Series on Knots and Everything — Vol. 39

TOPOLOGICAL LIBRARY

Part 1: Cobordisms and Their Applications

editors

S. P. Novikov **I. A. Taimanov**

translated by **V. O. Manturov**

TOPOLOGICAL LIBRARY

Part 1: Cobordisms and Their Applications

SERIES ON KNOTS AND EVERYTHING

Editor-in-charge: Louis H. Kauffman *(Univ. of Illinois, Chicago)*

The Series on Knots and Everything: is a book series polarized around the theory of knots. Volume 1 in the series is Louis H Kauffman's Knots and Physics.

One purpose of this series is to continue the exploration of many of the themes indicated in Volume 1. These themes reach out beyond knot theory into physics, mathematics, logic, linguistics, philosophy, biology and practical experience. All of these outreaches have relations with knot theory when knot theory is regarded as a pivot or meeting place for apparently separate ideas. Knots act as such a pivotal place. We do not fully understand why this is so. The series represents stages in the exploration of this nexus.

Details of the titles in this series to date give a picture of the enterprise.

Published:

Vol. 1: Knots and Physics (3rd Edition)
by L. H. Kauffman

Vol. 2: How Surfaces Intersect in Space — An Introduction to Topology (2nd Edition)
by J. S. Carter

Vol. 3: Quantum Topology
edited by L. H. Kauffman & R. A. Baadhio

Vol. 4: Gauge Fields, Knots and Gravity
by J. Baez & J. P. Muniain

Vol. 5: Gems, Computers and Attractors for 3-Manifolds
by S. Lins

Vol. 6: Knots and Applications
edited by L. H. Kauffman

Vol. 7: Random Knotting and Linking
edited by K. C. Millett & D. W. Sumners

Vol. 8: Symmetric Bends: How to Join Two Lengths of Cord
by R. E. Miles

Vol. 9: Combinatorial Physics
by T. Bastin & C. W. Kilmister

Vol. 10: Nonstandard Logics and Nonstandard Metrics in Physics
by W. M. Honig

Vol. 11: History and Science of Knots
edited by J. C. Turner & P. van de Griend

Vol. 12: Relativistic Reality: A Modern View
edited by J. D. Edmonds, Jr.

Vol. 13: Entropic Spacetime Theory
by J. Armel

Vol. 14: Diamond — A Paradox Logic
by N. S. Hellerstein

Vol. 15: Lectures at KNOTS '96
by S. Suzuki

Vol. 16: Delta — A Paradox Logic
by N. S. Hellerstein

Vol. 17: Hypercomplex Iterations — Distance Estimation and Higher Dimensional Fractals
by Y. Dang, L. H. Kauffman & D. Sandin

Vol. 19: Ideal Knots
by A. Stasiak, V. Katritch & L. H. Kauffman

Vol. 20: The Mystery of Knots — Computer Programming for Knot Tabulation
by C. N. Aneziris

Vol. 24: Knots in HELLAS '98 — Proceedings of the International Conference on Knot Theory and Its Ramifications
edited by C. McA Gordon, V. F. R. Jones, L. Kauffman, S. Lambropoulou & J. H. Przytycki

Vol. 25: Connections — The Geometric Bridge between Art and Science (2nd Edition)
by J. Kappraff

Vol. 26: Functorial Knot Theory — Categories of Tangles, Coherence, Categorical Deformations, and Topological Invariants
by David N. Yetter

Vol. 27: Bit-String Physics: A Finite and Discrete Approach to Natural Philosophy
by H. Pierre Noyes; edited by J. C. van den Berg

Vol. 28: Beyond Measure: A Guided Tour Through Nature, Myth, and Number
by J. Kappraff

Vol. 29: Quantum Invariants — A Study of Knots, 3-Manifolds, and Their Sets
by T. Ohtsuki

Vol. 30: Symmetry, Ornament and Modularity
by S. V. Jablan

Vol. 31: Mindsteps to the Cosmos
by G. S. Hawkins

Vol. 32: Algebraic Invariants of Links
by J. A. Hillman

Vol. 33: Energy of Knots and Conformal Geometry
by J. O'Hara

Vol. 34: Woods Hole Mathematics — Perspectives in Mathematics and Physics
edited by N. Tongring & R. C. Penner

Vol. 35: BIOS — A Study of Creation
by H. Sabelli

Vol. 36: Physical and Numerical Models in Knot Theory
edited by J. A. Calvo et al.

Vol. 37: Geometry, Language, and Strategy
by G. H. Thomas

Vol. 38: Current Developments in Mathematical Biology
edited by K. Mahdavi, R. Culshaw & J. Boucher

Vol. 39: Topological Library
Part 1: Cobordisms and Their Applications
edited by S. P. Novikov and I. A. Taimanov

Vol. 40: Intelligence of Low Dimensional Topology 2006
edited by J. Scott Carter et al.

K&E Series on Knots and Everything — Vol. 39

TOPOLOGICAL LIBRARY

Part 1: Cobordisms and Their Applications

editors

S. P. Novikov
Landau Institute for Theoretical Physics, Russia &
University of Maryland, USA

I. A. Taimanov
Sobolev Institute of Mathematics, Russia

translated by **V. O. Manturov**

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Topological library: cobordisms and their applications / edtied by S.P. Novikov & I.A. Taimanov ; translated by V.O. Manturov.
p. cm. -- (Series on knots and everything ; v. 39) (Topological Library)
Includes bibliographical references and index.
ISBN-13 978-981-270-559-4 (hardcover : alk. paper) -- ISBN-10 981-270-559-7
1. Cobordism theory. I. Novikov, Sergei Petrovich. II. Taimanov, I.A. (Iskander Asanovich), 1961–

QA613.66.C63 2007
514'.72--dc22

2007016753

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore.

TOPOLOGICAL LIBRARY

Volume I. Cobordisms and Their Applcations

1) Pontrjagin L. S. Smooth Manifolds and Their Applications in Homotopy Theory.
2) Thom R. Some General Properties of Differentiable Manfiolds.
3) Novikov S. P. Homotopy Properties of Thom Complexes.
4) Smale S. Generalized Poincaré conjecture in dimensions greater than four.
5) Smale S. On the Structure of Manfiolds.
6) Quillen D. On the formal group laws of unoriented and complex cobordism theory.
7) Buchstaber V.M., Mishchenko A.S., Novikov S.P. Formal Groups and Their Role in Algebraic Topology Approach.
8) Buchstaber V.M., Novikov S.P. Formal groups, power systems and Adams operators

Volume II. Characteristic classes and smooth structures on manifolds

1) Milnor J. On manifolds homeomorphic to the 7-sphere.
2) Kervaire M. A., Milnor J. Groups of homotopy spheres.

3) Novikov S.P. Homotopy equivalent smooth manifolds.

4) Novikov S.P. Rational Pontrjagin Classes. Homomorphism and Homotopy Type of Closed Manifolds.

5) Novikov S.P. On Manifolds with Free Abelian Fundamental Group.

6) Kirby R. Stable Homomorphisms and The Annulus Conjecture.

Volume III. Spectral Sequences in Topology

1) Serre J.-P. Singular Homology of Fibred Spaces.

2) Serre J.-P. Homotopy Groups and Classes of Abelian Groups.

3) Serre J.-P. Cohomology Mod 2 of Eilenberg-MacLane Complexes.

4) Borel A. On Cohomology of Principal Fibred Spaces and Homogeneous Spaces of Compact Lie Groups.

5) Borel A. Cohomology Mod 2 of Some Homogeneous Spaces.

6) Milnor J. The Steenrod Algebra and The Dual Algebra.

7) Adams J.F. On the Structure and Applications of the Steenrod Algebra.

8) Atiyah M., Hirzebruch F. Vector Bundles and Homogeneous Spaces.

9) Novikov S.P. Algebraic Topology Methods from The Point of View of Cobordism Theory.

Contents

Preface

Topology, created by H. Poincaré in the late 19th and early 20th century as a new branch of mathematics under the name "Analysis Situs" differed in its style and character from other parts of mathematics: it was less rigorous, more intuitive and visible than the other branches. It was not by chance that topological ideas attracted physicists and chemists of the 19th century, for instance, Maxwell, Kelvin and Betti, as well as other scientists residing at the junction of mathematics and physics, such as Gauss, Euler and Poincaré. Hilbert thought it necessary to make this beautiful part of mathematics more rigorous; as it was, it seemed to Hilbert alien.

As a result of the rapid development of 1930s–1960s, it was possible to make all achievements of previously known topology more rigorous and to solve many new deep problems, which seemed to be inaccessible before. This leads to the creation of new branches, which changed not only the face of topology itself, but also of algebra, analysis, geometry — Riemannian and algebraic, — dynamical systems, partial differential equations and even number theory. Later on, topological methods influenced the development of modern theoretical physics. A number of physicists have taken a great interest in pure topology, as in 19-th century.

How to learn classical topology, created in 1930s–1960s? Unfortunately, the final transformation of topology into a rigorous and exact section of pure mathematics had also negative consequences: the language became more abstract, its formalization — I would say, excessive, took topology away from classical mathematics. In the 30s and 40s of the 20-th century, some textbooks without artificial formalization were created: "Topology" by Seifert and Threlfall, "Algebraic Topology" by Lefschetz, "The topology of fibre bundles" by Steenrod. The monograph "Smooth manifolds and their applications in homotopy theory" by Pontrjagin written in early 50s and, "Morse Theory" by Milnor, written later, are also among the best examples. One should also recommend Atiyah's "Lectures on K-Theory" and Hirzebruch's "New Topological Methods in Algebraic Geometry", and also "Modern geometric structures and fields" by Novikov and Taimanov and Springer Encyclopedia Math Sciences, vol. 12, Topology-1 (Novikov) and vol. 24, Topology-2 (Viro and Fuchs), and Algebraic Topology by A.Hatcher (Cambridge Univ. Press).

However, no collection of existing textbooks covers the beautiful ensemble of methods created in topology starting from approximately 1950, that is, from Serre's celebrated "Singular homologies of fibre spaces". The description of this and following ideas and results of classical topology (that finished around 1970) in the textbook literature is reduced to impossible abstractly and to formally stated slices, and in the rest simply is absent. Luckily, the best achievements of this period are quite well described in the original papers — quite clearly and with useful proofs (after the mentioned period of time even that disappears — a number of fundamental "Theorems" is not proved in the literature up to now).

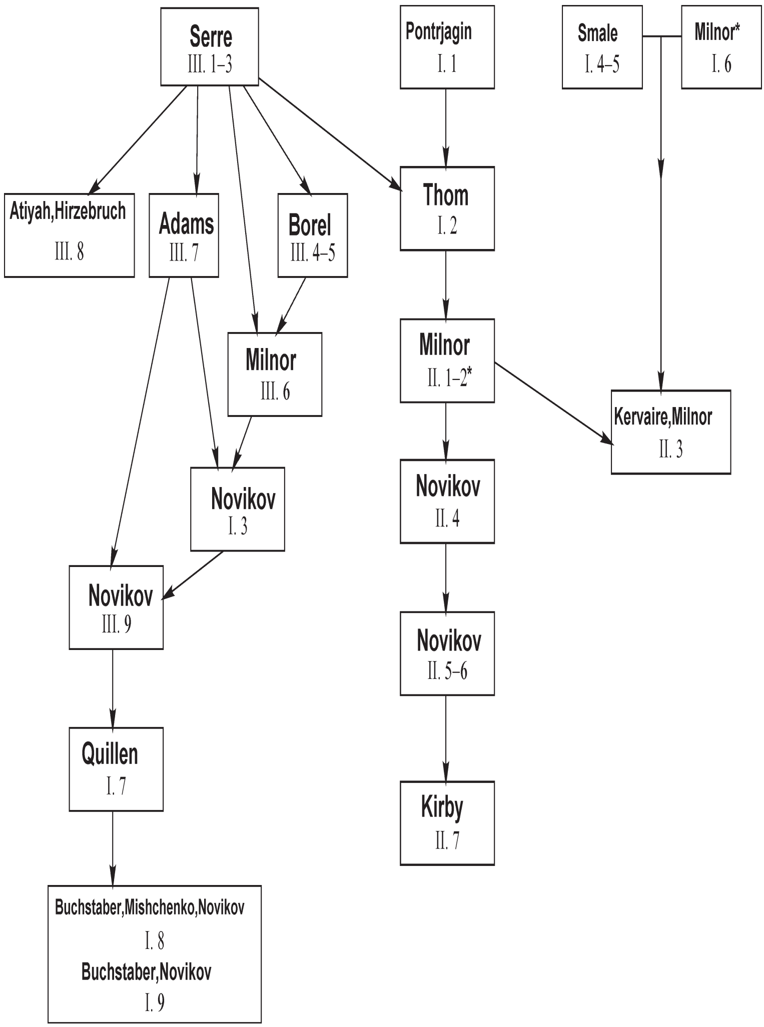
We have decided to publish this collection of works of 1950s–1960s, that allow one to learn the main achievements of the above mentioned period. Something similar was done in late 1950s in the USSR, when the celebrated collection "Fibre spaces" was published, which allowed one to teach topology to the whole new generation of young mathematicians. The present collection is its ideological continuation. We should remark that the English translations of the celebrated papers by Serre, Thom, and Borel which are well-known for the excellent exposition and which were included in the book of "Fibre spaces" were never published before as well as the English translation of my paper "Homotopical properties of Thom complexes".

Its partition into 3 volumes is quite relative: it was impossible to collect all papers in one volume. The algebraic methods created in papers published in the third volume are widely used even in many articles of the first volume, however, we ensured that several of the initial articles of the first volume employ more elementary methods. We supply this collection by the graph which demonstrates the interrelation of the papers: if one of them has to be studied after another this relation is shown by an arrow. We also present the list of additional references to books which will be helpful for studying topology and its applications.

We hope that this collection would be useful.

S. P. Novikov

The interrelation between articles listed in the Russian edition of the Topological Library looks as follows:



Milnor's books "Lectures on the h-cobordism Theorem" and "Lectures of Characteristic Classes" (Milnor I.6 and Milnor II.2) are not included into the present edition of the series.[1]

[1]Due to the omission of the two articles, the numerical order of the present edition has been shifted.

Complementary References:

Springer "Encyclopedia of Math. Sciences" books.

Topology I General Survey, Novikov, S.P., Vol. 12, 1996.

Topology II, Homotopy and Homology: Fuchs, D.B., Viro, O.Y.Rokhlin, V.A., Novikov, S.P. (Eds.), Vol. 24, 2004

Novikov, S.P., Taimanov, I.A., Modern Geometric Structures and Fields, AMS, 2006.

Milnor, J. W. Morse Theory. Princeton, NJ: Princeton University Press, 1963.

Atiyah, M.F. K-theory, W. A. Benjamin, New York, 1967.

S.Lefschetz, Algebraic Topology, AMS., 1942.

Algebraic Topology, to Appear, available from http://www.math.cornell.edu/ ~ hatcher/# ATI

Hirzebruch, F. Topological Methods in Algebraic Geometry, Springer, N.-Y., 1966.

1

Smooth manifolds and their applications in homotopy theory[1]

L. S. Pontrjagin

Introduction

The main goal of the present work is the homotopy classification of maps from the $(n+k)$-dimensional sphere Σ^{k+n} to the n-dimensional sphere S^n; here we solve this problem only for $k = 1, 2$. The method described below was published earlier in notes [1, 2]. It allowed V. A. Rokhlin [3] to solve the problem also for $k = 3$. One has not yet obtained the results for $k > 3$ in this way. The main obstruction comes from studying smooth (differentiable) manifolds of dimensions k and $k+1$. After [1–3], a series of works of French mathematicians [4] appeared, the authors succeeded much more in the classification of a sphere to a sphere of smaller dimensions. The methods of the French school principally differ from those applied here.

Smooth manifolds are the main, and, perhaps, the only subject of this research, thus we completely devote Chapter I to them; in that chapter we investigate them more widely, which is necessary for future applications. Besides main definitions, chapter I contains a simpler (resp. Whitney [5]) proof of embeddability of a smooth n-dimensional manifold into $(2n+1)$-dimensional Euclidean space; we also state and investigate the question concerning singular points of smooth mappings from an n-dimensional manifold to the Euclidean space of dimension less than $2n+1$.

In Chapter II we describe the way of applying smooth manifolds for solutions of homotopy problems. First of all, we show that for the homotopy

[1] Л. С. Понтрягин, Гладкие многообразия и их применения в теории гомотопий, Москва, Наука, 1976. Translated by V.O.Manturov

classification of mappings from one manifold to another one may restrict only to the case of smooth mappings and smooth deformations. Later on, we describe our method of applying smooth manifolds to the homotopy classification of mappings from the sphere Σ^{n+k} to the sphere S^n, which goes as follows.

A smooth closed manifold k-dimensional manifold M^k lying in $(n+k)$-dimensional Euclidean space E^{n+k} is called *framed* if for any point $x \in M^k$ a system $U(x) = \{u_1(x), \ldots, u_n(x)\}$ of n linearly independent vectors orthogonal to M^k and smoothly depending on x is given; notation: (M^k, U). Compactifying the space E^{n+k} by the infinite point q', we get the sphere Σ^{n+k}. Let $e_1, \ldots, e_n$ be a system of linearly independent vectors tangent to the sphere $S^n \subset E^{n+1}$ in its north-pole p. It turns out that there exists a smooth mapping f from Σ^{n+k} to the sphere S^n such that $f^{-1}(p) = M^k$, whence the mapping f_x obtained by linearisation of f at $x \in M^k$ maps the vectors $u_1(x), \ldots, u_n(x)$, to $e_1, \ldots, e_n$, respectively. The homotopy type of the mapping f enjoying these properties is uniquely defined by the framed manifold (M^k, U). For each homotopy type of the mapping of Σ^{n+k} to the sphere S^n there exists such a framed manifold that the corresponding mapping belongs to the prescribed homotopy type. Two framed manifolds (M_0^k, U_0) and (M_1^k, U_1) define one and the same homotopy type of mapping from the sphere Σ^{n+k} to the sphere S^n, when they are *homologous* in the following sense. Let $E^{n+k} \times E^1$ be the direct product of the Euclidean space E^{n+k} by the line E^1 of variable t. We think of the framed manifold (M_0^k, U_0) lying in the space $E^{n+k} \times 0$, and the framed manifold (M_1^k, U_1) lying in $E^{n+k} \times 1$. The framed manifolds (M_0^k, U_0) and (M_1^k, U_1) are thought to be homologous if in the strip $0 \leqslant t \leqslant 1$ there exists a smooth framed manifold (M^{k+1}, U) with boundary consisting of M_0^k and M_1^k, whose framing U coincides with the framings U_0 and U_1 on the boundary components.

The described construction allows one to reduce the homotopy classification question for mappings $\Sigma^{n+k} \to \Sigma^n$ to the homology classification of framed k-dimensional manifolds. The role of k-dimensional and $(k+1)$-dimensional manifolds is clear here. The homology classification of zero-dimensional framed manifolds is trivial; thus one easily classifies mappings from Σ^n to the sphere S^n. The homology classification of one- and two-dimensional manifolds is also not very difficult, and it leads to the homotopy classification of mappings from Σ^{n+k} to S^n for $k = 1, 2$. We describe this question in chapter IV of the present work. The homology classification of three-dimensional framed manifolds meets significant difficulties. It was obtained by V. A. Rokhlin [3].

For realising the homology classification of smooth manifolds in the present work, we use homology invariants of these manifolds. With a framed submanifold (M^k, U) of the Euclidean space E^{n+k} we associate a homology invariant of it, being at the same type a homotopy invariant of the corresponding mapping of the sphere Σ^{n+k} to the sphere S^n. For $n = k+1$ there is the well-known Hopf invariant γ for mappings Σ^{2k+1} to S^{k+1}. The invariant γ can be easily interpreted as a homology invariant of the framed manifold. In chapter III we give a definition of the invariant γ based on the smooth manifold theory, and also give its interpretation as a homologous invariant of framed manifolds. For $k = 1$, the Hopf invariant is a classifying one; this fact is proved (in a known way) in chapter IV. In chapter IV for $k = 1, 2$; $n \geqslant 2$ we construct an invariant δ. This invariant is a residue class modulo 2. From its existence, one deduces that the number of mapping classes $\Sigma^{n+k} \to S^n$ for $k = 1, 2$; $n \geqslant 2$ is at least two. The uniqueness of this invariant for all cases except $k = 1$, $n = 2$ is based on the uniqueness of γ for $k = 1$.

Chapter I

Smooth manifolds and their maps

§ 1. Smooth manifolds

Below, we first give the definition of smooth (differentiable) manifold of finite class and simplest relevant notions; besides, we consider some smooth manifolds playing an important role, more precisely: submanifolds of smooth manifolds, manifold of linear elements of a smooth manifold, the Cartesian product of two manifolds and the manifold of vector subspaces of a given dimension for a given vector space. Together with finite differentiable manifolds one can also define infinitely differentiable manifolds, for which the functions in questions are infinitely differentiable and also analytic manifolds where all functions in questions are analytic. In the present paper, infinitely differentiable and analytic manifolds play no role; thus they are out of question.

The notion of smooth manifold

A) Let E^k be a Euclidean space of dimension k provided with Cartesian coordinates $x^1, \ldots, x^k$. By a *half-space* of the space E^k we mean the set E_0^k, defined by the condition

$$x^1 \leqslant 0. \tag{1}$$

By a *boundary* of the half-space E_0^k we mean the hyperplane E^{k-1} defined as

$$x^1 = 0. \tag{2}$$

A *domain* of the half-space E_0^k is an open subspace of it (which might not be open for the whole space E^k). The points of the half-space E_0^k, belonging to the boundary E^{k-1} are called its *boundary points.* A Hausdorff topological space M^k with a finite base is a *topological manifold* if each point a of it admits a neighbourhood U^k homeomorphic to a domain W^k of the half-space E_0^k or of a space E^k. Obviously, each domain of the space E^k is homeomorphic to some domain of the half-space E_0^k, but for coordinate systems, it is more convenient to consider both domain types. If a point a corresponds to a boundary point of the domain W^k, then it is called a *boundary point* for the manifold M^k as well as for its neighbourhood U^k. It is known that the notion of boundary point is invariant. A manifold having boundary point is said to be a manifold *with boundary*, otherwise it is called a manifold *without boundary.* A compact manifold without boundary is said to be closed. It is easy to check that the set of all boundary points of a manifold M^k is a $(k-1)$-dimensional manifold.

Definition 1. Let M^k be a topological manifold of dimension k and let U^k be some neighbourhood (being a subset) of this manifold homeomorphic to a domain W^k of the half-space E_0^k or of a space E^k. Defining a homeomorphism between U^k and W^k is equivalent to providing a coordinate system $X = \{x^1, \ldots, x^k\}$ for U^k corresponding to the coordinate system of the Euclidean space E^k. Herewith, two different coordinate systems X and Y in U^k are always connected by a one-to-one continuous transformation

$$y^j = y^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, k. \tag{3}$$

Fix a positive integer m and assume that functions (3) are not just continuous, but also m times continuously differentiable in the domain U^k and the Jacobian $\left|\dfrac{\partial y^j}{\partial x^i}\right|$ is non-zero. With that, we say that the coordinate systems X and Y belong to the same *smoothness class* of order m. Obviously, different classes do not intersect and each class is defined by any coordinate system belonging to it. If there is a preassigned class, then the neighbourhood U^k is called m times *continuously differentiable.* Thus, two m times continuously differentiable neighbourhoods U^k, V^k of the manifold M^k always induce two coordinate classes for its intersection; if these classes coincide, we say that the neighbourhoods U^k and V^k are *compatibly* differentiable. If all neighbourhoods of some bases for a manifold M^k are m times continuously differentiable and the classes are mutually compatible, then the manifold M^k is called m times *continuously differentiable* or *smooth* of class m; sometimes we refer just to smooth manifold without indicating m which is always assumed to be sufficiently large for our purposes. [Analogously, if the functions (3) are analytic, then the manifold is called *analytic*.]

As seen from the definition given above, setting the differentiable structure for a manifold is obtained by setting some bases for any neighbourhood. If two bases for a manifold define two smooth structures, they are thought to be equivalent iff the union of these bases satisfies the condition 1. Indeed, to define a smooth structure for a manifold, one should define it for any neighbourhood of some covering of the manifold. Obviously, such a covering defines the topology of the manifold as well. If we restrict ourselves to connected neighbourhoods, which is always possible, then in each neighbourhoods all coordinate systems are split into two classes, such that the transformation (3) inside one class has a positive Jacobian. Each of these two classes is called an *orientation* of the given neighbourhood. Obviously, a smooth manifold is orientable if and only if there exists a compatible orientation for all neighbourhoods. With each such choice, one associates an orientation of the manifold.

B) The boundary M^{k-1} of a smooth manifold M^k is itself a smooth manifold of the same class; this results from the following construction. Let U^k be a neighbourhood in M^k provided with a fixed coordinate system X such that the intersection $U^{k-1} = U^k \cap M^{k-1}$ is non-empty. The equation defining the subset U^{k-1} in U^k, obviously, looks like $x^1 = 0$; thus it is natural to take $x^2, \ldots, x^k$ as preassigned coordinates in U^{k-1}. Let V^k be another neighbourhood in M^k (possibly, coinciding with U^k) with a fixed coordinate system Y for which the intersection $V^{k-1} = V^k \cap M^{k-1}$ is non-empty. For the common part of neighbourhood U^k and V^k we have

$$y^j = y^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, k, \tag{4}$$

from which at $x^1 = 0$ we obtain

$$y^j = y^j(0, x^2, \ldots, x^k), \qquad j = 2, \ldots, k. \tag{5}$$

From differentiability of relations (4) one obtains the differentiability of relation (5). Furthermore, from the relation $y^1(0, x^2, \ldots, x^k) = 0$ we get (for $U^{k-1} \cap V^{k-1}$)

$$\frac{\partial(y^1, \ldots, y^k)}{\partial(x^1, \ldots, x^k)} = \frac{\partial y^1}{\partial x^1} \frac{\partial(y^2, \ldots, y^k)}{\partial(x^2, \ldots, x^k)}; \tag{6}$$

herewith, since the left-hand side is non-zero, we get $\dfrac{\partial(y^2, \ldots, y^k)}{\partial(x^2, \ldots, x^k)} \neq 0$. If the system X is orienting for the neighbourhood U^k, then we may take $x^2, \ldots, x^k$ to be the orienting system for U^{k-1}. Because $\dfrac{\partial y^1}{\partial x^1} > 0$ then from positivity of $\dfrac{\partial(y^1, \ldots, y^k)}{\partial(x^1, \ldots, x^k)}$ we obtain the positivity of $\dfrac{\partial(y^2, \ldots, y^k)}{\partial(x^2, \ldots, x^k)}$. Thus, the boundary of a smooth orientable manifold gets a natural orientation.

C) Let a be a point of a smooth manifold M^k. Each coordinate system defined in a neighbourhood U^k of the point a belonging to the preassigned class is called a *local coordinate system* at the point a. Obviously, each point a of the manifold M^k can be treated as a base point of some local coordinate system. By a *vector* (countervariant) on the manifold M^k at a we mean a function associating with each local corrdinate system at a a system of k real numbers called *vector components* with respect to this coordinate system, in such a way that the components $u^1, \ldots, u^k$ and $v^1, \ldots, v^k$ of the same vector seen from two coordinate systems $x^1, \ldots, x^k$ and $y^1, \ldots, y^k$ are connected by the relation

$$v^j = \sum_{i=1}^{k} \frac{\partial y^j(a)}{\partial x^j} u^i. \tag{7}$$

Obviously, the vector is uniquely defined by its components given in one local coordinate systems. Defining linear operations over vectors as linear operations over their components, we define the k-dimensional vector space structure R_a^k on the set of all vectors on the manifold M^k at the point a; this space is called *tangent* to the manifold M^k at the point a. With each local coordinate system at the point a one associates a basis in the tangent space, where all vectors have the same components as with respect to the coordinate system. If a point a belongs to the boundary M^{k-1} of the manifold M^k, then besides the tangent space R_a^k, one also defines the space R_a^{k-1} tangent to the manifold M^{k-1}. Take the parameters $x^2, \ldots, x^k$ to be local coordinates for M^{k-1} (see sect. «B») and associate with the vector from R_a^{k-1} having components $u^2, \ldots, u^k$ the vector from R_a^{k-1} having components $0, u^2, \ldots, u^k$; thus we obtain a natural embedding of the space R_a^{k-1} to R_a^k.

Smooth mappings

D) Let M^k and N^l be two m-smooth manifolds and let φ be a continuous mapping of the first manifold to the second manifold. At the point $a \in M^k$, choose a local coordinate system X; at the point $b = \varphi(a) \in N^l$ choose a local coordinate system Y; then in the neighbourhood of the point a the mapping φ will look as

$$y^j = \varphi^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, l. \tag{8}$$

If the function φ is n times continuously differentiable, $n \leqslant m$, then it will be n times continuously differentiable for any other choice of local coordinates; thus, one may speak of the n-smoothness class of the mapping φ. Later on, while speaking of smooth mapping, we shall always assume that n is sufficiently large. If the rank of the matrix $\left\| \frac{\partial \varphi^j}{\partial x^i} \right\|$ at the point a equals k, then the mapping φ is called *regular* at a. It is easy to see that if the point a belongs to the boundary M^{k-1} of the manifold M^k, then from the regularity of the mapping φ at a follows its regularity at the point a of the manifold M^{k-1}. If the mapping φ is regular at each point $a \in M^k$, then it is called *regular*. It is easy to check that if the mapping φ is regular at a, then it is regular and homeomorphic in some neighbourhood of the point a. A regular homeomorphic mapping is called a *smooth embedding*. The mapping φ is called *proper* at the point $a \in M^k$, if the rank of the matrix $\left\| \frac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, l$; $i = 1, \ldots, k$, equals l. It is easy to see that the set of all nonproper points of the mapping φ is closed in M^k. A point $b \in N^i$ is called *proper* for the mapping φ if the mapping φ is proper at any point of the set $\varphi^{-1}(b) \subset M^k$. The point a is a *singular* point of the mapping f if it is non-regular and nonproper at the same time, i.e. if the rank of the matrix $\left\| \frac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, l$; $i = 1, \ldots, k$, is less than any of k and l.

E) Each smooth mapping φ of a smooth manifold M^k to a smooth manifold N^l induces at each point $a \in M^k$ a linear mapping φ_a of the vector space R^k_a tangent to the manifold M^k at a, to the vector space R^l_0 tangent to N^l at $b = \varphi(a)$. Namely, if the local coordinate systems at points a and b, are X and Y, respectively, then to the vector $u \in R^k_a$ with components $u^1, \ldots, u^k$ in the system X one associates the vector $v = \varphi_a(u) \in R^l_0$ with components

$$v^j = \sum_{i=1}^{k} \frac{\partial \varphi^j(a)}{\partial x^i} u^i, \qquad j = 1, \ldots, l, \tag{9}$$

in the coordinate system Y. It is not easy to see that this correspondence is well defined, i.e. for any choice of local coordinate it results in one and the same mapping φ_a. If the mapping φ is regular at a, then the mapping

φ_a is one-to-one and defines an embedding of the same R^k_a into R^l_b. If φ is proper at a, then $\varphi_a(R^k_a) = R^l_b$.

Definition 2. A smooth mapping φ of class n from an m-smooth manifold M^k onto the smooth m-manifold N^k, $m \geqslant n$, is called *smooth homeomorphism* if it is regular. Obviously, if the mapping φ is a smooth homeomorphism of class n then the inverse mapping φ^{-1} is also a smooth homeomorphism of class n. Two manifolds are called *smoothly isomorphic* if there exists a smooth homeomorphism from one manifold onto the other.

Certain ways of constructing smooth manifolds

F) Let P^r be a subset of a smooth manifold M^k of class m, defined in the neighbourhood of any point belonging to it by a system of k—r independent equation. This means that for each point $a \in P^r$ there exists a neighbourhood U^k in the manifold M^k with local system X that the intersection $P^r \cap U^k$ consists of all points with coordinates satisfying the equations

$$\psi^j(x^1, \ldots, x^k) = 0, \qquad j = 1, \ldots, k - r. \tag{10}$$

Herewith we assume that the function ψ^j is m times smoothly differentiable and the functional matrix $\left\| \dfrac{\partial \psi^j(a)}{\partial x^i} \right\|$, $j = 1, \ldots, k - r;\ i = 1, \ldots, k$, has rank $k - r$; if a is a boundary point of the manifold M^k then we assume that the reduced functional matrix $\left\| \dfrac{\partial \psi^j(a)}{\partial x^i} \right\|$, $j = 1, \ldots, k - r;\ i = 2, \ldots, k$ has rank $k - l$. With the conditions above, the set P^r turns out to have a natural smooth r-dimensional m-smooth manifold structure; this manifold is smoothly embedded into M^k. Such a manifold P^r is called a *submanifold* of the manifold M^k. Furthermore, it turns out that the boundaries P^{r-1} and M^{k-1} of the manifolds P^r and M^k enjoy the relation

$$P^{r-1} = P^r \cap M^{k-1}, \tag{11}$$

and if $a \in P^{r-1}$ and R^k_a, R^{k-1}_a, R^r_a, R^{r-1}_a are tangent spaces to the manifolds M^k, M^{k-1}, P^r, P^{r-1} at the point a, then

$$R^{r-1}_a = R^r_a \cap R^{k-1}_a. \tag{12}$$

Here the spaces R^{k-1}_a, R^r_a, R^{r-1}_a are considered as subspaces of R^k_a (see «C» and «E»).

To prove that P^r is an r-dimensional manifold and to define the differentiable structure on it, we change, if necessary, the enumeration of coordinate for the Jacobian $\left| \dfrac{\partial \psi^j(a)}{\partial x^i} \right|$, $j = 1, \ldots, k - r;\ i = r + 1, \ldots, k$ to be non-zero; in the case of boundary point we may not change the number of the coordinate x^1. Then the system (10) will be uniquely resolvable in variables $x^1, \dots, x^k$:

$$x^i = f^i(x^1, \dots, x^r), \qquad i = r+1, \dots, k. \tag{13}$$

In the case of boundary point, the coordinate x^1 is not among the independent variables. The functions f^i are defined, m times continuously differentiable in some domain W^r of the half-space E_0^r in variables $x^1, \dots, x^r$ and define a homeomorphic mapping of this domain onto some neighbourhood U^r of the point a in P^r. Thus we have proved that P^r is an r-dimensional manifold. The differentiability for the neighbourhood U^r is defined by coordinates $x^1, \dots, x^r$.

The natural inclusion of the manifold P^r in the manifold M^k is given in U^r by relations

$$\begin{aligned} &x^i = x^i, && i = 1, \dots, r; \\ &x^i = f^i(x^1, \dots, x^r), && i = r+1, \dots, k, \end{aligned} \tag{14}$$

where the parameters $x^1, \dots, x^r$ on the right-hand sides are thought to be coordinates in U^r and the parameters $x^1, \dots, x^r$ on the left-hand side be the coordinates in U^k. The relation (11) is evident. Now, let $a \in P^{r-1}$; let us prove the relation (12). To local coordinates X, there correspond a certain basis $e_1, \dots, e_k$ in R_a^k; the basis of the space R_a^{k-1} consists of vectors $e_2, \dots, e_k$; the basis of the space R_a^k consists of vectors $e_i + \sum_{j=r+1}^k \dfrac{\partial f^j}{\partial x^i} e_j$, $i = 1, \dots, r$; finally, the basis of the space R_a^{r-1} consists of the same vectors except for the first one. Considering these bases, we easily get to the relation (12).

To prove the compatibility of the coordinate systems we constructed for P^r consider together with the point a, another point $b \in P^r$ with local coordinates Y and neighbourhoods V^k and V^r analogous to the neighbourhoods U^k and U^r. The relations analogous to (13), will look like

$$y^i = g^i(y^1, \dots, y^r), \qquad i = r+1, \dots, k. \tag{15}$$

Suppose that U^r and V^r have a non-empty intersection. Then U^k and V^k also have a non-empty intersection; let

$$y^i = y^i(x^1, \dots, x^k), \qquad i = 1, \dots, k; \tag{16}$$

$$x^i = x^i(y^1, \dots, y^k), \qquad i = 1, \dots, k, \tag{17}$$

be the coordinate changes from X and Y and back. Substituting $x^{r+1}, \dots, x^k$ from (13) for (16), we get for the first r variables y

$$y^i = \overset{*}{y}{}^i(x^1, \ldots, x^r), \qquad i = 1, \ldots, r. \tag{18}$$

In the same way substituting $y^{r+1}, \ldots, y^k$ from (15) for (17) we get

$$x^i = \overset{*}{x}{}^i(y^1, \ldots, y^r), \qquad i = 1, \ldots, r. \tag{19}$$

The coordinate changes (18) and (19) are m times continuously differentiable; since they are inverse to each other, their Jacobians are both nonzero.

This completes the proof of Statement «F».

G) Let M^k be a smooth manifold of class $m \geqslant 2$ and let L^{2k} be the set of all tangent vectors to it (see «C»), i.e. pairs of type (a, u), where $a \in M^k$, $u \in R^k_a$. The set L^{2k} naturally turns out to be a $2k$-dimensional manifold of class $m-1$ according to the following construction. Let U^k be a certain neighbourhood in the manifold M^k with local coordinate system X. By U^{2k}, denote the set of all pairs $(x, u) \in L^{2k}$ satisfying the condition $x \in U^k$. Take the set U^{2k} to be the neighbourhood in L^{2k}; the fixed coordinate system in it is constructed as follows. Let $x^1, \ldots, x^k$ be the coordinates of the point x in the system X and let $u^1, \ldots, u^k$ be the components of the vector u in the local coordinate system X; then the coordinates of the pair (x, u) are defined to be the numbers

$$x^1, \ldots, x^k,\ u^1, \ldots, u^k. \tag{20}$$

If V^k is a neighbourhood in M^k (possibly coinciding with U^k) with a fixed system Y, for which $x \in V^k$ and the coordinates of the pair (x, u) in the neighbourhood V^{2k} defined by Y are

$$y^1, \ldots, y^k,\ v^1, \ldots, v^k, \tag{21}$$

then the coordinate change from (20) to (21) is, evidently, given by the relation

$$y^j = y^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, k; \tag{22}$$

$$v^j = \sum_{i=1}^{k} \frac{\partial y^j}{\partial x^i} u^i, \qquad j = 1, \ldots, k \tag{23}$$

[see (9)]. These relations are $m-1$ times differentiable and have the Jacobian equal to $\left|\dfrac{\partial y^j}{\partial x^i}\right|^2$; this Jacobian is, evidently, positive. Since the neighbourhoods of type U^{2k} cover L^{2k}, the described construction turns L^{2k} into a smooth manifold of class $m-1$.

H) Let R^k be a vector space of dimension k. By a ray u^* in R^k passing through the vector $u \neq 0$ we mean the set of all vectors tu where t is some positive real number. Fix some basis for R^k and denote by R_i^{k-1} the coordinate hyperplane $u^i = 0$. If the ray u^* does not lie in R_i^{k-1}, then there exists a unique vector u on it satisfying the condition $|u^i| = 1$; call this vector the *basic* vector with respect to the plane R_i^{k-1}. The set of all rays for which the basic vector with respect to R_i^{k-1} satisfies $u^i = +1$ or $u^i = -1$, denote by U_{i1}^{k-1} or, by U_{i2}^{k-1}, respectively. For coordinates of the ray $u^* \in U_{ip}^{k-1}$, $p = 1,\ 2$, we take the components $u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^k$ of the basic vector u of this ray with respect to R_i^{k-1}. Since the system of all sets U_{ip}^{k-1} covers the set S^{k-1} of all rays, the set S^{k-1} becomes a smooth manifold evidently homeomorphic to the $(r-1)$-sphere.

I) Let M^k be a smooth manifold of class m. *Linear element manifold* of it is the set L^{2k-1} of all pairs (x, u^*), where $x \in M^k$, and u^* is a ray in R_x^k; the natural differential structure is defined according to the following construction. Let U^k be a neighbourhood in M^k with a fixed system X. In the vector space R_x^k tangent to M^k at $x \in U^k$ we have a basis corresponding to the local system X; thus, in the set S_x^{k-1} of rays of the space R_x^k we have domains $U_{ip,x}^{k-1}$ (see «H») endowed with coordinate systems. By U_{ip}^{2k-1} denote the set of all pairs (x, u^*) satisfying the condition $x \in U^k$, $u^* \in U_{ip,x}^{k-1}$, where the coordinates of the pair (x, u^*) in U_{ip}^{2k-1} are taken to be the numbers

$$x^1, \dots, x^k,\ u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^k, \tag{24}$$

where $x^1, \dots, x^k$ are the coordinates of x in the system X, and $u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^k$ are the coordinates of the ray u^* in $U_{ip,x}^{k-1}$. It can be easily checked that the system of neighbourhoods U_{ip}^{2k-1} covers L^{2k-1} and that the introduced coordinate systems are compatible with each other; thus L^{2k-1} is a $(2k-1)$-dimensional smooth manifold of class $m-1$.

J) Let M^k and N^l be two smooth manifolds of class m; suppose M^k has empty boundary. The direct product (Cartesian product) $P^{k+l} = M^k \times N^l$, i.e. the set of all pairs (x, y), where $x \in M^k$, $y \in N^l$, is naturally a smooth manifold of class m according to the following construction. Let U^k and V^l be arbitrary coordinate neighbourhoods in the manifolds M^k and N^l with coordinate systems X and Y. Consider the set $U^k \times V^l \subset M^k \times N^l$ as the coordinate neighbourhood in the manifold P^{k+1}: here the coordinates of the point $(x, y) \in U^k \times V^l$ are set to be the numbers $x^1, \dots, x^k, y^1, \dots, y^l$, where $x^1, \dots, x^k$ are the coordinates of the point x

in the system X and $y^1, \ldots, y^l$ are the coordinates of y with respect to Y. It follows from a straightforward check that the coordinate neighbourhood system constructed above defines in P^{k+l} a smooth structure of class m. If M^k and N^l are orientable manifolds and the systems X and Y correspond to the orientations of these manifolds, we define the orientation of P^{k+l} by the system X, Y. Herewith, the Cartesian product acquires a natural orientation. If N^{l-1} is the boundary of the manifold N^l, then the boundary of the manifold $M^k \times M^l$ turns out to be $M^k \times M^{l-1}$.

K) Let E^{k+l} be a vector space of dimension $k+l$ and let $G(k,l)$ be the set of all k-dimensional vector subspaces of it. The set $G(k,l)$ is a smooth (even analytic) manifold with respect to the following construction. Let $E_0^k \in G(k,l)$ and let $e_1, \ldots, e_k$, $f_1, \ldots, f_l$ be a basis of the space E^{k+l} such that the vectors $e_1, \ldots, e_k$ lie in E_0^k. Denote the linear span of vectors $f_1, \ldots, f_l$ by E^l. Denote by U^{kl} the set of all vector subspaces $E^k \in G(k,l)$ the intersection of which with E^l consists of only the origin of coordinates. If $E^k \in U^{kl}$ then there exists a basis $e'_1, \ldots, e'_k$ of the vector space E^k defined by the relations

$$e'_i = e_i + \sum_{j=1}^{i} x_i^j f_j, \qquad i = 1, \ldots, k,$$

where $\left\| x_i^j \right\|$ is a real number matrix. Consider the elements x_i^j, $i = 1, \ldots, k$, $j = 1, \ldots, l$, of this matrix as coordinates of the element E^k in the coordinate neighbourhood U^{kl}. It can be checked straightforwardly that the set of coordinate neighbourhoods of type U^{kl} defines an analytic structure in $G(k,l)$; this $G(k,l)$ is an analytic manifold of dimension kl.

§ 2. Embedding of a manifold into Euclidean space

In the present subsection we show that any compact k-dimensional smooth manifold of class $m \geqslant 2$ can be regularly homeomorphically mapped into the Euclidean space R^{2k+1} of dimension $2k+1$ and can be regularly mapped into R^{2k}; here the smoothness class of these mappings equals m. These statements in a stronger form, i.e. for $m \geqslant 1$ and without compactness assumptions, were proved by Whitney [5]; the proof given below is somewhat easier.

In the proof, we shall rely on the following quite elementary Theorem 1.

Smooth mapping of a manifold to a manifold of larger dimension

Theorem 1. *Let M^k and N^l be two smooth manifolds of dimensions k and l, respectively, where $k < l$, and let φ be a smooth mapping of class 1 of the manifold M^k to the manifold N^l. It turns out that the set $\varphi(M^k)$ has the first category in N^1, i.e. it can be represented as a sum of countably many nowhere dense sets in N^l. In particular, if the manifold M^k is compact, then the set $\varphi(M^k)$ is compact as well; thus $N^l \setminus \varphi(M^k)$ is a domain everywhere dense in N^l.*

PROOF. Suppose $a \in M^k$, let $b = \varphi(a)$, V_b^i be some coordinate neighbourhood of the point b in N^l and let U_a^k be such a coordinate neighbourhood of the point a in M^k that $\varphi(U_a^k) \subset V_b^l$. Choose neighbourhoods U_{a1}^k and U_{a2}^k of the point a in M^k such that $\overline{U}_{a1}^k \subset U_a^k$, $\overline{U}_{a2}^k \subset U_{a1}^k$ and such that the set $\overline{U}_{a1}^k$ is compact. The domains $\overline{U}_{a2}^k$, $a \in M^k$, cover the manifold M^k. From this cover, one can take a countable subcover; thus, in order to prove the theorem it suffices to show that for any arbitrary choice of the point a from M^k, the set $\varphi(\overline{U}_{a2}^k)$ is nowhere dense in V_b^l. Since the domain U_a^k is the homeomorphic image of a domain of the Euclidean subspace E_0^k, we shall assume that U_{a2}^k itself is a domain of the subspace E_0^k. In the same way, we assume that V_b^l is a domain of the Euclidean subspace E_0^l. Thus the mapping φ can be treated as a smooth mapping of class 1 from a domain U_a^k to the Euclidean space E^l; thus it suffices to show that the set $\varphi(\overline{U}_{a2}^k)$ is nowhere dense in E^l. Let us prove this.

The smoothness of φ and compactness of the mapping $\overline{U}_{a1}^k$ result in the existence of a positive constant c such that for any two arbitrary points x and x' from $\overline{U}_{a1}^k$, the inequality

$$\rho(\varphi(x), \varphi(x')) < c\rho(x, x') \tag{1}$$

holds. Chose some ε-cubature of the Euclidean subspace E_0^k, i.e. tile the subspace E_0^k into right-angled cubes with edge ε. Denote the set of all cubes intersecting $\overline{U}_{a2}^k$ by Ω. As the set $\overline{U}_{a2}^k$ is compact and hence is bounded by a rather large cube, the number of cubes in Ω does not exceed c_1/ε^k where c_1 is some positive constant independent of ε. Let δ be the distance between the sets $E_0^k \setminus U_{a1}^k$ and $\overline{U}_{a2}^k$. Suppose that the diagonal length $\varepsilon\sqrt{k}$ of each cube from Ω is less than δ. Then each cube K_i from Ω lies in the domain U_{a1}^k and, by virtue of (1), the set $\varphi(K_i)$ is contained in some cube L_i of the space E^l with edge length $c\sqrt{k} \cdot \varepsilon$; the volume of the latter cube equals $c^l k^{l/2} \cdot \varepsilon^{l-k}$. Thus the whole set $\varphi(\overline{U}_{a2}^k)$ is contained in the union of cubes L_i, whose number does not exceed c_1/ε^k; thus the total volume of the set $\varphi(\overline{U}^k_{a2})$ does not exceed the number $c_1 c^l k^{l/2} \cdot \varepsilon^{l-k}$. Since ε is chosen arbitrarily small, from above it follows that the set $\varphi(\overline{U}^k_{a2})$ does not contain any domain and, being compact, it should be nowhere dense in E^l.

Thus, Theorem 1 is proved.

The projection operation in the Euclidean space

Later on, the projection operation will play a key role. Let C^r be a vector space and let B^q be its vector subspace. Regarding the space C^r as an additive group and the space B^q as a subgroup of it, we obtain a tiling of the space C^r into conjugacy classes according to B^q; these conjugacy classes form a vector space A^p of dimension $p = r - q$. Associating with any element $x \in C^r$ the corresponding conjugacy class $\pi(x) \in A^p$, we get a linear mapping π of the space C^r onto the space A^p called the *projection* along the projecting subspace B^q. More intuitively, the space A^p can be realized as a linear subspace of dimension p of the space C^r intersecting the space B^q only in the origin; then the operation π is just the original projection. If the space C^r is Euclidean, then defining B^q to be the orthogonal complement to the given space $A^p \subset C^r$, we get an orthogonal projection π of the space C^r to the subspace A^p.

A) Let φ be a smooth mapping of a smooth manifold M^k to some vector space C^r regular at a point $a \in M^k$, and let π be the projection of the space C^r along the one-dimensional subspace B^1 to the space A^{r-1}. It turns out that the mapping $\pi\varphi$ from M^k to A^{r-1} is not regular at a (see § 1, «D») if and only if the line $\varphi(a) + B^1$ passing through $\varphi(a)$ parallel to B^1 is tangent to $\varphi(M^k)$ at the point $\varphi(a)$.

To prove this, choose some local coordinates $x^1, \ldots, x^k$ in the neighbourhood of a; endow C^r with rectilinear coordinates $y^1, \ldots, y^r$ such that the last axis coincides with B^1. In the chosen coordinate system, the mapping φ looks like: $y^j = \varphi(x^1, \ldots, x^k)$, $j = 1, \ldots, r$, where the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, r$; $i = 1, \ldots, k$, at the point a, is, by regularity assumption, equal to k. With each vector u on M^k at a one associates the vector $v = \varphi_a(u) \in C^r$, which is tangent to $\varphi(M^k)$ at the point $\varphi(a)$ and has components $v^1, \ldots, v^r$ [i.e. defined by relations (9) § 1, $l = r$]. Now, if the mapping $\pi\varphi$ is not regular at the point a, then the rank of the matrix $\left\| \dfrac{\partial \varphi^j}{\partial x^i} \right\|$, $j = 1, \ldots, r-1$, $i = 1, \ldots, k$, is less than k; thus there exists a vector $u \neq 0$ such that for the vector $v = \varphi_a(u)$ we have $v^1 = \ldots = v^{r-1} = 0$, $v^r \neq 0$; the latter means that $v \in B^1$. If, on the contrary, there exists a vector $v = \varphi_a(u) \neq 0$ belonging to B^1 then the rank of the matrix $\left\|\dfrac{\partial \varphi^j}{\partial x^i}\right\|$, $j = 1, \ldots, r-1$; $i = 1, \ldots, k$ is less than k, i.e. the mapping $\pi\varphi$ is not regular in a.

B) Let φ be a smooth regular mapping of class 2 from a smooth manifold M^k to the vector space C^r of dimension $r > 2k$, and let $B^q \in\in G(q, r-q)$ be the subspace of dimension $q \leqslant r-2k$ of projection for C^r onto the space A^p. Denote the projection by π. By Ω'_q we denote the set of all such projecting spaces B^q for which the mapping $\pi\varphi$ is not regular. It turns out that the set Ω'_q has first category in the manifold $G(q, r-q)$ of all projecting directions.

Let (x, u^*) be an arbitrary linear element of the manifold M^k (see § 1, «I») and let u be some non-zero vector of the ray u^*. To the vector u, according to (9) § 1, there corresponds the vector $v = \varphi_x(u) \neq 0$. The ray v^* of the space C^r defined by the vector v depends only on the linear element (x, u^*), and we set $v^* = \Phi(x, u^*)$. It can be easily checked that the mapping Φ from the manifold L^{2k-1} (see § 1, «I») to the manifold S^{r-1} (see § 1, «H») has smoothness class one, thus $\Phi(L^{2k-1})$ is of first category in S^{r-1} (since $r-1 > 2k-1$, see Theorem 1). Thus, by virtue of «A», we get «B» for $q = 1$.

Applying this construction consequently, we get the proof of the statement «B» for any arbitrary $q \leqslant r-2k$.

C) Let φ be a smooth of class one one-to-one mapping from the smooth manifold M^k to the vector space C^r and let $B^q \in G(q, r-q)$ be the projecting subspace of the dimension $q \leqslant r-2k-1$. Denote the projection by π. By Ω''_q, denote the set of all projecting subspaces B^q such that the mapping $\pi\varphi$ is not one-to-one. It turns out that Ω''_q has first category in the manifold $G(q, r-q)$.

Let x and y be two arbitrary different points of the manifolds M^k. By $\Phi'(x, y)$ denote the ray consisting of all vectors of the type $t(\varphi(y) - \varphi(x))$, where t is a positive number. Thus we get a mapping Φ' from the manifold M^{2k} of all ordered pairs (x, y), $x \neq y$, to the manifold S^{r-1} of all rays of the space C^r. In the manifold M^{2k} one naturally introduces differentiability, and it can be easily checked that the mapping Φ' is smooth of class 1. Thus, $\Phi'(M^{2k})$ turns out to be of first category in S^{r-1} (see Theorem 1), from which follows «C» for $q = 1$. Applying this construction consequently we get the proof of «C» for arbitrary $q \leqslant r-2k-1$.

From «B» and «C» one straightforwardly gets

D) Let φ be a smooth one-to-one regular mapping of class 2 of a smooth manifold M^k to the vector space C^r and let $B^q \in G(q, r-q)$ be the projecting space of dimension $q \leqslant r-2k-1$. Denote the projection mapping by π, and denote by Ω_q the set of all projecting spaces B^q such that the mapping $\pi\varphi$ is not one-to-one and regular. Since $\Omega_q = \Omega'_q \cup \Omega''_q$ Ω_q has first category in the manifold $G(q, r-q)$.

The embedding theorem

E) Let $\varphi_1, \ldots, \varphi_n$ be smooth (of class m) mappings of the smooth manifold M^k to vector spaces $C_1, \ldots, C_n$, respectively. Denote by C the direct sum of the spaces $C_1, \ldots, C_n$ consisting of all systems $[u_1, \ldots, u_n]$, with $u_i \in C_i$. Define the direct sum φ of mappings $\varphi_1, \ldots, \varphi_n$ by $\varphi(x) = [\varphi_1(x), \ldots, \varphi_n(x)]$, $x \in M^k$. It is easy to see that φ is an m-smooth mapping of the manifold M^k to C. It can be easily checked that if at least one mapping $\varphi_1, \ldots, \varphi_n$ is regular in $a \in M^k$ then so is φ. Furthermore, it can be easily checked that if two points a and b from M^k are mapped to different points by one of the mappings $\varphi_1, \ldots, \varphi_n$ then they have different images under φ.

Theorem 2. *Let M^k be a smooth compact manifold of class $m \geqslant 2$. There exists a smooth embedding of class of the manifold M^k into a finite-dimensional Euclidean space.*

Proof. Denote by $\varkappa(t)$ some real function in the real variable t, which is infinitely differentiable and satisfies the following properties:

$$\varkappa(t) = 1 \text{ for } |t| \leqslant 1/2; \qquad \varkappa(t) = 0 \text{ for } |t| \geqslant 1;$$

for $-1 \leqslant t \leqslant -1/2$ the function $\varkappa(t)$ monotonously increases; for $1/2 \leqslant t \leqslant 1$, the function $\varkappa(t)$ monotonously decreases. Such a function can be easily constructed.

Set

$$\varkappa^i(t^1, t^2, \ldots, t^k) = t^i \cdot \varkappa(t^1) \cdot \varkappa(t^2) \ldots \varkappa(t^k),$$

for $i = 1, \ldots, k$ and

$$\varkappa^{k+1}(t^1, t^2, \ldots, t^k) = \varkappa(t^1) \cdot \varkappa(t^2) \ldots \varkappa(t^k).$$

Let R^k be the Euclidean space with Cartesian coordinates $t^1, \ldots, t^k$ and let R^{k+1} be the Euclidean space with Cartesian coordinates $y^1, \ldots, y^{k+1}$. Denote by Q the cube in the space R^k defined by the inequalities $|t^i| < 2$, denote by Q' the cube of the same space defined by the inequalities $|t^i| < 1$ and by Q'' the cube defined as $|t^i| < 1/2$. By Q_0 we denote the half-cube cut out from the cube Q by the inequality $t^1 \leqslant 0$. Now, define the mapping from R^k to the space R^{k+1} by the relations

$$y^j = \varkappa^j(t^1, t^2, \ldots, t^k), \qquad j = 1, \ldots, k+1. \tag{2}$$

It can be easily checked that this mapping is infinitely differentiable, maps the set $R^k \setminus Q'$ to the coordinate origin of the space R^{k+1}, its restriction to the cube Q' is a continuous and one-to-one mapping and its restriction to the cube Q'' is regular.

Now, let a be an arbitrary point of M^k and U_a^k be some coordinate neighbourhood of it endowed with a coordinate system X having origin at a; finally, let ε be a small positive number such that under the mapping

$$t^i = \frac{x^i}{\varepsilon}, \qquad i = 1, \ldots, k, \tag{3}$$

of the neighbourhood U_a^k to the space R^k the image of this neighbourhood covers the whole cube Q, whence a is an interior point of M^k or the whole half-cube Q_0, whence a is a boundary point of M^k. Denote the pre-images of the cubes Q' and Q'' under this mapping by Q'_a and Q''_a, respectively.

Define the mapping φ_a of the manifold M^k to the Euclidean space R^{k+1} by

$$y^j = \varkappa^j\left(\frac{x^1}{\varepsilon}, \frac{x^2}{\varepsilon}, \ldots, \frac{x^k}{\varepsilon}\right)$$

for the point $x \in U_a^k$ with coordinates $x^1, \ldots, x^k$ and by $y^j = 0$ for the point $x \in \in M^k \setminus U_a^k$. It can be easily checked that φ_a is an m-smooth mapping of M^k to R^{k+1}, which is homeomorphic on Q'_a and regular on Q''_a.

Selecting among neighbourhoods Q''_a a finite cover $Q''_{a_1}, \ldots, Q''_{a_n}$ of the manifold M^k and taking the direct sum of mappings corresponding to these cubes, $\varphi_{a_1}, \ldots, \varphi_{a_n}$ (see «E»), we get the desired mapping φ of the manifold M^k to a finite-dimensional Euclidean space.

From the statements proved above the theorem formulated earlier, follows straightforwardly. Indeed, the manifold M^k can be regularly and homeomorphically embedded into a vector space C of rather high dimension (see Theorem 2). Furthermore, in the space C^r there exists such a projecting direction B^{r-2k-1}, such that the obtained projection of the manifold M^k to the space A^{2k+1} is regular and homeomorphic (see «D»). In the same way, in the space C^r there exists a projecting direction B^{r-2k} such that the projection of the manifold M^k to the space A^{2k} is regular (see «B»). Below we prove a stronger Theorem 3 showing that for any smooth mapping of a manifold M^k to a Euclidean space C^{2k+1} there exists an arbitrarily close regular and homeomorphic mapping of the same manifold, and for any smooth mapping of M^k to the Euclidean space C^{2k} there exists an arbitrarily close regular mapping. For the precise formulation of Theorem 3, one needs to introduce the notion of m-neighbourhood for mappings, taking into account all derivatives up to order m, inclusively.

First note that if f is a smooth mapping of the domain W^k of the

Euclidean half-space E_0^k to a vector space C^r then the partial derivatives of the vector function $f(x) = f(x^1, \ldots, x^k)$ are vectors of the space C^r.

F) Let M^k be an m-smooth compact manifold and E^l be a vector space, P be the set of all m-smooth mappings of the manifold M^k to the space E^l. Introduce the topology for P by setting a metric depending on an arbitrary choice of some constructed elements. Let $U_s, V_s, s = 1, \ldots, n$, be a finite set of coordinate domains of the manifold M^k such that the domains $U_s,\ s = 1, \ldots, n$, cover M^k and the inclusions $\overline{U}_s \subset V_s,\ s = 1, \ldots, n$ hold, wherever in each domain V_s a preassigned coordinate system X_s is chosen. Furthermore, let Y be a Cartesian coordinate system of the space E^l. Define the distance $\rho(f, g)$ between two mappings f and g from P (depending on the choice of U_s, V_s, coordinate systems $X_s,\ s = 1, \ldots, n$, and the coordinate system Y). To do this, let us write the mappings f and g of the domain V_s in coordinate form by setting

$$y^j = f_s^j(x) = f_s^j(x^1, \ldots, x^k), \tag{4}$$

$$y^j = g_s^j(x) = g_s^j(x^1, \ldots, x^k). \tag{5}$$

Let $i_1, \ldots, i_k$ be a set of non-negative integers with sum not exceeding m. Set

$$\omega_s^j(x; i_1, \ldots, i_k) = \left| \frac{\partial^{i_1 + \ldots + i_k}(f_s^j(x) - g_s^j(x))}{(\partial x^1)^{i_1} \ldots (\partial x^k)^{i_k}} \right|.$$

Denote the maximum of the function $\omega_s^j(x; i_1, \ldots, i_k)$ in the variable x at $x \in \overline{U}_s$ by $\omega_s^j(i_1, \ldots, i_k)$, and define the distance $\rho(f, g)$ between f and g to be the supremum of all numbers $\omega_s^j(i_1, \ldots, i_k)$, where $i_1, \ldots, i_k$, s, j run over all admissible values. It can be easily checked that the topology of the space P does not depend on the arbitrary choice of the system of $U_s,\ V_s,\ s = 1, \ldots, n$, and coordinate systems $X_s,\ s = 1, \ldots, n,\ Y$. The topological space P is called the *class m mapping space* of the manifold M^k to the space E^l. The statement that for the map f there is an arbitrary close map enjoying some property A means that in any neighbourhood of the point f in the space P there exists a map enjoying the property A.

Theorem 3. *Let M^k be a class $m \geqslant 2$ smooth k-dimensional compact manifold, let A^p be a vector space of dimension p and let P be the class m mapping space from the manifold M^k to the space A^p. The set of all regular mappings from the set P denoted by Π'; denote the set of all regular and homeomorphic maps belonging to P by Π. It turns out that the sets Π' and Π are domains in the space P. Furthermore, if $p \geqslant 2k$ then the domain Π' is everywhere dense in P and if $p \geqslant 2k + 1$ then the domain Π is everywhere dense in P.*

PROOF. First, show that the sets Π' and Π are everywhere dense in the space P for the values of p indicated in the theorem. Let $f \in P$ and let e be a class m regular and homeomorphic mapping of the manifold M^k to a vector space B^q of sufficiently large dimension (see Theorem 2). Denote the direct sum of the vector spaces A^p and B^q by C^r; here we assume the spaces A^p and B^q to be linear subspaces of the space C^r. The mapping h, being a direct sum of the mappings f and e (see «E») is regular and homeomorphic, and its projection to A^p along B^q coincides with the mapping f. By virtue of statements «B» and «D», in any neighbourhood of the projecting direction B^q there exists a projecting direction B_1^q such that the projection g of the mapping h is regular if $p \geqslant 2k$; it is regular and homeomorphic if $p \geqslant 2k + 1$. Thus, for a given map f there exists an arbitrarily close map g enjoying the desired properties.

Let us show that Π' is a domain. Let $f \in \Pi'$. Since the mapping f is regular at $x \in U_s$ the rank of the matrix $\left\| \dfrac{\partial f_s^j}{\partial x^i} \right\|$ at this point equals k (see § 1, «F»). Consequently, the rank of a matrix close to the matrix $\left\| \dfrac{\partial f_s^j}{\partial x^i} \right\|$ also equals k. Thus, there exists such a small positive number ε' such that for $\rho(f, g) < \varepsilon'$ the mapping g is regular at the point x. Since the first derivatives of the functions $f_s^j(x)$ are continuous and the sets $\overline{U}_s$ are compact and one can choose a finite number of them to cover M^k, there exists a small positive number ε such that for $\rho(f, g) < \varepsilon$, the mapping g is regular at each point $x \in M^k$.

To prove that Π is a domain, first note the following:

a) In the set Q of all linear mappings of the Euclidean vector space E^k to the Euclidean vector space A^p, let us introduce the metrics according to some coordinate systems X and Y in these spaces. Let φ and ψ be elements from Q written in coordinates as

$$y^j = \sum_{i=1}^{k} \varphi_i^j x^i, \qquad j = 1, \ldots, p;$$

$$y^j = \sum_{i=1}^{k} \psi_i^j x^i, \qquad j = 1, \ldots, p.$$

Define the distance $\rho(\varphi, \psi)$ as the maximum of $|\varphi_j^i - \psi_j^i|$. It turns out that for any compact set F of non-degenerate mappings there exists a small positive δ such that for $\rho(F, \psi) < \delta$ we have

$$|\psi(x)| > \delta \cdot |x|,$$

where x is an arbitrary vector from E^k.

Taking into account the continuity, one easily proves this statement by reductio ad absurdum.

Let $f \in \Pi$. It turns out that there exist small numbers δ and ε such that for $\rho(f, g) < \varepsilon$ (see «F») the equality

$$\rho(g(a), g(x)) \geqslant \delta\rho(f(a), f(x)) \tag{6}$$

holds; here a and x are two arbitrary points from M^k.

Indeed, when $\rho(f(a), f(x)) < \alpha$, where α is a positive constant the mappings f and g in the neighbourhood of a are very exactly approximated by linear ones, herewith this can be done uniformly with respect to $a \in M^k$. In this case the inequality (6) easily follows from statement «A». In the case when $\rho(f(a), f(x)) \geqslant \alpha$, the inequality (6) follows from the bijectivity of f for ε being reasonably small. From inequality (6) and the bijectivity of f one gets the bijectivity for any map g reasonable close to f.

Thus, Theorem 3 is proved.

§ 3. Nonproper points of smooth maps

First, recall the definition of nonproper point for a map (see § 1, «D»). Let φ be a smooth mapping from a manifold M^k to a manifold N^l. A point a of the manifold M^k is called nonproper for the mapping φ if the functional matrix of the mapping φ at the point a has rank strictly less than l. A point b of the manifold N^l is called nonproper with respect to φ if the whole pre-image $\varphi^{-1}(b)$ of this point contains at least one nonproper point $a \in M^k$ of φ. Thus, one should distinguish between nonproper points of φ in M^k and nonproper points of φ in N^l. If F is the set of all nonproper points of φ in the manifold M^k, then $\varphi(F)$ is the set of all nonproper points of the mapping φ in the manifold N^l. Theorem 4 below due to Dubovitsky [6] states that the set $\varphi(F)$ has first category in the manifold N^l, i.e. it can be represented as a countable union of compact sets nowhere dense in N^l. It follows from this that the set $N^l \setminus \varphi(F)$ of all proper points of the mapping φ in the manifold N^l has second category N^l, i.e. «widely spread» and, in any case, everywhere dense. Informally speaking this can be formulated by saying that the points of the manifold N^l are, in general, proper. Theorem 4 has some important applications in smooth manifold theory; there are many corollaries saying that in general position some "good" property obtains. To prove any result of such type one should properly define the manifolds M^k and N^l together with a mapping φ. This choice can be described by Statement «A» given below: rather general and thus, not very formal.

General position argument

A) Let Q be a smooth manifold and let P be a set of operations over Q that constitutes a smooth manifold as well. While performing an operation $p \in P$ over Q some point $q \in Q$ can be singular in a certain sense, which should be clearly described. The pair (p, q), $p \in P$, $q \in Q$ is *marked* if the point q is singular with respect to the operation p. It is assumed that the set of all marked pairs (p, q) constitutes a smooth submanifolds M^k of the manifold $P \times Q$ (see § 1, «J», «E»). With each point $(p, q) \in M^k$, one associates the point $\varphi(p, q) = p$. Thus one gets a mapping φ from the manifold M^k to the manifold $N^l = P$. If the point $p_0 \in P$ is a proper point of the mapping φ in the manifold $P = N^l$, then any point $q \in Q$ singular with respect to p_0, is in some sense *typical*, and the set Q_0 of all points q of the manifold Q which are singular with respect to the operation p_0 consists of typical singular points.

There are many applications of the construction «A»; some of them are to be demonstrated in § 4. A very simple application of the construction «A» having illustrative character is given below as Statement «B».

B) Let A^r and B^s be two smooth submanifolds of the vector space E^n. One says that at a point $a \in A^r \cap B^s$ the manifolds A^r and B^s are in *general position* if tangent planes to the manifolds A^r and B^s have intersection of dimension $r + s - n$. One says that the manifolds A^r and B^s are in general position if they are in general position at any common point. It can be shown straightforwardly that if the manifolds A^r and B^s are in general position then their intersection $A^r \cap B^s$ is a submanifold of dimension $r+s-n$ in the space E^n. Let $p \in E^n$. Denote by A^r_p the manifold consisting of all points of type $p+x$, where $x \in A^r$. Thus the manifold A^r_p is obtained from the manifold A^r by shifting along the vector p. It turns out that the set of all vectors $p \in E^n$, for which the manifolds A^r_p and B^s are in general position, is the set of second category in E^n; thus there exist an arbitrarily small shift p for which the manifolds A^r_p and B^s are in general position.

To prove Statement «B», let us use construction «A» by setting $Q = A^r \times B^s$, $P = E^n$ and assuming the point $q = (a, b) \in A^r \times B^s$ to be singular with respect to the operation $p \in E^n$ if $p + a = b$. The set M^k of all marked pairs (p, q) where $p \in E^n$, $q = (a, b) \in A^r \times B^s$ is thus defined by $p = b - a$, i.e. the pair (p, q) is uniquely defined by the point $q = (a, b)$; thus there is a natural smooth homeomorphism of the manifolds M^k and $A^r \times B^s$ that allows us to identify these manifolds. The mapping φ of the manifold $M^k = A^r \times B^s$ to the manifold $P = E^n$ is defined according to the formula $\varphi(a,b) = b - a$. Simple calculations show that a point $q = (a,b) \in M^k$ is a proper point of the map φ if and only if the manifolds A^r_{b-a} and B^s are in general position at their intersection point b. Thus, the point $p_0 \in E^n$ is a proper point of the mapping φ if and only if the manifolds $A^r_{p_0}$ and B^s are in general position. From that and from Theorem 4 proved below, one gets the claim of «B».

The Dubovitsky Theorem

In the formulation of the Dubovitsky theorem, the smoothness class m of the map $\varphi : M^k \to N^l$ is defined as $m = k - l + 1$ and not as (1), as given below. In this sense Theorem 4 is weaker than Dubovitsky's theorem. Since the exact estimate for the smoothness class m is not important, below we give a weaker estimate (1), which allows us to simplify the proof.

Theorem 4. *Let M^k and N^l be two smooth manifolds of positive dimensions k and l and let φ be an*

$$m = m(k,l) = 2 + \frac{(k-l)(k-l+1)}{2} \tag{1}$$

class smooth mapping from M^k to N^l. It turns out that the set of all nonproper points of φ in the manifold N^l is of first category in N^l. In particular, if the manifold M^k is compact then the complement to this set is an everywhere dense domain in the manifold N^l.

Proof. First consider the case when the manifold M^k has no boundary. Let $a \in M^k$, $b = \varphi(a)$, and let V^l_b be some coordinate neighbourhood of the point b in the manifolds N^l; let U^k_a be a coordinate neighbourhood of the point a in the manifold M^k such that $\varphi(U^k_a) \subset V^l_b$. Let us choose neighbourhoods U^k_{a1} and U^k_{a2} of the point a in M^k that $\overline{U}^k_{a1} \subset U^k_a$, $\overline{U}^k_{a2} \subset U^k_{a1}$ and such that the set $\overline{U}^k_{a1}$ is compact. The domains U^k_{a2}, $a \in M^k$, cover the manifold M^k. Among them, one can select a finite cover, thus, to prove the theorem, it suffices to prove it for mappings φ from $U^k_{a2} \subset M^k$ to the manifold V^l_b. Since the domain U^k_a is a homeomorphic image of a domain in the Euclidean space E^k, we may just assume that U^k_a is a domain in the space E^k. Analogously, we assume that V^l_b is a domain in the Euclidean space E^l. From this point of view, φ is an m-smooth mapping of the domain U^k_a to the Euclidean space E^l, and it suffices to show that the set of nonproper points has first category in E^l. Let us do it.

Fix the point a and remove the index a from the notation. The mapping φ of the domain U^k of E^k to E^l has the following form in Cartesian coordinates:

$$y^j = \varphi^j(x) = \varphi^j(x^1, \dots, x^k), \qquad j = 1, \dots, l. \tag{2}$$

Here the functions φ^j are m times continuously differentiable. By F_0 we denote the set of all points $x \in U_2^k$ where the functional matrix $\left\|\dfrac{\partial\varphi^j}{\partial x^i}\right\|$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ has rank less than l. For $k < l$, Theorem 4 becomes Theorem 1 which has already been proved. Thus we will assume that $k \geqslant l$. Set $s = k - l + 1$. The function φ^l will play a special role. From (1) it follows that $m > s$; thus the function φ^l is $s+1$ times continuously differentiable. Let r be a positive integer less than or equal to s. Denote by F_r the set of all points from F_0, where all the partial derivatives of orders $1, 2, \ldots, r$ of the function φ^l equal zero. Then we evidently have

$$F_0 \supset F_1 \supset \ldots \supset F_s.$$

We will show that the images of all sets $F_0 \setminus F_1, \ldots, F_{s-1} \setminus F_s$ under φ have first category in E^l. This will prove that the set $\varphi(F_0)$ of nonproper points of the mapping φ is of first category in E^l as well.

First, let us consider the set F_s. The Taylor decomposition for φ^l at the point $p \in F_s$ does not contain terms of degrees $1, 2, \ldots, s$. From this and from compactness of the set $\overline{U}_1$, it follows that there exists a constant c such that for $p \in F_s, x \in \overline{U}_1$ we have

$$|\varphi^l(x) - \varphi^l(p)| < c \cdot (\rho(p, x))^{s+1}. \tag{3}$$

For the remaining functions φ^j, $j = 1, \ldots, l-1$, the equalities

$$|\varphi^j(x) - \varphi^j(p)| < c\rho(p, x) \tag{4}$$

hold; they result from the continuity of the first derivatives and the compactness of the set $\overline{U}_1$. The constant c in inequalities (3) and (4) is common for all functions φ^j, $j = 1, 2, \ldots, l$. Choose a certain ε-cubature for E^k, i.e. tile the space E^k into proper cubes with edge length ε, and denote by Ω the set of all closed cubes of this cubature intersecting the set F_s. Since the set $\overline{F}_s$ is compact, the number of cubes from Ω does not exceed $\frac{c_1}{\varepsilon^k}$, where c_1 is a positive constant independent of ε. Let δ be the distance between the sets $E^k \setminus U_1^k$ and $\overline{U}_2$. Assume that $\varepsilon < \delta/\sqrt{k}$; then each cube K_q from Ω is contained in U_1^k. From that and from the fact that K_q contains the point $p \in F_s$, and from inequalities (3), (4) it follows that the set $\varphi(K_q)$ is contained in some orthogonal parallelepiped L_q of the space E^l having one edge length equal to $2c\sqrt{k} \cdot \varepsilon^{s+1}$ and the remaining $l-1$ edges equal to $2c\sqrt{k} \cdot \varepsilon$. The volume of this parallelepiped L_q equals $2^l c^l k^{l/2} \cdot \varepsilon^{l+s}$. The compact set $\varphi(\overline{F}_s)$ is contained in the sum of closed parallelepipeds of type L_q; the number of them does not exceed c_1/ε^k. It follows that the volume of the set $\varphi(\overline{F}_s)$ does not exceed $c_1 \cdot \varepsilon^{l+s-k} = c_2 \cdot \varepsilon$ (c_2 does not depend on

ε). Thus, since ε is chosen arbitrarily small, the compact set $\varphi(\overline{F}_s)$ does not contain any domain of the space E^l, thus is nowhere dense in E^l.

If $k = 1$, then, since $k \geqslant l \geqslant 1$ we have $l = 1$, $s = 1$. In this case $F_s = F_0$, and we arrive at the statement of the theorem for $k = 1$. This gives us the induction hypothesis on k. We suppose that the theorem is true for the case when the source manifold has dimension less than k. Let us prove the theorem for dimension k.

Let us prove that for $0 \leqslant r < s$, the set $\varphi(F_r \setminus F_{r+1})$ is of first category in the space E^l. This is precisely the part of the proof to be done by induction. Let $p \in F_r \setminus F_{r+1}$. Since p does not belong to the set F_{r+1}, there exists a partial derivative of order $r+1$ of the function φ^l taking a non-zero value at p. Denote the value of this derivative at $x \in U^k$ by $\omega_1(x)$. Since $\omega_1(x)$ is a derivative of order $r+1$ then $\omega_1(x) = \partial\omega(x)/\partial x^i$, where $\omega(x)$ is the derivative of order r for $r > 0$ or the function $\varphi^l(x)$ itself for $r = 0$. For definiteness, assume $i = k$. Set

$$z^i = x^i,\ i = 1, \ldots, k-1; \quad z^k = \omega(x) = \omega(x^1, \ldots, x^k). \tag{5}$$

It follows from $\partial\omega(p)/\partial x^k \neq 0$ that the functional determinant of (5) is non-zero at p; thus, this transformation introduces in some neighbourhood W_p^k of p new coordinates $z^1, \ldots, z^k$. We shall assume that W_p^k does not intersect F_{r+1} and choose a neighbourhood W_{p1}^k of the point p such that its closure $\overline{W}_{p1}^k$ is compact and is contained in W_p^k. By varying the point p, we can cover the set $F_r \setminus F_{r+1}$ by a countable system of neighbourhoods of type W_{p1}^k. Thus, to prove that the set $\varphi(F_r \setminus F_{r+1})$ has first category, it is sufficient to show that $\varphi(F_r \cap \overline{W}_{p1}^k)$ is nowhere dense in E^l. Let us prove this fact.

Let us fix the point p and omit the index p in the notation. Substituting in (2) the expressions $x^1, \ldots, x^k$ in terms of $z^1, \ldots, z^k$, we get the expression for φ in coordinates $z^1, \ldots, z^k$ for the domain W^k. Suppose this expression looks like

$$y^j = \varphi^j(x) = \psi^j(z^1, \ldots, z^k). \tag{6}$$

Here $z^1, \ldots, z^k$ are the new coordinates of the point x. Consider the domain W^k with coordinates $z^1, \ldots, z^k$ as a smooth manifold. It follows from (5) that the mapping φ from W^k to the space E^l given by (6) has smoothness type $m(k,l) - r$. For $r = 0$ the smoothness class of the map φ equals $m(k,l) = m(k-1, l-1)$ [see (1)]. Choosing for $r > 0$ the worst estimate for the smoothness class, that is, $r = s - 1 = k - l$, we see that for $r > 0$ the smoothness class of the considered map φ equals $m(k,l) - (k-l) = m(k-1,l)$ [see (1)]. The set $H \subset W^k$

of all nonproper points of the mapping φ in the manifold W^k is defined by $H = W^k \cap F_0$. This follows from the non-degeneracy of (5) at W^k. Denote by W_t^{k-1} the submanifold of the manifold W^k defined by the equation $z^k = t$. Note that the smoothness class of the mapping from W_t^{k-1} to E^l equals $m(k-1, l-1)$ for $r = 0$ and it equals $m(k-1, l)$ for $r > 0$. Let us consider the cases $r = 0$ and $r > 0$ separately.

Assume $r = 0$. Then $\omega(x) = \varphi^l(x) = z^k$. Thus, the expression (6) for the mapping φ turns into

$$y^j = \psi^j(z^1, \dots, z^k), \quad j = 1, \dots, l-1; \quad y^l = z^k. \tag{7}$$

Denote by E_t^{l-1} the linear subspace of the space E^l defined by the equation $y^l = t$. It follows from the relations (7) that $\varphi(W_t^{k-1}) \subset E_t^{l-1}$. Denote by $H_t \subset W_t^{k-1}$ the set of all nonproper points of the mapping φ from the manifold W_t^{k-1} to the space E_t^{l-1}. It follows from the relations (7) that $H_t = H \cap W_t^{k-1}$. If the set $\varphi(F_0 \cap \overline{W}_1^k)$ contained a domain, then there would exist a value t such that the intersection $\varphi(F_0 \cap \overline{W}_1^k) \cap E_t^{l-1}$ would contain a domain in E_t^{l-1}. However, this is impossible because

$$\varphi(F_0 \cap \overline{W}_1^k) \cap E_t^{l-1} \subset \varphi(H) \cap E_t^{l-1} = \varphi(H \cap W_t^{k-1}) = \varphi(H_t),$$

and the set $\varphi(H_t)$ has first category in E_t^{l-1} according to the induction assumption. Thus, the set $\varphi(F_0 \cap \overline{W}_1^k)$ is nowhere dense in E^l; the case $r = 0$ is discussed completely.

Now assume $r > 0$. Then $\omega(x)$ is a derivative of order r of the function φ^l; thus $\omega(x) = 0$ for $x \in F_r$. Since for the neighbourhood W^k we have $\omega(x) = z^k$ then

$$F_r \cap W^k \subset W_0^{k-1}. \tag{8}$$

Let $H' \subset W_0^{k-1}$ be the set of all nonproper points of the mapping $\varphi : W_0^{k-1} \to E^l$. It is easy to see that $H \cap W_0^{k-1} \subset H'$ [see (6)] and, since $F_r \cap W_1^k \subset H$ then it follows from (8) that $F_r \cap W_1^k \subset H'$. By virtue of the induction hypothesis, the set $\varphi(H')$ has first category in E^l. Since $F_r \cap W_1^k \subset H'$ the set $\varphi(F_r \cap W_1^k)$ is nowhere dense in E^l. Thus we have completed the proof for the case $r > 0$.

Thus, Theorem 4 is proved when M^k has no boundary.

Finally, suppose the manifold M^k has a non-empty boundary M^{k-1}. Suppose $F' \subset M^{k-1}$ is the set of all nonproper points of the mapping $\varphi : M^{k-1} \to N^l$ and $F \subset M^k$ is the set of all nonproper points of the mapping $\varphi : M^k \to N^l$. It is easy to see that

$$F \cap M^{k-1} \subset F'.$$

Thus,

$$F \subset (F \setminus M^{k-1}) \cup F'.$$

The set $F \setminus M^{k-1}$ consists of all nonproper points of the mapping φ in the manifold $M^k \setminus M^{k-1}$ with boundary deleted. Analogously, the set F' consists of all nonproper points of the mapping φ on M^{k-1} without boundary. Thus, both sets $\varphi(F \setminus M^{k-1})$ and $\varphi(F')$ have first category in N^l. The set $\varphi(F)$ is contained in their union, thus it has the first category in N^l.

Therefore, Theorem 4 is proved.

§ 4. Non-degenerate singular points of smooth mappings

Let f be a smooth mapping from a manifold M^k to a manifold N^l. Let $a \in M^k$ and $b = f(a) \in N^l$ be interior (non-boundary) points of the manifolds M^k and N^l. In the neighbourhoods of a and b, let us introduce local coordinates $x^1, \ldots, x^k$ and $y^1, \ldots, y^l$ taking these points to be coordinate origins. Let

$$y^j = f^j(x) = f^j(x^1, \ldots, x^k)$$

be the coordinate expression for f in the chosen coordinate systems.

Suppose a is a regular point of f, i.e. that the rank of the matrix $\left\| \frac{\partial f^j(a)}{\partial x^i} \right\|$, $j = 1, \ldots, l$, $i = 1, \ldots, k$, equals k; to be more precise, we shall assume that the determinant $\left| \frac{\partial f^j(a)}{\partial x^i} \right|$, $i, j = 1, \ldots, k$ is non-zero. With this assumption the relations

$$\xi^i = f^i(x^1, \ldots, x^k), \qquad i = 1, \ldots, k,$$

may serve to define in the neighbourhood of a the new coordinates $\xi^1, \ldots, \xi^k$ of the point x. Let

$$\begin{aligned} y^j &= \xi^j, && j = 1, \ldots, k; \\ y^j &= \varphi^j(\xi^1, \ldots, \xi^k), && j = k+1, \ldots, l, \end{aligned}$$

be the expression of the mapping f in these new coordinates. Let us introduce in the neighbourhood of the point b the new coordinates $\eta^1, \ldots, \eta^l$, by setting

$$\begin{aligned} \eta^j &= y^j, && j = 1, \ldots, k; \\ \eta^j &= y^j - \varphi^j(y^1, \ldots, y^k), && j = k+1, \ldots, l. \end{aligned}$$

In coordinates $\xi^1, \dots, \xi^k, \eta^1, \dots, \eta^l$ the mapping f looks like

$$\eta^j = \xi^j,\ j = 1, \dots, k; \quad \eta^j = 0,\ j = k+1, \dots, l. \tag{1}$$

Now, let us assume that the point a is proper, i.e. the rank of the matrix $\left\| \dfrac{\partial f^j(a)}{\partial x^j} \right\|$, $j = 1, \dots, l$, $i = 1, \dots, k$ equals l, and assume for definiteness that the determinant $\left| \dfrac{\partial f^j(a)}{\partial x^i} \right|$, $i, j = 1, \dots, l$ is non-zero. Then the relations

$$\xi^i = f^i(x^1, \dots, x^k),\ i = 1, \dots, l; \quad \xi^i = x^i,\ i = l+1, \dots, k,$$

may serve for introducing in a neighbourhood of a the new coordinates $\xi^1, \dots, \xi^k$ of the point x. Furthermore, assuming

$$\eta^j = y^j, \quad j = 1, \dots, l,$$

we see that in coordinates $\xi^1, \dots, \xi^k, \eta^1, \dots, \eta^l$ the mapping f can be written as

$$\eta^j = \xi^j, \quad j = 1, \dots, l. \tag{2}$$

Thus, if the manifold M^k is closed and $b \in N^l$ is a proper point of the mapping f, then $f^{-1}(b)$ is a smooth $(k-l)$-dimensional submanifold of the manifold M^k with local coordinates $\xi^{l+1}, \dots, \xi^k$ in the neighbourhood of a. In the case when the manifolds M^k and N^l are oriented and their orientations are given by the coordinate systems $\xi^{l+1}, \dots, \xi^k, \xi^1, \dots, \xi^l$ and $\eta^1, \dots, \eta^l$, then the manifold $f^{-1}(b)$ gets a natural orientation given by the coordinate system $\xi^{l+1}, \dots, \xi^k$.

We see that both in the case of a regular point a and in the case of proper point a the mapping is written quite simply in the properly chosen coordinate systems [see (1), (2)].

It was shown in § 2 that in any neighbourhood of any arbitrary smooth mapping from M^k to the vector space A^{2k} there exists a regular mapping, and all mappings sufficiently close to a regular one, are regular as well (see Theorem 3). In this sense, singular points (see § 1, «D») of mappings $M^k \to A^{2k}$ are unbalanced, that is, they are removable by a small perturbation. For mappings from M^k to the vector space A^{2k-1} we have another situation: singular points that occur there are, generally, balanced: they cannot be removed by a small perturbation. This problem was solved by Whitney. Here we give a simpler proof of his theorem (see Theorem 6). We will not use this theorem in the sequel. The question about typical singular points is solved here also for mappings of a manifold M^k to the one-dimensional space A^1, i.e. to the line (see Theorem 5; it will have applications in the homotopy theory of mappings, see § 3, Chapter 4). Thus, the question about typical singular points of a mapping is solved for mappings from manifolds of dimension k to space of dimension $2k-1$ or 1. For other dimension, it remains a quite actual open problem.

Generally, a regular mapping from M^k to the vector space A^{2k} is not homeomorphic: it has self-intersections, which might be non-removable by small perturbation of the initial mapping. The question whether a self-intersection is typical is also solved here (see «A» and «B»); these statements will be used in the sequel.

For proving Theorems 5 and 6, and also Statement «B» we significantly use the construction «A» (see page 21) and Theorem 4.

Typical self-intersection points of mappings $M^k \to E^{2k}$

A) Let f be a regular smooth mapping of class $m \geqslant 1$ from a closed manifold M^k to the vector space A^{2k} and let a and b be two different points from M^k having the same image $f(a) = f(b) \in A^{2k}$. Furthermore, let U and V be neighbourhoods of points a and b in M^k such that the mapping f is homeomorphic for any of these neighbourhoods, and T_a^k and T_b^k are tangent planes at points $f(a)$ and $f(b)$ to the manifolds $f(U)$ and $f(V)$, respectively. Say that for a self-intersection pair (a, b) the mapping f is *typical* if the tangent planes T_a^k and T_b^k are in general position, i.e. they intersect precisely at one point $f(a) = f(b)$. Obviously, in this case for sufficiently small neighbourhoods U and V, the manifolds $f(U)$ and $f(V)$ have a unique common point $f(a) = f(b)$ as well (implicit function theorem), and small perturbations of the mapping preserve typical self-intersections. If f is typical for any self-intersection pair and, furthermore, no three pairwise different points have the same image, we say that f is *typical.* It follows from closeness of the manifold M^k that, for a mapping f typical for any self-intersection pair, there exists only a finite number of self-intersection pairs.

B) Let f be a closed homeomorphic mapping of a closed manifold M^k to a vector space C^{2k+1}. The set P^{2k} of all pairs (x, y), where $x \in M^k$, $y \in M^k$, $x \neq y$, naturally forms a smooth manifold of dimension $2k$. With each point $(x, y) = P^{2k}$, associate a point $\sigma(x, y) = (f(y) - f(x))^* \in S^{2k}$, i.e. the ray of the vector $f(y) - f(x)$ (see § 1, «H»). Let e be an arbitrary non-zero vector from the space C^{2k+1} and let π_e be the projection along the one-dimensional space e^{**} containing e. It turns out that the regular mapping $\pi_e f$ is typical for any self-intersection pair (see «A») if and only if the mapping σ from the manifold P^{2k} to the manifold S^{2k} is proper in the point $e^* \in S^{2k}$. From this, by virtue of Theorem 4, it follows that for any given one-dimensional projection direction there exists an arbitrarily closed projection direction e^{**} that the mapping $\pi_e f$ is typical for each self-intersection pair. Furthermore, it turns out that for any one-dimensional projection direction there is an arbitrarily close direction e_0^{**} that the mapping $\pi_{e_0} f$ is typical.

Let us prove Statement «B». Let $e_1, \ldots, e_{2k+1}$ be a basis of a vector space C^{2k+1}. Denote by W the set of all vectors $u = \sum_{n=1}^{2k+1} u^n e_n$ of the space C^{2k+1} for which $u^{2k+1} > 0$, and denote by W^* the set of all rays u^* for $u \in W$. For coordinates of the ray $u^* \in W^*$, we take the numbers $u^{*n} = u^n/u^{2k+1}$, $n = 1, \ldots, 2k$. Herewith we introduce local coordinates for the domain W^* of the manifold S^{2k} (see § 1,«H»). Now, let a and b be two different points of the manifold M^k. Choose a basis $e_1, \ldots, e_{2k+1}$ in such a way that $e_{2k+1} = e = f(b) - f(a)$. In neighbourhoods of points a and b of the manifold M^k, let us choose local coordinates $x^1, \ldots, x^k$ and $y^1, \ldots, y^k$; let

$$u^n = f_a^n(x^1, \ldots, x^k) = f_a^n(x), \qquad n = 1, \ldots, 2k+1; \tag{3}$$

$$u^n = f_b^n(y^1, \ldots, y^k) = f_b^n(y), \qquad n = 1, \ldots, 2k+1, \tag{4}$$

be a coordinate expression of the mapping f in the neighbourhoods of a and b, respectively. While projecting along the vector $e = f(b) - f(a)$, the points b and a merge: $\pi_e f(a) = \pi_e f(b)$; thus the condition that $\pi_e f$ is typical for the self-intersection pair (a, b), evidently, means that the determinant

$$\begin{vmatrix} \dfrac{\partial f_a^1(a)}{\partial x^1} & \cdots & \dfrac{\partial f_a^{2k}(a)}{\partial x^1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial f_a^1(a)}{\partial x^k} & \cdots & \dfrac{\partial f_a^{2k}(a)}{\partial x^k} \\ \dfrac{\partial f_b^1(b)}{\partial y^1} & \cdots & \dfrac{\partial f_b^{2k}(b)}{\partial y^1} \\ \dfrac{\partial f_b^1(b)}{\partial y^k} & \cdots & \dfrac{\partial f_b^{2k}(b)}{\partial y^k} \end{vmatrix} \tag{5}$$

is non-zero. For a neighbourhood of the point (a, b) of the manifold P^{2k} we may use the coordinate system consisting of numbers $x^1, \ldots, x^k$, $y^1, \ldots, y^k$; thus the mapping σ has the following coordinate form:

$$u^{*n} = \frac{f_b^n(y) - f_a^n(x)}{f_b^{2k+1}(y) - f_a^{2k+1}(x)}, \qquad n = 1, \ldots, 2k. \tag{6}$$

In these coordinates, the functional determinant of the mapping σ at the point (a, b), evidently, coincides with the determinant (5) up to sign. Thus we have proved that regular mapping $\pi_e f$ is typical for each self-intersection pair if and only if the mapping σ is proper at the point e^*.

Now, choose the ray e^* in such a way that the vector e is not parallel to any vector tangent to the manifold $f(M^k)$ and that the mapping σ is proper at the point $e^* \in S^{2k}$. By virtue of Theorems 1 and 4, the set of rays enjoying the above properties, is everywhere dense in the manifold S^{2k}. Suppose there exist three pairwise distinct points a, b, c of the manifold M^k such that $\pi_e f(a) = \pi_e f(b) = \pi_e f(c)$. In the neighbourhood of c in M^k, let us introduce the local coordinates $z^1, \ldots, z^k$, and let

$$u^n = f_c^n(z^1, \ldots, z^k) = f_c^n(z), \qquad n = 1, \ldots, 2k+1, \tag{7}$$

be the coordinate expression of the mapping f in the neighbourhood of c, analogous to the expressions (3) and (4). Now, if x, y, z are three points of the manifold M^k close to a, b, c, respectively, such that the points $f(x)$, $f(y)$, $f(z)$ lie on the same line then we have

$$\frac{f_a^n(x) - f_c^n(z)}{f_a^{2k+1}(x) - f_c^{2k+1}(z)} = \frac{f_b^n(y) - f_c^n(z)}{f_a^{2k+1}(y) - f_c^{2k+1}(z)}, \qquad n = 1, \ldots, 2k. \tag{8}$$

Here we have $2k$ equation. We may assume that these equations implicitly define the functions $x^1, \ldots, x^k$, $y^1, \ldots, y^k$ in independent variables $z^1, \ldots, z^k$. For the initial value $z = c$ we have the solution $x = a$, $y = b$. For these initial values of the functions and independent variables, the functional determinant of the system (8) is non-zero, since so is the determinant (5). Thus, the system (8) satisfies the condition of the implicit function theorem. It follows now that the set of triples x, y, z closed to the triple a, b, c and satisfying the condition that $f(x)$, $f(y)$, $f(z)$ lie on the same line, forms a k-dimensional manifold. Thus, by virtue of Theorem 1, we see that for the point e^* of the manifold S^{2k} there is an arbitrarily close point e_0^* satisfying the conditions of the Statement «B».

Typical critical points of a real-valued function on a manifold

C) Let f be a class m smooth mapping ($m \geqslant 2$) from a manifold M^k to the one-dimensional Euclidean space E^1, or, what is the same, to the line. By choosing a coordinate system on the line E^1, we write down the mapping f as $y^1 = f^1(x)$, $x \in M^k$, where f^1 is a real-valued function of class m, defined on M^k. In a neighbourhood of a certain point $a \in M^k$, let us introduce local coordinates $x^1, \ldots, x^k$ with the origin at a, and let

$$y^1 = f^1(x) = f^1(x^1, \ldots, x^k)$$

be the expression for f in these coordinates. The point a is called a *critical point* of the function f^1, and the number $f^1(a)$ is called the *critical value* of the function f^1 at the point a if all derivatives of the first order of the function f^1 are zeros at a or, which is the same, if a is a singular point of the function f (see § 1, «D»). Taking the Taylor decomposition for the function f^1 at the critical point a, we get

$$f^1(x) = f^1(a) + \sum_{i,j} a_{ij} x^i x^j + \dots . \tag{9}$$

If the determinant $|a_{i,j}| \neq 0$, then the critical point a is called *non-degenerate*. It can be checked straightforwardly that for a critical point a of the function f, any arbitrary coordinate change the matrix $\|a_{i,j}\|$ is transformed as coefficients of quadratic form. From this it follows, in particular, that the non-degeneration of the singular point is its invariant property, i.e. it does not depend on the choice of the coordinate system.

D) Let h be an m-smooth mapping ($m \geqslant 2$) of a manifold M^k to the Euclidean vector space C^{q+1}. Let u be a non-zero vector from C^{q+1} and let u^{**} be the one-dimensional subspace containing the vector u. Denote by π_u the orthogonal projection of the space C^{q+1} to the line u^{**}. The set N^q of all pairs (x, u^*), where $x \in M^k$, and u^* is a ray orthogonal to the manifold $h(M^k)$ at the point $h(x)$ can be naturally seen as an $(m-1)$-smooth manifold of dimension q. With each point $(x, u^*) \in N^q$, associate a point $\nu(x, u^*) = u^* \in S^q$ (see § 1, «H»). The mapping ν is a smooth mapping of class $m-1$ from N^q to S^q. It turns out that the point $a \in M^k$ is a singular point of the mapping $\pi_u h$ from M^k to u^{**} if and only if the ray u^* is orthogonal to the manifold $h(M^k)$ at the point $h(a)$. Furthermore, if the ray u^* is orthogonal to the manifold $h(M^k)$ at the point $h(a)$ then the singular point a of the mapping $\pi_u h$ is non-degenerate if and only if (a, u^*) is a proper point of the mapping ν.

Let us prove Statement «D». Denote the scalar product of vectors u and v from C^{q+1}, as usual, by (u, v). Let $u \in C^{q+1}$ and $(u, u) = 1$. Indeed, the real-valued function $(u, h(x))$ in variable $x \in M^k$ defined on M^k, corresponds to the mapping $\pi_u h$ of the manifold M^k to the axis u^{**}. In the local coordinates $x^1, \dots, x^k$ defined in a neighbourhood of a, one has

$$\frac{\partial}{\partial x^i}(u, h(a)) = \left(u, \frac{\partial h(a)}{\partial x^i}\right), \qquad i = 1, \dots, k. \tag{10}$$

The fact that the left-hand sides of all relations (10) are all zeros means that a is a singular point of the mapping $\pi_u h$; the fact that all right-hand sides are zeros means that the vector u is orthogonal to the manifold $h(M^k)$ at the point $h(a)$. Thus, we have proved that the point a is a singular point for the mapping $\pi_u h$ if and only if the ray u^* is orthogonal to $h(M^k)$ at the point $h(a)$.

To establish a criterion whether a singular point a of the mapping $\pi_{u_0}h$ is degenerate, let us choose in the space C^{q+1} such an orthonormal basis $e_1, \ldots, e_{q+1}$ that the vectors $e_1, \ldots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$, and the vector e_{q+1} coincides with u_0. In the corresponding coordinates $y^1, \ldots, y^{q+1}$ of the space C^{q+1} the map h in the neighbourhood of a looks like

$$y^j = h^j(x) = h^j(x^1, \ldots, x^k), \qquad j = 1, \ldots, q+1. \tag{11}$$

Since the vectors $e_1, \ldots, e_k$ are tangent to the manifold $h(M^k)$ at the point $h(a)$, it follows directly that

$$\left| \frac{\partial h^j(a)}{\partial x^i} \neq 0 \right|, \qquad i = 1, \ldots, k,$$

From that we see that the relations

$$\xi^i = h^i(x^1, \ldots, x^k), \qquad i = 1, \ldots, k$$

may serve for introducing new coordinates $\xi^1, \ldots, \xi^k$ of the point x in the neighbourhood of the point a in M^k. In these coordinates, the mapping h looks like

$$h(x) = \sum_{i=1}^{k} \xi^i e_i + \sum_{j=1}^{q+1-k} \varphi^j(x) \cdot e_{k+j}. \tag{12}$$

Because the vectors $e_1, \ldots, e_k$ are tangent to $h(M^k)$ at the point $h(a)$, it follows that

$$\frac{\partial \varphi^j(a)}{\partial \xi^i} = 0, \quad i = 1, \ldots, k; \; j = 1, \ldots, q+1-k. \tag{13}$$

Let (x, u^*) be a point of the manifold N^q close to the point $(a, u^*) = (a, e^*_{q+1})$. On the ray u^*, let us choose a vector u satisfying the condition

$$(u, e_{q+1}) = 1.$$

Denote the remaining q components of the vector u in the basis $e_1, \ldots, e_{q+1}$ by $u^1, \ldots, u^q$: $u^i = (u, e_i)$, $i = 1, \ldots, q$. The orthogonality condition for the vector u and $h(M^k)$ at the point $h(x)$ now looks like

$$0 = \left(u, \frac{\partial h(x)}{\partial \xi^i} \right) = u^i + \sum_{j=1}^{q-k} u^{k+j} \frac{\partial \varphi^j(x)}{\partial \xi^i} + \frac{\partial \varphi^{q+1}(x)}{\partial \xi^i}, \; i = 1, \ldots, k. \tag{14}$$

This relation shows that for coordinates of the element (x, u^*) of the manifold N^q we can choose the coordinates $\xi^1, \ldots, \xi^k$ of the point x and the components $u^{k+1}, \ldots, u^q$ of the vector u. For coordinates of the ray u^* in the manifold S^q, let us take the first q components of the vector u and denote these components by $v^1, \ldots, v^k$ in order not to mix them up with the coordinates $u^{k+1}, \ldots, u^q$ of the element (x, u^*) in the manifold N^q. Since $v^i = u^i$, $i = 1, \ldots, q$, then in the chosen coordinate system the mapping $\nu : N^q \to S^q$ looks like [see (14)]

$$v^i = -\sum_{j=1}^{q-k} u^{k+j} \frac{\partial \varphi^j(x)}{\partial \xi^i} - \frac{\partial \varphi^{q+1}(x)}{\partial \xi^i}, \; i = 1, \ldots, k,$$

$$v^{k+j} = u^{k+j}, \qquad j = 1, \ldots, q-k.$$

The direct calculation [see (13)] shows that the Jacobian of the mapping ν at the point (a, e^*_{q+1}) is equal to $(-1)^k \left|\dfrac{\partial^2 \varphi^{q+1}(a)}{\partial \xi^i \partial \xi^\alpha}\right|$, $\quad i, \alpha = 1, \ldots, k$. Thus, the point (a, u_0) is a proper point of the mapping ν if and only if the following equation holds:

$$\left|\frac{\partial^2 \varphi^{q+1}(a)}{\partial \xi^i \partial \xi^\alpha}\right| \neq 0. \tag{15}$$

Since the mapping $\pi_{u_0} h$ from M^k to the axis u_0^{**} is associated with the function $\varphi^{q+1}(x)$, the condition (14) coincides with the non-degeneracy condition for the singular point a of the mapping $\pi_{u_0} h$. This completes the proof of «D».

Theorem 5. *Let M^k be a smooth compact manifold of class $m \geqslant 3$ with boundary M^{k-1} consisting of two closed manifolds M_0^{k-1} and M_1^{k-1}, each of which possibly empty. Let f^1 be a real-valued function of class m defined on M^k. Suppose that the function f^1 takes the same value c_i, $i = 0, 1$ in all points of the manifold M_i^{k-1} and $c_0 < c_1$, and that for any non-boundary point $x \in M^k$ the inequality $c_0 < f(x) < c_1$ holds. Moreover, suppose that no critical point of f^1 lies on the boundary M^{k-1}. It turns out that for the function f^1 there exists an arbitrarily m-class close (see § 2, «F») function g^1 coinciding with f^1 in some neighbourhood of the boundary such that all critical points of the function g^1 are not degenerate and critical values in different critical points are pairwise distinct.*

Proof. With the function f^1, let us associate the mapping f from the manifold M^k to the one-dimensional vector space A^1. Let e be a homeomorphic regular class m mapping from M^k to the Euclidean space B^q

(see Theorem 2). Denote the direct sum of vector spaces A^1 and B^q by C^{q+1}; let us consider the spaces A^1 and B^q as orthogonal subspaces of the space C^{q+1}. Denote the direct sum of mappings f and e (see § 2, «E») by h. The mapping h is a regular homeomorphic class m mapping from the manifold M^k to the Euclidean space C^{q+1} such that the orthogonal projection π of h to the line A^1 coincides with f: $f = \pi h$. First of all, let us show that in any neighbourhood of the line A^1 there exists a line in the orthogonal projection to which generates a function having only non-degenerate critical values. The desired function in the formulation of the theorem is to be obtained from this by some modifications.

Let N^q be a manifold of all normal elements (x, u^*) of the manifold $h(M^k)$, as defined in «D», and let ν be the mapping from the manifold N^q to the manifold S^q also defined in «D». Let us show that if $u^* \in S^q$ is a proper point of ν then all singular points of $\pi_u h$ are non-degenerate. Indeed, if a is a singular point of the mapping $\pi_u h$, then the ray u^* is orthogonal to $h(M^k)$ at the point $h(a)$; thus $(a, u^*) \in N^q$. Since the mapping ν is proper at (a, u^*) of the manifold N^q, then the singular point a is non-degenerate (see «D»). Let ε be a given positive number and let u be such a unit vector of the spaces C^{q+1} that the function $h^1 = (u, h(x))$ is class m ε-close to f^1 and that $u^* \in S^q$ is a proper point of the mapping ν so that all critical points of the function h^1 are non-degenerate. By Theorem 4, such a vector u does exist.

Let δ be such a small positive number that for $f^1(x) < c_0 + 3\delta$ and for $f^1(x) > c_1 - 3\delta$ the point x is not a critical point of the function f^1. The existence of such δ follows from the conditions of the theorem, since neither the boundary M^{k-1} nor its small neighbourhood contains critical points of f^1. Furthermore, suppose $\chi(t)$ is a real-valued class m function in variable t equal to zero at $t \leqslant c_0 + \delta$ and $t \geqslant c_1 - \delta$ and equal to one at $c_1 - 2\delta \geqslant t \geqslant c_0 + 2\delta$. Set

$$h^2(x) = f^1(x) + \chi(f^1(x))(h^1(x) - f^1(x)). \tag{16}$$

It is easy to see that if ε that we have taken for constructing the function $h^1(x)$, is chosen to be reasonably small then all critical points of the function $h^2(x)$ defined by (16) coincide with the critical points of the function $h^1(x)$ and thus they are non-degenerate. Since for $t \leqslant c_0 + \delta$ and for $t \geqslant c_1 - \delta$ the function $\chi(t)$ equals zero it follows that for some neighbourhood of the boundary M^{k-1} the functions $h^2(x)$ and $f^1(x)$ coincide.

Typical singularities of mappings $M^k \to E^{2k-1}$

E) Let f be an m-class smooth ($m \geqslant 2$) mapping from the manifold M^k to the vector space A^{2k-1}. Let a be a singular point of the mapping f and let $x^1, \dots, x^k$ be a local coordinate system in its neighbourhood such that

$$\frac{\partial f(a)}{\partial x^1} = 0. \tag{17}$$

Such a coordinate system in a singular point neighbourhood always exists. If the system

$$\frac{\partial^2 f(a)}{\partial x^1 \partial x^i}, \ \frac{\partial f(a)}{\partial x^j}, \quad i = 1, \dots, k; \ j = 2, \dots, k, \tag{18}$$

of $2k-1$ vectors of the space A^{2k-1} is linearly independent, then the singular point a is called *non-degenerate*. Later on, we shall show that the non-degeneracy of a singular point is invariant, i.e. this notion is independent of the coordinate system: if some coordinate system $\xi^1, \dots, \xi^k$ defined in a neighbourhood of a satisfies the condition

$$\frac{\partial f(a)}{\partial \xi^1} = 0, \tag{19}$$

then the vector systems (18) and

$$\frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^i}, \quad \frac{\partial f(a)}{\partial \xi^j}, \quad i = 1, \dots, k; \ j = 2, \dots, k, \tag{20}$$

are either both linearly dependent or both linearly independent. It turns out that in a sufficiently small neighbourhood of a non-degenerate singular point there are no other singular points.

Let us prove that the non-degeneracy is invariant. Assume that the relations (17) and (19) hold and that the vector system (20) is linearly independent. Let us show that the system (18) is also linearly independent. We have

$$\frac{\partial f(a)}{\partial x^1} = \sum_\alpha \frac{\partial f(a)}{\partial \xi^\alpha} \frac{\partial \xi^\alpha(a)}{\partial x^1},$$

from which, according to the assumption above, we deduce

$$\frac{\partial \xi^\alpha(a)}{\partial x^1} = 0, \ \alpha = 2, \dots, k. \tag{21}$$

Since the Jacobian $\left| \frac{\partial \xi^\alpha(a)}{\partial x^i} \right|$, $\alpha, i = 1, \dots, k$, is non-zero, it follows from (21) that

$$\frac{\partial \xi^1(a)}{\partial x^1} \neq 0, \quad \left| \frac{\partial \xi^\alpha(a)}{\partial x^i} \right| \neq 0, \quad \alpha, i = 2, \dots, k. \tag{22}$$

From the relations (21) we get

$$\frac{\partial f(a)}{\partial x^j} = \sum_{\alpha=2}^{k} \frac{\partial f(a)}{\partial \xi^\alpha} \frac{\partial \xi^\alpha(a)}{\partial x^j}, \quad j = 2, \ldots, k. \tag{23}$$

Furthermore, taking into account (21) and (19), we get

$$\frac{\partial^2 f(a)}{\partial x^1 \partial x^i} = \sum_{\beta=1}^{k} \frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^\beta} \frac{\partial \xi^1(a)}{\partial x^1} \frac{\partial \xi^\beta(a)}{\partial x^i} + \sum_{\alpha=2}^{k} \frac{\partial f(a)}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha(a)}{\partial x^1 \partial x^i}, \quad i = 1, \ldots, k. \tag{24}$$

From the relations (23), (24), (22) and linear independence of the system (20) one gets the linear independence of the system (18).

Now, let us show that the singular point a is isolated. To do that, view a as the origin for the coordinate system $x^1, \ldots, x^k$ and consider the Taylor decomposition for vectors $\dfrac{\partial f(x)}{\partial x^i}$, $i = 1, \ldots, k$, in the neighbourhood of a in coordinates $x^1, \ldots, x^k$:

$$\frac{\partial f(x)}{\partial x^1} = \sum_{\alpha=1}^{k} \frac{\partial^2 f(a)}{\partial x^1 \partial x^\alpha} x^\alpha + \varepsilon_1, \tag{25}$$

$$\frac{\partial f(x)}{\partial x^i} = \frac{\partial f(a)}{\partial x^i} + \varepsilon_i, \quad i = 2, \ldots, k, \tag{26}$$

where ε_1 is second-order small with respect to $\varrho = \sqrt{(x^1)^2 + \ldots + (x^k)^2}$, and $\varepsilon_2, \ldots, \varepsilon_k$ are first-order small with respect to ϱ. Since the vectors of the system (19) are linearly independent, it follows from (25) and (26) that the vectors $\dfrac{\partial f(x)}{\partial x^1}, \ldots, \dfrac{\partial f(x)}{\partial x^k}$ are linearly independent for all points $x \neq a$ sufficiently close to a.

F) Let h be a regular class m mapping ($m \geqslant 2$) from the manifold M^k to the vector space C^{2k}. Denote the manifold of all rays u^* of the manifold C^{2k} by S^{2k-1} (see § 1, «H») and denote by L^{2k-1} the manifold of all linear elements of the manifold $h(M^k)$, i.e. the manifold of all pairs (x, u^*), where $x \in M^k$, and u^* is the ray tangent to $h(M^k)$ at $h(x)$. Define the mapping τ from the manifold L^{2k-1} to the manifold S^{2k-1} by setting $\tau(x, u^*) = u^*$. Denote the projection of the space C^{2k} along the line u^{**} containing u by π_u. As noticed before (see § 2, «A»), the point $a \in M^k$ is a singular point of $\pi_u h$ if and only if the ray u^* is tangent to $h(M^k)$ at the point $h(x)$, i.e. if $(a, u^*) \in L^{2k-1}$. It turns out that the singular point a of the mapping $\pi_u h$ is non-degenerate if and only if the mapping τ is proper at the point $(a, u^*) \in L^{2k-1}$.

Let us prove the last statement. Let a be a singular point of $f = \pi_{u_0} h$. Choose a basis $e_1, \dots, e_{2k}$ of the vector space C^{2k} in such a way that the vectors $e_1, \dots, e_k$ are tangent to $h(M^k)$ at the point $h(a)$ and so that the vector e_1 coincides with u_0. Let $y^j = h^j(x) = h^j(x^1, \dots, x^k)$ be the expression of the mappings h in the coordinates $y^1, \dots, y^{2k}$ with respect to the basis $e_1, \dots, e_{2k}$. Note that the absolute value of the Jacobian $\left| \dfrac{\partial h^j(a)}{\partial x^i} \right|$, $i, j = 1, \dots, k$, differs from zero, thus the relations

$$\xi^i = h^i(x^1, \dots, x^k),\ m = 1, \dots, k,$$

can be used to introduce new coordinates $\xi^1, \dots, \xi^k$ of x in the neighbourhood of a. In the new coordinates, the vector $h(x)$ will look like

$$h(x) = \sum_{i=1}^{k} \xi^i e_i + \sum_{i=1}^{k} \varphi^j(x) e_{k+j}, \tag{27}$$

where the functions $\varphi^j(x)$ satisfy the condition

$$\frac{\partial \varphi^j(x)}{\partial \xi^i} = 0, \quad i, j = 1, \dots, k. \tag{28}$$

Let (x, u^*) be an element of the manifold L^{2k-1} close to the element (a, u_0). The vector u is tangent to $h(M^k)$ at the point $h(x)$; thus it can be written as

$$u = \sum_{i=1}^{k} u^i \frac{\partial h(x)}{\partial \xi^i} = \sum_{i=1}^{k} u^i e_i + \sum_{i,j=1}^{k} u^i \frac{\partial \varphi^j(x)}{\partial \xi^i} e_{j+k}. \tag{29}$$

On the ray u^*, let us choose a vector u such that $u^1 = 1$; then the expression (29) looks like

$$u = e_1 + \sum_{i=2}^{k} u^i e_i + \sum_{j=1}^{k} \frac{\partial \varphi^j(x)}{\partial \xi^1} e_{k+j} + \sum_{j=1}^{k} \sum_{i=2}^{k} u^i \frac{\partial \varphi^j(x)}{\partial \xi^i} e_{j+k}. \tag{30}$$

For coordinates of the elements (x, u^*) in L^{2k-1} we may take the numbers $u^2, \dots, u^k, \xi_1, \dots, \xi^k$. Since the first component of the vector u in the space C^{2k} equals one [see (30)], the coordinates of the row u^* in the manifold S^{2k-1} can be set to be the remaining components $v^2, \dots, v^{2k}$ of the vector u in the space C^{2k}. In the chosen coordinates, the mapping τ is written (according to (30)) as

$$v^i = u^i, \quad i = 2, \dots, k;$$
$$v^{k+j} = \frac{\partial \varphi^j(x)}{\partial \xi^1} + \sum_{i=1}^{k} u^i \frac{\partial \varphi^j(x)}{\partial \xi^i}, \quad j = 1, \dots, k. \tag{31}$$

A simple calculation shows that the Jacobian of the mapping τ at the point (a, u_0) is equal to

$$\left| \frac{\partial^2 \varphi^j(a)}{\partial \xi^1 \partial \xi^i} \right|, \quad i, j = 1, \dots, k. \tag{32}$$

Consider now the mapping $\pi_{u_0} h$. Let us assume that it is a projection to some vector space A^{2k-1} with basis $e_2, \dots, e_{2k}$ along some line e_1^{**}. Then we have [see (27)]

$$f(x) = \pi_{u_0} h(x) = \sum_{i=2}^{k} \xi^i e_i + \sum_{\alpha=1}^{k} \varphi^\alpha(x) e_{k+\alpha}. \tag{33}$$

Thus we deduce

$$\frac{\partial^2 f(a)}{\partial \xi^1 \partial \xi^i} = \sum_{\alpha=1}^{k} \frac{\partial^2 \varphi^\alpha(x)}{\partial \xi^1 \partial \xi^i} \cdot e_{k+\alpha}, \quad i = 1, \dots, k,$$
$$\frac{\partial f(a)}{\partial \xi^j} = e_j, \quad j = 2, \dots, k.$$

Thus, in this case the vectors of the system (19) are linearly independent if and only if the Jacobian (32) is non-zero.

Statement «F» is proved.

Theorem 6. *Let f be an m-class smooth ($m \geqslant 3$) mapping from a compact manifold M^k of dimension k to the vector space A^{2k-1} of dimension $2k-1$. It turns out that for the mapping f there is an arbitrarily m-close mapping g with all singular points non-degenerate and not lying on the boundary M^{k-1} of the manifold M^k.*

Proof. Let us treat the vector space A^{2k-1} as a subspace of the vector space C^{2k} of dimension $2k$. Let B^1 be some one-dimensional subspace of the space C^{2k} not lying in A^{2k-1}. Denote the projection of the space C^{2k} to the space A^{2k-1} along B^1 by π. Fix a positive number ε; let h be a regular mapping of M^k to the vector space C^{2k} such that the mapping πh is ε-close to f (see Theorem 3). Let L^{2k-1} be the manifold of linear elements of the manifold $h(M^k)$ (see «F»); let L^{2k-2} be the submanifold of L^{2k-1} consisting of all elements of the type (x, u^*) where $x \in M^{k-1}$, and let τ be the mapping from L^{2k-1} to the sphere S^{2k-1} constructed in «F». It follows from «F» that if $u^* \in S^{2k-1}$ is not a singular point of the mapping τ and does not belong to the set $\tau(L^{2k-2})$ then all singular points of the mapping $\pi_u h$ are non-degenerate and do not belong to the boundary of the manifold M^k. By virtue of Theorems 4 and 1, there exists a vector u such that u^* satisfies the conditions described above and the mapping $\pi_u h$ is ε-close to πh. Thus, there is a 2ε-close to f mapping $g = \pi_u h$ satisfying the conditions of the theorem.

Theorem 6 is proved.

Canonical form of typical critical points and typical singular points

In Statements «C» and «E», several singular points of mappings from manifolds M^k to vector spaces A^1 and A^{2k-1}, were found to be non-degenerate. In Theorems 5 and 6, it was shown that all degenerate singular points of the considered mappings are not balanced, i.e. removable by small perturbations. However, we did not prove that those singular points called non-degenerate are balanced, i.e. they are preserved by small perturbations. The proof of this fact is not difficult, but we shall omit it. Also, we have not described the structure of the mapping in the neighbourhood of a non-degenerate singular point. It is not easy in the general situation; below we present the results without proving them.

G) Let a be a non-degenerate critical point of a real-valued function $f^1(x)$ defined on a manifold M^k. As noticed in Statement «A», the Taylor decomposition of the function $f^1(x)$ in the neighbourhood of the point a looks like (9). It turns out that (see [7]) by a coordinate change in the neighbourhood of a this Taylor decomposition can be transformed to that of the type

$$f^1(x) = f^1(a) + (x^1)^2 + \ldots + (x^s)^2 - (x^{s+1})^2 - \ldots - (x^k)^2, \qquad (34)$$

where the number s of positive squares is an invariant of the point a, i.e. does not depend on the coordinate choice in the neighbourhood of this point, and is not changed by a small perturbation. Thus, the function defined on a k-dimensional manifold has $k+1$ possible types of critical points ($s = 0, \ldots, k$). Since the mapping f of the manifold M^k does not define the function $f^1(x)$ directly, then the points of different type for the function may happen to be of different type for a mapping. Indeed, changing the sign of the function $f^1(x)$ interchanges the roles of s and $k - s$; thus the corresponding critical points belong to the same type of mapping critical points. It is worth mentioning that, in the general situation, one cannot get from the expression (9) to the expression (34) by a linear coordinate change, as it might seem. An evident linear transformation is just the first step of the transformation of (9) to (34). Under linear transformation the third-order and higher-order terms are preserved, whence they are absent in (34).

H) Let a be a non-degenerate critical point of $f : M^k \to A^{2k-1}$ (see «E»). It turns out (see [8]) that in the neighbourhoods of the points a and $f(a)$ one can change the coordinate systems (generally, the coordinate change is not linear) such that the mapping f in the neighbourhood of a has the following coordinate form:

$$\begin{aligned} &y^1 = (x^1)^2, \quad y^2 = x^1x^2, \dots, \quad y^k = x^1x^k, \\ &y^{k+1} = x^2, \quad y^{k+2} = x^3, \dots, \quad y^{2k-1} = x^k. \end{aligned} \tag{35}$$

Here the points a and $f(a)$ are taken to be the coordinate origins.

Statement «H» is quite a difficult theorem.

By using the expression (35), one can visualize the geometry of the mapping f in the neighbourhood of a, especially in the case when $k = 2$.

Chapter II

Framed manifolds

§ 1. Smooth approximations of continuous mappings and deformations

In the present section, we shall show that while studying the homotopy types of mappings from one manifold to another it is sufficient to consider only smooth mappings and smooth homotopies. This results from the following facts. Let M^k and N^l be two m-smooth closed manifolds. It turns out that in any homotopy class of mappings from N^l to M^k there exists an $(m-1)$-smooth mapping, and if two $(m-1)$-smooth mappings from the manifold N^l to the manifold M^k are homotopic, then there exists an $(m-3)$-smooth homotopy between these mappings. Thus, while studying mappings of smoothness class m, one has to consider the homotopies of class $m-3$. This loss of smoothness class can be avoided by using several tricks, but since the results of this section are to be used only for studying maps from sphere to sphere and the sphere is an analytic manifold, we need not worry about the loss of smoothness class; thus there is no sense in giving a more difficult proofs of a more precise statements.

The structure of neighbourhood of a smooth manifold

The statement given below will be used only for the case of closed manifolds; the proof in this case is much simpler than that in the general case, as seen from the proof itself. In the next section, we shall use the general case.

A) Let E^{n+k} be the Euclidean space with a fixed Cartesian coordinate system $y^1, \dots, y^{n+k}$, and let E_0, E_1 be two hyperplanes of the space E^{n+k} defined by $y^{n+k} = c_0$ and $y^{n+k} = c_1$, where $c_0 < c_1$ and E^{n+k} is the strip of the space E^{n+k} bounded by these hyperplanes, i.e. the set of points satisfying the conditions $c_0 \leqslant y^{n+k} \leqslant c_1$. Furthermore, let M^k be an m-class smooth ($m \geqslant 4$) compact submanifold (in the case of a closed manifold M^k it is sufficient to take $m \geqslant 2$) (see § 1, «F») of the strip E^{n+k}, with boundary M^{k-1}. Denote the total normal subspace at the point z to M^k by N_z. This subspace is an n-dimensional linear subspace of the Euclidean space E^{n+k}. We shall also assume that for any boundary point z of the manifold M^k, this manifold is orthogonal to the boundary of the strip E^{n+k}, i.e. that for $x \in M^{k-1}$ we have

$$N_x \subset E_0 \cup E_1. \tag{1}$$

For the Euclidean space N_z denote the open ball centred at z with radius $\delta > 0$, by $H(z) = H_\delta(z)$ and denote the union of all balls $H_\delta(z)$ over all $z \in P$, where $P \subset M^k$, by $H_\delta(P)$. It turns out that there exists such a small positive number δ that for $z \neq z'$ the balls $H_\delta(z)$ and $H_\delta(z')$ do not intersect each other, whence the set $W_\delta = H_\delta(M_k)$ forms a neighbourhood of the manifold M^k in E^{n+k}. Associating with each point $y \in W_\delta$ the unique point $z \in M^k$ for which $y \in H_\delta(z)$ we obtain a smooth mapping $y \to z = \pi(y)$ from the manifold W_δ to the manifold M^k; in the case when M^k is smooth, this mapping is of class $m-1$.

Let us prove Statement «A». Let $a \in M^k; x^1, \dots, x^k$ be some local coordinates defined in a neighbourhood of the point a taken to be the origin, and let

$$y^j = f^j(x) = f^j(x^1, \dots, x^k), \quad j = 1, \dots, n+k, \tag{2}$$

be the parametric equation defining the manifold M^k in the neighbourhood of a. The functions f^j are defined for those values of $x^1, \dots, x^k$ satisfying

$$\left|x^i\right| < \varepsilon, \quad i = 1, \dots, k, \tag{3}$$

in the case when a is an interior point of M^k; they satisfy (3) and

$$x^1 \leqslant 0 \tag{4}$$

if a is a boundary point of M^k. Thus the functions f^j define the mapping f_a from the open cube K_ε, defined by inequalities (3), or, respectively, from the half-cube K'_ε, defined by (3) and (4). In the case of boundary point a we extend the function f^j to the positive values x^1 by setting

$$\begin{aligned} f^j(x) &= f^j(x^1, \ldots, x^k) \\ &= f^j(0, x^2, \ldots, x^k) + \frac{\partial}{\partial x^1}(0, x^2, \ldots, x^k)x^1 \\ &\quad + \frac{\partial^2}{(\partial x^1)^2} f^j(0, x^2, \ldots, x^k) \cdot (x^1)^2, \quad x^1 \geqslant 0. \end{aligned}$$

The functions f^j defined in this way define a regular homeomorphic mapping f_a from the open cube K_ε (where ε is a small positive number) for any arbitrary point $a \in M^{k-1}$.

The equation of the normal space $N_{f_a(x)} = N_x$ to the manifold $f_a(K_\varepsilon)$ at the point $f_a(x)$ has the following vector form:

$$\left(\frac{\partial f_a(x)}{\partial x^i}, y - f_a(x) \right) = 0, i = 1, \ldots, k. \tag{5}$$

Here y is the vector describing the normal space N_x. We shall consider the system of relations (5) as a system of equations in unknown functions $x^1, \ldots, x^k$ of the independent variables $y^1, \ldots, y^{n+k}$, which are the components of the vector y. For the initial values $y = f_a(0) = a$, the system (5) has the evident solution $x^i = 0$, $i = 1, \ldots, k$. The functional determinant of the system (5) for these values is equal to $(-1)^k \left| \left(\frac{\partial f_a(a)}{\partial x^i}, \frac{\partial f_a(a)}{\partial x^j} \right) \right|$, $i, j = 1, \ldots, k$. This determinant is non-zero, since the mapping f_a is regular at 0. Thus the system (5) is solvable. Let $x = \sigma(y)$, or, in coordinate form,

$$x^i = \sigma^i(y^1, \ldots, y^{n+k}), \quad i = 1, \ldots, k, \tag{6}$$

be its solution defined for all points y belonging to some neighbourhood V_a of the point a in the neighbourhood E^{n+k}. For $y \in V_a$, there exists thus precisely one point $x \in K_\varepsilon$ satisfying the condition $y \in N_x$; this point x is defined as $x = \sigma(y)$. In other words, for each point $y \in V_a$ there passes a unique normal space N_x, where $x \in K_\varepsilon$. From the continuity of the function $\sigma(y)$, it easily follows that there exist small positive numbers δ_a and ε_a that for $\delta \leqslant \delta_a$, $\varepsilon \leqslant \varepsilon_a$ the set $H_\delta(f_a(K_\varepsilon))$ is completely contained in V_a and it is a neighbourhood of the point a in the space E^{n+k}.

Let us show that for a boundary point a there exist some small positive numbers δ'_a and ε'_a that for $\delta \leqslant \delta'_a$ and $\varepsilon \leqslant \varepsilon'_a$ the set $H_\delta(f_a(K'_\varepsilon))$ is a neighbourhood of the point a in the strip E^{n+k}. For definiteness let us assume that $a \in E_1$; then we have

$$\sigma^1(y^1, \dots, y^{n+k-1}, c_1) = 0. \tag{7}$$

It is clear that

$$\frac{\partial \sigma^1(y^1, \dots, y^{n+k-1}, c_1)}{\partial y^{n+k}} > 0. \tag{8}$$

From the above it follows that for any point y sufficiently close to a the sign of the function $\sigma^1(y^1, \dots, y^{n+k})$ coincides with the sign of the number $y^{n+k} - c_1$; this shows that for sufficiently small numbers at δ and ε we have

$$H_\delta(f_a(K'_\varepsilon)) = H_\delta(f_a(K_\varepsilon)) \cap E^{n+k}_*. \tag{9}$$

Since $H_\delta(f_a(K_\varepsilon))$ is a neighbourhood of the point a in the space E^{n+k} then, according to (9), the set $H_\delta(f_a(K'_\varepsilon))$ is a neighbourhood of the point a in the strip E^{n+k}_*.

For an interior point $a \in M^k$, set $\delta'_a = \delta_a$, $\varepsilon'_a = \varepsilon_a$. The set of all domains $U_a = f_a(K_{\varepsilon'_a}) \cap M^k$, $a \in M^k$, covers the manifold M^k. Suppose $U_{a_1}, \dots, U_{a_p}$ is a finite cover of the manifold M^k. There exists a small number $\eta > 0$ such that any two points M^k at distance less than η, are contained in a domain from this cover. Now let δ be the minimum amongst δ'_{a_n}, $n = 1, \dots, p$, and $\eta/2$. Since $H_\delta(M^k) = H_\delta(U_{a_1}) \cup \dots \cup H_\delta(U_{a_p})$ we see that $H_\delta(M^k)$ is a neighbourhood of the manifold M^k in the strip E^{n+k}_*. Furthermore, for two distinct points $z \in M^k$ and $z' \in M^k$, the balls $H_\delta(z)$ and $H_\delta(z')$ do not intersect. Indeed, if $\varrho(z, z') \leqslant 2\delta$ then the points z and z' belong to the same domain U_{a_n}, thus, as shown above, the balls $H_\delta(z)$ and $H_\delta(z')$ cannot intersect. If $\varrho(z, z') > \delta$ then these balls cannot intersect because the distance between their centres is greater than the sum of their radii.

Thus, Statement «A» is completely proved.

Smooth approximations

B) Let $f^1(x)$ be a *continuous* real-valued function defined on a class $m \geqslant 2$ smooth compact manifold M^k and let ε be a positive number. Then there exists a *smooth* real-valued function $g^1(x)$ of class m defined on M^k and satisfying the condition $|g^1(x) - f^1(x)| < \varepsilon$. In other words, a continuous function defined on M^k can be arbitrarily closely approximated by a smooth function.

