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Part 2: Characteristic Classes and Smooth Structures on Manifolds

editors

S. P. Novikov & **I. A. Taimanov**

translated by **V. O. Manturov**

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World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Topological library: cobordisms and their applications / edtied by S.P. Novikov & I.A. Taimanov ; translated by V.O. Manturov.
p. cm. -- (Series on knots and everything ; v. 39) (Topological Library)
Includes bibliographical references and index.
ISBN-13 978-981-270-559-4 (hardcover : alk. paper) -- ISBN-10 981-270-559-7
1. Cobordism theory. I. Novikov, Sergei Petrovich. II. Taimanov, I.A. (Iskander Asanovich), 1961–

QA613.66.C63 2007
514'.72--dc22

2007016753

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Series on Knots and Everything — Vol. 44
TOPOLOGICAL LIBRARY
Part 2: Characteristic Classes and Smooth Structures on Manifolds
eds. S.P. Navikov and I.A. Taimanov

ISBN-13 978-981-283-686-1
ISBN-10 981-283-686-1

Typeset by Stallion Press
Email: enquiries@stallionpress.com

Printed in Singapore.

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Contents

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Preface

Topology, created by H. Poincaré in the late 19th and early 20th century as a new branch of mathematics under the name "Analysis Situs" differed in its style and character from other parts of mathematics: it was less rigorous, more intuitive and visible than the other branches. It was not by chance that topological ideas attracted physicists and chemists of the 19th century, for instance, Maxwell, Kelvin and Betti, as well as other scientists residing at the junction of mathematics and physics, such as Gauss, Euler and Poincaré. Hilbert thought it necessary to make this beautiful part of mathematics more rigorous; as it was, it seemed to Hilbert alien.

As a result of the rapid development of 1930s–1960s, it was possible to make all achievements of previously known topology more rigorous and to solve many new deep problems, which seemed to be inaccessible before. This leads to the creation of new branches, which changed not only the face of topology itself, but also of algebra, analysis, geometry — Riemannian and algebraic, — dynamical systems, partial differential equations and even number theory. Later on, topological methods influenced the development of modern theoretical physics. A number of physicists have taken a great interest in pure topology, as in 19th century.

How to learn classical topology, created in 1930s–1960s? Unfortunately, the final transformation of topology into a rigorous and exact section of pure mathematics had also negative consequences: the language became more abstract, its formalization — I would say, excessive, took topology away from classical mathematics. In the 30s and 40s of the 20th century, some textbooks without artificial formalization were created: "Topology" by Seifert and Threlfall, "Algebraic Topology" by Lefschetz, "The topology of fiber bundles" by Steenrod. The monograph "Smooth manifolds and their applications in homotopy theory" by Pontrjagin written in early 50s and, "Morse Theory" by Milnor, written later, are also among the best examples. One should also recommend Atiyah's "Lectures on K-Theory" and Hirzebruch's "New Topological Methods in Algebraic Geometry", and also "Modern geometric structures and fields" by Novikov and Taimanov and Springer Encyclopedia Math Sciences, Vol. 12, Topology-1 (Novikov) and Vol. 24, Topology-2 (Viro and Fuchs), and Algebraic Topology by A. Hatcher (Cambridge Univ. Press).

However, no collection of existing textbooks covers the beautiful ensemble of methods created in topology starting from approximately 1950, that is, from Serre's celebrated "Singular homologies of fiber spaces". The description of this and following ideas and results of classical topology (that finished around 1970) in the textbook literature is reduced to impossible abstractly and to formally stated slices, and in the rest simply is absent. Luckily, the best achievements of this period are quite well described in the original papers — quite clearly and with useful proofs (after the mentioned period of time even that disappears — a number of fundamental "Theorems" is not proved in the literature up to now).

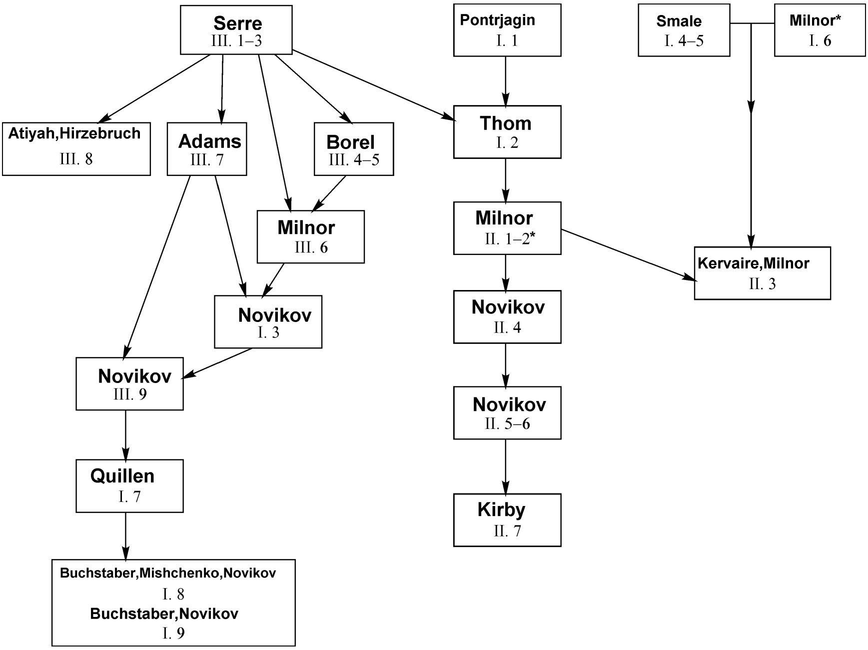
We have decided to publish this collection of works of 1950s–1960s, that allow one to learn the main achievements of the above-mentioned period. Something similar was done in late 1950s in the USSR, when the celebrated collection "Fiber spaces" was published, which allowed one to teach topology to the whole new generation of young mathematicians. The present collection is its ideological continuation. We should remark that the English translations of the celebrated papers by Serre, Thom, and Borel which are well-known for the excellent exposition and which were included in the book of "Fiber spaces" were never published before as well as the English translation of my paper "Homotopical properties of Thom complexes".

Its partition into three volumes is quite relative: it was impossible to collect all papers in one volume. The algebraic methods created in papers published in the third volume are widely used even in many articles of the first volume, however, we ensured that several of the initial articles of the first volume employ more elementary methods. We supply this collection by the graph which demonstrates the interrelation of the papers: if one of them has to be studied after another this relation is shown by an arrow. We also present the list of additional references to books which will be helpful for studying topology and its applications.

We hope that this collection would be useful.

S. P. Novikov

The interrelation between articles listed in the Russian edition of the Topological Library looks as follows:



Milnor's books "Lectures on the h-cobordism Theorem" and "Lectures of Characteristic Classes" (Milnor I.6 and Milnor II.2) are not included into the present edition of the series.[1]

[1]Due to the omission of the two articles, the numerical order of the present edition has been shifted.

Complementary References:

Springer "Encyclopedia of Math. Sciences" books.

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Topology II, Homotopy and Homology: Fuchs, D. B., Viro, O. Y. Rokhlin, V. A., Novikov, S. P. (Eds.), Vol. 24, 2004.

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1

On manifolds homeomorphic to the 7-sphere[1]

J. Milnor

The objective of this note will be to show that the 7-sphere possesses several distinct differentiable structures.

In § 1 an invariant λ is constructed for oriented, differentiable 7-manifold M^7 satisfying the hypothesis

$$H^3(M^7) = H^4(M^7) = 0, \tag{*}$$

(integer coefficients are to be understood). In § 2 a general criterion is given for proving that an n-manifold is homeomorphic to the sphere S^n. Some examples of 7-manifolds are studied in § 3 (namely, 3-sphere bundles over the 4-sphere). The results of the preceding two sections are used to show that some of these manifolds are topological 7-spheres, but not differentiable 7-spheres. Several related problems are studied in § 4.

All manifolds considered, with or without boundary, are to be differentiable, orientable and compact. The word differentiable will mean differentiable of class C^∞. A closed manifold M^n is oriented if one generator $\mu \in H_n(M^n)$ is distinguished.

[1] J. Milnor, On Manifolds Homeomorphic to The 7-Sphere, Annals of Mathematics, **64** (1956), 399–405 (Received June 14, 1956).

§ 1. The invariant $\lambda(M^7)$

For every closed, oriented 7-manifold satisfying (*), we will define a residue class $\lambda(M^7)$ modulo 7. According to Thom [5] every closed 7-manifold M^7 is a boundary of an 8-manifold M^8. The invariant $\lambda(M^7)$ will be defined as a function of the index τ and the Pontrjagin class p_1 of B^8.

An orientation $\nu \in H_8(B^8, M^7)$ is determined by the relation $\partial\nu = \mu$. Define a quadratic form over the group

$$H^4(B^8, M^7)/(\text{torsion})$$

by the formula $\alpha \to \langle \nu, \alpha^2 \rangle$. Let $\tau(B^8)$ be the index of this form (the number of positive terms minus the number of negative terms, when the form is diagonalized over the real numbers).

Let $p_1 \in H^4(B^8)$ be the first Pontrjagin class of the tangent bundle of B^8 (for the definition of Pontrjagin classes see [2] or [6]). The hypothesis (*) implies that the inclusion homomorphism

$$i : H^4(B^8, M^7) \to H^4(B^8)$$

is an isomorphism. Therefore we can define a "Pontrjagin number"

$$q(B^8) = \langle \nu, (i^{-1}p_1)^2 \rangle.$$

Theorem 1.1. *The residue class of* $2q(B^8) - \tau(B^8)$ *modulo* 7 *does not depend on the choice of the manifold* B^8.

