.$$

(iii) *For every $y \in C$, $f(x, y)$ is lower-semicontinuous in x on the intersection of C with finite-dimensional subspaces.*
(iv) *Whenever $x, y \in C$ and x is in the closure of a set G such that*

$$f(z, (1-t)x + ty) \le 0 \quad \text{for } 0 \le t \le 1$$

and all $z \in G$, then $f(x, y) \le 0$.
(v) *There is a compact subset L of E and a y_0 in $L \cap C$ such that $f(x, y_0) > 0$ for $x \in C, x \notin L$.*

[1] In applications we often take E to be a reflexive Banach space with its weak topology. Then any closed (in the norm topology), bounded convex set in E is compact in the weak topology.

Conclusion. There exists $x_0 \in L \cap C$ such that

$$f(x_0, y) \le 0 \quad \text{for all } y \in C .$$

The proof will be given below.

REMARK. Ky Fan assumed C to be compact and f lower-semicontinuous in x on all of C.

Applications.

(1) Suppose E and C are as above and f is defined on $C \times C$ satisfying (i), (iii), (v), and, in addition:
   (a) For every $x \in C$ and every $k \ge 0$, the set

$$\{y \in C \mid f(x, y) \ge K\} \quad \text{is closed and convex.}$$

   (b) For every $x, y \in C$ if $f(x, y) \le 0$, then $f(y, x) \ge 0$.
   (c) If $f(x, y_1) > f(x, y_2) \ge 0$, then

$$f(x, ty_1 + (1-t)y_2) > f(x, y_2) \quad \text{for } 0 < t \le 1 .$$

Conclusion. There exists $x_0 \in L \cap C$ such that $f(x_0, y) \le 0$ for all y.

PROOF: We have to verify that f satisfies the conditions of Theorem 5.2.3. Condition (a) implies condition (ii) of Theorem 5.2.3. So we have only to verify condition (iv). Suppose then x, y, and G are as in condition (iv) but $f(x, y) > 0$. By (b) we have

$$f((1-t)x + ty, z) \ge 0 \quad \text{for } 0 \le t \le 1 \text{ and all } z \in G ,$$

and by (a) it follows that

$$f((1-t)x + ty, x) \ge 0 \quad \text{for } 0 \le t \le 1 . \tag{5.3}$$

Since $f(v, y)$ is lower-semicontinuous as $v = (1-t)x + ty$ moves on the line between x and y, we see that for small positive t, $f(v, y) > 0$. From (5.3) we have $f(v, x) \ge 0$. Thus, from (a) it follows that $f(v, v) > 0$, a contradiction. For if $f(v, x) = f(v, y)$, this follows from (a), while if $f(v, x) \ne f(v, y)$, it follows from (c).

As an application of this result, we may derive the following generalization of Theorem 5.1.6 and the succeeding remark. □

(2) Suppose E and C are as above with C compact. Let A be a mapping of C into E^*, the dual of E, satisfying the following:
   (a) For x, y in C, whenever $(Ax, x - y) \le 0$, then $(Ay, y - x) \ge 0$.
   (b) A is continuous on finite-dimensional linear subspaces.
   Then there exists $x_0 \in C$ such that

$$(Ax_0, x_0 - y) \le 0 \quad \text{for all } y \in C .$$

PROOF: Set $f(x, y) = (Ax, x - y)$. Since f is linear in y, all the conditions of (1) are easily checked. □

The proof of Theorem 5.2.3 is based on an infinite-dimensional version of the lemma of Knaster, Kuratowski, and Mazurkiewicz of Section 1.6.4. (The version given here is a slight extension of Ky Fan's.)

LEMMA 5.2.4 *Let E be a Hausdorff topological vector space and X an arbitrary subset of E. To each $x \in X$, let a set $F(x)$ in E be assigned to satisfy the following:*

(i) *$\overline{F(x_0)} = L$ is compact for some $x_0 \in X$.*
(ii) *The convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_1^n F(x_i)$.*
(iii) *For every $x \in X$, the intersection of $F(x)$ with any finite-dimensional subspace is closed.*
(iv) *For every convex subset D of E we have*

$$\left[\overline{\bigcap_{x \in X \cap D} F(x)}\right] \cap D = \left[\bigcap_{x \in X \cap D} F(x)\right] \cap D .$$

Conclusion.

$$\bigcap_{x \in X} F(x) \neq \emptyset .$$

REMARK. Assumptions (iii) and (iv) clearly hold if $F(x)$ is closed for every $x \in X$; this is the case treated by Ky Fan.

PROOF: The result holds in finite dimensions; this is just the result of Section 1.6.4. We may assume $x_0 = 0$. Let $(E_i)_{i \in I}$ be the class of all finite-dimensional subspaces of E ordered by inclusion; i.e., $i \geq j$ means $E_j \subset E_i$. By the finite-dimensional result, it follows that for every $i \in I$ there is a $u_i \in L \cap E_i$ satisfying

$$u_i \in F(x) \quad \text{for all } x \in X \cap E_i .$$

Let $\Phi_1 = \bigcup_{j \geq i} \{u_j\}$, so $u \in F(z)$ for $u \in \Phi_i$ and $z \in X \cap E_i$; hence $\Phi_i \subset \bigcap_{z \in X \cap E_i} F(z)$.

Suppose $\tilde{x} \in \bigcap_{i \in I} \overline{\Phi_i}$, which is not empty by the compactness of L, and let i_0 be such that $\tilde{x} \in E_{i_0}$. For any $x \in X$, we can find $i \geq i_0$ and $x \in E_i$. We have, therefore,

$$\tilde{x} \in \overline{\Phi_i} \cap E_i \subset \left[\overline{\bigcap_{z \in X \cap E_i} F(x)}\right] \cap E_i = \left[\bigcap_{z \in X \cap E_i} F(z)\right] \cap E_i$$

by (iv). Therefore $\tilde{x} \in F(x)$ and consequently $\tilde{x} \in \bigcap_{x \in X} F(x)$. □

PROOF OF THEOREM 5.2.3: We shall apply the preceding to the assignment: For each $y \in C$, let

$$F(y) = \{x \in C \mid f(x, y) \leq 0\} .$$

The conclusion of Theorem 5.2.3 is equivalent to the assertion $\bigcap_{y \in C} F(y) \neq \emptyset$. Properties (i), (iii), and (iv) of Lemma 5.2.4 follow from (v), (iii), and (iv) of Theorem 5.2.3, respectively.

We have only to prove (ii); suppose that (ii) does not hold. Then for some choice of y_i and $\alpha_i \geq 0$, $1 \leq i \leq n$, with $\sum \alpha_i = 1$, we have

$$\sum \alpha_i y_i \notin \bigcup_1^n F(y_i), \qquad \text{i.e.,} \quad f\left(\sum_1^n \alpha_i y_i, y_j\right) > 0 \quad \text{for } 1 \leq j \leq n .$$

By (ii) of Theorem 5.2.3, it follows that $f(\sum_1^n \alpha_i y_i, \sum_1^n \alpha_j y_j) > 0$, contradicting Theorem 5.2.3(i). □

We omit the proof of the generalized min-max theorem, Theorem 5.2.2. It is in [**48**], referred to at the beginning of this section.

5.3. Dense Single-Valuedness of Monotone Operators

Lecture of N. Bitzenhofer

We have discussed single-valued monotone operators, but for certain problems, it is important to consider set-valued maps T. This section is a report on [**53**], showing that a monotone set-valued map is in fact single-valued at most points. Some related references are [**51, 52**]. We will be concerned with multi-valued monotone mappings of a separable Banach space X to its dual X^*, i.e., $T : X \to \mathcal{P}(X^*) \equiv 2^{X^*}$, the power set of X^*. If $x \in X$, $x^* \in X^*$, we denote the pairing $x^*(x)$ by $\langle x^*, x\rangle$.

DEFINITION 5.3.1 A set $M \subset X \times X^*$ is *monotone* if for all pairs (x_1, x_1^*), (x_2, x_2^*) in M, we have $\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0$. M is *maximal monotone* if M is not properly contained in any other monotone set. The set-valued map $T : X \to \mathcal{P}(X^*)$ is then *monotone* if its graph is monotone, i.e., for any $x, y \in X$ and any choice of Tx, $Ty \in X^*$, we have $\langle Tx - Ty, x - y\rangle \geq 0$. T is then *maximal monotone* if its graphs are maximal monotone. Note that T is not assumed to be defined on all of X; its domain of definition is denoted by $\mathcal{D}(T)$.

To prove our main result we will need the following definition and lemma (see [**6**]):

DEFINITION Let X be a locally convex, real Hausdorff topological vector space. Then a monotone operator $T : X \to \mathcal{P}X^*$ is *locally bounded* at $x \in X$ if x has a neighborhood U such that $T(U) \subset X^*$ is an equicontinuous set. Note that for X a Banach space, the equicontinuous sets are just the bounded sets.

LEMMA 5.3.2 *If X is a Banach space, $T : X \to \mathcal{P}X^*$ is maximal monotone, and* int $\mathcal{D}(T) \neq \emptyset$, *then*

(i) int $\mathcal{D}(T)$ *is convex,*
(ii) $\overline{\text{int}\, \mathcal{D}(T)} = \overline{\mathcal{D}(T)}$, *and*
(iii) T *is locally bounded at each point of* int $\mathcal{D}(T)$.

A particularly simple proof of (iii) can be found in [**50**].

Our principal result is the following:

THEOREM 5.3.3 *Let X be a separable Banach space, $T : X \to \mathcal{P}(X^*)$ monotone. Then the set of points Z where T is not single-valued has empty interior. If $\operatorname{int} \mathcal{D}(T) \neq \emptyset$, Z is an F_σ-set.*[2] *If X is finite-dimensional, Z has Lebesgue measure zero.*

PROOF: We assume $\operatorname{int} \mathcal{D}(T) \neq \emptyset$, or else there is nothing to prove. The theorem clearly holds for T if it holds for any extension of T, and, since any monotone operator has a maximal extension, we may assume without loss of generality that T is maximal monotone. By Lemma 5.3.2, $\operatorname{int} \mathcal{D}(T)$ is an open convex set whose closure contains $\mathcal{D}(T)$, and T is locally bounded at each point of $\operatorname{int} \mathcal{D}(T)$. In particular, the image Tx for $x \in \operatorname{int} \mathcal{D}(T)$ is a bounded set of functionals in X^*. By the maximality of T, this set is also closed and convex, for suppose $x_1^*, x_2^* \in Tx$, $0 < \alpha < 1$. Then for any $y \in \operatorname{int} \mathcal{D}(T)$ and $y^* \in Ty$,

$$\begin{aligned}
&\langle \alpha x_1^* + (1-\alpha)x_2^* - y^*, x - y \rangle \\
&\quad = \langle \alpha x_1^* + (1-\alpha)x_2^* - \alpha y^* - (1-\alpha)y^*, x - y \rangle \\
&\quad = \alpha \langle x_1^* - y^*, x - y \rangle + (1-\alpha)\langle x_2^* - y^*, x - y \rangle \\
&\quad \geq 0 .
\end{aligned}$$

Thus, $\alpha x_1^* + (1-\alpha)x_2^*$ must be in Tx too.

To prove the theorem, we must show that

$$Z = \{x \in \operatorname{int} \mathcal{D}(T) \mid Tx \text{ is not a singleton}\}$$

has empty interior. To do this, we consider the real-valued function $k(x, u) = \sup_{x^* \in Tx} \langle x^*, u \rangle$, $x \in \mathcal{D}(T)$, $u \in X$.

Claim 1. For $x \in \operatorname{int} \mathcal{D}(T)$, $k(x, u)$ is finite. This follows from the local boundedness of T.

Claim 2. For fixed $u \in X$, $k(x, u)$ is an upper-semicontinuous function of x on $\operatorname{int} \mathcal{D}(T)$: We must show that for $x \in \operatorname{int} \mathcal{D}(T)$,

$$k(x, u) \geq \overline{\lim_{y \to x}} k(y, u) .$$

So let $\{x_n\}_{n=1}^\infty$ be a sequence in $\operatorname{int} \mathcal{D}(T)$ converging to x, and pick a sequence $\{x_n^*\}_{n=1}^\infty$, with $x_n^* \in Tx_n$, so that $\langle x_n^*, u \rangle \to \overline{\lim}_{y \to x} k(y, u)$. By the local boundedness of T, we can assume $\{x_n^*\}_{n=1}$ is bounded in X^*, so that by conditional weak* compactness of bounded sets in X^*, we can extract a subsequence $\{x_{n_i}^*\}$ with $\langle x_{n_i}^*, u \rangle \to \langle x^*, u \rangle$ for some $x^* \in X^*$. By the maximality of T, $x^* \in Tx$, and so

$$k(x, u) \geq \langle x^*, u \rangle = \lim_{i \to \infty} \langle x_{n_i}^*, u \rangle = \lim_{n \to \infty} \langle x_n^*, u \rangle = \overline{\lim_{y \to x}} k(y. u) .$$

Thus claim 2 is established.

Now we need some inequalities based on the behavior of $k(x. u)$ along lines parallel to $u \in X$. If $x \in \operatorname{int} \mathcal{D}(T)$, the line $\{x + tu\}_{t=-\infty}^\infty$ intersects $\operatorname{int} \mathcal{D}(T)$ in an open segment. We will show that $k(x, u)$ is monotone increasing in t along such a line.

---

[2]That is, Z is the union of a denumerable number of closed sets, all without interiors.

Let $s < t$ be two real numbers such that $x + su$ and $x + tu$ are in $\operatorname{int} \mathcal{D}(T)$. Then if $x_t^* \in T(x + tu)$, $x_s^* \in T(x + su)$, we have

$$\langle x_t^*. u\rangle - \langle x_s^*. u\rangle = \frac{1}{t-s}\langle x_t^* - x_s^*. (x + tu) - (x + su)\rangle \geq 0.$$

by monotonicity. In particular, for any $x_t^* \in T(x + tu)$, $x_s^* \in T(x + su)$, we have $\langle x_t^*, u\rangle \geq \langle x_s^*. u\rangle$, so that

$$\inf_{x_t^* \in T(x+tu)} \langle x_t^* u.\rangle \geq \sup_{x_s^* \in T(x+su)} \langle x_s^*. u\rangle = k(x + su, u). \tag{5.4}$$

and the monotonicity follows.

Furthermore,

$$\begin{aligned} k(x + tu, u) = \sup_{x_t^* \in T(x+tu)} \langle x_t^*, u\rangle &\geq \inf_{x_t^* \in T(x+tu)} \langle x_t^*, u\rangle \\ &= - \sup_{x_t^* \in T(x+tu)} \langle x_t^*, -u\rangle = -k(x + tu, u) \\ &\geq \sup_{x_s^* \in T(x+su)} \langle x_s^*, u\rangle \quad (\text{by } (5.4) = k(x + su, u)). \end{aligned}$$

Hence

$$0 \leq k(x + tu, u) + k(x + tu, -u) \leq k(x + tu, u) - k(x + su, u)$$

and, letting $s \uparrow t$,

$$0 \leq k(x + tu, u) + k(x + tu, -u) \leq k(x + tu, u) - \lim_{s \uparrow t} k(x + su, u). \tag{5.5}$$

Now consider the quantity

$$k(x, u) + k(x, -u) = \sup_{x^* \in Tx} \langle x^*, u\rangle - \inf_{x^* \in Tx} \langle x^*, u\rangle,$$

and let $\{u_n\} \subset X$ be a sequence such that if $x^* \in X^*$and $\langle x^*, u_n\rangle = 0\ \forall n$, then $x^* \equiv 0$ (recall X is separable). It is easily seen that for $x \in \operatorname{int} \mathcal{D}(T)$,

$$Tx \text{ is not a singleton iff for some } n.\ k(x, u_n) + k(x, -u_n) > 0.$$

Thus, if we set

$$Z_n = \{x \in \operatorname{int} \mathcal{D}(T) \mid k(x, u_n) + k(x, -u_n) > 0\},$$

we have

$$Z = \bigcup_{n=1}^{\infty} Z_n.$$

From (5.5) we see that any point in Z_n that is also on the line $\{x + tu_n\}_{t=-\infty}^{\infty}$ is associated with a jump in the nondecreasing function $k(x + tu_n, u_n)$, and consequently Z_n intersects any line parallel to u_n in at most a countable number of points. Therefore $\operatorname{int} Z_n = \emptyset$. In the finite-dimensional case, we can immediately conclude from Fubini's theorem that each Z_n, and hence Z, has Lebesgue measure zero.

We still must show $\operatorname{int} Z = \emptyset$, which entails a Baire category argument. Let $Z_{n,m} = \{x \in \operatorname{int} \mathcal{D}(T) \mid k(x, u_n) + k(x, -u_n) \geq \frac{1}{m}\}$, $m \in \mathbb{Z}^+$; then $Z_n = \bigcup_{m=1}^{\infty} Z_{n,m}$, and by a characterization of upper-semicontinuity, each $Z_{n,m}$ is closed.

If int $Z \neq \emptyset$, we could find a closed nonempty sphere B in Z, and then $B = \bigcup_{n=1,m=1}^{\infty}(B \cap Z_{n,m})$. B is of the second category and each $B \cap Z_{n,m}$ is closed, so $Z_{n,m}$, and hence also Z_n, would have to have nonempty interior, which is a contradiction. Thus int $Z = \emptyset$, and from $Z = \bigcup_{n=1,m=1}^{\infty} Z_{n,m}$ we set that Z is an F_σ-set. □

COROLLARY 5.3.4 *If $\{T_n\}$ is a sequence of set-valued monotone operators defined on an open set $A \subset X$, then the set of points where all the T_n's are simultaneously single-valued is dense in A.*

PROOF: Since the countable union of F_σ-sets is an F_σ-set, the same Baire category argument can be used here. □

Application to Subdifferential Maps. Again X is a real, separable Banach space, and X^* its dual.

DEFINITION A *proper convex function* on X is a convex function $f : X \to R \cup \{\infty\}$ that is not identically infinite. Let f be a proper convex function on X, and let $x \in X$. An element $x^* \in X^*$ is called a *subgradient* of f at x if for all $y \in X$, $f(y) \geq f(x) + \langle x^*, y-x\rangle$. The set of all subgradients x^* of f at x is denoted $\partial f(x)$, and the map of $\partial f : X \to X^*$ which sends $x \mapsto \partial f(x)$ is called the *subdifferential* of f. Note that ∂f is multivalued. For example, if f is convex on the open convex set $U \subset \mathbb{R}^n$ and if the gradient $\nabla f(\vec{x}_0)$ exists, we know

$$f(\vec{x}) - f(\vec{x}_0) \geq \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0) \quad \forall x \in U\,.$$

EXERCISE Prove that $\partial f : X \to \mathcal{P} X^*$ is a monotone operator.

As our first example, let $K \subset X$ be a nonempty, closed convex set. Define the *indicator function* of K as follows:

$$\delta = \delta_K(x) = \begin{cases} 0, & x \in K\,, \\ \infty, & x \notin K\,, \end{cases}$$

δ_K is a proper convex function on X. If $x \notin K$, the inequality

$$\delta(y) \geq \delta(x) + \langle x^*, y - x\rangle \quad \forall y \in X$$

is not satisfied for any $x^* \in X^*$, whereas if $x \in K$, the inequality reduces to

$$\langle x^*, y - x\rangle \leq 0 \quad \forall y \in K\,.$$

Thus, $x^* \in \partial\delta_K(x)$ iff

$$x \in K \quad \text{and} \quad \langle x^*, y - x\rangle \leq 0 \quad \forall y \in K\,.$$

Such an $x^* \in X^*$ is called a *normal* to K at x.

We can similarly reverse the map and assign to each $x^* \in X^*$ the *face* of K perpendicular to x^*:

$$F_K(x^*) = \left\{x \in K \mid \langle x^*, x\rangle = \sup_{y \in K}\langle x^*, y\rangle\right\}.$$

Then x^* is an outer normal to K at x, and since F_K is the (multivalued) inverse of $\partial\delta_K$, F_x is also monotone, for the graphs are the same. Then if X^* is separable, our theorem yields the following:

COROLLARY 5.3.5 *The set of normals to a closed convex set in a Banach space with separable dual which exists at more than one point, has empty interior, and in finite dimensions has measure* 0.

Now consider the Minkowski functional or support function of a closed convex set K,

$$p_K(x) = \inf\{a > 0 \mid a^{-1}x \in K\}.$$

Then applying our theorem to the subdifferential map ∂p_K gives the following: If a convex subset of a separable Banach space has nonempty interior, it has a unique tangent functional at each point of a dense subset of its boundary. This follows from the fact that if $f \in \partial p_K(x)$, then

$$p_K(y) - p_K(x) \geq f(y) - f(x) \quad \forall y \in K,$$

and consequently f satisfies the conditions necessary to be a tangent functional at x (see [**49**, chap. V.9]).

