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Application of Lie groups and differentiable manifolds to general methods for simplifying systems of partial differential equations

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Abstract. General techniques are developed to obtain: (1) the completion of a system of nonlinear first-order partial differential equations (PDEs) which is an independent set of further PDEs derivable from the system by differentiation and elimination; and (2) simplifications of the system by choosing appropriate new independent and dependent variables using a result from Lie group theory. The number of dependent and independent variables is reduced to the minimum. The theory specializes to the classical theory of a single nonlinear PDE with one unknown and can be combined with the methods of Olver, Edelen and Estabrook and Wahlquist. Most of the methods appear to be sufficiently well defined for automation as are the techniques in Olver. A second-order nonlinear equation in n dimensions is given which is related to a functional differential equation in statistical mechanics. It is reducible to two dimensions for any value of $n \geq 2$.

1. Introduction

In this paper I will develop the idea of reduction of dimension for linear and nonlinear systems of partial differential equations (PDEs). It is an extension of Monge's method for tackling single PDEs of first order with one unknown. The method is applicable to any system defined with sufficiently differentiable functions but the result is not usually one-dimensional; in fact there may be no reduction of dimension giving no simplification at all. The result of the transformation is another system of PDEs having the same set of solutions with a possibly smaller number of independent variables but the number of dependent variables, which are the unknowns, may be increased initially but their number will afterwards be minimized. From the point of view of the general theory of systems of PDEs (called systems for brevity) the procedures indicated here should be applied initially, then symmetry methods should be applied if necessary. The best known of these are, firstly, looking for infinitesimal generators of geometrical symmetries [1] (isovectors of the differential ideal [2, 3]) from which group invariant solutions can be obtained and the generalized method of characteristics ([4, 5]). The latter method requires the initial data to satisfy an extra condition but perhaps more flexibility can be obtained by applying the method to a prolongation of the original system (including derivatives of the dependent variables as new unknowns).

Secondly there are the related methods of Estabrook and Wahlquist [6] originally applied to PDEs with two independent variables. They prolong the differential ideal

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in such a way that it remains closed and well posed introducing auxiliary variables known as pseudopotentials [7]. If this is possible it leads to a set of conservation laws and allows a calculation of Bäcklund transformations which, whether or not they form a group, allow a new solution to the PDE to be obtained from one or more known solutions. Denes and Finley [8] discuss the existence of Bäcklund transformations for a general PDE with any number of independent variables.

Many if not all of the methods presented here can be carried out mechanically, as can the calculation of symmetry groups, hence they can be done by computer. These methods must be equivalent to a method for finding a minimal basis of 1-forms (characteristic system) in which to express the closed differential ideal corresponding to the system [9]. This method requires finding the first integrals of the characteristic system. This may turn out to give an explicit procedure for carrying out the reduction which can be stated more concisely, treating the dependent and independent variables on the same footing. This is based on the Cartan theory of exterior differential systems [16], a good introduction to which has been given recently [10].

Several examples are given which motivate the general theory but by far the most important of these is the last example which is closely related to a functional differential equation in statistical mechanics. It shows that a second-order PDE in n independent variables can be reduced to a system in two independent variables for any value of $n \geq 2$. Hence there is a second-order functional differential equation closely related to those arising in the classical statistical mechanics of the one-dimensional fluid [11, 12] which can be expressed with two independent variables. The consequences of a generalization of this result will be explained in a future publication on statistical mechanics [13].

The layout of the paper is as follows. In section 2 I start with the general linear second-order PDE to motivate the general theory and to show some simple results giving a powerful simplification of a class of PDEs. I then show (as is well known) how any solution $u_1 \ldots u_q$ of any system can always be regarded as a subset of the unknowns in a solution $u_1 \ldots u_p$ $(p \geq q)$ of a corresponding first-order system introducing what I refer to as the 'standard' method of obtaining such a first-order system. This justifies restricting all further discussion to first-order systems but it raises the question of how the different ways of reducing a system to first order are related. I show that they are all related by a change of dependent variables.

A very important idea is how the solution of a system varies with the boundary or initial conditions. A small change in these conditions will give usually a small change in the solution, the difference satisfying, to first order, a linear system. I argue in section 3 that some properties of the original system also hold for the linearized system, hence this can be used for classification purposes. This provides a motivation for studying linear systems. For these systems I have formulated the minimization of dimension by first applying a completion procedure analogous to the method used for treating the system $\boldsymbol{f}_k \cdot \nabla \boldsymbol{u} = 0$ (the name being justified by Frobenius's theorem) followed by a change of dependent and independent variables applied to linear combinations of the system in such a way that the number of independent and dependent variables is minimized. This is straightforward provided one has familiarity with some of the essential concepts of differentiable manifolds and functions, vectors and forms defined on them, an excellent introduction to which is given by Boothby [14].

In section 4 I have extended the methods of section 3 to general nonlinear systems. The concept of local solvability mentioned by Olver [1] is introduced and the procedure for obtaining a locally solvable system is believed to require only the repeated elimination of the second derivatives from all the first total derivatives of each equation of the system. An extension of the argument to minimization of dimension for nonlinear systems holds. In section 5 the number of unknowns is minimized by a change of dependent variables, the number of them being determined by the rank of a matrix. In section 6 general conclusions are given about the simplification methods and it is shown that the 'standard' way to get a first-order system from a higher order system by introducing auxiliary variables preserves local solvability provided some extra equations are added, thus showing that the completion procedure is not necessary for higher order systems known to be locally solvable.

Finally in section 7 an example is given of a nonlinear equation of second order in n independent variables which is reducible to two dimensions by this method for any value of n. These equations can be regarded as a sequence of approximations to a second-order functional differential equation as n increases which is therefore also reducible in some sense to a two-dimensional system and it is consequently tractable numerically if not by further analysis.

2. Minimization of the dimension of a second-order PDE and expression of a general system as a first-order system

Consider the following class of PDEs

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + a(x)u(x) = 0 \tag{1}$$

where a_{ij} is a symmetric matrix of rank 1 which is a function of $x = (x_1, \ldots, x_n)$ so one can write $a_{ij} = b_i b_j$.

(Note that all the arbitrary functions will be assumed to be sufficiently many times differentiable for all the operations to be well defined and note that I have used the same symbol a for three different functions, being distinguished by the number of subscripts. I have done this throughout because it saves constantly having to find new symbols.)

Consider the curves $x_i(t)$ defined by the differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = b_i\,(x(t)) \tag{2}$$

then the first and second derivatives of u with respect to t can be written in terms of derivatives with respect to the x_i.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{i=1}^{n}\frac{\partial u}{\partial x_i} b_i$$

and

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}\frac{\partial u}{\partial x_i} b_i\right) b_j = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial b_i}{\partial x_j}. \tag{3}$$

Hence (1) can be written as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \left(a_i - \sum_{j=1}^{n} b_j \frac{\partial b_i}{\partial x_j} \right) + au = 0. \tag{4}$$

Suppose that the $(n-1)$ parameter set of curves $x_i(t)$ can be parametrized by $y_i = x_i|_{x_n=0}$ where t could be given by $t=0$ when $x_n = 0$. Suppose that (1) only relates u at points on the same surface and the whole space is filled with such surfaces, and suppose initially that these surfaces or manifolds are of dimension $n-1$ then there is a one parameter family of them. Hence *there is a function* $z_1(\boldsymbol{x})$ *whose level surfaces are* these manifolds which can be rewritten in terms of the y_i and t which are defined in the region of $\boldsymbol{x}$ of interest. A necessary condition on the manifolds is that $\mathrm{d}/\mathrm{d}t|_{y_1\ldots y_{n-1}}$ is an interior derivative to them

$$\left.\frac{\mathrm{d}z_1}{\mathrm{d}t}\right|_{y_1\ldots y_{n-1}} = 0$$

i.e. z_1 depends only on $y_1, \ldots, y_{n-1}$. Choose $z_2, \ldots, z_{n-1}$ also to be functions of $y_1, \ldots, y_{n-1}$ such that the Jacobian

$$\frac{\partial(z_1, \ldots, z_{n-1})}{\partial(y_1, \ldots, y_{n-1})} \neq 0$$

so the transformation can be locally inverted. Then the coordinates $\boldsymbol{x}$ can be replaced by $z_1, \ldots, z_{n-1}, t$. Making this change of variables in the second term of (4) gives

$$\frac{\partial u}{\partial t} \sum_{i=1}^{n} \frac{\partial t}{\partial x_i} \left(a_i - \sum_{j=1}^{n} b_j \frac{\partial b_i}{\partial x_j} \right) + \sum_{j=1}^{n-1} \frac{\partial u}{\partial z_j} \sum_{i=1}^{n} \frac{\partial z_j}{\partial x_i} \left(a_i - \sum_{l=1}^{n} b_l \frac{\partial b_i}{\partial x_l} \right). \tag{5}$$

It is now clear that I still do not have a sufficient condition for the reduction of dimension. It is also necessary that this expression does not involve $\partial u/\partial z_1$. This requires that

$$\sum_{i=1}^{n} \frac{\partial z_1}{\partial x_i} \left(a_i - \sum_{l=1}^{n} b_l \frac{\partial b_i}{\partial x_l} \right) = 0 \tag{6}$$

in addition to the condition obtained previously:

$$\sum_{i=1}^{n} \frac{\partial z_1}{\partial x_i} b_i = 0. \tag{7}$$

This is a system of the form $\boldsymbol{f}_k \cdot \boldsymbol{\nabla} \boldsymbol{u} = 0$ and general theory shows that this has a non-trivial solution for z_1 if and only if the pair of vector fields $\boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b}$ generate a Lie group whose action on the coordinate space of points $(x_1, \ldots, x_n)$ gives orbits of dimension at most $n-1$ or equivalently the Lie algebra generated by taking commutators of the vector fields repeatedly until closure yields at most $n-1$ linearly independent vector fields. From (6) and (7) it follows at once that the transformed equation is interior to the manifolds z_1 = constant i.e. the transformed equation does not involve the independent variable z_1 in the derivatives. It is now easy to show that if the dimension of these orbits is r then equation (1) can be expressed only in terms of $z_{n-r+1}, \ldots, z_{n-1}, t$ thus showing that it is reducible to r dimensions. This is a great simplification of the original equation (1) but extensions of it to higher order systems seem to be very complicated. The treatment of first-order equations is easier to describe. Hence, because any system is expressible as a first-order system by introducing auxiliary variables, I shall briefly discuss this and show how the various forms it can take are related. The remainder of the paper concerns the analogous technique of minimization of dimension for first-order systems, first the linear case and then the general nonlinear case.

Consider a general system of differential equations for the unknown functions $u_1, \ldots, u_p$ of the independent variables $x_1, \ldots, x_n$

$$F_k\left(x_1 \ldots x_n, u_1 \ldots u_p, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \ldots \frac{\partial u_1}{\partial x_n}, \frac{\partial u_2}{\partial x_1} \ldots \frac{\partial u_p}{\partial x_n}, \frac{\partial^2 u_1}{\partial x_1 \partial x_2} \ldots \left\{\frac{\partial^q u_p}{\partial x_{i_1} \ldots \partial x_{i_q}}\right\}\right) = 0 \tag{8}$$

for $1 \le k \le m$, which can be written more compactly as $F_k\left(\boldsymbol{x}, \boldsymbol{u}^{(q)}\right) = 0$ where $\boldsymbol{u}^{(q)}$ is the set of all qth and lower-order derivatives of $u_1, u_2 \ldots u_p$ including the undifferentiated variables. By the following 'standard' method of introducing auxiliary variables, the system can be expressed as a first-order system. Let

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} \qquad u_{i,jk} = \frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial u_{i,j}}{\partial x_k} \tag{9}$$

etc up to

$$u_{i,j_1 \ldots j_{q-1}} = \frac{\partial^{q-1} u_i}{\partial x_{j_1} \ldots \partial x_{j_{q-1}}} = \frac{\partial u_{i,j_1 \ldots j_{q-2}}}{\partial x_{j_{q-1}}} \qquad \text{for } 1 \le j_1, \ldots, j_{q-1} \le n; 1 \le i \le p. \tag{10}$$

Then the original system (8) becomes the first-order equations

$$F_k\left(\boldsymbol{x}, \boldsymbol{u}, \left\{u_{i,j}\right\}, \ldots \left\{u_{i,j_1 \ldots j_{q-1}}\right\}, \left\{\frac{\partial u_{i,j_1 \ldots j_{q-1}}}{\partial x_{j_q}}\right\}\right) = 0 \tag{11}$$

together with the auxiliary equations (9) up to (10). The significance of the transformation is that any solution $\boldsymbol{u}$ of (8), when differentiated gives $\boldsymbol{u}^{(q-1)}$ which satisfies (9) up to (10) and (11) and conversly $\boldsymbol{u}$ and its derivatives satisfying this system implies that $\boldsymbol{u}$ satisfies (8). Hence any solution $\boldsymbol{u}$ of the nonlinear system (8) is obtained by picking out $\boldsymbol{u}$ from the solution $\boldsymbol{u}^{(q)}$ of the first-order system (9) to (10) and (11). This justifies restricting attention to first-order systems.