Let us prove Statement «B». By virtue of Theorem 2 we may assume that the manifold M^k is embedded into the Euclidean space E^l of some high dimension. Let Q be a closed cube containing M^k. According to the well-known Urysohn theorem (see [9]) the function $f^1(x)$ given on M^k can be continuously extended to the whole cube Q. This function, defined on Q can be ε-approximated by a polynomial $g^1(x)$ in the Cartesian coordinates of the point $x \in Q$. The function $g^1(x)$ considered on M^k is a function we are interested in.

Theorem 7. *Let M^k be an m-smooth ($m \geqslant 2$) closed manifold, let N^l be an m-smooth compact manifold and let f be a continuous mapping from the manifold N^l to the manifold M^k. We shall treat M^k as a metric space. It turns out that for any positive ε there exists a mapping h of class $m-1$ from the manifold N^l to the manifold M^k, such that $\varrho(f(x), h(x)) < \varepsilon$, $x \in N^l$. In other words, a continuous mapping from the manifold N^l to M^k can be arbitrarily closely approximated by a smooth one.*

PROOF. By virtue of Theorem 2, we may assume that the manifold M^k is a submanifold of some Euclidean space E^{n+k}. Let δ be a number defined for this submanifold in Statement «A», and $\varepsilon' < \dfrac{\delta}{\sqrt{n+k}}$. Denote the components of the vector $f(x), x \in N^l$ by $f^1(x), \ldots, f^{n+k}(x)$. According to Statement «B», there exists a real-valued m-smooth function $g^i(x)$, $i = 1, \ldots, n+k$, defined on N^l and satisfying the inequality $|f^i(x) - g^i(x)| < \varepsilon$, $i = 1, \ldots, n+k$. Denote the vector with components $g^1(x), \ldots, g^{n+k}(x)$ by $g(x)$. The mapping g of the manifold N^l to E^{n+k} is m-smooth and $g(N^l) \subset W_\delta$ (see «A»). For sufficiently small ε', the mapping $h = \pi g$ (see «A») satisfies the conditions of the theorem.

C) A family of continuous mappings f_t, $0 \leqslant t \leqslant 1$ from a closed manifold N^l to a manifold M^k is called a *continuous family* or a *deformation* of the mapping f_0 to the mapping f_1 if $f_t(x)$ is a continuous function in two variables x, t. Let $N^l \times I$ be the Cartesian product of N^l and the real closed interval $I = [0, 1]$ (see §1, «I»). Set $f_*(x, t) = f_t(x)$. It is clear that the family f_t is continuous if and only if the mapping f_t of the manifold $N \times I$ is continuous. We shall call the family f_t *smooth* of class m (or f_t is an *m-smooth* deformation) if the mapping f_* is m-smooth. If the mappings f_0 and f_t are connected by a smooth deformation, they are called *smoothly homotopic*. It is quite evident that the relation of smooth homotopy is reflexive and symmetric. The transitivity of this relation is not totally obvious and requires a proof. Let us prove it.

Let f_{-1}, f_0, f_1 be three m-smooth mappings from N^l to M^k; let f_t, $-1 \leqslant t \leqslant 0$, be an m-smooth deformation of the mapping f_{-1} to the mapping f_0 and let f_t, $0 \leqslant t \leqslant 1$, be a smooth m-deformation of the mapping f_0 to the mapping f_1. The deformation f_t, $-1 \leqslant t \leqslant 1$, is, clearly, continuous, but at $t = 0$ it might not be smooth, thus it is necessary to reconstruct it to get an m-smooth deformation. Let n be an odd natural number such that $n \geqslant m$. Set $g_t(x) = f_{t^n}(x)$. It is easy to see that g_t, $-1 \leqslant t \leqslant 1$, is an m-smooth deformation of the mapping $g_{-1} = f_{-1}$ to the mapping $g_1 = f_1$. Thus, the transitivity of the m-smooth homotopy relation is proved.

D) Let M^k and N^l be two m-smooth closed manifolds, such that M^k is a metric space. Then there exists a small number ε such that if f_0 and f_1 are two m-smooth mappings from N^l to M^k with distance less than ε, i.e. satisfying the condition $\varrho(f_0(x), f_1(x)) < \varepsilon, x \in N^l$, then there exists an $(m-1)$-smooth deformation of the mapping f_0 to the mapping f_1.

Let us prove Statement «D». By virtue of Theorem 2, one may assume that M^k is a submanifold of the Euclidean space E^{n+k} of some high dimension. Let δ be a number defined for $M^k \subset E^{n+k}$ in Statement «A». We shall assume that the metrics in the manifold M^k is induced by $M^k \subset E^{n+k}$; choose ε to be so small that for $\varrho(x, x') < \varepsilon$ the interval connecting the points x and x' lies in W_δ. Set

$$f_t(x) = \pi(f_0(x)(1-t) + f_1(x)t).$$

It is evident that f_t, $0 \leqslant t \leqslant 1$, is an $(m-1)$-smooth $m-1$ deformation from the mapping f_0 to the mapping f_1 (see «A»).

Theorem 8. *Let f_t, $0 \leqslant t \leqslant 1$ be a continuous deformation of the closed manifold N^l to the closed manifold M^k such that the mappings f_0 and f_1 are m-smooth. Then there exists an $(m-2)$-smooth deformation from f_0 to f_1. In other words, if two smooth mappings can be connected by a continuous deformation then they can be connected by a smooth deformation.*

Proof. With the continuous deformation f_t, associate (see «C») the continuous mapping f_* from the manifold $N^l \times I$ to M^k. By virtue of Theorem 7, the continuous mapping f_* can be ε-approximated by an $(m-1)$-smooth mapping g_* from $N^l \times I$ to the manifold M^k. With the mapping g_*, one associates a smooth deformation g_t, $0 \leqslant t \leqslant 1$ of mappings from N^l to M^k. For ε sufficiently small, the mappings f_i and g_i, $i = 0, 1$, are close to each other, thus there exists an $m-2$-homotopy between them (see «D»). From the transitivity of smooth homotopy relation, it follows that there is an $(m-2)$-smooth homotopy between f_0 and f_1.

This completes the proof of Theorem 8.

§ 2. The basic method

In this section with each mapping from the $(n+k)$-dimensional sphere Σ^{n+k} to the n-dimensional sphere S^n we associate a *smoothly framed* submanifold M^k of the Euclidean space E^{n+k}. The framing of the manifold M^k means that at each point x we define a system $U(x) = \{u_1(x), \dots, u_n(x)\}$ of linearly independent vectors orthogonal to M^k, whence the vector $u_i(x)$ depends smoothly on $x \in M^k$. The framing is called *smooth* if the vectors $u_i(x)$ depend smoothly on x. The manifold M^k together with its framing U is called a *framed manifold* and is denoted by (M^k, U). It turns out that each smoothly framed manifold (M^k, U) corresponds to some mapping from the sphere Σ^{n+k} to the sphere S^n and that the mappings representing the same smoothly framed manifolds are homotopic. The smooth manifolds corresponding to two homotopic smooth mappings might not coincide and even not be homotopic. This leads to the definition of *homological equivalence* of two framed manifolds (M_0^k, U_0) and (M_1^k, U_1) located in the Euclidean space E^{n+k}. The two manifolds (M_0^k, U_0) and (M_1^k, U_1) are homological if in the Cartesian product $E^{n+k} \times I$ of the space E^{n+k} and the interval $I = [0, 1]$ there exists a compact framed submanifold (M^{k+1}, U) whose boundary consists of $M_0^k \times 0$ and $M_1^k \times 1$ and the framing U of that manifold restricted to the boundary coincides with the framings $U_0 \times 0$ and $U_1 \times 1$ defined on $M_0^k \times 0$ and $M_1^k \times 1$. It turns out that two mappings from the sphere Σ^{n+k} to the sphere S^n are homotopic if and only if the corresponding smoothly framed manifolds are homological (the framing generating this homological equivalence is not assumed to be smooth). Thus, the homotopy classification problem of mapping from a sphere to a sphere is reduced to the homological classification of smoothly framed manifolds. One should admit that the question of homology classification of framed manifolds is not simple.

Framed manifolds

Definition 3. Let E^{n+k} be the Euclidean spaces with Cartesian coordinates $y^1, \dots, y^{n+k}$ and let E_0 and E_1 be two hyperplanes of the space E^{n+k} defined by the equations $y^{n+k} = c_0$ and $y^{n+k} = c_1$, $c_0 < c_1$; let E_*^{n+k} be the strip consisting of all points of the space E^{n+k} for which $c_0 \leqslant y^{n+k} \leqslant c_1$. Furthermore, let M^k be an m-smooth compact submanifold (see § 1, «F») of the strip E_*^{n+k} with boundary M^{k-1}. If the manifold M^k is closed then the hyperplanes E_0 and E_1 play no role and we assume that $E_*^{n+k} = E^{n+k}$. Consider the total normal space N_x at the point $x \in M^k$ as a vector space having origin at x; suppose that

$$N_x \subset E_0 \cup E_1, \quad x \in M^{k-1},$$

i.e. that the manifold M^k is orthogonal at its boundary points to the boundary of the strip E_*^{n+k} (cf. § 1, Ch.2, «A»). Thus, lying inside the boundary

E_*^{n+k} manifold M^k is *framed*, if in any vector space N_x a basis

$$u_1(x), \dots, u_n(x),$$

is fixed, whence the vector $u_i(x)$, viewed as a vector of E^{n+k}, is a continuous function of $x \in M^k$. The system $U(x) = \{u_1(x), \dots, u_n(x)\}$ is to be called the *framing* of the manifold M^k, and the manifold M^k together with $U(x)$ will be denoted by $(M^k, U(x))$ and called a *framed manifold*. A framing $U(x)$ is called *orthonormal* if for any point $x \in M^k$ the basis $U(x)$ is orthonormal. The framing $U(x)$ is called class m *smooth* if each vector $u_i(x)$ is a class m smooth function of the point $x \in M^k$.

One should point out that any framed manifold is oriented and it inherits the *natural orientation* if the ambient Euclidean space E^{n+k} is oriented. Indeed, let $e_1, \dots, e_k$ be a linearly independent vector system tangent to the manifold M^k at some point x. We say that the system $e_1, \dots, e_k$ defines the *natural orientation* of the manifold M^k if the system $e_1, \dots, e_k, u_1(x), \dots, u_n(x)$ corresponds to the positive orientation of the space E^{n+k}.

The definitions given below are devoted to the notion of homological equivalence between two k-dimensional framed submanifolds of the Euclidean space E^{n+k}.

Definition 4. Let (M_0^k, U_0) and (M_1^k, U_1) be two smooth framed submanifolds of the Euclidean space E^{n+k}. Let $E^{n+k+1} = E^{n+k} \times E^1$, where E^1 is the real line with variable t. Set $E_t = E^{n+k} \times t$, $t = 0, 1$ and denote by E_*^{n+k+1} the strip of the space E^{n+k+1} bounded by hyperplanes E_0 and E_1. The framed manifolds (M_0^k, U_0) and (M_1^k, U_1) are *homological* if there exists a framed submanifold (M^{k+1}, U) of the strip E_*^{n+k+1} such that

$$M^{k+1} \cap E_0 = M_0^k \times 0,$$
$$M^{k+1} \cap E_1 = M_1^k \times 1,$$

whence the framing U coincides on $M_t^k \times t$ with the framing $U_t \times t$, $t = 0, 1$. A framed manifold (M_0^k, U_0) is called *null-homologous* if it is homological to the framed manifold (M_1^k, U_1) where M_1^k is empty. In this case the framed manifold (M^{k+1}, U) representing the homology has boundary M^k. It turns out that the homological equivalence is reflexive, symmetric and transitive; thus the set of all k-dimensional framed submanifolds of the Euclidean space E^{n+k} is split into classes of homological ones.

It is obvious that this relation is reflexive and symmetric. Let us show that it is transitive. Let $(M_{-1}^k, U_{-1}), (M_0^k, U_0)$ and (M_1^k, U_1) be three framed manifolds of the Euclidean space E^{n+k} for which the following relations hold:

$$(M^k_{-1}, U_{-1}) \sim (M^k_0, U_0),$$
$$(M^k_0, U_0) \sim (M^k_1, U_1).$$

Now, let $E^{n+k+1} = E^{n+k} \times E^1$ be the Cartesian product of the Euclidean space E^{n+k} and the real line E^1 with variable t, with two strips E_{*i}, $i-1 \leqslant t \leqslant i$, $i = 0, 1$ selected. Set $E_* = E_{*0} \cup E_{*1}$. We shall assume that the homology $(M^k_{i-1}, U_{i-1}) \sim (M^k_i, U_i)$ is realized in the strip E_{*i} by a manifold (M^{k+1}_i, U_{*i}), $i = 0, 1$. Now let m be a sufficiently large odd natural number. Define the mapping ψ from the strip E_* onto itself by setting $\psi(x,t) = (x, \sqrt[m]{t})$, $x \in E^{n+k}$. The mapping ψ from the strip E_* to itself, is clearly, homeomorphic. It is regular in all points (x,t), where $t \neq 0$. It is easy to check that $M^{k+1} = \psi(M^{k+1}_0 \cup M^{k+1}_1)$ is a smooth submanifold of the frame E_*. Denote the vector system $U_{*i}(x,t)$ by $U_*(x,t)$. Let N'_{xt} be the normal subspace to the manifold $M^{k+1}_0 \cup M^{k+1}_1$ at (x,t), $-1 \leqslant t \leqslant 1$ and let N_{xt} be the normal subspace to the manifold M^{k+1} at the point $\psi(x,t)$. It can be easily checked that the orthogonal projection of the space N'_{xt} to the space N_{xt} is non-degenerate. Thus, defining $U(x,t)$ to be the orthogonal projection of the system $U_*(x,t)$ to N_{xt}, we get a framed manifold (M^{k+1}, U) providing the homology $(M^k_{-1}, U_{-1}) \sim (M^k_1, U_1)$ in the strip E_*. Thus the transitivity of the homological equivalence is proved.

From mappings to framed manifolds

A) Let E^{r+1} be a Euclidean vector space. The sphere S^r of dimension r and radius $1/2$ is defined in E^{r+1} by

$$(x,x) = \frac{1}{4}.$$

Let p and q be two antipodal points of the sphere S^r. Call the first one the *north pole*, and call the second one the *south pole*. Furthermore, let T_p and T_q be the tangent spaces to the sphere S^r at p and q, respectively, and let $e_1, \ldots, e_r$ be an orthonormal basis of the space T_p generating the positive orientation of S^r. We obtain the corresponding basis for T_q by parallel transport of the vectors $e_1, \ldots, e_r$ from p to q. These bases define some coordinate systems for T_p and T_q. Now, let us introduce coordinates in $S^r \backslash q$, $S^r \backslash p$ by using $(p; e_1, \ldots, e_r)$. To do that, define by $\psi(x)$ the central projection of the point $x \in S^r \backslash q$ from the centre q to the space T_p and take coordinates $x^1, \ldots, x^r$ of the point $\psi(x)$ in T_p to be the coordinates of the point x in $S^q \backslash q$. In the same way, by using central projection from p to T_q, we define the coordinates $y^1, \ldots, y^r$ of the point $x \in S^r \backslash p$. It is easy to see that for $x \in S^r \backslash (p \cup q)$ we have

$$y^i = \frac{x^i}{(x^1)^2 + \ldots + (x^r)^2}, \tag{1}$$

$$x^i = \frac{y^i}{(y^1)^2 + \ldots + (y^r)^2}. \tag{2}$$

Thus S^r is an analytic manifold.

To each smooth mapping from the $(n+k)$-dimensional sphere Σ^{n+k} to the n-dimensional sphere S^n associate a certain closed framed manifold (M^k, U) of dimension k situated in the Euclidean space E^{n+k} of dimension $n+k$.

Definition 5. Let f be a smooth mapping from the smooth oriented sphere Σ^{n+k} to the smooth oriented sphere S^n. Fix the north pole p' of the sphere Σ^{n+k}; denote its south pole by q'; denote the tangent space at p' by E^{n+k} and denote the central projection of the domain $\Sigma^{n+k} \backslash q'$ to E^{n+k} from the point q' by φ. Define the north pole p of the sphere S^n to be an arbitrary (but fixed) proper point of the mapping f distinct from $f(q')$ (see Theorem 4). Let $e_1, \ldots, e_n$ be a certain orthonormal vector system tangent to S^n at the point p and defining the orientation of the sphere S^n. Denote the tangent space to the sphere S^n at p by T_p. Since p is a proper point of the mapping f, the set $f^{-1}(p)$ is a smooth k-dimensional submanifold of Σ^{n+k} (see § 1, «F»). Since, moreover, the set $f^{-1}(p)$ does not contain q', $M^k = \varphi f^{-1}(p)$ is a smooth closed submanifold of the Euclidean space E^{n+k}. The mapping $f\varphi^{-1}$ from the manifold E^{n+k} to the manifold S^n is proper at any point $x \in M^k$. Denote the tangent space at x to the manifold E^{n+k} by E^{n+k}_x (see § 1, «C»). Since the manifold E^{n+k} is the Euclidean space, the space E^{n+k}_x can be identified with the space E^{n+k}, taking point x to be the coordinate origin. Denote the total normal subspace and the total tangent subspace to the manifold M^k at the point x by N_x and T_x, respectively. Denote the linear mapping from the vector space E^{n+k}_x to the vector space T_p corresponding to the mapping $f\varphi^{-1}$ by f_x (see § 1, «E»). Since the mapping $f\varphi^{-1}$ is proper at the point x, $f_x(E^{n+k}) = T_p$, and, since $f\varphi^{-1}(M^k) = p$, we have $f_x(T_x) = p$. It thus follows that the mapping f_x from the vector space N_x to the vector space T_p is a non-degenerate mapping onto T_p. Denote the pre-image of the vector e_i in the space N_x under f_x by $u_i(x)$. The system $U(x) = \{u_1(x), \ldots, u_n(x)\}$, $x \in M^k$, generates a smooth framing of the manifold M^k. We associate the framed manifold (M^k, U) with the mapping f : $f \to (M^k, U)$. The correspondence $f \to (M^k, U)$ depends on the arbitrary choice of the system $p, e_1, \ldots, e_n$; thus, more completely $f \to (M^k, U)$ should be written as

$$(f; p, e_1, \dots, e_n) \to (M^k, U).$$

The pole p' of the sphere Σ^{n+k} is fixed, i.e. it will remain the same for all mappings from the sphere Σ^{n+k} to the sphere S^n. The pole p of the sphere S^n should be a proper point of the mapping f distinct from $f(q')$, thus it cannot be fixed.

The theorem given below showing that for homotopic mappings we get homological manifolds and particularly shows the independence of the arbitrary choice of the system $p, e_1, \dots, e_n$.

It will be shown later (see Theorem 10) that from the homological equivalence of framed manifolds we get the homotopy of the corresponding mappings.

Theorem 9. *Let f_0 and f_1 be two smooth mappings from the oriented sphere Σ^{n+k} to the oriented sphere S^n $(n \geqslant 2, k \geqslant 0)$ and let*

$$(f_0; p_0, e_{10}, \dots, e_{n0}) \to (M_0^k, U_0),$$

$$(f_1; p_1, e_{11}, \dots, e_{n1}) \to (M_1^k, U_1)$$

(*see Definition* 5). *It turns out that if for $n \geqslant 2$ the mappings f_0 and f_1 are homotopic then the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) are homologous.*

Proof. Since the orientations of the sphere S^n defined by the tangent systems $e_{10}, \dots, e_{n0}$ and $e_{11}, \dots, e_{n1}$ coincide, there exists an isometric mapping ϑ of the sphere S^n onto itself that can be realized by a continuous twisting and thus is homotopic to the identity; moreover, such a mapping maps $p_0, e_{10}, \dots, e_{n0}$ to the system $p_1, e_{11}, \dots, e_{n1}$. Set $g_0 = f_0$, $g_1 = \vartheta^{-1} f_1$. Since the mapping ϑ is homotopic to the identity the mappings g_0 and g_1 are homotopic. Moreover, it is easy to see that

$$(g_0; p, e_1, \dots, e_n) \to (M_0^k, U_0),$$

$$(g_1; p, e_1, \dots, e_p) \to (M_1^k, U_1),$$

where

$$(p, e_1, \dots, e_n) = (p_0, e_{10}, \dots, e_{n0}).$$

Since the smooth mappings g_0 and g_1 are homotopic then there exists a smooth homotopy g_t connecting them (see Theorem 8); for this deformation there corresponds a smooth mapping g_* from the manifold $\Sigma^{n+k} \times I$ to S^n (see § 1, Chapter 2, «C»). Define the mapping φ_* from $(\sigma^{n+k} \backslash q') \times I$ to the Cartesian product $E^{n+k} \times I$ by setting

$$\varphi_*(x, t) = (\varphi(x), t). \tag{3}$$

Let us consider the product $E^{n+k} \times I$ as the strip E_*^{n+k+1} in the space $E^{n+k+1} = E^{n+k} \times E^1$, where E^1 is the real line. Let us make the following assumption about g_*:

a) The point p is a proper point of the mapping g_* from the manifold $\Sigma^{n+k} \times I$ and it does not belong to $g_*(q' \times I)$.

It follows from «a» that the set $M^{k+1} = \varphi_* g_*^{-1}(p)$ is a smooth compact submanifold of the strip E_*^{n+k+1}. Denote by N_x the normal subspace in the space E^{n+k+1} to the manifold M^{k+1} at the point $x \in M^{k+1}$. Since the mapping $g_* \varphi_*^{-1}$ is proper at x then it is regular at x on N_x; thus, to the system of vectors $e_1, \ldots, e_n$ one naturally associates in N_x the system of vectors $U(x) = \{u_1(x), \ldots, u_n(x)\}$ (Cf. Definition 5). Let us make one more assumption about g_*.

b) The manifold M^{k+1} is orthogonal on its boundary to the boundary of the strip E_*^{n+k+1} (see Definition 3).

It follows from the assumption «b» that $U(x)$ is a framing of the manifold M^{k+1}; it is easy to see that the framed manifold (M^{k+1}, U) provides homology between the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) (Cf. Definition 4). Thus, to prove the theorem it suffices to construct such a smooth homotopy g_t connecting g_0 and g_1 for which the assumptions «a» and «b» hold. Let us do that.

Let h_t be an arbitrary smooth homotopy connecting g_0 and g_1. Let us correct it to make the assumption «a» hold. It is assumed that p is a proper point of the mappings g_0 and g_1 and that it does not coincide with the points $g_0(q')$ and $g_1(q')$. From this, it follows that there exists a positive ε satisfying the following conditions. For $p_* \in S^n$, $\varrho(p, p_*) < \varepsilon$ and $t \leqslant \varepsilon$ or $t \geqslant 1 - \varepsilon$ the point p_* is proper for both h_t and $\varrho(h_t(q'), p) > \varepsilon$. Let us fix ε satisfying the conditions above. Let p_* be a proper point for h_*, not belonging to $h_*(q' \times I)$ and satisfying the condition $\varrho(p, p_*) < \varepsilon$. By virtue of Theorems 4 and 1 such a point p_* does exist. We shall assume that the sphere S^n is situated in the Euclidean vector space E^{n+1}; let E^{n-1} be the linear subspace of the space E^{n+1} orthogonal to the vectors p and p_*. Denote the α-twisting of the sphere S^n around the axis E^{n-1} by ϑ_α; let $\vartheta_\beta(p) = p_*$, $0 < \beta < \pi$. Let $\chi(t)$ be a smooth real-valued curve parametrized by t defined on the interval $0 \leqslant t \leqslant 1$ and satisfying the following conditions:

$$0 \leqslant \chi(t) \leqslant 1, \quad \chi(0) = \chi(1) = 0;$$

$$\chi(t) = 1 \text{ for } \varepsilon \leqslant t \leqslant 1 - \varepsilon.$$

Set $\eta_t = \vartheta_{\beta\chi(t)}$. The twisting η_t of the sphere S^n around E^{n-1} depending on t defined as above moves p to p_* as t changes from 0 to ε, and further, returns the point p to the initial position as t changes from $1-\varepsilon$ to 1. Now, let us define the family of mappings l_t by setting

$$l_t = (\eta_t)^{-1} h_t.$$

This family is smooth and it connects g_0 and g_1. It turns out that for $g_t = l_t$ the assumption «a» holds. For $0 \leqslant t \leqslant \varepsilon$ or $1-\varepsilon \leqslant t \leqslant 1$ we have $l_t(g') \neq p$. For $\varepsilon \leqslant t \leqslant 1-\varepsilon$ the set $l_t^{-1}(p)$ coincides with the set $h_t^{-1}(p_*)$; thus $l_t(q')$ does not coincide with p in this case either. Let us prove that p is a proper point of the mapping l_*. Let $(a, t_0) \in l_*^{-1}(p)$ and let $x^1, \dots, x^{n+k}$ be the local coordinates in the neighbourhood of the point a. For the point (a, t_0) to be proper for the mapping l_*, it is necessary and sufficient that amongst the vectors

$$\frac{\partial \varphi l_*(a,t_0)}{\partial x^1}, \dots, \frac{\partial \varphi l_*(a,t_0)}{\partial x^{n+k}}, \frac{\partial \varphi l_*(a,t_0)}{\partial t}$$

there are n linearly independent ones. For $0 \leqslant t_0 \leqslant \varepsilon$ or $1-\varepsilon \leqslant t_0 \leqslant 1$ there are n such vectors even amongst the first $n+k$ ones because of the choice of ε. For $\varepsilon \leqslant t \leqslant 1-\varepsilon$, amongst the $n+k+1$ vectors in question there are n linearly independent ones because of the choice of the point p_*. Thus, for $g_t = l_t$ the assumption «a» holds.

In order to realize the condition «b», let us construct an integer-valued function $s(t)$ of the parameter t, $0 \leqslant t \leqslant 1$, satisfying the conditions:

$$s(t) = 0 \quad \text{for} \quad 0 \leqslant t \leqslant 1/3,$$

$$s(t) = 1 \quad \text{for} \quad 2/3 \leqslant t \leqslant 1,$$

$$\frac{ds}{dt} > 0 \quad \text{for} \quad 1/3 < t < 2/3,$$

and set

$$g_t = l_{s(t)}.$$

First of all, show that for the homotopy g_t the condition «a» holds as well. Since $l_{s(t)}(q') \neq p$, $g_t(q') \neq p$. Now let (a, t_0) be an arbitrary point of the set $g_*^{-1}(p) \subset \Sigma^{n+k} \times I$ and $x^1, \dots, x^{n+k}$ be the coordinates in the neighbourhood of the point a in σ^{n+k}. It follows from $(a, t_0) \in g_*^{-1}(p)$ that $(a, s(t_0)) \in l_*^{-1}(p)$. For the mapping g_* to be proper at (a, t_0), it is necessary and sufficient to have among vectors

$$\frac{\partial \varphi g_*(a,t_0)}{\partial x^1}, \dots, \frac{\partial \varphi g_*(a,t_0)}{\partial x^{n+k}}, \frac{\partial \varphi g_*(a,t_0)}{\partial t}$$

n linearly independent ones. For the point $(a, s(t_0))$ to be proper with respect to l_*, it is necessary and sufficient that among the vectors

$$\frac{\partial \varphi l_*(a, s(t_0))}{\partial x^1}, \ldots, \frac{\partial \varphi l_*(a, s(t_0))}{\partial x^{n+k}}, \frac{\partial \varphi l_*(a, s(t_0))}{\partial s}$$

there are n linearly independent ones. For $1/3 \leqslant t_0 \leqslant 2/3$ we have $\frac{\partial \varphi g_*(a, t_0)}{\partial t} = \frac{\partial \varphi l_*(a, s(t_0))}{\partial s} \cdot \frac{ds(t_0)}{dt}$, for $\frac{ds(t_0)}{dt} > 0$; thus, because the point $(a, s(t_0))$ is proper with respect to the l_*, it follows that the point (a, t_0) is a proper point of the mapping g_*. For $0 \leqslant t \leqslant 1/3$ or $2/3 \leqslant t \leqslant 1$ the point a belongs to $l_0^{-1}(p)$ or $l_1^{-1}(p)$, respectively; thus, even amongst the vectors

$$\frac{\partial \varphi g_*(a, t_0)}{\partial x^1}, \ldots, \frac{\partial \varphi g_*(a, t_0)}{\partial x^{n+k}}$$

there are n linearly independent ones. Thus, for the mapping g_t the assumption «a» holds.

Since for $0 \leqslant t \leqslant 1/3$ or $2/3 \leqslant t \leqslant 1$ we have $g_t = g_0$ or, respectively, $g_t = g_1$ then the orthogonality of the manifold M^{k+1} to the boundary of the strip E_*^{n+k+1} is evident.

Thus, Theorem 9 is proved.

Theorem 9 is proved here only for the case $n \geqslant 2$; it is not difficult to prove it for $n = 1$. However, in this case we have no interest of it since the classification of mappings from the sphere Σ^{k+1} to the sphere S^1 is quite elementary (see Theorem 12 concerning the case $k = 0$ and Theorem 18 concerning the case $k > 0$).

From framed manifolds to mappings

B) Let N^r be a vector space with a fixed basis $u_1, \ldots, u_r$. Denote by K'_α the domain of the space N^r generated by vectors $\xi = \xi^1 u_1 + \ldots + \xi^r u_r$ satisfying the inequality $(\xi^1)^2 + \ldots + (\xi^r)^2 < \alpha^2$. Define the mapping λ_α from the space N^r to the sphere S^r by taking any point $\xi \in K'_\alpha$ to the point S^r with coordinates

$$x^i = \frac{\xi^i \alpha^{2m}}{[\alpha^2 - (\xi^1)^2 - \ldots - (\xi^r)^2]^m}$$

(see «A») and taking the whole set $N^r \backslash K'_\alpha$ to the point $q \in S^r$. It follows directly from (1) and (2) that the mapping λ_α is m-smooth. Furthermore, it is checked straightforwardly that the functional matrix of the mapping

λ_α at $\xi = (0, \ldots, 0)$ is the identity matrix. Now, let N^r be the space T_p with basis $e_1, \ldots, e_r$ and set $\omega_\alpha(x) = \lambda_\alpha \varphi(x)$, $x \in S^r \backslash q$ and $\omega_\alpha(q) = q$. The m-smooth mapping ω_α of the sphere S^r to itself obtained in this way is homotopic to the identity mapping. This mapping is homeomorphic, it maps the ball neighbourhood $K_\alpha = \varphi^{-1}(K'_\alpha)$ onto $S^r \backslash q$ and maps the whole set $S^r \backslash K_\alpha$ to the point q.

Theorem 10. *Let Σ^{n+k} and S^n be two oriented spheres and let p' be a fixed point of Σ^{n+k} and E^{n+k} be the space tangent to Σ^{n+k} at the point p'. Furthermore, let p_0 and p_1 be two points of the sphere S^n and let $e_{10}, \ldots, e_{n0}; e_{11}, \ldots, e_{n1}$ be the orthonormal vector systems tangent to S^n at p_0 and p_1, respectively. Let (M_0^k, U_0) and (M_1^k, U_1) be some two homologous smoothly framed manifolds in E^{n+k}. It turns out that there exists a mapping g_0 from the sphere Σ^{n+k} to the sphere S^n such that*

$$(g_0; p_0, e_{10}, \ldots, e_{n0}) \to (M_0^k, U_0).$$

Moreover, it turns out that if f_0 and f_1 are two mappings from Σ^{n+k} to the sphere S^n such that

$$(f_0; p_0, e_{10}, \ldots, e_{n0}) \to (M_0^k, U_0),$$

$$(f_1; p_1, e_{11}, \ldots, e_{n1}) \to (M_1^k, U_1),$$

then the mappings f_0 and f_1 are homotopic.

Proof. Since the tangent vector systems $e_{10}, \ldots, e_{n0}$ and $e_{11}, \ldots, e_{n1}$ define the same orientation of the sphere S^n, there exists an isometric mapping ϑ from the sphere S^n onto itself obtained from the identity mapping by a continuous twisting, such that the system $p_0, e_{10}, \ldots, e_{n0}$ is transformed to the system $p_1, e_{11}, \ldots, e_{n1}$. The mappings f_1 and $\vartheta^{-1} f_1$ are homotopic and

$$(\vartheta^{-1} f_1; p_0, e_{10}, \ldots, e_{n0}) \to (M_1^k, U_1).$$

Thus, to prove the second part of the theorem, it is sufficient to consider only the case when

$$(p_0, e_{10}, \ldots, e_{n0}) = (p_1, e_{11}, \ldots, e_{n1}),$$

i.e. we have to show that the relations

$$(f_0; p, e_1, \ldots, e_n) \to (M^k, U), \tag{4}$$

$$(f_1; p, e_1, \ldots, e_n) \to (M^k, U) \tag{5}$$

imply the homotopy of the mappings f_0 and f_1.

First of all, let us show that if

$$(M_0^k, U_0) = (M_1^k, U_1) = (M^k, U), \tag{6}$$

then the mappings f_0 and f_1 are homotopic.

Let N_a be the normal subspace at the point a to the manifold M^k in the Euclidean space E^{n+k} and let $\eta^1, \ldots, \eta^n$ be the components of the vector $\eta \in N_a$ in the basis $u_1(a), \ldots, u_n(a)$ if the space N_α. For the neighbourhood $S^n \backslash q$ of the point p in the sphere S^n, let us introduce the coordinates $x^1, \ldots, x^n$ induced by the north pole p and the orthonormal system $e_1, \ldots, e_n$ given at p (see «A»). It follows from (4)–(6) that the coordinate form of the mappings f_0 and f_1 from N_a to S^n near the point a looks like

$$x^i = \eta^i + \ldots, \quad i = 1, \ldots, n,$$

$$x^i = \eta^i + \ldots, \quad i = 1, \ldots, n,$$

where only the first-order terms are given, and the higher-order terms are omitted. Thus, the mappings f_0 and f_1 from the space N_a to S^n near the point a coincide up to second-order terms. This implies that for $\eta \in W_\delta$, where δ is small enough (see § 1, Chapter 2, «A»), the geodesic interval $(f_0\varphi^{-1}(\eta), f_1\varphi^{-1}(\eta))$ on the sphere S^n connecting $f_0\varphi^{-1}(\eta)$ to $f_1\varphi^{-1}(\eta)$ does not pass through the point p. Set $W'_\delta = \varphi^{-1}(W_\delta)$. Since the domain W'_δ contains the set $f_0^{-1}(p) = f_1^{-1}(p) = \varphi^{-1}(M^k)$, the closed sets $f_0(S^n \backslash W'_\delta)$ and $f_1(S^n \backslash W'_\delta)$ do not contain the point p. For $\xi \in W'_\delta$, set $\sigma(\xi) = \varrho(\varphi(\xi), \pi\varphi(\xi))$. Now, let us make the point $f_0(\xi), \xi \in W'_\delta$, move uniformly along the geodesic interval $(f_0(\xi), f_1(\xi))$ in such a way that it passes this interval in the unit period of time. Denote the position of the moving point at the moment t, $0 \leqslant t \leqslant 1$, by $h(\xi, t)$. Let $\chi(\sigma)$ be a real-valued function of σ, defined on the interval $0 \leqslant \sigma \leqslant \delta$ and satisfying the following conditions:

$$\chi(\sigma) = 1 \ \text{ for } 0 \leqslant \sigma \leqslant \tfrac{1}{2}\delta, \quad \chi(\delta) = 0,$$

$$0 \leqslant \chi(\sigma) \leqslant 1 \ \text{ for } \ 0 \leqslant \sigma \leqslant \delta.$$

Set

$$h_t(\xi) = h(\xi, t_\chi(\sigma(\xi))) \ \text{ for } \ \xi \in W'_\delta,$$

$$h_t(\xi) = f_0(\xi) \ \text{ for } \ \xi \in S^n \backslash W'_\delta.$$

The family of mappings h_t, $0 \leqslant t \leqslant 1$, provides a continuous deformation from the mapping $f_0 = h_0$ to the mapping h_1. Here the mapping h_1

possesses the following properties. There exists a small ball neighbourhood K_α of the point p in the sphere S^n such that

$$h_1^{-1}(K_\alpha) = f_1^{-1}(K_\alpha) = V \subset W'_\delta,$$

and for $\xi \in V$ we have

$$h_1(\xi) = f_1(\xi). \tag{7}$$

Now, it is easy to show that the mappings f_0 and f_1 are homotopic. Indeed, it follows from (7) that the mappings $\omega_\alpha h_1$ and $\omega_\alpha f_1$ (see «B») coincide. Since the mapping ω_α is homotopic to the identity the mappings h_1 and f_1 are homotopic; thus so are f_0 and f_1.

Thus, it is proved that if (6) holds, then (4) and (5) yield the homotopy of the mappings f_0 and f_1.

Now, let us show that if the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) corresponding to the mappings f_0 and f_1, are null-homologous then these mappings are homotopic. Let (M^{k+1}, U) be a framed submanifold of the strip $E^{n+k} \times I \subset E^{n+k} \times E^1 = E^{n+k+1}$ providing a homotopy between the framed manifolds (M_0^k, U_0) and (M_1^k, U_1) (see Definition 4). Denote the normal subspace $a \in M^{k+1}$ to the manifold M^{k+1} in the space E^{n+k+1} by N_a and let W_δ be the neighbourhood of the manifold M^{k+1} in the strip $E^{n+k} \times I$ constructed as in § 1, Chapter 2, «A». In the vector space N_a we have a basis $U(a)$. Let us choose a positive number α in such a way that for any arbitrary point $a \in M^{k+1}$ the inclusion $\bar{K}_\alpha \subset W_\delta$ holds (see «B»). Now, define the mapping g_* from the manifold $\Sigma^{n+k} \times I$ to the sphere S^n by setting

$$g_*(\xi) = \lambda_\alpha(\varphi_*(\xi)) \;\text{ for }\; \varphi_*(\xi) \in H_\delta(a) \quad \text{(see § 1, Chapter 2, «A»)},$$

$$g_*(\xi) = q \qquad\qquad \text{for }\; \varphi_*(\xi) \notin W_\delta(a) \quad \text{[see (3)]}.$$

For the mapping g_* from the manifold $\Sigma^{n+k} \times I$ to the sphere S^n there corresponds a deformation g_t of mappings from the sphere Σ^{n+k} to the sphere S^n. It follows from the properties of the mapping λ_α (see «B») that the framed manifolds corresponding to the mappings g_0 and g_1 coincide with the given framed manifolds (M_0^k, U_0) and (M_1^k, U_1). Since the mappings f_0 and g_0 have the same corresponding framed manifold (M_0^k, U_0) then the mappings f_0 and g_0, are homotopic, as shown above. Reasoning as above, the mappings f_1 and g_1 are homotopic as well. Since the mappings g_0 and g_1 are connected by a homotopy g_t then they are homotopic as well. From that and from the transitivity of homotopy it follows that f_0 is homotopic to f_1.

Thus, the second part of the theorem is proved. The proof of the first part is contained in the last construction. Let us present this proof. We are given a framed manifold (M^k, U). Denote the normal subspace at the point $a \in M^k$ by N_a. In the vector space N_a we have a basis $U(a)$. Define a positive number α in such a way that for any point $a \in M^k$ the inclusion $\bar{K}_\alpha \subset W_\delta$ holds (see § 1, Chapter 2, «A»). Define the mapping g from the sphere Σ^{n+k} to the sphere S^n by the following relations:

$$g(\xi) = \lambda_\alpha(\varphi(\xi)) \quad \text{for} \quad \varphi(\xi) \in H_\delta(a),$$

$$g(\xi) = q \quad \text{for} \quad \varphi(\xi) \notin W_\delta(a).$$

It follows straightforwardly from the properties of λ_α that the framed manifold corresponding to g is (M^k, U). Thus, the first part of the theorem is proved.

Theorem 10 is completely proved.

It is easy to show that each framed submanifold (M^k, U) of the Euclidean space E^{k+1} is null-homologous. Thus, for $n = 1$, Theorems 9 and 10 are not interesting.

§ 3. Homology group of framed manifolds

In this section we first define the notion of deformation for framed manifolds. If the manifold in question is smooth, has no intersections and its framing varies continuously together with it, one says that one has a smooth deformation of the framed manifold. It can be easily shown that two framed manifolds obtained from each other by a deformation are homologous. Later on, we introduce the sum operation for the homology classes of framed manifolds in the given Euclidean space, so that the set of these classes is naturally endowed with a commutative group structure. If π_1 and π_2 are two homology classes and $(M_1^k, U_1) \in \pi_1$ and $(M_2^k, U_2) \in \pi_2$, then the sum $\pi_1 + \pi_2$ is defined as the class containing the union of these two framed manifolds. It is necessary here that the manifolds M_1^k and M_2^k do not intersect and that they are unknotted; the knottedness is possible if the dimension of the ambient Euclidean space is strictly less than $2k + 2$. The unknottedness means that the manifolds M_1^k and M_2^k can be moved away from each other by a deformation of each of them. To satisfy these conditions, it is assumed that the manifolds M_1^k and M_2^k lie on different sides of some hyperplane.

Homotopy of framed manifolds

A) Let E^r be a Euclidean space, let X be some compact metric space and let $N_{x,t}^n$ be a linear subspace of E^r with the fixed origin $O(x, t)$, that continuously depends on the pair (x, t), $x \in X$, $0 \leqslant t \leqslant 1$. Furthermore, let $U(x) = \{u_1(x), \dots, u_n(x)\}$ be a basis of the vector space $N^n_{x,0}$ continuously depending on $x \in X$. Then there exists a basis $U(x, t)$ of the space $N^n_{x,t}$ continuously depending on the pair (x, t) and coinciding with $U(x)$ at $t = 0$. If, moreover, the vector space $N^n_{x,t}$ does not depend on t for $x \in Y \subset X$, then we have $U(x, t) = U(x)$ for $x \in Y$.

Let us prove Statement «A». Let ε be a small positive number such that for $|t - t'| \leqslant \varepsilon$, $x \in X$, the orthogonal projection of the space $N^n_{x,t}$ to the space $N^n_{x,t'}$ is non-degenerate. Set $U(x, 0) = U(x)$. Suppose that the basis $U(x, t)$ is already constructed for $0 \leqslant t \leqslant p\varepsilon < 1$, $x \in X$ (p is a non-negative integer). For $p\varepsilon \leqslant t \leqslant (p+1)\varepsilon$, let us construct the basis $U(x, t)$ by transporting the basis $U(x, p\varepsilon)$ parallel to the point $O(x, t)$ and then projecting it orthogonally to $N^n_{x,t}$.

B) Let E^{n+k} be the Euclidean space endowed with the Cartesian coordinate system $y^1, \dots, y^{n+k}$; let E^{n+k}_* be the strip defined by $c_0 \leqslant y^{n+k} \leqslant c_1$, restricted by the hyperplanes E_0 and E_1, and let M^k be a smooth submanifold of the strip E^{n+k}_* orthogonal on the boundary to the boundary $E_0 \cup E_1$ of the strip E^{n+k}_* (see § 1, Chapter 2, «A»). A smooth family of mappings e_t, $0 \leqslant t \leqslant 1$, from the manifold M^k to the strip E^{n+k}_* is called a *smooth deformation* of the sumbanifold M^k of the strip E^{n+k}_* if e_0 is the identity mapping and e_t is a regular homeomorphic mapping from the manifold M^k to the submanifold $e_t(M^k)$ of the strip E^{n+k}_* orthogonal at the boundary to the boundary of the strip E^{n+k}_*. If U is a framing of M^k and there is a framing $e_t(U)$ of $e_t(M^k)$ depending continuously on t such that $e_0(U) = U$, we say that e_t is a deformation of the framed manifold (M^k, U). (In the case of closed M^k we assume that $E^{n+k}_* = E^{n+k}$.) If for arbitrary t, the mapping e_t of the manifold M^k is the identity, then e_t provides a deformation for framing U of the fixed manifold M^k. This deformation provides a homotopy of the framings $e_0(U)$ and $e_1(U)$ of the manifold M^k. It turns out that if e_t is a smooth deformation of the submanifold M^k of the strip E^{n+k}_* and some framing U of M^k is given, then there exists a deformation e_t of the framed manifold (M^k, U). Furthermore, if e_t preserves the boundary points of the manifold M^k fixed then the framing $e_t(U)$, $0 \leqslant t \leqslant 1$, of the manifold $e_t(M^k)$ can be constructed in such a way that on the boundary of the manifold $e_t(M^k)$ this framing coincides with the initial framing U.

Let us prove Statement «B». Let (M^k, U) be a framed submanifold of the strip E^{n+k}_* and let e_t be a given smooth deformation of the submanifold M^k in the strip E^{n+k}_*. Let us construct a framing $e_t(U)$ for the submanifold $e_t(M^k)$ depending continuously on the parameter t in such a way that $e_0(U) = U$. Denote the normal subspace at $e_t(x)$ to the manifold

$e_t(M^k)$ by N^n_{xt}. Taking the vector system $U(x)$ to be the initial basis of the space N^n_{x0} we get, according to «A», a basis $U(x,t)$ of the vector space N^n_{xt} with origin at the point $e_t(x)$. The vector systems $U(x,t)$, $x \in M^k$ at a fixed t provide the desired framing $e_t(U)$. This completes the proof of «B».

C) Let (M^k, U) be a smoothly framed submanifold of the Euclidean space E^{n+k} and let e_t be its deformation in E^{n+k}. It turns out that the framed manifolds $(e_0(M^k), e_0(U))$ and $(e_1(M^k), e_1(U))$ are homologous.

Let us prove this fact. Set $s(t) = 3t^2 - 2t^3$. It follows immediately that the function $s(t)$ satisfies the conditions

$$s(0) = 0, \quad s(1) = 1, \quad s'(0) = 0, \quad s'(1) = 0,$$

$$s'(t) > 0 \ \text{ for } \ 0 < t < 1.$$

Define the deformation f_t of the framed manifold (M^k, U) by setting $f_t = e_{s(t)}$. Obviously, we have

$$f_0 = e_0, \quad f_1 = e_1.$$

To prove that the manifolds $(f_0(M^k),\ f_0(U))$ and $(f_1(M^k), f_1(U))$ are homologous in the strip $E^{n+k+1}_* = E^{n+k} \times I \subset E^{n+k+1}$, define the manifold M^{k+1} as the set of all points of the type $(f_t(x), t)$, $x \in M^k$, $0 \leqslant t \leqslant 1$. Let N'_{xt} be the normal subspace at $f_t(x)$ to $f_t(M^k)$ in the space E^{n+k}. In the space E^{n+k+1}, denote the normal subspace to the manifold M^{k+1} at $(f_t(x), t)$ by N_{xt}. It is easy to see that $N_{xt} = (N'_{xt}, t)$ for $t = 0, 1$, i.e. in the boundary points, the manifold M^{k+1} is orthogonal to the boundary of the strip E^{n+k+1}_*. At those points $(f_t(x), t)$, where t is distinct from 1 and 0, the normal subspaces (N'_{xt}, t) and N_{xt} are distinct, and thus the system $f_t(U(x))$ does not lie in N_{xt}. To transform the system $f_t(U(x))$ to some system $U(x,t)$ lying in N_{xt}, let us project the system $f_t(U(x))$ orthogonally to the space N_{xt}. It is easy to see that this projection is non-degenerate. Thus, the system $U(x,t)$ is linearly independent, thus it constitutes a framing for M^{k+1}. Since on the boundary components $(f_0(M^k), 0)$ and $(f_1(M^k), 1)$, the framing $U(x,t)$ coincides with the given framings $(f_0(U), 0)$ and $(f_1(U), 1)$, the framed manifold (M^{k+1}, U) provides the homology between the framed manifolds

$$(f_0(M^k), f_0(U)) \ \text{ and } \ (f_1(M^k), f_1(U)).$$

D) Each m-smooth framing is homotopic to an $(m-1)$-smooth framing of the same manifold.

Let M^k be an m-smooth submanifold of the strip E^{n+k}_* and let $U(x) = \{u_1(x), \ldots, u_n(x)\}$ be some framing of it. Denote the components of the vector $u_i(x)$ in the space E^{n+k} by $u_i^1(x), \ldots, u_i^{n+k}(x)$. Let ε be a positive number and let $v_i^j(x)$ be a real-valued m-smooth function on M^k such that $|u_i^j(x) - v_i^j(x)| < \varepsilon$ (see § 1, Chapter 2, «B»). Denote the vector of the space E^{n+k} with components $v_i^1(x), \dots, v_i^{n+k}(x)$ by $v_i(x)$; let $w_i(x)$ be the orthogonal projection of the vector $v_i(x)$ to N_x. Set

$$W_t(x) = \{u_i(x) \cdot (1-t) + w_i(x) \cdot t\}, \quad i = 1, \dots, n.$$

The system $W_t(x)$ is non-degenerate for ε sufficiently small; thus it provides a deformation of the source framing $U(x) = W_0(x)$ to the $(m-1)$-smooth framing $W_1(x)$.

In § 2, Chapter 2, it was shown that the homotopy classification of mappings from the sphere Σ^{n+k} to the sphere S^n is equivalent to the homology classification of smoothly framed k-dimensional submanifolds of E^{n+k} (see Theorems 9 and 10). By virtue of statements «C» and «D» we may omit the smoothness assumption and consider arbitrary continuous framings of smooth manifolds. Indeed, each smooth framing of a smooth manifold is homotopic to a smooth one (see «D»), and (not necessarily smoothly) homotopic smooth framings of the same manifold are homologous (see «C»); thus they correspond to homotopic mappings of spheres.

The homology group Π_n^k of framed manifolds

E) Let (M^k, U) be a framed submanifold of the Euclidean space E^{n+k} and let f be a homothetic mapping of E^{n+k} onto itself. It is evident that $(f(M^k), f(U))$ is also a framed submanifold of the Euclidean space E^{n+k}. If the mapping f preserves the orientation of the space E^{n+k}, then it is easy to see that there exists a family f_t (smoothly depending on t, $0 \leqslant t \leqslant 1$) of the homothety mappings from E^{n+k} onto itself such that the mapping f_0 is identical and $f_1 = f$. The family f_t, $0 \leqslant t \leqslant 1$, provides a smooth deformation of the framed manifold (M^k, U) to the framed manifold $(f(M^k), f(U))$. Thus, these framed manifolds are homological (see «C»). From the above, it follows that if we move a framed manifold in the space as a rigid body and shrink it homothetically, we will preserve the homology class of the framed manifold.

Definition 6. Split the totality of all framed k-dimensional manifolds in the Euclidean space E^{n+k} into classes of pairwise homological ones. Denote the set of all homology classes by Π_n^k. Define the sum as follows. Let π_1 and π_2 be two elements from Π_n^k. Choose an arbitrary hyperplane $E^{n+k-1} \subset E^{n+k}$ and choose representatives (M_1^k, U_1) and (M_2^k, U_2) for each of the two classes π_1 and π_2 in such a way that the manifolds

M_1^k and M_2^k lie on different sides of the hyperplane E^{n+k-1}. According to Statement «E», this is always possible. Define the framed manifold $(M^k, U) = (M_1^k, U_1) \cup (M_2^k, U_2)$ as the union of the manifolds M_1^k and M_2^k taken together with their framings. It turns out that the homology class π of the framed manifold (M^k, U) does not depend on the arbitrary choice of the hyperplane E^{n+k-1}; it does not depend on the choice of representatives (M_1^k, U_1), (M_2^k, U_2) of homology classes π_1, π_2 either. By definition, set $\pi = \pi_1 + \pi_2$. It turns out that, according to this definition, the set Π_n^k becomes a commutative group. The zero of the group Π_n^k is the class of null-homologous framed manifolds. The element $-\pi$ opposite to the element π can be described as follows. Let E^{n+k-1} be an arbitrary hyperplane from E^{n+k}, and let $(\hat{M}^k, \hat{U})$ be some framed manifold representing the class π and let σ be the reflection of E^{n+k} in the hyperplane E^{n+k-1}. The homology class $-\pi$ contains the framed manifold $(\sigma(M^k), \sigma(U))$.

First, prove that the operation defined in this way for the set Π_n^k, is invariant. Choose, together with the hyperplane E^{n+k-1} and framed manifolds (M_1^k, U_1) and (M_2^k, U_2) in the space E^{n+k}, a hyperplane $\hat{E}^{n+k-1}$ and framed manifolds $(\hat{M}_1^k, \hat{U}_1)$ and $(\hat{M}_2^k, \hat{U}_2)$ representing the classes π_1 and π_2. Let us show that the framed manifolds $(M_1^k, U_1) \cup (M_2^k, U_2)$ and $(\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$ belong to the same homology class. This will show that the sum operation is well defined. Clearly, there exists an orientation-preserving isometric mapping f of the space E^{n+k} onto itself such that $f(\hat{E}^{n+k-1}) = E^{n+k-1}$ and the manifolds $f(\hat{M}_1^k)$ and M_1^k lie on the same side of the hyperplane E^{n+k-1}. By virtue of «E», we have

$$f(\hat{M}_i^k, \hat{U}_i) \sim (\hat{M}_i^k, \hat{U}_i), \quad i = 1, 2;$$

$$f((\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)) \sim (\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2).$$

Thus, we reduced the question to the case when $\hat{E}^{n+k-1} = E^{n+k-1}$ and both representatives $(\hat{M}_1^k, \hat{U}_1)$ and (M_1^k, U_1) of the class π_1 lie on the same side (with respect to E^{n+k-1}) in the half-space E_-^{n+k} whence both representatives $(\hat{M}_2^k, \hat{U}_2)$ and (M_2^k, U_2) of the class π_2 lie on the other side away from the hyperplane E^{n+k-1} in the half-space E_+^{n+k}. Let (M_1^{k+1}, U_1^*) be a framed submanifold of the strip $E^{n+k} \times I$ providing a homology $(\hat{M}_1^k, \hat{U}_1) \sim (M_1^k, U_1)$ and let (M_2^{k+1}, U_2^*) be a framed submanifold of $E^{n+k} \times I$, providing homology $(\hat{M}_2^k, \hat{U}_2) \sim (M_2^k, U_2)$. If $M_1^{k+1} \subset E_-^{n+k} \times I$ and $M_2^{k+1} \subset E_+^{n+k} \times I$ then the framed manifolds (M_1^{k+1}, U_1^*) and (M_2^{k+1}, U_2^*) do not intersect and their union, being a framed manifold, would provide a homology $(\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2) \sim (M_1^k, U_1) \cup (M_2^k, U_2)$.

Let e be the vector of the space E^{n+k} orthogonal to E^{n+k-1} and directed toward E_+^{n+k}. Denote by g_t the mapping of the space E^{n+k} onto itself and taking x to $x+te$. Choose the vector e to be so long that the following inclusions are obtained:

$$g_{-1}(M_1^{k+1}) \subset E_-^{n+k} \times I, \quad g_1(M_2^{k+1}) \subset E_+^{n+k} \times I.$$

Finally, note that the framed manifold $g_{-t}(\hat{M}_1^k, \hat{U}_1) \cup g_t(\hat{M}_2^k, \hat{U}_2)$ provides a deformation from the manifold $(\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$ to the manifold $g_{-1}(\hat{M}_1^k, \hat{U}_1) \cup g_1(\hat{M}_2^k, \hat{U}_2)$; thus, by virtue of «C», we have the homology $g_{-1}(\hat{M}_1^k, \hat{U}_1) \cup g_1(\hat{M}_2^k, \hat{U}_2) \sim (\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2)$. In the same way we get $g_{-1}(M_1^k, U_1) \cup g_1(M_2^k, U_2) \sim (M_1^k, U_1) \cup (M_2^k, U_2)$. Thus,

$$(\hat{M}_1^k, \hat{U}_1) \cup (\hat{M}_2^k, \hat{U}_2) \sim (M_1^k, U_1) \cup (M_2^k, U_2).$$

It follows from the independence of the sum operation on the choice of representatives that the zero of the group Π_n^k is represented by the class containing the empty manifold, i.e. the class of null-homologous framed manifolds. Let us prove that the opposite element $-\pi$ for the element π is described as follows.

Assume that the Euclidean space E^{n+k} lies in the Euclidean space E^{n+k+1} where it is defined as $y^{n+k+1} = 0$. Let us also assume that all points of the manifold M^k are at the distance less than one from the hyperplane E^{n+k-1} (see «C»). Let E_+^{n+k} and E_-^{n+k} be the half-spaces of the space E^{n+k} separated by E^{n+k-1}, so that $M^k \subset E_+^{n+k}$. Let us rotate the half-space E_+^{n+k} in the half-space $y^{n+k+1} \geqslant 0$ of the space E^{n+k+1} until it coincides with the half-space E_-^{n+k}; then during the process, the manifold (M^k, U) circumscribes the framed submanifold (M^{k+1}, U^*) of the half-space $y^{n+k+1} \geqslant 0$. The framed manifold (M^{k+1}, U^*) lies completely in the strip $0 \leqslant y^{n+k+1} \leqslant 1$ of the space E^{n+k+1} and provides the homology between the manifold $(M^k, U) \cup (\sigma(M^k), \sigma(U))$ and zero.

Split the set of all mappings from the sphere Σ^{n+k} to the sphere S^n into sets of pairwise homotopic classes; denote the set of such classes by $\pi^{n+k}(S^n)$. Since between elements of the group Π_n^k and elements of $\pi^{n+k}(S^n)$ there is a one-to-one correspondence (see § 2, Chapter 2), the sum operation defined in Π_n^k induces the sum operation in $\pi^{n+k}(S^n)$. It is easy to show that the sum operation on the set $\pi^{n+k}(S^n)$ defined in this way coincides with the usual sum operation of the homotopy group (see [10]). However, we shall neither prove nor use this fact. A reader familiar with elements of homotopy theory can easily prove this fact.

Orthogonalization of framings

Statement «G» given below shows that in the homology theory of framed manifolds, it is sufficient to restrict ourselves to orthonormal framings. Statement «H» provides an approach to the orthonormal framing homotopy classification question.

F) Let $U = \{u_1, \ldots, u_n\}$ be a linearly independent vector system of the space E^l. Let us undertake the orthogonalization process, i.e. let us find the orthonormal system $\bar{U} = \{\bar{u}_1, \ldots, \bar{u}_n\}$ obtained from the system U by formulae

$$\bar{u}_j = \sum_{i=1}^{n} a^i_j u_j, \quad j = 1, \ldots, n,$$

where the coefficients a^i_j satisfy the conditions

$$a^i_j = 0 \text{ at } i > j; \quad a^i_j > 0 \text{ at } i = j.$$

These conditions uniquely define the coefficients a^i_j; the latter can be expressed in terms of scalar products of vectors of the system U. If U is orthonormal, then $\bar{U} = U$. Set

$$U^t = \{u^t_1, \ldots, u^t_n\},$$

where

$$u^t_i = u_i(1-t) + \bar{u}_i t.$$

The system U^t is linearly independent since the matrix $\|(1-t)\delta^i_j - ta^i_j\|$ is non-degenerate. Thus, the system $\bar{U} = U^1$ is obtained from the system $U = U^0$ by using a continuous deformation uniquely defined by the system U.

G) Let $U(x)$ be some framing of the manifold M^k. The framing $U^t(x)$ (see «F») provides a continuous deformation from the initial framing $U(x)$ to the orthonormal framing $\bar{U}(x)$. If the initial framing is m-smooth, then the whole deformation U^t is so. Finally, if there exists a deformation $U_t(x)$, $0 \leqslant t \leqslant 1$ from some orthonormal framing $U_0(x)$ to some other orthonormal framing $U_1(x)$ then there exists an orthonormal deformation $\bar{U}_t(x)$ from $U_0(x)$ to $U_1(x)$.

H) Let (M^k, V) be an orthonormally framed submanifold of the orientable Euclidean space E^{n+k}. According to the remark for Definition 3, the manifold M^k has a fixed orientation, and we shall say that V is a framing of the *oriented* manifold M^k. Let U be a certain orthonormal framing of the oriented manifold M^k. Let us compare the framings V and U. For each normal subspace N_x to the manifold M^k, there are two orthonormal vector systems:

$$V(x) = \{v_1(x), \ldots, v_n(x)\} \quad U(x) = \{u_1(x), \ldots, u_n(x)\};$$

thus we have

$$u_i(x) = \sum_{j=1}^{n} f_{ij}(x)v_j(x), \quad i = 1, \dots, n,$$

where $f(x) = \|f_{ij}(x)\|$ is an orthogonal matrix with positive determinant. Thus, with each orthonormal framing U for V being fixed, we associate a mapping f from the manifold M^k to the manifold H_n of all orthogonal matrices with positive determinant: $U \to f$. Clearly, we have the converse as well: for each mapping f from M^k to H_n there corresponds a unique framing $U : f \to U$. Assume that together with a fixed framing V there are two orthonormal framings U_0 and U_1 of the oriented manifold M^k, and let $U_0 \to f_0, U_1 \to f_1$. It is easy to see that the framings U_0 and U_1 are homotopic if and only if the mappings f_0 and f_1 are homotopic. Thus, the homotopy classification of all framings of the oriented manifold M^k in the oriented Euclidean space E^{n+k} is equivalent to the homotopy classification of mappings from the manifold M^k to the manifold H_n of all orthogonal matrices of order n with positive determinant.

§ 4. The suspension operation

In this section, we shall define and investigate (to some extent) the suspension operation for framed manifolds; this operation plays an important role in the question of homotopy classification for mappings from sphere to sphere. Let (M^k, U) be a framed submanifold of the Euclidean space E^{n+k} situated in the Euclidean space E^{n+k+1}. For any point $x \in M^k$ construct in E^{n+k+1} the unit vector $u_{n+1}(x)$ perpendicular to the hyperplane E^{n+k} in such a way that all vectors $u_{n+1}(x)$, $x \in M^k$ have the same direction and set $EU(x) = \{u_1(x), \dots, u_n(x), u_{n+1}(x)\}$. The framed manifold $E(M^k, U) = (M^k, EU)$ of the Euclidean space E^{n+k+1} is called the *suspension* of the framed manifold (M^k, U). It turns out that the suspensions for homological framed manifolds are homological as well and that the mapping E from the group Π_n^k to the group Π_{n+1}^k (see Definition 6) is a homomorphism. It is proved in Theorem 11 that for $n \geqslant k+1$ the homomorphism E is an *epimorphism* and for $n \geqslant k+2$ it is an isomorphism, so that the groups Π_{n+2}^k, $\Pi_{n+3}^k, \dots$ are all naturally isomorphic.

In terms of sphere-mappings, the suspension operation can be described as follows. Let p' and q' be the poles of the sphere Σ^{n+k+1} and let Σ^{n+k} be its equator, i.e. the section by the hyperplane perpendicular to the interval $p'q'$ and passing through the centre of this interval. Analogously, let p and q be the poles of the sphere S^{n+1} and let S^n be its equator. With any mapping f from Σ^{n+k} to S^n, let us associate the mapping Ef from Σ^{n+k+1} to S^{n+1} which maps the meridian $p'xq'$, $x \in \Sigma^{n+k}$, of the sphere Σ^{n+k+1} to the meridian $pf(x)q$ of the sphere S^{n+1}. The suspension Ef of the mapping f in the form described above was introduced by Freudenthal [11]. We will not use it here, however. The fact that the suspension of a mapping and the suspension over a framed manifold correspond to each other in the sense of Definition 5 can be easily proved; however, we shall not prove it.

Definition 7. Let (M^k, U), $U(x) = \{u_1(x), \dots, u_n(x)\}$, be a framed submanifold of the oriented Euclidean space E^{n+k} and let E^{n+k+1} be an orientable Euclidean space containing E^{n+k}. Let $e_1, \dots, e_{n+k}$ be a basis of E^{n+k} generating its orientation and let e_{n+k+1} be a unit vector of the space E^{n+k+1} orthogonal to E^{n+k} such that the basis $e_1, \dots, e_{n+k}, e_{n+k+1}$ generates the orientation of the space E^{n+k+1}. Denote by $u_{n+1}(x)$ the vector emanating from the point $x \in M^k$ obtained from e_{n+k+1} by parallel transport. Set

$$EU(x) = \{u_1(x), \dots, u_n(x), u_{n+1}(x)\}.$$

Then $E(M^k, U) = (M^k, EU)$ is a framed submanifold of the Euclidean space E^{n+k+1}. The framed manifold $E(M^k, U)$ is called the *suspension* of the framed manifold (M^k, U). It turns out that from $(M_0^k, U_0) \sim (M_1^k, U_1)$ it follows that $E(M_0^k, U_0) \sim E(M_1^k, U_1)$. Thus, the correspondence $(M^k, U) \to E(M^k, U)$ generates the mapping from the group Π_n^k to the group Π_{n+1}^k. This mapping turns out to be a homomorphism. We shall denote it also by E.

Let us show that if $(M_0^k, U_0) \sim (M_1^k, U_1)$ then $E(M_0^k, U_0) \sim E(M_1^k, U_1)$. Let (M^{k+1}, U^*) be the framed submanifold of the strip $E^{n+k} \times I$ providing the homology $(M_0^k, U_0) \sim (M_1^k, U_1)$. In the strip $E^{n+k+1} \times I$ at the point $y \in M^{k+1}$, let us choose the unit vector $u_{n+1}^*(y)$ orthogonal to the strip $E^{n+k} \times I$ and collinear with the vector e_{n+k+1}. Set $EU^*(y) = \{u_1^*(y), \dots, u_n^*(y), u_{n+1}^*(y)\}$. Clearly, the framed submanifold $E(M^{k+1}, U^*) = (M^{k+1}, EU^*)$ of the strip $E^{n+k+1} \times I$ provides the homology $E(M_0^k, U_0) \sim E(M_1^k, U_1)$.

The fact that E is a homomorphism is even simpler. The homomorphism E from the group Π_n^k to the group Π_{n+1}^k is in several cases an *epimorphism* and even an *isomorphism*. Let us consider these cases. Before that, let us prove the following statement.

A) Let E^{n+k+1} be the oriented Euclidean space and let E^{n+k} be its oriented hyperplane. Furthermore, let (M^{k+1}, V) be an orthonormally framed submanifold of the strip $E^{n+k+1} \times I$ such that the manifold M^{k+1} itself lies in the frame $E^{n+k} \times I$. M^{k+1} might possibly be closed. Assume that the boundary of the manifold M^{k+1} consists of manifolds $M_0^k \times 0$ and $M_1^k \times 1$, so that $M_0^k \subset E^{n+k}$, $M_1^k \subset E^{n+k}$. Suppose that the framing V restricted to the boundary components $M_1^k \times 0$ and $M_1^k \times 1$ is a suspension, i.e.

$$V(x,\tau) = EU_\tau(x) \times \tau, \quad \tau = 0,1,$$

where U_τ is a framing of the manifold M_τ^k, $\tau = 0,1$, in the space E^{n+k}. At each point $x \in M^{k+1}$, let us choose the unit vector $u_{n+1}(x)$ to be orthogonal to $E^{n+k} \times I$ and directed in a proper way. In the normal subspace N_x to the manifold M^{k+1} at x in the space $E^{n+k+1} \times I$, fix a basis $v_1(x), \ldots, v_{n+1}(x)$. Thus, for the vector $u_{n+1}(x)$ also lying in N_x, we have

$$u_{n+1}(x) = \psi^1(x)v_1(x) + \ldots + \psi^{n+1}(x)v_{n+1}(x). \tag{1}$$

Let N be the Euclidean space of dimension $n+1$ with preassigned coordinate system and let $\mathfrak{S}^n$ be the unit sphere of this space centred at the origin of coordinates. Denote the point $(0,\ldots,0,1)$ of the sphere $\mathfrak{S}^n$ by $\mathfrak{P}$. Now, with each point $x \in M^{k+1}$, associate the point $\psi(x)$ of the sphere $\mathfrak{S}^n$ with coordinates $\psi^1(x),\ldots,\psi^{n+1}(x)$. Thus, ψ is a mapping from the manifold M^{k+1} to the sphere $\mathfrak{S}^n$ taking the whole boundary to the point $\mathfrak{P}$. Suppose there exists a continuous deformation ψ_t, $0 \leqslant t \leqslant 1$ from the mapping $\psi = \psi_0$ to the mapping ψ_1, so that ψ takes the whole manifold M^{k+1} to the point $\mathfrak{P}$, and ψ_t takes the boundary of the manifold M^{k+1} to $\mathfrak{P}$ for arbitrary t. It turns out that then there exists a deformation of the framing V to the framing EU where U is a framing of the submanifold M^{k+1} in $E^{n+k} \times I$, and during the whole deformation, the framing remains the same on the boundary of the manifold M^{k+1}. For the case of closed M^{k+1} this means that the framed manifold (M^{k+1}, V) is homologous to the framed manifold $E(M^{k+1}, U)$. For a non-closed manifold M^{k+1} this allows us to deduce from the homology $E(M_0^k, U_0) \sim E(M_1^k, U_1)$ provided by (M^{k+1}, V) the homology $(M_0^k, U_0) \sim (M_1^k, U_1)$.

Let us prove «A» now. Introduce in N_x the Cartesian coordinates corresponding to the basis $v_1(x),\ldots,v_{n+1}(x)$. Let λ_x be the coordinate-wise mapping from N onto N_x. Set $\psi(x,t) = \lambda_x\psi_{1-t}(x)$. The vector $\psi(x,t)$ of the space $E^{n+k+1} \times I$ lies in N_x and depends continuously on the variables (x,t) so that $\psi(x,0) = v_{n+1}(x)$, and $\psi(x,1) = u_{n+1}(x)$. Denote the subspace of N orthogonal to $\psi(x,t)$, by P_{xt}. Since $\psi(x,0) = v_{n+1}(x)$ the vectors $v_1(x),\ldots,v_{n+1}(x)$ form a basis of the space P_{x0}. Taking it to be the initial basis and applying to the variable vector space P_{xt} Statement «A» § 3, Chapter 2, we get a basis $U(x,t)$ of this space. Together with the vector $\psi(x,t)$ this basis gives us the desired deformation for the framing V. Thus, Statement «A» is proved.

Theorem 11. *The homomorphism E from Π_n^k to Π_{n+1}^k is an epimorphism for $n \geqslant k+1$ and an isomorphism for $n \geqslant k+2$. Thus, the groups $\Pi_{k+2}^k, \Pi_{k+3}^k, \ldots$ are all naturally isomorphic.*

PROOF. Let $n \geqslant k+1$, $\hat{\pi} \in \Pi^k_{n+1}$ and let $(\hat{M}^k, \hat{U})$ be a framed submanifold of the Euclidean space E^{n+k+1} representing the homology class $\hat{\pi}$. According to Statement «D» §2, there exists such a one-dimensional projecting direction E^1 along which the manifold $\hat{M}^k$ is projected regularly without intersection to the manifold M^k. We shall project along E^1 to the hyperplane E^{n+k} of the space E^{n+k+1} orthogonal to E^1 in such a way that $M^k \subset E^{n+k}$. We shall make each point $x \in \hat{M}^k$ move in a straightforward line towards E^1 until it coincides with its projection to M^k in such a way that it passes the whole way in unit time. This gives a deformation of the manifold $\hat{M}^k$ to the manifold M^k. According to Statements «B» and «G» §7, there exists a deformation of the framed manifold $(\hat{M}^k, \hat{U})$ to the orthonormally framed manifold (M^k, U). Since $n \geqslant k+1$, the mapping ψ from M^k to the sphere $\mathfrak{S}^n$, constructed in Statement «A», is homotopic to the mapping of the manifold M^k to the point $\mathfrak{P}$; thus the framing V of the manifold M^k is homotopic to the framing EU of the same manifold. By virtue of Statement «C» §7 we have $(\hat{M}^k, \hat{U}) \sim E(M^k, U)$. Let $\pi \in \Pi^k_n$ be the homology class of the framed manifold (M^k, U); then we have $\hat{\pi} = E\pi$. Thus it is proved that $\Pi^k_{n+1} = E\Pi^k_n$ for $n \geqslant k+1$.

Suppose now that $n \geqslant k+2$; let us show that E is an isomorphism, i.e. that for $\pi_0 \in \Pi^k_n, \pi_1 \in \Pi^k_n$ the relation $E\pi_0 = E\pi_1$ implies that $\pi_0 = \pi_1$. Let (M^k_0, U_0) and (M^k_1, U_1) be orthonormally framed manifolds in the Euclidean space $E^{n+k} \subset E^{n+k+1}$ belonging to homology classes π_0 and π_1. Furthermore, let $(\hat{M}^{n+k}, \hat{U})$ be a framed submanifold of the strip $E^{n+k+1} \times I$ providing the homology $E(M^k_0, U_0) \sim E(M^k_1, U_1)$. Denote by $\hat{E}^1$ the one-dimensional direction in the space $E^{n+k+1} \times I$ orthogonal to the strip $E^{n+k} \times I$. By virtue of Statement «D» §2 there exists an arbitrarily close to $\hat{E}^1$ projecting direction E^1 such that the projection of M^{k+1} along it is regular without intersection. Choose E^1 to be so close to $\hat{E}^1$ that the projection M^{k+1} of the manifold $\hat{M}^{k+1}$ along E^1 lies in the strip $E^{n+k} \times I$. The deformation of the manifold $\hat{M}^{k+1}$ to M^{k+1} preserves the boundary pointwise fixed, thus the deformation of the framed manifold $(\hat{M}^{k+1}, \hat{U})$ to the orthonormally framed manifold (M^{k+1}, V) (which exists according to Statements «B» and «G» §3, Chapter 2) preserves the framing on the boundary. Now, the homology $E(M^k_0, U_0) \sim E(M^k_1, U_1)$ is represented by the framed manifold (M^{k+1}, V); here $M^{k+1} \subset E^{n+k} \times I$, i.e. the conditions of Statement «B» hold; thus, the framed manifolds (M^k_0, U_0) and (M^k_1, U_1) are homologous. Thus $\pi_0 = \pi_1$.

Thus, Theorem 11 is proved.

Chapter III

The Hopf invariant

§ 1. Homotopy classification of mappings of n-manifolds to the n-sphere

Here we present a homotopy classification of mappings of smooth closed orientable n-manifolds to the n-sphere. This result is well known even for non-smooth manifolds, however, this plays an auxiliary role here. The proof is performed by using specific methods for smooth manifolds. This simplifies the ways of applying the result in the sequel. First, we define the mapping degree and prove its simplest properties. Later, based on the theory constructed, we present a classification of mappings from the n-dimensional sphere to itself; this gives an illustration of general results presented in previous sections. Finally, we reduce the classification of mappings from an n-manifold to the n-sphere to the classification of mappings from the n-sphere to itself.

Mapping degree

Definition 8. Let f be a smooth mapping from an r-dimensional oriented manifold P^r to an r-dimensional oriented manifold Q^r and let b be an interior point of the manifold Q^r which is a proper point of f, such that the full pre-image of this point is compact and does not intersect the boundary of the manifold P^r. With the assumptions above, the full pre-image $f^{-1}(b)$ consists of a finite number of points $a_1, \ldots, a_p$ and the functional determinant of f is non-zero; thus, it has a well-defined sign (the manifolds P^r and Q^r are oriented). Denote the sign of the functional determinant of the mapping f at the point a by $\varepsilon_i (= \pm 1)$, $i = 1, \ldots, p$. Call this determinant the *degree of f at a_i*. The sum $\varepsilon_1 + \ldots + \varepsilon_p$ is called the *mapping degree of f at b*. Now, if both manifolds P^r and Q^r are closed, then the set G of all those points b for which the conditions above hold is an everywhere dense domain in Q^r (see Theorem 4). It will be shown later (see «B») that if, furthermore, the manifold Q^r is connected then for all points $b \in G$ the degree of f is the same; it is called *the mapping degree of* $1f$. It will also be shown later (see «B») that the degrees of homotopic mappings coincide. Thus, in the case of closed P^r and connected closed Q^r, the mapping degree is an invariant of the homotopy class of mappings; thus, it is well defined for any mapping.

A) Let Q^r be a connected closed manifold, let P^{r+1} be a compact manifold with boundary P^r; let f be a smooth mapping from the manifold P^{r+1} to the manifold Q^r; finally, let $b \in Q^r$ be a proper point of the mapping f from P^r to Q^r. It turns out that the degree of f at b is zero.

Let us prove this fact. Let V be a connected neighbourhood of the point b in Q^r consisting of proper points of the mapping $f : P^r \to Q^r$. It is easy to see that for all points $b' \in V$, the mapping degree of $f : P^r \to Q^r$ is the same. Thus, without loss of generality, we may assume the point b to be a proper point of the mapping f from P^{r+1} to Q^r (see Theorem 4). Thus, $f^{-1}(b)$ is a one-dimensional submanifold M^1 of the manifold P^{r+1}, consequently, it consists of finitely many components, some of which are homeomorphic to the circle, the remaining ones being homeomorphic to the interval. All points of the full pre-image of the point b in P^r are endpoints of components of M^1. Let L^1 be a component of M^1 homeomorphic to the interval; denote its endpoints by a_0 and a_1. According to the results of § 4 [see (2)], for a given coordinate system $y^1, \dots, y^r$ with the origin at b defined in some neighbourhood of b, one can choose such coordinates $x^1, \dots, x^{r+1}$ in the neighbourhood of $a \in L^1$ that the mapping f looks like

$$y^i = x^i, \quad i = 1, \dots, r.$$

We shall assume that the coordinates $y^1, \dots, y^r$ generate the orientation of the manifold Q^r. In coordinates $x^1, \dots, x^{r+1}$, the curve L^1 is defined by the equations $x^1 = 0, \dots, x^r = 0$, i.e. x^{r+1} can be treated as a variable parameter on L^1. We shall assume that as the parameter x^{r+1} increases, the point on the curve L^1 moves from a_0 to a_1. With that assumption, the coordinates $x^1, \dots, x^{r+1}$ might not define the orientation of the manifold P^{r+1}; denote by $\varepsilon(= \pm 1)$ the corresponding sign that distinguishes the orientation fixed for P^{r+1} from the orientation defined by the coordinates $x^1, \dots, x^{r+1}$. It can be easily checked that ε does not depend on the arbitrary choice of coordinates $x^1, \dots, x^{r+1}$ and does not change while moving the point a along L^1. It follows from the definition of the orientation for the boundary (see § 1, «B») that the mapping degree of f defined on P^r equals $-\varepsilon \cdot (-1)^r$ at the point a_0 and equals $\varepsilon \cdot (-1)^r$ at the point a_1. Assuming that all components of M^1 are homeomorphic to the interval, we see that the mapping degree of f at b equals zero.

B) Let f_0 and f_1 be two homotopic mappings from a closed oriented manifold P^r to a closed oriented manifold Q^r; let G_t be the set of all proper points $b \in Q^r$ of $f_t, t = 0, 1$. It turns out that for $b \in G_0 \cap G_1$, the degrees of f_0 and f_1 at b, are equal. Furthermore, it turns out that if b_0 and b_1 are two points from G_0 then the mapping degrees of the mapping f_0 for the points b_0 and b_1 are equal, too.

Let us prove Statement «B». Since the mappings f_0 and f_1 are homotopic, there exists a smooth family f_t connecting these maps (see Theorem 8). For the family f_t, we have the corresponding mapping f_* from the product $P^r \times I$ (see § 1, Chapter 2, «C»). The boundary of the manifold $P^r \times I$ consists of manifolds $P^r \times 0$ and $P^r \times 1$. Choose the orientation for the manifold $P^r \times I$ in such a way that the manifold $P^r \times 0$ is represented in the boundary of the product $P^r \times I$ with the positive sign; then the manifold $P^r \times 1$ in the boundary of $P^r \times I$ will have the minus sign. From this and from Statement «A» it follows now that the degrees of mappings f_0 and f_1 at b, coincide.

Let us show now that the degrees of f_0 coincide for all points $b \in G_0$. Let X be a coordinate system having origin at the point $c \in Q^r$; let V be a ball neighbourhood of the point c in this coodinate system. Furthermore, let b_0 and b_1 be two points from $V \cap G_0$. It is easy to construct a regular homeomorphic mapping φ of the manifold Q^r to itself that fixes all points $Q^r \backslash V$ and takes the point b_0 to the point b_1. Such a mapping φ is, clearly, homotopic to the identity. Clearly, the degree of φf_0 at the point b_1 equals the degree of f_0 at b_0; and since the mappings φf_0 and f_0 are homotopic, their degrees at the point b_1 coincide. Thus, the degrees of the mapping f_0 coincide for all points $b \in V \cap G$. Now, since the manifold Q^r is connected and the set G_0 is everywhere dense in Q^r, it follows that the degree of f_0 is the same for all points $b \in G_0$.

Mappings from S^n to S^n

C) Let (M^0, U) be a zero-dimensional framed submanifold of the framed Euclidean space E^n. Since M^0 is a compact submanifold; it consists of finitely many points $a_1, \ldots, a_r$. Associate with a_i the index $+1$ if the vectors $u_1(a_i), \ldots, u_n(a_i)$ generate the positive orientation of the space E^n, and the index -1, otherwise. Call the sum $I(M^0, U)$ of indices of all points $a_1, \ldots, a_r$ *the index* of the framed manifold. It is clear that the index of the framed manifold (M^0, U) equals the degree of the corresponding mapping (see Definition 5) from the oriented sphere Σ^n to the oriented sphere S^n.

Theorem 12. *If two mappings f_0 and f_1 from the oriented sphere Σ^n to the oriented sphere S^n are of the same degree, then they are homotopic. Moreover, there exists a mapping with any preassigned degree.*

Proof. From Statement «C» and Theorem 10 it follows that to prove this theorem, it is sufficient to prove that any two framed zero-dimensional manifolds having the same index are homologous and that there exist zero-dimensional manifolds with any preassigned index. It is easy to see that the two framed manifolds (M_0^0, U_0) and (M_1^0, U_1) each consisting of one point and having index equal to $+1$ can be obtained from each other by a deformation (see § 3, Chapter 2, «B»); thus they belong to the same homology class (see § 3, Chapter 2, «C»), Thus, all one-point framed manifolds having index $+1$ belong to one and the same homology class ε. In the same way, all one-point framed manifolds with index -1 belong to one and the same homology class ε'. Since after symmetry in each hyperplane, the space E^n changes the orientation, we have $\varepsilon' = -\varepsilon$ (see Definition 6). Since, moreover, each zero-dimensional framed manifold (M^0, U) is a union of finite number of one-point framed manifolds, some of them with index $+1$, the other ones having index -1, then ε is a generator of the group Π_n^0, and (M^0, U) belongs to the class $I(M^0, U) \cdot \varepsilon$. Thus, two framed zero-dimensional manifolds with the same index are homologous. Obviously, there exist framed zero-dimensional manifolds with any preassigned index.

Thus, Theorem 12 is proved.

It follows from Theorem 12 and «C» that the group Π_n^0, or, what is the same, the group $\pi^n(S^n)$, is free cyclic.

D) Let f be a smooth mapping from the oriented sphere Σ^{n+k} to the oriented sphere S^n and let g be a smooth mapping of the sphere Σ^{n+k} onto itself having degree ν. Denote the element of the group Π_n^k corresponding to the mapping f by π and denote the element of Π_n^k corresponding to fg by π'. Then it turns out that

$$\pi' = \nu\pi. \tag{1}$$

Let us prove Statement (1). Let p' and q' be the north pole and the south pole of Σ^{n+k}; let E^{n+k} be the tangent space at p' to the sphere Σ^{n+k} and let φ be the central projection from the point q' of the domain $\Sigma^{n+k} \backslash q'$ to the space E^{n+k}. For $\nu = 1$, the mapping g is homotopic to the identity (see Theorem 12), thus, in this case, the relation (1) holds. Let us prove it for $\nu = -1$. Since all mappings of the sphere Σ^{n+k} onto itself having degree -1 are homotopic to each other, it is sufficient to consider one concrete mapping g of degree -1. Let E^{n+k-1} be a hyperplane of the space E^{n+k} passing through the point p' and let σ be a symmetry of the space E^{n+k} in this hyperplane. The mapping

$$g = \varphi^{-1}\sigma\varphi$$

of the domain $\Sigma^{n+k} \backslash q'$ onto itself extended by $g(q') = q'$ is a degree -1 mapping of the sphere Σ^{n+k} onto itself. For the mapping g constructed in this way, the relation (1) is evident.

Now let g be a smooth mapping of the sphere Σ^{n+k} onto itself, for which the set $g^{-1}(p')$ consists of proper points of the mapping g and does not contain the point g'; this can be reached by a small perturbation of the given mapping g. Let

$$\varphi g^{-1}(p) = \{a_0, \ldots, a_r\};$$

denote the sign of the functional determinant of the mapping g in $\varphi^{-1}(a_i)$ by ε_i. Let V_i be the ball of radius δ in the space E^{n+k} centred at a_i. We assume δ to be so small that there exists a hyperplane E_1^{n+k-1} of the space E^{n+k} disjoint from the balls V_i such that any preassigned part of the set $\{a_1, \ldots, a_r\}$ lies on one side with the remaining part lying on the other side. Choose a small positive α such that for the ball neighbourhood K_α of the point p', the full pre-image $g^{-1}(K_\alpha)$ consists only of proper points of the mapping g and is split into domains $A_1, \ldots, A_r$; $a_i \in A_i$ each of which maps diffeomorphically to K_α by g. Furthermore, suppose α to be so small that $\varphi(A_i) \subset V_i$. Now, define the mapping h_i of the sphere Σ^{n+k} to coincide with $\omega_\alpha g$ (see «B») on A_i and taking the set $\Sigma^{n+k} \backslash A_i$ to the point q'. Since the degree of h_i is equal to ε_i, the framed manifold E_1^{n+k-1} corresponding to fh_i belongs to the homology class $\varepsilon_i \pi$. It is clear that $M_i^k \subset V_i$ and that the mapping $f\omega_\alpha g$ generates the framed manifold $(M_1^k, U_1) \cup \ldots \cup (M_r^k, U_r)$. Since the mappings $\omega_\alpha g$ and g are homotopic, the existence of a hyperplane E_1^{n+k-1} with the properties described above yields the relation (1).

Mappings from n-dimensional manifold to the n-sphere

Theorem 13 below resolves completely the classification question for mappings from orientable closed n-manifolds to the n-sphere. Theorem 13 follows from Theorem 12 on the classification of mappings from the n-sphere to the n-sphere.

Theorem 13. *Two continuous mappings f_0 and f_1 of a smooth oriented manifold M^n to the smooth oriented sphere S^n are homotopic if and only if they have the same degree (see Definition 8). If the degree equals zero, then the mapping is zero-homotopic, i.e. contractible to a point. Thus, there exist mappings of any arbitrary degree.*

Proof. To prove the first part of the theorem, it suffices to show that if two mappings f_0 and f_1 are smooth and have the same degree then they are homotopic. To reduce the proof of this fact to Theorem 12, show that for any finite set Q of points of the manifold M^n there exists a smoothly homeomorphic to the open ball domain $B \subset Q$ of the manifold M^n.

It is easy to construct a simple closed curve K embedded in M^n and containing all points of Q. Let us assume that M^n is a submanifold of the Euclidean space E^{2n+1}; denote by N_x the normal space to K in E^{2n+1} at the point $x \in K$. As in Statement «A» § 1, Chapter 2, denote by $H_\delta(x)$ the ball of the Euclidean space N_x centred at x and having radius δ. Then there exists a small positive δ such that the set $W_\delta = H_\delta(K)$ is a neighbourhood of the curve K in E^{2n+1} and, if we take any point $y \in W_\delta$ to the point $x = \pi(y)$ for which $y \in H_\delta(x)$, we get a smooth mapping π from the manifold W_δ to the curve K (see § 1, Chapter 2, «A»). Take a closed interval L of the curve K containing all the points Q. Let us introduce a smooth parameter t for this interval $-1 \leqslant t \leqslant 1$. Thus, for each value of the parameter t, $-1 \leqslant t \leqslant 1$, there corresponds a point $x(t) \in K$. Denote the tangent space to the manifold M^k in $x \in K$, by T_x; set $N'_t = N_{x(t)} \cap T_{x(t)}$. In the vector space N'_t, let us choose an orthonormal basis $e_1(t), \ldots, e_{n-1}(t)$. By using «A» § 3, Chapter 2, and the orthogonalization process described in § 3, Chapter 2, «G», it is possible to choose the basis $e_1(t), \ldots, e_{n-1}(t)$ to depend smoothly on t. Let $W'_\delta = M^n \cap W_\delta$ and let π' be the mapping π restricted to W'_δ. Denote the full pre-image of the point $x(t) \in L$ in W'_δ under the mapping π' by H'_t. Let ε be a positive number. Denote by H^*_t the ball of radius $\varepsilon\sqrt{1-t^2}$ in N'_t centred at $x(t)$. The orthogonal projection χ_t of the manifold M^n to the space $T_{x(t)}$ takes some neighbourhood of the point $x(t)$ in M^n smoothly regularly and homeomorphically to some neighbourhood of the point $x(t)$ in $T_{x(t)}$. From this, it follows that for δ small enough, the projection χ_t is smooth, regular and homeomorphic mapping of the manifold H'_t to some neighbourhood of the point $x(t)$ in N'_t; thus, there exists a small ε such that

$$H^*_t \subset \chi_t(H'_t), \quad -1 \leqslant t \leqslant 1.$$

Denote the coordinates of $z \in H^*_t$ in the basis $e_1(t), \ldots, e_{n-1}(t)$ by $\varepsilon z^1, \ldots, \varepsilon z^{n-1}$; take the numbers $z^1, \ldots, z^{n-1}$, t to be the coordinates of $\chi_t^{-1}(z)$. The set B of all points $\chi_t^{-1}(z)$, $-1 \leqslant t \leqslant 1$, $z \in H^*_t$, constitutes a domain in M^n endowed with smooth coordinates $z^1, \ldots, z^{n-1}$, t satisfying the condition

$$(z^1)^2 + \ldots + (z^{n-1})^2 + t^2 < 1.$$

Thus, the domain B is smoothly homeomorphic to the open n-ball.

Choose a point p of the sphere S^n in such a way that the set $f_t^{-1}(p) = P_t$, $t = 0, 1$, consists of proper points of the mapping f_t (see Theorem 4). Set $Q = P_0 \cup P_1$; let B be a ball domain of the manifold M^n containing the finite set Q. Take p to be the north pole of S^n and denote the south pole of this sphere by q. Let α be a small positive number such that the ball neighbourhood K_α (see § 2, Chapter 2, «B») of the point p satisfies the conditions

$$\bar{A}_t \subset B, \quad \text{where} \quad A_t = f_t^{-1}(K_\alpha), \quad t = 0, 1, \tag{2}$$

and let ω_α be a mapping of the sphere S^n onto itself corresponding to the chosen α (see § 2, Chapter 2, «A»). Since the mapping ω_α is homotopic to the identity, the mappings $\omega_\alpha f_t$ and f_t, $t = 0, 1$ are homotopic. Assume that B is a unit ball of some Euclidean space R^n. Then there exists such a positive number $\beta < 1$ that the ball B_β of radius β concentric to B contains the sets $\bar{A}_t, t = 0, 1$. Let λ_β be the mapping of the space R^n to the sphere S^n described in Statement «B» § 2, Chapter 2. Define the mapping ϑ from M^n to the sphere S^n to coincide with λ_β on the ball B and to take the set $M^n \backslash B$ to the point q. Since the set $\bar{A}_t, t = 0, 1$ is contained in B_β the mapping ϑ is homeomorphic on $\bar{A}_t$.

Now, let us define the mapping g_t, $t = 0, 1$ of the sphere S^n onto itself as follows. On the set $\vartheta(A_t)$, we set $g_t = \omega_\alpha f_t \vartheta^{-1}$, and for $x \in S^n \backslash \vartheta(A_t)$ we put $g_t(x) = q$. From this definition of g_t it follows that

$$g_t \vartheta = \omega_\alpha f_t, \quad t = 0, 1. \tag{3}$$

The mappings f_t and $\omega_\alpha f_t$, clearly, have the same degree at p, and from (3) it follows that the mappings g_t and f_t have the same degree p as well. Since the mappings f_0 and f_1 have the same degree, so do the mappings g_0 and g_1 of the sphere S^n onto itself. Thus, the mappings g_0 and g_1 of the sphere S^n onto itself are homotopic (see Theorem 12). This yields the mappings $g_0 \vartheta$ and $g_1 \vartheta$ from M^n to S^n are homotopic, thus, so are the mappings $\omega_\alpha f_0$ and $\omega_\alpha f_1$ from M^n to S^n [see (3)]. Since the mappings $\omega_\alpha f_0$ and $\omega_\alpha f_1$ are homotopic to f_0 and f_1, respectively, the latter two are homotopic.

One can easily construct a mapping $M^n \to S^n$ of any preassigned degree.

Thus, Theorem 13 is proved.

§ 2. The Hopf invariant of mappings $\Sigma^{2k+1} \to S^{k+1}$

The Hopf invariant plays an important role in the homotopy classification of mappings from sphere to sphere. This invariant was first introduced for constructing infinitely many mapping classes of $S^3 \to S^2$ [12]. Later, this invariant was defined by Hopf for mappings from the $(2k+1)$-sphere to the $(k+1)$-sphere. However, for even k, this invariant always equals zero. The Hopf invariant is defined to be the link coefficient of the preimages of two different points of the sphere S^{k+1} in the sphere Σ^{2k+1}. In the present section, we first give the definition of the linking coefficient for two manifolds according to Brouwer [13], i.e. by means of the mapping degree, and not by means of the intersection index, as is usually done now. The form presented here is more convenient for this work. Later, we define the Hopf invariant and, finally, this invariant is described in terms of framed manifolds corresponding to the mapping. Besides, we establish several connection between properties of framed manifolds and properties of the Hopf invariant. These connections play a key role for the classification of mappings from S^{n+2} to S^n.

The linking coefficient

Definition 9. Let M^k and N^l be two closed smooth oriented manifolds of dimensions k and l, respectively, and let f and g be their continuous mappings to the oriented Euclidean space E^{k+l+1} of dimension $k+l+1$ so that the sets $f(M^k)$ and $g(N^l)$ are disjoint. Furthermore, let S^{k+1} be the unit sphere of the space E^{k+l+1} centred at an arbitrary point O taken with the orientation representing it as the boundary of the ball, and let $M^k \times N^l$ be the oriented direct product (see § 1, Chapter 1, «K») of the manifolds M^k and N^l. To each point $(x, y) \in M^k \times N^l, x \in M^k, y \in N^l$, there corresponds a non-zero interval $(f(x), g(y))$ in the space E^{k+l+1}, going from the point $f(x)$ to the point $g(y)$. Construct a ray emanating from O and parallel to $(f(x), g(y))$. Denote the intersection of this ray with S^{k+1} by $\chi(x, y)$. The mapping degree of $\chi : M^k \times N^l \to S^{k+l}$ (see Definition 8) is called the *linking coefficient* of the manifolds (f, M^k) and (g, N^l); it is denoted by $\mathfrak{v}((f, M^k), (g, N^l))$. It is evident that if we continuously deform the mappings f and g: $f = f_t$, $g = g_t$ in such a way that the sets $f_t(M^k)$ and $g_t(N^l)$ remain disjoint for arbitrary t, then the mapping $\chi = \chi_t$ is deformed continuously as well, thus, the linking coefficient does not change. In the partial case when M^k and N^l are submanifolds of the space E^{k+l+1}, and the mappings f and g are identical, the linking coefficient is defined as well; it is then denoted by $\mathfrak{v}(M^k, N^l)$. It turns out that

$$\mathfrak{v}((f, M^k), (g, N^l)) = (-1)^{(k+1)(l+1)}\mathfrak{v}((f, M^k), (g, N^l)). \qquad (1)$$

Let us prove (1). Let χ' be the mapping from $N^l \times M^k$ to S^{k+1} analogous to χ constructed above. Denote by λ the mapping from $N^l \times M^k$ to $M^k \times N^l$ taking (y, x) to (x, y); let μ be the mapping of the sphere S^{k+1} onto itself taking each point to its antipode. It is clear that the mapping degree of λ is equal to $(-1)^{kl}$, and the degree of μ equals $(-1)^{k+l+1}$. It is easy to see that $\chi' = \mu\chi\lambda$. From the above we get (1).

A) Suppose instead of one manifold (g, N^l) we have two mapped manifolds (g_0, N_0^l) and (g_1, N_1^l). Furthermore, assume there exists an oriented bounded compact manifold N^{l+1} with oriented boundary consisting of the manifolds N_0^l and $-N_1^l$, and there exists a mapping g from N^{l+1} to E^{k+l+1} coinciding with g_0 on N_0^l and coinciding with g_1 on N_1^l such that the sets $f(M^k)$ and $g(N^{l+1})$ are disjoint. Then it turns out that

$$\mathfrak{v}((f, M^k), (g_0, N_0^l)) = \mathfrak{v}((f, M^k), (g_1, N_1^l)). \qquad (2)$$

Let us prove this fact. The boundary of the manifold $M^k \times N^{l+1}$ is $M^k \times N_0^l - M^k \times N_1^l$. To each point $(x, y) \in M^k \times N^{l+1}$, there corresponds the interval $(f(x), g(y))$. Let us draw a ray from O parallel to the interval $(f(x), g(y))$. Denote the intersection of this ray with the sphere S^{k+1} by $\chi(x, y)$. Thus, we get a continuous mapping χ from $M^k \times N^{l+1}$ to the sphere S^{k+1}; thus the degree of χ on its boundary equals zero (see § 1, «A»). Thus yields (2).

The Hopf invariant

Definition 10. Let f be a smooth mapping from the oriented sphere Σ^{2k+1} of dimension $2k+1$ to the oriented sphere S^{k+1} of dimension $k+1$, $k \geqslant 1$. Let p' and q' be the north pole and the south pole of the sphere Σ^{2k+1}; let E^{2k+1} be the tangent space to the sphere Σ^{2k+1} at the point p' and let φ be the central projection from $\Sigma^{2k+1} \backslash q'$ to the space E^{2k+1}. On the sphere S^{k+1}, let us choose (see Theorem 4) two distinct proper points a_0 and a_1 from $f(q')$ of the mapping f; then $M_0^k = \varphi f^{-1}(a_0)$ and $M_1^k = \varphi f^{-1}(a_1)$ are closed oriented submanifolds of the Euclidean space E^{2k+1} (see Introduction to § 4, Chapter 1: orientation for the preimage of a point). Set

$$\gamma(f) = \gamma(f, p', a_0, a_1) = \mathfrak{v}(M_0^k, M_1^k). \tag{3}$$

It turns out that $\gamma(f)$ is a homotopy invariant of the mapping f, which does not depend on the choice of points p', a_0 and a_1, and that for even k this invariant is equal to zero.

Let us prove the invariance of $\gamma(f)$.

Let f_0 and f_1 be two smooth homotopic mappings from Σ^{2k+1} to S^{k+1} and let f_t be a smooth deformation connecting them. For the deformation f_t, we have the corresponding mapping f_* from the product $\Sigma^{2k+1} \times I$ to S^{k+1} (see § 1, Chapter 2, «C»). Note that for a small enough movement of a_0 and a_1, the number $\gamma(f_t)$, $t = 0, 1$, does not change since the manifolds $\varphi f_t^{-1}(a_i)$ are not drastically deformed. Thus, we may assume that the curve $f_t(q')$, $0 \leqslant t \leqslant 1$ does not pass through a_0 or a_1. Let r be such a large positive integer that for $|t' - t| < \frac{1}{r}$, the sets $f_t^{-1}(a_0)$ and $f_t^{-1}(a_1)$ are disjoint. Now, let us move the points a_0 and a_1 in such a way that they become proper points of the mapping f_* and the mappings

$$f_t;\ t = 0, \frac{1}{r}, \ldots, \frac{r-1}{r}, 1.$$

Let us prove that

$$\gamma(f_1) = \gamma(f_0).$$

Denote the part of I, consisting of those points for which $\frac{s}{r} \leqslant t \leqslant \frac{s+1}{r}$, by I_s. Let $M^{k+1}_{s,i}$ be the full pre-image of the point a_i in the strip $M^k \times I_s$ under f_*. According to the conditions on a_0 and a_1, the set $M^{k+1}_{s,i}$ is an oriented submanifold of the manifold $\Sigma^{2k+1} \times I$, having the manifold $-f^{-1}_{s/r}(a_i) + f^{-1}_{(s+1)/r}(a_i)$ as its boundary. Denote the projection operator from $\Sigma^{2k+1} \times I$ along I to the sphere Σ^{2k+1} by π. The mapping $\varphi\pi$ of the manifold $M^{k+1}_{s,i}$ defines the mapped manifold $(\varphi\pi, M^{k+1}_{s,i})$ with boundary $-\varphi f^{-1}_{s/r}(a_i) + \varphi f^{-1}_{(s+1)/r}(a_i)$. Since the sets $\varphi\pi(M^{k+1}_{s,0})$ and $\varphi\pi(M^{k+1}_{s,1})$ are disjoint, it follows from «A» that

$$\gamma(f_{(s+1)/r}) = \gamma(f_{s/r});$$

thus, $\gamma(f_1) = \gamma(f_0)$.

Let us prove now that $\gamma(f, p', a_0, a_1)$ does not depend on the choice of the points a_0 and a_1. Suppose instead of a_0 and a_1, we have chosen b_0 and b_1. Then there exists a smooth homeomorphism λ of the sphere S^{k+1} onto itself homotopic to the identity, such that $\lambda(a_i) = b, i = 0, 1$. Clearly, $\gamma(\lambda f, p', b_0, b_1) = \gamma(f, p', a_0, a_1)$, and since the mappings λf and f are homotopic, according to what we have proved above, we get $\gamma(f, p', b_0, b_1) = \gamma(f, p', a_0, a_1)$.

Analogously, it can be proved that $\gamma(f, p', a_0, a_1)$ does not depend on the choice of p', since there exists a twisting of the sphere Σ^{2k+1} taking p' to any preassigned point of the sphere Σ^{2k+1}.

Finally, let us show that for even k the invariant $\gamma(f)$ is equal to zero. Since $\gamma(f)$ does not depend on the choice p_0 and p_1, we may change their roles; thus we have

$$\mathfrak{v}(M^k_0, M^k_1) = \mathfrak{v}(M^k_1, M^k_0).$$

Since, moreover, we have (1),

$$\mathfrak{v}(M^k_1, M^k_0) = (-1)^{(k+1)^2}\mathfrak{v}(M^k_0, M^k_1),$$

then for even k we get $\mathfrak{v}(M^k_0, M^k_1) = 0$.

The Hopf invariant of a framed manifold

Since homotopy classes of mappings from the $(2k+1)$-sphere to the $(k+1)$-sphere are in one-to-one correspondence with homology classes of k-dimensiobnal framed manifolds of the Euclidean $(2k+1)$-space, the invariant $\gamma(f)$ can be interpreted as an invariant of homology classes of framed k-dimensional manifolds in the $(2k+1)$-space. Let us give this interpretation of $\gamma(f)$ explicitly.

B) Let (M^k, U), $U(x) = \{u_1(x), \ldots, u_{k+1}(x)\}$, be a framed submanifold of the oriented Euclidean space E^{2k+1} and let N_x be the normal subspace to the manifold M^k at $x \in M^k$. The normal subspace is a vector space having origin at x; thus $U(x)$ is a basis of the space N_x. Let us choose an arbitrary vector $c = \{c^1, \ldots, c^{k+1}\}$ of the Euclidean space N (with Cartesian coordinates fixed) and associate with any point $x \in M^k$ the point $c(x) = c^1 u_1(x) + \ldots + c^{k+1} u_{k+1}(x)$ of the space N_x. For c reasonably small, the map c is a homeomorphism from M^k to the space E^{2k+1} (see § 1, Chapter 2, «A»). It is evident that for $c \neq 0$ the manifolds M^k and $c(M^k)$ are disjoint and that for two distinct non-zero vectors c and c', the manifolds $c(M^k)$ and $c'(M^k)$ are homotopic in the space $E^{2k+1} \backslash M^k$. Thus, for c reasonably small (but non-zero) the linking coefficient $\mathfrak{v}(M^k, c(M^k))$ does not depend on c; set

$$\gamma(M^k, U) = \mathfrak{v}(M^k, c(M^k)).$$

It turns out that if $f \to (M^k, U)$ (see Definition 5) then

$$\gamma(f) = \gamma(M^k, U). \tag{4}$$

Since $\gamma(f)$ is a homotopy invariant of f, $\gamma(M^k, U)$ is a homology invariant of the framed manifold (M^k, U).

Let us prove (4). Let f be a smooth mapping from the sphere Σ^{2k+1} to the sphere S^{k+1} and let $p \in S^{k+1}$ be a proper point of f, distinct from $f(q')$. Then, in order to construct the manifold (M^k, U) corresponding to the map f, one should take p to be the north pole of the sphere S^{k+1} (see Definition 5). Let $e_1, \ldots, e_{k+1}$ be an orthonormal basis of the plane, tangent at the point p to the sphere S^{k+1} and let $x^1, \ldots, x^{k+1}$ be the coordinates corresponding to this basis in the domain $S^{k+1} \backslash q$ (see § 2, Chapter 2, «A»). In order to construct the invariant $\gamma(f)$, we take the point a_0 to be p, and set a_1 to be the point with coordinates $x^1 = c^1, \ldots, x^{k+1} = c^{k+1}$. Such a choice of the points a_0 and a_1 means that the manifold M_0^k coincides with the manifold M^k, whence the manifold M_1^k is second-order close to the manifold $c(M^k)$ with respect to the length of the vector c. Thus, $\mathfrak{v}(M^k, c(M^k)) = \mathfrak{v}(M_0^k, M_1^k)$, and (4) is proved.

C) Let Π_{k+1}^k be the homology group of framed k-dimensional manifolds of the Euclidean space E^{2k+1}. With each element $\pi \in \Pi_{k+1}^k$ we associate the integer $\gamma(\pi) = \gamma(M^k, U)$, where (M^k, U) is a framed manifold representing the class π. As shown above (see «B»), the number $\gamma(\pi)$ depends only on π and does not depend on the arbitrary choice of the manifold (M^k, U). It turns out that γ is a homeomorphism from the group Π_{k+1}^k to the additive group of integers. Thus, it follows that the set $\hat{\Pi}_{k+1}^k$ of all elements $\pi \in \Pi_{k+1}^k$ for which $\gamma(\pi) = 0$ is a subgroup of Π_{k+1}^k.

Let us prove statement «C». Let π_1 and π_2 be any two elements of the group Π^k_{k+1} and let (M^k_1, U_1) and (M^k_2, U_2) be the framed manifolds representing the classes π_1 and π_2, respectively, and lying on different sides of some hyperplane E^{2k} of the space E^{2k+1}. Furthermore, let S^{2k} be the unit sphere of the space E^{2k+1} centred at $O \subset E^{2k}$. Let us choose an arbitrarily small vector c defining the shift of the manifold $M^k_1 \cup M^k_2$ (see «B»). We have

$$\gamma(\pi_1 + \pi_2) = \mathfrak{v}(M^k_1 \cup M^k_2, c(M^k_1 \cup M^k_2)).$$

The linking coefficient in the right-hand side is defined as the degree of the mapping χ from $(M^k_1 \cup M^k_2) \times c(M^k_1 \cup M^k_2)$ to the sphere S^{2k}; herewith the mapping χ is constructed as in Definition 9. Let us define the degree of χ in some point p of the sphere S^{2k}, lying close to the hyperplane E^{2k}. Such a choice of the point p guarantees that the interval $(x, c(y))$, where $x \in M^k_1$, $y \in M^k_2$, is not parallel to the interval (O, p). Analogously, the interval $(x, c(y))$, where $x \in M^k_2$, $y \in M^k_1$, is not parallel (O, p). This yields that

$$\mathfrak{v}(M^k_1 \cup M^k_2, c(M^k_1 \cup M^k_2)) = \mathfrak{v}(M^k_1, c(M^k_1)) + \mathfrak{v}(M^k_2, c(M^k_2)),$$

that is,

$$\gamma(\pi_1 + \pi_2) = \gamma(\pi_1) + \gamma(\pi_2).$$

Thus, «C» is proved.

D) Let f be a smooth mapping from the oriented sphere Σ^{2k+1} to the oriented sphere S^{k+1}, and let g be the mapping of Σ^{2k+1} onto itself having degree σ, and let h be the mapping of the sphere S^{k+1} onto itself of degree m. Set $f' = hfg$. It turns out that

$$\gamma(f') = \sigma\tau^2\gamma(f). \tag{5}$$

It is sufficient to prove statement «D» separately for the case when h is the identity and for the case when the mapping g is the identity. The relation (5) in the case of h being the identity follows from Statement «C» of the present section and from Statement «D» § 1, Chapter 3. Let us consider the case when g is the identity, i.e. when $f' = hf$. Let a_0 and a_1 be two different points of S^{k+1} distinct from $f'(q')$, which are proper points of the mappings h and hf. Then $h^{-1}(a_t) = \{a_{t1}, \ldots, a_{tr_t}\}, t = 0, 1$, whence the mapping f is proper at any of the points a_{ti}, $t = 0, 1$; $i = 1, 2, \ldots, r_t$. Denote the sign of the functional determinant of the mapping h at the point a_{ti} by ε_{ti}, $i = 1, \ldots, r_t$; $t = 0, 1$. Denote the tangent subspace at the north pole p' of the sphere Σ^{2k+1} by E^{2k+1} and denote the central projection mapping from the set $\Sigma^{2k+1} \backslash q'$ to the tangent space of the point q' by φ. Set $\varphi f'^{-1}(a_t) = M^k_t$, $t = 0, 1$; $\varphi f^{-1}(a_{ti}) = M^k_{ti}$. It is easy to see that

$$M^k_t = \varepsilon_{t1} M^k_{t1} \cup \varepsilon_{t2} M^k_{t2} \cup \ldots \cup \varepsilon_{tr_t} M^k_{tr_t}, \tag{6}$$

where the signs ε_{ti} agree with the orientations of the pre-images. Since a_{0i} and a_{1j} are two distinct points of the sphere S^{k+1} which are proper points of the mapping f then the invariant $\gamma(f)$ can be defined as $\mathfrak{v}(M^k_{0i}, M^k_{1j})$. From this and from (6) we have

$$\gamma(f) = \mathfrak{v}(\varepsilon_{01}M^k_{01} \cup \ldots \cup \varepsilon_{0r_0}M^k_{0r_0}, \varepsilon_{11}M^k_{11} \cup \ldots \cup \varepsilon_{1r_1}M^k_{1r_1}) \\ = \sum_{i=1}^{r_0}\sum_{j=1}^{r_1} \varepsilon_{0i}\varepsilon_{1j}\gamma(f) = \gamma(f)(\sum_{i=1}^{r_0}\varepsilon_{0i})(\sum_{j=1}^{r_1}\varepsilon_{1j}) = \tau^2\gamma(f).$$

Thus, Statement «D» is proved.

E) Let (M^k, V), $V(x) = \{v_1(x), \ldots, v_{k+1}(x)\}$, be an orthonormally and smoothly framed submanifold of the oriented Euclidean space E^{2k+1}, and assume that the manifold M^k lies in a hyperplane E^{2k} of the ambient space. Denote by $u(x)$ the unit vector emanating from $x \in M^k$ and perpendicular to E^{2k}. Then we have

$$u(x) = \psi^1(x)v_1(x) + \ldots + \psi^{k+1}(x)v_{k+1}(x). \tag{7}$$

Here $\psi(x) = \{\psi^1(x), \ldots, \psi^{k+1}(x)\}$ is a unit vector of the coordinate Euclidean space N such that ψ maps the manifold M^k to the unit sphere $\mathfrak{S}^k$ of the space N (the mapping ψ was considered in Statement «A» § 4, Chapter 2). It turns out that the degree of the mapping ψ is equal to $\varepsilon\gamma(M^k, V)$, where $\varepsilon = \pm 1$ and the sign depends only on k.

Let us prove «E». Assume that $\mathfrak{P} = (0, \ldots, 0, 1) \in \in \mathfrak{S}^k$ is a proper point of ψ. If this were not so, we could easily obtain it by an orthogonal transformation of all systems $V(x)$, $x \in M^k$. To calculate $\gamma(M^k, V)$, let us choose in the space E^{2k+1} the unit sphere S^{2k} centred at some point O and take the vector c to be the vector $\{0, \ldots, 0, \delta\}$. If we move the vector $u(x)$ parallel to the point O then its end will lie on the sphere S^{2k} at some point; denote the latter point by u. Draw a ray from O, parallel to the interval $(x, c(y))$; x, $y \in M^k$; denote the intersection of this ray with S^{2k} by $\chi(x, y)$. By definition, $\gamma(M^k, V)$ is the degree of the mapping χ from $M^k \times M^k$ to S^{2k}. We shall calculate the degree of this mapping at the point u. While calculating, we will show that u is a proper point of the mapping χ. Let $\chi(a, b) = u$, then the interval $(a, c(b))$ is orthogonal to the hyperplane E^{2k} and directed along the vector u in such a way that $c(b) \in \overline{H_\delta(a)}$ (see § 1, Chapter 2, «A»). Since, furthermore, $c(b) \in \overline{H_\delta(b)}$, it follows that for δ small enough, we have $b = a$ (see § 1, Chapter 2, «A»). Thus, for $\chi(a, b) = u$ we have $b = a$ and $\psi(a) = \mathfrak{P}$. Conversely, if $\psi(a) = \mathfrak{P}$, then $\chi(a, a) = u$. Take a to be the origin of coordinates O of the space E^{2k+1}; take its basis to consist of vectors $u_1 = u_1(a), \ldots, u_{k+1} = u_{k+1}(a), u_{k+2}, \ldots, u_{2k+1}$, where $u_{k+2}, \ldots, u_{2k+1}$ is an orthonormal vector system tangent to the manifold M^k at the point a. Denote the coordinates of $x \in M^k$ in this basis by $z^1(x), \dots, z^{2k+1}(x)$. In a neighbourhood of a in M^k, it is easy to introduce coordinates $x^1, \dots, x^k$ of the point x such that the equation defining the manifold M^k looks like

$$z^1 = z^1(x), \dots, z^{k+1} = z^{k+1}(x), z^{k+2} = z^{k+2}(x) = x^1, \dots \\ \dots, z^{2k+1} = z^{2k+1}(x) = x^k, \tag{8}$$

where $z^i(x)$, $i = 1, \dots, k+1$, is second-order small with respect to $\rho(a, x)$. Transporting the system $V(y)$ parallel to the point $O = a$, we express its vectors in terms of $u_1, \dots, u_{2k+1}$:

$$v_j(y) = \sum_{\alpha=1}^{k+1} a_{j\alpha}(u)u_\alpha + \sum_{\beta=k+2}^{2k+1} b_{j\beta}(y)u_\beta. \tag{9}$$

Here $b_{j\beta}$ are second order small with respect to $\rho(a, y)$ and $a_{j\alpha}$, $\alpha \neq j$, are first-order small with respect to $\rho(a, y)$. Thus, since $V(y)$ is orthonormal, we see that, up to second-order small values (resp. to $\rho(a, y)$) the following equalities hold:

$$a_{ji}(y) = 1, \quad a_{ji}(y) = -a_{ji}(y), \quad i \neq j. \tag{10}$$

Since $a_{ij}(y) = (u_i, v_j(y))$, i, $j = 1, \dots, k+1$ then, by (7), (9) and (10), we have (up to second order) $\rho(a, y)$ $\psi^j(y) = -a_{k+1,j}(y)$, $j = 1, \dots, k$; $\psi^{k+1}(y) = 1$. Thus, up to the second order in $\rho(a, y)$ the point $c(y)$ has in the basis $u_1, \dots, u_{2k+1}$ the following coordinates: $-\delta\psi^1(y), \dots, -\delta\psi^k(y), \delta, y^1, \dots, y^k$. Analogously, the point x (up to second order) has in the basis $u_1, \dots, u_{2k+1}$ the coordinates [see (8)]

$$0, \dots, 0, \quad x^1, \dots, x^k.$$

Thus, the components of the interval $(x, c(y))$ in the basis $u_1, \dots, u_{2k+1}$ are

$$-\delta\psi^1(y), \dots, -\delta\psi^k(y), \delta, y^1 - x^1, \dots, y^k - x^k$$

up to second order with respect to $\rho(a, x) + \rho(a, y)$. From this, it follows that at the point (a, a) the sign of the functional determinant of the mapping χ differs from that of the functional determinant of ψ in a by a factor $\varepsilon = \pm 1$, which depends only on k. Thus, «E» is proved.

§ 3. Framed manifolds with Hopf invariant equal to zero

The main goal of this section is to prove Theorem 16 that any framed manifold having Hopf invariant equal to zero is homologous to a suspension.

Thus the theorem is a sequel to Theorem 11. Since the Hopf invariant of an even-dimensional manifold always equals zero, it follows from Theorem 16 that each even-dimensional framed sumbanifold (M^k, U) of E^{2k+1} is homologous to a suspension. This statement will be used in the present work only for the case $k = 2$ while classifying mappings $\Sigma^{n+2} \to S^n$. From this and Theorem 11 it follows that the number of mapping classes for $\Sigma^{n+2} \to S^n$, $n \geqslant 2$, does not exceed the number of mapping classes $\Sigma^4 \to S^2$.

While proving Theorem 16 as well as in some other cases it is desirable to deal with connected framed manifolds. Theorem 14 says that each framed manifold is homologous to a smooth one. To prove this theorem, it is sometimes necessary to perform a surgery of a manifold in order to make it connected. Such a surgery has a rather bergthy description in the following Statement «A», but its geometrical sense is clear and means the following.

The equation

$$x^2 + y^2 - z^2 = -t$$

represents a two-sheeted hyperboloid for $t > 0$ and a one-sheeted hyperboloid for $t < 0$. In the strip of the space of the variables x, y, z, t, defined by the inequality $-1 \leqslant t \leqslant 1$, the above equation defines a submanifold with boundary consisting of the two parts: the disconnected one lying in $t = -1$, and the connected part lying in $t = 1$.

In Statement «A», the surgery described above, is performed for a pair of parallel planes. In these planes, we get "dents", which move towards each other like sheets of the two-sheeted hyperboloid, then form a tube connecting the holes in the planes. To perform the operation described above to an arbitrary manifold, we prove an almost obvious Statement «C» that in a neighbourhood of any point, the manifold can be deformed to a plane. Making the manifold planar in the neighbourhood of two points, we may perform the surgery «A» connecting the two components into a single one. Since we have to reconstruct framed manifolds, one should also care about what happens to the framings. These constructions are discussed in Statements «B» and «D». The surgery «A» can be applied not only in order to get a connected manifold but also in order to embed a k-dimensional manifold in $2k$-space.

The surgery

A) Let E^{k+2} be the Euclidean space with coordinates $\xi^1, \ldots, \xi^k$, η, τ; let E_*^{k+2} be the strip defined by inequalities $-1 \leqslant \tau \leqslant +1$, with boundary consisting of two hyperplanes E_{-1}^{k+1} and E_{+1}^{k+1} defined by $\tau = -1$ and $\tau = +1$. Let H^{k+2} be the part of E^{k+2} defined by the inequalities

$$(\xi^1)^2 + \ldots + (\xi^k)^2 \leqslant 1, \quad -1 \leqslant \eta \leqslant 1.$$

It turns out that in the strip E_*^{k+2} there exists a smooth submanifold P^{k+1} orthogonal to the boundary of the strip E_*^{k+2} in its boundary points and possessing the following properties (see Fig. 3.1):

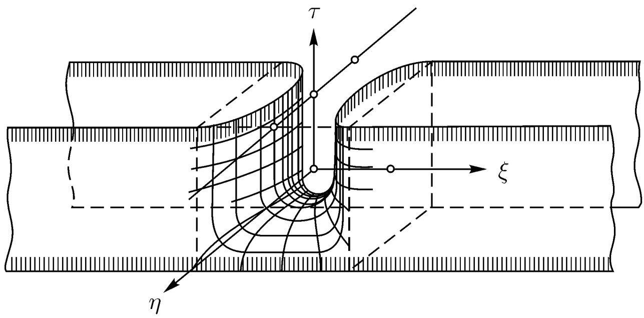


Figure 3.1.

a) outside H^{k+2} the manifold P^{k+1} consists of all points satisfying $|\eta| = 1$;

b) the manifold $P_{-1}^k = P^{k+1} \cap E_{-1}^{k+1}$ consists of all points of the hyperplane E_{-1}^{k+1} satisfying $|\eta| = 1$;

c) the intersection of $P_1^k = P^{k+1} \cap E_{+1}^{k+1}$ with the hyperplane defined by $\eta = \alpha$, for $|\alpha| < 1$ is a sphere of radius $\varrho(\alpha) < 1$, defined in the plane $\eta = \alpha$, $\tau = 1$ by the equation $(\xi^1)^2 + \ldots + (\xi^k)^2 = \varrho^2(\alpha)$, where $\varrho(\alpha)$ tends to 1 as $|\alpha|$ tends to 1. Thus, the set $P_1^k \cap H^{k+2}$ intersects the line $\xi^1 = 0, \ldots, \xi^k = 0; \tau = 1$; moreover, this set is connected for $k > 1$ and consists of two simple arcs for $k = 1$.

For constructing P^{k+1}, let us first consider the case $k = 1$. Rename the coordinates ξ^1, η, τ in E^3 as x, y, t. Let

$$\varphi(x, y, t) = y^2 - (1 + t)x^2 + t.$$

Consider the surface Q^2 given by the equation $\varphi(x, y, t) = 0$. It can be checked straightforwardly that this surface has no singular points, i.e. that the equations

$$\frac{\partial \varphi}{\partial x} = 0, \quad \frac{\partial \varphi}{\partial y} = 0, \quad \frac{\partial \varphi}{\partial t} = 0, \quad \varphi = 0$$

are not compatible. Consider the section C_β of the surface Q^2 by the plane $t = \beta$ ($|\beta| \leqslant 1$). The curve C_{-1} then becomes a pair of parallel lines $y = \pm 1$. For $-1 < \beta < 0$, the curve C_β is a hyperbola whose real axis is the line $t = \beta$, $x = 0$. The curve C_0 represents a pair of intersecting lines $y = \pm x$. Finally, for $0 < \beta < +1$ the curve C_β is a hyperbola with real axes being $t = \beta$, $y = 0$. For all values of β, the curve C_β passes through the points $(\pm 1, \pm 1, \beta)$; furthermore, it is symmetric with respect to $x = 0$ and $y = 0$. In our case, the set H^3 is a cube defined by $|x| \leqslant 1$, $|y| \leqslant 1$, $|t| \leqslant 1$. Let us complete the part Q^2_* of the surface Q^2, lying in the cube H^3, by the points satisfying $|y| = 1$, $|x| \geqslant 1$, $|t| \leqslant 1$. Denote the obtained surface by $\hat{P}^2$. The surface $\hat{P}^2$ satisfies the conditions «(a)»—«(c)», but it is not smooth; moreover, it is not orthogonal to the strip of the boundary E^3_* ($|t| \leqslant 1$).

Now, consider the case of arbitrary k. Define the function $\varphi(x^1, \ldots, x^k, y, t)$ by setting

$$\varphi(x^1, \ldots, x^k, y, t) = y^2 - (1 + t)((x^1)^2 + \ldots + (x^k)^2) + t.$$

It follows straightforwardly that the hypersurface Q^{k+1} defined in the space E^{k+2} with coordinates $x^1, \ldots, x^k, y, t$ by $\varphi(x^1, \ldots, x^k, y, t) = 0$ has no singular points, i.e. the equations

$$\frac{\partial \varphi}{\partial x^1} = 0, \ldots, \frac{\partial \varphi}{\partial x^k} = 0, \frac{\partial \varphi}{\partial y} = 0, \frac{\partial \varphi}{\partial t} = 0, \varphi = 0$$

are not compatible. The hypersurface Q^{k+1} can be intuitively represented if we note that its section by any 3-space containing (y, t) is the surface Q^2 described above. Set $Q^{k+1}_* = Q^{k+1} \cap H^{k+2}$. Now, complete the set Q^{k+1}_* by the points satisfying $|y| = 1$, $(x^1)^2 + \ldots + (x^k)^2 > 1$, $|t| \leqslant 1$. The set $\hat{P}^{k+1}$, obtained, is a manifold satisfying the conditions «(a)»—«(c)», but it is not smooth in the points where it intersects the boundary of H^{k+2}. Moreover, it is not orthogonal to the boundary of the strip E^{k+2}_*. Let us now correct the manifold $\hat{P}^{k+1}$.

Let $\chi(s)$ be an m-smooth ($m \geqslant 1$), odd, monotonically increasing function of one variable s, defined on the interval $-1 \leqslant s \leqslant 1$, enjoying the following properties:

$$\chi(-1) = -1, \quad \chi(1) = 1,$$

$$\chi'(-1) = \chi''(-1) = \ldots = \chi^{(m)}(-1)$$

$$= \chi'(1) = \chi''(1) = \ldots = \chi^{(m)}(1) = 0,$$

$$\chi'(s) > 0 \quad \text{for} \quad |s| < 1.$$

Clearly, such a function exists. Now, let us define the mapping σ

$$\sigma(x^1, \ldots, x^k, y, t) = (\xi^1, \ldots, \xi^k, \eta, \tau)$$

of H^{k+2} onto itself by

$$\xi^1 = x^1, \dots, \xi^k = x^k, \eta = \chi(y), \tau = \chi^{-1}(t),$$

where χ^{-1} is the function inverse to χ. Obviously the mapping σ of H^{k+2} onto itself is homeomorphic and the mappings σ and σ^{-1} are smooth at all points of the set H^{k+2}. The map σ is not smooth only for $t = \pm 1$, and σ^{-1} is not smooth only for $\eta = \pm 1$. It is easy to check that if we replace the part Q_*^{k+1} of the manifold $\hat{P}^{k+1}$ by the set $\sigma(Q_*^{k+1})$, we get a manifold P^{k+1} satisfying all properties of Statement «A».

B) Let W^{k+2} be the ε-neighbourhood of the set H^{k+2} in the Euclidean space E^{k+2} (see «A») and let ϑ be a smooth homeomorphism to the Euclidean space E^{n+k+1}. Then there exists a framing $V(\zeta) = \{v_1(\zeta), \dots, v_n(\zeta)\}$ of the manifold $\vartheta(P^{k+1} \cap W^{k+2})$ in the space E^{n+k+1}, inducing the given orientation of the manifold.

Let us prove Statement «B». Let O be the centre of the figure H_{k+2} and let X be the boundary of the convex set W^{k+2}. Furthermore, let ζ be an arbitrary point from W^{k+2} and let (O, x) be the interval passing through ζ and connecting the point O with a boundary point $x \in X$. Denote the ratio of the lengths (O, ζ) and (O, x) by t and set $\zeta = (x, t)$. Thus, we have introduced a polar coordinate system in the domain W^{k+2}; here $(x, 0) = O$. Denote the normal subspace at the point $\vartheta(\zeta)$ to the manifold $\vartheta(W^{k+2})$ in the space E^{n+k+1} by N_{xt}. In the space N_{x0}, let us choose an arbitrary basis $v_1, \dots, v_{n-1}$. By virtue of Statement «A» §3, Chapter 2 the basis $v_1(x,t), \dots, v_{n-1}(x,t)$ of the normal N_{xt} can be chosen to depend smoothly on the pair (x,t) and to coincide for $t = 0$ with the basis $v_1, \dots, v_{n-1}$. Set $v_1(\zeta) = v_i(x,t)$, $i = 1, \dots, n-1$. The vector $v_n(\zeta)$ at the point $\vartheta(\zeta)$, where $\zeta \in P^{k+1} \cap W^{k+2}$, is then chosen to be the unit vector normal to the manifold $\vartheta(P^{k+1})$ at the point $\vartheta(\zeta)$ and tangent to the manifold $\vartheta(W^{k+2})$. These conditions define the vector $v_n(\zeta)$ up to sign. Since the manifold $P^{k+1} \cap W^{k+2}$ is connected, the whole field $v_n(\zeta)$ is uniquely defined up to sign;, thus, choosing the direction of $v_n(\zeta)$ in a proper way, we may guarantee that the constructed framing $V(\zeta)$, $\zeta \in P^{k+1} \cap W^{k+2}$ induces the given orientation on $\vartheta(P^{k+1} \cap W^{k+2})$.

C) Let M^k be a smooth submanifold of the Euclidean space E^{n+k}, $a \in M^k$; and let T^k be the tangent subspace to M^k at a and let δ be a positive number. It turns out that there exists a smooth deformation τ_t, $0 \leqslant t \leqslant 1$, of the manifold M^k, enjoying the following properties. Let $x \in M^k$; then:

a) for $\varrho(a, x) \geqslant \delta$ we have $\tau_t(x) = x$;

b) for $\varrho(a, x) \leqslant \delta$ the value $\varrho(x, \tau_t(x))$ is second-order small with respect to $\varrho(x, a)$, i.e. $\varrho(x, \tau_t(x)) < c\varrho^2(x, a)$, where c is a constant independent of δ;

c) for $\varrho(x,a) < \delta/2$ we have $\tau_1(x) \in T^k$.

Let us prove Statement «C». Assume δ to be so small that the orthogonal projection π to the plane T^k maps the δ-neighbourhood of the point a in M^k to T^k smoothly, regularly and homeomorpichally. Then let $\mu(s)$ be a smooth even function of the parameter s, $-\infty < s < +\infty$, equal to zero at $0 < s < \delta/2$, monotonically increasing for $\delta/2 \leqslant s \leqslant \delta$ and equal to 1 for $s > \delta$. Then the desired deformation τ_t is defined by

$$\tau_t(x) = x\lambda t + \pi(x)(1-\lambda)t + x(1-t),$$

where $\lambda = \mu(\varrho(x,a))$.

D) Let (M^k, U) be a framed submanifold of the strip E_*^{n+k} of the Euclidean space E^{n+k}, and let K' be such a neighbourhood of some interior point $a \in M^k$ that its closure $\bar{K}'$ is homeomorphic to the k-ball. We assume $\bar{K}'$ to be a ball centered at a; and let K be a smaller ball concentric with K'. If for a part $\bar{K}'$ of the manifold M^k, there is some framing V, inducing the same orientation of K' as the framing U then there exists a framing U' of the whole manifold M^k, homotopic to U and coinciding with it on $M^k \backslash K'$ and coinciding with the framing V on K.