Define $\lambda(M^7)$ as this residue class.[1] As an immediate consequence we have:

Corollary 1.2. *If* $\lambda(M^7) \neq 0$, *then* M^7 *is not the boundary of any* 8-*manifold having fourth Betti number zero.*

PROOF OF THEOREM 1.1. Let B_1^8, B_2^8 be two manifolds with boundary M^7. (We may assume they are disjoint: $B_1^8 \bigcap B_2^8 = M^7$.) Then $C^8 = B_1^8 \bigcup B_2^8$ is a closed 8-manifold which possesses a differentiable structure compatible with that of B_1^8 and B_2^8. Choose that orientation ν for C^8 which is consistent with the orientation ν_1 of B_1^8 (and therefore consistent

[1] Similarly for $n = 4k-1$ a residue class $\lambda(M^n) \bmod s_k\mu(L_k)$ could be defined (see [2], p. 14). For $k = 1, 2, 3, 4$ we have $s_k\mu(L_k) = 1, 7, 62, 381$ respectively.

with $-\nu_2$). Let $q(C^8)$ denote the Pontrjagin number $\langle \nu, p_1^2(C^8)\rangle$. According to Thom [5] or Hirzebruch [2] we have

$$\tau(C^8) = \left\langle \nu, \frac{1}{45}(7p_2(C^8) - p_1^2(C^8))\right\rangle$$

and therefore

$$45\tau(C^8) + q(C^8) = 7\langle \nu, p_2(C^8)\rangle \equiv 0 \pmod 7.$$

This implies

$$\lambda = 2q(C^8) - \tau(C^8) \equiv 0 \pmod 7. \tag{1}$$

Lemma 1.3. *Under the above conditions we have*

$$\tau(C^8) = \tau(B_1^8) - \tau(B_2^8), \tag{2}$$

and

$$q(C^8) = q(B_1^8) - q(B_2^8). \tag{3}$$

Formulas (1)–(3) clearly imply that

$$2q(B_1^8) - \tau(B_1^8) \equiv 2q(B_2^8) - \tau(B_2^8) \pmod 7,$$

which is just the assertion of Theorem 1.1.

Proof of Lemma 1.3. Consider the diagram

$$\begin{array}{ccc} H^n(B_1^8, M^7) \oplus H^n(B_2^8, M^7) & \underset{\approx}{\overset{h}{\longleftarrow}} & H^n(C^8, M^7) \\ \downarrow {\scriptstyle i_1 \oplus i_2} & & \downarrow {\scriptstyle j} \\ H^n(B_1^8) \oplus H^n(B_2^8) & \overset{k}{\longleftarrow} & H^n(C^8). \end{array}$$

Note that for $n = 4$ these homomorphisms are all isomorphisms. If $\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C^8)$, then

$$\begin{aligned} \langle \nu, \alpha^2\rangle &= \langle \nu, jh^{-1}(\alpha_1^2 \oplus \alpha_2^2)\rangle = \langle \nu_1 \oplus (-\nu_2), \alpha_1^2 \oplus \alpha_2^2\rangle \\ &= \langle \nu_1 \alpha_1^2\rangle - \langle \nu_2 \alpha_2^2\rangle. \end{aligned} \tag{4}$$

Thus the quadratic form of C^8 is the "direct sum" of the quadratic form of B_1^8 and the negative of the quadratic form of B_2^8. This clearly implies formula (2).

Define $\alpha_1 = i_1^{-1} p_1(B_1^8)$ and $\alpha_2 = i_2^{-1} p_1(B_2^8)$. Then the relation

$$k(p_1(C^8)) = p_1(B_1^8) \oplus p_1(B_2^8)$$

implies that

$$jh^{-1}(\alpha_1 \oplus \alpha_2) = p_1(C^8).$$

The computation (4) now shows that

$$\langle \nu, p_1^2(C^8) \rangle = \langle \nu_1 \alpha_1^2 \rangle - \langle \nu_2 \alpha_2^2 \rangle,$$

which is just formula (3). This completes the proof of Theorem 1.1.

The following property of the invariant λ is clear.

Lemma 1.4. *If the orientation of M^7 is reversed, then $\lambda(M^7)$ is multiplied by -1.*

As a consequence we have

Corollary 1.5. *If $\lambda(M^7) \neq 0$, then M^7 possesses on orientation reversing diffeomorphism onto itself.*[1]

§ 2. A partial characterization of the n-sphere

Consider the following hypothesis concerning a closed manifold M^n (where R denotes real numbers).

Hypothesis (H). There exists a differentiable function $f\colon M^n \to R$, having only two critical points x_0, x_1. Furthermore these critical points are non-degenerate.

(That is if $u_1, \ldots, u_n$ are local coordinates in a neighborhood of x_0 (or x_1) then the matrix $\|\partial^2 f / \partial u_i \partial u_j\|$ is non-singular at x_0 (or x_1).)

Theorem 2.1. *If M^n satisfies the hypothesis* (H) *then there exists a homeomorphism of M^n onto S^n which is a diffeomorphism except possibly at a single point.*

Added in proof. This result is essentially due to Reeb [7].

The proof will be based on the orthogonal trajectories of the manifolds $f = \text{const}$.

---

[1] A diffeomorphism f is a homeomorphism onto such that both f and f^{-1} are differentiable.

Normalize the function f so that $f(x_0) = 0, f(x_1) = 1$. According to Morse [3] (Lemma 4) there exists local coordinates $v_1, \ldots, v_n$ in a neighborhood V of x_0 so that $f(x) = v_1^2 + \cdots + v_n^2$ for $x \in V$. (Morse assumes that f is of class C^3, and constructs coordinates of class C^1; but the same proof works in the C^∞ case.) The expression $ds^2 = dv_1^2 + \cdots + dv_n^2$ defines a Riemannian metric for M^n which coincides with this in some neighborhood V of x_0. Choose a differentiable Riemannian metric for M^n which coincides with this in some neighborhood V' of x_0.[1] Now the gradient of f can be considered as a contravariant vector field.

Following Morse, we consider the differential equation

$$\frac{dx}{dt} = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|^2}.$$

In the neighborhood V' this equation has solutions

$$(v_1(t), \ldots, v_n(t)) = (a_1 t^2, \ldots, a_n t^2)$$

for $0 \leq t \leq \varepsilon$ (where $a = (a_1, \ldots, a_n)$ is any n-tuple with $a_1^2 + \cdots + a_n^2 = 1$). These can be extended uniquely to solutions $x_a(t)$ for $0 \leq t \leq 1$. Note that these solutions satisfy the identity

$$f(x_a(t)) = t.$$

Map the interior of the unit sphere of R^n into M^n by the map

$$(a_1 t^{\frac{1}{2}}, \ldots, a_n t^{\frac{1}{2}}) \to x_a(t).$$

It is easily verified that this defines a diffeomorphism of the open n-cell onto $M^n \backslash \{x_1\}$. The assertion of Theorem 2.1 now follows.

Given any diffeomorphism $g : S^{n-1} \to S^{n-1}$, an n-manifold can be obtained as follows.

Construction (C). Let $M^n(g)$ be the manifold obtained from two copies of R^n by matching the subsets $R^n \backslash \{0\}$ under the diffeomorphism

$$u \to v = \frac{1}{\|u\|} g\left(\frac{u}{\|u\|}\right).$$

(Such a manifold M^n is clearly homeomorphic to S^n. If g is the identity map then $M^n(g)$ is diffeomorphic to S^n.)

Corollary 2.2. *A manifold M^n can be obtained by the construction* (C) *if and only if it satisfies the hypothesis* (H).

[1] This is possible by [4] (Secs. 6.7 and 12.2).

If $M^n(g)$ is obtained by using the construction (C) then the function

$$f(x) = \frac{\|u\|^2}{1+\|u\|^2} = \frac{1}{1+\|v\|^2},$$

will satisfy the hypothesis (H). The converse can be established by a slight modification of the proof of Theorem 2.1.

§ 3. Examples of 7-manifolds

Consider 3-sphere bundles over the 4-sphere with the rotation group $SO(4)$ as structural group. The equivalence classes of such bundles are in one-to-one correspondence with elements of the group $\pi_3(SO(4)) \approx Z+Z$. A specific isomorphism between these groups is obtained as follows. For each $(h,j) \in Z+Z$ let $f_{hj} : S^3 \to SO(4)$ be defined by $f_{hj}(u)\cdot v = u^h v u^j$, for $u \in S^3, v \in R^4$ quaternion multiplication is understood on the right.

Let $\imath$ be the standard generator for $H^4(S^4)$. Let ξ_{hj} denote the sphere bundle corresponding to $(f_{hj}) \in \pi_3(SO(4))$.

Lemma 3.1. *The Pontrjagin class $p_1(\xi_{hj})$ equals $\pm 2(h-j)\imath$.*

(The proof will be given later. One can show that the characteristic class $\bar{c}(\xi_{hj})$ (see [4]) is equal to $(h+j)\imath$.)

For each odd integer k let M_k^7 be the total space of the bundle ξ_{hj} where h and j are determined by the equations $h+j=1, h-j=k$. This manifold M_k^7 has a natural differentiable structure and orientation, which will be described later.

Lemma 3.2. *The invariant $\lambda(M_k^7)$ is the residue class modulo 7 of k^2-1.*

Lemma 3.3. *The manifold M_k^7 satisfies the hypothesis* (H).

Combining these, we have:

Theorem 3.4. *For $k^2 \equiv 1 \mod 7$ the manifold M_k^7 is homeomorphic to S^7 but not diffeomorphic to S^7.*[1]

---

[1] From Theorem 2.2 it easily follows that every manifold satisfying the hypothesis (H) is combinatorially equivalent to the sphere. Thus, Theorem 3.4 can be reformulated as follows: for $k^2 \equiv 1 \mod 7$ the manifold M_k^7 is combinatorially equivalent to the sphere, but not diffeomorphic to it. — *Editor's remark.*

(For $k = \pm 1$ the manifold M_k^7 is diffeomorphic to S^7; but it is not known whether this is true for any other k.)

Clearly any differentiable structure on S^7 can be extended through $R^8 \backslash \{0\}$. However:

Corollary 3.5. *There exists a differentiable structure S^7 which cannot be extended throughout R^8.*

This follows immediately from the preceding assertions, together with Corollary 1.2.

Proof of Lemma 3.1. It is clear that the Pontrjagin class $p_1(\xi_{hj})$ is a linear function of h and j. Furthermore it is known that it is independent of the orientation of the fiber. But if the orientation of S^3 is reversed, then ξ_{hj} is replaced by $\xi_{-j,-h}$. This shows that $p_1(\xi_{hj})$ is given by an expression of the form $c(h-j)\imath$. Here c is a constant which will be evaluated later.

Proof of Lemma 3.2. Associated with each 3-sphere bundle $M^7 \to S^4$ there is a 4-cell bundle ρ_k: $B_k^8 \to S_4$. The total space B_k^8 of this bundle is a differentiable manifold with boundary M_k^7. The cohomology group $H^4(M_k^8)$ is generated by the element $\alpha = \rho_k^*(\imath)$. Choose orientations μ and ν for M_k^7 and B_k^8 so that

$$\langle \nu, (i^{-1}\alpha)^2 \rangle = +1.$$

Then the index $\tau(B_k^8)$ will be $+1$.