Consider the most general possible way to introduce auxiliary variables into (8) involving only first-order equations. Let $v_1 = g_1(\boldsymbol{x}, \boldsymbol{u}^{(1)})$ then introduce $v_2 = g_2(\boldsymbol{x}, \boldsymbol{u}^{(1)}, v_1^{(1)})$ etc and in general $v_i = g_i(\boldsymbol{x}, \boldsymbol{u}^{(1)}, v_1^{(1)}, \ldots, v_{i-1}^{(1)})$ for $1 \le i \le r$. The auxiliary variables $v_1 \ldots v_r$ can now be related to the standard auxiliary variables $u_{i,j_1 \ldots j_k}$. Since $v_1 = g_1\left(\boldsymbol{x}, \boldsymbol{u}, \{u_{i,j}\}\right)$ it follows that

$$\frac{\partial v_1}{\partial x_l} = \frac{\partial g_1}{\partial x_l} + \sum_{i=1}^{p} \frac{\partial g_1}{\partial u_i} u_{i,l} + \sum_{i=1}^{p}\sum_{j=1}^{n} \frac{\partial g_1}{\partial u_{i,j}} u_{i,jl}$$

and

$$v_2 = g_2\left(\boldsymbol{x}, \boldsymbol{u}^{(1)}, g_1\left(\boldsymbol{x}, \boldsymbol{u}, \{u_{i,j}\}\right), \left\{\frac{\partial v_1}{\partial x_j}\right\}\right)$$

which gives an expression depending on just $\boldsymbol{x}, \boldsymbol{u}, \{u_{i,j}\}$ and $\{u_{i,jl}\}$. In the same way, if one has $v_3 = g_3\left(\boldsymbol{x}, \boldsymbol{u}^{(1)}, v_1^{(1)}, v_2^{(1)}\right)$ differentiation of v_2 gives

$$\frac{\partial v_2}{\partial x_l} = \frac{\partial g_2}{\partial x_l} + \sum_{i=1}^{p} \frac{\partial g_2}{\partial u_i} u_{i,l} + \sum_{i=1}^{p}\sum_{j=1}^{n} \frac{\partial g_2}{\partial u_{i,j}} u_{i,jl} + \sum_{i=1}^{p}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial g_2}{\partial u_{i,jk}} u_{i,jkl}$$

which is now a function of $\boldsymbol{x}, \{u_i\}, \{u_{i,j}\}, \{u_{i,jk}\}, \{u_{i,jkl}\}$ so by substitution v_3 can be expressed in terms of these variables etc. Hence any new set of auxiliary dependent variables $\boldsymbol{v}$ reducing the system to first order, can be expressed as $v_i = h_i(\boldsymbol{x}, \boldsymbol{u})$ for some known functions h_i where $\boldsymbol{u}$ is now the standard set of dependent variables (the original ones and the auxiliary ones) making the system first order i.e. the $\boldsymbol{u}^{(\varsigma)}$ in this argument. There are cases when the number of variables $\boldsymbol{u}$ to express the system can be less than the standard method gives. Then the argument in section 5 shows how their number can be obtained and their explicit forms.

3. Linearization and minimization of dimension for linear systems

It is of fundamental importance in the study of systems of PDEs to find the type of boundary conditions under which a system has a unique solution or more generally how the domain of uniqueness B of the solution is related to the set or manifold $A = \{(\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}(\boldsymbol{s})) : \boldsymbol{s} \in S\}$ of initial data. The domain B is defined to be the region over which all possible solutions of the system consistent with the initial data A coincide. In general one can ask how the solution of a system is altered if the initial data are varied by $O(\epsilon)$. On the assumption that this is also $O(\epsilon)$ which one would expect if the F_k are C^∞ this gives rise to a linearized form of the system and the argument following shows that B for the original system with a given A is the same as B for the linearized system with the initial data $\phi(\boldsymbol{x})$ given on the same set $\{\boldsymbol{x}(\boldsymbol{s}) : \boldsymbol{s} \in S\}$ upon which A was defined showing that the equations are of the same character.

Let $\boldsymbol{u}$ satisfy the system

$$F_k\left(\boldsymbol{x}, \boldsymbol{u}^{(1)}\right) = 0 \qquad \text{for } 1 \le k \le m \quad \forall \boldsymbol{x} \in D \tag{12}$$

where D is some open subset of R^n which may depend on $\boldsymbol{u}$. Unless otherwise specified the coordinate point $\boldsymbol{x} \in D$. Let the initial data be $\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}(\boldsymbol{s})$, where $\boldsymbol{s} = (s_1, \ldots, s_n)$. Let $\boldsymbol{u} + \delta\boldsymbol{u}$ satisfy the same system with initial data $\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}(\boldsymbol{s}) + \delta\boldsymbol{u}(\boldsymbol{s})$. Then

$$F_k\left(\boldsymbol{x}, \boldsymbol{u}^{(1)} + \delta\boldsymbol{u}^{(1)}\right) = 0. \tag{13}$$

Let $\delta\boldsymbol{u}$ be small say $O(\epsilon)$ as $\epsilon \to 0$ so put $\delta\boldsymbol{u} = \epsilon\boldsymbol{\phi}$ and subtract (12) from (13) giving

$$F_k\left(\boldsymbol{x}, \boldsymbol{u}^{(1)} + \epsilon\boldsymbol{\phi}^{(1)}\right) - F_k\left(\boldsymbol{x}, \boldsymbol{u}^{(1)}\right) = 0. \tag{14}$$

Taking the leading terms which are $O(\epsilon)$ as $\epsilon \to 0$ by differentiating with respect to ϵ gives

$$\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{\partial F_k}{\partial u_{i,j}}\frac{\partial \phi_i}{\partial x_j} + \sum_{i=1}^{p}\frac{\partial F_k}{\partial u_i}\phi_i = 0. \tag{15}$$

This is the equation satisfied for small changes $\boldsymbol{\phi}$ in $\boldsymbol{u}$ resulting from small changes in the boundary conditions. The derivatives $\partial F_k/\partial u_i$ and $\partial F_k/\partial u_{i,j}$ must be evaluated for $\boldsymbol{u}$ equal to the original solution.

Suppose for definiteness that $\boldsymbol{u}$ is determined uniquely for $\boldsymbol{x} \in B$ by the system (12) and the initial conditions A. Now if $\boldsymbol{u}$ is altered to $\boldsymbol{u}(\boldsymbol{s}) + \epsilon\boldsymbol{\phi}(\boldsymbol{s})$ and $\boldsymbol{x}(\boldsymbol{s})$ is kept fixed for $\boldsymbol{s} \in S$ then $\boldsymbol{u}(\boldsymbol{x})$ may be altered for $\boldsymbol{x} \in B$. But if $\boldsymbol{u}$ is altered in such a way that $\boldsymbol{u}(\boldsymbol{s})$ for $\boldsymbol{s} \in S$ is unchanged, then $\boldsymbol{u}(\boldsymbol{x})$ for $\boldsymbol{x} \in B$ is unchanged. Hence obtaining this change approximately by linearization, $\boldsymbol{\phi}$ must clearly be 0 for $\boldsymbol{x} \in B$ provided $\boldsymbol{\phi}(\boldsymbol{s}) = 0$ for $\boldsymbol{s} \in S$. If C is the set of $\boldsymbol{x}$ for which the linearized equation has a unique solution, i.e. 0 when $\boldsymbol{\phi}(\boldsymbol{s}) = 0$ for all $\boldsymbol{s} \in S$, since $x \in B$ then $\boldsymbol{x} \in C$ so $B \subseteq C$. Now conversely suppose $\boldsymbol{x} \in C$ so that $\boldsymbol{\phi}(\boldsymbol{x}) = 0$ satisfies the linearized equation uniquely when $\boldsymbol{\phi} = 0$ for $\boldsymbol{s} \in S$. I need to consider an arbitrary change in the data $\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}(\boldsymbol{s})$ leaving them fixed for $\boldsymbol{s} \in S$. Let $\boldsymbol{u}_I(\boldsymbol{x})$ satisfy (12) and boundary conditions $\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}_I(\boldsymbol{s}), \boldsymbol{s} \in T$ and let $\boldsymbol{u}_F(\boldsymbol{x})$ satisfy (12) and boundary conditions $\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}_F(\boldsymbol{s}), \boldsymbol{s} \in T$ where $\boldsymbol{u}_I(\boldsymbol{s}) = \boldsymbol{u}_F(\boldsymbol{s})$ for $\boldsymbol{s} \in S \subset T$. Let $\boldsymbol{u}_i(\boldsymbol{s}) = \boldsymbol{u}_I(\boldsymbol{s}) + (i/N)\left(\boldsymbol{u}_F(\boldsymbol{s}) - \boldsymbol{u}_I(\boldsymbol{s})\right), \boldsymbol{s} \in T$ for $1 \le i \le N$ and consider $\boldsymbol{u}_i(\boldsymbol{x})$ determined by (12) and the boundary conditions $\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}_i(\boldsymbol{s}), \boldsymbol{s} \in S$. Let $\epsilon = 1/N$ then $\epsilon\boldsymbol{\phi}_i(\boldsymbol{x}) = \boldsymbol{u}_i(\boldsymbol{x}) - \boldsymbol{u}_{i-1}(\boldsymbol{x})$ is obtained to $O(\epsilon^2)$ by linearizing (12) about $\boldsymbol{u}_{i-1}(\boldsymbol{x})$ with boundary conditions $\epsilon\boldsymbol{\phi}_i(\boldsymbol{s}) = (1/N)\left(\boldsymbol{u}_F(\boldsymbol{s}) - \boldsymbol{u}_I(\boldsymbol{s})\right)$ which is 0 if $\boldsymbol{s} \in S$. For $i = 1$, since $\boldsymbol{x} \in C$, then $\boldsymbol{\phi} = 0$ and so $\boldsymbol{u}_1(\boldsymbol{x}) - \boldsymbol{u}_0(\boldsymbol{x}) = O(\epsilon^2)$. For $i = 2, \boldsymbol{u}_2(\boldsymbol{x}) - \boldsymbol{u}_1(\boldsymbol{x})$ is obtained by linearizing (12) about $\boldsymbol{u}_1(\boldsymbol{x})$ and applying the boundary condition which is again $\epsilon\boldsymbol{\phi}(\boldsymbol{s}) = 0$ for $\boldsymbol{s} \in S$. The equation for $\boldsymbol{\phi}$ from (15) differs from that for $i = 1$ by $O(\epsilon^2)$ so $\boldsymbol{u}_2(\boldsymbol{x}) - \boldsymbol{u}_1(\boldsymbol{x}) = O(\epsilon^2)$. In the same way $\boldsymbol{u}_i(\boldsymbol{x}) - \boldsymbol{u}_{i-1}(\boldsymbol{x}) = O(\epsilon^2)$ for $1 \le i \le N$ so $\boldsymbol{u}_N(\boldsymbol{x}) - \boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{u}_F(\boldsymbol{x}) - \boldsymbol{u}_I(\boldsymbol{x}) = \sum_{i=1}^{N} O(\epsilon^2) = O(\epsilon) \to 0$ as $N \to \infty$. Hence $\boldsymbol{u}_F(\boldsymbol{x}) = \boldsymbol{u}_I(\boldsymbol{x})$ and since $\boldsymbol{u}_I(\boldsymbol{s})$ and $\boldsymbol{u}_F(\boldsymbol{s})$ are arbitrary, $\boldsymbol{x} \in B$, hence $C \subseteq B$ and finally $C = B$. It follows that the domain B of dependence of $\boldsymbol{u}$ on the initial data $A = \{(\boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}(\boldsymbol{s})), \boldsymbol{s} \in S\}$ for the nonlinear system (12) is the same as the domain of dependence C of $\boldsymbol{\phi}$ for the linearized equation, (evaluated about the solution $\boldsymbol{u}$ of (12) with the same initial data A) with initial data $\boldsymbol{\phi}(\boldsymbol{s}) = 0$ for $\boldsymbol{s} \in S$. B will depend on A and the relationship gives the qualitative properties of the system. A consequence of this is that the full range of types of behaviour is exhibited locally by systems which are of the form of the linearized system above i.e.