We derive one more corollary. Let $T : X \to \mathcal{P}X^*$ be monotone as above. Let L be a closed linear subspace of X, and let $L^\perp$ denote its annihilator on X^*. Then $X^*/L^\perp \simeq L^*$, and for arbitrary $x_0 \in X$ define $T_L : L \to X^*/L^\perp$ by $T_L(x + x_0) =$ the coset in $X^*/L^\perp$ containing $T(x + x_0)$; i.e., T_L is defined on the affine manifold $L+x_0$. T_L is monotone since T is monotone. Applying our theorem, one can derive the following:

COROLLARY 5.3.6 *Let $T : X \to X^*$ be a monotone operator, and let M be a separable affine manifold in X. Then the set of points in M where Tx is not orthogonal to M has no interior in M (if M is finite dimensional, the set has measure* 0).

Here orthogonality means that the difference of any two points in Tx annihilates the difference of any two points in M. That is, since T_L is essentially single-valued, two functionals in Tx will lie in the same coset of $X^*/M^\perp$, and hence their difference is in $M^\perp$.

CHAPTER 6

Generalized Implicit Function Theorems

Lecture and Notes by E. Zehnder

The classical implicit function theorem is concerned with the solvability of the equation

$$\mathcal{F}(f, u) = 0$$

where $\mathcal{F}$ is a smooth map of a neighborhood of (f_0, u_0) in $X \times Y$ into Z; X, Y, and Z are Banach spaces. Assuming

$$\mathcal{F}(f_0, u_0) = 0$$

and f close to f_0, we wish to solve $\mathcal{F}(f, u) = 0$ for $u(f)$. If $\mathcal{F}_u(f_0, u_0) = D_2\mathcal{F}(f_0, u_0)$ has a bounded inverse $\mathcal{F}_u(f_0, u_0)^{-1} : Z \to Y$, then there is a unique solution $u(f)$ with $u(f_0) = u_0$. We shall use the notation $L(Z, Y)$ to denote the space of bounded linear maps of a Banach space Z to a Banach space Y.

Since Nash's work [**58**], there has been great interest in such problems in situations where $D_2\mathcal{F}(f_0, u_0)^{-1}$ is unbounded. For example, $\mathcal{F}$ may act on functions f, u defined in a compact manifold, with $X = C^{\ell+u}$, $Y = C^{\ell+\beta}$, $Z = C^{\ell-\alpha}$ for every $\ell \geq \alpha$ (in the notation of Section 2.5). $(D_2\mathcal{F})^{-1}$ may exist but may lose derivatives, say $D_2\mathcal{F}^{-1} \in (C^{\ell-\alpha}, C^{\ell+\beta-\delta})$ for some $\delta > 0$. In such a case the classical implicit function theorem does not apply, and the usual Picard iteration scheme for solving the equation $\mathcal{F}(f, u) = 0$ does not work. In his important paper [**56**], J. Moser developed a general approach to such problems. This assumes the invertibility of $D_2\mathcal{F}(f, u)$ for (f, u) "near" (f_0, u_0) and replaces the usual Picard iteration scheme by a more rapidly convergent one (of the type of Newton's) which is used in conjunction with smoothing operators. J. Schwartz presents a form of this result in [**11**, chap. II]. A related technique, connected with earlier work of Kolmogorov and Arnold on small-divisor problems in mechanics, works with analytic approximation of the functions; a variety of applications may be found in the beautiful papers by Moser [**57**].

There are many way in which one can present the generalized implicit function theorems, usually called Nash-Moser implicit function theorems, depending on the application one has in mind. Often the difficulty occurs in showing that the conditions are satisfied in some particular problem. In this chapter we will present several forms of the method, some operating in a framework modeled after analytic functions, others using the analogue of C^∞ smoothing operators. In the last section we will present an application to the conjugacy problem of vector fields in a torus. In our treatment we will not assume that $D_2\mathcal{F}(f, u)$ has an inverse, but will make the weaker requirement that there exists an approximate inverse $\eta(f, u)$; η is to be such that, in terms of suitable norms,

$$|D_2\mathcal{F}(f, u) \cdot \eta(f, u) - I| \leq \text{const } |\mathcal{F}(fu, u)| .$$

Therefore, η is required to be a precise inverse only at points (f, u) that satisfy $\mathcal{F}(f, u) = 0$.

In these lectures we describe some of the results of E. Zehnder [**60**]. We will first take up the case of analytic functions, though placed in a more abstract setting.

6.1. C^ω Smoothing: The Analytic Case

We begin with the abstract setup. This is somewhat similar to the setup used in [**54, 55, 59**]. We consider three one-parameter families of Banach spaces X_σ, Y_σ, and Z_σ in the closed unit interval $0 \leq \sigma \leq 1$ such that for $0 \leq \sigma' \leq \sigma \leq 1$,

$$X_0 \supset X_{\sigma'} \supseteq X_\sigma \supseteq X_1 , \tag{6.1}$$

and with norms $|\ |_\sigma$ satisfying

$$|f|_{\sigma'} \leq |f|_\sigma \tag{6.2}$$

for all $f \in X_\sigma$ and $0 \leq \sigma' \leq \sigma$ (analogously for Y_σ and Z_σ).

An example of such spaces X_σ is the following:

Let $T_\sigma = \{\text{complex strip of points } x \in \mathbb{C}^n \mid |\text{Im } x_j| < \sigma,\ j = 1, \ldots, n\}$.

For an integer $m \geq 0$, set

> $A(\sigma, C^m) =$ the set of holomorphic functions on $T_\sigma u(x)$ which are real for real arguments (i.e., which satisfy $\overline{u(x)} = u(\bar{x})$) and which are periodic in each x_j of period 1.

Introduce norms

$$|u|_{\sigma, C^m} = \sup_{\substack{\text{strip} \\ |\alpha| \leq m}} |D^\alpha u(x)| ,$$

i.e., the sup of all derivatives of u up to order m in the strip. From the Cauchy integral formula, one has

$$|u|_{\sigma', C^m} \leq \text{const } |\sigma - \sigma'|^{\ell - m} |u|_{\sigma, C^\ell} \quad \text{for } \ell < m \text{ and } \sigma' < \sigma .$$

Set $X_\sigma = A(\sigma, C^m)$ for $\sigma > 0$ and $X_0 = C^m(\mathbb{T}^n)$ where $\mathbb{T}^n$ is the real torus (corresponding to period 1 in each x_j). Other examples will occur later.

Let $\mathcal{F}$ be a mapping defined in $X_0 \times Y_0$ and with range in Z_0 such that

$$\mathcal{F}(f_0, u_0) = 0 \tag{6.3}$$

for $(f_0, u_0) \in X_1 \times Y_1$ (the smallest spaces!). In order to define the domain of definition of $\mathcal{F}$, we introduce the open balls $B_\sigma \subset X_\sigma \times Y_\sigma$,

$$B_\sigma = \left\{(f, u) \in X_\sigma \times Y_\sigma \mid |f - f_0|_\sigma < N, |u - u_0|_\sigma < R\right\}$$

for some fixed $N > 0$ and $0 < R \leq 1$. Assume $\mathcal{F}$ is defined for $(f, u) \in B_0$ and $\mathcal{F}(B_\sigma) \subset Z_\sigma$ for all $0 \leq \sigma \leq 1$ with

$$\mathcal{F} : B_\sigma \to Z_\sigma \tag{6.4}$$

continuous for every $0 \leq \sigma \leq 1$. For given $(f, u) \in B_\sigma$, $\sigma > 0$, our aim is to solve the equation $\mathcal{F}(f, v) = 0$ for v close to u in some larger space $Y_{\sigma'}$, $\sigma' < \sigma$, assuming that $|\mathcal{F}(f, u)|_\sigma$ is sufficiently small. We make the following three assumptions, in which $M \geq 1$, $\gamma > 0$, and $\alpha \geq 0$ are fixed constants.

Hypotheses.

(H1) *Taylor Estimate.* For every $0 < \sigma \leq 1$ and every $f \in X_\sigma \cap B_\sigma$, the mapping $\mathcal{F}(f, \cdot)$ from $Y_\sigma \cap B_\sigma$ into $Z_{\sigma'}$, $\sigma' < \sigma$, is differentiable. Denote its Frechet derivative at $(f, u) \in B_\sigma$ by $d\mathcal{F}(f, u)$. For $(f, u), (f, v) \in B_\sigma$, the quantity

$$Q(f; u, v) \equiv \mathcal{F}(f, u) - \mathcal{F}(f, v) - d\mathcal{F}(f, v)(u - v)$$

satisfies

$$|Q(f; u, v)|_{\sigma'} \leq \frac{M}{(\sigma - \sigma')^{2\alpha}} |u - v|_\sigma^2 \quad \text{for all } \sigma' < \sigma .$$

(H2) *Uniform Lipschitz Condition in First Argument.* For every $0 < \sigma \leq 1$, if $(f, u), (g, u) \in B_\sigma$, then

$$|\mathcal{F}(f, u) - \mathcal{F}(g, u)|_\sigma \leq M |f - g|_\sigma .$$

(H3) *Approximate Right Inverse of Loss γ.* For every $0 < \sigma \leq 1$ and $(f, u) \in B_\sigma$, there is a linear map $\eta(f, u) \in L(Z_\sigma, Y_{\sigma'})$ for all $\sigma' < \sigma$ such that for all $z \in Z_\sigma$:

$$|\eta(f, u)(z)|_{\sigma'} \leq \frac{M}{(\sigma - \sigma')^\gamma} |z|_\sigma$$

and

$$|(d\mathcal{F}(f, u) \cdot \eta(f, u) - 1)(z)|_{\sigma'} \leq \frac{M}{(\sigma - \sigma')^{2(\alpha+\gamma)}} |\mathcal{F}(f, u)|_\sigma |z|_\sigma .$$

Actually, we will need these estimates only for $z = \mathcal{F}(f, u)$.

THEOREM 6.1.1 *Let $\mathcal{F}$ satisfy* (H1) *and* (H3). *Then there exists a constant $C > 0$, depending on M, α, and γ, such that if $(f, u) \in B_\sigma$, for some $\sigma > 0$, satisfies $|u - u_0|_\sigma \leq r < R$ and*

$$|\mathcal{F}(f, u)|_\sigma \leq C(R - r)\sigma^q$$

for some $q \geq 2(\alpha + \gamma)$, then there exists a $u_f \in Y_{\sigma/2} \cap B_{\sigma/2}$ such that

(i) $\mathcal{F}(f, u_f) = 0$ *and*

(ii) $|u_f - u|_{\sigma/2} \leq C^{-1} \cdot |\mathcal{F}(f, u)|_\sigma \cdot^{-\gamma}$.

PROOF: We shall use Newton's iteration method but with the approximate right inverse $\eta(f, u)$ of (H3) in place of the inverse of $d\mathcal{F}(f, u)$, which need not exist. We will define inductively a sequence (u_n), $n \geq 0$, which will converge in $Y_{\sigma/2}$ to a solution of $\mathcal{F}(f, u) = 0$. Starting with $u_0 = u$ (u as in the formulation of the theorem), we set for $n = 0, 1, \ldots,$

$$u_{n+1} = u_n - \eta(f, u_n)(\mathcal{F}(f, u_n)) . \tag{6.5}$$

To carry out the induction step below, we introduce beforehand a sequence $(\varepsilon_n)_{n\geq 0}$ of small numbers as follows:

(6.6) $$\varepsilon_{n+1} = a \cdot b^n \varepsilon_n^\kappa , \quad 1 < \kappa \leq 2 ,$$

where $a = M^3 2^{6(\alpha+\gamma)+2}$ and $b = 2^{2(\alpha+\gamma)}$. For ε_0 sufficiently small, this sequence converges exponentially to zero, for if $\delta_n = a^{(\kappa-1)^{-1}} b^{n(\kappa-1)^{-1}+(\kappa-1)^{-2}} \varepsilon_n$, then $\delta_{n+1} = \delta_n^\kappa$, hence $\delta_n = \delta_0^{(\kappa^n)}$, and we can write $\varepsilon_n = a^{-(\kappa-1)^{-1}} b^{-n(\kappa-1)^{-1}} - (\kappa-1)^{-2}\delta_0^{(\kappa^n)}$; ε_0 will be chosen sufficiently small during the proof. We shall make use of the following estimates:

(6.7) $$\varepsilon_n^2 \leq ab^n \varepsilon_n^2 \leq \varepsilon_{n+1} < 1 .$$

To label the spaces, we introduce for $\sigma > 0$ the sequences $(\sigma_n)_{n\geq 0}$ and $(\tau_n)_{n\geq 1}$, as $\sigma_n = \frac{\sigma}{2}(1 + 2^{-n})$ and $\tau_{n+1} = \frac{1}{2}(\sigma_{n+1} + \sigma_n)$ for $n = 0, 1, \ldots$. Note $\sigma_0 = \sigma$, $\lim \sigma_n = \frac{\sigma}{2}$ as $n \to \infty$, and $\sigma_{n+1} < \tau_{n+1} < \sigma_n$. Choosing $q \geq 2(\alpha + \gamma)$, we are going to prove that there is a constant $C > 0$ such that, if

$$|\mathcal{F}(f, u)|_\sigma \leq \nu(R - r)\sigma^q C$$

for some $0 \leq \nu \leq 1$, then the following statements S_n for the sequence $(u_n)_{n\geq 0}$, defined inductively by (6.4), hold for all $n \geq 0$:

(S_n1) $(f, u_n) \in B_{\sigma_n} |\mathcal{F}(f, u_n)|_{\sigma_n} \leq \nu(R - r)\sigma^q \varepsilon_n^4$,
(S_n2) $(f, u_{n+1}) \in B_{\tau_{n+1}} |u_{n+1} - u_n|_{\tau_{n+1}} \leq \nu(R - r)\sigma^{q-\gamma} \varepsilon_n^3$, and
(S_n3) $|u_{n+1} - u|_{\tau_{n+1}} \leq (R - r)(1 - \varepsilon_n)$.

The parameter ν has been introduced for the following reason: If $\mathcal{F}(f, u)|_\sigma < (R - r)C \cdot \sigma^q$, then there exists a $0 \leq \nu \leq 1$ such that $|\mathcal{F}(f, u)|_\sigma = \nu(R - r)C\sigma^q$, which then will allow us to estimate the solution in terms of $|\mathcal{F}(f, u)|_\sigma$. From ($S_n$1) one concludes that $\mathcal{F}(f, u_n) \to 0$ in $Z_{\sigma/2}$ as $n \to \infty$. By means of (S_n2), the sequence $(u_n)_{n\geq 0}$ is a Cauchy sequence $Y_{\sigma/2}$. Calling its limit $u_f = \lim u_n$ we conclude from the continuity of $\mathcal{F}$ that $\mathcal{F}(f, u_f) = 0$. The statements (S_n3) guarantee that we stay in the domain of definition of $\mathcal{F}$ and keep the induction going, namely,

$$|u_{n+1} - u_0|_{\tau_{n+1}} \leq |u_{n+1} - u|_{\tau_{n+1}} + |u - u_0|_{\tau_{n+1}} \leq (R - r)(1 - \varepsilon_n) + r < R .$$

The proof of the statements S_n is by induction. The statement S_0 follows from the smallness condition on $|\mathcal{F}(f, u)|_\sigma$, namely, $|\mathcal{F}(f, u)|_\sigma \leq \nu(R - r)C\sigma^q$, if $C \leq \varepsilon_0^4$. Here one uses the same estimates as in the induction step below. Assuming now the validity of S_j for $1 \leq j \leq n$, we shall prove the statement S_{n+1}. We know $(f, u_n), (f, u_{n+1}) \in B_{\tau_{n+1}} \subset B_{\sigma_{n+1}}$, and using the definition (6.5) of u_{n+1}, we can write:

(6.8) $$\mathcal{F}(f, u_{n+1}) = -(d\mathcal{F}(f, u_n) \circ \eta(f, u_n) - 1)(\mathcal{F}(f, u_n)) + Q(f, u_{n+1}, u_n) .$$

Using (H3) and (H1) we reach the following estimate, in which we do not indicate the dependence on f:

$$|\mathcal{F}(u_{n+1})|_{\sigma_{n+1}} \leq \frac{M}{(\sigma_n - \sigma_{n+1})^{2(\alpha+\gamma)}}|\mathcal{F}(u_n)|^2_{\sigma_n} \frac{M}{(\tau_{n+1} - \sigma_{n+1})^{2\alpha}}|\eta(u_n)(\mathcal{F}(u_n))|^2_{\tau_{n+1}}$$
$$\leq \left(\frac{M}{(\sigma_n - \sigma_{n+1})^{2(\alpha+\gamma)}} + \frac{M^3}{(\tau_{n+1} - \sigma_{n+1})^{2\alpha}(\sigma_n - \tau_{n+1})^{2\gamma}}\right)|\mathcal{F}(u_n)|^2_{\sigma_n}.$$

Inserting the definitions of the sequences (σ_n) and (τ_n), we get

$$|\mathcal{F}(u_{n+1})|_{\sigma_{n+1}} \leq \sigma^{-2(\alpha+\gamma)} ab^n \nu^2 (R-r)^2 \sigma^{2q} \varepsilon_n^8 .$$

Since $R, \nu, \sigma \leq 1$, and $q \geq 2(\alpha+\gamma)$, this can be estimated by $\nu(R-r)\sigma^q ab^n \varepsilon_n^8 \leq \nu(R-r)\sigma^q \varepsilon_{n+1}^4$, where we have used (6.7); hence we have proved $S_{n+1}1$. Calling $u_{n+2} - u_{n+1} \equiv v_{n+1}$, we get by means of (6.5) and (H3) the estimate

$$|v_{n+1}|_{\tau_{n+2}} \leq \frac{M}{(\sigma_{n+1} - \tau_{n+2})^{\gamma}}|\mathcal{F}(u_{n+1})|_{\sigma_{n+1}} .$$

Using $S_{n+1}1$ and (6.6) we find

$$|v_{n+1}|_{\tau_{n+2}} \leq \nu(R-r)\sigma^{q-\gamma} ab^n \varepsilon_{n+1}^4 < \nu(R-r)\sigma^{q-\gamma}\varepsilon_{n+1}^3 ,$$

which proves $S_{n+1}2$; that (f, u_{n+2}) is in $B_{\tau_{n+2}}$ follows easily from

$$|u_{n+2} - u|_{\tau_{n+2}} \leq |u_{n+1} - u|_{\tau_{n+1}} + |v_{n+1}|_{\tau_{n+2}}$$
$$\leq (R-r)(1 - \varepsilon_n + \varepsilon_{n+1}^3) < (R-r)(1 - \varepsilon_{n+1})$$

for ε_0 sufficiently small; here we have used S_n3. To prove estimate (ii) of the theorem, observe that from the statements S_n2, we conclude that for all $n \geq 1$,

$$|u_n - u|_{\sigma/2} \leq \sum_{n=0}^{\infty} |v_n|_{\sigma/2} \leq \nu \cdot (R-r)\sigma^{q-\gamma} \sum_{n\geq 0} \varepsilon_n^3 ,$$

which can be estimated by $\nu(R-r)\sigma^{q-\gamma}$, by choosing ε_0 so small that $\sum_{n\geq 0}\varepsilon_n^3 < 1$; therefore $|u_f - u|_{\sigma/2} \leq \nu \cdot (R-r)\sigma^{q-\gamma}$. Finally we choose $C = \varepsilon_0^4$; if now $|\mathcal{F}(f,u)|_\sigma < (R-r)C \cdot \sigma^q$, we take $\nu = |\mathcal{F}(f,u)|_\sigma \cdot C^{-1} \cdot (R-r)^{-1} \cdot \sigma^{-q}$ and find $|u_f - u|_{\sigma/2} \leq C^{-1}|\mathcal{F}(f,u)|_\sigma \cdot \sigma^{-\gamma}$. □

Observe that the approximate right inverse $\eta(f,u)$ of $d\mathcal{F}(f,u)$ is an exact right inverse for every solution of $\mathcal{F}(f,u) = 0$ in the following sense: If $(f,u) \in B_\sigma$ for $\sigma > 0$ is such a solution, then, since $\eta(f,u) \in L(Z_\sigma, Y_{\sigma'})$, $\sigma' < \sigma$, $d\mathcal{F}(f,u) \cdot \eta(f,u)$ mapping Z_σ into $Z_{\sigma'}$, $\sigma' < \sigma$, is the continuous injection $Z_{\sigma'} \hookrightarrow Z_\sigma$. We proceed by briefly discussing uniqueness, parameter dependence, and some modifications.