Let us prove «D». Let

$$U(x) = \{u_1(x), \dots, u_n(x)\}, \quad V(x) = \{v_1(x), \dots, v_n(x)\};$$

then we have

$$u_i(x) = \sum_{j=1}^{n} \lambda_{ij}(x) v_j(x), \quad x \in \bar{K}',$$

where $\lambda(x) = \|\lambda_{ij}(x)\|$ is a positive determinant matrix depending continuously on the point $x \in \bar{K}'$ in such a way that λ is a continuous mapping from the ball $\bar{K}'$ to the manifold L_n of all $n \times n$ having positive determinant. We shall consider $\bar{K}'$ as a ball of the Euclidean space E^k, the latter being a hyperplane of the space E^{k+1}; let L be a linear interval of E^{k+1} perpendicular to the hyperplane E^k with one end at the centre a of $\bar{K}'$. Denote the other end of the interval L by b. One can easily construct a deformation ψ_t of the mapping from $\bar{K}'$ to the set $\bar{K}' \cup L$, under which all points of the boundary of $\bar{K}'$ remain fixed and as a result of this deformation the ball K is taken to the point b, i.e. $\psi_1(K) = b$. Since L_n is connected, then the mapping λ defined on $\bar{K}'$ can be extended to a continuous mapping λ from $\bar{K}' \cup L$ to L_n taking the point b to the identity matrix. Thus, $\|\mu_{ij}(x)\| = \mu(x) = \lambda\psi_1(x)$ is a matrix with positive determinant depending continuously on $x \in \bar{K}'$. Denote the framing U' for the ball $\bar{K}'$ by setting

$$u_i'(x) = \sum_{j=1}^{n} \mu_{ij}(x) v_j(x).$$

On the set $M^k \backslash K'$, we define the framing U' to coincide with U. Clearly, U' is precisely what we need.

Manifolds with zero Hopf invariant

Theorem 14. *Each framed submanifold of a Euclidean space is homologous to a connected framed submanifold of the same Euclidean space.*

Proof. Let (M^k_{-1}, U) be an oriented framed submanifold of the oriented space E^{n+k}, and let $n \geqslant 2$. The case $n = 1$ is not interesting since in this case the framed manifold is always null-homologous (see the end of § 2, Chapter 2). Suppose M^k_{-1} is not connected. Let us show that there exists a framed manifold (M^k_1, U_*) homologous to the initial one with one fewer connected component than M^k_{-1}. This will prove the Theorem. Let a_{-1} and a_1 be two points of M^k_{-1} belonging to different components of it. By virtue of Statement «C», we may assume that in the neighbourhood of a_{-1} and a_1 the manifold M^k_{-1} is planar. Since $n \geqslant 2$, the manifold M^k_{-1} does not divide the space E^{n+k}. This yields that in E^{n+k} there exists a simple closed curve L given by the parametric equation

$$y = y(\eta), \quad -2 \leqslant \eta \leqslant 2; \quad y(-2) = y(2),$$

intersecting M^k only in a_{-1} and a_1 for $\eta = -1$ and 1, respectively. Furthermore, suppose that L is orthogonal to M^k_{-1} at a_{-1} and a_1. By using Statement «A» § 3, Chapter 2, and the orthogonalization process, we may endow the interval $-1.5 \leqslant \eta \leqslant 1.5$ of L with an orthonormal framing, i.e. for each point $y(\eta)$ of this segment, construct a system of vectors $e_1(\eta), \dots, e_{n+k-1}(\eta)$, orthogonal to L at $y(\eta)$ and depending smoothly on η. We shall assume the vectors $e_1(-1), \dots, e_k(-1)$ to be tangent to the manifold M^k_{-1} at a_{-1} and define the orientation of this manifold, whence the vectors $e_1(1), \dots, e_k(1)$ are tangent to M^k_{-1} at a_1 and define the orientation opposite to the orientation of M^k_{-1}. This can be reached by applying an orthogonal transformation smoothly depending on η to the vectors $e_1(\eta), \dots, e_{n+k-1}(\eta)$. Let $E^{n+k+1}_* = E^{n+k} \times I$, where I is the interval $-1 \leqslant t \leqslant 1$; consider the Cartesian product E^{n+k+1}_* as a strip in the Euclidean space E^{n+k+1}. Now, let us construct the mapping ϑ of the subset H^{k+2} (see «A») of E^{k+2} to E^{n+k+1}, depending smoothly on positive parameter ϱ, and mapping the point $(\xi^1, \dots, \xi^k, \eta, \tau) \in H^{k+2}$ to the point $(z, t) \in E^{n+k+1}_*$:

$$\begin{aligned} z &= y(\eta) + \sum_{i=1}^{k} \varrho \xi^i e_i(\eta), \\ t &= \tau. \end{aligned} \tag{1}$$

Here $z, y(\eta)$ are vectors in E^{n+k}. The relations above define the mapping ϑ not only on the set H^{k+2} but also on some ε-neighbourhood W^{k+2} of this set in E^{k+2}. Obviously, for ϱ small enough the mapping ϑ is a smooth regular homeomorphism of the manifold W^{k+2}. For ϱ small enough the intersection of $\vartheta(W^{k+2})$ with the manifold $M^k_{-1} \times (-1)$ is contained in neighbourhoods of points $a_{-1} \times (-1)$ and $a_1 \times (-1)$. We assume ϱ so small that this intersection is contained in those neighbourhoods of $a_{-1} \times (-1)$ and $a_1 \times (-1)$ where the manifold $M^k_{-1} \times (-1)$ is planar. In the strip $E^{n+k} \times I$ we have the submanifold $M^k_{-1} \times I$. In this submanifold, let us replace its part lying in $\vartheta(H^{k+2})$ by $\vartheta(P^{k+1} \cap H^{k+2})$ (see «A»); namely, we set

$$M^{k+1} = (M^k_{-1} \times I \backslash \vartheta(H^{k+2})) \cup \vartheta(P^{k+1} \cap H^{k+2}).$$

Then M^{k+1} is a smooth submanifold of the strip E^{n+k+1}_* orthogonal at its boundary points to the boundary of this strip; moreover, the part of the boundary of M^{k+1} lying in the hyperplane $E^{n+k} \times (-1)$ coincides with the manifold $M^k_{-1} \times (-1)$, and the part $M^k_1 \times 1$ lying in $E^{n+k} \times 1$ has one fewer component than the manifold M^k_{-1}.

Now, let us construct the framing V of the manifold M^{k+1} that we need to prove the homology $(M^k_{-1}, U) \sim (M^k_1, U_*)$. The framing V of $\vartheta(P^{k+1} \cap H^{k+2})$ we chose in «B» is in such a way that in $a_{-1} \times (-1)$ the vectors $v_1, \dots, v_n$ have positive determinant with respect to the vectors $u_1 \times (-1), \dots, u_n \times (-1)$. By virtue of Statement «D», we may assume that the vectors $u_1 \times (-1), \dots, u_n \times (-1)$ coincide with the vectors $v_1, \dots, v_n$ in $M^k_{-1} \times (-1) \cap \vartheta(H^{k+2})$. Thus, we have constructed V for part of the manifold M^{k+1}, namely, for $\vartheta(P^{k+1} \cap H^{k+2})$. For the part, $M^{k+1} \backslash \vartheta(P^{k+1} \cap H^{k+2})$, at a point (x, t), $x \in M^{k+1}$, $t \in I$, we define $v_1, \dots, v_n$ to be parallel to the vectors $u_1 \times (-1), \dots, u_n \times (-1)$. Thus, the framed manifold (M^{k+1}, V) is constructed.

Theorem 14 is proved.

Theorem 15. *Let (M^k_{-1}, U) be a framed manifold of E^{n+k}, $n \geqslant k+1$. Then there exists a framed submanifold (M^k, W) of E^{n+k} homologous to (M^k_{-1}, U) such that the manifold M^k is connected and lies in a $2k$-dimensional linear subspace E^{2k} of the space E^{n+k}.*

PROOF. By virtue of Theorems 11 and 14, it is sufficient to prove Theorem 15 only for the case when $n = k+1$ and the manifold M^k_{-1} is connected. According to Statement «B» § 4, Chapter 1, there exists a hyperplane E^{2k} of the space E^{2k+1} such that the orthogonal projection π of the manifold M^k_{-1} to this plane is typical. Let a_{-1} and a_1 be two distinct points from M^k_{-1} satisfying $\pi(a_{-1}) = \pi(a_1)$. There exist only finitely many such pairs in M^k_{-1} (see § 4, Chapter 1, «A»). Let us perform a surgery of M^k_{-1} in a neighbourhood of (a_{-1}, a_1). One should perform analogous surgeries for any intersection pair of the mapping π of M^k_{-1}.

By virtue of «C» we may assume that the manifold M^k_{-1} is planar in the neighbourhoods of a_{-1} and a_1. Let $e_1, \ldots, e_k$ be a system of linearly independent vectors tangent to M^k_{-1} in a_{-1} that generate the orientation of M^k_{-1} and let $e_{k+1}, \ldots, e_{2k}$ be a system of linearly independent vectors tangent to M^k_{-1} in a_1 and generating the orientation opposite to the orientation of M^k_{-1}. Denote by e_{2k+1} the vector emanating from the middle point O of the segment (a_{-1}, a_1) and with the endpoint at a_1. Taking the point O to be the origin of coordinates and transporting all the vectors to it, we get the basis $e_1, \ldots, e_{2k+1}$ of the vector space E^{2k+1}. Let $E^{2k+2}_* = E^{2k+1} \times I$ where I is the interval $-1 \leqslant t \leqslant 1$; we shall consider the product E^{2k+2}_* as a strip of the Euclidean space E^{2k+2}. Let us construct a mapping ϑ from the subset H^{k+2} (see «A») of the space E^{k+2} to the space E^{2k+2} such that the mapping depends on the positive-valued parameter ϱ, the latter being small enough for further construction; let ϑ take $(\xi^1, \ldots, \xi^k, \eta, \tau) \in H^{k+2}$ to the point $(z, t) \in E^{2k+2}_*$:

$$z = \eta e_{2k+1} + \varrho \sum_{i=1}^{k} \xi^i \left(\cos\left(\frac{\pi}{4}\eta + \frac{\pi}{4}\right) e_i + \sin\left(\frac{\pi}{4}\eta + \frac{\pi}{4}\right) e_{i+k} \right), \qquad t = \tau.$$

The relations above define the mapping ϑ not only on the set H^{k+2} but also for some ε-neighbourhood W^{k+2} of this set in the Euclidean space E^{k+2}. Here z is a mapping from the set $H^{k+1}_{\tau_0}$ of points $(\xi^1, \ldots, \xi^k, \eta, \tau_0)$ satisfying (1) to the vector space E^{2k+1}. Note that the mapping πz is regular and homeomorphic everywhere except for the points of the interval $\xi^i = 0$, $|\eta| \leqslant 1$, so that the mapping πz from the manifold $P^k_1 \subset H^{k+1}_1$ to the space E^{2k} is regular and homeomorphic. Now, in the submanifold $M^k_{-1} \times I$ of the strip $E^{2k+1} \times I$, we replace the part lying in $\vartheta(H^{k+2})$ by $\vartheta(P^{k+1} \cap H^{k+2})$ (see «A»); namely, we set

$$M^{k+1} = (M^k_{-1} \times I \backslash \vartheta(H^{k+2})) \cup \vartheta(P^{k+1} \cap H^{k+2}).$$

It can be easily seen that M^{k+1} is a smooth submanifold of the strip E^{2k+2}_* which is orthogonal to the boundary of the strip; herewith the part of the boundary of M^{k+2} lying in $E^{2k+1} \times (-1)$ coincides with $M^k_{-1} \times (-1)$, whence the part of $M^k \times 1$, lying in the hyperplane $E^{2k+1} \times 1$ is such that the projection π makes for M^k_1 is one intersection pair less than for M^k_{-1}. If $k > 1$, then the connectedness of the manifold M^k_{-1} yields the connectedness of the manifold M^k_1. For $k = 1$, Theorem 15 will follow immediately from Statement «B» § 2, Chapter 4; the proof given here is not valid for $k = 1$ since the constructed manifold M^1_1 might not be connected.

Now, let us construct the framing V of the manifold M^{k+1} needed for the proof of the homology $(M^k_{-1}, U) \sim (M^k_1, U_*)$.

We construct V for the manifold $\vartheta(P^{k+1} \cap H^{k+2})$ as shown in «B» (here $n = k + 1$), in such a way that at the point $a_{-1} \times (-1)$ the vectors $v_1, \ldots, v_{k+1}$ are obtained from the vectors $u_1 \times (-1), \ldots, u_{k+1} \times (-1)$ by a positive determinant transformation. By virtue of Statement «D», one may assume that in the intersection $M^k_{-1} \times (-1) \cap \vartheta(H^{k+2})$, the vectors $u_1 \times (-1), \ldots, u_{k+1} \times (-1)$ coincide with the vectors $v_1, \ldots, v_{k+1}$ previously constructed. The framing V of M^{k+1} is already constructed for $\vartheta(P^{k+1} \cap H^{k+2})$. For the part $M^{k+1} \backslash \vartheta(P^{k+1} \cap H^{k+2})$, in any point (x, t), $x \in M^{k+1}$, $t \in I$, define the vectors $v_1, \ldots, v_{k+1}$ to be parallel to the vectors $u_1 \times (-1), \ldots, u_{k+1} \times (-1)$. Thus, the framed manifold (M^{k+1}, V) is constructed.

We shall assume that the surgery of M^k_{-1} described above is performed for all self-intersection pairs of the mapping π. Then the obtained manifold M^k_1 is projected by π regularly and homeomorphically to the submanifold $M^k = \pi(M^k_1)$ of the space E^{2k}. This projection might be performed as a deformation of the smooth submanifold M^k_1 to the smooth manifold M^k. By virtue of Statement «B» § 3, Chapter 2, this deformation can be extended to get a deformation of the framed manifold. In this way we obtain the desired submanifold (M^k, W) of the space E^{2k+1}.

Thus, Theorem 15 is proved.

Theorem 16. *Let (M^k_0, U_0) be a framed submanifold of the Euclidean space E^{2k+1} for which $\gamma(M^k_0, U_0) = 0$* (this is always true for even k, see Definition 10) (see § 2, Chapter 3, «B»). *Then, in the hyperplane E^{2k} of the space E^{2k+1} there exists a framed submanifold (M^k_1, V_1) such that $(M^k_0, U_0) \sim E(M^k_1, U_1)$* (see Definition 7)*.*

PROOF. By virtue of Theorems 14 and 15, there exists a connected framed submanifold (M^k_1, U_1) of the space E^{2k+1} homologous to the given one (M^k_0, U_0), such that $M^k_1 \subset E^{2k}$. By virtue of Statement «B» § 2, Chapter 3, we have $\gamma(M^k_1, U_1) = 0$. Thus, Statement «E» § 2, Chapter 3, yields that the degree of $\psi : M^k_1 \to \mathfrak{S}^k$ equals zero; thus the mapping ψ is null-homotopic (see Theorem 13). By virtue of Statement «A» § 4, Chapter 2, the framed manifold (M^k_1, U_1) is homologous to the framed manifold $E(M^k_1, V)$, where (M^k_1, V) is a framed submanifold of the space E^{2k}.

Thus, Theorem 16 is proved.

Chapter IV

Classification of mappings of the $(n+1)$-sphere and $(n+2)$-sphere to the n-sphere

§ 1. The Euclidean space rotation group

The main goal of this section is to establish the basic topological properties of the group H_n of all rotations of the n-dimensional Euclidean space E^n, needed for the classification of mappings $\Sigma^{n+k} \to S^n$ for $k = 1, 2$. It will be proved (see Theorem 17) that manifold H_n is connected and that for $n \geqslant 3$ there exist precisely two homotopy classes of mappings $S^1 \to H_n$. In order to prove these properties of H_n, we use a well-known covering homotopy lemma; this lemma is of independent interest; besides, we use the description of H_3 by employing quaternions; this also has independent interest and it is used in the future.

Quaternions

We recall the notion of *quaternion*, which we shall need in the rest of this work.

A) Let K be the four-space with fixed Cartesian coordinate system. Let us write arbitrary vector $x = (x^1, x^2, x^3, x^4) \in K$ of the space as $x = x^1 + ix^2 + kx^3 + kx^4$, where i, j, k are the *unit quaternions*. Define the multiplication law for the set K according to the following axioms: it is distributive; real numbers commute with unit quaternions and the multiplication of quaternion units looks like follows

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad ii = jj = kk = -1. \quad (1)$$

It is easy to see that the multiplication defined in K in this way is associative. Define the *adjoint quaternion* $\bar{x}$ to the quaternion x by setting $\bar{x} = x^1 - ix^2 - jx^3 - kx^4$. It can be easily checked that

$$\overline{xy} = \bar{y}\bar{x}. \quad (2)$$

Define the *modulus* of x as the non-negative real number

$$|x| = \sqrt{x\bar{x}} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}.$$

We have $|xy|^2 = xy\overline{xy} = xy\bar{y}\bar{x} = x \cdot |y|^2 \cdot \bar{x} = |y|^2 \cdot x\bar{x} = |x|^2 \cdot |y|^2$. Thus,

$$|xy| = |x| \cdot |y|. \tag{3}$$

If $x \neq 0$, then $|x| \neq 0$, thus there exists a quaternion x^{-1}, which is the *inverse* of the quaternion x, namely, $x^{-1} = \bar{x}/|x|^2$. Thus, the set of all quaternions K constitutes an algebraic skew field. The skew field K contains the field D of real numbers that consists of all quaternions of the type $x = x^1 + 0 \cdot i + 0 \cdot j + 0 \cdot k$. The set G of all quaternions x satisfying $|x| = 1$ constitutes a group with respect to the multiplication operation, according to (3). The set G is the 3-sphere of the Euclidean space K. The quaternions of the type $x^2 i + x^3 j + x^4 k$ are called *purely imaginary*. The set J of all such quaternions forms a 3-space in K, which is orthogonal to the line D.

B) Let K be the skew field of quaternions containing the field D of real numbers, and let J be the set of all imaginary quaternions; let G be the group of quaternions having absolute unit value (see «A»). With each quaternion $g \in G$ we associate a mapping ψ_g of K onto itself by setting

$$\psi_g(x) = gxg^{-1}. \tag{4}$$

Thus, by virtue of (3), we have $|gxg^{-1}| = |x|$, so, the mapping ψ_g, being linear, is a rotation of the space K. Since $\psi_g(D) = D$, the orthogonal complement to D (the space J) is mapped by ψ_g to itself, i.e. we get a rotation of J. It turns out that, by associating with each quaternion $g \in G$ the corresponding rotation $v(g) = \psi_g$ of the space J, we get a homomorphic mapping v of the group G to the group H_3 of all rotations of the space J. The kernel of v consists of the two elements, 1 and -1. Furthermore, it turns out that the subgroup S^1 of all quaternions $g \in G$ satisfying $\psi_g(i) = i$ consists of all quaternions of the type $\cos\alpha + i\sin\alpha$.

Let us prove Statement «B». First of all, we have

$$\psi_{gh}(x) = ghxh^{-1}g^{-1} = \psi_g(hxh^{-1}) = \psi_g\psi_h(x),$$

thus, v is a homomorphic map from G to H_3. Let us show that $v(G) = H_3$. Let $l = aj + bk$, where $a^2 + b^2 = 1$. It is easy to see that

$$l^2 = -1, \quad li = -il. \tag{5}$$

Now let $g = \cos\beta + l\sin\beta$. From (5) it follows that

$$\begin{aligned}\psi_g(i) &= (\cos\beta + l\sin\beta)i(\cos\beta - l\sin\beta)\\ &= (\cos\beta + l\sin\beta)^2 i = (\cos 2\beta + l\sin 2\beta)i\\ &= i\cos 2\beta + (bj - ak)\sin 2\beta,\end{aligned} \tag{6}$$

thus, with an appropriate choice of a, b and β, we can map the quaternion i by ψ_g to any preassigned quaternion of the set $S^2 = J \cap G$. Furthermore, by setting $a = 0, b = 1$, we get from (6)

$$\psi_g(i) = i \cos 2\beta + j \sin 2\beta, \tag{7}$$

and, since g commutes with k, then by using transformations like ψ_g we can perform any rotation of J around k. Since G is a group, this yields that by using transformations ψ_g, $g \in G$, we may perform any rotation of the space J. Note that from the multiplication law (1) it follows that the only quaternions which commute with i are just those of the type $\cos\alpha + i \sin\alpha$; thus the group S^1 consists of all quaternions of this type. Analogously, j commutes with only those quaternions from G of the type $\cos\alpha + j \sin\alpha$. Thus, the kernel of v consists precisely of $+1$ and -1.

This completes the proof of «B».

Covering homotopy

Lemma 1. *Let φ be a smooth mapping from a closed manifold P^p to a closed manifold Q^q, $p \geqslant q$, which is proper in all points. Later let f be a continuous mapping from a compact metric space R to P^p and let g_t, $0 \leqslant t \leqslant 1$ be a deformation of mappings from R to Q^q such that $g_0 = \varphi f$. Then there exists a continuous deformation f_t of mappings from R to the manifold P^p such that $f_0 = f$ and $\varphi f_t = g_t$. Then the deformation f_t is called a* covering *for the deformation g_t. If for some point $x \in R$ we have $g_t(x) = g_0(x)$ for all t, $0 \leqslant t \leqslant 1$ then $f_t(x) = f_0(x)$. Furthermore, if R is a smooth manifold and f is a smooth mapping and g_t is a smooth deformation then the mapping f_t is smooth as well.*

Proof. Denote the full preimage of $y \in Q^q$ in P^p under φ by $M_y : M_y = \varphi^{-1}(y)$. It follows from (2) § 4, Chapter 1 that M_y is a $(p-q)$-dimensional submanifold of the manifold P^p. By virtue of Theorem 2, we may assume that P^p is a smooth submanifold of some high-dimensional Euclidean space A. Denote the normal subspace at $x_0 \in M_{y_0}$ to the manifold M_{y_0} in A, by N_{x_0}. Now, let us show that if y is close enough to y_0 then there exists precisely one point $\gamma(x_0, y)$ of the intersection between the normal subspace N_{x_0} and the manifold M_y, which is close to x_0. To prove this fact, let us introduce in the neighbourhoods of points x_0 and y_0 in P^p and Q^q such local coordinates $x^1, \ldots, x^p$ and $y^1, \ldots, y^q$ with origins at x_0 and y_0, where the mapping φ looks like

$$y^1 = x^1, \ldots, y^q = x^q \tag{8}$$

[see § 4, Chapter 1, formula (2)]. Let $x = \vartheta(x^1, \ldots, x^p)$ be the parametric equation of the manifold P^p in a neighbourhood of x_0. Then N_{x_0} in A is defined by the system of equations

$$\left(x - x_0, \frac{\partial\vartheta(0,\dots,0)}{\partial x^i}\right) = 0, \quad i = q+1,\dots,p, \tag{9}$$

where x is the radius vector with endpoint running over the space N_{x_0}. The parametric equation of the manifold N_y looks like

$$x = \vartheta(y^1,\dots,y^q,x^{q+1},\dots,x^p), \tag{10}$$

where $y^1,\dots,y^q$ are the coordinates of the point y and $x^{q+1},\dots,x^p$ are the local coordinates in M_y. Thus, in order to find the point $\gamma(x_0,y)$, one should substitute the value of x from (10) into the equations (9), and then solve the obtained system in the unknown variables $x^{q+1},\dots,x^p$. The above substitution leads to

$$\Big(\vartheta(y^1,\dots,y^q,x^{q+1},\dots,x^p) - \vartheta(y_0^1,\dots,y_0^q,x_0^{q+1},\dots,x_0^p), \frac{\partial\vartheta(0,\dots,0)}{\partial x^i}\Big) = 0, \quad i = q+1,\dots,p. \tag{11}$$

Here we have a system of $p-q$ equations in $p-q$ unknown variables $x^{q+1},\dots,x^p$. With the initial conditions $y^1=0,\dots,y^q=0$, the system (11) has the evident solution $x^{q+1}=0,\dots,x^p=0$. The functional determinant of the system (10) then equals the determinant

$$\left|\left(\frac{\partial\vartheta(0,\dots,0)}{\partial x^j}, \frac{\partial\vartheta(0,\dots,0)}{\partial x^i}\right)\right|, \quad i,j = q+1,\dots,p,$$

which is non-zero since the vectors $\dfrac{\partial\vartheta(0,\dots,0)}{\partial x^i}$, $i = q+1,\dots,p$ are linearly independent. Thus, for a point y sufficiently close to y_0, there exists precisely one point x close to x_0 and satisfying the condition

$$x = \gamma(x_0,y) \in N_{x_0} \cap M_y.$$

From the compactness of P^p we see that there exists a small positive δ such that for $\varrho(y,\varphi(x_0)) < \delta$ the function $\gamma(x_0,y)$ is defined and is continuous with respect to its arguments $x_0 \in P^p$ and $y \in Q^q$. This function enjoys the following two properties:

$$\gamma(x_0,\varphi(x_0)) = x_0, \tag{12}$$

$$\varphi(\gamma(x_0,y)) = y. \tag{13}$$

These are just those properties we shall need in the sequel.

Let us now construct the deformation f_t; for this purpose, we shall use the function $\gamma(x_0, y)$. Define f_0 by setting $f_0 = f$. Let ε be a small positive number such that for $|t - t'| \leqslant \varepsilon$ we have $\varrho(g_t(u), g_{t'}(u)) < \delta$, $u \in R$. Assume that the mapping f_t is defined for all values of t satisfying $0 \leqslant t \leqslant n\varepsilon < 1$, where n is a non-negative integer. Define f_t for t, satisfying $n\varepsilon \leqslant t \leqslant (n+1)\varepsilon$, by setting

$$f_t(u) = \gamma(f_{n\varepsilon}(u), g_t(u)). \tag{14}$$

It follows from (12) and (13) that the mapping f_t defined in this way gives a continuous deformation and satisfies the condition $g_t = \varphi f_t$.

Thus, Lemma 1 is proved.

Torsion group of Euclidean space

C) Let E^n be the Euclidean vector space and let S^{n-1} be the sphere in this space defined by $(x, x) = 1$; let H_n be the rotation group of E^n and let a be a fixed point from S^{n-1}. It turns out that H_n is a smooth manifold of dimension $\dfrac{n(n-1)}{2}$; moreover, if we associate with each element h the point $\chi(h) = h(a)$, we get a smooth, everywhere proper mapping χ from the manifold H_n to the manifold S^{n-1}.

Let us prove Statement «C». Let $e_1, \ldots, e_n$ be some orthonormal basis of the space E^n. If $h \in H_n$, then

$$h(e_j) = \sum_i h_{ij} e_i. \tag{15}$$

Thus, to each rotation h of the space E^n there corresponds some orthogonal matrix $\|h_{ij}\|$ with positive determinant: $h \to \|h_{ij}\|$; conversely, to each orthogonal matrix $\|h_{ij}\|$ with positive determinant there corresponds, by (14), a certain rotation of E^n. The correspondence $h \to \|h_{ij}\|$ identifies the group H_n with the group of all orthogonal matrices of order n having positive determinant. It is well known that the orthogonality conditions for a matrix look like

$$F_{ij} = \delta_{ij}, \quad \text{where} \quad F_{ij} = \sum_\alpha h_{i\alpha} h_{j\alpha}. \tag{16}$$

Let us show that in a neighbourhood of the identity matrix $\|\delta_{ij}\|$ we can take the numbers $H_{ij}, i > j$ for local coordinates of the matrix $h \in H_n$. To do it, it is sufficient to show that for the initial values $h_{ij} = \delta_{ij}$ the system (16) is solvable in h_{ij}, where $i \leqslant j$. Note that since $F_{ij} = F_{ji}$, we may consider only those F_{ij} for $i \leqslant j$; thus the number of equation equals the number of variables. We have

$$\frac{\partial F_{ij}}{\partial h_{kl}} = \sum_\alpha (\delta_{ik}\delta_{\alpha l} h_{j\alpha} + h_{i\alpha}\delta_{jk}\delta_{\alpha l});$$

for $h_{ij} = \delta_{ij}$ this gives $\dfrac{\partial F_{ij}}{\partial h_{kl}} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}$. If at least one of the inequalities $i \leqslant j$, $k \leqslant l$ is strict then the equalities $j = k$, $i = l$ are impossible, thus the second summand equals zero. Thus, the functional matrix of the system F_{ij}, $i \leqslant j$ in the variables h_{kl}, $k \leqslant l$, is a diagonal matrix with all ones and twos on the diagonal. Thus, the system (2) is solvable. Let U be a neighbourhood of the identity matrix where this resolvability takes place, and where we can take h_{ij}, $i > j$ to be the coordinates. Let $h_0 \in H_n$; then Uh_0 is a neighbourhood of the matrix h_0, and we define the coordinates of the element $hh_0 \in Uh_0$ in the neighbourhood Uh_0 as the coordinates of the element h in the neighbourhood U. Let Uh_0 and Uh_1 be two disjoint neighbourhoods. It is easy to see that the coordinate transformation from those coordinates in Uh_0 to the coordinates in Uh_1, is smooth. Thus, H_n is a smooth manifold.

Since H_n is a group, which can map the point a to any arbitrary point of the sphere, then $\chi(h_n) = S^{n-1}$; thus, it is sufficient to prove that the mapping χ is proper in one point of the manifold H_n, e. g., in $\|\delta_{ij}\|$. For $a = e_1$, with the matrix $\|h_{ij}\|$, we associate (according to χ), the point of the sphere S^{n-1} with coordinates h_i, $i = 1, \dots, n$. Because $h_{21}, h_{31}, \dots, h_{n1}$ are the coordinates of the element h_{ij} in U, and the coordinates of the point $\chi(h) \in S^{n-1}$ can be defined as $h_{21}, \dots, h_{n1}$, then the properness of the mapping χ in $\|\delta_{ij}\|$ is evident.

Thus, Statement «C» is proved.

Theorem 17. *Let H_n be the rotation group of the Euclidean vector space E^n, $n \geqslant 3$. It turns out that H_n is a closed manifold and there exist precisely two homotopy classes of mappings from S^1 to H_n, one of which consists of all null-homotopic mappings, the other one consisting of all mappings not homotopic to zero. The latter ones can be described as follows. Let E^2 be an arbitrary two-dimensional subspace of the vector space E^n and let E^{n-2} be its orthogonal complement. It is natural to consider the group H_2 of rotations of the Euclidean plane E^2 (this group is homeomorphic to the circle) as a subgroup of the group H_n if we extend any rotation of the plane to the whole space E^n, assuming this rotation to be the identity on E^{n-2}. It turns out that the mapping g from the circle S^1 to the circle H_2 is null-homotopic in H_n if and only if the mapping degree of g is even. It turns out that each mapping h from the circle S^1 to the manifold H_n can be continuously transformed to a mapping g from S^1 to H_2 in such a way that during the whole deformation the images of all points x, for which $h(x) \in H_2$ remain fixed.*

Proof. Let S^{n-1} be the unit sphere of the space E^n, $a \in S^{n-1}$, and let χ be a mapping from the manifold H_n to the sphere S^{n-1} constructed in

«C». It is clear that the set $\chi^{-1}(a)$ is a subgroup H_{n-1} of H_n, representing all rotations of the space E^{n-1} orthogonal to the vector a.

Let f_0 be a smooth mapping from a compact manifold M^r, $r \leqslant n-2$, to H_n. Let us show that there is a deformation $f_t, 0 \leqslant t \leqslant 1$ of the mapping f_0 which fixes all points of the manifold M^r mapped to H_{n-1}, and maps the whole manifold M^r to H_{n-1}: $f_1(M^r) \subset \subset H_{n-1}$. By virtue of Theorem 1, the set $\chi f_0(M^r)$ is nowhere dense in S^{n-1}; thus, there exists a smooth deformation g_t of the mapping $g_0 = \chi f_0$ for which each point of M^r, mapped to a, remains fixed, and the mapping g_1 takes the whole manifold M^r to a. Then the deformation f_t covering the deformation g_t is just what we wanted (see Lemma 1).

Applying the above consideration to the case $M^r = S^1$, we see that any mapping $S^1 \to H_n$ is homotopic to some mapping of the circle to H_{n-1}. If $n-1 \geqslant 3$, then, repeating the same argument, we see that any mapping of the circle S^1 to H_n is homotopic to some mapping of the circle to H_{n-2}, where H_{n-2} is the rotation group of some subspace E^{n-2} of the space E^{n-1}. Arguing as above, we conclude that each mapping from S^1 to H_n is homotopic to some mapping of the circle to $H_2 \subset H_n$.

Let us show that if a mapping $g : S^1 \to H_2$ is null-homotopic in H_n, then it is null-homotopic in H_3 as well, where $H_2 \subset H_3 \subset H_n$. Let K^2 be a certain disc bounded by the circle S^1. Since the mapping g of the circle S^1 is null-homotopic in H_2 then it can be extended to a mapping g of the whole disc K^2 to H_n. Applying this argument to the case $M^r = K^2$, we see that the mapping $g : S^1 \to H_2$ is null-homotopic in H_3. To conclude the proof, it remains to show that the mapping $g : S^1 \to H_2$ is null-homotopic in H_3 if and only if the degree σ of the mapping g is even.

To prove this fact, let us use a homomorphism v of the group G to the group H_3 (see «B»). The mapping v is smooth, it is everywhere proper and maps precisely two points from G to any preassigned point from H_3. Furthermore, note that $\Sigma^1 = v^{-1}(H_2)$ is a circle, which is taken by v to the circle H_2 with mapping degree two [see (7)].

Assume that $\sigma = 2\varrho$, and let v be a mapping of S^1 to Σ^1 of degree ϱ. Then the mapping $v\nu$ of the circle S^1 to the circle H_2 has degree $2\varrho = \sigma$, thus being homotopic to the mapping g. Since the mapping ν is null-homotopic in the sphere G, then $v\nu$ is null-homotopic in H_3. Thus, the mapping g is null-homotopic in H_3 as well.

Now assume that $g : S^1 \to H_2$ is null-homotopic in H_3, so that there exists such a continuous deformation g_t, $0 \leqslant t \leqslant 1$, of mappings $S^1 \to H_3$ that $g_1 = g$ and the image $g_0(S^1)$ consists of precisely one point from H_3. Let p be a point from G such that $v(p) = g_0(S^1)$, and let f_0 be the mapping taking all of S^1 to the point p; then $vf_0 = g_0$, and, according to Lemma 1, there exists a covering deformation f_t for the deformation g_t.

Thus, $vf_1 = g_1$, and, respectively, f_1 is a mapping from the circle S^1 to the circle Σ^1; since $vf_1 = g_1$, the degree of f_1 should be even (because the degree of v equals two).

The connectedness of the manifold H_n is proved straightforwardly. It follows from the fact that there exists precisely one class of null-homotopic mappings from S^1 to H_n.

Thus, Theorem 17 is proved.

D) With each mapping h from M^1 to the group H_n of rotations of the n-space, $n \geqslant 2$, we associate the residue class $\beta(h)$ modulo 2. For $n \geqslant 3$ for one-component M^1, the residue class $\beta(h)$ is assumed to be zero when h is null-homotopic in H_n, and equal to one, otherwise. For multicomponent M^1 we define $\beta(h)$ to be the sum of residue classes $\beta(h)$ over all connected components. For $n = 2$, define the residue class $\beta(h)$ as the degree of the mapping from M^1 to H_2 taken modulo two. Having two mappings, f and g of S^1 to the group H_n, define their *group product* $h = fg$ by setting

$$h(x) = f(x)g(x), \quad x \in S^1,$$

where on the right-hand side we have the group product of the elements $f(x)$ and $g(x)$ from the group H_n. It turns out that

$$\beta(h) = \beta(f) + \beta(g). \tag{17}$$

Let us prove (17). Let $T^2 = S^1 \times S^1$ be the direct product of two copies of the circle S^1, i.e. the set of all pairs x, y, where $x \in S^1$, $y \in S^1$. Define the mapping φ of the torus T^1 to H_n by setting

$$\varphi(x, y) = f(x)g(y).$$

Now, let a be a fixed point of S^1. Without loss of generality, we may assume that $f(a) = g(a) = e \in H_n$. Let us define the three mappings f', g', h' from S^1 to the torus T^2 by setting

$$f'(x) = (x, a), \quad g'(x) = (a, x), \quad h'(x) = (x, x).$$

It is evident that

$$\varphi f' = f, \quad \varphi g' = g, \quad \varphi h' = h.$$

It is well known (and it can be easily checked) that the mapping h' from the circle S^1 to the torus T^2 is homotopic to the mapping $\hat{h}$ from S^1 to the lemniscate $S^1 \times a \cup a \times S^1$, that maps S^1 with degree one both for $S^1 \times a$ and for $a \times S^1$. Thus, the mappings $\varphi h'$ and $\varphi\hat{h}$ are homotopic. Besides, for the mapping $\varphi\hat{h}$ it can be checked straightforwardly that $\beta(\varphi\hat{h}) = \beta(\varphi h') + \beta(\varphi g')$. Thus, formula (17) is proved.

§ 2. Classification of mappings of the three-sphere to the two-sphere

In this section, we give a homotopy classification of mappings from Σ^3 to S^2; namely, we prove that the Hopf invariant γ (see §2, Chapter 3) is *the only* homotopy invariant and it can take any integer value. We use the Hopf mapping ω from Σ^3 to S^2 as an important tool for this classification. This mapping can be well described in terms of quaternions. Let K be the skew field of quaternions and let G be the set of all quaternions with absolute value equal to one, and let J be the set of all purely imaginary quaternions (see § 1, «A»). We consider the group G as the sphere Σ^3 and the intersection $G \cap J$ as the sphere S^2. With each element $g \in G$ we associate the element $\omega(g)$ by setting $\omega(g) = gig^{-1}$, where i is the quaternionic unit. It turns out that the mapping ω defined in this way is everywhere proper and has Hopf invariant equal to one. We shall use these two properties of the mapping ω for the classification of mappings $\Sigma^3 \to S^2$. To perform this classification, we shall also use the fact that any mapping from S^n, $n \geqslant 2$, to the circle S^1, is null-homotopic. The proof of this elementary theorem is also given here.

Sphere-to-circle mappings

Theorem 18. *Every mapping $S^n \to S^1$ for $n \geqslant 2$ is null-homotopic.*

PROOF. Let p and q be the north pole and the south pole of the sphere S^n and let S^{n-1} be the equator of this sphere, i.e. its section by the hyperplane perpendicular to the segment pq and passing through its middle point. For any point $x \in S^{n-1}$ there exists a unique meridian pxq of the sphere S^n passing through the point x, i.e. a great half-circle of the sphere S^n connecting the poles p and q and passing through x. Let us introduce the angle coordinate α on pxq; we count the angle from p. Define the point of pxq with coordinate α, by (x, α). We have $(x, 0) = p$, $(x, \pi) = q$, and any point $y \in S^n \backslash (p \cup q)$, can be uniquely presented as $y = (x, \alpha)$, where $0 < \alpha < \pi$.

Let f be an arbitrary mapping from S^n to S^1. Let us introduce an angular coordinate β for the circle S^1 taking $f(p)$ to be the base point. The coordinate β of $f(x, \alpha)$ is a number defined up to a multiple of 2π. Now, let us define a continuous function $g(x, \alpha)$ which is equal to $f(x, \alpha)$ when reduced modulo 2π. To do this, set $g(x, 0) = 0$ and for any fixed point $x \in S^{n-1}$ we define the function $g(x, \alpha)$ to be continuous with respect to α, $0 \leqslant \alpha \leqslant \pi$. It is evident that the function $g(x, \alpha)$ constructed in this way is a continuous function of variables x, α. Let us show that $g(x, \pi)$ is a constant. Let x_0 and x_1 be two arbitrary points from S^{n-1} and let x_t be a point

S^1 depending continuously on the parameter t, $0 \leqslant t \leqslant 1$. The numerical function $g(x_t, \pi)$ of the parameter t is continuous and, being reduced modulo 2π it does not depend on t, because $f(x_t, \pi) = f(q)$; thus the function $g(x_t, \pi)$ does not depend on t either. Thus, $g(x_t, \pi)$ is a constant. Reducing the function $(1-t)g(x, \alpha)$ modulo 2π, we get the angular function $f_t(x, \alpha)$ in two variables, x and α, satisfying the conditions $f_0(x, \alpha) = f(x, \alpha)$, $f_1(x, \alpha) = 0$. The function $f_t(x, \alpha)$ defines a deformation of the mapping $f = f_0$ to the mapping f_1; the latter maps the whole sphere S^n to a point.

Thus, Theorem 18 is proved.

Hopf mapping of the 3-sphere to the 2-sphere

A) In the Euclidean space E^3 with coordinates y^1, y^2, y^3 and origin O, for a given integer r, let us construct a framed manifold $(S^1, V_{(r)})$, where S^1 is a circle parametrically defined by

$$y^1 = \cos x, \quad y^2 = \sin x, \quad y^3 = 0, \tag{1}$$

and $\gamma(S^1, V_{(r)}) = r$ (see § 2, Chapter 3, «B»). Define the normal subspace N_x^2 at the point x of the circle S^1 by the parametric equation

$$y^1 = (1 + t^1) \cos x, \quad y^2 = (1 + t^1) \sin x, \quad y^3 = t^2, \tag{2}$$

where t^1, t^2 are the Cartesian coordinates in the plane N_x^2 with origin x. Denote the basis vectors in this coordinate system in N_x^2 by $u_1(x)$ and $u_2(x)$: $u_1(x) = \{1, 0\}$, $u_2(x) = \{0, 1\}$. Define the vectors $v_1(x)$ and $v_2(x)$ of the framing $V_{(r)}$ by the relations

$$\begin{aligned} v_1(x) &= u_1(x) \cos rx + u_2(x) \sin rx, \\ v_2(x) &= -u_1(x) \sin rx + u_2(x) \cos rx. \end{aligned} \tag{3}$$

To calculate $\gamma(S^1, V_{(r)})$, let us use Statement «E» § 2, Chapter 3. We have

$$u_2(x) = v_1(x) \sin rx + v_2(x) \cos rx,$$

which yields that the degree of the mapping ψ from S^1 to $\mathfrak{S}^1$ is equal to $\pm r$. Thus, for an appropriate orientation of the space E^3, we get $\gamma(S^1, V_{(r)}) = +r$.

Lemma 1. *There exists a smooth mapping ω from Σ^3 to S^2, which is proper in all points, such that the preimage $\omega^{-1}(y)$ of each point $y \in S^2$ is homeomorphic to the circle, and $\gamma(\omega) = +1$.*

PROOF. Let K be the skew field of quaternions, and let G be the group of all quaternions with unit absolute value; let J be the set of all purely imaginary quaternions (see § 1, «A»). Let $\Sigma^3 = G$, $S^2 = J \cap G$; define ω by setting $\omega(g) = \psi_g(i) = gig^{-1}$ (see § 1, «B»). Since for each element $y \in S^2$ there is such an element $g \in G$ that $\psi_g(i) = y$, then the preimages of all points of the sphere S^2 under ω are homeomorphic, and since the preimage of the point i is homeomorphic to the circle (see § 1 «B»), then each of them is homeomorphic to the circle.

The mapping ω is proper because $\omega = \chi v$ (see § 1 «B» «C») and each of the mappings χ, v is proper at every point.

Now, let us construct the framed manifold (S^1, V) corresponding (see Definition 5) to the mapping ω. To do this, we take the north pole p' of the sphere $\Sigma^3 = G$ to be the quaternion k and define the mapping φ as the projection of $G \backslash k$ from k to the space E^3, the latter consisting of quaternions of the type $y^1 + iy^2 + jy^3$. Though this plane is not tangent to the sphere G at k, it is parallel to the latter; thus, such a replacement results in a homothetic transformation of the framed manifold, which does not change its homology class. For the pole p of the sphere S^2 we take the quaternion i; then $\omega^{-1}(p) = S^1$, whence $\varphi(S^1) = S^1$ (see § 1 «B»). Here S^1 consists of all quaternions of the type $\cos x + i \sin x$.

Let I be the subspace of the vector space K with basis j, k. Denote by P^3_x the tangent space to the sphere G at the point $\cos x + i \sin x \in S^1$, and denote by R^2 the tangent space to the sphere S^2 at the point i. Associating with $\xi \in I$ the point $q_x(\xi) = \cos x + i \sin x + \xi$, we get an isometric mapping q_x from the plane I to the plane Q_x, the latter contained in P^3_x. Analogously, associating with $\xi \in I$ the point $r(\xi) = i + \xi i$, we get an isometric mapping of the plane I to the plane R^2. The mapping ω from G to S^2 then corresponds to the linear mapping ω_x of the tangent space P^3_x to the tangent space R^2 (see § 1, Chapter 1, «E») and, in particular, the mapping ω_x from Q_x to R^2. In the sequel, we consider the mapping ω_x only restricted to Q_x and, for the study of this map, we set $\tilde{\omega}_x = r^{-1}\omega_x q_x$. Thus, $\tilde{\omega}_x$ is a linear mapping from the vector space I to itself. Let us calculate the mapping $\tilde{\omega}_x$. Let

$$g = 1 + x^3 j + x^4 k + \varepsilon$$

be the element of the group G, close to the identity, where ε is a quaternion, which is second order small with respect to $\sqrt{(x^3)^2 + (x^4)^2}$. From the formula (6) § 1 we see that, up to second-order terms, we have

$$\omega(g) = i + 2(x^3 j + x^4 k) \cdot i. \tag{4}$$

Now, let

$$h = \cos x + i \sin x + x^3 j + x^4 k$$

$$= [1 + (x^3 j + x^4 k)(\cos x - i \sin x)](\cos x + i \sin x)$$

be an element of the group G which is close to $\cos x + i \sin x$; we omit the second order terms. Since the element $\cos x + i \sin x$ commutes with i, from (4), omitting second-order terms, we get

$$\omega(h) = i + 2(x^3 j + x^4 k)(\cos x - i \sin x)i. \tag{5}$$

From this we get

$$\tilde{\omega}_x(\xi) = 2\xi(\cos x - i \sin x). \tag{6}$$

Thus, if we write the quaternions ξ in the polar coordinates ϱ, β, i.e. if we set

$$\xi = j\varrho(\cos\beta - i \sin\beta),$$

we see that $\tilde{\omega}_x$ is an x-rotation of the plane I together with a homothety of coefficient two.

The normal space N_x^2 at $\cos x - i \sin x$ to the circle S^1 in the space E^3 is described parametrically by (2). As in Statement «A», set $u_1(x) = \{1, 0\}$, $u_2(x) = \{0, 1\}$. To the mapping φ, there corresponds a mapping φ_x of the tangent space P_x^3 to the sphere G at x to the linear space E^3 (see § 1, Chapter 1, «E»). It can be readily seen that

$$\varphi_x q_x(j) = u_2(x), \quad \varphi_x q_x(k) = u_1(x). \tag{7}$$

According to Definition 5, in order to construct the framing $V = \{v_1(x), v_2(x)\}$ corresponding to ω, we have to choose in R^2 two vectors e_1 and e_2; for the linear mapping $(\omega\varphi^{-1})_x$ from N_x^2 to R^2 we have to find vectors $v_1(x)$ and $v_2(x)$ in N_x^2 such that $e_1 = (\omega\varphi^{-1})_x v_1(x)$, $e_2 = (\omega\varphi^{-1})_x v_2(x)$. In order to choose the vectors e_1 and e_2 and calculate the vector $v_1(x)$ and $v_2(x)$, note that

$$(\omega\varphi^{-1})_x^{-1} = \varphi_x \omega_x^{-1} = \varphi_x q_x q_x^{-1} \omega_x^{-1} r r^{-1} = (\varphi_x q_x)\tilde{\omega}_x^{-1} r^{-1}. \tag{8}$$

Thus, taking $e_1 = r\left(\frac{k}{2}\right)$, $e_2 = r\left(\frac{j}{2}\right)$, we get, according to (6)–(8),

$$\begin{aligned} v_1(x) &= (\varphi_x q_x)\tilde{\omega}_x^{-1}\left(\frac{k}{2}\right) = \varphi_x q_x(k(\cos x + i \sin x)) \\ &= \varphi_x q_x(k \cos x + j \sin x) = u_1(x)\cos x + u_2(x)\sin x, \\ v_2(x) &= (\varphi_x q_x)\tilde{\omega}_x^{-1}\left(\frac{j}{2}\right) = \varphi_x q_x(j(\cos x + i \sin x)) \\ &= \varphi_x q_x(-k \sin x + j \cos x) = -u_1(x)\sin x + u_2(x)\cos x. \end{aligned}$$

Thus, by virtue of «A», we get $\gamma(S^1, V) = 1$, which yields $\gamma(\omega) = 1$. Thus, Lemma 1 is proved.

Classification of mappings $S^3 \to S^2$

Lemma 2. *Let $\pi_n(S^r)$ be the set of all homotopy classes of mappings from S^n to S^r, $n \geqslant 3$, $r = 2, 3$, and let ω be the mapping from Σ^3 to S^2, constructed in Lemma 1. Obviously, if f_0 and f_1 are two homotopic mappings from S^n to Σ^3 then the mappings ωf_0 and ωf_1 from S^n to S^2 are also homotopic. Thus, for $\alpha \in \pi_n(\Sigma^3)$, the set $\omega\alpha$ belongs to the same class $\hat{\omega}(\alpha) \in \pi_n(S^2)$. It turns out that $\hat{\omega}$ is a mapping of the set $\pi_n(\Sigma^3)$ to the whole set $\pi_n(S^2)$ and that the pre-image of the zero element of $\pi_n(S^2)$ under ω consists only of the zero element of the set $\pi_n(\Sigma^3)$.*

It follows from the definition of the sum operation in $\pi_n(S^r)$ (which is not given in the present work) that $\hat{\omega}$ is a homomorphism from the group $\pi_n(\Sigma^3)$ to the group $\pi_n(S^2)$. Thus, from Lemma 2 it follows that ω is an isomorphism of the group $\pi_n(\Sigma^3)$ to the group $\pi_n(S^2)$. However, we shall not use this result.

PROOF. First, let us show that the only element which is mapped to the zero element of the set $\pi_n(S^2)$ is the zero element of the set $\pi_n(\Sigma^3)$. Let f be a mapping from S^n to the sphere Σ^3 such that ωf is a null-homotopic mapping from the sphere S^n to the sphere S^2. Then there exists a continuous family of mappings g_t, $0 \leqslant t \leqslant 1$ from S^n to S^2 such that $g_0 = \omega f$, and g_1 is a mapping of the sphere S^n to a fixed point c of the sphere S^2. By virtue of Lemma 1, there exists a continuous family f_t of mappings from S^n to Σ^3 such that $f_0 = f$ and $\omega f_t = g_t$ (see Lemma 1 § 1). Since $g_1(S^n) = c$ then $f_1(S^n) \subset \omega^{-1}(c)$, and, by virtue of Lemma 1, the set $\omega^{-1}(c)$ is homeomorphic to the circle. Thus, the mapping f_1 is null-homotopic by Theorem 18; consequently, so is f_0.

Let us show now that for any element $\beta \in \pi_n(S^2)$ there exists such an element $\alpha \in \pi_n(\Sigma^3)$ that $\hat{\omega}(\alpha) = \beta$. We shall think of S^n as a sphere of unit radius centred at the origin of coordinates of the Euclidean space E^{n+1} with some fixed coordinate system $x^1, \ldots, x^{n+1}$. Denote the set of all points of S^n satisfying the condition $x^{n+1} \leqslant 0$, by E_-, and denote the set of all points of S^n satisfying $x^{n+1} \geqslant 0$ by E_+; finally, denote the set of points of S^n satisfying $x^{n+1} = 0$ by S^{n-1}. We take the point $p = (0, 0, \ldots, 0, 1)$ to be the north pole of the sphere S^n, and the south pole to be the point $q = (0, 0, \ldots, 0, -1)$. It is evident that there exists a mapping of S^n onto itself homotopic to the identity, which takes the half-sphere E_- to the point q. This yields that in the mapping class β, one may choose such a mapping g taking the half-sphere E_- to one point $c \in S^2$. Let P_x^2 be the half-plane of the space E^{n+1}, bounded by the line passing through p, q, and containing the point $x \in S^{n-1}$. Denote the intersection of the half-plane P_x^2 of the sphere S^n and the hyperplane $x^{n+1} = 1 - t$ by (x, t).

Thus, to each pair (x,t), $x \in S^{n-1}$, there corresponds a point $(x,t) \in E_+$, and each point $y \in E_+$ can be written in the form $y = (x,t)$; this pair is unique for $y \neq p$, at $p = (x,0)$, where x is an arbitrary point of the sphere S^{n-1}. Set $g_t(x) = g(x,t)$. Thus, we have defined the family g_t, $0 \leqslant t \leqslant 1$ of mappings from S^{n-1} to the sphere S^2, here $g_1(S^{n-1}) = c$, and $g_0(S^{n-1}) = g(p) = b$. Let a be an arbitrary point of the circle $\omega^{-1}(b)$ and let f be a mapping from S^{n-1} to Σ^3, taking the whole sphere to one point a. By virtue of Lemma 1 § 1, there exists a deformation f_t, $0 \leqslant t \leqslant 1$ of mappings $S^{n-1} \to \Sigma^3$ such that $f_0 = f$, and $\omega f_t = g_t$. Now, set $f(x,t) = f_t(x)$. This defines the mapping f from the half-sphere E_+ that takes S^{n-1} to the circle $\omega^{-1}(c)$. Since the mapping of the sphere S^{n-1} is zero-homotopic in the circle $\omega^{-1}(c)$ (see Theorem 18), the mapping f from the half-sphere E_+ can be deformed to a mapping f of the whole sphere S^n such that $f(E_-) \subset \omega^{-1}(c)$. Such a mapping f satisfies $\omega f = g$.

Thus, Lemma 2 is proved.

Theorem 19. *The homomorphism γ of the group Π^1_2 to the group of integers is epimorphic (see § 2, Chapter 3, «C»). From this we have that two mappings f_0 and f_1 from Σ^3 to S^2 are homotopic if and only if $\gamma(f_0) = \gamma(f_1)$, and, furthermore, for any integer c there exists a mapping f from Σ^3 to the sphere S^2 such that $\gamma(f) = c$.*

PROOF. First of all show that the kernel of the homomorphism γ contains only the zero element of the group Π^1_2. It is sufficient to show that the mapping $g : \Sigma^3 \to S^2$, satisfying $\gamma(g) = 0$, is null-homotopic. By virtue of Lemma 2, there exists such a mapping f of the sphere Σ^3 to itself such that the mapping ωf is homotopic to g, and, consequently, $\gamma(\omega f) = 0$. By virtue of Statement «D» § 2, Chapter 3, the degree σ of mapping f of the sphere Σ^3 to itself is defined as $\gamma(\omega f) = 1^2 \cdot \sigma$, so that $\sigma = 0$. Thus (see Theorem 12), the mapping f of Σ^3 to itself is null-homotopic, thus, so are ωf and g.

Let us show that for any integer σ there exists a mapping g from Σ^3 to S^2 such that $\gamma(g) = \sigma$, i.e. that γ is an epimorphism. Indeed, let f be a mapping of Σ^3 to itself having degree σ. Then for the mapping $g = \omega f$, we have, according to «D» § 2, Chapter 3, $\gamma(\omega f) = \sigma \cdot 1 = \sigma$.

Thus, Theorem 19 is proved.

B) Comparing Statement «A» and Theorem 19, we see that each one-dimensional framed manifold of the three-dimensional Euclidean space is homologous to the framed manifold $(S^1, V_{(r)})$ constructed in «A», for some r.

§ 3. Classification of mappings from $(n+1)$-sphere to n-sphere

In this section we prove that for $n \geqslant 3$ there exist precisely two homotopy classes of mappings from Σ^{n+1} to S^n. The proof is based on a homological invariant $\delta(M^1, U)$ of framed manifolds in E^{n+1}, $n \geqslant 2$, which is a mod 2 residue class and can take both values, 0 and 1. Thus, even from the existence of δ, it follows that there exist at least two mapping classes of $\Sigma^{n+1} \to S^n$ for $n \geqslant 2$. The invariant δ is described as follows. Let $U(x) = \{u_1(x), \ldots, u_n(x)\}$ be an orthonormal framing of M^1 and let $u_{n+1}(x)$ be the unit vector tangent to M^1 at some point $x \in M^1$. The system $U'(x) = \{u_1(x), \ldots, u_{n+1}(x)\}$ can be obtained from some fixed orthonormal basis of the space E^{n+1} by the rotation $h(x)$. Thus we get a continuous mapping h from the manifold M^1 to the manifold H_{n+1} of all rotations of the space E^{n+1}. In the case of one-component curve M^1, the invariant δ is defined to be zero if the mapping h is *not* zero-homotopic, and equal to one, otherwise. In the case of a multicomponent curve, the invariant δ is defined as the mod 2 sum of the values of δ on the components.

To prove the invariance of δ, we preliminarily prove the general Lemma 1, where we improve the framed manifold (M^{k+1}, U) realizing the homology. The improved manifold (M^{k+1}, U) enjoys the property that its section by the plane $E^{n+k} \times t$ is a framed manifold (M_t^k, U_t) for all values of the parameter t, except for a finite number of critical ones. Since for non-critical values of the parameter t, the framed manifold (M_t^k, U_t) depends continuously on t, the invariance of the residue class δ should be proved only when passing of t passes through a critical value. When it passes through a critical value, the manifold (M_t^k, U_t) is reconstructed rather easily, thus, it is possible to prove the invariance of δ.

For a one-dimensional framed submanifold of the 3-space we define the invariants γ and δ; it turns out that δ is the residue class obtained by taking the integer γ modulo 2. Since every framed one-manifold is obtained by means of suspensions from a one-submanifold of the three-space (see Theorem 11), then for the classification of mappings $\Sigma^{n+1} \to S^n$ for $n \geqslant 3$ one may use the classification of mappings from the 3-sphere to the two-sphere. Namely, in this vein we prove that if $\gamma(M^1, U)$ is even then $E(M^1, U) \sim 0$. Thus, we show that there exist no more than two mapping classes for $\Sigma^{n+1} \to S^n$, $n \geqslant 3$.

Improving the framed manifold that performs the homology

Lemma 1. *Let (M_0^k, U_0) and (M_1^k, U_1) be two homologous framed submanifolds of the Euclidean space E^{n+k}, in such a way that there exists a* *framed submanifold* (M^{k+1}, U) *of the strip* $E^{n+k}\times I$, *realizing the homology* $(M_0^k, U_0) \sim (M_1^k, U_1)$. *A point* (x_0, t_0) *of the manifold* M^{k+1} *is called* CRITICAL *(then we call the corresponding value* t_0 *of the parameter* t *critical) if the tangent space* T_0^{k+1} *to the manifold* M^{k+1} *at the point* (x_0, t_0) *lies in the hyperplane* $E^{n+k}\times t_0$. *It turns out that the framed manifold* (M^{k+1}, U) *realizing homology between framed manifolds can be chosen in such a way that:*

a) *there exist only finitely many critical points of the manifold* M^{k+1} *with different critical values of the parameter* t *(for distinct points)*

b) *for each critical point* (x_0, t_0) *of* M^{k+1} *one may choose an orthonormal basis* $e_1, \ldots, e_{n+k}$ *such that in the corresponding coordinates* $x^1, \ldots, x^{n+k}$ *with origin at* x_0 *the manifold* M^{k+1} *in a neighbourhood of* x_0 *is given by the equations*

$$t = t_0 + \sum_{i=1}^{k+1} \sigma^i (x^i)^2, \quad \sigma^i = \pm 1; \quad x^{k+2} = \ldots = x^{n+k} = 0, \tag{1}$$

and the framing $U = \{u_1(x,t), \ldots, u_n(x,t)\}$ *in a neighbourhood of the point* (x_0, t_0) *is given by*

$$\begin{aligned} u_1(x,t) &= \sigma \left(e - \sum_{i=1}^{k} 2\sigma^i x^i e_i \right), \quad \sigma = \pm 1; \\ u_2(x,t) &= e_{k+2}, \ldots, u_n(x,t) = e_{n+k}, \end{aligned} \tag{2}$$

where e *is the unit vector of the strip* $E^{n+k}\times I$ *directed along the* t*-axis.*

PROOF. Let (N_0^{k+1}, V_0) be some framed manifold realizing the homology $(M_0^k, U_0) \sim (M_1^k, U_1)$. To each point $y = (x,t)$ of the manifold N_0^{k+1} we associate the number $f(y) = f(x,t) = t$. By virtue of Theorem 5 there exists such a real-valued function $q(y)$ defined on the manifold N^{k+1}, coinciding with $f(y)$ near the boundary of N_0^{k+1}, and being first-order close (with respect to ε) to the function f, with all critical points non-degenerate and all critical values distinct. Now, we associate with each point $y = (x,t)$ of the manifold N_0^{k+1} the point $\varphi_s(y) = (x, t + s(g(y) - f(y)))$, where s is a fixed number, $0 \leqslant s \leqslant 1$. For ε small enough, the mapping φ_s is regular and homeomorphic (see Theorem 3). Thus, φ_s is a deformation of the smooth submanifold N_0^{k+1} to the submanifold $N_1^{k+1} = \varphi_1(N_0^{k+1})$.

It turns out that the critical points of the function $g(y)$ coincide with the critical points of the manifold N_1^{k+1}. Thus, the condition «a» holds for the manifold $M^{k+1} = N_1^{k+1}$.

Now, let us make a further improvement of the manifold N_1^{k+1} in order to make the condition «b» hold.

Let $y_0 = (x_0, t_0)$ be an arbitrary critical point of the manifold N_1^{k+1} and let T_0^{k+1} be the tangent space to the manifold N_1^{k+1} at the point y_0. The plane T_0^{k+1} lies in $E^{n+k} \times t_0$; thus $T_0^{k+1} = T^{k+1} \times t_0$; $T^{k+1} \subset E^{n+k}$. In the space E^{n+k}, let us choose a basis $e_1, \dots, e_{n+k}$ in such a way that the vectors $e_1, \dots, e_{k+1}$ lie in T^{k+1}. In a neighbourhood of (x_0, t_0), the manifold N_1^{k+1} is described in these coordinates as

$$t = t_0 + \varphi(x^1, \dots, x^{k+1}) + \psi(x^1, \dots, x^{k+1}), \tag{3}$$

$$x^{k+j} = \psi^j(x^1, \dots, x^{k+1}), \quad j = 2, \dots, n, \tag{4}$$

where φ is a non-degenerate quadatic form in the variables $x^1, \dots, x^{k+1}$; here ψ is third-order small with respect to $\xi = \sqrt{(x^1)^2 + \ldots + (x^{k+1})^2}$, and ψ^j is second-order small with respect to ξ. Choosing the axes in the plane T^{k+1} in a proper way, we can transform φ to the form

$$\varphi = \sum_{i=1}^{k+1} \lambda^i (x^i)^2, \tag{5}$$

where λ^i are non-zero real numbers. Let us improve the manifold N_1^{k+1} in a neighbourhood of the point (x_0, t_0). Let $\chi(\eta)$ be a smooth real-valued monotonic function in the variable $\eta \geqslant 0$, satisfying the conditions

$$\chi(\eta) = 0 \ \text{ for } \ 0 \leqslant \eta \leqslant \frac{1}{2}; \quad \chi(\eta) = 1 \ \text{ for } \ \eta \geqslant 1.$$

Set

$$\chi(\xi, s) = s\chi\left(\frac{\xi}{\alpha}\right) + (1 - s),$$

where α is a small positive number. Define the manifold N_{1+s}^{k+1} by

$$t = t_0 + \sum_{i=1}^{k+1} \lambda^i (x^i)^2 + \chi(\xi, s)\psi(x^1, \dots, x^{k+1}),$$
$$x^{k+j} = \chi(\xi, s)\psi^j(x^1, \dots, x^{k+1}), \quad j = 2, \dots, n,$$

for $\xi \leqslant \alpha$, $|t - t_0| \leqslant \alpha$; we shall assume that $N_{1+s}^{k+1} = N_1^{k+1}$ in the remaining points. It is evident that the submanifold N_{1+s}^{k+1} realizes a smooth deformation of the submanifold N_1^{k+1} to the submanifold N_2^{k+1}, so that the latter one for $|t - t_0| \leqslant \alpha$ is defined by

$$t = t_0 + \sum_{i=1}^{k+1} \lambda^i (x^i)^2 + \chi\left(\frac{\xi}{\alpha}\right)\psi(x^1, \dots, x^{k+1}), \tag{6}$$

$$x^{k+j} = \chi\left(\frac{\xi}{\alpha}\right)\psi^j(x^1,\dots,x^{k+1}),\quad j=2,\dots,n. \tag{7}$$

It is evident that in a small neighbourhood of the point (x_0, t_0), more precisely, for $\xi < \frac{\alpha}{2}$, the manifold N_2^{k+1} is given by the equations

$$t = t_0 + \sum_{i=1}^{k+1}\lambda^i(x^i)^2;\quad x^{k+j}=0,\quad j=2,\dots,n. \tag{8}$$

Let us show that the manifold N_2^{k+1} has no critical points distinct from those of the manifold N_1^{k+1}. For that, it is sufficient to study those points of N_2^{k+1}, defined by (6), (7) and satisfying $\xi \leqslant \alpha$, and to show that among them the only critical point is $\xi = 0$.

We have

$$\frac{dt}{dx^i} = 2\lambda^i(x^i+\vartheta^i),$$

where

$$2\lambda^i\vartheta^i = \frac{\chi'\left(\frac{\xi}{\alpha}\right)}{\alpha}\frac{x^i}{\xi}\psi(x^1,\dots,x^{k+1}) + \chi\left(\frac{\xi}{\alpha}\right)\frac{\partial\psi(x^1,\dots,x^{k+1})}{\partial x^i}.$$

Thus,

$$|\vartheta^i| \leqslant \frac{c_1}{\alpha}\xi^3 + c_2\xi^2.$$

It is clear that for α small enough, we have

$$|\vartheta^i| \leqslant \frac{1}{k+1}\xi,\quad \xi\leqslant\alpha.$$

Now, if for $\xi \leqslant \alpha$ we have $\dfrac{dt}{dx^i} = 0$, $i = 1,\dots,k+1$, then

$$x^i = -\vartheta^i. \tag{9}$$

Thus, by summing the squares of the equalities (9), we get $\xi^2 = \sum\vartheta_i^2 < \dfrac{k+1}{(k+1)^2}\xi^2$, so that $\xi^2 \leqslant \dfrac{1}{k+1}\xi^2$, which is possible only for $\xi = 0$.

Let us now perform further corrections of N_2^{k+1} in such a way that its equations in the neighbourhood of the critical point (x_0, t_0) look like (1). Let us write the numbers λ^i as $\lambda^i = \dfrac{\sigma^i}{a_i^2}$, where a_i is a positive number and $\sigma^i = \pm 1$. Let α' be such a small positive number that for $|x^i| \leqslant \alpha'$, $|t - t_0| \leqslant$

α' the manifold N_2^{k+1} is defined by (8). Let us now define the smooth function $\varkappa_a(\eta) > 0$ of η, $|\eta| \leqslant \alpha'$ depending on the positive parameter a and satisfying $\varkappa_a'(\eta) > 0$; $\varkappa_a(\eta) = a\eta$ for $|\eta| < \beta$; $\varkappa_a(\eta) = \eta$ for $|\eta| > \frac{\alpha'}{2}$. Here β is a small positive number such that the function $\varkappa_a(\eta)$, satisfying the conditions above, exists. Now, define the manifold N_{2+s}^{k+1} for $|x^i| \leqslant \alpha$ as

$$t = t_0 + \sum_{i=1}^{k+1} \lambda^i \cdot [(1-s)x^i + s\varkappa_{a_i}(x^i)]^2; \qquad x^{k+j} = 0, \quad j = 2, \ldots, n, \tag{10}$$

elsewhere this manifold coincides with N_2^{k+1}. It can be easily checked that N_{2+s}^{k+1} realizes a smooth deformation of N_2^{k+1} to N_3^{k+1} and that the critical points of N_2^{k+1} coincide with those of N_3^{k+1}. Here the equations of the manifold N_3^{k+1} near the point (x_0, t_0) look like (1).

Performing such a surgery near each critical point of N_1^{k+1}, we construct the manifold M^{k+1}, and, since it is obtained from N_0^{k+1} as a result of several smooth deformations then there exists a framing V of M^{k+1} such that the framed manifold (M^{k+1}, V) realizes the homology $(M_0^k, U_0) \sim (M_1^k, U_1)$. Taking an appropriate value of $\sigma = \pm 1$ in (2), we can make the framings U and V define the same orientation of M^{k+1}; thus near the critical point (x_0, t_0), one can deform the framing V in order to get the framing U (see § 3, Chapter 3, «D»). Performing such an improvement of the framing near each critical point of M^{k+1}, we get the desired framing U.

Thus, Lemma 1 is proved.

Invariant δ of mappings $\Sigma^{n+1} \to S^n$

Theorem 20. *Let (M^1, U) be a one-dimensional framed submanifold of the oriented Euclidean space $E^{n+1}, n \geqslant 2$, $U(x) = \{u_1(x), \ldots, u_n(x)\}$. At each point $x \in M^1$, let us draw the unit vector $u_{n+1}(x)$ tangent to the curve M^1 and directed in such a way that the vectors $u_1(x), \ldots, u_n(x), u_{n+1}(x)$ determine the positive orientation of E^{n+1}. Furthermore, let $e_1, \ldots, e_{n+1}$ be some orthonormal basis of the space E^{n+1} defining its positive orientation. Then*

$$u_i(x) = \sum_{j=1}^{n+1} h_{ij}(x) e_j, \quad i = 1, \ldots, n+1, \tag{11}$$

where $h(x) = \|h_{ij}(x)\|$ is an orthogonal matrix with positive determinant depending continuously on $x \in M^1$. Thus, h is a continuous mapping of the curve M^1 to the manifold H_{n+1} of all rotations of the Euclidean space

E^{n+1}. Set

$$\delta(M^1, U) \equiv \beta(h) + r(M^1) \pmod 2,$$

where $r(M^1)$ is the number of components of M^1, and the residue class $\beta(h)$ is defined as in Statement «D» § 1. It turns out that the residue class $\delta(M^1, U)$ is a homology class invariant of the framed manifold (M^1, U), so that if the mapping f from Σ^{n+1} to S^n is associated with the framed manifold (M^1, U), then, by setting

$$\delta(f) = \delta(M^1, U),$$

we get an invariant $\delta(f)$ of the homotopy class of f. The residue class $\delta(M^1, U)$ does not depend on the orientation of E^{n+1} and on the arbitrariness in the choice of the basis $e_1, \ldots, e_{n+1}$.

Proof. First, let us prove the invariance of $\delta(M^1, U)$ under the change of $e_1, \ldots, e_{n+1}$. Let $e'_1, \ldots, e'_{n+1}$ be some orthonormal basis defining the positive orientation of E^{n+1} obtained from the initial one; then

$$e_j = \sum_{k=1}^{n+1} a_{jk} e'_k, \quad j = 1, \ldots, n+1,$$

where $a = \|a_{jk}\|$ is an orthogonal matrix with positive determinant. In the basis $e'_1, \ldots, e'_{n+1}$, the matrix corresponding to the manifold (M^1, U) is not $h(x)$, but $h'(x) = h(x) \cdot a$. Since the manifold H_{n+1} is connected then there exists a matrix $a_t \in H_{n+1}$ depending continuously on the parameter t, $0 \leqslant t \leqslant 1$ such that $a_1 = a$, and a_0 is the identity matrix. The mapping $h_t = ha_t$ performs a continuous deformation of h to the mapping h', so that these mappings are homotopic, thus $\delta(M^1, U)$ does not depend on the choice of $e_1, \ldots, e_{n+1}$.

Let us now prove the independence of $\delta(M^1, U)$ on the orientation of E^{n+1}. To change the orientation of E^{n+1}, let us replace the vector $u_{n+1}(x)$ with the vector $-u_{n+1}(x)$; in the basis $e_1, \ldots, e_{n+1}$ we can perform the orientation change by replacing the vector e_{n+1} with the vector $-e_{n+1}$. Thus, instead of $h(x)$ we get the matrix $h'(x)$, which is obtained from $h(x)$ by multiplying both the last row and the last column by -1. Associate with each matrix $l \in H_{n+1}$ the matrix l' obtained from l by multiplying the last row and the last column of it by -1. If we take the plane E^2 to be the plane with basis e_1, e_2, then we see that the mapping $l \to l'$ takes the curve, H_2 from Theorem 17, which is not zero-homotopic in H_{n+1}, identically to itself. Thus, the orientation change of the space E^{n+1} does not change the residue class $\delta(M^1, U)$.

Finally, let us prove the main property of $\delta(M^1, U)$, i.e. its invariance under the choice of the framed manifold (M^1, U) from the given homology class.

Let (M_0^1, U_0) and (M_1^1, U_1) be two framed submanifolds of the space E^{n+1} and let (M^2, U) be a framed submanifold of the strip $E^{n+1} \times I$ realising the homology $(M_0^1, U_0) \sim (M_1^1, U_1)$ and chosen in such a way that the conditions «a» and «b» of Lemma 1 hold. The intersection $M^2 \cap (E^{n+1} \times t)$ lies in $E^{n+1} \times t$; thus, it looks like $M_t^1 \times t$, where $M_t^1 \subset E^{n+1}$. It is easy to see that if the point (x, t) is not a critical point of the surface M^2 (see Lemma 1) then the set M_t^1 represents a smooth curve in the neighbourhood of the point x; thus when t is not a critical value of the parameter, M_t^1 is a smooth submanifold of the space E^{n+1}. Let us construct the framing V_t of the manifold M_t^1. Let (x, t) be a non-critical point of M^2; let $V(x, t) \times t$ be the orthogonal projection of the vector system $U(x, t)$ to the hyperplane $E^{n+1} \times t$ and let $V_t(x)$ be the system obtained from $V(x, t)$ by using the orthogonalization process (see § 3, Chapter 2, «G»). Since all vectors of the system $U(x, t)$ are orthogonal to M^2 at the point (x, t), all vectors of the system $V(x, t)$ are orthogonal to M_t^1 at x. Since the point (x, t) is not critical, we see that the vectors of the system $V(x, t)$ are linearly independent. Thus, the system $V_t(x)$ gives a framing of the manifold M_t^1 for any non-critical value of t. To the framed manifold (M_t^1, V_t), there corresponds a mapping h_t of the curve M_t^1 to H_{n+1}. From the continuity argument it follows that when the parameter t changes continuously without passing through critical values, the residue class $\delta(M_t^1, V_t)$ remain unchanged. Let us prove that it remains unchanged while passing through a critical value t_0 of the parameter t. From that, by virtue of the relation $V_0 = U_0$, $V_1 = U_1$, we shall get the invariance of $\delta(M^1, U)$.

Let (x_0, t_0) be the unique critical point of the manifold M^2, where the parameter t has critical value $t = t_0$. Near the point (x_0, t_0), the manifold M^2 is defined by the equations

$$t = t_0 + \sigma^1 (x^1)^2 + \sigma^2 (x^2)^2, \quad \sigma^1 = \pm 1, \quad \sigma^2 = \pm 1,$$

$$x^3 = \ldots = x^{n+1} = 0$$

[see (1)]. From this we have that for t close to t_0, the equations of the manifold M_t^1 near x_0 look like

$$\sigma^1 (x^1)^2 + \sigma^2 (x^2)^2 = t - t_0, \quad x^3 = \ldots = x^{n+1} = 0. \tag{12}$$

Furthermore, it follows from (2) that the system $V_t(x)$ for $|t - t_0|$ small enough and for x close to x_0 is defined by the formulae

$$\begin{aligned} (v_t)_1(x) &= \sigma \left(\sigma^1 \frac{x^1}{\xi} e_1 + \sigma^2 \frac{x^2}{\xi} e_2 \right); \\ (v_t)_j(x) &= e_{j+1}, \quad j = 2, \ldots, n, \end{aligned} \tag{13}$$

where $\xi = \sqrt{(x^1)^2 + \ldots + (x^2)^2}$. To find $\delta(M_t^1, V_t)$ we take the plane E^2 (see Theorem 17) to be the plane with the basis e_1, e_2. Now, let us consider the following cases: 1) $\sigma^1 = \sigma^2$ and 2) $\sigma^1 = -\sigma^2$.

In the first case, we may assume $\sigma^1 = \sigma^2 = -1$. According to this assumption, the manifold M_t^1 for $t < t_0$ contains a component defined by (12) and representing the usual metric circle of small radius. Denote this component by S^1. It can be readily seen that the mapping h_t takes the circle S^1 to the circle H_2 with degree one. For $t > t_0$ the component defined by the equations (12) becomes imaginary, i.e. it disappears, since all other components of the curve M_t^1 together with their framings are changed continuously. Thus, in the first case as the parameter t passes through the value t_0, the residue class $\beta(h_t)$ is changed by one as well as the number of components of the manifold M_t^1; thus, the residue class $\delta(M_t^1, V_t)$ does not change.

In the second case the set $M_{t_0}^1$ near the point x_0 is given by $(x^1)^2 - (x^2)^2 = 0$, i.e. it is a cross K_{t_0}, which is a union of two intervals intersecting at a point. From this we see that the component L_{t_0} of the set $M_{t_0}^1$, containing the cross K_{t_0}, is homeomorphic to the lemniscate. Since the surface M^2 is orientable, the neighbourhood of the lemniscate L_{t_0} in M^2 is homeomorphic to a 2-connected plane domain; thus, the part L_t of the set M_t^1 located near the lemniscate L_{t_0}, consists of two components S_1^1 and S_2^1 for those values of t, lying on one side from t_0, and of one component of $\hat{S}$ for those values of t lying on the other side of t_0. We shall assume that L_t consists of two components for $t < t_0$ and of one component for $t > t_0$. If we denote the residue classes $\beta(h_t)$ corresponding to the components S_1^1, S_2^1 and $\hat{S}$ $\beta(h_t)$ by $\beta_1, \beta_2, \hat{\beta}$ then for the invariance of $\delta(M^1, U)$ it is sufficient to show that $\beta_1 + \beta_2 \equiv \hat{\beta} + 1 \pmod 2$. Let us prove this fact. Denote the part of L_t lying near the cross K_{t_0} by K_t. This part is described by the equation $(x^1)^2 - (x^2)^2 = \sigma_1(t - t_0)$, i.e. it represents the hyperbola. From the formulae (13) we see that $h_t(K_t) \subset H_2$; here we see that for $t < t_0$ the set $h_t(K_t)$ covers two fourth parts of the circle H_2, and for $t > t_0$ it covers the remaining part of H_n. By virtue of Theorem 17, the mapping h_t of the curve L_t can be replaced with a homotopic mapping h_t' in such a way that $h_t'(L_t) \subset H_2$ and the mappings h_t' and h_t coincide for K_t. From the above it follows that the sum of the degrees of h_t' for S_1^1 and S_2^1 when $t < t_0$ differs from the degree of h_t' for $\hat{S}$ with $t > t_0$, by one. Thus, $\beta_1 + \beta_2 \equiv \hat{\beta} + 1 \pmod 2$, and the invariance of $\delta(M^1, U)$ is proved completely.

Thus, Theorem 20 is proved.

Let us mention some properties of $\delta(M^1, U)$, which are easy to check.

A) Let Π_n^1 be the group of homology classes of framed one-dimensional submanifolds of the Euclidean space E^{n+1}. Since $\delta(M^1, U)$ is an invariant of the homology class, we may set $\delta(\pi) = \delta(M^1, U)$, where (M^1, U) is the framed manifold of the class $\pi \in \Pi_n^1$. It can be easily checked that δ is a homomorphism from the group Π_n^1 to the group of modulo 2 residue classes. Furthermore, it is clear that if $E\pi$ is a suspension over the class π, i.e. $E(M^1, U) \in E\pi$ (see § 4), then $\delta(E\pi) = \delta(\pi)$.

The classification of mappings $\Sigma^{n+1} \to S^n$

Theorem 21. *For $n \geqslant 3$, the homomorphism δ of the group Π_n^1 to the group of residue classes modulo 2 is an epimorphism, since the group Π_n^1 is cyclic of the order two. Thus, there exist precisely two homotopy classes of mappings from the sphere Σ^{n+1} to the sphere S^n ($n \geqslant 3$). Furthermore, the homomorphism δ from the group Π_2^1 to the group of residue classes modulo 2 is an epimorphism and, since the group Π_2^1 is mapped isomorphically to the group of integers under γ (see Theorem 19), then the homomorphism $\delta\gamma^{-1}$ from the group of integers to the group of residue classes is just the reduction modulo 2.*

PROOF. Let (S^1, U) be some orthonormally framed submanifold of the Euclidean space S^{n+1} homeomorphic to the circle, $U(x) = \{u_1(x), \ldots, u_n(x)\}$. To calculate the invariant $\delta(S^1, U)$, denote by $u_{n+1}(x)$ the corresponding unit vector tangent to S^1 in x (with appropriate direction), and let $e_1, \ldots, e_{n+1}$ be a basis of E^{n+1}. We have

$$u_i(x) = \sum_{j=1}^{n+1} h_{ij}(x) e_j, \quad i = 1, \ldots, n+1, \tag{14}$$

so that $h(x) = \|h_{ij}(x)\|$ is an orthogonal matrix with positive determinant and h is a continuous mapping from S^1 to H_{n+1}. By definition of $\delta(M^1, U)$ (see Theorem 20), we have

$$\delta(S^1, U) \equiv \beta(h) + 1 \pmod 2. \tag{15}$$

Furthermore, let $g(x) = \|g_{ij}(x)\|$ be the orthogonal matrix of order n with positive determinant, so that g is a continuous mapping from S^1 to H_n. Set

$$v_i(x) = \sum_{j=1}^{n} g_{ij}(x) u_j(x); \quad i = 1, \ldots, n,$$

and denote by $g[U]$ the framing $V(x) = \{v_1(x), \ldots, v_{n+1}(x)\}$. In order to calculate $\delta(S^1, g[U])$, set $v_{n+1}(x) = u_{n+1}(x)$ and denote by $g'(x)$ the matrix of order $n+1$ obtained from the matrix $g(x)$ by adding the elements

$g_{i,n+1}(x)$ and $g_{n+1,i}(x)$; here only one of these elements, $g_{n+1,n+1}(x)$, is non-zero; it is equal to one. Clearly, for $n \geqslant 2$ we have

$$\beta(g') = \beta(g) \tag{16}$$

(see § 1 «D»). Later, we have

$$v_i(x) = \sum_{j,k=1}^{n+1} g'_{ij}(x) \cdot h_{jk}(x) \cdot e_k, \quad i = 1, \dots, n+1.$$

Thus, by virtue of Statement «D» § 1, we have

$$\begin{aligned} \delta(S^1, g[U]) &= \beta(g'h) + 1 = \beta(g') + \beta(h) + 1 \\ &= \delta(S^1, U) + \beta(g). \end{aligned} \tag{17}$$

From Theorem 11 and Statement «B» § 2, it follows straightforwardly that for any framed manifold (M^1, W) of the Euclidean space E^{n+1}, the following homology occurs:

$$(M^1, W) \sim E^{n-2}(S^1, V_{(r)}), \tag{18}$$

where $(S^1, V_{(r)})$ is the framed submanifold of the 3-space E^3 constructed in Statement «A» § 2, and E^{n-2} is the $(n-2)$-fold suspension operation. We have

$$V_{(r)} = g_{(r)}[V_{(0)}], \tag{19}$$

where

$$g_{(r)}(x) = \begin{Vmatrix} \cos rx & \sin rx \\ -\sin rx & \cos rx \end{Vmatrix}$$

(see § 2, «A»). Thus,

$$\beta(g_{(r)}) \equiv r \pmod 2. \tag{20}$$

It can be easily checked that $\delta(S^1, V_{(0)}) = 0$. Thus, by virtue of (17), (19) and (20), it follows that

$$\delta(S^1, V_{(r)}) \equiv r \pmod 2. \tag{21}$$

Since $\gamma(S^1, V_{(r)}) = r$ (see § 2 «A»), it follows from (21) that the homomorphism $\delta\gamma^{-1}$ from the group of integers to the group of residue classes modulo 2 is the modulo 2 reduction. This completes the proof of the second part of Theorem 21.

Furthermore, we have

$$EV_{(r)} = g'_{(r)}[EV_{(0)}],$$

where

$$g'_{(r)}(x) = \begin{Vmatrix} \cos rx & \sin rx & 0 \\ -\sin rx & \cos rx & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

thus $\beta(g'_{(r)}) \equiv r \,(\mathrm{mod}\ 2)$. Since $\gamma(S^1, V_{(0)}) = 0$, then we have $(S^1, V_{(0)}) = 0$ (see Theorem 19); thus $E(S^1, V_{(0)}) \sim 0$. Consequently, $E(S^1, V_{(r)}) \sim 0$, if the mapping $g'_{(r)}$ from S^1 to H_3 is null-homotopic (see § 3, Chapter 2, «H»); this is true for even r. So, $E(S_1, V_{(r)}) \sim 0$ if $\delta(E(S^1, V_{(r)})) = 0$. From this and from (18) we see that for $n \geqslant 3$ the equality $\delta(M^1, W) = 0$ yields $(M^1, W) \sim 0$. Because $\delta(E^{n-2}(S^1, V_{(1)})) = 1$, the framed manifold $E^{n-2}(S^1, V_{(1)})$ is not null-homologous. Thus we have shown that the homomorphism δ from the group Π^1_n to the group of residue classes modulo 2 is an epimorphism.

Thus, Theorem 21 is completely proved.

§ 4. Classification of mappings from the $(n+2)$-sphere to the n-sphere

In this section we show that for $n \geqslant 2$ there exist precisely two homotopy classes of mappings from Σ^{n+2} to S^n. This proof is based on a homological invariant $\delta(M^2, U)$ of framed manifolds (M^2, U) of the Euclidean space E^{n+2}, which is a residue class modulo 2, and may take any of the two values, 0 and 1. Thus, the existence of δ yields the existence of at least two mapping classes $\Sigma^{n+2} \to S^n$. The invariant δ is described as follows. Let $U(x) = \{u_1(x), \dots, u_n(x)\}$ be the orthonormal framing of M^2, and let C be a smooth simple closed curve on M^2. Denote the unit normal to C tangent to M^2 at $x \in C$ by $u_{n+1}(x)$ and set $V(x) = \{u_1(x), \dots, u_{n+1}(x)\}$. For the one-dimensional framed manifold (C, V) the invariant $\delta(C, V)$ (see § 3) is well defined. In this case, we denote this invariant by $\delta(C)$. First assume that M^2 is a connected surface; denote its genus by p. Then there exists on M^2 a system $A_1, \dots, A_p, B_1, \dots, B_q$ of closed simple closed curves such that the curves A_i and B_i, $i = 1, \dots, p$, have a unique non-tangent intersection point, and any two other curves do not intersect at all. It turns out that the residue class

$$\delta(M^2, U) = \sum_{i=1}^{p} \delta(A_i)\delta(B_i)$$

does not depend on the arbitrariness; it is a homological invariant of the framed manifold (M^2, U). In the case of a multicomponent surface, the invariant δ is defined as the sum of its values over the components.

From Theorems 11 and 16 it follows that the number of mapping classes for $\Sigma^{n+2} \to S^n$ does not exceed the number of mappings from Σ^4 to S^2. The number of mapping classes from the sphere Σ^4 to the sphere S^2, by Lemma 2 § 2, does not exceed the number of mapping classes of $\Sigma^4 \to S^3$; the latter is equal to two by virtue of Theorem 21. Thus, we see that the number of mapping classes of $\Sigma^{n+2} \to S^n$ does not exceed two.

A) Let M^2 be an orientable surface, i.e. smooth orientable manifold of dimension two, and let M^1 be a curve, i.e. a smooth one-dimensional manifold. Furthermore, let f be a smooth regular mapping from the curve M^1 to the surface M^2 such that no three distinct points of M^1 are mapped to the same point of the surface M^2. We shall also assume that if two distinct points a and b of the curve M^1 are taken by f to one point $c = f(a) = f(b)$ of the surface M^2 then the neighbourhoods of points a and b on the curve M^1 are mapped by f to curves having different tangent vectors at the point c. Under the above conditions, the set $C = f(M^1)$ is called a *smooth curve* on M^2. If the manifold M^1 is orientable, then the curve $C = f(M^1)$ is also said to be *orientable.* The points of type $c = f(a) = f(b)$ where $a \neq b$, are called *double* points of the curve C. It is easy to see that a curve on a surface can have only finitely many double points. If $C = f_1(M_1^1) = f_2(M_2^1)$, i.e. if the curve C is obtained from two different maps f_1 and f_2 of two different curves M_1^1 and M_2^1, and for f_1 and f_2 the conditions above hold, then there exists a smooth homeomorphism φ of M_1^1 to M_2^1 such that $f_2\varphi = f_1$. Thus, components of C can be defined as images of the components of M^1. We shall deal with empty curves as well. It is easy to see that if $C = f(M^1)$ is a curve on a surface, then if f' is sufficiently close to f (with respect to class 1), then the set $C' = f'(M^1)$ is also a curve on the surface. We shall say that C' is obtained from C by a *small shift.*

B) A curve C on M^2 is *zero-homologous* (more, precisely, mod 2 - zero homologous, if there exists on M^2 an open set G that $C = \bar{G} \setminus G$ and such that in any neighbourhood of any point $x \in C$ there are points belonging to M^2, and not belonging to $\bar{G}$; notation, $C = \triangle G, C \sim 0$. Obviously, a small shift of a zero-homologous curve is a zero-homologous curve. Let C_1 and C_2 be two such curves on M^2 such that double points of each of them do not belong to the other one and in any intersection point of the two curves the tangent vectors to them are distinct. In this case we say that $C_1 \cup C_2$ is a curve and we shall also say that C_1 and C_2 *admit a summation*; denote their sum $C_1 \cup C_2$ by $C_1 + C_2$. It is easy to see that if we have any two curves on M^2, then, after a small shift of one of them, we get two curves admitting summation. If two curves C_1 and C_2 on M^2 admit a summation and each of them is zero-homologous then their sum is also zero-homologous. Indeed, let $C_1 = \triangle G_1$, $C_2 = \triangle G_2$.

Put $G = (G_1 \cup G_2) \backslash (\bar{G}_1 \cap \bar{G}_2)$. It is easy to see that $C_1 + C_2 = \triangle G$. We shall also write $C_1 + C_2 \sim 0$ as $C_1 \sim C_2$. Thus, the relation $C_1 \sim C_2$ makes sense only in the case when the curves C_1 and C_2 admit a summation. If the curves C_1 and C_2 do not admit summation, then, applying a small translation to one of them, say, to C_1, we get two curves C_1' and C_2 admitting summation. If, furthermore, $C_1' \sim C_2$, then one says that $C_1 \sim C_2$. This is well defined since the definition is invariant upon passing from C_1 to the curve C_1'. The relation $C_1 \sim C_2$ turns out to be reflexive, symmetric and transitive; thus the set of all curves on the surface M^2 is divided into classes of homologoical curves. Denote the set of these classes by $\triangle^1(M^2) = \triangle^1$. In the set $\triangle^1$, we have the well-defined sum operation. If z_1, z_2 are two elements from $\triangle^1$, and $C_1 \in z_1$ and $C_2 \in z_2$ so that the curves C_1 and C_2 admit a summation then the class z containing the curve $C_1 + C_2$, is, by definition, set to be the sum of the classes z_1 and z_2, $z = z_1 + z_2$. This rule does not depend on the arbitrariness of the choice of C_1 and C_2 representing the classes z_1 and z_2. The group $\triangle^1$ is called *connection group* of the surface M^2. All elements of this group are of order two. A finite system of curves $C_1, \ldots, C_q$ on M^2 is called the *homology base* if for any curve C on the surface M^2 the relation

$$C \sim \sum_{i=1}^{q} \varepsilon_i C_i$$

holds, where $\varepsilon_i \equiv 0$ or $1 \pmod 2$, and if the relation

$$C \sim 0$$

yields that all residue classes ε_i are equal to zero.

C) Let C_1 and C_2 be two curves on the surface M^2, admitting summation. The number of intersection points of these curves taken modulo two is denoted by $J(C_1, C_2)$ and called the *intersection index.* It is easy to see that

$$J(C_1 + C_2, C_3) = J(C_1, C_3) + J(C_2, C_3)$$

and that from $C_1 \sim 0$ it follows that $J(C_1, C_2) = 0$. From this we have that if $C_1 \sim D_1$, $C_2 \sim D_2$ then $J(C_1, C_2) = J(D_1, D_2)$. Thus, by setting $J(z_1, z_2) = J(C_1, C_2)$, where $C_1 \in z_1$, $C_2 \in z_2$, we get a definition of the intersection index $J(z_1, z_2)$ for two homology classes. It turns out that on any surface M^2 there exists a homology basis consisting of curves $A_1, \ldots, A_p, B_1, \ldots B_p$, for which the equations

$$J(A_i, A_j) = J(B_i, B_j) = 0, \quad J(A_i, B_j) = \delta_{ij}, \qquad i, j = 1, \ldots, p \tag{1}$$

hold. Any such basis is called *canonical*. From this, it follows immediately that for any homology class $z \in \triangle^1$ the relation

$$J(z, z) = 0,$$

holds, and, furthermore, if z_1 is a non-zero homology class then there exists a homology class z_2 such that

$$J(z_1, z_2) = 1.$$

Since M^2 is connected, we can set the curves $A_1, \dots, A_p, B_1, \dots B_p$ to be those curves giving the *canonical cut* of the surface M^2. In this case, p is the genus of the surface. In the case of a disconnected surface the homology basis is obtained as a union of the bases of connected components. In this case, p is the sum of the genera of the connected components constituting M^2.

Theorem 22. *Let (M^2, U) be an orthonormally framed surface of the oriented Euclidean space E^{n+2} with basis $e_1, \dots, e_{n+2}$, defining its orientation, $U(x) = \{u_1(x), \dots, u_n(x)\}$, and let $C = f(M^1)$ be an oriented curve on M^2.*

Let $y \in M^1$. Denote by $\hat{u}_{n+2}(y)$ the unit vector tangent at $f(y)$ to the curve $f(M^1)$ and corresponding to the orientation of it, and denote by $\hat{u}_{n+1}(y)$ the unit vector tangent to M^2 at $f(y)$, orthogonal to $\hat{u}_{n+2}(y)$ and directed in such a way that the vectors $u_1(f(y)), \dots, u_n(f(y)), \hat{u}_{n+1}(y)$, $\hat{u}_{n+2}(y)$ give the positive orientation of the space E^{n+2}. For notational convenience we set $\hat{u}_i(y) = u_i(f(y))$, $i = 1, \dots, n$. We have

$$\hat{u}_i(y) = \sum_{j=1}^{n+2} h_{ij}(y) \cdot e_j, \quad i = 1, \dots, n+2,$$

where $h(y) = \|h_{ij}(y)\|$ is an orthogonal matrix with positive determinant, so that h is a continuous mapping from M^1 to the group H_{n+2}. Set

$$\delta(M^2, U, C) = \delta(C) \equiv \beta(h) + r(C) + s(C), \tag{2}$$

where $\beta(h)$ is defined in Statement «D» § 1, and $r(C)$ is the number of components of the curve C, and $s(C)$ is the number of double points of C. It turns out that $\delta(C)$ is an invariant of the homology class $z \in \triangle_1$ containing the curve C; thus we may set $\delta(M^2, U, z) = \delta(z) = \delta(C)$. Furthermore, it turns out that for any two arbitrary homology classes z_1 and z_2 of the surface M^2 we have

$$\delta(z_1 + z_2) = \delta(z_1) + \delta(z_2) + J(z_1, z_2). \tag{3}$$

PROOF. First of all, let us prove that $\beta(h)$ does not depend on the basis choice, $e_1, \dots, e_{n+2}$, nor on the orientation of $C = f(M^1)$. If instead of $e_1, \dots, e_{n+2}$ we take another basis $e'_1, \dots, e'_{n+2}$ then we shall get

$$e_j = \sum_{k=1}^{n+2} l_{jk} e'_k, \quad j = 1, \dots, n+2,$$

where $l = \|l_{jk}\|$ is an orthogonal matrix with positive determinant. Such a substitution will lead to the matrix $h'(y) = h(y)l$ instead of $h(y)$. Since the manifold H_{n+2} is connected, the mappings h and h' are homotopic; this yields the independence of $\beta(h)$ from the choice of $e_1, \dots, e_{n+2}$. Now, if we change the orientation of the one-component curve $C = f(S^1)$ to the opposite one, then the vectors $\hat{u}_{n+1}(y)$ and $\hat{u}_{n+2}(y)$ are to be replaced by the vectors $-\hat{u}_{n+1}(y)$ and $-\hat{u}_{n+2}(y)$. This leads to the replacement of $h(y)$ by the matrix $h'y = l \cdot h(y)$, where $l_{ij} = 0$ for $i \neq j$,

$$l_{11} = \ldots = l_{nn} = 1, \quad l_{n+1,n+1} = l_{n+2,n+2} = -1.$$

Since the matrix $l = \|l_{ij}\|$ belongs to the manifold H_{n+2}, then the mappings h and h' are homotopic; thus $\beta(h)$ does not depend on the orientation of the one-component curve C. Clearly, the same is true for any arbitrary curve.

To prove that $\delta(C)$ is an invariant of the homology class z containing the curve C, let us introduce the following surgery operation for an orientable curve C near its double point a; as a result the curve C will be transformed to the oriented curve $C_a = f_a(M^1_a)$. We shall denote the mapping from M^1_a to H_{n+2}, corresponding to the curve C_a, by h_a. The surgery operation will be defined in such a way that the curve C_a has one double point less than the curve C; moreover, the conditions

$$C_a \sim C, \quad \delta(C_a) = \delta(C)$$

hold.

By virtue of Statements of «C» and «D» § 3, Chapter 3, one may assume that near the point a the surface M^2 coincides with the plane E_2, the curve C coincides with two intersecting lines and the vectors $u_i(x)$ coincide with e_i, $t = 1, \dots, n$. We take these lines to be the coordinate axes of the coordinate system x^1, x^2 defined near the point a on M^2. We choose the direction of axes in such a way that as any coordinate increases, we move along the curve in the positive direction. We shall assume that the curves C_a and C coincide outside a neighbourhood of the point a and that near the point a the curve C_a is given by the equation $x^1 \cdot x^2 = -\varepsilon$, where $\varepsilon > 0$ (see Fig. 4.1). Thus, the orientation of the curve C naturally generates the

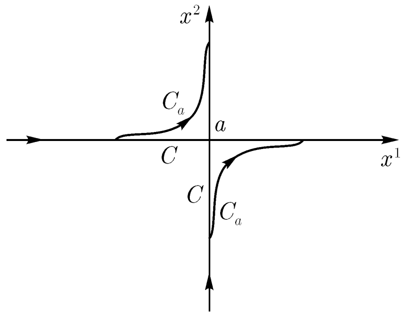


Figure 4.1.

orientation of C_a. It is easy to see that if both branches of the curve C passing through a belong to the same component of it then, after performing the surgery on this component, we get two different components of the curve C_a. Conversely, if two branches of C passing through a belong to different components of C, then, as a result of the surgery, instead of these two components of C, we get one component of C_a. Thus, in both cases $r(C) + s(C) \equiv r(C_a) + s(C_a) \pmod 2$.

Let us show that $\beta(h) = \beta(h_a)$. Indeed, the mapping hf^{-1} takes the neighbourhood of the point a on the curve C to two points on the circle H_2 (see Theorem 12), and the mapping $h_a f_a^{-1}$ takes the parts of the curve C_a close to a to the circle H_2 with mapping degree equal to zero. From this it follows that $\delta(C) = \delta(C_a)$. Now, it is evident that the curves C and C_a are homologous.

As a result of finitely many surgeries as above, we transform C to a curve $O(C)$ without double points, for which the relations

$$O(C) \sim C, \quad \delta(O(C)) = \delta(C) \tag{4}$$

hold.

Let us show now that if the curve C without double points is null-homologous on M^2 then

$$\delta(C) = 0. \tag{5}$$

Let $C = \triangle G$; then $\bar{G}$ is a smooth surface bounded by the curve C. It is easy to define on $\bar{G}$ a smooth function χ, which is positive and less than 1 in G and equal to zero on C, so that the full differential of this function is non-zero on C. Inside the strip $E^{n+2} \times I$, where I is the unit interval $0 \leqslant t \leqslant 1$, consider the surface P^2, defined by the equation

$$t = +\sqrt{\chi(x)}, \quad x \in \bar{G}.$$

It is easy to see that the boundary of this surface is the curve $C \times 0$; moreover, the surface is orthogonal to the boundary $E^{n+2} \times 0$ of the strip $E^{n+2} \times I$. Since the surface P^2 is homeomorphic to the orientable surface $\bar{G}$, we assume the former to be orientable. Since vectors of the system $U(x)$ are orthogonal to the surface $\bar{G}$ at x, the vectors of the system $U(x) \times t$ are orthogonal to P^2 at $x \times t$. Let us complete the vectors of the system $U(x) \times t$ by the unit vector $u_{n+1}(x,t) \times t$ in such a way that the obtained system $\tilde{U}(x,t) \times t$ contains an orthonormal framing of the oriented surface P^2 in the oriented strip $E^{n+2} \times I$. The vector $u_{n+1}(x,0)$ obtained in this way, is orthogonal to the curve C and tangent to M^2 at the point x. Thus, by completing the system $U(x)$ by $u_{n+1}(x,0)$, we get a framing $V(x)$ of the curve C such that the framed curve (C,V) is zero-homologous. Furthermore, by completing the system $V(x)$ by the vector $u_{n+2}(x)$ tangent to C at x, we get precisely the system $\hat{U}(x)$ needed to calculate $\delta(C)$. Comparing the construction of the residue class of $\delta(C)$, given here, with the construction of $\delta(C,V)$ (see Theorem 20), we see that

$$\delta(C) = \delta(C,V).$$

Since the framed manifold (C,V) is null-homologous, then $\delta(C) = \delta(C,V) = 0$ (see Theorem 20). Thus, (5) is proved.

Let C_1 and C_2 be two arbitrary curves on M^2 admitting summation. We have

$$s(C_1 + C_2) \equiv s(C_1) + s(C_2) + J(C_1, C_2) \pmod 2,$$

$$r(C_1 + C_2) = r(C_1) + r(C_2),$$

$$\beta(C_1 + C_2) = \beta(C_1) + \beta(C_2).$$

From this we have that

$$\delta(C_1 + C_2) = \delta(C_1) + \delta(C_2) + J(C_1, C_2). \tag{6}$$

If, in particular, $C_1 \sim C_2$ then $J(C_1, C_2) = 0$ and from the relations (6), (5) we get

$$\delta(C_1) + \delta(C_2) = \delta(C_1 + C_2) = \delta(O(C_1 + C_2)) = 0.$$

Thus, we have proved that $\delta(C)$ is a homology invariant. From this and from the relation (6) applied to arbitrary curves C_1 and C_2 admitting summation, we obtain (3).

Thus, Theorem 22 is proved.

Theorem 23. *Let (M^2, U) be an orthonormally framed submanifold of the Euclidean space E^{n+2} and let*

$$A_1, \dots, A_p, B_1, \dots, B_p \tag{7}$$

be any arbitrary canonical basis of the surface M^2. It turns out that the residue class

$$\delta = \delta(M^2, U) = \sum_{i=1}^{p} \delta(A_i)\delta(B_i) \tag{8}$$

does not depend on the arbitrary choice of the canonical basis (7) *and it is an invariant of the framed manifold* (M^2, U).

PROOF. Let us consider some canonical basis

$$A'_1, \dots, A'_p, B'_1, \dots, B'_p \tag{9}$$

of the surface M^2 and show that

$$\sum_{i=1}^{p} \delta(A_i)\delta(B_i) = \sum_{i=1}^{p} \delta(A'_i)\delta(B'_i). \tag{10}$$

The direct proof of the equality (10) in the case of arbitrary canonical bases (7) and (9) has several technical difficulties; thus we shall consider three particular types of transformations of the canonical basis; for any of these types the proof of formula (10) is quite easy. Finally, we shall show that any canonical basis transformation from (7) to (9) can be obtained as a sequence of the partial cases described above. This will complete the invariance of the residue class δ.

Transformation 1. Let j be a positive integer which does not exceed p. Set

$$A'_j = B_j, \quad B'_j = A_j, \quad A'_i = A_i, \quad B'_i = B_i, \quad i \neq j. \tag{11}$$

Clearly, the basis $A'_1, \dots, A'_p, B'_1, \dots, B'_p$ defined by these relations, is canonical and the relation (10) holds in this case.

Transformation 2. Set

$$A'_i = \sum_{k=1}^{p} a_{ik} A_k, \quad i = 1, \dots, p, \tag{12}$$

$$B'_i = \sum_{k=1}^{p} b_{jk} B_k, \quad j = 1, \dots, p, \tag{13}$$

where a_{ik} and b_{jk} are residue classes modulo two. In order for the basis (12)–(13) to be canonical, it is necessary that the matrix $a = \|a_{ij}\|$ be non-degenerate, i.e. it should have determinant $+1$, and that the matrix $b = \|b_{ij}\|$ be connected with a by

$$\sum_{k} a_{ik} b_{jk} = \delta_{ij}, \tag{14}$$

i.e. if we denote the unit matrix by e and the matrix obtained from b by transposing, by b', then $ab' = e$, or, equivalently $a^{-1} = b'$; thus $b'a = e$. The latter relation gives

$$\sum_{i=1}^{p} a_{ij} b_{ik} = \delta_{jk}. \tag{15}$$

Thus,

$$\begin{aligned}\sum_i \delta(A'_i)\delta(B'_i) &= \sum_i \delta\left(\sum_{j=1}^{p} a_{ij} A_j\right) \delta\left(\sum_{k=1}^{p} b_{ik} B_k\right) \\ &= \sum_{i,j,k=1}^{p} a_{ij} b_{ik} \delta(A_j)\delta(B_k) = \sum_{j,k=1}^{p} \delta_{jk}\delta(A_j)\delta(B_k) \\ &= \sum_{j=1}^{p} \delta(A_j)\delta(B_j)\end{aligned}$$

[see (3)], and the relation (10) in the case of transformation 2 holds. Note that transformation 2 is completely defined by the matrix a, giving the transformation (12). The transformation (13), by virtue of formula (14), is uniquely defined by (12). We shall say that the transformations (12) and (13) are coordinated if the relation holds (14).

Transformation 3. Set

$$A'_i = A_i + \sum_k c_{ik} B_k, \quad i = 1, \ldots, p, \tag{16}$$

$$B'_i = B_i, \quad i = 1, \ldots, p. \tag{17}$$

In order to get $J(A'_i, A'_j) = 0$, $i, j = 1, \ldots, p$, it is necessary and sufficient that

$$c_{ij} = c_{ji}. \tag{18}$$

Indeed

$$\begin{aligned}J(A'_i, A'_j) &= J\left(A_i, \sum_k c_{jk} B_k\right) + J\left(\sum_k c_{ik} B_k, A_j\right) \\ &= \sum_k c_{jk}\delta_{ik} + \sum_k c_{ik}\delta_{jk} = c_{ji} + c_{ij}.\end{aligned}$$

If the relation (18) holds then the basis (16)–(17) is canonical. Let us show that under the transformation 3, the relation (10) holds. We have

$$\sum_i \delta(A'_i)\delta(B'_i) = \sum_i \delta\left(A_i, \sum_k c_{ik} B_k\right)\delta(B_i)$$
$$= \sum_i \left(\delta(A_i) + \sum_k c_{ik}\delta(B_k) + \sum_k c_{ik} J(A_i, B_k)\right)\delta(B_i)$$
$$= \sum_i \delta(A_i)\delta(B_i) + \sum_{i,k} c_{ik}\delta(B_k)\delta(B_i) + \sum_{i,k} c_{ik}\delta_{ik}\delta(B_i).$$

Then,

$$\sum_{i,k} c_{ik}\delta(B_i)\delta(B_k) = \sum_i c_{ii}\delta(B_i)\delta(B_i) = \sum_i c_{ii}\delta(B_i)$$

(since we deal with residue classes modulo 2) and

$$\sum_{i,k} c_{ik}\delta_{ik}\delta(B_i) = \sum_i c_{ii}\delta(B_i).$$

Thus, relation (10) holds.

Now, let us consider an arbitrary canonical basis transformation from (7) to the basis (9). We have

$$A'_i = \sum_j r_{ij} A_j + \sum_k s_{ik} B_k. \tag{19}$$

The rank of the rectangular matrix of p rows and $2p$ columns, defining this transformation, is equal to p, i.e. one of its minors of order p is non-zero. Applying to the basis (7) the transformation 1 several times, we may arrive at such a new basis (which we shall denote again by $A_1, \dots, A_p, B_1, \dots, B_p$) that in the formula (19) the minor $|r_{ij}|$ will be non-zero. Applying to the obtained basis $A_1, \dots, A_p, B_1, \dots, B_p$ the transformation 2 with matrix $\|a_{ij}\| = \|r_{ij}\|$, we shall make the transformation (19) look like (16). Now, let us introduce a new canonical basis $A''_1, \dots, A''_p, B''_1, \dots, B''_p$, by applying the transformation 2:

$$A''_i = A_i + \sum_{k=1}^{p} c_{ik} B_k, \quad B''_i = B_i.$$

From (9), we get this basis by the following formulae

$$A_i' = A_i'',$$
$$B_i' = \sum_{j=1}^{p} r_{ij}' A_j'' + \sum_{k=1}^{p} s_{ik}' B_k''. \tag{20}$$

The relation $J(A_i', B_j') = \delta_{ij}$ gives $\sum_k s_{jk}' \delta_{ik} = \delta_{ij}$, or $s_{ij}' = \delta_{ij}$. Thus, the transformation (20) will look like

$$A_i' = A_i'',$$
$$B_i' = B_i'' + \sum_{j=1}^{p} r_{ij}' A_j'',$$

i.e. it is a transformation of the third type where the roles of the curves A_i and B_i are interchanged. Thus, we can get from the basis (7) to the basis (9) by a sequence of transformations 1–3.

Thus, Theorem 23 is proved.

Theorem 24. *If two framed submanifolds (M_0^2, U_0) and (M_1^2, U_1) of the Euclidean space E^{n+2} are homologous, then we have*

$$\delta(M_0^2, U_0) = \delta(M_1^2, U_1) \tag{21}$$

[*see* (8)]. *Thus, to each element π of the group Π_n^2 there corresponds a unique residue class $\delta(\pi)$ defined by the relation $\delta(\pi) = \delta(M^2, U)$, where (M^2, U) is a framed manifold of class π. It turns out that for $n \geqslant 2$, δ is an isomorphism of the group Π_n^2 to the group of residue classes modulo two. From the above it follows that there exist precisely two classes of mappings $\Sigma^{n+2} \to S^n$, $n \geqslant 2$.*

PROOF. First, let us prove the relation (21). Let (M^3, U) be the framed submanifold of the strip $E^{n+2} \times I$, realising the homology $(M_0^2, U_0) \sim (M_1^2, U_1)$, which is constructed in Lemma 1 § 3. Set $M_t^2 \times t = M^3 \cap (E^2 \times t)$. If the point $(x, t) \in M^3$ is not a critical point of the manifold M^3 then the neighbourhood of the point x in the set M_t^2 is a smooth surface, so that for a non-critical value of the parameter t the set M_t^2 is a surface. In this case, if (x_0, t_0) is a critical point of the manifold M^3 then the set M_t^2 for a small value of $|t - t_0|$ is defined by

$$\sigma^1(x^1)^2 + \sigma^2(x^2)^2 + \sigma^2(x^3)^2 = t - t_0,$$
$$x^4 = \ldots = x^{n+2} = 0 \tag{22}$$

near x_0 [see § 3, formula (3)]. If $(x, t) \in M^3$ is not a critical point of M^3 then the orthogonal projection of $U(x, t)$ to the plane $E^{n+2} \times t$ is a linearly independent system of vectors. Denote the system obtained from it by the orthogonalization process, by $V_t(x) \times t$. For a non-critical value of the parameter t, the system V_t gives an orthonormal framing of the manifold M_t^2. As t continuously increases without passing through critical values, the framed manifold (M_t^2, V_t) is continuously deformed; thus, it follows from the continuity argument that $\delta(M_t^2, V_t)$ does not change. Thus, to prove Statement (21), it suffices to show that $\delta(M_t^2, V_t)$ does not change as t passes through the critical point $t = t_0$. Let us do this. We have two different cases.

CASE 1. Assume $\sigma^1 = \sigma^2 = \sigma^3$. For definiteness, we shall assume that $\sigma^1 = \sigma^2 = \sigma^3 = +1$. Under this assumption, after passing through a critical value, the surface M_t^2 acquires a new component, which is a small sphere, and the remaining part of M_t^2 is transformed continuously together with its framing. Since the attachment of a sphere as a separate component does not increase the genus of the surface, then the canonical basis may be thought to remain the same thus $\delta(M_t^2, V_t)$ does not change.

CASE 2. Assume that among $\sigma^1, \sigma^2, \sigma^3$ there are distinct numbers. For definiteness, we shall think that $\sigma^1 = \sigma^2 = +1, \sigma^3 = -1$. Under this assumption, the surface M_t^2 for $t < t_0$ near the point x_0 looks like a two-sheeted hyperboloid and for $t > t_0$ it looks like a one-sheeted hyperboloid. Thus surgery is identical to a tube attachment to the surface M_t^2, $t < t_0$. If the tube connects two different components of the surface M_t^2, $t < t_0$, then the basis of the surface M_t^2 does not change while passing through t_0; thus the residue $\delta(M_t^2, V_t)$ remains invariant. If the tube is attached to one component, then the basis of the surface should be completed by two curves. Let us be more detailed. Let $A_1, \dots, A_p, B_1, \dots, B_p$ be the canonical basis of the surface M_t^2, $t < t_0$. We may assume that the curve composing this basis avoids from the point x_0; thus, as the parameter t passes through the critical value t_0, the basis changes continuously, so that the residue classes $\delta(A_i)$ and $\delta(B_i)$, $i = 1, \dots, p$, remain unchanged. We define the curve A_{p+1} on M_t^2 to be the circle cut from the part of M_t^2 close to x_0, by the hyperplane $x^3 = \varepsilon$, where ε is a small positive number. For $t < t_0$, it is evidently null-homologous on M_t^2, and, since the framing of it changes continuously as t passes through t_0; then $\delta(A_{p+1}) = 0$. Now let B'_{p+1} be an arbitrary curve on M_t^2, $t > t_0$ having with A_{p+1} intersection index equal to one. Clearly, such a curve exists. Now, set

$$B_{p+1} = B'_{p+1} + \sum_{j=1}^{p} J(B_i, B'_{p+1})A_i + \sum_{i=1}^{p} J(A_i, B'_{p+1})B_i.$$

Obviously, the curves $A_1, \dots, A_{p+1}, B_1, \dots, B_{p+1}$ form a canonical basis of the surface M_t^2, $t > t_0$, and since $\delta(A_{p+1}) = 0$ then $\delta(A_{p+1})\delta(B_{p+1}) = 0$.

Thus, the residue class $\delta(M_t^2, V_t)$ remains unchanged as t passes through the critical value t_0; thus, $\delta(M_0^2, V_0) = \delta(M_1^2, V_1)$. Since $U_0 = V_0, U_1 = V_1$, the relation (21) holds.

From the above argument it follows that δ is a mapping from the group Π_n^2 to the group of residue classes modulo two. It follows from the definition of the sum operation in Π_n^2 that δ is a homomorphic mapping.

Now let us prove that δ is an epimorphic mapping to the whole group of residue classes modulo two. For that, it suffices to show that there exists a framed manifold (M^2, U) for which $\delta(M^2, U) = 1$. Since, evidently,

$$\delta(E(M^2, U)) = \delta(M^2, U), \quad n \geqslant 2,$$

where E is the suspension operation, it suffices to consider the case $n = 2$.

Let E^4 be the Euclidean space having orthonormal basis e_1, e_2, e_3, e_4 and coordinates x^1, x^2, x^3, x^4; let E^3 be the linear subspace of E^4 defined by the equation $x^4 = 0$ and let M^2 be the usual metric torus, lying in E^3 and having the rotation axis e_3. Let us introduce on M^2 the usual cyclic coordinates φ, ψ and define the surface M^2 by the equations

$$\begin{aligned} x^1 &= (2 + \cos\varphi)\cos\psi, \\ x^2 &= (2 + \cos\varphi)\sin\psi, \\ x^3 &= \sin\varphi. \end{aligned} \tag{23}$$

Denote by A_1 the curve on M^2 defined by $\psi = 0$ and denote by B_1 the curve defined by $\varphi = 0$. It is evident that the system A_1, B_1 forms a canonical basis of the surface M^2. Denote by $v_1(x)$ the unit vector in E^3 which is normal to M^2 at $x = (\varphi, \psi)$ and directed outwards, and denote by $v_2(x)$ the vector emanating from x and parallel to e_4. Define the framing $U(x) = \{u_1(x), u_2(x)\}$ by the relations

$$\begin{aligned} u_1(x) &= v_1 \cos(\varphi - \psi) - v_2(x)\sin(\varphi - \psi), \\ u_2(x) &= v_1 \sin(\varphi - \psi) + v_2(x)\cos(\varphi - \psi) \end{aligned} \tag{24}$$

and let us show that

$$\delta(M^2, U) = 1. \tag{25}$$

Let C be any simple closed curve on M^2. Denote the unit tangent vector to C at the point $x = (\varphi, \psi) \in C$, by $v_4(x)$, and denote the unit vector tangent to M^2 at x and orthogonal to $v_4(x)$, by $v_3(x)$. Let us add to (24) the relations

$$u_3(x) = v_3(x), \quad u_4(x) = v_4(x). \tag{26}$$

The relations (24) and (26), taken together, transform the system $V(x)$ to the system $U(x)$. Denote the corresponding matrix by $f(x), x \in C$. It is easy to see that for $C = A_1$ and for $C = B_1$ we have

$$\beta(f) = 1. \tag{27}$$

Furthermore, for $C = A_1$ we have

$$x = (\varphi, 0), \quad v_1(x) = e_1 \cos\varphi + e_3 \sin\varphi, \quad v_2(x) = e_4,$$

$$v_3(x) = e_2, \quad v_4(x) = -e_1 \sin\varphi + e_3 \cos\varphi,$$

thus, the transfer from e_1, e_2, e_3, e_4 to the system $V(x)$ is defined by the orthogonal matrix $g(x)$, so that

$$\beta(g) = 1. \tag{28}$$

For $C = B_1$, we have analogously

$$x = (0, \psi), \quad v_1(x) = e_1 \cos\psi + e_2 \sin\psi, \quad v_2(x) = e_4,$$

$$v_3(x) = -e_3, \quad v_4(x) = -e_1 \sin\psi + e_2 \cos\psi,$$

so that the transfer from e_1, e_2, e_3, e_4 to $V(x)$ is defined by $g(x)$, herewith

$$\beta(g) = 1. \tag{29}$$

In both cases for $C = A_1$ and for $C = B_1$, the transfer from e_1, e_2, e_3, e_4 to $U(x)$ is defined by the matrix $h(x) = g(x)f(x)$, where

$$\delta(h) = \delta(g) + \delta(f) = 0$$

[see (27)–(29) and «D» § 1]. Thus, by virtue of (2) and (8), we have

$$\delta(A_1) = 1, \quad \delta(B_1) = 1, \quad \delta(M^2, U) = 1.$$

Thus, (25) is proved.

Finally, let us show that δ is an isomorphism. To do this, it suffices to show that the group Π_n^2 contains no more than two elements, since it is mapped to the whole group of residue classes modulo 2. From Theorems 11 and 16 it follows that for any framed manifold (M^2, U) of the Euclidean space E^{n+2} we have

$$(M^2, U) \sim E^{n-2}(N^2, V),$$

where (N^2, V) is a framed manifold of the four-space, and E^{n-2} is the $(n-2)$-times suspension operation. Thus, it suffices to show that Π_2^2 contains no more than two elements, i.e. there exist no more than two mapping classes $\Sigma^4 \to S^2$. By virtue of Lemma 2 § 2, the number of mapping classes from the sphere Σ^4 to the sphere S^2 does not exceed the number of mapping classes from Σ^4 to S^3; the latter equals two by virtue of Theorem 21.

Thus, Theorem 24 is proved.

References

[1] Pontrjagin L. S. *A classification of continuous transformations of a complex into a sphere*, 1. Doklady Akad. Nauk SSSR (N.S.) 19 (1938), 361–363.

[2] Pontrjagin L. S. *Homotopy classification of mappings of an (n + 2)-dimensional sphere on an n- dimensional one*, Doklady Akad. Nauk SSSR (N.S.) 19 (1950), 957–959

[3] Rokhlin V. A. *The classification of mappings of the* $(n+3)$*-sphere to the n-sphere.* — Doklady Akad Nauk SSSR (N.S.), 1951, **81**, No. 1, c. 19–22.

[4] Serre J. P. *Sur les groupes d'Eilenberg — Maclane.* — C. r. Acad. Sci. Paris, 1952, **234**, No. 12, p. 1243–1245; Serre J. P. *Sur la suspension de Freudenthal.* — C. R. Acad. Sci. Paris, 1952, **234**, No. 13, p. 1340–1342.

[5] Whitney H. *Differentiable manifolds.* — Ann. Math., 1936, **37**, p. 645–680.

[6] Dubovistsky. Ya. *Sur les transformations differentiables des cubes* (en russe) (Math. Sb., 1953, No. 2., pp. 443-464)

[7] Morse M. *The calculus of variations in the large.* — Amer. Math. Soc. Colloq. Publ., 1934, **18**.

[8] Whitney H. *The general type of singularity of a set of* $(2n-1)$*-smooth functions of n variables.* — Duke Math. J., 1943, **10**, p. 162–172.

[9] Hausdorff. *Set Theory.* — AMS Chelsea Publishing, 2005.

[10] Rokhlin V. A. *Homotopy Groups.* — Uspekhi Mat. Nauk (Russ. Math. Surveys), 1946, **1**, No. 5–6, c. 175–223.

[11] Freudenthal H. *Über die Klassen der Sphärenabbildungen.* — Compositio math., 1937, **5**, S. 299.

[12] Hopf H. *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche.* — Math. Ann., 1931, **104**, S. 639.

[13] Brouwer L. E. J. *On looping coefficients.* — Proc. Kon. Akad. Wetensch. Amsterdam, 1912, **15**, p. 113.

2

Some "global" properties of differentiable manifolds[1]

R. Thom

Introduction

The present work is devoted to the proof of results announced in the author's note [28]. It is divided into four chapters. In Chapter I, we consider some questions of approximation of differentiable manifolds. The theorems proved there, which are analogous to the topological simplicial approximation theorem, allow one to avoid references to the triangulation theorem for manifolds. Chapter II is devoted to the realisation problem of homology classes of manifolds by means of cycles. The main results obtained in this chapter are the following: *a mod 2 homology class of any manifold is realisable by means of a submanifold if the dimension of this class is less than half the dimension of the manifold. For any integer homology class z of an orientable manifold V there exists a non-zero integer number N such that the class Nz is realisable by a submanifold.* In Chapter III, the results obtained in Chapter II are applied for the solution of the following Steenrod problem: can any homology class of a finite polyhedron be represented as an image of the fundamental class of some manifold? It is shown that for mod 2 homologies the answer is affirmative. *On the contrary, in any dimension $\geqslant 7$ there exist integral homology classes which are not representable as images of fundamental classes of compact differentiable manifolds.* Finally Chapter IV is devoted to the conditions sufficient for a manifold to be a boundary and to the conditions under which two manifolds are cobordant.

[1]Thom R., Quelques proprietés globales des variétés differentiables, *Comm. Math. Helv.*, 28 (1954), 17–86. Reprinted with permission from Birkhäuser. *Translated by V.O.Manturov with M.M.Postnikov's comments (1958).*

Here we obtain quite complete results only for mod 2 cobordism classes, when we do not pay attention to the orientability of the manifolds. On the contrary, for the groups Ω^k, which appear in the classification of orientable manifolds, we have obtained only partial results. The difficulties which appear in this direction are of an algebraic nature, and they are connected with the behaviour of Steenrod's degrees in spectral sequences of bundles. The results of Chapter IV are closely connected with the question of the topological value of Pontrjagin's characteristic numbers.

The methods used in this work are based almost always on the consideration of some auxiliary polyhedra $M(SO(k))$ and $M(O(k))$. The study of geometric properties of these polyhedra uses methods of H.Cartan and J.-P.Serre. In particular, the Eilenberg-MacLane polyhedra play a key role. I wish to thank them for communicating these results to me before the publication. In particular, I would like to mention J.-.P.Serre's help both in editing the manuscript and for improving many proofs.

Chapter I

Properties of differentiable mappings

In the sequel, by V^n we mean any paracompact[1] differentiable n-manifold of class C^∞.

1. Definitions

Let f be a mapping of class C^m, $m \geqslant 1$, from V^n to some manifold M^p. By a *critical point* we mean a point x of V^n where the rank of f is *strictly less* than the dimension p of the manifold M^p. The set Σ of critical points x or *the critical set* of the mapping f, is closed in V^n. Any point y in the image $f(\Sigma) \subset M^p$ of this set is called a *critical value* of f. Any point y of M^p not belonging to $f(\Sigma)$ is a *regular value*[2].

2. Pre-image of a regular value

The pre-image $f^{-1}(y)$ of a regular value $y \in M^p$ might be empty. For instance, if the dimension n of the manifold V^n is strictly less than the

[1] Recall that a connected paracompact manifold can be defined as a manifold which is a union of countably many compacta.

[2] Note that this definition of critical values differs from the usual one: if the dimension n of the manifold V is strictly less than the dimension p of the manifold M then any point of the image $f(V)$ is a critical value, even for mapping f having maximal rank in any point of $f^{-1}(x)$. Conversely, any point not belonging to the image $f(V)$ is a regular value.

dimension p of the manifold M^p then the pre-image $f^{-1}(y)$ is empty for any regular value y. Assuming that the pre-image $f^{-1}(y)$ is non-empty, let us consider an arbitrary point $x \in f^{-1}(y)$. Let $y_1, y_2, \ldots, y_p$ be some local coordinates of the manifold M^p in some neighbourhood of the point y. Since the mapping f has rank p at the point x, then in some reasonably small neighbourhood U_x of the point x in the manifold V^n, there exists a local coordinate system of the type $(y_1, \ldots, y_p, x_{p+1}, \ldots, x_n)$. In these coordinates, the pre-image $f^{-1}(y)$ is defined in the neighbourhood of U_x by the equations $y_1 = y_2 = \ldots = y_p = 0$. Consequently, the point x has a neighbourhood in $f^{-1}(y)$, which is homeomorphic to the Euclidean space R^{n-p}. Since this is true for any point $x \in f^{-1}(y)$, the pre-image $f^{-1}(y)$ is a differentiable class C^m manifold V^n. Below, we denote this manifold by W^{n-p}.

Let V_x be a tangent space at the point x of V^n, and let W_x be the subspace of V_x consisting of those vectors tangent to the submanifold W^{n-p}. Furthermore, let M_y be the tangent space to M^p at y. The condition that f has rank p in x means that the tangent space mapping $\bar{f}$ defines an isomorphism mapping from the factor-space V_x/W_x to the space M_y. Admitting some non-exact terminology, we call the factor-space V_x/W_x *transverse* to the manifold W^{n-p} in x. If the ambient manifold V^n is endowed with a Riemannian metric then we have a well-defined space H_x, *normal* to W^{n-p} at x. It is clear that the spaces V_x/W_x and H_x are isomorphic, and the isomorphism can be defined globally, i.e. for all points of the manifold W^{n-p}. All normal (resp., transversal) vectors to the submanifold W^{n-p} form a normal (resp., transversal) fibre. According to the arguments above, these spaces are isomorphic.

Collecting the arguments above, we have: *the pre-image $f^{-1}(y) = W^{n-p}$ of any regular value y of the function f is a submanifold of the manifold V^n and the corresponding mapping $\bar{f}$ induces a natural isomorphism between the normal fibre for W^{n-p} and the Cartesian product $W^{n-p} \times M_y$, where $M_y \approx R^p$ is the tangent space to M^p at the point y.*

Remark. The statement above is true even in the case when y is a *limit point* of the set of critical values. Note that if f is *proper*[1] (in particular, if the manifold V^n is *compact*) then the set $f(\Sigma)$ of critical values is *closed* in M^p. In the latter case, any regular value y has a neighbourhood U_y, for which the mapping f is *locally fibre.* This is the local form of Ehresmann's theorem [10].

[1]A mapping f is called proper if the full pre-image $f^{-1}(K)$ of any compact subset $K \subset M^p$ is a compact subset of the manifold V^n. — *Editor's remark.*

3. Properties of the critical values set $f(\Sigma)$

The *subset of interior points* of the set $f(\Sigma)$ might be non-empty. For example, Whitney constructed in [30] a numerical C^1-function defined on the square for which any value is critical. However, this phenomenon can take place for C^m-mappings f only if m is strictly less than the dimension n of the mapped manifold. Indeed, Morse [16] has proved the following theorem.

Theorem I.1. *The set of critical values of a numerical function of class C^m, defined in R^n where $m \geqslant n$, has measure zero.*

(For functions of class C^r, $r < n$, this theorem is not true.)

Any paracompact manifold V^n can be covered by countably many neighbourhoods homeomorphic to the space R^n; and, since the union of countably many sets of measure zero is a set of measure zero, we get the following

Theorem I.2. *If $m \geqslant n$ then the set of critical points of any numerical class C^m function defined on an n-dimensional manifold V^n has measure zero.*

Now, let us prove the following Theorem[1].