$$\sum_{i=1}^{p}\sum_{j=1}^{n}\frac{\partial u_i}{\partial x_j}a_{ijk}(\boldsymbol{x}) + \sum_{i=1}^{p} u_i a_{ik}(\boldsymbol{x}) = 0 \qquad \text{for } 1 \le k \le m. \tag{16}$$

If the system (12) is analytic the relationship between A and B is expected to be smooth (except when the topology of $\boldsymbol{x}(\boldsymbol{s})$ changes for example for an elliptic system in two independent variables $B = A$ when A is given on an open curve but B is the interior of the curve when A is defined on a closed curve). For linear systems i.e. systems for which $\partial F_k/\partial u_i$ and $\partial F_k/\partial u_{i,j}$ are independent of $\boldsymbol{u}$, the relationship between A and B is dependent only on $\boldsymbol{x}$. This implies that the standard classification of PDEs in two independent variables and systems as elliptic, parabolic or hyperbolic is *independent of the solution* $\boldsymbol{u}$ *for the linear case i.e. the classification can be obtained* in terms of $\boldsymbol{x}$ in advance of calculating the solution. But for nonlinear systems the result will be dependent on the particular solution which in turn is dependent on the boundary conditions. A similar remark holds for more than two independent variables when the classification is more complicated [15]. With this notation (16) for the general linear system, I shall now discuss them showing one parameter describing different types of behaviour and the simplest form the equations take. This was mainly inspired by the classification and canonical forms of second-order PDEs in two variables [17]. The type of behaviour in example (1) is that a set of linear combinations of the system can be expressed in a family of submanifolds i.e. as another linear system with fewer independent variables such that only derivatives interior to the manifolds appear. If it is a complete set of linear combinations of the original system this will result in the domain of dependence on the initial data being a subset of the manifold on which the data appears. I refer to this as a complete reduction of dimension of the system. In the example (27) some partial results are obtained in fewer dimensions.

Before looking for reduction of dimension note that the theory of linear systems (16) must include the theory of systems of the form $\boldsymbol{f}_k \cdot \nabla \boldsymbol{u} = 0$. In this case extra linearly independent (LI) equations obtained by differentiation and eliminating the higher derivatives as described in Chester (i.e. forming the commutators of the $\boldsymbol{f}_k$) must be found before the general solution can be described geometrically. A more comprehensive account of the method with very compact notation is found in Schouten and Kulk [18] where this is shown to be equivalent to the 'outer problem'. Returning to the system (16), combinations of the first derivatives of the system are sought which involve only the first derivatives of $\boldsymbol{u}$ and which are linearly independent of the original system. This procedure is then repeated starting with the augmented system and continued until there are no new results after one step. Finally from this derived system of the form (16) the systems with reduced dimension are sought.

From (16) applying the differential operator $\boldsymbol{h}_k \cdot \nabla$ to equation k and summing over k gives (dropping the argument $\boldsymbol{x}$)

$$\sum_{k=1}^{m} \boldsymbol{h}_k \cdot \nabla \left(\sum_{i=1}^{p} \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} a_{ijk} + \sum_{i=1}^{p} u_i a_{ik} \right) = 0. \tag{17}$$

The second-order terms can only arise from the terms

$$\sum_{i=1}^{p} \sum_{j=1}^{n} \sum_{k=1}^{m} \boldsymbol{h}_k \cdot \nabla \left(\frac{\partial u_i}{\partial x_j} \right) a_{ijk}. \tag{18}$$

Let $(\boldsymbol{h}_k)_l = h_{kl}$ then the vanishing of the second-order terms i.e. those involving $u_{i,jl}$ is obtained by equating all their coefficients to zero which gives the equations

$$\sum_{k=1}^{m} \left(h_{kl} a_{ijk} + h_{kj} a_{ilk} \right) = 0 \qquad \text{for } 1 \le j, l \le n; 1 \le i \le p \tag{19}$$

i.e. the matrix

$$(c_i)_{jl} = \sum_{k=1}^{m} h_{kl} a_{ijk} \tag{20}$$

is skew symmetric for $1 \le i \le p$. Then the derived equation can be written as

$$\sum_{i=1}^{p}\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j}\left[\sum_{k=1}^{m}\sum_{l=1}^{n} h_{kl}\frac{\partial a_{ijk}}{\partial x_l} + \sum_{k=1}^{m} a_{ik} h_{kj}\right] + \sum_{i=1}^{p} u_i \left[\sum_{k=1}^{m}\sum_{l=1}^{n} h_{kl}\frac{\partial a_{ik}}{\partial x_l}\right] = 0 \tag{21}$$

which is a linear equation of the form (16) dependent on the matrix h_{kl} regarded as an mn dimensional vector. Hence at each step of the method, one must find a complete set of linearly independent vectors h_{kl} satisfying (20) then the corresponding equations (21) are the extension obtained after one step. This is repeated until no linearly independent equations appear and the resulting system I shall refer to as the completion of (16) which is a basis of a vector space of linear PDEs which is again of the form (16).

Now look for subspaces of this vector space of equations which is expressible in $r < n$ dimensions. For simplicity of notation I shall again use the notation (16) for such a subspace. Let $z_1, \ldots, z_n$ be a new set of independent variables,then

$$\frac{\partial u_i}{\partial x_j} = \sum_{l=1}^{n} \frac{\partial u_i}{\partial z_l}\frac{\partial z_l}{\partial x_j}.$$

Changing variables from x to z, I will arrange that $z_1, \ldots, z_{n-r}$ are absent from the derivatives in the system. These variables may still appear undifferentiated, in this case the reduced form of the system will have only parametric dependence on them. The system (16) becomes

$$\sum_{i=1}^{p}\sum_{l=1}^{n} \frac{\partial u_i}{\partial z_l}\left[\sum_{j=1}^{n} \frac{\partial z_l}{\partial x_j} a_{ijk}\right] + \sum_{i=1}^{p} u_i a_{ik} = 0 \qquad \text{for } 1 \le k \le m \tag{22}$$

so the previous condition gives

$$\sum_{j=1}^{n} \frac{\partial z_l}{\partial x_j} a_{ijk} = 0 \qquad \text{for } 1 \le l \le n-r; 1 \le i \le p; 1 \le k \le m \tag{23}$$

i.e. there are $n - r$ functionally independent functions z satisfying

$$\sum_{j=1}^{n} \frac{\partial z}{\partial x_j} a_{ijk} = 0 \qquad \text{for } 1 \le i \le p; 1 \le k \le m. \tag{24}$$

Functional independence ensures that no equation of the form $f(z_1, \ldots, z_{n-r}) = 0$ holds identically so that the z_i can vary independently. This system is of the form $\boldsymbol{f}_{ik} \cdot \boldsymbol{\nabla} z = 0$ where $a_{ijk} = (\boldsymbol{f}_{ik})_j$, hence the set of vectors $\boldsymbol{f}_{ik}$ generate a Lie algebra with the corresponding Lie group having orbits of dimension r. Roughly speaking this is because in the orbits z is constant and the equations give no relationship between the values of z on different orbits so the most general solution is an arbitrary function of the orbits which is an $n-r$ parameter set, hence there are $n-r$ independent solutions if the orbits have dimension r. It follows that the $\boldsymbol{f}_{ik}$ are tangent to the set of r-dimensional manifolds and if coordinate frames $\partial/\partial s_j, 1 \leq j \leq r$ are defined on them giving a basis at each point,the $\boldsymbol{f}_{ik}$ are linear combinations of these i.e.

$$\boldsymbol{f}_{ik} = \sum_{l=1}^{r} \lambda_{ilk} \boldsymbol{b}_l \tag{25}$$

where $\boldsymbol{b}_l \cdot \boldsymbol{\nabla} = \partial/\partial s_l$ or equivalently $a_{ijk} = \sum_{l=1}^{r} \lambda_{ilk} b_{lj}$. Hence given a system of the form (16), after the completion and choice of a linear subspace again of the form (16) one should consider the a_{ijk} to find out the dimension r of the orbits of the Lie group generated by the vector fields $\boldsymbol{f}_{ik}$. This gives the reduced dimension r of the system. The transformed system is easily found by substituting for the a_{ijk} in (16) using (25). This gives

$$\sum_{i=1}^{p} \sum_{l=1}^{r} \lambda_{ilk} \frac{\partial u_i}{\partial s_l} + \sum_{i=1}^{p} u_i a_{ik} = 0 \qquad \text{for } 1 \leq k \leq m. \tag{26}$$

The variables $s_1 \ldots s_r$ are arbitrary coordinates which parametrize the r-dimensional manifolds. The reduced system (26) is again of the same form as (16) with coefficients λ_{ilk} which correspond to a set of vector fields which must have r as the dimension of the orbits of the Lie group otherwise further reduction of dimension would have been possible.

The system (26) is the result of a change of variables starting with a system of the form (16) and exploiting the property that it is effectively a system involving $r \leq n$ independent variables. The systems (16) and (26) therefore have the same set of solutions. Hence considering equations of the form

$$\sum_k h_k^* \cdot \boldsymbol{\nabla} \left(\sum_{i=1}^{p} \sum_{l=1}^{r} \lambda_{ilk} \frac{\partial u_i}{\partial s_l} + \sum_{i=1}^{p} u_i a_{ik} \right) = 0$$

which are first order where $h_k^*.\boldsymbol{\nabla}$ acts within the r-dimensional manifolds one sees at once that it is equivalent to

$$\sum_k h_k^* \cdot \boldsymbol{\nabla} \left(\sum_{i=1}^{p} \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} a_{ijk} + \sum_{i=1}^{p} u_i a_{ik} \right) = 0$$

which must also be first order, so it must be in the vector space of equations which are the completion of the original system (16). Hence there are no new equations obtained by repeating the completion procedure starting with the system (26) so the general procedure to apply to a system of the form (16) is to do the completion, and look for subspaces of this vector space of equations which have reduced dimension 1,2,3 etc in turn. Each subspace is represented by a basis of equations.

As a very simple example of these ideas suppose that

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1} - u_1 + u_2 = 0 \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2} - u_1 + u_2 = 0. \end{aligned} \tag{27}$$

Then by differentiation and elimination of second derivatives

$$\frac{\partial}{\partial x_2}(u_2 - u_1) = \frac{\partial}{\partial x_1}(u_2 - u_1) \tag{28}$$

so $\partial v_1/\partial x_2 = \partial v_1/\partial x_1$ where $v_1 = u_2 - u_1$. From (27) by subtraction

$$\frac{\partial}{\partial x_1}(u_1 + u_2) = \frac{\partial}{\partial x_2}(u_1 + u_2) \tag{29}$$

combining this with (28) gives $\partial u_1/\partial x_1 = \partial u_1/\partial x_2$. Hence the general solution of (27) is of the form $u_1 = g_1(x_1 + x_2), u_2 = g_2(x_1 + x_2)$ but substituting back gives

$$(g_1 + g_2)' = g_1 - g_2. \tag{30}$$

This form of the general solution results from two independent equations derivable from (27) which have $r = 1$ and are reduced in the same set of one-dimensional manifolds, namely $x_1 + x_2 =$ constant. The extra condition (30) relates the solution on different manifolds to each other, hence (27) is not completely reducible to one dimension. This shows an example of the significance of partial reductions.