6.1.1. Uniqueness. Since we have no left inverse of $d\mathcal{F}(f,u)$, uniqueness of the solution cannot be expected. As a natural condition, which we show guarantees local uniqueness, we assume the existence of an approximate left inverse: For every

$\sigma > 0$ and $(f, u) \in B_\sigma$, there is a linear map $\xi(f, u) \in L(Z_\sigma, Y_{\sigma'})$ for all $\sigma' < \sigma$ such that for all $z \in Z_\sigma$, $\hat{v} \in Y_\sigma$,

$$|\xi(f, u)(z)|_{\sigma'} \leq \frac{M}{(\sigma - \sigma')^\gamma} |z|_\sigma \tag{6.9}$$

and

$$|(\xi(f, u) \circ d\mathcal{F}(f, u) - 1)\hat{v}|_{\sigma'} \leq \frac{M}{(\sigma - \sigma')^{2(\alpha+\gamma)}} |\mathcal{F}(f, u)|_\sigma |\hat{v}|_\sigma . \tag{6.10}$$

Let $\mathcal{F}$ satisfy (H1), and assume we have an approximate left inverse. Let (f, u), $(f, v) \in B_\sigma$, $\sigma > 0$, such that $\mathcal{F}(f, u) = \mathcal{F}(f, v) = 0$. If $|u - v|_\sigma \leq C \cdot \sigma^q$, C and q as in Theorem 6.1.1, then $u = v$ in $Y_{\sigma/2}$.

Indeed, denoting $u - v \equiv w \in Y_\sigma$, we get $d\mathcal{F}(f, u)w = Q(f; u, v)$ and therefore

$$|d\mathcal{F}(f, u)w|_{\sigma'} \leq \frac{M}{(s - \sigma')^{2\alpha}} |w|_s^2 \tag{6.11}$$

for all $\sigma' < s \leq \sigma$. On the other hand, from (6.10) we get in $Y_{\sigma'}$, $\sigma' < \sigma$, $\xi(f, u) \circ d\mathcal{F}(f, u)w = w$. This leads with (6.9) and (6.11) to the estimates, for all $\sigma' < s \leq \sigma$.

$$|w|_{\sigma'} \leq \frac{M^2 2^{2(\alpha+\gamma)}}{(s - \sigma')^{2(\alpha+\gamma)}} |w|_s^2 , \tag{6.12}$$

Inductively it then follows from (6.12) that if $|w|_\sigma \leq \sigma^q C$, the estimates $|w|_{\sigma_n} \leq \sigma^q \varepsilon_n^4$ hold, $\sigma_n = (\sigma/2)(1 + 2^{-n})$, and hence $w = 0$ in $Y_{\sigma/2}$.

6.1.2. Parameter Dependence. We assume the approximate right inverse η in (H3) to be continuous, which means that for every (σ', σ), $0 < \sigma' < \sigma$, the mapping $\eta : B_\sigma \to L(Z_\sigma, Y_{\sigma'})$, $(f, u) \to \eta(f, u)$ is continuous.

COROLLARY 6.1.2 *Let $\mathcal{F}$ be as in Theorem* 6.1.1, *and let η be continuous. If ϕ*

$$\phi : D \to B_\sigma , \quad \sigma > 0 ,$$

$D \ni w \mapsto \phi(w) = (\phi_1(w), \phi_2(w)) \in X_\sigma \times Y_\sigma$, is a continuous map defined on an open set $D \subset W$ of some Banach space W, satisfying for all $w \in D$ the two estimates $|\phi_2(w)|_\sigma \leq r < R$ and $|\mathcal{F} \circ \phi(w)|_\sigma < C(R - r)\sigma^q$, C and q as in Theorem 6.1.1. *Then there exists a continuous function $\theta : D \to Y_{\sigma/2} \cap B_{\sigma/2}$ such that, for all $w \in D$,*

(i) *$\mathcal{F}(\phi_1(w), \theta(w)) = 0$, and*
(ii) *$|\theta(w) - \phi_2(w)|_{\sigma/2} \leq C^{-1} \cdot |\mathcal{F} \circ \phi(w)|_\sigma \cdot \sigma^{-\gamma}$.*

PROOF: Define, as in Theorem 6.1.1, θ by $\theta(w) = \lim_{j\to\infty} u_j(w)$ in $Y_{\sigma/2}$, where $u_0(w) = \phi_2(w) \in Y_\sigma \cap B_\sigma$ and

$$u_{j+1}(w) = u_j(w) - \eta(\phi_1(w), u_j(w))(\mathcal{F}(\phi_1(w)), u_j(w))) \in Y_{\tau_{j+1}} \cap B_{\sigma_{j+1}}$$

to show that θ is a uniform limit of continuous functions and therefore continuous. Since η is continuous and $\eta(\phi_1(w), u_{j-1}(w))$ is linear, the functions $u_j : D \to B_{\sigma_j} \to Y_{\sigma/2}$ are continuous. Furthermore, we have the uniform estimates

$\sup_D |u_{j+1}(w) - u_j(w)|_{\sigma_{j+1}} \le (R-r)\sigma^{q-\gamma}\varepsilon_j^3$; hence θ is continuous. Estimate (ii) follows as in Theorem 6.1.1. □

6.1.3. Modifications. There is, of course, a great amount of arbitrariness in the formulation of the assumptions of generalized implicit function theorems like Theorem 6.1.1. They are dictated by the type of problems one chooses to look at. Our particular assumptions, chosen for their simplicity, cover many small-divisor problems arising in celestial mechanics. However, various far-reaching modifications lead to the same type of existence statement. We mention just one.

The continuity condition in the setup (6.4) can be replaced by the condition $\mathcal{F} : B_0 \to Z_0$, $\mathcal{F} : B_\sigma \to Z_{\sigma'}$, $0 \le \sigma' < \sigma$ continuous. In (H1) the smoothness can be replaced by the following assumption: For every $\sigma > 0$ and $(f, u) \in B_\sigma$, there is a mapping $\phi(f, u)$ from Y_σ into $Z_{\sigma'}$ for all $\sigma' < \sigma$ satisfying the following two estimates replacing (H1) and the second estimate in (H3): For $(f, u), (f, v) \in B_\sigma$, set $Q(f; u, v) \equiv \mathcal{F}(f, u) - \mathcal{F}(f, v) - \phi(f, v)(u - v)$; then for all $\sigma' < \sigma$,

$$|Q(f; u, v)|_{\sigma'} \le \frac{M}{(\sigma - \sigma')^{2\alpha}} \left(|\mathcal{F}(f, v)|_\sigma^{\beta_1} |u - v|_\sigma + |u - v|_\sigma^{1+\beta_2} \right) \tag{6.13}$$

and

(6.13′)

$$|(\phi(f, v) \circ \eta(f, v) - 1)(z)|_{\sigma'} \le \frac{M}{(\sigma - \sigma')^{2(\alpha+\gamma)}} \left(|\mathcal{F}(f, v)|_\sigma^{\beta_3} |z|_\sigma + |z|_\sigma^{1+\beta_4} \right)$$

for all $z \in Z_\sigma$, where $\beta_i > 0$, $1 \le i \le 4$, are fixed constants. Under these modified assumptions the same statement as in Theorem 6.1.1 holds true with a possibly larger q (but the same loss γ!). The proof is completely analogous; one just replaces $d\mathcal{F}(f, u)$ by $\phi(f, u)$. The speed of the convergence of the iteration, however, measured by $\kappa > 1$ in (6.6), is slower: $1 < \kappa < 2$ unless $\beta_i = 1$ for $1 \le i \le 4$. (Observe that we can choose $\kappa = 2$ in the proof of Theorem 6.1.1.)

Theorem 6.1.1 is quantitative in nature, and we shall use it in an iterative way to extend the existence statement to larger spaces in order to deal later on with spaces of differentiable functions. We shall squeeze in between X_0 and X_σ, $\sigma > 0$, an entire family $(X_0^\ell)_{\ell > 0}$ of Banach spaces that are subspaces of X_0 satisfying for all $0 < \ell' \le \ell < \infty$ and all $\sigma > 0$,

$$X_0 \supset X_0^{\ell'} \supseteq X_0^\ell \supset \left(X_0^\infty \equiv \bigcap_{\ell > 0} X_0^\ell \right) \supset X_\sigma \supset X_1 .$$

These new spaces will be characterized in a natural way by their approximation properties with respect to the smaller spaces X_σ, $\sigma > 0$, the characterization being quantitative in nature.

6.1.4. The Spaces X_0^ℓ, $0 < \ell < \infty$. For $0 < \ell < \infty$ and $0 < \sigma \le 1$, we shall call a sequence $(h_j)_{j \ge 0} \subset X_0$ a (σ, ℓ)-sequence if $h_0 = 0$, $h_j \in X_{\sigma \cdot 2^{-j}}$, and $\sup_{j \ge 1}(2^{j\ell}|h_j - h_{j-1}|_{\sigma \cdot 2^{-j}}) < \infty$. For such sequences we define

$$[(h_j)] \equiv \sup_{j \ge 1}(2^{j\ell}|h_j - h_{j-1}|_{\sigma \cdot 2^{-j}}) . \tag{6.14}$$

Note that every (σ, ℓ)-sequence (h_j) satisfying $|h_j - h_{j-1}|_0 \leq 2^{-j\ell}[(h_j)]$ for all $j \geq 1$ and some $\ell > 0$ is a Cauchy sequence in X_0–ℓ indicating the speed of the convergence. Denoting its limits by $h = \lim_{j\to\infty} h_j \in X_0$, we can write $h = \sum_{j\geq 1}(h_j - h_{j-1})$ and read off the estimate

(6.15) $$|h|_0 \leq (2^\ell - 1)^{-1}[(h_j)].$$

For all $0 < \ell < \infty$ and $0 < \sigma \leq 1$, the linear subspaces $X_0^{(\sigma,\ell)} \subset X_0$ are now defined as follows:

$$X_0^{(\sigma,\ell)} \equiv \{h \in X_0 \mid \exists \text{ a } (\sigma,\ell)\text{-sequence } (h_j) \text{ with } h_j \to h \text{ in } X_0\}.$$

Clearly $X_{\sigma'} \subset X_0^{(\sigma,\ell)}$ for all $\sigma' > 0$. Indeed, if $h \in X_{\sigma'}$, take $j_0 \in Z$ such that $\sigma \cdot 2^{-j_0} \leq \sigma'$ and define (h_j) by $h_j = 0$ for $0 \leq j \leq j_0 - 1$ and $h_j = h \in X_{\sigma\cdot 2^{-j}}$ for $j \geq j_0$. It then follows that $h_j \to h$ in X_0 and $[(h_j)] = 2^{j_0\ell}|h|_{\sigma\cdot 2^{-j_0}}$.

Denoting $S(\sigma,\ell;h)$ the equivalence class of (σ,ℓ)-sequences (h_j) with $\lim_{j\to\infty} h_j = h$ on X_0, we introduce in $X_0^{(\sigma,\ell)}$ the following norm:

(6.16) $$\|h\|_{(\sigma,\ell)} \equiv \inf_{S(\sigma,\ell;h)} [(h_j)], \quad h \in X_0^{(\sigma,\ell)};$$

note that $\|h\|_{(\sigma,\ell)} \geq (2^\ell - 1)|h|_0$, which follows from (6.15).

LEMMA 6.1.3 *The space $X_0^{(\sigma,\ell)}$ with the norm $\| \ \|_{(\sigma,\ell)}$ is a Banach space.*

PROOF: To shorten the notation, we write for $\|h\|_{(\sigma,\ell)}$ simply $\|h\|_\ell$. We let $(k^{(n)})_n \subset X_0^{(\sigma,\ell)}$ be a Cauchy sequence. We choose a subsequence, which we call $(h^{(n)})_n$, such that

(6.17) $$\|h^{(n)} - h^{(n+1)}\|_\ell < 2^{-(n+1)}.$$

It is sufficient to show that there is an $h \in X_0^{(\sigma,\ell)}$ such that for every $\varepsilon > 0$, $\|h^{(n)} - h\|_\ell < \varepsilon$ for $n > N(\varepsilon)$. We can choose (σ,ℓ)-sequences $(h_j^{(n)})_{j\geq 0} \in S(\sigma,\ell;h^{(n)})$ such that, because of (6.17),

(6.18) $$[(h_j^{(n)} - h_j^{(n+1)})] < 2^{-n}$$

for all n. In order to show that for fixed $j \geq 0$, $(h_j^{(n)})_{n\geq 0} \subset X_{\sigma\cdot 2^{-j}}$ is a Cauchy sequence in $X_{\sigma\cdot 2^{-j}}$, we prove for $n = 0, 1, \ldots,$

(6.19) $$|h_j^{(n)} - h_j^{(n+1)}|_{\sigma\cdot 2^{-j}} \leq 2^{-n}\theta(1-\theta)^{-1},$$

where $\theta = 2^{-\ell}$. (6.19) follows from the estimates $|h_j^{(n)} - h_j^{(n+1)}|_{\sigma\cdot 2^{-j}} \leq 2^{-n}\sum_{s=1}^{j}\theta^s$, $\theta = 2^{-\ell}$, which will be proved by induction. For $j = 0$ we have $h_0^{(n)} = 0$ for all $n \geq 0$. For the induction step from j to $j+1$, note that from (6.18),

(6.20) $$\left|\left(h_{j+1}^{(n)} - h_{j+1}^{(n+1)}\right) - \left(h_j^{(n)} - h_j^{(n+1)}\right)\right|_{\sigma\cdot 2^{-(j+1)}} < 2^{-(j+1)\ell}\cdot 2^{-n};$$

therefore

$$
\begin{aligned}
|h_{j+1}^{(n)} - h_{j+1}^{(n+1)}|_{\sigma\cdot 2^{-(j+1)}} \\
&\leq |(h_{j+1}^{(n)} - h_{j+1}^{(n+1)}) - (h_j^{(n)} - h_j^{(n+1)}|_{\sigma\cdot 2^{-(j+1)}} + |h_j^{(n)} - h_j^{(n+1)}|_{\sigma\cdot 2^{-(j+1)}} \\
&\leq 2^{-(j+1)\ell}\cdot 2^{-n} + 2^{-n}\sum_{s=1}^{j}\theta^s = 2^{-n}\sum_{s=1}^{j+1}\theta^s .
\end{aligned}
$$

Denoting by $h_j = \lim_{n\to\infty} h_j^{(n)} \in X_{\sigma\cdot 2^{-j}}$, the limit of the Cauchy sequence $(h_j^{(n)})_{n\geq 0} \subset X_{\sigma\cdot 2^{-j}}$, we conclude from $h_j^{(n)} - h_j = \sum_{s\geq n}(h_j^{(s)} - h_j^{(s+1)})$ and (6.19) that

$$
|h_j^{(n)} - h_j|_{\sigma\cdot 2^{-j}} \leq 2^{-(n-1)}\theta(1-\theta)^{-1} . \tag{6.21}
$$

We next show that $(h_j)_{j\geq 0}$ is a (σ, ℓ)-sequence. From (6.18) we conclude that $[(h_j^{(n)})] \leq M$ for all $n \geq 0$ and some $M > 0$. Writing $h_j - h_{j-1} \equiv (h_j - h_j^{(n)}) + (h_j^{(n)} - h_{j-1}^{(n)}) + (h_{j-1}^{(n)} - h_{j-1})$, we obtain the estimate

$$
\begin{aligned}
2^{j\ell}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}} \leq 2^{j\ell}|h_j - h_j^{(n)}|_{\sigma\cdot 2^{-j}} &+ 2^{j\ell}|h_j^{(n)} - h_{j-1}^{(n)}|_{\sigma\cdot 2^{-j}} \\
&+ 2^{j\ell}|h_{j-1} - h_{j-1}^{(n)}|_{\sigma\cdot 2^{-j}} ,
\end{aligned}
$$

which is $\leq 2M$ for n sufficiently large, since $h_j^{(n)} \to h_j$ on $X_{\sigma\cdot 2^{-j}}$; hence

$$
[(h_j)] = \sup_{j\geq 1}(2^{j\ell}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}}) \leq 2M < \infty
$$

and (h_j) is a (σ, ℓ)-sequence $\in S(\sigma, \ell; h)$ for a unique $h \in X_0^{(\sigma,\ell)}$. Finally, we show that $\|h^{(n)} - h\|_\ell \leq 2^{-n}(1-\theta)^{-1}$. Using the identity

$$
\begin{aligned}
h_j^{(n)} - h_{j-1}^{(n)} - (h_j - h_{j-1}) = \sum_{s=n}^{m}&\left\{h_j^{(s)} - h_j^{(s+1)} - \left(h_{j-1}^{(s)} - h_{j-1}^{(s+1)}\right)\right\} \\
&+ h_j^{(m+1)} - h_{j-1}^{(m+1)} - (h_j - h_{j-1}) ,
\end{aligned}
$$

we find with the aid of (6.20),

$$
\begin{aligned}
2^{j\ell}|h_j^{(n)} &- h_{j-1}^{(n)} - (h_j - h_{j-1})|_{\sigma\cdot 2^{-j}} \\
&\leq 2^{-n}\sum_{s=0}^{m}\theta^s + 2^{j\ell}\left(|h_j^{(m+1)} - h_j|_{\sigma\cdot 2^{-j}} + |h_{j-1}^{(m+1)} - h_{j-1}|_{\sigma\cdot 2^{-j}}\right) \\
&\leq 2^{-n}(1-\theta)^{-1}
\end{aligned}
$$

by letting $m \to \infty$. According to the definition of the norm, the last estimate leads to $\|h^{(n)} - h\|_\ell \leq [(h_j^{(n)} - h_j)] \leq 2^{-n}(1-\theta)^{-1}$; hence the Cauchy sequence $h^{(n)}$ converges to h in $X_0^{(\sigma,\ell)}$. □

For later use we add the following trivial but typical lemma:

LEMMA 6.1.4 *Let* $(h_j)_{j\geq 0}$ *be a* (σ,ℓ)*-sequence with* $\lim h_j = h$ *in* X_0. *Then* $(h_j)_{j\geq 0} \subset X_0^{(\sigma,\ell')}$ *is a Cauchy sequence in* $X_0^{(\sigma,\ell')}$ *for all* $\ell' < \ell$, *and*

$$\|h_n - h_{n-1}\|_{\ell'} \leq 2^{-n(\ell-\ell')}[(h_j)]$$

and $\lim_{n\to\infty} h_n = h$ *in* $X_0^{(\sigma,\ell')}$.

PROOF: To estimate $\|h_n - h_{n-1}\|_{(\sigma,\ell')}$, we pick a sequence

$$(g_j)_{j\geq 0} \in S(\sigma,\ell',h_n - h_{n-1})$$

as follows: $g_j = 0$, $0 \leq j < n$, and $g_j = h_n - h_{n-1} \in X_{\sigma\cdot 2^{-j}}$ for $j \geq n$. Then

$$\|h_n - h_{n-1}\|_{(\sigma,\ell')} \leq [(g_j)] = \sup_{j\geq 1}(2^{j\ell'}|g_j - g_{j-1}|_{\sigma\cdot 2^{-j}}) = 2^{n\ell'}|h_n - h_{n-1}|_{\sigma\cdot 2^{-n}}$$
$$\leq 2^{-n(\ell-\ell')}[(h_j)].$$

Assume $\lim_{n\to\infty}\|h_n - h^*\|_{\ell'} = 0$; it follows from $\|h_n - h^*\|_{\ell'} \geq (2^{\ell'}-1)|h_n - h^*|_0$ that $h^* = h$. □

From the definition one sees immediately that for all $0 < \ell' \leq \ell < \infty$ and for all $h \in X_0^{(\sigma,\ell)}$,

(6.22) $$X_0^{(\sigma,\ell')} \supseteq X_0^{(\sigma,\ell)}, \qquad \|h\|_{(\sigma,\ell')} \leq 2^{-(\ell-\ell')}\|h\|_{(\sigma,\ell)}.$$

Obviously $X_0^{(\sigma,\ell)} = X_0^{(\sigma',\ell)}$ for all $0 < \sigma',\sigma \leq 1$, the corresponding norms being equivalent. We shall therefore write $X_0^\ell \equiv X_0^{(\sigma,\ell)}$ for all $\sigma > 0$ and $\| \ \|_\ell$ for some choice of norm fixed from now on. We also introduce the notation

(6.23) $$X_0^\infty \equiv \bigcap_{\ell>0} X_0^\ell.$$

With these spaces in mind, we define the concept of analytic (C^ω-)smoothing.

DEFINITION 6.1.5 An *analytic smoothing* in $(X_\sigma)_{\sigma\geq 0}$ with respect to $(X_0^\ell)_{\ell>0}$ is a family $(S_t)_{t>0}$ of linear operators $S_t \in L(X_0, X_1)$ together with constants $k(\ell) > 0$ for every $0 < \ell < \infty$ such that the following three conditions are satisfied:

(6.24) $$\lim_{t\to\infty}|(S_t - 1)u|_0 = 0, \qquad u \in X_0,$$

(6.25) $$|S_t u|_{t^{-1}} \leq k(\ell)\|u\|_\ell, \qquad u \in X_0^\ell,$$

(6.26) $$|(S_\tau - S_t)u|_{\tau^{-1}} \leq t^{-\ell}k(\ell)\|u\|_\ell, \quad u \in X_0^\ell \quad \text{for } \tau \geq t \geq 1.$$

From (6.24) it follows in particular, that $X_1 \subset X_0$ is dense in X_0. (6.26) says that the convergence $S_t u \to u$ as $t \to \infty$ is faster, the smaller the space X_0^ℓ to which u belongs. (6.25) and (6.26) are estimates in the spaces X_σ, $\sigma > 0$.