The tangent bundle of B_k^8 is the "Whitney sum" of (1) the bundle of vectors tangent to the fiber, and (2) the bundle of vectors normal to the fiber. The first bundle (1) is induced (under ρ_k) from the bundle ξ_{hj}, and therefore has Pontrjagin class $p_1 = \rho_k^*(c(h-j)\imath) = ck\alpha$. The second is induced from the tangent bundle of S^4, and therefore has first Pontrjagin class zero. Now by the Whitney product theorem ([2] or [6])

$$p_1(B_k^8) = ck\alpha + 0.$$

For the special case $k = 1$ it is easily verified that B_1^8 is the quaternion projective plane $P_2(K)$ with an 8-cell removed. But the Pontrjagin class $p_1(P_2(K))$ is known to be twice the generator of $H^4(P_2(K))$ (see Hirzebruch [1]). Therefore the constant c must be ± 2, which completes the proof of Lemma 3.1.

Now $q(B_k^8) = \langle \nu, (i^{-1}(\pm 2k\alpha))^2 \rangle = 4k^2$ and $2q - \tau = 8k^2 - 1 \equiv k^2 - 1$ (mod 7). This completes the proof of Lemma 3.2.

Proof of Lemma 3.3. As coordinate neighborhoods in the base space S^4 take the complement of the north pole, and the complement to the south pole. These can be identified with Euclidean space R^4 under stereographic projection. Then a point which corresponds to $u \in R^4$ under one projection will correspond to $u' = \frac{u}{\|u\|^2}$ under the other.

The total space M_k^7 can be obtained as follows.[1] Take two copies of $R^4 \times S^3$ and identify the subsets $(R^4 \backslash \{0\} \times S^3)$ under the diffeomorphism

$$(u, v) \to (u', v') = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|} \right)$$

(using quaternion multiplication). This makes the differentiable structure of M_k^7 precise.

Replace the coordinates (u', v') by (u'', v'), where $u'' = u'(v')^{-1}$. Consider the function $f : M_k^7 \to R$ defined by

$$f(x) = \frac{\operatorname{Re} v}{(1 + \|u\|)^{1/2}} = \frac{\operatorname{Re} u''}{(1 + \|u''\|)^{1/2}},$$

where $\operatorname{Re} v$ denotes the real part of the quaternion v. It is easily verified that f has only two critical points (namely, $(u, v) = (0, \pm 1)$) and that these are non-degenerate. This completes the proof.

§ 4. Miscellaneous results

Theorem 4.1. *Either* (a) *there exists a closed topological* 8*-manifold which does not possess any differentiable structure*; *or* (b) *the Pontrjagin class* p_1 *of an open* 8*-manifold is not a topological invariant.*

(The author has no idea which alternative holds.)

PROOF. Let X_k^8 be the topological 8-manifold obtained from B_k^8 by collapsing its boundary (a topological 7-sphere) to a point x_0. Let $\bar{\alpha} \in H^4(X_k^8)$ correspond to the generator $\alpha \in H^4(B_k^8)$. Suppose that X_k^8, possesses a differentiable structure, and that $p_1(X_k^8 \backslash \{x_0\})$ is a topological invariant. Then $p_1(X_k^8)$ must equal $\pm 2k\bar{\alpha}$, hence

$$2q(X_k^8) - \tau(X_k^8) = 8k^2 - 1 \equiv k^2 - 1 \pmod 7.$$

But for $k^2 \equiv 1 (\mathrm{mod}\ 7)$ this is impossible.[2]

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[1] See [4], § 18.

[2] The manifold X_k^8 admits a natural triangulation. One can show that a combinatorial manifold X_k^8 is not combinatorially equivalent to a C^1-triangulation of a smooth manifold (see V.A. Rokhlin and A.S. Shvarč. The combinatorial invariance of Pontrjagin classes. *Dokl. Akad. Nauk SSSR*, **114** (1957), 490–493). — *Editor's remark.*

Two diffeomorphisms $f, g : M_1^n \to M_2^n$ will be called differentiably isotopic if there exists a diffeomorphism $M_1^n \times R \to M_2^n \times R$ of the form $(x,t) \to (h(x,t),t)$ such that

$$h(x,t) = \begin{cases} f(x), & t \leq 0, \\ g(x), & t \geq 1. \end{cases}$$

Lemma 4.2. *If the diffeomorphisms $f, g : S^{n-1} \to S^{n-1}$ are differentiably isotopic, then the manifolds $M^n(f)$ and $M^n(g)$ obtained by the construction (C) are diffeomorphic.*

The proof is straightforward.

Theorem 4.3. *There exists a diffeomorphism $f : S^6 \to S^6$ of degree $+1$ which is not differentiably isotopic to the identity.*[1]

PROOF. By Lemma 3.3 and Corollary 2.2 the manifold M_3^7 is diffeomorphic to $M^7(f)$ for some f. If f were differentiably isotopic to the identity then Lemma 4.2 would impy that M_3^7 was diffeomorphic to S^7. But this is false by Lemma 3.2.

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[1] It is not difficult to show that two such homeomorphisms of the same degree are topologically isotopic. Thus Theorem 4.3 yields that there exist two topologically isotopic diffeomorphisms which are not smoothly isotopic. — *Editor's remark.*

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2

Groups of homotopy spheres. I[1]

M. Kervaire and J. Milnor

§ 1. Introduction

All manifolds, with or without boundary, are to be compact, oriented, and differentiable of class C^∞. The boundary of M will be denoted by bM. The manifold M with orientation reversed is denoted by $-M$.

Definition. The manifold M is a *homotopy n-sphere* if M is closed (that is, $bM = \emptyset$) and has the homotopy type of the sphere S^n.

Definition. Two closed n-manifolds M_1 and M_2 are *h-cobordant*[2] if the disjoint sum $M_1 + (-M_2)$ is the boundary of some manifold W, where both M_1 and $(-M_2)$ are deformation retracts of W. It is clear that this is an equivalence relation.

The *connected sum* of two connected n-manifolds is obtained by removing a small n-cell from each, and then pasting together the resulting boundaries. Details will be given in § 2.

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[1]Groups of homotopy spheres. I, Annals of Math., **77** (1963), 504–537 (Received April 19, 1962).

[2]The term "J-equivalence" has previously been used for this relation. Compare [15], [16] and [17].

Theorem 1.1. *The h-cobordism classes of homotopy n-sphere form an abelian group under the connected sum operation.*

This group will be denoted by Θ_n and called the n-th homotopy sphere cobordism group. It is the object of this paper (which is divided into 2 parts) to investigate the structure of Θ_n.

It is clear that $\Theta_1 = \Theta_2 = 0$. On the other hand, these groups are not all zero. For example, it follows easily from Milnor [14] that $\Theta_7 \neq 0$.

The main result of the present Part I will be

Theorem 1.2. *For $n \neq 3$ the group Θ_n is finite.*

(Our methods break down for the case $n = 3$). However, if one assumes the Poincaré hypothesis, then it can be shown that $\Theta_3 = 0$.

More detailed information about these groups will be given in Part II. For example, for $n = 1, 2, 3, \ldots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

Partial summaries of results are given in § 4 and § 7.

Remark. S. Smale [25] and J. Stallings [27], C. Zeeman [33] have proved that every homotopy n-sphere, $n \neq 3, 4$, is actually homeomorphic to the standard sphere S^n. Furthermore, Smale has proved [26] that two homotopy n-spheres ($n \neq 3, 4$), are h-cobordant if and only if they are diffeomorphic. Thus for $n \neq 3, 4$ (and possibly for all n) the group Θ_n can be described as the set of all diffeomorphic classes of differentiable structures on the topological n-sphere. These facts will not be used in the present paper.

§ 2. Construction of the group Θ_n

First we give a precise definition of the connected sum $M_1 \# M_2$ of two connected n-manifolds M_1 and M_2 (compare Seifert [22] and Milnor [15], [16]). The notation D^n will be used for the unit disk in Euclidean space. Choose imbeddings

$$i_1 : D^n \to M_1, \quad i_2 : D^n \to M_2$$

so that i_1 preserves orientation and i_2 reverses it. Now obtain $M_1 \# M_2$ from the disjoint sum

$$(M_1 - i_1(0)) + (M_2 - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1-t)u)$ for each unit vector $u \in S^{n-1}$ and each $0 < t < 1$. Choose the orientation for $M_1 \# M_2$ which is compatible with that of M_1 and M_2. (This makes sense since the correspondence $i_1(tu) \to i_2((1-t)u)$ preserves orientation.)

It is clear that the sum of two homotopy n-spheres is a homotopy n-sphere.

Lemma 2.1. *The connected sum operation is well defined, associative, and commutative up to orientation preserving diffeomorphism. The sphere S^n serve as identity element.*

PROOF. The first assertions follow easily from the lemma of Palais [20] and Cerf [5] which asserts that any two orientation preserving imbeddings i, $i' : D^n \to M$ are related by the equation $i' = f \cdot i$, for some diffeomorphism $f : M \to M$. The proof that $M \# S^n$ is diffeomorphic to M, will be left to the reader.

Lemma 2.2. *Let M_1, M_1' and M_2 be closed and simply connected.*[1] *If M_1 is h-cobordant to M_1' then $M_1 \# M_2$ is h-cobordant to $M_1' \# M_2$.*

PROOF. We may assume that the dimension n is ≥ 3. Let $M_1 + (-M_1') = bW_1$, where M_1 and $-M_1'$ are deformation retracts of W_1. Choose a differentiable arc A from a point $p \in M_1$ to a point $p' \in -M_1'$ within W_1 so that a tubular neighborhood of this arc is diffeomorphic to $R^n \times [0,1]$. Thus we obtain an imbedding

$$i : R^n \times [0,1] \to W_1$$

with $i(R^n \times 0) \subset M_1$, $i(R^n \times 1) \subset M_1'$ and $i(0 \times [0,1]) = A$. Now form a manifold W from the disjoint sum

$$(W_1 - A) + (M_2 - i_2(0)) \times [0,1]$$

by identifying $i(tu, s)$ with $i_2((1-t)u) \times s$ for each $0 < t < 1$, $0 \leq s \leq 1$, $u \in S^{n-1}$. Clearly W is a compact manifold bounded by the disjoint sum

$$M_1 \# M_2 + (-(M_1' \# M_2)).$$

We must show that both boundaries are deformation retracts of W.