As the example shows some of the reduced systems may be further simplified by a choice of dependent variables which minimizes the number of them appearing in the derivatives. Consider again a system of the form (16) and introduce new variables v_i by the equations $u_i = \sum_{l=1}^{p} d_{il} v_l$ where d_{il} are functions of $\boldsymbol{x}$ and $\det(d_{il}) \neq 0$. Then

$$\sum_{l=1}^{p}\sum_{j=1}^{n}\frac{\partial v_l}{\partial x_j}\left[\sum_{i=1}^{p} d_{il}a_{ijk}\right] + \sum_{l=1}^{p} v_l\left[\sum_{i=1}^{p}\sum_{j=1}^{n}\frac{\partial d_{il}}{\partial x_j}a_{ijk} + \sum_{i=1}^{p} d_{il}a_{ik}\right] = 0 \quad \text{for } 1 \le k \le m. \tag{31}$$

Suppose that the system is independent of the derivatives of v_l then

$$\sum_{i=1}^{p} d_{il}a_{ijk} = 0 \qquad \text{for } 1 \le k \le m; 1 \le j \le n. \tag{32}$$

If the system is also independent of v_l itself then

$$\sum_{i=1}^{p}\sum_{j=1}^{n}\frac{\partial d_{il}}{\partial x_j}a_{ijk} + \sum_{i=1}^{p} d_{il}a_{ik} = 0 \qquad \text{for } 1 \le k \le m. \tag{33}$$

Using (32) equations (33) can be rewritten as

$$\sum_{i=1}^{p} d_{il}\left(a_{ik} - \sum_{j=1}^{n}\frac{\partial a_{ijk}}{\partial x_j}\right) = 0 \qquad \text{for } 1 \le k \le m \tag{34}$$

so first one should find as many $(p-p')$ LI solution vectors d_i as possible satisfying the equations (32). There may be a subspace of these satisfying (34) also, which should be identified. Suppose it is $(p-t)$-dimensional then $0 \le p-t \le p-p' \le p$ and suppose that $d_{i,1}\dots d_{i,p-t}$ satisfy (32) and (34) and $d_{i,p-t+1}\dots d_{i,p-p'}$ satisfy (32) only, then the set of vectors is completed so that d_{ij} is a non-singular square matrix and the system (31) simplifies to

$$\sum_{l=p-p'+1}^{p}\sum_{j=1}^{n}\frac{\partial v_l}{\partial x_j}\left[\sum_{i=1}^{p} d_{il}a_{ijk}\right]+\sum_{l=p-t+1}^{p} v_l\left[\sum_{i=1}^{p}\sum_{j=1}^{n}\frac{\partial d_{il}}{\partial x_j}a_{ijk}+\sum_{i=1}^{p} d_{il}a_{ik}\right]=0$$

$$\text{for } 1\le k\le m. \tag{35}$$

As shown later for the case $p=m$ at this stage there can be no further first-order PDEs amongst the variables; hence in particular the variables $v_{p-t+1},\dots,v_{p-p'}$, which appear only undifferentiated in (35), are independent unless any linear combination of (35) gives such an equation. Hence they can be specified beforehand provided the original system was consistent. Systems of low dimensionality n obtained in this way will be particularly important, especially if p' is small.

4. Nonlinear systems

The earlier argument dealing with linearization strongly suggests that a similar kind of analysis also works in the general nonlinear case. In this section I develop this theory which includes these results and the general theory of first-order PDEs as special cases but note that in the introduction I made some remarks to the effect that there is probably an equivalent method based on exterior differential systems. All the results are now dependent not only on the point $\boldsymbol{x}$ but also on the solution $\boldsymbol{u}$ there. Return to consideration of the system (12). The first step should be the completion which requires finding the first-order equations which are functions of the members of the system and all their first total derivatives. (A total derivative is a derivative with respect to any of the independent variables while regarding the $\boldsymbol{u}$ as fixed functions of $\boldsymbol{x}$.) This generalizes Jacobi's method [17] for over determined first-order systems with a single unknown which must only appear in the derivatives. From the chain rule the total derivatives are

$$\frac{\mathrm{d}F_k}{\mathrm{d}x_l}=\frac{\partial F_k}{\partial x_l}+\sum_{i=1}^{p}\frac{\partial F_k}{\partial u_i}\frac{\partial u_i}{\partial x_l}+\sum_{i=1}^{p}\sum_{j=1}^{n}\frac{\partial F_k}{\partial u_{i,j}}u_{i,jl}\qquad \text{for } 1\le k\le m; 1\le l\le n. \tag{36}$$

Consider a function $h(\boldsymbol{x},\boldsymbol{u},\{F_k\},\{\mathrm{d}F_k/\mathrm{d}x_l\})$ which is independent of the second derivatives $u_{i,jl}$. From the chain rule

$$\frac{\partial h}{\partial u_{i,jl}}=\sum_{k=1}^{m}\sum_{l^*=1}^{n}\frac{\partial h}{\partial(\mathrm{d}F_k/\mathrm{d}x_{l^*})}\frac{\partial(\mathrm{d}F_k/\mathrm{d}x_{l^*})}{\partial u_{i,jl}}=0.$$

The last derivative can be evaluated from (36) as

$$\frac{\partial(\mathrm{d}F_k/\mathrm{d}x_{l^*})}{\partial u_{i,jl}}=\sum_{i^*=1}^{p}\sum_{j^*=1}^{n}\frac{\partial F_k}{\partial u_{i^*,j^*}}\delta(\{j^*l^*\},\{jl\})\delta_{i^*i}$$

where the first δ function is 1 if and only if the two sets are the same and 0 otherwise; (actually the concept of a set with multiplicity is needed here, each element can appear any number of times.) If $j = l$, $\delta(\{jl\}, \{j^*l^*\})$ can be written as $\delta_{j^*l}\delta_{l^*l}$ but if $j \neq l$ it is non-zero if and only if j^*, l^* are equal to j, l in either order giving $\delta_{jj^*}\delta_{ll^*} + \delta_{jl^*}\delta_{lj^*}$; hence

$$\begin{aligned}\delta(\{jl\}, \{j^*l^*\}) &= \delta_{jl}(\delta_{j^*l}\delta_{l^*l}) + (1 - \delta_{jl})(\delta_{jj^*}\delta_{ll^*} + \delta_{jl^*}\delta_{lj^*}) \\ &= \delta_{jj^*}\delta_{ll^*} + \delta_{jl^*}\delta_{lj^*} - \delta_{l^*j^*}\delta_{jl^*}\delta_{lj^*}\end{aligned}$$

and the equation for h becomes

$$\sum_{k=1}^{m}\sum_{l^*=1}^{n} \frac{\partial h}{\partial\left(\mathrm{d}F_k/\mathrm{d}x_{l^*}\right)} \sum_{j^*=1}^{n} \frac{\partial F_k}{\partial u_{i,j^*}} \delta(\{jl\}, \{j^*l^*\}) = 0.$$

These equations can be written as

$$\sum_{k=1}^{m}\left[\frac{\partial h}{\partial\left(\mathrm{d}F_k/\mathrm{d}x_l\right)}\frac{\partial F_k}{\partial u_{i,j}} + \frac{\partial h}{\partial\left(\mathrm{d}F_k/\mathrm{d}x_j\right)}\frac{\partial F_k}{\partial u_{i,l}}\right] = 0$$
$$\text{for } 1 \leq j \leq l \leq n \text{ and } 1 \leq i \leq p \tag{37}$$

by considering the cases $j = l$ and $j \neq l$ separately. The equations (37) are the necessary and sufficient conditions for h to involve no second derivatives of $\boldsymbol{u}$.

In the first step of the completion procedure a functionally independent complete set of solutions $\boldsymbol{h}$ of (37) which are zero for any solution $\boldsymbol{u}$ of (12) must be found. They can be found from

$$\boldsymbol{h} = \hat{\boldsymbol{h}}\left(\boldsymbol{x}, \boldsymbol{u}, \{F_k\}, \left\{\frac{\mathrm{d}F_k}{\mathrm{d}x_l}\right\}\right) - \hat{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u}, \mathbf{o}, \mathbf{o})$$

where $\hat{\boldsymbol{h}}$ satisfies (37). These include the original equations since if $h = F_k$ then $\partial h/\partial\left(\mathrm{d}F_k/\mathrm{d}x_j\right)$ are all zero and (37) is satisfied. As with the linear equations this should be repeated, starting with the F_k replaced by a complete independent set of solutions $\boldsymbol{h}$ of (37) and continue to be repeated until no new functionally independent results are obtained. The result of this is a set of functions $h_i(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})$ for $1 \leq i \leq m'$ which are zero for any solution of the system (12) and such that any function of $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{h}, \{\mathrm{d}\boldsymbol{h}/\mathrm{d}x_l\}$ necessarily involves some second derivatives of $\boldsymbol{u}$ after substituting for $\boldsymbol{h}$ provided some of the first derivatives $\mathrm{d}h_i/\mathrm{d}x_l$ actually appear. Hence one expects that a function involving second derivatives of $\boldsymbol{h}$ to involve third derivatives of $\boldsymbol{u}$. I will now show that this is true and a straightforward generalization of it leads to the conclusion that no new results can be obtained from the completion procedure by allowing higher derivatives of F_k to appear in the expressions for $\boldsymbol{h}$. The argument is similar to the preceding one to get the equation for $\boldsymbol{h}$ but it is generalizable more easily. Consider $g(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{h}, \{\mathrm{d}\boldsymbol{h}/\mathrm{d}x_l\})$, substituting for the $\boldsymbol{h}$s in terms of $(\boldsymbol{x}, \boldsymbol{u}^{(1)})$, g becomes a function of $(\boldsymbol{x}, \boldsymbol{u}^{(2)})$. From the chain rule

$$\frac{\partial g}{\partial u_{i,jl}} = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} \frac{\partial g}{\partial\left(\mathrm{d}h_\alpha/\mathrm{d}x_\beta\right)} \frac{\partial\left(\mathrm{d}h_\alpha/\mathrm{d}x_\beta\right)}{\partial u_{i,jl}} \tag{38}$$

and from $\boldsymbol{h}$ similarly

$$\frac{\mathrm{d}h_\alpha}{\mathrm{d}x_\beta} = \frac{\partial h_\alpha}{\partial x_\beta} + \sum_{i=1}^{p}\frac{\partial h_\alpha}{\partial u_i}\frac{\partial u_i}{\partial x_\beta} + \sum_{i^*=1}^{p}\sum_{j^*=1}^{n}\frac{\partial h_\alpha}{\partial u_{i^*,j^*}} u_{i^*,\beta j^*}. \tag{39}$$

Hence

$$\frac{\partial\left(\mathrm{d}h_\alpha/\mathrm{d}x_\beta\right)}{\partial u_{i,jl}} = \sum_{j^*=1}^{n} \frac{\partial h_\alpha}{\partial u_{i,j^*}} \delta\left(\{\beta j^*\},\{jl\}\right). \tag{40}$$

This is zero except when β is j or l so, to fix the order of j and l, let $j = \beta$. Then if (40) is non-zero, $j \neq \beta$ implies $j \neq l$ and $\beta = l$, a contradiction so $\delta_{j\beta}$ is a factor. If $j = \beta$ this simplifies to

$$\sum_{j^*=1}^{n} \frac{\partial h_\alpha}{\partial u_{i,j^*}} \delta_{j^* l}$$

so (40) can be written as

$$\frac{\partial\left(\mathrm{d}h_\alpha/\mathrm{d}x_\beta\right)}{\partial u_{i,jl}} = \delta_{j\beta} \frac{\partial h_\alpha}{\partial u_{i,l}} \tag{41}$$

and the equations (38) become

$$\frac{\partial g}{\partial u_{i,jl}} = \sum_{\alpha=1}^{m} \frac{\partial g}{\partial\left(\mathrm{d}h_\alpha/\mathrm{d}x_j\right)} \frac{\partial h_\alpha}{\partial u_{i,l}} \tag{42}$$

which is a set of equations of the form $\boldsymbol{W}_j = M\boldsymbol{V}_j$ for $1 \le j \le n$ where each $\boldsymbol{W}$ is a vector of dimension pn and each $\boldsymbol{V}$ is a vector of dimension m. Once the completion has been done if at least one of the vectors $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_n$ say $\boldsymbol{V}_j$ is non-zero, at least one of the vectors $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_n$ is non-zero which must be $\boldsymbol{W}_j$. Hence M has the property that a non-zero argument gives a non-zero image so M has rank m, the dimension of the space of $\boldsymbol{V}$.