THEOREM 6.1.6 *Let* $\mathcal{F}$ *satisfy the setup and hypotheses* (H1)–(H3). *Assume there exists an analytic smoothing* $(S_t)_{t>0}$ *in* X_σ *with respect to* $(X_0^\ell)_{\ell>0}$. *Let* $q \geq 2(\alpha+\gamma)$. *Then there exist an open neighborhood* D *of* f_0 *in* X_0^q *and a mapping* $\psi : D \to Y_0^{q-\gamma}$ *such that*

(i) $\mathcal{F}(f,\psi(f)) = 0$ *for all* $f \in D$ *and*
(ii) $\psi(D \cap X_0^\ell) \subset Y_0^{\ell-\gamma}$ *for all* $\ell \geq q$.

In particular, $\psi(D \cap X_0^\infty) \subset Y_0^\infty$. Furthermore, if $f \in D \cap X_0^\ell$, $\ell \geq q$, then

(iii) $\|\psi(f) - u_0\|_m \leq C_{m\ell}\|f - f_0\|_\ell$

for all $m < \ell - \gamma$. Here $C_{m\ell} > 0$ are constants depending on m and ℓ. For $\ell \geq q$, set $D^\ell \equiv D \cap X_0^\ell$ with the induced topology and $D^\infty \equiv D \cap X_0^\infty$, and denote the restrictions $\psi_\ell \equiv \psi \mid D^\ell$. If η is also continuous, then the mappings ψ_ℓ, $\ell \geq q$,

$$\psi_\ell : D^\ell \to Y_0^m$$

for $m < \ell - \gamma$ are continuous; in particular, $\psi_\infty : D^\infty \to Y_0^\infty$ is continuous.

REMARK. Statement (i) and estimate (iii) for $\ell = q$ follow without the assumption of the existence of an analytic smoothing.

PROOF: The proof uses an idea of J. Moser that was elaborated by H. Jacobowitz (see the references). Instead of working directly in the spaces (X_0^ℓ), (Y_0^ℓ), and (Z_0^ℓ), we go by means of the C^ω smoothing into the smaller spaces X_σ. The method, quantitative in nature, is a double approximation: We are going to solve, exactly, infinitely many approximate problems in the smaller spaces by repeated use of Theorem 6.1.1. Doing so we retain maximal smoothness during the iteration at the expense of accuracy. We start with the unperturbed solution (f_0, u_0) of (6.3),

$$\mathcal{F}(f_0, u_0) = 0\,, \tag{6.27}$$

which by assumption already belongs to the smallest spaces $X_1 \times Y_1$. We pick a $q \geq 2(\alpha + \gamma)$ and define the neighborhood $D \subset X_0^q$ of f_0 by

$$D = \{f \in X_0^q \mid \|f - f_0\|_q < \delta\} \tag{6.28}$$

for some $0 < \delta < \delta_0$ sufficiently small, to be determined during the proof. We define a sequence of mappings $(\phi_j)_{j\geq 0}$

$$\phi_j : D \to X_1$$

by means of the smoothing as follows: For $j = 0$, $\phi_0(f) = f_0$, while for $j \geq 1$

$$\phi_j(f) - f_0 = S_{t_j}(f - f_0)\,. \tag{6.29}$$

Here $t_j = \sigma_{j-1}^{-1}$, $j \geq 1$, and $\sigma_j = \sigma_0 \cdot 2^{-j}$ for $j \geq 0$ and for some positive $\sigma_0 \leq 1$ fixed from now on. Note $2\sigma_{n+1} = \sigma_n$ and $\sigma_n \downarrow 0$ as $n \to \infty$. Note also that $\phi_j(f) - f = (S_{t_j} - 1)(f - f_0)$, and therefore by (6.24)

$$\lim_{j\to\infty} |\phi_j(f) - f|_0 = 0\,.$$

Using Theorem 6.1.1, we shall construct inductively a sequence of mappings $(\psi_j)_{j\geq 0}$

$$\psi_j : D \to Y_{\sigma_j} \cap B_{\sigma_j}$$

starting with $\psi_0(f) = u_0$ such that, for δ_0 sufficiently small, the following statements S_n hold for all $n \geq 1$ and $f \in D$:

(S_n1) $(\phi_n(f), \psi_n(f)) \in B_{\sigma_n}$, $\mathcal{F}(\phi_n(f), \psi_n(f)) = 0$,

(S_n2) $|\psi_n(f) - \psi_{n-1}(f)|_{\sigma_n} \leq C^{-1} \cdot \sigma_{n-1}^{-\gamma} |\mathcal{F}(\phi_n(f), \psi_{n-1}(f))|_{\sigma_{n-1}}$

with the constant $C > 1$ of Theorem 6.1.1. For $f \in D$ we introduce the notation

$$f_j \equiv \phi_j(f)\,, \qquad u_j = \psi_j(f)\,.$$

Step 1. We first check that for δ_0 sufficiently small $f_j \in X_{\sigma_{j-1}} \cap B_{\sigma_{j-1}}$ for all $j \geq 1$. From the definition of $(S_t)_{t>0}$, (6.26), we get for $j = 1$,

$$|f_1 - f_0|_{\sigma_0} = |S_{t_1}(f - f_0)|_{t_1^{-1}} \leq k(q)\|f - f_0\|_q ,$$

and for all $j \geq 2$ using (6.26)

$$|f_j - f_{j-1}|_{\sigma_{j-1}} = |(S_{t_j} - S_{t_{j-1}})(f - f_0)|_{t_j^{-1}} \leq k(q)\sigma_{j-2}^q\|f - f_0\|_q ,$$

and therefore for all $j \geq 1$

$$|f_j - f_0|_{\sigma_{j-1}} \leq \sum_{n=1}^{\infty} |f_n - f_{n-1}|_{\sigma_{n-1}} \leq C_1\|f - f_0\|_q ,$$

with $C_1 = k(q)(1 + \sigma_0^q(1 - 2^{-q})^{-1})$. Hence in order to get $|f_j - f_0|_{\sigma_{j-1}} < N$ as required in the setup, we have to choose $\delta_0 \leq C_1^{-1} \cdot N$.

Step 2. Now we prove the induction statement S_1. We know $\mathcal{F}(f_0, u_0) = 0$ and $(f_1, u_0) \in B_{\sigma_0}$. Using now (H2) for the first time, we can estimate

$$\begin{aligned} |\mathcal{F}(f_1, u_0)|_{\sigma_0} &= |\mathcal{F}(f_1, u_0) - \mathcal{F}(f_0, u_0)|_{\sigma_0} \\ &\leq M \cdot |f_1 - f_0|_{\sigma_0} \leq Mk(q)\|f - f_0\|_q < C\sigma_0^q \frac{R}{2} , \end{aligned}$$

if $\delta_0 \leq C_2$, $C_2 = C \cdot \sigma_0^q (R/2)k(q)^{-1}M^{-1}$. The assumptions of Theorem 6.1.1 are satisfied for the pair $(f, u) \equiv (f_1, u_0) \in B_{\sigma_0}$, $\sigma = \sigma_0$ and $r = R/2$, and we get $u_1 \in Y_{\sigma_1} \cap B_{\sigma_1}$ such that $\mathcal{F}(f_1, u_1) = 0$, and $|u_1 - u_0|_{\sigma_1} \leq C^{-1}\sigma_0^{-\gamma}|\mathcal{F}(f_1, u_0)|_{\sigma_0}$.

Step 3. Assuming now the validity of the statements S_j for $1 \leq j \leq n$, we shall prove the validity of S_{n+1}. We know from S_n that $(f_{n+1}, u_n) \in B_{\sigma_n}$ and $\mathcal{F}(f_n, u_n) = 0$; again by (H2) we estimate

$$\begin{aligned} |\mathcal{F}(f_{n+1}, u_n)|_{\sigma_n} = |\mathcal{F}(f_{n+1}, u_n) - \mathcal{F}(f_n, u_n)|_{\sigma_n} &\leq M|f_{n+1} - f_n|_{\sigma_n} \\ &= M|(S_{t_{n+1}} - S_{t_n})(f - f_0)|_{t_{n+1}^{-1}} . \end{aligned}$$

Using the fact that $f - f_0 \in X_0^q$, this can be estimated further by means of (6.26),

$$\leq Mk(q)2^q\sigma_n^q\|f - f_0\|_q \leq C\sigma_n^q \frac{R}{2} ,$$

if only $\delta_0 \leq C_3$, $C_3 = C(R/2)M^{-1} \cdot k(q)^{-1} \cdot 2^{-q}$. In order to prove $|u_n - u_0|_{\sigma_n} < R/2$, we make use of $S_j 2$ for all $1 \leq j \leq n$ and estimate

$$\begin{aligned} |u_n - u_0|_{\sigma_n} \leq \sum_{j=1}^{n} |u_j - u_{j-1}|_{\sigma_j} &\leq C^{-1} \sum_{j=1}^{n} \sigma_{j-1}^{-\gamma}|\mathcal{F}(f_j, u_{j-1})|_{\sigma_{j-1}} \\ &\leq C^{-1} \cdot M \cdot k(q)2^q\|f - f_0\|_q \sum_{n\geq 0} \sigma_n^{q-\gamma} \\ &= C_4 \cdot \|f - f_0\|_q . \end{aligned}$$

We have to choose $\delta_0 \leq c_4^{-1}R/2$ to get $|u_n - u_0|_{\sigma_n} \leq R/2$. Recalling the above estimates, we can apply Theorem 6.1.1 to the pair $(f, u) \equiv (f_{n+1}, u_n) \in B_{\sigma_n}$, with $\sigma = \sigma_n$, $r = R/2$, and conclude the existence of $u_{n+1} \in Y_{\sigma_{n+1}} \cap B_{\sigma_{n+1}}$ such that

$\mathcal{F}(f_{n+1}, u_{n+1}) = 0$ and $|u_{n+1} - u_n|_{\sigma_{n+1}} \le C^{-1}\sigma_n^{-\gamma}|\mathcal{F}(f_{n+1}, u_n)|_{\sigma_n}$. Hence we have proved the induction statement S_{n+1}.

Step 4. Here we consider the consequences of S_n, $n \ge 1$. From the induction statements (S_n2) we reach the following estimates for the sequence $(\psi_j)_{j\ge 0}$ of mappings $\psi_j : D \to Y_{\sigma_j} \cap B_{\sigma_j}$ with $\psi_0(f) = u_0$,

$$|\psi_1(f) - u_0|_{\sigma_1} \le C^{-1}\sigma_0^{-\gamma}|S_{t_1}(f - f_0)|_{t_1^{-1}},$$

$$|\psi_j(f) - \psi_{j-1}(f)|_{\sigma_j} \le C^{-1} \cdot M \cdot \sigma_{j-1}^{-\gamma}|(S_{t_j} - S_{t_{j-1}})(f - f_0)|_{t_j^{-1}}.$$

Therefore, if $f \in D \cap X_0$, $\ell \ge q$, and hence $f - f_0 \in X_0^\ell$, we get by means of (6.26) for all $j \ge 1$,

$$|\psi_j(f) - \psi_{j-1}(f)|_{\sigma_j} \le K(\ell)\sigma_j^{\ell-\gamma}\|f - f_0\|_\ell, \tag{6.30}$$

where $K(\ell) = C^{-1} \cdot M \cdot k(\ell)2^\ell\sigma_0^{-(\ell-\gamma)}$. Therefore, for each $f \in D \cap X_0^\ell$, the sequence $(\omega_j(f))_{j\ge 0}$, $\omega_j(f) = \psi_j(f) - u_0$ is a $(\sigma_0, \ell - \gamma)$-sequence in Y_0, and

$$\sup_{j\ge 1}\left(2^{(\ell-\gamma)j} \mid \psi_j(f) - \psi_{j-1}(f)|_{\sigma_j}\right) \le K(\ell) \cdot \|f - f_0\|_\ell. \tag{6.31}$$

Observe $\ell \ge q > \gamma$; hence $(\omega_j(f))_{j\ge 0}$ is a Cauchy sequence in Y_0. Denoting its limit by $\lim_{j\to\infty}\omega_j(f) = \psi(f) - u_0$, we have, according to definition, $\psi(f) - u_0 \in Y_0^{\ell-\gamma}$; hence $\psi(f) \in Y_0^{\ell-\gamma}$. On the other hand, we know $\lim_{j\to\infty}|\phi_j(f) - f|_0 = 0$, and using the fact that $\mathcal{F} : X_0 \times Y_0 \to Z_0$ is continuous, we conclude from S_n1 that $\mathcal{F}(f, \psi(f)) = 0$ for all $f \in D$. Moreover, as we have just seen, $\psi(D \cap X_0^\ell) \subset Y_0^{\ell-\gamma}$ for all $\ell \ge q$, hence we have proved (i) and (ii) of Theorem 6.1.6. Moreover, by Lemma 6.1.4, the sequence $(\psi_j(f))_{j\ge 0}$, $f \in D \cap X_0^\ell$, is a Cauchy sequence in Y_0^m for all $m < \ell - \gamma$, and by means of (6.30) we have

$$\|\psi_j(f) - \psi_j(f_0)\|_m \le 2^{-\varepsilon j}\widehat{K}(\ell)\|f - f_0\|_\ell \tag{6.32}$$

for $\varepsilon = \ell - \gamma - m > 0$, from which the required estimate (iii) easily follows.

Step 5. Let η be continuous. We shall prove $\psi_\ell : D^\ell \to Y_0^m$, $m < \ell - \gamma$, is continuous. Since $S_{t_j} \in L(X_0^\ell, X_{t_j^{-1}})$, the mappings $\phi_j : D^\ell \to B_{\sigma_{j-1}} \cap X_{\sigma_{j-1}}$ are continuous. Inductively applying the corollary to Theorem 6.1.1, we conclude that the mappings $\psi_j : D^\ell \to Y_{\sigma_j} \cap B_j \to Y_0^m$ are continuous. Here we have used the fact that the injection $Y_\sigma \subset Y_0^m$ is continuous. Since according to (6.32), $(\psi_j(f))_{j\ge 0}$ is (if $f \in D^\ell$) a locally uniform Cauchy sequence in X_0^m for all $m < \ell - \gamma$, the mapping ψ, defined by $\psi(f) = \lim_{j\to\infty}\psi_j(f)$, is continuous from D^ℓ into Y_0^m. □

6.2. Analytic Smoothing on Function Spaces and Analytic Mappings

In order to be able to apply Theorems 6.1.1 and 6.1.6 to mappings $\mathcal{F}$ between function spaces on compact analytic manifolds, we have to realize the setup of these theorems. Since the general case can be reduced to the situation in which the manifold is a torus, we restrict ourselves to function spaces on an n-dimensional torus T. These are simply functions on $\mathbb{R}^n$ that are periodic with period 1 in each argument. We start with some notation and definitions.

We denote $D_j = \partial/\partial x_j$, $D^k = D_1^{k_1} \circ D_2^{k_2} \circ \cdots \circ D_n^{k_n}$, $|k| = \sum_{i=1}^n k_i$. For integers $p \geq 0$ we introduce the seminorms

(6.33) $$\|u\|_{C^p} = \sup_{\substack{x \in T \\ |k|=p}} |D^k u(x)|.$$

In $C^p(T)$ we have the norms $|u|_{C^p} = \sup_{0 \leq n \leq p} \|u\|_{C^n}$. The Hölder spaces $C^\ell(T) \subset C^p(T)$. $\ell = p + \alpha$. p an integer ≥ 0. and $\alpha \in (0, 1)$. consist of functions $u \in C^p(T)$, such that $|u|_{C^\ell} \equiv |u|_{C^p} + \|u\|_{C^\ell} < \infty$, where

(6.34) $$\|u\|_{C^\ell} = \sup_{\substack{x \neq y \\ |k|=p}} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\alpha}.$$

For p an integer ≥ 1, we introduce a Banach space $\widehat{C}^p$ such that $C^p \subset C^{p-1,1} \subset \widehat{C}^p \subset C^\ell$ for all $\ell < p$, where C^{p-1} is the space of functions whose derivatives of order $p - 1$ are Lipschitz-continuous. $\widehat{C}^p$ is defined by the following Zygmund condition: $\widehat{C}^p(T) \equiv \{f \in C^{p-1}(T) \mid \|f\|_{\widehat{C}^p} < \infty\}$, where the seminorm $\|f\|_{\widehat{C}^p}$ is defined by the symmetric difference

(6.35) $$\|f\|_{\widehat{C}^p} = \sup_{\substack{x \neq y \\ |k|=p-1}} \frac{|D^k f(x) + D^k f(y) - 2D^k f\left(\frac{1}{2}(x+y)\right)|}{|x - y|}.$$

In $\widehat{C}^p(T)$ we introduce the norm $|f|_{\widehat{C}^p} = |f|_{C^{p-1}} + \|f\|_{\widehat{C}^p}$. Note $C^{p-1,1} \neq \widehat{C}^p \neq C^\ell$ for all $\ell < p$.

In order to realize the setup of Theorem 6.1.1, the following spaces, described earlier, of (real) holomorphic functions defined on complex neighborhoods of T are important. For some fixed $r > 0$, we define the complex strips T_σ for all $\sigma > 0$ as follows:

(6.36) $$T_\sigma = \{x \in \mathbb{C}^n \mid |\mathrm{Im}\, x_i| < r\sigma,\ 1 \leq i \leq n\}$$

for $\ell \geq 0$; the Banach spaces $A(\sigma, C^\ell)$, for $\sigma > 0$, are then defined as spaces of real holomorphic functions u defined on T_σ (u real means $\overline{u(x)} = u(\bar{x})$), with period 1 in each variable and such that $|u|_{\sigma, C^\ell} < \infty$, where the norms $|u|_{\sigma, C^\ell}$ are defined as above, the supremum, however, being taken over the open neighborhood T_σ of T. We have the following well-known Cauchy estimates for $0 \leq \ell < m$,

(6.37) $$|u|_{\sigma', C^m} \leq C_{m\ell} \cdot (\sigma - \sigma')^{-(m-\ell)} |u|_{\sigma, C^\ell}$$

for all $\sigma' < \sigma$, where $C_{m\ell}$ are constants depending only on m, ℓ, and r (r as in the definition of the domain T_σ). The estimates for m, ℓ integers follow simply from the Cauchy formula

$$D^k u(z) = \frac{k!}{(2\pi i)^n} \int_{\partial_1} \cdots \int_{\partial_n} \frac{u(\zeta) d\zeta}{(\zeta_1 - z_1)^{k_1+1} \ldots (\zeta_n - z_n)^{k_n+1}},$$

with $\partial_i = \{\zeta \in P(z, \rho) \mid |\zeta_i - z_i| = \rho\}$, $k! = (k_1!) \ldots (k_n!)$, and $0! = 1$. The generalization to m, ℓ not integers is then straightforward.

The one-parameter family $(X_\sigma)_{\sigma \geq 0}$ defined by $X_0 = C^m(T)$ and $X_\sigma = A(\sigma, C^m)$ for $\sigma > 0$ and for some fixed $m \geq 0$ clearly satisfies $X_0 \supset X_{\sigma'} \supseteq X_\sigma \supset$

X_1 and $|u|_0 \leq |u|_{\sigma'} \leq |u|_\sigma$ for all $0 \leq \sigma' \leq \sigma$ and $u \in A(\sigma, C^m)$. We have used the abbreviated notation $|u|_\sigma \equiv |u|_{\sigma, C^m}$. The question arises: What are the subspaces $X_0^\ell \subset X_0$, $\ell > 0$, characterized by their approximation properties with respect to real holomorphic functions in X_σ? One might conjecture that $X_0^\ell = C^{\ell+m}(T)$, $\ell > 0$; however, this is not true if ℓ is an integer. We therefore define the family $\widehat{C}^\ell(T)$, $\ell > 0$, by $\widehat{C}^\ell = C^\ell$ if ℓ is not an integer, and $\widehat{C}^p$ for $p \geq 1$ an integer. We shall prove the following characterization of $\widehat{C}^\ell$:

PROPOSITION 6.2.1 *Let $m \geq 0$, and let $(X_\sigma)_{\sigma \geq 0}$ be defined by $X_0 = C^m(T)$ and $X_\sigma = A(\sigma, C^m)$ for $\sigma > 0$; then*

$$X_0^\ell = \widehat{C}^{\ell+m}(T),$$

the corresponding norms being equivalent. There exists an analytic smoothing in $(X_\sigma)_{\sigma \geq 0}$ with respect to $(\widehat{C}^\ell)_{\ell > 0}$ (and with respect to $(C^\ell)_{\ell > 0}$).

The proof of Proposition 6.2.1 follows from Lemmas 6.2.2 and 6.2.3 below. For notational convenience we shall assume $n = 1$ (the dimension of the torus T), and we also assume at first $m = 0$; the statement for $m > 0$ will then follow by means of Cauchy estimates (6.37).