Theorem I.3. *If $m \geqslant n$ then for each mapping f of class C^m from a manifold V^n to a manifold M^p, in any open set of the manifold M^p there exist regular values of the mapping f.*

In other words, the *set $f(\Sigma)$ of critical values of the mapping f has no interior points.*

Since this theorem deals with a local property of the manifold M^p, we may take the Euclidean space R^p instead of this manifold. Note that for $p = 1$ the theorem is a straightforward corollary of Theorem I.2. Let us prove it by induction on p. Thus, supposing that if the theorem is true for manifolds of dimension less than or equal to $p-1$, consider an arbitrary mapping f of class C^n from V^n to the space R^p. Let $y_1, y_2, \ldots, y_p$ be the coordinates in the space R^p and let U be an arbitrary open set in the space R^p and let (a, b) be some open interval of values that the function y_p takes on the set U. Since on V^n the coordinate y_p is a smooth function of class C^n, then, by Theorem I.2, the interval (a, b) contains some regular value c of the function y_p. We may assume that the pre-image $W^{n-1} = y_p^{-1}(c)$, being an $(n-1)$-dimensional submanifold of the manifold V^n, is non-empty (otherwise the Theorem is trivial). Let x be an arbitrary point of the

[1] As de Rham communicated to me, this result is a partial case of a theorem by Sard (A. Sard. The measure of the critical values of differentiable maps. Bull. Amer. Math. Soc., **48** (1942), 883–890).

submanifold W^{n-1}. In a small neighbourhood V_x of x in the manifold V^n there exists a local coordinate system $(x_1, x_2, \ldots, x_{n-1}, y_p)$ that contains the coordinate y_p. Let U_c be a section of the domain U by the hyperplane $\{y_p = c\}$. The restriction f_c of f to the manifold W^{n-1} is a mapping of class C^n. Consequently, according to the induction hypothesis, for the mapping $f_c : W^{n-1} \to R^{p-1}$, where R^{p-1} is the hyperplane $\{y_p = c\}$, there exists in U_c some regular value a. Let $x \in V^n$ be an arbitrary point of the pre-image $f_c^{-1}(a) = f^{-1}(a, c)$ (we may assume that this pre-image is non-empty). Since a is a regular value of f_c then in the neighbourhood V_x of x this mapping has rank $p-1$. On the other hand, let $f(x_1, x_2, \ldots, x_{n-1}, c) = (y_1, y_2, \ldots, y_{p-1})$. Then for $y_p = c$ at least one of the minors $\left|\dfrac{\partial y_i}{\partial x_j}\right|$ is non-zero. Consequently, by continuity argument, this minor is non-zero for all values of the coordinate y_p close to c. From this it follows that in a neighbourhood $V'_x \subset V_x$, the functions $x_1, x_2, \ldots, x_{n-p}, y_1, y_2, \ldots, y_p$ form a local coordinate system. In other words, in x the mapping f has maximal rank. Since this is true for any point x of the pre-image $f^{-1}(a, c)$, the point $(a, c) \in U$ is a regular value of the mapping f. Thus, Theorem I.3 is proved.

If f is a *proper mapping* (in particular if the manifold V^n is *compact*) then the set $f(\Sigma)$ of critical values of the mapping f is a closed set *without interior points*, i.e. a *sparse*, according to Bourbaki's terminology, [6], IX, set of the manifold M^p. For an arbitrary mapping f, consider first a covering $V^n = \cup_j K_j$ of the manifold V^n by some compacta K_j. Each intersection $\Sigma_j = K_j \cap \Sigma$ is a compact set, thus $f(\Sigma_j)$ is a *sparse* compact set of the manifold M^p. Thus, the set $f(\Sigma) = \cup_j f(\Sigma_j)$ is a continuous union of sparse closed sets, i.e. according to [6], IX, it is a *thin subset of the manifold M^p*.

3a. Pre-image of a manifold[1]

Definition. *Tubular neighbourhood of a manifold.* Let N^{p-q} be some compact class C^∞ submanifold of the manifold M^p. Assume that M^p is endowed with a class C^∞ Riemannian metric, and consider the set T of all points of the manifold M^p located at distance $\leqslant \varepsilon$ from N^{p-q}. If ϵ is small enough, then through any point $x \in T$ there passes a unique geodesic normal to the submanifold N^{p-q}. Denote the intersection point of this normal with the submanifold N^{p-q} by $y = p(x)$. Thus, we get a mapping $p : T \to N^{p-q}$, which is indeed a bundle whose fibres $p^{-1}(y)$ are q-dimensional geodesic balls. The boundary F of T is a $(p-1)$-dimensional manifold fibred by p into $(q-1)$-sphere (with the base space N^{p-q}). The above neighbourhood T of the submanifold N^{p-q} is called the *normal tubu-*

[1] In the original text, this subsection was numbered wrongly as 3. — *Editor's remark*

lar neighbourhood of the manifold N^{p-q}. Note that the structure group of p is some subgroup of the orthogonal group $O(q)$. The fibre space T obtained in this way is naturally isomorphic to the normal (and thus, transversal) bundle of the manifold N^{p-q} in M^p.

On differentiable homeomorphisms of balls. Let B^q be the closed q-ball centred at O, and A a homeomorphic class C^∞ mapping of this ball onto itself. If the inverse homeomorphism A^{-1} is differentiable then the mapping A has the same rank q at each point of B^q. Denote the group of all homeomorphisms satisfying this condition and coinciding with the identity on the boundary S^{q-1} of the ball B^q by $\boldsymbol{G}$.

For any interior point c of the ball B^q, one may construct a homeomorphism $A \in \boldsymbol{G}$ for which $A(c) = O$. Furthermore, one can show that there exists a homeomorphism with this property, which is homotopic in $\boldsymbol{G}$ to the identity mapping, i.e. in $\boldsymbol{G}$ one can construct a homeomorphism A_t depending continuously on the parameter t $(0 \leqslant t \leqslant 1)$, for which $A_0 = A$, and A_1 is the identity mapping. Here, we mean that in the group $\boldsymbol{G}$ there is a topology in which some sequence of mappings is thought to converge if and only if this sequence and all sequences obtained from it by (partial) differentiation up to order n are uniformly convergent as well as all sequences obtained from the above ones by passing to inverse maps.

The group $\boldsymbol{H}$ of the normal tubular neighbourhood homeomorphisms. Let T be a normal tubular neighbourhood of a submanifold N^{p-q} of M^p. Consider the group $\boldsymbol{H}$ of C^n-homeomorphisms of T satisfying the following conditions:

1) any homeomorphism $A \in \boldsymbol{H}$ maps every fibre $p^{-1}(y)$ onto itself;

2) any element of the group $\boldsymbol{H}$ is the identical mapping on the boundary F of T.

In the group $\boldsymbol{H}$, one introduces a topology analogous to the one introduced above in the group $\boldsymbol{G}$. (In order to define partial derivatives of the mapping $A: T \to T$, one may embed T into the Euclidean space R^k; the resulting topology of the group $\boldsymbol{H}$, as one may see, does not depend on this embedding.) With respect to the topology defined in this way, the group $\boldsymbol{H}$ is a *Bär space* [1] and even a *complete metric space*. Indeed, let (A_ι) be an arbitrary Cauchy filter in the group $\boldsymbol{H}$. Then for each point $x \in T$ the points $A_\iota(x)$ form a Cauchy filter in T. Let $J(x) \in T$ be a limit point of this filter. It is easy to see that the obtained limit mapping J belongs to the class C^n. Analogously, the Cauchy filter $A_\iota^{-1}(x)$ converges and defines a mapping J^{-1} of class C^n; this mapping is the inverse for J. Thus, any

[1] I.e. a second category space. — *Editor's remark*

Cauchy filter (A_ι) of the group $\boldsymbol{H}$ converges to the homeomorphism J, that evidently belongs to the group $\boldsymbol{H}$. Thus, the statement is proved.

Definition. *Mapping, t-regular on a submanifold.* Let f be a differentiable mapping from a manifold V^n to a manifold M^p and let y be an arbitrary point of the submanifold $N^{p-q} \subset M^p$. Denote by M_y the tangent space at y to the manifold M^p and denote by N_y the subspace of this tangent space consisting of vectors tangent to the submanifold N^{p-q}. Furthermore, let x be an arbitrary point of the full pre-image $f^{-1}(y)$ and let V_x be the tangent space to V^n at x. One says that y is a *t-regular value* of f if at any point $x \in f^{-1}(y)$ the composition map $\bar{f} : V_x \to M_y \to M_y/N_y$ has rank q and is an epimorphism[1].

4. Pre-image of a manifold under a *t*-regular mapping

We say that the mapping $f : V^n \to M^p$ is t-regular on a submanifold $N^{p-q} \subset M^p$ if any point $y \in N^{p-q}$ is a t-regular value of f. In a neighbourhood of y, choose a local coordinate system $y_1, y_2, \ldots, y_p$ where the submanifold N^{p-q} is defined (locally) by the equations $y_1 = y_2 = \ldots = y_q = 0$. Let x be any arbitrary point from the pre-image $f^{-1}(y)$ (we assume that this pre-image is not empty). If y is a t-regular value then on the manifold V^n in some neighbourhood U_x of x there exists a local coordinate system of the type $(x_1, x_2, \ldots, x_{n-q}, y_1, y_2, \ldots, y_q)$. In this case, the pre-image $f^{-1}(N^{p-q})$ is defined in the neighbourhood U_x by the equations $y_1 = y_2 = \ldots = y_q = 0$. Consequently, x has in the pre-image $f^{-1}(y)$ a neighbourhood which is homeomorphic to R^{n-q}. In other words, the pre-image $f^{-1}(N^{p-q})$ is a differentiable (C^n) *submanifold* W^{n-q}.

Let V_x be the tangent space at x to V^n, and let W_x be the tangent space at x to the manifold W^{n-q}. Because $y = f(x)$ is a t-regular value, then, by definition, the linear part $\bar{f}$ of f generates an isomorphism of the transversal space V_x/W_x to the space M_y/N_y transverse to the submanifold N^{n-q} in y. *Consequently, the transverse (or normal) bundle of the submanifold W^{n-q} in V^n is naturally isomorphic (by the map induced by f) to the transverse bundle of N^{p-q} in M^p.*

Let y be an arbitrary point of the submanifold N^{p-q}. In N^{p-q}, consider the open ball X centred in y and having radius r and the ball concentric to it X' of radius $2r$. In order for X' to be really a ball, assume r to be sufficiently small. Since any bundle over the ball is trivial then the subsets $D = p^{-1}(X)$ and $D' = p^{-1}(X')$ of the tubular neighbourhood T are homeomorphic to the products $X \times B^q$ and $X' \times B^q$, respectively. Such a homeomorphism generates a mapping $k : D'(\text{or D}) \to B^q$. Let us prove the following lemma.

[1]The pre-image $f^{-1}(y)$ of a t-regular value $y \in N^{p-q}$ might be empty; in this case one says that y is a trivial t-regular value.

Lemma I.4. *For any class C^n mapping $f : V^n \to M^p$, the set $A \in \boldsymbol{H}$ of homeomorphisms of the tubular neighbourhood T for which the composite mapping $A \circ f$ is not t-regular on X, is a thin subset of the Bär subspace $\boldsymbol{H}$.*

The fact that the mapping $G : V^n \to M^p$ is not t-regular on X means that the composite mapping $k \circ G$ defined on $g^{-1}(D)$, has a critical value at the centre O of the fibre B^q. Indeed, the linear part $\bar{k}$ of k in $y \in M^p$, is, by definition, a mapping from the tangent space M_y to the factor-space by the space N_y which is tangent to the submanifold N^{p-q}.

Let K_i be compact subsets whose union forms a manifold V^d. A homeomorphism $A \in \boldsymbol{H}$ is *i-critical* if the composite mapping $k \circ A \circ f$ defined on $f^{-1}(D)$ has in K_i at least one critical point x, for which[1] $f(x) \in X$. Let σ_i be the set of all i-critical homeomorphisms $A \in \boldsymbol{H}$. Let us show that σ_i is *closed in $\boldsymbol{H}$ and has no interior points.*

1) *σ_i is closed.* Let A be an arbitrary element of the group $\boldsymbol{H}$ not belonging to the set σ_i, i.e. an element such that for the composite mapping $k \circ A \circ f$ on $K_i \cap f^{-1}(D)$, the point O is a regular value. Let $y_1, y_2, \dots, y_q$ be some coordinates a q-ball B^q. By assumption, in the intersection $K_i \cap f^{-1} \circ A^{-1}(N^{p-q})$ the absolute values of the Jacobians $|\partial y_i/x_k|$ of order q have a positive lower bound; denote it by $3B$, $B > 0$. Consequently, in K_i there exists a closed, thus compact, neighbourhood J of the set $K_i \cap f^{-1} \circ A^{-1}(N^{p-q})$, where the absolute values of the Jacobians $|\partial y_i/x_k|$ are greater than $2B$.

Now, let us consider the set of all homeomorphisms $A' \in H$ close to A such that:

a) the intersection $K_i \cap f^{-1} \circ A'^{-1}(N^{p-q})$ is contained in J. This condition will hold if we restrict the distance from A to A' (in M^p) in a proper way. For instance, it is sufficient to assume that $\|A'(y) - A(y)\|$ is less than the distance from O to the boundary of the set $k \circ Af(J)$;

b) In J, the absolute values of the Jacobians $|\partial y_j/\partial x_k|$, corresponding to $k \circ A' \circ f$, are greater than $B > 0$. We may obtain this condition if we choose first order partial derivatives of the mapping A' close enough to the corresponding partial derivatives of the mapping A. Indeed, the Jacobians $|\partial y_j/\partial x_k|$ are continuous functions in the first order partial derivatives of A.

For all homeomorphisms A' so close to A that these two conditions hold, the Jacobians $|\partial y_j/\partial x_k|$ are non-zero on $K_i \cap f^{-1}A'^{-1}(N^{p-q})$, and, consequently, the mapping $A' \circ f$ is regular on the ball X.

[1]I.e. point $O \in B^q$ is a critical value of $k \circ A \circ f$ considered on $K_i \cap f^{-1}(D)$. — *Editor's remark.*

2) σ_i *has no interior points.* Let $A \in \sigma_i$. Then for the composite map $k \circ A \circ f$ the point O is a critical value. On the other hand, this mapping, being a mapping of class C^n, admits, by Theorem I.3, a regular value c which can be chosen arbitrarily close to the point O. Let G be a homeomorphism of the q-ball B^q taking c to O and being identical on the boundary S^{q-1} of the ball B^q, and let G_t be a homeomorphism depending continuously on the parameter $t \in J$, for which $G_0 = G$, and G_1 is the identity mapping. Furthermore, let d be a function of class C^∞ that vanishes on $\bar{X}$, equals one on the boundary of the ball X' and increases from zero to one as the geodesic distance from the centre y of X increases from r to $2r$. By using the homeomorphism $D' \approx X' \times B^q$, where $D' = p^{-1}(X')$, let us define the homeomorphism E of D' onto itself by setting:

$$E(y_1, z) = (y_1, G_{d(y)}(z)), \quad y_i \in X', \quad z \in B^q.$$

The homeomorphism E preserves the fibres $p^{-1}(y)$ and it reduces to the identity on the boundary of the set D'. Consequently, it can be extended to a homeomorphism of the whole tubular neighbourhood T onto itself; to do that, outside D', it is sufficient to take the identity mapping. The homeomorphism E defined in such a way, clearly belongs to H.

Furthermore, the mapping $E \circ A \circ f$ is t-regular on X, thus, according to the construction, the point O is a regular value of the composite mapping $k \circ E \circ A \circ f$. Thus, the mapping $A' \circ f'$, where $A' = E \circ A$, is t-regular on X and can be chosen to be arbitrarily close to $A \circ f$. Indeed, we can choose E arbitrarily close to the identity; to do that, it is sufficient to take a regular value c close enough to the point O.

(Note that in the present (second) part of the proof, the compact set K_i was not used. Thus, we have shown that the set A of homeomorphisms with $A \circ f$ not t-regular on X *has no interior points in* $\boldsymbol{H}$.)

Since the manifold V^n is a countable union of compacta K_i, then the set σ of homeomorphisms A such that $A \circ f$ is not t-regular on X, is a countable union of *sparse* sets σ_i, i.e. it is a *thin* subset of $\boldsymbol{H}$. Thus, Lemma I.4 is proved.

The submanifold N^{p-q}, that we assume to be paracompact, can be covered by countably many open balls X (by the way, note that the normal tubular neighbourhood can be defined for any paracompact submanifold if we admit tubular neighbourhoods having variable radius). This yields that the set of homeomorphisms A for which the mapping $A \circ f$ is not t-regular on N^{p-q} is a countable union of thin subsets of $\boldsymbol{H}$; thus, it is *thin* set without interior points. Thus, we have proved the following

Theorem I.5. *Let f be an arbitrary class C^n mapping from a manifold V^n to a manifold M^p, let N^{p-q} be an arbitrary paracompact submanifold of the manifold M^p and let T be a normal tubular neighbourhood of N^{p-q} in M^p. Then, there is such a homeomorphism A of T onto itself arbitrarily close to the identical such that*

1) *the pre-image $f'^{-1}(N^{p-q})$ of the submanifold N^{p-q} under $f' = A \circ f$ is a smoothly embedded $(n-q)$-dimensional submanifold W^{n-q} of V^n of class C^n;*

2) *the normal bundle of the submanifold W^{n-q} in V^n is naturally isomorphic to the space induced by the normal bundle of the submanifold N^{p-q} in M^p.*

5. The isotopy theorem

The property proved in this subsection will be needed only in Chapter IV and only for the case of *compact* V^n. Thus we prove it only for the compact case.

Let f be a class C^n mapping from a manifold V^n to a manifold M^p. Assume that t is regular on a compact submanifold N^{p-q}. Suppose in a neighbourhood of each point y from N^{p-q} there is a chosen local coordinate system $y_1, y_2, \ldots, y_p$ such that in a neighbourhood of the point y the submanifold N^{p-q} is defined by $y_1 = y_2 = \ldots = y_q = 0$. Since, by assumption, N^{p-q} is compact, it can be covered by finitely many coordinate neighbourhoods of this type. By means of a Riemannian metric, introduced arbitrarily on V^n, define the tubular neighbourhood Q of $W^{n-q} = f^{-1}(N^{p-q})$. We choose the radius ϵ of this neighbourhood so small that the following condition holds.

Let x be any point from W^{n-q}, and let B_x be the geodesic q-ball centred in x, and normal to W^{n-q}. We require that the coordinates $y_1, y_2, \ldots, y_q$, transported to V^n by means of $y = f(x)$, represent a coordinate system in the q-ball B_x. Since f is, by assumption, t-regular, and the submanifold W^{n-q} is *compact*, we can easily meet this condition.

Let A be any element of the group $\boldsymbol{H}$ which is close to the identity. Consider the pre-image $g^{-1}(N^{p-q})$, where $g = A \circ f$. It is clear that if A is close enough to the identity then the mapping g is also t-regular on N^{p-q}. Indeed, if the distance $\|A(y) - y\|$ in M^q is strictly less than the distance from the manifold N^{p-q} to the boundary of $f(Q)$ then the pre-image $g^{-1}(N^{p-q})$ is contained in Q. Furthermore, assume that all partial derivatives of A are close to the partial derivatives of the identity. Then the mapping g, as well as f, has rank q on the q-ball B^q; thus, it is t-regular. Now, let us show that if A is close enough to the identity then the submanifolds $W^{n-q} = f^{-1}(N^{p-q})$ and $W' = g^{-1}(N^{p-q})$ are isotopic in V^n. We shall prove this fact according to Seifert's scheme [21].

Let $y = f(x)$ be a point of N^{p-q}. With each point $z \in B_x$ we associate the point $L(z)$ of R^q with coordinates $y_i(g(z))$, where y_i are local coordinates in some neighbourhood of y in the normal space to the submanifold N^{p-q}. For the mapping L defined in such a way, the preimage $L^{-1}(O)$ is the intersection of the ball B_x with the submanifold $W' = g^{-1}(N^{p-q})$.

Let, as above, ϵ be the radius of the ball B_x. If the mapping A is sufficiently close to the identity, then for any coordinate system (y_j) in $y = f(x)$, we have: $\|L(z) - z\| < \epsilon$ uniformly with respect to x. This yields that the pre-image $L(S^{q-1})$ of the boundary S^{q-1} of the ball B_x is homotopic to the sphere S^{q-1} in the space $R^q \backslash O$ compactified by O. Thus the degree of L with respect to the point O is equal to the degree of the identity, i.e. it is equal to $+1$. Furthermore, L has maximal rank in any point of the ball B_x; thus in a neighbourhood of any point z of B_x the mapping L is a local homeomorphism; thus, the pre-image $L^{-1}(O)$ consists only of isolated points. Since at any point of B_x the degree of L is equal to $+1$ (this is equal to the sign of the corresponding Jacobian) then the pre-image $L^{-1}(O)$ consists of a unique point x'. Thus, the submanifold W' intersects B_x only at the point x'. Thus, the correspondence $x \to x'$ is a homeomorphic mapping from the submanifold W^{n-q} to the submanifold W'^{n-q}. Let us connect x' in the ball B_x to the point x by a geodesic arc $s(x, x')$. The motion along this arc defines an isotopy that deforms the submanifold W^{n-q} to the submanifold W'^{n-q}. This proves the following

Theorem I.6. *Let f be an arbitrary class C^n mapping from a compact manifold V^n to a manifold M^p and assume this mapping is t-regular on a certain compact submanifold N^{p-q}. Then for any homeomorphism $A \in \boldsymbol{H}$ of the tubular neighbourhood of N^{p-q}, which is close to the identity, the composite mapping $g = A \circ f$ is t-regular on N^{p-q} and the submanifolds $W^{n-q} = f^{-1}(N^{p-q})$, $W'^{n-q} = g^{-1}(N^{p-q})$ are isotopic in the manifold V^n.*

Chapter II

Submanifolds and homology classes of a manifold

1. Formulation of the problem

Let V^n be an orientable manifold. In order to *orient* V^n, one should indicate in the group $H_n(V^n; Z)$ of integral homology some generator. This generator is called the *fundamental class* of the oriented manifold V^n. For an oriented manifold V^n, the homology group $H_{n-k}(V^n; Z)$ is naturally isomorphic to the cohomology group $H^k(V^n; Z)$ (Poincaré duality). We call those classes paired by Poincaré duality *corresponding* or *Poincaré dual* classes. If we take the group Z_2 of residue classes modulo two to be the group of coefficients then the fundamental class of $H_n(V^n; Z)$ is well defined even when V^n is not orientable. Furthermore, there is a well defined Poincaré-Veblen isomorphism between the groups $H_{n-k}(V^n; Z_2)$ and $H^k(V^n; Z_2)$. For the sake of simplicity, we assume the manifold V^n to be *compact*; we shall just touch on the case of paracompact manifolds which are not compact.

Let W^p be a p-dimensional submanifold of a manifold V^n and let i_* be the homomorphism from $H_p(W^p)$ to $H_p(V^n)$, defined by the embedding map $i : W^p \to V^n$. Let z be the image of the fundamental class of W^p under i_*. We say that the class z is *realised* by means of a submanifold W^p. In the present work we address the following question: is the given homology class z of the manifold V^n realisable by means of a submanifold? The answers to this question are, as we shall see, quite different for the cases of Z and Z_2 as the coefficient ring. In the first case we assume, not necessarily saying this exactly that the manifold V^n in question is *orientable and endowed with an arbitrary but fixed orientation.*

2. The space adjoint to a subgroup of the orthogonal group

Let G be a closed subgroup of the orthogonal group $O(k)$ of order k. It is well known that that any fibre space with fiber sphere S^{k-1} and with structure group G can be obtained from some universal fibre space E_G. The base B_G of this universal bundle is a compact manifold (we restrict ourselves to fibre spaces with bases of finite dimension $\leqslant N$). Denote by A_G the cylinder of the fibre map $E_G \to B_G$. This cylinder is, on the one hand, a space fibred by k-balls with base B_G and, on the other hand, a *manifold with boundary* E_G. The corresponding open manifold $A'_G = A_G \backslash E_G$ is a fibre space with k-ball as a fibre, associated with the fibre space E_G (see [27]).

Definition. The space $M(G)$ obtained from the manifold A_G by contracting its boundary E_G to one point A is called the space, *adjoint* to the subgroup G of $O(k)$. The space $M(G)$ can also be considered as a *one-point compactification (in Alexandrov's sense)* of the fibre space A'_G with open ball as a fibre.

Cohomology of $M(G)$. The cohomology group $H^r(M(G))$ for any $r > 0$ can be identified with the cohomology group H^r_K *with compact support*, and also with the relative cohomology group $H^r(A_G, E_G)$. On the other hand, as follows from fibre spaces theory (when the fibre is an open ball), there exists an isomorphism[1] (see [26])

$$\varphi_G^* : H^{r-k}(B_G) \to H_K^r(A'_G) \approx H^r(M(G)).$$

In this case, we should, in general, restrict ourselves to Z_2 as the coefficient group. However, if the fibre space E_G *is orientable* (the group G is connected), then we can deal with the group Z. Thus, in dimensions $r > 0$, the cohomology ring $H^*(M(G))$ is obtained from the cohomology ring $H^*(B_G)$ of the classifying space B_G by increasing all dimensions by k. In particular, in dimensions greater than zero, the first non-trivial cohomology group is the group $H^k(M(G))$. This group is cyclic. Its generator $U \in H^k(M(G))$ is defined as

$$U = \varphi_G^*(\omega_G),$$

where $\omega_G \in H^0(B_G)$ — *the unit class.* The class U is called the *fundamental class* of $M(G)$. Note that U is an integral class if E_G is orientable (G is connected), and a modulo 2 class if E_G is non-orientable (G is not connected).

3. The main theorem

Definition. We say that a cohomology class $u \in H^k(A)$ of a topological space A is *realisable with respect to* $G \subset O(k)$ or *admits a G-realisation* if there exists a mapping $f\colon A \to M(G)$ such that the homomorphism f^* generated by this mapping takes the fundamental class U of $M(G)$ to the class u.

Then the following Theorem holds:

Theorem II.1. *In order for the class $z \in H_{n-k}(V^n)$, $k > 0$, to be realisable by means of a submanifold W^{n-k} with normal space having*

[1] Consider an arbitrary cellular decomposition of B_G. It is evident that the full pre-images $p^{-1}(\sigma)$ of cells σ of this decomposition under the map $p : A_G \to B_G$ form a cellular decomposition $A'_G = A_G \backslash E_G$. The isomorphism considered by the author $\varphi_G^* : H^{r-k}(B_G) \to H^r(A_G, E_G)$ for modulo two homology groups is generated by the correspondence $\sigma \to p^{-1}(\sigma)$. (It is easy to see that this correspondence is one-to-one, it preserves the incidence relation and increases the dimension by k.) If the group G is connected then the cells σ and $p^{-1}(\sigma)$ can be considered coordinatewise, i.e. in such a way that the correspondence $\sigma \to p^{-1}(\sigma)$ preserves the incidence coefficients. Consequently, in this case the correspondence $\sigma \to p^{-1}(\sigma)$ generates an isomorphism of integral homology groups.

The isomorphism φ_G^* can also be constructed by using spectral sequences theory. Indeed, by Leray theorem, the second term $(E_2^{p,q})$ of the spectral sequence for homology with finite support of the bundle $A'_G \to B_G$ with the restrictions on coefficients described above is isomorphic to $H^p(B_G) \otimes H_K^q(E^k)$, consequently, it is zero if $q \neq k$. Thus $d_r = 0$ for all $r \geqslant 2$, i.e. ${}^rE_2 \approx {}^rE_\infty \approx \approx H_K^r(A'_G)$. On the other hand, ${}^rE_2 = E_2^{r-k,k} \approx H^{r-k,k}(B_G) \otimes H_K^k(E^k) \approx H^{p-q}(B_G)$. Thus, $H^{p-q}(B_G) \approx H_K^r(A'_G)$. — *Editor's remark* *structure group G, it is necessary and sufficient that the cohomology class $u \in H^k(V^n)$ corresponding to z be realisable with respect to G.*

1) *Necessity.* Assume that there exists a submanifold W^{n-k} with fundamental class representing z in V^n. Let N be the normal tubular neighbourhood of W^{n-k} and let T be its boundary. The normal geodesic bundle $p : N \to W^{n-k}$ admits, by the assumption of the theorem, G as a structure group, thus, it is induced by a certain mapping $g : W^{n-k} \to B_G$ from its base to the base of the universal space A_G. With this mapping, one associates a mapping $F : N \to A_G$ (taking fibre to fibre) for which the diagram

$$\begin{array}{ccc} N & \xrightarrow{F} & A_G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_G} \\ W^{n-k} & \xrightarrow{g} & B_G \end{array}$$

is commutative.

The mapping F takes the boundary T of N to the boundary E_G of the manifold A_G. Let φ^* and φ^*_G be the isomorphisms mentioned above corresponding to k-ball fibre spaces N and A_G, respectively. Obviously, the diagram

$$\begin{array}{ccc} H^k(N,T) & \xrightarrow{F} & H^k(A_G, E_G) \\ {\scriptstyle \varphi^*}\uparrow & & \uparrow{\scriptstyle \varphi^*_G} \\ H^0(W^{n-k}) & \xrightarrow{g} & H^0(B_G) \end{array} \tag{1}$$

is commutative as well.

On the other hand, let $j_* : H^k(N,T) \to H^k(V^n)$ be the natural inclusion homomorphism. It is known that in the open manifold $N' = N \backslash T$ the class $\varphi^*(\omega)$ corresponds, by the Poincaré duality, to the fundamental homology class of the base W^{n-k} (see [27], Theorem I.8)[1]. Consequently, the class $j_*\varphi^*(\omega) \in H^k(V^n)$ coincides with the class u, the latter corresponding to z.

Denote by $h : A^G \to M(G)$ the natural identification mapping taking the boundary E_G of A_G to one point a. The composite mapping $h \circ g$ takes the boundary T of N to a. Consequently, the mapping $h \circ g$ can

[1]It is sufficient to check that the scalar product of the classes $\varphi^*(\omega)$ and W^{n-k} is non-zero; this follows from the arguments given in page 143, where the isomorphism φ^* is constructed. — *Editor's remark*

be extended over the whole manifold V^n; to do this, it is sufficient to map the complement $V^n \backslash N$ to the point a. Thus, we define the mapping f from V^n to $M(G)$, for which, according to the commutative diagram (1),

$$f^*(U) = f\varphi_G^*(\omega_G) = j_*\varphi^*(\omega) = u.$$

2) *Sufficiency.* Suppose there exists a mapping f from V^n to $M(G)$ such that $f^*(U) = u$. If we delete the point a from $M(G)$, we get a differentiable manifold. Consequently, the mapping f on the complement $V^n \backslash f^{-1}(a)$ can be regularised, i.e. we can construct a differentiable mapping f_1 close to f which is of class C^n on $V^n \backslash f^{-1}(a)$. Let us apply Theorem I.5 to f_1. As a result we get a mapping F arbitrarily close to f such that the pre-image $F^{-1}(B_G)$ is a submanifold W^{n-k} of the manifold V^n. Since the normal bundle of the submanifold W^{n-k} is induced by the space A_G, then the structure group of this space is G. As we have seen 1), the class $u = f^*(U) = F^*(U)$ coincides with the class $j_*\varphi^*(\omega)$, where φ^* is the isomorphism corresponding to some normal tubular neighbourhood of the submanifold W^{n-k} in V^n, and ω is the unit class of the submanifold W^{n-k}. This means precisely that the class u corresponds, by Poincaré duality, to the class of the fundamental cycle of W^{n-k}.

Generalisation of Theorem II.1 for paracompact manifolds which are not compact. Recall that for paracompact but not compact manifolds there are as many duality theorems as families (Φ) of closed subsets used for the definition of homology and cohomology groups (see [28], Theorem 0.3). Thus, we are interested in the following question: given a class $z \in H_{n-k}^{\Phi}(V^n)$ with support in Φ, can it be realised by a certain manifold W^{n-k}? To study this question, the proof given above should be slightly modified.

First, the normal tubular neighbourhood can be defined even for paracompact submanifolds if we allow its radius to vary. Then, a mapping $f : V \to M$ Φ-is *proper* if the pre-image $f^{-1}(K)$ of any compact subset $K \subset M$ belongs to the family (Φ) (if (Φ) is the family $\boldsymbol{K}$ of all compacta of the manifold V, then we get the classical definition of proper mappings). Then we have

Theorem II.1′. *In order for the class $z \in H_{n-k}^{\Phi}(V^n)$ to be realised by a submanifold with normal bundle having structure group G, it is necessary and sufficient that there exists a (Φ)-proper mapping $f : V^n \to M(G)$ to $M(G)\backslash a$ such that the class $f^*(U) \in H_{\Phi}^k(V^n)$ is Poincaré-dual to the class z.*

4. The case when G reduces to the unit element $e \in O(k)$

In this case the classifying space B_G consists of one point and the space A_G is the closed k-ball, and $M(e)$ is the sphere S^k. An integral cohomology class u of the space A is *spherical* if there exists a mapping $f : A \to S^k$ such that $u = f^*(s^k)$, where s^k is the fundamental class of the group $H^k(S^k, Z)$. From Theorem II.1′ we get

Theorem II.2. *In order for the homology class $z \in H_{n-k}(V^n, Z)$ of an orientable manifold V^n to be realisable by means of a submanifold with trivial normal bundle it is necessary and sufficient that the cohomology class $u \in H^k(V^n, Z)$, which is Poincaré-dual to z, be spherical.*

In algebraic topology, no sphericality criteria for a cohomology class are found. The only general result was obtained by Serre [22]:

Theorem II.3. *If k is odd and $n < 2k$ then for any k-dimensional homology class $x \in H^k(A, Z)$ of an n-dimensional polyhedron A there exists a non-zero integer N depending only on k and n such that the class Nx is sphericalal.*

This yields

Theorem II.4. *Let k be odd or $n < 2k$. Then there exists a non-zero integer N depending only on k and n such that for any integral homology class $z \in H_{n-k}(V^n, Z)$ of any orientable manifold V^n the class Nz is realisable by a submanifold with trivial normal bundle.*

5. The structure of spaces $M(O(k))$ and $M(SO(k))$

From Theorem II.1, one gets immediately the following Theorems, which underline the role of the spaces $M(O(k))$ and $M(SO(k))$.

Theorem II.5. *In order for a homology class $z \in H_{n-k}(V^n, Z)$ of an orientable manifold V^n to be realisable by a certain submanifold, it is necessary and sufficient that the Poincaré-dual class u be realisable with respect to the rotation group.*

Theorem II.5′. *In order for a modulo 2 homology class $z \in H_{n-k}(V^n, Z_2)$ of V^n to be realisable by means of a certain submanifold, it is necessary and sufficient that the Poincaré dual cohomology class u be realisable with respect to the orthogonal group.*

Denote by G_k the Grassmann manifold of k-dimensional planes in the Euclidean space R^m. The dimension m of the Euclidean space is assumed to be rather large. It is well known that G_k is the classifying space $B_{O(k)}$

of the orthogonal group $O(k)$. Let $\widehat{G}_k$ be the Grassmann manifold of k-dimensional *oriented* planes in the space R^m. This is the classifying space $B_{SO(k)}$ of $SO(k)$. The manifold $\widehat{G}_k$ is a 2-fold covering over G_k.

Associating with every k-dimensional plane from $\widehat{G}_k$ its intersection S^{k-1} with the unit sphere of the space R^m, we get the universal fibre space $E_{SO(k)}$. Thus, the space $E_{SO(k)}$ can be considered as the space of pairs each consisting of an oriented k-dimensional plane and unit vectors lying in the corresponding planes. Associate with each such pair the $(k-1)$-dimensional plane lying in the k-dimensional plane of the pair and orthogonal to the vector of this pair. This defines a fibration of the space $E_{SO(k)}$ to spheres S^{m-k} with base $\widehat{G}_{k-1}$. This yields that in dimensions less than the classifying dimension $m-k$, the space $E_{SO(k)}$ has the same homotopy type as the Grassmann manifold $\widehat{G}_{k-1}$. Moreover, the inclusion $E_{SO(k)} \to A_{SO(k)}$, from the homotopy theory point of view, coincides with the natural mapping $\widehat{G}_{k-1} \to \widehat{G}_k$, generated by the inclusion of the subgroup $SO(k-1)$ to $SO(k)$[1].

Cohomologies of $M(SO(k))$. Thus, in dimensions $r > 0$, the cohomology algebra $H^*(M(SO(k)))$ can be identified with the algebra of relative cohomology $H^*(\widehat{G}_k, \widehat{G}_{k-1})$. The later algebra can be defined from the exact sequence

$$\to H^r(\widehat{G}_k) \xrightarrow{i^*} H^r(\widehat{G}_{k-1}) \xrightarrow{\delta^*} H^{r+1}(\widehat{G}_k, \widehat{G}_{k-1}) \to H^{r+1}(\widehat{G}_k) \to, \qquad (2)$$

since the homomorphism i^* is well studied.

Cohomologies modulo 2. It is known ([3]) that the cohomology algebra $H^*(\widehat{G}_k; Z_2)$ is the polynomial algebra in $(k-1)$ generators $W_2, W_3, \ldots, W_k$. The generator W_i has degree i and represents the *i-th Stiefel-Whitney class.* The homomorphism i^* maps the classes W_j to themselves. Thus, the algebra $H^*(\widehat{G}_k, \widehat{G}_{k-1})$ of relative homologies is isomorphic to the *ideal of the polynomial algebra $H^*(\widehat{G}_k, Z_2)$ generated by the class W_k.* This result can be obtained straightforwardly, if we consider (as in 2) the isomorphism φ^*.

Cohomologies modulo p, for odd prime p We should note the difference between the following two cases:

1) k is *odd*, $k = 2m+1$. In this case, $H^*(\widehat{G}_k, Z_p)$ is a polynomial algebra in generators $P^4, P^8, \ldots, P^{4m}$ of dimensions divisible by 4 (these generators are the *Pontrjagin classes* reduced modulo p).

[1] It is sufficient to note that the natural inclusion $\widehat{G}_k \to A_{SO(k)}$ is a homotopy equivalence. — *Editor's remark*

2) k is even, $k = 2m'$. In this case the algebra $H^*(\widehat{G}_k, Z_p)$ is the polynomial algebra generated by the Pontrjagin classes reduced modulo p, $P^4, P^8, \dots, P^{4m'-4}$ and the fundamental class $X^{2m'}$.

The homomorphism i^* of cohomology algebras generated by the natural mapping $i : \widehat{G}_k \to \widehat{G}_{k+1}$, takes the Pontrjagin classes P^{4i} to the Pontrjagin classes, except for $P^{4m} \in H^{4m}(\widehat{G}_{k+1})$ of maximal dimension (this class exists for even k). This class is taken by i^* to the square $(X^{2m})^2$ of the fundamental class $X^{2m} \in H^k(\widehat{G}_k)$ (see [5]). This yields that for *even* k the algebra $H^*(\widehat{G}_k, \widehat{G}_{k-1})$ is isomorphic to the ideal of $H^*(\widehat{G}_k)$ generated by the class X^{2m}; for k odd, the algebra $H^*(M(SO(k)))$ is isomorphic to the exterior algebra with generator $\delta(X^{2m'})$.

Cohomologies of $M(O(k))$. Let us use the exact sequence (2), where we replace $\widehat{G}_k$ with G_k.

Cohomologies modulo 2. The cohomology algebra $H^*(G_k; Z_2)$ is the polynomial algebra in k variables $W_1, W_2, W_3, \dots, W_k$. As before, we see that the algebra $H^*(G_k, G_{k-1}; Z_2)$ is isomorphic to the ideal $\boldsymbol{J}$ of the algebra $H^*(G_k; Z_2)$ generated by the class W_k.

Cohomologies modulo p with p odd prime Denote by g the automorphism group of the two-fold covering $\widehat{G}_k \to G_k$. It is easy to see that the Pontrjagin classes P^{4i} are invariant under the action of g. Conversely (for even k), the group g takes the fundamental class X_k to the opposite class $(-X_k)$. According to the classical theorems on cohomology of the covering space (Eckmann [9]) this yields that for odd k, the algebra $H^*(G_k; Z_p)$ is isomorphic to the algebra $H^*(\widehat{G}_k, Z_p)$, and for even $k = 2m$ the algebra $H^*(G_k; Z_p)$ is the polynomial algebra in the Pontrjagin classes $P^4, P^8, \dots, P^{4m'-4}$ and the square $(X_k)^2$ of the fundamental class X_k (indeed, though the class X_k is not invariant under g, its square $(X_k)^2$ is invariant).

Using an exact sequence analogous to the above one (2), consider, as above, the two cases:

1) $k = 2m + 1$ *is odd.* Since the algebras $H^*(G_k; Z_p)$ and $H^*(G_{2m}; Z_p)$ are isomorphic and the isomorphism is generated by i^* (recall that $i^*(P^{4m}) = (X_{2m})^2$), then

$$H^r(G_k, G_{k-1}; Z_p) = 0 \quad \text{for all} \quad r > 0.$$

2) $k = 2m$ *even.* In this case, the algebra $H^*(G_k, G_{k-1})$ is isomorphic to the ideal of the polynomial algebra $H^*(G_k; Z_p)$ generated by the class $(X_{2m})^2$.

The fundamental group. According to the general theory, the fundamental group of $M(G)$ is a quotient group of the fundamental group of A_G (or B_G) by the image of the group $\pi_1(E_G)$ under $E_G \to A_G$. Consequently:

a) because $\pi_1(\widehat{G}_k) = 0$, we have $\pi_1(M(SO(k))) = 0$;

b) since the homomorphism i^* maps the group $\pi_1(G_{k-1}) \simeq Z_2$ onto $\pi_1(G_k) \simeq Z_2$, $\pi_1(M(O(k))) = 0$ for $k \geqslant 1$.

Thus, the spaces $M(O(k))$ and $M(SO(k))$ are *simply connected*. This yields that these spaces are *aspherical* up to dimension $k-1$ inclusive, since in positive dimensions, the first non-trivial homology group of these spaces is H^k. The first non-trivial homotopy groups are $\pi_k(M(O(k))) = Z_2$ and $\pi_k(M(SO(k))) = Z$.

Let us now prove a theorem from general topology which allows us to define homotopic properties of spaces by using their cohomological properties. Basically, this theorem is due to J.H.C.Whitehead [29].

Theorem II.6. *Let X and Y be simply connected cellular decompositions and let f be a mapping from the decomposition X to the decomposition Y such that for any coefficient group Z_p, the homomorphism $f^* : H^k(Y) \to H^k(X)$ generated by f is an isomorphism when $r < k$, and is a monomorphism when $r = k$. Then there exists a mapping g taking the k-frame of the decomposition Y to the decomposition X such that the mappings $f \circ g$ and $g \circ f$ (considered on the corresponding $(k-1)$-skeletons) are homotopic to the identity.*

In particular, this yields that the decompositions X and Y have *the same k-type*: their homotopy groups are isomorphic in dimensions $\leqslant k-1$.

Replace Y by *the cylinder Y' of f*. This is legal because Y is a deformation retract of the cylinder Y'; thus, it has the same homotopy type. Consider the exact sequence

$$H^r(Y') \xrightarrow{\ f\ } H^r(X) \to H^{r+1}(Y', X) \to H^{r+1}(X) \to H^{r+1}(Y').$$

The conditions imposed on f are equivalent to $H^r(Y', X; Z_p) = 0$ for all prime p and $r \leqslant k$. Thus, by duality argument, we get that $H_r(Y', X; Z_p) = 0$ for $r \leqslant k$. Consequently, by the universal coefficient formula, $H_r(Y', X; Z) = 0$ for $r \leqslant k$. Since X and Y are simply connected, one may apply the relative Hurewicz theorem [15]. By this theorem, $\pi_r(Y', X) = 0$ for $r \leqslant k$. Consequently, on the k-skeleton of Y' one may define the mapping $g : Y' \to X$ inverse to f, such that $g \circ f$ and $f \circ g$ are homotopic to the identity mappings on the corresponding $(k-1)$-skeletons of X.

The above description of the cohomology groups of $M(O(k))$ and $M(SO(k))$ yields that the cohomology groups $H^{k+i}(M(O(k)))$ and $H^{k+i}(M(SO(k)))$ do not depend on k if $i < k$. It turns out that the analogous property takes place for homotopy groups as well.

Theorem II.7. *If $i < k$ then the homotopy groups $\pi_{k+i}(M(O(k)))$ and $\pi_{k+i}(M(SO(k)))$ do not depend on k.*

This theorem is quite analogous to the Freudenthal theorem on homotopy groups of spheres.

Let $A'_{O(k-1)}$ be the fibre space described above with base G_{k-1} and fibres $(k-1)$-dimensional open balls. Denote by $A' \otimes I$ the join (after Whitney)[1] of the fibre space $A'_{O(k-1)}$ and the open interval I considered as the fibre space with one point as the base. By definition, $A' \otimes I$ is a k-ball fibre space with base G_{k-1}. Clearly, the mapping $i : G_{k-1} \to G_k$ that induces this space coincides with the mapping i, considered above, corresponding to the inclusion $O(k-1) \subset O(k)$. Let f be the mapping from $A' \otimes I$ to $A'_{O(k)}$ corresponding to i. Consider the compact space X obtained from the space $A' \otimes I$ by one-point compactification by x, and extend f to $F : X \to M(O(k))$. Obviously, the homomorphism F^* generated by F is an isomorphism from $H^{k+i}(M(O(k)))$ to the group $H^{k+i}(X)$ for $i < k-1$. For $i = k-1$ the mapping F^* is a monomorphism. Besides that, the spaces X and $M(O(k))$ are simply connected. Thus, one can apply Theorem II.6 which says that the homotopy groups π_{k+i} of X and $M(O(k))$ are isomorphic for $i < k-1$.

Let $T(k-1)$ be the *suspension*[2] over $M(O(k-1))$ with poles p and p', a is the "infinite" point of the space $M(O(k))$ and g is the mapping contracting the whole segment $[pap']$ to x. The resulting space is nothing but X. Since g satisfies the conditions of Theorem II.6 (one may even show

[1]Let $(A^i, B^i, F^i, G^i, U^i_{\alpha_1}, g^i_{\alpha_1\beta_1})$, $i = 1, 2, \ldots$ be two fibre spaces. Here B^i are bases, F^i are fibres, G^i are the structure groups, and $U^i_{\alpha_1}$ are the coordinate neighbourhoods, $g^i_{\alpha_1\beta_1} : U^i_{\alpha_1} \cap U^i_{\beta_1} \to G^i$ are the corresponding coordinate transformations. Then the group $G^1 \times G^2$ is naturally an effective group of transformations of the space $F^1 \times F^2$, and the family of sets $U^1_{\alpha_1} \times U^2_{\alpha_1}$ makes a covering of $B^1 \times B^2$, and the mapping

$$g^1_{\alpha_1\beta_1} \times g^2_{\alpha_1\beta_1} : (U^1_{\alpha_1} \times U^2_{\alpha_1}) \cap (U^1_{\beta_1} \times U^2_{\beta_1}) \to G^1 \times G^2$$

forms a system of coordinate transformations in the sense of [33], 3.1. The corresponding fibre space

$$(A, B^1 \times B^2, F^1 \times F^2, G^1 \times G^2, U^1_{\alpha_1} \times U^2_{\alpha_1}, g^1_{\alpha_1\beta_1} \times g^2_{\alpha_1\beta_1})$$

is called the *join* of the given fibre spaces. — *Editor's remark.*

[2]Let X be an arbitrary space and let p be a point not belonging to X. The cylinder of the trivial map $X \to p$ is called the cone over X with vertex (pole) p. The set-theoretic union of two cones over X with vertices p and p' (it is assumed that these cones meet only at X) is called the suspension over X with poles p and p'. — *Editor's remark*

that g is a homotopy equivalence), then the homotopy groups of the spaces X and $T(k-1)$ are isomorphic.

There is a well-known theorem: let K be a polyhedron which is aspherical up to dimension $n-1$ inclusive, and let $T(K)$ be the suspension over K. Then for $j < 2n$, the Freudenthal homomorphism $E : \pi_j(K) \to \pi_{j+1}(T(K))$ is an isomorphism (see Blakers–Massey [2])[1].

Considering the sequence of isomorphisms

$$\pi_{k-1+i}(M(O(k-1))) \xrightarrow{E} \pi_{k+i}(T(k-1)) \\ \longrightarrow \pi_{k+i}(X) \simeq \pi_{k+i}(M(O(k))), \qquad i < k-1,$$

we get the desired result. For $M(SO(k))$ the proof is analogous.

6. The homotopy type of $M(O(k))$

To compute the homotopy groups of $M(O(k))$, it is necessary to use some general properties of Eilenberg-MacLane spaces on the one hand, and some properties of Grassmann spces on the other hand.

Eilenberg-MacLane space. Let π be an abelian group. An *Eilenberg-MacLane space* $K(\pi, n)$ is a connected space with all homotopy groups in positive dimensions trivial except the group $\pi_n(K(\pi, n)) \approx \pi$. All such spaces have the same homotopy type, and at least one of them is a simplicial decomposition. If π is of *finite* type then there exists a space $K(\pi, n)$ being a simplicial decomposition whose finite-dimensional frames are *finite* decompositions. Since this fact is not assumed to be well known, we give a short proof here. The proof is by induction on q (q is the dimension of the frame). The induction base is guaranteed by the fact that the n-skeleton of the complex $K(\pi, n)$ consists of finitely many spheres S^n. Now, let the frame K^q of some dimension $q \geqslant n$ be finite. According to Serre's theorems [24], the homotopy group $\pi_q(K^q)$ has finite type. Consequently, this group can be made zero by attaching to K^q *finitely many* $(q+1)$-balls, with boundary spheres mapped to K^q. Since these mappings can be assumed simplicial, we get, as a result, a *finite decomposition* K^{q+1} for which all groups $\pi_i(K^{q+1})$

---

[1]Let K_1 and K_2 be two cones over K, whose union is the suspension $T(K)$ (see the previous remark). Since the cones K_1 and K_2 are contractible to a point, it is evident that the boundary homomorphism $\partial : \pi_{j+1}(K_2, K) \to \pi_j(K)$ and the inclusion homomorphism $r : \pi_{j+1}(T(K)) \to \pi_{j+1}(T(K), K_1)$ are indeed isomorphisms. The Freudenthal homomorphism $E : \pi_j(K) \to \pi_{j+1}(T(K))$ considered here can be defined as a composition $r \circ e \circ \partial^{-1}$, where $e : \pi_{j+1}(K_2, K) \to \pi_{j+1}(T(K), K_1)$ is the natural inclusion homomorphism. Thus, the Blakers-Massey theorem used here is equivalent to the statement that for $i < 2n-1$ the homomorphism $e : \pi_i(K_2, K) \to \pi_i(T(K), K_1)$ is an isomorphism. — *Editor's remark*

for $n < i \leqslant q$, are trivial. This decomposition is the desired $(q+1)$-dimensional frame.

The homology groups $K(\pi, n)$ with G-coefficients are $H^r(\pi, n; G)$. Recall that the group $H^n(G, n; G)$ contains the so-called "*fundamental class*[1]" ι. For any homology class $u \in H^n(A; G)$ of some polyhedron A there exists a mapping $f : A \to K(G, n)$ that[2] $u = f^*(\iota)$.

The homology groups of the spaces $K(Z, n)$ and $K(Z_p, n)$ with coefficients in Z_p are computed by J.-P. Serre and H. Cartan. We recall some remarks on their work.

Cohomology of $K(Z_2, n)$ (cf. Serre [23]). The cohomology algebra $H^*(Z_2, k; Z_2)$ is generated by *iterated Steenrod squares* of the fundamental class $\iota \in H^k(Z_2, k; Z_2)$, and their products. For $h < k$ (the *stationary part* of the algebra $H^*(Z_2, k; Z_2)$) the group $H^{k+h}(Z_2, k; Z_2)$ is generated by iterated squares $Sq^{i_1}\ Sq^{i_2} \dots\ Sq^{i_r}(\iota)$, where $\sum\limits_m i_m = h$. As a basis of this group, one may choose the set of iterated squares

$$Sq^{i_1} Sq^{i_2} \dots Sq^{i_r}(\iota), \text{ where } i_1 \geqslant 2i_2,\ i_2 \geqslant 2i_3,\ i_3 \geqslant 2i_4, \dots, i_{r-1} \geqslant 2i_r.$$

The sequence $I = \{i_1, i_2, \dots, i_r\}$ satisfying these inequalities is an *admissible sequence*, according to [23]. The corresponding iterated square $Sq^{i_1}\ Sq^{i_2} \dots Sq^{i_r}$ is denoted by Sq^I. The rank $c(h)$ of the group $H^{k+h}(Z_2, k; Z_2)$ is equal to *the number of decompositions of h into summands of type* $2^m - 1$ (here the order does not matter).

Analogous results hold for $H(Z, k; Z_2)$.

Homology of $K(Z, k)$ over Z_p, $p > 2$. We shall use only the following result by H. Cartan: the algebra $H^*(Z, k; Z_p)$ is generated by *iterated Steendor powers* of the fundamental class ι.

The Grassmann manifold G_k. As mentioned above, the algebra $H^*(G_k; Z_2)$ is the polynomial algebra generated by Stiefel–Whitney classes W_i, $1 \leqslant i \leqslant k$. It is often useful to consider the classes W_i as *elementary symmetric functions* in variables $t_1, t_2, \dots, t_r$ of the first degree.

---

[1]The isomorphism $H^n(G, n; G) \approx \mathrm{Hom}(G, G)$ takes this class to the identity map $G \to G$. — *Editor's remark*

[2]Let z be an arbitrary cocycle of the class u in some cellular decomposition of the polyhedron A. Consider the mapping of the n-skeleton of this decomposition to the space $K(G, n)$, taking the $(n-1)$-skeleton to some point of $K(G, n)$ and defining on any n-cell σ the element $z(\sigma)$ of $G = \pi_n(K(G, n))$. Since the cochain z is a cocycle then this mapping is null-homologous on the boundary of any $n+1$-cell of the cellular decomposition; thus, it can be extended over the $(n+1)$-skeleton, and, consequently, to the whole polyhedron A (since for $i > n$ the groups $\pi_i(K(G, n))$ are trivial by the assumption). The constructed mapping $f : A \to K(G, n)$ evidently satisfies the condition $u = f^*(\iota)$.

From this construction it follows that up to homotopy the mapping f is well defined — *Editor's remark*

The variables, t_r intoduced formally by W.T. Wu, found their topological interpretation in the Borel–Serre theory [4, 5]. By using the variables t_r, one can easily prove the following Wu formulae [34] for Steenrod squares of classes W_i:

$$Sq^r W_i = \sum_t \binom{r-i+t-1}{t} W_{r-t} W_{i+t}. \tag{3}$$

The following lemma, whose proof due to Serre, shows that the Grassmann manifold G_k can, to some extent, replace the Eilenberg–MacLane space $K(Z_2, k)$.

Lemma II.8. *Any linear combination of iterated Steenrod squares Sq^I of total degree $h \leqslant k$, which vanishes on the class $W_k \in H^k(G_k; Z_2)$, is identically equal to zero.*

First, note that any class of the type $Sq^I(W_k)$, where the sequence I is not necessarily admissible, looks like[1] $W_k \cdot Q_i$, where $Q_i \in H^h(G_k)$ is a polynomial of total weight h with respect to W_i. Consequently, the class $Sq^I(W_k)$ belongs to the ideal $\boldsymbol{J}$ of the algebra $H^*(G_k)$ generated by W_k.

Let us introduce on the set of monomials W_i the lexicographic ordering (R) by setting $W_m \prec W_n$ if $m < n$. For instance,

$$W_4 \prec W_4(W_1)^2 \prec W_4 W_2 W_1 \prec W_4 W_3.$$

Let $Sq^I = Sq^{i_1} Sq^{i_2} \dots Sq^{i_r}$, where the system $I = i_1, i_2, \dots, i_r$ is *admissible*, $(i_{m-1} \geqslant 2i_m)$, and let $Sq^I W_k = W_k \cdot Q_I$. It turns out that $Q_I = W_{i_1} W_{i_2} \dots W_{i_r} +$ monomials preceding $W_{i_1} W_{i_2} \dots W_{i_r}$ with respect to (R). This fact is proved by induction on r. If $r = 1$ then, by formula (3), $Sq^i W_k = W_k W_i$, thus $Q_i = W_i$. Assume the statement holds for $r - 1$ and consider the class

$$Sq^I W_k = Sq^{i_1}(Sq^{i_2} \dots Sq^{i_r} W_k) = Sq^{i_1}(W_k \cdot P),$$

where, by assumption, the polynomial P looks like $W_{i_2} W_{i_3} \dots W_{i_r} +$ lower order monomials in (R). This product is equal to

$$Sq^i(W_k \cdot P) = \sum_{0 \leqslant m \leqslant i_1} Sq^m(P) \cdot Sq^{i-m} W_k = \sum_{0 \leqslant m \leqslant i_1} Sq^m(P) \cdot W_{i-m} W_k.$$

Consequently, by setting $Sq^I W_k = W_k \cdot Q_I$, we get

$$Q_I = \sum_{0 \leqslant m \leqslant i_1} Sq^m(P) \cdot W_{i_1 - m}.$$

[1] This is proved by iterating (3). — *Editor's remark*

In this sum the term with $m = 0$ looks like $P \cdot W_{i_1} = W_{i_1} W_{i_2} \dots W_{i_r} +$ monomials of lower order in (R). On the other hand, no term in the decomposition of $Sq^m(P)$, $m > 0$, can contain the class W_i greater than or equal to W_{i_1} with respect to (R). Indeed, by formula (3), the square $Sq^m W_s$ contains only those classes W_i for which $i < 2s$. This yields that the square $Sq^m(P)$, $m > 0$, contains only those classes W_i for which $i < 2_{i_2} \leqslant i_1$. Thus, all these terms are strictly less than $W_{i_1} W_{i_2} \dots W_{i_r}$.

Thus, we see that all the classes $Sq^I(W_k)$, where I is any admissible sequence of total degree h, are linearly independent in the group $H^{k+h}(G_k)$. Indeed, if there were a non-trivial linear relation between these classes, then, taking the highest term with respect to (R), we would see that this term can be linearly expressed as a combination of strictly lower terms (in (R)), which is impossible[1].

Treating the classes W_i as symmetric functions in k variables t_m of first degree, formulate the lemma we have proved, as follows.

Lemma II.8′. *The classes $Sq^I(t_1 t_2 \dots t_k)$, where I runs over the set of admissible sequences of total degree $h \leqslant k$, are linearly independent symmetric functions in t_i.*

We have seen that the cohomology algebra $H^*(M(O(k)); Z_2)$ is isomorphic to the ideal $\boldsymbol{J}$ of the algebra $H^*(G_k; Z_2)$ generated by the class W_k. On the other hand, the basis of the group $H^h(G_k; Z_2)$ is generated by symmetrised monomials

$$\sum (t_1)^{a_1} (t_2)^{a_3} \dots (t_r)^{a_r}, \tag{4}$$

where the sum of exponents a_i is equal to h, and the symmetrisation sign $\sum$ means the summation over all permuations essential for (4), i.e. over representatives of conjugacy classes of the full symmetric group of k variables by the subgroup of permutations fixing the monomial (4) [2]. For instance,

$$\sum (t_1)(t_2)\dots(t_k) = t_1 t_2 \dots t_k.$$

For any decomposition (ω) of h into summands $h = \sum_i a_i$ we denote by $\boldsymbol{S}_\omega$ the system of corresponding essential permutations. Furthermore, we assume that the $\sum$ sign before the monomial of type (4), means, unless

[1] See remark 2 at the end of the article, page 203. — *Editor's remark*

[2] Every element of $H^h(G_k; Z_2)$ is a symmetric polynomial of degree h in $t_1, \dots, t_k$. Let $\alpha t_1^{\alpha_1} \dots t_r^{\alpha_r}$ be the leading term of this polynomial. Subtracting the symmetrised monomial $\sum (t_1)^{\alpha_1} \dots (t_r)^{\alpha_r}$, we evidently get a symmetric polynomial with a smaller leading term. Repeating this process, we can express each element of the group $H^h(G_k; Z_2)$ as a linear combination of independent symmetrised monomials (4). — *Editor's remark*

otherwise specified, that the sum is taken only over the system of essential permutations.

In dimension $k + h$, the base of the ideal $\boldsymbol{J}$ consists of symmetrised monomials

$$\sum (t_1)^{\alpha_1+1}(t_2)^{\alpha_1+1}\dots(t_r)^{\alpha_r+1}t_{r+1}\dots t_k, \tag{5}$$

obtained from the elements of the base (4) by multiplication by the class $W_k = t_1t_2\dots t_k$. Indeed, any permutation which is essential for the monomial (4), is essential for (5), and vice versa.

Definition. Let P be an arbitrary polynomial in variables t_i. A variable t_i is a *dyadic* variable for the polynomial P, if the exponent of this variable in terms of the polynomial P is either zero or[1] 2^m.

Lemma II.9. *Any variable t_n, which is dyadic for the polynomial P, is dyadic for Sq^iP as well.*

Indeed, it is known that[2] $Sq^a(t_n)^m = \binom{m}{a}(t_n)^{m+a}$. On the other hand, if m is non-zero and it is a power of 2 then the binomial coefficient $\binom{m}{a}$ is congruent to zero[3], except to those cases when $a = 0$ or $a = m$. (Indeed, $\binom{m}{a} = 1 \; mod \; 2$ if and only if the binary decomposition of p contains the binary decomposition of q (see [26]).) In these cases, the new exponent $m + a$ is also a power of two.

Definition. By a *non-dyadic factor* of the monomial $(t_1)^{a_1}(t_2)^{a_2}\dots(t_r)^{a_r}$ we mean the monomial consisting of all non-dyadic variables; denote the number of these variables by u, and denote the total degree of the non-dyadic factor by ν. For the set of monomials in (t_i) variables we define a quasi-order relation as follows[4] (Q): a monomial X is greater than the monomial Y with respect to (Q) if $u(X) > u(Y)$ or if $u(X) = u(Y)$ and $\nu(X) < \nu(Y)$.

---

[1] The case $m = 0$ is also possible. — *Editor's remark*

[2] If $a + m = 1$ then the relation is evident. Assuming that it holds for $a + m < N$, we get for $a + m = N$:

$$Sq^a t_n^m = Sq^a(t_n^{m-1}\cdot t) = Sq^a t_n^{m-1}\cdot t_n + Sq^{a-1}t_n^{m-1}\cdot t_n^2$$

$$= \left(\binom{m-1}{a} + \binom{m-1}{a-1}\right) t_n^{a+m} = \binom{m}{a} t_n^{a+m}.$$

— *Editor's remark*

[3] modulo 2. — *Editor's remark*

[4] Here "quasi" means the following: if X is not greater than Y and Y is not greater than X, it does not yield, in general, that $X = Y$. — *Editor's remark*

For any number $h \leqslant k$ consider the classes

$$X_\omega^h = \sum (t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1} t_{r+1} \dots t_k, \tag{6}$$

where $\omega = \{a_1, a_2, \dots, a_r\}$ is an arbitrary decomposition of h into summands, *with no summand of type* $2^m - 1$ (*non-dyadic decomposition of* h). Denote the number of such decompositions by $d(h)$.

For any dimension $m \leqslant k$, consider the classes

$$X_{\omega_m}^m,\ Sq^1 X_{\omega_{m-1}}^{m-1},\ Sq^2 X_{\omega_{m-2}}^{m-2}, \dots,\ Sq^{I_h} X_{\omega_h}^h, \dots,\ Sq^I W_k, \tag{7}$$

where Sq^{I_h} is an admissible sequence of total degree $(m-h)$, and ω_h is a non-dyadic decomposition of h.

It turns out that all classes (7) are *linearly independent.*

To prove it, take a term of the polynomial (6), apply the operation Sq^I and take the leading monomial. It turns out that the sum of all such leading monomials looks like

$$\sum_{\boldsymbol{S}_\omega} (t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1} \cdot Sq^I(t_{r+1} \dots t_k), \tag{8}$$

where the sum is taken over all permutations $\boldsymbol{S}_\omega$ which are essential for the monomial (6) corresponding to the decomposition ω. Indeed, the index u of any term of the polynomial

$$Sq^I((t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1} t_{r+1} \dots t_k),$$

is less than or equal to r, since, by Lemma II.9, the variables $(t_{r+1}, \dots, t_k)$ are dyadic. For $u = r$ we have two cases: either the monomial in question enters the polynomial

$$((t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1}) \cdot Sq^I(t_{r+1} \dots t_k), \tag{9}$$

or it enters the polynomial

$$Sq^{I'}(t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1} \cdot Sq^{I''}(t_{r+1} \dots t_k).$$

In the first case we have $\nu = u+h$ and in the second case ν is strictly greater than $r+h$. This yields that all terms of the polynomial (9) are greater (with respect to (Q)) than any other term of the polynomial $Sq^I X_\omega^h$. On the other hand, no term of the polynomial (9) can vanish as a result of symmetrisation from (8). Indeed, any permutation of variables t_i, which is essential for (6), is essential for its non-dyadic factor $(t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1}$, which is a non-dyadic factor for any term of the polynomial (9). Consequently, transforming (9) by permutations of the system S_ω, we obtain the relations not containing non-dyadic factors. Thus their sum is non-zero.

Since no term can be expressed as a linear combination of strictly lower terms (in (Q)), the above arguments yield that any linear dependence between classes (7) is a consequence of those linear dependencies containing those classes $Sq^I X^h_\omega$, leading with respect to (Q), whose members are of the same index $u = r$ and of the same index $\nu = r + h$, this, having the same degree h. Furthermore, the decompositions ω of h by means of which we constructed X^h_ω, for which this linear dependence holds, should be the same. Otherwise, non-dyadic factors of higher terms (in (Q)) of the decompositions of the squares $Sq^I X^h_\omega$ should all be different, and their sum should be non-zero. Thus, any linear dependence between classes (7) is a corollary of linear relations of the type $\sum\limits_\lambda c_\lambda Sq^{I_\lambda} X^h_\omega = 0$, containing only one class X^h_ω.

Let us write down the (Q)-leading terms of this relation:

$$\sum_\lambda c_\lambda (t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1} Sq^{I_\lambda}(t_{r+1}\dots t_k) = 0.$$

All members of this relation containing a fixed factor $(t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1}$, should sum to zero. Thus,

$$(t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1}\sum_\lambda c_\lambda Sq^{I_\lambda}(t_{r+1}\dots t_k) = 0.$$

But, according to Lemma II.8′, all classes $Sq^I(t_{r+1}\dots t_k) = 0$ are linearly independent if the degree $m - h$ of the sequence I does not exceed $k - r$. Since, evidently, $h \geqslant 2r$, this inequality holds for all $m \leqslant k$. Consequently, the coefficients c_λ are equal to zero. Thus, the classes (7) are not connected by any non-trivial linear dependence.

The rank of $H^{k+m}(M(O(k)))$, i.e. the rank of the ideal $\boldsymbol{J}$, is equal to the total number $p(m)$ of decompositions of m into summands. On the other hand, the number of classes (7) is equal to $\sum\limits_{h\leqslant m} c(m-h)d(h)$. It is easy to see that

$$p(m) = \sum_{h\leqslant m} c(m-h)d(h).$$

Indeed, to each decomposition of m there correspond two decompositions: the decomposition of $(m-h)$ consisting of summands of the type $2^n - 1$ and the decomposition of h, consisting of the remaining summands. Thus, the classes (7) *form a base* of the group $H^{k+m}(M(O(k)))$.

Associate with each class X^h_ω a mapping

$$F_\omega : M(O(k)) \to K(Z_2, k+h),$$

such that $F^*_\omega(\iota) = X^h_\omega$, where F^*_ω is the homomorphism generated by the mapping F_ω. The mappings F_ω define the mapping F from $M(O(k))$ to the product

$$Y = K(Z_2, k) \times K(Z_2, k+2) \times \dots \\ \times (K(Z_2, k+h))^{d(h)} \times \dots \times (K(Z_2, 2k))^{d(k)}. \quad (10)$$

Since the classes (7) from the base of $H^{k+h}(M(O(k)))$, then the homomorphism F^* generated by F, is an isomorphic mapping from the group $H^{k+m}(Y; Z_2)$ to the group $H^{k+m}(M(O(k)))$ for all $m \leqslant k$. Considered modulo p, $p > 2$, the cohomology algebra of Y is trivial[1], and the cohomology algebra of the space $(M(O(k)))$ is trivial in dimensions less than $2k$. Consequently, the homomorphism F^* is in this case an isomorphism in dimensions less than $2k$; it is a monomorphism in dimension $2k$. Thus, for $M(O(k))$ and Y, one can apply Theorem II.6. According to this theorem, there exists a mapping g from the $2k$-frame of the cellular decomposition Y to the cellular decomposition $M(O(k))$, for which the mapping $g \circ F$ is homotopic to the identity mapping on the $(2k-1)$-frame of the decomposition $M(O(k))$.

Consequently, we get

Theorem II.10. *The space $M(O(k))$ has the same homotopy $2k$-type as the product Y (10) of Eilenberg-MacLane polyhedra.*

Corollary II.11. *The stable homotopy group $\pi_{k+h}(M(O(k)))$, $h < k$, is isomorphic to the direct sum of $d(h)$ groups Z_2.*

Consider g restricted to the first factor, we get

Corollary II.12. *There exists a mapping g from the $2k$-skeleton of the decomposition $K(Z_2, k)$ to the decomposition $M(O(k))$ such that $g^*(U) = \iota$, where ι is the fundamental class of the decomposition $K(Z_2, k)$.*

Since every class $u \in H^k(A; Z_2)$ of any space A is an image of the fundamental class ι under some mapping $f : A \to K(Z_2, k)$, then we get

Corollary II.13. *Any k-dimensional modulo 2 cohomology class of any space of dimension $\leqslant 2k$ admits an orthogonal realisation.*

7. The space $M(O(k))$ for small k

$k = 1$. The space of unoriented 1-vectors is the real projective space $PR(N)$ of some very high dimension N; the corresponding universal fibre space $A_{O(1)}$ coincides with the cylinder of the two-fold covering $S^N \to PR(N)$. Contracting in $A_{O(1)}$ the boundary sphere S^N to a point, we get as $M(O(1))$ the real projective space $PR(N+1)$. Thus, both spaces $K(Z_2, 1)$ and $M(O(1))$ coincide with the real projective space $PR(\infty)$ of infinite

[1]Because, according to statement 8 of Chapter II of [21], all integral cohomology groups of the space Y are 2-groups. — *Editor's remark*

dimension. Consequently, *every one-dimensional mod 2 cohomology class admits an orthogonal realisation.*

$k = 2$. The cohomology of the space $M(O(2))$ is described as follows[1].

In dimension 2 there is one modulo 2 class — the fundamental class U.

In dimension 3 — the integral[2] class $Sq^1U = UW_1$.

In dimension 4 — the integral class X and the class $U(W_1)^2$ mod 2. The class X is the square of the fundamental class of the space $M(SO(2))$, and the modulo 2 class is the square U^2 of the class U.

In dimension 5 there is an integer class of order two

$$Sq^1(U(W_1)^2) = U(W_1)^3$$

and the class U^2W_1 modulo 2.

For the natural mapping $F : M(O(2)) \to K(Z_2, 2)$, we have for Z_2 coefficients:

$$F^*(\iota) = U; \quad F^*(Sq^1\iota) = UW_1; \quad F^*(Sq^2\iota) = U^2;$$
$$F^*(Sq^2Sq^1\iota) = Sq^2(UW_1) = U^2W_1 + U(W_1)^3; \quad F^*(\iota \cdot Sq^1\iota) = U^2W_1.$$

Consider, as in the proof of Theorem II.6, the cylinder K of F. This cylinder K contains as a closed subset the space $M(O(2))$; for conciseness, we denote it by M. From the exact sequence corresponding to the inclusion $F : M \to K$, it follows that

$$H^r(K, M; Z_p) = 0 \text{ for } r < 5,$$
$$H^5(K, M; Z_p) = Z_p \text{ for all prime } p.$$

[1]The algebra $H^*(M(O(2), Z_2))$ is an ideal of the polynomial algebra $H^*(G_2, Z_2)$ in W_1 and W_2. This ideal is generated by the element W_2. Consequently, by setting $U = W_2$, we get: $H^2 = (U)$, $H^3 = (UW_1)$, $H^4 = (U^2, U(W_1)^2)$, $H^5 = (U(W_1)^3, U^2W_1)$ and so on. On the other hand, by Serre's theorem, $H^3(Z_2, 2; Z_2) = Sq^1(\iota)$. Since $F^*(\iota) = U$, where $F : M(O(2)) \to K(Z_2, 2) \times K(Z_2, 4)$ is the mapping constucted in the proof of Theorem II.10, then

$$Sq^1U = Sq^1F^*(\iota) = F^*Sq^1(\iota) = UW_1,$$

consequently,

$$Sq^1(U(W_1)^2) = U \cdot Sq^1((W_1)^2) + Sq^1(U) \cdot (W_1)^2 = U(W_1)^3,$$

because $Sq^1((W_1)^2) = W_1 \cdot Sq^1(W_1) + Sq^1(W_1) \cdot W_1 = 0$. — *Editor's remark*

[2]That is, obtained from an integral cohomology class by reduction modulo 2. Here and in the sequel, one should note that the classes Sq^ix for odd i are integer in this sense. — *Editor's remark*

From here, by a duality argument, we get

$$H_r(K, M; Z_p) = 0 \text{ for } r < 5,$$
$$H_5(K, M; Z_p) = Z_p \text{ for all prime } p.$$

Thus, by the universal coefficient formula,

$$H_r(K, M; Z_p) = 0 \text{ for } r < 5,$$
$$H_5(K, M; Z_p) = Z.$$

Applying the relative version of the Hurewicz theorem, we get:

$$\pi_4(K, M) = 0, \quad \pi_5(K, M) = Z.$$

Thus,

$$\pi_3(M) = 0, \quad \pi_4(M) = Z.$$

The mapping g, which is homotopically inverse to F, can be defined on the 4-skeleton of the decomposition K. While extending g to the 5-skeleton of $K(Z_2, Z)$, we get an obstruction in the group $\pi_4(M) = Z$. According to the general second obstruction theory [14], this class belonging to the group $H^5(Z_2, 2; Z)$ is nothing but the *Eilenberg-MacLane invariant* corresponding to the second non-trivial homotopy group $\pi_4(M)$. This class generates the kernel of the homomorphism $F^* : H^5(Z_2, 2; Z) \to H^5(M; Z)$[1].

The group $H^5(Z_2, 2; Z)$ is the *cyclic group of order four.* It is generated by the element $\frac{1}{4}\delta p(\iota)$, which is the image of the *Pontrjagin square* $p(\iota)$ of the fundamental class ι under the Bockstein homomorphism $\frac{1}{4}\delta$. The group $H^5(M; Z)$ is the *cyclic group of order two*; it is generated by the class $Sq^1(U(W_1)^2)$. Reducing this class modulo 2, we obtain the class $U(W_1)^3$. It turns out that the homomorphism F^* takes the generator of the first group to the generator of the second group. It is clear that it is sufficient to check this statement only modulo 2. To do this, let us calculate the modulo 2 reduction of the class $\frac{1}{4}\delta p(\iota)$. Let u be a cocycle of the class ι and let $\nu = \frac{1}{2}\delta p(u)$ be a cocycle of the class $Sq^1\iota$. The Pontrjagin square is defined by the formula[2] $p(u) = u \smile u + u \smile_1 \delta u$. Consequently, according to the coboundary formula, we have

[1]Only this property of the obstruction will be used in the sequel. Thus, in the case in question we may define the obstruction as the class generating the kernel of the homomorphism F^*. — *Editor's remark.*

[2]The Pontrjagin product associates with a mod τ-cycle X for even τ a mod 2τ cycle as follows. We take X and define the operation $\smile_1$ for u and δu. We have $\delta u = \tau a$; then $u \smile_1 \delta u = u * a$.

Here the operation $*$ is defined as follows. Let K be a cellular complex and let u and

$$\delta p(u) = \delta u \smile u + u \smile \delta u + \delta u \smile_1 \delta u + u \smile \delta u - \delta u \smile u.$$

From here, dividing by 4 and reducing modulo 2, we get

$$\frac{1}{4}\delta p(u) = u \smile \nu + \nu \smile_1 \nu = \iota \cdot Sq^1\iota + Sq^2 Sq^1 \nu.$$

The homomorphism F^* takes this class to the class

$$U^2 W_1 + U^2 W_1 + U(W_1)^3 = U(W_1)^3,$$

i.e. to the generator of the group $H^5(M; Z)$, reduced modulo 2.

Since the generator of $H^5(M; Z)$ has order 2, the mapping F^* takes the class $\frac{1}{2}\delta p(\iota)$ to zero; this class is just the desired obstruction. (Note that though this obstruction is a second order class, if we reduce it modulo 2, we get zero. This yields that it cannot be expressed by means of operations Sq^i.) Thus, we have proved the following

Theorem II.14. *In order for a class $x \in H^2(A; Z_2)$ of a certain 5-dimensional space A to admit an orthogonal realisation, it is necessary and sufficient that the class $\frac{1}{2}\delta p(x)$ equals zero, where $p(x)$ is the Pontrjagin square of x.*

Note that the conditions of this theorem hold if and only if there exists a class $X \in H^4(A; Z_2)$ such that $Sq^2 Sq^1 x + x \cdot Sq^1 x = Sq^1 X$.

$k = 3$. Compare the polyhedron $M(O(3))$ with the product Y of Eilenberg-MacLane spaces, given in Theorem II.10. It turns out that the homomorphism F^* is an isomorphism not only in dimension 6, but also in dimension 7. On the contrary, in dimension 8, this homomorphism is presumably not an isomorphism. Indeed, we have:

in dimension 3

$$F^*(i) = U;$$

in dimension 4

$$F^*(Sq^1\iota) = UW_1;$$

in dimension 5

$$F^*(Sq^2\iota) = UW_2 \text{ and}$$

$$F^*(X^2) = U(W_1)^2 \text{ (new generator);}$$

v be two modulo τ cohomology classes of dimensions r and s, respectively. Then for an $(r+s-1)$-simplex, we define

$$(u * v)(T) = \sum_{i=1}^{s} u(a_i, a_{i+1}, \ldots, a_{i+r}) v(a_1, \ldots, a_{i+r}, \ldots, a_{i+s})$$

— Translator's remark.

in dimension 6

$$F^*(Sq^3\iota) = U^2,\ F^*(Sq^2Sq^1\iota) = U(W_2W_1 + (W_1)^3),$$
$$F^*(Sq^1X^2) = U(W_1)^3;$$

in dimension 7

$$F^*(\iota \cdot Sq^1\iota) = U^2W_1,$$
$$F^*(Sq^2X^2) = U(W_2(W_1)^2 + (W_1)^4).$$

In dimension 7, we have two new generators, $F^*(X^4) = U(W_1)^4$ and $F^*(X^{22}) = U(W_2)^2$. The homomorphism F^* is not an isomorphism in dimension 8 because $F^*(Sq^3X^2 + (Sq^1\iota)^2 + Sq^1X^4) = 0$. From the above we get (see Theorem II.6)

Theorem II.15. *Every three-dimensional class over Z_2 of any space of dimension less than 8 admits an orthogonal realisation.*

REMARK. I do not know what the obstruction in dimension 8 is equal to. Possibly, it vanishes.

We conclude this subsection with the following general remark: for $k > 1$ there is no map

$$g : K(Z_2, k) \to M(O(k)),$$

homotopy inverse to the map $F : M(O(k)) \to K(Z_2, k)$. Indeed, as Serre has pointed out to me, the cohomology algebra of the space $K(Z_2, k)$ is for $k > 1$ the polynomial algebra *in infinitely many variables.* The cohomology of $M(O(k))$, is, on the contrary, isomorphic (up to grading change) to the cohomology of the Grassmann manifold G_k; thus, it is of *finite type.* For sufficiently large dimensions, the rank of the algebra $H^*(Z_2, k)$ is strictly greater than the rank of $H^*(M(O(k)))$, so that the kernel of F^* is distinct from zero; thus, the mapping g cannot exist. Thus, *for any $k > 1$ there exist spaces of high dimension (greater than $2k$), for which some modulo 2 k-dimensional cohomology classes admit no orthogonal realisation.*

8. The complex $M(SO(k))$. Stationary case

In Section 6 we have described the "stationary" homotopy type of the space $M(O(k))$. For the space $M(SO(k))$ we cannot do this because the homotopy type of this space is much more complicated. Indeed, the polyhedron Y, equivalent to $M(O(k))$, is a *topological product* of the polyhedra $K(Z_2, k)$; the polyhedron equivalent to the space $M(SO(k))$ is not a product, but an iterated fibre space, where all the fibres are polyhedra of types $K(Z_2, r)$ and $K(Z, m)$ (and, possibly, even $K(Z_p, n)!$), and the resulting bundles are, in general, non-trivial. Thus, we restrict ourselves with the description of an equivalent polyhedron only in dimensions $k + i$, where $i \leqslant 7$.

The definition of "Silber's polyhedron" K. It is known that the polyhedron $K(Z, k+4)$ is a fibre of some asphericalal space A, whose base is $K(Z, k+5)$ (see Serre, [24][1]). Let u be the fundamental class of the base $K(Z, k+5)$. There exists a mapping f taking $K(Z, k)$ to $K(Z, k+5)$ such that $f^*(u) = St_3^5(\nu)$, where St_3^5 is the "Steenrod cube" of dimension 5, which is an integer class of order three[2]. Denote by K the space induced by the mapping f and the fibre space A. The space K is a fibre space with base $K(Z, k)$ and fibre $K(Z, k+4)$. The only non-trivial homotopy groups of K are the groups π_k and π_{k+4}, each isomorphic to Z. The corresponding *Eilenberg-MacLane invariant* $k \in H^{k+5}(Z, k; Z)$ coincides with $St_3^5(\nu)$. The necessary and sufficient condition for $F : M \to K$ (M is a cellular decomposition), defined on the $(k+4)$-skeleton of M, to be extended over M, is the triviality of the cube $St_3^5(x)$, where x is the image of the class ι under F^*.

The cohomology of K

1. Cohomology modulo 2. Let F^3 be a mapping of $K(Z, k)$ to itself such that $(F^3)^*(\iota) = 3\iota$. The fibre space generated by the space K under this mapping coincides with the product $K(Z, k) \times K(Z, k+4)$, because the Eilenberg-MacLane invariant of this space is equal to zero:

$$F^{3*}(St_3^5(\iota)) = St_3^5(F^3(\iota)) = St_3^5(3\iota) = 0.$$

Consequently, there exists a mapping G coordinated with the bundles over $K(Z, k+4)$, that takes the product $K(Z, k) \times K(Z, k+4)$ to the space K, for which the corresponding mapping of the bases $K(Z, k)$ coincides with F^3.

The mapping G generates a homomorphism of the cohomology spectral sequence of the bundle K to the trivial spectral sequence of cohomology of the product $K(Z, k) \times K(Z, k+4)$.

The homomorphism G^* is an isomorphism for the E^2 terms of these spectral sequences. Moreover, the homomorphism $(F^3)^*$ is an automorphism of the algebra $H^*(Z, k; Z_2)$. Consequently, the Leray differential d_2 of the term E_2 of the spectral sequence for the fibre K is trivial, because it is trivial in the spectral sequence of the product. The same is true for all sequences of differentials d_i. Thus we get: the cohomology algebra $H^*(K; Z_2)$ is isomorphic to the cohomology algebra of the product $K(Z, k) \times K(Z, k+4)$.

---

[1] Here A is the space of paths with fixed initial point of the polyhedron $K(Z, k+5)$. — *Editor's remark*

[2] Here the author uses the fact that the classes $St_p^{2k(p-1)+1}(x)$ can be considered as integral. — *Editor's remark*

2. Cohomology modulo p, where p is prime $\geqslant$ 5. Arguing as above, we arrive at an analogous conclusion: the cohomology algebra $H^*(K; Z_p)$ is isomorphic to the cohomology algebra of the product $K(Z, k) \times K(Z, k+4)$.

3. Cohomology modulo 3. Here we shall need a more detailed inspection of the spectral sequence of K. Denote by ν the fundmental class of the fibre $K(Z, k+4)$. By construction of K, the transgression (which coincides here with d_{k+5}) takes the fundamental class ν to the class $St_3^5(\iota)$. Since Steenrod's powers commute with the transgression (up to a non-zero coefficient), the class $St_3^4\nu$ maps to a[1] class $St_3^4 \circ St_3^5(\iota) = St_3^9(\iota)^2$, and the class $St_3^5(\nu)$ maps to the class $St_3^5 \circ St_3^5(\iota) = 0$. Consequently, the cohomology algebra $H^*(K; Z_p)$ has the following generators:

in dimension k — generator corresponding to the class ι (by abuse of notation, we denote it also by ι);

in dimension $k+4$ the class $St_3^4(\iota)$;

in dimension $k+8$ the class $St_3^8(\iota)$;

in dimension $k+9$ the element generated by the class[3] $St_3^5(\nu)$.

The space equivalent to $M(SO(k))$. This space Y is the product of the space K defined above and the Eilenberg-MacLane space $K(Z_2, k+5)$. The corresponding mapping $F: M(SO(k)) \to Y$ is defined according to the following arguments.

There exists a mapping f from the $(k+4)$-skeleton of the cellular decomposition of $M(SO(k))$ to the space K such that $f^*(\iota) = U$. Since $St_3^5 U = 0$ (because the cohomology groups of $M(SO(k))$, as well as those of $\widehat{G}_k$, have no elements of order 3), then the mapping f can be extended to a mapping

$$f: M(SO(k)) \to K$$

from $M(SO(k))$ to K. On the other hand, there exists a mapping $g: M(SO(k)) \to K(Z_2, k+5)$ such that $g^*(\iota') = UW_2W_3$, where ι' is the fundamental class of the space $K(Z_2, k+5)$. The pair f and g defines the desired mapping $F: M(SO(k)) \to Y$.

Let us calculate the homomorphism F^* generated by F.

[1] Non-zero. — *Editor's remark*

[2] Here and later the author uses Adem's formulae (see remark on page 203) for Steenrod's powers. (Recall that up to a factor the operation $\mathscr{D}_p^a$ coincides with $St_p^{2a(p-1)}$, and the operation $\beta\mathscr{D}_p^a$ coincides with $St_p^{2a(p-1)+1}$.) — *Editor's remark*

[3] Indeed, since $d_{k+5}(\nu) = St_3^5(\iota)$ and $d_{k+5}(St_3^4(\nu)) = St_3^9(\iota)$, when passing from E_2 to E_∞ the elements ν, $St_3^5(\iota)$, $St_3^4(\nu)$ and $St_3^9(\iota)$ vanish. On the other hand, according to H.Cartan (see page 152), in the dimensions $\leqslant k+9$ the only generators of $E_2 = H^*(Z, k; Z_3) \otimes H^*(Z, k+1, Z_3)$, except the vanishing ones, are the elements indicated by the author. — *Editor's remark*

Calculation modulo 2. Consider the dimensions $k+i$, where $0 \leqslant i \leqslant 8$.

Denote by ν the image of the generator of the cohomology algebra of $K(Z, k+4)$ under

$$H^*(K; Z_2) \approx H^*(Z, k; Z) \otimes H^*(Z, k+4; Z_2),$$

we have:

$$\begin{array}{lll}
i=0; & F^*(\iota) & = U. \\
i=1; & F^*(0) & = 0. \\
i=2; & F^*(Sq^2\iota) & = UW_2. \\
i=3; & F^*(Sq^3\iota) & = UW_3. \\
i=4; & F^*(Sq^4\iota) & = UW_4, \\
 & F^*(\nu) & = U(W_2)^2. \\
i=5; & F^*(Sq^5\iota) & = UW_5, \\
 & F^*(\iota') & = UW_2W_3. \\
i=6; & F^*(Sq^6\iota) & = UW_6, \\
 & F^*(Sq^4Sq^2\iota) & = U(W_2W_4 + (W_3)^2 + (W_2)^3), \\
 & F^*(Sq^2\nu) & = U((W_2)^3 + (W_3)^2), \\
 & F^*(Sq^1\iota') & = U(W_3)^2. \\
i=7; & F^*(Sq^7\iota) & = UW_7, \\
 & F^*(Sq^5Sq^2\iota) & = U(W_5W_2 + W_4W_3 + W_3(W_2)^2), \\
 & F^*(Sq^3\nu) & = UW_3(W_2)^2, \\
 & F^*(Sq^2\iota') & = UW_2(W_5 + W_3W_2), \\
i=8; & F^*(Sq^8\iota) & = UW_8, \\
 & F^*(Sq^6Sq^2\iota) & = U(W_6W_2 + W_5W_3 + W_4(W_2)^2), \\
 & F^*(Sq^4\nu) & = U(W_4(W_2)^2) + W_2(W_3)^2 + (W_2)^4), \\
 & F^*(Sq^3\iota') & = UW_5W_3, \\
 & F^*(Sq^2Sq^1\iota') & = UW_2(W_3)^3.
\end{array}$$

It is easy to see that for $i \leqslant 8$ the elements of the algebra $H^*(M(SO(k)); Z_2)$ given in the table above, are *linearly independent.* Moreover, for $i \leqslant 7$ these elements *form a basis* of the group $H^{k+i}(M(SO(k)); Z_2)$. Consequently, the mapping F^* for $i \leqslant 7$ is an isomorphism from the group $H^{k+i}(Y)$ to the group $H^{k+i}(M(SO(k)); Z_2)$, and for $i = 8$ the mapping F^* is a monomorphism.

REMARK. According to the canonical type (Serre, [23]) of the generators of $H^*(Z, k; Z_2)$, we may proceed with our calculations. In dimension 8, we get two new generators corresponding to the Pontrjagin classes $(W_2)^4$ and $(W_4)^2$.

Calculation modulo 3. The factor $K(Z_2, k+5)$ gives nothing. Thus,

$$\begin{array}{lll}
i=0; & F^*(\iota) & = U, \\
i=4; & F^*(St_3^4\iota) & = UP_4, \\
i=8; & F^*(St_3^8\iota) & = U((P_4)^2 + 2P_8).
\end{array}$$

modulo[1] 5:

$$i=0;\ F^*(\iota)=U,\ F^*(\nu)=UP_4,\ F^*(St_5^8\iota)=U((P_4)^2-2P_8).$$

Calculation modulo p, $p>5$:

$$\begin{aligned} i=0;\ F^*(\iota)&=U,\\ i=4;\ F^*(\nu)&=UP_4,\\ i=8;\ F^*(0)&=0. \end{aligned}$$

Thus, for any field of coefficients the homomorphism F^* for $i\leqslant 7$ is an isomorphism from the group $H^{k+i}(Y)$ to the group $H^{k+i}(M(SO(k)))$ and for $i=8$ it is an isomorphism. Since the spaces Y and $M(SO(k))$ are simply connected, one may apply Theorem from 6 to them. According to this theorem, the spaces $M(SO(k))$ and Y have the same $(k+8)$-type. Thus, we get

Theorem II.16. *For $i\leqslant 7$ the stationary homotopy groups $\pi_{k+i}(M(SO(k)))$ are defined by the following formulae:*

$$\pi_{k+1}=\pi_{k+2}=\pi_{k+3}=0;$$
$$\pi_{k+4}=Z;\quad \pi_{k+5}=Z_2;\quad \pi_{k+6}=\pi_{k+7}=0.$$

Theorem II.17. *For $k>8$ an integral k-dimensional cohomology class x of a $(k+8)$-dimensional space is realisable with respect to the torsion group if and only if the integral class $St_3^5(x)$ vanishes.*

9. The space $M(SO(k))$ for small k

In the present subsection, we define the first obstruction for the mapping $g: K(Z,k)\to M(SO(k))$ for $k<5$. This obstruction is, as in the stationary case, *the Steenrod cube $St_3^5(\iota)$* of the fundamental class.

$k=1$. The space $M(SO(1))$ is a product $S^\infty\times S^1$, where some sphere of the type $S\times t$ is contracted to a point. This space has the homotopy type of the circle S^1. On the other hand, S^1 is a realisation of the space $K(Z,1)$. This yields that *any one-dimensional integral cohomology class is realisable with respect to the rotation group* (this group, however, consists of one element).

$k=2$. The Grassmann manifold $\widehat{G}_2$ of two-dimensional planes is the classifying space of the group $SO(2)=SU(1)=S^1$. Thus, this manifold can be identified with the complex projective space $PC(N)$ of high

[1]The calculation of Steenrod's operations $\boldsymbol{St_p^i}$ of the fundamental class $\boldsymbol{U}$ can be found in Borel-Serre [5] and Wu [35].

dimension. The universal fibre space $A_{SO(2)}$ over $\widehat{G}_2$ can be identified with the normal tubular neighbourhood of the space $PC(N)$, the latter considered as a projective hyperplane in $PC(N+1)$. This yields that the space $M(SO(2))$ can be identified with the space $PC(N+1)$. Thus, the space $M(SO(2))$, *as well as* $K(Z,2)$, can be realised by a projective space of "high dimension[1]". Thus, *any two-dimensional cohomology class is realisable with respect to the rotation group.*

$k=3$. As usual, denote by ι the fundamental class of $K(Z,3)$. It is known that the class $St_3^5(\iota)$ is non-zero[2]. Let us construct the "Silber space" K, whose Eilenberg-MacLane invariant k is equal to $St_3^5(\iota)$. This space is a fibre space with base $K(Z,3)$ and fibre $K(Z,7)$. As above, we see that for any prime $p \neq 3$, the cohomology algebra $H^*(K;Z_p)$ is isomorphic to the cohomology algebra of the product $K(Z,3)\times K(Z,7)$. Let ν be the fundamental class of the space $K(Z,7)$. It is easy to see that the group $H^3(K,Z_3)$ has a unique generator, which is the image of ι under the fibration map $K \to K(Z,3)$; we shall denote this generator also by ι. Furthermore, $H^4 = H^5 = H^6 = 0$, the group H^7 is generated by the element $St_3^4(\iota)$, and the group H^8 is trivial.

Since $St_3^5(U)=0$ there exists a mapping $F: M(SO(3))\to K$ such that:

mod 2
$$F^*(\iota)=U,$$
$$F^*(Sq^2\iota)=UW_2,$$
$$F^*(Sq^3\iota)=UW_3=U^2,$$
$$F^*(\nu)=U(W_2)^2,$$
$$F^*(\iota\cdot Sq^2\iota)=U^2W_2.$$

mod 3
$$F^*(\iota)=U,$$
$$F^*(St_3^4\iota)=UP_4,$$
and nothing more up to dimension 11.

mod $p, p\geqslant 5$
$$F^*(\iota)=U,$$
$$F^*(\nu)=UP_4,$$
and nothing more up to dimension 11.

Thus, for any coefficient field, the homomorphism F^* is an isomorphism from $H^*(K)$ onto $H^*(M(SO(3)))$ in dimensions $\leqslant 7$, and in dimension 8, the homomorphism F^* is a monomorphism. Thus, by Theorem II.6, the spaces K and $M(SO(3))$ have the same 8-type. This yields

Theorem II.18. *An integral 3-dimensional cohomology class x of any space of dimension $\leqslant 8$ is realisable with respect to the rotation group if and only if the integral class $St_3^5(x)$ is equal to zero.*

---

[1]More precisely, of infinite dimensional projective space $PC(\infty)$. — *Editor's remark*

[2]Otherwise the operation St_3^5 would be trivial in any space. — *Editor's remark*

$k = 4$. Let us construct the Silber space K for $M(SO(4))$. The homomorphism F^*, generated by $F : M(SO(4)) \to K$, is described by the following formulae (the notation is as above):

$$\begin{array}{ll}
\text{mod } 2 & F^*(\iota) = U, \\
 & F^*(Sq^2\iota) = UW_2, \\
 & F^*(Sq^3\iota) = UW_3, \\
 & F^*(Sq^4\iota) = U^2, \\
 & F^*(\nu) = U(W_2)^2, \\
 & \text{and nothing more up to dimension 9.} \\
\text{mod } 3 & F^*(\iota) = U, \\
 & F^*(St^4\iota) = UP_4, \\
 & F^*(\iota^2) = U^2, \\
 & \text{and nothing more up to dimension 12.} \\
\text{mod } p, p \geqslant 5 & F^*(\iota) = U, \\
 & F^*(\iota^2) = U^2, \\
 & F^*(\nu) = UP_4, \\
 & \text{and nothing more up to dimension 12.}
\end{array}$$

Thus, the homomorphism F^* is an isomorphism in dimension $\leqslant 8$; it is a monomorphism in dimension 9. Consequently, the spaces $M(SO(4))$ and K have the same 9-type. This yields

Theorem II.19. *An integral four-dimensional cohomology class x of a space of dimension $\leqslant 9$ is realisable with respect to rotation group if and only if the integer class $St_3^5(x)$ vanishes.*

10. The multiplication theorem

In this subsection we describe several general theorems about classes, realisable with respect to the rotation group. First, let us prove the following necessary condition:

Theorem II.20. *A necessary condition for an integer cohomology class x to be realisable with respect to the rotation group is that all Steenrod powers $St_p^{2m(p-1)+1}(x)$ vanish for all prime p.*

Indeed, for an odd prime p all Steenrod powers $St_p^{2m(p-1)+1}U$ of the fundamental class U of $M(SO(k))$ vanish because the Grassmann manifold $\widehat{G}_k$ for $p > 2$ has no p-torsion (cf. [3]).

To prove the following theorem, we shall use some lemmas on the Eilenberg-MacLane spaces $K(Z, n)$.

NOTATION: Let F_N be the mapping of $K(Z, n)$ to itself (up to homotopy), for which $F_N^*(\iota) = N\iota$.

Lemma II.21. *Let* $0 \to G' \to G \to G'' \to 0$ *be an exact sequence of abelian groups. Assume the endomorphisms* $(F_{N'})^* : H^*(Z,k;G')$ *and* $(F_{N''})^* : H^*(Z,k;G'')$ *are trivial. Then the endomorphism* $(F_N)^* : H^*(Z,k;G)$, *where* $N = N'N''$ *is also trivial.*

Indeed, consider the corresponding exact sequence of cohomology groups

$$\to H^r(Z,k;G') \xrightarrow{f} H^r(Z,k;G) \xrightarrow{g} H^r(Z,k;G'') \to .$$

For any integer m the homomorphisms f and g from this sequence commute with the endomorphism $(F_m)^*$. Let $x \in H^r(Z,k;G)$. Then, by assumption, $g(F^*_{N''}(x)) = F^*_{N''}(g(x)) = 0$.

Consequently, $F^*_{N''}(x) = f(y)$, where $y \in H^r(Z,k;G')$, thus,

$$F^*_N(x) = F^*_{N'} \circ F^*_{N''}(x) = F^*_{N'}(f(y)) = f(F^*_{N'}(y)) = f(0) = 0.$$

The lemma is proved.

From this lemma we get the following

Lemma II.22. *For any abelian group* G *of finite order the endomorphism* $(F_N)^* : H^*(Z,k;G)$ *is trivial.*

Since G is a direct sum of its p-primary components, it is sufficient, by previous lemma, to prove that for any prime p the endomorphism $(F_p)^*$ of the algebra $H^*(Z,k;Z_p)$ is trivial. But this follows from the fact that the algebra $H^*(Z,k;Z_p)$ is generated, as shown in point 6, by iterated p-powers St^i_p of the fundamental class ι.

Lemma II.23. *Let* G *be an abelian group of finite type and let all elements of the group* $H^r(Z,k;G)$ *have finite order* N. *Then there exists a non-zero integer* m *such that the endomorphism* $(F_m)^* : H^r(Z,k;G)$ *is trivial.*

Decompose the group G into a direct sum of a free group F and a finite group T. Then

$$H^r(Z,k;G) \approx H^r(Z,k;F) + H^r(Z,k;T).$$

All elements of $H^r(Z,k;F)$ have order N. It is clear that it suffices to prove the lemma only for $H^r(Z,k;F)$, because the group $H^r(Z,k;T)$ satisfies the conditions of Lemma II.22, because T is finite. Consider the exact sequence

$$0 \to F \xrightarrow{(N)} F \to F' \to 0,$$

where the homomorphism (N) is multiplication by a non-zero integer N. Since the group F is of finite type, the group F' is a finite group of some order N'. Let $x \in H^r(Z,k;F)$ and let g be a homomorphism of the exact sequence of the homology groups

$$\ldots \to H^r(Z,k;F) \xrightarrow{(N)} H^r(Z,k;F) \xrightarrow{g} H^r(Z,k;F') \to \ldots.$$

Then, by Lemma II.22, $g \circ F^*_{N'}(x) = F^*_{N'}(g(x)) = 0$.

Thus, the element $F^*_{N'}(x)$ has type Ny, where $y \in H^r(Z,k;F)$, and thus it is equal to zero. The lemma is proved.

Lemma II.24. *Let Y be an arbitrary space for which the free component of the k-dimensional homotopy group $\pi_k(Y)$ is isomorphic to Z, and let t be the generator of this free component. If for all $q \geqslant k$ the homotopy groups $\pi_q(Y)$ are of finite type and the cohomology groups $H^{q+1}(Z,k;\pi_q(Y))$ are finite then for any $q \geqslant k$ there exists a mapping G_q from the q-skeleton K^q of the cellular decomposition $K(Z,k)$ to the space Y that takes the generator of the group $\pi_k(K(Z,k)) \approx Z$ to the element $N(q,k)t$, where the non-zero integer $N(q,k)$ depends only on k, q and Y.*

Indeed, the k-skeleton of the cellular decomposition $K(Z,k)$ can be thought of as a sphere S^k. Then the corresponding mapping $G_k : S^k \to Y$ is defined as the mapping generating the element t of $\pi_k(Y)$. Assume for some $q \geqslant k$ we have already defined the mapping G_q from the q-frame K^q of the decomposition $K(Z,k)$. When extending G_q to the $(q+1)$-frame of the decomposition $K(Z,k)$ we get an obstruction w', which is a cocycle whose class is an element of the group $H^{q+1}(Z,k;\pi_q(Y))$, which is finite by assumption. Consider the mapping $F_m : K(Z,k) \to K(Z,k)$, corresponding (by Lemma II.23) to the finite group $H^{q+1}(Z,k;\pi_q(Y))$. The composite map $G_q \circ F_m$

$$K(Z,k) \xrightarrow{F_m} K(Z,k) \xrightarrow{G_q} Y$$

is defined on the q-skeleton $K(Z,k)$. When extending it over the $(q+1)$-frame, we get an obstruction $w = (F_m)^*(w')$. By Lemma II.23, the cohomology class of the cocylce w is zero. Thus, after a possible deformation the composite mapping $G_q \circ F_m$ is extended over the $(q+1)$-frame of the decomposition $K(Z,k)$ thus defining a mapping G_{q+1}. The corresponding number $N(q+1,k)$ is, evidently, defined by the formula $N(q+1,k) = mN(q,k)$, thus, it is non-zero. Lemma II.24 is completely proved.

Let us apply Lemma II.24 to the Grassmann manifold $\widehat{G}_k$ of k-planes, or, more exactly, to the universal space $A_{SO(k)}$, which is homotopy equivalent to this manifold. The number k is assumed to be even. Recall that the cohomology algebra $H^*(\widehat{G}_k)$ over the real numbers is a polynomial algebra. The generators of this algebra are the Pontrjagin classes P^{4i}, $i < [k/2]$, and the fundamental k-dimensional class X_k. This together with Serre's

$\mathscr{C}$-theory [22] yields that the integral cohomology group of the manifold $\widehat{G}_k$ is $\mathscr{C}$-isomorphic, where $\mathscr{C}$ is the class of finite groups, to the integral cohomology algebra of the product

$$K(Z,4) \times K(Z,8) \times \ldots \times K(Z,k)$$

of the Eilenberg-MacLane polyhedra[1]. Consequently[2], the only homotopy groups of $\widehat{G}_k$ which are not finite are those of dimensions $4i$ and k (i.e. the dimensions of the generators given above).

Let $t \in \pi_k(\widehat{G}_k)$ be a generator of the free component corresponding to the class[3] X_k (the arbitrariness choice of this component plays no role). For the corresponding mapping $t : S^k \to \widehat{G}_k$ we have $t^*(X_k) = N^0 s^k$, where N^0 is some non-zero integer.

It is clear that the conditions of Lemma II.24 hold. Indeed, if $q \not\equiv 0 \mod 4$ then the homology group $H^{q+1}(Z,k;\pi_q(\widehat{G}_k))$ is finite because so is the group $H^{q+1}(Z,k;Z)$[4].

Thus, for any $q \geqslant k$ one can define a mapping G_q from the q-frame of the cellular decomposition $K(Z,k)$ to the space $A_{SO(k)}$ and hence, to $M(SO(k))$. Consider the composite mapping

$$K^q \xrightarrow{G_q} \widehat{G}_k \xrightarrow{h} M(SO(k)).$$

Let U be the fundamental class of the space $M(SO(k))$. Then if k is even then $h^*(U) = X_k$ and, consequently, $G_q^* \circ h^*(U) = N\iota$, where the non-zero number N depends only on q and k. Thus, we have proved

Theorem II.25. *For any integral k-dimensional homology class x of some polyhedron of finite dimension q there exists a positive non-zero N, depending only on q and k for which the class Nx is realisable in the rotation group*[5].

REMARK. The arguments above are applicable not only to the real Grassmann manifold $\widehat{G}_k$ but also to the complex one, and even (for $k \equiv 0 \mod 4$) to

---

[1]From arguments on page 151 it follows that there exists a continuous mapping $f : \widehat{G}_k \to \to K(Z,4) \times K(Z,8) \times \ldots \times K(Z,k)$, for which $f^*(u_{4i}) = P^{4i}, i = 1,2,\ldots, f^*(u_k) = X$, where u_{2p} is the fundamental class of the polyhedron $K(Z,2p)$. Since, by [24], the real cohomology algebra of the polyhedron $K(Z,2p)$ is the polynomial algebra in u_{2p}, then f^* is an isomorphic in the case of real coefficients. Thus, by [22], f^* is an isomorphism over integers. — *Editor's remark*

[2]By generalised J. H. C. Whitehead theorem. — *Editor's remark*

[3]That is of the component taken to $\pi_k(K(Z,k))$ by the mapping f constructed in the previous remark — *Editor's remark*

[4]This follows from [24] that the group $H^p(Z,k;Z)$ is infinite if and only if p is divisible by k. — *Editor's remark*

[5]If A has dimension q, then for any $x \in H^k(A;Z)$ there exists such a mapping $f : A \to K^q$ that $x = f^*(\iota)$; see page 151. — *Editor's remark*

the classifying space of the symplectic group. Consequently, the analogous theorem holds not only for $SO(k)$ but also for the unitary (resp., symplectic) group. However, for these groups the coefficient N is much larger.

11. Summary of results

Below we formulate the results concerning our initial problem: to realise a given homology class by means of a submanifold. According to Theorems II.5 and II.5′, this problem is reduced to the question whether the corresponding cohomology class admits an orthogonal realisation. A partial answer to this question was given in §§ 7–10.

1. Modulo 2 classes. From Theorems II.13–II.15, we get

Theorem II.26. *For any differentiable manifold V, all elements of the following homology groups are realisable by submanifolds:*

$$H_{n-1}(V^n) \text{ for all } n;$$
$$H_{n-2}(V^n) \text{ for all } n < 6;$$
$$H_{n-3}(V^n) \text{ for all } n < 8;$$
$$H_i(V^n) \text{ for } i \leqslant n/2 \text{ and all } n.$$

Note that in the case of $H_{n-2}(V^n)$, the obstruction $(1/2)\delta p(u)$ (see Theorem II.14) necessarily vanishes on the fundamental class of any five-manifold V^5. Indeed, for an orientable manifold V^5 this is trivial and for a non-orientable one it follows from the fact that the fundamental class of the group $H^5(V^5, Z)$ is a Steenrod square Sq^1, thus, it is non-zero when considered modulo two. We cannot say anything about realisability of elements of the group $H_4(V^6)$. This is the simplest example of homology groups such that the realisation-by-submanifold question cannot be solved by using results obtained here.

2. Integral cohomology classes. From Theorems II.17–II.19 we get

Theorem II.27. *For any orientable manifold V^n, all elements of the following integral homology groups are realisable by orientable submanifolds: $H_{n-1}(V^n); H_{n-2}(V^n)$ for all n; $H_i(V^n)$ for $i \leqslant 5$ and all n.*

In the limit case $H_5(V^8; Z)$ the corresponding obstruction, which is the Steenrod cube $St_3^5(u)$, has order three thus, it vanishes on the fundamental class. Thus, we get

Corollary II.28. *For any orientable manifold of dimension* $\leqslant 8$ *all integral homology classes are realisable by submanifolds.*

Here the simplest case not covered by our theorems, is the group $H_6(V^9; Z)$ (see, however, remark 1).

Furthermore, note that, by Theorem II.17, in order for a 8-dimensional homology class of any manifold of dimension $\geqslant 17$ to be realisable by a submanifold, it is necessary and sufficient that the cube St_3^5 of the dual cohomology class vanishes.

Finally, from the "multiplication" Theorems II.4 and II.25, we get

Theorem II.29. *For any integral homology class* z *of some orientable manifold* V^n *there exists a non-zero* N *such that the class* Nz *is realisable by a submanifold.*

This theorem has an interesting corollary, that deals with homology group over the integers or rational numbers.

Corollary II.30. *The integral (rational) homology groups of any orientable manifold* V^n *have a basis consisting of elements realisable by submanifolds.*

REMARK. One should not think that any integral homology class of some manifold can be realised by a submanifold. In Chapter III, we give an example of a homology class of dimension 7 (in a manifold of dimension 14), which is not realisable by means of a submanifold. Moreover, it turns out that for each dimension $\geqslant 7$ there exist (in some manifold of arbitrarily large dimension) *non-realisable integral homology classes.*

I don't know whether there exist non-realisable homology classes of dimension 6[1].

The realisability of classes z and z' does not yield, in general, the realisability of $z + z'$. This is true, in general, when the dimensions of z and z' are strictly less than half the dimension of the given manifold. Conversely, the intersection of two realisable homology classes is realisable. It follows almost immediately from Theorem I.5.

Necessity of the differentiability assumption. All theory described here relies on method, where the ambient manifold and all submanifolds are endowed with a differentiable structure. However, for the realisation of classes modulo 2 one can show that some conditions of Theorem II.1 have an intrinsic topological meaning. For instance, let $F : M(O(k)) \to K(Z_2, k)$

[1] One may show that any integral homology class of dimension 6 is realizable. The corresponding obstruction defined by the homomorphism $St_3^5 : H^{n-6}(V^n; Z) \to H^{n-1}(V^n; Z)$, is identically zero. Analogously, one can improve the results of Theorems II.18 and II.19. In Corollary II.28, one can replace 8 with 9. Thus the simplest homology group for which the question is open is $H_7(V^{10}; Z)$.

be the canonical mapping for which $F^*(\iota) = U$ and let $c = T(\iota)$ be some element from $H^*(Z_2, k; Z_2)$ belonging to the kernel of F^* (here T is a certain sum of iterated Steenrod squares Sq^i). Clearly, the cohomology class $x \in H^k(V^n)$ corresponds to the class of some smoothly embedded manifold *if and only if* $T(x) = 0$. On the other hand, one may show that if $T(x)$ is not equal to zero modulo two then the homology classes corresponding to x, cannot be realised *even by a topologically embedded manifold*. Indeed, as I showed in [27], with any topologically embedded manifold one can associate generalised normal characteristic classes W^i, which have formal properties of the Stiefel-Whitney class of normal bundles of some smoothly embedded manifold. Moreover, for these classes the Wu formulas (3) (that can be proved by using relation given on page 203 for the iterated squares Sq^i). With any operation of type T increasing the dimension by i, we may associate a certain polynomial in W_j of total degree i. If the class $T(\iota)$ belongs to the kernel of F^* then this polynomial is identically zero. Consequently, $T(x)$ should be equal to zero, which contradicts the initial assumption. All these calculations can be performed explicitly for the operation T, defined as

$$T(\iota) = (Sq^2 Sq^1 \iota^2) \cdot \iota^2 + (Sq^1 \iota)^3 + Sq^1 \iota \cdot \iota^3,$$

where ι is the fundamental class of $K(Z_2, 2)$. This example was communicated to me by Serre.

Chapter III

On Steenrod's problem

1. Statement of the problem

Steenrod [12] has stated the following problem: define whether for a given homology class $z \in H_r(K)$ of some finite polyhedron K there exists a *compact* manifold M^r and a mapping $f : M^r \to K$ such that the class z is the image of the fundamental class of M^r under f_*. For solving this problem, we shall also require that M^r is *differentiable*. As we shall see, the answers to this question are quite different depending on the coefficient group (Z or Z_2). It turns out that Steenrod's problem is closely connected with the submanifold realisation problem considered in Chapter II.

2. Definition. Manifolds associated with a given finite polyhedron *K*

Let K be a finite m-dimensional polyhedron. It is known that K can be linearly embedded in a Euclidean space R^n of dimension $n \geqslant 2m + 1$. Let us now define, e.g., as a solution of Dirichlet's problem, an integral function f of class C^∞, which is equal to zero on K and is strictly positive on the complement $R^n \setminus K$. Since K is an absolute neighbourhood retract, then for some open neighbourhood U of K there exists a retraction mapping $r : U \to K$. Let c be a small *retraction value* of f such that the pre-image $f^{-1}(0,c)$ is contained in the neighbourhood U (such a c exists by Theorem I.1). This pre-image $M^n = f^{-1}(0,c)$ will be a neighbourhood of the polyhedron K, whose boundary (i.e. the pre-image $f^{-1}(0,c)$) is a differentiable manifold in R^n. It is evident that the polyhedron K is a retract of the neighbourhood M^n. The corresponding retraction mapping is a part of the mapping r.

Remark. If on the interval $[0,c]$ there is no critical value of f then the polyhedron K is a *deformation retract* of the neighbourhood M^n. The corresponding deformation $M^n \to K$ can be defined as a flow along the integral curves of the gradient function f. However, I don't know whether there always exists a function f, not having arbitrarily small critical values.

From the neighbourhood M^n with boundary $T^{n-1} = f^{-1}(c)$ one can get, by using the classical "doubling" construction some *compact* submanifold V^n. This manifold is obtained by gluing two isomorphic copies of the neighbourhood M^n along their common boundary T^{n-1}. Denote by $g : M^n \to V^n$ the inclusion mapping and by $h : V^n \to M^n$ the mapping obtained by identifying the two components M_1^n and M_2^n. The manifold V^n is called the *manifold associated* with the finite polyhedron K. It is clear that the polyhedron K is a retract of any associated manifold; thus, for any coefficient group the homomorphism $h^* \circ r^* : H^r(K) \to H^r(V^n)$ generated by $r \circ h : V^n \to K$ is a *monomorphism*. Indeed, the composition of the mappings $r \circ h$ and $g \circ i$, where i is the inclusion mapping for $K \to M^n$, is the identity.

Now, let us prove the following theorem that establishes the connection between the problems of this chapter and the problems of the previous chapter.

Theorem III.1. *In order for the homology class $z \in H_r(K)$ to be the image of a compact differentiable manifold, it is necessary and sufficient that for some large enough n, the image of the class z in the manifold V^n associated with the polyhedron K can be realised as a submanifold.*

The sufficiency is evident. Indeed, if z is realized by a compact submanifold W^r in V^n then this class is the image of the fundamental class of W^r under the homomorphism generated by the retraction $V^n \to K$.

The condition is necessary. Assume that z is the image of the fundamental class of some smooth manifold W^r under f. Consider

a) a regular embedding g of W^r to some R^n, *b*) a linear embedding i of K in some R^m.

Denote by Y the cylinder of the mapping f. For every point x of W^r, denote by (x,t) the point of Y dividing the segment $[f(x), x]$ with ratio t $(0 \leqslant t \leqslant 1)$.

Finally, let a be a real non-negative parameter. Define an embedding F_a of Y into the Euclidean space $R^{n+m+1} \approx R^n \times R^m \times R$ by setting

$$F_a(x,t) = (atg(x), (1-t)i \circ f(x), at),$$
$$F_a(y) = (0, i(y), at),\ y \in K.$$

Let M be a certain neighbourhood of the polyhedron described above embedded to R^{n+m+1} by means of F_0. By compactness argument, for some small value of the parameter a, the image $F_a(Y)$ for $a < c$ is contained in M. Then the image $F_a(W^r, 1)$ is a submanifold of the neighbourhood M and, consequently, it is a submanifold of the associated manifold V. The fundamental cycle of this submanifold belongs to the image $k(z)$ of z under the inclusion mapping $k : K \to V$, corresponding to F_a. Thus, Theorem III.1 is completely proved.

3. Applications. The case of modulo 2 coefficients

Whenever the class $k(z)$ is realisable by a submanifold of the associated manifold V^n, the Steenrod problem has a positive solution. The dimension n of the associated manifold can always be made greater than $2r$. Consequently, considering the case of modulo 2 coefficients and taking Theorem II.26, we get:

Theorem III.2. *Any modulo 2 homology class of any finite polyhedron is an image of the fundamental class of some compact differentiable manifold.*

For the case of integer coefficients, Theorem II.27 yields

Theorem III.3. *Any integral homology class of dimension* $\leqslant 5$ *of any finite polyhedron is the image of the fundamental class of some compact orientable manifold.*

From Theorem II.29 we get the following "multiplication theorem":

Theorem III.4. *For any integer p-dimensional homology class z of some finite polyhedron K there exists a non-zero integer N depending only on p such that the class Nz is the image of the fundamental class of some differentiable manifold.*

In order to get more exact results in the case of integer coefficients, one should introduce new operations on homology classes of K.

4. Operations ϑ_i^p

Let K be a finite polyhedron, topologically embedded into R^n. Consider the projective limit $H^*(U)$ of the finite support cohomology algebra for open neighbourhoods of K in R^n. According to the Poincaré duality law (see [27], Theorem III.4), for any coefficient group there exists an isomorphism χ from the group $H_r(K)$ to the group $H^{n-r}(U)$.

For any even i, define a homomorphism $\vartheta_i^p : H_r(K; Z_p) \to H_{r-i}(K; Z)$ by setting

$$\vartheta_i^p = \chi^{-1} St_p^i \chi,$$

where St_i^p is Steenrod's power[1] of index i.

The operations ϑ_i^p corresponding to St_p^i with odd index are defined by the formula

$$\vartheta_{2r+1}^p = \vartheta_1^p \circ \vartheta_{2r}^p,$$

where ϑ_1^p is the Bockstein homomorphism $(\frac{1}{p}\delta)$. (This definition allows us to avoid the signs $\pm$, which appear because the operator St_p^1 does not commute with the suspension.)

The following properties of the operators ϑ_i^p, proved in [27] for the case $p = 2$, can be easily extended for the case $p > 2$.

1) The operations ϑ_i^p are topologically invariant, i.e. they do not depend on the way of embedding of K into Euclidean space.

2) The operations ϑ_i^p commute with the homomorphisms f^* generated by continuous mappings $f : K \to K'$.

3) Over the field Z_p, the operations ϑ_i^p can be expressed via St_p^i. Let $Q_p^i : H^{r-i}(K, Z_p) \to H^r(K, Z_p)$ be the homomorphism which is dual to the homomorphism ϑ_i^p considered over Z_p. It turns out that the homomorphisms Q_p^i with odd indices i are connected with the operations St_p^i by the following formulae:

$$\sum_i Q_p^{m-i} St_p^i = 0 \quad m, i \equiv 0 \mod 2(p-1), \quad Q_p^0 = 1.$$

The proof of these formulae is quite analogous to the proof of formula (60) of Theorem II.3 of [27][2]. Homomorphisms Q_p^i with odd indices are obtained

[1] Here we assume that Steenrod's powers $St_p^{2k(p-1)}$ are endowed with normalising coefficients introduced by Serre [5]. (Serre denoted the operation $St_p^{2k(p-1)}$ by $\mathscr{P}_p^k$.) Odd index powers are obtained from the powers $\mathscr{P}_p^k$ with even index by using Bockstein's homomorphism $\dfrac{1}{p}\delta$.

[2] See Editor's remark on page 204 at the end of the article. — *Editor's remark*

from homomorphisms Q_p^i with even indices according to the following formula (which is dual to the formula defining ϑ_i^p):

$$Q_p^{2r+1} = Q_p^{2r} \circ Q_p^1,$$

where Q_p^1 is the Bockstein homomorphism $(1/p)\delta$, with the image reduced modulo p.

The relations 3) indeed allow us to express Q_p^i via the operations St_p^i. For instance,

$$Q_3^4 = -St_3^4, \quad Q_3^5 = -St_3^4 \circ Q_3^1 = St_3^4 St_3^1.$$

Now, let us return to the polyhedron K, embedded into the associated manifold V^n. Let z be an element of $H_r(K; Z)$. We shall denote the image u of the cohomology class $\chi(z)$ under the natural mapping from $H^{n-r}(U; Z)$ to $H^{n-r}(V^n)$, also by $\chi(z)$ (abusing notation). The class u is Poincaré-dual to the homology class $i_*(z) \in H_r(V^n)$. Since the homomorphism $i_* : H_r(K) \to H_r(V^n)$ is a monomorphism, so is $\chi : H_r(K) \to H^{n-r}(V^n)$. Besides, by definition,

$$St_p^i \chi(z) = \pm\chi(\vartheta_i^p(z)).$$

As we know, the homology class $i_*(z)$ is realisable in V^n by a submanifold if and only if all Steenrod powers St_p^i (i, p odd) of the corresponding cohomology class $\chi(z)$ are equal to zero (Theorem II.20). Thus, the following theorem holds.

Theorem III.5. *In order for an integral homology class z to be the image of the fundamental class of some compact differentiable manifold, it is necessary and sufficient that all homology classes $\vartheta_i^p(z)$ for odd p and i vanish.*

In dimensions $\leqslant 8$ this condition will be sufficient. Indeed, from Theorem II.17 we get

Theorem III.6. *In order for an integer homology class z of dimension $\leqslant 8$ of a finite polyhedron to be the image of the fundamental class of some compact differentiable orientabel manifold, it is necessary and sufficient that the integral homology class $\vartheta_5^3(z)$ vanish.*

For $r \leqslant 5$, this result is known from Theorem III.3. Consider the case $r = 6$, when the class $\vartheta_5^3(z)$ is a third order element of $H_1(K; Z)$. If this class is non-zero then for some integer m, divisible by three, there exists a cohomology class $u \in H^1(K; Z_m)$, whose scalar product with $\vartheta_5^3(z)$ is non-zero. Let f be the mapping from K to $K(Z_m, 1)$, such that $f^*(\iota) = u$, where ι is the fundamental class of $K(Z_m, 1)$. Consider the commutative diagram

$$\begin{array}{ccc} H_6(K;Z) & \longrightarrow & H_6(Z_m,1;Z) \\ {\scriptstyle\vartheta_5^3}\downarrow & & \downarrow{\scriptstyle\vartheta_5^3} \\ H_1(K;Z) & \xrightarrow{f_*} & H_1(Z_m,1;Z). \end{array}$$

According to well-known results [11] on homology groups of cyclic groups, the group $H_6(Z_m, 1; Z)$ is trivial. Thus, $f_*(\vartheta_5^3(z)) = 0$ and the scalar product of the elements $\vartheta_5^3(z)$ and u is equal to zero. Since this is true for any integer m, then the *integral* homology class $\vartheta_5^3(z)$ is equal to zero. Thus we have

Corollary III.7. *Every six-dimensional integral homology class of any finite polyhedron is an image of the fundamental class of a certain compact differentiable manifold.*

Let us show now that this result cannot be improved. Preliminarily, let us prove the following lemma on Eilenberg-MacLane polyhedra.

Lemma III.8. *For $r \geqslant 2$, the cohomology class $St_3^5 St_3^1(\iota)$ of $K(Z_3, r)$ is non-zero.*

First of all note that if the class $St_3^5 St_3^1(\iota)$ is non-zero in $K(Z_3, n)$ then it is non-zero in all other polyhedra $K(Z_3, m)$ for $m > n$, because, up to sign, Steenrod's powers commute with the suspension. Thus, it suffices to show that the class $St_3^5 St_3^1(\iota)$ is non-zero in $K(Z_3, 2)$. This is really true, but the direct proof is rather complicated. Thus, it would be convenient to replace the complex $K(Z_3, 2)$ with the product of two complexes $K(Z_3, 1)$. Let ν_1 and ν_2 be the fundamental cycles of the two complexes $K(Z_3, 1)$, and let $u_1 = St_3^1\nu_1$ and $u_2 = St_3^1\nu_2$ be the generators of the two-dimensional cohomology groups of these complexes over Z_3. Then

$$\begin{aligned} St_3^5 St_3^1(\nu_1 \cdot \nu_2) &= St_3^1 St_3^4(u_1 \cdot \nu_2 - \nu_1 \cdot u_2) \\ &= St_3^1((u_1)^3 \cdot \nu_2 - (u_2)^3 \cdot \nu_1) = (u_1)^3 \cdot u_2 - u_1 \cdot (u_2)^3 \neq 0. \end{aligned}$$

Thus, the lemma is proved[1].

Since the integral class

$$St_3^5 St_3^1(\iota) \in H^{r+6}(Z_3, r; Z), \quad r \geqslant 2,$$

is non-zero, then, by the duality argument, there exists a cohomology class $z \in H_{r+5}(Z_3, r; Z)$, whose scalar product with the class $St_3^4 St_3^1(\iota)$ is non-

[1] Indeed, consider a continuous mapping $f : K(Z_3, 1) \times K(Z_3, 1) \to K(Z_3, 2)$, such that $f^*(\iota) = \nu_1\nu_2$. Then $f^* St_3^5 St_3^1(\iota) = St_3^5 St_3^1(\nu_1\nu_2) \neq 0$, and hence, $St_3^5 St_3^1(\iota) \neq 0$. — *Editor's remark*

zero modulo[1] 3, i.e. $\langle z, Q_3^5(\iota)\rangle \neq 0$. Thus, $\langle \vartheta_5^3(z), \iota\rangle \neq 0$, so that $\vartheta_5^3(z) \neq 0$. Thus, we have proved the following

Theorem III.9. *In any dimension $r \geqslant 7$ in some finite polyhedron there exists an integral homology class that cannot be represented as the image of the fundamental class of a compact smooth differentiable orientable manifold.*

EXAMPLE. Let us realise complexes $K(Z_3, 1)$ by means of lens spaces. Here it is sufficient to consider the spaces L^7 of dimension 7, which are the quotient spaces of the sphere S^7 by the group Z_3 acting freely on this sphere. Let L_1 and L_2 be two copies of L. Denote by $\nu_1, \nu_2, u_1 = St_3^1\nu_1$ and by $u_2 = St_3^1\nu_2$ the generators of $H^1(L_1; Z_3)$, $H^1(L_2; Z_3)$, $H^2(L_1; Z_3)$ and $H^2(L_2; Z_3)$, respectively. Consider the product V^{14} of L_1 and L_2. Let

$$X = u_1 \cdot \nu_2 \cdot (u_2)^2 - \nu_1 \cdot (u_2)^3.$$

This is an integral class because $X = St_3^1(\nu_1 \cdot \nu_2 \cdot (u_2)^2)$.

Let $z \in H_7(V^{14}; Z)$ be the homology class which is Poincaré-dual to the class X. It turns out that the homology class $\vartheta_5^3(z)$ reduced modulo 3 is non-zero. Indeed, consider the scalar product

$$\langle \vartheta_5^3(z), \nu_1 \cdot \nu_2\rangle = \langle z, Q_3^5(\nu_1 \cdot \nu_2)\rangle = \langle z, St_3^4 St_3^1(\nu_1 \cdot \nu_2)\rangle \mod 3.$$

This scalar product is equal to the Kolmogorov-Alexander product

$$X \cdot St_3^4 St_3^1(\nu_1 \cdot \nu_2) = X \cdot ((u_1)^3 \cdot \nu_2 - \nu_1 \cdot (u_2)^3) = \nu_1 \cdot \nu_2(u_1 \cdot u_2)^3 \neq 0.$$

Consequently, $\vartheta_5^3(z)$ is non-zero, hence the homology class z is not the image of the fundamental class of any compact differentiable manifold. Thus, z cannot be represented in V^{14} by means of a submanifold. This fact can be checked directly, as well:

$$St_3^5 X = St_3^1((u_1)^3 \cdot \nu_2 \cdot (u_2)^2) = (u_1 \cdot u_2)^3 \neq 0.$$

One can show analogous examples of non-realisable (by submanifolds) seven-dimensional homology classes in manifolds of arbitrarily high dimensions.

5. Steenrod's powers in cohomology algebras of differentiable manifolds

Let V^n be a compact differentiable manifold and let (V^n) be its fundamental class. By Theorem III.5, all integral classes $\vartheta_i^p(V^n)$ for odd prime p and for $i \equiv 1 \mod 2(p-1)$ are equal to zero. Consequently, by the duality argument, we have:

[1]Here one should note that the class $St_3^5 St_3^1(\iota)$ is the image of the class $St_3^4 St_3^1(\iota)$ under Bockstein's homomorphism. — *Editor's remark*

Theorem III.10. *For every compact differentiable orientable manifold V^n the homomorphisms*

$$Q_p^i : H^{n-i}(V^n; Z_p) \to H^n(V^n; Z_p)$$

are trivial (p and i are odd).

For instance, the homomorphism

$$Q_3^5 = St_3^4 St_3^1 : H^{n-5}(V^n) \to H^n(V^n; Z_3)$$

is trivial.

These relations between Steenrod's powers can be obtained by applying Theorem II.20 to the *diagonal class* of $V^n \times V^n$ which is realisable by a submanifold. Note that the relations above take place not only in differentiable manifolds but also in arbitrary manifolds which are images of differentiable manifolds under mappings of degree 1. However, in this case they possibly cannot be obtained from Poincaré duality. There is an open question, whether these relations can be proved for any topological manifold without any differentiability assumption?