Next consider a function

$$g\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{h}, \{h_{,i}\}, \{h_{,ij}\}\right)$$

where I have used the notations $h_{,i_1\ldots i_s}$ and $\mathrm{d}^s h/\mathrm{d}x_{i_1}\ldots\mathrm{d}x_{i_s}$ for the *total* derivatives of h interchangeably. Then

$$\frac{\partial g}{\partial u_{i,jkl}} = \sum_{\alpha=1}^{m} \sum_{1<\beta\le\gamma\le n} \frac{\partial g}{\partial h_{\alpha,\beta\gamma}} \frac{\partial h_{\alpha,\beta\gamma}}{\partial u_{i,jkl}}. \tag{43}$$

The last derivative can be found by picking out the third-order terms from $h_{\alpha,\beta\gamma}$ which are

$$\sum_{i^*=1}^{p} \sum_{j^*=1}^{n} \left[\frac{\partial h_\alpha}{\partial u_{i^*,j^*}} u_{i^*,\beta\gamma j^*}\right]. \tag{44}$$

Hence

$$\frac{\partial h_{\alpha,\beta\gamma}}{\partial u_{i,jkl}} = \sum_{j^*=1}^{n} \frac{\partial h_\alpha}{\partial u_{i,j^*}} \delta\left(\{\beta\gamma j^*\},\{jkl\}\right). \tag{45}$$

If $\{\beta, \gamma\} \not\subseteq \{j, k, l\}$ the result is zero and if the inclusion does hold then choose indices such that $\beta = j, \gamma = k$ where $\beta \leq \gamma$. This fixes all the indices uniquely. Then if either $\beta \neq j$ or $\gamma \neq k$ the right-hand side is zero. But if $j = \beta$ and $k = \gamma$ then it becomes $\partial h_\alpha / \partial u_{i,l}$; hence from (43)

$$\frac{\partial g}{\partial u_{i,jkl}} = \sum_{\alpha=1}^{m} \frac{\partial g}{\partial h_{\alpha,jk}} \frac{\partial h_\alpha}{\partial u_{i,l}} \tag{46}$$

where $j \leq k$. Comparing (42) with (46) one sees that (46) can also be written as

$$(\boldsymbol{W}'_{jk})_{il} = \sum_{\alpha=1}^{m} M_{\alpha il} (\boldsymbol{V}'_{jk})_\alpha \qquad \text{or} \qquad \boldsymbol{W}'_{jk} = M\boldsymbol{V}'_{jk} \qquad \text{for } 1 \leq j \leq k \leq n \tag{47}$$

where M is the same as in (42) with rank m; $\boldsymbol{V}'$ has dimension m and $\boldsymbol{W}'$ has dimension pn. Hence one non-zero component of $\boldsymbol{V}'_{jk}$ will give at least one non-zero component of $\boldsymbol{W}'_{jk}$ (with the same j and k) i.e. one non-zero value of $\partial g / \partial h_{\alpha,jk}$ will give at least one non-zero value of $\partial g / \partial u_{i,jkl}$ (with the same j and k). This shows that if the h_α are obtained from the F_k by the completion procedure then any function g involving $h_{\alpha,jk}$ must involve some third derivatives of $\boldsymbol{u}$ so it cannot be first order. This shows that after completing the completion procedure previously described, no new first-order equations can be derived by considering second-order differential functions of the $\boldsymbol{h}$. It is fairly straightforward to generalize this to any order.

Reasoning as before shows that at least one non-zero value of $\partial g / \partial h_{\alpha,i_1 \ldots i_s}$ will give rise to at least one non-zero value of $\partial g / \partial u_{i,i_1 \ldots i_{s+1}}$ so if g involves any sth derivatives of any of the h_α, when expressed in terms of $\boldsymbol{u}$ and its derivatives, g will involve an $(s+1)$th order derivative of at least one of the u_i. It shows that if the completion procedure is extended to involve taking higher derivatives and trying to eliminate all but first-order derivatives of $\boldsymbol{u}$ no new results can be obtained. The $\boldsymbol{h}$ obtained at each step are a functionally independent set of solutions of (37) which is of the form $\boldsymbol{f}_k \cdot \boldsymbol{\nabla} \boldsymbol{h} = 0$ where the independent variables are $\{\mathrm{d}F_k / \mathrm{d}x_l\}$ which can be written in terms of $(\boldsymbol{x}, \boldsymbol{u}^{(2)})$. Let the $\tilde{F}_k$ be the $\boldsymbol{h}$ obtained in the last step of the completion which does give new functionally independent results from (37) (which could be the original functions F_k if the system is already complete). In the following step, done to check completion, giving no functionally independent $\boldsymbol{h}$, the $\boldsymbol{h}$ are some set of functions related to the $\tilde{F}_k$ by a non-singular transformation.

The procedure gives all independent first-order equations derivable from the system by repeated differentiation and elimination of the higher derivatives. This presumably generates a complete set of independent first-order equations derivable from the original system i.e. any other first-order equation derivable from the system can be obtained in the form $l(\boldsymbol{x}, \boldsymbol{u}, \{\tilde{F}_k\}) = l(\boldsymbol{x}, \boldsymbol{u}, \mathrm{o})$ i.e. purely algebraically from the set of equations $\tilde{F}_k = 0$ obtained by this procedure. The concept of completeness is here based on the somewhat vague notion of derivability which can be made precise by the concept of local solvability (Olver [1] p 162) which results from considering the problem geometrically in the space with coordinates $(\boldsymbol{x}, \boldsymbol{u}^{(1)})$.

The original system (12) defines the submanifold S_1 of points $(\boldsymbol{x}, \boldsymbol{u}^{(1)})$. Each solution $\boldsymbol{u}(\boldsymbol{x})$ of (12) has a first prolongation which is the set of points $(\boldsymbol{x}, \boldsymbol{u}^{(1)})$ which is always a submanifold S_3 of S_1. Let the union of the S_3 for all solutions $\boldsymbol{u}(\boldsymbol{x})$ be

S_2 then through any point of S_2 there passes the prolongation of a solution $\boldsymbol{u}(\boldsymbol{x})$ of (12) so $S_2 \subseteq S_1$. The system (12) is said to be locally solvable at $(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \in S_1$ if $(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \in S_2$ and locally solvable if $(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \in S_1$ implies $(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \in S_2$ i.e. $S_1 \subseteq S_2$ so $S_1 = S_2$. This is equivalent to requiring that every point $(\boldsymbol{x}, \boldsymbol{u}^{(1)})$ satisfying (12) corresponds to at least one solution $\boldsymbol{u}(\boldsymbol{x})$ in a neighbourhood of $\boldsymbol{x}_0$. Any first-order equations derivable from (12) not by algebra alone will force S_2 to be a proper subset of S_1. The equations defining S_2 which must be first order and deducible from (12) may be called the completion of (12). Therefore a locally solvable system must be the same as its completion and therefore the same as the result of the procedure above i.e. differentiation and elimination of higher derivatives. The completion of (12) must be locally solvable and no further functionally independent equations are derivable from them (this would further reduce the dimension of S_2). It is therefore natural to conjecture that the completion of the system (12) is the same as the system obtained from (12) by the procedure above i.e. the deducibility referred to previously is just repeated differentiation and elimination of the higher derivatives of $\boldsymbol{u}(\boldsymbol{x})$. This justifies the term completion used above. This follows from the conjecture that this procedure always generates a locally solvable system. This can be established for $m = p$ for analytic systems using Finzi's theorem (Olver [1] p 172). Put $n = k = 1$ and $q = m$ and taking its negation on both sides gives: Let $F_k(\boldsymbol{x}, \boldsymbol{u}^{(1)})$ be a first-order system. Then F has a non-characteristic direction at $(\boldsymbol{x}_0, \boldsymbol{u}_0^{(1)})$ if and only if there do not exist operators

$$D_k = \sum_{i=1}^{m} P_k^i \frac{\mathrm{d}}{\mathrm{d}x_i} \qquad \left(\frac{\mathrm{d}}{\mathrm{d}x_i} = \text{total derivative}\right)$$

such that $\sum_{k=1}^{m} D_k F_k = Q(\boldsymbol{x}_0, \boldsymbol{u}_0^{(1)})$. The latter condition follows from the result of the completion procedure. Hence after the completion has been carried out the system is *normal*. If the system is also analytic so is its completion and by corollary 2.80 it is locally solvable.

Having carried out this procedure one obtains another system of the form (12). Consider a set of linear combinations of them

$$\sum_{k=1}^{m} F_k h_{\alpha k}(\boldsymbol{x}, \boldsymbol{u}) = 0 \tag{48}$$

and ask what are the conditions under which equations (48) can be expressed with fewer independent variables. This reduces to the corresponding treatment of the linear case above and I will show how the theory of the single first-order nonlinear PDE in one unknown comes out of this argument.

Suppose that the new independent variables are $z_1, z_2 \ldots z_n$ and each equation of (48) is independent of derivatives with respect to $z_1 \ldots z_{n-r}$. Introducing the new variables into (48) the chain rule must be used to substitute for $\partial u_i / \partial x_j$ thus

$$\frac{\partial u_{i\bullet}}{\partial x_{j\bullet}} = \sum_{l=1}^{n} \frac{\partial u_{i\bullet}}{\partial z_l} \frac{\partial z_l}{\partial x_{j\bullet}}. \tag{49}$$

I require that, applying the chain rule again with these derivatives regarded as variables,

$$0 = \frac{\partial}{\partial(\partial u_i / \partial z_l)} \left(\sum_{k=1}^{m} F_k h_{\alpha k}(\boldsymbol{x}, \boldsymbol{u}) \right)$$

$$= \sum_{i^*=1}^{p}\sum_{j^*=1}^{n} \frac{\partial\left(\partial u_{i^*}/\partial x_{j^*}\right)}{\partial\left(\partial u_i/\partial z_l\right)} \frac{\partial}{\partial\left(\partial u_{i^*}/\partial x_{j^*}\right)} \left(\sum_{k=1}^{m} F_k h_{\alpha k}(\boldsymbol{x},\boldsymbol{u})\right)$$

for $1 \le i \le p; 1 \le l \le n-r$ and from (49)

$$\frac{\partial\left(\partial u_{i^*}/\partial x_{j^*}\right)}{\partial\left(\partial u_i/\partial z_l\right)} = \delta_{ii^*} \frac{\partial z_l}{\partial x_{j^*}}.$$

Hence

$$\sum_{j=1}^{n} \frac{\partial z_l}{\partial x_j} \left(\sum_{k=1}^{m} \frac{\partial F_k}{\partial\left(\partial u_i/\partial x_j\right)} h_{\alpha k}(\boldsymbol{x},\boldsymbol{u})\right) = 0. \tag{50}$$

This holds for each value of α say $1 \le \alpha \le m' \le m$. Denoting the term in parentheses by $(\boldsymbol{f}_{i\alpha})_j$ which depends only on $\boldsymbol{x}$ for fixed $\boldsymbol{u}(\boldsymbol{x})$, the equations take the form

$$\boldsymbol{f}_{i\alpha} \cdot \boldsymbol{\nabla} z = 0 \qquad \text{for } 1 \le i \le p \text{ and } 1 \le \alpha \le m' \tag{51}$$

which is satisfied by the functionally independent variables $z_1, \ldots, z_{n-r}$. Hence the Lie algebra generated by the set of vector fields $\boldsymbol{f}_{i\alpha}$ generates a Lie group with orbits of dimension r. Let the $\boldsymbol{b}_l = \partial/\partial s_l$ define new coordinates $s_1 \ldots s_r$ which parametrize the orbits which are surfaces of constant $z_1 \ldots z_{n-r}$ so that $z_1 \ldots z_{n-r}, s_1 \ldots s_r$ are a new set of coordinates related to $x_1, \ldots, x_n$ by a non-singular transformation. Hence there is a set of r commuting vector fields $\boldsymbol{b}_l$, $1 \le l \le r$ spanning the tangent space of the orbits at each point and the $\boldsymbol{f}_{i\alpha}$ are linear combinations of them:

$$\boldsymbol{f}_{i\alpha} = \sum_{t=1}^{r} \lambda_{it\alpha} \boldsymbol{b}_t. \tag{52}$$

For this to hold it is necessary that $h_{\alpha k}$ is such that for each fixed α, the corresponding set of vectors $\boldsymbol{f}_{i\alpha}$ have a completion which spans a space of dimension at most r. From (50) and (52)

$$f_{i\alpha j} = \sum_{t=1}^{r} \lambda_{it\alpha}(\boldsymbol{x},\boldsymbol{u}) b_{tj}(\boldsymbol{x},\boldsymbol{u}) = \sum_{k=1}^{m} \frac{\partial F_k}{\partial\left(\partial u_i/\partial x_j\right)} h_{\alpha k}(\boldsymbol{x},\boldsymbol{u})$$
$$\text{for } 1 \le \alpha \le m'; 1 \le i \le p; 1 \le j \le n. \tag{53}$$