First it is necessary to show that the inclusion map

$$M_1 - p \xrightarrow{j} W_1 - A$$

is a homotopy equivalence. Since $n \geq 3$, it is clear that both of these manifolds are simply connected. Mapping the homology exact sequence of

[1]This hypothesis is imposed in order to simplify the proof. It could easily be eliminated.

the pair $(M_1, M_1 - p)$ into that of the pair $(W_1, W_1 - A)$, we see that j induces isomorphisms of homology groups, and hence is a homotopy equivalence. Now it follows easily, using a Mayer-Vietoris sequence, that the inclusion

$$M_1 \# M_2 \to W$$

is a homotopy equivalence; hence that $M_1 \# M_2$ is a deformation retract of W. Similarly $M_1' \# M_2$ is a deformation retract of W, which completes the proof of Lemma 2.2.

Lemma 2.3. *A simply connected manifold M is h-cobordant to the sphere S^n if and only if M bounds a contractible manifold.*

(Here the hypothesis of simple connectivity cannot be eliminated.)

PROOF. If $M + (-S^n) = bW$ then filling in a disk D^{n+1} we obtain a manifold W' with $bW' = M$. If S^n is a deformation retract of W, then it clearly follows that W' is contractible.

Conversely if $M = bW'$, then removing the interior of an imbedded disk we obtain a simply connected manifold W with $bW = M + (-S^n)$. Mapping the homology exact sequence of the pair (D^{n+1}, S^n) into that of the pair (W', W), we see that the inclusion $S^n \to W$ induces a homology isomorphism; hence S^n is a deformation retract of W. Now applying the Poincaré duality isomorphism

$$H_k(W, M) \simeq H^{n+1-k}(W, S^n),$$

we see that the inclusion $M \to W$ also induces isomorphisms of homology groups. Since M is simply connected, this completes the proof.

Lemma 2.4. *If M is a homotopy sphere, then $M \#(-M)$ bounds a contractible manifold.*

PROOF. Let $H^2 \subset D^2$ denote the half-disk consisting of all $(t \sin\theta, t\cos\theta)$ $0 \le t \le 1$, $0 \le \theta \le \pi$, and let $\frac{1}{2}D^n \subset D^n$ denote the disk of radius $\frac{1}{2}$. Given an imbedding $i : D^n \to M$, form W from the disjoint union

$$\left(M - i\left(\frac{1}{2}D^n\right)\right) \times [0,\pi] + S^{n-1} \times H^2$$

by identifying $i(tu) \times \theta$ with $u \times ((2t-1)\sin\theta, (2t-1)\cos\theta)$ for each $\frac{1}{2} < t \le 1$, $0 \le \theta \le \pi$. (Intuitively we are removing the interior of $i(\frac{1}{2}D^n)$ from M and then "rotating" the result through 180° around the resulting $bW = M \# -M$.) Furthermore W contains $(M - \text{interior}\, i(\frac{1}{2}D^n))$ as deformation retract, and therefore is contractible. This proves Lemma 2.4.

Proof of Theorem 1.1. Let Θ_n denote the collection of all h-cobordism classes of homotopy n-spheres. By Lemmas 2.1 and 2.2 there is a well-defined, associative, commutative addition operation in Θ_n. The sphere S^n serves as zero element. By Lemmas 2.3 and 2.4, each element of Θ_n has an inverse. Therefore Θ_n is an additive group.

Clearly Θ_1 is zero. For $n \leq 3$, Munkres [19] and Whitehead [31] had proved that a topological n-manifold has a differentiable structure which is unique up to diffeomorphism. It follows that $\Theta_2 = 0$. If the Poincaré hypothesis were proved, it would follow that $\Theta_3 = 0$, but at present the structure of Θ_3 remains unknown. For $n > 3$ the structure of Θ_n will be studied in the following sections.

Addendum. There is a slight modification of the connected sum construction which is frequently useful. Let W_1 and W_2 be $(n+1)$-manifolds with connected boundary. Then the sum $bW_1 \# bW_2$ is the boundary of a manifold W constructed as follows. Let H^{n+1} denote the half-disk consisting of all $x = (x_0, x_1, \ldots, x_n)$ with $|x| \leq 1$, $x_0 \geq 0$ and let D^n denote the subset $x_0 = 0$. Choose imbeddings

$$i_q : (H^{n+1}, D^n) \to (W_q, bW_q), \quad q = 1, 2,$$

so that $i_2 \cdot i_1^{-1}$ reverses orientation. Now form W from

$$(W_1 - i_1(0)) + (W_2 - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1-t)u)$ for each $0 < t < 1$, $u \in S^n \cap H^{n+1}$.

It is clear that W is a differentiable manifold with $bW = bW_1 \# bW_2$. Note that W has homotopy type of $W_1 \vee W_2$: the union with a single point in common.

W will be called the *connected sum along the boundary* of W_1 and W_2. The notation $(W, bW) = (W_1, bW_1) \# (W_2, bW_2)$ will be used for this sum.

§ 3. Homotopy spheres are s-parallelizable

Let M be a manifold with tangent bundle $\tau = \tau(M)$, and let ε^1 denote a trivial line bundle over M.

Definition. M will be called *s-parallelizable* if the Whitney sum $\tau \oplus \varepsilon^1$ is a trivial bundle.[1] The bundle $\tau \oplus \varepsilon^1$ will be called the *stable tangent*

[1]The authors have previously used the term "π-manifold" for an s-parallelizable manifold.

bundle of M. It is a stable bundle in the sense of [10]. (The expression s-parallelizable stands for stably parallelizable.)

Theorem 3.1. *Every homotopy sphere is s-parallelizable.*

In the proof, we will use recent results of J. F. Adams [1], [2].

PROOF. Let Σ be a homotopy n-sphere. Then the only obstruction to the triviality of $\tau \oplus \varepsilon^1$ is a well-defined cohomology class

$$o_n(\Sigma) \in H^n(\Sigma; \pi_{n-1}(SO_{n+1})) = \pi_{n-1}(SO_{n+1}).$$

The coefficient group may be identified with the stable group $\pi_{n-1}(SO)$. But these stable groups have been computed by Bott [4], as follows, for $n \geq 2$:

The mod 8 residue class:	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO)$	Z	Z_2	Z_2	0	Z	0	0	0.

(Here Z, Z_2, 0 denote the cyclic groups of orders ∞, 2, 1 respectively.)

Case 1. $n \equiv 3, 5, 6$ or 7 (mod 8). Then $\pi_{n-1}(SO) = 0$, so that $o_n(\Sigma)$ is trivially zero.

Case 2. $n \equiv 0$ or 4 (mod 8). Say that $n = 4k$. According to [18], [10], some non-zero multiple of the obstruction class $o_n(\Sigma)$ can be identified with the Pontrjagin class $p_k(\tau \oplus \varepsilon^1) = p_k(\tau)$. But the Hirzebruch signature[1] theorem implies that $p_k[\Sigma]$ is a multiple of the signature $\sigma(\Sigma)$, which is zero since $H^{2k}(\Sigma) = 0$. Therefore every homotopy $4k$-sphere is s-parallelizable.

Case 3. $n \equiv 1$ or 2 (mod 8), so that $\pi_{n-1}(SO)$ is cyclic of order 2. For each homotopy sphere Σ the residue class modulo 2

$$o_n(\Sigma) \in \pi_{n-1}(SO) \simeq Z_2$$

is well defined. It follows from an argument of Rokhlin that

$$J_{n-1}(o_n) = 0,$$

where J_{n-1} denotes the Hopf–Whitehead homomorphism

$$J_{n-1} : \pi_{n-1}(SO_k) \to \pi_{n+k-1}(S^k)$$

in the stable range $k > n$ (compare [18, Lemma 1]). But J_{n-1} is a monomorphism for $n \equiv 1$ or 2 (mod 8). For the case $n = 2$ this fact is well known, and for $n = 9, 10$ it has been proved by Kervaire [11]. For $n = 17$, 18

[1]We will substitute the word "signature" for index as used in [7, 14, 17, 18, 28], since this is more in accord with the usage in other parts of mathematics. The *signature* of the form $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+l}^2$ is defined as $\sigma = k - l$.

it has been verified by Kervaire and Toda in unpublished computations. A proof that J_{n-1} is injective for all $n \equiv 1$ or $2 \pmod 8$, has recently been given by J. F. Adams [1], [2]. Now the relation $J_{n-1}(o_n) = 0$, together with the information that J_{n-1} is a monomorphism implies that $o_n = 0$. This completes the proof of Theorem 3.1.

In conclusion, here are two lemmas which clarify the concept of s-parallelizability. The first is essentially due to J. H. C. Whitehead [32].

Lemma 3.2. *Let M be an n-dimensional submanifold of S^{n+k}, $n < k$. Then M is s-parallelizable if and only if its normal bundle is trivial.*

Lemma 3.3. *A connected manifold with non-vacuous boundary is S-parallelizable if and only if it is parallelizable.*

The proofs will be based on the following lemma (compare Milnor [17, Lemma 4]).

Let ξ be a k-dimensional vector space bundle over an n-dimensional complex $k > n$.

Lemma 3.4. *If the Whitney sum of ξ with a trivial bundle ε^r is trivial then ξ itself is trivial.*

PROOF. We may assume that $r = 1$, and that ξ is oriented. An isomorphism $\xi \oplus \varepsilon^1 \approx \varepsilon^{k+1}$ gives rise to a bundle map f from ξ to the bundle γ^k of oriented k-planes in $(k+1)$-space. Since the base space of ξ has dimension n, and since the base space of γ^k is the sphere S^k, $k > n$, it follows that f is null-homotopic; and hence ξ is trivial.

PROOF OF LEMMA 3.2. Let τ, ν denote the tangent and normal bundles of M. Then $\tau \oplus \nu$ is trivial hence $(\tau \oplus \varepsilon^1) \oplus \nu$ is trivial. Applying Lemma 3.5, the conclusion follows.

PROOF OF LEMMA 3.3. This follows by a similar argument. The hypothesis on the manifold guarantees that every map into a sphere of the same dimension is null-homotopic.