Chapter IV

Cobordant differentiable manifolds

Let V^n be an orientable compact manifold. One says that M^{n+1} is a *manifold with boundary* V^n if the following conditions hold:

a) the complement $M^{n+1} \setminus V^n$ is an $(n+1)$-dimensional open manifold;

b) for any point x of V^n there exist a neighbourhood U of this point in M^{n+1} and differentiable functions $x_0, x_1, \ldots, x_n$, defined in this neighbourhood such that

1) the functions $x_0, x_1, \ldots, x_n$ are local coordinates in M^{n+1}, i.e. they realise a homeomorphic mapping from U to the half-space of the space R^{n+1}, defined as $x_0 \geqslant 0$;
2) the functions $x_1, \ldots, x_n$ are local coordinates in V^n, i.e. they are differentiable in V^n, and they provide a homeomorphism from the intersection $U' \cap V^n$ to the hyperplane $x_0 = 0$.

If $M^{n+1} \setminus V^n$ is *orientable* then the boundary V^n is *orientable* as well and every orientation of M^{n+1} naturally generates an orientation for the boundary V^n. The correspondence for these orientations is defined by the boundary operator $\delta : H_{n+1}(M^{n+1}, V^n) \to H_n(V^n)$.

One says that an *orientable* compact manifold V^n is *null-cobordant* if there exists a compact orientable manifold M^{n+1} with boundar V^n, such that in M^{n+1} one can introduce an orientation inducing the given orientation of V^n. The present chapter is devoted to the solution of the following problem by Steenrod [12]: find necessary and sufficient conditions for a given manifold V^n to be null-cobordant. In [27], I indicated some conditions necessary for a manifold to be null-cobordant or null-cobordant modulo 2 (i.e. to be the boundary of some manifold without any orientability assumption). Generalising this problem, we can address the question about sufficient conditions.

Definition. *Cobordant manifolds.* Two *oriented* compact manifolds V and V' of the same dimension k are called *cobordant* (notation: $V \simeq V'$), if the manifold $V' \cup (-V)$ which is the disjoint union of V' and V, the latter taken with the opposite orientation, is *null-cobordant*.

If V and V' are cobordant to the same manifold V'' then they are cobordant. For the proof, it is sufficient to identify along V'' the boundaries of the manifolds defining the inner homology $V \simeq V''$ and $V' \simeq V''$. This yields that the set of all compact oriented manifolds of dimension k is divided into equivalence classes. We shall denote the class of a manifold V by $[V]$.

For these classes, let us define a commutative summation by setting $[V] + [V'] = [V \cup V']$. Denoting by $-V$ the manifold V with the opposite orientation, we have $[V] + [-V] = 0$, where by 0 we denote the class of null-cobordant manifolds. Indeed, the manifold $V \cup (-V)$ is the boundary of the product $V \times I$. Thus, the set of classes $[V]$ of k-dimensional manifolds is an abelian group, which we denote by Ω^k (the cobordism group in dimension k).

If a manifold V is cobordant to a manifold V' then it is easy to check that for any orientable compact manifold W the product $V \times W$ is cobordant to the product $V' \times W$. This yields that for the classes $[V]$ one can define a *multiplication*. This multiplication is anticommutative and distributive with respect to the summation. Thus, the direct sum Ω of the groups Ω^k is defined as a ring.

If we omit all orientation arguments, then we get *modulo 2 cobordant manifolds*; we shall denote modulo 2 cobordism classes by $[V]_2$. Denote the group of modulo 2 cobordisms in dimension k by $\mathfrak{N}^k$; denote by $\mathfrak{N}$ the ring of cobordism classes modulo 2. It is clear that in the ring $\mathfrak{N}$, each element has order 2.

1. Invariants of cobordism classes

It is evident that any condition that some manifold is null-cobordant can be reformulated as a condition that two manifolds are cobordant. Thus, Theorem V.11 of [27], saying that for a null-cobordant oriented manifold V^{4k} the signature τ of the quadratic form defined by means of the cohomology on the space $H^{2k}(V^{4k})$ is equal to zero, gives the following theorem:

Theorem IV.1. *For two oriented cobordant manifolds V and V' of dimension $4k$ the quandratic forms defined by using the cohomology on $H^{2k}(V)$ and $H^{2k}(V')$, respectively, have the same signature τ.*

(Recall that the signature of a quadratic form is the difference between the number of positive squares and the number of negative squares in the canonical representation of the quadratic form over the field of real or rational numbers.)

It is easy to see that the invariant τ of cobordism classes, whose dimensions are divisible by 4, is additive and multiplicative. Thus, it defines a homomorphism from the ring Ω to the ring Z of integers.

Furthermore, Pontrjagin's theorem [18], cited in [27], which says that characteristic classes of null-cobordant manifolds are equal to zero, yields the following Theorem:

Theorem IV.2. *For cobordant oriented manifolds V and V' of dimension $4k$, Pontrjagin's characteristic classes $\Pi(P^{4i})$ coincide.*

These invariants are additive, and the characteristic number corresponding to the class P^{4k} of maximal dimension, is moreover, multiplicative. (This follows from the fact that Pontrjagin's classes are defined according to the tensor law for sphere-fibre spaces, which are products of two given fibre spaces.)

For Stiefel-Whitney classes, the following theorem holds

Theorem IV.3. *Two modulo two cobordant manifolds V and V' of the same dimension k have equal Stiefel-Whitney characteristic classes.*

Here the invariants are also additive, and the only multiplicative number is the number corresponding to the class of maximal dimension. The latter invariant coincides with the Euler-Poincaré characteristic taken modulo two.

2. Differentiable mappings of manifolds with boundary

Let X^{n+1} be a compact manifold with boundary V^n and let f be an arbitrary differentiable mapping from the manifold X^{n+1} to some manifold M^p, containing a compact submanifold N^{p-q}. The mapping f is *t-regular* on the submanifold N^{p-q}, if so are (in sense of I.3) the restriction of f to the interior $X^{n+1} \setminus V^n$ and to the boundary V^n.

The pre-image of a *t*-regular mapping. According to general properties of t-regular mappings given in I.4, the intersection of V^n with the pre-image $A^{n+1-q} = f^{-1}(N^{p-q})$ is a submanifold C^{n-q} of V^n. Analogously, the intersection of the pre-image A^{n+1-q} with the interior $X^{n+1} \setminus V^n$ is a submanifold $A^{n+1-q} \setminus C^{n-q}$ of dimension $n+1-q$. Let us show that A^{n+1-q} is a *manifold with boundary* C^{n-q}. Let x be an arbitrary point of the manifold C^{n-q}, let $y = f(x)$ be its image in N^{p-q} and let $y_1, y_2, \ldots, y_q$ be some local coordinates defined in the q-ball which is geodesically normal to N^{p-q} at y. Furthermore, let $(x_1, x_2, \ldots, x_n, t)$ be a local coordinate system defined in a certain neighbourhood of x, for which the last coordinate t takes only positive values and $t = 0$ is the equation of the boundary V^n; then the t-regularity of f means that on V^n the mapping $(x_1, x_2, \ldots, x_n, 0) \to (y_1, y_2, \ldots, y_q)$ has rank q in x. In other words, there exists a Jacobian $|\partial y_r / \partial x_i|$ of order q, which is non-zero for $x_i = 0$ and $t = 0$. By continuity, this Jacobian is non-zero for x_i and t small enough. Thus, the variables

$$(y_1, y_2, \ldots, y_q, x_{q+1}, \ldots, x_n, t)$$

are local coordinates in some neighbourhood of x. In this neighbourhood, the pre-image A^{n+1-q} is defined by the linear equations $y_1 = y_2 = \ldots y_q = 0$, and the submanifold C^{n-q} is defined by the same equations and the equation $t = 0$. Thus, for the point x there is a neighbourhood of A^{n+1-q} which is homeomorphic to the half-space of the space R^{n+1-q} (with coordinates $x_{q+1}, \ldots, x_n, t$), restricted by the space R^{n-q} (with coordinates $x_{q+1}, \ldots, x_n$), which is the image of the manifold C^{n-q}. The statement is proved.

Definition. *Induced orientation of the submanifold.* Let f be a mapping from an *orientable* manifold V^n to a manifold M^p, which is t-regular on a submanifold N^{p-q}, and let C^{n-q} be the pre-image of N^{p-q}. Assume that the normal fibred neighbourhood of N^{p-q} in M^p is orientable. Then for the normal tubular neighbourhood of N one can define the "fundamental" class $U = \varphi^*(\omega) \in H^q(T; Z)$. The tubular neighbourhood of the submanifold C^{n-q} is then oriented as well. The corresponding "fundamental" class U will be the image of the class U of the neighbourhood T under f^*. We say that the manifold C^{n-q} is endowed with an *orientation induced by the orientation of the manifold* V^n, if its fundamental cycle (C^{n-q}) is defined in the normal tubular neighbourhood of the submanifold C^{n-q} by the formula[1]

[1]By $\frown$ we denote the Whitney product. — *Editor's remark*

$$(C^{n-q}) = (V^n) \frown U,$$

where (V^n) is the fundamental n-dimensional closed-support homology class of the tubular neighbourhood of C^{n-q} that induces the given orientation of the manifold V^n.

Let f, as above, be some mapping from an *oriented* compact manifold X^{n+1} with boundary V^n to a manifold M^p, so that f is t-regular on some submanifold N^{p-q}. Furthermore, assume the normal bundle of N^{p-q} is *orientable.* Then the pre-images $A^{n+1-q} = f^{-1}(N^{p-q})$ and $C^n = A^{n+1-q} \cap V^n$ are *orientable* as well; this can be shown, for instance, by using the Whitney duality theorem [32]. Let us endow V^n with the orientation induced by the orientation of X^{n+1}. Thus, $(V^n) = \partial(X^{n+1}, V^n)$, where ∂ is the boundary operation. Under these assumptions, the orientation of the submanifold C^{n-q}, induced by its embedding in V^n, coincides with the orientation C^{n-q} considered as the boundary A^{n+1-q}, the latter being endowed with the orientation induced by X^{n+1}. Indeed, $V^n \frown U = \partial(X^{n+1}) \frown U = \partial(X^{n+1} \frown U)$. Now, let us prove the following:

Theorem IV.4. *Let f and g be two mappings of class C^m, $m \geqslant n$ of an oriented compact manifold V^n to a manifold M^p, which are t-regular on $N^{p-q} \subset M^p$, the latter having orientable normal neighourhood. Let $W^{n-q} = f^{-1}(N^{p-q})$, $W'^{(n-q)} = g^{-1}(N^{p-q})$ be the pre-images of the submanifolds N^{p-q}, which are, as is easy to see, orientable manifolds. Let us endow them with the orientation induced by V^n. If f is homotopic to g then the manifolds W^{n-q} and W'^{n-q} are cobordant.*

Omitting the orientability assumption in the condition of the theorem, we get that the *pre-images $f^{-1}(N^{p-q})$, $g^{-1}(N^{p-q})$ are cobordant modulo 2.*

First, let us prove the following theorem. The idea of the proof was communicated to me by Whitney.

Lemma IV.5. *If two mappings f and g of class C^m from a manifold V^n to a manfiold M^p are homotopic, then they can be connected by a C^m-class deformation.*

Let $F : V \times I \to M^p$ be a homotopy of class C^m, connecting f to g. Replace it with the homotopy $G : V \times I \to M^p$, defined as follows: $G(V,t) = f = F(V,0)$ if $0 \leqslant t \leqslant \frac{1}{4}$; $G(V,t) = F(V, \dfrac{4t-1}{2})$ if $\frac{1}{4} \leqslant t \leqslant \frac{3}{4}$; $G(V,t) = g$ if $\frac{3}{4} \leqslant t \leqslant 1$. Endow the product $V \times I$ with a Riemannian metric, which is the product of some metric of V^n and the Euclidean metric of the interval I. Then we smooth the mapping G, by replacing G by its average over the geodesic balls of radius r. We let r be a constant less than

$\frac{1}{8}$ for $\frac{1}{8} \leqslant t \leqslant \frac{7}{8}$; for the boundary fibres $t < \frac{1}{8}$ and $\frac{7}{8} < t$ we let r be an increasing function of class C^∞ of variable t (resp., $(1-t)$), which is equal to zero for $t = 0$ and for $t = 1$. Thus, in the fibre $\frac{1}{8} \leqslant t \leqslant \frac{7}{8}$ the differentiability class of G is increased by one, and it is not decreased in the fibres $0 \leqslant t \leqslant \frac{1}{8}$ and $\frac{7}{8} \leqslant t \leqslant 1$. Furthermore, on $(V, 0)$ and on $(V, 1)$ the smoothed mapping G coincides with f and g, respectively. Reiterating this construction several times, we get a mapping of class C^m from $V \times I$ to M^p, that coincides on $(V, 0)$ and $(V, 1)$ with f and g, respectively. The lemma is proved.

The constructed mapping $F : V \times I \to M^p$ of class C^m might not be t-regular on N^{p-q}. But the set $\boldsymbol{H}_0$ of such homeomorphisms $h \in \boldsymbol{H}$ of the tubular neighbourhood T of N^{p-q} for which the mapping $h \circ F$ restricted to the interior of $V \times I$, is not t-regular on N^{p-q}, is a *thin* subset of H. An analogous statement holds for homomorphisms h for which the mapping $h \circ F$, considered over $(V, 0) \cup (V, 1)$, is not t-regular on N^{p-q}. Thus we see that Theorems I.5 and I.6 hold for mappings of manifolds with boundary. For F, considered on $(V, 0) \cup (V, 1)$, choose a homomorphism h close enough to the identity and satisfying Theorem I.6. Let $F' = h \circ F$ and let f', g' be the parts of F' restricted to $(V, 0)$ and $(V, 1)$, respectively, By Theorem I.6, the manifolds $C^{n-q} = f'^{-1}(N^{p-q})$, $C'^{(n-q)} = g'^{-1}(N^{p-q})$ are isotopic to the manifolds $W^{n-q} = f^{-1}(N^{p-q})$ and $W'^{(n-q)} = g^{-1}(N^{p-q})$, respectively. This isotopy preserves the induced orientations, so that the oriented manifolds C^{n-q} and $C'^{(n-q)}$ together form the boundary of the manifold $A = F'^{-1}(N^{p-q})$. Thus, the manifolds C^{n-q} and $C'^{(n-q)}$ are cobordant, hence, so are W^{n-q} and $W'^{(n-q)}$. Theorem IV.4 is proved completely.

3. *L*-equivalent manifold

In II.2, we associate with each oriented submanifold W^{n-k} of orientable manifold V^n some mapping $f : V^n \to M(SO(k))$. Here we make this dependence between submanifolds and mappings more precise.

Assume a manifold V^n is immersed to the space R^{n+m}. For any point x of the submanifold W^{n-k}, denote by $H(x)$ the k-plane tangent to V^n at x and normal to the submanifold W^{n-k} (in any Riemannian metric). Associate with the plane $H(x)$ the plane of R^{n+m} parallel to it and passing through the origin O. Thus we get some mapping

$$g : W^{n-k} \to \hat{G}_k,$$

where by $\hat{G}_k$, we denote, as above, the Grassmann manifolds of oriented k-planes. Let N be an arbitrary tubular neighbourhood of the submanifold

W^{n-k} in V^n. Associate with any geodesic normal passing through $x \in W^{n-k}$, its tangent vector at x, we get, after a parallel transport, a mapping

$$F : N \to A_{SO(k)},$$

for which the diagram

$$\begin{array}{ccc} N & \xrightarrow{F} & A_{SO(k)} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ W^{n-k} & \xrightarrow{g} & \widehat{G}_k \end{array}$$

is commutative (here p and p' are canonical k-ball bundles).

As in II.2, we may extend F to a mapping

$$f : V^n \to M(SO(k)).$$

If we replace the initial embedding of V^n into R^{n+m} by another embedding or replace the metric by another metric, then instead of f we get some *homotopic* mapping. Indeed, since any two Riemannian metrics of V^n can be continuously deformed to each other, then the corresponding tubular neighbourhoods N and N' are isotopic, hence, so are the mappings $F : N \to A_{SO(k)}$, and, finally, so are the mappings $f : V^n \to M(SO(k))$. To prove the independence of homotopy type of f on the embedding of V^n into R^{n+m}, we shall need one more lemma, that we shall use several times in the sequel.

Let Q^{n+1} be a manifold with boundary V^n and let X^{k+1} be a submanifold with boundary W^k, the latter contained in V^n. Assume that at any point $x \in W^k$ the half-space R^{k+1} tangent to X^{k+1} is transverse to the boundary V^n in the sense that the intersection of this half-space with the space tangent to V^n coincides with the space tangent to W^k. Finally, assume that Q^{n+1} is endowed with a Riemannian metric. This metric allows us to consider the normal neighbourhood of the boundary V^n in Q^{n+1} as the product $V^n \times I$, where the rays $(x, t), t \in I$, are geodesic normals of the boundary V^n. Then the following lemma holds:

Lemma IV.5′. *There exists a homeomorphism Φ of Q^{n+1} onto itself taking the submanifold X^{k+1} to a submanifold which is orthogonal to the boundary V^n.*

To prove this, let us consider the homeomorphism Φ of the manifold Q^{n+1}, which is the identity outside $V^n \times I$ and represents on $V^n \times I$ the motion along the normals defined by the function $t' = \varphi(t)$, for which $0 = \varphi(0),\ 1 = \varphi(1)$; $d\phi'/dt = +\infty$ for $t = 0$ and $dt'/dt = 1$ for $t = 1$.

It is easy to check that any vector tangent at x to the manifold $\Phi(X^{k+1})$ is tangent to the cylinder $W^k \times X$ and orthogonal to the boundary V^n. The lemma is proved.

Now, let us return to the manifold V^n, immersed to the space R^{n+m} by means of two different immersions i_0 and i_1. If $m > n+2$, then we may assume the images $i_0(V^n)$ and $i_1(V^n)$ don't intersect. By a theorem of Whitney, one may (possibly, after a small translation of i_1) find such an embedding i of $V^n \times I$ to R^{n+m}, whose restriction to $(V, 0)$ and $(V, 1)$ coincides with i_0 and i_1, respectively. The embedded manifold $i(V^n \times I)$ contains a submanifold of the type $W^{n-k} \times I$, that intersects transversely the boundary of the manifold $V^n \times I$.

Given a Riemannian metric in $V^n \times I$. According to the lemma above, we may assume the manifold $V^{n-k} \times I$ orthogonal to the boundaries $(V^n, 0)$ and $(V^n, 1)$ of the product $V^n \times I$. Let N be the normal tubular neighbourhood of the submanifold $W^{n-k} \times I$ in the product $V^n \times I$. Then the intersections $N_0 = N \cap i_0(V^n)$ and $N_1 = N \cap i_1(V^n)$ are normal tubular neighbourhoods of the submanifolds $i_0(W^{n-k})$ and $i_1(W^{n-k})$ in $i_0(V^n)$ and $i_1(V^n)$, respectively. Let us construct, by using parallel transport, the canonical mapping

$$F : N \to A_{SO(k)}.$$

Extending it to the whole product, we get a mapping

$$F : V^n \times I \to M(SO(k)),$$

whose restriction to $(V^n, 0)$ and $(V^n, 1)$ evidently coincides with the canonical mappings f_0 and f_1 corresponding to the embeddings i_0 and i_1 of V_n. Thus, f_0 and f_1 are indeed homotopic.

Definition. *L-equivalent submanifolds.* Let W_0^{n-k} and W_1^{n-k} be two oriented submanifolds of the same dimension $n-k$, embedded into an orientable manifold V^n. *The submanifolds W_0^{n-k} and W_1^{n-k} are L-equivalent if there exists an oriented manifold X^{n-k+1} with boundary $W_0^{n-k} \cup W_1^{n-k}$, which is embedded into $V^n \times I$ in such a way that*

$$X^{n-k+1} \cap (V^n, 0) = W_0^{n-k},$$
$$X^{n-k+1} \cap (V^n, 1) = W_1^{n-k},$$

provided that this manifold admits such an orientation that $\partial X^{n-k+1} = W_1^{n-k} \cup (-W_0^{n-k})$.

From the above and from Lemma IV.5′ we immediately get that if two submanifolds are L-equivalent to a third manifold, then they are L-equivalent. The set of L-equivalent manifolds of dimension $n-k$ of V^n is thus split into L-equivalence classes. Denote by $L_{n-k}(V^n)$ the set of such classes.

If in previous definitions we omit all orientability assumptions, we shall define modulo 2 L-equivalent submanifolds and the set $L_{n-k}(V^n; Z_2)$ of modulo 2 L-equivalence classes.

It is clear that two L-equivalent submanifolds are *homologous and cobordant* to each other. If two submanifolds W_0 and W_1 form in V^n the boundary of some submanifold X, then these submanifolds are L-equivalent.

Consider the natural mapping from the set $L_{n-k}(V^n)$ to the homology group $H_{n-k}(V^n; Z)$. The image set $L_{n-k}(V^n)$ under this mapping is the set of homology classes realisable by submanifolds. The "kernel" of this mapping is, in general, non-trivial, we shall see this later. It is natural to address the question whether the set $L_{n-k}(V^n)$ can be represented as a group by introducing an operation compatible with the mapping above. It turns out that it is possible if $n-k < n/2-1$. In this case the summation operation of L-equivalence classes is generated by the usual disjoint union of representatives of these classes. Indeed, these representatives for $n-k < n/2$ can be thought of as *disjoint*, and for $n-k < n/2-1$ the L-class defined in this way does not depend on the way of embedding of these manifolds. On the other hand, the sum $[W]+[-W]$ is the zero class since it is always possible to embed (locally) the product $V \times I$ into a normal tubular neighbourhood of W.

According to the above argument, with each submanifold W^{n-k} of V^n, one can associate a certain class of mappings from V^n to $M(SO(k))$. It can be easily checked that two *L-equivalent* submanifolds W and W' generate two *homotopic* mappings $f : V^n \to M(SO(k))$.

Indeed, if submanifolds W_0 and W_1 are L-equivalent then there exists a manifold X embedded into $V^n \times I$, with boundary being the union of W_0 embedded into $(V^n, 0)$ and W_1, embedded into $(V^n, 1)$. By Lemma IV.5′, we may assume that X is orthogonal to $(V^n, 0)$ and to $(V^n, 1)$. Consider the normal tubular neighbourhood Q of X in $V^n \times I$ and the corresponding mapping $F : Q \to A_{SO(k)}$. The mapping F can be extended to $F_1 : V^n \times I \to M(SO(k))$, the latter being the above homotopy between the canonical mappings

$$F|_{(V^n,0)} = f_0, \quad F|_{(V^n,1)} = f_1,$$

associated with the manifolds W_0 and W_1, respectively.

This defines a mapping J from the set $L_{n-k}(V^n)$ of L-equivalence classes to the set $C^k(V)$ of homotopy classes of mappings $f : V^n \to M(SO(k))$. The mapping J is *bijective.* Indeed, if two manifolds W_0 and W_1 generate homotopic mappings, then, by Theorem IV.4, the homotopy $F : V^n \times I \to M(SO(k))$, connecting these mappings, can be thought of as smooth. Furthermore, after a possible isotopy (which is also an L-equivalence), we may assume that the pre-images $f_0^{-1}(\widehat{G}_k) = W_0$ and $f_1^{-1}(\widehat{G}_k) = W_1$ form the boundary $A = F^{-1}(\widehat{G}_k)$.

Note that J takes the class of manifolds L-equivalent to zero to the zero class of inessential mappings $f : V^n \to M(SO(k))$. If $k > (n/2) + 1$ then, according to the general cohomotopy group theory[1, 2], the set $C^k(V)$ of homotopy classes of mappings of V to the space $M(SO(k))$ (which is aspherical up to dimension k) can be considered as an abelian group. It is easy to check that J is then a *homomorphism*. It is sufficient to check that, by definition of the sum $f+g$ of two mappings, the pre-image $(f+g)^{-1}(\widehat{G}_k)$ is, up to L-equivalence, the union of the preimages $f^{-1}(\widehat{G}_k)$ and $g^{-1}(\widehat{G}_k)$.

Now, let us show that J takes the set $L_{n-k}(V^n)$ to the set $C^k(V^n)$.

Let c be some homotopy class belonging to $C^k(V)$ and let h be an arbitrary mapping of class c. By Theorem I.5, one may assume that the mapping h is t-regular on the Grassmann manifold $\widehat{G}_k$, which is embedded in $M(SO(k))$. Let $W^{n-k} = h^{-1(\widehat{G}_k)}$ be the pre-image of $\widehat{G}_k$ and let N be the normal tubular neighbourhood of the manifold W^{n-k} in the manifold V^n. One may assume that the mapping h is *normalised* in such a way that it maps the interior of the neighbourhood N to $M(SO(k)) \setminus a$, takes open k-balls to open k-balls, and takes the complement $Q = V^n \setminus N$ to the point a. Now, denote by $i : V^n \to R^p$ an arbitrary embedding of the manifold V^n to the space R^p, and denote by

$$g : W^{n-k} \to \widehat{G}_k, \quad F : N \to A_{SO(k)}, \quad f : V^n \to M(SO(k))$$

the mappings which are naturally defined by this embedding by parallel transport. It is clear that the mappings

$$h : W^{n-k} \to \widehat{G}_k \text{ (i.e. the mapping } h/W^{n-k})$$

and

$$g : W^{n-k} \to \widehat{G}_k$$

both generate the normal bundle of W^{n-k} in V^n. Consequently, by the fibre space classification theorem, these mappings are homotopic. Let us

[1] In the case of spheres cohomotopy groups were studied by Spanier (Ann. Math., **50** (1949), 203–245). Their generalisations to the case of arbitrary aspherical spaces form the content of the unpublished paper by Steenrod, Spanier and J. H. C. Whitehead. The proof sketched below can easily be completely restored for the typical case of spheres. Its generalisation for the general case gives no new difficulties.

[2] See remark on page 205 at the end of the article. — *Editor's remark*

consider again the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{h} & A_{SO(k)} \\ \downarrow & & \downarrow \\ W^{n-k} & \xrightarrow{h} & \widehat{G}_k. \end{array}$$

By the covering homotopy theorem, there exists a mapping h_1 homotopic to h such that

$$\begin{array}{ccc} N & \xrightarrow{h_1} & A_{SO(k)} \\ \downarrow & & \downarrow \\ W^{n-k} & \xrightarrow{g} & \widehat{G}_k. \end{array}$$

Thus, the new mapping h_1 differs from F by an isomorphism α of the tubular neighbourhod N. In other words, in the neighbourhood N we have $h_1 = F \circ \alpha$.

It turns out that one can construct (increasing the dimension of the ambient Euclidean space, if necessary) a new embedding i' of V^n, for which in a neighbourhood N we have

$$i' = \alpha \circ i.$$

Indeed, let us prove the following lemma:

Lemma. *Let Q be a manifold with boundary T. Assume that there is an embedding i of some neighbourhood of the boundary T (of type $T \times I$) to R^p. Then this embedding can be extended to an embedding of the whole manifold Q to the space R^{p+q} with some large q.*

Indeed, let $y_1, y_2, \dots, y_p$ be the coordinates in R^p. In some neighbourhood U of $T \times I$, the coordinates y_i, extended to the whole of Q, are distinct for distinct points. Now let $(x_1, x_2, \dots, x_q)$ be some functions which are equal to zero on $T \times I$ and do not take the same values for distinct points on the complement $Q \setminus U$. (Such functions always exist if q is greater than $2n+1$, where n is the dimension of the manifold Q.) The system of functions y_i, x_j defines the desired embedding of Q to R^{p+q}.

Let us apply this lemma to the complement $Q = V^n \setminus N$. The embedding of the boundary T of N and the neighbourhood of type $T \times I$ is given by $i' = \alpha \circ i$. It is evident that the mapping $F' : N \to A_{SO(k)}$ corresponding to the embedding i' can be identified with h_1. Consequently, the mapping

$$f_1 : V^n \to M(SO(k))$$

corresponding to h_1 can be identified with the natural mapping corresponding to the embedding i'. On the other hand, since the mapping

$h_1 : N \to A_{SO(k)}$ is homotopic to the restriction of h to N, then the "complete" mapping f_1 is homotopic to h. (Recall that the mapping h is "normalised", hence it takes the manifold Q to the "special point" a from $M(SO(k))$.) Thus, we have proved

Theorem IV.6. *The set $L_{n-k}(V^n)$ of L-equivalence classes of some manifold V^n can be identified with the set $C^k(V)$ of mapping classes from V^n to $M(SO(k))$. If $k > (n/2) + 1$ then this identification preserves the group operations defined for L_{n-k} and for $C^k(V)$.*

In an analogous theorem for the set $L_{n-k}(V^n; Z_2)$, the polyhedron $M(SO(k))$ should be replaced with $M(O(k))$.

Applications. The maximal number of L-classes contained in some homology class z corresponding to the class $u \in H^k(V^n; Z)$ is equal to the number of mapping homotopy classes of f from V^n to $M(SO(k))$ for which[1] $f^*(U) = u$. Since the polyhedra $M(SO(1))$ and $M(SO(2))$ can be identified with the polyhedra $K(Z, 1)$ and $K(Z, 2)$, then we see that *homologous oriented submanifolds of dimension $n-1$ in an orientable manifold of dimension are always L-equivalent; the same is true about manifolds of dimension $n-2$.*

Corollary. *All null-homologous $(n-2)$-dimensional manifolds are null-cobordant* (for $n-1$ this is trivial).

For modulo two homology the analogous statement holds for $(n-1)$-dimensional submanifolds.

Finally, from Chapter II, we know (Theorem II.16), the second non-zero homotopy group of the space $M(SO(k))$ appears in dimension $k+4$. This yields that two mappings from a manifold V^n to $M(SO(k))$ which are homotopic on the k-skeleton of the manifold V^n, are also homotopic on the $(k+3)$-skeleton. Thus, *oriented homologous manifolds of dimension $\leqslant 3$ are always L-equivalent.*

4. The basic theorem

Let us apply the previous theorem to the case when V^n is the sphere S^n.

Lemma IV.7. *If $n > 2k+2$ then the group $L_k(S^n)$ of L-equivalence classes of the sphere S^n can be identified with the cobordism group Ω^k.*

Consider the natural mapping of the group $L_k(S^n)$ to the group Ω^k, taking each representative of some L-class to its cobordism class. Clearly, this mapping is a homeomorphism, since the summation is well defined in both groups as a union of representatives. Thus, homomorphism takes the

[1]In the original work, this statement is wrongly formulated. — *Editor's remark*

group $L_k(S^n)$ onto the group Ω^k. Indeed, let c be some element of the group Ω^k and let W^k be some manifold of class c. Since $n \geqslant 2k+2$, the manifold W^k can be immersed in R^n, hence, in S^n. Thus c is the image of some L-class of S^n. Thus, in order to prove the lemma, it remains to show that the kernel of the homomorphism $L_k(S^n) \to \Omega^k$ is trivial. In other words, we have to show that two manifolds W^k and W'^k, immersed in S^n, are L-equivalent if they are cobordant.

Let X^{k+1} be a manifold with boundary $W'^k \setminus W^k$. Since $n > 2k+2$ then the manifold X^{k+1} is embeddable into R^n.

On X^{k+1}, define a function t (of class C^∞), that takes values from 0 to 1, and such that the equations $t=0$ and $t=1$ define the submanifolds W^k and W'^k, respectively. Completing the spaces (R^n, t) and (S^n, t) by the "infinite point", we get an embedding of the manifold X^{k+1} to the product $S^n \times I$. This embedding defines the desired L-equivalence. Finally, note that for $n \geqslant 2k+2$ two arbitrary embeddings of W^k to S^n are always L-equivalent. This completes the proof that the correspondence between the groups $L_k(V^n)$ and Ω^k is an isomorphism.

Now, we are ready to formulate the main theorem of the present chapter.

Theorem IV.8. *The group Ω^k of cobordisms and the group $\mathfrak{N}^k$ of cobordisms are isomorphic to the stable homotopy groups $\pi_{n+k}(M(SO(n)))$ and $\pi_{n+k}(M(O(n)))$, respectively.*

To prove it suffices to apply Theorem IV.6 for the case when the manifold V^n is S^n, and use the isomorphism $L_k(S^n) \simeq \Omega^k$, indicated in Lemma IV.7. Furthermore, it is necessary to use a classical theorem of the cohomotopy group theory, saying that the mapping class cohomotopy group of

$$f : S^{n+k} \to M(SO(n))$$

is isomorphic to the homotopy group $\pi_{n+k}(M(SO(n)))$[1].

5. Modulo 2 class groups $\mathfrak{N}^k$

In Chapter II, stable homotopy groups $\pi_{n+k}(M(O(n)))$ were defined. As we know (Theorem II.10), in dimensions $< 2n$, the space $M(O(n))$ has the same homotopy type as the product Y of the following Eilienberg-MacLane polyhedra:

$$Y = K(Z_2, n) \times K(Z_2, n+2) \times \ldots \times (K(Z_2, n+h))^{d(h)} \times \ldots, \quad h \leqslant n,$$

[1]Indeed, both groups consist of the same set of elements. To prove that the operations agree it is sufficient to note that (both in homotopy group theory and in cohomotopy group theory) the sum of $f, g : S^{n+k} \to M(SO(n))$ can be defined as the composite mapping $S^{n+k} \to S^{n+k} \vee S^{n+k} \to M(SO(n))$, where the first mapping has degree $+1$ on each sphere of the wedge $S^{n+k} \vee S^{n+k}$, and the second mapping coincides with f on one sphere of the wedge, and with g on the other sphere. — *Editor's remark*

where $d(h)$ is the number of *non-dyadic decompositions* h, i.e. of such decompositions *not containing integers of type* $2^m - 1$. Consequently, we get

Theorem IV.9. *For any dimension k, the group $\mathfrak{N}^k$ is the direct sum of $d(k)$ groups each isomorphic to Z_2 where $d(k)$ is the number of non-dyadic decompositions of k.*

Thus, we have defined the additive structure of the group $\mathfrak{N}^k$.

From Theorem II.10, it follows that for $n > k$ any homologcially trivial (modulo 2) mapping from S^{n+k} to $M(O(n))$ is homotopic to the trivial mapping. This result can be justified according to the argument below.

For any non-dyadic decomposition ω of k, consider the mapping

$$F_\omega : M(O(n)) \to K(Z_2, k+n),$$

for which $F_\omega^*(\iota) = X_\omega$, where X_ω is the element of $H^{k+n}(M(O(n)))$ corresponding to the symmetric function

$$\sum (t_1)^{a_1+1}(t_2)^{a_2+1} \dots (t_r)^{a_r+1} t_{r+1} \dots t_n$$

((a_i) is the given non-dyadic decomposition ω of the number k). Let

$$Y_\omega = \sum (t_1)^{a_1}(t_2)^{a_2} \dots (t_r)^{a_r}$$

be the corresponding element of the cohomology group $H^k(G_k, Z_2)$. Then, in the notation of II.2, we have: $X_\omega = \varphi_G^*(Y_\omega)$.

Let f_{ω^1} be mappings $S^{n+k} \to M(O(n))$ such that

$$f_{\omega^1}^* F_\omega^*(\iota) = \delta_{\omega^1}^\omega(s), \tag{1}$$

where s is the fundamental class of the group $H^{k+n}(S^{k+n}, Z_2)$, and $\delta_{\omega^1}^\omega$ is the Kronecker symbol in its classical interpretation, but with decompositions ω taken instead of numerical indices. Homotopy classes of mappings f_{ω^1}, clearly, form a basis of the group $\pi_{n+k}(M(O(n)))$.

The mappings f_{ω^1} can be thought of as t-regular over the Grassmann manifold G_n contained in the space $M(O(n))$. Let V_{ω^1} be the pre-image of the manifold G_n under f_{ω^1}. Consider the normal tubular neighbourhood N of V_{ω^1} in S^{n+k}. Let $\varphi^* : H^{r-k}(V_{\omega^1}) \to H^r(N)$ be the corresponding isomorphism of cohomology groups. Denote by Y_ω^1 the image of Y_ω in the cohomology group of V_{ω^1} under the homomorphism $f_{\omega^1}^*$ induced by the mapping f_{ω^1} considered on the manifold V_{ω^1}. The classes Y_{ω^1} are expressed via Stiefel-Whitney characteristic classes $\overline{W}_i$ of the normal bundle of the manifold V_{ω^1} in S^{n+k}. By formula (1) and commutative diagram (1) from II.3 we have

$$\varphi^*(Y_{\omega^1}) = \varphi^* f_{\omega^1}^*(Y_\omega) = f_{\omega^1}^* \varphi_G^*(Y_\omega) = f_{\omega^1}^*(X_\omega) = \delta_{\omega^1}^\omega(s). \tag{2}$$

The *normal characteristic numbers* of $V = V_\omega$ are values of the polynomial of total degree k in $\overline{W}_i$, on the fundamental class of the manifold V. From (2), it follows that for any mapping $f : S^{n+k} \to M(O(n))$ which is not null-homotopic, there exists a non-trivial linear combination of classes X_ω whose image under f^* is non-zero in the algebra $H^*(S^{n+k}, Z_2)$ [1]. Consequently, at least one normal characteristic number of the corresponding manifold[2] V is non-zero. This yields the following theorem, inverse to Pontrjagin's theorem

Theorem IV.10. *If all Stiefel-Whitney numbers of some manifold V^k are equal to zero then this manifold is modulo 2 null-cobordant.*

Indeed, if all characteristic numbers defined for the classes W_i of the tangent bundle are equal to zero then the normal characteristic numbers are equal to zero. Indeed, by Whitney's relation[3] $\sum\limits_i W_i \overline{W}_{r-i} = 0$, the classes $\overline{W}_r$ are polynomials in the classes W_i.

Corollary IV.11. *If two manifolds V and V' have equal Stiefel-Whitney characteristic classes then these manifolds are cobordant modulo 2.*

REMARK. This result yields that in the group of (tangent) characteristic numbers[4] of k-manifolds V^k (this group is isomorphic to the group $H^k(G_n)$) there are precisely $d(k)$ linearly independent numbers. For low dimensions ($k \leqslant 6$), this result can be checked by means of Wu relations [33] for classes W_i of the tangent bundle of the manifold. This leads to the question whether Wu relations give *all* relations between the classes W_i of the tangent bundle of an arbitrary manifold.

6. Multiplicative structure of the groups $\mathfrak{N}_k$

Let $k = r + s$, and let ω_1 be a non-dyadic decomposition of s. Then the union (ω_1, ω_2) is a non-dyadic decomposition of k.

Above, we have defined the manifold V^k_ω. Recall that all normal characteristic classes[5] $Y_{\omega'}$ of V^k_ω are equal to zero except for the number Y_ω. Let us show that the manifold V^k_ω is cobordant modulo 2 to the product $V^r_{\omega_1} \times V^8_{\omega_2}$. To do this, it is sufficient to prove, according to the Corollary

[1] One should note that any mapping f is homotopic to a linear combination of mappings f_{ω^1}. — *Editor's remark*

[2] I.e. of pre-image of G_n under f. — *Editor's remark*

[3] This relation follows immediately from Whitney's duality theorem because the sum of the normal bundle and the tangent bundle of any manifold, has trivial characteristic classes. — *Editor's remark*

[4] I.e. in the additive group of polynomials of total weight k in variables W_i. — *Editor's remark.*

[5] Here we deal with values of classes $Y_{\omega'}$ on the fundamental class of V_ω. — *Editor's remark*

IV.11 that all Stiefel-Whitney characteristic classes of manifolds V^k_ω and $V^r_{\omega_1} \times V^s_{\omega_2}$ are equal.

The last statement follows immediately from the formula that we are going to prove:

$$Y_\omega = \sum_{(\omega_1,\omega_2)} (Y_{\omega_1}) \cdot (Y_{\omega_2}), \tag{3}$$

where (ω_1, ω_2) are all possible decompositions ω of k represented as a decomposition ω_1 of r and a decomposition ω_2 of s. Indeed, from formula (3) it follows that all numbers Y_ω of the product $V_{\omega_1} \times V_{\omega_2}$ are equal to zero except for the number corresponding to the decomposition $\omega = (\omega_1, \omega_2)$.

Recall that the normal bundle of the product $V^r_{\omega_1} \times V^s_{\omega_2}$ is the sum (union) of the normal bundles of manifolds $V^r_{\omega_1}$ and $V^s_{\omega_2}$. Denote by $\overline{W}_i$ the normal classes of the product $V_{\omega_1} \times V_{\omega_2}$, denote by U_i the normal classes of $V^r_{\omega_1}$ and by V_i the normal classes of $V^s_{\omega_2}$. Then, by Whitney's "duality theorem", the following symbolic formula holds:

$$\sum_i \overline{W}_i t^i = \sum_i U_i t^i \times \sum_j V_j t^j.$$

Denote by u_i the symbolic roots of the first factor and denote by v_j the symbolic roots of the second factor. Let us substitute in

$$Y_\omega = \sum (t_1)^{a_1} (t_2)^{a_2} \dots (t_q)^{a_q}$$

the roots u_i and v_j instead of t_i's. Then it follows from the dimension argument that all terms having total degree in (u_i)'s not equal to r, and the total degree in (v_j)'s not equal to s, should be equal to zero. The remaining terms can be grouped as follows:

$$Y_\omega = \sum_{(\omega_1,\omega_2)} \sum (u_1)^{a_1} (u_2)^{a_2} \dots (u_m)^{a_m} \cdot \sum (v_1)^{b_1} (v_2)^{b_2} \dots (v_n)^{b_n}, \tag{4}$$

where ω_1 is the decomposition $(a_1, a_2, \dots, a_m)$ of r generated by the decomposition ω, and ω_2 is the decomposition $(b_1, b_2, \dots, b_n)$ of s composed of the remaining numbers of the decomposition ω. The first sum is taken over all possible decompositions ω into a decomposition ω_1 of r and a decomposition ω_2 of s. The remaining two signs $\sum$ mean the symmetrisation in the sense described on page 154. Note that any decomposition (ω_1, ω_2) of ω occurs in (4) exactly once even in the case when this decomposition can be obtained in different ways. Indeed, assume (ω_1, ω_2) can be obtained in two different ways. Then there exists a permutation of variables (t_i) transforming the typical monomial

$$(t_1)^{a_1}(t_2)^{a_2}\dots(t_m)^{a_m}(t_{m+1})^{b_1}\dots(t_k)^{b_n}$$

of (ω_1, ω_2) to itself. Consequently, this transposition is not essential and it is not used in symmetrisation. Thus, formula (4) coincides with formula (3), as we had to prove.

As it was shown, from formula (3) it follows that for any decomposition of a non-dyadic decomposition ω of k into ω_1 of r and ω_2 of s, the cobordism classes of the corresponding manifolds V^k_ω satisfy the following

$$[V^k_\omega] = [V^k_{\omega_1}] \times [V^s_{\omega_2}]. \tag{5}$$

Thus, the only irreducible classes $[V^k_\omega]$ are $[V^k_{(k)}]$, where (k) is the decomposition of k consisting only of k itself. (It is assumed that k is not of type $2^m - 1$.) Any other class is uniquely represented as a sum of products of such irreducible classes. This proves the following theorem:

Theorem IV.12. *The ring $\mathfrak{N}$ of modulo 2 cobordisms is isomorphic to some polynomial algebra over Z_2. This algebra has generators of type $[V^k_{(k)}]$, where k runs over all numbers not equal to $2^m - 1$.*

Corollary. *The topological product of two manifolds each not null-cobordant modulo 2 is not null-cobordant modulo 2.*

Generators in low dimensions. The first generator appears for $k = 2$. The corresponding characteristic number is equal to

$$\sum(t^2) = (\sum t)^2 = (\overline{W}_1)^2 = (W_1)^2.$$

As a representative of this class $[V^2_{(2)}]$, we can take the real projective plane $PR(2)$.

For $k = 3$ the group $\mathfrak{N}^3$ is trivial.

For $k = 4$, a new generator, corresponding to the normal characteristic number $(t)^4 = (\overline{W}_1)^4 = (W_1)^4$, appears. Here $PR(4) + (PR(2))^2$ represents this number. The group $\mathfrak{N}^4$ is isomorphic to the direct sum $Z_2 + Z_2$. Note that the complex projective plane $PC(2)$ is cobordant modulo 2 to the square of the real projective plane $PR(2)$.

For $k = 5$, the group $\mathfrak{N}^5$ is isomorphic to Z_2 with generator $[V_{(5)}]$. The corresponding *tangent* characteristic number is equal to W_2W_3. For representative $[V_{(5)}]$, we take the Wu space [33][1] which is a circle-bundle fibre space over the complex projective plane $PC(2)$.

[1]The Wu space is obtained from the product $PC(2) \times I$ where $I = [0,1]$ by identifying the points $(\overline{x}_0, x_1, x_2) \times 0$ and $(\overline{x}_0, \overline{x}_1, \overline{x}_2) \times 1$, respectively. — *Editor's remark*

For $k = 6$, the group $\mathfrak{N}^6$ is isomorphic to $(Z_2)^3$. It has two reducible representations, $(PR(2))^3$, $PR(4) \times PR(2)$ and a primitive class $[V_{(6)}]$ corresponding to the normal characteristic number

$$\sum(t^6) = (\sum(t)^3)^2 = (\overline{W}_3)^2 + (\overline{W}_2\overline{W}_1)^2 + (\overline{W}_1)^6.$$

The projective space $PR(6)$ represents the last class.

For $k = 7$, all classes are reducible, because k is equal to $2^3 - 1$. The group $\mathfrak{N}^7$ is isomorphic to Z_2 with generator $[V_{(5)}] \times [V_{(2)}]$.

For $k = 8$, reducible classes can be easily found. Besides the reducible classes, there is an irreducible class $[V_{(8)}]$, with the corresponding characteristic number $(W_1)^8$. Every manifold of this class is (up to reducible manifolds) cobordant modulo 2 to the projective space $PR(8)$.

The last statement is of a general nature. Namely:

for any even dimension $n = 2r$ the primitive class $[V^n_{(n)}]$ is the sum of the class $[PR(n)]$ and some reducible classes.

It suffices to show that for the manifold $PR(n)$, the normal characteristic number $\sum(\overline{t}_i)^n$ is non-zero (here $\overline{t}_i$ are symbolic variables corresponding to the normal classes $\overline{W}_i$). Let

$$\sum(\overline{t}_1)^{a_1}(\overline{t}_2)^{a_2}\dots(\overline{t}_m)^{a_m}$$

be an arbitrary non-zero normal characteristic number of the manifold $PR(n)$, which is distinct from $\sum(\overline{t}_i)^n$. Here $a_1, a_2, \dots, a_m$ forms some non-dyadic decomposition ω_i of n. Consider the sum $PR(n) + U_i V^n_{\omega_i}$. All normal characteristic numbers of this manifold corresponding to non-dyadic decompositions of n, are equal to zero except for $\sum(\overline{t})^n$. Consequently, this manifold belongs to the primitive class $[V^n_{(n)}]$. On the other hand, by formula (5), all classes $[V^n_{\omega_i}]$ for $\omega_i \neq (n)$ are reducible.

Note that for any manifold V^n the normal characteristic number $\sum(\overline{t})^n$ is equal to the tangent characteristic number $\sum(t^n)$. Indeed, by the Whitney duality Theorem [32], the variables t corresponding to the tangent bundle are connected with the variables $\overline{t}_i$ corresponding to the normal bundle by the following relation:

$$\sum W_i t^i \times \sum \overline{W}_j \overline{t}^j = 1.$$

This relation yields that any symmetric function in the variables t_i and $\overline{t}_j$, which is not a non-zero constant vanishes. In particular,

$$\sum(t_i)^n + \sum(\overline{t}_j)^n = 0.$$

The Stiefel-Whitney polynomial of the manifold $PR(n)$ looks like

$$1 + \binom{n+1}{1} dt + \binom{n+1}{2} d^2t^2 + \dots + \binom{n+1}{p} d^pt^p + \dots + \binom{n+1}{n} d^nt^n,$$

where d is the generator of the group $H^1(PR(n); Z_2)$. Since $d^{n+1} = 0$, this manifold can be symbolically written as

$$(1 + dt)^{n+1}.$$

One may assume that this polynomial has $n + 1$ roots each equal to $t = -1/d$. Since n is even, the sum $\sum(t_i)^n$ is equal to $1/d^n$. Consequently, the corresponding characteristic number is equal to one.

As for generators in odd dimension, I do not know any analogous construction.

7. The groups Ω^k

In the general case, the groups

$$\pi_{n+k}(M(SO(n)))$$

are unknown. For small values of k, these groups are indicated in Theorem II.16. Consequently, by Theorem IV.8, we have:

Theorem IV.13. *For $k < 8$ the groups Ω^k are defined as:*

$$\Omega^0 = Z;\ \Omega^1 = \Omega^2 = \Omega^3 = 0;$$
$$\Omega^4 = Z;\ \Omega^5 = Z_2;\ \Omega^6 = \Omega^7 = 0.$$

This result is trivial for $k \leqslant 2$. The groups Ω^3 and Ω^4 were found by V. A. Rokhlin [19], [20]. The generator of Ω^4 is the complex projective plane $PC(2)$. Particularly, this yields

Corollary IV.14. *The fourth Pontrjagin number P^4 of an oriented four-manifold is equal to 3τ, where τ is the signature of the quadratic form defined by homological multiplication on $H^2(V^4, R)$.*

To prove this, it suffices to apply Theorems IV.1 and IV.2 and use the equality $\Omega^4 = Z$. The coefficient 3 is equal to the characteristic number P^4 of the complex projective plane $PC(2)$, for which $\tau = 1$. This result was proposed by Wu, who proved that P^4 is divisible by 3 [35]. It was first proved by V. A. Rokhlin [20] and by me, in a way quite different from the described one[1].

Note that the equality $P^4 = 3\tau$ yields topological invariance of the characteristic number P^4 for any manifold V^4. It would be very interesting to find a direct proof of this relation.

From the topological invariance of P^4 it follows that the cobordism class of the manifold V^4 does not depend on the differentiable structure of the manifold.

As it was shown in II.5, the cohomology algebra $H^*(M(SO(n)))$ over the field of rational numbers is isomorphic to the cohomology algebra of the product Y of the following Eilenberg-MacLane polyhedra:

$$Y = K(Z,k) \times K(Z,k+4) \times (K(Z,k+8))^2 \times \ldots \times (K(Z,k+4m))^{c(m)} \ldots, \quad m \leqslant k,$$

where $c(m)$ is the rank of $H^{4m}(\widehat{G}_k; R)$, here the above isomorphism is generated by some mapping $F : M(SO(k)) \to Y$. Thus, by using Serre's $\mathscr{C}$-theory results [22], for the case when $\mathscr{C}$ is the class of finite groups, we get:

Theorem IV.15. *If $i \not\equiv 0 \mod 4$ then the group Ω^i is finite. The rank of the free component of Ω^{4m} is equal to $c(m)$, that is the $4m$-th Betti number of the Grassmann manifold $\widehat{G}_k$.*

Corollary IV.16. *If all the Pontrjagin characteristic numbers of an orientable manifold V^k are equal to zero then for some non-zero integer N, the manifold NV^k is null-cobordant.*

Note that as a generator of the group $\Omega^5 \simeq Z_2$ we may take the Wu manifold defined in [33].

Multiplicative structure of groups $\boldsymbol{\Omega^k}$. Let Ω^T be the set of all finite-order elements of the ring Ω. The set Ω^T forms an ideal of the ring Ω, so that there is a quotient ring Ω/Ω^T. We know (by Theorem IV.15) that the $4m$-dimensional component of this quotient ring is a direct sum of $c(m)$ free cyclic groups. On the other hand,

$$\Omega^{4m} \otimes Q \simeq \pi_{k+4m}(M(SO(k))) \otimes Q,$$

where Q is the field of rational numbers. Since the latter group is dual (over Q) to the cohomology group

$$H^{k+4m}(M(SO(k)); Q) \simeq H^{4m}(\widehat{G}_k; Q),$$

[1]See my note in Colloque de Topologie de Strasbourg (June, 1952). Rokhlin's note also contains results concerning groups $\mathfrak{N}$. One of these results is false, namely, Rokhlin states that $\mathfrak{N}^4 = Z_2$ (instead of $Z_2 + Z_2$)[2].

[2]The correct result is given by V. A. Rokhlin in the following note: Doklady Mathematics, **89** (1953), 789–792. — *Editor's remark*

then any $4m$-dimensional element of the ring Ω/Ω^T is completely characterised by the values of the normal characteristic numbers

$$\langle \Pi(P^{4r}), V^{4m} \rangle,$$

defined by an arbitrary embedding of some manifold V^{4m} of the given class to the Euclidean space. To make this statement more precise, it is important to note that, in general, there is no manifold whose characteristic numbers would have any prefixed values n_i. However, one may say that for some non-zero integer N, the products Nn_i are normal (or tangent) characteristic numbers of some manifold V^{4m}.

Now, we can construct for the tensor product $\Omega \otimes Q$, a theory analogous to the one constructed above for the ring $\mathfrak{N}$. Recall that, by Borel and Serre, Pontrjagin classes are in one-to-one correspondence with symmetric functions in squares $(x_i)^2$ of some two-dimensional variables x_i (if there exists a unitary fibre space adjoint to the given orthogonal fibre space then its Chern classes are given by symmetric functions in x_i). Thus, the base of the group $H^{4m}(\widehat{G}_k)$ consists of symmetrised monomials of the type

$$P_\omega = \sum (x_1^2)^{a_1} (x_2^2)^{a_2} \dots (x_r^2)^{a_r},$$

where $a_1, a_2, \dots, a_r$ is an arbitrary decomposition (ω) of m.

Normal characteristic classes of the product $X^p \times Y^q$ of two oriented manifolds X^p and Y^q are defined by

$$P_\omega(X^p \times Y^q) = \sum_{(\omega_1, \omega_2)} P_{\omega_1}(X^p) \cdot P_{\omega_2}(Y^q), \tag{3'}$$

where the sum is taken over all complementary decompositions ω_1, ω_2, for which $\deg \omega_1 = p,\ \deg \omega_2 = q$.

According to the remark above, in each dimension $4m$ there exist manifolds V^{4m} for which all normal characteristic numbers are equal to zero except for the number $\langle \sum (x_i)^{2m}, V^{4m} \rangle$. Let $Y_{(4m)}$ be the corresponding class of the group $\Omega^{4m} \otimes Q$. From formula (3′) and Corollary IV.16, it follows that the classes $Y_{(4m)}$ are irreducible and that any other element of the tensor product $\Omega \otimes Q$ can be uniquely represented as a sum of products of $Y_{(4m)}$. Thus, we get

Theorem IV.17. *The algebra $\Omega \otimes Q$ is a polynomial algebra. In any dimension divisible by 4, there is a unique generator $Y_{[4m]}$ of this algebra.*

Now, let us show that up to some non-zero factor, the class $Y_{[4m]}$ is a sum of the complex projective space $PC(2m)$ class and some reducible classes.

In other words, the classes of spaces $PC(2m)$ can be viewed as generators of the algebra $\Omega \otimes Q$. To prove this statement, it suffices to show that the normal characteristic number of the space $PC(2m)$ corresponding to the class $\sum(x_i)^{2m}$ is non-zero. But, by the duality theorem between normal and tangent classes, the normal characteristic number corresponding to the sum $\sum(x_i)^{2m}$ differs from the corresponding tangent characteristic number only in sign. On the other hand, it is known that the Chern polynomial of the complex projective space $PC(2m)$ looks like

$$C(x) = 1$$

$$+ \binom{2m+1}{1} dx + \ldots + \binom{2m+1}{i} d^i x^i + \ldots + \binom{2m+1}{2m} d^{2m} x^{2m},$$

where d is the cohomology class of the projective line. Symbolically, this polynomial can be written as:

$$C(x) = (1 + dx)^{2m+1}.$$

Consequently, all symbolic roots of this manifold are equal to -1.
Thus, the characteristic number $\langle \sum(x_i)^{2m}, PC(2m)\rangle$ is equal to

$$\sum \langle (-1/d)^{2m}, d^{2m}\rangle = 2m + 1.$$

The normal characteristic number of the manifold $PC(2m)$ corresponding to the class $\sum(x_i)^{2m}$ is hence equal to $-(2m+1)$, thus, it is non-zero. This proves the property formulated above. This yields

Corollary IV.18. *For any oriented manifold V^n there exists a non-zero integer N such that the manifold NV^n is cobordant to some linear integer combination of products of even-dimensional projective spaces. The coefficients of this linear combination are linear homogeneous functions of Pontrjagin's characteristic numbers NV^n.*

REMARK. It is natural to address the question whether the products of the spaces $PC(2j)$ form a basis of the Z-module Ω/Ω^T? This is true for dimension 4, because the class of $PC(4)$ generates the group Ω^4. One can show that this is true for dimension 8 as well. Indeed, in this dimension the characteristic numbers P^8 and $(P^4)^2$ enjoy the following relations[1]:

$$(P^4)^2 - 2P^8 \equiv 0 \mod 5,$$
$$7P^8 - (P^4)^2 = 45\tau.$$

[1]Indeed, for $PC(4)$ we have $p = (1 + u^2)^5$, so that $P^4 = 5u^2$; $(P^4)^2 = 25u^4$; $P^8 = 10u^4$, for $PC(2) \times PC(2)$ we have $p = (1+u_1^2)^3(a+u_2^2)^3$, so that $P^4 = 3(u_1^2+u_2^2)$; $(P_4)^2 = 18u_1^2u_2^2$; $P^8 = 9u_1^2u_2^2$. This yields the desired formulas. — Translator's remark.

The first relation follows from the equality $St_5^8\Delta = 0$, that occurs in a topological product of manifolds (cf. Wu [35]); the second one is obtained if we write the index τ as a linear homogeneous function of the classes P^8 and $(P^4)^2$ and define the coefficients for the typical manifolds $PC(4)$ and $(PC(2))^2$. Let V^8 be an arbitrary manifold and let τ be the signature of the quadratic form defined by the cohomology product on $H^4(V^8, R)$. It is easy to check that the manifold V^8 and the manifold

$$q \cdot PC(4) + (\tau - q) \cdot (PC(2))^2,$$

where q is defined by $(P^4)^2 - 2P^8 + 5q$, have the same Pontrjagin numbers and are cobordant (modulo Ω^T). The study in higher dimensions requires a more exact consideration of arithmetical and topological properties of Pontrjagin's number[1].

Editor's remarks

(to page 154)

As Serre has mentioned, this lemma allows to prove the following Adem-Wu formulae:

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c, \quad a < 2b,$$

that allow us to express any iterated square as a linear combination of iterated squares corresponding to admissible sequence.

To reduce the calculations, set

$$C_{a,b} = Sq^a Sq^b - \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c,$$

$$T_{a,b} = Sq^{a-1} Sq^b + Sq^a Sq^{b-1} - \sum_c \binom{b-c-1}{a-2c} \left(Sq^{a+b-c-1} Sq^c + Sq^{a+b-c} Sq^{c-1} \right).$$

From the evident formula

$$\binom{b-c-1}{a-2c} + \binom{b-c-1}{a-2c-1} + \binom{b-c-2}{a-2c-2} + \binom{b-c-2}{a-2c} \equiv 0 \pmod 2$$

it follows that

$$T_{a,b} = C_{a-1,b} + C_{a,b-1}.$$

[1] About this, see the recent work of Hirzebruch (mimeographed notes of Princeton University, July–August, 1953).

On the other hand, it is easy to see (by applying Cartan's formula twice), that for any $x \in H^*(X, Z_2)$ and for any $t \in H^1(X, Z_2)$ (where X is an arbitrary space) we have

$$Sq^i Sq^j(xt) = Sq^i Sq^j(x) \cdot t + \big(Sq^{i-1}Sq^j(x) + Sq^i Sq^{j-1}(x)\big) \cdot t^2 + Sq^{i-2}Sq^{j-1}(x) \cdot t^4.$$

This yields that

$$C_{a,b}(xt) = C_{a,b}(x) \cdot t + T_{a,b}(x) \cdot t^2 + C_{a-2,b-1}(x) \cdot t^4.$$

Substituting the expression for $T_{a,b}$, we get: if $C_{a,b} = 0$ for $a + b < n$ then $C_{a,b}(xt) = C_{a,b}(x) \cdot t$ for any a, b satisfying $a + b = n$.

Now, let $X = G_k$, where $k > a + b$. Since $W_k = t_1, \dots, t_k$ then from the relation $C_{a,b}(xt) = C_{a,b}(x) \cdot t$ it follows that $C_{a,b}(W_k) = C_{a,b}(e) \cdot W_k$, where e is the unit class. Since $\dim e = 0$, $C_{a,b}(e) = 0$ and hence, $C_{a,b}(W_k) = 0$, i.e. by the lemma we have proved, $C_{a,b} = 0$. Thus, if $C_{a,b} = 0$ for $a + b < n$ then $C_{a,b} = 0$ for $a + b = n$. To complete the proof of the Adem-Wu formula, it remains to note that for $a + b = 1$ this formula is evident.

Analogously, one can prove Adem's formulae for Steenrod's powers:

$$\mathscr{P}_p^a \mathscr{P}_p^b = \sum_c (-1)^{c-a} \binom{(p-1)(b-c)-1}{a-pc} \mathscr{P}_p^{a+b-c} \mathscr{P}_p^c.$$

$$\mathscr{P}_p^{a+1} \beta \mathscr{P}_p^b = \sum_c (-1)^{c-a} \binom{(p-1)(b-c)-1}{a-pc} \mathscr{P}_p^{a+b-c} \beta \mathscr{P}_p^c + \sum_c (-1)^{c-a+1} \binom{(p-1)(b-c)}{a-pc+1} \beta \mathscr{P}_p^{a+b-c} \mathscr{P}_p^c, \ a < pb.$$

(to page 177)

The operations ϑ_i^p and Q_p^i are connected, by definition, by

$$x \frown \vartheta_i^p y = Q_p^i x \frown y, \quad x \in H^{r-i}(K, Z_p), \ y \in H_r(K, Z_p),$$

where $\frown$ is the Whitney multiplication. Consequently,

$$\sum_i Q_p^{m-i} St_p^i(x) \frown y = \sum_i St_p^i x \frown \vartheta_{m-i}^p y, \quad x \in H^{r-m}(K, Z_p), \ y \in H_r(K, Z_p).$$

Applying χ and taking into account that

$$\chi(u \frown v) = u \smile \chi v,$$

we get

$$\chi \sum_i Q_p^{m-i} St_p^i(x) \frown y = \sum_i St_p^i x \smile St_p^{m-i} \chi y.$$

But, by Cartan's formula

$$\sum_i St_p^i x \smile St_p^{m-i} \chi y = St_p^m (x \smile \chi y).$$

Thus,

$$\chi \sum_i Q_p^{m-i} St_p^i(x) \frown y = St_p^m \chi(x \frown y),$$

i.e.

$$\sum_i Q_p^{m-i} St_p^i(x) \frown y = \vartheta_p^m (x \frown y).$$

But composite $x \frown y \in H_m(K, Z_p)$. Furthermore, it is easy to see that for any $u \in \in H_m(K, Z_p), m > 0$,

$$\vartheta_m^p(u) = 0.$$

Indeed, consider an arbitrary embedding f of the polyhedron K to some Euclidean space R^N. According to property 2 of the operations, ϑ_i^p

$$f_*^0 \vartheta_m^p(u) = \vartheta_m^p(f_*^m u) = 0,$$

because $f_*^m u = 0$. On the other hand, the mapping f_*^0, is, evidently, isomorphic (the polyhedron K is assumed to be connected).

Thus,

$$\sum_i Q_p^{m-i} St_p^i(x) \frown y = 0$$

for all $y \in H_r(K, Z_p)$. Consequently,

$$\sum_i Q_p^{m-i} St_p^i(x) = 0.$$

(to page 190)

Let Y be a piecewise-connected simply connected topological space, which is aspherical up to dimension $n-1$, inclusively ($n > 2$), let $Y \times Y$ be the topological product of Y with itself and let $Y \vee Y$ be the subset of the space $Y \times Y$ of type $Y \times y_0 \cup y_0 \times Y$, where y_0 is some fixed point of Y. By the Künneth formula,

$$H_p(Y \times Y) \approx \sum_{i+j=p} H_i(Y) \otimes H_j(Y) + \sum_{i+j=p-1} H_i(Y) * H_j(Y).$$

On the other hand, it is clear that

$$H_p(Y \vee Y) \approx H_p(Y) \otimes H_0(Y) + H_0(Y) \otimes H_p(Y),$$

and the natural inclusion $H_p(Y \vee Y) \to H_p(Y \times Y)$ is an isomorphism. Thus, by using the homological exact sequence of the pair $(Y \times Y, Y \vee Y)$, we get that

$$H_p(Y \times Y, Y \vee Y) \approx \sum_{\substack{i+j=p \\ i \neq 0, j \neq 0}} H_i(Y) \otimes H_j(Y) + \sum_{i+j=p-1} H_i(Y) * H_j(Y). \tag{1}$$

Since $\pi_i(Y) = 0$, if $0 < i < n$, then (by the Hurewicz theorem) $H_i(Y) = 0$ if $0 < i < n$. Thus, by using formula (1), we get that $H_p(Y \times Y, Y \vee Y) = 0$ if $1 \leqslant p \leqslant 2n - 1$, thus (by the relative Hurewicz theorem),

$$\pi_p(Y \times Y, Y \vee Y) = 0. \tag{2}$$

By using (2) and by using an obstruction theory argument, one easily obtains the following lemma.

Any mapping from a finite polyhedron of dimension less than or equal to $2n - 1$ *to the space* $Y \times Y$ *is homotopic to a mapping from the same polyhedron to* $Y \vee Y$.

Now let V be an arbitrary finite polyhedron of dimension less than $2n-1$ and let $f : Y \to \Omega$ and $g : V \to Y$ be arbitrary continuous mappings of V to some space Y. Denote their multiplication $f \times g : V \to Y \times Y$ (i.e. $(f \times g)(x) = (f(x)g(x))$). According to the lemma proved above, the mapping $f \times g$ is homotopic to some mapping $h : V \to Y \vee Y$. Let us define $\varphi : Y \vee Y \to Y$ by setting

$$\varphi(y \times y_0) = y,$$
$$\varphi(y_0 \times y) = y.$$

The composition $\phi \circ h : V \to Y$ is called the *sum* of the mappings f and g and is denoted by $f + g$. It can be easily checked that the homotopy class of $f + g$ depends only on the homotopy classes of f and g. Thus, one may speak about the sum of mapping homotopy classes $V \to Y$. It turns out that, with respect to this summation operation, the set $Y(V)$ of all homotopy classes of mappings $V \to Y$ is an abelian group. It is called the *Y-cohomotopy group of the polyhedron V.* In the case when $Y = S_n$, the group $S_n(V)$ is denoted by $\pi^n(V)$; it is then called the *n-th cohomotopy group* of V.

References

[1] J. Adem. The iteration of the Steenrod squares in algebraic topology. Proc. Nat. Acad. Sci. U.S.A., **38**, 1952, 8, 720–726.

[2] A.L. Blakers, W.S. Massey. The homotopy groups of a triad. Ann. Math. **53**, 1951, 161–204.

[3] A. Borel. Sur la cohomologie des éspaces fibrés principaux et des éspaces homogènes des groupes Lie compacts. Ann. Math. **57**, 1953, 115–207.

[4] A. Borel. La cohomologie modulo 2 de certains éspaces homogènes. Comm. Math. Helv. **27**, 1953, 165–197.

[5] A. Borel, J.P. Serre. Groupes de Lie et puissances réduites de Steenrod. Amer. J. of Math. **75**, 1953, 409–448.

[6] N. Bourbaki. Topologie Générale. Chap. IX. p. 75. Act. Sci. Ind. Paris.

[7] H. Cartan. Sur les groupes d'Eilenberg–MacLane (à paraitre ultérieurement).

[8] S. Chern. On the multiplication of the characteristic ring of a sphere-bundle. Ann. Math. **49**, 1948, 2.

[9] B. Eckmann. Coverings and Betti numbers. Bull. Amer. Math. Soc. **55**, 1949, 95–101.

[10] C. Ehresmann. Sur les espaces fibrés différentiables. C. R. Acad. Sci. Paris **224**, 1947, 1611–1612.

[11] S. Eilenberg. Topological methods in abstract algebra; cohomology theory of groups. Bull. Amer. Math. Soc. **55**, 1949, 3–27.

[12] S. Eilenberg. Problems in topology. Ann. Math. **50**, 1949, 246–260.

[13] S. Eilenberg, S.M. MacLane. Cohomology theory of abelian groups and homotopy groups. I. Proc. Nat. Acad. Sci. U.S.A. **36**, 1950, 443–447.

[14] S. Eilenberg, S.M. MacLane. Cohomology theory of abelian groups and homotopy groups. IV. Proc. Nat. Acad. Sci. U.S.A. **38**, 1952, 1340–1342.

[15] W. Hurewicz. Beiträge zur Topologie der Deformationen. Nederl. Akad. Wetensch. Proc. Ser. A, 1935 — 6.

[16] A.P. Morse. The behavior of a function on its critical set. Ann. Math. **40**, 1939, 62–70.

[17] L. S. Pontrjagin. Mappings of the 3-dimensional sphere into an n-dimensional complex, Doklady Akad. Nauk SSSR, **34**, 1942, 35–37.

[18] L. S. Pontrjagin. Characteristic cycles on differentiable manifolds, Mat. Sb.21(63)(1947), 233–284.

[19] V. A. Rokhlin A three-dimensional manifold is the boundary of a four-dimensional one. (Russian) Doklady Akad. Nauk SSSR (N.S.) 81, (1951) 355–357.

[20] V. A. Rokhlin. New results in the theory of 4-manifolds, Doklady Acad. Nauk. SSSR (N.S.) 84 (1952) 221–224.

[21] H. Seifert. Algebraische Approximation von Mannigfaltigkeiten. Math. Z. **41**, 1936, 1–17.

[22] J.P. Serre. Groupes d'homotopie et classes des groupes abéliens. Ann. Math. **52**, 1953, 258–294.

[23] J.P. Serre. Cohomologie modulo 2 des complexes d'Eilenberg–MacLane. Comm. Math. Helv. **27**, 1953, 198–231.

[24] J.P. Serre. Homologie singulière des éspaces fibrés. Applications. Ann. Math. **54**, 1951, 425–505.

[25] N.E. Steenrod. Cyclic reduced powers of cohomology classes. Proc. Nat. Acad. Sci. U.S.A. **39**, 1953, 213–223.

[26] N.E. Steenrod, J.H.C. Whitehead. Vector fields of the n-sphere. Proc. Nat. Acad. Sci. U.S.A. **37**, 1951, 58–63.

[27] R. Thom. Espaces fibrés en sphères et carrés de Steenrod. Ann. Sci. Ecole Norm. Sup. **69**, 1952, 109–181.

[28] R. Thom. Sous-variétés et classes d'homologie des variétés différentiables. C. R. Acad. Sci. Paris **236**, 453–454 et 573–575; Sur un problème de Steenrod. Ibid. **236**, 1128–1130; Variétés différentiables cobordantes. Ibid. **236**, 1733–1735.

[29] J.H.C. Whitehead. Combinatorial homotopy. Bull. Amer. Math. Soc. **55**, 1949, 213–245 and 453–496.

[30] H. Whitney. A function not constant on a connected set of critical points. Duke Math. J. **4**, 1935, 514–517.

[31] H. Whitney. Analytic extensions of differentiable functions defined in closed sets. Trans. Amer. Math. Soc. **36**, 1934, 63–89.

[32] H. Whitney. Topology of differentiable manifolds. Michigan Lectures, 101–141.

[33] W.T. Wu. Classes caractéristiques et i-carrés d'une variété C. R. Acad. Sci. Paris. **230**, 1950, 508–509.

[34] W.T. Wu. Les i-carrés dans une variété grassmannienne. C. R. Acad. Sci. Paris. **230**, 1950, 918.

[35] W.T. Wu. Sur les puissances de Steenrod. Colloque de Topologie de Strasbourg, Juillet 1951. (Miméographié.)

3

Homotopy properties of Thom complexes

S. P. Novikov [1]

Introduction

In the present work, we give detailed proofs of results published in the note [19]. Our goal is to investigate the rings of inner homology ("cobordisms") corresponding to classical Lie groups: $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, in the sequel to be denoted by V_{SO}, V_U, V_{SU}, V_{Sp}, and also to investigate the realisability of k-dimensional integral cycles in manifolds of dimension $\geqslant 2k+1$ by smooth orientable submanifolds. As Thom proved [16],

[1]Homotopy properties of Thom complexes. Translated by V.O.Manturov
As it is well-known, calculation of the multiplicative structure of the orientable cobordism ring modulo 2-torsion was announced in the works of J.Milnor (see [18]) and of the present author (see [19]) in 1960. In the same works the ideas of cobordisms were extended. In particular, very important unitary ("complex") cobordism ring was invented and calculated; many results were obtained also by the present author studying special unitary and symplectic cobordisms. Some western topologists (in particular, F.Adams) claimed on the basis of private communication that J.Milnor in fact knew the above mentioned results on the orientable and unitary cobordism rings earlier but nothing was written. F.Hirzebruch announced some Milnors results in the volume of Edinburgh Congress lectures published in 1960. Anyway, no written information about that was available till 1960; nothing was known in the Soviet Union, so the results published in 1960 were obtained completely independently. Let us make some comments concerning the proof. There exists a misunderstanding of that question in the topological literature. Contrary to the Adams claims, the Milnor's work [18] did not contain proof of the theorem describing multiplicative structure of the cobordism ring and its complex analogue. It used the so-called Adams Spectral Sequence only for calculation of the additive structure and proved "no torsion theorem". For the orientable case it was done independently by my friend B.Averbukh [2] using the standard Cartan-Serre technique; it was Averbukh's work that attracted me to this area: I decided to apply here the Adams Spectral Sequence combined with the homological theory of Hopf algebras and coalgebras instead of the standard Cartan-Serre method because my approach worked very well for the mul-

for any cycle $z_k \in H^k(M^n)$ there exists a number α such that the cycle αz_k is realizable by a submanifold and for $k \leqslant 5$ any cycle z_k is realizable (i.e. $\alpha = 1$).

In § 1 of Chapter III, we prove the following theorem.

Assume for $n \geqslant 2k+1$ that the groups $H_i(M^n)$ have no p-torsion for $i = k - 2t(p-1) - 1$ for all $t \geqslant 1$, $p \geqslant 3$. Then any cycle z_k of dimension k is realizable by a submanifold.

The proof of this theorem relies on the well-known Thom construction and it is based on new results on the homotopy group of the complexes constructed by him, which enjoy several remarkable properties. In fact, these properties allow one to reduce many problems of manifold topology to homotopy problems. One can construct examples showing that the above theorem gives a final criterion in terms of homology groups.

In Chapter II, we explicitly find the algebraic structure of the rings V_{SO}/T, V_U and $V_{Sp} \otimes Z_{p^h}$ for $p > 2$; we also prove that the ring V_{Sp}/T is not polynomial (here T is the ideal consisting of finite order elements; one may assume that all orders of elements look like[1] z^s). The known information about V_{SU} is given in Appendix 1; we did not include it into formulations of the main theorems. It turns out that the algebraic struc-

---

tiplicative problems. The present article was presented in 1959/60 as my diploma work at the Algebra Chair in the Moscow State University. In the Introduction (see below) I made mistakable remark that Milnor also calculated the ring structure using Adams Spectral Sequence (exactly as I did myself). However, it was not so: as it was clearly written by Milnor in [18], his plan was completely different; he intended to prove this theorem geometrically in the second part but never wrote it. I cannot understand why F.Adams missed this fundamental fact in his review in the Math Reviews Journal on my Doklady note ([19]). Does it mean that he never looked carefully in these works? As I realized later after personal meeting in Leningrad with Milnor (and Hirzebruch) in 1961 during the last Soviet Math Congress, his plan was to use some specific concrete algebraic varieties in order to construct the additive basis and apply Riemann-Roch Theorem. I described his manifolds in the Appendix (they are very useful) but never realized his plan of the proof: my own purely Hopf-algebraic homotopy-theoretical proof was so simple and natural that I believe until now that Milnor lost interest in his geometric proof after seeing my work. I added the Appendix in 1961 but forgot to change the Introduction written in 1960, so the mistakable remark survived. It is interesting that in 1965 Stong and Hattori published a work dedicated to this subject. They claimed that they found a "first calculation of the complex cobordism ring avoiding the Adams Spectral Sequence" not mentioning exactly where this theorem was proved first. Let me point out that their work was exactly realization of the Milnor's original plan but Stong and Hattori never mentioned that. — *S. P. Novikov's remark* (2004)

[1]The algebra $V_{SO} \otimes Q$ was first found by Thom [16]. V. A. Rokhlin [13] and Wall [20],

ture of V_U was found by Milnor slightly earlier [18][1] (also, by using Adams' method), who also found the geometric generators of the rings V_{SO} and V_U and finally solved the Pontrjagin (Chern) characteristic number problem for smooth (complex-analyitic, almost complex) manifolds, i.e. gave a necessary and sufficient condition for a set of numbers to be a set of Pontrjagin (Chern) classes of a smooth (almost complex) manifold. The geometric generators weren't known before Milnor's works and, because of their interest, we indicated them in Appendix 2 (we knew only (see § 5 Ch II) that for prime numbers $p \geqslant 2$ for generators in dimensions $2p-2$ we may take $P^{p-1}(C)$). The author's results about the multiplicative structure of the ring V_{Sp} and about the ring V_{SU} were not previously known.

Chapter I contains some geometric and algebraic information about Thom complexes.

Chapter II is devoted to the calculation of integral homology rings. We study several questions concerning these rings there (see also Appendix 1).

In Chapter III, we consider different types of realisations of cycles by submanifolds.

Chapter I

Thom's spaces

§ 1. *G*-framed submanifolds. *L*-equivalence submanifold classes

Consider a smooth compact closed manifold M^n of dimension n, endowed with a Riemannian metric, and fix a subgroup G of $O(n-i)$, where $i < n$. Assume furthermore that the manifold M^n is orientable and that the subgroup G of the group $O(n-i)$ is connected. Orient the manifold M^n in a certain way. Consider a compact closed manifold W^i smoothly embedded into M^n. We assume the submanifold W^i of the manifold M^n to be orientable. In this case, the normal $SO(n-i)$-bundle ν^{n-i} of W^i in M^n is defined. We consider only such submanifolds W^i of M^n with normal bundle ν^{n-i} admitting the subgroup G of $SO(n-i)$ as the structure group.

by using the well-known Rokhlin theorem on the kernel of the homomorphism $V_{SO} \to V_O$ (see [12]), found the structure of 2-torsion of the ring V_{SO} (independently). B. G. Averbuch and J. Milnor (independently) proved that there is no p-torsion for $p > 2$ in the ring V_{SO} (see [2], [10], [18]). In [10], the structure of the ring V_{SO}/T was found.

[1] Milnor [18] also considered the ring V_{Spin}; however, he did not get complete results about it.

Definition 1.1. A submanifold W^i of the manifold M^n is called G-framed if in the normal bundle ν^{n-i} of the submanifold W^i in the manifold M^n a G-bundle structure is fixed.

Now let N^{n+1} be a smooth compact manifold with boundary M^n and let V^{i+1} be a compact smoothly embedded submanifold with boundary $W^i = M^n \cap V^{i+1}$, so that the manifold V^{i+1} is orthogonal to the boundary M^n of the manifold N^{n+1}. In this case, one may also speak about the normal bundle τ^{n-i} of the submanifold V^{i+1} in the manifold N^{n+1}. All manifolds mentioned here are assumed to be oriented unless otherwise specified. Thus the bundle τ^{n-i} can be considered as an $SO(n-i)$-bundle of planes R^{n-i}, and, analogously to Definition 1.1, one may define G-framed submanifolds with boundary (clearly, the boundary is a closed G-framed submanifold of the boundary M^n of N^{n+1}).

Following Thom, let us introduce the L-equivalence relation in the set of G-framed closed submanifolds of a closed manifold M^n. Consider the direct product $M^n \times I$ of the manifold M^n and the oriented closed interval $I = [0, 1]$. Then the manifold with boundary $N^{n+1} = M^n \times I$ gets a natural orientation. Let W_1^i and W_2^i be two G-framed closed submanifolds of the manifold M^n. The submanifolds $W_1^i \times 0$ and $W_2^i \times 1$ are naturally oriented as well, in the manifolds $M^n \times 0$ and $M^n \times 1$, and the oriented submanifold $W_1^i \times 0 \cup W_2^i \times 1$ of the manifold $M^n \times 0 \cup M^n \times 1$ is G-framed.

Definition 1.2. Two G-framed submanifolds W_1^i and W_2^i of a manifold M^n are L-equivalent if there exists a G-framed submanifold V^{i+1} of the manifold $M^n \times I$ with boundary $W_1^i \times 0 \cup W_2^i \times 1$.

It can be easily checked that the L-equivalence of G-framed submanifolds is symmetric, transitive and reflexive thus the set of G-framed submanifolds of a given manifold M^n is divided into classes of L-equivalent submanifolds. Denote the set of such classes by $V^i(M^n, G)$. Note that every element of the set $V^i(M^n, G)$ defines an integer cycle $z_i \in H_i(M^n)$, i.e. there is a well-defined mapping

$$\lambda_G \colon V^i(M^n, G) \to H_i(M^n).$$

Definition 1.3. A cycle $z_i \in H_i(M^n)$ is G-realizable, if it belongs to the image of the mapping λ_G.

Definition 1.3 is evidently equivalent to the G-realizability definition after Thom (see [16]).

§ 2. Thom spaces. Classifying properties of Thom spaces

In § 1, we have fixed a connected subgroup of the group $O(n-i)$. Assume this subgroup is closed in $O(n-i)$. Let B_G be the classifying space of the group G. Without loss of generality, we may assume that B_G is a manifold of high enough dimension. Denote by $\eta(G)$ the classifying G-bundle of spheres S^{n-i-1}. Denote the total space by E_G; denote the projection function by p_G. The projection cylinder p_G is a manifold T_G with boundary E_G. The cylinder T_G can be considered as the space of the classifying G-bundle of closed balls E^{n-i}. Let us contract the boundary E_G of the manifold T_G to a point.

Denote the obtained manifold by M_G; let us call it the Thom space of the subgroup G of $O(n-i)$. It easily follows from the general theory that for $n-i>1$, the space M_G is simply connected (see, e.g., [16] for $G=SO(n-i)$). As for the cohomology of the space M_G, as Thom has shown, there exists a natural isomorphism $\varphi\colon H^k(B_G)\to H^{k+n-i}(M_G)$.

Denote by $u_G\in H^{n-i}(M_G)$ the element equal to $\varphi(1)$. Then the following Theorem holds

Thom's Theorem. *An integral cycle $z_i\in H^i(M^n)$ is G-realizable if and only if there exists a mapping $f\colon M^n\to M_G$ such that the cohomology class $f^*(u_G)$ is Poincaré-dual to the cycle z_i.*

(If G is not connected then Thom's theorem holds for modulo 2 cycles.)

Thom found a connection between the sets $V^i(M^n, SO(n-i))$ and the sets of homotopy classes $\pi(M^n, M_G)$ of mappings $M^n\to M_G$ ([16], Theorem IV.6). From the proof of Theorem IV.6 it follows that, substituting SO to G, one can easily get the following lemma.

Lemma 2.1. *Elements of the set $V^i(M^n, G)$ are in one-to-one correspondence with elements of the set $\pi(M^n, M_G)$.*

For $i<[n/2]$, both sets have natural abelian group structures. Their natural one-to-one correspondence, established in Theorem IV.6 by Thom, is in this case a group isomorphism. This takes place also when $M^n=S^n$ (for arbitrary i). However, we are not interested in this case in the sequel.

Let us define the pairing for the groups:

$$V^{i_1}(S^{n_1}, G_1)\otimes V^{i_2}(S^{n_2}, G_2)\to V^{i_1+i_2}(S^{n_1+n_2}, G_1\times G_2). \qquad (I)$$

The group $G_1\times G_2$ is assumed to be embedded into $SO(n_1+n_2-i_1-i_2)$. This inclusion is defined by the natural decomposition of the Euclidean space $R^{n_1+n_2-i_1-i_2}$ into the direct product $R^{n_1-i_1}\times R^{n_2-i_2}$. In order to define the pairing (I), let us choose representatives for the two given elements $x_1\in V^{i_1}(S^{n_1}, G_1)$, $x_2\in V^{i_2}(S^{n_2}, G_2)$. These representatives, are, clearly, G_1- and G_2-framed submanifolds $W^{i_1} \subset S^{n_1}$ and $W^{i_2} \subset S^{n_2}$, respectively.

The direct product $W^{i_1} \times W^{i_2}$ is naturally embedded into $S^{n_1+n_2}$ and it is $G_1 \times G_2$-framed because the normal bundle of the direct product is decomposed into the direct product of the normal bundles of the manifolds W^{i_1} and W^{i_2}. We assume the group $G_1 \times G_2$ is embedded into $SO(n_1 + n_2 - i_1 - i_2)$ precisely as shown above. Then the following lemma holds.

Lemma 2.2. *There exists a homeomorphism*

$$M_{G_1} \times M_{G_2}/M_{G_1} \vee M_{G_2} \to M_{G_1 \times G_2} \qquad (II)$$

such that the diagram

$$\begin{array}{ccccc} V^{i_1}(S^{n_1}, G_1) \otimes V^{i_2}(S^{n_2}, G_2) & \to & V^{i_1+i_2}(S^{n_1+n_2}, G_1 \times G_2) \\ \downarrow \wr \qquad\qquad \downarrow \wr & & \downarrow \wr \\ \pi_{n_1}(M_{G_1}) \otimes \pi_{n_2}(M_{G_2}) & \to & \pi_{n_1+n_2}(M_{G_1 \times G_2}) \end{array} \qquad (III)$$

is commutative. (Here the upper row corresponds to the pairing (I), and the lower row represents the pairing of homotopy groups defined by (II).)

PROOF. To prove the existence of the homeomorphism (II) note that the classifying space $B_{G_1 \times G_2}$ of the group $G_1 \times G_2$ is decomposed into the direct product $B_{G_1} \times B_{G_2}$. The classifying $G_1 \times G_2$-bundle of planes $R^{n_1+n_2-i_1-i_2}$ is also decomposed into the direct product of classifying bundles, namely, the G_1- and the G_2-bundle. The classifying G-bundle of spheres is obtained from the plane bundle by taking in each fibre the set of all vectors of length 1. The set of all vectors of lengths not greater than 1 gives us the classifying bundle of closed balls. Now, taking B_{G_1} and B_{G_2} to be some manifolds of some high dimension, and recalling the definition of Thom spaces via projection cylinders of classifying sphere-bundles T_{G_1} and T_{G_2}, we get a natural homeomorphism $T_{G_1 \times G_2} = T_{G_1} \times T_{G_2}$.

The cylinders T_G are manifolds with boundary E_G. For constructing Thom spaces, the boundary E_G is identified to a point. Clearly, one gets a homeomorphism $E_{G_1 \times G_2} \approx T_{G_1} \times E_{G_2} \cup E_{G_1} \times T_{G_2}$. This yields the existence of the homeomorphism (II). As for the commutativity of the diagram (III), it follows from the geometric meaning of the vertical isomorphisms (see proof of Theorem IV.4 in [16]). The lemma is proved.

Thom showed that the space M_e is homeomorphic to the sphere S^{n-i} if the unit group e is considered as a subgroup of $O(n-i)$. On the other hand, for each polyhedron K, the polyhedron $K \times S^{n-i}/K \vee S^{n-i}$ is homeomorphic to the iterated suspension $E^{n-i}K$ over K. Let the group G coincide with one of the classical Lie groups: $SO(k)$, $U(k)$, $SU(k)$, $Sp(k)$. The natural embeddings:

$$SO(k) \times e \subset SO(k+1), \quad \text{where} \quad e \in SO(1),$$
$$U(k) \times e \subset U(k+1), \quad \text{where} \quad e \in U(1),$$
$$SU(k) \times e \subset SU(k+1), \quad \text{where} \quad e \in SU(1),$$
$$Sp(k) \times e \subset Sp(k+1), \quad \text{where} \quad e \in Sp(1),$$

evidently define the mappings

$$\begin{aligned} EM_{SO(k)} \to M_{SO(k+1)}, \qquad & E^2 M_{U(k)} \to M_{U(k+1)}, \\ E^2 M_{SU(k)} \to M_{SU(k+1)}, \qquad & E^4 M_{Sp(k)} \to M_{Sp(k+1)}. \end{aligned} \tag{IV}$$

(To construct these mappings, one should apply Lemma 2.2. Recall that the group inclusions $G \subset \overline{G} \subset SO(k)$ produce natural mappings $M_G \to M_{\bar{G}}$.)

It is easy to show that the mappings (IV) in the stable dimensions, are homotopy equivalences. Indeed (IV) commutes with the Thom isomorphism $\varphi\colon H^i(B_G) \to H^{k+i}(M_G)$, where $G \subset SO(k)$. But it is well known that for small i the mapping of the group classifying spaces from formulae (IV) (Grassmann manifolds), gives an isomorphism of cohomology and cohomology groups. The Thom spaces are simply connected, thus the mappings (IV) are homotopy equivalences in stable range dimensions.

Denote the groups $V^i(S^n, SO(n-i))$ by V^i_{SO} if $i < [n/2]$; denote the groups $V^i\Big(S^n, U\Big(\dfrac{n-i}{2}\Big)\Big)$ by V^i_U if $n-i$ is odd and $i < [n/2]$; denote the groups $V^i\Big(S^n, SU\Big(\dfrac{n-i}{2}\Big)\Big)$ by V^i_{SU} under the same assumptions, denote the groups $V^i\Big(S^n, Sp\Big(\dfrac{n-i}{4}\Big)\Big)$, for $n-i \equiv 0 \pmod 4$ and $i < [n/2]$, by V^i_{Sp}. From the above, it follows that this is well defined because of stabilisation of homotopy groups of Thom spaces.

By means of the pairing (I), the direct sums $V_{SO} = \sum_{i\geqslant 0} V^i_{SO}$, $V_U = \sum_{i\geqslant 0} V^i_U$, $V_{SU} = \sum_{i\geqslant 0} V^i_{SU}$, $V_{Sp} = \sum_{i\geqslant 0} V^i_{Sp}$ naturally acquire a graded ring structure. We shall denote these rings also by V_{SO}, V_U, V_{SU} and V_{Sp}, respectively.

§ 3. Cohomology of Thom spaces modulo p, where $p > 2$

Consider the classifying spaces $B_{SO(2k)}$, $B_{U(k)}$, $B_{SU(k)}$, $B_{Sp(k)}$. Their modulo p cohomology algebras are well known (see [4]). Namely, $H^*(B_{SO(2k)})$

is the polynomial algebra in the Pontrjagin classes $p_{4i} \in H^{4i}(B_{SO(2k)}, Z_p)$, where $0 \leqslant i < k$, and the class $W_{2k} \in H^{2k}(B_{SO(2k)}, Z_p)$. The algebra $H^*(B_{U(k)}, Z_p)$ is isomorphic to the polynomial algebra in the generators $c_{2i} \in H^{2i}(B_{U(k)}, Z_p)$, where $0 \leqslant i \leqslant k$; the algebra $H^*(B_{SU(k)}, Z_p)$ is isomorphic to the polynomial algebra in $c_{2i} \in H^{2i}(B_{SU(k)}, Z_p)$, where $i \neq 1$; the algebra $H^*(B_{Sp(k)}, Z_p)$ is isomorphic to the polynomial algebra in $k_{4i} \in H^{4i}(B_{Sp(k)}, Z_p)$, where $0 \leqslant i \leqslant k$. Here the generators c_{2i} are the Chern classes reduced modulo p and the generators k_{4i} are the symplectic Borel classes (see [5]), reduced modulo p. Thom has shown that the algebra $H^*(M_{SO(2k)}, Z_p)$ is isomorphic to the ideal of the algebra $H^*(B_{SO(2k)}, Z_p)$ generated by the element W_{2k} (in positive dimensions). We wish to prove the analogous statements for other classical Lie groups. The following lemma holds:

Lemma 3.3. *The homomorphism*

$$j^* \colon H^*(M_G, Z_p) \to H^*(B_G, Z_p),$$

generated by the natural inclusion $j \colon B_G \subset M_G$, enjoys the following properties for the classical Lie groups:

$$G = SO(2k), \quad G = U(k), \quad G = SU(k), \quad G = Sp(k)\colon$$

a) the homomorphism j^* is a monomorphism;

b) $\operatorname{Im} j^*$ *is equal to the ideal generated by the element w_{2k} for the group $SO(2k)$, by the element c_{2k} for the groups $U(k)$ and $SU(k)$ and by the element k_{4k} for $Sp(k)$.*

Proof. Consider the space E_G of the classifying G-bundle of spheres S^{2k-1} if the group coincides with one of the groups $SO(2k)$, $U(k)$ or $SU(k)$, and the bundle of spheres S^{4k-1} if $G = Sp(k)$.

By construction of the Thom space M_G (see § 2), its cohomology algebra $H^*(M_G)$ can be identified in low dimensions with the algebra $H^*(T_G, E_G)$, where T_G is the cylinder of the projection P_G of the classifying sphere bundle. We can write down the exact cohomology sequence of the pair (T_G, E_G):

$$\ldots \to H^i(T_G) \xrightarrow{p_G^*} H^i(E_G) \xrightarrow{\delta} H^{i+1}(T_G, E_G) \xrightarrow{j^*} H^{i+1}(T_G) \to \ldots. \quad (V)$$

The space T_G is homotopy equivalent to the space B_G, and the homomorphisms $H^*(T_G) \to H^*(E_G)$ and $H^*(T_G, E_G) \to H^*(E_G)$, generated by inclusions, evidently coincide with the homomorphisms $p_G^* \colon H^*(B_G) \to H^*(E_G)$ and $j^* \colon H^*(M_G) \to H^*(B_G)$.

One can study the homomorphism p_G^* by using the spectral sequence of the classifying G-bundle of spheres. But this spectral sequence in our case is well studied (see [3]). In the spectral sequence of this spherical bundle, the following relations hold:

$$E_2 \approx H^*(S^{2k-1}, Z_p) \otimes H^*(B_G, Z_p)$$

if $G = SO(2k)$, $G = U(k)$, $G = SU(k)$, and

$$E_2 \approx H^*(S^{4k-1}, Z_p) \otimes H^*(B_G, Z_p)$$

if $G = Sp(k)$. Denote the generator of the group $H^*(S^{2k-1}, Z_p)$ by V^{2k-1} and denote that of the group $H^*(S^{4k-1}, Z_p)$ by V^{4k-1} (we choose the generator coinciding with the integral generator reduced modulo p). It is well known (see [3]) that $d_{2k}(v^{2k-1} \otimes 1) = 1 \otimes w_{2k}$, that $d_{4k}(v^{4k-1} \otimes 1) = 1 \otimes k_{4k}$ for the group $Sp(k)$ and $d_{2k}(v^{2k-1} \otimes 1) = 1 \otimes c_{2k}$ for the groups $U(k)$ and $SU(k)$ in the corresponding spectral sequences. Clearly, $E_\infty \approx E_{4k+1}$ for the group $Sp(k)$ and $E_\infty \approx E_{2k+1}$ for other Lie groups. Addressing the spectral sense of the homomorphism p_G^*, we see that in all the cases above the homomorphism p_G^* is an epimorphism to the algebra $H^*(E_G, Z_p)$ with the desired ideal being the kernel. The homomorphism δ in the exact sequence (V) is trivial. Lemma 3.1 is proved.

Remark. It is easy to see that the proof of Lemma 3.1 works even for $p = 2$ for all groups except for $SO(2k)$. Thus, we shall not describe these special cases in the next subsection.

Our main goal is to study the action of Steenrod's powers on the homology of Thom spaces. Following [16], we shall use "Wu's generators" defined in [4].

Consider symbolic two-dimensional elements $t_1, \dots, t_k$. We shall not make any assumptions about them. In the polynomial algebra $P(t_1, \dots, t_k)$, we select the subalgebra of symmetric polynomials. Set $c_{2i} = \sum t_1 \circ \dots \circ t_i$, $p_{4i} = k_{4i} = \sum t_1^2 \circ \dots \circ t_i^2$ for $i < k$, $w_{2k} = c_{2k} = t_1 \circ \dots \circ t_k$ and $k_{4k} = w_{2k}^2$. We can calculate completely the Steenrod operation for any arbitrary polynomial by using Cartan's formulae. Furthermore, set $\beta(t_i) = 0$ $(i = 1, \dots, k)$, where β is the Bockstein homomorphism. Now, with each decomposition ω of a positive integer q into positive integer summands $q_1, \dots, q_s$ (unordered) we associate the symmetrised monomial $\sum t_1^{q_1} \circ \dots \circ t_s^{q_s}$. We denote this monomial by v_ω. Analogously, with each decomposition $\overline{\omega}$ of an even positive integer $2q$ into even positive summands $2q_1, \dots, 2q_s$ (unordered) we associate the symmetrised monomial $\sum t_1^{2q_1} \circ \dots \circ t_s^{2q_s} = v_{\bar{\omega}}$. It is known that the polynomial algebra in c_{2i} (considered as symmetric polynomials of "Wu's generators") is isomorphic, as a module over the Steenrod algebra, to the cohomology algebra $H^*(B_{U(k)}, Z_p)$ (analogously for the cohomology algebras $H^*(B_{SO(2k)}, Z_p)$ and $H^*(B_{Sp(k)}, Z_p)$).

The algebra $H^*(B_{SU(k)}, Z_p)$ is isomorphic, as a module over the Steenrod algebra, to the quotient of the algebra $H^*(B_{U(k)}, Z_p)$ by the ideal generated by c_2. Applying Lemma 3.3 and the results of Borel-Serre [4], we shall represent the cohomology of Thom's spaces via symmetric polynomials of "Wu's generators". We call the decompositions ω (or $\overline{\omega}$) p-adic, if at least one of the summands is equal to $p^i - 1$. Recall that in [6], Cartan associated with cohomology operation a certain number, the *type* in Cartan's sense (the number of occurrences of the Bockstein homomorphism in the iterated operation).

Lemma 3.4. *All zero-type Steenrod operations in modulo p cohomologies of the spaces $M_{SO(2k)}$, $M_{U(k)}$, $M_{SU(k)}$, $M_{Sp(k)}$ are trivial. The Steenrod operations of zero type of the elements W_{2k} and $V_{\overline{\omega}} \circ W_{2k}$ for all non-p-adic decompositions $\overline{\omega}$ are independent in dimensions less than $4k$, and form a Z_p-basis in these dimensions of the algebra $H^*(M_{SO(2k)}, Z_p)$. The zero-type Steenrod operations of elements c_{2k} and $V_\omega \circ c_{2k}$ for all non-p-adic decompositions ω are independent and form a Z_p-basis of the algebra $H^*(M_{U(k)}, Z_p)$ in dimensions less than $4k$. The Steenrod operations of zero-tpye of the elements k_{4k} and $V_{\overline{\omega}} \circ k_{4k}$ for all non-p-adic decompositions $\overline{\omega}$ are independent and form a Z_p-basis of the algebra $H^*(M_{Sp(k)}, Z_p)$ in dimensions less than $8k$.*

(We recall again that in the above text we denoted by $\overline{\omega}$ decompositions of even numbers into even summands, and by ω decompositions of arbitrary positive integers into integer summands; and we defined the polynomials $V_{\overline{\omega}}$ and V_ω.)

We do not give a proof of the above lemma. It repeats the arguments of Thom (see [16]) and Cartan (see [6]). For the group $SO(2k)$, it is given [2].

For convenience of further formulations, introduce graded modules over the Steenord algebra (to be denoted by $H_{SO}(p)$, $H_U(p)$, $H_{SU}(p)$ and $H_{Sp}(p)$), whose homogeneous summands are stable cohomology groups of the correponding Thom spaces modulo p (here p might be equal to two). Lemma 3.4, is, actually, a statement about these modules.

§ 4. Cohomology of Thom spaces modulo 2

It follows from Remark in § 3 that we can apply the same method for studying the cohomology of the Thom spaces $M_{U(k)}$ and $M_{Sp(k)}$ modulo 2. However, the method of [16] does not work for the groups $SO(2k)$ and $SU(k)$ because the cohomology of the classifying spaces for these groups (viewed as modules over the Steenord algebra) are not described in terms of polynomial subalgebras in "Wu's generators" (see § 3).

Consider the iterated Steenrod squares Sq^J corresponding to admissible sequences $J = (i_1, \ldots, i_s)$ in the sense of Serre (see [14]). We shall write $J \equiv 0 \pmod q$ if all $i_j \equiv 0 \pmod q$, and $J \equiv m \pmod q$, if at least one $i_j \equiv m \pmod q$, where m and q are positive integers. We shall say that the Steenrod operation Sq^J has type m modulo q if $J \equiv m \pmod q$.

The description of the cohomology of the classifying spaces $B_{U(k)}$ and $B_{Sp(k)}$, given in § 3, is applicable here even modulo 2 (it is applicable for integers). Following § 3, let us introduce the symmetrised monomials v_ω for every decomposition $\omega = (a_1, \ldots, a_t)$ of a positive integer into positive summands, and the symmetrised monomials $v_{\overline{\omega}}$ for every decomposition $\overline{\omega} = (2a_1, \ldots, 2a_t)$ of an even positive integer into even summands. The decompositions ω and $\overline{\omega}$ are assumed unordered.

Lemma 4.5. *All non-zero type Steenrod operations modulo two act trivially on modulo two cohomology 2 of the Thom spaces $M_{U(k)}$, $M_{SU(k)}$ and $M_{Sp(k)}$. The non-zero type Steenrod operations modulo 4 act trivially on the cohomology of the Thom space $M_{Sp(k)}$ modulo 2. The Steenrod operations Sq^J of zero type modulo 2 of arbitrary elements $V_\omega \circ c_{2k}$ and c_{2k}, where the decompositions ω contain no summands of type $2^t - 2$, are independent and form a Z_2-basis of the algebra $H^*(M_{U(k)}, Z_2)$ in dimensions less than $4k$. The Steenrod operations Sq^J of zero type modulo 4 of arbitrary elements $V_{\overline{\omega}} \circ k_{4k}$ and k_{4k}, where $\overline{\omega}$ contain no summands of type $2^t - 4$, are independent and form a Z_2-basis of the algebra $H^*(M_{Sp(k)}, Z_2)$ in dimensions less than $8k$.*

The proof of this lemma is quite analogous to the proof of Lemma 3.4, one can just repeat Thom's arguments (see [16], Lemma II.8, Lemma II.9, Theorem II.10).

Now, let us try to study the modulo 2 cohomology of the Thom spaces $M_{SO(k)}$ and $M_{SU(k)}$. As above, we shall consider only cohomology in stable dimensions. Analogously to the previous lemmas, introduce unordered decompositions $\overline{\omega} = (a_1, \ldots, a_t)$ of a positive integer $a = \sum a_i$, $a \equiv 0 \pmod 4$, into summands a_i $(i = 1, \ldots, t)$, also positive, $a_i \equiv 0 \pmod 4$.

Lemma 4.6. *In the algebra $H^*(M_{SO(k)}, Z_2)$, one may choose a system of elements $u_{\bar{\omega}} \in H^{k+a}(M_{SO(k)}, Z_2)$ for all decompositions $\overline{\omega} = (a_1, \ldots, a_t)$ of the numbers $a \equiv 0 \pmod 4$ into summands $a_i \equiv 0 \pmod 4$ and a system of elements $x_l \in H^{k+i_l}(M_{SO(k)}, Z_2)$ such that:*

a) all Steenrod operations Sq^J of elements x_l are independent in dimensions less than $2k$;

b) all Steenrod operations Sq^J of $u_{\bar{\omega}}$ and $w_k \in H^k(M_{SO(k)}, Z_2)$ are independent with the operations of x_l, if $J = (i_1, \ldots, i_s)$, where $i_s > 1$, and the dimensions of the elements $Sq^J(u_{\bar{\omega}})$ and $Sq^J(w_k)$ are less than $2k$;

c) the elements $Sq^J(u_{\bar\omega})$ and $Sq^J(w_k)$ are equal to zero if $J = (i_1, \ldots, i_s)$, where $i_s = 1$;

d) in the algebra $H^*(M_{SO(k)}, Z_2)$, all elements of type $Sq^J(u_{\bar\omega})$, $Sq^J(w_k)$, $Sq^J(x_l)$ form a Z_2-basis in dimensions less than $2k$.

Before proving Lemma 4.2, note that the results of this lemma allow one to describe the action of Steenrod squares in the modulo 2 cohomology of the space $M_{SU(k)}$. To perform this deed, recall the description of the algebras $H^*(B_{O(k)}, Z_2)$ and $H^*(B_{SO(k)}, Z_2)$ via one-dimensional "Wu generators" $y_1, \ldots, y_k$. Set $w_i = \sum y_1 \circ \ldots \circ y_i$, where $i \leqslant k$. It is evident that $w_k = y_1 \circ \ldots \circ y_k$, and all Steenrod operations of w_i's are then calculated by Cartan's formulae. Wu has shown that the algebra $H^*(B_{O(k)}, Z_2)$ is isomorphic, as a module over the Steenrod algebra, to the algebra $P(w_1, \ldots, w_k)$ and that the algebra $H^*(B_{SO(k)}, Z_2)$ is isomorphic, as a module over the Steenrod algebra, to the quotient of the algebra $P(w_1, \ldots, w_k)$ by the ideal generated by w_1. Here is an evident analogy with the description of algebras $H^*(B_{U(k)}, Z_2)$ and $H^*(B_{SU(k)}, Z_2)$ via two-dimensional Wu generators $t_1, \ldots, t_k$ (see § 3).

In [1], the author defines an endomorphism of the Steenrod algebra $A = A_2$ over Z_2

$$h\colon A \to A \qquad (VI)$$

such that $h(Sq^{2i}) = Sq^i$ and $h(Sq^{2i+1}) = 0$.

Consider the isomorphism $\mu\colon P(t_1, \ldots, t_k) \to P(y_1, \ldots, y_k)$ of graded algebras over Z_2 decreasing the dimension twice. Clearly, the isomorphism μ enjoys the following property:

$$\mu(Sq^J(x)) = h(Sq^J)(\mu(x)), \qquad (VII)$$

for $x \in P(t_1, \ldots, t_k)$. The isomorphism μ induces isomorphisms

$$\mu_1\colon H^*(B_{U(k)}, Z_2) \to H^*(B_{O(k)}, Z_2)$$

and

$$\mu_2\colon H^*(B_{SU(k)}, Z_2) \to H^*(B_{SO(k)}, Z_2)$$

that also satisfy (VII), and are isomorphic to

$$\left.\begin{aligned} \lambda_1\colon H^*(M_{U(k)}, Z_2) \to H^*(M_{O(k)}, Z_2), \\ \lambda_2\colon H^*(M_{SU(k)}, Z_2) \to H^*(M_{SO(k)}, Z_2) \end{aligned}\right\} \qquad (VIII)$$

making the dimension two times smaller and possessing the property (VII). Clearly, $\lambda_1(U_{U(k)}) = U_{O(k)}$ and $\lambda_2(U_{SU(k)}) = U_{SO(k)}$. Thus, we get

Corollary 4.4. *There exists an isomorphism* $\lambda_2\colon H^*(M_{SU(k)}, Z_2) \to H^*(M_{SO(k)}, Z_2)$ *making the dimension two times smaller and such that* $\lambda_2(U_{SU(k)}) = U_{SO(k)}$ *and* $h(Sq^J \lambda_2(x)) = \lambda_2(Sq^J(x))$ *for all* $x \in H^i(M_{SU(k)}, Z_2)$, $j < 2k$.

Now, let us prove Lemma 4.6. It follows from Rokhlin's work [12] that the kernel of the mapping $i_*\colon \pi_m(M_{SO(k)}) \to \pi_m(M_{O(k)})$, generated by the inclusion $i\colon SO(k) \subset O(k)$, consists of all elements divisible by 2, for $m < 2k-1$. Denote by $\pi_m^{(2)}(M_{SO(k)})$ the quotient of the group $\pi_m(M_{SO(k)})$ by the subgroup consisting of all elements of odd order (in [2], it is shown that the groups $\pi_m(M_{SO(k)})$ do not contain elements of odd order, but we shall not rely on this result). It follows from [16] that one may choose systems of generators $x_i^{(m)}$ of the groups $\pi_m^{(2)}(M_{SO(k)})$ and $y_j^{(m)}$ for the groups $\pi_m(M_{O(k)})$ such that the mapping i_* takes the set $\{x_i^{(m)}\}$ to a subset of $\{y_j^{(m)}\}$. As Thom has shown, (see [16], II.6–II.10), the space $M_{O(k)}$ can be thought of as homotopy equivalent (in stable dimensions) to the direct product of Eilenberg-MacLane complexes. This defines a mapping i_1 from $M_{SO(k)}$ to the direct product Π of Eilenberg-MacLane complexes of types $K(Z_2, n_j)$, for which the generators of the homotopy groups are in one-to-one correspondence with the elements $x_i^{(m)}$, so that the mapping i_{1*} takes the element $x_i^{(m)}$ to the generator of this product corresponding to it. The fundamental classes of factors of this direct product, $u_i^{(m)}$, can be defined by the equalities $(u_i^{(m)}, i_1 x_i^{(m)}) = 1$, $\quad (u_i^{(m)}, x_{i'}^{(m')}) = 0$, if $i \neq i'$ or $m \neq m'$.

Denote by Π_m the subproduct of the product Π of Eilenberg-MacLane complexes, defined by elements of homotopy groups of dimensions greater than or equal to m. Denote by $i_1^{(m)}$ the projection of the mapping i_1 to Π_m. The following Serre fibre spaces are well known (see [14]): $p_m\colon \widehat{M}_{SO(k)} \xrightarrow{M_m} M^{(m)}$, where by $\widehat{M}_{SO(k)}$ we denote some space which is homotopy equivalent to $M_{SO(k)}$, and by $M^{(m)}$ we denote the space obtained from $M_{SO(k)}$ by "killing" all homotopy groups starting with the m-th. The fibres of these bundles are m-killing spaces for $M_{SO(k)}$. Denote them by $M_{(m)}$. We shall denote the generators of the groups $\pi_m^{(2)}(M_m)$ also by $x_i^{(m)}$. The group $H^m(M_{(m)} Z_2)$ is generated by the elements $v_i^{(m)}$, defined by

$$(v_i^{(m)}, x_i^{(m)}) = 1, \qquad (v_i^{(m)}, x_{i'}^{(m)}) = 0, \qquad i' \neq i.$$

We consider the space Π_m as a fibre space with base consisting of one point. The mapping $i_1^{(m)}$ induces the mapping $\hat{i}_1^{(m)}\colon M_{(m)} \to \Pi_m$.

Obviously, $\hat{i}_1^{(m)^*}(u_i^{(m)}) = v_i^{(m)}$. From the last statement, it follows immediately that in the Serre fibre space $p_m\colon \widehat{M}_{SO(k)} \xrightarrow{M_{(m)}} M^{(m)}$ the transgression is trivial for all elements $v_i^{(m)}$ because the mapping $i_1^{(m)}$ can be thought of as a mapping from this fibre space to the trivial one described above. This yields that all factors of the space $M_{SO(k)}$ in the sense of M.M. Postnikov (see [11]), reduced modulo 2, are trivial. The statement of Lemma 4.6 now follows from Theorem IV.15 of Thom (see [16]) and the fact that the cohomology groups $H_i(M_{SO(k)})$ have no elements of order 4 for $i < 2k - 1$. Here we mean cohomology groups with integer coefficients.

§ 5. Diagonal homomorphisms

Let K be an arbitrary polyhedron. Denote by $H^+(K, Z_p)$ its module over the Steenrod algebra $A = A_p$, with homogeneous summands being the cohomology groups of positive dimensions. Let K_1 and K_2 be two polyhedra. There is a well-known isomorphism

$$H^+(K_1 \times K_2/K_1 \vee K_2,\, Z_p) \approx H^+(K_1,\, Z_p) \otimes H^+(K_2,\, Z_p).$$

This is an A-module isomorphism (which makes sense because A is a Hopf algebra). This yields that the homeomorphism (II) from Lemma 2.2 defines diagonal homomorphisms generated by the above inclusions $SO(m) \times SO(n) \subset SO(m+n)$, $U(m) \times U(n) \subset U(m+n)$, $SU(m) \times SU(n) \subset SU(m+n)$, $Sp(m) \times Sp(n) \subset Sp(m+n)$:

$$\left.\begin{aligned} H^+(M_{SO(m+n)}) &\to H^+(M_{SO(m)}) \otimes H^+(M_{SO(n)}),\\ H^+(M_{U(m+n)}) &\to H^+(M_{U(m)}) \otimes H^+(M_{U(n)}),\\ H^+(M_{SU(m+n)}) &\to H^+(M_{SU(m)}) \otimes H^+(M_{SU(n)}),\\ H^+(M_{Sp(m+n)}) &\to H^+(M_{Sp(m)}) \otimes H^+(M_{Sp(n)}). \end{aligned}\right\} \qquad (IX)$$

We shall define all these homomorphisms by $\Delta_{m,n}$. The homomorphisms $\Delta_{m,n}$ denote for the modules $H_{SO}(p)$, $H_U(p)$, $H_{SU}(p)$, $H_{Sp}(p)$, (see § 3) the following homomorphisms Δ:

$$\left.\begin{aligned} H_{SO}(p) &\to H_{SO}(p) \otimes H_{SO}(p),\\ H_U(p) &\to H_U(p) \otimes H_U(p),\\ H_{SU}(p) &\to H_{SU}(p) \otimes H_{SU}(p),\\ H_{Sp}(p) &\to H_{Sp}(p) \otimes H_{Sp}(p). \end{aligned}\right\} \qquad (X)$$

The aim of this section is to calculate the homomorphisms (X). The following lemma holds.

Lemma 5.7. *For generators u_ω and $u_{\bar\omega}$ of the modules $H_{SO}(p)$, $H_U(p)$, $H_{Sp}(p)$ for all $p \geqslant 2$, the homomorphisms Δ look like:*

$$\left.\begin{aligned} \Delta(u_\omega) &= \sum_{\substack{(\omega_1,\omega_2)=\omega\\ \omega_1\neq\omega_2}} [u_{\omega_1}\otimes u_{\omega_2} + u_{\omega_2}\otimes u_{\omega_1}] + \sum_{(\omega_1,\omega_2)=\omega} u_{\omega_1}\otimes u_{\omega_1},\\ \Delta(u_{\bar\omega}) &= \sum_{\substack{(\bar\omega_1,\bar\omega_2)=\bar\omega\\ \bar\omega_1\neq\bar\omega_2}} [u_{\bar\omega_1}\otimes u_{\bar\omega_2} + u_{\bar\omega_2}\otimes u_{\bar\omega_1}] + \sum_{(\bar\omega_1,\bar\omega_2)=\bar\omega} u_{\bar\omega_1}\otimes u_{\bar\omega_1}. \end{aligned}\right\} \quad (XI)$$

Note that in formulas (XI) we admit decomposition ω_1 ($\overline{\omega}_1$), consisting of the empty set of summands. In this case, the generator u_{ω_1} ($u_{\bar\omega_1}$) for the empty summand ω_1 ($\overline{\omega}_1$) corresponds to the elements w_{2k}, c_{2k} or k_{4k}, as in previous sections. (By $u_{\bar\omega}$ we denote the generator of the module corresponding to the product $v_{\bar\omega} \circ w_{2k}$ or $v_{\bar\omega} \circ k_{4k}$ from Lemma 3.4. Here by u_ω we denote the generator of the module corresponding to the product $v_\omega \circ c_{2k}$, and also generators from Lemma 4.6.)

Proof. First consider the modules $H_{SO}(p)$ for $p > 2$. $H_U(p)$ for $p \geqslant 2$ and $H_{Sp}(p)$ for $p \geqslant 2$. Let us return to the description of Thom spaces by ideals in the cohomology of classifying spaces (see Lemma 3.3) and "Wu's generators". We are going to calculate the homomorphism (IX), by using Whitney's fomulae for Pontrjagin's, Chern's, and Borel's symplectic classes.

Let m and n be large enough and let $x_1, \ldots, x_m$, $y_1, \ldots, y_n$ be the symbolic two-dimensional "Wu generators". In the algebra $P(x_1, \ldots, x_m, y_1, \ldots, y_n)$, choose elementary symmetric polynomials in $x_1, \ldots, x_m$, $y_1, \ldots, y_n$ and $x_1^2, \ldots, x_m^2$, $y_1^2, \ldots, y_n^2$. The topological meaning of these polynomials was described above. Analogously, we take the elementary symmetric polynomials in the algebras $P(x_1, \ldots, x_m)$ and $P(y_1, \ldots, y_n)$ over the field Z_p. Note that the homomorphisms $\Delta_{m,n}$ should satisfy Whitney's formulae, and these formulae uniquely define the homomorphisms (IX). We set formally:

$$\Delta_{m,n}(x_i) = x_i \otimes 1, \qquad \Delta_{m,n}(y_j) = 1 \otimes y_j \qquad (XII)$$

for all $i \leqslant m$, $j \leqslant n$. We treat the set of elements x_i as the "Wu generators" for the algebras

$$H^*(B_{SO(2m)}, Z_p), \quad H^*(B_{U(m)}, Z_p) \quad \text{and} \quad H^*(B_{Sp(m)}, Z_p)$$

and the elements y_j as the "Wu generators" for the algebras

$$H^*(B_{SO(2n)}, Z_p), \quad H^*(B_{U(n)}, Z_p), \quad H^*(B_{Sp(n)}, Z_p).$$

If we apply (XII) to elementary symmetric polynomials, we can easily see that from (XII) one gets the Whitney formula for all characteristic classes mentioned above. Thus the homomorphisms $\Delta_{m,n}$ calculated according to (XI) and introduced formally coincide on the symmetric polynomials with the "geometric" homomorphisms $\Delta_{m,n}$. Now, let us apply formulae (XII) to the polynomials $v_{\bar{\omega}} \circ w_{2k}$, $v_\omega \circ c_{2k}$, $v_{\bar{\omega}} \circ k_{4k}$, w_{2k}, c_{2k} and k_{4k}. It is easy to see that they lead to the desired result. In order to prove (XI) for the modulo $H_{SO}(2)$, note that Pontrjagin's classes satisfy Whitney's formulae (without torsions). We take u_ω from Lemma 4.6, the modulo 2 reductions of polynomials in Pontrjagin classes and the class $w_{2(m+n)}$, corresponding to the symmetrised monomial

$$\sum x_1^{a_1+1} \circ \ldots \circ x_k^{a_s+1} \circ \ldots \circ x_m \circ y_1 \circ \ldots \circ y_n,$$

where $\omega = (a_1, \ldots, a_s)$ is an arbitrary decomposition of an odd number into odd summands, in elementary symmetric polynomials of squares x_1^2, y_j^2 and the polynomial $w_{2(m+n)} = x_1 \circ \ldots \circ x_m \circ y_1 \circ \ldots \circ y_n$. Arguing as above, we see that formula (XI) for these elements is valid up to some elements belonging to the image of Sq^1 (in integral homology with 2-torsion omitted). The lemma is proved.

Now, let us give the conclusion of this Chapter. In § 2 we associated with the group sequences $\{G_i = SO(i)\}$, $\{G_i = U(i)\}$, $\{G_i = SU(i)\}$, $\{G_i = Sp(i)\}$, graded rings V_{SO}, V_U, V_{SU}, V_{Sp}. We shall call them inner homology rings. On the other hand, in §§ 3–5 we associated with the same sequences the graded modules over the Steenrod algebra: $H_{SO}(p)$, $H_U(p)$, $H_{SU}(p)$, $H_{Sp}(p)$ for all prime $p \geqslant 2$. These modules were calculated in §§ 3–4. In § 5 we associated with these modules the diagonal mappings (X).

The aim of the next chapter is to calculate the inner homology rings by using Adams' spectral method.

Chapter II

Inner homology rings

This chapter, as mentioned above, is devoted to the calculation of inner homology rings. The main theorems of this chapter are formulated in §§ 4–5. In the first three sections, we study modules extensions over the Steenrod algebra.

§ 1. Modules with one generator[1]

Let A be the graded associative algebra $A = \sum_{i \geqslant 0} A^{(i)}$ over Z_p. As usual, we assume that $A^{(0)} = Z_p$ and $A^{(i)}$ are finite-dimensional linear spaces over the ground field. Consider the graded A-module M with one generator u of dimension 0 endowed with some homogeneous Z_p-basis $\{x_i^{(m)}\}$, $x_i^{(m)} \in M^{(m)}$. The free A-module with one generator is also to be denoted by A if this generator has dimension 0, in this case, this module is identified with the algebra A. We denote the generator by 1 and identify it with the unit of A. Clearly, there is a well-defined canonical A-modules mapping $\varepsilon\colon A \to M$ such that $\varepsilon(1) = u$. Denote by $\overline{A}$, as usual, the ideal of A generated by all elements of positive dimension.

Let B be a graded subalgebra of A, $B = \sum_{i \geqslant 0} B^{(i)}$, $B^{(0)} = Z_p$ and $B^{(i)} = B \cap A^{(i)}$. Denote by M_B the one-generator module equal to $A/A \circ \overline{B}$. Denote its Z_p-basis, as before, by $x_i^{(m)}$, and denote its generator by $u = \varepsilon(1)$, where ε is the natural homomorphism $A \to A/A \circ \overline{B}$. Denote by $\{y_j^{(k)}\}$ any homogeneous Z_p-basis of B, and in each set $\varepsilon^{-1}(x_i^{(m)})$ choose and fix one element $z_i^{(m)} \in \varepsilon^{-1}(x_i^{(m)})$. The elements $z_i^{(m)}$ are assumed homogeneous of the same dimension as $x_i^{(m)}$.

Definition 1.1. The subalgebra B of A is special if all possible products $z_i^{(m)} \circ y_j^{(k)}$ form a homogeneous Z_p-basis of the algebra A, and they are independent.

As usual, we shall endow the field Z_p with the trivial A-module structure.

Lemma 1.1. *Assume the subalgebra B of A is special. In this case we have an isomorphism:*

$$\mathrm{Ext}_A^{s,t}(M_B, Z_p) \approx \mathrm{Ext}_B^{s,t}(Z_p, Z_p). \qquad (XIII)$$

PROOF. Let $C_B(Z_p)$ denote the B-free standard complex of the algebra B (see [8]). To prove the isomorphism (XIII), we shall construct an A-free acyclic complex $C_A(M_B)$, such that there exists a differential isomorphism

$$\mathrm{Hom}_B^{s,t}(C_B(Z_p), Z_p) \approx \mathrm{Hom}_A^{s,t}(C_A(M_B), Z_p) \qquad (XIV)$$

for all pairs s, t. From the isomorphism (XIV) that for all s, t commutes with the differential, we easily get (XIII). For constructing the complex $C_A(M_B)$, we shall use Z_p-bases of our algebras and of the module M_B, given in Definition 1.1 of this section, with the same notation.

[1] We consider only left A-modules.

For the module $C^0_A(M_B)=\sum_t C^{0,t}_A(M_B)$, we take the free A-module, and define the mapping $\varepsilon\colon A\to M_B$ such as it is defined in the beginning of this section. Obviously, for generators of the A-module $\operatorname{Ker}\varepsilon\subset C^0_A(M_B)$ we may take all possible elements $y^{(k)}_j$ for $k>0$. Each element $y\in\operatorname{Ker}\varepsilon$ looks like $\sum_q z^{(m_q)}_{i_q}\circ y^{(k_q)}_{j_q}$, where all $k_q>0$. It follows immediately from Definition 1.1 that all A-relations between the generators $y^{(k)}_j$ of the module $\operatorname{Ker}\varepsilon$ follow from the multiplicative relations in B. We shall construct the complex $C_A(M_B)=\sum_{s,t}C^{s,t}_A(M_B)$ by induction on s. Assume that:

a) the complex $C_A(M_B)$ is constructed for all $s\leqslant n$;

b) generators of the A-module $\operatorname{Ker}d_{n-1}\colon C^n_A(M_B)\to C^{n-1}_A(M_B)$ are in one-to-one correspondence with sequences of homogeneous elements $(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$ of positive dimension; denote these generators by $u(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$, and denote for each element $y_i=\sum q_l y^{(k_{i,l})}_{j_{i,l}}$ by $u(y^{(k_1)}_{j_1},\dots,y_i,\dots,y^{(k_{n+1})}_{j_{n+1}})$ the linear combination $\sum_l q_l u(y^{(k_1)}_{j_1},\dots,y^{k_{i,l}}_{j_{i,l}},\dots,,y^{(k_{n+1})}_{j_{n+1}})$ of generators of the kernel $\operatorname{Ker}d_{n-1}$ (for each $1\leqslant i\leqslant n+1$);

c) generators of the A-module $\operatorname{Ker}d_{n-1}$ satisfy the following relations:

$$y\circ u(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})=u(y\circ y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})+\sum_{i=1}^{n}(-1)^i u(y,y^{(k_1)}_{j_1},\dots,y^{(k_i)}_{j_i}\circ y^{(k_{i+1})}_{j_{i+1}},\dots,y^{(k_{n+1})}_{j_{n+1}}),$$

where $y\in\overline{B}$.

d) all relations are linear combinations of right-hand sides of relations from c), multiplied from the left by $z^{(m)}_i\circ y^{(k)}_j$ (and trivial relations).

e) the dimension of $u(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$ is equal to the sum of the dimensions of $y^{(k_i)}_{j_i}$ $(i=1,\dots,n+1)$.

Clearly, these assumptions are proved above $n=0$. Now, let us construct a module $C^{n+1}_A(M_B)=\sum_t C^{n+1,t}_A(M_B)$ and a mapping $d_n\colon C^{n+1}_A(M_B)\to C^n_A(M_B)$. We choose A-generators of the free module $C^{n+1}_A(M_B)$ in such a way that they are in one-to-one correspondence with the elements $u(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$. We denote these free generators by $v(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$. We set the dimension of $v(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$ to be equal to $\sum k_i$, as well as that of the generator $u(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}})$. We set:

$$d_{n+1}(v(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}}))=u(y^{(k_1)}_{j_1},\dots,y^{(k_{n+1})}_{j_{n+1}}).$$

Let us prove that the kernel $\operatorname{Ker}d_{n+1}$ satisfies the assumptions b)–e).

We set

$$u(y_{j_1}^{(k_1)}, \ldots, y_{j_{n+2}}^{(k_{n+2})}) = y_{j_1}^{(k_1)} \circ v(y_{j_2}^{(k_2)}, \ldots, y_{j_{n+2}}^{(k_{n+2})}) - \sum_{i=1}^{n+1} (-1)^i v(y_{j_1}^{(k_1)}, \ldots, y_{j_i}^{(k_i)} \circ y_{j_{i+1}}^{(k_{i+1})}, \ldots, y_{j_{n+2}}^{(k_{n+2})}).$$

The properties c) and e) can be checked straightforwardly. To prove d), let us use the properties of bases of the algebra A. Let us compose a relation, a linear combination of elements of type $z_j^{(m)} \circ y_q^{(l)} \circ u(y_{j_1}^{(k_1)}, \ldots, y_{j_{n+2}}^{(k_{n+2})})$, equal to zero. Then we see that d) easily follows from the properties of bases and the induction hypotheses.

The resolvent $C_A(M_B)$ that we have constructed, evidently, satisfies (XIV) by definition of the standard complex $C_B(Z_p)$ of the algebra B. The lemma is proved.

Now let A be a Hopf algebra and let B be a special subalgebra which is closed with respect to the diagonal mapping $\psi\colon A \to A \otimes A$. In this case, the A-module M_B has also a diagonal mapping

$$\widetilde{\psi}\colon M_B \to M_B \otimes M_B,$$

induced by ψ and which is an A-module homomorphism. The homomorphism $\widetilde{\psi}$ can be considered, as well as the homomorphism from the A-module M_B to the $A \otimes A$-module $M_B \otimes M_B$, such that: $\widetilde{\psi}(a \circ x) = \psi(a) \circ \widetilde{\psi}(x)$, for $a \in A$, $x \in M_B$. The homomorphism $\widetilde{\psi}$ endows the direct sum

$$\mathrm{Ext}_A(M_B, Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(M_B, Z_p)$$

with a bigraded algebra structure over Z_p.

Lemma 1.2. *Let A be a Hopf algebra with a special subalgebra B such that B is closed with respect to the diagonal mapping. Then (XIII) is a graded algebra isomorphism.*

The proof of Lemma 1.2 follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{B\otimes B}(Z_p, Z_p) & \approx & \mathrm{Ext}_{A\otimes A}(M_{B\otimes B}, Z_p)^1 \\ \downarrow \psi^* & & \downarrow \widetilde{\psi^*} \\ \mathrm{Ext}_B(Z_p, Z_p) & \approx & \mathrm{Ext}_A(M_B, Z_p) \end{array}$$

[1]It remains to note that the algebra $B \otimes B$ is special in $A \otimes A$ and the $A \otimes A$-modules $M_B \otimes M_B$ and $M_{B\otimes B}$ are canonically isomorphic.

§ 2. Modules over the Steenrod algebra. The case of prime $p > 2$

In [1], the families of elements e'_r, $r \geqslant 0$, and $e_{r,k}$, $r \geqslant 1$, $k \geqslant 0$, in the Steenrod algebra A over Z_p are defined. These elements possess the following properties:

a) $e'_r \in A^{(2p^r-1)}$, $e_{r,k} \in A^{k(2p^r-2)}$, $e_{r,0} = 1$;

b) $\psi(e'_r) = e'_r \otimes 1 + 1 \otimes e'_r$, $\psi(e_{r,k}) = \sum_{i+j=k} e_{r,i} \otimes e_{r,j}$,
where ψ is a Hopf homomorphism of Steenrod algebras;

c) if we order the set of elements e'_r and positive-integer valued functions $f_r(k) = e_{r,k}$ somehow, then the set of monomials which are products of e'_r and $e_{r,k}$ with arbitrary arguments k_r substituted, forms a basis of A;

d) the elements $e_{r,k}$ have Cartan zero type (see [6]), the elements e'_r are of Cartan type 1;

e) The elements e'_r anti-commute.