LEMMA 6.2.2 *There exists an analytic smoothing in the family $(X_\sigma)_{\sigma \geq 0}$ with respect to $(\widehat{C}^\ell)_{\ell > 0}$ (and $(C^\ell)_{\ell > 0}$): i.e., a family of linear continuous operators $S_t \in L(C^0, A(1, C^0))$ together with constants $k(\ell)$, $0 < \ell < \infty$, such that for all $\ell > 0$ and $t > 0$,*

$$\lim_{t \to \infty} |(S_t - 1)u|_{C^0} = 0, \qquad u \in C^0(T), \tag{6.38}$$

$$|S_t u|_{t^{-1}} \leq k(\ell)|u|_{\widehat{C}^\ell}, \qquad u \in \widehat{C}^\ell, \tag{6.39}$$

$$\tau \geq t, \ |(S_\tau - S_t)u|_{\tau^{-1}} \leq t^{-\ell} k(\ell)|u|_{\widehat{C}^\ell}, \quad u \in \widehat{C}^\ell, \tag{6.40}$$

and in (6.39) *and* (6.40) *the spaces $\widehat{C}^\ell$ can be replaced by the usual C^ℓ spaces with their norms $|u|_{C^\ell}$. S_t is a convolution operator: $S_t u = s_t * u$, $S_t(z) = ts(tz)$, where $s(\cdot)$ is an entire real holomorphic function.*

PROOF: Take a function $\tilde{s} \in C_0^\infty(R)$ vanishing outside a compact set and identically equal to 1 in a neighborhood of 0, and let s be its Fourier transform. Clearly for any n, N we have the estimate $|D^n s(x)| \leq A_{n,N}(1 + |x|)^{-N}$. Moreover, since $\tilde{s}$ is identically equal to 1 near 0, we have

$$\int s(x) P(x) dx = P(0) \tag{6.41}$$

for every polynomial P. In addition, s has an analytic continuation to an entire real holomorphic function on C, which we shall denote by the same letter s. From the definition of s, we see immediately that for any n, N there is a $C_{nN} > 0$ such that

$$|D^n s(z)| \leq C_{nN}(1 + |z|)^{-N} e^{c|\operatorname{Im} z|} \tag{6.42}$$

for all $z \in \mathbb{C}$, where $c > 0$ is a bound for the support of $\tilde{s}$: $\operatorname{supp}(\tilde{s}) \subset \{x \in R \mid |x| < c\}$. Shifting the path of integration and using the Cauchy integral formula, we get from (6.41) and (6.42)

$$\int s(\xi - i\eta)P(\xi)d\xi = P(i\eta) \tag{6.43}$$

for all real η and every polynomial P. For later use we define for $\alpha \geq 0$ the real valued function $\phi_\alpha(\cdot) : R \to R_+$ by

$$\phi_\alpha(\eta) = \frac{1}{([\alpha]-1)!}\int |s(\xi - i\eta)|\,|\xi|^\alpha\, d\xi\,, \tag{6.44}$$

with the convention $(-1)! = 0! = 1$, $\phi_\alpha \in C^0(R)$. With $\psi_\alpha(\cdot)$, we denote the function from $\mathbb{R}^+$ onto $\mathbb{R}^+$: $\psi_\alpha(\rho) = 2\sup\phi_\alpha(\eta)$, the supremum being taken over $|\eta| < \rho$. On $C^0(T)$ we now introduce the family of linear operators S_t for $t > 0$, $S_t \in L(C^0, A(1, C^0))$, by means of the convolution $S_t u = s_t * u$, $s_t(z) = ts(tz)$:

$$S_t u(z) = t\int s(t(y-z))u(y)dy\,, \tag{6.45}$$

which can be written, by the change of variable $\xi = t\,\mathrm{Re}(y - z) = ty - t\,\mathrm{Re}\,z$, as

$$S_t u(z) = \int s(\xi - it\,\mathrm{Im}\,z)u\left(\mathrm{Re}\,z + \frac{\xi}{t}\right)d\xi\,. \tag{6.46}$$

From (6.45) it is plain that $S_t u$ is an entire real holomorphic function on $\mathbb{C}$. From (6.46) we conclude that $S_t u$ has period 1 since u has period 1. Using (6.41), we have for all $x \in R\, u(x) - S_t u(x) = \int s_t(\xi)[u(x) - u(x+\xi)]d\xi$, from which it follows that $\lim_{t\to\infty} |(S_t - 1)u|_{C^p} = 0$ if $u \in C^p$, since $s \in \mathcal{S}(R)$ and $D^p u$ is uniformly continuous on R. Our aim, however, is to prove the following estimates, which give more information about the speed of the above convergence: For all $\ell > 0$, if $u \in C^\ell$, then

$$|(S_t - 1)u|_{C^0} \leq t^{-\ell}\phi_\ell(0)\|u\|_{C^\ell}\,, \qquad |(S_\tau - S_t)u|_{\tau^{-1}} \leq t^{-\ell}\psi_\ell(1)\|u\|_{C^\ell}\,, \tag{6.47}$$

for all $\tau \geq t > 0$; and if $u \in \widehat{C}^p$, $p = 1, 2, \ldots$, then

$$\begin{aligned} |(S_t - 1)u|_{C^0} &\leq t^{-p}\left(\frac{1}{2}\phi_{p-1}(0) + 2\phi_{p+1}(0)\right)\|y\|_{\widehat{C}^p} \\ |(S_\tau - S_t)u|_{\tau^{-1}} &\leq t^{-p}\left(\frac{1}{2}\psi_{p-1}(1) + 2\psi_{p+1}(1)\right)\|u\|_{\widehat{C}^p} \end{aligned} \tag{6.48}$$

for all $\tau \geq t > 0$. In order to prove these estimates we shall make use of Taylor's formula with integral remainder in the following form: Let $u \in C^\ell$, $\ell = p + \alpha$, p an integer, and $\alpha \in [0, 1)$; then

$$\begin{aligned} u(x+\eta) &= \sum_{n=0}^{p}\frac{1}{n!}\eta^n D^n u(x) + R_u(x,\eta)\,, \\ R_u(x,\eta) &= \frac{1}{(p-1)!}\eta^p\int_0^1 d\mu(1-\mu)^{p-1}\{D^p u(x+\mu\eta) - D^p u(x)\}\,. \end{aligned} \tag{6.49}$$

Applying this to $u(\operatorname{Re} z + \xi/t)$ in (6.46), we obtain, by means of (6.43), the following identity for all $z \in \mathbb{C}$:

$$S_t u(z) = \sum_{n=0}^{p} \frac{1}{n!}(i \operatorname{Im} z)^n D^n u(\operatorname{Re} z) + \widehat{R}_u(z,t) \tag{6.50}$$

$$\widehat{R}_u(z,t) = \int s(\xi - it \operatorname{Im} z) R_u\left(\operatorname{Re} z, \frac{\xi}{t}\right) d\xi \,.$$

Observe that the first part of $S_t u$ in (6.50) is independent of t. Inserting the estimate $|D^p u(x + \mu\xi/t) - D^p u(x)| \leq \|u\|_{C^\ell} \circ |\mu\xi/t|^\alpha$ into (6.50), we end up with the following estimate for the remainder term $\widehat{R}_u(z,t)$:

$$|\widehat{R}_u(z,t)| \leq t^{-\ell}\phi_\ell(t \operatorname{Im} z)\|u\|_{C^\ell} \tag{6.51}$$

for all $(z,t) \in \mathbb{C} \times R_+$ and $u \in C^\ell$. For $z = x \in R$, it follows from (6.50) that $S_t u(x) = u(x) + \widehat{R}_u(x,t)$, and using (6.51), we have $|(S_t - 1)u|_{C^0} \leq t^{-\ell}\phi_\ell(0) \cdot \|u\|_{C^\ell}$; hence, the first estimate of (6.47). To prove the second, observe that by means of (6.50),

$$|(S_\tau - S_t)u(z)| \leq |\widehat{R}_u(z,\tau)| + |\widehat{R}_u(z,t)| \,. \tag{6.52}$$

Estimates (6.51) and (6.52) lead to $|(S_\tau - S_t)u(z)| \leq t^{-\ell}\psi_\ell(1)\|u\|_{C^\ell}$ for all z with $\tau|\operatorname{Im} z| < 1$, $\tau \geq t > 0$, hence we have proved (6.47).

We proceed to prove the estimates (6.48) for $u \in \widehat{C}^p$, $p = 1, 2, \ldots$. Since $\widehat{C}^p \subset C^{p-1}$, we can first obtain estimates in the C^{p-1} norm; from these we will derive the required $\widehat{C}^p$ estimates by a simple trick. We define the family of linear operators $M_t \in L(C^p, C^{p+1})$, $p = 0, 1, \ldots$, for all $t > 0$ as follows: $M_t u = m_t * u$, where $m_t(x) = t$ for $-(2t)^{-1} \leq x \leq (2t)^{-1}$ and $m_t(x) = 0$ otherwise. From the definition of M_t, we immediately get the following estimates for $u \in C^\ell$, $\ell = p + \alpha$, p an integer ≥ 0 and $\alpha \in (0,1)$:

$$\|M_t u\|_{C^{p+1}} \leq t^{1-\alpha}\|u\|_{C^\ell}\,, \qquad \|(1 - M_t)u\|_{C^p} \leq t^{-\alpha}\|u\|_{C^\ell}\,, \tag{6.53}$$

and if $u \in \widehat{C}^p \subset C^{p-1}$, then

$$\|M_t^2 u\|_{C^{p+1}} \leq 2t\|u\|_{\widehat{C}^p}\,, \qquad \|(1 - M_t^2)u\|_{C^{p-1}} \leq (2t)^{-1}\|u\|_{\widehat{C}^p}\,. \tag{6.54}$$

Now let $u \in \widehat{C}^p$, and put $u_t \equiv M_t^2 u$; then from the linearity of S_t and by means of (6.47) and (6.54) for $u = (u - u_t) + u_t$, we find the estimates

$$\begin{aligned}
|(S_t - 1)u|_{C^0} &\leq |(S_t - 1)(u - u_t)|_{C^0} + |(S_t 1)u_t|_{C^0} \\
&\leq t^{-(p-1)}\phi_{p-1}(0)\|u - u_t\|_{C^{p-1}} + t^{-(p+1)}\phi_{p+1}(0)\|u_t\|_{C^{p+1}} \\
&\leq t^{-p}\left(\frac{1}{2}\phi_{p-1}(0) + 2\phi_{p+1}(0)\right)\|u\|_{\widehat{C}^p}\,,
\end{aligned}$$

and analogously,

$$
\begin{aligned}
|(S_\tau - S_t)u|_{\tau^{-1}} &\le |(S_\tau - S_t)(u-u_t)|_{\tau^{-1}} + |(S_\tau - S_t)u_t|_{\tau^{-1}} \\
&\le t^{-(p-1)}\psi_{p-1}(1)\|u-u_t\|_{C^{p-1}} + t^{-(p+1)}\psi_{p+1}(1)\|u_t\|_{C^{p+1}} \\
&\le t^{-p}\left(\frac{1}{2}\psi_{p-1}(1) + 2\psi_{p+1}(1)\right)\|u\|_{\widehat{C}^p};
\end{aligned}
$$

hence we have proved the required estimates (6.48). From definition (6.46), we get immediately for all $u \in C^0(T)$

$$
|S_t u|_{t^{-1}} \le \psi_0(1)|u|_{C^0},
$$

and therefore $|S_t u|_{t^{-1}} \le \psi_0(1)|u|_{\widehat{C}^\ell}$ if $u \in \widehat{C}^\ell$, hence (6.39), and the lemma is proved.

□

LEMMA 6.2.3 *For every* $0 < \sigma \le 1$ *there exist two functions* $\gamma_\sigma(\cdot)$ *and* $\delta_\sigma(\cdot)$ *from* R_+ *into* R_+ *such that the following statements hold:*

(i) *Let* $h \in C^0(T)$; *then there is a sequence* $(h_j)_{j\ge 0}$, $h_0 = 0$ *and* $h_j \in A(\sigma 2^{-j}, C^0)$ *for* $j \ge 1$, *such that* $\lim_{j\to\infty}|h_j - h|_{C^0} = 0$, *and if* $h \in \widehat{C}^\ell$ *(or* $h \in C^\ell$*) for some real number* $\ell > 0$, *then* $(h_j)_{j\ge 0}$ *is a* (σ,ℓ)*-sequence with*

$$
\|h\|_\ell \le \sup_{j\ge 1}(2^{j\ell}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}}) \le \gamma_\sigma(\ell)\cdot|h|_{\widehat{C}^\ell}
$$

and $\lim_{j\to\infty}|h_j - h|_{C^{\ell-\varepsilon}} = 0$ *for all* $0 < \varepsilon \le \ell$.

(ii) *Conversely, let* $(h_j)_{j\ge 0}$ *be a* (σ,ℓ)*-sequence with* $h_0 = 0$, $h_j \in A(\sigma\cdot 2^{-j}, C^0)$, *and* $\sup_{j\ge 1}(2^{j\ell}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}}) = M < \infty$; *then there is an* $h \in \widehat{C}^\ell$ *(not necessarily* $h \in C^\ell$*) such that* $\lim_{j\to\infty}|h_j - h|_{C^{\ell-\varepsilon}} = 0$ *for all* $0 < \varepsilon \le \ell$, *and*

$$
|h|_{\widehat{C}^\ell} \le \delta_\sigma(\ell)\cdot M.
$$

PROOF: (i) Let $h \in C^0(T)$ be given. Define the sequence $(h_j)_{j\ge 0}$ as follows: $h_0 = 0$, $h_j = S_{t_j}(h)$, $j \ge 1$, with $t_j = \sigma^{-1}2^j$ and $S_t \in L(C^0, A(1, C^0))$ according to Lemma 6.2.2. Then by (6.38) $\lim_{j\to\infty}|h_j - h|_{C^0} = 0$. If now $h \in \widehat{C}^\ell$, then $2^\ell|h_1|_{\sigma\cdot 2^{-1}} \le 2^\ell k(\ell)|h|_{\widehat{C}^\ell}$ by (6.39) and $z^{j\ell}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}} \le \sigma^\ell k(\ell)|h|_{\widehat{C}^\ell}$ for $j \ge 2$ by (6.40). Define the function $\gamma_\sigma(\cdot)$ by $\gamma_\sigma(\ell) = 2^\ell k(\ell)$.

(ii) Let $\sup_{j\ge 1}(2^{j\ell}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}}) = M$ for some $\ell > 0$. Using the Cauchy estimates (6.37), we get $|h_j - h_{j-1}|_{C^{\ell-\varepsilon}} \le M\cdot C_{\ell-\varepsilon,0}\cdot\sigma^{-(\ell-\varepsilon)}2^{-j\varepsilon}$ for all $0 < \varepsilon \le \ell$; hence $(h_j)_{j\ge 0}$ is a Cauchy sequence in $C^{\ell-\varepsilon}$. To prove the rest of the statement, it suffices to consider a sequence (h_j) satisfying $\sup_{j\ge 1}(2^{\alpha j}|h_j - h_{j-1}|_{\sigma\cdot 2^{-j}}) = M < \infty$ for some α, $0 < \alpha \le 1$, and to prove that the limit $h = \sum_{j\ge 1}(h_j - h_{j-1})$ in C^0, which as we already know exists, actually belongs to $\widehat{C}^\alpha$. However, we do not claim that $h_j \to h$ in

$\widehat{C}^{\alpha}$. Clearly $|h|_{C^0} \leq M \cdot \sum_{j\geq 1} 2^{-\alpha j} = M \cdot (2^{\alpha} - 1)^{-1}$. For $N \in Z_+$, we consider x, y such that

$$2^{-(N+1)} < |x - y| < 2^{-N}. \tag{6.55}$$

Considering first the case $0 < \alpha < 1$, we define, for $j \geq 1$, $H_j(x, y) \equiv (h_j - h_{j-1})(x) - (h_j - h_{j-1})(y)$ and write $h(x) - h(y) = \sum_{j=1} H_j(x, y) + \sum_{j\geq N+1} H_j(x, y)$. By means of the mean value theorem, we therefore get

$$|h(x) - h(y)| \leq \sum_{j=1}^{N} |h_j - h_{j-1}|_{C^1}|x - h| + 2 \sum_{j\geq N+1} |h_j - h_{j-1}|_{C^0}. \tag{6.56}$$

Using our assumption, we have $|h_j - h_{j-1}|_{C^0} \leq |h_j - h_{j-1}|_{\sigma \cdot 2^{-j}} \leq 2^{-\alpha j} \cdot M$, and by means of (6.37), $|h_j - h_{j-1}|_{C^1} \leq \sigma^{-\alpha} 2^{j(1-\alpha)} C_{\alpha,0} \cdot M$. From (6.56), we therefore obtain the estimate $|h(x) - h(y)| \leq M \cdot \sigma^{-\alpha} C_{\alpha,0} 2^{N(1-\alpha)}(1 - 2^{-(1-\alpha)})^{-1}|x - y| + 2M2^{-(N+1)\alpha}(1 - 2^{-(1-\alpha)})^{-1}$. Observing now (6.55), we end up with

$$|h(x) - h(y)| \leq \delta_{\sigma}(\alpha) \cdot M|x - y|^{\alpha}, \tag{6.57}$$

where $\delta_{\sigma}(\alpha) = (\sigma^{-\alpha} C_{\alpha,0} + 2)(1 - 2^{-(1-\alpha)})^{-1}$. Turning to the case $\alpha = 1$, we define, for $j \geq 1$, $G_j(x, y) \equiv (h_j - h_{j-1})(x) + (h_j - h_{j-1})(y) - 2(h_j - h_{j-1})\left(\frac{x+y}{2}\right)$ and get the estimate

$$\begin{aligned} \left| h(x) + h(y) - 2h\left(\frac{x+y}{2}\right) \right| &\leq \frac{1}{2} \sum_{j=1}^{N} |h_j - h_{j-1}|_{C^2} \cdot |x - y|^2 \\ &\quad + 2 \sum_{j\geq N+1} |h_j - h_{j-1}|_{C^0}. \end{aligned} \tag{6.58}$$

Here we have used the fact that for $u \in C^2$, the estimate $|u(x) + u(y) - 2u(\frac{x+y}{2})| \leq \frac{1}{4}|u|_{C^2}|x - y|^2$ holds. This is easily verified by applying the mean value theorem twice. Since $\alpha = 1$, we have $|h_j - h_{j-1}|_{C^0} \leq M \cdot 2^{-j}$, and by Cauchy, $|h_j - h_{j-1}|_{C^2} \leq \sigma^{-2} 2^j C_{2,0} \cdot M$. Therefore, inserting these estimates into (6.58) and observing (6.55), we end up with the estimate

$$\left| h(x) + h(y) - 2h\left(\frac{x+y}{2}\right) \right| \leq \delta_{\sigma}(1) \cdot M \cdot |x - y|, \tag{6.59}$$

with $\delta_{\sigma}(1) \equiv (\sigma^{-2} C_{2,0} + 4)$. We can prove estimates (6.58) and (6.59) for all integers N, and since $\delta_{\sigma}(\alpha)$ and $\delta_{\sigma}(1)$ are independent of N, the result follows. ☐

From Lemmas 6.2.2 and 6.2.3, the proposition follows for $m = 0$. To estimate the C^m norms for some $m > 0$ and fixed, just observe that for $u \in C^{\ell+m}$ we have proved $|(S_{\tau} - S_t)u|_{\tau^{-1}} \leq t^{-(m+\ell)} k(\ell + m)\|u\|_{\widehat{C}^{m+\ell}}$, and by means of (6.37) we get

$$\begin{aligned} |(S_{\tau} - S_t)u|_{\tau^{-1}, C^m} \leq |(S_{\tau} - S_t)u|_{2\tau^{-1}, C^m} &\leq \tau^m \cdot C_{m,0}|(S_{\tau} - S_t)u|_{\tau^{-1}} \\ &\leq \left(\frac{\tau}{t}\right)^m C_{m,0} \cdot k(\ell + m)t^{-\ell}\|u\|_{\widehat{C}^{2+m}} \\ &\leq t^{-\ell}\hat{k}(\ell)\|u\|_{\widehat{C}^{\ell+m}} \end{aligned}$$

for all $\tau \le 2t$, where $\hat{k}(\ell) = 2^m C_{m,0} k(\ell + m)$. This is all we need to prove the statement for $m > 0$ (we have to restrict the smoothing estimates (6.40) to $2t \ge \tau > t$).

The new feature of the analytic smoothing is the fact that functions in $C^0(T)$ are not only approximated by C^∞ functions but by real holomorphic functions defined in complex strips; this allows estimates of the differences $(S_\tau - S_t)u$ in the analytic spaces $A(\sigma, C^m)$. For later use we shall also introduce at this point the standard C^∞ smoothing by proving the following well-known lemma.