From (48), taking the total derivatives gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x_j} \left(\sum_{k=1}^{m} F_k h_{\alpha k}(\boldsymbol{x},\boldsymbol{u})\right) \\ &\quad = \sum_{k=1}^{m} \left(\left[\frac{\partial F_k}{\partial x_j} + \sum_{i=1}^{p} \frac{\partial F_k}{\partial u_i} \frac{\partial u_i}{\partial x_j} + \sum_{l=1}^{n}\sum_{i=1}^{p} \frac{\partial F_k}{\partial\left(\partial u_i/\partial x_l\right)} \frac{\partial^2 u_i}{\partial x_l \partial x_j} \right] h_{\alpha k} \right) \\ &\quad = 0 \end{aligned} \tag{54}$$

since $F_k = 0$. The third term becomes, using (53),

$$\sum_{l=1}^{n}\sum_{i=1}^{p}\frac{\partial^2 u_i}{\partial x_l \partial x_j}\sum_{t=1}^{r}\lambda_{it\alpha}b_{tl} = \sum_{i=1}^{p}\sum_{t=1}^{r}\lambda_{it\alpha}\frac{\partial}{\partial s_t}\left(\frac{\partial u_i}{\partial x_j}\right)$$

giving the equations

$$\sum_{k=1}^{m}\left(\left[\frac{\partial F_k}{\partial x_j}+\sum_{i=1}^{p}\frac{\partial F_k}{\partial u_i}\frac{\partial u_i}{\partial x_j}\right]h_{\alpha k}\right)+\sum_{i=1}^{p}\sum_{t=1}^{r}\lambda_{it\alpha}\frac{\partial}{\partial s_t}\left(\frac{\partial u_i}{\partial x_j}\right)=0 \quad (55)$$

for $1 \le j \le n$ and $1 \le \alpha \le m'$. From (53) once the $f_{i\alpha j}$ and one set of vector fields $\boldsymbol{b}_j$ (independent of the $h_{\alpha k}$) have been found for a particular example, such that $[\boldsymbol{b}_i, \boldsymbol{b}_j] = 0$ for all i, j (I am not sure how hard this will be to arrange in general) the $\lambda_{it\alpha}$ can be written down as linear combinations of the $h_{\alpha k}$. Sets of values of $r, m', \boldsymbol{\lambda}, \boldsymbol{b}, \boldsymbol{h}$ are obtained by searching first for cases with $r = 1$ then $r = 2, 3$ etc. $\boldsymbol{h}$ is a vector space of dimension m' any spanning set of which gives a set of $\boldsymbol{f}$ which are in involution. These are substituted into (55) which results in nm' equations for the $np + p + n$ unknowns $x_j, u_i, \partial u_i/\partial x_j$ in terms of $s_1, \ldots, s_r$ as independent variables. In addition to these results one has two further sets of equations:

$$\frac{\partial x_j}{\partial s_i} = b_{ij} \qquad \frac{\partial u_l}{\partial s_i} = \sum_{j=1}^{n}\frac{\partial u_l}{\partial x_j}b_{ij}. \quad (56)$$

If $m' = m$ the reduction of dimension from n to r will be called complete. In this case all the coefficients of the $h_{\alpha k}$ can be independently equated to zero as happens in the last example in this paper. In the resulting system α appears only in the unknowns λ, b, h and is therefore to be treated as a parameter. Hence using the notation

$$\frac{\partial F_k}{\partial u_{i,j}} = G_{kij}\left(\boldsymbol{x}, \boldsymbol{u}^{(1)}\right) \quad (57)$$

equation (53) can be written as

$$\sum_{t=1}^{r}\lambda_{it}b_{tj} = \sum_{k=1}^{m}G_{kij}h_k. \quad (58)$$

Now a matrix $A_{jk} = (\boldsymbol{A}_j)_k$ has rank $\le r$ if and only if there exist r linearly independent vectors $\boldsymbol{c}_i$ such that

$$\boldsymbol{A}_j = \sum_{i=1}^{r}\alpha_{ij}\boldsymbol{c}_i \qquad \text{i.e. } A_{jk} = \sum_{i=1}^{r}\alpha_{ij}c_{ik} \quad (59)$$

hence from (58) the condition on h_k that the b_{tj} exist is that

$$\operatorname{rank}\left(\sum_{k=1}^{m}G_{kij}h_{\alpha k}\right) \le r \qquad \text{for } 1 \le \alpha \le m'. \quad (60)$$

These are necessary conditions on $h_{\alpha k}$ but they are clearly not sufficient unless $m' = 1$ because the b_{lj} do not depend on α. In fact the rank of $\sum_{k=1}^{m} G_{kij} h_{\alpha k}$ is the least dimension for expressing equation α of (48). Further from (53) rank $\left(\sum_{k=1}^{m} (\partial F_k / \partial u_{i,j}) h_{\alpha k}\right) = r$ where the argument is regarded as a matrix with indices $(i, \alpha), j$ and r is the least dimension for expressing the set of linear combinations $h_{\alpha k}$ of the original equations. This is clearly sufficient as well as necessary for the existence of $\boldsymbol{\lambda}$ and $\boldsymbol{b}$ satisfying (53) but it is not sufficient for the $\boldsymbol{b}$ to commute.

After all these calculations have been done for a system it is obvious from the arguments that if $\boldsymbol{u}$ satisfies the original system then $\boldsymbol{x}, \boldsymbol{u}^{(1)}$ satisfy the derived system. The converse can be assured by including the original system, now regarded as a set of algebraic equations amongst the new unknowns, with the derived system. Since (54) is satisfied everywhere it follows that $\sum_{k=1}^{m} F_k h_{\alpha k}(\boldsymbol{x}, \boldsymbol{u}) = 0$. Hence one set of solutions of (53), as above, contributes m' linear combinations of the original system, and when a total of m independent linear combinations of the original system are obtained the original system must hold. This shows that the method does not alter the set of solutions provided care is taken ensure that a complete set of derived equations is obtained. To look for one-dimensional systems resulting from (12), the most useful for numerical calculation, put $r = 1$ then equation (53) becomes

$$\lambda_{i\alpha} b_j = \sum_{k=1}^{m} G_{kij} h_{\alpha k}$$

so for fixed $\boldsymbol{b}$ one seeks all possible solutions $\boldsymbol{\lambda}, \boldsymbol{h}$ of

$$\lambda_i b_j = \sum_{k=1}^{m} G_{kij} h_k .$$

This implies that

$$\lambda_i = \frac{\sum_{k=1}^{m} G_{kij} h_k}{b_j}$$

which is independent of j i.e.

$$\lambda_i = \frac{\sum_{k=1}^{m} G_{ki1} h_k}{b_1} = \frac{\sum_{k=1}^{m} G_{ki2} h_k}{b_2} \cdots = \frac{\sum_{k=1}^{m} G_{kin} h_k}{b_n}. \tag{61}$$

If the ratios of the numerators are independent of i, this determines $\boldsymbol{b}$ up to a scalar factor, otherwise there are no solutions for $\boldsymbol{b}$. If such a $\boldsymbol{b}$ does exist, $\boldsymbol{\lambda}$ is found at once, and the equations can be solved for (some of) the h_k. The system (53), (55) and (56) are a generalization of the equations for the integral strips in the general theory of first-order nonlinear PDEs with one unknown. For if $m = p = 1$ then $m' = 1$ and trying $r = 1$, there is a single vector field $\boldsymbol{f}$ and λ can be taken to be 1. Also h and k are not needed. So put $h = \lambda = 1$ and equation (53) gives simply

$$\frac{\partial F}{\partial \left(\partial u / \partial x_j\right)} = b_j$$

and (55) gives

$$\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial u} \frac{\partial u}{\partial x_j} + \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x_j} \right) = 0 \qquad \text{for } 1 \le j \le n.$$

Hence introducing $p_j = \partial u/\partial x_j$ and eliminating b_j from (56) one obtains

$$\frac{\partial p_j}{\partial s} = -\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial u}p_j \qquad \frac{\partial x_j}{\partial s} = \frac{\partial F}{\partial p_j}$$

and hence

$$\frac{\partial u}{\partial s} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial s}$$

which are the equations for an integral strip for a single nonlinear first-order PDE.

5. Minimizing the number of dependent variables

Returning to the general case, the resulting system (53), (55) and (56) of the form (12) may have an unnecessarily large number of unknowns. If there is a simpler way than the standard method for writing the original higher-order system as a first-order system i.e. involving fewer unknowns, then from section 1, these new unknowns are functions of all the variables in the standard first-order system. To minimize their number look for a change of dependent variables $u_i \to v_i$ for $1 \le i \le p$ given by $u_i = e_i(\boldsymbol{x}, \boldsymbol{v})$ such that upon substituting for u_i and $\partial u_i/\partial x_j$ using

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial e_i}{\partial x_j} + \sum_{l=1}^{p} \frac{\partial e_i}{\partial v_l}\frac{\partial v_l}{\partial x_j}.$$

I obtain

$$0 = \frac{\partial F_k}{\partial\left(\partial v_l/\partial x_j\right)} = \sum_{i=1}^{p}\sum_{j^*=1}^{n} \frac{\partial F_k}{\partial\left(\partial u_i/\partial x_{j^*}\right)}\frac{\partial\left(\partial u_i/\partial x_{j^*}\right)}{\partial\left(\partial v_l/\partial x_j\right)} = \sum_{i=1}^{p} \frac{\partial F_k}{\partial\left(\partial u_i/\partial x_j\right)}\frac{\partial e_i}{\partial v_l} \tag{62}$$

for: $1 \le j \le n$; $p' + 1 \le l \le p$; $1 \le k \le m$; p' chosen to be as small as possible. Hence there are $p - p'$ linearly independent vectors $\partial/\partial v_l$ acting on the manifold with coordinates $u_1, \ldots, u_p$ which satisfy the system

$$\sum_{i=1}^{p} \frac{\partial F_k}{\partial\left(\partial u_i/\partial x_j\right)}\frac{\partial e_i}{\partial v} = 0 \qquad \text{for } 1 \le j \le n; 1 \le k \le m. \tag{63}$$

Hence the matrix $\partial F_k/\partial u_{i,j} = f_{(j,k)i}$ has rank p'. Since the derivatives $\partial e_i/\partial v$ are each taken with all the other v_i held constant, the vectors $X_l = \partial/\partial v_l$ for $p' + 1 \le l \le p$ are in involution with $v_1, \ldots, v_{p'}$ as invariants of the $(p - p')$ dimensional integral manifolds. The equations (63) can be written as

$$\boldsymbol{f}_{(j,k)} \cdot \boldsymbol{X} = 0. \tag{64}$$

From these equations a basis for the X_l must be found and a complete set of invariants of their integral manifolds gives the required new variables. Suppose for example that the given system is

$$F_1 = \frac{\partial}{\partial x_1}(u_1 + u_4 + x_1 u_2) + (u_1 u_2 + u_3)\frac{\partial}{\partial x_2}(u_1 + x_1 u_2 + u_4) = 0 \tag{65}$$

$$F_2 = \frac{\partial}{\partial x_1}(u_1u_2 + u_3) + x_1(u_1 + x_1u_2 + u_4)\frac{\partial}{\partial x_2}(u_1 + x_1u_2 + u_4) = 0. \tag{66}$$

Here $p = 4, m = 2$ and $n = 2$ and note that this system could have been disguised so as to make the choice of new variables not immediately obvious. First the derivatives $\partial F_k/\partial u_{i,j}$ must be found which are as follows where the column headings are (k, j) and i goes from 1 to 4 down each column:

$$\begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \\ 1 & u_1u_2 + u_3 & u_2 & x_1(u_1 + x_1u_2 + u_4) \\ x_1 & x_1(u_1u_2 + u_3) & u_1 & x_1^2(u_1 + x_1u_2 + u_4) \\ 0 & 0 & 1 & 0 \\ 1 & u_1u_2 + u_3 & 0 & x_1(u_1 + x_1u_2 + u_4). \end{array} \tag{67}$$