Later on, we shall use these properties of Adams' elements in the Steenrod algebra. Let us return to our modules $H_{SO}(p)$, $H_U(p)$, $H_{Sp}(p)$. The formulae (X) § 5 Ch. I define diagonal homomorphisms for these modules. Thus, the bigraded groups

$$\mathrm{Ext}_A(H_{SO}(p),\, Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(H_{SO}(p),\, Z_p),$$

$$\mathrm{Ext}_A(H_U(p),\, Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(H_U(p),\, Z_p),$$

$$\mathrm{Ext}_A(H_{Sp}(p),\, Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(H_{Sp}(p),\, Z_p).$$

have a natural structure of bigraded algebras. Furthermore, recall that in § 3 of Chapter I we introduced decompositions ω and $\overline{\omega}$ of some numbers of the same type; we call twice the sum of this numbers the dimension of the decomposition ω ($\overline{\omega}$) and denote it by $R(\omega)$ ($R(\overline{\omega})$); also, we introduced the notion of p-adic decomposition ω ($\overline{\omega}$).

Theorem 2.1. *The algebras* $\mathrm{Ext}_A(H_{SO}(p), Z_p)$ *and* $\mathrm{Ext}_A(H_{Sp}(p), Z_p)$ *are isomorphic and they are polynomial algebras with the following generators:*

$$\left.\begin{aligned} &1 \in \mathrm{Ext}_A^{0,0}(H_{Sp}(p),\, Z_p),\\ &z_{4k} \in \mathrm{Ext}_A^{0,4k}(H_{Sp}(p),\, Z_p), \quad 2k \neq p^i - 1,\\ &h'_r \in \mathrm{Ext}_A^{1,2p^r-1}(H_{Sp}(p),\, Z_p), \quad r \geqslant 0. \end{aligned}\right\} \qquad (XV)$$

The algebra $\operatorname{Ext}_A(H_U(p), Z_p)$ *is a polynomial algebra of*

$$\left.\begin{aligned} &1 \in \operatorname{Ext}_A^{0,0}(H_U(p), Z_p),\\ &z_{2k} \in \operatorname{Ext}_A^{0,2k}(H_U(p), Z_p), \quad k \neq p^i - 1,\\ &h'_r \in \operatorname{Ext}_A^{1,2p^r-1}(H_U(p), Z_p), \quad r \geqslant 0. \end{aligned}\right\} \tag{XVI}$$

PROOF. Denote by M_β the module over the Steenrod algebra with one generator u of dimension 0, and the only non-trivial relation being $\beta(x) = 0$ for all $x \in M_\beta$.

Obviously, the module M_β has the diagonal mapping $\Delta\colon M_\beta \to M_\beta \otimes M_\beta$, and the group $\operatorname{Ext}_A(M_\beta, Z_p) = \sum_{s,t} \operatorname{Ext}_A^{s,t}(M_\beta, Z_p)$ is an algebra.

Lemma 2.3. *The algebra* $\operatorname{Ext}_A(M_\beta, Z_p)$ *is a polynomial algebra with the following generators:*

$$1 \in \operatorname{Ext}_A^{0,0}(M_\beta, Z_p), \qquad h'_r \in \operatorname{Ext}_A^{0,2p^r-1}(M_\beta, Z_p), \qquad r \geqslant 0.$$

PROOF. To prove this lemma, let us use properties a)–e) of the Adams elements and Lemma 1.2 § 1, Chapter II (see above). We define B to be the subalgebra of A generated by the elements e'_r, $r \geqslant 0$, and $e_{r,0} = 1$. Now, let us order the Adams elements in such a way that all elements e'_r precede from the left all elements $e_{r,k}$, and define the basis of the Steenrod algebra by property c), for this ordering. Clearly, for the algebra B all assumptions of Lemma 1.2 § 1, Chapter II, hold. The algebra B is the exterior algebra with generators $e'_r \in B^{(2p^r-1)}$; thus we see that its cohomology algebra $H^*(B) = \operatorname{Ext}_B(Z_p, Z_p)$ is a polynomial algebra. On the other hand, it follows from d) that in this case $M_B = M_\beta$, which yields the conclusion of the lemma. The lemma is proved.

Now, let us use Lemmas 3.4 and 5.7 of Chapter I. We see that the modules $H_{SO}(p)$, $H_U(p)$ and $H_{Sp}(p)$ are direct sums of modules of type M_β, the only difference being that their generators u_ω and $u_{\bar\omega}$, except one, have non-zero dimension which is equal to $R(\omega)$ or $R(\overline{\omega})$, respectively. Denote by $z_\omega \in \operatorname{Ext}_A^{0,R(\omega)}(H_U(p), Z_p)$, by $z_{\overline{\omega}} \in \operatorname{Ext}_A^{0,R(\overline{\omega})}(H_{SO}(p), Z_p)$ and by $z_{\overline{\omega}} \in \operatorname{Ext}_A^{0,R(\overline{\omega})}(H_{Sp}(p), Z_p)$ the elements of these algebras defined by the following equalities:

$$\left.\begin{aligned} &(z_\omega, u_\omega) = 1, \qquad (z_\omega, u_{\omega_1}) = 1, \qquad \omega_1 \neq \omega,\\ &(z_{\overline{\omega}}, u_{\overline{\omega}}) = 1, \qquad (z_{\overline{\omega}}, u_{\overline{\omega}_1}) = 1, \qquad \overline{\omega}_1 \neq \overline{\omega}. \end{aligned}\right\} \tag{XVII}$$

From Lemma 5.7, Chapter I it follows that in the algebras $\operatorname{Ext}_A(H_{SO}(p), Z_p)$, $\operatorname{Ext}_A(H_U(p), Z_p)$, $\operatorname{Ext}_A(H_{Sp}(p), Z_p)$ the following relations hold:

$$z_\omega \circ z_{\omega_1} = z_{(\omega,\omega_1)}, \qquad z_{\overline{\omega}} \circ z_{\overline{\omega}_1} = z_{(\overline{\omega},\overline{\omega}_1)}$$

for all non p-adic decompositions ω, ω_1 and $\overline{\omega}$, $\overline{\omega}_1$. Now, it suffices to define the generators z_{4k} and z_{2k} to be the elements $z_{\overline{\omega}}$ and z_ω, where $\overline{\omega}$ and ω consist of one summand each. The theorem is proved.

§ 3. Modules over the Steenrod algebra. The case $p = 2$

In [1], modulo 2 bases of the Steenrod algebra were studied. Namely, Adams defined a family of elements $e_{r,k} \in A^{(k\cdot 2^r - k)}$ possessing properties analogous to a)–e) § 2. In this case, the elements $e_{r,1}$ commute with each other for any r and $e_{r,1}^2 = 0$ (also for any r).

Analogously to Theorem 2.1 § 2 Chapter II, one can prove the following

Theorem 3.2. *The algebra* $\mathrm{Ext}_A(H_U(2), Z_2)$ *is a polynomial algebra with generators:*

$$\left.\begin{array}{l} 1 \in \mathrm{Ext}_A^{0,0}(H_U(2), Z_2), \\ z_{2k} \in \mathrm{Ext}_A^{0,2k}(H_U(2), Z_2), \quad k \neq 2^t - 2, \\ h'_r \in \mathrm{Ext}_A^{1,2p^r-1}(H_U(2), Z_2), \quad r \geqslant 0. \end{array}\right\} \qquad (XVIII)$$

Proof. The proof is analogous to the proof of Theorem 2.1 (in this case $\beta = Sq^1$). We indicate the only difference: in the Steenrod algebra over Z_2 there are no Cartan types. As before, denote by M_β the module over the Steenrod algebra with the only generator of dimension 0, and the only relation $\beta(x) = 0$ for all $x \in M_\beta$. We define the subalgebra B of A to be the commutative algebra generated by the elements $e_{r,1}$, $r \geqslant 1$, and $e_{r,0} = 1$. Obviously, it is special. Let us prove that $M_B = M_\beta$. We shall use the dividing Adams homomorphism h (see formulae (VI) § 4 of Chapter I). Clearly, the homomorphism h annihilates all iterations of type 1 modulo 2 (see § 4 of Chapter I). Adams showed that the following relations hold:

$$\left.\begin{array}{l} h(e_{r,2k}) = e_{r,k}, \\ h(e_{r,2k+1}) = 0. \end{array}\right\} \qquad (XIX)$$

From (XIX), it follows that $h(e_{r,1}) = 0$. Instead of elements $e_{r,k}$, we shall consider now only $e_{r,2^i}$ and construct bases of type c) § 2 Chapter II only by using such elements (see [1]). Consider the ordering where all elements of the type $e_{r,1}$ precede all elements of the type $e_{r,2^i}$ for $i > 0$ we see that the homomorphism h annihilates only those monomials of the basis having $e_{r,1}$ on the left side. This yields that $M_\beta = M_B$. The theorem is proved.

Denote by $\widetilde{H}_{SO}(2)$ the quotient of the module $H_{SO}(2)$ by its A-free part generated by the generators x_l from Lemma 4.6 § 4, Chapter I. From Lemmas 4.6 and 5.7 of Chapter I, analogously to Theorem 2.1 and Theorem 3.2 of this chapter, we get

Lemma 3.4. *The algebra* $\mathrm{Ext}_A(\widetilde{H}_{SO}(2), Z_2)$ *is a polynomial algebra with generators:*

$$\left.\begin{aligned} 1 \in \mathrm{Ext}_A^{0,0}, z_{4k} \in \mathrm{Ext}_A^{0,4k}, \quad k \geqslant 1 \\ h_0 \in \mathrm{Ext}_A^{1,1}. \end{aligned}\right\} \qquad (XX)$$

The proof of this lemma is analogous to the previous ones. For B, we may take the subalgebra generated by $e_{1,1} = Sq^1$.

Denote by $\widetilde{M}_\beta$ the module over the Steenrod algebra with one generator u of dimension 0 and one non-trivial relation $\beta(x) = 0$ for $x \in \widetilde{M}_\beta$ and the relation $Sq^2(u) = 0$. From Lemma 4.6 and Corollary 4.4 of Chapter I it follows that the module $H_{SO}(2)$ is a direct sum of modules of types M_β and $\widetilde{M}_\beta$. The following theorem holds

Theorem 3.3. *The algebra* $\mathrm{Ext}_A(\widetilde{M}_\beta, Z_2)$ *admits a system of generators*

$$\left.\begin{aligned} &1 \in \mathrm{Ext}_A^{0,0}(\widetilde{M}_\beta, Z_2), h_0 \in \mathrm{Ext}^{1,1}(\widetilde{M}_\beta, Z_2), h_1 \in \mathrm{Ext}^{1,2}(\widetilde{M}_\beta, Z_2), \\ &x \in \mathrm{Ext}^{3,7}(\widetilde{M}_\beta, Z_2), y \in \mathrm{Ext}^{4,12}(\widetilde{M}_\beta, Z_2), h'_r \in \mathrm{Ext}^{1,2^r-1}(\widetilde{M}_\beta, Z_2), r \geqslant 3, \end{aligned}\right\} \qquad (XXI)$$

satisfying the relations

$$h_0 h_1 = 0, h_1^2 = 0, x^2 = h_0^2 y, h_1 x = 0, \qquad (XXII)$$

and all the relations follow from (XXII).

Proof. To prove Theorem 3.3 is, we shall, as above, find a special subalgebra B of the Steenrod algebra A, that should correspond to the module $\widetilde{M}_\beta$. For B we take the subalgebra generated by the element $e_{1,2} = Sq^2$ and all elements of the type $e_{r,1}$. It follows trivially from the description of the elements $e_{r,k}$ in [1] that $[e_{1,1}; e_{1,2}] = e_{2,1}$ and that $[e_{r,1}; e_{1,2}] = 0$ for $r > 1$. (By $[a; b]$ here we denote the element $ab - ba$ (commutator).) Let us calculate the cohomology algebra $H^*(B)$. Clearly, the subalgebra generated in B by $e_{r,1}$ for $r > 1$, is a central subalgebra in B. Denote it by C. It is easy to see that the algebra $B//C$ is commutative, with any element squared being equal to zero (because in B we have $e_{1,2}^2 = e_{1,1} \circ e_{2,1}$). The algebra $H^*(C)$ is isomorphic to the polynomial algebra with generators $h'_r \in H^{1,2^r-1}(C)$ for all $r \geqslant 2$. The algebra $H^*(B//C)$ is isomorphic to the polynomial algebra with generators $h_0 \in H^{1,1}(B//C)$ and $h_1 \in H^{1,2}(B//C)$.

Consider the Serre-Hochschild spectral sequence for the central subalgebra C of B (see [15]). We know the E_2 term of it, namely,

$E_2^{p,q} = H^p(B//C) \otimes H^q(C)$. Simple calculations show that in the Serre-Hochschild spectral sequence the following relations hold:

$$\left.\begin{array}{ll} d_2(1 \otimes h_2') = h_0 h_1 \otimes 1, \quad d_i(1 \otimes h_r') = 0, \quad r \geqslant 3, \quad i \geqslant 2, \\ d_3(1 \otimes {h_2'}^2) = h_1^3 \otimes 1, \quad d_i(1 \otimes {h_2'}^4) = 0, \quad i \geqslant 2. \end{array}\right\} \qquad (XXIII)$$

Setting $x = h_0 \otimes {h_2'}^2$, $y = 1 \otimes {h_2'}^4$ and preserving the previous notation, we easily get the desired result. The theorem is proved.

It remains thus to study only the module $H_{Sp}(2)$. Denote by $M_{1,2}$ the module with one generator over the Steenrod algebra with two identical relations $Sq^1(x) = 0$ and $Sq^2(x) = 0$ for all $x \in M_{1,2}$. The dimension of the generator is assumed to be zero. Lemmas 4.5 and 5.7 of Chapter I reduce the study of the algebra $\mathrm{Ext}_A(H_{Sp}(2), Z_2)$ to the study of $\mathrm{Ext}_A(M_{1,2}, Z_2)$. Arguing as above, it is easy to show that the algebra $\mathrm{Ext}_A(M_{1,2}, Z_2)$ is isomorphic to the algebra $H^*(B)$, where B is the subalgebra of the Steenrod algebra A generated by all elements of the type $e_{r,1}$ and $e_{r,2}$. Recalling the description in [1] of the elements $e_{r,k}$, it is easy to show by simple calculations that the elements $e_{r,1}$ and $e_{r,2}$ satisfy the following relations:

$$\left.\begin{array}{l} [e_{r_1,1}; e_{r_2,1}] = 0, \quad e_{r,1}^2 = 0, \quad [e_{r,2}; e_{1,1}] = e_{r+1,1}, \\ [e_{r_1,2}; e_{r_2,1}] = 0, \quad r_2 > 1, \quad [e_{r,2}; e_{1,2}] = e_{r+1,1} \circ e_{1,1}, \\ [e_{r_1,2}; e_{r_2,2}] = 0, \quad r_1 > 1, \quad r_2 > 1, \quad e_{1,2}^2 = e_{2,1} \circ e_{1,1}, \\ e_{r,2}^2 = 0, \quad r > 1, \end{array}\right\} \qquad (XXIV)$$

and all relations follow from (XXIV).

Let us choose in this algebra the central subalgebra C, generated by the elements $e_{r,1}$ for $r \geqslant 2$. The cohomology algebra $H^*(C)$ is isomorphic to the polynomial algebra in $h_r' \in H^{1,2^r-1}(C)$ for all $r \geqslant 2$, as is easy to see from (XXIV). The cohomology algebra $H^*(B//C)$ is isomorphic to the polynomial algebra in $h_0 \in H^{1,1}(B//C)$ and $h_{r,1} \in H^{1,2^{r+1}-2}(B//C)$ for all $r \geqslant 1$. From (XXIV), one can easily deduce that in the Serre-Hochschild spectral sequence for the subalgebra C of B, the following relations hold:

$$\left.\begin{array}{l} d_2(1 \otimes h_r') = h_0 h_{r-1,1} \otimes 1, \quad r \geqslant 2, \\ d_3(1 \otimes {h_r'}^2) = h_{1,1} h_{r-1,1}^2 \otimes 1, \quad r \geqslant 2, \\ d_i(1 \otimes {h_r'}^4) = 0, \quad i \geqslant 2, \quad r \geqslant 2. \end{array}\right\} \qquad (XXV)$$

Set $x = h_0 \otimes {h_2'}^2$ and $y = 1 \otimes {h_2'}^4$. Obviously $d_i(x) = 0$ and $d_i(y) = 0$ for all $i \geqslant 2$. Thus there exist elements $x \in \mathrm{Ext}_A^{3,7}(H_{Sp}(2), Z_2)$ and $y \in \mathrm{Ext}_A^{4,12}(H_{Sp}(2), Z_2)$ satisfying the following relation:

$$x^2 = h_0^2 y, \qquad (XXVI)$$

where $h_0 \in \operatorname{Ext}_A^{1,1}(H_{Sp}(2), Z_2)$ (such an element h_0, clearly, exists). Besides, from (XXV), it evidently follows that in the algebra $\operatorname{Ext}_A(H_{Sp}(2), Z_2)$ for every n we have $h_0^n x \neq 0$ and $h_0^n y \neq 0$.

§ 4. Inner homology rings

Theorem 4.4. *The quotient ring of V_{SO} by 2-torsion is isomorphic to the ring of polynomials with generators u_{4i} of dimension $4i$ for all $i \geqslant 0$. The ring V_U is isomorphic to the ring of polynomials with generators V_{2i} of dimension $2i$ for all $i \geqslant 0$. The algebras $V_{Sp} \otimes Z_p$ are isomorphic to the algebras of polynomials with generators t_{4i} of dimension $4i$ for all $i \geqslant 0$ for every $p > 2$. The ring V_{Sp} has no p-torsion for $p > 2$. The quotient of the ring V_{Sp} by 2-torsion is not a polynomial ring. There are elements $x \in V_{Sp}^4$, $y \in V_{Sp}^8$ such that $x^2 - 4y \equiv 0$ (mod 2-torsion), so that the elments x and y are generators of the groups V_{Sp}^4 and V_{Sp}^8 of infinite order.*

The proof of Theorem 4.4 uses the Adams spectral sequence method; thus, it is necessary to give the precise formulation of the main theorem from [1].

Let K be an arbitrary finite complex. Denote by $\pi_n^s(K)$ the groups $\pi_{n+i}(E^i K)$, where E is the suspension and i is large enough. If $K = K_1 \times K_2 / K_1 \vee K_2$, where K_1 and K_2 are finite polyhedra, then there is a well-defined pairing of groups: $\pi_{n_1}(K_1) \otimes \pi_{n_2}(K_2) \to \pi_{n_1+n_2}(K)$. From the properties of the operation $K_1 \times K_2 / K_1 \vee K_2$ it follows that the above pairing induces some pairing

$$\pi_{n_1}^s(K_1) \otimes \pi_{n_2}^s(K_2) \to \pi_{n_1+n_2}^s(K). \qquad (XXVII)$$

On the other hand, it is well known that $H^+(K_1, Z_p) \otimes H^+(K_2, Z_p) \approx \approx H^+(K, Z_p)$. The last isomorphism, by algebraic reasoning defines a pairing

$$\begin{aligned} &\operatorname{Ext}_A^{s,t}(H^+(K_1, Z_p), Z_p) \otimes \operatorname{Ext}_A^{\bar{s},\bar{t}}(H^+(K_2, Z_p), Z_p) \\ &\qquad \to \operatorname{Ext}_A^{s+\bar{s},t+\bar{t}}(H^+(K, Z_p), Z_p). \end{aligned} \qquad (XXVIII)$$

Theorem 4.5. (Adams) *For every polyhedron K there exists a spectral sequence $\{E_r(K), d_r\}$ such that:*

a) $E_r(K) \approx \sum_{s,t} E_r^{s,t}$, $\quad d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}$, $\quad E_r^{s,t} = 0$, $\quad s > t$*;*

b) $E_2^{s,t} \approx \operatorname{Ext}_A^{s,t}(H^+(K, Z_p), Z_p)$*;*

c) The group $\sum_{t-s=m} E_\infty^{s,t}$ is adjoint to the quotient of the group $\pi_m^s(K)$ by a subgroup consisting of elements of order coprime with p;

d) if $K = K_1 \times K_2 / K_1 \vee K_2$ then there are pairings

$$Q_r \colon E_r^{s,t}(K_1) \otimes E_A^{\bar{s},\bar{t}}(K_2) \to E_A^{s+\bar{s},t+\bar{t}}(K) \qquad (XXVIII')$$

such that

$$d_r Q_r(x \otimes y) = Q_r(d_r(x) \otimes y) + (-1)^{t-s} Q_r(x \otimes d_r(y)); \qquad (XXVIII'')$$

e) the pairing Q_2 coincides (up to signs) with the pairing (XXVIII), and Q_∞ is adjoint to the pairing (XXVII).

Evidently, from Adams' theorem and Lemmas 2.2 and 5.7 of Chapter I we get the following

Lemma 4.5. *There exist exact sequences of algebras $\{E_r(SO), d_r\}$, $\{E_r(U), d_r\}$, $\{E_r(SU), d_r\}$, $\{E_r(Sp), d_r\}$ such that:*

a)

$$E_2(SO) \approx \mathrm{Ext}_A(H_{SO}(p), Z_p),$$
$$E_2(U) \approx \mathrm{Ext}_A(H_U(p), Z_p),$$
$$E_2(SU) \approx \mathrm{Ext}_A(H_{SU}(p), Z_p),$$
$$E_2(Sp) \approx \mathrm{Ext}_A(H_{Sp}(p), Z_p),$$

and, for $E_r^{(m)} = \sum_{t-s=m} E_r^{s,t}$,

b) the graded algebras $E_\infty = \sum_m E_\infty^{(m)}$ are adjoint to the quotient rings of V_{SO}, V_U, V_{SU}, V_{Sp} by some ideals consisting of elements of order coprime with p, for all sequences of groups: $\{SO(n)\}$, $\{U(n)\}$, $\{SU(n)\}$, $\{Sp(n)\}$.

It follows from Adams' theorem, a) that $d_r(E_r^{(m)}) \subset E_r^{(m-1)}$. From Theorems 2.1, 3.2 and Lemma 3.4 it follows that in our case the groups $E_2^{s,t}(SO)$, $E_2^{s,t}(U)$ and $E_2^{s,t}(Sp)$ are zero for $t - s \equiv 1 \pmod 2$ (except for the groups $E_2^{s,t}(Sp)$ and groups $E_2^{s,t}(SO)$ for $p = 2$). This yields that in the Adams spectral sequence, defined by Lemma 4.5, all differentials are trivial. Together with Lemma 3.4, this leads to the following isomorphisms

$$\left.\begin{aligned} E_\infty(SO) &\approx \mathrm{Ext}_A(H_{SO}(p), Z_p), && p > 2,\\ E_\infty(U) &\approx \mathrm{Ext}_A(H_U(p), Z_p), && p \geqslant 2,\\ E_\infty(Sp) &\approx \mathrm{Ext}_A(H_{Sp}(p), Z_p), && p > 2. \end{aligned}\right\} \qquad (XXIX)$$

These algebras, as we have shown above, are polynomial algebras (see §§ 2,3). Consider the elements $h_0 \in \mathrm{Ext}_A^{1,1}$ for all our algebras. It is well known that the multiplication by such an element is adjoint to the multiplication by p in homotopy groups (see [1]). Comparing the obtained results for all prime p, we get all statements of the theorem except the last one. From (XXV) and (XXVI) it follows that the elements $x \in \mathrm{Ext}_A^{3,7}(H_{Sp}(2), Z_2)$ and $y \in \mathrm{Ext}_A^{4,12}(H_{Sp}(2), Z_2)$ are cycles of all differentials in the Adams spectral sequence and that in the ring V_{Sp}: $x^2 - 4y \equiv 0$ (mod 2-torsion). The absence of p-torsion in V_{SO}, V_U and V_{Sp} follows from the fact that in the corresponding algebras $E_\infty(SO)$, $E_\infty(U)$ and $E_\infty(Sp)$ the relations $h_0^n z = 0$ do not hold for any n and z.

The theorem is proved.

§ 5. Characteristic numbers and the image of the Hurewicz homomorphism in Thom spaces

Consider the Thom space M_G corresponding to some subgroup G of the group $SO(n)$. Let W^i be a smooth compact oriented manifold smoothly embedded into S^{n+i}. We have an $SO(n)$-bundle of planes R^n, normal to the submanifold W^i in the sphere S^{n+i}. Denote it by ν^n and assume that it is endowed with a G-bundle structure, as in § 1 of Chapter I. Denote by p the classifying mapping of this bundle equal to the G-bundle of planes over B_G. Let $x \in H^i(B_G, Z)$ be an arbitrary cohomology class. We call the scalar product $(p^*x, [W^i])$ a characteristic number of the manifold W^i, corresponding to the element x (here $[W^i]$ denotes the fundamental cycle of the manifold W^i with the given orientation). We shall denote this number by $x[W^i]$ or by $x[\nu^n]$. Recall the Thom isomorphism $\varphi\colon H^i(B_G) \to H^{n+i}(M_G)$ and the Thom construction: for a manifold W^i, embedded into S^{n+i} as above, one gets a map $f(\nu^n, W^i)\colon S^{n+i} \to M_G$. The following lemma holds.

Lemma 5.6. *If $[S^{n+i}]$ is the fundamental cycle of the sphere, whose orientation is compatible with the submanifold W^i, G-framed in S^{n+i}, then the following equality holds*

$$(f(\nu^n, W^i)_*[S^{n+i}], \varphi(x)) = x[W^i] \qquad (XXX)$$

for all $x \in H^i(B_G, Z)$.

PROOF. Consider the closed tubular ε-neighbourhood $\widetilde{T}(W^i)$ of the G-framed submanifold W^i of the sphere S^{n+i} for sufficiently small ε. We assume the mapping $f(\nu^n, W^i)\colon S^{n+i} \to M_G$ to be t-regular in the tubular neighbourhood $\widetilde{T}(M^i)$ (see [16]). Thus we get Thom isomorphism $\varphi\colon H^l(W^i) \to H^{n+l}(\widetilde{T}(W^i), \partial\widetilde{T}(W^i))$ for all $l \geqslant 0$.

Denote by $m_0 \in M_G$ the point of the Thom space obtained by contracting the boundary E_G of the cylinder T_G to a point. Denote by E_δ^{n+i} the complement in the sphere of a small cell-neighbourhood of radius δ of some point which is inside $\widetilde{T}(W^i)$. Clearly, there is an embedding of pairs:

$$j\colon (\widetilde{T}(W^i), \partial\widetilde{T}(W^i)) \subset (S^{n+i}, E_\delta^{n+i}).$$

The mapping of pairs $f(\nu^n, W^i)\colon (S^{n+i}, s_0) \to (M_G, m_0)$ can be equivalently replaced with the mapping of pairs

$$\widetilde{f}(\nu^n, W^i)\colon (S^{n+i}, E^{n+i}_\delta) \to (M_G, m_0), \qquad s_0 \in E^{n+i}_\delta.$$

Denote the composition $\widetilde{f}(\nu^n, W^i)\circ j$ by g. On the other hand, by definition of a t-regular mapping $f(\nu^n, W^i)$, it induces the mapping of pairs

$$f_1(\nu^n, W^i)\colon (\widetilde{T}(W^i), \partial\widetilde{T}(W^i)) \to (M_G, m_0),$$

which commutes with φ. Denote by μ the fundamental cycle of the manifold $\widetilde{T}(W^i)$ modulo boundary. It follows from regularity of our mappings that

$$(f_1(\nu^n, W^i)_*(\mu), \varphi(x)) = (\varphi^{-1}(\mu), p^*(x)) = x[W^i],$$

where $p\colon W^i \to B_G$ is the mapping induced by $f_1(\nu^n, W^i)$ on the subspace W^i. Clearly,

$$(g^*(\mu), \varphi(x)) = (\widetilde{f}(\nu^n, W^i)_* \circ j_*(\mu), \varphi(x)) = (\widetilde{f}(\nu^n, W^i)_*[S^{n+i}], \varphi(x)).$$

It remains to note that $f(\nu^n, W^i)_*[S^{n+i}]=g_*(\mu)$, since $j_*(\mu)=[S^{n+i}]$ and $\widetilde{f}(\nu^n, W^i)_*[S^{n+i}] = f(\nu^n, W^i)_*[S^{n+i}]$.

The lemma is proved.

Let us return to the case when G is one of the classical Lie groups. As in Chapter I, we shall consider the cohomology of Thom's spaces and of the spaces $B_{SO(2k)}$, $B_{U(k)}$, $B_{Sp(k)}$ in terms of two-dimensional "Wu generators" $t_1, \ldots, t_k$. From Milnor's lectures on characteristic classes (see [9]) it follows that $j^*(p_{4i}) = \sum_{m+l=2i} c_{2m} \circ c_{2l}$, where $j\colon U(k) \to SO(2k)$ is the natural group inclusion. Thus the quotient of the ring $H^*(B_{SO(2k)})$ by 2-torsion can be thought as the polynomial ring with generators $\sum_{m+l=2i} c_{2m} \circ c_{2l} = \sum t_1^2 \cdots t_i^2$, being the elementary symmetric polynomials in squares of Wu generators, and the polynomial $w_{2k} = t_1 \circ \ldots \circ t_k$ (for $H^*(B_{U(k)})$ and $H^*(B_{Sp(k)})$ this is evident because these rings have no torsion). Let ω and $\overline{\omega}$ be decompositions as in Chapters I and II, v_ω and $v_{\bar\omega}$ be the symmetrized monomials corresponding to ω and $\overline{\omega}$ (see § 3 Chapter I). As above, denote by $R(\omega)$ and $R(\overline{\omega})$ the dimension of elements v_ω and $v_{\bar\omega}$ in the rings of symmetric polynomials with generators $t_1, \ldots, t_k$. Clearly, $v_\omega \in H^{R(\omega)}(B_{U(k)})$ and $V_{\bar\omega} \in H^{R(\bar\omega)}(B_{SO(2k)})$ or $v_{\bar\omega} \in H^{R(\bar\omega)}(B_{Sp(k)})$ (more precisely, in the quotient of the group $H^{R(\bar\omega)}(B_{SO(2k)})$ by 2-torsion). We shall call the characteristic numbers of the framed manifold corresponding to the elements v_ω and $v_{\bar\omega}$, ω (respectively $\overline{\omega}$) — numbers of manifolds[1]. We shall be especially interested in the case when $\omega = (k)$

[1]In Milnor's lectures [9] $\omega\,(\overline{\omega}$, respectively) numbers of manifolds are denoted by $S_\omega(S_{(\overline{\omega})})$. The properties of $\omega(\overline{\omega})$-numbers are also described in [9].

and $R(\omega) = 2k$ and when $\overline{\omega} = (2k)$ and $R(\overline{\omega}) = 4k$. From Lemma 5.7 of Chapter I, it follows that the characteristic numbers are well defined for elements of the rings V_{SO}, V_U, V_{SU}, V_{Sp}.

Theorem 5.6. *The $(\overline{2k})$-number of the $4k$-dimensional polynomial generator of the quotient of V_{SO} by 2-torsion is equal to p if $2k = p^i - 1$, where $p > 2$, and it is equal to 1, if $2k \neq p^i - 1$, for all $p > 2$; the (k)-number of the $2k$-dimensional polynomial generator of the ring V_U is equal to p for $k = p^i - 1$, where $p \geqslant 2$, and to one, if $k \neq p^i - 1$, for all $p \geqslant 2$. The minimal non-zero $(\overline{2k})$-number of a $4k$-dimensional symplectic framed manifold is equal to $2^s p$, if $k = p^i - 1$, where $p > 2$, and it is equal to 2^s if $k \neq p^i - 1$, for all $p > 2$, where $s \geqslant 0$.*

(By minimal number we mean the number with minimal absolute value.)

PROOF. We shall prove this theorem by homotopy means, based on the equality (XXX). Since the proofs are quite analogous for all rings V_{SO}, V_U and V_{Sp}, we shall carry it out only for the ring V_U.

Consider the module $H_U(p)$. As shown above (see §§ 2–3, Chapter II), the module $H_U(p)$ for all $p \geqslant 2$ is a direct sum of modules M_β^ω of type M_β with generators U_ω for all non-p-adic decompositions ω (and the generator U of dimension 0). While proving Theorem 3.2 of Chapter II we have shown that the module M_β corresponds to the special subalgebra B of the Steenrod algebra A, generated by all elements e'_r and 1 (see § 2 Chapter II) for $p > 2$, and by all elements $e_{r,1}$ and 1 (see § 3 Chapter II) for $p = 2$, i.e. $M_\beta = M_B$. Consider the reslovent $C_A(M_B)$, constructed when proving the isomorphism (XIV). Denote by $C_A(H_U(p))$ the direct sum $\sum_\omega C_A(M_\beta^\omega)$, where the resolvents $C_A(M_\beta^\omega)$ are constructed analogously to the resolvent $C_A(M_\beta)$, the only difference being that the dimension of all elements is shifted by $R(\omega)$; the resolvents $C_A(M_\beta^\omega)$ coincide with the minimal resolvents of the special subalgebra; the sum is taken over all non p-adic decompositions of ω.

Later on, we shall study only the mappings

$$\begin{aligned} \varepsilon\colon C_A^0(H_U(p)) &\to H_U(p), \\ d_0\colon C_A^1(H_U(p)) &\to C_A^0(H_U(p)). \end{aligned}$$

Take the Thom space $M_{U(k)}$ for k large enough. It is aspherical in dimensions less than $2k$. Following [1], consider the realisation $Y = \{Y_{-1} \supset Y_0 \supset \ldots \supset Y_n\}$ of the free acyclic resolvent $C_A(H_U(p))$ (see [1, Chapter II]). We assume the realisation of Y polyhedral. Here n is large enough. By definition of a resolvent realisation, the space Y_{-1} is homotopy equivalent to the space $M_{U(k)}$ (k is large), the A-modules $H^*(Y_{i-1}, Y_i; Z_p)$ are isomorphic (up to some high dimension) to the A-modules $C_A^i(H_U(p))$,

$i \geqslant 0$, the mappings $\delta_i^*\colon H^*(Y_{i-1}, Y_i; Z_p) \to H^*(Y_{i-2}, Y_{i-1}; Z_p)$ coincide with $d_{i-1}\colon C_A^i(H_U(p)) \to C_A^{i-1}(H_U(p))$ for all $i \geqslant 1$, and the mapping $\delta_0^*\colon H^*(Y_{-1}, Y_0; Z_p) \to H^*(Y_{-1}; Z_p)$ coincides with $\varepsilon\colon C_A^0(H_U(p)) \to H_U(p)$. (Here we assume that cohomology is taken over Z_p.) In our case, evidently, $\pi_{2k+1}(Y_{i-1}, Y_i) \approx \mathrm{Hom}_A^t(C_A^i(H_U(p)), Z_p)$ for all $t < 4k-1$, $i \geqslant 0$. Now, consider the cohomology exact sequence of the pair (Y_{-1}, Y_0):

$$\ldots \to H^q(Y_{-1}, Y_0; Z_p) \xrightarrow{\delta_0^*} H^q(Y_{-1}; Z_p) \xrightarrow{j^*} H^q(Y_0; Z_p) \to \ldots, \qquad (XXXI')$$

$$\ldots \to H^q(Y_{-1}, Y_0; Z) \xrightarrow{\bar\delta_0^*} H^q(Y_{-1}; Z) \xrightarrow{\bar j^*} H^q(Y_0; Z) \to \ldots. \qquad (XXXI'')$$

In the sequence (XXXI′), the homomorphism δ_0^* is indeed an epimorphism for $q < 4k-1$, and thus the homomorphism j^* is trivial. But the groups $H^q(Y_{-1}; Z)$ have no torsion, thus in the sequence (XXXI″) the homomorphism $\overline{\delta}_0^*$ is trivial because all groups $H^q(Y_{-1}, Y_0; Z)$ are finite and direct sums of the groups Z_{p_i}. From the triviality of j^* in (XXXI′) it follows that in the group $H^q(Y_0; Z)$ the image $\mathrm{Im}\,\bar j^*$ is divisible by p. Since the factor-group $H^q(Y_0; Z)/\mathrm{Im}\,\bar j^* \subset H^{q+1}(Y_{-1}, Y_0; Z)$, it follows that the image $\mathrm{Im}\,\bar j^*$ is not divisible by any ap, where $|a| > 1$, because the groups $H^t(Y_{-1}, Y_0; Z)$ for $t < 4k-1$ are direct sums of the groups Z_{p^i} (see [7]). This yields that the image $\mathrm{Im}\,\bar j_*\colon H_q(Y_0, Z) \to H_q(Y_{-1}, Z)$ consists of all elements of the type px, $x \in H_q(Y_{-1}, Z)$. Now, consider the elements $\widetilde{Z}_{2l} \in \mathrm{Hom}_A^{2l}(C_A^0(H_u(p), Z_p))$, defining the elements (XVI) or (XVIII) in $\mathrm{Ext}_A^{0,2l}(H_u(p), Z_p)$ for $l \neq p^i - 1$. We may assume, by definition of resolvent, that $\widetilde{Z}_{2l} \in \pi_{2l+2k}(Y_{-1}, Y_0)$. Moreover, since the differential in the Adams spectral sequence are trivial, we may assume that the elements $\widetilde{Z}_{2l} \in \pi_{2l+2k}(Y_{-1}, Y_0)$ belong to the image of the homomorphism $\delta_{0*}(\pi_{2l+2k}(Y_{-1}))$. Denote by $H\colon \pi_i(K) \to H_i(K, Z)$ the Hurewicz homomorphism and denote by $\widetilde{\widetilde{z}}_{2l} \in \pi_{2l+2k}(Y_{-1})$ an element such that the scalar product $(H\widetilde{\widetilde{z}}_{2l}, v_l \circ c_{2k})$ has minimal absolute value. It is evident that $\delta_{0*}\widetilde{\widetilde{z}}_{2l} = \lambda \widetilde{z}_{2l}$, moreover λ is coprime to p, since the mapping δ_{0*} is an epimorphism for homotopy groups, and the cycle $H\widetilde{z}_{2l}$ is the image under δ_{0*} of a cycle x_{2l} such that $(x_{2l}, V_{(l)} \circ c_{2k}) \neq 0$ by construction of the resolvent. This yields that the scalar product $(H\widetilde{\widetilde{z}}_{2l}, V_{(l)} \circ c_{2k})$ is coprime to p if $l \neq p^i - 1$. Comparing the obtained results for different p, we get that the scalar product $(H\widetilde{\widetilde{z}}_{2l}, V_{(l)} \circ c_{2k}) = \pm 1$, if $l \neq p^i - 1$ for any $p \geqslant 2$, and that this scalar product is equal to $\pm p^s$ if $l \neq p^i - 1$. We do not know the value of s yet. It remains to find it. To do this, we have to consider the cohomology exact sequences of the pair (Y_0, Y_1):

$$\ldots \to H^q(Y_0, Y_1; Z_p) \xrightarrow{i^*} H^q(Y_0; Z_p) \to H^q(Y_1; Z_p) \to \ldots, \qquad (XXXII')$$

$$\ldots \to H^q(Y_0, Y_1; Z) \xrightarrow{\bar{i}^*} H^q(Y_0; Z) \to H^q(Y_1; Z) \to \ldots . \qquad (XXXII'')$$

Note that the module $H^q(Y_0; Z_p) \approx \operatorname{Ker}\varepsilon$ is calculated in detail for a similar case in § 1, Chapter II while proving Lemma 1.1 we are going to apply it here. The homomorphism i^* from (XXXII′) is an epimorphism as well as that from (XXXI″). Arguing as above, it is easy to show that there exists an element $\tilde{\tilde{z}}_{2l} \in \pi_{2k+2l}(Y_0)$ such that the scalar product $(H\tilde{\tilde{z}}_{2l}, y_{2l})$ is coprime to p, where $y_{2l} \in H^{2k+2l}(Y_0)$ is such an element that $pj^*(c_{2k} \circ v_{(l)}) - y_{2l} = \sum \lambda_i j^*(v_{\omega_i} \circ c_{2k})$, where $\omega_i \neq (l)$ (j^* is the homomorphism $H^*(Y_{-1}) \to H^*(Y_0)$).

Comparing this result with the previous one, we get the desired statement. To do that, it suffices to apply Lemma 5.6 about characteristic numbers and scalar products. To conclude the proof of the theorem, it remains to show that the (l)-number of an element of the ring V_U is 0 if this element is decomposable into a linear combination of elements of smaller dimensions. Thus, the polynomial generator of dimension $2l$ the (l)-number is minimal, and every element $x \in V_u^{2l}$ with minimal absolute value of the (l)-number can be treated as a polynomial generator[1]. We have analogous properties of $(\overline{2l})$-numbers in the rings V_{SO} and V_{Sp}. The theorem is proved.

It is well known that in V_{SO} (see [9]) the set of complex projective plane $P^{2k}(C)$ forms a polynomial sub-ring (more precisely, $P^{2k}(C)$ with their natural normal framings can be considered as representatives of such elements x_{4k} that form a polynomial subring of the ring V_{SO}, so that the quotient group $V_{SO}/P(x_4, x_8, \ldots)$ consists of finite-order elements). By multiplicity of an element $x \in V_{SO}^{4k}$ we mean the coefficient of the $4k$-dimensional generator in the decomposition of x. The absolute value of the multiplicity does not depend on the choice of polynomial generators in V_{SO} (its quotient ring by 2-torsion). We call the multiplicity x the multiplicity of its representatives, that is, multiplicity of manifolds. Theorem 5.5 yields

Corollary 5.2. *The multiplicity of the complex projective plane $P^{2k}(C)$ in the ring V_{SO} is equal to $2k+1$ if $2k+1 \neq p^i$ for any $p > 2$, and it is equal to $\frac{2k+1}{p}$ if $2k+1 = p^i$ for $p > 2$.*

Complex analytic manifolds are embeddable into real affine even-dimensional spaces of some dimension. Such an embedding induces a

[1]This easily follows from the Whitney formulae for the Pontrjagin, Chern, and symplectic classes):

$$\omega(\xi \otimes \eta) = \sum_{\substack{(\omega_1, \omega_2) = \omega \\ \omega_1 \neq \omega_2}} [\omega_1(\xi)\omega_2(\eta) + \omega_2(\xi)\omega_1(\eta)] + \sum_{(\omega_1, \omega_1) = \omega} \omega_1(\xi)\omega_1(\eta). \qquad (XXXII)$$

complex framing, which is inverse to the tangent bundle. Thus, one may speak about the multiplicity of a complex analytic manifold in the ring V_U.

Corollary 5.3.[1] *In the ring V_U, the multiplicities of the projective planes $P^k(C)$ is equal to $k+1$, if $k+1 \neq p^i$ for any $p \geqslant 2$, otherwise it is equal to $\frac{k+1}{p}$ if $k+1 = p^i$, where $p \geqslant 2$.*

To prove Corollaries 5.3 and 5.4, it suffices to show that the $(\overline{2k})$-numbers $P^{2k}(C)$ in V_{SO} and the (k)-numbers $P_k(C)$ in V_U are equal to $2k+1$ and $k+1$, respectively, which leads to our statements. (Evidently, the $(\overline{2k})$- and (k)-numbers, respectively, are some polynomials in Pontrjagin (Chern) classes of the normal bundles, which are inverse to the tangent ones. These polynomials in this case are trivially calculated via symmetric polynomials in "Wu's generators". For tangent Pontrjagin (Chern) classes this is done in [9]. The $(\overline{2k})$-number and the (k)-number of the normal bundle is equal to the $(\overline{2k})$-number (respecitvely, (k)-number with minus sign). This easily follows from the Whitney formula written in $\overline{\omega}$ (ω)-numbers.)

Chapter III

Realization of cycles

§ 1. Possibility of G-realization of cycles

Let M^n be a compact closed oriented manifold.

Definition 1.1. A dimension i for M^n is called p-regular for p prime if $2i < n$ and all groups $H_{i-2q(p-1)-1}(M^n, Z)$ have no p-torsion for $q \geqslant 1$.

Theorem 1.1. *If a dimension i for M^n is p_s-regular for some (finite or infinite) number of odd prime $\{p_s\}$ then for any integral cycle $z_i \in H_i(M^n, Z)$ there exists an odd number α, which is coprime all p_s's such that the cycle αz_i is realizable by a submanifold. If a cycle $z_i \in H_i(M^n, Z)$ is realizable by a submanifold, the dimension i is 2-regular and $n-i \equiv 0 \pmod 2$ then the cycle z_i is $U\left(\frac{n-i}{2}\right)$-realizable. If a cycle $z_i \in H_i(M^n, Z)$ is realizable by a submanifold, $2i < n$ and $n-i \equiv 0 \pmod 4$, then the cycle $2^t z_i$ is $Sp\left(\frac{n-i}{4}\right)$-realizable for t large enough.*

---

[1] Milnor has found manifolds $H_{r,t} \subset P^r(C) \times P^t(C)$, $r>1$, $t>1$, of dimension $2k = 2(r+t-1)$ such that $(k)[H_{r,t}] = -\binom{r+t}{r}$. These manifolds are algebraic (see [17]).

Proof. We shall use the Thom method and use the homotopy structure of the spaces $M_{SO(n-i)}$, $M_{U(\frac{n-i}{2})}$ and $M_{Sp(\frac{n-i}{4})}$, studied before.

Let us prove the first statement of the theorem. Consider the cohomology class $z^{n-i} \in H^{n-i}(M^n, Z)$, which is dual to the cycle z_i. Consider mapping $q\colon M^n \to M_{SO(n-i)}$, $q^* U_{SO(n-i)} = \alpha z^{n-i}$, where α is some odd number coprime to all p_s. From Lemma 4.6 of Chapter I it follows that the k-th Postnikov factor $M^{(k)}$ (see [11]) of the space $M_{SO(n-i)}$ for $k = 2(n-i)-2$ is homotopy equivalent to the direct product of some space $\widetilde{M}_{SO(n-i)}$ and Eilenberg-MacLane complexes of type $K(Z_2, l)$. Moreover, from Theorem 4.1 of Chapter II it follows that the factor $\widetilde{M}_{SO(n-i)}$ of this product can be chosen in such a way that all groups $\pi_t(\widetilde{M}_{SO(n-i)})$ are free abelian and they are all zero except when $t \equiv n-i \pmod 4$. Denote the Postnikov complexes of the space $\widetilde{M}_{SO(n-i)}$ by $\widetilde{M}^{(q)}$. Clearly, $\pi_t(\widetilde{M}^{(q)}) = 0$ for $t < n-i$ or for $t > q$. Denote the space of type $K(\pi_q(\widetilde{M}_{SO(n-i)}), q)$ by K_q. From [6], it follows that the groups $H^{q+t}(Z, q; Z)$ for $t \leqslant q-1$ are finite, and they are direct sums of groups Z_p, where $p \geqslant 2$.

From here and from natural bundles $\eta_q\colon \widetilde{M}^{(q)} \to \widetilde{M}^{(q-1)}$ with fibres K_q one easily gets that the groups $H^{q+t}(\widetilde{M}^{(q)}, Z)$ are finite for $0<t<2(n-i)$ and that the Postnikov factors $\Phi_q \in H^{q+2}(\widetilde{M}^{(q)}, \pi_{q+1}(\widetilde{M}_{SO(n-i)}))$ are homology classes of finite order with coefficients in an abelian group. Denote the order of the factor Φ_q by λ_q. From Lemma 4.6 of Chapter I it follows that λ_q is even. Denote the fundamental cohomology class of the complex K_q by U_q. We shall now construct a family of mappings $g_q\colon M^n \to \widetilde{M}^{(q)}$ such that $\eta_q(g_q) = g_{q-1}$ and $g^*_{n-i}(U_{n-i}) = \alpha z^{n-i}$. Recall that, under our assumptions, the sets of mapping homotopy classes of $\pi(M^n, \widetilde{M}^{(q)})$ form abelian groups, and for each pair of elements $h_1 \in \pi(M^n, \widetilde{M}^{(q)})$, $h_2 \in \pi(M^n, \widetilde{M}^{(q)})$ we have $(h_1 + h_2)^* = h_1^* + h_2^*$ for the induced homomorphisms of cohomology groups. Denote by $\widetilde{H}^{q,t} \subset H^{q+t}(\widetilde{M}^{(q)}, Z)$ the subgroup of $H^{q+t}(\widetilde{M}^{(q)}, Z)$, consisting of elements of finite order coprime to all p_s's, and of elements of order coprime to λ_q. Given a mapping $f_q\colon M^n \to \widetilde{M}^{(q)}$ such that $f_q^*(\Phi_q) = 0$. We shall denote the homotopy class of the mapping f by $\{f\}$.

Lemma 1.1. *There is a mapping $f_{q+1}\colon M^n \to \widetilde{M}^{(q+1)}$ and an odd number α_q, relatively prime to all numbers p_s such that $\{\eta_{q+1} f_{q+1}\} = \alpha_q \{f_q\}$ and $f^*_{q+1}(\widetilde{H}^{q+1,t}) \subset \operatorname{Im} f_q^*$ for $t+q+1 \leqslant q+n-i$.*

Proof. Consider the spectral sequence of the bundle η_{q+1} with coefficients in $\pi_{q+1}(\widetilde{M}_{SO(n-i)})$; this sequence reduces to the exact sequence in low dimensions. As usual, we denote the transgression in this bundle by τ. Clearly, $\tau(u_{q+1}) = \Phi_q$ and $\tau(\lambda_q u_{q+1}) = 0$. It is also evident that $\tau(x) = 0$ if x is an element of finite order coprime to λ_q and of dimension less than $n - i + q$. Every element $\widetilde{x} \in H^*(\widetilde{M}^{(q+1)}, \pi_{q+1}(\widetilde{M}_{SO(n-i)}))$ adjoint to x is a stable primary cohomology operation up to some element $y \in \eta^*_{q+1}(H^*(\widetilde{M}^{(q)}))$ for $\widetilde{u}_{q+1}$ adjoint to the element $\lambda_q u_{q+1}$. It is well known that for any mapping $\widetilde{f}_{q+1} \colon M^n \to \widetilde{M}^{(q+1)}$ such that $\eta_{q+1}\widetilde{f}_{q+1} = f_q$ and any element $z \in H^{q+1}(M^n, \pi_{q+1}(\widetilde{M}_{SO(n-i)}))$, there exists a mapping $\widetilde{f}'_{q+1} \colon M^n \to \widetilde{M}^{(q+1)}$ such that $\widetilde{f}'^*_{q+1}(\widetilde{u}_{q+1}) - \widetilde{f}^*_{q+1}(\widetilde{u}_{q+1}) = \lambda_q z$. This yields that there exists a mapping $\widetilde{f}_{q+1} \colon M^n \to \widetilde{M}^{(q+1)}$, such that $\widetilde{f}'^*_{q+1}(\widetilde{x}) \subset f^*_q(H^*(\widetilde{M}^{(q)}))$ and $\eta_{q+1}\widetilde{f}_{q+1} = f_q$, where $\widetilde{x}$ is the element adjoint to $x \in H^t(K_{q+1}; \pi_{q+1}(\widetilde{M}_{SO(n-i)}))$ for $t < n - i + q$. (The order of x is assumed relatively prime to λ_q.)

Now denote by α_q the number of elements of the quotient group

$$\Big[\widetilde{f}^*_{q+1}\Big(\sum_{t=q+2}^{n-1} \widetilde{H}^{q+1,t}\Big)\Big] \Big/ \Big[\operatorname{Im} f^*_q \cap \widetilde{f}^*_{q+1}\Big(\sum_{t=q+2}^{n-1} \widetilde{H}^{q+1,t}\Big)\Big].$$

Since λ_q is odd, it follows from the construction of $\widetilde{f}^*_{q+1}$ that α_q is odd and coprime to all p_s's. Setting $\{f_{q+1}\} = \alpha_q\{\widetilde{f}_{q+1}\}$, we get the desired statement. The lemma is proved.

Now, let us construct a family of mappings $f_q \colon M^n \to \widetilde{M}^{(q)}$, such that $f^*_{n-i}(u_{n-i}) = z^{n-i}$ and $\{\eta_{q+1} f_{q+1}\} = \alpha_q\{f_q\}$, where α satisfies the assumption of Lemma 1.1. We shall prove that such a construction exists by induction on q.

Note that $\lambda_{n-i-1} = 1$. This evidently yields that $\operatorname{Im} \alpha_{n-i-1} f^*_{n-i} = 0$ in dimensions greater than $n - i$ because we have no p_s-torsion in the groups $H^{n-i+2q(p-1)+1}(M^n)$ for $q \geqslant 1$[1]. Now assume the mappings f_j are constructed for all $j \leqslant m$ and $f^*_j(H^t(\widetilde{M}^{(j)}, Z)) = 0$ for $j < t < 2(n - i)$. Without loss of generality assume that $m - n + i \equiv 3 \pmod 4$. We distinguish between two classes of p_s's: those of the first class are those for which λ_q and p_s are coprime, the second class contains all other numbers.

It is easy to see that for the numbers p_s of the first class, the mapping $f_{m+1} \colon M^n \to \widetilde{M}^{(m+1)}$ satisfies the induction hypothesis as well, i.e. $f^*_{m+1}(\widetilde{x}) = 0$ if $\widetilde{x} \in H^t(\widetilde{M}^{(m+1)}, Z)$ for $t > m + 1$ and the order of $\widetilde{x}$ is divisible by p_s (this trivially follows from the lemma, the assumptions of the theorem and the structure of $H^t(K_{m+1}, Z)$). Consider the case when p_s belongs to the second class. In this case the num-

[1]For generators of the Steenrod algebra of stable cohomology primary operations $\theta^j \in H^{n+j}(Z, n; Z)$ for $j \leqslant n + 1$ we may take elements of dimension $2q(p-1) + 1$ for $q \geqslant 1$.

ber λ_m is divisible by p_s. The factor $\Phi_m \in H^{m+2}(\widetilde{M}^{(m)}, \pi_{m+1}(\widetilde{M}_{SO(n-i)}))$ can be viewed as a partial operation $\Phi(\eta_m^* \circ \ldots \circ \eta_{n-i+1}^*(u_{n-i}))$ on the element $\eta_m^* \circ \ldots \circ \eta_{n-i+1}^*(u_{n-i})$. Let us decompose the element Φ_m as a sum $\Phi_m = \Phi_m^{(1)} + \Phi_m^{(2)}$, where the order of $\Phi_m^{(2)}$ is a number coprime to p_s and the order of $\Phi_m^{(1)}$ is a number of type p_s^l. Both elements $\Phi_m^{(1)}$ and $\Phi_m^{(2)}$ can be viewed as partial operations of one and the same element $\eta_m^* \circ \ldots \circ \eta_{n-i+1}^*(u_{n-i})$, which are defined on the same kernels.

The following lemma holds

Lemma 1.2. *Let Φ be a partial stable cohomology operation of the element $\eta_m^* \circ \ldots \circ \eta_{n-i+1}^*(u_{n-i})$ that increases the dimension by $m-n+i+2$ which is defined on and takes value in subgroups of the cohomology groups with coefficients in abelian groups. If for some p the cohomology operation $p^l\Phi$ is trivial, $\eta_{m+1}^*\Phi(\eta_m^* \circ \ldots \circ \eta_{n-i+1}^*(u_{n-i})) = 0$, $m-n+i \not\equiv -1 \pmod{2p-2}$, where p is an odd prime then the operation Φ is also trivial.*

The statement of the Lemma follows easily from the homotopy structure of Thom spaces studied in Chapter II.

From Lemma 1.2 it follows that the partial cohomology operation $\Phi_m^{(1)}$ is trivial. Now we can find a mapping f_{m+1} such that $\{\eta_{m+1} f_{m+1}\} = \{f_m\} \cdot \alpha_m$, where α_m satisfies the conditions of Lemma 1.2 and the image f_{m+1}^* is trivial in dimensions greater than $m+1$. To do that, it suffices to apply Lemmas 1.1 and 1.2 for all prime numbers p_s of the second class. Thus, f_q is constructed. It defines a family of mappings $\widetilde{\widetilde{f}}_q\colon M^n \to M_{SO(n-i)}^{(q)}$ such that $\widetilde{\widetilde{\eta}}_{q+1}\widetilde{\widetilde{f}}_{q+1} = \alpha_q\{\widetilde{\widetilde{f}}_q\}$ and $\widetilde{\widetilde{f}}_{n-i}(u_{n-i}) = z^{n-i}$, where $M_{SO(n-i)}^{(q)}$ is the Postnikov complex of the space $M_{SO(n-i)}$ and $\widetilde{\widetilde{\eta}}\colon M_{SO(n-i)}^{(q+1)} \to M_{SO(n-i)}^{(q)}$ is the natural projection.

Set $\{g_q\} = \alpha_{n-1} \circ \ldots \circ \alpha_q\{\widetilde{\widetilde{f}}_q\}$. It is evident that $\{\widetilde{\widetilde{\eta}}_{q+1} g_{q+1}\} = \{g_q\}$, $g_{n-i}^*(u_{n-i}) = \alpha_{n-1} \circ \ldots \circ \alpha_{n-i+1} \circ u_{n-i}$. The family of mappings g_q is thus constructed and satisfies the desired properties. Thus, the first statement of the theorem is proved. The remaining statements are proved analogously.

From the proof of Theorem 1.1 and the structure of $H^t(B_{U(m)}, Z)$ it follows that for any cocycle $z^{2i} \in H^{2i}(B_{U(m)}, Z)$ there exists a mapping of the $(4i-1)$-skeleton $g\colon \widetilde{B}_{U(m)}^{(4i-1)} \to M_{U(i)}$ such that $g^*(u_{U(i)}) = z^{2i}$. This yields

Corollary 1.2. *The homology class, Poincaré dual to the polynomial $P(c_2, c_4, \ldots)$ in the Chern classes of an arbitrary $U(m)$-bundle* *over M^n, admits a $U(i)$-realization if the dimension of this polynomial is $2i$, where $2i > \left[\frac{n+1}{2}\right]$.*

Corollary 1.3. *A cycle $Z_i \subset H_i(M^n)$ for $i < \left[\frac{n}{2}\right]$ is realizable by a submanifold if $2^k Z_i = 0$.*

Appendix 1. On the structure of V_{SU}

It is well known that the cohomology algebras $H^*(B_{SU(k)})$ with any coefficients can be described via symmetric polynomials in "Wu's generators" $t_1, \ldots, t_k$, taking into account the relation $t_1 + \ldots + t_k = 0$. The same is true about the algebras $H^*(M_{SU})$.

Let $\omega = (a_1, \ldots, a_s)$, $v_\omega = \sum t_1^{a_1} \circ \ldots \circ t_s^{a_s}$,

$$u_\omega = v_\omega \circ c_{2k} = \sum t_1^{a_1+1} \circ \ldots \circ t_s^{a_s+1} \circ t_{s+1} \circ \ldots \circ t_k,$$

as in Chapter I. Assume also $\sum a_i < k$.

Definition. A decomposition $\omega = (a_1, \ldots, a_s)$ is p-admissible if the number of indices i such that $a_i = p^l$ is divisible by p for every $l \geqslant 0$. (Note that for a characteristic zero field this means that $a_i \neq 1$ $(i = 1, \ldots, s)$.)

Lemma 1. *The module $H_{SU}(p)$ for $p > 2$ is isomorphic to the direct sum $\sum_\omega M_\beta^\omega$ of modules M_β^ω of the type M_β with generators u_ω corresponding to p-admissible and non-p-adic decompositions $\omega = (a_1, \ldots, a_s)$. The dimension of the generator u_ω is equal to $2\left(\sum a_i\right)$.*

The module $H_{SU}(p)$ has a diagonal mapping

$$\Delta\colon H_{SU}(p) \to H_{SU}(p) \otimes H_{SU}(p),$$

on u_ω and looks like the one constructed in § 5, Chapter I, with respect to the notion of p-admissibility. Moreover, the formulae (XI) will hold not absolutely, but modulo some reducible elements. Thus, the following lemma holds.

Lemma 2. *The algebra $\operatorname{Ext}_A(H_{SU}(p), Z_p)$ is isomorphic to the polynomial algebra with the following generators:*

$$1 \in \operatorname{Ext}_A^{0,0}(H_{SU(p)}, Z_p), \quad h'_r \in \operatorname{Ext}_A^{1,2p^r-1}(H_{SU(p)}, Z_p), \quad r \geqslant 0,$$
$$z_{(k)} \in \operatorname{Ext}_A^{0,2k}(H_{SU(p)}, Z_p), \quad k \neq p^r, \quad p^r - 1, \quad r \geqslant 0,$$

$$z_{(\omega_r)} \in \operatorname{Ext}_A^{0,2p^{(r+1)}}(H_{SU(p)}, Z_p), \quad r \geqslant 0, \quad \omega_r = \frac{1}{p^{l_r}}(p^{r+1})^1.$$

Arguing as in § 4 Chapter II, from these lemmas we get

Theorem 1. *The ring $V_{SU} \otimes Z_{p^h}$ is isomorphic to the polynomial ring with generators v_{2i} ($i = 0, 2, 3, 4, \dots$) for all $p > 2$, $h = 0$. The ring V_{SU} has no p-torsion for $p > 2$.*

From Lemma 5.6 of Chapter II, arguing as in Theorem 5.5 of Chapter II, one may prove (taking into account the p-admissibility) the following:

Theorem 2. *For a sequence of SU-framed manifold $M^4, M^6, M^8, \dots$ to form a system of polynomial generators of the ring $V_{SU} \otimes Z_{p^h}$, it is necessary and sufficient that the following conditions concerning ω-Chern numbers of SU-framings hold:*

$$(k)[M^{2k}] \not\equiv 0 \pmod p, \quad k \neq p^i, \quad p^i - 1,$$

$$\frac{1}{p}(k)[M^{2k}] \not\equiv 0 \pmod p, \quad k = p^i - 1,$$

$$\frac{1}{p^{l_s}}(p^{s+1})[M^{2p^{s+1}}] \not\equiv 0 \pmod p, \quad s \geqslant 0, \quad l_s \geqslant 1.$$

(Note that $(p^{s+1})[M^{2p^{s+1}}] \equiv 0 \pmod p$, since $c_2 = (1) = 0$.)

Now, consider the case $p = 2$.

From Corollary 4.4 of Chapter I it follows that $H_{SU}(2) = \sum_i M_\beta^{(i)} + \sum_\omega \widetilde{M}_\beta^\omega$, where $\widetilde{M}_\beta^\omega$ are the quotient modules of type M_β^ω by the relations $Sq^2(u_\omega) = 0$. The dimension of u_ω is equal to $8a$, where ω is an arbitrary decomposition of $8a$ into summands $(8a_1, \dots, 8a_s)$, $a_i > 0$. The dimensions of the generators of $M_\beta^{(i)}$ are even. Set $N_\beta = \sum M_\beta^{(i)}$, $\widetilde{N}_\beta = \sum_\omega \widetilde{M}_\beta^\omega$. Clearly, one has:

$$\operatorname{Ext}_A^{s,t}(H_{SU}(2), Z_2) \approx \operatorname{Ext}_A^{s,t}(N_\beta, Z_2) + \operatorname{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2).$$

In Chapter II, the algebras $\operatorname{Ext}_A(M_\beta, Z_2)$ and $\operatorname{Ext}_A(\widetilde{M}_\beta, Z_2)$ are calculated (for the algebra $\operatorname{Ext}_A(\widetilde{M}_\beta, Z_2)$ see Theorem 3.3). As above, denote by

$$h_0 \in \operatorname{Ext}_A^{1,1}(H_{SU}(2), Z_2), \qquad h_1 \in \operatorname{Ext}_A^{1,2}(H_{SU}(2), Z_2)$$

the known elements satisfying $h_0 h_1 = 0$, $h_1^2 \neq 0$, $h_1^3 = 0$ (see Theorem 3.3).

[1]Since $\sum t_i = 0$, then $\sum t_i^{p^{r+1}} = p^{l_r} \sum \lambda_{r,i} \circ u_{\omega_{r,i}}$, where l_r is maximum possible moreover $\omega_{r,1} = (p^r, \dots, p^r)$, $\lambda_{r,1} \neq 0$.

From § 3, Chapter II one easily gets

Lemma 3. *Let $h_0^k x = 0$, where $k > 0$, $x \in \mathrm{Ext}_A^{s,t}(H_{SU}(2), Z_2)$. Then $x = h_1 y$. Let $x \in \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2)$. Then $h_1^2 x = 0$ yields $h_1 x = 0$ if $t - s = 2k$. If $t - s = 2k + 1$ then we always have $x = h_1 y_1$. Let $x \in \mathrm{Ext}_A^{s,t}(N_\beta, Z_2)$. Then $h_1 x = h_1 y$, where $y \in \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2)$ and $t - s = 2k$, if $x \neq 0$.*

Now, consider the Adams spectral sequence $E_r^{s,t} = E_r^{s,t}(SU)$, described in § 4 of Chapter II (see also [1]).

From 3 and multiplicative properties of the Adams spectral sequence we get

Theorem 3. *If $x \in \mathrm{Ext}_A^{s,t}(N_\beta, Z_2)$ then $d_i(x) = d_i(y)$, $y \in \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2)$ for all $i \geqslant 2$. The elements $h_0 \in \mathrm{Ext}_A^{1,1}(H_{SU}(2), Z_2)$ and $h_1 \in \mathrm{Ext}_A^{1,2}(H_{SU}(2), Z_2)$ are cycles for all differentials. If $h_0 x \neq 0$, $x \in \mathrm{Ext}_r^{s,t}(H_{SU}(2), Z_2)$, then $x \neq d_r(y)$ for any $y \in \mathrm{Ext}_r^{s-r,t-r+1}(H_{SU}(2), Z_2)$, $r \geqslant 2$. If $t - s = 2k + 1$, $x \in E_r^{s,t}$ then $x = h_1 y$ and $h_1 x \neq 0$ for all $r \geqslant 2$. If $x \in E_2^{s,t}$, $t - s = 2k$ and $x = h_1^2 y$ then $d_i(x) = 0$, $i \geqslant 2$.*

Since the multiplication by h_0 in E_∞ is adjoint to the multiplication by 2 in the ring V_{SU}, and the element $h_1 \in E_\infty^{1,2}$ defines in V_{SU} an element $\bar{h}_1$ such that $2\bar{h}_1 = 0$, $\bar{h}_1^2 \neq 0$, $\bar{h}_1^3 = 0$; so from Theorem 3 we get

Corollary 1. *The groups V_{SU}^{2k+1} have no elements of order 4 for all $k \geqslant 2$. Moreover, if $x \in V_{SU}^{2k+1}$, then $2x = 0$ and $x = \bar{h}_1 y$ where the element $y \in V_{SU}^{2k}$ can be thought of as finite order and $\bar{h}_1^2 y = \bar{h}_1 x \neq 0$, and $x \neq 0$.*

From Theorem 3.3 of Chapter II we see that in the algebras $E_2 = E_2(SU) = \mathrm{Ext}_A(H_{SU}(2), Z_2)$ and $E_\infty = E_\infty(SU)$ the relation $h_0^2 x = 0$ yields $h_0 x = 0$. This leads to the question: do any of V_{SU}^{2k} contain an element of order 4?

Appendix 2. Milnor's generators of the rings V_{SU} and V_U

Consider the algebraic submanifold $H_{r,t} \subset P^r(C) \times P^t(C)$ realising the cycle $P^{r-1}(C) \times P^t(C) + P^r(C) \times P^{t-1}(C)$ without singularities. It is easy to show that

$$(r + t - 1)[H_{r,t}] = -\binom{r+t}{r}.$$

It is known that $(r + t - 1)[P^{r+t-1}(C)] = +(r + t)$. Note that the GCD of $\left\{\binom{k}{i}\right\}$ $(i = 1, \ldots, k - 1)$ is equal to 1 if $k \neq p^l$ for any prime $p \geqslant 2$, and it is equal to p if $k = p^l$. Thus, taking a linear combination of manifolds $H_{r,t}$,

$P^{r+t-1}(C)$ for $r+t = \text{const.}$, one may get a manifold Σ^{r+t-1} such that

$$(r+t-1)[\Sigma^{r+t-1}] = \begin{cases} 1, \, r+t \neq p^l, \\ p, \, r+t = p^l. \end{cases}$$

From Theorem 5.5 of Chapter II (see also [17]) we see that the sequence of manifolds

$$\Sigma^1, \, \Sigma^2, \, \dots, \, \Sigma^k, \, \dots$$

gives a system of polynomial generators of the ring V_U. Now, consider the natural ring homomorphism $V_U \to V_{SO}/T$, where V_{SO}/T is the quotient ring of V_{SO} by 2-torsion. It is easy to show that the composition

$$V_U \to V_{SO} \to V_{SO}/T$$

is an epimorphism. Thus the manifolds Σ^{2k} generate the ring V_{SO}/T. The characteristic numbers of manifolds Σ^k can be easily calculated, thus the question of which set of numbers can be the set of numbers of a certain manifold was solved completely by Milnor (analogously one solves the question about Chern numbers of algebraic, complex-analytic, almost complex, and U-framed manifolds).

References

[1] J. F. Adams. On the structure and applications of the Steenrod algebra, Comm. Math. Helv., 32, No. 3 (1958), 180–214.

[2] B. G. Averbukh, The algebraic structure of inner homology groups, Dokl. Akad. Nauk SSSR, 125, No. 1 (1959), 11–14.

[3] A. Borel, Sur la cohomologie des éspaces fibrés principaux et des éspaces homogenes de groupes de Lie compacts, Ann. of Math., 57 (1953), 115–207.

[4] A. Borel, J. P. Serre. Groupes de Lie et puissances réduites de Steenrod, Amer. Math. Journ., 75 (1953), 409–448.

[5] A. Borel, F. Hirzebruch, Characteristic classes and homogeneous spaces. I, Amer. Math. Journ., 80 (1958), 459–538.

[6] H. Cartan, Sur l'itération des opérations de Steenrod, Comm. Math. Helv., 29, No. 1 (1955), 40–58.

[7] H. Cartan, Seminaire, 1954/55.

[8] H. Cartan, S. Eilenberg, Homological algebra, Princeton Univ. Press, 1956, 1–390.

[9] J. Milnor, Lectures on characteristic classes (mimeographed), Princeton University, 1958.

[10] J. Milnor, On the cobordism ring, Notices of Amer. Math. Soc., 5 (1958), 457.

[11] M. M. Postnikov, A study on the homotopy theory of continuous maps, Trudy Mat. Inst. im. V. A. Steklova, XXXXVI (1955).

[12] V. A. Rokhlin, Inner homology, Dokl. Akad. Nauk SSSR, 89, No. 5 (1953), 789–792.

[13] V. A. Rokhlin, Inner homology, Dokl. Akad. Nauk SSSR, 119, No. 5 (1958), 876–879.

[14] J. P. Serre, Cohomologie modulo 2 des complèxes d'Eilenberg–MacLane, Comm. Math. Helv., 27, No. 3 (1953), 198–232.

[15] J. P. Serre, G. Hochschild, Cohomologies of group extensions, Trans. Amer. Math. Soc., 74 (1953), 110–134.

[16] R. Thom, Quelques propriétés globales de variétés différentiables, Comm. Math. Helv., 28 (1954), 17–86.

[17] F. Hirzebruch, Die komplexe Mannigfaltigkeiten, Proc. Intern. Congress in Edinburgh, 1960.

[18] J. Milnor, On the cobordism ring and a complex analogue. I, Amer. Math. Journ., 82, No. 3 (1960), 505–521.

[19] S. P. Novikov, On some problems of the topology of manifolds related to the theory of Thom spaces, Dokl. Akad. Nauk SSSR, vol. 132, No. 5 (1960), 1031–1034.

[20] G. T. G. Wall, Determination of the cobordism ring, Ann. of Math., 72, No. 2, (1960), 292–311.

4

Generalized Poincaré's conjecture in dimensions greater than four

S. Smale [1]

Poincaré has posed the problem as to whether every simply connected closed 3-manifold (triangulated) is homeomorphic to the 3-sphere (see, [18]). This problem, still open, is usually called Poincaré's conjecture. The generalized Poincaré conjecture (see [11] or [28] for example) says that every closed n-manifold which has the homotopy type of the n-sphere S^n is homeomorphic to the n-sphere. One object of this paper is to prove that this is indeed the case if $n \geqslant 5$ (for differentiable manifolds in the following theorem and combinatorial manifolds in Theorem B).

Theorem A. *Let M^n be a closed C^∞ manifold which has the homotopy type of S^n, $(n \geqslant 5)$. Then M^n is homeomorphic to S^n.*

Theorem A and many of the other theorems of this paper were announced in [20]. This work is written from the point of view of differential topology, but we are also able to obtain a combinatorial version of Theorem A.

Theorem B. *Let M^n be a combinatorial manifold which has the homotopy of S^n $(n \geqslant 5)$. Then M^n is homeomorphic to S^n.*

J. Stallings has obtained a proof of Theorem B (and hence Theorem A) for $n \geqslant 7$ using different methods (*Polyhedral Homotopy-spheres*, Bull. Amer. Math. Soc., 66 (1960), 485–488).

The basic theorems of this paper, Theorems C and I below, are much stronger than Theorem A.

A *nice* function f on a closed C^∞ manifold is a C^∞ function with nondegenerate critical points and, at each critical point β, $f(\beta)$ equals the index of β. These functions were studied in [21].

[1] Annals of Mathematics, **74** (1961), 391–406 (Received October 11, 1960, Revised March 27, 1961). Reprinted with permission from Princeton University Press

Theorem C. *Let M^n be a closed C^∞ manifold which is $(m-1)$-connected, and $n \geqslant 2m, (n,m) \neq (4,2)$. Then there is a nice function f on M with type numbers satisfying $M_0 = M_n = 1$ and $M_i = 0$ for $0 < i < m$, for $0 < i < n, n-m < i < n$.*

Theorem C can be interpreted as stating that a cellular structure can be imposed on M^n with one 0-cell, one n-cell and no cells in the range $0 < i < m, n-m < i < n$. We will give some implications of Theorem C. First, by letting $m = 1$ in Theorem C, we obtain a recent theorem of M.Morse [13].

Theorem D. *Let M^n be a closed connected C^∞ manifold. There exists a (nice) non-degenerate function on M with just one local maximum and one local minimum.*

On p. 1, the handlebodies, elements of $\mathscr{H}(n,k,s)$, are defined. Roughly speaking, if $H \in \mathscr{H}(n,k,s)$ then H is defined by attaching s-disks, k in number, to the n-disk and "thickening" them. By taking $n = 2m+1$ in Theorem C, we will prove the following theorem, which in the case of 3-dimensional manifolds gives the well-known Heegaard decomposition.

Theorem F. *Let M be a closed C^∞ $(2m+1)$-manifold which is $(m-1)$-connected. Then $M = H \bigcup H'$, $H \bigcap H' = \partial H = \partial H'$ where $H, H' \in \mathscr{H}(2m+1,k,m)$ are handlebodies (∂V means the boundary of the manifold V).*

By taking $n = 2m$ in Theorem C we will get the following

Theorem G. *Let M^{2m} be a closed $(m-1)$-connected C^∞ manifold, $m \neq 2$. Then there is a nice function on M whose type numbers equal the corresponding Betti numbers of M. Furthermore M, with the interior of a $2m$-disk deleted, is a handlebody, an element of $\mathscr{H}(2m,k,m)$ where k is the m-th Betti number of M.*

Note that the first part of Theorem G is an immediate consequence of the Morse relation that the Euler characteristic is the alternating sum of the type numbers [12], and Theorem C.

The following is a special case of Theorem G.

Theorem H. *Let M^{2m} be a closed C^∞ manifold $m \neq 2$ of the homotopy type of S^{2m}. Then there exists on M a non-degenerate function with one maximum, one minimum, and no other critical point. Thus M is the union of two $2m$-disks whose intersection is a submanifold of M, diffeomorphic to S^{2m-1}.*

Theorem H implies the part of Theorem A for even dimensional homotopy spheres.

Two closed C^∞ oriented n-dimensional manifolds M_1 and M_2 are *J-equivalent* (according to Thom see [25] or [10]) if there exists an oriented manifold V with ∂V diffeomorphic to the disjoint union and each M_i is a deformation retract of V.

Theorem I. *Let M_1 and M_2 be $(m-1)$-connected oriented closed C^∞ $(2m+1)$-dimensional manifolds which are J-equivalent, $(m \neq 1)$. Then M_1 and M_2 are diffeomorphic.*

We obtain an orientation preserving diffeomorphisms. If one takes M_1 and M_2 J-equivalent disregarding the orientation one finds that M_1 and M_2 are diffeomorphic.

In studying manifolds under the relation of J-equivalence, one can use the methods of cobordism and homotopy theory, both of which are fairly developed. The importance of Theorem I is that it reduces diffeomorphis problems to J-equivalence problems for a certain class of manifolds. It is an open question as to whether arbitrary J-equivalent manifolds are diffeomorphic (see [10], Problem 5) (since this was written, Milnor has found a counter-example).

A short argument of Milnor ([10], p. 33) using Mazur's theorem (see [7]) applied to Theorem I yields the odd dimensional part of Theorem A. In fact, it implies that, if M^{2m+1} is a homotopy sphere $(m \neq 1)$ then M^{2m+1} minus a point is diffeomorphic to the Euclidean $(2m+1)$-space (see also [9], p. 440).

Milnor [10] has defined a group $\mathscr{H}^n$ of C^∞-homotopy n-spheres under the relation of J-equivalence. From Theorems A and I, and the work of Milnor [10] and Kervaire [5], the following is an immediate consequence

Theorem J. *If n is odd, $n \neq 3$, $\mathscr{H}^n$ is the group of all differentiable structures on S^n under the equivalence of diffeomorphism. For n odd there are a finite number of differentiable structures on S^n. For example:*

n	3	5	7	9	11	13	15
Number of Differentiable Structures on S^n	0	0	28	8	992	3	16256

Previously it was known that there are a countable number of differentiable structures on S^n for all n (Thom [22], see also [9], p. 442); and unique structures on S^n for $n \leqslant 3$ (e.g. Munkres [14]). Milnor [8] has also established lower bounds for the number of differentiable structures on S^n for several values of n.

A group Γ^n has defined by Thom [24] (see also Munkres [14] and Milnor [9]). This is the group of all diffeomorphisms of S^{n-1} modulo those which can be extended to the n-disk. A group A^n has been studied by Milnor as those structures on the n-sphere which, minus a point, are diffeomorphic to Euclidean space [9].

The group Γ^n can be interpreted (by Thom TOM [22] and Munkres [14]) as the group of differentiable structures on S^n, which admit a C^∞ function with the non-degenerate critical points and hence one has the inclusion map $i : \Gamma^n \to A^n$ defined.

Also, by taking J-equivalence classes, one gets a map $p : A^n \to \mathscr{H}^n$.

Theorem K. *With notations in the preceding paragraph, the following sequences are exact:*

(a) $A^n \xrightarrow{p} \mathscr{H}^n \to 0$, *$n \neq 3, 4$;*

(b) $\Gamma^n \xrightarrow{i} A^n \to 0$, *$n$ even, $n \neq 4$;*

(c) $0 \to A^n \xrightarrow{p} \mathscr{H}^n$, *$n$ odd, $n \neq 3$.*

Hence, if n is even and $n \neq 4$, $\Gamma^n = A^n$, if n is odd and $n \neq 3$, $A^n = \mathscr{H}^n$.

Here (a) follows from Theorem A, (b) from Theorem H, and (c) from Theorem I.

Kervaire [4] has also obtained the following result.

Theorem L. *There exists a manifold with no differentiable structure at all.*

Take the manifold W_0 of Theorem 4.1 of Milnor [10] for $k = 3$. Milnor shows ∂W_0 is a homotopy sphere. By Theorem A, ∂W_0 is homeomorphic to S^{11}. We attach a 12-disk to W_s by a homeomorphism of the boundary onto ∂W_0 to obtain a closed 12-dimensional manifold m. Starting with a triangulation of W_0, one can easily obtain a triangulation of M. If m possessed a differentiable structure it would be almost parallelizable, since the obstruction to almost parallelizability lies in $H^6(M, \pi_5(SO(12))) = 0$. But the index of M is 8 and hence by Lemma 3.7 of [10] M cannot possess any differentiable structure. Using Bott's results on the homotopy groups of Lie groups [1], one can similarly obtain manifolds of arbitrarily high dimension without a differentiable structure.

Theorem M. *Let C^{2m} be a contractible manifold, $m \neq 4$, whose boundary is simply connected. Then C^{2m} is diffeomorphic to the $2m$-disk. This implies that differentiable structures on disks of dimension $2m$, $(m \neq 2)$, are unique. Also the closure of the bounded component C of a C^∞ imbedded $(2m-1)$-sphere in the Euclidean $2m$-space, $(m \neq 2)$, is diffeomorphic to a disk.*

For these dimensions, the last statement of Theorem M is a strong version of the Schoenflies problem for the differentable case. Mazur's theorem [7] had already implied C was homeomorphic to the $2m$-disk.

Theorem M is proved as follows from Theorems C and I by Poincaré duality and the homology sequence of the pair $(C, \partial C)$, it follows that ∂C is a homotopy sphere and J-equivalend to zero since it bounds C. By Theorem I, then, ∂C is diffeomorphic to S^n. Now attach to C^{2m} a $2m$-disk by a diffeomorphism of the boundary to obtain a differentiable manifold V. One shows easily that V is a homotopy sphere and, hence by Theorem H, V is the union of two $2m$-disks. Since any two $2m$ sub-disks of V are equivalent under a diffeomorphism of V (for example see Palais [17]), the original $C^{2m} \subset V$ must already have been diffeomorphic to the standard $2m$-disk.

To prove Theorem B, note that $V = (M$ with the interior of a simplex deleted) is a contractible manifold, and hence possesses a differentiable structure (Munkres [15]). The double W of V is a differentiable manifold which has the homotopy type of a sphere. Hence by Theorem A, W is a topological sphere. Then according to Mazur [7], ∂V, being a differentiable submanifold and a topological sphere, divides W into two topological cells. Thus V is topologically a cell and M a topological sphere.

Theorem N. *Let $C^{2m} \neq 2$, be a contractible combinatorial manifold whose boundary is simply connected. Then C^{2m} is combinatorially equivalent to a simplex. Hence the Hauptvermutung (see [11]) holds for combinatorial manifolds which are closed cells in these dimensions.*

To prove Theorem N, one first applies a recent result of M.W.Hirsch [3] to obtain a compatible differentiable structure on C^{2m}. By Theorem M, this differentiable structure is diffeomorphic to the $2m$-disk D^{2m}. Since the standard $2m$-simplex σ^{2m} is a C^1 triangulation of D^{2m}, Whitehead's theorem [27] applies to yield that C^{2m} must be combinatorially equivalent to σ^{2m}.

Milnor first pointed out that the following theorem was a consequence of this theory.

Theorem O. *Let M^{2m}, $(m \neq 2)$, be a combinatorial manifold which has the same homotopy type as S^{2m}. Then M^{2m} is combinatorially equivalent to S^{2m}. Hence, in these dimensions, the Hauptvermutung holds for spheres.*

For even dimensions greater than four, Theorems N and O improve recent results of Gluck (cf. [2]).

Theorem O is proved by applying Theorem N to the complement of the interior of a simplex of M^{2m}.

Our program is the following. We introduce handlebodies, and then prove the "handlebody theorem" and a variant. These are used together with a theorem on the existence of "nice functions" from [21] to prove

Theorems C and I, the basic theorems of the paper. After that, it remains only to finish the proof of Theorems F and G of the Introduction.

The proofs of Theorems C and I are similar. Although they use a fair amount of the technique of differential topology, they are, in a certain sense, elementary. It is in their application that we use many recent results.

A slightly different version of this work was mimeographed in May 1960. In this paper J. Stallings pointed out a gap in the proof of the handlebody theorem (for the case $s = 1$). This gap happened not to affect our main theorems.

Everything will be considered from the C^∞-point of view. All imbeddings will be C^∞. A *differentiable isotopy* is homotopy of imbeddings with continuous differential.

$$E^n = \{x = (x_1, \dots, x_n)\}, \quad \|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}.$$

$$D^n = \{x \in E^n, \|x\| \leqslant 1\}, \quad \partial D^n = S^{n-1} = \{x \in E^n, \|x\| = 1\};$$

D_i^n etc. are copies of D^n.

A. Wallace's recent article [26] is related to some of this paper.

1. Let M^n be a compact manifold, Q a component of ∂M and

$$f_i \colon \partial D_i^s \times D_i^{n-s} \longrightarrow Q, \quad i = 1, \dots, k,$$

imbeddings with disjoint images, $s \geqslant 0, n \geqslant s$. We define a new compact C^∞-manifold $V = \chi(M, Q; f_1, \dots, f_k; s)$ as follows. The underlying topological space of V is obtained from M, and the $D_i^s \times D_i^{n-s}$ by identifying points which correspond under some f_i. The manifold thus defined has a natural differentiable structure except along corners $\partial D_i^s \times D_i^{n-s}$ for each i. The differentiable structure we put on V is obtained by the process of "straightening the angle" along these corners. This is carried out by Milnor [10] for the case of the product of manifolds W_1 and W_2 with a corner along $\partial W_1 \times \partial W_2$. Since the local situation for the two cases is essentially the same, his construction applies to give a differentiable structure on V. He shows that this structure is well-defined up to diffeomorphism.

If $Q = \partial M$ we omit it from the notation $\chi(M, Q; f_1, \dots, f_k; s)$, and we sometimes also omit the s. We can consider the "handle" $D_i^s \times D_i^{n-s} \subset V$ as differentiably embedded.

The next lemma is a consequence of the definition

Lemma 1.1. *Let $f_i \colon \partial D_i^s \times D_i^{n-s} \to Q$ and $f_i' \colon \partial D_i^s \times D_i^{n-s} \to Q$, $i = 1, \dots, k$, be two sets of imbeddings each with disjoint images Q, M as above. Then $\chi(M, Q; f_1, \dots, f_k; s)$ and $\chi(M, Q; f_1', \dots, f_k'; s)$ are diffeomorphic if*

(a) *there is a diffeomorphism* $h\colon M \to M$ *such that* $f'_i = hf_i$, $i = 1, \ldots, k$; *or*

(b) *there exist diffeomorphisms* $h_i{:}D^s \times D^{n-s} \to D^s \times D^{n-s}$ *such that* $f'_i = f_i h_i$, $i = 1, \ldots, k$; *or*

(c) *the* f'_i *are permutations of the* f_i*'s.*

If V is the manifold $\chi(M, Q; f_i, \ldots, f_k; s)$, we say $\sigma = (M, Q; f_1, \ldots, f_k; s)$ is *a presentation* of V.

A *handlebody* is a manifold which has a presentation of the form $(D^n; f_1, \ldots, f_k; s)$. Fixing n, k, s the set of all handlebodies is denoted by $\mathscr{H}(n, k, s)$. For example, $\mathscr{H}(n, k, 0)$ consists of one element, the disjoint union of $(k+1)$ n-disks; and one can show $\mathscr{H}(2, 1, 1)$ consists of $S^1 \times I$ and the Möbius strip, and $\mathscr{H}(3, k, 1)$ consists of the classical handlebodies [19; Henkelkörper], orientable and non-orientable, or at least differentiable analogues of them.

Theorem 1.2 (Handlebody theorem). *Let* $n \geqslant 2s + 2$, *and, if* $s = 1, n \geqslant 5$ *let* $H \in \mathscr{H}(n, k, s)$, $V = \chi(H; f_1, \ldots, f_r; s + 1)$ *and* $\pi_s(V) = 0$. *Also if* $s = 1$, *assume* $\pi_1(\chi(H; f_1, \ldots, f_{r-k}; 2)) = 1$. *Then* $V \in \mathscr{H}(n, r-k, s+1)$. *(We do not know if the special assumption for* $s = 1$ *is necessary.)*

The next three sections 2–4 are devoted to a proof of this Theorem.

2. Let $G_r{=}G_r(s)$ be the free group on r generators $D_1, \ldots, D_r$ if $s = 1$ and the free abelian group on r generators $D_1, \ldots, D_r$, if $s > 1$. If $\sigma = (M, Q; f_1, \ldots, f_r; s + 1)$ is a presentation of a manifold V, define a homomorphism $f_\sigma\colon G_r \to \pi_s(Q)$, by $f_\sigma(D_i) = \varphi_i$, where $\varphi_i \in \pi_s(Q)$ is the homotopy class of $\bar{f}_i\colon \partial D^{s+1} \times 0 \to Q$, the restriction of f_i. To take care of base points in case $\pi_1(Q) \neq 1$, we will fix $x_0 \in D_i^{s+1} \times 0$, $y_0 \in Q$. Let U be some cell neighborhood of y_0 in Q, and assume $\bar{f}_i(x_0) \in U$. We say that the homomorphism f_σ is *induced* by the presentation σ.

Suppose now that $F\colon G_r(s) \to \pi_s(Q)$ is a homomorphism where Q is a component of the boundary of a compact n-manifold M. Then we say that a manifold V realizes F if some presentation of V induces F. Manifolds realizing a given homomorphism are not necessarily unique.

The following theorem is the goal of this section

Theorem 2.1. *Let* $n \geqslant 2s + 2$ *and if* $s = 1$, $n \geqslant 5$*; let* $\sigma = (M, Q; f_1, \ldots, f_r; s + 1)$ *be a presentation of a manifold* V, *and assume* $\pi_1(Q) = 1$, *if* $n = 2s + 2$. *Then for any automorphism* $\alpha\colon G_r \to G_r$, V *realizes* $f_\sigma \alpha$.

Our proof of Theorem 2.1 is valid for $s = 1$, but we have application for the theorem only for $s > 1$. For the proof, we will need some lemmas.

Lemma 2.2. *Let Q be a component of the boundary of a compact manifold M^n and $f_1\colon \partial D^s \times D^{n-1} \to Q$ an imbedding. Let $\bar{f}_2\colon \partial D^s \times 0 \to Q$ be an imbedding, differentiably isotopic in Q to the restriction $\bar{f}_1$ of f_1 to $\partial D^s \times 0$. Then there exists an imbedding $f_1\colon \partial D^s \times D^{n-s} \to Q$, extending $\bar{f}_2$ and a diffeomorphism $h\colon M \to M$ such that $hf_2 = f_1$.*

PROOF. Let $\bar{f}_t : \partial D^s \times 0 \to Q$, $1 \leqslant t \leqslant 2$, be a differentiable isotopy between $\bar{f}_1$ and $\bar{f}_2$. Then by the covering homotopy property for spaces of differentiable embeddings (see Thom [23], and R.Palais, Comment. Math. Helv., 34 (1960), 305–312), there is a differentiable isotopy $F_t\colon \partial D^s \times D^{n-s} \to Q$, $1 \leqslant t \leqslant 2$, with $F_1 = f_1$ and F_f restricted to $\partial D^s \times 0 = \bar{f}_t$. Now by applying this theorem again, we obtain a differentiable isotopy $G_t\colon M \to M$, $1 \leqslant t \leqslant 2$, with G_1 equal the identity, and G_t, restricted to image of F_1 equal $F_t F_1^{-1}$. Then taking $h = G_2^{-1}$, F_2 satisfies the requirements of f_2 of (2.2); i.e. $hf_2 = G_2^{-1}F_2 = F_1 F_2^{-1} F_2 = f_1$.

Theorem 2.3 (H.Whitney, W.T.Wu). *Let $n \geqslant \max(2k+1, 4)$ and $f, g : M^k \to X^n$ be two imbeddings, M closed and connected and X simply connected if $n = 2k+1$. Then, if f and g are homotopic, they are differentiably isotopic.*

Whitney [29] proved this theorem for $n \geqslant 2k+2$. W.T.Wu [30] (using methods of Whitney) proved it where X^n was Euclidean space, $n = 2k+1$. His proof also yields Theorem 2.3 as stated.

Lemma 2.4. *Let Q be a component of the boundary of a compact manifold M^n, $n \geqslant 2s+2$ and if $s = 1$, $n \geqslant 5$, and $\pi_1(Q) = 1$ if $n = 2s+2$. Let $f_1\colon \partial D^{s+1} \times D^{n-s-1} \to Q$ be imbedding homotopic and $\bar{f}_2\colon \partial D^{s+1} \times 0 \to Q$ be an imbedding homotopic in Q to $\bar{f}_1$, the restriction of f_1 to $\partial D^{s+1} \times 0$. Then there exists an imbedding $f_2\colon \partial D^{s+1} \times D^{n-s-1} \to Q$, extending $\bar{f}_2$ such that $\chi(M, Q; f_2)$ is diffeomorphic to $\chi(M, Q; f_1)$.*

PROOF. By Theorem 2.3, there exists a differentiable isotopy between $\bar{f}_1$ and $\bar{f}_2$. Apply Lemma 2.2 to get $f_2\colon \partial D^{s+1} \times D^{n-s-1} \to Q$, extending $\bar{f}_2$, and a diffeomorphism $h\colon M \to M$ with $hf_2 = f_1$. Application of Lemma 1.1 yields the desired conclusion.

Lemma 2.5 (Nielsen [16]). *Let G be a free group on r generators $D_1, \ldots, D_r$ and $\mathscr{A}$ be the group of automorphisms of G. Then $\mathscr{A}$ is generated by the following automorphisms*

$$R: \quad D_1 \to D_1^{-1}, \quad D_i \to D_i, \quad i > 1,$$

$$T_i: \quad D_1 \to D_i, \quad D_i \to D_1,\ D_j \to D_j, \quad j \neq 1,\ i \neq j,\ i = 2, \ldots, r,$$

$$S: \quad D_1 \to D_1 D_2, \quad D_i \to D_i, \quad i > 1.$$

The same is true for the free abelian case (well-known).

It is sufficient to prove Theorem 2.1 with α replaced by the generators of $\mathscr{A}$ of Lemma 2.5.

First take $\alpha = R$. Let $h\colon D^{s+1} \times D^{n-s-1} \to D^{s+1} \times D^{n-s-1}$ be defined by $h(x, y) = (rx, y)$, where $r\colon D^{s+1} \to D^{s+1}$ is a reflection through an equatorial s-plane. Then let $f_1' = f_1 h$. If $\sigma' = (M, Q; f_1', f_2, \ldots, f_r; s+1)$, $\chi(\sigma')$ is diffeomorphic to V by Lemma 1.1. On the other hand, $\chi(\sigma')$ realizes $f_{\sigma'} = f_\sigma \alpha$.

The case $\alpha = T_i$ follows immediately from Lemma 1.1. So now we proceed with the proof of Theorem 2.1 with $\alpha = S$.

Define V_1 to be the manifold $\chi(M, Q; f_2, \ldots, f_r; s+1)$ and let $Q_1 \subset \partial V_1$ be $Q_1 = \partial V_1 \setminus (\partial M \setminus Q)$. Let $\varphi_i \in \pi_s(Q)$, $i = 1, \ldots, r$, denote the homotopy class of $\bar{f}_i\colon \partial D_i^{s+1} \times 0 \to Q$, the restriction of f_i. Let $\gamma : \pi_s(Q \cap Q_1) \to \pi_s(Q)$ and $\beta\colon \pi_s(Q \cap Q_1) \to \pi_s(Q_1)$ be the homomorphisms induced by the respective inclusions.

Lemma 2.6. *With notations and conditions as above, $\varphi_2 \in \gamma \operatorname{Ker} \beta$.*

PROOF. Let $q \in \partial D_2^{n-s-1}$ and $\psi\colon \partial D_2^{s+1} \times q \to Q \cap Q_1$ be the restriction of f_2. Denote by $\bar{\psi} \in \pi_s(Q \cap Q_1)$ the homotopy class of ψ. Since ψ and $\bar{f}_2$ are homotopic in Q, $\gamma\bar{\psi} = \varphi_2$. On the other hand $\beta\bar{\psi} = 0$, thus proving Lemma 2.6.

By Lemma 2.6, let $\bar{\psi} \in \pi_s(Q \cap Q_1)$ with $\gamma\bar{\psi} = \varphi_2$ and $\beta\bar{\psi} = 0$. Let $g = y + \bar{\psi}$ (or $g = y\bar{\psi}$ in case $s = 1$, our terminology assumes $s > 1$), where $y \in \pi_s(Q \cap Q_1)$ is the homotopy class of $\bar{f}_1 : \partial D_1^{s+1} \times 0 \to \to Q \cap Q_1$. Let $\bar{g}\colon \partial D^{s+1} \times 0 \to Q \cap Q_1$ be an imbedding realizing g (see [29]).

If $n = 2s + 2$ then from the fact that $\pi_1(Q) - 1$, it follows that also $\pi_1(Q_1) = 1$. Then since $\bar{g}$ and $\bar{f}_1$ are homotopic in Q_1, i.e. $\beta g = \beta y$, Lemma 2.4 applies to yield an imbedding $e\colon \partial D^{s+1} \times D^{n-s-1} \to Q_1$, extending $\bar{g}$ such that $\chi(V_1, Q_1; f_1)$ and $\chi(V_1, Q_1; e)$ are diffeomorphic.

On one hand $V = \chi(M, Q; f_1, \ldots, f_r) = \chi(V_1, Q_1; f_1)$ and, on the other hand, $\chi(M, Q; e, f_2, \ldots, f_r) = \chi(V_1, Q_1; e)$, so by the preceding statement, V and $\chi(V, Q; e, f_2, \ldots, f_r)$ are diffeomorphic. Since $\gamma g = g_1 + g_2$, $f_\sigma \alpha(D_1) = f_\sigma(D_1 + D_2) = g_1 + g_2$, $f_{\sigma'} = g D_1 = g_1 + g_2$, to $f_\sigma \alpha = f_{\sigma'}$, where $\sigma' = (V, Q; e, f_2, \ldots, f_r)$. This proves Theorem 2.1.

3. The goal of this section is to prove the following theorem.

Theorem 3.1. *Let $n \geqslant 2s + 2$ and, if $s = 1$, $n \geqslant 5$. Suppose $H \in \mathscr{H}(n, k, s)$. Then given $r \geqslant k$, there exists an epimorphism $g\colon G_r \to \pi_s(H)$ such that every realization of g is in $\mathscr{H}(n, r - k, s + 1)$.*

For the proof of Theorem 3.1, we need some lemmas.

Lemma 3.2. *If $H \in \mathscr{H}(n,k,s)$, then $\pi_s(H)$ is*

(a) *a set of $k+1$ elements if $s=0$;*

(b) *a free group on k generators if $s=1$;*

(c) *a free abelian group on k generators if $s>1$. Furthermore if $n \geqslant 2s+2$, then $\pi_i(\partial H) \to \pi_i(H)$ is an isomorphism for $i \leqslant s$.*

PROOF. We can assume $s>0$ since, if $s=0$, H is a set of n-disks $k+1$ in number. Then H has a deformation retract in an obvious way the wedge of k s-spheres. Thus (b) and (c) are true. For the last statement of Lemma 3.2, from the exact homotopy sequence of the pair $(H, \partial H)$, it is sufficient to show that $\pi_i(H, \partial H) = 0$, $i \leqslant s+1$.

Thus let $f\colon (D^i, \partial D^i) \to (H, \partial H)$ be a given continuous map with $i \leqslant s+1$. We want to construct a homotopy $f_t\colon (D^i, \partial D^i) \to (H, \partial H)$ with $f_0 = f$ and $f_1(D^i) \subset \partial H$.

Let $f_1\colon (D^i, \partial D^i) \to (H, \partial H)$ be a differentiable approximation of f. Then by a radial projection from a point in D^n not in the image of f_1, f_1 is homotopic to a differentiable map $f_2\colon (D^i, \partial D^i) \to (H, \partial H)$, with the image of f_2 not intersecting the interior of $D^n \subset H$. Now for dimensional reasons f_2 can be approximated by a differentiable map $f_3\colon (D^i, \partial D^i) \to (H, \partial H)$ with the image of f_3 not intersecting any $D_i^s \times 0 \subset H$. Then by other projections, one for each i, f_3 is homotopic to a map $f_4\colon (D^i, \partial D^i) \to (H, \partial H)$ which sends all of D^i into ∂H. This shows $\pi_i(H, \partial H) = 0$, $i \leqslant s+1$, and proves Lemma 3.2.

If $\beta \in \pi_{s-1}(O(n-s))$, let H_β be the $(n-s)$-cell bundle over S^s determined by β.

Lemma 3.3. *Suppose $V = \chi(H_\beta; f; s+1)$ where $\beta \in \pi_{s-1}(O(n-s))$, $n \geqslant 2s+2$ or if $s=1$, $n \geqslant 5$. Let also $\pi_s(V) = 0$. Then V is diffeomorphic to D^n.*

PROOF. The zero-cross-section $\sigma\colon S^s \to H_\beta$ is homotopic to zero, since $\pi_s(V) = 0$, and so regularly homotopic in V to a standard s-sphere S_0^s, contained in a cell neighborhood by dimensional reasons [29]. Since a regular homotopy preserves the normal bundle structure, $\sigma(S^s)$ has a trivial normal bundle and thus $\beta = 0$. Hence H_β is diffeomorphic to the product of S^s and D^{n-s}.

Let $\sigma_1\colon S^s \to \partial H_\beta$ be a differentiable cross section and $\bar{f}\colon \partial D^{s+1} \times 0 \to \partial H_\beta$ the restriction of $f\colon \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$. Then σ_1 and $\bar{f}$ are homotopic in ∂H_β (perhaps after changing f by a diffeomorphism of $D^{s+1} \times D^{n-s-1}$ which reverses the orientation of $\partial D^{s+1} \times 0$). Thus we can assume $\bar{f}$ and σ_1 are the same.

Let f_ε be the restriction of f to $\partial D^{s+1} \times D_\varepsilon^{n-s-1}$, where D_ε^{n-s-1} denotes the disk $\{x \in D_\varepsilon^{n-s-1},\ \|x\| \leqslant \varepsilon\}$, and $\varepsilon > 0$. Then the imbedding g_ε :

$\partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$i differentiably isotopic to f where $g_\varepsilon(x,y) = f_\varepsilon r\varepsilon(x,y)$ and $r_\varepsilon(x,y) = (x, \varepsilon y)$. Define $k_\varepsilon \colon \partial D^{s+1} \times D^{n-s-1} \to F_x$ by $p_x g_\varepsilon(x,y)$ where $p_x \colon g_\varepsilon(x \times D^{n-s-1}) \to F_x$ is projection into the fibre F_x of ∂H_β over $\sigma^{-1} g_\varepsilon(x,0)$. If ε is small enough, k_ε is well-defined and an embedding. In fact if ε is small enough one can even suppose that for each x, k_ε maps $x \times D^{n-s-1}$ linearly onto image $k_\varepsilon \cap F_x$ where image $k_\varepsilon \cap F_\varepsilon$ has a linear structure induced from F.

It can be proved that k_ε and g_ε are differentiably isotopic. (The referee has remarked that there is a theorem, Milnor's "tubular neighborhood theorem", which is useful in this connection and can indeed be used to make this proof clearer in general.)

We finish the proof of Lemma 3.3 as follows.

Suppose V is as in Lemma 3.3 and $V' = \chi(H_\beta; f'; s+1)$, $\pi_s(V') = 0$. It is sufficient to prove V and V' are diffeomorphic since it is clear that one can obtain D^n by choosing f' properly and using the fact that H_β is a product of S^s and D^{n-s}. From the previous paragraph, we can replace f and f' by k_ε and k'_ε with those properties listed. We can also suppose without loss of generality that the images of k_ε and k'_ε coincide. It is now sufficient to find a diffeomorphism h of H_β with $hf = f'$. For each x, define h on image $f \cap F_x$ to be the linear map which has this property. One can now easily extend h to all of H_β and thus we have finished the proof of Lemma 3.3.

Suppose now M_1^n and M_2^n are compact manifolds and $f_i \colon D^{n-1} \times i \to \partial M_i$, are imbeddings for $i = 1$ and 2. Then $\chi(M_1 \cup M_2; f_1 \cup f_2; 1)$ is a well defined manifold, where $f_1 \cup f_2 \colon \partial D^1 \times \partial D^{n-1} \to \partial M_1 \cup \partial M_2$ is defined by f_1 and f_2, the set of which, as the f_i vary, we denote by $M_1 + M_2$. (If we pay attention to orientation, we can restrict $M_1 + M_2$ to have but one element.)

The following lemma is easily proved.

Lemma 3.4. *The set $M^n + D^n$ consists of one element, namely M^n.*

Lemma 3.5. *Suppose an imbedding $f \colon \partial D^s \times D^{n-s} \to dM^n$ is null-homotopic where M is a compact manifold, $n \geqslant 2s+2$ and, if $s = 1$, $n \geqslant 5$. Then $\chi(M; f) = M + H_\beta$ for some $\beta \in \pi_{s-1}(O(n-s))$.*

PROOF. Let $\bar{f} \colon \partial D^s \times q \to \partial M$ be the restriction of f, where q is a fixed point in ∂D^{n-s}. Then by dimensional reasons [29], $\bar{f}$ can be extended to an imbedding $\varphi \colon D^s \to \partial M$ where the image of ϕ intersects the image of f only on $\bar{f}$. Next let T be a tubular neighborhood of $\phi(D^s)$ in M. This can be done so that T is a cell, $T \cup f(\partial D^s \times D^{n-s})$ is of the form H_β and $V \in M + H_\beta$. We leave the details to the reader.

To prove Theorem 3.1, let $H = \chi(D^n;\ f_1, \ldots, f_k; s)$. Then f_i defines a class $\bar{\gamma}_i \in \pi_s(H, D^n)$. Let $\gamma_i \in \pi_s(\partial H)$ be the image of $\bar{\gamma}_i$ under the inverse of the composition of the isomorphisms $\pi_s(\partial H) \to \pi_s(H) \to \pi_s(H, D^n)$; (using Lemma 3.2). Define g of Theorem 3.1 by $gD_i = \gamma_i,\ \ i \leqslant k$ and $gD_i = 0,\ i > k$. That g satisfies Theorem 3.1, follows by induction from the following lemma.

Lemma 3.6. $\chi(H; g_i; s+1) \in \mathscr{H}(n, k-1, s)$, *if the restriction of* g_i *to* $\partial D^{s+1} \times 0$ *has homotopy class* $\gamma_i \in \pi_s(\partial H)$.

Now Lemma 3.6 follows from Lemmas 3.3–3.5 and the fact that g_i is differentiably isotopic to g_i' whose image is in $\partial H_\beta \subset \partial H$, where H_β is defined by Lemma 3.5 and f_1.

4. We prove here Theorem 1.2. First suppose $s = 0$. Then $H \in \mathscr{H}(n, k, 0)$ is the disjoint union of n-disks, $k+1$ in number, and $V = \chi(H; f_1, \ldots, f_r; 1)$. Since $\pi_0(V) = 1$, there exists a permutation of $1, \ldots, r$, $i_1, \ldots, i_r$, such that $Y = \chi(H; f_{i_s}, \ldots, f_{i_k}; 1)$ is connected. By Lemma 3.3, Y is diffeomorphic to D^n. Hence $V = \chi(Y; f_{i_{k+1}}, \ldots, f_{i_r}; 1)$ is in $\mathscr{H}(n, r-k, 1)$.

Now consider the case $s = 1$. Choose, by Theorem 3.1, $g\colon G_k \to \pi_1(\partial H)$ such that every manifold derived from g is diffeomorphic to D^n. Let $Y = \chi(H; f_1, \ldots, f_{r-k})$. Then $\pi_1(Y) = 1$ and by the argument of Lemma 3.2, $\pi_1(\partial Y) = 1$. Let $\bar{g}_i\colon \partial D^2 \times 0 \to \partial H$ be disjoint imbeddings realizing the classes $g(D_i) \in \pi_1(\partial H)$ which are disjoint from the images of all f_i, $1, \ldots, k$. Then by Lemma 2.4, there exist imbeddings $g_1, \ldots, g_k\colon \partial D^2 \times D^{n-2} \to \partial H$, extending the $\bar{g}_i$ such that $V = \chi(Y; f_{r-k+1}, \ldots, f_r)$ and $\chi(Y; g_1, \ldots, g_k)$ are diffeomorphic. But

$$\begin{aligned}\chi(Y; g_1, \ldots, g_k) &= \chi(H; g_1, \ldots, g_k,\ f_1, \ldots, f_{r-k}) \\ &= \chi(D^n; f_1, \ldots, f_{r-k}) \in H(n, r-k, 2).\end{aligned}$$

Hence, so does V.

For the case $s > 1$, we can use an algebraic lemma.

Lemma 4.1. *If* $f, g\colon G \to G'$ *are epimorphisms where* G *and* G' *are finitely generated free abelian groups, then there exists an automorphism* $\alpha\colon G \to G$ *such that* $f\alpha = g$.

PROOF. Let G'' be a free abelian group of rank equal to $rank G - rank G'$ and let $p\colon G''{+}G'{\to}G'$ be the projection. Then, identifying elements of G and $G' + G''$ under some isomorphism, it is sufficient to prove the existence of α for $g = p$. Since the groups are free, the following exact sequence splits

$$0 \to f^{-1}(0) \to G \xrightarrow{f} G' \to 0$$

Let $h\colon G \to f^{-1}(0)$ be the corresponding projection and let $k\colon f^{-1}(0) \to G''$ be some isomorphism. Then $\alpha\colon G \to G' + G''$ defined by $f + kh$ satisfies the requirement of Lemma 4.1.

REMARK. Using Grushko's Theorem [6], one can also prove Lemma 4.1 when G and F′ are free groups.

Now take $\sigma = (H; f_1, \ldots, f_r; s+1)$ of Theorem 1.2 and $g\colon G_r \to \pi_s(\partial H)$ of Theorem 3.1. Since $\pi_s(V) = 0$ and $s > 1$, $f_\sigma : G_r \to \pi_s(\partial H)$ is an epimorphism. By Lemmas 3.2 and 4.1 there is an automorphism $\alpha\colon G_r \to G_r$ such that $f_\sigma \alpha = g$. Then Theorem 2.1 implies that V is in $\mathscr{H}(n, r-k, s+1)$ using the main property of g.

5. The goal of this section is to prove the following analogue of Theorem 1.2.

Theorem 5.1. *Let $n \geqslant 2s + 2$ or if $s = 1$, $n \geqslant 5$, M^{n-1} be a simply connected $(s-1)$-connected closed manifold and $\mathscr{H}_M(n, k, s)$ the set of all manifolds having presentations of the form $(M \times [0,1], M \times 1; f_1, \ldots, f_k; s)$. Now let $H \in \mathscr{H}_M(n, k, s)$, $Q = \partial H \setminus (M \times 0)$, $V = \chi(H, Q; g_1, \ldots, g_r; s+1)$, and suppose $\pi_s(M \times 0) \to \pi_s(V)$ is an isomorphism. Also suppose if $s = 1$, that $\pi_s(\chi(H, Q; g_1, \ldots, g_{r-k}; 2) = 1)$. Then $V \in \mathscr{H}_M(n, r-k, s+1)$.*

One can easily obtain Theorem 1.2 from Theorem 5.1 by taking for M, the $(n-1)$-sphere. The following Lemma is easy, following Lemma 3.2.

Lemma 5.2. *With definitions and conditions as in Theorem 5.1, $\pi_s(Q) = G_k$, if $s = 1$, and if $s > 1$, $\pi_s(Q) = \pi_s(M) + G_k$.*

Let $p_1\colon \pi_s(Q) \to \pi_s(M)$, $p_2\colon \pi_s(Q) \to G_k$ be the respective projections.

Lemma 5.3. *With definitions and conditions as in Theorem 5.1, there exists a homomorphism $g : G_r \to \pi_s(Q)$ such that $p_1 g$ is trivial, $p_2 g$ is an epimorphism, and every realization of g is in $\mathscr{H}_M(n, r-k, s+1)$, each $r \geqslant k$.*

The proof follows Lemma 3.1.

We now prove Theorem 5.1. The cases $s = 0$ and $s = 1$ are proved similarly to these cases in the proof of Theorem 1.2. Suppose $s > 1$. From the fact that $\pi_s(M \times 0) \to \pi_s(V)$ is an isomorphism, it follows that $p_1 f_\sigma$ is trivial and $p_2 f_\sigma$ is an epimorphism where $\sigma = (H, Q; g_1, \ldots, g_r; s+1)$. Then apply Lemma 4.1 to obtain an automorphism $\alpha\colon G_r \to G_r$ such that $p_2 f_\sigma \alpha = p_2 g$ where g is as in Lemma 5.3. Then $f_\sigma \alpha = g$, hence using Theorem 2.1, we obtain the conclusion of Theorem 5.1.

6. The goal of this section is to prove the following two theorems.

Theorem 6.1. *Suppose f is a C^∞ function on a compact manifold W with no critical points on $f^{-1}[-\varepsilon,\varepsilon] = N$ except k non-degenerate ones on $f^{-1}(0)$, all of index λ, and $N \cap \partial W = \varnothing$. Then $f^{-1}[-\infty,\varepsilon]$ has a presentation of the form*

$$(f^{-1}[-\infty,-\varepsilon]; f^{-1}(-\varepsilon); f_1,\dots,f_k;\lambda).$$

Theorem 6.2. *Let $(M,Q;f_1,\dots,f_k;s)$ be a presentation of a manifold V and g be a C^∞ function on M, regular, in a neighborhood of Q, and constant with its maximum value on Q. Then there exists a C^∞ function G on V which agrees with g outside a neighborhood of Q, is constant and regular on $\partial V\backslash(\partial M\backslash Q)$, and has exactly k new critical points, all non-degenerate, with the same value and with index s.*

SKETCH OF PROOF OF THEOREM 6.1. Let β_i denote the critical points of f at level zero, $i=1,\dots,k$, with disjoint neighborhoods V_i. By a theorem of Morse [13] we can assume V_i has a coordinate system $(x_1,\dots,x_n)$, such that for $\|x\| \leqslant \delta$, some $\delta > 0$, $f(x) = -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$. Let E_1 be the $(x_1,\dots,x_\lambda)$ plane of V_i and E_2 be the $(x_{\lambda+1},\dots,x_n)$ plane. Then for $\varepsilon_1 > 0$ sufficiently small $E_1 \cap f^{-1}[-\varepsilon_1,\varepsilon_1]$ is diffeomorphic to D^λ. A sufficiently small tubular neighborhood T of E_1 will have the property that $T' = T \cap f^{-1}[-\varepsilon_1,\varepsilon_1]$ is diffeomorphic to $D^\lambda \times D^{n-\lambda}$ with $T \cap f^{-1}(-\varepsilon_1)$ corresponding to $\partial D^\lambda \times D^{n-\lambda}$.

As we pass from $f^{-1}[-\infty,-\varepsilon_1]$ to $f^{-1}[-\infty,\varepsilon_1]$, it happens that one such T' is added for each i, together with a tubular neighborhood of $f^{-1}(-\varepsilon_1)$ so that $f^{-1}[-\infty,\varepsilon_1]$ is diffeomorphic to a manifold of the form $\chi(f^{-1}[-\infty,-\varepsilon_1], f^{-1}(-\varepsilon_1); f_1,\dots,f_k;\lambda)$. Since there are no critical points between $-\varepsilon$ and $-\varepsilon_1$, ε_1 and ε, ε_1 can be replaced by ε in the preceding statement thus proving Theorem 6.1.

Theorem 6.2 is roughly a converse of Theorem 6.1 and the proof can be constructed similarly.

7. In this section we prove Theorems C and I of the Introduction.

The following theorem was proved in [21].

Theorem 7.1. *Let V^n be a C^∞ compact manifold with ∂V the disjoint union of V_1 and V_2, each V_i closed in ∂V. Then there exists a C^∞ function f on V with non-degenerate critical points, regular on ∂V, and such that $f(V_1) = -1/2$, $f(V_2) = n+1/2$, and at a critical point β of f, $f(\beta) = index \beta$.*

Functions of such sort are called *proper* functions.

Suppose now M^n is a closed C^∞-manifold and f is the function of Theorem 7.1. Let $X_s = f^{-1}[0, s+1/2]$, $s = 0, 1, \dots, n$.

Lemma 7.2. *For each s, the manifold X_s has a presentation of the form $(X_{s-1}; f_1, \dots, f_k; s)$.*

This follows from Theorem 6.1.

Lemma 7.3. *If $H \in \mathscr{H}(n,k,s)$, then there exists a C^∞ non-degenerate function f on H, $f(\partial H) = s + 1/2$; f has one critical point β_0 of index 0 value 0, k critical points of index s, value s and no other critical points.*

This follows immediately from Theorem 6.2.

The proof of Theorem C then goes as follows. Take a nice function f on M by 7.1, with X_s defined as above. Note that $X_0 \in \mathscr{H}(n,q,0)$ and $\pi_0(X_1) = 0$, hence by Lemma 7.2 and Theorem 1.2, $X_1 \in \mathscr{H}(n,k,1)$. Suppose now that $\pi_1(M) = 1$ and $n \geqslant 6$.

The following argument suggested by H.Samelson simplifies and replaces a complicated one of the author. Let X_2' be the sum of X_2 and k copies $H_1, \dots, H_k$ of $D^{n-2} \times S^2$. Then since $\pi_1(X_2) = 0$, Theorem 1.2 implies that $X_2' \in \mathscr{H}(n,r,2)$. Now let $f_i \colon \partial D^s \times D^{n-s} \to \partial H_i \cap \partial X_2'$ for $i = 1, \dots, k$ be differentiable imbeddings such that the composition

$$\pi_2(\partial D^s \times D^{n-s}) \to \pi_2(\partial H_i \times \partial X_2') \to \pi_2(\partial H_i)$$

is an isomorphism. Then by Lemmas 3.3 and 3.4, $\chi(X_2', f_1, \dots, f_k; 3)$ is diffeomorphic to X_2. Since $X_3 = \chi(X_2'; g_1, \dots, g_l; 3)$, we have

$$X_3 = \chi(X_2', f_1, \dots, f_k, g_1, \dots, g_l; 3)$$

and another application of Theorem 1.2 yields that $X_3 \in \mathscr{H}(n, k+l-r; 3)$.

Iteration of the argument yields that $X_m' \in \mathscr{H}(n,r,m)$. By applying Lemma 7.3, we can replace g by a new nice function h with type numbers satisfying $M_0 = 1$ and $M_i = 0$, $0 < i < m$. Now apply the preceding arguments to $-h$, to yield that $h^{-1}[n-m-1/2, n] = X_m^* \in \mathscr{H}(n,k_1,m)$. Now we modify h by Lemma 7.3 on X_m^* to get a new nice function on M agreeing with h on $M - X_m^*$ and satisfying the conditions of Theorem C.

The proof of Theorem I goes as follows. Let V^n be a manifold with $\partial V = V_1 - V_2, n = 2m+2$. Take a nice function f on V by Lemma 7.1 with $f(V_1) = -1/2$ and $f(V_2) = n + 1/2$.

Following the proof of Theorem C, replacing the use of Theorem 1.2 with Theorem 5.1, we obtain a new nice function g on V with $g(V_1) = -1/2$ and $g(V_2) = n + 1/2$ and no critical points except possibly of index $m+1$. The following lemma can be proved by the standard methods of Morse theory [12].

Lemma 7.4. *Let V be as in Theorem 7.1 and f be a C^∞ non-degenerate function on V with the same boundary conditions as in Theorem 7.1. Then*

$$\chi_V = \sum(-1)^q M_q + \chi_{V_1},$$

where χ_V and χ_{V_1} are the Euler characteristics, and M_q denote the q-th type number of f.

This lemma implies that our function g has no critical points, and hence V_1 and V_2 are diffeomorphic.

8. We have to prove Theorems F and G. For Theorem F, observe by theorem C, there is a nice function f on M with vanishing type numbers except in dimensions $M_0, M_m, M_{m+1}, M_n = 1$, and $M_0 = M_n = 1$. Also, by the Morse relation, observe that the Euler characteristic is the alternating sum of the type numbers, $M_m = M_{m+1}$. Then by Lemma 7.2, $f^{-1}[0, m + 1/2], f^{-1}[m + 1/2, 2m + 1] \in \mathscr{H}(2m + 1, M_m, m)$ proving Theorem F.

All but the last statement of Theorem G has been proved. For this just note that $M \setminus D^{2m}$ is diffeomorphic to $f^{-1}[0, m + 1/2]$, which by Lemma 7.2 is in $\mathscr{H}(2m, k, m)$.

References

[1] Bott R. The stable homotopy of the classical groups, *Ann. Math.*, 70 (1959), 313–337.

[2] Gluck H. The weak Hauptvermutung for cell and spheres, *Bull. Amer. Math. Soc.*, 66 (1960), 282–284.

[3] Hirsch M. W. On combinatorial submanifolds of differentiable manifolds, *Comment. Math. Helv.*, 34 (1961), 103–111.

[4] Kervaire M. A manifold which does not admit any differentiable structure, *Comment. Math. Helv.*, 34 (1960), 304–312.

[5] Kervaire M., Milnor J. Groups of homotopy spheres, *Ann. Math.*, **77**, №3 (1963), 504–537.

[6] Курош А. Г. Теория групп, 2-е изд., Гостехиздат, М., 1953.

[7] Mazur B. On embeddings of spheres, *Bull. Math. Soc.*, 65 (1959), 19–65.

[8] Milnor J. Differentiable structures on spheres, *Amer. J. Math.* (1959), 862–972.

[9] Milnor J. Sommes des varietes differentiables et structures differentiables des soheres, *Bull. Soc. Math. France*, 87 (1959), 439–444.

[10] Milnor J. Differentiable manifolds which are homotopy spheres, Princeton University, 1956 (мимеограф.)

[11] Moise E. E. Certain classical problem of Euclidean topology, Lectures of Summer Institute on Set theoretic topology, Madison, 1955.

[12] Morse M. The calculus of variations in the large, Amer. Math. Soc. Colloq. publications, v. 18, New York, 1934.

[13] Morse M. The existence of polar non-degenerate functions on differentiable manifolds, *Ann. Math.*, 71 (1960), 352–383.

[14] Munkres J. Obstructions to the smoothing of piecewise differentiable homeomorphisms, *Bull. Amer. Math. Soc.*, 65 (1965), 332–334.

[15] Munkres J. Obstructions to imposing differentiable structures, Abstract. *Notices of Amer. Math. Soc.*, 7 (1960), 204.

[16] Nielsen J. Über die Isomorphismen unendlicher Gruppen ohne Relationen, *Math. Ann.*, 9 (1919), 269–272.

[17] Palais R. Extending diffeomorphisms, *Proc. Amer. Math. Soc.*, 11 (1960), 27–277.

[18] Parakyriakopoulos C. D. Some problems on 3-dimensional manifolds, *Bull. Amer. Math. Soc.*, 64 (1958), 317–335.

[19] H. Seifert - W. Threllfall, Lehrbuch der Topologie. Teubner, Leipzig - Berlin, 1934.

[20] Smale S. The generalized Poincaré conjecture in higher dimensions, to appear.

[21] Smale S. On gradient dynamical system, *Ann. Math.,* 74 (1961), 199–206.

[22] Thom R. Les structures différentiables des boules et des spheres (в печати).

[23] Thom R. La classification des immersions, Seminaire Bourbaki, Paris, 1957.

[24] Thom R. Des variétés triangulees aux variétés differentiables, Proc. International Congress Math., 1958, Cambridge University Press, 1960.

[25] Thom R. Les classes caracteristiques de Pontrjagin des variétés triangulées, Topología Algebráica, México, 1958, pp. 54–67.

[26] Wallace A. H. Modifications and cobounding manifolds, *Canad. J. Math.*, 12 (1960), 503–528.

[27] Whitehead J. H. C. On C^1-complexes, *Ann. Math.*, 41 (1940), 809–824.

[28] Whitehead J. H. C. On the homotopy type of manifolds, *Ann. Math.*, 41 (1940), 825–832.

[29] Whitney H. Differentiable manifolds, *Ann. Math.*, 37 (1936), 645–680.

[30] Wu W. T. On the isotopy of C^r-manifolds of dimension n in euclidean $(2n+1)$-space, *Science Record.* New Ser., 2 (1958), 271–275.

5

On the structure of manifolds[1]

S. Smale

In this paper, we prove a number of theorems which give some insight into the structure of differentiable manifolds.

The methods, results and some notation of [15], hereafter referred to as GPC, and [14] will be used. These two papers and [16] can be considered as a starting point for this. The main theorems in these papers are special cases of the theorems here.

Among the most important theorems in this paper are 1.1 and 6.1.

Some conversations with A. Haefliger were helpful in the preparation of parts of this paper.

Everything will be considered from the differentiable, equivalently, C^∞, point of view; manifolds, imbeddings, and isotopies will be in C^∞.

§ 1

We give a necessary and sufficient condition for two closed simply connected manifolds of dimension greater than four to be diffeomorphic. The condition is *h-cobordant*, first defined by Thom [18] for the combinatorial case, and developed by Milnor [9], and Kervaire and Milnor [7] for the differentiable case (sometimes previously h-cobordant has been called J-equivalent). It involves a combination of homotopy theory and cobordism theory. More precisely, two closed connected oriented manifolds M_1^n and M_2^n are h-cobordant if there exists an oriented compact manifold W, with ∂W (the boundary of W) diffeomorphic to the disjoint union of M_1 and $-M_2$, and each component of ∂W is a deformation retract of W.

[1] Amer. J. Math., 84 (1962), No. 3, 387–399 (received July 18, 1961). Reprinted with permission of The Johns Hopkins University Press.

Theorem 1.1. *If $n \geqslant 5$, and two closed oriented simply connected manifolds M_1^n and M_2^n are h-cobordant, then M_1 and M_2 are diffeomorphic by an orientation preserving diffeomorphism.*

It has been asked by Milnor whether h-cobordant manifolds in general are diffeomorphic, problem 5, [9]. Subsequently Milnor himself has given a counter-example of 7-dimensional manifolds with fundamental group Z_7, h-cobordant but not diffeomorphic [10]. Thus the condition of simple-connectedness is necessary in Theorem 1.1.

Theorem 1.1 was proved in special cases in [15] and [16]. These special cases were applied to show that every sphere not of dimension four or six has a finite number of differentiable structures. The six-dimensional case is taken care of by the following.

Corollary 1.2. *Every homotopy 6-sphere is diffeomorphic to S^6.*

This follows from 1.1 and the result of Kervaire and Milnor [7] that every homotopy 6-sphere is h-cobordant to S^6.

Corollary 1.3. *The semigroup of 2-connected closed 6-manifolds is generated by $S^3 \times S^3$.*

This follows from 1.2 and [17].

Haefliger [2] has extended the notion of h-cobordant to the relative case. Let V_1, V_2, M_1, M_2 be closed oriented, connected manifolds with $V_i \subset M_i, i = 1, 2$. According to Haefliger (M_1, V_1), (M_2, V_2) are *h-cobordant* if there is a pair (M, V) (i.e. $V \subset M$) with $\partial M = M_1 - M_2$, $\partial V = V_1 - V_2$ and $M_1 \to M, V_i \to V$ homotopy equivalences. Then Theorem 1.1 can be extended to the relative case.

Theorem 1.4. *Suppose (M_1^n, V_1^k) and (M_2^n, V_2^k) are h-cobordant, $k \geqslant 5$, $\pi_1(V_i) = \pi_1(M_i - V_i) = 1$. Then there is an orientation preserving diffeomorphism of M_1 onto M_2 sending V_1 to V_2.*

By taking V_i empty (the proof of 1.4 is valid for this case also), one can consider 1.1 as a special case of 1.4.

Actually we obtain much stronger theorems which will imply 1.4. The proof of 1.4 is completed in § 3.

It would not be surprising if the hypothesis of simple connectedness in these theorems could be weakened using torsion invariants (see [10], for example).

Theorem 1.4 has application to the theory of knots except in codimension two.

§ 2

The main theorem we prove in this section is the following. Here we use the notation of GPC.

Theorem 2.1. *Let M^n be a compact manifold with a simple connected boundary component Q. Let $V = (\chi * M, Q; f, m)$ where $f : \partial D_0^m \times D_0^{n-m} \to Q$ is an imbedding, $m > 2$, $n - m > 3$. Suppose $W = \chi(V; Q_1; g_1, \dots, g_r; m+1)$ where Q_1 is the component of ∂V corresponding to Q and suppose that $H_m(W, M)$ is zero. Then W is of the form*

$$\chi(M; Q; g_1', \dots, g_{r-1}'; m+1).$$

Note that an example of Mazur [8] shows that dimensional restrictions are necessary here.

For the proof we use several lemmas.

Lemma 2.2. *Let M^n be a compact manifold, Q a component of ∂M $n - m > 1$. Let*

$$V = \chi(M, Q; f; m),$$
$$W = \chi(V, Q_1; g_1, \dots, g_n),$$

where Q_1 is the component of ∂V corresponding to Q, and

$$f : \partial D_0^m \times D_0^{n-m} \to Q,$$
$$g_i : \partial D_i^{m+1} \times D_i^{n-m-1} \to Q_1$$

are imbeddings. Let $F = q \times D_0^{n-m} \subset V$ with $q \in \partial D_0^m$. Suppose ∂F does not intersect $g_i(\partial D_i^{m+1} \times 0)$, $i = 1, \dots, r-1$, and $g_r(\partial D_r^{m+1} \times 0)$ intersects ∂F transversally in a single point. Then W is of the form

$$\chi(M, Q_1; g_1', \dots, g_{r-1}', m+1).$$

Proof of Lemma 2.2. In the proof of Lemma 2.2, we use without further mention, the fact that the diffeomorphism type of an n-manifold is not changed when an n-disk is adjoined by identifying an $(n-1)$ disk on the boundary of each under a diffeomorphism. See GPC, 3.4, and also [11], [12].

We may assume, using the uniqueness of tubular neighborhoods that ∂F does not intersect $g_i(\partial D_i^{m+1} \times D_i^{n-m-1})$, $i = 1, 2, \dots, r-1$.