§ 4. Which homotopy spheres bound parallelizable manifolds?

Define a subgroup $bP_{n+1} \subset \Theta_n$ as follows. A homotopy n-sphere M represents an element of bP_{n+1} if and only if M is a boundary of a parallelizable manifold. We will see that this condition depends only on the

h-cobordism class of M, and that bP_{n+1} does form a subgroup. The object of this section will be to prove the following:

Theorem 4.1. *The quotient group Θ_n/bP_{n+1} is finite.*

PROOF. Given an s-parallelizable closed manifold M of dimension n, choose an imbedding

$$i : M \to S^{n+k}$$

with $k > n+1$. Such an imbedding exists and is unique up to differentiable isotopy. By Lemma 3.3 the normal bundle of M is trivial. Now choose a specific field φ of normal k-frames. Then the Pontrjagin–Thom construction yields a map

$$p(M, \varphi) : S^{n+k} \to S^k$$

(see Pontrjagin [21, pp. 41–57] and Thom [28]). The homotopy class of $p(M, \varphi)$ is a well-defined element of the stable homotopy group

$$\Pi_n = \pi_{n+k}(S^k).$$

Allowing the normal frame field φ to vary, we obtain a set of elements

$$p(M) = \{p(M, \varphi)\} \subset \Pi_n.$$

Lemma 4.2. *The subset $p(M) \subset \Pi_n$ contains the zero element of Π_n if and only if M bounds a parallelizable manifold.*

PROOF. If $M = bW$ with W parallelizable then the imbedding $i\colon M \to S^{n+k}$ can be extended to an imbedding $W \to D^{n+k+1}$, and W has a field ψ of normal k-frames. We set $\varphi = \psi|_M$. Now the Pontrjagin–Thom map $p(M, \varphi) : S^{n+k} \to S^k$ extends over D^{n+k+1}, hence is null-homotopic.

Conversely if $p(M, \varphi) \simeq 0$, then M bounds a manifold $W \subset D^{n+k+1}$, where φ extends to a field ψ of normal frames over W. It follows from Lemmas 3.3 and 3.4 that W is parallelizable. This completes the proof of Lemma 4.2.

Lemma 4.3. *If M_0 is h-cobordant to M_1, then $p(M_0) = p(M_1)$.*

PROOF. If $M_0 + (-M_1) = bW$, we choose an imbedding of W in $S^{n+k} \times [0, 1]$ so that $M_q \to S^{n+k} \times \{q\}$ for $q = 0, 1$. Then a normal frame field φ_q on M_q extends to a normal frame field ψ on W which restricts to some normal frame field φ_{1-q} on M_{1-q}. Clearly (W, ψ) gives rise to a homotopy between $p(M_0, \varphi_0)$ and $p(M_1, \varphi_1)$.

Lemma 4.4. *If M and M' are s-parallelizable, then*

$$p(M) + p(M') \subset p(M \# M') \subset \Pi_n.$$

PROOF. Start with the disjoint sum

$$M \times [0,1] + M' \times [0,1]$$

and join the boundary components $M \times 1$ and $M' \times 1$ together, as described in the addendum at the end of § 2. Thus we obtain a manifold W bounded by the disjoint sum

$$(M\#M') + (-M) + (-M').$$

Note that W has the homotopy type $M \vee M'$, the union with a single point in common.

Choose an imbedding of W in $S^{n+k} \times [0,1]$ so that $(-M)$ and $(-M')$ go into well separated submanifolds of $S^{n+k} \times 0$, and so that $M\#M'$ goes into $S^{n+k} \times 1$. Given fields φ and φ' of normal k-frames on $(-M)$ and $(-M')$, it is not hard to see that there exists an extension defined throughout W. Let ψ denote the restriction of this field to $M\#M'$. Then clearly $p(M,\varphi) + p(M',\varphi')$ is homotopic to $p(M\#M',\psi)$. This completes the proof.

Lemma 4.5. *The set $p(S^n) \subset \Pi_n$ is a subgroup of the stable homotopy group Π_n. For any homotopy sphere Σ the set $p(\Sigma)$ is a coset of this subgroup $p(S^n)$. Thus the correspondence $\Sigma \to p(\Sigma)$ defines a homomorphism p' from Θ_n to the quotient group $\Pi_n/p(S^n)$.*

PROOF. Combining Lemma 4.4 with the identities

(1) $S^n\#S^n = S^n$,
(2) $S^n\#\Sigma = \Sigma$,
(3) $\Sigma\#(-\Sigma) \sim S^n$,

we obtain

$$p(S^n) + p(S^n) \subset p(S^n), \tag{1}$$

which shows that $p(S^n)$ is a subgroup of Π_n;

$$p(S^n) + p(\Sigma) \subset p(\Sigma), \tag{2}$$

which shows that $p(\Sigma)$ is a union of cosets of this subgroup; and

$$p(\Sigma) + p(-\Sigma) \subset p(S^n), \tag{3}$$

which shows that $p(\Sigma)$ must be a single coset. This completes the proof of Lemma 4.5.

By Lemma 4.2 the kernel of $p : \Theta_n \to \Pi_n/p(S^n)$ consists exactly of all h-cobordism classes of homotopy n-spheres which bound parallelizable manifolds. Thus these elements form a group which we will denote by $bP_{n+1} \subset \Theta_n$. It follows that Θ_n/bP_{n+1} is isomorphic to a subgroup of $\Pi_n/p(S^n)$. Since Π_n is finite (Serre [24]), this completes the proof of Theorem 4.1.

Remark. The subgroup $p(S^n) \subset \Pi_n$ can be described in more familiar terms as the image of the Hopf–Whitehead homomorphism

$$J_n : \pi_n(SO_k) \to \pi_{n+k}(S^k)$$

(see Kervaire [9, p. 349]). Hence $\Pi_n/p(S^n)$ is the cokernel of J_n. The actual structure of these groups for $n \leq 8$ is given in the following table. For details, and for higher values of n the reader is referred to Part II of this paper.

n	1	2	3	4	5	6	7	8
Π_n	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{240}	$Z_2 + Z_2$
$\Pi_n/p(S^n)$	0	Z_2	0	0	0	Z_2	0	Z_2
Θ_n/bP_{n+1}	0	0	0	0	0	0	0	Z_2

The prime $q \geq 3$ first divides the order of Θ_n/bP_{n+1} for $n = 2q(q-1) - 2$.

Using Theorem 4.1, the proof of the main theorem (Theorem 1.2), taking that Θ_n is finite for $n \neq 3$, reduces now to proving that bP_{n+1} is finite for $n \neq 3$.

We will prove that the group bP_{n+1} is zero for n even (§ 5, 6) and is finite cyclic for n odd, $n \neq 3$ (see § 7, 8). The first few groups can be given as follows:

n	1	3	5	7	9	11	13	15	17	19
Order of bP_{n+1}	1	?	1	28	2	992	1	8128	2	130816

(Again see Part II for details). The cyclic group bP_{n+1} has order 1 or 2 for $n \equiv 1 \pmod 4$, but the order grows more than exponentially for $n \equiv 3 \pmod 4$.

§ 5. Spherical modifications

This section and § 6 which follows, will prove that the groups bP_{2k+1} are zero.[1] That is:

Theorem 5.1. *If a homotopy sphere of dimension $2k$ bounds an s-parallelizable manifold M, then it bounds a contractible manifold M_1.*

[1] An independent proof of this theorem has been given by C. T. C. Wall [29].

For the case $k = 1$, this assertion is clear since every homotopy 2-sphere is actually diffeomorphic to S^2. The proof for $k > 1$ will be based on the technique of "spherical modifications" (see Wallace [30] and Milnor [15, 17]).[1]

Definition. Let M be a differentiable manifold of dimension $n = p + q + 1$ and let

$$\varphi : S^p \times D^{q+1} \to M$$

be a differentiable imbedding. Then a new differentiable manifold $M' = \chi(M, \varphi)$ is formed from the disjoint sum

$$(M - \varphi(S^p \times 0)) + D^{p+1} \times S^q$$

by identifying $\varphi(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$, $0 < t \leq 1$. We will say that M' is obtained from M by the *spherical modification* $\chi(\varphi)$. Note that the boundary of M' is equal to the boundary of M.

In order to prove Theorem 5.1 we will show that the homotopy groups of M can be completely killed by a sequence of such spherical modifications. The effect of a single modification $\chi(\varphi)$ on the homotopy groups of M can be described as follows.

Let $\lambda \in \pi_p M$ denote the homotopy class of the map $\varphi|_{S^p \times 0}$ from $S^p \times 0$ to M.

Lemma 5.2. *The homotopy groups of M' are given by*

$$\pi_i M' \simeq \pi_i M \quad \textit{for } i < \min(p, q)$$

and

$$\pi_p M' \simeq \frac{\pi_p M}{\Lambda},$$

provided that $p < q$; where Λ denotes a certain subgroup of $\pi_p M$ containing λ.

The proof is straightforward (compare [17, Lemma 2]).

Thus, if $p < q$ (that is, if $p \leq n/2 - 1$), the effect of the modification $\chi(\varphi)$ is to kill the homotopy class λ.

Now suppose that some homotopy class $\lambda \in \pi_p M$ is given.

Lemma 5.3. *In M^n is s-parallelizable and if $p < n/2$, then the class λ is represented by some imbedding $\varphi : S^p \times D^{n-p} \to M$.*

---

[1]The term "surgery" is used for this concept in [15, 17].

Proof. (Compare [17, Lemma 3]) Since $n \geq 2p + 1$ it follows from a well-known theorem of Whitney that λ can be represented by an imbedding

$$\varphi_0 : S^p \to M.$$

It follows from Lemma 3.5 that the normal bundle of $\varphi_0 S^p$ in M is trivial. Hence φ_0 can be extended to the required imbedding $S^p \times D^{n-p} \to M$.

Thus Lemmas 5.2 and 5.3 assert that spherical modifications can be used to kill any required element $\lambda \in \pi_p M^n$ provided that $p \leq n/2 - 1$. There is one danger however. If the imbedding φ is chosen badly then the modified manifold $M' = \chi(M, \varphi)$ may no longer be s-parallelizable. However the following was proven in [17]. Again let $n \geq 2p + 1$.

Lemma 5.4. *The imbedding $\varphi : S^p \times D^{n-p} \to M$ can be chosen within its homotopy class so that the modified manifold $\chi(M, \varphi)$ will also be s-parallelizable.*

For the proof, the reader may either refer to [17, Theorem 2], or make use of the sharper Lemma 6.2 which will be proved below.