LEMMA 6.2.4 *There exists a C^∞ smoothing in the family $(C^\ell)_{\ell \ge 0}$, i.e., a family $(S_t)_{t>0}$ of linear mappings $S_t : C^0(T) \to C^\infty(T) = \bigcap_{\ell>0} C^\ell$, together with constants $C_{\lambda\mu}$, $0 \le \lambda, \mu < \infty$, satisfying the following three conditions:*

$$\lim_{t \to \infty} |(S_t - 1)u|_{C^0} = 0\,, \quad u \in C^0(T)\,, \tag{6.60}$$

$$|S_t u|_{C^m} \le t^{(m-\ell)} C_{\ell m} |u|_{C^\ell}\,, \tag{6.61}$$

for all $u \in C^\ell(T)$ and all $0 \le \ell \le m$, and

$$|(S_t - 1)u|_{C^\ell} \le t^{-(m-\ell)} C_{\ell m} |u|_{C^m} \tag{6.62}$$

for all $u \in C^m(T)$ and all $0 \le \ell \le m$.

PROOF: Define S_t as in Lemma 6.2.2 by $S_t u = s_t * u \in C^\infty(T)$ without, however, going into the complex. We have already proved (6.60) and (6.62) for integral ℓ, observing that S_t commutes with partial differential operators D^k. (6.61) follows immediately from the definition if ℓ, m are integers.

To estimate the Hölder norms, write $u = (1 - M_t)u + M_t u$, use the estimates (6.53), and observe the following trivial fact: For $0 < \lambda < \mu \le 1$ there exists a constant $C > 0$ such that for any ε with $0 < \varepsilon < 1$ and any $u \in C^\mu$, we have the estimate $\|u\|_{C^\lambda} \le \varepsilon \|u\|_{C^\mu} + \varepsilon^{-\alpha} \cdot C \cdot |u|_{C^0}$, where $\alpha = \lambda(\mu - \lambda)^{-1}$. □

Analytic Mappings. Knowing Proposition 6.2.1, one can apply the abstract generalized implicit function theorems to so-called analytic mappings. As a simple example, we shall consider a mapping $\mathcal{F}$,

$$\mathcal{F} : C^\mu(\mathbb{T}^{n_1}, \mathbb{R}^{m_1}) \times C^\beta(\mathbb{T}^{n_2}, \mathbb{R}^{m_2}) \to C^0(\mathbb{T}^{n_3}, \mathbb{R}^{n_3})\,,$$

continuous and defined in a certain open neighborhood of a solution (f_0, u_0) of $\mathcal{F}(f_0, u_0) = 0$. We assume that (f_0, u_0) are analytic, and we assume that $\mathcal{F}$ is analytic, meaning it maps analytic functions into such and moreover has a continuation to the families (X_σ), (Y_σ), (Z_σ), $0 \le \sigma \le 1$, satisfying the setup and the hypotheses (H1)–(H3) of Theorem 6.1.1 with some constants $N, R, M, \alpha \ge 0$ and $\gamma > 0$, where

$$\begin{aligned} X_\sigma &: X_0 = C^\mu\,, \quad X_\sigma = A(r_1\sigma, c^\mu)\,, \quad \sigma > 0\,, \\ Y_\sigma &: Y_0 = C^\beta\,, \quad Y_\sigma = A(r_2\sigma, C^\beta)\,, \quad \sigma > 0\,, \\ Z_\sigma &: Z_0 = C^0\,, \quad Z_\sigma = A(r_3\sigma, C^0)\,, \quad \sigma > 0\,, \end{aligned}$$

for some $r_1, r_2, r_3 > 0$. Theorem 6.1.1 and Proposition 6.2.1 now lead immediately to the following statement:

THEOREM 6.2.5 *Let $(\mathcal{F}, f_0, u_0)$ be analytic as above and let $q = 2(\alpha+\gamma)$. Then there exist a $\widehat{C}^{q+\mu}$ neighborhood of f_0, call it D, and a mapping $\psi : D \to \widehat{C}^{q+\beta-\gamma}$ such that*

(i) $\mathcal{F}(f, \psi(f)) = 0$, $f \in D$, *and*
(ii) $\psi(D \cap \widehat{C}^{\ell+\mu}) \subset \widehat{C}^{\ell+\beta-\gamma}$

for all $\ell \geq q$, in particular, $\psi(D \cap C^\infty) \subset C^\infty$. For $f \in D \cap \widehat{C}^{\ell+\mu}$, $\ell \geq q$, we have the estimates $|\psi(f) - u_0|_{\widehat{C}^m} \leq C_{m,\ell}|f - f_0|_{\widehat{C}^{\ell+\mu}}$ for all $m < \ell+\beta-\gamma$. Moreover, if the approximate right inverse η is continuous, denoting $\psi_{\ell+\mu} \equiv \psi | D^{\ell+\mu}$, $D^{\ell+\mu} \equiv D \cap \widehat{C}^{\ell+\mu}$, then for $\ell \geq q$,

$$\psi_{\ell+\mu} : D^{\ell+\mu} \to \widehat{C}^m$$

is continuous for all $m < \ell+\beta-\gamma$; in particular, $\psi_\infty : D^\infty \to C^\infty$ is continuous.

This theorem can cover the situation $\mathcal{F} : C^{\ell+\mu} \times C^{\ell+\beta} \to C^{\ell-\alpha}$, $\ell \geq \alpha$, differentiable in a $C^\mu \times C^\beta$ neighborhood of an analytic solution (f_0, u_0) of $\mathcal{F}(f_0, u_0) = 0$, and where we have only an approximate right inverse $\eta(f, u)$ of $D_2\mathcal{F}(f, u)$ mapping C^ℓ into $C^{\ell+\beta-\gamma}$ for some $\gamma > 0$, and where the ordinary implicit function theorem does not apply. The result is satisfying in the following sense. The loss of derivatives (γ) in solving the nonlinear problem agrees with the loss γ in approximately solving the *linear* problem. We achieved this optimal result by means of the very strong assumption that the unperturbed solution (f_0, u_0) and the mapping $\mathcal{F}$ itself are analytic by using the analytic smoothing. This assumption is certainly a shortcoming from a differentiable point of view. But it seems, as we shall show next, working with the much cruder C^∞ smoothing technique, that if $(\mathcal{F}, f_0, u_0)$ are only of finite but sufficiently higher order, the loss of derivatives is higher. Studying the dependence of this loss on the order, we shall see however, that the situation improves with increasing order.

6.3. C^∞ Smoothing and Mapping of Finite Order

Let $(X_\sigma)_{\sigma\geq 0}$ be a one-parameter family of Banach spaces over the reals $0 \leq \sigma < \infty$, with norms $|\ |_\sigma$, such that for all $0 \leq \sigma' \leq \sigma < \infty$,

$$X_0 \supseteq X_{\sigma'} \supseteq X_\sigma \supseteq X_\infty \equiv \bigcap_{\sigma>0} X_\sigma\,, \tag{6.63}$$

$$|u|_{\sigma'} \leq |u|_\sigma\,, \tag{6.64}$$

for all $u \in X_\sigma$, $\sigma' \leq \sigma$.

DEFINITION 6.3.1 A C^∞ smoothing in (X_σ) is a one-parameter family $(S_t)_{t>0}$ of linear mappings $S_t : X_0 \to X_\infty$, together with constants $C_{\lambda\mu}$ for $0 \leq \lambda, \mu < \infty$, satisfying the following three conditions:

$$\lim_{t\to\infty} |(S_t - 1)u|_0 = 0\,, \quad u \in X_0\,, \tag{6.65}$$

$$|S_t u|_\mu \leq t^{(\mu-\lambda)} C_{\lambda\mu} |u|_\lambda\,, \tag{6.66}$$

for all $u \in X_\lambda$ and $0 \leq \lambda \leq \mu$;

$$|(S_t - 1)u|_\lambda \leq t^{-(\mu-\lambda)} C_{\lambda,\mu} |u|_\mu \tag{6.67}$$

for all $u \in X_\mu$ and $0 \leq \lambda \leq \mu$.

From (6.65) it follows in particular, that $X_\infty \subset X_0$ is dense in X_0. (6.67) says that $S_t u \in X_\infty$ approximates u in X_λ more closely, the smaller the subspace X_μ, $\mu > \lambda$, to which u belongs. (6.66) measures quantitatively for $u \in X_\lambda$ how $S_t u$ blows up in higher norms. Such C^∞ smoothings exist for function spaces $C^{m+\sigma}(M)$, $0 \leq \sigma < \infty$, and some $m \geq 0$ fixed. Over C^∞ compact manifolds (Lemma 6.2.4 of Section 6.2) a trivial consequence of the existence of a C^∞ smoothing is the following well-known convexity statement, which will be the main tool later on:

LEMMA 6.3.2 *Assume* $(X_\sigma)_{\sigma \geq 0}$ *has a* C^∞ *smoothing. Then for all* $0 \leq \lambda_1 \leq \lambda_2$, $\alpha \in [0, 1]$, *and* $u \in X_{\lambda_2}$,

$$|u|_\lambda \leq A_{\alpha,\lambda_1,\lambda_2} |u|_{\lambda_1}^{1-\alpha} |u|_{\lambda_2}^{\alpha}, \qquad \lambda = (1-\alpha)\lambda_1 + \alpha\lambda_2 ,$$

where $A_{\alpha,\lambda_1,\lambda_2} \equiv \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} C_{\lambda,\lambda}^{1-\alpha} C_{\lambda\lambda_2}^{\alpha}$.

PROOF: For all $t > 0$, $u = S_t u + (1 - S_t)u$ and therefore, if $u \in X_{\lambda_2}$,

$$|u|_\lambda \leq |S_t u|_\lambda + |(1 - S_t)u|_\lambda \leq t^{\lambda-\lambda_1} C_{\lambda_1\lambda} |u|_{\lambda_1} + t^{-(\lambda_2-\lambda)} C_{\lambda\lambda_2} |u|_{\lambda_2} .$$

Computation of the minimum of the function in t on the right-hand side of this inequality leads immediately to the result. □

We shall use Lemma 6.3.2 in order to estimate norms of u that are between two known norms.

Setup. In the following we consider three one-parameter families of Banach spaces $X_\sigma, Y_\sigma, Z_\sigma$, $0 \leq \sigma < \infty$, each with a C^∞ smoothing denoted with the same letter $(S_t)_{t>0}$ and a mapping $\mathcal{F}$ with domain of definition in $X_0 \times Y_0$ and with range in Z_0 such that

$$\mathcal{F}(f_0, u_0) = 0 \tag{6.68}$$

for some $(f_0, u_0) \in X_0 \times Y_0$. We assume $\mathcal{F} : B_0 \to Z_0$ to be continuous, where for $\sigma \geq 0$, $B_\sigma \equiv \{(f, u) \in X_\sigma \times Y_\sigma \mid |f - f_0|_\sigma, |u - u_0|_\sigma < 1\}$. Our aim is to solve for given $f \in X_0 \cap B_0$ the equation $\mathcal{F}(f, u) = 0$, assuming f is sufficiently close to f_0. We shall make the following hypotheses:

Hypotheses.

(H1) *Smoothness.* Assume that $\mathcal{F}(f, \cdot) : Y_0 \to Z_0$ is two times differentiable, with the uniform estimate for all $(f, u) \in B_0$,

$$|D_f \mathcal{F}(f, u)|_0 , \qquad |D_2^2 \mathcal{F}(f, u)|_0 \leq M_0 , \tag{6.69}$$

for some $M_0 \geq 1$.

(H2) $\mathcal{F}$ *Uniformly Lipschitz in* X_0. For all $(f, u), (g, u) \in B_0$,

$$|\mathcal{F}(f, u) - \mathcal{F}(g, u)|_0 \leq M_0 |f - g|_0 . \tag{6.70}$$

(H3) *Order.* The triple $(\mathcal{F}, f_0, u_0)$ is of order s, $s > \gamma \geq 1$ (s will be specified later on, γ appears in H4). Here we use the following definition:

DEFINITION $(\mathcal{F}, f_0, u_0)$ is called of *order* s, $1 \leq s < \infty$, if the following three conditions are satisfied:

(i) $(f_0, u_0) \in X_s \times Y_s$,

(ii) $\mathcal{F}(B_0 \cap (X_\sigma \times Y_\sigma)) \subset Z_\sigma$, $1 \leq \sigma \leq s$, and

(iii) there exist constants M_σ, $1 \leq \sigma \leq s$, such that if $(f, u) \in (X_\sigma \times Y_\sigma) \cap B_1$ satisfies $|f - f_0|_\sigma, |u - u_0|_\sigma \leq K$, then

(6.71) $$|\mathcal{F}(f, u)|_\sigma \leq M_\sigma \max\{K, K^\delta\}$$

for some fixed δ, $1 \leq \delta < 2$. If $(\mathcal{F}, f_0, u_0)$ is of order s for all $1 \leq s < \infty$, then we call the triple of order ∞; in this case clearly $(f_0, u_0) \in X_\infty \times Y_\infty$.

(H4) *Existence of an Approximate Right Inverse of Loss* γ, $1 \leq \gamma < s$. For every $(f, u) \in B_\gamma$ there exists a linear map $\eta(f, u)(\cdot) \in L(Z_\gamma, Z_0)$ such that for all $z \in Z_\gamma$

(6.72) $$|\eta(f, u)(z)|_0 \leq M_0 |z|_\gamma ,$$

(6.73) $$|(D_2\mathcal{F}(f, u) \circ \eta(f, u) - 1)(z)|_0 \leq M_0 |\mathcal{F}(f, u)|_\gamma \cdot |z|_\gamma .$$

Moreover, for all $\gamma \leq \sigma \leq s$, if $(f, u) \in B_\gamma \cap (X_\sigma \times Y_\sigma)$, then $\eta(f, u) \in L(Z_\sigma, Y_{\sigma-\gamma})$, and if $|f - f_0|_\sigma, |u - u_0|_\sigma \leq K$, then

(6.74) $$|\eta(f, u)(\mathcal{F}(f, u))|_{\sigma-\gamma} \leq M_\sigma \max\{K, K^\delta\}$$

with δ as in (H3). Actually, we will need estimates (6.72) and (6.73) only for $z = \mathcal{F}(f, u)$.

We call η continuous if $\eta : \beta_\gamma \cap (X_\sigma \times Y_\sigma) \to L(Z_\sigma, Y_{\sigma-\gamma})$ is continuous for all $\gamma \leq \sigma \leq s$.

REMARK. Differential operators on function spaces over C^∞ compact manifolds are included in the setup and hypotheses above, since one just chooses $X_\sigma = C^{m+\sigma}(M)$ for some fixed $m \geq 0$, and so on. The condition (6.71) will normally hold with $\delta = 1$ for any map $\mathcal{F}$ involving partial differentiation or functional substitution. For instance, $\Phi(x, D^n f(x), D^m u(x))$ grows at most linearly with $|f|_{s+n}, |u|_{s+m}$. Indeed, if $|f|_0, |u|_0 \leq 1$, then there is for every s a constant C, such that $|\Phi(D^n f, D^m u)|_s \leq C(1 + |f|_{s+n} + |u|_{s+m})$. This at first glance seems surprising, but it follows easily using the chain rule and Lemma 6.3.2. Analogously for compositions, if $|u_j|_1 \leq M$, $1 \leq j \leq n$, then for each $s \geq 1$ there is a constant $C > 0$ depending only on M such that $|f \circ u|_s \leq C(|f|_s + |f|_1 \sum_{j=1}^n |u_j|_s)$. Note that we have to hold down the $|\ |_1$ norms; for this reason the balls B_1 are introduced in (H3)(iii).

THEOREM 6.3.3 *Let* $\alpha, \kappa, \lambda, \rho, \gamma, \delta$, *and* s *be positive real numbers satisfying the following set of inequalities*:

(6.75) $$1 \leq \delta \leq \kappa < 2, \quad \delta < \alpha, \quad 1 \leq \gamma \leq \rho < \lambda < s .$$

(6.76) $$\lambda > \max\{2\kappa\gamma\delta(2-\kappa)^{-1}, \ \kappa(\gamma\delta + \rho\kappa)\} .$$

(6.77) $$s > \max\{\alpha\gamma(\alpha\delta)^{-1}, \lambda + \alpha\gamma(\kappa - \delta)^{-1}\} .$$

Let $(\mathcal{F}, f_0, u_0)$ *be of order* s *and satisfy* (H1)–(H4) *with a loss* γ, *and with* δ *as in* (H3). *Then there exist an open neighborhood* $D_\lambda \subset X_\lambda$ *of* f_0, $D_\lambda = \{f \in X_\lambda \mid |f - f_0|_\lambda < C\}$, *and a mapping* $\psi : D_\lambda \to Y_\rho$ *such that*

(i) $\mathcal{F}(f, \psi(f)) = 0,\ f \in D_\lambda$,

(ii) $|\psi(f) - u_0|_\rho \leq C^{-1}|f - f_0|_\lambda$.

Moreover, if η is continuous, then $\psi : D_\lambda \to Y_\rho$ is continuous.

PROOF: The proof uses again an iteration technique similar to the Newton method in which one replaces the inverse, which need not exist, by the approximate right inverse, modified by a double C^∞ smoothing. The first smoothing is standard; it will be introduced in Y_0 in order to catch up with the loss γ at each iteration step. The second smoothing, however, approximates elements in $D_\lambda \subset X_\lambda$ by smoother ones (in analogy with our procedure in the C^ω smoothing) in order to retain maximal smoothness during the iteration in the X_0-space. This is at the expense of the accuracy in approximately solving the linearized equation. In a different context, such a procedure is suggested in the Pisa lectures of J. Moser [**57**].

Our guiding principle will be the following: We shall estimate the lowest norms $|\ |_0$ rather carefully to keep them down, but the highest norms $|\ |_s$ only crudely by letting them grow, using the crude assumptions (6.71) and (6.74) on the growth of the higher norms. The norms in between are then taken care of by the convexity lemma, Lemma 6.3.2. It is essential to suppose $\delta < 2$ in order to achieve a contraction. Let $M \geq \max\{A_{\alpha\lambda\mu}, M_\sigma, C_{\lambda\mu}$ for $0 \leq \alpha \leq 1$, $0 \leq \lambda, \mu \leq s$, and $0 \leq \sigma \leq s\}$, $A_{\alpha\lambda\mu}$ as in Lemma 6.3.2, M_σ as in (H1)–(H4), and $C_{\lambda\mu}$ as in the definition of the C^∞ smoothing.

Define

$$D_\lambda = \{f \in X_\lambda \mid |f - f_0|_\lambda < \varepsilon\} \tag{6.78}$$

for some $0 < \varepsilon < \varepsilon_0$ sufficiently small, to be determined later on. We define a sequence $(\phi_j)_{j\geq 0}$ of linear mappings $\phi_j : D_\lambda \to X_\infty$ by means of the C^∞ smoothing as follows: $j = 0 : \phi_0(f) = f_0$ and for $j \geq 1$,

$$\phi_j(f) - f_0 = S_{\tau_j}(f - f_0)\,, \tag{6.79}$$

where $\tau_j = Q^{(\kappa^j)}$ for some $Q > 1$ sufficiently large to be chosen later. Observe $\tau_j \to \infty$, since $\kappa > 1$. From $\phi_j(f) - f = (S_{\tau_j} - 1)(f - f_0)$, we conclude $\lim_{j\to\infty} |\phi_j(f) - f|_\mu = 0$ for all $0 \leq \mu < \lambda$ by means of (6.67).

We shall construct inductively a sequence $(\psi_j)_{j\geq 0}$ of mappings $\psi_j : D_\lambda \to Y_\infty$, starting with $\psi_0(f) = u_0$, and for $j \geq 0$

$$\psi_{j+1}(f) - \psi_j(f) = S_{t_{j+1}}\eta\big(\phi_{j+1}(f), \psi_j(f)\big)\big(\mathcal{F}\big(\phi_{j+1}(f), \psi_j(f)\big)\big) \tag{6.80}$$

with $t_j = \tau_j^\alpha = Q^{\alpha\kappa^j}$. Note that we use two different rates of approximations, employing S_τ (in X_σ) and S_{t_j} (in Y_σ). We shall show by induction that if ε_0 is sufficiently small and $f \in D_\lambda$ satisfies $|f - f_0|_\lambda \leq \nu\varepsilon_0$ for some $0 \leq \nu \leq 1$, then the following statements S_n hold for $n \geq 1$:

(S_n1) $(\phi_n(f), \psi_n(f)) \in B_\gamma \cap (X_\infty \times Y_\infty)$ and

$$|\mathcal{F}(\phi_n(f), \psi_n(f))|_0 \leq \frac{\nu}{2} Q^{-\lambda\kappa^n}\,,$$

(S_n2) $|\psi_n(f) - \psi_{n-1}(f)|_0 \leq \nu \cdot 4M^4 Q^{-(\lambda-\kappa\gamma\delta)\kappa^{n-1}}$, and

(S_n3) $|\psi_n(f) - \psi_{n-1}(f)|_s \leq \nu \cdot Q^{(s-\lambda)\kappa^{n+1}}$.