Since $g_r(\partial D_i^{m+1} \times 0)$ is transversal to ∂F in ∂V, there exists a disk neighborhood L of $\sigma = g_r(\partial D_i^{m+1} \times 0) \bigcap F$, $L = A^m \times D^{n-m-1}$, where $A^m \times 0$ is a disk neighborhood of σ in $g_r(\partial D_i^{m+1} \times 0)$, $0 \times D^{n-m-1}$ a disk neighborhood of σ in ∂F with $(0,0)$ corresponding to σ.

Now there exists a disk neighborhood D_a^m of the point $F \bigcap D_0^m \times 0$ in $D_0^m \times 0$ so small that if $N = D_a^m \times D_0^{n-m}$, then

$$N \bigcap \operatorname{im} g_i = \emptyset, \quad i = 1, \ldots, r-1 \tag{1}$$

and

$$N \bigcap \operatorname{im} g_r \subset L. \tag{2}$$

Since both $D_0^m \times 0$ and $A^m \times 0$ (i.e. $A^m \times 0$) are transversal to ∂F in ∂V, we may assume using a diffeomorphism of V, and restricting L, that $A^m \times 0$ and $D_0^m \times 0$ coincide and that L coincides with image $g_r \bigcap N$.

The following statements are made under the assumptions that the corners are smoothed via "straightening the angle", § 1 of GPC, or better [9]. Let $K = N \bigcup D_r^{m+1} \times D^{n-m-1} \subset W$.

We claim that $K \bigcap \mathrm{Cl}\,(W{-}K)$[1] is diffeomorphic to an $(n{-}1)$-disk. First, $K \bigcap \mathrm{Cl}\,(W-K)$ is

$$\partial D_a^m \times D_0^{n-m} \bigcup \{(\partial D_r^{m+1} \times D_r^{n-m-1}) \setminus \text{ interior} L\}$$

or $\partial D_a^m \times D_0^{n-m} \bigcup \partial D_b^m \times D^{n-m-1}$, where D_b^m is ∂D_r^{m+1} minus the interior of an m-disk. Furthermore $K \bigcap \mathrm{Cl}\,(W-K)$ can be described as $\partial D_a^m \times D_0^{n-m}$ with $\partial D_b^{m+1} \times D_r^{n-m-1}$ attached by an embedding $h : \partial D_b^m \times D_r^{n-m-1} \to \partial D_a^m \times D_0^{n-m-1}$ with the property that $h(\partial D_b^m \times 0)$ coincides with $\partial D_b^m \times c$ for some point $c \in D_0^{n-m}$. This is the situation in the proof of 3.3 of GPC, where it was shown that the resulting manifold was a disk. Thus $K \bigcap \mathrm{Cl}\,(W-K)$ is indeed an $(n-1)$-disk.

Since K is an n-disk, $K \bigcap \mathrm{Cl}\,(W-K)$ an $(n-1)$-disk, we have that W is diffeomorphic to $\mathrm{Cl}\,(W-K)$. On the other hand it is clear from the previous considerations that $\mathrm{Cl}\,(W-K)$ is of the form $\chi(M, Q; g_1', \ldots, g_{r-1}', m+1)$. This proves Lemma 2.2.

The next lemma follows from the method of Whitney [20] of removing isolated intersection points. The paper of A. Shapiro [13] makes this apparent (apply 6.7, 6.10, 7.1 of [13]).

Lemma 2.3. *Suppose N^{n-m} is a closed submanifold of the closed manifold X^n and $f : M^m \to X^n$ is an embedding of a closed manifold. Suppose also that M, N are connected, X simply connected, $n - m > 2, m > 2$ and $b = f(M^m) \circ N^{n-m}$ is the intersection number of $f(M)$ and N. Then there exists an imbedding $f' : M^m \to X^n$ isotopic to f such that $f'(M^m)$ intersects N^{n-m} in b points, each with transversal intersection.*

Lemma 2.4. *Let F_0^{n-m-1} be a submanifold of Q where Q is a component of the boundary of a compact manifold V^n, $n - m > 2$. Let*

[1] The Cl means the closure.

$W = \chi(M, Q; g; m+1)$, where $g : \partial\ D_0^{m+1} \times D^{n-m-1} \to Q$ is an imbedding with b the intersection number $g(\partial D_0^{m+1} \times 0) \circ F_0$. For an imbedding $h : S^m \to Q \bigcap \partial W$, there is an imbedding $h' : S^m \to Q \bigcap \partial W$, isotopic to h in ∂W, with

$$h'(S^m) \circ F_0^{n-m-1} = h(S^m) \circ F_0^{n-m-1} \pm b,$$

sign prescribed.

PROOF. Let D be the closed upper hemisphere of S^m, $\chi_0 \in \partial D_0^{n-m-1}$ and H^+, H^- be the closed upper, lower hemisphere respectively of $g(\partial D_0^{m+1} \times \chi_0)$. Then h is isotopic in $\partial W \bigcap Q$ to an imbedding $\overline{h}' : S^m \to Q \bigcap \partial W$, with $h'(S^m) \bigcap (\partial D_0^{m+1} \times \chi_0)$ equal H^+ with the orientation determined by the $\pm b$ of 2.4. This follows from Palais [12], (Theorem 13, Corollary 1).

Next let $\overline{h}$ be $\overline{h}'$ followed by the reflection map $H^+ \to H^-$, so that $\overline{h}, \overline{h}' : D \to \partial W$ are naturally topologically isotopic. However $\overline{h}$ is an angle on ∂D. By the familiar process of "straightening the angle" we modify $\overline{h}' : S^m \to \partial W \bigcap Q$ to an embedding $h' : S^m \to \partial W \bigcap Q$. Our construction makes it clear that h' and h are isotopic in ∂W, and that h' has the desired property of 2.4.

We now prove 2.1. Let F be as in 2.2 and b_i be the algebraic intersection number $g_i(\partial D_i^{m+1} \times 0) \circ \partial F$, $i = 1, \dots, r$. We first note that the b_i are relatively prime. This in fact follows from the homology hypothesis of the theorem.

The proof proceeds by induction on $\sum_{i=1}^r |b_i|$ and is started by 2.3 and 2.2. Suppose 2.1 is true in case $\sum_{i=1}^r |b_i|$ is $p - 1 > 0$. We can say from the homotopy structure of W that $H_m(W, M)$ is $H_m(V, M)$ with the added relations $[\partial D_i^{m+1}] = 0$, $i = 1, \dots, r$, where $[\partial D_i^{m+1}] \subset H_m(V, M) = Z$ and $H_m(V, M)$ is generated by $(D_0^m, \partial D_0^m)$.

Since $H_m(W, M) = 0$, $[D_i^{m+1}]$ are relatively prime. On the other hand, since $(D_0^m \times 0) \circ F = 1$, we have that $[\partial D_i^{m+1}] = b$. So the b_i, $i = 1, \dots, r$, are relatively prime.

Since the b_i are relatively prime, there exist, $i_0, i_1, i_0 \neq i_1$ with $|b_{i_0}| \geqslant |b_{i_1}| > 0$. One now applies 2.4 to reduce $|b_{i_0}|$ by $|b_{i_1}|$ using the covering homotopy property as in §2 of GPC. The induction hypothesis applies and we have proved 2.1.

Lemma 2.5. *Let $n \geqslant 2m+1$, $(n, m) \neq (4,1), (3,1), (5,2), (7,3)$, M^n be a compact manifold with a simply connected boundary component Q and $V = \chi(M, Q; f; m)$ where $f : \partial D^m \times D^{n-m} \to Q$ is a contractible imbedding. Let Q_1 be the component of ∂V corresponding to Q and $W = \chi(V, Q_1; g; m+1)$ where $g : \partial D_1^{m+1} \times D_1^{n-m-1} \to Q_1$. Then if* *the homomorphism* $\pi_m(V, M) \to \pi_m(W, M)$ *induced by inclusion is zero,* W *is diffeomorphic to* M*.*

We use the following for the proof of 2.5.

Lemma 2.6. *Let* Y *be a simply connected polyhedron and* Z *an* $(m-1)$*-connected polyhedron. Then* $\pi_m(Y \bigvee Z) = \pi_m(Y) + \pi_m(Z)$*.*

This is a standard fact in homotopy theory. For example it follows from [6], Ch.3.1 and the relative Hurewicz theorem.

Using 2.6 it follows easily that $\pi_m(Q_1) = \pi_m(Q) + \pi_m(S^m)$.

Then from the homotopy hypothesis it follows that the homotopy class γ if g restricted to $\partial D_1^{m+1} \times 0$ is of the form $a + g_1$, where $a \in \pi_m(Q)$ and g_1 generates $\pi_m(S^m)$. Since Q is contractible, $V = M + H$, where H is an $(n-m)$-bundle over S^m, and also $Q_1 = Q + \partial H$. Then let $g_1' : \partial D^{m+1} \to Q$ be an imbedding representing a and $g_2' : \partial D^{m+1} \to \partial H$ an imbedding intersecting ∂F transversally in a single point where F is the same as in 2.2. Then by the sum construction we obtain $g' : \partial D^{m+1} \times 0 \to Q_1$ realizing γ with the property that $g'(\partial D^{m+1} \times 0)$ intersects ∂F transversally in a single point where F is the same as in 2.2. Application of Lemma 2.4 of GPC and 2.2 finishes the proof.

§ 3

Among other things, we apply the theory of §2 to obtain Theorem 1.4.

Theorem 3.1. *Let* W^n *be a manifold (not necessarily compact),* $n > 5$*, with* ∂W *the disjoint union of simply-connected manifolds* M_1 *and* M_2 *where the inclusion* $M_i \to W$ *are homotopy equivalences. Suppose* $j : V_0 \to M_1$ *is the inclusion of a compact manifold* V_0 *into* M_1 *which is a homotopy equivalence and there is an imbedding* $\alpha : \mathrm{Cl}\,(M_1 - V_0) \times [1, 2] \to W$ *such that: a) the complement of the image of* α *has compact closure and; b)* $\alpha(\mathrm{Cl}\,(M_1 - V_0) \times n) \subset M_n$, $n = 1, 2$, α *restricted to* $\mathrm{Cl}\,(M_1 - V_0)$ *is* j*. Then* α *can be extended to a diffeomorphism* $M_1 \times [1, 2] \to W$*.*

Proof of Theorem 3.1. Let $I_0 = \left[-\frac{1}{2}, n + \frac{1}{2}\right]$ and replace $[1, 2]$ in Theorem 3.1 by I_0, denoting the projection $\mathrm{Cl}\,(M_1 - V_0) \times I_0 \to I_0$ by f_0.

We may assume that points under α have been identified so that $\mathrm{Cl}\,(M_1 - V_0) \times I_0 \subset W$. Then by the results of [14] one can find a non-degenerate C^∞ real function f on W such that a) f restricted to $\mathrm{Cl}\,(M_1 - V_0) \times I_0$ is f_0; b) at a critical point the value of f is the index and c) $f(M_1) = -\frac{1}{2}$, $f(M_2) = n + \frac{1}{2}$.

Let $X_p = f_0\left[-\frac{1}{2}, p+\frac{1}{2}\right]$. We will show inductively that by suitable modifications of f which also satisfy a), b), c), we can assume X_p is a product $M_1 \times I$ (or equivalently the modified f has no critical points of index $\leqslant p$).

First by 5.1 of GPC, note that we may assume that the function f has no critical points of index 0. Next by the method in §7 of GPC, using the fact that $\pi_1(M_1) = \pi_1(W) = 1$, we can similarly assume that there are no critical points of f of index 1.

We are not quite yet in the dimension range where Theorem 2.1 applies, but we apply Lemma 2.5 to eliminate a critical point of 2 if it occurs, as follows.

We have that $X_2 = \chi(X_1, Q_1; f_1, \ldots, f_k; 2)$. $X_3 = \chi(X_2, Q_2; g_1, \ldots, g_r; 3)$, where $Q_1 = f^{-1}\left(1\frac{1}{2}\right)$, $Q_2 = f^{-1}\left(2\frac{1}{2}\right)$. It follows from the homotopy hypothesis that each f_i is contractible in Q_1 so that X_2 is of the form $X_1 + H$, $H \in \mathscr{H}(n, k, 2)$ (following notation of GPC[1]).

The g_i's induce a homomorphism $G_r \to \pi_2(Q_2)$. Let φ be the composition

$$G_r \to \pi_2(Q_2) \to \pi_2(X_2) \to \pi_2(H),$$

where the last homomorphism is obtained by identifying X_1 to a point in X_2.

Assertion. φ is an epimorphism.

Assume the assertion is false and $\alpha \in \pi_2(H)$ is not in the image of φ. Then since

$$\pi_2(X_2) = \pi_2(X_1) + \pi_2(H)$$

(by Lemma 2.6), the image of α under $\pi_2(H) \to \pi_2(X_2) \to \pi_2(X_3)$ is not in the image of

$$\pi_2(X_1) \to \pi_2(X_2) \to \pi_2(X_3).$$

But the last composition is an isomorphism since $X_1 = (M_1 \times I)$, thus contradicting the existence of such an α. Hence the assertion is true.

Let $\gamma_1, \ldots, \gamma_k$ be the generators of $\pi_2(H)$ corresponding to $f_1, \ldots, f_k$. Then by Lemma 4.1 of GPC, there is an automorphism β of G_r such that $\varphi\beta(g_i) = \gamma_i$, $i \leqslant k$ and $\varphi\beta(g_i) = 0$, $i > k$. By Theorem 2.1 of GPC it can be assumed that the g_i are such that $\varphi(g_i) = \gamma_i$, $i \leqslant k$ and $\varphi(g_i) = 0$, $i > k$.

Now apply Lemma 2.5 with W, V, M corresponding to $\chi(X_2, Q; g_k)$, $\chi(X_1, Q_1; f_1, \ldots, f_k)$ and $\chi(X_1, Q_1; f_1, \ldots, f_{k-1})$. This eliminates the critical point of f corresponding to f_k and by induction all the critical points of index 2.

[1]The set $\mathscr{H}(n, k, s)$, consists of manifolds of the type $\chi(M, Q; f_1, \ldots, f_r, s)$, where $f \in \mathscr{H}(n, k, s-1)$ and $\mathscr{H}(n, k, 0) = D^n$. — *Editor's remark.*

Applying some of the previous considerations to $n-f$ we eliminate the critical points of f of index n, $n-1$.

Now more generally suppose f on X_{p-1} has no critical points where $p \leqslant n-3$. Then since $H_p(X_{p+1}, X_p) = 0$, Theorem 2.1 applies to eliminate the critical points of index p. Thus we obtain by induction a function f on W with critical points only of index $n-2$ and which satisfies the conditions a)-c) above. By Lemma 7.5 of GPC, f has no critical points at all. This proves Theorem 3.1.

Corollary 3.2. *Suppose W^n is compact, $n > 5$, ∂W the disjoint union of closed manifolds M_1, M_2, with each $M_i \to W$ a homotopy equivalence. Suppose also $V \subset W$ with $\partial V = V_1 \bigcup V_2$, $V_i \subset M_i$, $V = V_1 \times I$ and $\pi_1(M_i - V_i) = 1$. Then $i : V \to W$ can be extended to a diffeomorphism of $M_1 \times I$ onto W.*

PROOF. First i may be extended to $T \times I$ where T is a tubular neighborhood of V_1 in M_1. Then apply Theorem 3.1 to $W - V$ to get 3.2.

Now we can prove Theorem 1.4. First by Corollary 3.2 with V empty applied to V of Theorem 1.4 yields that V is diffeomorphic to $V_1 \times I$. Now Corollary 3.2 applies to yield Theorem 1.4.

§ 4

The following is quite a general theorem and in fact contains Theorem 1.1 as a special case with $k = n-1$.

Theorem 4.1. *Suppose $W^n \supset M^k$ where W is a compact connected manifold and M is a closed manifold. Furthermore suppose*

a) $\pi_1(\partial W) = \pi_1(M) = 1$;

b) $n > 5$;

c) The inclusion of M into W is a homotopy equivalence.

Then W is diffeomorphic to a closed cell bundle over M, in particular to a tubular neighborhood of M in W.

We need a lemma.

Lemma 4.2. *Suppose B is a compact connected n-dimensional submanifold of a compact connected manifold V^n with $\partial B \bigcap \partial V = \varnothing$, $\pi_1(\partial B) = \pi_1(\partial V) = 1$ and $H_*(B) \to H_*(V)$, induced by the inclusion is bijective. Then $Q = \mathrm{Cl}\,(V - B)$ has boundary consisting of ∂V ∂B, with the inclusions of ∂V and ∂B into Q homotopy equivalences.*

For the proof of Lemma 4.2 we use the following version of the Poincaré Duality Theorem, which follows from the Lefschetz Duality Theorem.

Theorem 4.3. *Suppose W^n is a compact manifold ∂W, the disjoint union of manifolds M_1 and M_2 (possibly either or both empty). Then for all i, $H^i(W, M_1)$ is isomorphic to $H_{n-i}(W, M_2)$.*

To prove Lemma 4.2 note

$$H_i(Q, \partial B) = H_i(V, B) = 0 \text{ and } H^i(Q, \partial B) = H^i(V, B) = 0$$

for all i. By Theorem 4.3, $H_i(Q, \partial V) = 0$ for all i also. By the Whitehead theorem we get 4.2.

The proof of Theorem 4.1 then goes as follows. We can first suppose that M is disjoint from the boundary of W. Now let T be the tubular neighborhood of M which is also disjoint from ∂W. Now apply Lemma 4.2 and Corollary 3.2 to $\mathrm{Cl}\,(W - T)$ with V of Corollary 3.2 empty. This yields that $\mathrm{Cl}\,(W - T)$ is diffeomorphic to $\partial T \times I$ and hence W is diffeomorphic to T. We have proved Theorem 4.1.

Theorem 4.4. *Suppose $2n \geqslant 3m + 3$ and a compact manifold W^n has the homotopy type of a closed manifold $M^m, n > 5$ with $\pi_1(\partial W) = \pi_1(M) = 1$. Then W is diffeomorphic to a cell-bundle over M.*

Proof. Let $f : M \to W$ be a homotopy equivalence. By Haefliger [1], f is homotopic to an embedding $g : M \to W$. Now Theorem 4.1 applies to yield Theorem 4.4.

§ 5

We continue with some consequences of Theorem 4.1. The next theorem is a strong form of the Generalized Poincaré Conjecture for $n > 5$ and it was first proved in [16] except for $n = 7$. This theorem follows from Theorem 4.1 by taking M to be a point.

Theorem 5.1. *Suppose C^n is a compact contractible manifold with $\pi_1(\partial C) = 1$ and $n > 5$. Then C is diffeomorphic to the n-disk D^n.*

For $n = 5$, if one knows in addition that ∂C is diffeomorphic to S^4, then using the theorem of Milnor $\Theta^5 = 0^1$.[1]

The following is a weak unknotting theorem in the differentiable case.

[1] Θ^5 is the h-cobordis class group for 5-dimensional homotopy spheres with the connected sum operation. — *Editor's remark.*

Haefliger [2] has given an imbedding (differentiable) of S^3 in S^6 which does not bound an imbedded D^4. On the other hand we have:

Theorem 5.2. *Suppose $S^k \subset S^n$ with $n-k > 2$. Then the closure of the complement of a tubular neighborhood T of S^k in S^n is diffeomorphic to $S^{n-k-1} \times D^{k+1}$.*

PROOF. The proof of Theorem 5.2 is as follows (the case $n \leqslant 5$ is essentially contained in Wu Wen Tsun [21]). It is well-known and easy to prove that if $X = \mathrm{Cl}\,(S^n - T)$, X has the homotopy type of S^{n-k-1}. In fact T is diffeomorphic to a cell bundle over S^k and the inclusion of the boundary of a fiber S_0^{n-k-1} into X induces the equivalence. Furthermore the normal bundle of S_0^{n-k-1} in S^n is trivial because S_0^{n-k-1} bounds a disk in S^n. Now Theorem 4.1 applies to yield Theorem 5.2.

One can also prove some recent theorems of M. Hirsch [5], replacing his combinatorial arguments by application of the above theorems.

Theorem 5.3. (Hirsch) *If $f : M_1^n \to M_2^n$ is a homotopy equivalence of simply connected closed manifolds such that the tangent bundle of M_1 is equivalent to the bundle over M_1 induced from the tangent bundle of M_2 by f, then $M_1 \times D^k$ and $M_2 \times D^k$ are diffeomorphic if $k > n$.*

One obtains Theorem 5.3 by imbedding M_1 in $M_2 \times D^k$ approximating the homotopy equivalence and applying Theorem 4.1. The tangential property of f is used to conclude that a tubular neighborhood of M_1 in $M_2 \times D^k$ is a product neighborhood.

Theorem 5.4. (Hirsch) *If the homotopy sphere M^n bounds a parallelizable manifold then $M^n \times D^3$ is diffeomorphic to $S^n \times D^3$.*

One first proves that M^n can be imbedded in S^{n+3} with trivial normal bundle by following Hirsch [4] or using "handlebody theory". Then apply the argument in Theorem 5.2 to obtain the complement of a tubular neighborhood of M^n is diffeomorphic to $S^2 \times D^{n+1}$. The closure of the complement $S^2 \times D^{n+1}$ in S^{n+3} is $S^n \times D^3$, thus proving 5.4.

§ 6

The main goal of this section is the following theorem.

Theorem 6.1. *Let M be a simply connected closed manifold of dimension greater than 5. Then on M there is a non-degenerate C^∞ function* *with the minimal number of critical points consistent with the homology structure.*

One actually obtains such a function with the additional property that at a critical point its value is the index.

Statement 6.2. *We make more explicit the conclusion of Theorem 6.1. Suppose for each $i, 0 \leqslant i \leqslant n$, the set $\sigma_{i1}, \ldots, \sigma_{ip(i)}$, τ_{i1}, …, $\tau_{ip(i)}$ is the set of generators for a corresponding direct sum decomposition of $H_i(M)$, σ_{i_j} free, τ_{i_j} of finite order. Then one can obtain the function of Theorem 6.1 with type numbers satisfying $M_i = p(i) + q(i) + q(i-1)$. By taking the $q(i)$ minimal, the M_i becomes minimal.*

In the case there is no torsion in the homology of M, Theorem 6.1 becomes

Theorem 6.3. *Let M be a simply connected closed manifold of dimension greater than five with no torsion in the homology of M. Then there is a non-degenerate function on M with type numbers equal the Betti numbers of M.*

We start the proof of Theorem 6.1 with the following Lemma.

Lemma 6.4. *Let M^n be a simply connected compact manifold, $n > 5$, $n \geqslant 2m$. Then there is an n-dimensional simply connected manifold X_m such that:*

a) *$H_j(X_m) = 0$, $j > m$;*

b) *There is a "nice" function on X_m, minimal with respect to its homology structure. In other words there is a C^∞ non-degenerate function on X_m, value at a critical point equal the index, equal to $m+\frac{1}{2}$ on ∂X_m regular on the neighborhood of ∂X_m and the k-th type number M_k is minimal in the sense of 6.2;*

c) *There is an imbedding $i : X_m \to M^n$ such that $i(\partial X_m) \bigcap \partial M = \varnothing$ $i : H_j(X_m) \to H_j(M^n)$ is bijective for $j < m$ and surjective for $j = m$.*

The proof goes by induction on m starting by taking X_1 to be an n-disk. Suppose $X_{k-1}, i_0 : X_{k-1} \to M$ have been constructed satisfying a)–c). For convenience we identify points under i_0, so that $X_{k-1} \subset M$. We now construct X_k, $i : X_k \to M$, satisfying a)–c).

By the relative Hurewicz theorem the Hurewicz homomorphism $h : \pi_k(M, X_{k-1}) \to H_k(M, X_{k-1})$ is bijective.

For the structure of $H_k(M, X_{k-1})$ consider the exact sequence

$$0 \to H_k(M) \to H_k(M, X_{k-1}) \to H_{k-1}(X_{k-1}) \xrightarrow{j} H_{k-1}(M) \to 0.$$

Let $\gamma_1 \dots, \gamma_p$ be the set of generators of $H_k(M, X_{k-1})$ corresponding to a minimal set of generators of $H_k(M)$ together with a minimal set for $\ker j$.

Represent the elements $h^{-1}(\gamma_1), \dots, h^{-1}(\gamma_p)$ by imbeddings $\overline{g}_i : (D^k, \partial D^k) \to (\mathrm{Cl}\,(M - X_{k-1}), \partial X_{k-1})$ with $\overline{g}_i(D^{k'})$ transversal to ∂X_{k-1} along $\overline{g}_i(\partial D^k)$, for example following Wall [19], proof of Theorem 1.

In the extreme case $n = 2k$, the images of $\overline{g}_i$ generically intersect each other in isolated points. These points can be removed by pushing them along arcs past the boundaries. Still following [19], the $\overline{g}_i$ can be extended to tubular neighborhoods,

$$g_i : (D^k, \partial D^k) \times D^{n-k} \to (\mathrm{Cl}\,(M - X_{k-1}), \partial X_{k-1}).$$

Then we take X_k to be $\chi(X_{k-1}; g_1', \dots, g_p'; k)$ where $g_i' : \partial D^k \times D^{n-k} \to \partial X_{k-1}$ is the restriction of g_i. It is not difficult to check that X_k has the desired properties a)–c). This proves Lemma 6.4.

To prove 6.1, let M^n be as in 6.1 with $n = 2m$ or $2m + 1$. Let $X_m \subset M$ as in Lemma 6.4, f the nice function on X_m and $K = \mathrm{Cl}\,(M - X_m)$. Then $H_i(M, X_m) = 0$, $i \leqslant m$, so by the duality $H^j(K) = 0$, $j \geqslant n - m$. By the Universal Coefficient Theorem this implies that $H_{n-m-1}(K)$ is torsion free. Let $Y_{n-m-1} \subset K$ be again given by Lemma 6.4 with g the nice function on Y_{n-m-1}. By 4.2 and 3.2 we can in fact assume that K and Y_{n-m-1} are the same, so $M = X_m \bigcup Y_{n-m-1}$. Let f_0 be the function on M which is f on X_m and $n - g$ on Y_{n-m-1}. By smoothing f_0 along ∂X_m we obtain a C^∞ function f'. It is not difficult using the Universal Coefficient Theorem and Poincaré Duality to show that f' may be taken as the desired function of the Theorem.

The previous results of this section may be extended to manifolds with boundary.

By the previous methods one may prove the following generalization of Theorem 6.1. We leave the details to the reader.

Theorem 6.5. *Suppose W^n is a simply connected manifold with simply-connected boundary, $n > 5$. Then there is a nice function f on W^n (non-degenerate, value $n + \frac{1}{2}$ on ∂W, regular in a neighborhood of ∂W, value at a critical point is the index) with type numbers minimal with respect to the homology structuire of $(W, \partial W)$.*

§ 7

The goal of this section is to prove the following.

Theorem 7.1. *Let $f : W_1^n \to W_2^n$ be a homotopy equivalence between two manifolds such that the tangent bundle T_1 of M_1 is equivalent to $f^{-1}T_2$. Suppose also that $n > 5$, $n \geqslant 2m+1$, $H^i(W_1) = 0$, $i > m$, $\pi_1(W_1) = \pi_1(\partial W_1) = \pi_1(\partial W_2) = 1$. Then W_1 and W_2 are diffeomorphic by a diffeomorphism homotopic to f.*

Let g be a nice function on W_1 with no critical points of index greater than m, whose existence is implied by Theorem 6.5. Then we let $X_k = g^{-1}\left[0, k+\frac{1}{2}\right]$, $k = 0, \ldots, m$ with $X_m = W_1$. By Corollary 3.2 and Lemma 4.2 it is sufficient to imbed X_m in W_2 by a map homotopic to f.

Suppose inductively we have defined a map $f_{k-1} : X_k \to W_2$ homotopic to f with the property that f_{k-1} is an imbedding $k \geqslant m$. Let X_k be written in the form

$$\chi(X_{k-1}; g_1, \ldots, g_p; k),\ \ g_i : \partial D^k \times D^{n-k} \to \partial X_{k-1}.$$

Using the Whitney imbedding theory we can find $f'_{k-1} : X_k \to W_2$ homotopic to f_{k-1}, which is an imbedding on X_{k-1} and on the images $g_i(D^* \times 0)$ in X_k as well. It remains to make f_{k-1} an imbedding on a tubular neighborhood of each of the $g_i(D^k \times 0)$, or equivalently one each of the $g_i(D^k \times D^{n-k})$.

This can be done for a given i if and only if an element γ_i in $\pi_{k-1}(O(n-k))$ defined by f'_{k-1} in a neighborhood of $g_i(\partial D^k \times 0)$, is zero. But the original tangential assumptions on f insure $\gamma_i = 0$ in this dimension range. The arguments in proving these statements are so close to the arguments in Hirsch [3] Section 5, that we omit them. This finishes the proof of 7.1.

§ 8

We note here the following theorem.

Theorem 8.1. *Let M^{2m+1} be a closed simply connected manifold, $m > 2$, with $H_m(M)$ torsion free. Then there is a compact manifold W^{2m+1}, uniquely determined by M and a diffeomorphism $h : \partial W \to \partial W$ such that M is a union of two copies of W with points identified under h.*

Proof. Let $W_1^{2m+1} \subset M$ be the manifold given by Lemma 5.4. Let $W_2^{2m+1} \subset \mathrm{Cl}\,(M - W_1)$ be also given by Lemma 6.4. Then it is not difficult using homotopy theory to show that W_1, W_2 satisfy the hypothesis of 7.1. Also by previous arguments W_2 is diffeomorphic to $\mathrm{Cl}\,(M - W_1)$. The uniqueness of $W_1 = W_2$ is also given by 7.1. Putting these facts together, we get Theorem 8.1.

Remark. I don't believe the condition on $H_m(M)$ is really necessary here. Also in a different spirit, Theorem 8.1 is true for the cases $m = 1,\ m = 2$.

References

[1] Haefliger A. Plongement differentiable des varietes (to appear).

[2] Haefliger A. Knotted $(4k-1)$ spheres in $6k$ space, *Ann. Math.*, **75** (1961), No. 3, 452–466.

[3] Hirsch M.W. Immersions of manifolds, *Trans. Amer. Math. Soc.*, **93** (1959), 242–276.

[4] Hirsch M.W. Of manifolds which bound in Euclidean space (to appear).

[5] Hirsch M.W. Diffeomorphisms of tabular neighborhoods (mimeographed). Berkeley 1961.

[6] Hu S. T. Homotopy theory, New York, 1959.

[7] Kervaire M., Milnor J. Groups of homotopy spheres, I and II (to appear).

[8] Mazur B. A note on some contractible 4-manifolds, *Ann. Math.*, **73** (1961), 221–228.

[9] Milnor J. Differentiable manifolds which are homotopy spheres (mimeographed), Princeton, 1959.

[10] Milnor J. Two complexes which are homeomorphic but combinatorial distinct, *Ann. Math.*, **74** (1961), 575–590.

[11] Milnor J. Sommes des variétés differentiables et structure differentiables des spheres *Bull. Soc. Math. France*, **87** (1959), 439–444.

[12] Palais R. Extending diffeomorphism, *Proced. Amer. Math. Soc.*, **11** (1960), 274–277.

[13] Shapiro A. Obstructions to the imbedding of a complex in a Euclidian space, 1. The first obstruction, *Ann. Math.*, **66** (1957), 256–269.

[14] Smale S. On gradient dynamical systems, *Ann. Math.*, **74** (1961), No. 1, 199.

[15] Smale S. Generalized Poincaré's s conjecture in dimensions greater than four, *Ann. Math.*, **74** (1961), No. 2, 391–406.

[16] Smale S. Differentiable and combinatorial structures of manifolds, *Ann. Math.*, **74** (1961), No. 3, 498.

[17] Smale S. On the structure of five manifolds, *Ann. Math.*, **75** (1962), No. 1, 38.

[18] Thom R. Les classes caracteristiques de Pontrjagin des variétés triangulées, Topología Algebráica, México, 1958, pp. 54–67.

[19] Wall C.T.C. Classification of $(n-1)$-connected $2m$-manifolds, *Ann. Math.*, **75** (1962), No. 1, 163.

[20] Whitney H. The self-intersections of a smooth m-manifold in $2n$-space, *Ann. Math.*, **45** (1944), 220–246.

[21] Wu Wen Tsun. On the isotopy of C^r manifolds of dimension n in Euclidean $(2n+1)$-space, *Science Record* (New Ser.), **11** (1958), 271–275.

This page intentionally left blank

6

On the formal group laws of unoriented and complex cobordism theory

D. Quillen [1]

In this note we outline a connection between the generalized conhomology theories of unoriented cobordism and (weakly-) complex cobordism and the theory of formal commutative groups of one variable [4], [5]. This connection allows us to apply Cartier's theory of typical group laws to obtain an explicit decomposition of complex cobordism theory localized at a prime p into a sum of Brown-Peterson cohomology theories [1] and to determine the algebra of cohomology operations in the latter theory.

1. Formal group laws. If R is a commutative ring with unit, then by a *formal* (commutative) *group* over R one means a power series $F(X, Y)$ with coefficients in R such that

(i) $F(X, 0) = F(0, X) = X$;

(ii) $F(F(X, Y), Z) = F(X, F(Y, Z))$;

(iii) $F(X, Y) = F(Y, X)$.

We let $I(X)$ be the "inverse" series satisfying $F(X, I(X)) = 0$, and let

$$\omega(X) = dX/F_2(X, 0)$$

be the normalized invariant differential form where the subscript 2 denotes differentiation with respect to the second variable. Over $R \otimes Q$, there is a unique power series $l(X)$ with leading term X such that

$$l(F(X, Y)) = l(X) + l(Y). \tag{1}$$

[1]Bull. Amer. Math. Soc., **75** (1969), 1293–1298 (Communicated by F. Peterson, May, 16, 1969).

The series $l(X)$ is called the *logarithm* of F and is determined by the equations

$$l'(X)dX = \omega(X), \qquad l(0) = 0. \tag{2}$$

2. The formal group law of complex cobordism theory. By *complex cobordism theory* $\Omega^*(X)$ we mean the generalized cohomology theory associated with the spectrum MU. If E is a complex vector bundle of dimension n over X, we let $c_i^\Omega(E) \in \Omega^{2i}(X)$, $1 \leqslant i \leqslant n$, be the Chern classes of E in the sense of Conner-Floyd [3]. Since $\Omega^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega^*(\mathrm{pt})[[x, y]]$, where $x = c_1^\Omega(O(1)) \otimes 1$, $y = 1 \otimes c_1^\Omega(O(1))$ and $O(1)$ is the canonical line bundle on $\mathbb{C}P^\infty$, there is a unique power series $F^\Omega(X, Y) = \sum a_{kl}X^kY^l$ with $a_{kl} \in \Omega^{2-2k-2l}(\mathrm{pt})$ such that

$$c_1^\Omega(L_1 \otimes L_2) = F^\Omega(c_1^\Omega(L_1), c_1^\Omega(L_2)) \tag{3}$$

for any two complex line bundles with the same base. The power series F^Ω is a formal group law over $\Omega^{ev}(pt)$.

Theorem 1. *Let E be a complex vector bundle of dimension n, let $f : \mathbb{P}E' \to X$ be the associated projective bundle of lines in the dual E' of E, and let $O(1)$ be the canonical quotient line bundle on $\mathbb{P}E'$. Then the Gysin homomorphism $f_* : \Omega^q(\mathbb{P}E') \to \Omega^{q-2n+2}(X)$ is given by the formula*

$$f_*(u(\xi)) = \operatorname{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^{n} F^\Omega(Z, I(\lambda_j))}. \tag{4}$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^\Omega(O(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^Ω, and λ_j are the dummy variables of which $c_q^\Omega(E)$ is the q-th elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7]. Applying the theorem to the map $f : \mathbb{C}P^n \to \mathrm{pt}$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of $\mathbb{C}P^n$ in $\Omega^{-2n}(\mathrm{pt})$. From (2) we obtain the

Corollary (Mishchenko [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geqslant 0} P_n \frac{X^{n+1}}{n+1}. \tag{5}$$

3. The universal nature of cobordism group laws.

Theorem 2. *The group law F^{Ω} over $\Omega^{ev}(\mathrm{pt})$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(\mathrm{pt}) \to R$ carrying F^{Ω} to F.*

PROOF. Let F_u over L be a universal formal group law [5] and let $h : L \to \Omega^{ev}(pt)$ be the unique ring homomorphism sending F_u to F^{Ω}. The law F_u over $L\otimes\mathbb{Q}$ is universal for laws over $\mathbb{Q}$-algebras. Such a law is determined by its logarithm series which can be any series with leading term X. Thus if $\sum p_n X^{n+1}/n+1$ is the logarithm of F_u, $L\otimes\mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ with generators p_i. By (5) $hp_n = P_n$, so as $\Omega^*(pt)\otimes\mathbb{Q} \cong \mathbb{Q}[P_1, P_2, \dots]$ it follows that $h\otimes\mathbb{Q}$ is an isomorphism.

By Lazard [5, Theorem II], L is a polynomial ring over $\mathbb{Z}$ with infinitely many generators; in particular L is torsion-free and hence h is injective. To prove surjectivity we show $h(L)$ contains generators for $\Omega^*(pt)$. First of all $hp_n = P_n \in h(L)$ because $p_n \in L$ is the n-th coefficient of the invariant differential F_u. Secondly we must consider elements of the form $[M_n]$ where M_n is a nonsingular hypersurface of degree $k_1, \dots, k_r$ in $\mathbb{C}P^{n_1}\times\dots\times\mathbb{C}P^{n_r}$. Let π be the map of this multiprojective space to a point. Then $[M_n] = \pi_* c_1^{\Omega}(L_1^{k_1}\otimes\dots\otimes L_r^{k_r})$, where L_j is the pull-back of the canonical line bundle on the j-th factor. The Chern classes of this tensor product may be written using the formal group law F^{Ω} in the form $\sum \pi^* a_{i_1\dots i_r} z_1^{i_1}\dots z_r^{i_r}$, where $0 \leqslant i_j \leqslant n_j$, $1 \leqslant j \leqslant r$, $z_i = c_1^{\Omega}(L_i)$ and where $a_{i_1\dots i_r} \in h(L)$. Since

$$\pi_* z_1^{i_1}\dots z_r^{i_r} = \prod_{j=1}^{r} P_{n_j - i_j}$$

also belongs to $h(L)$, it follows that $[M_n] \in h(L)$. Thus h is an isomorphism and the theorem is proved.

We can also give a description of the unoriented cobordism ring using formal group laws. Let $\eta^*(X)$ be the unoriented cobordism ring of a space X, that is, its generalized cohomology with values in the spectrum MO. There is a theory of Chern (usually called Whitney) classes for real vector bundles with $c_i(E) \in \eta^i(X)$. The first Chern class of a tensor product of line bundles gives rise to a formal group law F^{η} over the commutative ring $\eta^*(\mathrm{pt})$. Since the square of a real line bundle is trivial, we have the identity

$$F^{\eta}(X, X) = 0. \tag{6}$$

Theorem 3. *The group law F^{η} over $\eta^*(\mathrm{pt})$ is a universal formal (commutative) group law over a ring of characteristic two satisfying (6).*

4. Typical group laws (after Cartier [2]). Let F be a formal group law over R. Call a power series $f(X)$ with coefficients in R and without constant term a *curve* in the formal group defined by the law. The set of curves forms an abelian group with addition $(f +^F g)(X) = F(f(X), g(X))$ and with operators

$$([r]f)(X) = f(rX), \quad r \in R,$$

$$(V_n f)(X) = f(X^n), \quad n \geqslant 1,$$

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}), \quad n \geqslant 1,$$

where ζ_i are the n-th roots of unity. The set of curves is filtered by the order of a power series and is separated and complete for the filtration.

If R is an algebra over $Z_{(p)}$, the integers localized at the prime p, then a curve is called *typical* if $F_q f = 0$ for all prime $q \neq p$. If R is torsion-free then it is the same to require that the series $l(f(X))$ over $R \otimes \mathbb{Q}$ has only terms of degree a power of p, where l is the logarithm of F. The group law F is said to be a *typical law* if the curve $\gamma_0(X) = X$ is typical. There is a canonical change of coordinates rendering a given law typical. Indeed let c_F be the curve

$$c_F^{-1} = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n \gamma_0, \tag{7}$$

where the division by n prime to p is taken in the filtered group of curves and where μ is the Möbius function. Then the group law $(c_{F*}F)(X, Y) = c_F(F(c_F^{-1}X, c_F^{-1}Y))$ is typical.

5. Decomposition of $\Omega^*_{(p)}$. For the rest of this paper p is a fixed prime. Let $\Omega^*_{(p)}(X) = \Omega^*(X) \otimes \mathbb{Z}_{(p)}$ and let $\xi = c_F\Omega$. Then $\xi(Z)$ is a power series with leading term Z with coefficients in $\Omega^*_{(p)}(\mathrm{pt})$, so there is a unique natural transformation $\hat{\xi} : \Omega^*_{(p)}(X) \to \Omega^*_{(p)}(X)$ which is stable, a ring homomorphism, and such that

$$\hat{\xi} c_1^\Omega(L) = \xi(c_1^\Omega(L))$$

for all line bundles L.

Theorem 4. *The operation $\hat{\xi}$ is homogeneous, idempotent, and its values on $\Omega^*_{(p)}(\mathrm{pt})$ are:*

$$\hat{\xi}(P_n) = \begin{cases} P_n, & \text{if } n = p^a - 1 \text{ for some } a \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Omega T^*(X)$ be the image of $\hat{\xi}$. Then there are canonical ring isomorphisms

$$\Omega T^*(\mathrm{pt}) \underset{\Omega^*_{(p)}(\mathrm{pt})}{\otimes} \Omega^*_{(p)}(X) \cong \Omega T^*(X), \tag{8}$$

$$\Omega^*_{(p)}(\mathrm{pt}) \underset{\Omega T^*(\mathrm{pt})}{\otimes} \Omega T^*(X) \cong \Omega^*_{(p)}(X) \tag{9}$$

and ΩT^* is the generalized cohomology theory associated to the Brown-Peterson spectrum [1] *localized at p.*

It is also possible to apply typical curves to unoriented cobordism theory where the prime involved is $p = 2$. One defines similarly an idempotent operator $\hat{\xi}$ whose image now is $H^*(X, \mathbb{Z}/2\mathbb{Z})$; there is also a canonical ring isomorphism

$$\eta^*(\mathrm{pt}) \otimes H^*(X, \mathbb{Z}/2\mathbb{Z}) \simeq \eta^*(X),$$

analogous to (9).

6. Operations in ΩT^*. If $\pi : \Omega^*_{(p)} \to \Omega T^*$ is the surjection induced by $\hat{\xi}$, then π carries the Thom class in $\Omega^*_{(p)}(MU)$ into one for ΩT^*. As a consequence ΩT^* has the usual machinery of characteristic classes with $c_i^{\Omega T}(E) = \pi c_i^{\Omega}(E)$ and $F^{\Omega T} = \pi F^{\Omega}$. Let $t = (t_1, t_2, \dots)$ be an infinite sequence of indeterminates and set

$$\varphi_t(X) = \sum_{n \geqslant 0}^{F^{\Omega T}} t_n X^{p^n}, \quad t_0 = 1,$$

where the superscript on the summation indicates that the sum is taken as curves in the formal group defined by $F^{\Omega T}$. There is a unique stable multiplicative operation

$$(\varphi_t^{-1})\hat{}\colon \Omega^*(X) \to \Omega T^*(X)[t_1, t_2, \dots]$$

such that

$$(\varphi_t^{-1})\hat{}\, c_1^{\Omega}(L) = \varphi_t^{-1}(c_1^{\Omega T}(L))$$

for all line bundles L. This operation can be shown using (8) to kill the kernel of π and hence it induces a stable multiplicative operation

$$r_t : \Omega T^*(X) \to \Omega T^*(X)[t_1, t_2, \dots].$$

Writing

$$r_t(x) = \sum_{\alpha} r_\alpha(x) t^\alpha, \quad x \in \Omega T^*(X),$$

where the sum is taken over all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ of natural numbers all but a finite number of which are zero, we obtain stable operations

$$r_\alpha : \Omega T^*(X) \to \Omega T^*(X).$$

Theorem 5. *(i) r_α is a stable operation of degree $2\sum_i \alpha_i(p^i - 1)$. Every stable operation may be uniquely written as an infinite sum*

$$\sum_\alpha u_\alpha r_\alpha, \quad u_\alpha \in \Omega T^*(\mathrm{pt}),$$

and every such sum defines a stable operation.

(ii) If $x, y \in \Omega T^*(X)$, then

$$r_\alpha(xy) = \sum_{\beta+\gamma=\alpha} r_\beta(x) r_\gamma(y).$$

(iii) The action of r_α on $\Omega T^*(\mathrm{pt})$ is given by

$$r_t(P_{p^n-1}) = \sum_{h=0}^{n} p^{n-h} P_{p^h-1} t_{n-h}^{p^h}.$$

(iv) If $t' = (t'_1, t'_2, \dots)$ is another sequence of determinates, then the compositions $r_\alpha \circ r_\beta$ are found by comparing the coefficients of $t^\alpha t'^\beta$ in

$$r_t \circ r_{t'} = \sum_\gamma \Phi(t, t')^\gamma r_\gamma,$$

where $\Phi = (\Phi_1 = (t_1; t'_1), \Phi_2 = (t_1, t_2; t'_1, t'_2), \dots)$ is the sequence of polynomials with coefficients in $\Omega T^*(\mathrm{pt})$ in the variables t_i and t'_i obtained by solving the equations

$$\sum_{h=0}^{N} p^{N-h} P_{p^h-1} \Phi_{N-h}^{p^h} = \sum_{k+m+n=N} p^{m+n} P_{p^k-1} t_m^{p^k} t_n'^{p^{k+m}}.$$

This theorem gives a complete description of the algebra of operations in ΩT^*. The situation is similar to that for Ω^* except the set of $\mathbb{Z}_{(p)}$-linear combinations of the r_α's is not closed under composition.

References

[1] E. H. Brown Jr. and F. P. Peterson, *A spectrum whose Z_p cohomology is the algebra of reduced p-th powers.* Topology **5** (1966), 149–154.

[2] P. Cartier, *Comptes rendus* **265** (1967), 49, 129.

[3] P. E. Conner and E. E. Floyd, *The relation of cobordism to K-theories*, Lectures Notes in Math., vol. 28, 1966, Springer–Verlag, Berlin.

[4] A. Fröhlich, *Formal groups*, Lecture Notes in Math., vol. 74, 1968, Springer–Verlag, Berlin.

[5] M. Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France **83** (1955), 251–274.

[6] S. P. Novikov, Methods of algebraic topology from the point of view of cobordism theory. Izv. Akad. Nauk SSSR, 1967, v. 31, N 4, 885–951.

[7] J. T. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. **1** (1968), 149–159.

This page intentionally left blank

7

Formal groups and their role in algebraic topology approach[1]

V. M. Buchstaber, A. S. Mishchenko, S. P. Novikov

Dedicated to
Ivan Georgievich PETROVSKY
on his 70-th birthday.

Introduction

This review is closely connected to the review by S. P. Novikov [13], we recommend to read them simultaneously. Here we touch on in general, the results of development of the cobordism theory based on the work of us, D. Quillen and some others. In the appendix, we describe the beautiful idea of Sullivan idea concerning the so-called Adams conjecture in K-theory.

§ 1. Formal groups

The theory of formal groups plays a large role in the modern approach to topology based on cobordism theory. Below, we describe

Let A be a commutative associative unital ring, $A[x_1, \ldots, x_n]$ the polynomial ring in $x_1, \ldots, x_n$ with coefficients in A and $A[[x_1, \ldots, x_n]]$ the corresponding power series ring.

[1]Формальные группы и их роль в аппарате алгебраической топологии. Успехи математических наук, 1971, Т. 26, вып. 2, с. 130–154 (поступила в редакцию 3 декабря 1970 г.).— Translated by V.O.Manturov

Definition 1.1. A one-dimensional *commutative formal group over A* is a power series $F(u,v) \in A[[u,v]]$ such that $F(F(u,v),w) = F(u,F(v,w))$ and $F(u,v) = F(v,u)$, whence $F(u,0) = u$.

Note that the existence of an "inverse element" $\varphi(u) \in A[[u]]$ such that $F(u,\varphi(u)) = 0$ follows from Definition 1.1.

Definition 1.2. A *homomorphism* Ψ *of formal groups* $G \xrightarrow{\Psi} F$ defined over a ring A is such a power series $\psi(u)$ that $F(\psi(u),\psi(v)) = \psi(G(u,v))$. If $\psi(u) = u + O(u^2)$ then ψ is a *strong isomorphism* (invertible variable change).

The basic rings A considered hitherto in basic examples are the ring $\mathbf{Z}$ of integer, the ring $\mathbf{Z}_p$ of p-adic integers, modulo p residue classes: $Z_p = \mathbf{Z}/p\mathbf{Z}$, integer elements in some field of algebraic numbers or p-adic completions. In topology, those are rings Ω of some cobordism type, especially, the unitary cobordism ring, which is algebraically isomorphic to the graded ring of polynomials over $\mathbf{Z}$ with polynomial generators in all even dimensions.

Various examples of finite groups over numeric rings can be found in an excellent paper by Honda [17].

Simplest examples. a) The linear group over $\mathbf{Z}$, where $F_0(u,v) = u+v$.

b) The multiplicative group over $\mathbf{Z}$, where $F_m(u,v) = u + v \pm uv$; the variable change $\psi(u) = \pm\ln(1 \pm u)$, transforming $F_m(u,v)$ to the linear form, lies in the ring $Q \supset \mathbf{Z}$, thus, over $\mathbf{Z}$, this group is not isomorphic to the linear one.

c) *The Lazard group.* Consider the ring $B = \mathbf{Z}[x_1, \dots, x_n, \dots]$ of integer polynomials in infinitely many variables and the series $g(u) = u + \sum_{n\geqslant 1} \dfrac{u^{n+1}x^n}{n+1}$. Then one defines a formal group

$$F(u,v) = g^{-1}(g(u) + g(v)),$$

where $g^{-1}(g(u)) = u$. The coefficients α_{ij} of the series $F(u,v)$ belong to the ring $B \bigotimes Q$ and generate over $\mathbf{Z}$ a subring $A \subset B \bigotimes Q$, where $F(u,v) = u + v + \sum_{i\geqslant 1, j\geqslant 1} \alpha_{ij} u^i v^i$.

The following Lazard's theorems take place.

Theorem 1.1. *The ring A of coefficients of Lazard's group is a polynomial ring over* $\mathbf{Z}$ *in finitely many generators.*

Theorem 1.2. *For every one-dimensional commutative formal group over every ring A' there exists a unique homomorphism $A \to A'$, taking the Lazard group to the given group («the universality of Lazard's group»).*

Theorem 1.3. *For every commutative one-dimensional formal group $F(u,v)$ over every ring A' there exists a series $\varphi(u) \in A'[[u]] \bigotimes Q$ such that*

$$\varphi(u) = u + O(u^2) \quad \text{and} \quad F(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)) \in A'[[u,v]] \bigotimes Q.$$

Thus, all groups can be linearized over the rational numbers. The series $\varphi(u)$ is called the *logarithm* of the formal group $F(u,v)$. Note that the coefficients of the formal differential $d\varphi(u) = (\sum_{n\geqslant 0} \varphi_n u^n)du$ belong to the ring A', where $\varphi_0 = 1, \varphi(u) = u + \sum_{n\geqslant 0} \frac{\varphi_n}{n+1} u^{n+1}$. The differential $d\varphi$ is called the *invariant differential* of the group $F(u,v)$, and it is calculated as follows: $d\varphi = du/(\frac{\partial}{\partial v} F(u,v))_{v=0}$ (see Honda [17]). Over the ring $A' \bigotimes Q$ we also have $\varphi(u) = [\frac{1}{k}\varphi^{-1}(k\varphi(u))]_{k=0}$.

A proof of Theorems 1.1–1.3 can be found in [4]; the expressions of type $\frac{1}{k}\varphi^{-1}(k\varphi(u)) = \frac{1}{k}F(u, F(u, ...)...) = \Psi^k(u)$ are connected with "Adams operations" in topology.

It is a remarkable fact that the "geometric cobordism formal group" introduced by A. S. Mishchenko and S. P. Novikov in [14], which plays an important role and has a simple geometric sense, turns out to coincide with the universal group of Lazard. This was first mentioned by Quillen in [7]; he proposed further important applications of this group in topology. The invariant differential of this group looks like $dg(u) = (\sum_{n\geqslant 0}[CP^n]u^n)du$, where $[CP^n]$ are the unitary cobordism classes of complex projective spaces; the coefficient ring A of Lazard's group coincides with the ring Ω of unitary cobordisms. Later, we shall use the notion of "power system", which is weaker transformal group.

Definition 1.3. A *type* $s \geqslant 1$ *power system* over a ring A is a sequence of series $f_k(u) \in A[[u]]$ such that $f_k(u) = k^s u + O(u^2)$ and $f_k(f_l(u)) = f_{kl}(u)$, where k, l are any integers (over rings A with torsion it is useful to require that the coefficients of $f_k(u)$ are algebraic in k).

One has the following fact (V. M. Buchstaber and S. P. Novikov [4]). Over the ring $A \bigotimes Q$, there exists a series $B(u) \in A[[u]] \bigotimes Q$ such that $f_k(u) = B^{-1}(k^s B(u))$, where $B^{-1}(B(u)) = u$ and $B(u) = u + O(u^2)$.

Every group generates a power system over the same ring; the reverse is, however, not true, since the coefficient ring of a power system is much smaller. A series of examples of power systems and their properties can be found in [4].

Note that in [4], [21] and in further sections of this review, we shall also see "double-valued" analogues of formal groups, which are defined by the equations (having no solutions over $A[[u,v]]$)

$$Z^2 - \Theta_1(u,v)Z + \Theta_2(u,v) = 0.$$

Here Θ_1, Θ_2 are in some sense the "sum" and the "product" of the values of the group $F_{\pm}(u,v)$ not belonging to the initial ring. The "inverse element" for u is a series $\varphi(u)$ such that $\Theta_2(u,\varphi(u)) = 0$. An important special case is $\varphi(u) = u$ (see § 4).

§ 2. Cobordism and bordism theories

I. Axiomatics of bordism theories. General properties. Given a class of smooth manifolds [closed with boundary], possibly, with extra structures, such that

a) the boundary of a manifold from the class belongs to this class;

b) the Cartesian product of two manifolds from the class belongs to the class ("multiplicativity");

c) any closed region with smooth boundary of a manifold belonging to the class itself belongs to the class (the closed interval belongs to the class as well) (the "cutting axiom" and homotopy invariance).

One says that such a class defines a cobordism (and bordism) theory. Denote this class by P.

Cycles (singular bordism of class P) for every complex K are such pairs (M,f) where $M \in P, f : M \to K$ is a continuous map and M is a closed manifolds. By *singular strip* we mean a pair (N,g), where $N \in P$ has a boundary as above, and $g : N \to K$. In an evident way, one defines the group of n-dimensional cycles factorized by the boundary of films in the class P, for any prefixed complex K; this group is denoted by $\Omega_n^P(K)$; it is called the bordism group of the complex K with respect to the class P. Analogously, one defines the relative bordism group $\Omega_n^P(K,L)$; one has the exact sequence of the pair: $\ldots \to \Omega_n^P(L) \to \Omega_n^P(K) \to \Omega_n^P(K,L) \xrightarrow{\partial} \Omega_{n-1}^P(L) \to \ldots$. For mappings $K_1 \xrightarrow{\varphi} K_2$ there exists a homomorphism $\varphi_* : \Omega_n^P(K_1) \to \Omega_n^P(K_2)$. The group Ω_n^P together with the homomorphisms φ_* are homotopy invariant (it is assumed that $I^1 \in P$). For the Euclidean space R^q (equivalently, for the point), the groups $\Omega_n^P(R^q)$ for $n > 0$, are, in general, not trivial. The direct sum $\Omega_*^P = \sum_n \Omega_n^P(R^q)$ forms the "scalar ring of the bordism theory".

For finite complexes K, we shall define the cobordism groups $\Omega_P^n(K)$ according to the Alexander–Pontrjagin duality law: if $K \subset S^N$, where S^N is the sphere, N is large enough, then, setting by definition $\Omega_P^n(K) = \Omega_{N-n}^P(S^N, S^N \backslash K)$, and this definition does not depend on N nor on the embedding $K \subset S^N$. The groups Ω_P^i enjoy the properties of homology groups; also, the relative groups $\Omega_P^i(K, L)$ are defined. The sum $\Omega_P^* = \sum_n \Omega_P^n(K, L)$ forms the "cobordism ring" of the pair $K \supset L$. For the space R^q (for the point) the ring $\Omega_P^* = \sum_n \Omega_P^n(R^q)$ is an analogue of the scalar ring.

By definition, for a point x we have $\Omega_P^n = \Omega_{-n}^P(x)$.

For some classes of manifolds the following Poincaré–Atiyah duality law holds: $D : \Omega_P^i(M^n) \xrightarrow{\approx} \Omega_{n-i}^P(M^n)$.

Examples. The most important examples of classes P are connected with some structure on the stabilized tangent bundle τ_M to the manifold M; for example, an orientation in the fibration $\tau \times R^k$ for some $k \geqslant 0$, a complex structure in $\tau_M \times R^q$, a symplectic structure in $\tau_M \times R^q$ or a trivialization of $(-\tau_M) \times R^q$ (framing or Pontrjagin's structure) etc. Thus, the classes P of this type are connected with some class Q of vector bundles over arbitrary complexes, i.e. $P = P(Q)$.

The Thom isomorphism. For classes P connected with a class Q of vector bundles, we require one more property, in addition to a), b), and c) described above:

d) the total fibre space of class Q with fibre disk and base $M \in P$, is a manifold belonging to the class P.

If the base of η is K then the fibre space with fibre disk D^n is E_η, its boundary E_η° being a fibration with fibre S^{n-1}; then it follows from the definitions and d) that we have the so-called "Thom isomorphism" φ_P: $\Omega_i^P(K) \xrightarrow{\approx} \Omega_{n+i}^P(E_\eta, E_\eta^\circ)$, which is defined by means of spaces of induced bundles $f^*\eta$. The Thom isomorphism generates the Poincaré-Atiyah duality for all manifolds from the class $P : D : \Omega_P^i(M^n) \xrightarrow{\approx} \Omega_{n-i}^P(M^n)$. One can define the fundamental cycle $[M^n] \in \Omega_n^P(M^n)$, the Čeh operation $x \bigcap y \in \Omega_{n-q}^P(K)$ for $x \in \Omega_P^q, y \in \Omega_n^P$, and prove that the Poincaré duality is defined by the Čeh operation. Moreover, for all continuous mappings f one has $f_*(f^*x \bigcap y) = x \bigcap f_* y$, where $x \in \Omega_P^q, y \in \Omega_n^P$.

II. Unitary cobordisms. The main class P we are interested in is the class of stable almost complex manifolds and the class Q of complex vector bundles. In this case the groups $\Omega_*^P(K)$ *and* $\Omega_P^*(K)$ are usually denoted by $U_*(K)$ *and* $U^*(K)$ and called "unitary bordisms and cobordisms". The ring $U_*(\text{point}) = \Omega_*^U$ is the polynomial ring over $\mathbf{Z}$ with even-dimensional generators, one in each even dimension.

Other bordisms of classes P which are connected with other types of manifolds: orientable, special unitary, unitary, stable symplectic or framed ones, etc., are usually denoted by $\Omega_*^O, \Omega_*^{SO}, \Omega_*^U = U_*, \Omega_*^{SU}, \Omega_*^{Sp}, \Omega_*^1 =$ (bordisms of framed manifolds). In S. P. Novikov's review [13] one may find information about these groups.

The operation ring.

Definition 2.1. A (stable) *homology operation* is an additive homomorphism $\theta : \Omega_*^P(K, L) \to \Omega_*^P(K, L)$ defined simultaneously for all dimensions and all complexes and commuting with continuous mappings and also commuting with the boundary homomorphism $\partial : \Omega_*^P(K, L) \to \Omega_*^P(L), K \supset L$.

Such operations form a ring, the "Steenrod ring" A^P, which is denoted by A^U in the case of unitary bordisms U_*. For cobordisms, the operation ring is defined analogously and coincides with the operation ring A^U.

If U_N is the unitary group and BU_N is the base of some fibration, η_N the total space (with fibre the disk), then denote by MU the spectrum (MU_N) of the Thom space $MU_N = E_{\eta N}/E^\circ_{\eta N}$, where $E_{\eta N}$ is the total space of η_N. Stable homotopy classes of mappings $[K, MU]$ coincide with the ring $U^*(K)$. In particular, $A^U = [MU, MU]$, and there is a well-defined Thom isomorphism $\varphi : U^*(BU_N) \xrightarrow{\approx} U^*(MU_N)$ (see review [13]).

Example. Multiplication by a "scalar" $\lambda \in U^*(\text{point})$ is, clearly, a cohomology operation. Note that for U-cobordisms, we have $\Omega_*^U = \mathbf{Z}[x_1, ..., x_n, ...]$. For cohomology operations in classical homology and cohomology theories scalars are just usual numbers and commute with all other operations. In cobordisms, the situation is more difficult.

The ring A^U was calculated by S. P. Novikov in [14]. It is described as follows. For every symmetric splitting $k = \dim \omega = \sum k_i, k_i \geqslant 0$, one defines operators $S_\omega \in A^U$ such that $S_{(0)} = 1$, and every element from A^U looks like a formal series $\sum_i \lambda_i S_{\omega_i}$ that $\dim \omega_i \to \infty$ for $i \to \infty$ and $\lambda_i \in \Omega_*^U$. The superposition formulae $S_{\omega_1} \circ S_{\omega_2}$ are given in [14]; ultimately, they result from the Leibniz formula $S_\omega(xy) = \sum_{(\omega_1, \omega_2) = \omega} S_{\omega_1}(x) S_{\omega_2}(y)$[1]. The superposition of the type $S_\omega \circ \lambda$ is equal to

$$\lambda \circ S_\omega + \sum_{\substack{(\omega_1, \omega_2) = \omega; \\ \dim \omega_1 > 0}} \sigma^*_{\omega_1}(\lambda) S_{\omega_2},$$

[1]The description of the ring A^U without the superposition formula $S_\omega \circ \lambda$ was also obtained by P. Landweber in [22].

where additive homomorphisms $\sigma^*_\omega(\lambda)$ on Ω^U_* are calculated by using the geometry of manifolds representing $\lambda \in \Omega^U_*$. For instance, $\sigma^*_{(q)}([CP^n]) = -(n+1)[CP^{n-q}]$. In particular, a representation * such that $S_\omega \xrightarrow{*} \sigma^*_\omega$ and $\lambda \xrightarrow{*}$ (multiplication by λ), of the operation ring A^U on the bordism ring of the point $U^*(\text{point}) = \Omega^*_U$, is exact.

Geometrical bordisms. One should indicate important subset of "geometric cobordisms" $V(K) \subset U^2(K)$ in every complexes K, and the dual sets $V(M^n) \subset U_{2n-2}(M^n)$ for almost complex manifolds ("geometrical bordisms"), consisting of submanifolds of complex codimension 1. If $u \in V(K)$ then $S_\omega(u) = 0$ for $\omega \neq (q)$ and $S_{(q)}u = u^{q+1}$. This property completes the set of axioms for the operations S_ω together with the multiplication formula $S_\omega(xy) = \sum_{(\omega_1,\omega_2)=\omega} S_{\omega_1}(x)S_{\omega_2}(y)$.

Various multiplicative operations $\alpha \in A^U$, i.e. such that $\alpha(xy) = \alpha(x)\alpha(y)$ for all x, $y \in U^*(K)$ and for all K, can be defined by one series $\alpha(u) \in U^*(CP^\infty)$, where $u \in V(CP^\infty)$ and CP^∞ is the infinite-dimensional projective space. One should note that the ring $U^*(CP^\infty)$ is just the ring of formal series $U^*(CP^\infty) = \Omega^*_U[[u]]$, where $\Omega^*_U = U^*(\text{point})$.

Characteristic classes. Formal group. Having operations S_ω and the Thom isomorphism, one can construct in the usual way the analogous of "Chern classes" $C_\omega(\eta)$ (where, by definition, $C_k = C_{(1,\dots,1)}$) for every U_N bundle η (see [8])[1]. Herewith $C_\omega(\eta) \in U^*$ (base). For U_1-bundles ξ *and* η product $\xi \bigotimes \eta$ is a U_1-bundle. The class $C_1(\xi \bigotimes \eta) = F(C_1(\xi), C_1(\eta))$ is calculated as a formal series with coefficients in Ω^*_U (see [14], Appendix 1). Thus one get a formal group of "geometric cobordisms" $F(u,v) = F(C_1(\xi), C_1(\eta)) = C_1(\xi \bigotimes \eta) = u + v - [CP^1]uv + \dots$. A. S. Mishchenko showed that $F(u,v) = g^{-1}(g(u) + g(v))$, where $g(u) = \sum_{n\geqslant 0} \frac{CP^n}{n+1} u^{n+1}$ and $dg(u) = (\sum_{n\geqslant 0} CP^n u^n)du = CP(u)du$.

Formal groups and operations. Analogues of Adams' operations $\Psi^k \in A^U \bigotimes \boldsymbol{Z}[\frac{1}{k}]$ are defined from multiplicativity $\Psi^k(xy) = \Psi^k(x)\Psi^k(y)$ and $\Psi^k(u) = \frac{1}{k} g^{-1}(kg(u))$ for $u \in V(CP^\infty) \subset U^2(CP^\infty)$, i.e. they are connected with taking the k-th power in the formal group $F(u,v)$. They generate a power system. Furthermore, $\Psi^0 \in A^U \bigotimes Q$ is defined as $\Psi^0(u) = g(u) = [\frac{1}{k} g^{-1}(kg(u))]_{k=0}$ and it defines a projector of the cobordism theory $U^* \bigotimes Q$ to the usual homology $H^*(Q)$ (see [14]).

[1] Note that in the cobordism theory, characteristics classes were introduced first (Conner and Floyd), and, based on them, cohomology operations were defined and the calculation of the algebra A^U (S. P. Novikov) was performed.

Strictly speaking, in the rings $A^U \bigotimes \mathbf{Z}_p$ for prime p there exist many multiplicative projectors (see [7], [14]). A canonical projector π_p is proposed by Quillen ([7]), namely, $\pi_p^*[CP^n] = 0$ for $n \neq p^h - 1$ and $\pi_p^*[CP^{p^h-1}] = [CP^{p^h-1}]$. This projector has found by Quillen by using the formal group approach. Projection operators are important because they select smaller homology theories, more convenient for calculation, for instance, the homotopy groups by using the Adams-type spectral sequence introduced to cobordism theory [14]. It is, however, necessary, to calculate the homology groups of these smaller theories; here one may use the known structure of the operation ring A^U in unitary cobordisms, if the projection is simple. This program has realized by Quillen in [7] by finding an appropriate projector. The role of formal groups in constructing such an operation became evident; moreover, it is confirmed also by the results of the authors and G. G. Kasparov concerning fixed points of maps. Here one should especially mention the results of A. S. Mishchenko [11] (see also [4] and § 5) concerning a fixed manifold with trivial bundle with respect to some group actions.

Chern characters. Note that formal groups are closely connected to analogues of the so-called "Chern character". The classical Chern character ch is an additive-multiplicative function of the bundle valued in rational cohomology. S. P. Novikov in [14] showed that such a cobordism-valued function of bundles is defined by its value on U_1-bundles η, where it is equal to $\exp(g(u))$, where $u = C_1(\eta), g(u) = \sum_{n\geqslant 0} \dfrac{CP^n}{n+1} u^{n+1}$. Another notion of the Chern character, which is abstract (introduced by Dold) is not connected with fibrations: it is just an isomorphism of the theories $\mathrm{ch}_U : U^* \bigotimes Q \to H^*(\Omega_U^* \bigotimes Q)$, which is the identity on the homology of the point. Here the series $g(u)$ appears as well. As V. M. Buchstaber showed [2], for the basic element $t \in H^2(CP^\infty)$, we have $\mathrm{ch}_U^{-1}(t) = g(u)$. In [2], he studied the general Chern-Dold character in unitary cobordisms, and described several applications, further developed in [3], [4], [21].

Hirzebruch genera. As Novikov indicated in [15], the so-called "mutliplicative Hirzebruch genera" $Q(z)$ or homomorphisms $Q : \Omega_*^U \to \mathbf{Z}$ such that $Q(CP^n) = [Q(z)^{n+1}]_n = \dfrac{1}{2\pi i}\int_{|z|=\varepsilon} \dfrac{Q^{n+1}}{z^{n+1}} dz$, are calculated via $g^{-1}(z)$, namely, $Q(z) = \dfrac{z}{g_Q^{-1}(z)}$, where

$$g_Q(u) = \sum_{n\geqslant 0} \frac{Q(CP^n)}{n+1} u^{n+1} \quad \text{and} \quad g_Q^{-1}(g_Q(u)) = u.$$

Thus, *all basic notions and facts in the unitary cobordism theory, both modern and classical, can be expressed in terms of the Lazard formal group.*

Facts from K-theory. Let us concentrate on the usual complex K-theory $K^*(X)$, where $K^i(X) = K^{i+2}(X)$ for all i (Bott periodicity), $K^0(X)$ are stable classes of complex bundles over X and $K^1(X)$ are homotopy classes of mappings $X \to U_N$, for $N > \dim X$. If λ^i denote exterior powers, $\lambda_t = \sum_{i\geqslant 0} \lambda^i t^i$ then $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$ is an exponential operation. For symmetric powers S_i we have $S_t = \sum_{i\geqslant 0} S_i t^i = \dfrac{1}{\lambda_t}$. Furthermore, if $Q_k = \sum_{i=1}^N t_i^k (N \to \infty)$ and $Q_k = Q_k(\sigma_1, \dots, \sigma_k)$, where $\sigma_k(t_1, \dots, t_n)$ are elementary symmetric functions then the virtual representation, called the Adams operation is just $Q_k(\lambda^1, \dots, \lambda^k) = \Psi^k$. It turns out that $\Psi^k(x+y) = \Psi^k(x) + \Psi^k(y)$, $\Psi^k(xy) = \Psi^k(x)\Psi^k(y)$ and $\Psi^k \circ \Psi^l = \Psi^{kl}$. Later, for U_1-bundles $\eta \in K^0(X)$ we have $\Psi^k \eta = \eta^k$. The Adams operator Ψ^k does not commute with the Bott periodicity operator $\beta : \ K^i \to K^{i-2}$. The following formula holds: $\Psi^k \cdot \beta = k\beta \cdot \Psi^k$. Thus the operators Ψ^k are defined in the theory $K^* \bigotimes \mathbf{Z}\left[\dfrac{1}{k}\right]$. The cohomology of a point in K-theory looks like $K^*(\text{point}) = \mathbf{Z}[\beta, \beta^{-1}]$ and $\Psi^k\beta = k\beta\Psi^k$. This completes the description of operations in K-theory. The analogues of “geometric cobordisms” in K-theory are U_1-bundles, more precisely, these are elements $u = \beta^{-1}(\xi - 1) \in K^2(X)$ for U_1-bundles $\xi = \beta u + 1$. We have $k\Psi^k(u) = \beta^{-1}((\beta u + 1)^k - 1)$ and $F(u, v) = u + v - \beta uv = \beta^{-1}((\beta u + 1)(\beta v + 1) - 1)$ is a multiplicative group. Thus, the well-known Riemann-Roch-Grothendieck homomorphism $r : U^*(X) \to K^*(X)$ corresponds to the homomorphism of the universal Lazard group to the multiplicative group, where $r(\lambda) = T(\lambda \cdot \beta^{\overline{\frac{\dim \lambda}{2}}}))$, T is the Todd genus, $\lambda \in \Omega_U^*$.

§ 3. The formal group of geometrical cobordisms

Multiplication law in the geometrical cobordism formal group. Let $\eta \to CP^n, n \leqslant \infty$, be the canonical complex one-dimensional bundle over the projective space CP^n. As mentioned in § 2, the formal series $c_1(\eta_1 \bigotimes \eta_2) = F(u, v) \in U^2(CP^\infty \times CP^\infty) = \Omega[[u, v]]$, $u = c_1(\eta_1)$, $v = c_1(\eta_2)$, defines the multiplication law in the one-dimensional formal group of geometrical cobordisms over the ring Ω_U.

Theorem 3.1. *a) One has*

$$F(u,v) = \frac{u + v + \sum [H_{r,t}] u^r v^t}{CP(u) \cdot CP(v)},$$

where $H_{r,t}$ is the algebraic submanifold of complex codimension 1 in $CP^r \times CP^t$, representing the zero set of the bundle $\eta_1 \bigotimes \eta_2 \to CP^r \times CP^t$; this manifold realizes the cycle $[CP^{r-1} \times CP^t + CP^r \times CP^{t-1}] \in H_{2(r+t-1)}(CP^r \times CP^t)$.

b) The logarithm of the group $F(u,v)$ looks like $g(u) = \sum \frac{[CP^n]}{n+1} u^{n+1}$.

Proof. We have $F(u,v) = u + v + \sum e_{ij} u^i v^j$ and $\lambda : CP^r \times CP^t \to CP^\infty \times CP^\infty$ is the standard embedding. Then $\varepsilon D \lambda^* F(u,v) = [H_{r,t}] = [CP^{r-1}][CP^t] + [CP^r][CP^{t-1}] + \sum e_{i,j} [CP^{r-i}][CP^{t-j}]$, where D the Poincaré-Atiyah duality operator, $\varepsilon : U^*(CP^r \times CP^t) \to \Omega_U$ is the augmentation to the point, and $[CP^{r-i}] = \varepsilon D u^i$ if $u = c_1(\eta) \in U^2(CP^r)$. Thus, $u + v + \sum [H_{r,t}] u^r v^t = F(u,v) CP(u) CP(v)$. The statement a) is proved. One has

$$dg(u) = \frac{du}{\frac{\partial F(u,v)}{\partial v}|_{v=0}},$$

consequently,

$$dg(u) = \frac{CP(u) du}{1 + \sum ([H_{r,1}] - [CP^1][CP^{r-1}]) u^r}.$$

It is easy to show, e.g., by comparing the Chern numbers, that $[H_{r,1}] = [CP^1] \times [CP^{r-1}]$. Consequently, $dg(u) = CP(u)$. The theorem is proved.

The universality of the geometric cobordism formal group. As shown in [9], [12], the ring Ω_U is multiplicatively generated by the elements $[H_{r,t}]$, and, by [16], the ring $\Omega_U \bigotimes Q$ is multiplicatively generated by $[CP^n], n \geqslant 0$. Now, from Theorem 3.1 we get that the subring of Ω_U generated by the coefficients of the geometrical cobordism formal group coefficients, coincides with Ω_U and the coefficients of the logarithm expansion of this group are algebraically independent, and generate the ring $\Omega_U \bigotimes Q$. Let us show now, that these facts trivially yield the universality of the group $F(u,v)$ over Ω_U for the category of commutative rings without torsion. Let $G(u,v)$ be any arbitrary formal group over a ring R without torsion, and let $g_G(u) = \sum \frac{a_n}{n+1} u^{n+1}, a_n \in R$, denote its logarithm. Consider the ring homomorphism $r : \Omega_U \to R \bigotimes Q, r[CP^n] = a_n$. Since $G(u,v) = g^{-1}(g(u) + g(v))$, we have that $r(F(u,v)) = \sum r(e_{i,j}) u^i v^j = G(u,v)$. Consequently, $r(e_{i,j}) \in R$, i.e. $\mathrm{Im}(r : \Omega_U \to R \bigotimes Q) \subset R$. Since the universal

Lazard group is defined over a torsion-free ring, we have proved the following Theorem.

Theorem 3.2. *The formal group of geometrical cobordisms coincides with the universal formal Lazard group, i.e. the homomorphism of the Lazard ring A to Ω^U, corresponding to the group $F(u,v)$ (see § 1), is an isomorphism.*

Hirzebruch genera from the formal group point of view. By virtue of Theorem 3.2, every integer-valued Hirzebruch genus, or, which is the same, any homomorphism $\Omega_U \to Z$, defines a formal group over Z; conversely, every formal group over Z generates a Hirzebruch genus. Here, the Hirzebruch genus defining the homomorphism $\Omega_U \to Z$ can have rational coefficients. Equivalent (strongly isomorphic) formal groups are defined by Hirzebruch series $Q(z), Q'(z)$, which are connected by the formula $\left(\frac{z}{Q(z)}\right) = \varphi^{-1}\left(\frac{z}{Q'(z)}\right)$, where $\varphi^{-1}(u) = u + \sum \lambda_i u^{i+1}, \lambda_i \in \mathbf{Z}$. This follows from the fact that the logarithms of the formal groups are equal to $g_Q(z) = (\frac{z}{Q(z)})^{-1}$; thus, we see by definition, that $g_Q(z) = g_{Q'}(\varphi(z))$. Given an integer-valued Hirzebruch genus, generated by a rational series $g_Q(u)$, then Q' is a genus such that $q_Q(u) = g_{Q'}(\varphi(u)), \varphi(u) = u + \sum \mu_i u^{i+1}, \mu_i \in \mathbf{Z}$, this genus also takes integer values on Ω_U. This is the sense of equivalence of Hirzebruch genera as formal groups. Now, let us consider the formal groups corresponding to previously known multiplicative genera c, T, L, A. Consider the T_y genus (see [18]). Since $T_y([CP^n]) = \sum_{i=0}^n (-y)^i$, then the corresponding formal group over $\mathbf{Z}[[y]]$ looks like $F_{T_y} = \dfrac{u+v+(y-1)uv}{1+uvy}$. For $y = -1, 0, 1$ we get formal group, corresponding to the Euler characteristic c, the Todd genus T and Hirzebruch's L-genus. For all y, the group $F_{T_y}(u,v)$ is equivalent either to the linear group or to the multiplicative group. Now, note that the A-genus is equivalent to the L-genus as a formal group; thus we see that all Hirzebruch genera considered hitherto in topology are connected either with the linear group or with the multiplicative group[1].

[1]The series $g^{-1}(t) = \dfrac{t}{Q(t)}$ is called the exponent of the formal group $f(u,v) = g^{-1}(g(u)+g(v))$. For the Hirzebruch genera T, L and A we have:

$$g_T^{-1}(t) = 1 - \exp(-t); \quad g_L^{-1}(t) = \tanh t; \quad g_A^{-1}(t) = 2\sinh t/2.$$

Obviously, there exists series $\varphi(z) \in \mathbf{Z}\left[\frac{1}{2}\right]$ such that $g_A^{-1}(t) = \varphi^{-1}(g_L^{-1}(t))$, i.e. the A-genus and the L-genus are equivalent over the ring $\mathbf{Z}\left[\frac{1}{2}\right]$. In the end of 1980-s, the

Multiplicative cohomology operations and Hirzebruch genera. Every multiplicative cohomology operation in cobordisms is, on the one hand, uniquely given by a ring homomorphism $\varphi^* : \Omega_U \to \Omega_U$, that it induces cobordism substituting as in § 1, and, on the other hand, by its value on the geometrical cobordism $u \in U^2(CP^\infty)$, i.e. the formal series $\varphi(u) = u + O(u^2) \in U^2(CP^\infty) = \Omega_U[[u]]$. Note that the series $\varphi(u)$ generate a strong isomorphism of the universal group $F(u,v) = u + v + \sum e_{i,j} u^i v^j$ and the group

$$\varphi(F(u,v)) = u + v + \sum \varphi^*(e_{i,j}) u^i v^j.$$

In the characteristic class theory, the ring homomorphisms $\Omega_U \to \Omega_U$ are given by Hirzebruch series, $K(1+u) = Q(u)$,

$$Q(u) = \frac{u}{a(u)}, \quad a(u) = u + \sum \lambda_i u^i, \quad \lambda_i \in \Omega_U \bigotimes Q.$$

From the point of view of Hirzebruch series, the action of the series $a(u)$ on the ring Ω_U is given by the formula $a([CP^n]) = [(\frac{u}{a(u)})^{n+1}]_n$, where $[f(u)]_n$ is the n-th coefficient of $f(u)$. Thus, every formal series $a(u) = u + O(u^2)$ generates a ring homomorphism $a^* : \Omega_U \to \Omega_U$ as a multiplicative operation in cobordisms, as well as a ring homomorphism $a : \Omega_U \to \Omega_U$ defined by the Hirzebruch series $Q(u) = \frac{u}{a(u)}$. The two actions of $a(u) = u + O(u^2)$ on the ring Ω_U, indicated above, do not coincide. For instance, for $a(u) = u$ we have $a^*([CP^n]) = [CP^n], a([CP^n]) = 0, n > 0$. As V. M. Buchstaber and S. P. Novikov showed (see [4]), the following result holds[1]

Theorem 3.3. *The mapping $g : a(u) \to a(g(u))$ of the series ring possesses the property $a(u)[x] = a(g(u))^*[x]$ for each x, where $g(u) = \sum_{n \geqslant 0} \frac{CP^n}{n+1} u^{n+1}$ is the logarithm of the formal group of geometrical cobordisms.*

---

elliptic genus E was introduced into topology (see S. Ochanine. Sur les genres multiplicatifs difinis par der integrales elliptiques. Topology **26** (1987), 143–151), the exponent $g_E^{-1}(t) = \operatorname{sn} t$ given by the elliptic sinus function. The formal group corresponding to the genus E is

$$f(u,v) = \frac{u\sqrt{1 - 2av^2 + bv^4} + v\sqrt{1 - 2au^2 + bu^4}}{1 - bu^2v^2}.$$

V. M. Buchstaber's remark (2004).

[1]This result in a similar formulation was obtained by J. Adams [23].

The generalized characteristic Todd class.

Definition 3.1. A *generalized Todd class* of a complex bundle ξ over X is a characteristic class $T(\xi) \in H^*(X, \Omega_U \bigotimes Q)$ corresponding to the series $Q(u) = \frac{u}{g^{-1}(u)}, g^{-1}(g(u)) = u$.

Consider now a continuous mapping $f : M^{2n} \to M^{2m}$ of almost complex mappings, and denote by $\tau(f)$ the element $(\tau(M^{2n}) - f^*\tau(M^{2m})) \in \widetilde{(K)}(M^{2n})$, where τ is the tangent bundle.

Theorem 3.4. (see [2]) *One has* $\mathrm{ch}_U\, D[f] = f_! T(\tau(f))$, *where* $[f]$ *is the bordism class of the mapping* f, ch_U *is the Chern–Dold character* (*see* § 2) *and* $f_!$ *is the Gysin homomorphism in cohomology.*

V. M. Buchstaber [2] has indicated several formulae expressing the generalized Todd class $T(\xi)$ in classical characteristic classes of the bundle ξ. In some cases, these formulae allow one to calculate effectively the bordism class of the mapping f. Let us give the simplest of them.

Theorem 3.5. *Let* η *be a one-dimensional bundle over* X *and let* $u = c_1(\eta) \in H^2(X, Z)$; *then* $T(\eta) = \frac{u}{g^{-1}(u)}$ *and* $g^{-1}(u) = \mathrm{ch}_U\, \sigma_1(\eta) = u + \sum [M^{2n}] \frac{u^{n+1}}{(n+1)!}$, *where* $[M^{2n}] = \sigma_1(\xi_{n+1}) \in U^2(S^{2n+2}) \approx \Omega_U^{-2n}$, *here* $s_\omega(-\tau(M^{2n})) = 0, \omega \neq (n), s_{(n)}(M^{2n}) = -(n+1)!, \sigma_1(\xi_{n+1})$ *is the first Chern class in cobordisms of the generator* $\xi_{n+1} \in K(S^{2n+2})$ *and* s_ω *are the Chern numbers corresponding to* ω.

§ 4. Two-valued formal groups and power systems

The notion of two-valued formal group. Let $F(u, v) = u + v + \ldots$ be a one-dimensional formal group over a commutative ring R with unit 1, $\bar{u} = -u + o(u^2) \in R[[u]]$ be the formal series generating the inverse element in the group $F(u, v)$, i.e. $F(u, \bar{u}) = 0$, and $g_F(u)$ be the logarithm of the group $F(u, v)$. It is shown in [4] that the formal series $F(u, v) \cdot F(\bar{u}, \bar{v}) + F(u, \bar{v}) \cdot F(\bar{u}, v) = |F(u, v)|^2 + |F(u, \bar{v})|^2$ and $|F(u, v)|^2 \cdot |F(u, \bar{v})|^2$ from the ring $R[[u, v]]$ indeed belong to the ring $R[[x, y]] \subset R[[u, v]]$, where $x = u\bar{u} = |u|^2, y = |v|^2$, i.e. they look like $\Theta_1(x, y)$ and $\Theta_2(x, y)$, respectively. Over $R[[x, y]]$, consider the quadratic equation $\mathscr{Y}(x, y) = Z^2 - \Theta_1(x, y)Z + \Theta_2(x, y) = 0$ and denote by $B(x) = x + O(x^2) \in R[[x]] \bigotimes Q$ the series, which in $R[[u]] \bigotimes Q \supset R[[x]] \bigotimes Q$ looks like $g_F(u) g_F(\bar{u}) = -g_F^2(u)$. As Novikov has shown in [4], over a torsion-free ring R the solutions of $\mathscr{Y}(x,y)=0$ look like

$$F^{\pm}(x,y)=B^{-1}((\sqrt{B(x)}\pm\sqrt{B(y)})^2). \tag{1}$$

These solutions are evidently not formal series in x and y, but, as (4.1) shows, they satisfy a certain associativity. Such quadratic equations were called *two-valued formal groups* in [4].

Two-valued formal groups and symplectic cobordisms. Consider the two-valued formal group in cobordisms constructed from the formal group of geometrical cobordisms. As shown in [4], the series $B^{-1}(z)$ from (4.1) coincides with the formal series $\mathrm{ch}_U(x)=z+\sum_{n=2}^{\infty}[N^{4n-4}]\frac{z^n}{(2n)!}\in H^*(CP^\infty,\Omega_U\bigotimes Q)$, where z is the generator of the group $H^4(CP^\infty,\mathbf{Z})$ and $s_{(2n-2)}[N^{4n-4}]=(-1)^n 2\cdot(2n)!\neq 0$.

Theorem (see [4]). *For each $n\geqslant 2$, the bordism classes $[N^{4n-4}]$ belong to the image of homomorphisms $\Omega_{Sp}^{-4n+4}\to\Omega_U^{-4n+4}$. For $n\equiv 1$* mod 2 *the group* $\mathrm{Im}\,(\Omega_{Sp}\to\Omega_U)$ *contains elements $[N^{4n-4}]/2\in\Omega_U$.*

The canonical mapping of spectra $\omega: MSp\to MU$ corresponding to the group inclusion $Sp(n)\subset U(2n)$, defines an epimorphism $A^U\to U^*(MSp(n))$, and, consequently, an embedding of the ring $\mathrm{Hom}_{A^U}(U^*(MSp),\Omega_U)$ into Ω_U. Later on, we identify $\mathrm{Hom}_{A^U}(U^*(MSp),\Omega_U)$ with its image in Ω_U. There is an embedding $i:\mathrm{Im}(\Omega_{Sp}\to\Omega_U)\subset\mathrm{Hom}_{A^U}(U^*(MSp),\Omega_U)$; moreover, the homomorphism $i\bigotimes Z\left[\frac{1}{2}\right]$ is an isomorphism. This easily follows from [12]. In addition to Theorem 4.1 note that the elements $[N^{8n-4}]/2$ belong to the group $\mathrm{Hom}_{A^U}(U^*(MSp),\Omega_U)$ but do not belong to $\mathrm{Im}(\Omega_{Sp}\to\Omega_U)$ (see [4]). It is shown in [4] that Theorem 4.1 together with properties of the Chern character described in [2], yield

Theorem 4.1. *Let $\Lambda\subset\Omega_U$ be the ring generated by coefficients of the two-valued formal group in cobordisms:*

$$a)\ \Lambda\subset\mathrm{Hom}_{A^U}(U^*(MSp),\Omega_U);$$

$$b)\ \Lambda\left[\frac{1}{2}\right]\approx\Omega^*_{Sp}(*)\otimes Z\left[\frac{1}{2}\right].$$

Since the ring Λ is smaller than the ring Ω_U, then over $\Omega_U\bigotimes Q$ there are many one-dimensional formal groups, for which the squares of absolute values define a two-valued formal group in cobordisms. V. M. Buchstaber showed in [21] that the minimal one-dimensional group (with respect to coefficients) of such type over $\Omega_U \bigotimes Q$ is uniquely defined by the multiplicative projector $\varkappa^* : \Omega_U[\frac{1}{2}] \to \Omega_U[\frac{1}{2}]$, whose value on the geometrical cobordism $u \in U^2(CP^\infty)$ is equal to $\varkappa(u) = -\sqrt{-u\bar{u}} = u + O(u^2) \in U^2(CP^\infty)[\frac{1}{2}]$. V. M. Buchstaber proved in [21] the following

Theorem 4.2. *a) In order for an element $\sigma \in \Omega_U$ to belong to the group $Hom_{A^U}(U^*(MSp), \Omega_U) \subset \Omega_U$, it is necessary and sufficient that $\varkappa^*(\sigma) = \sigma$.*

b) $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) \cong \mathrm{Im}\,\varkappa \bigcap \Omega_U$.

Corollary 4.1. *The composition*

$$Sp^*(X)\left[\frac{1}{2}\right] \xrightarrow{\omega} U^*(X)\left[\frac{1}{2}\right] \xrightarrow{\varkappa} \mathrm{Im}\left(\varkappa U^*(X)\left[\frac{1}{2}\right]\right)$$

establishes an isomorphism between the cohomology theory $Sp^*\left[\frac{1}{2}\right]$ and the theory extracted from $U^*\left[\frac{1}{2}\right]$ by the projection operator $\varkappa$.

Algebraic properties of two-valued formal groups. V. M. Buchstaber [21] gave an axiomatic definition of a two-valued formal group $\mathscr{Y}(x, y) = Z^2 - \Theta_1(x, y)Z + \Theta_2(x, y) = 0$, with a partial case being the quadratic equation defined by the square of the absolute value of a one-dimensional formal group. We are not giving this definition because of inconvenience; we just note that this definition requires the existence of formal series $\varphi(x)$ such that $\Theta_2(x, \varphi(x)) = 0$. The series $\varphi(x)$ has the meaning of inverse element and it plays an important role in the classification of two-valued formal groups. For instance, the following theorem takes place

Theorem 4.3. *The two-valued formal group in cobordisms considered over the ring $\Lambda \subset \Omega_U$ of coefficients of $\Theta_1(x, y)$ and $\Theta_2(x, y)$ is universal for two-valued groups over torsion-free rings R, for which $\varphi(x) = x$, i.e. $\Theta_2(x, x) = 0$.*

Formal power systems not lying in formal groups. Let $\mathscr{Y}(x, y) = Z^2 + \Theta_1(x, y)Z + \Theta_2(x, y) = 0$ be a two-valued formal group over a ring $R[[x, y]]$, which is defined by the module-square of a one-dimensional group $F(u, v) \in R[[u, v]]$. Consider the sequence of formal series $\varphi_k(x) \in R[[x]] : \varphi_0(x) = 0, \varphi_1(x) = x, \varphi_2(x) = \Theta_1(x, x), \ldots, \varphi_n(x) = \Theta_1(x, \varphi_{n-1}(x)) - \varphi_{n-2}(x), \ldots$. The series $\varphi_k(x) = k^2x + O(x^2)$ considered in the ring $R[[u]] \supset R[[x]]$ look like $[u]_k, [\bar{u}]_k$ where $[u]_k$ is the k-th power of the element u in the group $F(u, v)$. Consequently, the sequence of series $\varphi_k(x)$ forms a formal power system of type $s = 2$. In the case when $R = \Omega_U$ and $F(u, v)$ is a formal group of geometrical cobordisms, it is easy to show (see [14]) that the system $\varphi_k(x)$ does not correspond to exponential in any formal group over Ω_U. Note that the system $\varphi_k(x)$ has important topological applications; it first appeared implicitly in Novikov's work [15] for describing fixed points of actions of generalized quaternions 2-groups on almost complex manifolds.

Finally, let us show that the power system $\varphi_k(x)$ has a natural generalization. Let $F(u,v)$ be a formal group over the ring R without torsion, and $g_F(u)$ be its logarithm. Consider the full set $(\xi_0 = 1, \dots, \xi_{m-1})$ of m-th roots of unity. Set $B_m^{-1}(-y) = \prod_{j=0}^{m-1} g_F^{-1}(\xi_j \sqrt[m]{y})$, $x = \prod_{j=0}^{m-1} g_F^{-1}(\xi_j g_F(u))$. Then $-B_m(x) = g_F(u)^m$, and we get the formal power system

$$F_k^{(m)}(x) = B_m^{-1}(k^m B_m(x)) = \prod_{j=0}^{m-1} g_F^{-1}(k\xi_j g_F(u))$$

of type $s = m$. Coefficients of the series $F_k^{(m)}(x)$ a fortiori lie in the ring R for formal groups $F(u,v)$ with complex multiplication on ξ_j (ξ_j-exponential). This construction originates from the formal group of geometrical cobordisms over the ring $\Omega_U \bigotimes \mathbf{Z}_p$ and $m = p-1$, where $\mathbf{Z}_p$ is the ring of integer p-adic numbers. As in the case $m = 2$, one can consider the m-valued formal group given by an algebraic equation of degree m whose solution looks like $F(x,y) = B_m^{-1}(\sqrt[m]{B_m(x)} + \sqrt[m]{B_m(y)})^m$.

§ 5. Fixed points of periodic transformations in terms of formal groups

Conner and Floyd [6] first showed that the bordism theory language is very convenient for studying the fixed points of smooth periodic transformations. The use of formal groups allowed them to systematize and generalize the results in this direction.

Basic constructions and notions. Let M^n be an almost complex smooth manifold, let T be a smooth transformation of M^n, $T^p = \mathrm{id}$, and p be some prime number; assume T preserves the almost complex structure of the manifold M^n. It is easy to show that the set $X \subset M^n$ of fixed points of T, i.e. points such that $x \in M^n$, $Tx = x$ forms a disjoint union of finitely many closed submanifolds N_i with a natural structure on them. Here, one can exhibit tubular neighborhoods U_i of manifolds N_i in such a way that U_i are total spaces of normal bundles corresponding to the embeddings of N_i into M, where the action of T is linear on U_i and free outside the zero-sections $N_i \subset U_i$. Thus, the boundaries of tubular neighborhoods ∂U_i are almost complex manifolds with free action of Z_p; thus they define an element of bordisms of the infinite lens space BZ_p, $\alpha(T) \in U_{n-1}(BZ_p)$. The element $\alpha(T)$ is defined only by the behavior of T near the fixed manifolds N_i. It is clear that $\alpha(T) = 0$ because $\bigcup_i \partial U_i = \partial(M^n \backslash \bigcup_i U^i)$ and the action of T on the manifold $M^n \backslash \bigcup_i U_i$ is free. Consequently, the classification problem for almost complex manifolds with Z_p action in terms of bordisms is reduced to the following two problems: a) description of the Z_p action near the set of fixed points and b) finding sets of fixed manifolds such that $\alpha(T) = 0$.

Stating the problem. We first justify what we mean by classification under Z_p action in terms of bordisms. We say that an almost complex manifold M^n with Z_p is bordant to zero if there exists such an almost complex manifold with boundary W and an almost complex action T' on it such that $(T')^p = \mathrm{id}, \partial W = M, T'|_{\partial W} = T$. We shall study classes of bordant manifolds in the sense indicated above. The behavior of T near the fixed submanifolds can be easily described. It is known that if on a complex bundle ξ the group Z_p acts as the identity on the base then the bundle ξ can be represented as a sum $\xi = \bigoplus_{i=1}^{p} \xi_i$, and the action of Z_p is defined on ξ_i by one of the irreducible unitary representations of the group Z_p. Thus, if T is a generator of $Z_p, \zeta = \exp(\frac{2\pi}{p})$ then $T(x) = \zeta^i x$ for $x \in \xi_i$. In the class of bordant manifolds with Z_p action a fixed component N_i determines a bordisms of the sum of $(p-1)$ fibers, i.e.

$$\beta(N_i) \in U_{k_i}(\prod_{j=1}^{p-1} BU(l_j^i)), \qquad \text{where} \qquad \dim N_i = k_i, \quad \dim \xi_i = l_j^i.$$

Thus, if $\Omega_{U,p}^m$ is the group of n-dimensional bordisms with Z_p action on it, then there exists a mapping

$$\beta : \ \Omega_{U,p}^n \to \bigoplus_{k+2\sum l_i = n} U_k(\prod BU(l_i)),$$

that associates with a manifold with Z_p action the set of bordisms generated by components of the fixed submanifold.

The second problem is to determine for every bordism

$$x \in \bigoplus_{k+2\sum l_i = n} U_k(\prod BU(l_i))$$

whether the element x is realizable as a set of fixed points of some almost complex action of Z_p. As mentioned above, there exists a mapping

$$\alpha : \bigoplus_{k+2\sum l_i = n} U_k(\prod BU(l_i)) \to U_{n-1}(BZ_p)$$

here if $\alpha(x) = 0$ then the element x is realizable as the set of fixed points under Z_p action. In other words, if A is the ring of all bordisms

$$A = \bigoplus_{k,l_1,\dots,l_{p-1}} U_k(\prod BU(l_i)),$$

then the sequence $\Omega^*_{U,p} \xrightarrow{\beta} A \xrightarrow{\alpha} U_*(BZ_p)$ is exact. It is easy to see that α is epimorphic and $\operatorname{Ker}\beta \approx p\Omega^*_U$. It is interesting to consider such an action of Z_p where the fixed submanifold consists only of isolated points or manifolds with trivial normal bundle. In the last case the fixed submanifold is determined by the bordism $x \in \Omega^k_U$ and the weight system $x_1, \dots, x_{\frac{n-k}{2}}$ of the representation of Z_p in the normal bundle.

Basic formulae. Interesting connections with the formal group in cobordisms are connected to the description of the homomorphism α (for detailed description, see [4]). It is known that the cobordism ring of the space BZ_p can be represented as

$$U^*(BZ_p) = \Omega_U[[u]]/p\Psi^p(u) = 0. \tag{1}$$

Then, for an isolated fixed point with weights $(x_1, \dots, x_n)$ one has the following formula obtained by G. G. Kasparov [5], A. S. Mishchenko [10], S. P. Novikov [15]:

$$\alpha(x_1, \dots, x_n) = \prod_{j=1}^{n} \frac{u}{g^{-1}(x_j g(u))} \bigcap \alpha(1, \dots, 1), \tag{2}$$

where $g(u)$ is the logarithm of the formal group. From (1), it follows that the right hand of the formula (2) makes sense. In the general case, the multiplicative basis of the ring A over the ring Ω_U form manifolds CP^k with 1-dimensional Hopf bundle with weight x. This means that the elements of the ring A are defined by the sequence of numbers $((k_1, x_1), \dots, (k_l, x_l))$, $\sum(k_i + 1) = x$.

Consider the following meromorphic differential Ω with poles at $z = u$ defined on the formal group $f(u, v)$, where $\Omega = \Omega(u, z)dz = \dfrac{dg(z)}{f(u, \bar{z})}, \bar{z} = g^{-1}(-g(z))$; it is invariant under the shift $u \to f(u, \omega), z \to f(z, \omega), \Omega \to \Omega$. This differential is the analogue of $dz/(u-z)$ on the linear group. Let $t = \frac{z}{u}$ and $dt = \frac{dz}{u}$, where u is a parameter. We have $\Omega = \Omega(u, z)dz = G(u, t)dt$, whence G has a pole for $t = 1$ for all z, u. Then, as shown in [11] (see also [4]), the following formula holds

$$\alpha((k_1, x_1), \ldots, (k_l, x_l))$$
$$= [\prod_{q=1}^{l} G(g^{-1}(x_q g(u)), t_q) \frac{u}{g^{-1} x_q g(u)}]_{k_1,\ldots,k_l} \bigcap \alpha_{2n-1}(1, \ldots, l),$$

where $[\ ]_{k_1,\ldots,k_l}$ denotes the coefficient of $t_1^{k_1}, \ldots, t_l^{k_l}$.

The connection to the Atiyah-Bott formula. Besides the description of admissible sets of fixed points, it is also interesting to discuss the question, on which manifold such an admissible set can be realized, i.e. we have to describe the map $\operatorname{Ker} \alpha \to \Omega \bigotimes Z/pZ$. It turns out that the admissible set

$$(x_1, \ldots, x_n) - \prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)} \bigcap (1_1, \ldots, 1_n),$$

where

$$u^k \bigcap (1_1, \ldots, 1_n) = (1_1, \ldots, 1_{n-k}),$$

is realizable on a manifold from the class

$$\left[\prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)}\right]_n \in \Omega_U^{2n} \otimes Z/pZ.$$

From the work of Atiyah–Bott [1], one can extract the following formula for the Todd genus of the manifold M^n in terms of weights of the transformation at fixed points

$$-T(M^n) \equiv \sum_j \operatorname{Tr} \left(\prod_{k=1}^{n} (1 - \exp(\frac{2\pi i x_k^j}{p}))^{-1} \right) \mod p, \tag{3}$$

where $\operatorname{Tr} : Q(\sqrt[p]{1}) \to Q$ is the number theoretic trace, and the sum in (5.3) is taken over all fixed points. It would be interesting to get an analogous Atiyah-Bott result in cobordisms. This problem is connected with the construction of a homomorphism $\gamma : A \to \Omega_U \otimes Q_p$, coinciding with $\prod_j \frac{u}{x_j \Psi^{x_j}(u)}$ on $\operatorname{Ker} \alpha$. Here by A we mean only fixed points of a submanifold with trivial bundle. As shown in [4], the formula for the homomorphism γ looks like

$$\gamma(x_1, \ldots, x_n) = \left[\frac{1}{x_1, \ldots, x_n} (\prod_{j=1}^{n} \frac{u}{\Psi^{x_j}(u)}) \frac{u}{\Psi^p(u)}\right]_n. \tag{4}$$

Applying to (5.4) the Todd genus $T:\Omega_U\to Z$, we get a numerical function

$$\gamma(x_1,\dots,x_n)=\left[\frac{pu}{1-(1-u)^p}\prod_{j=1}^{n}\frac{u}{1-(1-u)^{x_k}}\right]_n. \tag{5}$$

However, it does not coincide with the Atiyah-Bott function

$$A\mathit{Б}(x_1,\dots,x_n)=\mathrm{Tr}\left(\prod_{j=1}^{n}(1-\exp(-\frac{2\pi i x_k}{p}))^{-1}\right). \tag{6}$$

The functions (5.5) and (5.6) coincide only on $\operatorname{Ker}\alpha$. More precisely, let $\mathrm{K\Phi}(x_1,\ \dots,\ x_n)_m, 0\leqslant m\leqslant n-1$, be the composition of functions

$$\left[\frac{u}{\Psi^p(u)}\prod_{k=1}^{n}\frac{u}{x_k\Psi^{x_k}(u)}\right]_m \tag{7}$$

with the Todd genus. Note that for an admissible set of fixed points, the functions (5.7) become zero.

Theorem 5.1. (See [4].)

$$\mathrm{AБ}(x_1,\dots,x_n)=\gamma(x_1,\dots,x_n)+\sum_{m=0}^{n-1}K\varPhi(x_1,\dots,x_n)_m \mod p\mathbf{Z}p.$$

Theorem 5.1 yields that the results of Atiyah and Bott on the Todd genus of manifolds by using fixed point invariants are a reduction of an analogous result in cobordisms by using the Todd genus. It is interesting to note (as D. K. Faddeev showed) that the Atiyah-Bott formula has an expression in terms of the formal group corresponding to the multiplicative homomorphism $T:\Omega_U\to Z$, which is called the multiplicatve formal group. Namely (see [4]),

$$A\mathit{Б}(x_1,\dots,x_n)$$
$$=\sum_{m=0}^{n}\left[\frac{pu}{\langle u\rangle_p}\prod_{k=1}^{n}\frac{u}{\langle u\rangle_{x_k}}\right]_m=-\left[\frac{p\langle u\rangle_{p-1}}{\langle u\rangle_p}\prod_{k=1}^{n}\frac{u}{\langle u\rangle_{x_k}}\right]_n \mod p\,\mathbf{Z}_p,$$

where $\langle u\rangle_q$ is the q-th exponent of u in the formal group $f(u,v)=u+v-uv$.

Circle action on almost complex manifolds. In the last few years, S.Gusein–Zade studied fixed points of the circle action S^1 on almost complex manifolds. As in the case of Z_p, one can construct the Conner–Floyd exact sequence $0\to U_*(S^1)\xrightarrow{\gamma}\oplus U_*(\prod BU(n_i))\xrightarrow{\alpha}U_*(S^1,\{Z_s\}_s)\to 0$, where the middle term describes the structure of the S^1 action near the fixed points, and the last term means the bordism group with S^1 action without fixed points (stationary points are admitted). A remarkable result of S.Gusein–Zade describes the last term of this sequence. Namely,

$$U_*(S^1, \{Z_s\})_s \approx \oplus U_*(\prod_i BU(n_i) \times BU(1)). \tag{8}$$

After proving (5.8), the description of the homomorphism α can be easily reduced to the algebraic problem by using the formal group language. For convenience, we do not give these formulae here (see the description of S.Gusein−Zade's results in [21]).

Appendix I[1]

Steenrod Powers in Cobordisms and a New Method of Calculation of the Cobordism Ring of Quasicomplex Manifolds[2]

The Thom isomorphism in cobordisms. For every complex bundle ξ over X, $\dim \xi = n$, we have a well-defined Thom class $u(\xi) \in U^{2n}(M(\xi))$ corresponding to the classifying map $M(\xi) \to MU(n)$, where $M(\xi)$ is the Thom complex of the bundle ξ. The multiplication by $u(\xi)$ defines the factorial Thom isomorphism $\varphi(\xi) : U^q(X) \to \tilde{U}^{q+2n}(M(\xi)), \varphi(\xi)(\alpha) = u(\xi)\alpha$. Consider the pair of complexes $i : Y \subset X$ and denote by ξ' the restriction of ξ to Y. We have a well-defined homomorphism

$$\varphi(\xi, \xi') : \tilde{U}^q(X/Y) \to \tilde{U}^{q+2n}(M(\xi)/M(\xi')), \quad \varphi(\xi, \xi')(\alpha) = u(\xi)\alpha.$$

Since $i^* u(\xi) = u(\xi')$ and $\varphi(\xi), \varphi(\xi')$ are isomorphisms then $\varphi(\xi, \xi')$ is an isomorphism. Let ξ and η be fibers over X. Consider the composition of the mappings

$$\begin{aligned}\Delta : M(\xi + \eta)/M(\xi' + \eta') &\xrightarrow{j} M(\xi \times \eta)/M(\xi' \times \eta') \\ &\xrightarrow{\approx} (M(\xi) \wedge M(\eta))/(M(\xi') \wedge M(\eta')) \to M(\xi) \wedge (M(\eta)/M(\eta')),\end{aligned}$$

where $M(\xi \times \eta)$ is the Thom complex of the bundle $\xi \times \eta$ over $X \times X$; the map j is defined by the diagonal $X \to X \times X$ and $(X \times Y)/X \times * \cup * \times Y = X \wedge Y$, are the fixed points. We have a well-defined homomorphism

$$\Phi(\xi) : \widetilde{U}^q(M(\eta)/M(\eta')) \to \widetilde{U}^{q+2n}(M(\xi + \eta)/M(\xi' + \eta')),$$
$$\Phi(\xi)\alpha = \Delta^*(u(\xi) \cdot \alpha).$$

Since $u(\xi \times \eta) = u(\xi) \cdot u(\eta) \in U^*(M(\xi \times \eta))$ we see that $\Phi(\xi)\varphi(\eta) = \varphi(\xi + \eta, \xi' + \eta')$, consequently, $\Phi(\xi)$ is an isomorphism.

[1] The appendix is written by V. M. Buchstaber after T. Dieck [19] and D. Quillen [20].

[2] The bordism ring of almost complex manifolds was computed a long ago (Milnor, Novikov) by using Adams' spectral sequence. The aim of the new method (due to Quillen) for calculating this ring is to do without Adams' spectral sequences.

Exterior Steenrod's powers. Let $S^\infty = \lim S^{2n+1}$ be the infinite-dimensional sphere and let $S^\infty \to BZ_p = L_p^\infty$ be the universal Z_p-bundle. For every X with fixed point * denote by $E(X)$ the space $(S^\infty \cup *) \wedge \underbrace{X \wedge \ldots \wedge X}_{p\ times}$. Over $E(X)$ we have a well-defined canonical action of Z_p, acting on $X \wedge \ldots \wedge X$ by permutations. Set $E_p(X) = E(X)/Z_p$. The correspondence $X \mapsto E_p(X)$ is, evidently, factorial with respect to the mappings $X \to Y$. Over the complex $V = S^\infty \times X \wedge \ldots \wedge X$, consider the bundle $\xi \wedge \ldots \wedge \xi$ lifted from $X \wedge \ldots \wedge X$. Since the action of Z_p on V is free, the bundle $\xi_{(p)} = (\xi \wedge \ldots \wedge \xi)/Z_p \to V/Z_p$ is well-defined. One has the equation $E_p(M(\xi)) = M(E_{(p)})$.

Definition I.1. *Steenrod's exterior powers in U-coboridsms* is the set $P_e = \{P_e^{2n}, n \in \boldsymbol{Z}\}$ of natural maps $P_e^{2n} : \widetilde{U}^{2n}(X) \to \widetilde{U}^{2np}(E_p(X))$ such that:

1) $i^* P_e^{2n}(a) = a^p \in \widetilde{U}^{2np}(X \wedge \ldots \wedge X)$, where $i : X \wedge \ldots \wedge X \to E_p(X), i(x_1, \ldots, x_p) = (e, x_1, \ldots, x_p), e \in S^\infty$ is the inclusion;

2) $P_e^{2(n+m)}(ab) = T^*(P_e^{2n}(a) P_e^{2m}(b)) \in \widetilde{U}^{2(n+m)p(E_p(X \wedge Y))}$, where $a \in \widetilde{U}^{2n}(X)$, $b \in \widetilde{U}^{2m}(Y)$, $ab \in \widetilde{U}^{2(n+m)}(X \wedge Y)$ and $T : E_p(X \wedge Y) \to E_p(X) \wedge E_p(Y), T(e, x_1, y_1, \ldots, x_p, y_p) = (e, x_1, \ldots, e, y_1, \ldots, y_p)$;

3) $P_e^{2n}(u(\xi)) = u(\xi_{(p)}) \in \widetilde{U}^{2np}(M(\xi_{(p)}))$, where ξ is a bundle over X, $\dim X = n$.

It follows from the axioms that for the canonical element $u_n \in U^{2n}(MU(n))$ we have $P_e^{2n} u_n = u(\eta_{n,(p)})$, where η is the universal $U(n)$-bundle over $BU(n)$. Now, let the element $a \in \widetilde{U}^{2n}(X)$ be represented by a map $f : S^{2k} X \to MU(k+n)$. Since $S^{2k} X = M(k)/M(k')$, where k is the trivial k-dimensional bundle over X and k' is its restriction to $* \in X$ then $E_p(S^{2k} X) = M(k_{(p)})/M(k'_{(p)})$ where $k'_{(p)}$ is the restriction of $k_{(p)}$ to the subcomplex $Y \subset (S^\infty \times X \times \ldots \times X)/Z_p$ generated by points $(e, x_1, \ldots, x_p)$ for which at least one coordinate $x_i = * \in X$. Since $E_p(X) = M(0)/M(0')$ then we have a well-defined mapping $\Delta : E_p(S^{2k} X) = M(k_{(p)})/M(k'_{(p)}) \to E_p(X) \wedge E_p(S^{2k} X)$, inducing an isomorphism $\Phi(k_{(p)}) : U^*(E_p(X)) \to U^*(E_p(S^{2k} X))$, $\Phi(k_{(p)})(a) = \Delta^*(u(k_{(p)}) \cdot a)$.

Since $f^* u_{k+n} = u(k) \cdot a$ then we have $E_p(f)^*(u(\eta_{k+n,(p)})) = \Phi(k_{(p)})(P_e^{2n} a)$. From the properties of the isomorphism $\Phi(k_{(p)})$ it easily follows that the above formula defines uniquely the element $P_e^{2n}(a) \in \widetilde{U}^{2np}(E_p(X))$. Thus, the exterior Steenrod powers in cobordisms exist and they are unique.

Steenrod powers in cobordisms. The diagonal mapping $X \to X \wedge \ldots \wedge X$ defines an inclusion $i : (L_p^\infty \cup *) \wedge X = (S^\infty \cup *) \wedge X/Z_p \to E_p(X)$.

Definition I.2. By *Steenrod power* is meant the set of natural transformations $P = \{P^{2n} : \widetilde{U}^{2n}(X) \to \widetilde{U}^{2np}((L_p^\infty \cup *) \wedge X), n \in Z\}$ such that $P^{2n}(a) = i^* P_e^{2n} a$.

Let $j : BU(n) \to BU(n) \times \ldots \times BU(n)$ be the diagonal.

The inclusion $i : (L_p^\infty \cup *) \wedge MU(n) \to E_p(MU(n))$ is decomposed as $(L_p^\infty \cup *) \wedge MU(n) \xrightarrow{\lambda} M((j^*\eta_n)_{(p)}) \xrightarrow{\bar{j}} M(\eta_{n,(p)}) = E_p(MU(n))$. Let C^p be the p-dimensional complex linear space, where Z_p acts by permutations. Consider the complex bundle $\widetilde{v} = S^\infty \times_{Z_p} C^p \to L_p^\infty$. It follows straightforwardly that $M((j^*\eta_n)_{(p)})$ is the Thom space of the bundle $\widetilde{v} \otimes \eta_n \to \to L_p^\infty \times BU(n)$. Let us calculate the Chern class $\sigma_{np}(\widetilde{v} \otimes \eta_n) \in U^{2np}(L_p^\infty \times BU(n))$. Decomposing the representation of Z_p over C^p into one-dimensional factors, we see that $\widetilde{v}$ is isomorphic to the sum of bundles $1 + \sum_{q=1}^{p-1} \eta^q$, where η is the canonical bundle over L_p^∞. Now, represent η_n as a sum of formal one-dimensional bundles $\sum_{l=1}^n \mu_l$:

$$\sigma_{np}(\widetilde{v} \otimes \eta_n) = \prod_{l=1}^{n} \prod_{q=0}^{p-1} \sigma_1(\eta^q \otimes \mu_l)$$

$$= \prod_{l=1}^{n} \prod_{q=0}^{p-1} (\sigma_1(\eta^q) + \sigma_1(\mu_l) + \sum e_{i,j} \sigma_1(\eta^q)^i \sigma_1(\mu_l)^j),$$

where $e_{i,j} \in \Omega_U^{-2(i+j-1)}$ are the coefficients of geometric cobordism formal group. Denote the ring generated by the elements $e_{i,j}$, by $A \subset \Omega_U$. Since all elements $\sigma_1(\eta^q) \in U^2(L_p^\infty)$ are formal rows of $u = \sigma_1(\eta)$ with coefficients from a subring $A \subset \Omega_U$, we see that

$$\text{(I.1)} \qquad \sigma_{np}(\widetilde{v} \otimes \eta_n) = \sigma_{np}(\eta_n)(w^n + \sigma_n(\eta_n)^{p-1} + \sum w^{n-|\omega|} \alpha_\omega(u) \sigma_\omega(\eta_n)),$$

where $w = \sigma_{p-1}\left(\sum_{q=1}^{p-1} \eta^q\right)$, $\sigma_\omega(\eta_n)$ is the characteristic class corresponding to the decomposition $\omega = (i_1, \ldots, i_n)$, $|\omega| = \sum i_k$ and $\alpha_\omega(u) \in U^*(L_p^\infty)$ is a polynomial in $u = \sigma_1(\eta)$ with coefficients in A. Note that the space $(L_p^\infty U^*) \wedge MU(n)$ is the Thom complex of the bundle $\eta_n \to L_p^\infty BU(n)$, whereas the mapping of the Thom complexes $\lambda : (L_p^\infty U^*) \wedge MU(n) \to M((j^*\eta_n)_{(p)})$ is the identity on the base. Recall that the cohomology operations $S_\omega(u_n)$ can be defined as $S_\omega(u_n) = u_n \cdot \sigma_\omega(\eta_n)$. We have $P^{2n} u_n = i^* P_e^{2n} u_n = \lambda^* j^* u(\eta_{n,(p)}) = \lambda^* u((j^*\eta_n)_{(p)}) = w^n u_n + u_n^p + \sum w^{n-|\omega|} \alpha_\omega(u) S_\omega(u_n)$. Here we used the fact that the restriction of the Thom class $u(\xi)$ to the zero section of the bundle ξ gives, by definition, the characteristic class $\sigma_n(\xi)$, where $n = \dim$.

Theorem I.1. *Assume the element* $a \in U^{2n}(X)$ *is represented by the mapping* $f : S^{2k}X \to MU(k+n)$*; then in the ring* $U^*(L_p^\infty \times X)$ *the following formula holds:*

$$w^k P^{2n} a = w^{n+k} a + \sum w^{n-|\omega|} \alpha_\omega(u) S_\omega(a),$$

where $w = \sigma_{p-1}(\sum_{q=1}^{p-1} \eta^q) \in U^*(L_p^\infty)$ *and* $\alpha_\omega(u) \in U^*(L_p^\infty)$ *are polynomials in* u *with coefficients from the ring* A*, generated by formal group multiplication laws of the geometric cobordism group.*

Proof. Let $u(k) \in \widetilde{U}^{2k}(S^{2k}) = Z$ be a generator. We have $f^* u_{k+n} = u(k) \cdot a$. Consequently, $f^* P^{2(k+n)} u_{k+n} = T^*(P^{2k} u(k) \times P^{2n}(a))$, where $T : (L_p^\infty \cup *) \wedge S^{2k} X \to (L_p^\infty \cup *) \wedge S^{2k} \wedge (L_p^\infty \cup *) \wedge X$. The element $u(k)$ is represented by the inclusion $S^{2k} \subset MU(k)$, thus, $P^{2k} u(k) = w^k u(k)$. Now, using the formula for $P^{2(k+n)} u_{k+n}$, we get the proof of the theorem.

Calculating the bordism ring for almost complex manifolds. Standard arguments from homotopy theory not using any information about the ring Ω_U show that if a canonical mapping $\mu : U^*(X) \to H^*(X, Z)$ is an epimorphism and the group $H^*(X, Z)$ is torsion-free then the group $U^*(X)$ is a free Ω_U-module (see [8], Appendix). Since the construction of characteristic classes $\sigma_\omega(\xi)$ in cobordisms is also independent of the results concerning the ring Ω_U and, moreover, $\mu\sigma_\omega(\xi) = c_\omega(\xi) \in H^{2|\omega|}(X, Z)$, where c_ω are Chern characteristic classes (see [6], Appendix), we see that the groups $U^*(BU(n) \times BU(k))$ and $U^*(MU(n))$, $n \geqslant 1$, $k \geqslant 1$ are free Ω_U-modules. In particular, $U^*(CP^\infty \times CP^\infty) = \Omega_U[[u, v]]$, where u, v are the first Chern classes in cobordisms of canonical one-dimensional bundles η_1 and η_2. Let $A = \mathbf{Z}[y_1, \ldots, y_n]$ be the ring of coefficients of the universal Lazard group and let $\varphi : A \to \Omega_U$ be the ring homomorphism corresponding to the formal group of the geometric cobordisms

$$F(u, v) = \sigma_1(\eta_1 \otimes \eta_2) \in U^2(CP^\infty \times CP^\infty).$$

In § 3, a direct calculation shows that the coefficients of the logarithm $g(u)$ of $F(u, v)$ algebraically independent. But since the coefficients of the logarithm of the Lazard group generate the ring $A \otimes Q$, we get that φ is a monomorphism.

Consider the formal series $\Theta_p(u) = \dfrac{[u]_p}{u} = p + \alpha_1 u + \ldots$ over $\Omega_U[[u]]$, where $[u]_p$ is p-th power of the element u in the formal group.

Lemma I.1. *The following sequence is exact:*

$$\Omega_U \xrightarrow{\Theta_p(u)} U^q(L_p^\infty) \xrightarrow{u} U^{q+2}(L_p^\infty),$$

where $u = \sigma_1(\eta), \eta$ *is the canonical fiber bundle over* L_p^∞, *and the homomorphisms in the sequence are multiplications by* $\Theta_p(u)$ *and* u.

PROOF. Denote by η the canonical bundle over CP^∞. Besides, consider the bundle $\eta^p \to CP^\infty$ and denote by $E \to CP^\infty$ the D^2-bundle associated with η^p. We have $\partial E = L_p^\infty, E/\partial E = M(\eta^p)$. A detailed consideration of the homomorphism $\mu : U^*(L_p^\infty) \to H^*(L_p^\infty, Z)$ yields that the homomorphism $U^*(E) = U^*(CP^\infty) \to U^*(L_p^\infty)$ is an epimorphism, and since $\sigma_1(\eta^p) = [u]_p$, we get that there is an exact sequence $0 \leftarrow U^*(L_p^\infty) \leftarrow U^*(CP^\infty) \xleftarrow{[u]_p} \widetilde{U}^*(M(\eta^p)) \leftarrow 0$. The proof of the lemma follows now from the fact that the multiplication by u homomorphism in $U^*(CP^\infty) = \Omega_U[[u]]$ is indeed a monomorphism.

Theorem I.2. *The homomorphism* $\varphi : Z[y_1, \dots, y_n, \dots] \to \Omega_U$ *from the Lazard group to* Ω_U *corresponding to the geometrical cobordism group is an isomorphism.*

PROOF. It remains to show that φ is an epimorphism. Set $C = \operatorname{Im}\varphi \subset \Omega_U$; let us show that for any $n \geqslant 1$ there is an isomorphism $U^*(S^n) = C\sum_{q\geqslant 0} U^q(S^n)$. First note that because of the isomorphism $U^q(S^n) \simeq U^{q+1}(S^{n+1})$ and the fact that $U^q(S^n)$ is finitely generated for every q, it suffices to show that for every prime p there exists an isomorphism $\widetilde{U}^{ev}(S^n) \otimes Z_p = C \cdot \sum_{q\geqslant 0} U^{2q}(S^n) \otimes Z_p$. Set $R_p = C \cdot \sum_{q\geqslant 0} U^{2q}(S^n) \otimes Z_p$. Assume for all $j < q$ we have already proved the isomorphism $R_p^{-2j} = \widetilde{U}^{-2j}(S^n) \otimes Z_p$. For $j = 0$ this isomorphic is evident. Assume the element $a \in \widetilde{U}^{-2q}(S^n)$ is represented by a mapping $f : S^{2k}S^n \to MU(k-q)$; then, by Theorem I.1, the following formula takes place:

$$w^k P^{-2q} a = w^{k-q} a + \sum w^{n-|\omega|} \alpha_\omega(u) \cdot S_\omega(a). \tag{I.2}$$

The element $w \in U^{2np}(L_p^\infty)$ is a formal series of type $(p-1)! \times u^{p-1} + O(u^p)$ with coefficients in C. By the induction hypothesis, $S_\omega(a) \in R_p, |\omega| > 0$, and we get from (I.2) that for some m the following holds:

$$u^m(w^q P^{-2q} a - a) = \psi(u) \in U^*(L_p^\infty \times S^n) \approx U^*(L_p^\infty) \otimes_{\Omega_U} U^*(S^n), \tag{I.3}$$

where $\psi(u) \in R_p[[u]]$. Assume that $m \geqslant 1$ is the least such number for which the formula (I.3) holds. Since $\psi(0) = 0$ then $\psi(u) = u\psi_1(u)$, $\psi_1(u) \in R_p[[u]]$, and we get $u(u^{m-1}(w^q P^{-2q} a - a) - \psi_1(u)) = 0$. Then by lemma I.1 there exists an element $y \in U^*(S^q)$ such that

$$u^{m-1}(w^q P^{-2q} a - a) = \psi_1(u) + y\Theta_p(u) \in U^*(L_p^\infty \times S^n). \tag{I.4}$$

Considering the restriction of this inequality to $U^*(L_p^\infty)$, we get $y'\Theta_p(u)=0$, where $y' = \varepsilon(y), \varepsilon : U^*(S^n) \to U^*(*)$. Consequently, if $m > 1$ then, by the induction hypothesis, $y \cdot \Theta_p(u) \in R_p[[u]]$, which contradicts the minimality of m. But, if $m = 1$, then, considering the restriction of (I.4) to $U^*(S^n)$, we get $-a = \psi_1(0) + py$, i.e. $a \in R_p$.

Appendix II[1]

The Adams Conjecture

The Adams conjecture concerns the calculation of the image of the J-homomorphism in the real K-theory (see [13]). The exact formulation says: for every bundle ξ there exists such an integer N that $J(k^N(\Psi^k(\xi)-\xi)) = 0$ for any $k \geqslant 1$. The Adams conjecture allows one to construct an upper bound for the image of the J-homomorphism. It has been known that the Adams conjecture is true for one-dimensional and orientable two-dimensional bundles, and their connected sums. To prove the Adams conjecture, it suffices to check it only for classifying bundles on Grassmann manifolds. We shall give a proof outline of the Adams conjecture, following Sullivan[2]. The main idea is to map the K-functor to some other functor for which the Adams operations Ψ^k, preserve the dimension of "the bundle". The exact sense is as follows.

Lemma II.1. *Let B_n be a sequence of complexes, $\gamma_n : E_n \to B_n$ be spherical bundles with fiber S^{n-1}, $f_n : B_n \to B_{n-1}$ be mappings such that $f_n^*(\gamma_{n+1}) = \gamma_n \oplus 1, f_n \sim \gamma_{n+1} \cdot h_n$, where $h_n : B_n \to E_{n+1}$ is a homotopy equivalence. Let $a_n : B_n \to B_n$ be a stable operation of the functor $\varinjlim[\,, B_n]$, i.e. $f_n a_n \sim a_{n-1} f_n$, which is invertible. Then if $J_n : B_n \to BG_n$ is a natural J-map, $G_n \approx (\Omega^{n-1}S^{n-1})_0$, then $J_n \sim J_n a_n$.*

Note that the operations Ψ^k for $B_n = BO(n)$ do not satisfy the assumptions of the lemma. Sullivan found an acceptable theory $K(\widehat{X})$, where some analogues of the operations Ψ^k satisfy the conditions of the lemma. Let X be an arbitrary CW-complex. By the completion $\widehat{X}$ of X we mean such a (unique) complex for which the condition $[Y, (\widehat{X})] = \lim_{\{F,f\}}[Y, F]$ holds. Here $\{F, f\}$ is the category of all mappings $f : X \to F$, where F runs over

[1] Appendix II is written by A. S. Mishchenko.

[2] The proof of Adams' conjecture started three years ago from Quillen's idea (Quillen D., Some remarks on etale homotopy, theory and a conjecture of Adams. Topology (1968), 7, No. 2, 111–116): apply the properties of etale-topology of Grassmann manifolds. In 1970, together with Sullivan's proof given below, Quillen constructed a proof based on the reduction of the Adams conjecture for the bundles with finite structure group; this proof differs from his first original idea.

complexes with all homotopy groups finite. Then we set $K(\widehat{X}) = [X, BO\widehat{\ }]$. It is easy to see that the space $BO(n)\widehat{\ }$ satisfies the conditions of the lemma if we take for γ_n the fibration with fiber $(S^{n-1})\widehat{\ }$. On the other hand, since all homotopy groups BG_n are finite then there exists a natural mapping $J\widehat{\ } : BO(n)\widehat{\ } \to BG(n)$, and thus the diagram

$$\begin{array}{ccc} BO & \xrightarrow{\widehat{\ }} & BO\widehat{\ } \\ J \big\downarrow & & \big\downarrow J\widehat{\ } \\ & BG & \end{array}$$

is commutative. Finally, one can define the operations $(\Psi^k)\widehat{\ }$ in the groups $K\widehat{\ }(X)$ in such a way that $(\Psi^k x)\widehat{\ } = (\Psi^k)\widehat{\ }(x\widehat{\ })$. If we prove that the operations $(\Psi^k)\widehat{\ }$ preserve the geometrical dimension of fibres, i.e. that there exist mappings $(\Psi^k_n)\widehat{\ } : BO(n)\widehat{\ } \to BO(n)\widehat{\ }$ such that $(\Psi^k)\widehat{\ } = \lim(\Psi^k_n)\widehat{\ }$, then the lemma yields the Adams conjecture. For the proof of the last statement, Sullivan uses the fact that the Grassmann manifolds $G_{n,k}$ are algebraic manifolds over the field of rational numbers. Thus, on the manifold $G_{n,k}$ we get a Galois group $\mathrm{Gal}(C, Q)$ action. It turns out that the induced action in the etale homology with coefficients in the finite group is defined only by representation of the group $\mathrm{Gal}(C, Q)$ in the permutation group of all roots of unity, i.e. by the homomorphism $\mathrm{Gal}(C, Q) \to (\widehat{Z})^*$. Together with the Artin theorem on the isomorphism of etale cohomology with coefficients in finite group with the usual cohomology of the manifold, we get the actions of the group $(\widehat{Z})^*$ on the space $(G_{n,k})\widehat{\ }$. It can be easily checked that the action of the element $(k) \in (\widehat{Z})^*$ coincides with the operation $(\Psi^k)\widehat{\ }$.

References

[1] M. F. Atiyah and R. Bott. Notes on the Lefschetz fixed point for elliptic complexes, Cambridge, Mass. (1964).

[2] V. M. Buchstaber. The Chern-Dold Character in Cobordisms. I, Mat. Sb., 83(125),(1970), 575–595 (Math. USSR Sb., **12** (1970)).

[3] V. M. Buchstaber. Spectral sequences connected with the Chern-Dold Character, Uspekhi Mat. Nauk (157) (1971), 575–595.

[4] V. M. Buchstaber, S. P. Novikov. Formal groups, power systems and Adams operators. Mat. Sb. 84, (1971), 81–118.