Now combining Lemmas 5.2, 5.3 and 5.4, one obtains the following (compare [17, p. 46]).

Theorem 5.5. *Let M be a compact, connected s-parallelizable manifold of dimension $n \geq 2k$. By a sequence of spherical modifications on M one can obtain an s-parallelizable manifold M_1, which is $(k-1)$-connected.*

Recall that $bM_1 = bM$.

Proof. Choosing a suitable imbedding $\varphi : S^1 \times D^{n-1} \to M$, one can obtain an s-parallelizable manifold $M' = \chi(M, \varphi)$ such that $\pi_1 M'$ is generated by fewer elements than $\pi_1 M$. Thus after a finite number of steps, one can obtain a manifold M'' which is 1-connected. Now, after a finite number of steps, one can obtain an s-parallelizable manifold M''' which is 2-connected, and so on until we obtain a $(k-1)$-connected manifold. This proves Theorem 5.5.

In order to prove Theorem 5.1, where $\dim M = 2k+1$, we must carry this argument one step further obtaining a manifold M_1 which is k-connected. It will then follow from the Poincaré duality theorem that M_1 is contractible.

The difficulty in carrying out this program is that Lemma 5.2 is no longer available. Thus if $M' = \chi(M, \varphi)$ where φ embeds $S^k \times D^{k+1}$ in M, the group $\pi_k M'$ may actually be larger than $\pi_k M$. It is first necessary to describe in detail what happens to $\pi_k M$ under such a modification. Since we may assume that M is $(k-1)$-connected with $k > 1$, the homotopy group $\pi_k M$ may be replaced by the homology group $H_k M = H_k(M; Z)$.

Lemma 5.6. *Let $M' = \chi(M, \varphi)$ where φ embeds $S^k \times D^{k+1}$ in M, and let*

$$M_0 = M - \text{interior } \varphi(S^k \times D^{k+1}).$$

Then there is a commutative diagram

$$\begin{tikzcd}
 & & H_{k+1}M' \arrow[d, "\cdot\lambda'"] & & \\
 & & Z \arrow[d, "\varepsilon"] \arrow[dr, "\lambda"] & & \\
H_{k+1}M \arrow[r, "\cdot\lambda"] & Z \arrow[r, "\varepsilon'"] \arrow[dr, "\lambda'"'] & H_kM_0 \arrow[r, "i"] \arrow[d, "i'"] & H_kM \arrow[r] & 0 \\
 & & H_kM' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

such that the horizontal and vertical sequences are exact. It follows that the quotient group $H_kM/\lambda(Z)$ is isomorphic to $H_kM'/\lambda'(Z)$.

Here the following notations are to be understood. The symbol λ denotes the element of H_kM which corresponds to the homotopy class $\varphi|_{S^k \times 0}$, and λ also denotes the homomorphism $Z \to H_kM$ which carries 1 into λ. On the other hand, $\cdot\lambda : H_{k+1}M \to Z$ denotes the homomorphism which carries each $\mu \in H_{k+1}M$ into the intersection number $\mu \cdot \lambda$. The symbols λ' and $\cdot\lambda'$ are to be interpreted similarly. The element $\lambda' \in H_kM'$ corresponds to the homotopy class $\varphi'|_{0 \times S^k}$ where

$$\varphi' : D^{k+1} \times S^k \to M'$$

denotes the canonical imbedding.

PROOF OF LEMMA 5.6. As horizontal sequences take the exact sequence

$$H_{k+1}M \to H_{k+1}(M, M_0) \xrightarrow{\varepsilon'} H_kM_0 \xrightarrow{i} H_kM \to H_k(M, M_0)$$

of the pair (M, M_0). By excision, the group $H_j(M, M_0)$ is isomorphic to

$$H_j(S^k \times D^{k+1}, S^k \times S^k) \simeq \begin{cases} Z & \text{for } j = k+1 \\ 0 & \text{for } j < k+1. \end{cases}$$

Thus we obtain

$$H_{k+1}M \to Z \xrightarrow{\varepsilon'} H_kM_0 \xrightarrow{i} H_kM \to 0,$$

as asserted. Since a generator of $H_{k+1}(M, M_0)$ clearly has intersection number ± 1 with the cycle $\varphi(S^k \times 0)$ which represents λ, it follows that the homomorphism $H_{k+1}M \to Z$ can be described as the homomorphism $\mu \to \mu \cdot \lambda$. The element $\varepsilon' = e'(1) \in H_k M_0$ can clearly be described as the homology class corresponding to the "meridian" $\varphi(x_0 \times S^k)$ of the torus $\varphi(S^k \times S^k)$ where x_0 denotes a base point in S^k.

The vertical exact sequence is obtained in a similar way. Thus $\varepsilon = \varepsilon(1) \in H_k M_0$ is the homology class of the "parallel" $\varphi(S^k \times x_0)$ of the torus. Clearly $i(\varepsilon) \in H_k M$ is equal to the homology class λ of $\varphi(S^k \times 0)$. Similarly $i'(\varepsilon') = \lambda'$.

From this diagram the isomorphisms

$$\frac{H_k M}{\lambda(Z)} \simeq \frac{H_k M_0}{\varepsilon(Z)} + \varepsilon'(Z) \simeq \frac{H_k M'}{\lambda'(Z)},$$

are apparent. This completes the proof of Lemma 5.6.

As an application, suppose that one chooses an element $\lambda \in H_k M$ which is *primitive* in the sense that $\mu \cdot \lambda = 1$ for some $\mu \in H_{k+1}M$. It follows that

$$i : H_k M_0 \to H_k M$$

is an isomorphism, and hence that

$$H_k M' \simeq \frac{H_k M}{\lambda(Z)}.$$

Thus:

Assertion. Any primitive element of $H_k M$ can be killed by a spherical modification.

In order to apply this assertion we assume the following:

Hypothesis. M is compact, s-parallelizable manifold of dimension $2k+1$, $k > 1$ and is $(k-1)$-connected. The boundary bM is either vacuous or a homology sphere.

This hypothesis will be assumed for the rest of § 5 and for § 6.

Lemma 5.7. *Subject to this hypothesis, the homology group $H_k M$ can be reduced to its torsion subgroup by a sequence of spherical modifications. The modified manifold M_1 will still satisfy the hypothesis.*

PROOF. Suppose that $H_k M \simeq Z \oplus \cdots \oplus Z \oplus T$, where T is the torsion subgroup. Let λ generate one of the infinite cyclic summands. Using the Poincaré duality theorem one sees that $\mu_1 \cdot \lambda = 1$ for some element $\mu_1 \in H_{k+1}(M, bM)$. But the exact sequence

$$H_{k+1}M \to H_{k+1}(M, bM) \to H_k(bM) = 0$$

shows that μ_1 can be lifted back to $H_{k+1}M$. Therefore λ is primitive, and can be killed by a modification. After finitely many such modifications one obtains a manifold M_1 with $H_k M_1 \simeq T \subset H_k M$. This completes the proof of Lemma 5.7.

Let us specialize to the case k even. Let M be as above, and let $\varphi : S^k \times D^{k+1} \to M$ be any imbedding.

Lemma 5.8. *If k is even then the modification $\chi(\varphi)$ necessarily changes k-th Betti number of M.*

The proof will be based on the following lemma (see Kervaire [8, Formula (8.8)]).

Let F be a fixed field and let W be an orientable homology manifold of dimension $2r$. Define the *semi-characteristic* $e^*(bW;F)$ to be the following residue class modulo 2:

$$e^*(bW;F) \equiv \sum_{i=0}^{r-1} \operatorname{rank} H_i(bW;F) \pmod 2.$$

Lemma 5.9. *The rank of the bilinear pairing*

$$H_r(W;F) \otimes H_r(W;F) \to F,$$

given by the intersection number, is congruent modulo 2 to $e^*(bW;F)$ plus the Euler characteristic $e(W)$.

[For the convenience of the reader, here is a proof. Consider the exact sequence

$$H_r W \xrightarrow{h} H_r(W, bW) \to H_{r-1}(bW) \to \cdots \to H_0(W, bW) \to 0,$$

where the coefficient group F is to be understood. A counting argument shows that the rank of the indicated homomorphism h is equal to the alternating sum of the ranks of the vector spaces to the right of h in this sequence. Reducing modulo 2 and using the identity

$$\operatorname{rank} H_i(W, bW) = \operatorname{rank} H_{2r-i} W,$$

this gives

$$\begin{aligned}\operatorname{rank} h &\equiv \sum_{i=0}^{r-1} \operatorname{rank} H_i(bW) + \sum_{i=0}^{2r} \operatorname{rank} H_i(W) \\ &\equiv e^*(bW;F) + e(W) \pmod 2.\end{aligned}$$

But the rank of

$$h : H_r W \to H_r(W, bW) \simeq \operatorname{Hom}_F(H_r W, F)$$

is just the rank of the intersection pairing. This completes the proof.]

PROOF OF LEMMA 5.8. First suppose that M has no boundary. As shown in [15] or [17], the manifolds M and $M' = \chi(M, \varphi)$, suitably oriented, together bound a manifold $W = W(M, \varphi)$ of dimension $2k + 2$. For the moment, since no differentiable structure on W is needed, we can simply define W to be the union

$$(M \times [0, 1]) \cup (D^{k+1} \times D^{k+1}),$$

where it is understood that $S^k \times D^{k+1}$ is to be pasted onto $M \times 1$ by the imbedding φ. Clearly W is a topological manifold with

$$bW = M \times 0 + M' \times 1.$$

Note that W has the homotopy type of M with a $(k+1)$-cell attached. Since the dimension $2k + 1$ of M is odd, this means that the Euler characteristic

$$e(W) = e(M) + (-1)^{k+1} = (-1)^{k+1}.$$

Since k is even, the intersection pairing

$$H_{k+1}(W; Q) \otimes H_{k+1}(W; Q) \to Q$$

is skew symmetric, hence has even rank. Therefore Lemma 5.9 (with rational coefficients) asserts that

$$e^*(M + M'; Q) + (-1)^{k+1} \equiv 0 \pmod 2$$

and hence that

$$e^*(M; Q) \not\equiv e^*(M'; Q).$$

But $H_i M \simeq H_i M' \simeq 0$ for $0 < i < k$, so this implies that

$$\operatorname{rank} H_k(M; Q) \not\equiv \operatorname{rank} H_k(M'; Q).$$

This proves Lemma 5.8 provided that M has no boundary.