This is easily seen to have rank 2, hence $p' = 2$ and the equations (64) become

$$\begin{aligned} &X^1 + x_1X^2 + X^4 = 0 \\ &u_2X^1 + u_1X^2 + X^3 = 0. \end{aligned} \tag{68}$$

To get the analytic form for the integral manifolds N it is convenient to let u_1 and u_2 be parameters within each member of N, hence choose $X_3^1 = 1, X_3^2 = 0$ then $X_3^3 = -u_2$ and $X_3^4 = -1$. Similarly choose $X_4^1 = 0, X_4^2 = 1$ then $X_4^3 = -u_1$ and $X_4^4 = -x_1$ so a basis of solutions is $X_3 = (1, 0, -u_2, -1)$ and $X_4 = (0, 1, -u_1, -x_1)$. The integral curves of X_3 are given by

$$\frac{\mathrm{d}u_1}{\mathrm{d}v_3} = 1 \qquad \frac{\mathrm{d}u_2}{\mathrm{d}v_3} = 0 \qquad \frac{\mathrm{d}u_3}{\mathrm{d}v_3} = -u_2 \qquad \text{and} \qquad \frac{\mathrm{d}u_4}{\mathrm{d}v_3} = -1$$

which can immediately be integrated to give

$$\boldsymbol{u} = (v_3 + c_1, c_2, -c_2v_3 + c_3, -v_3 + c_4)$$

where $v_3 = 0$ gives $u_i = c_i$ so $\boldsymbol{u}$ is the translation of $\boldsymbol{c}$ a parameter distance v_3 along X_3. Similarly the integral curves of X_4 may be found giving $\boldsymbol{u} = (d_1, v_4 + d_2, -d_1v_4 + d_3, -x_1v_4 + d_4)$ is the image of $\boldsymbol{d}$ after translation by v_4 along X_4. Hence applying both the mappings (which must commute) to $\boldsymbol{c}$ gives

$$\boldsymbol{u} = \phi^3_{v_3} o \phi^4_{v_4}(\boldsymbol{c}) = (v_3 + c_1, v_4 + c_2, -v_3v_4 - c_1v_4 - c_2v_3 + c_3, -x_1v_4 - v_3 + c_4). \tag{69}$$

The problem is now to determine which functions are constant within the manifolds obtained from the two families of integral curves. If from (69), v_3 and v_4 are eliminated by the relations

$$v_3 = u_1 - c_1 \qquad v_4 = u_2 - c_2$$

one obtains

$$u_3 = -u_1u_2 + c_1c_2 + c_3 \qquad u_4 = -u_1 - x_1u_2 + x_1c_2 + c_1 + c_4.$$

Hence the two functions $c_1c_2 + c_3$ and $x_1c_2 + c_1 + c_4$ together with the parameters u_1, u_2 determine $\boldsymbol{u}$ and the two functions therefore parametrize the set of manifolds so from the previous argument one should choose

$$v_1 = u_1u_2 + u_3 \qquad v_2 = x_1u_2 + u_1 + u_4$$

as new variables in (65) and (66) giving the simplified equations

$$\frac{\partial v_2}{\partial x_1} + v_1\frac{\partial v_2}{\partial x_2} = 0 \qquad \frac{\partial v_1}{\partial x_1} + x_1v_2\frac{\partial v_2}{\partial x_2} = 0.$$

This argument generalizes the corresponding argument for linear systems. In general one can expect parametric dependence on some of the variables in the set $\{v_{p'+1}, \ldots, v_m\}$ although their derivatives have been eliminated.

6. General conclusions

The procedure for analysing a first-order system now seems to be clear. Take the completion of the system so that it becomes locally solvable (at least for the case $m = p$). Then look for reduction of dimension of linear combinations of the equations (coefficients depending on $\boldsymbol{u}$ and $\boldsymbol{x}$) giving systems of dimension $r = 1, 2, 3$ etc in turn. For each case look for a minimal set of unknowns as above. Each set of equations obtained is a potentially useful consequence of the original problem (especially if $m' = p'$ and r is small) whether the solution is obtained finally by numerical or analytic means. If the original system (12) is inconsistent i.e. there are no solutions $\boldsymbol{u}$, every PDE for $\boldsymbol{u}$ is vacuously satisfied by every solution $\boldsymbol{u}$ of (12), hence any PDE should be derivable from (12) using the completion procedure thus the inconsistency of (12) would be expected to be made manifest. Another reason for wanting to use this procedure is that for the case $m = p$ it generates a locally solvable system therefore the necessary and sufficient conditions for an infinitesimal geometrical symmetry operation can be written down some uses of which are mentioned in the introduction.

Finally, it could be thought that the procedure can be repeated, giving results not obtainable from one application of it, by applying it to the derived r-dimensional system obtained from (55) and (56) as previously described. To show that this is not so, regard (55) and (56) as a system of PDEs for the unknowns $x_j, u_i, u_{i,j}$ for $1 \leq i \leq p$ and $1 \leq j \leq n$ with independent variables $s_1, \ldots, s_r$. Write down their first total derivatives with respect to s_β as one would do for the first step of the completion. The results are

$$\frac{\partial^2 x_j}{\partial s_i \partial s_\beta} = \frac{\partial b_{ij}}{\partial s_\beta} \tag{70}$$

$$\frac{\partial^2 u_l}{\partial s_i \partial s_\beta} = \sum_{j=1}^{n}\left[\frac{\partial}{\partial s_\beta}\left(\frac{\partial u_l}{\partial x_j}\right)b_{ij} + \frac{\partial u_l}{\partial x_j}\frac{\partial b_{ij}}{\partial s_\beta}\right] \tag{71}$$

$$\frac{\partial}{\partial s_\beta}\frac{\mathrm{d}}{\mathrm{d}x_j}\left(\sum_{k=1}^{m} F_k h_{\alpha k}(\boldsymbol{x}, \boldsymbol{u})\right)$$

$$= \frac{\partial}{\partial s_\beta} \sum_{k=1}^{m} \left[\frac{\partial F_k}{\partial x_j} + \sum_{i=1}^{p} \frac{\partial F_k}{\partial u_i} \frac{\partial u_i}{\partial x_j} \right] h_{\alpha k}$$

$$+ \sum_{t=1}^{r} \sum_{i=1}^{p} \left[\frac{\partial \lambda_{it\alpha}}{\partial s_\beta} \frac{\partial}{\partial s_t} \left(\frac{\partial u_i}{\partial x_j} \right) + \lambda_{it\alpha} \frac{\partial^2}{\partial s_\beta \partial s_t} \left(\frac{\partial u_i}{\partial x_j} \right) \right] = 0. \qquad (72)$$

Equations (72) have second derivatives of the unknowns with respect to s_β only in the last term because F_k and their derivatives are now functions of the new unknowns only, not their derivatives with respect to s. Can there be a linear combination of (70), (71) and (72) involving only the first derivatives of the unknowns i.e. only up to the second derivatives of u? Such a linear combination could clearly not involve either of the sets of equations (70) or (71) because the second derivatives in them appear once in each equation and nowhere else in the system. But the third set of equations (72) is obtained algebraically from the second derivatives of the original system after *completion*, hence from the completion procedure described in section 4 any linear combinations of them must involve third derivatives of u i.e. second derivatives of the new unknowns. This disproves this possibility of further reduction on the assumption that the completion of the original system was found in the first step.

In the case of a single higher order equation which is locally solvable (this includes all equations expressible in general Kovalevskaya form) the procedure can be simplified because of the following argument. For simplicity I shall only prove the result for second-order equations but generalization is straightforward.

If the equation

$$F\left(x, u, \left\{ \frac{\partial u}{\partial x_i} \right\}, \left\{ \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} \right) = 0 \qquad (73)$$

is locally solvable so is the system (74)+(75). A corresponding first-order system derived from (73) is

$$F\left(x, u, \{p_i\}, \left\{ \frac{\partial p_i}{\partial x_j} \right\} \right) = 0 \qquad p_i = \frac{\partial u}{\partial x_i} \qquad \text{for } 1 \le i \le n \qquad (74)$$

to which the extra equations

$$\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i} \qquad (75)$$

must be added. Suppose that

$$\left(x_0, u_0, \{p_i|_0\}, \left\{ \left. \frac{\partial u}{\partial x_j} \right|_0 \right\}, \left\{ \left. \frac{\partial p_i}{\partial x_j} \right|_0 \right\} \right)$$

satisfies (74) and (75), then

$$\left(x_0, u_0, \{p_i|_0\}, \left\{ \left. \frac{\partial p_i}{\partial x_j} \right|_0 \right\} \right)$$

satisfies the first equation of (74) and (75) and the same values of $(\boldsymbol{x}_0, u_0, \{\partial u/\partial x_i|_0\},$ $\{\partial^2 u/\partial x_i \partial x_j|_0\})$ satisfy (73) and hence there is a neighbourhood of $\boldsymbol{x}_0$ in which u satisfies (73) and at $\boldsymbol{x} = \boldsymbol{x}_0$,

$$\frac{\partial u}{\partial x_i} = \left.\frac{\partial u}{\partial x_i}\right|_0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \left.\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_0 .$$

From u compute $\partial u/\partial x_i = p_i$ then (74) and (75) are satisfied by u and p_i in the same neighbourhood, showing that (74) and (75) are locally solvable provided (73) is also.

A similar argument shows that any first-order system obtained from a higher order locally solvable equation by the standard method is itself locally solvable, provided the extra equations resulting from the commutativity of the partial derivatives are included. This result can be further extended to any reduction of the equation to first order because the new variables are then functions of $(\boldsymbol{x}, u, \{p_i\})$ and local solvability is a geometrical property independent of any particular coordinate system. This result also holds for any number of equations. Its significance is that systems in Kovalevskaya form can be directly written as a first-order system which is locally solvable which will not require the completion procedure previously described.

7. An application connected with statistical mechanics

This is a standard problem in statistical mechanics. The objective is to calculate thermodynamic properties (for example the relationship between the pressure, volume and temperature for unit mass of fluid) and correlation functions, describing the distribution of distances between the atoms or molecules, from a knowledge of the potential energy functions describing the law of force between the particles of the fluid. Usually in applications one is concerned with three-dimensional systems (in contrast to the one-dimensional systems treated here) for which many approximate methods have been developed both by analytic means and computer simulation [19]. The general statistical mechanics theory is based on Newton's laws of motion together with the standard statistical assumption of the Grand Canonical Distribution [20]. I have shown in [11] that these, for the one-dimensional case, give rise to a functional equation from which a hierarchy of approximations can be introduced obtained by truncating the functional Taylor expansion of this equation after N terms. The first-order equation was shown to lead to a set of integral equations which are numerically tractable. Later I attempted to solve the second-order equation [12]. The methods developed in the present paper should be applicable to any member of such a hierarchy after a discretizing approximation similar to those used in [12]. I will show here a related example of a system with n independent variables which is *completely* reducible to a system with of dimension $r = 2$ i.e. there are no more independent equations obtained by looking for derived systems with $r \geq 3$. Hence going to the limit $n \to \infty$ it gives a PDE in infinite dimensions (a functional differential equation) which is reducible to two dimensions. In a forthcoming paper I will report the details of the application of these ideas to statistical mechanics.