[5] G. G. Kasparov. Invariants of classical lens manifolds in cobordism theory. Math USSR Izv, 3 (1969), 695–706.

[6] P. Conner and E. Floyd. Differentiable periodic maps, Ergebnisse der Mathematik und Ihrer Grenzgebiete, N.F., Band 33, Springer-Verlag, Berlin, 1964.

[7] D. Quillen. On the formal group law of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. 75:6 (1969), 1293–1298.

[8] J. Milnor, "Lectures on characteristic classes" , Princeton Univ. Press (1957) (Notes by J. Stasheff).

[9] J. Milnor. Cobordism ring and complex analogue, Amer. J. Math. 82:3 (1960), 505–521.

[10] A. S. Mishchenko. Manifolds under the action of the group Z_p and fixed points, Mat. Zametki 4, No. 4 (1968), 381–386 (in Russian).

[11] A. S. Mishchenko. Bordisms with the action of the group Z_p and fixed points, Math USSR Sb., 9 (1969), 291–296.

[12] S. P.,Novikov. Homotopy properties of Thom complexes. Mat. Sb. 1962, v. 57, N 4, 406–442.

[13] S. P. Novikov. New ideas in algebraic topology (K-theory and its applications). Uspekhi Mat. Nauk, 1965, v. 20, N 3, 41–66.

[14] S. P. Novikov. Methods of algebraic topology from the point of view of cobordism theory. Izv. Akad. Nauk SSSR, 1967, v. 31, N 4, 885–951.

[15] S. P. Novikov, Adams operators and fixed points// Izv. Akad. Nauk SSSR, V. 32, 1968, P. 1245–1263.

[16] Thom R., Quelques proprietés globales des variétés differentiables, *Comm. Math. Helv.*, 28 (1954), 17–86.

[17] T. Honda. Formal Groups and zeta-functions, Osaka J. Math. **5** (1968), 199–213.

[18] Chern S.-S. Complex manifolds.-Chicago: Univ. Press, 1956.-195 p.

[19] T. Dieck. Steenrod – Operationen in Kobordism-Theorien, Math. Z. 107 (1968).

[20] D. Quillen. Elementary proofs of some results of cobordism theory using Steenrod operation (preprint).

[21] V. M. Buchstaber, Two-valued formal groups. Some applications to cobordism, Uspekhi Mat. Nauk 26:3. (1971), 195–196.

[22] P. S. Landweber. Cobordism operations and Hopf algebras, Trans. Amer. Math. Sos. 129:1 (1967), 94–110.

[23] J. F. Adams. Quillen's Work on Formal Group Law and Complex Cobordism, University of Chicago, Lecture Notes Ser., 1970.

This page intentionally left blank

8

Formal groups, power systems and Adams operations

V. M. Buchstaber, S. P. Novikov [1]

The theory of one-dimensional commutative formal groups at present consists of three parts:

1) The general theory based on Lazard's theorem [8] on the existence of a universal formal group; the coefficient ring of this formal group is the polynomial ring over integers.

2) Formal groups over arithmetic rings and fields of finite characteristic — for a survey of this theory see [4].

3) Commutative formal groups in cobordism theory and in the theory of cohomology operations and characteristic classes [5], [9], [12], [13], [14].

Quillen has recently shown that the formal group $f(u, v)$ of "geometrical cobordisms" is universal [14]. His proof makes use of Lazard's theorem on the existence of a universal formal group whose ring of coefficients is a torsion-free polynomial ring.

In the first section of this paper we prove the universality of the group of "geometrical cobordisms" directly by starting from its structure, as investigated in Theorem 4.8 of [2], without recourse to Lazard's theorem. Moreover, in § 1 we give formulas for calculating the cohomology operations in cobordism by means of the Hirzebruch index.

In connection with the theory of Adams operations in cobordism, the operation of "raising to powers" in formal groups is of particular importance (see [12], [13]). This operation can be axiomatized and studied for its own sake; in addition there are topologically important power systems which

[1] Math. USSR. Sbornik, vol 13 (1971), No.1., originally: В.М.Бухштабер, С.П.Новиков, Формальные группы, степенные системы и операторы Адамса, Математический сборник, 1971, T. 84 (126), No. 1, c. 81–118 (поступила в редакцию 9.06.1970). Translated by M.L. Glasser with further edition by V.O.Manturov

do not lie within formal groups, see § 2a. In § 2b we examine a distinctive "two-valued formal group" which is closely connected with simplicial cobordism theory. § 3 and the Appendix are devoted to the systematization and development of the application of formal groups to fixed point theory.

§ 1. Formal groups

First of all we introduce some definitions and general facts concerning the theory of formal groups. All rings considered in this paper are presumed to be commutative with unit.

Definition 1.1. A *one-dimensional formal commutative group* F over a ring R is a formal power series $F(x,y) \in R[[x,y]]$ which satisfies the following conditions:

a) $F(x,0) = F(0,x) = x$;

b) $F(F(x,y),z) = F(x,F(y,z))$;

c) $F(x,y) = F(y,x)$.

In the following a formal series $F(x,y)$ which satisfies axioms a), b) and c), will simply be called a *formal group*.

Definition 1.2. A *homomorphism* $\varphi\colon F \to G$ *of formal groups* over R is a formal series $\varphi(x) \in R[[x]]$ such that $\varphi(0) = 0$ and $\varphi(F(x,y)) = G(\varphi(x),\varphi(y))$.

If a formal series $\varphi_1(x)$ determines the homomorphism $\varphi_1\colon F \to G$ and a formal series $\varphi_2(x)$ determine the homomorphism $\varphi_2\colon G \to H$, it follows immediately from Definition 1.2 that the formal series $\varphi_2(\varphi_1(x))$ determines the composite homomorphism $\varphi_2 \cdot \varphi_1\colon F \to H$.

For formal groups F and G over R we denote by $\mathrm{Hom}_R(F,G)$ the set of all homomorphisms from F into G. With respect to the operation

$$(\varphi_1 + \varphi_2)(x) = G(\varphi_1(x), \varphi_2(x)), \quad \varphi_1, \varphi_2 \in \mathrm{Hom}_R(F,G),$$

the set $\mathrm{Hom}_R(F,G)$ is an abelian group.

By $T(R)$ we denote for any ring R the category of all formal groups over R and their homomorphisms. It is not difficult to verify that the category $T(R)$ is semi-additive, i.e. for any $F_1, F_2, F_3 \in T(R)$ the mapping

$$\mathrm{Hom}_R(F_1,F_2) \times \mathrm{Hom}_R(F_2,F_3) \to \mathrm{Hom}_R(F_1,F_2),$$

defined by composition of homomorphisms, is bilinear.

Let $F(x,y) = x + y + \sum \alpha_{i,j} x^i y^j$ be a formal group over R_1 and let $r\colon R_1 \to R_2$ be a ring homomorphism. Let $r[F]$ be the formal series

$$r[F](x,y) = x + y + \sum r(\alpha_{i,j}) x^i y^j,$$

which is clearly a formal group over R_2. If the series $\varphi(x) = \sum \alpha_i x^i$ gives the homomorphism $\varphi\colon F \to G$ of formal groups over R_1 then the formal series $r[\varphi](x) = \sum r(\alpha_i) x^i$ gives the homomorphism $r[\varphi]\colon r[F] \to r[G]$ of formal groups over R_2. Thus any ring homomorphism $r\colon R_1 \to R_2$ provides a functor from the category $T(R_1)$ to the category $T(R_2)$. Summing up, we may say that over the category of all commutative rings with unit we have a functor defined which associates with each ring R the semi-additive category $T(R)$ of all one-dimensional commutative formal groups over R.

Let R be a torsion-free ring and $F(x, y)$ a formal group over it. As was shown in [8] (see also [4]), there exists a unique power series $f(x) = x + \sum \frac{a_n}{n+1} x^{n+1}$, $a_n \in R$, over the ring $R \otimes Q$ such that

$$F(x, y) = f^{-1}(f(x) + f(y)). \tag{1}$$

Definition 1.4. The power series $f(x) = x + \sum \frac{a_n}{n+1} x^{n+1}$, $a_n \in R$ satisfying (1), is called the *logarithm of the group* $F(x, y)$ and is denoted by $g_F(x)$.

In [4] the notion of an invariant differential on a formal group $F(x, y)$ over a ring R was introduced, and it is shown there that the collection of all invariant differentials is the free R-module of rank 1 generated by the form $\omega = \psi(x)dx$, where $\psi(x) = \left(\left[\frac{\partial F(x,y)}{\partial y}\right]_{y=0}\right)^{-1}$. By the invariant differential on the group $F(x, y)$ we shall mean the form ω.

If the ring R is torsion-free then $\omega = dg_F(x)$. We point out that it was demonstrated in [12] that the logarithm of the formal group $f(u, v)$ of "geometrical cobordisms" is the series $g(u) = u + \sum \frac{[CP^n]}{n+1} u^{n+1}$. Consequently, for the group $f(u, v)$ we have

$$\omega = dg(u) = \left(\sum_{n=0}^{\infty} [CP^n] u^n\right) du = CP(u)du.$$

Let $f(u, v) = u + v + \sum e_{i,j} u^i v^j$, $e_{i,j} \in \Omega_U^{-2(i+j-1)}$, be the formal group of geometrical cobordism.

Lemma 1.5. *The elements $e_{i,j}$, $1 \leqslant i < \infty$, $1 \leqslant j < \infty$, generate the whole cobordism ring Ω_U.*

Proof. From the formula for the series $f(u, v)$ given in [2] (Theorem 4.8), we obtain

$$e_{1,1} = [H_{1,1}] - 2[CP^1], \quad e_{1,i} \approx [H_{1,i}] - [CP^{i-1}], \quad i > 1,$$
$$e_{i,j} \approx [H_{i,j}], \quad i > 1,\ j > 1,$$

where the sign $\approx$ denotes equality modulo factorizable elements in the ring Ω_U. Since $s_1([H_{1,1}]) = 2$, we have $e_{1,1} = -[CP^1]$; since $s_{i-1}([H_{1,i}]) = 0$ for any $i > 1$, we have $e_{1,i} \approx -[CP^{i-1}]$. According to the results in [10], [11], the elements $[H_{i,j}]$, $i > 1$, $j > 1$, and $[CP^i]$ generate the ring Ω_U. This proves the lemma.

Theorem 1.6. (Lazard–Quillen). *The formal group of geometrical cobordisms $f(u, v)$ over the cobordism ring Ω_U is a universal group, i.e. for any formal group $F(x, y)$ over any ring R there is a unique ring homomorphism $r\colon \Omega_U \to R$ such that $F(x, y) = r[f(u, v)]$.*

We show first that for a torsion-free ring R Theorem 1.6 is an easy consequence of Lemma 1.5. Let R be a torsion-free ring and let F be an arbitrary formal group over it, and let

$$g_F(x) = x + \sum \frac{a_n}{n+1} x^{n+1}, \quad a_n \in R.$$

Consider the ring homomorphism $r\colon \Omega_U \to R \otimes Q$ such that $r([CP^n]) = a_n$. We have $r[g_f] = g_F$, and, since $F(x, y) = g_F^{-1}(g_F(x) + g_F(y))$ and $f(x, y) = g_f^{-1}(g_f(x) + g_f(y))$, also $r[f(x, y)] = F(x, y)$. Consequently $r(e_{i,j}) \in R$. By now applying Lemma 1.5, we find that $\operatorname{Im} r \subset R \subset R \otimes Q$. This proves Theorem 1.6 for torsion-free rings.

Proof of Theorem 1.6. Recall that by $s_n(e)$, $e \in \Omega_U^{-2n}$, we denote the characteristic number corresponding to the characteristic class $\sum t_i^n$. It follows from the proof of Lemma 1.5 that for any $i > 1$, $j > 1$ we have the formula $s_{i+j-1}(e_{i,j}) = -C_{i+j}^i$. It is known that the greatest common divisor of the numbers $\{C_n^i\}_{i=1,\dots,(n-1)}$ is equal to 1, if $n \neq p^l$ for any prime $p \geqslant 2$, and is equal to p, if $n = p^l$. Consequently, a number $\lambda_{i,n}$ exists such that

$$\sum \lambda_{i,n} C_n^i = \begin{cases} 1, & \text{if } n \neq p^l, \\ p, & \text{if } n = p^l. \end{cases} \tag{2}$$

For each n, let us consider a fixed set of numbers $(\lambda_{i,n})$ which satisfy (2). From [10] and [11] we have that the elements $y_n = \sum \lambda_{i,n} e_{i,n-i} \in \Omega_U^{-2(n-1)}$, $n = 2, 3, \dots$, form a multiplicative basis for the ring Ω_U.

Let R be an arbitrary ring and let $F(x, y) = x + y + \sum \alpha_{i,j} x^i y^j$ be a formal group over this ring. We define the ring homomorphism $r\colon \Omega_U \to R$ by the formula $r(y_n) = \sum \lambda_{i,n} \alpha_{i,n-i}$; we shall show that $r(e_{i,n-i}) = \alpha_{i,n-i}$

for any i, n. From the commutative property of the formal group F it follows that $\alpha_{i,n-i} = \alpha_{n-i,i}$; from associativity we have

$$C^i_{i+j}\alpha_{i+j,k} - C^j_{j+k}\alpha_{j+k,i} = P(\alpha_{m,l}),$$

where $P(\alpha_{m,l})$ is a polynomial in the elements $\alpha_{m,l}$, $m + l < i + j + k$. It is clear that the form of the polynomial P does not depend on the formal group $F(x, y)$. Since $e_{1,1} = y_1$, we have $r(e_{1,1}) = \alpha_{1,1}$. We assume that for any number $n < n_0$ the equation $r(e_{i,n-i}) = \alpha_{i,n-i}$ is already proved. We have

$$C^i_{i+j}e_{i+j,k} - C^j_{j+k}e_{j+k,i} = P(e_{m,l}), \quad m + l < n_0 = i + j + k,$$

$$r(P(e_{m,l})) = P(r(e_{m,l})) = P(\alpha_{m,l}) = C^i_{i+j}\alpha_{i+j,k} - C^j_{j+k}\alpha_{j+k,i}.$$

It follows from the number-theoretical properties of C^q_p that for any $i_0 \geqslant 1$ and $n_0 = i_0 + j_0 + k_0$ the element α_{i_0,j_0+k_0} can be represented as an integral linear combination of the elements $r(y_{n_0}) = \sum \lambda_{i,n_0}\alpha_{i,j+k}$ and $r(P(e_{m,l})) = C^i_{i+j}\alpha_{i+j,k} - C^j_{j+k}\alpha_{j+k,i}$. Since the form of this linear combination depends neither on the ring R nor on the formal group F, we find that $r(e_{i_0,n_0-i_0}) = \alpha_{i_0,n_0-i_0}$. This concludes the induction, and Theorem 1.6 is proved.

It will be useful to indicate several important simple consequences of Theorem 1.6.

1. *In the class of rings R over $\mathbf{Z}_p$ the formal group $f(u, v) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ over the ring $\Omega_U \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is universal.*

2. *In the class of formal groups over graded rings the formal group of geometrical cobordisms $f(u, v)$, considered as having the natural grading of cobordism theory, is universal.*

In this case $\dim u = \dim v = \dim f(u, v) = 2$. Therefore the above refers to the class of formal groups F over commutative even-graded rings R, where $R = \sum\limits_{i \geqslant 0} R^{-2i}$, and all components of the series $F(u, v)$ have dimension 2. Of course, the general case of a graded ring reduces to the latter through the multiplication of the grading by a number.

3. The semigroup of endomorphisms of the functor T, which assigns to a commutative ring R the set $T(R)$ of all commutative one-dimensional formal groups over R, is denoted by A^T. *This semigroup A^T coincides with the semigroup of all ring automorphisms $\Omega_U \to \Omega_U$. In the graded case we refer to the functor as T_{gr}, and the semigroup as A^T_{gr}, which coincides with the semigroup of all dimension preserving homomorphisms $\Omega_U \to \Omega_U$. The "Adams operators" $\Psi^k \in A^T_{gr}$ form the center of the semigroup A^T_{gr}. The application of these operators Ψ^k to a formal group $F(x, y)$ over any ring*

proceeds according to the formula

$$\Psi^k F(x,y) = x + y + \sum k^{i+j-1}\alpha_{i,j}x^i y^j,$$

where $F(x,y) = x + y + \sum \alpha_{i,j}x^i y^j$.

We note that the semigroup A^0 of all multiplicative operations in U^*-theory is naturally imbedded in the semigroup A^T_{gr} (see [12], Appendix 2) by means of the representation (*) of the ring A^U over Ω_U; the elements of A^T are given in the theory of characteristic classes by rational "Hirzebruch series"

$$K(1+u) = Q(u), \quad Q(u) = \frac{u}{a(u)}, \quad a(u) = u + \sum_{i\geqslant 1}\lambda_i u^i, \quad \lambda_i \in \Omega_U \otimes Q.$$

What sort of Hirzebruch series give integer homomorphisms $\Omega_U \to \Omega_U$, i.e. belong to A^T? How does one distinguish $A^0 \subset A^T_{gr}$?

From the point of view of Hirzebruch series the action of a series $a = a(u) = u + \sum \lambda_i u^{i+1}$, $\lambda_i \in \Omega_U \otimes Q$, on the ring Ω_U is determined by the formula

$$a([CP^n]) = \left[\left(\frac{u}{a(u)}\right)^{n+1}\right]_n,$$

where $[f(u)]_n$ denotes the nth coefficient of the series $f(u)$. Note that $a^{-1}(u) = u + \sum \dfrac{a([CP^n])}{n+1}u^{n+1}$, where $a^{-1}(a(u)) = u$. This formula is proved in [13] (see also [2]) for series $a(u)$ giving homomorphisms $\Omega_U \to \mathbf{Z}$, and carries over with no difficulty to all series which give homomorphisms $\Omega_U \to \Omega_U$.

One should point out that the indicated operation $\Bigl($ in the "Hirzebruch genus" $Q(u) = \dfrac{u}{a(u)}$ sense $\Bigr)$ of a series $a(u)$ on Ω_U does not coincide with the operation (*) of the series $a(u) \in \Omega \otimes Q[[u]]$ on the ring $\Omega_U = U^*$(point), which defines a multiplicative cohomology operation in U^*-theory (see [12]). For example, for $a(u) = u$ we have $a([CP^n]) = 0$, $n \geqslant 1$, and $a^*([CP^n]) = [CP^n]$; as is proved in cobordism theory (see [12], [2]), for the series $a(u) = g(u) = \sum \dfrac{[CP^n]}{n+1}u^{n+1}$ we have the formula $a^*([CP^n]) = 0$, $n \geqslant 1$, and for the series $a(u) = g^{-1}(u)$ the formula $a([CP^n]) = [CP^n]$.

There arises the transformation of series of the ring $\Omega \otimes Q[[u]]$

$$\varphi\colon a(u) \to \varphi a(u),$$

defined by the requirement $a[x] = (\varphi a)^*[x]$ for all $x \in \Omega_U$, where we already know that $\varphi u = g(u)$ and $\varphi(g^{-1}(u)) = u$. We have

Theorem 1.7. *The transformation of series of the ring* $\Omega_U \otimes Q[[u]]$

$$g\colon a(u) \to a(g(u))$$

has the same properties as

$$a[x] = a(g)^*[x]$$

for any element $x \in \Omega_U$, *where* $g(u) = \sum \frac{[CP^n]}{n+1} u^{n+1}$, $a[CP^n] = \left[\left(\frac{u}{a(u)}\right)^{n+1}\right]_n$ *and* $b^*[x]$ *is the result of application to the element* $x \in U^*(\text{point}) = \Omega_U$ *of the multiplicative operation* b *from* $A^U \otimes Q$, *given by its value* $b(u) = u + \sum \lambda_i u^i$, $\lambda_i \in \Omega_U \otimes Q$, *on the geometrical cobordism* $u \in U^2(CP^\infty)$.

Proof. Let $b \in A^U \otimes Q$ be a multiplicative operation and let $\tilde{b}(\xi)$ be the exponential characteristic class of the fiber ξ with values in U^*-theory, which on the Hopf fiber η over CP^n, $n \leqslant \infty$, is given by the series $\tilde{b}(\eta) = \left(\frac{b(u)}{u}\right)^{-1}$; let $u \in U^2(CP^n)$ be a geometrical cobordism. As was shown in [12], for any U-manifold X^n we have the formula

$$b^*([X]) = \varepsilon D\tilde{b}(-\tau(X^n)),$$

where $[X^n] \in \Omega_U^{-2n}$ is the bordism class of the manifold X^n, $\varepsilon\colon X \to$(point), D is the Poincaré-Atiyah duality operator, and τ is the tangent bundle.

By using the formulas $Du^k = (CP^{n-k}) \in U_{2n-k}(CP^n)$, $n < \infty$, and $\tau(CP^n) + 1 = (n+1)\eta$, we obtain

$$\begin{aligned} b^*([CP^n]) &= \sum_{k=0}^{n} [CP^k] \left[\left(\frac{u}{b(u)}\right)^{n+1}\right]_{n-k} \\ &= \sum_{k=0}^{\infty} \frac{[CP^k]}{2\pi i} \int\limits_{|u|=\varepsilon} \left(\frac{u}{b(u)}\right)^{n+1} \frac{du}{u^{n+1-k}}; \end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{b^*([CP^n])}{n+1}t^{n+1} &= \int_0^t \sum_{n=0}^{\infty} b^*([CP^n])t^n dt \\
&= \int_0^t \left(\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{[CP^k]}{2\pi i}\int\limits_{|u|=\varepsilon}\frac{u^k du}{b(u)^{n+1}}\right)t^n dt \\
&= \frac{1}{2\pi i}\int\limits_{|u|=\varepsilon}\sum_{k=0}^{\infty}[CP^k]u^k\left(\int_0^t\sum_{n=0}^{\infty}\left(\frac{t}{b(u)}\right)^n\frac{dt}{b(u)}\right)du \\
&= \frac{1}{2\pi i}\int\limits_{\substack{|u|=\varepsilon\\ |t|<|b(u)|}} -\ln\left(1-\frac{t}{b(u)}\right)\left(\sum_{k=0}^{\infty}[CP^k]u^k du\right) \\
&= \frac{1}{2\pi i}\int\limits_{\substack{|u|=\varepsilon\\ |t|<|b(u)|}} -\ln\left(1-\frac{t}{b(u)}\right)dg(u),
\end{aligned}$$

where $dg(u)$ is the invariant differential of the formal group $f(u,v)$. By setting $g(u)=v$, we obtain from the formula for the inversion of series

$$\int\limits_{\substack{|u|=\varepsilon\\ |t|<|b(g^{-1}(v))|}} -\ln\left(1-\frac{t}{b(g^{-1}(v))}\right)dv = (b(g^{-1}(v)))^{-1}(t) = g(b^{-1}(t)).$$

Thus

$$\sum_{n=0}^{\infty}\frac{b^*([CP^n])}{n+1}t^{n+1} = g(b^{-1}(t)).$$

On the other hand, as was indicated above, we have the formula

$$\sum\frac{a([CP^n])}{n+1}t^{n+1} = a^{-1}(t).$$

Consequently, if $b(u)=a(g(u))$, then $b^*([CP^n])=a([CP^n])$ for any n. Since the elements $\{[CP^n]\}$ generate the entire ring $\Omega_U\otimes Q$, the theorem is proved.

Another proof of Theorem 1.7 can be obtained from the properties of the Chern–Dold character ch_U (see [2]). Let $\varphi\colon \Omega_U\to\Omega_U$ be a ring homomorphism and let $a(u)=u+\sum\lambda_i u^i$ be the corresponding Hirzebruch genus. We shall show that if the multiplicative operation $b\in A^U\otimes Q$ acts on the ring $U^*(\text{point})=\Omega_U$ as a homomorphism φ, then its value on the geometrical cobordism $u\in U^2(CP^\infty)$ is equal to the series $a(g(u))$, where

$g(u) = u + \sum \frac{[CP^n]}{n+1} u^{n+1}$. We have

$$\mathrm{ch}_U(u) = t + \sum \alpha_i t^{i+1} \in \mathscr{H}^*(CP^\infty, \Omega_U \otimes Q),$$

$\alpha_i \in \Omega_U^{-2i} \otimes Q, t \in H^2(CP^\infty; \mathbf{Z})$, $\mathrm{ch}_U(g(u)) = t$. Since $a^{-1}(t) = \sum \frac{\varphi([CP^n])}{n+1} t^{n+1}$, we have $a(t) = t + \sum \varphi(\alpha_i) t^{i+1} = t + \sum b^*(\alpha_i) t^{i+1}$. Thus,

$$a(g(u)) = g(u) + \sum b^*(\alpha_i) g(u^{i+1}),$$

$$\mathrm{ch}_U(a(g(u))) = t + \sum b^*(\alpha_i) t^{i+1} = b^*(\mathrm{ch}_U(u)) = \mathrm{ch}_U(b(u)).$$

Since the homomorphism $\mathrm{ch}_U : U^*(CP^\infty) \otimes Q \to \mathscr{H}^*(CP^\infty, \Omega_U \otimes Q)$ is a monomorphism, we find that $a(g(u)) = b(u)$. This proves the theorem.

By Theorem 1.6 any integer Hirzebruch genus, or, equivalently, any homomorphism $Q\colon \Omega_U \to \mathbf{Z}$, defines a formal group over $\mathbf{Z}$, and conversely (similarly for the ring $\mathbf{Z}_p$). In this connection the Hirzebruch genus, which defines this homomorphism, can be rational. Equivalent (or strongly isomorphic in the terminology of [4]) formal groups are defined by the Hirzebruch series $Q(z)$, $Q'(z)$, which are connected by the formula $\frac{z}{Q(z)} = \varphi^{-1}\left(\frac{z}{Q'(z)}\right)$, where $\varphi^{-1}(u) = u + \sum_{i\geqslant 1} \lambda_i u^{i+1}$, $\lambda_i \in \mathbf{Z}$. This follows from the fact that the logarithms of the formal groups are equal to $g_Q(z) = \left(\frac{z}{Q(z)}\right)^{-1}$, and by definition we have $g_Q(z) = g_{Q'}(\varphi(z))$.

Let us consider the integer Q-genus given by the rational series $g_Q(u)$. Then the Q'-genus such that $g_Q(u) = g_{Q'}(\varphi(u))$, $\varphi(u) = u + \sum_{i\geqslant 1} \lambda_i u^{i+1}$, $\lambda_i \in \mathbf{Z}$, also has integer values on Ω_U. In this connection the meaning of the equivalence for Hirzebruch genera is the same as for formal groups. What sort of examples of formal groups are considered in topology in connection with the well-known multiplicative genera c, T, L, A?

1. The Euler characteristic $c\colon \Omega_U \to \mathbf{Z}$. We have

$$f_c(u,v) = \frac{u+v-2uv}{1-uv}, \quad g_c(u) = \frac{u}{1-u}.$$

As a formal group, this genus is equivalent to the trivial one.

2. The Todd genus $T\colon \Omega_U \to \mathbf{Z}$. Here we have the law of multiplication

$$f_T(u,v) = u+v-uv, \quad g_T(u) = -\ln(1-u), \quad T(z) = \frac{-z}{1-e^{-z}} = \frac{-z}{g_T^{-1}(z)}.$$

3. The L-genus $\tau\colon \Omega_U \to \mathbf{Z}$ and the A-genus $A\colon \Omega_U \to \mathbf{Z}$, where $g_L^{-1}(z) = \tanh z$ and $g_A^{-1}(z) = \frac{1}{2}\sinh(2z)$. It is easily seen that these are strongly isomorphic to formal groups; both of them are strongly isomorphic over $\mathbf{Z}_2$ to a linear group, and over $\mathbf{Z}\left[\frac{1}{2}\right]$ to a multiplicative one (the Todd genus).

4. The T_y-genus (see [3]) $T_y([CP^n]) = \sum_{i=0}^{n}(-y^i)$. Here the law of multiplication is defined over the ring $\mathbf{Z}[[y]]$ and has the form

$$f_{T_y}(u,v) = \frac{u+v+(y-1)uv}{1+uvy}, \quad g_{T_y} = \frac{1}{(y+1)}\ln\left(1+(1+y)\frac{u}{1-u}\right).$$

We have for $y=-1,0,1$ the genera c, T, and L, respectively. The simple integral change of variables $u=\varphi(u')$ allows us to put f_{T_y} into the form

$$\varphi^{-1} f_{T_y}(\varphi(u'),\varphi(v')) = u' + v'(y-1)u'v'.$$

For all values of y this group either reduces to a linear one or to a multiplicative one over p-adic integers $\mathbf{Z}_p$.

Thus we see that in topology the multiplicative genera connected with other non-trivial formal groups over $\mathbf{Z}$, $\mathbf{Z}_p$ or $\mathbf{Z}/p\mathbf{Z}$ have not been considered previously.

§ 2. Formal power systems and Adams operators

Definition 2.1. A *formal power system* over a ring R is a collection of power series $\{f_k(u), k=\pm1,\pm2,\ \dots,\ f_k(u)\in R[[u]]\}$ such that $f_k(f_l(u)) = f_{kl}(u)$.

Consider on the ring $R[[u]]_0$ the operation of inserting one formal power series into another. With respect to this operation $R[[u]]_0$ is an associative (noncommutative) semigroup with unit. The role of the unit is played by the element u. Let $\mathbf{Z}^*$ denote the multiplicative semigroup of nonzero integers.

Definition 2.2. Any homomorphism $f\colon \mathbf{Z}^* \to R[[u]]$ will be called a *formal power system.*

Definition 2.3. We shall say that a formal power system is of type s if for any number k the series $f_k(u)$ has the form

$$f_k(u) = k^s u + \sum_{i\geqslant 1}\mu_i(k)u^{i+1}, \quad \mu_i(k)\in R.$$

We shall always assume the number s to be positive. Not every power system has type $s \geqslant 1$. For example, $f_k(u) = u^{k^s}$. More generally, the case

$$f_k(u) = \lambda_0(k)u^{k^s} + O(u^{k^s+1}) = \sum_{i\geqslant 0} \lambda_i(k)u^{k^s+i}$$

is possible.

Here it is especially important to distinguish two cases: 1) $\lambda_0(k) \equiv 1$, 2) $\lambda_0(k) \not\equiv 1$, but R does not have zero divisors. In the first case there exists a substitution $v = B(u) \in R[[u]] \otimes Q$, $v = u + O(u^2)$, in the ring such that $B(f_k(B^{-1}(v))) = v^{k^s}$ (the argument is similar to the proof of Lemma 2.4 below). 2) is the more general case, where $\lambda_0(k) \not\equiv 1$. Here a similar substitution exists and is correct over a field of characteristic zero which contains the ring R. Examples of such power systems may be found readily in the theory of cohomology operations in U^*-theory, by composing them out of series of operations $s_\omega \in A^U$ with coefficients in Ω_U. We are interested principally in Adams operations, and shall therefore consider only systems of type $s \geqslant 1$.

As in the theory of formal groups, an important lemma concerning "rational linearization" also plays a role in the theory of power systems. We note that the proof of this lemma presented below is similar to the considerations of Atiyah and Adams in K-theory (see [1]).

Lemma 2.4. *For any formal power system of type s there exists a series, $B(u) \in R[[u]] \otimes Q$, not depending on k, such that the equation $f_k(u) = B^{-1}(k^s B(u))$, where $B^{-1}(B(u)) = u$, is valid in the ring $R[[u]]\otimes Q$.*

The series $B(u)$ is uniquely defined by the power system, and is called its logarithm[1].

Proof. We shall show that for a given power system $f = \{f_k(u)\}$ of type s we are able to reconstruct, by an inductive process, the series $B(u) = u + \lambda_1 u^2 + \dots$. Assume that we have already constructed the series $v_n = B_n(u) \in R[[u]] \otimes Q$ such that for the formal power system $\{f_k^{(n)}(v_n)\} = \{B_n f_k(B_n^{-1}(v_n))\}$ we have the formula $f_k^{(n)}(v_n) = k^s v_n + \mu(k)v_n^{n+1} + O(v_n^{n+2})$, $\mu(k) \in R$. By using the relation $f_l^{(n)}(f_k^{(n)}(v_n)) = f_k^{(n)}(f_l^{(n)}(v_n))$, we obtain for all k and l

$$(kl)^s v_n + (l^s\mu(k) + \mu(l)k^{(n+1)s})v_n^{n+1} = (kl)^s v_n + (k^s\mu(l) + \mu(k)l^{(n+1)s})v_n^{n+1}.$$

[1]We point out that a formula for the logarithm of a formal group was given in [5] for the cobordism theory of power systems of type $s = 1$ for these groups. However, there it is necessary to make use of important additional information concerning the coefficients of the power systems of u^k as functions of k.

Consequently,

$$\frac{\mu(k)}{k^s(k^{ns}-1)} = \frac{\mu(l)}{l^s(l^{ns}-1)} = \lambda \in R\otimes Q,$$

where λ does not depend on k and l. Let us set $B_{n+1}(u) = v_n - \lambda v_n^{n+1}$. Direct substitution now shows that

$$B_{n+1}(f_k(B_{n+1}^{-1}(v_{n+1}))) = k^s v_{n+1} + O(v_{n+1}^{n+2}).$$

This completes the inductive step. We set $B(u) = \varinjlim B_n(u)$. Thus $B(f_k(B^{-1}(B(u)))) = k^s B(u)$, i.e. $f_k(u) = B^{-1}(k^s B(u))$, which completes the proof.

An important example of a formal power system is the operation of raising to the power s in the universal formal group $f(u,v)$ over the ring Ω_U. The operations of raising to a power in $f(u,v)$ have the form $k^s\Psi^{k^s}$, $\Psi^{k^s} \in A$. Let us denote by $\Lambda(s)$ the subring of Ω_U generated by all the coefficients of the formal series

$$k^s\Psi^{k^s}(u) = k^s u + \sum \mu_i^{(s)}(k)u^{i+1} \in \Omega_U[[u]] = U^*(CP^\infty)$$

for all k. In Theorem 4.11 of [2] the coefficients of the series $k^s\Psi^{k^s}$ are described in terms of the manifolds $M_{k^s}^{n-1} \subset CP^n$, $k = \pm 1, \quad \pm 2, \ldots$, which are the zero cross-section of the k^s-th tensor power of the Hopf fiber η over CP^n. In particular, from this theorem it follows that modulo factorizable elements in the ring Ω_U have the equation

$$\mu_i^{(s)}(k) \approx [M_{k^s}^i] - k^s[CP^i].$$

Since $\tau(M_{k^s}^i) = \varphi^*((i+1)\eta - \eta^{k^s})$, where $\varphi\colon M_{k^s}^i \subset CP^{i+1}$ is an imbedding map, we have $s_i([M_{k^s}^i]) - s_i(k^s[CP^i]) = k^s(1-k^{si})$. The calculation of the Chern numbers s_i (t-characteristic in the terminology of [13]) of the elements $\mu_i^{(s)}(k)$ is easily performed by the method of [13].

Lemma 2.5. *Let $\Lambda(s) = \sum \Lambda_n$ be the ring generated by the elements $\mu_i^{(s)}(k)$ for all k and i. The smallest value of the t-characteristic on the group Λ is equal to the greatest common divisor of the numbers $k^s(k^{ns}-1)$, $k = 2, 3, \ldots$. In particular, the ring $\Lambda(s)$ does not coincide with the ring Ω_U for any s, but the rings $\Lambda(s)\otimes Q$ and $\Omega_U\otimes Q$ are isomorphic.*

Theorem 2.6. *The formal power system of type s generated by the Adams operations $f_U(u) = \{k^s\Psi^{k^s}\}$, $k = \pm 1, \pm 2, \ldots$, and considered over* *the ring $\Lambda(s)$, is a universal formal system of type s on the category of torsion-free rings, i.e. for any formal power system $f = \{f_k(u)\}$ of type s over any torsion-free ring R there exists a unique ring homomorphism $\varphi\colon \Lambda(s) \to R$ such that $f = \varphi[f_U]$.*

PROOF. Let $B(u) = u + \sum \lambda_i u^{i+1}$, $\lambda_i \in R \otimes Q$, be the logarithm of the formal power system $f = \{f_k\}$. Consider the ring homomorphism $\varphi\colon \Omega_U \otimes Q \to R \otimes Q$, $\varphi\left(\frac{[CP^n]}{n+1}\right) = \lambda_n$. Since the coefficients of the formal power system $\{k^s \Psi^{k^s}\}$ generate the entire ring $\Lambda(s)$, we see that the homomorphism φ, restricted to the ring $\Lambda(s) \subset \Omega_U \otimes Q$, is integral, i.e. $\operatorname{Im} \varphi(\Lambda(s)) \subset R \subset R \otimes Q$. This proves the theorem.

With each formal power system $f = \{f_k(u)\}$ of type s over a torsion-free ring R we may associate a formal one-parameter group $B^{-1}(B(u) + B(v))$ over the ring $R \otimes Q$, where $B(u)$ is the logarithm of the power system. From Theorem 2.6 we get

Corollary 2.7. *Let $\varphi\colon \Lambda(s) \to R$ be the homomorphism corresponding to the formal power system $f = \{f_k(u)\}$. In order for the group $B^{-1}(B(u)+B(v))$ to be defined over the ring R, it is necessary and sufficient that the homomorphism φ extend to a homomorphism $\hat{\varphi}\colon \Omega_U \to R$.*

Thus the question of the relation of the concepts of a formal power system and a formal one-parameter group over a torsion-free ring R is closely related to the problem of describing the subrings $\Lambda(s)$ in Ω_U.

We shall demonstrate that the series $B^{-1}(k^s B(u))$ has the form $B^{-1}(k^s B(u)) = k^s u + k^s(k^s - 1)\lambda u^2 + \dots$, $\lambda \in R \otimes Q$, where $B(u) = u \pm \lambda u^2 + \dots$. Since the expression $k^s(k^s - 1)\lambda$ is integer valued for all k, it follows that an element $\lambda \in R \otimes Q$ can have in its denominator the Milnor–Kervaire–Adams constant $M(s)$, equal to the greatest common divisor of the numbers $\{k^s(k^s - 1)\}$. For example, $M(1) = 2$, $M(2) = 12$. For the series $B(u)$ obtained from a formal group over R the second coefficient λ can only have 2 in the denominator. For all $s > 1$ a realization of the universal system indicated in Theorem 2.6 does not, of course, occur naturally. A natural realization would be one over a subring of the ring Ω_U, where the second coefficient λ of the logarithm $B(u) = u + \lambda u^2 + \dots$ for the system of type $s = 2l$ would coincide with the well-known Milnor-Kervaire [6] manifold $V^s \in \Omega_U^{-4s}$, where $\lambda = \pm \frac{V^s}{M(s)}$, as follows from our considerations on the integer of $\lambda \cdot M(s)$. For $s = 2$ such a system will be given below.

It is simplest to describe the connection between the notion of a formal power system of type s and of a formal one-parameter group for $s = 1$. We consider the category of torsion-free rings which are modules over the

p-adic integers. The system $f_U = \{k^s \Psi^{k^s}(u)\}$, considered over the ring $\Lambda(s) \otimes \mathbf{Z}_p$, is a universal formal system of type s for systems over such rings. Consider in the ring Ω_U some fixed multiplicative system of generators $\{y_i\}$, $\dim y_i = -2i$; let us denote by $\Lambda_p \subset \Omega_U$ the subring generated by the elements $y_{(p^j-1)}$, $j = 0, 1, \ldots$, and by $\pi_p \colon \Omega_U \to \Omega_U$ the projection such that

$$\pi_p(y_i) = \begin{cases} y_i, & \text{if } i = p^j - 1 \\ 0 & \text{otherwise.} \end{cases}$$

According to Lemma 2.5, the minimum value of the t-characteristic on the group $\Lambda(1)_n \subset \Lambda(1)$ is equal to the greatest common divisor of the numbers $k(k^n - 1)$, $k = 2, 3, \ldots$. In the canonical factorization of the number $\{k(k^n - 1)\}$ into prime factors only first powers can appear, and since $t(y_{(p^j-1)}) = p$, $j > 0$ (see [11]), it follows that the homomorphism $\pi_p \colon \Lambda(1) \to \Lambda_p$ is an epimorphism. Let us define $f_U^{(p)} = \{\pi_p^*[k\Psi^k]\}$.

Corollary 2.8. *For any projection of type π_p the coefficients of the series $f_U^{(p)}$ generate the entire ring Λ_p, which coincides with the ring of coefficients of the formal group $\pi_p^*(f_U(u, v)) = f_U^{(p)}(u, v)$.*

We consider now the spectral projection $\overline{\pi}_p \colon \Omega_U \otimes \mathbf{Z}_p \longrightarrow \Omega_U \otimes \mathbf{Z}_p$ such that $\overline{\pi}_p^*([CP^i]) = 0$ if $i + 1 \neq p^h$, and $\overline{\pi}_p^*([CP^i]) = [CP^i]$, if $i + 1 = p^h$. This projection was given in [14], starting from the Cartier operation over formal groups. As was indicated § 1, the projection $\overline{\pi}_p$ can be considered as a "cohomology" operation on the set of all formal one-parameter over any commutative a $\mathbf{Z}_p$-ring R.

We shall say that the formal group $F(u, v)$ over the $\mathbf{Z}_p$-ring R *belongs to the class* P if $\overline{\pi}_p^*(F(u, v)) = F(u, v)$. Note that the group $\overline{\pi}_p^* f_U(u, v)$ is a universal formal group for groups of class P over the ring $\Lambda_p = \operatorname{Im} \overline{\pi}_p^*(\Omega_U)$.

From the description of the operator $\overline{\pi}_p^*$ and the definition of the projection $\overline{\pi}_p^*$ on the collection of groups it follows easily that for a torsion-free $\mathbf{Z}_p$-ring R the group $F(u, v)$ belongs to the class P if and only if its logarithm has the form $g_F(u) = u + \sum\limits_{i=1}^{\infty} \alpha_i u^{p^i}$.

We shall say that a formal power system $f(u)$ over $\mathbf{Z}_p$-ring R belongs to the class P if its logarithm has the form $B(u) = u + \sum \lambda_i u^{p^i}$.

Lemma 2.9. *The power system $\overline{\pi}_p^*[k\Psi^k(u)]$ is a universal formal power system of type 1 for the class P over the ring $\Lambda_p = \operatorname{Im} \overline{\pi}_p^*(\Omega_U)$.*

Proof easily follows from Lemma 2.4 and Corollary 2.8.

From Lemmas 2.4, 2.5 and 2.9 we have

Theorem 2.10. *Let R be a torsion-free $\mathbf{Z}_p$-ring, $f(u)$ a formal power system of type* 1 *of the class P over R, and $B(u)$ the logarithm of $f(u)$.*

Then a formal one-parameter group $F(u,v) = B^{-1}(B(u)+B(v))$ in class P is defined over the ring R, and, moreover, the mapping $f(u) \to F(u,v) = B^{-1}(B(u)+B(v))$ sets up a one-to-one correspondence between the collection of all formal power systems of type 1 of class P over R and the collection of all one-parameter formal groups of class P over R.

We shall now show that for a power system over a ring with torsion, as distinct from the case of formal groups, the theorem that any system can be lifted to a system over a torsion-free ring is not true. It will follow from this, in particular, that the formal system $\{k^s\Psi^{k^s}(u)\}$ over the ring $\Lambda(s)$ is not universal on the category of all rings.

EXAMPLE. Consider the ring $R = Z_p = \mathbf{Z}/p\mathbf{Z}$; we shall display a power system which cannot be lifted to a system over the ring $\mathbf{Z}_p$ of p-adic integers. Let $f(u) = \left\{ f_k(u) = ku + \sum_{i\geqslant 1} \mu_i(k)u^{p^i} \right\}_{k=\mu_0(k)}$ be a formal power system. Note that in R we have the identity $x^p = x$. Since $f_k(f_i(u)) = f_{kl}(u)$, we have

$$\mu_1(kl) = k\mu_1(l) + l\mu_1(k), \ldots, \mu_i(kl) = \sum_{\substack{j+q=i \\ j\geqslant 0, q\geqslant 0}} \mu_j(k)\mu_q(l).$$

Consequently the value of the function $\mu_i(k)$ for all $i \geqslant 1$ and prime numbers k can be given arbitrarily. For example, the values of the function $\mu_1(k)$ for the primes $k = 2, 3, 5, \ldots$ are arbitrary. Such functions $\mu_1(k)$ form a continuum. For formal systems of type $s = 1$, obtained from a system over $\mathbf{Z}_p$ by means of the homomorphism of reduction modulo p, by Lemma 2.4 the function $\mu_1(k)$ has the form $\left(\frac{k(k^{p-1}-1)}{p}\right) \cdot \gamma = \mu_1(k)$, where γ is a p-adic unit. Reduction of $\mu_1(\text{mod } p)$ gives a monomial over $\mathbf{Z}_p$. From this we have

Theorem 2.11. *There exists a continuum of formal power systems over the ring $R = \mathbf{Z}_p$ which are not homomorphic images of any power system over the p-adic integers (and in general over any torsion-free ring).*

§ 2a

We shall indicate another geometrical realization of a universal power system of type $s = 2$ which has an interesting topological meaning. In the universal formal group $f(u,v)$ the operation $u \to \bar{u} = -\Psi^{-1}(u)$, $f(u,\bar{u}) = 0$, is the lifting of the operation of complex conjugation into the cobordism of K-theory. Therefore the combination of the form $u\bar{u} = -u\Psi^{-1}(u)$ for geometrical cobordism has the sense of "square of the absolute value" $|u|^2 = u\bar{u}$.

Let $F(u,v)$ be a formal group over the ring R and let $\bar{u}$ be the element inverse to u, i.e. $F(u,\bar{u}) = 0$. Consider the element $x = u\bar{u} \in R[[u]]$, and let $[u]_k = F(u,\dots,u)(k \text{ places})$, where $F(u,\dots,u) = F(u,F(u,\dots))$. Define $\varphi_k(x) = [u]_k \cdot [\bar{u}]_k$ where the product is the ordinary product in the ring $R[[u]]$. We have

Lemma 2.12. *For a formal group $F = F(u,v)$ the values of the series $\varphi_k(x) = [u]_k[\bar{u}]_k$ lie in the ring $R[[x]] = R[[u\bar{u}]]$ and define a power system of type $s = 2$ over the ring R.*

PROOF. Let $f_U = f(u,v)$ be the universal group over the ring $R = \Omega_U$, and let $f(u,v) = g^{-1}(g(u)+g(v))$. Define $B^{-1}(-y) = g^{-1}(-\sqrt{y})g^{-1}(\sqrt{y})$. Since $[u]_k = g^{-1}(kg(u))$, we have $\varphi_k(x) = g^{-1}(kg(u)) \times g^{-1}(-kg(u)) = B^{-1}(-k^2g(u)^2)$. Furthermore $x = g^{-1}(g(u))g^{-1}(-g(u)) = B^{-1}(-g(u)^2)$. Therefore $B(x) = -g(u)^2$ and $\varphi_k(x) = B^{-1}(-k^2g(u)^2) = B^{-1}(k^2B(x))$. Consequently $\varphi_k(x)$ is a formal power system of type $s = 2$ over the ring Ω_U, with logarithm $B(x)$. In view of universality of the group f_U over Ω_U this completes the proof of the lemma in the general case.

We shall give a topological interpretation of Lemma 2.12. Consider the Thom spectrum $MSp = (MSp(n))$ of the symplectic group Sp. In particular $MSp(1) = KP^\infty$ is infinite dimensional quaternionic projective space. The canonical embedding $S^1 \to Sp(1) \to SU(2)$ defines a mapping $\varphi\colon CP^\infty \to KP^\infty$, and consequently a mapping $\varphi^*\colon U^*(KP^\infty) \to U^*(CP^\infty)$, where $U^*(KP^\infty) = \Omega_U[[x]]$, $\dim_R(x) = 4$, $U^*(CP^\infty) = \Omega_U[[u]]$, $\dim_R u = 2$ and $\varphi^*(x) = u\bar{u}$. This follows from the fact that the canonical $Sp(1)$-bundle γ over KP^∞ restricted to CP^∞ goes to $\eta + \bar{\eta}$, and $x = \sigma_2(\gamma) \to \sigma_1(\eta)\sigma_1(\bar{\eta}) = u\bar{u}$, where σ_i is the Chern class in cobordism theory.

We set

$$\varphi_k(x) = (k^2\Psi^k)x = k^2x + \sum_{i=1}^{\infty} \mu_i(k)x^{i+1}, \quad x \in U^4(KP^\infty),\ \mu_i(k) \in \Omega_U^{-4i}.$$

From the properties of the operations Ψ^k (see [12]) we obtain

$$\begin{aligned}\varphi_k(\varphi_l(x)) &= k^2\varphi_l(x) + \sum_{i=1}^{\infty}\mu_i(k)(\varphi_l(x))^{i+1}\\ &= k^2(l^2\Psi^l(x)) + \sum_{i=1}^{\infty}\mu_i(k)l^{2i+2}\Psi^l(x^{i+1})\\ &= l^2\Psi^l(k^2\Psi^k(x)) = l^2k^2\Psi^{lk}(x) = \varphi_{kl}(x).\end{aligned}$$

Here we used the formula $\Psi^l(\mu_i(k)) = l^{2i}\mu_i(k)$. Consequently the set of functions $\varphi(x) = \{\varphi_k(x)\}$ is a formal power system of type $s = 2$ over the ring Ω_U. Since $k^2\Psi^k(x) = k^2\Psi^k(u\bar{u}) = [u]_k[\bar{u}]_k$, by this means we obtain a topological proof of Lemma 2.12.

REMARK. We note that in § VII of the paper of Novikov [13] in the proof of Theorem 1b in Example 3 the case of groups of generalized quaternions was analyzed and the "square modulus" system arose there; the properties of this system are required for carrying out a rigorous proof for this example, without which Theorem 1b cannot be proved. Indeed, we used the fact that $k^2\Psi^k(\omega)$ is a series in the variable ω with coefficients in Ω_U, where $\omega = \sigma_2(\Delta_1)$. Moreover, for carrying out the proof of Theorem 1b, in analogy with Theorem 1 we require the fact that Δ_i are all obtained from Δ_1 by means of Adams operations, where the Δ_i are the 2-dimensional irreducible representations of the group of generalized quaternions.

Let us consider in more detail the logarithm $B(x) = -g(u)^2$ of the formal type $s = 2$ power system introduced in Lemma 2.12. Let t and z be the generators of the cohomology groups $H^2(CP^\infty; \mathbf{Z})$ and $H^4(KP^\infty; \mathbf{Z})$, respectively. Since $\varphi^*(c_2(\gamma)) = c_1(\eta)c_1(\bar{\eta})$, $\varphi^*(z) = -t^2$. We have $\mathrm{ch}_U(g(u)) = t$, $\mathrm{ch}_U(B(x)) = -t^2 = z$. Consequently,

$$B^{-1}(x) = \mathrm{ch}_U(x)|_{z=x} \in \mathscr{H}^*(KP^\infty; \Omega_U \otimes Q) = \Omega_U \otimes Q[[z]].$$

Let Ψ^0 be the multiplicative operation in $U^* \otimes Q$ theory, given by the series $\Psi^0(u) = \lim_{k\to 0}\left(\frac{1}{k}g^{-1}(kg(u))\right) = g(u) = u + \sum \frac{[CP^n]}{n+1}u^{n+1}$. Recall that in [2], [12] the operation Ψ^0 was denoted by Φ. We have $\mathrm{ch}_U\,\Psi^0(x) = \Psi^0\,\mathrm{ch}_U(x) = z$; here we have used the fact that $\Psi^0(y) = 0$, where $y \in \Omega_U^{-2n}$, $n > 0$. Since the homomorphism $\mathrm{ch}_U\colon U^*(KP^\infty) \to \mathscr{H}^*(KP^\infty; \Omega_U \otimes Q)$ is a monomorphism, it follows from the equation $\mathrm{ch}_U(B(x)) = z = \mathrm{ch}_U(\Psi^0(x))$ that $B(x) = \Psi^0(x)$.

According to Theorem 2.3 of [2], we have the formula

$$\mathrm{ch}_U(u) = t + \sum_{n=1}^{\infty}[M^{2n}]\frac{t^{n+1}}{(n+1)!},$$

for an element $u \in U^2(CP^\infty)$, where $s_\omega(-\tau(M^{2n})) = 0$, $\omega \neq (n)$ and $s_{(n)}(M^{2n}) = -(n+1)!$. Consequently,

$$\operatorname{ch}_U(u\bar{u}) = \left(t + \sum_{n=1}^{\infty}[M^{2n}]\frac{t^{n+1}}{(n+1)!}\right)\left(-t + \sum_{n=1}^{\infty}(-1)^{n+1}[M^{2n}]\frac{t^{n+1}}{(n+1)!}\right)$$
$$= -t^2 + \sum_{n=2}^{\infty}\left(\sum_{\substack{i+j=2n\\ i\geqslant 1, j\geqslant 1}}(-1)^i\frac{[M^{2i-2}][M^{2j-2}]}{i!j!}\right)t^{2n},$$

and we obtain the formula

$$B^{-1}(x) = x + \sum_{n=2}^{\infty}[N^{4n-4}]\frac{x^n}{2n!}, \quad [N^{4n-4}] \in \Omega_U^{-4n+4},$$

where $[N^{4n-4}] = \sum\limits_{\substack{i+j=2n\\ i\geqslant 1, j\geqslant 1}}(-1)^{n+i}C_{2n}^i[M^{2i-2}][M^{2j-2}]$ and $[M^{2m}] \in \Omega_U^{-2m}$ are bordism classes which are uniquely defined by the conditions

$$s_\omega(-\tau(M^{2m})) = 0, \quad \omega \neq (m) \quad \text{and} \quad s_{(m)}(\tau(M^{2m})) = -(m+1)!.$$

We have

Theorem 2.13. *The type $s = 2$ power system constructed in Lemma* 2.12 *for the group $f(u,v)$ of geometrical cobordisms is universal in the class of torsion-free rings if considered over the minimal ring of its coefficient $\Lambda \subset \Omega_U$.*

The proof follows easily from the fact that all the coefficients of the series $B^{-1}(x)$ and $B(x)$ are not zero and are algebraically independent in $\Omega_U \otimes Q$.

From the preceding lemma we have

Corollary 2.14. *For any complex X the image of the mapping $[X, KP^\infty] \xrightarrow{\alpha} U^4(X)$, which associates with the mapping $\varphi\colon X \to KP^\infty$ its fundamental class $\varphi^*(\sigma_2(\gamma))$ in U^*-theory, is the domain of definition of the type 2 power system that looks like $B^{-1}(k^2B(x))$, where $B^{-1}(x) = g^{-1}(\sqrt{x})g^{-1}(-\sqrt{x})$. The Adams operations on this image are given by $k^2\Psi^k(x) = B^{-1}(k^2B(x))$, $\Psi^0(x) = B(x) \in U^*(X) \otimes Q$.*

Questions. *Is a type 2 power system defined directly on quaternionic Sp-cobordisms $[X, KP^\infty] \to Sp^4(X)$? Is the image* $\operatorname{Im}\alpha$ *closed with respect to power operations?*

What are the inter-relations between the ring of coefficients of the power system $B^{-1}(k^2B(x))$ with the image $\Omega_{Sp} \to \Omega_U$?

Note that the restriction $U^*(MSp(n)) \to U^*(MU(n)) \to U^*(CP_1^\infty \times \ldots \times CP_n^\infty)$ consists of all elements of the form $F(|u_1|^2, \ldots, |u_n|^2) \prod_{i=1}^{n} |u_i|^2$, where F is any symmetric polynomial (in distinction from classical cohomology, where we have symmetric functions of squares of Wu's generators).

As was pointed out above, for the series $B^{-1}(x)$ we have

$$B^{-1}(x) = x + \sum_{n=2}^{\infty} [N^{4n-4}] \frac{x^n}{(2n)!}, \quad [N^{4n-4}] \in \Omega_U^{-4n+4},$$

where $[N^{4n-4}] = \sum_{\substack{i+j=2n \\ i \geqslant 1, j \geqslant 1}} (-1)^{n+i} C_{2n}^i [M^{2i-2}][M^{2j-2}]$. In particular,

$$[N^4] = -8[M^4] + 6[M^2]^2 = (2K) \in \operatorname{Im}(\Omega_{Sp} \to \Omega_U),$$

where $K = 8[CP^2] - 9[CP^1]^2$.

Theorem 2.15. *For $n \geqslant 2$ the bordism classes $[N^{4n-4}]$ belong to the image of the homomorphism $\Omega_{Sp}^{-4n+4} \to \Omega_U^{-4n+4}$. In addition, for $n \equiv 1 \mod 2$ the elements $[N^{4n-4}]/2 \in \Omega_U^{-4n+4}$ already belong to the group $\operatorname{Im}(\Omega_{Sp} \to \Omega_U)$.*

Proof. Let $v \in Sp^4(KP^\infty)$ be the canonical element. As is well known, $p_1(\gamma) = v$ (see [13]) and $\omega^*(p_1(\gamma)) = \sigma_2(\gamma) \in U^4(KP^\infty)$, where p_1 is the first Pontrjagin class in the symplectic cobordism of the canonical $Sp(1)$-bundle γ over KP^∞ and $\omega : Sp^* \to U^*$ is the natural transformation of cobordism theory. We shall calculate the coefficients of the series

$$\operatorname{ch}_{Sp}(p_1(\gamma)) = z + \sum_{n=1}^{\infty} \frac{C_n}{\lambda_n} z^{n+1} \in H^4(KP^\infty, \Omega_{Sp}^* \otimes Q) = \Omega_{Sp}^* \otimes Q[[z]],$$

where ch_{Sp} is the Chern–Dold character in Sp-theory (see [2]), $C_n \in \Omega_{Sp}^{-4n}$ are indivisible elements in the group Ω_{Sp}^{-4n}, $\lambda_n \in \mathbf{Z}$. Since $\operatorname{ch} \gamma = \operatorname{ch}(\eta + \bar{\eta}) = e^t + e^{-t} = 2 + t^2 + \ldots + \frac{2t^{2n}}{(2n)!} + \ldots$ and $z \to -t^2, t \in H^2(CP^\infty, \mathbf{Z})$, we have

$$\operatorname{ch}_{Sp}(p_1(\gamma)) = -\operatorname{ch}_2 \gamma + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+2)!}{2\lambda_n} C_n \operatorname{ch}_{2n+2}(\gamma).$$

By making use of the decomposition principle for quaternion fibrations and the additivity of the first Pontrjagin class we now find that for any $Sp(m)$-fiber bundle ζ over any complex X we have

$$\mathrm{ch}_{Sp}(p_1(\zeta)) = -\,\mathrm{ch}_2(\zeta) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n+2)!}{2\lambda_n}C_n\,\mathrm{ch}_{2n+2}(\lambda).$$

By Bott's theorem we have the isomorphism

$$\beta : \tilde{K}Sp(S^{4n}) \xrightarrow{\simeq} KO^4(S^{4n}), \quad \beta(\zeta) = (1-\gamma_1)\otimes_H \zeta,$$

where γ_1 is the $Sp(1)$-Hopf fiber bundle over S^4.

We shall next identify the elements $\zeta \in \tilde{K}Sp(S^{4n})$ with their images in the group $K(S^{4n})$. The formula $\mathrm{ch}(c\beta(\zeta)) = \mathrm{ch}(\zeta)$ is easily verified, where $c : KO^4 \to K^4$ is the complexification homomorphism.

Let ξ_n and z_n denote the generators of the groups $\tilde{K}Sp(S^{4n}) = \mathbf{Z}$ and $H^{4n}(S^{4n}; Z) = \mathbf{Z}$, respectively. From Bott's results concerning the homomorphism of complexification it follows that $\mathrm{ch}\,\xi_n = a_n z_n$, where

$$a_n = \begin{cases} 1, & \text{if } n \equiv 1 \mod 2,\\ 2, & \text{if } n \equiv 0 \mod 2.\end{cases}$$

Thus

$$\mathrm{ch}_{Sp}(p_1(\xi_n)) = (-1)^n\frac{(2n)!}{2\lambda_{n-1}}C_{n-1}\,\mathrm{ch}_{2n}\,\xi_n = (-1)^n\frac{(2n)!}{2\lambda_{n-1}}a_nC_{n-1}\cdot z_n.$$

Since $\mathrm{ch}_{Sp}(p_1(\xi_n)) \in \mathscr{H}^4(S^{4n}; \Omega^*_{Sp}) \subset \mathscr{H}^4(S^{4n}; \Omega^*_{Sp}\otimes Q)$, we find that the number $\dfrac{(2n)!}{2\lambda_{n-1}}a_n$ is an integer for any n.

It follows from [7] under the composition of homomorphisms

$$\tilde{K}Sp(X) \xrightarrow{p_1} \Omega^4_{Sp}(X) \xrightarrow{\omega} \Omega^4_U(X) \xrightarrow{\mu} \tilde{K}(X)$$

the element $\zeta \in \tilde{K}Sp(X)$ goes into the element $-\zeta \in \tilde{K}(X)$, where μ is the "Riemann-Roch" homomorphism. We have

$$\begin{aligned} -\,a_n z_n &= \mathrm{ch}(-\xi_n) = \mathrm{ch}(\mu\omega p_1(\xi_n)) = \mu\omega ch_{Sp}(p_1(\xi_n)) \\ &= \mu\omega\left((-1)^n\frac{(2n)!}{2\lambda_{n-1}}a_nC_{n-1}z_n\right) = (-1)^n\frac{(2n)!}{2\lambda_{n-1}}a_nTd(\omega(C_{n-1}))z_n,\end{aligned}$$

where $Td(\omega(C_{n-1}))$ is the Todd genus of the quasicomplex manifold $\omega(C_{n-1})$. Since the Todd genus of any $(8m+4)$-dimensional SU-manifold is even, we find that $Td(\omega(C_{n-1})) = a_n\delta_n$ for any n, where δ_n is an integer. We have

$$(-1)^{n-1}\frac{(2n)!}{2\lambda_{n-1}}a_n\delta_n = 1.$$

Thus the number $\frac{2\lambda_{n-1}}{(2n)!a_n}$ is an integer. On the other hand, it was shown earlier that the number $\frac{(2n)!a_n}{2\lambda_{n-1}}$ is also an integer. Consequently $\frac{2\lambda_{n-1}}{(2n)!a_n} = \pm 1$. Without loss of generality, we may assume that $\frac{2\lambda_{n-1}}{(2n)!a_n} = 1$. Since $a_{n-1} \cdot a_n = 2$ for any $n > 0$, it follows that $\frac{a_{n-1} \cdot \lambda_{n-1}}{(2n)!} = 1$, and we find

$$\lambda_{n-1} = \frac{(2n)!}{a_{n-1}}, \quad C_{n-1} = (-1)^n p_1(\xi_n) \in \Omega_{Sp}^{-4n+4} \cong Sp^4(S^{4n}),$$
$$Td(\omega(C_{n-1})) = (-1)^{n-1}a_n.$$

We have therefore proved the following lemma.

Lemma 2.16. *For the canonical element $v = p_1(\gamma) \in Sp^4(KP^\infty)$ and the Chern–Dold character in symmetric cobordism we have the formula*

$$\mathrm{ch}_{Sp}(p_1(\gamma)) = z + \sum_{n=2}^{\infty} a_{n-1}C_{n-1}\frac{z_n}{(2n)!}.$$

From the formula $\omega ch_{Sp}p_1(\gamma) = \mathrm{ch}_U\,\sigma_2(\gamma)$ we obtain

$$B^{-1}(z) = \mathrm{ch}_U\,\sigma_2(\gamma) = z + \sum_{n=2}^{\infty}[N^{4n-4}]\frac{z^n}{(2n)!}$$
$$= \omega ch_{Sp}p_1(\gamma) = z + \sum_{n=2}^{\infty} a_{n-1}\omega(C_{n-1})\frac{z^n}{(2n)!}.$$

Consequently in the group Ω_U^{-4n+4} we have the identity

$$a_{n-1} \cdot \omega(C_{n-1}) = [N^{4n-4}]$$

for any n. This theorem is proved.

Corollary 2.17. *The rational envelope of the ring of coefficients of the power system $B^{-1}(k^2B(x))$ of type $s = 2$ coincides with the group $\mathrm{Hom}^*_{A^U}(U^*(MSp), \Omega_U)$, which is the rational envelope of the image $\Omega_{Sp} \to \Omega_U$, where A^U is the ring of operations of the U^*-cobordism theory.*

We note that the element $(u+\bar{u}) = \sigma_1(\xi+\bar{\xi}) \in U^2(CP^\infty)$ can be expressed in terms of $x = u\bar{u} = \sigma_2(\xi+\bar{\xi})$. We have $(u+\bar{u}) = g^{-1}(g(u)) + g^{-1}(-g(u)) = F(g(u)^2) = F(-B(x)) = G(x)$, where $F(\alpha^2) = g^{-1}(\alpha) + g^{-1}(-\alpha) = -[CP^1]\alpha^2 + \sum 2\frac{[M^{4n+2}]}{(2n+2)!}\alpha^{2n+2}$.

Lemma 2.18. *For any k the series $G_k(x) = F(-k^2B(x))$ lie in $\Omega_U[[x]]$ and determine over the ring $\Omega_U\left[\frac{1}{[CP^1]}\right]$ a formal type $s=2$ power system by means of the formula*

$$\varphi_k(\omega) = F(k^2F^{-1}(\omega)),$$

where $\omega = u+\bar{u} = G(x)$.

The first assertion of the lemma follows from the fact that $G_k(x) = [u]_k + [\bar{u}]_k = \sigma_1\left(\xi^k+\bar{\xi}^k\right)$. The second assertion follows from the invertibility of the series $F(\alpha^2)$ in the ring $\Omega_U\left[\left[\frac{1}{[CP^1]},\alpha^2\right]\right]$, as a consequence of which $-B(x) = F^{-1}(G(x))$ and $G_k(x) = F(k^2F^{-1}(G(x)))$.

Corollary 2.19. *Let $F(u,v) = u+v+\alpha_{1,1}uv+\dots$ be a formal group over the ring R. If the element $\alpha_{1,1}$ is invertible in R, then the formal power system of type $s=2$, defined by the series $\varphi_k(\omega) = [u]_k+[\bar{u}]_k \in R[[\omega]], \omega = u+\bar{u}$, is defined over the ring R.*

Let $\varphi(x) = \{\varphi_k(x)\}$ be a type $s=2$ power system over a torsion-free ring Λ. It is natural to state the following problem

(*) Describe all rings R such that 1) $\Lambda \subset R$; 2) there exists over the ring R a one-dimensional formal group $F(u,v)$ from which the original formal power system $\{\varphi_k(x)\}$ is obtained as a system of the form $\{[u]_k[\bar{u}]_k\}, x = u\bar{u}$.

We note that the set of all such pairs $(R, F(u,v))$ forms a category in which the morphisms $(R_1,F_1) \to (R_2,F_2)$ are the ring homomorphisms $R_1 \to R_2$, which preserve the ring Λ and take the group F_1 into the group F_2. Next we shall present a universal formal group in this category, and by this means we shall obtain a complete solution to the problem stated above.

We consider first the case where $\Lambda = \Omega_U$ and $\varphi(x) = \{\varphi_k(x) = [u]_k[\bar{u}]_k = B^{-1}(k^2B(x))\}, x = u\bar{u}$.

Lemma 2.20. *The power system $\varphi(x) = \{B^{-1}(k^2B(x))\}$ together with the series $G(x) = F(-B(x)) = u+\bar{u}$ completely determines the original formal group $f(u,v) = g^{-1}(g(u)+g(v))$.*

PROOF. By knowing the series $G(x)$ we can calculate the series $\bar{u} = \theta(u)$ from the equation $u + \theta(u) = G(u \cdot \theta(u))$. Then, knowing the series $B(x)$, we can calculate the series $g(u)$ from the equation $B(u \cdot \theta(u)) = -g(u)^2$.

REMARK. The proof of Lemma 2.20 actually uses the fact that the elements u and $\bar{u}$ are the roots of the equation

$$y^2 - (u + \bar{u})y + u\bar{u} = y^2 - G(x)y + x = 0$$

over $\Omega_U[[x]]$.

From the formula introduced above it follows that the coefficients of the series $F(x)$ and $B(x)$ are algebraically independent and generate the entire ring $\Omega_U \otimes Q$. We have

$$F(x) = \sum_{i \geqslant 0} y_i x^{i+1}, \quad y_i \in \Omega_U^{-4i-2} \otimes Q; \quad B(x) = \sum_{i \geqslant 0} z_i x^{i+1}, z_i \in \Omega_U^{-4i} \otimes Q$$

and

$$\Omega_U \otimes Q = Q[y_i] \otimes Q[z_i].$$

Now let $\varphi(x) = \{\varphi_k(x)\}$ be an arbitrary type $s = 2$ formal power system over a torsion-free ring Λ and let $B(x) = \sum \beta_i x^{i+1}$ be its logarithm. Consider the ring homomorphism $\chi : \Omega_U \otimes Q \to \Lambda \otimes Q[y_i]$, defined by the equation $\chi(z_i) = \beta_i, \chi(y_i) = y_i$, and let R denote the subring of $\Lambda \otimes Q[y_i]$ which is generated by the ring Λ and the image of the ring $\Omega_U \subset \Omega_U \otimes Q$ under the homomorphism χ. The one-dimensional formal group $F(u, v)$, which is the image of the group $f(u, v)$ over Ω_U, is defined over R. From the universality of the group $f(u, v)$ and from Lemma 2.20 it follows easily that the group $F(u, v)$ over R is a universal solution of problem (*) for the system $\{\varphi_k(x)\}$ over $\Lambda \subset R$.

We note that from the proof of Lemma 2.20 there follows a direct construction for the formal group $F(u, v)$ over R from the system $\{\varphi_k(x)\} = \{B^{-1}(k^2 B(x))\}$ over Λ. Indeed, it is necessary to carry out the following procedure. Consider the ring $\Lambda \otimes Q[y_i]$ and over it the series $F(x) = \sum y_i x^{i+1}$ and the corresponding series $G(x) = F(-B(x))$; then, as in Lemma 2.20, with respect to the series $B(x)$ and $G(x)$, find the series $g_F(u) \in \Lambda \otimes Q[[u, y_i]]$. The ring R is then the minimal extension of the ring Λ in $\Lambda \otimes Q[y_i]$, which contains the ring of coefficients of the group $F(u, v) = g_F^{-1}(g_F(u) + g_F(v))$.

§ 2b

We next turn our attention to the case where the power system $B^{-1}(k^2 B(x)) = k^2 \Psi(u\bar{u})$ is related to a distinctive "two-valued formal group"

$$F^{\pm}(x,y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2),$$

in which the operation of raising to a power is single valued, and indeed $B^{-1}(k^2 B(x)) = F^{\pm}\underbrace{(x,\dots,x)}_{k \text{ times}}$. If $x = u\bar{u}$, $y = v\bar{v}$, then $F^{\pm}(x,y) = \{|f(u,v)|^2; |f(u,\bar{v})|^2\}$, and for the $U(1)$-bundles ξ, η over $CP^\infty \times CP^\infty$, where $u = \sigma_1(\xi), \sigma = \sigma_1(\eta)$, we have $F^{\pm}(x,y) = \{\sigma_2(\xi\eta + \overline{\xi\eta}); \sigma_2(\xi\overline{\eta} + \overline{\xi}\eta)\}, x = \sigma_2(\xi + \overline{\xi}), y = \sigma_2(\eta + \overline{\eta})$.

Lemma 2.21. *The sum $F^+(x,y) + F^-(x,y)$ and the product $F^+(x,y) \cdot F^-(x,y)$ of values for the two-valued group do not contain roots and lie in the ring $\Omega_U[[x,y]]$.*

PROOF. Consider the mapping $CP^\infty \times CP^\infty \to KP^\infty \times KP^\infty$, whose image $U^*(KP^\infty \times KP^\infty) \to U^*(CP^\infty \times CP^\infty)$ is precisely $\Omega_U[[x,y]] \subset \Omega_U[[u,v]], x = u\bar{u}, y = v\bar{v}$. Since $x = \sigma_2(\xi + \bar{\xi}), y = \sigma_2(\eta + \bar{\eta})$, we have that $\sigma_2((\xi + \bar{\xi})(\eta + \bar{\eta})) = a$ lies in $\Omega_U[[x,y]]$; moreover $a = \sigma_2(\xi\eta + \overline{\xi\eta}) + \sigma_2(\xi\overline{\eta} + \bar{\xi}\eta) + \sigma_1(\xi\eta + \overline{\xi\eta})\sigma_1(\xi\overline{\eta} + \bar{\xi}\eta) = F^+(x,y) + F^-(x,y) + \sigma_1{\sigma'}_1$. Next, $\sigma_1(\xi\overline{\eta} + \overline{\xi\eta}) = g^{-1}(g(u) + g(v)) + g^{-1}(-g(u) - g(v)), \sigma_1(\xi\overline{\eta} + \bar{\xi}\eta) = g^{-1}(g(u) - g(v)) + g^{-1}(g(v) - g(u))$. Let $g(u) = \gamma, g(v) = \delta$. Therefore

$$\begin{aligned}\sigma_1 \cdot \sigma_1' &= \sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta) \\ &= [g^{-1}(\gamma + \delta) + g^{-1}(-\gamma - \delta)][g^{-1}(\gamma - \delta) + g^{-1}(\delta - \gamma)],\end{aligned}$$

i.e. $\sigma_1 \cdot \sigma_1'$ is a function of γ^2 and δ^2. Also, since $\gamma^2 = g(u)^2 = -B(x)$ and $\delta^2 = g(v)^2 = -B(y)$, the product $\sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta)$ is a function of x and y. Since $F^+(x,y) + F^-(x,y) = \sigma_2(\xi + \bar{\xi})(\eta + \bar{\eta}) - \sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta)$, it follows that $F^+(x,y) \in \Omega_U[[x,y]]$.

We conclude the proof by noting that $F^+(x,y) \cdot F^-(x,y) = \sigma_2(\xi\eta + \bar{\xi}\bar{\eta})\sigma_2(\xi\bar{\eta} + \bar{\xi}\eta) = \sigma_4((\xi + \bar{\xi})(\eta + \bar{\eta})) \in \Omega_U[[x,y]]$.

Let us set $F^+(x,y) + F^-(x,y) = \Theta_1(x,y)$, $F^+(x,y) \cdot F^-(x,y) = \Theta_2(x,y)$. It now follows from Lemma 2.21 that the law of multiplication in the two-valued formal group $F^{\pm}(x,y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2)$ is given by solving the quadratic equation

$$Z^2 - \Theta_1(x,y)Z + \Theta_2(x,y) = 0$$

over the ring $\Omega_U[[x,y]]$. Let $\Lambda \subset \Omega_U$ denote the minimal subring in Ω_U generated by the coefficients of the series $\Theta_1(x,y)$ and $\Theta_2(x,y)$. We have $\Lambda = \sum_{n \geqslant 0} \Lambda_{4n}, \Lambda_{4n} \subset \Omega_U^{-4n}$.

Our next problem is to describe the ring Λ, which is natural to look upon as the ring of coefficients of the two-valued formal group $F^{\pm}(x,y)$. In the ring Λ it is useful to distinguish the two subrings Λ' and Λ'' which are generated by the coefficients of the series $\Theta_1(x,y)$ and $\Theta_2(x,y)$ respectively. As will be shown next, neither of the rings Λ' and Λ'' coincides with Λ. It is interesting to note that the ring of coefficients of the formal power system $\varphi(x) = \{\varphi_k(x)\} = \{B^{-1}(k^2B(x))\}$ lies in, but does not coincide with, the ring Λ'. This follows from the facts that $\varphi_1(x) = x$, $\varphi_2(x) = \Theta_1(x,x)$ and for any $k \geqslant 3$ the formula

$$\varphi_k(x) = \Theta_1(\varphi_{k-1}(x), x) - \varphi_{k-2}(x)$$

is valid.

The canonical mapping of the spectra $MSp \to MU$, which corresponds to the inclusion mapping $Sp(n) \subset U(2n)$, defines an epimorphism $A^U \to U^*(MSp)$, and consequently the inclusion of the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ in Ω_U. We shall next identify the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ with its image in Ω_U.

Theorem 2.22. *The quadratic equation*

$$Z^2 - \Theta_1(x,y)Z + \Theta_2(x,y) = 0,$$

which determines the law of multiplication in the two-valued formal group $F^{\pm}(x,y)$, is defined over the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$, and, moreover,

$$\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) \otimes Z\left[\frac{1}{2}\right] \cong \Lambda \otimes Z\left[\frac{1}{2}\right],$$

where Λ is the ring of coefficients of the group $F^{\pm}(x,y)$.

Remark 2.23. Apparently the rings $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ and Λ are isomorphic, but at the present time the authors do not have a rigorous proof of this fact[1].

Let $\Omega_U(Z)$ be the subring of $\Omega_U \otimes Q$ which is generated by the elements all of whose Chern numbers are integers. As was shown in [2], the ring $\Omega_U(Z)$ is isomorphic to the ring of coefficients of the logarithm of the universal formal group $f(u,v)$, i.e. $\Omega_U(Z) = Z\left[\frac{[CP^1]}{2}, \dots, \frac{[CP^n]}{n+1}, \dots\right]$. The Chern–Dold characteristic ch_U for any complex X defines a natural transformation

$$\mathrm{ch}_U : H_*(X) \to \mathrm{Hom}_{A^U}(U^*(X), \Omega_U(Z))$$

[1]These rings are not isomorphic. The manifold M^{4n}, whose complex cobordism class belongs to $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$, but not to Λ, appears starting from $n \geqslant 3$. For details see [В. М. Бухштабер, Топологические приложения теории двузначных групп, Изв. АН СССР, сер. матем., 1978, Т. 42, N. 1, 130–184]. — *V. M. Buchstaber's remark* (2004).

(see [2], Theorem 1.9), which, as is easily shown, is an isomorphism for torsion-free complexes in the homology. We have

$$\mathrm{ch}_U : H_*(MSp) \xrightarrow{\approx} \mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U(Z)).$$

The inclusion mapping $\Omega_U \subset \Omega_U(Z)$ and the canonical homomorphism $A^U \to U^*(MSp)$ lead to the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) & \longrightarrow & \Omega_U \\ \downarrow & & \downarrow \\ H_*(MSp) & \xrightarrow{\lambda} & \Omega_U(Z), \end{array}$$

in which all the homomorphisms are monomorphisms.

Since $\lambda(h) = (\mathrm{ch}_U\, v, h)$ and $\mathrm{ch}_U\, x = B^{-1}(x)$, where $h \in H_*(MSp)$, v is the generator of the A^U-module $U^*(MSp)$, and x is the generator of the group $U^4(KP^\infty)$, it follows that the ring $\mathrm{Im}\,\lambda \subset \Omega(Z)$ coincided with the ring of coefficients of the logarithm of the power system $\{B^{-1}(k^2B(x))\}$. Thus it follows from the diagram that the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ coincides with the subring of Ω_U whose elements are polynomials in the elements $y_i \in \Omega_U(Z)$ with integral coefficients, where $B(x) = x + \sum y_i x^{i+1}$.

As an immediate check it is easy to see that the coefficients of the series $\Theta_1(x,y) = F^+(x,y) + F^-(x,y)$ and $\Theta_2(x,y) = F^+(x,y) \cdot F^-(x,y)$, where $F^\pm(x,y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2) = B^{-1}\left(\left(x\sqrt{\frac{B(x)}{x}} \pm y\sqrt{\frac{B(y)}{y}}\right)^2\right)$. are polynomials with integral coefficients from among the coefficients of the series $B(x)$. The proof of the first part of the theorem is therefore complete.

For the proof of the second part of the theorem we require a lemma, which is itself of some interest.

Lemma 2.24. *Let $\Lambda = \sum \Lambda_{4n}$ be the ring of coefficients of a two-valued formal group. The minimum positive value of the t-characteristic on the group Λ_{4n} is equal to $2^{s(n)}p$ if $2n = p^i - 1$, where p is prime, and is equal to $2^{s(n)}$ if $2n \neq p^i - 1$ for all p, where*

$$s(n) = \begin{cases} 3, & \text{if } n = 2^j - 1, \\ 2, & \text{if } n \neq 2^j - 1. \end{cases}$$

Now, since $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) \otimes Z\left[\frac{1}{2}\right] \subset \Omega_U\left[\frac{1}{2}\right]$ is a polynomial ring, the proof of the second part of the theorem is easily obtained, via a standard argument concerning the t-characteristic, from the results of [11] and Lemma 2.24.

Proof of Lemma 2.24. Let x and y be the generators of the group $U^4(KP^\infty \times KP^\infty)$. We have

$$\Theta_1(x,y) = 2x + 2y + \sum \beta_{i,j} x^i y^j, \quad \beta_{i,j} = \beta_{j,i} \in \Omega_U^{-4(i+j-1)},$$

$$\Theta_2(x,y) = x^2 - 2xy + y^2 + \sum \alpha_{i,j} x^i y^j, \quad \alpha_{i,j} = \alpha_{j,i} \in \Omega_U^{-4(i+j-2)}.$$

Let z_1 and z_2 be the generators of the group $H^4(KP^\infty \times KP^\infty)$. By using Corollary 2.4 of [2] we obtain immediately from the definitions of the series $\Theta_1(x,y)$ and $\Theta_2(x,y)$ that

$$\operatorname{ch}_U \Theta_1(x,y) = 2z_1 + 2z_2 + 4 \sum_{m \geqslant 1} (-1)^m \frac{[M^{4m}]}{(2m+1)!} \sum_{l=0}^{m+1} C_{2m+2}^{2l} z_1^l z_2^{m-l+1},$$

$$\operatorname{ch}_U \Theta_2(x,y) = z_1^2 - 2z_1 z_2 + z_2^2 + 4 \sum_{m \geqslant 1} (-1)^m \frac{[M^{4m}]}{(2m+1)!} \sum_{l=0}^{m+1} (C_{2m}^{2l} - 2C_{2m}^{2l-2} + C_{2m}^{2l-4}) \cdot z_1^l z_2^{m-l+2},$$

where $s_{2m}[M^{4m}] = -(2m+1)!$. On the other hand,

$$\operatorname{ch}_U x = B^{-1}(x) = z_1 + \sum_{m \geqslant 1} [N^{4m}] \frac{z_1^{m+1}}{(2m+2)!},$$

where $s_{2m}([N^{4m}]) = (-1)^{m+1} \cdot 2(2m+2)!$. By combining these formulas we find

a) $s_{2m}(\beta_{m+1,0}) = 0, s_{2m}(\beta_{l,m-l+1}) = (-1)^{m+1} 4C_{2m+2}^{2l}, 0 < l < m+1,$

b) $s_{2m}(\alpha_{m+2,0}) = 0, s_{2m}(\alpha_{m+1,1}) = (-1)^{m+1} 4(C_{2m}^2 - C_{2m}^0),$
$s_{2m}(\alpha_{l,m-l+2}) = (-1)^{m+1} 4(C_{2m}^{2l} - 2C_{2m}^{2l-2} + C_{2m}^{2l-4}), 1 < l < m+1.$

We set $\varphi_{n,i} = C_{2n}^{2i} - C_{2n}^{2i-2}$. From equations a) and b) we obtain that the smallest value of the t-characteristic on the group Λ_{4n} is equal to the greatest common divisor of the numbers $\{4C_{2n+2}^{2l}, 4\varphi_{n,l}\}_{i=1,\dots,n}$. Since the greatest common divisor of the numbers $\{C_{2n+2}^{2l}\}_{l \neq 0, n+1}$ is even for $n+1 = 2^j$, and odd for the remaining n, by using the formula $\varphi_{n,l} + C_{2n+2}^{2l} = 2C_{2n+1}^{2l}$, we complete the proof of the lemma.

Remark. It follows from a) that the coefficients of the series $\Theta_1(x,y) = F^+(x,y) + F^-(x,y)$ do not generate the entire ring of coefficients of the two-valued formal group. From b) there follows a similar assertion for the series $\Theta_2(x,y) = F^+(x,y) \cdot F^-(x,y)$.

Let $F(u,v)$ be a formal group over the ring R, and let $g_F(u)$ be its logarithm. Consider the complete set $(\xi_0 = 1, \xi_1, \dots, \xi_{m-1})$ of m-th roots of unity. Let $B_m^{-1}(-y) = \prod_{j=0}^{m-1} g_F^{-1}(\xi_j \sqrt[m]{y}), x = \prod_{j=0}^{m-1} g_F^{-1}(\xi_j g_F(u)) \in R \otimes Q[[u]]$. Then $B_m(x) = g_F(u)^m$ and we obtain the formal power system $F_k^{(m)}(x) = B_m^{-1}(k^m B_m(x)) = \prod_{j=0}^{m-1} g_F^{-1}(k\xi_j g_F(u))$ of type m. The coefficients of the series $F_k^{(m)}(x) = B_m^{-1}(k^m B_m(x))$ automatically lie in the ring R for a formal group $F(u, v)$ with complex multiplication by ξ_j (raising to the power ξ_j). The particular case $m = 2$ of this construction was examined in detail in Lemma 2.12[1].

Example. Consider the formal group $f^{(p)}(u, v) = \bar{\pi}_p^*(f(u, v))$, where $\bar{\pi}_p^*$ is Quillens' p-adic geometric cobordism projector and $f(u, v)$ is the universal formal group over Ω_U. As we have already noted, the logarithm $g^{(p)}(u)$ of the group $f^{(p)}(u, v)$ has the form $g^{(p)}(u) = \bar{\pi}_p^* g(u) = \sum_{h \geqslant 0} \frac{[CP^{p^h-1}]}{p^h} u^{p^h}$. Let $m = (p-1)$, then $\xi_j^{p^h} = \xi_j$ and $g^{(p)}(\xi_j u) = \xi_j g^{(p)}(u), (g^{(p)})^{-1}(\xi_j g^{(p)}(u)) = \xi_j u$. We have $x = -u^{p-1} = \prod_{j=0}^{p-2} (g^{(p)})^{-1}(\xi_j g^{(p)}(u)), B_{p-1}(x) = -(g^{(p)}(u))^{p-1} = B_{p-1}(-u^{p-1})$. Thus formal raising to a power $F_k^{(p-1)}(x) = B_{p-1}^{-1}(k^{p-1} B_{p-1}(x))$ for the group $f^{(p)}(u, v)$ is "integer valued", and $F_k^{(p-1)}(x) = k^{p-1}\Psi^k(-u^{p-1})$. Consequently in U_p^*-theory the $(p-1)$-th powers of geometrical cobordisms are the range of definition of a power system of type $s = p - 1$.

We now note that the roots of unity of degree $p-1$ lie in the ring p-adic integers $\mathbf{Z}_p$. Therefore $g^{-1}(\xi_j g(u)) \in \Omega_U[[u]] \otimes \mathbf{Z}_p$ and $\prod_{i=0}^{p-2} g^{-1}(\xi_j g(u)) = x \in \Omega_U[[u]] \otimes \mathbf{Z}_p, \prod_{j=0}^{p-2} g^{-1}(k\xi_j g(u)) \in \Omega_U[[u]] \otimes \mathbf{Z}_p$, and the series $B_m(x)$ defines a power system of type $m = p-1$, whose p-adic projector was given in the above example.

The Adams operators are evaluated for an element x by the formula $k^{p-1}\Psi^k(x) = B_{p-1}^{-1}(k^{p-1} B_{p-1}(x))$ in $U^* \otimes \mathbf{Z}_p$-theory.

In analogy with Theorem 2.13 we have

Theorem 2.25. *The power system $B_{p-1}^{-1}(k^{p-1} B_{p-1}(x))$ of type $s = p - 1$, considered over the minimal ring of its coefficients, is universal in the class of all power systems of type $(p-1)$ over torsion-free $\mathbf{Z}_p$-rings.*

The proof, as did for Theorem 2.13, follows from the fact that the coefficients of the series $B_{p-1}(x)$ are all non-zero and algebraically independent in $\Omega_U \otimes Q_p$, where Q_p is the field of p-adic numbers.

[1]Here it is also appropriate to speak of the "multi-valued formal group" $(F(x, y) = \{B_m^{-1}[(\sqrt[m]{B_m(x)} + \xi_k \sqrt[m]{B_m(y)})^m], k = 0, \dots, m-1\}$. It would be interesting to know the nature of the ring of coefficients in this case.

§ 3. Fixed points of transformations of order p

We turn now to a different question which is also connected with the formal group of geometrical cobordisms and a type $s = 1$ system associated with it; namely, to the theory of fixed points of transformations $T (T^0 = 1)$ of quasicomplex manifolds (see [5], [9], [13]), which act so that the manifolds of fixed points have trivial normal bundle (or, for example, only isolated fixed points $\mathscr{P}_1, \cdots, \mathscr{P}_q \in M^n, T(\mathscr{P}_j) = \mathscr{P}_j$. If the transformation $dT|_{\mathscr{P}_j}$ has eigenvalues $\lambda_k^{(j)} = \exp\left\{\frac{2\pi i x_k^{(j)}}{p}\right\}, k = 1, \ldots, n, j = 1, \ldots, q$, then the "Conner-Floyd invariants" $\alpha_{2n-1}(x_1^{(j)}, \ldots, x_n^{(j)}) \in U_{2n-1}(BZ_p)$, and it is known that

$$U^*(BZ_p) = \Omega_U[[u]]/(p\Psi^p(u)) \quad \text{(see [12])},$$

$$\alpha_{2n-1}(x_1, \ldots, x_n) = \prod_{j=1}^{n} \frac{u}{g^{-1}(x_j g(u))} \bigcap \alpha_{2n-1}(1, \ldots, 1),$$

where $u^k \bigcap \alpha_{2n-1}(1, \ldots, 1) = \alpha_{2(n-k)-1}(1, \ldots, 1)$ (see [5], [9], [13]) and $g^{-1}(xg(u)) = x\Psi^x(u)$. Here it is already clear that only the coefficients of the power system enter into the expression for $U^*(BZ_p)$ and $\alpha_{2n-1}(x_1, \ldots, x_n)$. There is still one further question: on which classes of Ω_U can the group $Z_p = \mathbf{Z}/p\mathbf{Z}$ act? As is shown in [5], [9], the basis relations $0 = \alpha_{2n-1}(x_1, \ldots, x_n) - \prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)} \bigcap \alpha_{2n_1}(1, \ldots, 1)$ and $0 = p\frac{\Psi^p(u)}{u} \bigcap \alpha_{2n-1}(1, \ldots, 1)$, are realized on the manifolds $M^n(x_1, \ldots, x_n)$ and $M^n(p)$, and determine the elements $\left[\prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)}\right]_n \in \Omega_U^{2n} (\text{mod } p\Omega_U)$ and $\left[p\frac{\Psi^p(u)}{u}\right]_n \in \Omega_U^{2n} (\text{mod } p\Omega_U)$, whence it follows that the cobordism class of the manifold with Z_p-action of this sort coincides $(\text{mod}\, p\Omega_U)$ with the Ω_U-module $\bar{\Lambda}(1) = \Omega_U \cdot \Lambda^+(1)$, where $\Lambda^+(1)$ is the positive part of the ring $\Lambda(1)$ of coefficients of the power system $g^{-1}(kg(u))$. On the other hand, from Atiyah and Bott's results [15] for the complex d'' on forms of type $(0, q)$ and holomorphic transformations $T : M^n \to M^n$ we may introduce the following formula for the Todd genus $T(M^n) \mod p$, for example.

Lemma 3.1. *Let* $\lambda_k^{(j)} = \exp\left(\frac{2\pi i x_k^{(j)}}{p}\right)$ *be the eigenvalues of the* *transformation* dT *on the fixed points* $\mathscr{P}_j, j = 1, \dots, q, k = 1, \dots, n$. *Then*

$$-T(M^n) \equiv \sum_{j=1}^{q} \frac{p-1}{\prod_{k=1}^{n} -x_k^{(j)}}$$
$$\times \sum_{l=-\left[\frac{n}{p-1}\right]}^{\infty} (-p)^l \left[\prod_{k=1}^{n} \frac{-x_k^{(j)} z}{1 - \exp\left\{-x_k^{(j)}\left(z + \frac{z^p}{p}\right)\right\}}\right]_{n+l(p-1)} \mod \beta.$$

This formula and its proof were communicated by D. K. Faddeev.

PROOF. For the Euler characteristic $\chi(T)$ of the indicated elliptic complex we have the Atiyah-Bott formula:

$$\chi(T) = \sum_{j=1}^{q} \frac{1}{\det(1 - \overline{dT})\mathscr{P}_j} = \sum_{j=1}^{n} \prod_{k=1}^{q} \frac{1}{1 - \exp\left\{-\frac{2\pi i x_k^{(j)}}{p}\right\}}.$$

Since $\frac{1}{p} \sum_{l \in Z_p} \chi(T^l) = \varphi$ is the alternating sum of the dimensions of the invariant spaces of the action T on the homology of the complex $\chi(1) = T(M^n)$, we have

$$\chi(1) = T(M^n) = -\sum_{j=1}^{q} \sum_{l=1}^{p-1} \prod_{k=1}^{n} \frac{1}{1 - \exp\left\{-\frac{2\pi x_k^{(j)} l}{p}\right\}} + p\varphi.$$

If $\mathrm{Tr} : Q(\sqrt[p]{1}) \to Q$ is the number-theoretic trace, then by definition we have

$$-T(M^n) \equiv \sum_{j=1}^{q} \mathrm{Tr}\left(\prod_{k=1}^{n} \frac{1}{1 - \exp\left\{\frac{2\pi i x_k^{(j)}}{p}\right\}}\right) \mod p.$$

The field $Q(\sqrt[p]{1})$ and the field Q are embedded in the p-adic completions of $k = Q_p(\varepsilon), \varepsilon = \sqrt[p]{1}$, and Q_p. There exists in the field k an element λ such that $\lambda^{p-1} = -p$ and $k = Q_p(\lambda)$. Next, $\mathrm{Tr}(\lambda^s) = 0$ for $s \not\equiv 0 \mod (p-1)$ and $\mathrm{Tr}(\lambda^{k(p-1)}) = (-1)^k p^k (p-1)$. Since $\varepsilon = \exp\left(z + \frac{z^p}{p}\right)\Big|_{z=\lambda}$, we have

$$\exp\left\{-\frac{2\pi i x_k}{p}\right\} = \varepsilon^{-x_k} = \exp\left\{-\left(x_k z + x_k \frac{z^p}{p}\right)\right\}$$

(in k). Therefore

$$\prod_{k=1}^{n}\frac{1}{1-\varepsilon^{-x_k}}=\prod_{k=1}^{n}\frac{1}{1-\exp\left\{-\left(x_k z+x_k\frac{z^p}{p}\right)\right\}}\Bigg|_{z=\lambda}$$
$$=\frac{1}{z^n\prod_{k=1}^{n}-x_k}\prod_{k=1}^{n}\frac{x_k z}{1-\exp\left\{-x_k\left(z+\frac{z^p}{p}\right)\right\}}$$
$$=\frac{(-1)^n}{z^n\prod_{k=1}^{n}x_k}\left(1+\sum_{s=1}^{\infty}P_s(x_1,\dots,x_n)\lambda^s\right).$$

Thus,

$$\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-\varepsilon^{-x_k}}\right)=\frac{p-1}{\prod_{k=1}^{n}-x_k}$$
$$\times\sum_{l=-\left[\frac{n}{p-1}\right]}^{\infty}(-p)^k\left[\prod_{k=1}^{n}\frac{x_k z_k}{1-\exp\left\{-x_k\left(z+\frac{z^p}{p}\right)\right\}}\right]_{n+l(p-1)}.$$

The proof of the lemma is concluded by summing over the fixed points.

For $p>n+1$ this gives the formula

$$T(M^n)=\sum_{j=1}^{q}\frac{(-1)^n}{x_1^{(j)}\dots x_n^{(j)}}\left[\prod\frac{-x_k^{(j)}z}{1-\exp\{-x_k^{(j)}z\}}\right]_n,$$

proved in [13] as a consequence of Tamura's results.

We see that by Atiyah and Bott's procedure each fixed point is assigned a rational invariant. How does the analogous procedure look in bordism theory?

Let us define the functions $\gamma_p(x_1,\dots,x_n)\in\Omega_U\left[\frac{1}{p}\right]$ such that under the action of T on $M^n, T^p=1$, with isolated fixed points $\mathscr{P}_1,\dots,\mathscr{P}_q$ having weights $x_k^{(j)}, j=1,\dots,q, k=1,\dots,n$, the relation

$$\sum_{j=1}^{q}\gamma_p(x_1^{(j)},\dots,x_n^{(j)})\equiv[M^n]\mod p\Omega_U$$

is valid. Consider the $\Omega_U\otimes\mathbf{Z}_p$-free resolvent of the module $U_*(BZ_p,\text{point})$:

$$0\longrightarrow F_1\xrightarrow{d}F_0\longrightarrow(BZ_p,\text{point})\longrightarrow 0,$$

where for the generators of $U_*(BZ_p, \text{point})$ we take the elements $\alpha_{2n-1}(x_1, \ldots, x_n) \in \in U_{2n-1}(BZ_p, *)$ and the minimal module of relations is spanned by the relations $a(x_1, \ldots, x_n) = \alpha_{2n-1}(x_1, \ldots, x_n) - \left(\prod_{i=1}^n \frac{u}{x_i \Psi^{x_i}(u)}\right) \bigcap \alpha_{2n-1}(1, \ldots, 1)$ and $a_n = p\alpha_{2n-1}(1, \ldots, 1) + \left(\frac{p\Psi^p(u)}{u}\right) \bigcap \alpha_{2n-1}(1, \ldots, 1)$. Let $\Phi : F_1 \to \Omega_U \otimes \mathbf{Z}_p$ denote the $\Omega_U \otimes \mathbf{Z}_p$-module such that $\Phi(a(x_1, \ldots, x_n)) = \left[\prod_{i=1}^n \frac{u}{x_i \Psi^{x_i}(u)}\right] \in \Omega_U \otimes \mathbf{Z}_p$ and $\Phi(a_n) = -\left[\frac{p\Psi^p(u)}{u}\right]_n$. As we pointed out above, for any set of weights $(x_1, \ldots, x_n)$ we have the congruence $\Phi(a(x_1, \ldots, x_n)) \equiv [M^n] (\mathrm{mod}\ p)$, where M^n is a quasi-complex manifold on which the relation $a(x_1, \ldots, x_n)$ is realized. Relative to the multiplication operation, out of the relations in $U_*(BZ_p)$ the group F_1 is a ring, and, as is clear, the homomorphism $\Phi \mod p : F_1 \to \Omega_U$ $(\mathrm{mod}\ p\Omega_U)$ coincides with the well-known ring homomorphism which associates with each relation in F_1 the bordism class $\mod p$ of the manifold on which this relation is realized. The homomorphism Φ can be extended to a homomorphism

$$\gamma_p : F_0 \to \Omega_U \otimes Q_p, \quad \gamma_p(dF_1) = \Phi.$$

Lemma 3.2. *For any set of weights $(x_1, \ldots, x_n)$ we have the formula*

$$\gamma_p(x_1 \ldots x_n) = \left[\frac{1}{x_1 \ldots x_n} \left(\prod_{j=1}^n \frac{u}{\Psi^{x_j}(u)}\right) \frac{u}{\Psi^p(u)}\right]_n.$$

In particular,

$$\gamma_p(1, \ldots, 1) = \left[\frac{u}{\Psi^p(u)}\right]_n.$$

Proof. In the free $\Omega_U \otimes \mathbf{Z}_p$-module F_0 we have the identity

$$\alpha_{2n-1}(x_1, \ldots, x_n) = a(x_1, \ldots, x_n) + \sum_{k=0}^{n-1} \left[\prod_{j=1}^n \frac{u}{x_j \Psi^{x_j}(u)}\right]_k \alpha_{2n-2k-1},$$

$\left[\frac{u}{x_i \Psi^{x_i}(u)}\right]_k \in \Omega_U^{2k} \otimes \mathbf{Z}_p, \alpha_{2n-2k-1} = \alpha_{2n-2k-1}(1, \ldots, 1)$. Also, since $\gamma_p : F_0 \to \Omega_U \otimes Q_p$ is an $\Omega_U \otimes \mathbf{Z}_p$-module homomorphism and $\gamma_p(a(x_1, \ldots, x_n)) = \left[\prod_{i=1}^n \frac{u}{x_i \Psi^{x_i}}\right]_n$, it is sufficient to prove the lemma for the set of weights $(1, \ldots, 1)$. We have

$$-a_n + \sum_{k=0}^{n-1} \left[\frac{p\Psi^p(u)}{u}\right]_k \alpha_{2n-2k-1} = 0, \quad n \geqslant 1,$$

$$\gamma_p(a_n) + \sum_{k=0}^{n-1} \left[\frac{p\Psi^p(u)}{u}\right]_k \gamma_p(\alpha_{2n-2k-1}) = 0,$$

$$\left[\frac{p\Psi^p(u)}{u}\right]_n + \sum_{k=0}^{n-1} \left[\frac{p\Psi^p(u)}{u}\right]_k \gamma_p(\alpha_{2n-2k-1}) = 0,$$

$$\left(\frac{p\Psi^p(u)}{u}\right)\left(1 + \sum_{j=1}^{\infty} \gamma_p(\alpha_{2j-1})u^j\right) = p,$$

$$1 + \sum_{j=1}^{\infty} \gamma_p(\alpha_{2j-1})u^j = \frac{u}{\Psi^p(u)}$$

and the lemma is proved.

It follows immediately from the definition of the homomorphism $\Phi : F_1 \to \Omega_U \otimes \mathbf{Z}_p$ that $\operatorname{Im}\Phi(F_1) = \tilde{\Lambda}(1) \otimes \mathbf{Z}_p \subset \Omega_U \otimes \mathbf{Z}_p$, where $\tilde{\Lambda}(1) = \Lambda^+(1) \cdot \Omega_U$ and $\Lambda(1)$ is the ring of coefficients of the power system $\{k\Psi^k(u)\}_{k=\pm1,\pm2,\ldots}$.

Lemma 3.3. *The group* $\operatorname{Im}\gamma_p(F_0) \subset \Omega_U \otimes Q_p$ *coincides with the* $\Omega_U \otimes \mathbf{Z}_p$*-module spanned by the system of polynomial generators* $\delta_{n,p}$ *of the ring* $\Omega_U(\mathbf{Z}) \otimes \mathbf{Z}_p$ *of coefficients of the logarithm for the formal group* $f(u,v) \otimes \mathbf{Z}_p$, *where* $1 + \sum_{n=1}^{\infty} \delta_{n,p} t^n = \dfrac{t}{\Psi^p(t)}$.

The proof of the lemma follows easily by evaluating the t-characteristic of the coefficients of the series $\Psi^p(u) = \dfrac{g^{-1}(pg(u))}{p}$, by means of the fact that all the Chern numbers of the coefficients of the series $p\Psi^p(u)$ are divisible by p, and from the form of the functions $\gamma_p(x_1,\ldots,x_n)$, given in Lemma 3.2.

From the exactness of the sequence

$$0 \to F_1 \to F_0 \to U_*(BZ_p, \text{point}) \to 0$$

we now find that a $\Omega_U \otimes \mathbf{Z}_p$-module homomorphism

$$\gamma_p : U_*(BZ_p, \text{point}) \to \gamma(F_0)/\Phi(F_1),$$

is defined, where $\Phi(F_1) = \tilde{\Lambda}(1) \otimes \mathbf{Z}_p$ and $\gamma(F_0)/\Phi(F_1) \subset \Omega_U(\mathbf{Z})/\tilde{\mathbf{\Lambda}}(\mathbf{1}) \otimes \mathbf{Z_p}$, which is clearly an epimorphism. By collecting these results together, we arrive at the following Theorem.

Theorem 3.4. *Functions* $\gamma_p(x_1,\ldots, x_n)$, *of the fixed points are defined which take on values in the ring* $\Omega_U(\mathbf{Z}) \otimes \mathbf{Z_p}$ *of coefficients of the logarithm* $g(u) = \sum \dfrac{[CP^n]}{n+1} u^{n+1}$ *of the formal group* $f(u,v) \otimes \mathbf{Z}_p$ *for which* *a) for the action of T on the quasicomplex manifold $M^n, T^p = 1$, with the fixed manifold of classes $\lambda_j \in \Omega_U$, having weights $(x_k^{(j)}) \in Z_p^*$ in the (trivial) normal bundles, we have the relations*

$$[M^n] \equiv \sum_j \lambda_j \gamma_p(x_1^{(j)}, \ldots, x_{m_j}^{(j)}) \bmod p\,\Omega_U, \quad [M^n] \in \tilde{\Lambda}(1), m_i + \dim \lambda_i = n,$$

and

$$\gamma_p(x_1, \ldots, x_m) = \left[\frac{u}{\Psi^p(u)} \prod_{j=1}^m \frac{u}{x_j \Psi^{x_j}(u)}\right]_m;$$

b) the quotient module over $\Omega_U \otimes \mathbf{Z}_p$, equal to $\Omega_U(\mathbf{Z})/\tilde{\mathbf{\Lambda}}(\mathbf{1}) \otimes \mathbf{Z_p}$, contains the non-trivial image of the module $U^*(BZ_p$, point) under γ_p, coinciding with the quotient module $\tilde{\Lambda}(1)$ if the submodule in $\Omega_U(\mathbf{Z}) \otimes \mathbf{Z_p}$, which is spanned by the system of polynomial generators $\delta_{n,p}$.

Here $\tilde{\Lambda}(1)$ is the Ω_U-module which is generated by the ring $\Lambda^+(1)$ of coefficients of the power system $\{g^{-1}(kg(u))\}$ of type $s = 1$, and $1 + \sum_{n \geqslant 1} \delta_{n,p} t^n = \dfrac{t}{\Psi^p(t)}$.

Remark 1. If one deals with the action of a transformation $T, T^p = 1$, having isolated fixed points, then we have the group $U_{\mathrm{isol}}(Z_p) \subset U_*(BZ_p)$, spanned by all the elements $\alpha_{2n-1}(x_1, \ldots, x_n)$ (without the structure of an Ω_U-module), with the resolvent Z_p:

$$0 \to G_1 \xrightarrow{d} G_0 \to U_{isol}(Z_p) \to 0,$$

where G_0, G_1 are free and the generator G_1 is a formal relation. As above, homomorphisms

$$\Phi : G_1 \to \Omega_U \otimes \mathbf{Z}_p \quad \text{and} \quad \Phi' : G_0 \to \Omega_U \otimes Q_p,$$

are defined, where $\Phi' d = \Phi$. The quotient group $\Phi'(G_0)/\Phi(G_1)$ is a p-group and there exists a homomorphism

$$\gamma : U_{\mathrm{isol}}(Z_p) \to \Phi'(G_0)/\Phi(G_1).$$

We now consider the mappings $U_* \to K_*$ and $U^* \to K^*$ generated by the Todd genus. For the T-genus we have

$$T(\gamma_p(x_1, \ldots, x_n)) = \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^n \frac{u}{1-(1-u)^{x_k}}\right]_n \in Q_p.$$

For example, $T(\gamma_2(1, \ldots, 1)) = \frac{1}{2^n}$.

Under the action of the group Z_p on the manifold M^n with isolated fixed points $\mathscr{P}_1, \ldots, \mathscr{P}_q$ having weights $x_k^{(j)}, k = 1, \ldots, n, j = 1, \ldots, q$, we have the formula

$$T(M^n) \equiv \sum_{j=1}^{q} T(\gamma_p(x_1^{(j)}, \dots, x_n^{(j)})) \mod p\mathbf{Z}_p,$$

where $p\mathbf{Z}_p \subset Q_p$. At first sight this formula differs from the Atiyah-Bott formula given in Lemma 3.1. The question arises of how to reconcile these two formulas[1]? Another question, similar to the subject of the Stong–Hattori theorem [7], is: does the set of relations given by Atiyah–Bott–Singer for the action of Z_p on all possible elliptic complexes define an extension $\Omega_v(\mathbf{Z}) \otimes \mathbf{Z}_p$ of the cobordism ring (more precisely, the module $\tilde{\Lambda}(1)$ and the ring $\Lambda(1)$ in Ω_U)?

We now show that the results of [17] permit us to generalize our construction to the case of the action of a transformation $T, T^p = 1$, for which the manifolds of fixed points have arbitrary normal bundle.

Let T be a transformation of order p on the manifold M^n. As was shown in [16], the normal bundle ν_j at any pointwise-fixed manifold $N_j \subset M^n$ can be represented in the form $\nu_j = \bigoplus_{k=1}^{p-1} \nu_{jk}$, where the action of the group Z_p on the fiber ν_{jk} given by multiplication by the number $e^{\frac{2\pi i}{p}k}$. Thus the set of all fixed point submanifolds of the transformation T together with their normal bundles defines an element of the group $A = \sum U_*\left(\prod_{k=1}^{p-1} BU(l_k)\right)$, where the sum extends $l_1, \dots, l_{p-1}, l \geqslant 0$. By using the mapping $BU(n) \times BU(m) \to BU(n+m)$ (Whitney sum), a multiplication can be introduced into A. It is not difficult to show that A becomes a polynomial ring $\Omega_U[a_{j,k}], j \in Z_p^*, k \geqslant 0$, where $a_{j,k}$ is the bordism class of the embedding $CP^k \subset CP^\infty = BU(1)$, considered together with the action of the transformation $T = e^{\frac{2\pi i}{p}j}$ on the Hopf fiber bundle over CP^k. We introduce a grading into A by setting $\dim a_{j,k} = 2(k+1)$. We next describe the fixed point submanifolds N_m in terms of the generators $a_{j,k}$. For example, a fixed point with weights $(x_1, \dots, x_n)$ is described by the monomial $a_{0,x_1} \dots, a_{0,x_n}$. Consider the canonical homomorphism $\alpha : A \to U_*(BZ_p,\text{point})$, corresponding to the free action of the group Z_p on the sphere bundle associated with the normal fiber at a fixed point manifold. Denote by $\alpha((x_1, k_1), \dots, (x_l, k_l))$ the image under α of the monomial $a_{x_1,k_1} \dots, a_{x_l,k_l}, \alpha((x_1, k_1), \dots, (x_l, k_l)) \in U_{2n-1}(BZ_p,\ \text{point})$, where $n = \sum_{m=1}^{l} (k_m + 1)$. From [17] we take the following description of the elements $\alpha((x_1, k_1), \dots, (x_l, k_l))$.

[1] An answer to this question is given in the Appendix.

$$G(u,t) = \frac{\frac{\partial}{\partial t} g(ut)}{f(u, \overline{ut})} = 1 + \sum_{n=1}^{\infty} G_n(u) t^n,$$

where $g(ut) = \sum_{n=0}^{\infty} \frac{[CP^n]}{n+1}(ut)^{n+1}$ is the logarithm of the formal group $f(u,v)$ and $\overline{ut} = g^{-1}(-g(ut))^1$. Clearly $G_n(0) = 1$ for any $n \geqslant 1$. We set

$$\Psi^{x,n}(u) = \frac{\Psi^x(u)}{G_n(x\Psi^x(u))}.$$

We have $\Psi^{x,0}(u) = \Psi^x(u)$, $\Psi^{1,n}(u) = \frac{u}{G_n(u)}$. From [17] we find that for any set $((x_1,k_1),\dots,(x_l,k_l))$, $\sum_{m=1}^{l}(k_m+1) = n$, we have

$$\alpha((x_1,k_1),\dots,(x_l,k_l)) = \left(\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j,k_j}(u)}\right) \bigcap \alpha_{2n-1}(1,\dots,1).$$

Since $\Psi^{1,0}(u) = u$, from [16] we find that the relation $\alpha((x_1,k_1),\dots,(x_l,k_l)) = \left(\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j \cdot k_j}(u)}\right) \bigcap \alpha_{2n-1}(1,\dots,1)$ is realized in the manifold M^n, determined by the element

$$\left[\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j,k_j}(u)}\right]_n \in \Omega_v^{2n} \mod p\,\Omega_U.$$

By repeating the proof of Lemma 3.2, we obtain the following theorem.

Theorem 3.5. *A homomorphism $\gamma_p : A \otimes \mathbf{Z}_p \to \Omega_U \otimes Q_p$ is defined such that for any set $((x_1,k_1),\dots,(x_l,k_l))$ we have the formula*

$$\gamma_p((x_1,k_1),\dots,(x_l,k_l)) = \left[\frac{1}{x_1\dots x_l}\left(\prod_{j=1}^{n} \frac{u}{\Psi^{x_j,k_j}(u)}\right)\frac{u}{\Psi^p(p)}\right]_n,$$

$$n = \sum_{m=1}^{l}(k_m+1),$$

and if the element $a \in A$ corresponds to the union of all the fixed point submanifolds of the Z_p action on M^n, then $\gamma_p(a) \equiv [M^n] \mod p$.

[1] Note that under the substitution $t \to \frac{z}{u}$ (u is a parameter) the differential $G(u,t)dt$ goes into the meromorphic differential $\frac{dg(z)}{f(u,\bar z)}$ on the group $f(u,v)$, which is invariant with respect to the shift $u \to f(u,w), z \to f(z,w)$.

Appendix

The Atiyah–Bott formula, the functions $\gamma_p(x_1, \dots, x_n)$ of fixed points in bordism and the Conner–Floyd equation

Let ε be a primitive p-th root of unity, and let Tr: $Q(\varepsilon) \to Q$ be the number-theoretic trace.

Definition 1. The *Atiyah–Bott* function $\mathrm{AB}(x_1, \dots, x_n)$ of *fixed points* is the function which associates with each set of weights $(x_1, \dots, x_n), x_j \in Z_p$, the rational number

$$\mathrm{AB}(x_1, \dots, x_n) = -\operatorname{Tr}\left(\prod_{k=1}^{n} \frac{1}{1 - \exp\left\{\frac{2\pi}{p} x_k\right\}}\right).$$

As a corollary of the Atiyah–Bott formula for fixed points, we have

Theorem 2. *Let $f : M^n \to M^n$ be a holomorphic transverse mapping of period p of a compact complex manifold M^n and let $\mathscr{P}_1, \dots, \mathscr{P}_q$ be its fixed points. The mapping $df|\mathscr{P}_j$ in the tangent space at the fixed point $\mathscr{P}_j$ has the eigenvalue $\lambda_k^{(j)} = \exp\left\{\frac{2\pi i}{p} x_k^{(j)}\right\}, k = 1, \dots, n$, then the number*

$$\sum_{j=1}^{q} AB(x_1^{(j)}, \dots, x_n^{(j)})$$

is an integer and coincides modulo p with the Todd genus $T(M^n)$ of the manifold M^n.

PROOF. According to the Atiyah–Bott theorem for an elliptic complex d'', for forms of type $(0, l)$ we have

$$\chi(f) = \sum_{j=1}^{q} \prod_{k=1}^{n} \frac{1}{1 - \exp\left\{\frac{2\pi i}{p} x_k^{(j)}\right\}},$$

where

$$\chi(f) = \sum_{m=0}^{\infty} (-1)^m \operatorname{Tr} f^*|_{H^{0,m}(M^n)}.$$

As is known, $\chi(1) = T(M^n)$ and $\frac{1}{p} \sum_{m \in Z_p} \chi(f^m) = \varphi$ is the alternating sum of the dimensions of the invariant subspaces under the action of the transformation f^* on the cohomology $H^{0,m}(M^n)$. Consequently,

$$T(M^n) = -\sum_{j=1}^{q} \sum l_{m=1}^{p-1} \prod_{k=1}^{n} \frac{1}{1 - \exp\left\{\frac{2\pi i}{p} x_k^{(j)} \cdot m\right\}} + p\varphi.$$

By now making use of the definition of the number-theoretic trace and the Atiyah–Bott function, the theorem is proved.

We shall calculate $\operatorname{Tr}\left(\prod_{k=1}^{n} \frac{1}{1-\zeta^{x_k}}\right)$, where $\zeta = e^{\frac{2\pi i}{p}}$. Let us set $\theta = 1-\zeta$. We shall perform all calculations in the field $Q_p(\theta)$. By $\simeq$ we mean equality modulo the group $pZ_p \subset Q_p(\theta)$. The following lemma, like Lemma 3.1, has been provided at our request by D. K. Faddeev.

Lemma 3. *For the Atiyah-Bott function $AB(x_1,\dots,x_n)$ we have the formulas*

$$AB(x_1,\dots,x_n) \simeq \left[\frac{p < u >_{p-1}}{< u >_p} \prod_{k=1}^{n} \frac{u}{< u >_{x_k}}\right]_n,$$

$$AB(x_1,\dots,x_n) \simeq \sum_{m=0}^{n} \left[\frac{pu}{< u >} \prod_{k=1}^{n} \frac{u}{< u >_{x_k}}\right]_m,$$

where $< u >_q = 1-(1-u)^q$ is the q-th power of the element u in the formal group $f(u,v) = u+v-uv$ and $[\varphi(u)]_k$ is the coefficient of u^k in the power series $\varphi(u)$.

Proof. First of all note that $\operatorname{Tr}(\theta^k) \simeq 0$ for all $k > 1$. We have

$$\prod_{k=1}^{n} \frac{1}{1-\zeta^{x_k}} = \prod_{k=1}^{n} \frac{1}{1-(1-\theta)^{x_k}} = \frac{1}{\theta^n} \prod_{k=1}^{n} \frac{\theta}{1-(1-\theta)^{x_k}} = \frac{1}{\theta^n} \sum_{k=0}^{\infty} A_k \theta^k,$$

where $A_k \in Z_p$, and

$$\operatorname{Tr}\left(\frac{1}{\theta^n} \sum_{k=0}^{\infty} A_k \theta^k\right) \simeq \operatorname{Tr}\left(\frac{1}{\theta^n} \sum_{k=0}^{n} A_k \theta^k\right) = \sum_{k=0}^{n} A_k \cdot Tr(\theta^{k-n}).$$

Let us set $\operatorname{Tr}\theta^{-s} = B_s$ and introduce the two formal series

$$A(u) = \sum_{k=0}^{\infty} A_k u^k \quad \text{and} \quad B(u) = \sum_{k=0}^{\infty} B_k u^k.$$

Thus we must calculate the coefficient of u^k in the series $A(u) \times B(u)$. We have

$$B(u) = \operatorname{Tr}\left(1 + \sum_{s=1}^{\infty} \theta^{-1} u^s\right) = \operatorname{Tr}\left(\frac{1}{1-\theta^{-1}u}\right) = \operatorname{Tr}\left(\frac{\theta}{\theta-u}\right)$$
$$= \operatorname{Tr}\left(1 + \frac{u}{\theta-u}\right) = (p-1) + u \operatorname{Tr}\left(\frac{1}{\theta-u}\right).$$

First note that if $\varphi_a(u)$ is the minimal polynomial of the element α with respect to the extension $Q_p(\theta)|Q_p$, then

$$\operatorname{Tr}\left(\frac{1}{\alpha-u}\right) = -\frac{\varphi'_\alpha(u)}{\varphi_\alpha(u)}.$$

Since

$$\frac{\zeta^p-1}{\zeta-1} = \frac{(1-\theta)^p-1}{-\theta} = \frac{1-(1-\theta)^p}{\theta},$$

it follows that

$$\varphi_\theta(u) = \frac{1-(1-u)^p}{u}.$$

We have

$$-\operatorname{Tr}\left(\frac{1}{\theta-u}\right) = \frac{\varphi'_\theta(u)}{\varphi_\theta(u)} = \frac{p(1-u)^{p-1}}{1-(1-u)^p} - \frac{1}{u}.$$

Thus

$$B(u) = (p-1) - u\left(\frac{p(1-u)^{p-1}}{1-(1-u)^p} - \frac{1}{u}\right) = \frac{p(1-(1-u)^{p-1})}{1-(1-u)^p},$$

$$\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-\zeta^{x_k}}\right) \simeq [A(u)\cdot B(u)]_n$$

$$\simeq \left[\frac{p(1-(1-u)^{p-1})}{1-(1-u)^p}\prod_{k=1}^{n}\frac{u}{1-(1-u)^{x_k}}\right]_n,$$

and we obtain the first formula

$$-\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-\zeta^{x_k}}\right) = \left[\frac{p < u >_{p-1}}{< u >_p}\prod_{k=1}^{n}\frac{u}{< u >_{x_k}}\right]_n.$$

Next

$$\frac{p(1-(1-u)^{p-1})}{1-(1-u)^p} = p\frac{(1-u)-(1-u)^p}{(1-u)(1-(1-u)^p)}$$

$$= \frac{p}{1-u} - \frac{pu}{(1-u)(1-(1-u)^p)} \simeq -\frac{pu}{1-(1-u)^p}(1+u+u^2+\ldots),$$

and we obtain the second formula

$$-\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-\zeta^{x_k}}\right) \simeq \sum_{m=0}^{n}\left[\frac{pu}{1-(1-u)^p}\prod_{k=1}^{n}\frac{u}{1-(1-u)^{x_k}}\right]_m.$$

The lemma is therefore proved.

In § 3 the functions of the fixed points $\gamma_p(x_1, \ldots, x_n)$ having values in the ring $\Omega_U \otimes Q$

$$\gamma_p(x_1, \ldots, x_n) = \left[\frac{u}{\Psi^p(u)} \prod_{k=1}^{n} \frac{u}{x_k \Psi^{x_k}(u)} \right]_n$$

were considered. By considering the composition of the function γ_p with the Todd genus $T : \Omega_U \to \mathbf{Z}$, we obtain a function (which we continue to denote by $\gamma_p(x_1, \ldots, x_n)$), which associates with a set of weights the rational number $\mod p\, \mathbf{Z}_p$

$$\gamma_p(x_1, \ldots, x_n) = \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k}} \right]_n = \left[\frac{pu}{< u >_p} \prod_{k=1}^{n} \frac{u}{< u >_{x_k}} \right]_n$$

which is such that under the conditions of Theorem 2 the number

$$\sum_{j=1}^{n} \gamma_p(x_1^{(j)}, \ldots, x_n^{(j)})$$

is a p-adic integer and coincides modulo p with the Todd genus.

We now recall the Conner–Floyd equation introduced in [13]. If the group Z_p acts complexly on a manifold M^n with fixed points $\mathscr{P}_1, \ldots, \mathscr{P}_q$, where it has the set of weights $(x_i^{(j)}, \ldots, x_n^{(j)})$, $j = 1, \ldots, q$, then the Conner–Floyd equation

$$\sum_{j=1}^{q} u \prod_{k=1}^{n} \frac{u}{x_k^{(j)} \Psi^{x_k^{(j)}}(u)} = 0,$$

is satisfied, where u is the formal variable which generates the ring $\Omega_U[[u]]$ under the relations $p\Psi^p(u) = 0$ and $u^n = 0$. Consequently there is an element $\varphi \in \Omega_U[u]$ such that the equation $\Omega_U[[u]] \otimes Q$ is valid in the ring

$$\sum_{j=1}^{q} \frac{u}{\Psi^p(u)} \left(\prod_{k=1}^{n} \frac{u}{x_k^{(j)} \Psi^{x_k^{(j)}}(u)} \right) = p\varphi.$$

Thus, if $(x_1^{(j)}, \ldots, x_n^{(j)})$ are the sets of weights of the action of the group Z_p on the manifold M^n, then they are related by the Conner–Floyd equation

$$\sum_{j=1}^{q} \left[\frac{u}{\Psi^p(u)} \prod_{k=1}^{n} \frac{u}{x_k^{(j)} \Psi^{x_k^{(j)}}(u)} \right]_m \simeq 0, \quad m = 0, \ldots, n-1.$$

By considering the Todd genus $T : \Omega_U \to \mathbf{Z}$, we obtain the Conner–Floyd equation

$$\sum_{j=1}^{q} \left[\left(\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right) \right]_m \simeq 0, \quad m = 0, \ldots, n-1,$$

corresponding to the Todd genus.

Definition 4. The *Conner–Floyd functions* $\mathrm{CF}(x_1, \ldots, x_n)_m$, $m = 0, \ldots, n-1$, of *fixed points* are the functions which associate with each set of weights $(x_1, \ldots, x_n)$ the rational numbers

$$\mathrm{CF}(x_1, \ldots, x_n)_m = \left[\frac{pu}{< u >_p} \prod_{k=1}^{n} \frac{u}{< u >_{x_k}} \right]_m, \quad m = 0, \ldots, n-1.$$

Summing up, we obtain the following theorem.

Theorem 5. *The Atiyah–Bott and Conner–Floyd functions of fixed points and the functions $\gamma_p(x_1, \ldots, x_n)$ are related by the equation*

$$AB(x_1, \ldots, x_n) - \gamma_p(x, \ldots, x_n) \simeq \sum_{m=0}^{n-1} CF(x_1, \ldots, x_n)_m.$$

We can now answer the question about the relation of the formulas for fixed points taken from the Atiyah-Bott theory and cobordism theory.

Let $f : M^n \to M^n$ be a holomorphic transverse mapping of period p of the compact complex manifold M^n and let $\mathscr{P}_1, \ldots, \mathscr{P}_q$ be its fixed points. Let the mapping $df|\mathscr{P}_j$ in the tangent space at the fixed point $\mathscr{P}_j$ have eigenvalues $\lambda_k^{(j)} = \exp\left\{\frac{2\pi i}{p} x_k^{(j)}\right\}, k = 1, \ldots, n$. Then the formula which expresses the Todd genus in terms of the weights $(x_1^{(j)}, \ldots, x_n^{(j)})$, taken from the Atiyah–Bott theorem, has the form

$$T(M^n) \simeq \sum_{j=1}^{q} \sum_{m=0}^{n} \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right]_m \tag{1}$$

(see Theorem 2 and Lemma 3). A similar formula, from cobordism theory, has the form

$$T(M^n) \simeq \sum_{j=1}^{q} \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right]_n, \tag{2}$$

and the difference between the first and the second formulae is exactly the sum in the Conner–Floyd equation, expressed for the Todd genus $T : \Omega_U \to \mathbf{Z}$,

$$\sum_{m=0}^{n-1}\left(\sum_{j=1}^{q}\left[\frac{pu}{1-(1-u)^p}\prod_{k=1}^{n}\frac{u}{1-(1-u)^{x_k^{(j)}}}\right]_m\right)\simeq 0,$$

(see § IV of [13])[1].

In conclusion the authors wish to point out that out of the fundamental results of this paper the two different proofs of the theorem concerning the relation of the cohomology operations to the Hirzebruch series were obtained independently (and in the text of § 1 both proofs are presented).

The basic concepts, the general assertions about formal power systems and of the principal examples given of them, particularly the "square modulus" systems of type 2, to a large measure are due to Novikov, while the investigation of the logarithms of these systems by means of the Chern–Dold characters, the precise definition and investigation of the ring of coefficients of the "two-valued formal groups" and their connection with Sp-cobordisms are for the most part due to Buchstaber.

The remaining results were obtained in collaboration, while the important Lemma of §3, and also Lemma 3 of the Appendix, were proved at our request by D. K. Faddeev, to whom the authors express their deep gratitude. We also thank Yu.I. Manin and I.R. Shafarevich for discussions and valuable advice concerning the theory of formal groups and algebraic number theory.

References

[1] M. F. Atiyah. *K*-theory, Lecture Notes, Harvard University, Cambridge, Mass., 1964, 2nd Ed., Benjamin, New York, 1967.

[2] V. M. Buchstaber. The Chern–Dold Character in Cobordisms. I, Mat. Sb., 83(125),(1970), 575–595 (Math. USSR Sb., **12** (1970).

[3] F. Hirzebruch. Neue topologische Methoden in der algebraischen Geometrie, Springer–Verlag, Berlin–Göttingen–Heidelberg, 1956.

[1] Analogous results for the general Hirzebruch genus are obtained in [Т. Е. Панов. Вычисление родов Хирцебруха многообразий, несущих действие группы $\mathbf{Z}/p$, через инварианты действия, Известия РАН, сер. матем., 1998, Т. 62, N. 3, 87–120]. In the same work, the cases of the most important genera are considered: L-genus, $\hat{A}$-genus, χ_y-genus and the elliptic genus. — *V. M. Buchstaber's remark (2004).*

[4] T. Honda. Formal Groups and zeta-functions, Osaka J. Math. **5** (1968), 199–213.

[5] G. G. Kasparov. Invariants of classical lens manifolds in cobordism theory. Math USSR Izv, 3 (1969), 695–706.

[6] M. Kervaire, J. Milnor. Bernoulli numbers, homotopy groups and the theorem of Rokhlin, Proc. Internat. Congr. Math., 1958, Cambridge Univ. Press, 1960, 454–458.

[7] P. Conner, E. Floyd. The relation of cobordism to K-theories, Lecture notes in mathematics, No. 28., Springer-Verlag, New York, 1966.

[8] M. Lazard, Sur les groupes de Lie formels a un parametre, Bull.Soc.Math. France 83 (1955), 251–274.

[9] A. S. Mishchenko. Manifolds under the action of the group Z_p and fixed points, Mat. Zametki 4, No. 4 (1968), 381–386 (in Russian).

[10] J. Milnor. Cobordism ring and complex analogue, Amer. J. Math., 82, No. 3 (1960) 505–521.

[11] S. P. Novikov. Homotopy properties of Thom complexes. Mat. Sb. 1962, v. 57, N 4, 406–442.

[12] S. P. Novikov, Methods of algebraic topology from the point of view of cobordism theory. Izv. Akad. Nauk SSSR, 1967, v. 31, N 4, 885–951.

[13] S. P. Novikov. Adams operators and fixed points. Math USSR Izv., 2 (1968), 1139–1212.

[14] D. Quillen. On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc., 75, No. 6 (1969), 1293–1298.

[15] M. F. Atiyah and R. Bott. Notes on the Lefschetz fixed point for elliptic complexes, Cambridge, Mass. (1964).

[16] P. Conner and E. Floyd. Differentiable periodic maps, Ergebnisse der Mathematik und Ihrer Grenzgebiete, N.F., Band 33, Springer-Verlag, Berlin, 1964.

[17] A. S. Mishchenko. Bordisms with the action of the group Z_p and fixed points, Math USSR Sb., 9 (1969), 291–296.

Index