We introduce the following abbreviated notation:

(6.81) $$f_j = \phi_j(f) \quad \text{and} \quad u_j = \psi_j(f), \quad j \geq 0.$$

Step 1. Check that $f_j \in B_\gamma \cap X_\infty$ if ε_0 is sufficiently small. Recalling $1 \leq \gamma < \lambda < s$, we get immediately from definition (6.79) by means of property (6.66) of S_t the following: For $j \geq 1$

(6.82) $$|f_j - f_0|_\gamma \leq M|f - f_0|_\gamma \leq M|f - f_0|_\lambda,$$

(6.83) $$|f_j - f_0|_s \leq M|f - f_0|_\lambda \cdot Q^{(s-\lambda)\kappa^j},$$

and therefore $|f_j - f_0|_1 \leq |f_j - f_0|_\gamma < 1$ if $\varepsilon_0 \leq M^{-1}$.

Step 2. Statement S_1 follows from the smallness condition by choosing ε_0 sufficiently small. We assume now the validity of the statements S_n for $1 \leq n \leq j$ and prove S_{j+1}. We first prove $S_{j+1}2$. We already know $(f_{j+1}, u_j) \in B_\gamma$ from step 1 and S_j1, and from definition (6.80) we conclude by means of property (6.66) and (6.72)

$$\begin{aligned} |u_{j+1} - u_j|_0 &= |S_{t_j+1}\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_0 \\ &\leq M|\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j)|_0 \\ &\leq M^2|\mathcal{F}(f_{j+1}, u_j)|_\gamma. \end{aligned} \qquad (6.84)$$

In order to estimate this γ-norm, we shall estimate the 0-norm and the s-norm and then use the convexity lemma, Lemma 6.3.2. We can write $\mathcal{F}(f_{j+1}, u_j) = \mathcal{F}(f_{j+1}, u_j) - \mathcal{F}(f_j, u_j) + \mathcal{F}(f_j, u_j)$ and find by means of (H2)

$$|\mathcal{F}(f_{j+1}, u_j)|_0 \leq M|f_{j+1} - f_j|_0 + |\mathcal{F}(f_j, u_j)|_0.$$

Using (6.79) and property (6.67), we can estimate

$$\begin{aligned} |f_{j+1} - f_j|_0 &\leq |S_{t_j+1} - 1)(f - f_0)|_0 + |(S_{t_j} - 1)(f - f_0)|_0, \\ &\leq 2M|f - f_0|_\lambda \cdot Q^{-\lambda\kappa^j}. \end{aligned}$$

Therefore we find, using S_j1, the estimate

$$\mathcal{F}(f_{j+1}, u_j)|_0 \leq 2M^2|f - f_0|_\lambda Q^{-\lambda\kappa^j} + \left(\frac{\nu}{2}\right)Qa^{-\lambda\kappa^j} \leq \left(2M^2\nu\varepsilon_0 + \frac{\nu}{2}\right)Q^{-\lambda\kappa^j},$$

and hence choosing $\varepsilon_0 \leq (2M)^{-2}$, we have

(6.85) $$|\mathcal{F}(f_{j+1}, u_j)|_0 \leq \nu Q^{-\lambda\kappa^j}.$$

Observe next that

(6.86) $$|u_j - u_0|_s, |f_{j+1} - f_0|_s \leq 2\nu Q^{(s-\lambda)\kappa^{j+1}}.$$

Indeed, from the induction statements S_n3, $1 \leq n \leq j$, we conclude $|u_j - u_0)|_s \leq \sum_{n=1}^{j} |u_n - u_{n-1}|_s \leq 2\nu Q^{(s-\lambda)\kappa^{j+1}}$ provided Q is sufficiently large. Here we have used $s > \lambda$. Using now (6.71), we conclude from (6.86), since $2^\delta \leq 4$, $\nu^\delta \leq \nu$, and $Q \leq Q^\delta$,

(6.87) $$|\mathcal{F}(f_{j+1}, u_j)|_s \leq 4M \cdot \nu Q^{\delta(s-\lambda)\kappa^{j+1}}.$$

We have assumed the existence of a C^∞ smoothing in (Z_σ); therefore, we get by means of Lemma 6.3.2, together with estimates (6.85) and (6.87),

$$
\begin{aligned}
|\mathcal{F}(f_{j+1}, u_j)|_\gamma &\le M|\mathcal{F}(f_{j+1}, u_j)|_0^{1-\gamma/s}|\mathcal{F}(f_{j+1}, u_j)|_s^{\gamma/s}, \\
&\le 4M^2\nu \cdot Q^{-(\lambda-\kappa\gamma\delta+\frac{1}{2}\lambda\gamma(\kappa\delta-1))\kappa^j}, \\
&\le 4M^2\nu \cdot Q^{-(\lambda-\kappa\gamma\delta)\kappa^j}. \qquad (6.88)
\end{aligned}
$$

since $\kappa\delta > 1$. Using (6.88) and (6.84), we obtain the estimate $|u_{j+1} - u_j|_0 \le 4M^4 \cdot \nu Q^{-(\lambda-\kappa\gamma\delta)\kappa^j}$, and so we have proved $S_{j+1}2$.

We now turn to the proof of $S_{j+1}3$. Here we make use of (6.74) from (H4), estimate by means of property (6.66), and recall (6.86) as follows:

$$
\begin{aligned}
|u_{j+1} - u_j|_s &\le Mt_{j+1}^\gamma|\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_{s-\gamma} \qquad (6.89)\\
&\le 4M^2 \cdot \nu Q^{\alpha\delta\kappa^{j+1}} \cdot Q^{\delta(s-\lambda)\kappa^{j+1}}.
\end{aligned}
$$

Since according to (6.77) $\alpha\gamma + \delta(s-\lambda) < (s-\lambda)\kappa$, the right-hand side can be further estimated by $\le \nu \cdot Q^{(s-\lambda)\kappa^{j+2}}$ if Q is sufficiently large (independent of j); hence we have proved $S_{j+1}3$. In order to prove $S_{j+1}1$, we first show that $|u_{j+1} - u_0|_\gamma < 1$. Calling $v_{j+1} \equiv u_{j+1} - u_j$ for $j \ge 0$, we get from $S_{j+1}2$ and $S_{j+1}3$, by means of Lemma 6.3.2,

$$
|v_{j+1}|_\gamma \le M|v_{j+1}|_0^{1-\gamma/s}|v_{j+1}|_s^{\gamma/s} \le \nu 4M^4 Q^{-\xi\kappa^j},
$$

with $\xi = \lambda - \kappa(\gamma\delta + \gamma\kappa) > \lambda - \kappa(\gamma\delta + \rho\kappa) > 0$, where we have used (6.76) and $\gamma \le \rho$ in (6.75). Therefore, for Q sufficiently large, we can estimate

$$
|u_{j+1} - u_0|_\gamma \le \sum_{n=0}^{j+1} |v_{j+1}|_\gamma < 1.
$$

Setting now

$$
Q(f_{j+1}, u_{j+1}, u_j) \equiv \mathcal{F}(f_{j+1}, u_{j+1}) - \mathcal{F}(f_{j+1}, u_j) - D_2\mathcal{F}(f_{j+1}, u_j)(u_{j+1}, u_j),
$$

we have by (H1) the estimate $|Q(f_{j+1}, u_{j+1}, u_j)|_0 \le M|u_{j+1} - u_j|_0^2$, where we have used the Taylor estimate.

We now write

$$
\begin{aligned}
\mathcal{F}(f_{j+1}, u_j) = &-(D_2\mathcal{F}(f_{j+1}, u_j) \circ \eta(f_{j+1}, u_j) - 1)(\mathcal{F}(f_{j+1}, u_j)) \\
&+ D_2\mathcal{F}(f_{j+1}, u_j)(1 - S_{t_{j+1}})\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j)) \\
&+ Q(f_{j+1}, u_{j+1}, u_j),
\end{aligned}
$$

which will be estimated as follows:

$$
\begin{aligned}
|\mathcal{F}(f_{j+1}, u_j)|_0 \le\, &|(D_2\mathcal{F}(f_{j+1}, u_j) \circ \eta(f_{j+1}, u_j) - 1)(\mathcal{F}(f_{j+1}, u_j)|_0 \\
&+ |D_2\mathcal{F}(f_{j+1}, u_j)|\; |(1 - S_{t_{j+1}})\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_0 \\
&+ M|S_{t_{j+1}}\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_0^2.
\end{aligned}
$$

We bound the first term on the right-hand side by $M|\mathcal{F}(f_{j+1}, u_j)|_\gamma^2$, using (6.73). We bound the second term, using (6.67) and (6.69), by

$$Mt_{j+1}^{-(s-\gamma)}|\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_{s-\gamma}$$

and the third term, using (6.66) and (6.72), by

$$M^2|\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_0^2 \leq M^4|\mathcal{F}(f_{j+1}, u_j)|_\gamma^2 .$$

We thus find

$$|\mathcal{F}(f_{j+1}, u_j)|_0 \leq 2M^4|\mathcal{F}(f_{j+1}, u_j)|_\gamma^2 + Mt_{j+1}^{-(s-\gamma)}|\eta(f_{j+1}, u_j)(\mathcal{F}(f_{j+1}, u_j))|_{s-\gamma} ,$$

which is bounded, using (6.88) and again (6.74) with (6.86), by

$$32M^8 \cdot \nu \cdot Q^{-2(\lambda-\kappa\gamma\delta)\kappa^j} + 4M^2 \cdot \nu \cdot Q^{-\alpha(s-\gamma)\kappa^{j+1}} Q^{\delta(s-\lambda)\kappa^{j+1}} .$$

But this is bounded in turn by using (6.76) and (6.77) by $(\nu/2)Q^{-\lambda\kappa^{j+1}}$ if Q is sufficiently large; hence we have proved $\mathrm{S}_{j+1}1$.

Step 3. We now consider the consequences of S_n, $n \geq 1$. The sequence $(u_j)_{j\geq 0}$ is a Cauchy sequence in Y_ρ; indeed, from S_j2 and S_j3 for all $j \geq 1$, we conclude for $v_{j+1} \equiv u_{j+1} - u_j$, $j \geq 0$,

$$|v_{j+1}|_\rho \leq M|v_{j+1}|_0^{1-\rho/s}|v_{j+1}|_s^{\rho/s} \leq \nu \cdot 4M^5 Q^{-\eta\kappa^j} , \tag{6.90}$$

with $\eta \equiv \lambda - \kappa(\gamma\delta + \kappa\rho) > 0$ according to (6.76). We define $\psi : D_\lambda \to Y_\rho$ by $\psi(f) = \lim_{j\to\infty} \psi_j(f)$ in Y_ρ. We know $\lim_{j\to\infty} \phi_j(f) = f$, and since $\mathcal{F}(f_j, u_j) \to 0$ as $j \to \infty$ according to S_j1, we conclude from the continuity of $\mathcal{F}$ that $\mathcal{F}(f, \psi(f)) = 0$ for all $f \in D_\lambda$. Moreover, (6.90) gives for Q sufficiently large for all $f \in D_\lambda$ such that $|f - f_0|_\lambda \leq \nu\varepsilon_0$,

$$|\psi(f) - u_0|_\rho \leq \sum_{j\geq 1} |v_j|_\rho \leq \nu . \tag{6.91}$$

Therefore, if $|f - f_0|_\lambda \leq \varepsilon_0$, we choose $\nu = \varepsilon_0^{-1}|f - f_0|_\lambda$ and get the required estimate (ii) for the solution. If η is continuous, then the functions $\psi_j : D_\lambda \to Y_\rho$ are continuous, and since the limit $\psi(f) = \lim_{j\to\infty} \psi_j(f)$ in Y_ρ is uniform in $f \in D_\lambda$, ψ is continuous. The proof of Theorem 6.3.3 is complete. □

Observe we make a distinction between the orders of $(\mathcal{F}, f_0, u_0)$ and the smoothness assumption represented by λ, which allows us to study the dependence of λ and the loss $\lambda - \rho$ of the order s.

COROLLARY 6.3.4 *If* $\delta = 1$, *then for all* $s, s \geq 8\gamma$, *the following holds: Let* $\lambda(s) \equiv 2\gamma + 6a\gamma^2 s^{-1}$, *with* $a = \frac{7}{3}$; *there is in* $X_{\lambda(s)}$ *a neighborhood* $D_{\lambda(s)} = \{f \in X_{\lambda(s)} \mid |f - f_0|_{\lambda(s)} < C(s)\}$ *and a mapping* $\psi_s : D_{\lambda(s)} \to Y_\gamma$ *such that for all* $f \in D_{\lambda(s)}$,

(i) $\mathcal{F}(f, \psi_s(f)) = 0$ *and*
(ii) $|\psi_s(f) - u_0|_\gamma \leq C(s)^{-1}|f - f_0|_{\lambda(s)}$.

PROOF: Take $\delta = 1$, $\alpha = \frac{7}{6}$, $\kappa = 1 + a\gamma s^{-1}$, $a = \frac{7}{3}$, $\rho = \gamma$, and $\lambda = 2\gamma + 6a\gamma^2 s^{-1}$. These numbers satisfy the set of inequalities (6.75)–(6.77) if $s \geq 8\gamma$; hence the result follows from Theorem 6.3.3. □

If $\delta = 1$. then we have for the minimal order 8γ : $3\gamma < \lambda(8\gamma) < 4\gamma$, and with increasing order s. the loss of derivatives. $\lambda - \rho$, in solving the nonlinear problem tends to the loss γ in approximately solving the linearized problem

$$\lambda(s) - \gamma = \gamma + O(s^{-1}) . \tag{6.92}$$

The neighborhood $D_\lambda(s)$. however. depends on s. and we cannot conclude that $\lambda(\infty) - \gamma = \gamma$. in contrast to the result for analytic mappings.

COROLLARY 6.3.5 *Let $\delta = 1$ and let $(\mathcal{F}, f_0, u_0)$ be of order ∞. Then for every small $\varepsilon > 0$, there is a neighborhood $D \subset X_{2\gamma+\varepsilon}$ of f_0, $D = \{f \in X_{2\gamma+\varepsilon} \mid |f - f_0|_{2\gamma+\varepsilon} < C_\varepsilon\}$ and a mapping $\psi D \to Y_\gamma$ such that for $f \in D$,*

(i) $\mathcal{F}(f, \psi(f)) = 0$,
(ii) $|\psi(f) - u_0|_\gamma \le C_\varepsilon^{-1}|f - f_0|_{2\gamma+\varepsilon}$, *and moreover,*
(iii) $\psi(D \cap X_\infty) \subset Y_\infty$.

PROOF: It remains to prove (iii). Let $\varepsilon > 0$ be fixed. We shall show, since the order of $(\mathcal{F}, f_0, u_0)$ is ∞, that there is a τ, $0 < \tau < 1$. such that for every $\mu > 2\gamma + \varepsilon$ the following holds true: If $f \in D \cap X_\mu$. then $\psi(f) \in Y_\nu$, with $\nu = \tau(\mu - 1)$ and $|\psi(f) - u_0|_\nu \le C_\mu|f - f_0|_\mu^\tau$. Let $f \in D \cap X_\mu$, and let $f_j - f_0 = S_{\tau_j}(f - f_0)$ be the sequence (6.79) involved in the construction of $\psi(f)$, with $\tau_j = Q^{\kappa^j}$, $Q > 1$ and fixed. By means of property (6.67), we then have $|f_{j+1} - f_j|_{\mu-1} \le C_1 \cdot Q^{-\kappa^j}|f - f_0|_\mu$ for some constant C_1 depending on μ; hence we find $|f_j - f_0|_{\mu-1} \le C_2 \cdot |f - f_0|_\mu$ for some C_2 depending on μ.

We shall first show that there is a constant $C > C_2$ depending on μ such that for all $n \ge 1$,

$$|u_n - u_0|_{\mu-1} \le C|f - f_0|_\mu Q^{\beta\kappa^n} \tag{6.93}$$

for some β. $\beta > \gamma\alpha\kappa(\kappa - 1)^{-1}$, with u_n as in (6.80) and (6.81),

$$u_{n+1} = u_n - S_{t_{n+1}}\eta\big(f_{n+1}, u_n\big)\big(\mathcal{F}(f_{n+1}, u_n)\big) .$$

Having such a constant $C > C_2$ for all $n \le N$, we get for $n > N$ by means of (6.66) and (6.74),

$$\begin{aligned}|u_{n+1} - u_0|_{\mu-1} &\le |u_n - u_0|_{\mu-1} + t_{n+1}^\gamma C_3\big|\eta(f_{n+1}, u_n)(\mathcal{F}(f_{n+1}, u_n)\big|_{\mu-1-\gamma}\\ &\le |u_n - u_0|_{\mu-1} + C_4 \cdot C \cdot Q^{\alpha\gamma\kappa^{n+1}}|f - f_0|_\mu Q^{\beta\kappa^n}\\ &\le (1 + C_4 Q^{\alpha\gamma\kappa^{n+1}})C|f - f_0|_\mu Q^{\beta\kappa^n} .\end{aligned}$$

If $\beta > \alpha\gamma\kappa(\kappa - 1)^{-1}$, we can choose N so large that for $n > N$, $(1 + C_4 Q^{\alpha\gamma\kappa^{n+1}})Q^{\beta\kappa^n} \le Q^{\beta\kappa^{n+1}}$; hence we have (6.93) for all $n \ge 1$. Putting $v_n = u_n - u_{n-1}$, we get for all $\nu < \mu - 1$ by means of Lemma 6.3.2, (6.93), and the induction statement S_n2, for $\tau = \nu(\mu - 1)^{-1}$,

$$\begin{aligned}|v_n|_\nu &\le C_5|v_n|_0^{1-\tau}|v_n|_{\mu-1}^\tau \le C_5(C|f - f_0|_\mu)^\tau Q^{-(\lambda-\kappa\gamma)(1-\tau)\kappa^{n-1}} Q^{\beta\kappa\tau\kappa^{n-1}}\\ &\le C_6|f - f_0|_\mu^\tau Q^{-\xi\kappa^{n-1}} .\end{aligned} \tag{6.94}$$

where $\xi = (\lambda - \kappa\gamma) - \tau(\lambda - \kappa\gamma + \beta\kappa)$, which is > 0 if we choose

$$\tau < (\lambda - \kappa\gamma)(\lambda - \kappa\gamma + \beta\kappa)^{-1} .$$

Here C_5 and C_6 depend on μ. From (6.94) we conclude that v_n is a Cauchy sequence in Y_ν, $\nu = \tau(\mu - 1)$, and therefore $\psi(f) \in Y_\nu$ for $f \in D \cap X_\mu$, and since τ is independent of μ, the statement follows. □

6.4. A Theorem of Kolmogorov, Arnold, and Moser

We shall apply Theorems 6.1.1 and 6.1.6 to a model problem. Before introducing the problem, let's recall a result for vector fields on a 2-dimensional torus $\mathbb{T}^2$, contrasting with the problem considered later on. If X is any smooth vector field on $\mathbb{T}^2$ such that

(i) X is not singular on $\mathbb{T}^2$ (i.e., $X(x) \neq 0$, $x \in \mathbb{T}^2$), and
(ii) X has no periodic orbits.

Then there is a homeomorphism of the torus that maps the flow of X without parametrization into a linear flow; i.e., there is a smooth function $h \in C^\infty(\mathbb{T}^2)$, $h > 0$, such that the flow ϕ_s belonging to $h \cdot X$ is topologically conjugate to a linear flow by a homeomorphism ψ

$$\psi^{-1} \circ \phi_s \circ \psi : (x_1, x_2) \to (x_1 + s, x_2 + \rho \cdot s)\,. \tag{6.95}$$

ρ is the so-called rotation number of X and is a topological invariant.