If M is bounded by a homology sphere, then attaching a cone over bM, one obtains a homology manifold M_* without boundary. The above argument now shows that

$$\operatorname{rank} H_k(M_*; Q) \not\equiv \operatorname{rank} H_k(M'_*; Q).$$

Therefore the modification $\chi(\varphi)$ changes the rank of $H_k(M; Q)$ in this case also. This completes the proof of Lemma 5.8.

It is convenient at this point to insert an analogue of Lemma 5.8 which will only be used later (see the end of § 6). Let M be as above, with k even or odd, and let $W = W(M, \varphi)$.

Lemma 5.10. *Suppose that every* mod 2 *homology class*

$$\xi \in H_{k+1}(W; Z_2)$$

has self-intersection number $\xi \cdot \xi = 0$. *Then the modification* $\chi(\varphi)$ *necessarily changes the rank of the* mod 2 *homology group* $H_k(M; Z_2)$.

The proof is completely analogous to that of Lemma 5.8. The hypothesis, $\xi \cdot \xi = 0$ for all ξ, guarantees that the intersection pairing

$$H_{k+1}(W; Z_2) \otimes H_{k+1}(W; Z_2) \to Z_2$$

will have even rank.

We now return to the case k even.

PROOF OF THEOREM 5.1. (for k even) According to Lemma 5.6, we can assume that $H_k M$ is a torsion group. Choose

$$\varphi : S^k \times D^{k+1} \to M,$$

as in Lemma 5.4, so as to represent a non-trivial $\lambda \in H_k M$. According to Lemma 5.6 we have

$$\frac{H_k M}{\lambda(Z)} \simeq \frac{H_k M'}{\lambda'(Z)}.$$

Since the group $\lambda(Z)$ is finite, it follows from Lemma 5.8 that $\lambda'(Z)$ must be infinite. Thus the sequence

$$0 \to Z \xrightarrow{\lambda'} H_k M' \to \frac{H_k M'}{\lambda'(Z)} \to 0$$

is exact. It follows that the torsion subgroup of $H_k M'$ maps monomorphically into $H_k M'/\lambda'(Z)$; and hence is definitely smaller than $H_k M$. Now according to Lemma 5.7, we can perform a modification on M' so as to obtain a new manifold M'' with

$$H_k M'' \simeq \text{Torsion } H_k M' < H_k M.$$

Thus in two steps one can replace $H_k M$ by a smaller group. Iterating this construction a finite number of times, the group $H_k M$ can be killed completely. This completes the proof of Theorem 5.1 for k even.

§ 6. Framed spherical modifications

This section will complete the proof of Theorem 5.1. by taking care of the case k odd. This case is somewhat more difficult than the case k even (which was handled in § 5), since it is necessary to choose the imbeddings

φ more carefully, taking particular care not to lose s-parallelizability in the process. Before starting the proof, it is convenient to sharpen the concepts of s-parallelizable manifold, and of spherical modification.

Definition. A framed manifold (M, f) will mean a differentiable manifold M together with a fixed trivialization f of the stable tangent bundle $\tau_M \oplus \varepsilon_M$.

Now consider a spherical modification $\chi(\varphi)$ of M. Recall that M and $M' = \chi(M, \varphi)$ together bound a manifold

$$W = (M \times [0, 1]) \cup (D^{p+1} \times D^{q+1}),$$

where the subset $S^p \times D^{q+1}$ of $D^{p+1} \times D^{q+1}$ is pasted onto $M \times 1$ by the imbedding φ (compare Milnor [17]). It is easy to give W a differentiable structure, except along the "corner" $S^p \times S^q$. A neighborhood of this corner will be "diffeomorphic" with $S^p \times S^q \times Q$ where

$$Q \subset R^2$$

denotes the three-quarter disk consisting of all $(r \cos\theta, r \sin\theta)$ with $0 \le r < 1$, $0 \le \theta \le 3\pi/2$. In order to "straighten" this corner, map Q onto the half-disk H, consisting of all $(r \cos\theta', r \sin\theta')$ with $0 \le r < 1$, $0 \le \theta' \le \pi$; by setting $\theta' = 2\theta/3$. Now carrying the differentiable structure of H back to Q, this makes Q into a differentiable manifold. Carrying out the same transformation on the neighborhood of $S^p \times S^q$, this makes $W = W(M, \varphi)$ into the required differentiable manifold. Note that both boundaries of W get the correct differentiable structures.

Now identify M with $M \times 0 \subset W$ and identify the stable tangent bundle $\tau_M \oplus \varepsilon_M$ with the restriction $\tau_W|_M$. Thus a framing f of M determines a trivialization $\tau_W|M$.

Definition. A framed spherical modification $\chi(\varphi, F)$ of the framed manifold (M, f) will mean a spherical modification $\chi(\varphi)$ of M together with a trivialization F of the tangent bundle of W, satisfying the condition

$$F|M = f.$$

Note that the modified manifold $M' = \chi(M, \varphi)$ automatically acquires a framing

$$f' = F|_{M'}.$$

It is only necessary to identify $\tau_W|_{M'}$ with the stable tangent bundle $\tau_{M'} \oplus \varepsilon_{M'}$. To do this, we identify the positive direction in $\varepsilon_{M'}$ with the outward normal direction in $\tau_W|_{M'}$.

The following question evidently arises. Given a modification $\chi(\varphi)$ of M and a framing f of M, does f extend to a trivialization F of τ_W? The obstructions to such an extension lie in the cohomology groups

$$H^{r+1}(W, M; \pi_r(SO_{n+1})) \simeq \begin{cases} \pi_p(SO_{n+1}) & \text{for } r = p, \\ 0 & \text{for } r \neq p. \end{cases}$$

Thus the only obstruction to extending f is a well-defined class

$$\gamma(\varphi) \in \pi_p(SO_{n+1}).$$

The modification $\chi(\varphi)$ can be framed if and only if this obstruction $\gamma(\varphi)$ is zero.

Now consider the following alteration of the imbedding φ. Let

$$\alpha : S^p \to SO_{q+1}$$

be a differentiable map, and define

$$\varphi_\alpha : S^p \times D^{q+1} \to M$$

by

$$\varphi_\alpha(u, v) = \varphi(u, v \cdot \alpha(u)),$$

where the dot denotes the usual action of SO_{q+1} on D^{q+1}. Clearly φ_α is an imbedding which represents the same homotopy class $\lambda \in \pi_p M$, as φ.

Lemma 6.1. *The obstruction $\gamma(\varphi_\alpha)$ depends only on $\gamma(\varphi)$ and on the homotopy class (α) of α. In fact*

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha),$$

where $s_* : \pi_p(SO_{q+1}) \to \pi_p(SO_{n+1})$ is induced by the inclusion $s : SO_{q+1} \to SO_{n+1}$.

PROOF. (compare [17], proof of Theorem 2) Let W_α be the manifold constructed as W above, now using φ_α. There is a natural differentiable imbedding

$$i_\alpha : D^{p+1} \times \operatorname{int} D^{q+1} \to W_\alpha,$$

and $i_\alpha|_{S^p \times D^{q+1}}$ coincides with $\varphi_\alpha : S^p \times D^{q+1} \to M$ followed by the inclusion $M \to M \times 1 \subset W_\alpha$.

$\gamma(\varphi_\alpha)$ is the obstruction to extending $f|_{\varphi_\alpha(S^p \times 0)}$ to a trivialization of $\tau(W_\alpha)$ restricted to $i_\alpha(D^{p+1} \times 0)$. Let $t^{n+1} = e^{p+1} \times e^{q+1}$ be the standard framing on $D^{p+1} \times D^{q+1}$. Then $i'_\alpha(t^{n+1})$ is a trivialization of the tangent bundle of W_α restricted to $i_\alpha(D^{p+1} \times D^{q+1})$, and $\gamma(\varphi_\alpha)$ is the homotopy class of the map $g : S^p \to SO_{n+1}$, where $g(u)$ is the matrix $\langle f^{n+1}, i'_\alpha(t^{n+1})\rangle$ at $\varphi_\alpha(u,0)$.

Since $i_\alpha|_{D^{p+1}\times 0}$ is independent of α, and $i_\alpha|_{S^p\times D^{q+1}} = \varphi_\alpha$, we have

$$\varphi'_\alpha(t^{n+1}) = \varphi'(e^{p+1}) \times \varphi'_\alpha(e^{q+1})$$

at every point $(u,0) \in S^p \times D^{q+1}$.

Since

$$\varphi'_\alpha(e^{q+1}) = \varphi'_\alpha(e^{q+1}) \cdot \alpha(u)$$

at $(u,0)$, it follows that

$$i'_\alpha(t^{n+1}) = i'(t^{n+1}) \cdot s(\alpha).$$

Hence

$$\langle f^{n+1}, i'_\alpha(t^{n+1})\rangle = \langle f^{n+1}, i'(t^{n+1})\rangle \cdot s(\alpha)$$

and the lemma follows.

Now suppose (as usual) $p \le q$. Then the homomorphism

$$s_* : \pi_p(SO_{q+1}) \to \pi_p(SO_{n+1})$$

is onto. Hence α can be chosen so that

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$$

is zero. Thus we obtain:

Lemma 6.2. *Given $\varphi : S^p \times D^{q+1} \to M$ with $p \le q$, a map α can be chosen so that the modification $\chi(\varphi)$ can be framed.*

In particular, it follows that the manifold $\chi(M,\varphi)$ will be s-parallelizable. Thus we have proved Lemma 5.4 in a sharpened form.

We note however that α is not always uniquely determined. In the case $p = q = k$ odd, the homomorphism

$$s_* : \pi_k(SO_{k+1}) \to \pi_k(SO_{n+1})$$

has an infinite cyclic kernel. This freedom in the choice of α will be the basis of the proof of Theorem 5.1 for k odd.