Consider the following equation:

$$\frac{\partial u}{\partial x_0} = \exp\left[\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + \frac{1}{2}\left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j}\right)\right] \tag{76}$$

where the a_i are constants. Equations (76) can be written as a first-order system thus

$$F_1 = u_2 - \sum_{i=1}^{n} a_i \frac{\partial u_1}{\partial x_i} = 0$$
$$F_2 = \frac{\partial u_1}{\partial x_0} - \exp\left(u_2 + \frac{1}{2}\sum_{i=1}^{n} a_i \frac{\partial u_2}{\partial x_i}\right) = 0. \tag{77}$$

To show that (77) is locally solvable, differentiation gives

$$\frac{\partial F_1}{\partial u_{i,j}} = \begin{cases} -a_j & i=1;\ 1\le j\le n \\ 0 & i=1;\ j=0 \\ 0 & i=2 \end{cases} \tag{78}$$

and

$$\frac{\partial F_2}{\partial u_{i,j}} = \begin{cases} 1 & i=1;\ j=0 \\ 0 & i=1;\ 1\le j\le n \\ 0 & i=2;\ j=0 \\ -\frac{1}{2}Ea_j & i=2;\ 1\le j\le n \end{cases} \tag{79}$$

where I have written $\exp\left(u_2 + \frac{1}{2}\sum_{i=1}^{n} a_i \partial u_2/\partial x_i\right)$ as $E(\boldsymbol{x})$. Using the notation

$$\frac{\partial h}{\partial\left(\mathrm{d}F_k/\mathrm{d}x_j\right)} = \tilde{h}_{kj}$$

I obtain from (37) for $j = l$ and $i = 2$

$$\tilde{h}_{2j}\begin{Bmatrix} 0 & j=0 \\ -\frac{1}{2}Ea_j & 1\le j\le n \end{Bmatrix} = 0 \tag{80}$$

and likewise for $i = 1$

$$\tilde{h}_{1j}\begin{Bmatrix} -a_j & 1\le j\le n \\ 0 & j=0 \end{Bmatrix} + \tilde{h}_{2j}\begin{Bmatrix} 1 & j=0 \\ 0 & 1\le j\le n \end{Bmatrix} = 0. \tag{81}$$

From (80) and (81) it follows immediately that $\tilde{h}_{ij} = 0$ for $0 \le j \le n$ and $i = 1, 2$ except $\tilde{h}_{10}$, provided all the a_i are non-zero, but then (37) reduces to $\tilde{h}_{10}\partial F_1/\partial u_{i,l} = 0$ for $1 \le l \le n; 1 \le i \le p$ so $\tilde{h}_{10} = 0$, hence h is independent of dF_k/dx_j giving only a trivial solution. This shows that (77) is complete and locally solvable since $m = p$.

Next look for linear combinations of (77) which give reduction of dimension so consider

$$h = h_1(\boldsymbol{x}, \boldsymbol{u})F_1 + h_2(\boldsymbol{x}, \boldsymbol{u})F_2 \tag{82}$$

hence

$$\frac{\partial h}{\partial u_{i,j}} = h_1\frac{\partial F_1}{\partial u_{i,j}} + h_2\frac{\partial F_2}{\partial u_{i,j}}. \tag{83}$$

First suppose that $r = 1$ then changing independent variables to $z_0, z_1, \ldots, z_n$ and suppose that derivatives with respect to $z_1, \ldots, z_n$ do not appear. Then from (50)

$$\sum_{j=0}^{n} \frac{\partial h}{\partial u_{i,j}} \frac{\partial z_l}{\partial x_j} = 0 \qquad \text{for } 1 \leq l \leq n \text{ and } 1 \leq i \leq 2. \tag{84}$$

Writing down the derivatives of h explicitly from (78), (79) and (83) gives

$$\frac{\partial h}{\partial u_{i,j}} = \begin{cases} -h_1(\boldsymbol{x})a_j & i = 1; 1 \leq j \leq n \\ -\frac{1}{2}h_2(\boldsymbol{x})E(\boldsymbol{x})a_j & i = 2; 1 \leq j \leq n \\ h_2(\boldsymbol{x}) & i = 1; j = 0 \\ 0 & i = 2; j = 0 \end{cases} \tag{85}$$

hence (84) gives

$$h_2(\boldsymbol{x})\frac{\partial z_l}{\partial x_0} - \sum_{j=1}^{n} h_1(\boldsymbol{x})a_j \frac{\partial z_l}{\partial x_j} = 0 \qquad \text{for } i = 1$$

and

$$-\frac{1}{2}\sum_{j=1}^{n} \frac{\partial z_l}{\partial x_j} a_j h_2(\boldsymbol{x})E(\boldsymbol{x}) = 0 \qquad \text{for } i = 2$$

for $1 \leq l \leq n$. If there is a one-dimensional derived system, the Lie algebra generated by the $\boldsymbol{f}_k$ must have dimension one so they are parallel which implies that $h_2(\boldsymbol{x}) = 0$, hence $h_1(\boldsymbol{x}) = 0$ so only a trivial result is obtained showing that no reduction to one dimension is possible. Before searching systematically for reduced systems for small r one should first identify the vector fields $\boldsymbol{f}$ of equation (51) which from (85) are

$$(f_1)_j = f_{1j} = h_2\delta_{j0} - h_1 a_j(1 - \delta_{j0}) \tag{86}$$

$$(f_2)_j = f_{2j} = -\frac{1}{2}h_2 E a_j(1 - \delta_{j0}). \tag{87}$$

It is now straightforward to compute the derivatives of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ and hence their commutator

$$[\boldsymbol{f}_1, \boldsymbol{f}_2]_j = \boldsymbol{f}_1(\boldsymbol{f}_2(x_j)) - \boldsymbol{f}_2(\boldsymbol{f}_1(x_j)) = \sum_{i=0}^{n}\left(f_{1i}\frac{\partial}{\partial x_i}f_{2j} - f_{2i}\frac{\partial}{\partial x_i}f_{1j}\right).$$

The result is

$$\begin{aligned} [\boldsymbol{f}_1, \boldsymbol{f}_2]_0 &= \frac{1}{2}Eh_2(\boldsymbol{x})\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}h_2(\boldsymbol{x}) \\ [\boldsymbol{f}_1, \boldsymbol{f}_2]_j &= -\frac{1}{2}h_2 a_j \frac{\partial}{\partial x_0}(h_2 E) + \sum_{i=1}^{n}\left[\frac{1}{2}a_j a_i h_1 \frac{\partial}{\partial x_i}(h_2 E) - \frac{1}{2}h_2 E a_i a_j \frac{\partial h_1}{\partial x_i}\right] \\ & 1 \leq j \leq n. \end{aligned} \tag{88}$$

If $[\boldsymbol{f}_1, \boldsymbol{f}_2] = \lambda \boldsymbol{f}_1 + \mu \boldsymbol{f}_2$, (the necessary and sufficient condition for $r = 2$ for a single value of k in (52)), then for component 0,

$$\frac{1}{2} h_2 E \sum_{i=1}^{n} a_i \frac{\partial h_2}{\partial x_i} = \lambda h_2$$

so

$$\lambda = \frac{1}{2} E \sum_{i=1}^{n} a_i \frac{\partial h_2}{\partial x_i}$$

and for the other components, after dividing by a_j, I get a single equation from which μ may be determined thus

$$-\frac{1}{2} h_2 \frac{\partial}{\partial x_0} (h_2 E) + \frac{1}{2} \sum_{i=1}^{n} a_i \left[h_1 \frac{\partial}{\partial x_i} (h_2 E) - h_2 E \frac{\partial h_1}{\partial x_i} \right] = -\frac{1}{2} E h_1 \sum_{i=1}^{n} a_i \frac{\partial h_2}{\partial x_i} - \frac{1}{2} \mu h_2 E. \tag{89}$$

This shows that there is a non-trivial set of equations for $r = 2$ for arbitrary h i.e. any two LI vectors $\boldsymbol{h}_\alpha$ could be chosen each pair giving a complete set of two equations, each member of the pair being reducible to two dimensions. But to get a closed system in two dimensions requires the stronger condition that the set of vectors $\boldsymbol{f}$, for each choice of $\boldsymbol{h}$, spans the same space i.e. equation (52) must hold

$$f_{i\alpha j} = \sum_{t=1}^{2} \lambda_{it\alpha} b_{tj} = \begin{cases} h_{\alpha 2} \delta_{j0} - h_{\alpha 1} a_j (1 - \delta_{j0}) & i = 1 \\ -\frac{1}{2} h_{\alpha 2} E a_j (1 - \delta_{j0}) & i = 2 \end{cases} \qquad \text{for } 0 \le j \le n; \alpha = 1, 2. \tag{90}$$

Hence regarding this as a set of linear combinations of the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, (90) implies that the four vectors on the right for $i = 1, 2$ and $\alpha = 1, 2$ have rank 2. This is easily shown to be true and the obvious choice of the b and λ is

$$b_{1j} = \delta_{j0} \qquad b_{2j} = a_j (1 - \delta_{j0})$$

and

$$\begin{aligned} \lambda_{1t1} &= (h_{12}, -h_{11}) \qquad & \lambda_{1t2} &= (h_{22}, -h_{21}) \\ \lambda_{2t1} &= (0, -\tfrac{1}{2} h_{12} E) \qquad & \lambda_{2t2} &= (0, -\tfrac{1}{2} h_{22} E). \end{aligned} \tag{91}$$

Now substituting into (55) and (56) gives all the reduced equations.

Denoting for simplicity the derivative of F_k with respect to x_j taken at constant $u_{i,j}$ but varying u_i appearing in (55) by $\mathrm{d}F_k/\mathrm{d}x_j$ equation (55) becomes

$$\sum_{k=1}^{2} \frac{\mathrm{d}F_k}{\mathrm{d}x_j} h_{\alpha k} + \sum_{i=1}^{2} \sum_{t=1}^{2} \lambda_{it\alpha} \frac{\partial}{\partial s_t} \left(\frac{\partial u_i}{\partial x_j} \right) = 0.$$

Inserting the values of λ and because the $\boldsymbol{b}$ are independent of the $\boldsymbol{h}$, the coefficients of the $\boldsymbol{h}$ can be equated giving just two independent equations which are

$$\frac{\mathrm{d}F_1}{\mathrm{d}x_j} - \frac{\partial}{\partial s_2}\left(\frac{\partial u_1}{\partial x_j}\right) = 0 \qquad \frac{\mathrm{d}F_2}{\mathrm{d}x_j} + \frac{\partial}{\partial s_1}\left(\frac{\partial u_1}{\partial x_j}\right) - \frac{1}{2}E\frac{\partial}{\partial s_2}\left(\frac{\partial u_2}{\partial x_j}\right) = 0. \tag{92}$$

Using the notation $p_{ij} = u_{i,j}$ and from (77)

$$\frac{\mathrm{d}F_1}{\mathrm{d}x_j} = p_{2j} \qquad \frac{\mathrm{d}F_2}{\mathrm{d}x_j} = -Ep_{2j}$$

where

$$E = \exp\left(u_2 + \frac{1}{2}\sum_{i=1}^{n} a_i p_{2i}\right). \tag{93}$$

Hence the complete set of equations for the reduction of (76) to $r = 2$ is

$$\begin{aligned}
&\frac{\partial u_l}{\partial s_1} = p_{l0} \qquad \frac{\partial u_l}{\partial s_2} = \sum_{j=1}^{n} p_{lj} a_j \\
&p_{2j} - \frac{\partial}{\partial s_2} p_{1j} = 0 \qquad -Ep_{2j} + \frac{\partial}{\partial s_1} p_{1j} - \frac{1}{2}E\frac{\partial p_{2j}}{\partial s_2} = 0 \\
&\frac{\partial x_j}{\partial s_1} = \delta_{j0} \qquad \frac{\partial x_j}{\partial s_2} = a_j(1-\delta_{j0})
\end{aligned} \tag{94}$$

together with (93). The $3n+5$ unknowns are now x_j, u_i, p_{ij} for $i = 1, 2; 0 \le j \le n$ but there are $4n+8$ equations for them where E has been introduced as an abbreviation in (79). First consider the equations for $\boldsymbol{x}(\boldsymbol{s})$ which can be immediately integrated giving

$$x_j = \delta_{j0}s_1 + c_j \qquad x_j = a_j(1-\delta_{j0})s_2 + d_j. \tag{95}$$

These can be regarded as the result of translating the points $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively through parameter distances s_1,s_2 along the respective integral curves. Consistency, i.e. $\boldsymbol{x}$ is uniquely determined by (s_1, s_2) for a given value of $\boldsymbol{x}(0,0)$, requires that these mappings commute (as guaranteed by the general theory) which is obviously true. Applying the mappings in succession gives

$$x_j = a_j(1-\delta_{j0})s_2 + \delta_{j0}s_1 + c_j \qquad \text{for } 0 \le j \le n. \tag{96}$$

From the second and third equations of (94), p_{2j} can be eliminated giving

$$E\left(\frac{\partial p_{1j}}{\partial s_2} + \frac{1}{2}\frac{\partial^2 p_{1j}}{\partial s_2^2}\right) - \frac{\partial p_{1j}}{\partial s_1} = 0. \tag{97}$$

Hence p_{1j} should be found in terms of E from (97), then differentiated to obtain p_{2j}. Then $\boldsymbol{u}$ can be obtained from these by integration. Finally (93) allows the calculation of E which gives a closed set of equations to be solved by iteration or possibly by further analysis.

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