The proof is reduced to the Denjoy theorem of circle mappings. The homeomorphism ψ is unique up to a linear map, and it makes sense to ask whether ψ is smooth. One can show that there are irrational ρ such that ψ is not even absolutely continuous despite the fact that ϕ_s is smooth. In contrast to this result for $\mathbb{T}^2$, we shall consider a perturbation problem for vector fields on a torus $\mathbb{T}^n$, $n \geq 2$, given by

$$\sum_{k=1}^{n} \phi_k(x) \frac{\partial}{\partial x_k}\,, \qquad \phi = (\phi_1, \ldots, \phi_n)\,, \tag{6.96}$$

with ϕ_k functions on $\mathbb{R}^n$ periodic with period 2π. The vector fields are close to constant vector fields

$$\sum_{k=1}^{n} \omega_k \frac{\partial}{\partial x_k}\,, \qquad \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^k\,. \tag{6.97}$$

We shall look at the question of structural stability of such constant vector fields under the group of diffeomorphisms of $\mathbb{T}^n$. In other words, we ask: Given a vector field $\phi = \omega + f$, f small, does there exist a diffeomorphism g of $\mathbb{T}^n$, $x = g(\xi) = \xi + v(\xi)$, v a vector on $\mathbb{R}^n$ periodic with period 2π, transforming ϕ into the constant vector field ω? This means

$$dg(\xi)^{-1} \cdot \phi \circ g(\xi) = \omega\,. \tag{6.98}$$

This is clearly impossible, in general, for the following simple reason: Even if $\beta \in \mathbb{R}^n$ is a constant vector field close to ω, it cannot be transformed into ω unless $\beta = \omega$; otherwise the flow $\xi = \beta t$ would be transformed into the flow of ω by $\omega t + \text{const} = \beta t + v(\beta t)$, all $t > 0$, and therefore, since v is periodic, $\omega = \beta$. We shall therefore admit changes of the given vector ϕ by a constant vector $\lambda \in \mathbb{R}^n$ and ask the modified and artificial question: Does there exist for a given vector field

$\phi = \omega + f$. f small. a constant vector $\lambda \in \mathbb{R}^n$ and a diffeomorphism $g \in \mathrm{Diff}(\mathbb{T}^n)$ such that

(6.99) $$dg(\xi)^{-1}(\omega + f + \lambda) \circ g(\xi) = \omega\,?$$

We reformulate the problem in terms of a functional. Observing $g(\xi) = \xi + v(\xi)$. for given f we seek a solution $u = (v.\ \lambda)$ of the mapping

(6.100) $$\mathcal{F}(f.\ u) = f \circ (\mathrm{id} + v) + \lambda - \partial v\,.$$

where ∂ is the following partial differential operator with constant coefficients $\omega \in \mathbb{R}^n$:

(6.101) $$\partial = \sum_{k=1}^{n} \omega_k \frac{\partial}{\partial x_k}\,.$$

Clearly $\mathcal{F}(0.\ 0) = 0$; $\mathcal{F}$ is continuous as a map

$$\mathcal{F} : C^0(\mathbb{T}^n, \mathbb{R}^n) \times \left(C^1(\mathbb{T}^n, \mathbb{R}^n) \times \mathbb{R}^n\right) \to C^0(\mathbb{T}^n. \mathbb{R}^n)$$

and differentiable for all $\ell \geq 1$, $C^\ell \times (C^{\ell+1} \times \mathbb{R}^n) \to C^{\ell-1}$.

In order to apply any kind of implicit function theorem, we have to look at $D_2\mathcal{F}(f, u)$ for $(f.\ u) = (0.\ 0)$. We have

(6.102) $$D_2\mathcal{F}(0, 0)\hat{u} = \hat{\lambda} - \partial\hat{v}$$

where $\hat{u} = (\hat{v}.\ \hat{\lambda})$. The following well-known small-divisor lemma says that for certain $\omega \in \mathbb{R}^n$, the operator (6.102) has a right inverse, which, however, is unbounded:

LEMMA 6.4.1 *Let ω satisfy the following infinite set of inequalities:*

(6.103) $$|(\omega.\ k)|^{-1} \leq C_0|k|^\tau$$

for all integer vectors $|k| = \sum_{i=1}^n |k_i| > 0$. Here C_0 is a positive constant and τ some number $> n-1$. Assume $g \in A(\sigma, C^0)$ with mean value $[g] = 0$. Then there is a unique $v \in A(\sigma', C^0)$ for all $\sigma' < \sigma$ such that $[v] = 0$, satisfying $\partial v = g$. Moreover,

$$|v|_{\sigma'} \leq \frac{C}{(\sigma - \sigma')^\nu}|g|_\sigma$$

for all $\sigma' < \sigma$, and $\nu = \tau + 1 > n$. Here C denotes a constant depending on τ, n, and C_0 only.

PROOF: We merely prove the statement for $\nu = \tau + n$; the estimate stated in the lemma is more delicate and is based on the observation that only a few denominators $(\omega.\ k)$ actually are small. The solution can easily be found by Fourier expansion. Let $g(x) = \sum_{k\neq 0} g_k e^{i(k.x)}$; then we have the solution $v(x) = \sum_{k \neq 0} v_k e^{i(k,x)}$, with

(6.104) $$v_k = \frac{g_k}{i(\omega.\ k)}\,.$$

In order to estimate v, we use the fact that $g \in A(\sigma, C^0)$, which gives for the Fourier coefficients $|g_k| \leq e^{-|k|\sigma} \cdot |g|_\sigma$, by shifting the surface of integration to

$|\mathrm{Im}\, x_\nu| = \pm\sigma$. Therefore we can estimate the solution v by means of (6.103) in $|\mathrm{Im}\, x| \le \sigma' < \sigma$.

$$\begin{aligned}|v(x)| &\le \sum_{k\neq 0} \frac{|g_k|}{|(\omega, k)|} e^{|k|\sigma'} \le C_0 \sum_{k\neq 0} |k|^\tau e^{-|k|(\sigma-\sigma')} |g|_\sigma \\ &\le C_0 \int_{\mathbb{R}^n} |x|^\tau e^{-|x|(\sigma-\sigma')}\, dx \cdot |g|_\sigma \\ &\le C_1(\sigma-\sigma')^{-(\tau+n)} \cdot |g|_\sigma \,.\end{aligned}$$

v is real analytic; indeed, $\bar{g}_k = g_{-k}$ and hence by (6.104) $\bar{v}_k = v_{-k}$. □

Since the spectrum of ∂ has 0 as a cluster point, it is not clear at all whether the operator $D_2\mathcal{F}(f, u)$, $(f, u) \neq 0$ has an unbounded right inverse. But here the approximate right inverse will come in. First, we state the result.

THEOREM 6.4.2 (Kolmogorov, Arnold, Moser) *Let ω satisfy (6.103), $q = 2\tau + 6$, and $\gamma = \tau + 2$. Then there exist an open neighborhood D of 0 in $\widehat{C}^q$ (see definition on p. 120) and a mapping $\psi : D \to (\widehat{C}^{q+1-\gamma} \times \mathbb{R}^n)$ such that*

(i) *$\mathcal{F}(f, \psi(f)) = 0$ for all $f \in D$ and*
(ii) *$\psi(D \cap \widehat{C}^\ell) \subset (\widehat{C}^{\ell+1-\gamma} \times \mathbb{R}^n)$ for all $\ell \ge q$.*

In particular, $\psi(D \cap C^\infty) \subset (C^\infty \times \mathbb{R}^n)$. Moreover, the mappings $\psi_\ell : D^\ell \to \widehat{C}^m \times \mathbb{R}^n$ are continuous for all $\ell \ge q$ and $m < \ell + 1 - \gamma$.

PROOF: We shall show that $\mathcal{F}$ satisfies the setup and the assumptions (H1)–(H3) of Theorem 6.1.1 with $\alpha = 1$ and $\gamma = \tau + 2$. The statement then follows immediately from Theorem 6.1.6 and Proposition 6.2.1; $\mathcal{F}$ maps analytic functions into analytic functions, but we have to extend $\mathcal{F}$ to families $X_\sigma, Y_\sigma, Z_\sigma$ of real holomorphic functions defined in complex strips. Define

$$\begin{aligned} &X_\sigma : X_0 = C^0\,, && X_\sigma = A(2\sigma, C^0)\,, && \sigma > 0 \\ &Y_\sigma : Y_0 = C^1 \times \mathbb{R}^n\,, && Y_\sigma = A(\sigma, C^1) \times \mathbb{R}^n\,, && \sigma > 0 \qquad (6.105) \\ &Z_\sigma : Z_0 = C^0\,, && Z_\sigma = A(\sigma, C^0)\,, && \sigma > 0\,. \end{aligned}$$

We know $\mathcal{F}(0, 0) = 0$, and we define the open neighborhoods B_σ of (0,0) as follows:

$$B_\sigma = \left\{ (f, u) \in X_\sigma \times Y_\sigma \mid |f|_\sigma < 1, |u|_\sigma < R < \frac{1}{3n} \right\}. \qquad (6.106)$$

Note that $|u|_\sigma$ stands for $|u|_{\sigma, C^1}$.

We first show that for $\sigma \ge 0$, $\mathcal{F} : B_\sigma \to Z_\sigma$ and is continuous. Clearly $\partial v \in Z_\sigma$ if $v \in Y_\sigma$. Next we show that if $f \in X_\sigma$ and $v \in Y_\sigma$, then $f\circ(\mathrm{id}+v) \in Z_\sigma$, which is a question of domain of definition. We claim $|\mathrm{Im}(\xi + v(\xi))| < \frac{3}{2}\sigma$ if $|\mathrm{Im}\, \xi| < \sigma$. To show that $|\mathrm{Im}\, v(\xi)| < \frac{\sigma}{2}$ for $|\mathrm{Im}\, \xi| < \sigma$, we use the fact that v is real; $\mathrm{Im}\, v(\xi) = \frac{1}{2i}(v(\xi) - \overline{v(\xi)}) = \frac{1}{2i}(v(\xi) - v(\bar{\xi}))$; applying the mean value theorem and using the estimate $|dv(\xi)| < \frac{1}{2}$ from (6.106), we have

$$|\mathrm{Im}\, v(\xi)| = \left| \int_0^1 d\mu\, dv(\bar{\xi} + \mu(\xi - \bar{\xi}))\mathrm{Im}\, \xi \right| < \frac{1}{2}|\mathrm{Im}\, \xi| . \tag{6.107}$$

To get (H1), one observes that $\mathcal{F}(f, \cdot) : B_\sigma \to Z_{\sigma'}, \sigma' < \sigma$ is differentiable,

$$D_2\mathcal{F}(f, u)\hat{u} = df_{\circ(id+v)}\hat{v} + \hat{\lambda} - \partial \hat{v} , \tag{6.108}$$

$\hat{u} = (\hat{v}, \hat{\lambda})$. By the Cauchy estimate (6.37),

$$|df_{\circ(\mathrm{id}+v)}\hat{v}|_{\sigma'} \leq |f|_{\sigma' C^1}|\hat{v}|_\sigma \leq (\sigma - \sigma')^{-1}|f|_\sigma|\hat{v}|_\sigma \leq (\sigma - \sigma')^{-1}|\hat{v}|_\sigma .$$

In order to estimate $Q(f; u, v) = \mathcal{F}(f, u) - \mathcal{F}(f, v) - D_2\mathcal{F}(f, v)(u - v)$, we use the Taylor formula for functions and get for $(f, u), (f, v) \in B_\sigma$,

$$\begin{aligned} &|Q(f, u, v)|_{\sigma'} \\ &= \sup_{|\mathrm{Im}\, \xi|<\sigma'} \left| \frac{1}{2} \int_0^1 d\mu(1 - \mu)d^2 f(\xi + \mu v(\xi) + (1 - \mu)u(\xi))(v(\xi) - u(\xi))^2 \right| \\ &\leq \frac{1}{2}|f|_{\sigma' C^2}|v - u|_\sigma^2 \leq (\sigma - \sigma')^{-2}|v - u|_\sigma^2 . \end{aligned}$$

Hence (H1) is met.

(H2) is clear. In order to construct the approximate right inverse of $D_2\mathcal{F}(f, u)$, we shall first prove the following simple but crucial functional identity for equation (6.108):

$$D_2\mathcal{F}(f, u)\hat{u} = -(1 + dv)\partial(1 + dv)^{-1}\hat{v} + \hat{\lambda} + d\mathcal{F}(f, u)(1 + dv)^{-1}\hat{v} , \tag{6.109}$$

where d denotes differentiation of functions in x. Differentiating the function $\mathcal{F}(f, u)$, we get $d\mathcal{F}(f, u) = df_{\circ(\mathrm{id}+v)}(1 + dv) - \partial\, dv$, since ∂ has constant coefficients. Observe now, for $\hat{\omega}$ a vector function,

$$(\partial\, dv)\hat{\omega} = \partial(1 + dv) \cdot \hat{\omega} = \partial((1 + dv)\hat{\omega}) - (1 + dv) \cdot \partial\hat{\omega} .$$

We therefore get

$$\begin{aligned} d\mathcal{F}(f, u) \cdot (1 + dv)^{-1}\hat{v} &= df_{\circ(\mathrm{id}+v)}\hat{v} - \partial\hat{v} + (1 + dv)\partial(1 + dv)^{-1}\hat{v} \\ &= D_2\mathcal{F}(f, u)\hat{u} - \hat{\lambda} + (1 + dv)\partial(1 + dv)^{-1}\hat{v} , \end{aligned}$$

and formula (6.109) follows.

The existence of such an identity is no accident, as we shall see later on. It is related to the fact that we deal with conjugacy problems; this is the important algebraic feature of our problem and makes a solution possible. We write

$$D_2\mathcal{F}(f, u)\hat{u}) = L_u(\hat{u}) + \mathcal{R}_{(f,u)}(\hat{u}) , \tag{6.110}$$

where $L_u(\hat{u}) = -(1+dv)\partial(1+dv)^{-1}\hat{v}+\hat{\lambda}$ and $\mathcal{R}_{(f,u)}(\hat{u}) = d\mathcal{F}(f, u)(1+dv)^{-1}\hat{v}$. From the small-divisor lemma, Lemma 6.4.1, we conclude that the operator L_u has an unbounded right inverse of loss $\gamma = \tau + 2$, denoted by $\eta_u = L_u^{-1} \in L(Z_\sigma, Y_{\sigma'})$, and given by

$$(6.111)\quad \eta_u(z) = (\hat{v}, \hat{\lambda}),$$
$$\hat{v} = -(1 + dv)\eta\big((1 + dv)^{-1}\{z - [(1 + dv)^{-1}]^{-1}[(1 + dv)^{-1}z]\}\big),$$
$$\hat{\lambda} = [(1 + dv)^{-1}]^{-1}[(1 + dv)^{-1}z],$$

for all $z \in Z_\sigma$, where η denotes the right inverse of ∂ in the space of functions with mean value zero given by the lemma. We have used the constant vector $\hat{\lambda}$ to balance the mean values.

From (6.111) we find with the aid of the lemma and (6.37),

$$(6.112)\qquad |\eta_u(z)|_{\sigma'} \le \frac{M}{(\sigma - \sigma')^{\tau+2}} |z|_\sigma$$

for some constant M independent of u if $(f, u) \in B_\sigma$. The mapping $u \to \eta_u : B_\sigma \to (Z_\sigma, Y_{\sigma'})$ is clearly continuous for all $\sigma' < \sigma$. We have so far

$$(6.113)\qquad D_2\mathcal{F}(f, u) \circ \eta_u(z) - z = {}_{(f,u)} \cdot \eta_u(z).$$

By using (6.112) and (6.37), the right-hand side is estimated as follows:

$$\begin{aligned} |\mathcal{R}_{(f,u)}\eta_u(z)|_{\sigma'} &\le |d\mathcal{F}(f, u)|_{\sigma'} |(1 + dv)^{-1}|_{\sigma'} |\eta_u(z)|_{\sigma'} \\ &\le M(\sigma - \sigma')^{-(\tau+3)} |\mathcal{F}(f, u)|_\sigma \cdot |z|_\sigma \\ &\le M(\sigma - \sigma')^{-2(\alpha+\gamma)} |\mathcal{F}(f, u)|_\sigma \cdot |z|_\sigma ; \end{aligned}$$

hence η_u is an approximate right inverse satisfying assumption (H3) for all $(f, u) \in B_\sigma$ and some $M > 0$. The proof of the theorem is finished. □

6.5. Conjugacy Problems

We give a heuristic argument showing why one should expect to have an approximate right inverse in certain conjugacy problems. Consider an infinite-dimensional manifold B since everything is local, B may be assumed to be a Banach space. Also consider a differentiable group action Φ, $\Phi : B \times G \to B : (f, g) \mapsto \Phi(f, g)$, where G, however, is an infinite-dimensional group. Φ satisfies

$$(6.114)\qquad \Phi(f, \mathrm{id}) = f, \quad \Phi(f, g \circ g_0) = \Phi(\Phi(f, g), g_0).$$

For example, B is the space of functions on a manifold M, and G a subgroup of the group of diffeomorphisms of M, the group action being the composition $f \circ g$. Another example is the space of vector fields over a manifold M, the group action being the transformation law under a group of diffeomorphisms of M. The situation we want to study is the following. In B we single out a subset $N \subset B$ and ask whether there exists an open neighborhood U of N on B that belongs to the orbits of N under G; in other words, for f sufficiently close to N, does there exist a group element $g \in G$ such that $\Phi(f, g) = n \in N$?

If the answer is yes, we shall say: The subset N is stable in B under the group G. In order to formulate the assumptions on the group action Φ, we parametrize an open neighborhood of id $\in G$ by a chart exp: $V \subset T_{\mathrm{id}}(G) \to G$, $\exp(0) = \mathrm{id}$, and write simply $\Phi(f, \exp(\gamma)) \equiv \Phi(f, \gamma)$. The mapping $\mathcal{F}$ is then defined as follows:

$$\mathcal{F}(f, u) \equiv \Phi(f, \gamma) - n\,, \tag{6.115}$$

where u stands for $u = (\gamma, n) \in V \times N$.

For a given f sufficiently close to N, we look for a solution u of $\mathcal{F}(f, u) = 0$ under the following assumptions: We assume we can solve the linearized equation

$$D_f \mathcal{F}(f, u)\hat{u} = D_2\Phi(f, \gamma)\hat{\gamma} - \hat{n} = \hat{f} \in B \tag{6.116}$$

for all given $\hat{f}$, where $\hat{u} = (\hat{\gamma}, \hat{n}) \in T_{\mathrm{id}} \times N$ if $f \in N$ and $\gamma = 0$. In other words, we assume there is a right inverse

$$\eta_n = D_2\mathcal{F}(n, (0, n))^{-1} \tag{6.117}$$

for all $n \in N$. Note that we do not assume the existence of a right inverse of $D_2\mathcal{F}(f, (\gamma, n))$ for f in a full neighborhood of N and γ in a neighborhood of 0. The assumption will be rephrased by saying that the subset N is infinitesimally stable in B under G. In case the right inverse η_n (6.117) is bounded, one would apply the classical implicit function theorem. However, in the cases we are interested in, the right inverse η_n is unbounded (due, say, to the small divisors), and it is impossible to solve (6.116) for f not in N and $\gamma \neq 0$.

The main observation now is that due to the conjugacy identity (6.114), we can construct an approximate right inverse η_n of $D_2\mathcal{F}(f, u)$ for f in a full neighborhood of N and γ in a full neighborhood of 0 if we have a right inverse η_n, $n \in N$. Namely, we claim that there exists a linear map $\eta_u : B \to T_{\mathrm{id}} \times N$ such that for all $\hat{f} \in B$ (and suitable norms)

$$|(D_2\mathcal{F}(f, u) \circ \eta_u - 1)(\hat{f})| \leq \mathrm{const}\ |\mathcal{F}(f, u)|\,|\hat{f}|\,. \tag{6.118}$$

In order to construct η_u, we introduce the function χ_γ by

$$\exp(\gamma) \circ \exp(\gamma_0) = \exp\big(\chi_\gamma(\gamma_0)\big)$$

and with the conjugacy identity (6.114), $\Phi(f, \chi_\gamma(\gamma_0)) = \Phi(\Phi(f, \gamma), \gamma_0)$. Differentiation of this identity in γ_0 at $\gamma_0 = 0$ gives

$$D_2\Phi(f, \gamma) d\chi_\gamma(0)\hat{\gamma} = D_2\Phi(\Phi(f, \gamma), 0)\hat{\gamma}\,.$$

Introducing the linear operator L_γ, $L_\gamma(\hat{u}) = (d\chi_\gamma(0)\hat{\gamma}, \hat{n})$, $\hat{u} = (\hat{\gamma}, \hat{n})$, we can write $D_2\mathcal{F}(f, u) \circ L_\gamma \hat{u} = D_2\mathcal{F}(\Phi(f, \gamma), (0, n))\hat{u}$, and by means of the Taylor formula we get

$$D_2\mathcal{F}(f, u) \circ L_\gamma(\hat{u}) = D_2\mathcal{F}(n, (0, n))\hat{u} + \mathcal{B}_{(f,u)}(\Phi(f, \gamma) - n, \hat{u}) \tag{6.119}$$

with some bilinear operator $\mathcal{B}_{(f,u)}$.

Now, setting for $u = (\gamma, n)$, $\eta_u \equiv L_\gamma \circ \eta_n$ (η_n for $n \in N$ being the right inverse of $D_2\mathcal{F}(n, (0, n))$, which exists by assumption), we get the identity

$$(D_2\mathcal{F}(f, u) \circ \eta_u - 1)(\hat{f}) = \mathcal{B}_{(f,u)}(\mathcal{F}(f, u), \eta_n(\hat{f}))\,, \tag{6.120}$$

which, with an estimate like $|\eta_n(\hat{f})| \leq \mathrm{const}\ |\hat{f}|$, gives the required estimate (6.118). This is only a guiding principle in dealing with conjugacy problems. We have been vague about the topology, different norms, etc. Theorems 6.1.1, 6.1.6, and 6.3.3 (which postulate the existence of an approximate right inverse) give precise conditions under which the following statement holds true:

$$N \text{ infinitesimally stable in } B \text{ under } G \Rightarrow N \text{ stable in } B \text{ under } G.$$

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