Let us study the homology of the manifold

$$M'_\alpha = \chi(M, \varphi_\alpha),$$

where φ is now chosen, by Lemma 6.1, so that the spherical modification $\chi(\varphi)$ can be framed. Clearly the deleted manifold

$$M_0 = M - (\text{interior}\, \varphi_\alpha(S^k \times D^{k+1}))$$

does not depend on the choice of α. Furthermore the meridian $\varphi_\alpha(x_0 \times S^k)$ of the torus $\varphi_\alpha(S^k \times S^k) \subset M_0$ does not depend on the choice of α; hence the homology class

$$\varepsilon' \in H_k M_0$$

does not depend on α. On the other hand, the parallel $\varphi_\alpha(S^k \times x_0)$ does depend on α. In fact it is clear that the homology class $\varepsilon_\alpha \in H_k M_0$ of this parallel is given by

$$\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon',$$

where the homomorphism

$$j_* : \pi_k(SO_{k+1}) \to Z \simeq \pi_k(S^k)$$

is induced by the canonical map

$$\rho \xrightarrow{j} x_0 \cdot \rho$$

from SO_{k+1} to S^k.

The spherical modification $\chi(\varphi_\alpha)$ can still be framed provided α is an element of the kernel of

$$s_* : \pi_k(SO_{k+1}) \to \pi_k(SO_{n+1}).$$

Identifying the stable group $\pi_k(SO_{n+1})$ with the stable group $\pi_k(SO_{k+2})$, there is an exact sequence

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{s_*} \pi_k(SO_{k+2}),$$

associated with the fibration $SO_{k+2}/SO_{k+1} = S^{k+1}$. It is well known that the composition

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{j_*} \pi_k(S^k)$$

carries a generator of $\pi_{k+1}(S^{k+1})$ onto twice a generator of $\pi_k(S^k)$, provided that k is odd. Therefore the integer $j_*(\alpha)$ can be any multiple of 2.

Let us study the effect of replacing ε by $\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon'$ on the homology of the modified manifold. Consider the exact sequence

$$0 \to Z \xrightarrow{\varepsilon'} H_k M_0 \xrightarrow{i} H_k M \to 0$$

of Lemma 5.6, where i carries ε into an element λ of order $l > 1$. Evidently $l\varepsilon$ must be a multiple of ε', say:

$$l\varepsilon + l'\varepsilon' = 0.$$

Since ε' is not a torsion element, these two elements can satisfy no other relation. Since $\varepsilon_\alpha = \varepsilon + j_*(\alpha)\varepsilon'$ it follows that

$$l\varepsilon_\alpha + (l' - lj(\alpha))\varepsilon' = 0.$$

Now using the sequence

$$Z \xrightarrow{\varepsilon_\alpha} H_k M_0 \xrightarrow{i'_\alpha} H_k M'_\alpha \to 0,$$

we see that the inclusion homomorphism i'_α carries ε' into an element

$$\lambda'_\alpha \in H_k M'_\alpha$$

of order $|l' - lj(\alpha)|$. Since $H_k M'_\alpha / \lambda'_\alpha(Z)$ is isomorphic to $H_k M/\lambda(Z)$, we see that the group $H_k M'_\alpha$ is smaller than $H_k M_\alpha$ if and only if

$$0 < |l' - lj(\alpha)| < l.$$

But $j(\alpha)$ can be any even integer. Thus $j(\alpha)$ can be chosen so that

$$-l < l' - lj(\alpha) \leq l.$$

This choice of $j(\alpha)$ will guarantee an improvement except in the special case where l' happens to be divisible by l.

Our progress so far can be summarized as follows.

Lemma 6.3. *Let M be a framed $(k-1)$-connected manifold of dimension $2k+1$ with odd k, $k > 1$, such that $H_k M$ is finite. Let $\chi(\varphi, F)$ be a framed modification of M which replaces the element $\lambda \in H_k M$ of order $l > 1$ by an element $\lambda' \in H_k M'$ of order $\pm l'$. If $l' \not\equiv 0 \bmod l$, then it is possible to choose $(\alpha) \in \pi_k(SO_{k+1})$ so that the modification $\chi(\varphi)$ can still be framed, and so that the group $H_k M'_\alpha$ is definitely smaller than $H_k M$.*

Thus one must study the residue class of l' modulo l. Recall the definition of linking numbers (compare Seifert–Threlfall [23, § 7]).

Let $\lambda \in H_pM$, $\mu \in H_qM$ be homology classes of finite order, with $\dim M = p+q+1$. Consider the homology sequence

$$\cdots \to H_{p+1}\left(M; \frac{Q}{Z}\right) \xrightarrow{\beta} H_pM \xrightarrow{i_*} H_p(M;Q) \to \cdots ,$$

associated with the coefficient sequence

$$0 \to Z \xrightarrow{i} Q \to \frac{Q}{Z} \to 0.$$

Since λ is of finite order $i_*\lambda = 0$ and $\lambda = \beta(\nu)$ for some $\nu \in H_{p+1}(M; Q/Z)$. The pairing

$$\frac{Q}{Z} \otimes Z \to \frac{Q}{Z},$$

defined by multiplication induces a pairing

$$H_{p+1}\left(M; \frac{Q}{Z}\right) \otimes H_qM \to \frac{Q}{Z},$$

defined by the intersection of homology classes. We denote this pairing by a dot.

Definition. The *linking number* $L(\lambda, \mu)$ is the rational number modulo 1 defined by

$$L(\lambda, \mu) = \nu \cdot \mu.$$

This linking number is well defined, and satisfies the symmetry relation

$$L(\mu, \lambda) + (-1)^{pq} L(\lambda, \mu) = 0$$

(compare Seifert and Threlfall [23]).

Lemma 6.4. *The ration l'/l modulo* 1 *is, up to sign, equal to the self-linking number $L(\lambda, \lambda)$.*

Proof. Since

$$l\varepsilon + l'\varepsilon' = 0$$

in H_kM_0, we see that the cycle $l\varepsilon + l'\varepsilon'$ on bM_0 bounds a chain c on M_0. Let $c_1 = \varphi(x_0 \times D^{k+1})$ denote the cycle in $\varphi(S^k \times D^{k+1}) \subset M$ with boundary ε'. Then the chain $c - l'c_1$, has boundary $l\varepsilon$; hence $(c - l'c_1)/l$ has boundary ε, representing the homology class λ in H_kM. Taking the intersection of this chain with $\varphi(S^k \times 0)$, representing λ, we obtain $\pm l'/l$, since c is disjoint and c_1 has intersection number ∓ 1. Thus $L(\lambda, \lambda) = \pm l'/l \bmod 1$.

Now if $L(\lambda, \lambda) \neq 0$, then $l' \not\equiv 0 \pmod l$, hence the class λ can be replaced by an element of smaller order under a spherical modification. Hence, unless $L(\lambda, \lambda) = 0$ for all $\lambda \in H_kM$, this group can be simplified.

Lemma 6.5. *If H_kM is a torsion group with $L(\lambda, \lambda) = 0$ for every $\lambda \in H_kM$, and if k is odd, then this group H_kM must be a direct sum of cyclic groups of order* 2.

PROOF. The relation

$$L(\eta, \xi) + (-1)^{pq} L(\xi, \eta) = 0$$

with $p = q \equiv 1 \pmod 2$ implies that

$$L(\eta, \xi) = L(\xi, \eta).$$

Now if self-linking numbers are all zero, the identity

$$L(\xi + \eta, \xi + \eta) = L(\xi, \xi) + L(\eta, \eta) + L(\xi, \eta) + L(\eta, \xi)$$

implies that

$$2L(\xi, \eta) = 0$$

for all ξ and η. But, according to the Poincaré duality theorem for torsion groups (see [23, p. 245]), L defines a completely orthogonal pairing

$$T_pM \otimes T_qM \to \frac{Q}{Z}.$$

Hence the identity $L(2\xi, \eta) = 0$ for all η implies $2\xi = 0$. This proves Lemma 6.5.

It follows that, by a sequence of modifications, one can reduce H_kM to a group of the form $Z_2 \oplus \cdots \oplus Z_2 = sZ_2$.

Now let us apply Lemma 5.10. Since the modification $\chi(\varphi_\alpha)$ is framed, the corresponding manifold $W = W(M, \varphi_\alpha)$ is parallelizable. It follows from the formulas of Wu that the Steenrod operation

$$\mathrm{Sq}^{k+1} : H^{k+1}(W, bW; Z_2) \to H^{2k+2}(W, bW; Z_2)$$

is zero (see Kervaire [8, Lemma 7.9]). Hence every $\xi \in H_{k+1}(W; Z_2)$ has self-intersection number $\xi \cdot \xi = 0$. Thus, according to Lemma 5.10, the modification $\chi(\varphi_\alpha)$ changes the rank $H_k(M; Z_2)$.

But the effect of $\chi(\varphi_\alpha)$ on $H_k(M; Z)$, provided that α is chosen properly, will be to replace the element λ of order $l = 2$ by an element λ'_α of order l'_α where

$$-2 < l'_\alpha \leq 2, \quad l'_\alpha \equiv 0 \pmod 2.$$

Thus l'_α must be 0 or 2. Now using the sequence

$$0 \to Z_{l'_\alpha} \to H_k M'_\alpha \to \frac{H_k M_\alpha}{\lambda_\alpha(Z)} \to 0,$$

where the group on the right is isomorphic to $(s-1)Z_2$, we see that $H_k M'_\alpha$ is given by one of the following:

$$H_k M'_\alpha \simeq \begin{cases} Z + (s-1)Z_2, \\ Z_2 + (s-1)Z_2, \\ Z + (s-2)Z_2 & \text{or} \\ Z_4 + (s-2)Z_2. \end{cases}$$

But the first two possibilities cannot occur, since they do not change the rank of $H_k(M; Z_2)$. In the remaining two cases, a further modification will replace $H_k M'_\alpha$ by a group which is definitely smaller than $H_k M$. Thus in all cases $H_k M$ can be replaced by a smaller group by a sequence of framed modifications.

This completes the proof of Theorem 5.1. Actually we have proved the following result which is slightly sharper.

Theorem 6.6. *Let M be a compact, framed manifold of dimension $2k+1$, $k > 1$, such that bM is either vacuous or a homology sphere. By a sequence of framed modifications, M can be reduced to a k-connected manifold M_1.*

If bM is vacuous then the Poincaré duality theorem implies that M_1 is a homotopy sphere. If bM is a homology sphere, then M_1 is contractible.

The proof of Theorem 6.6 is contained in the above discussion, provided that M is connected. But using [17, Lemma 2′] it is easily seen that a disconnected manifold can be connected by framed modifications. This completes the proof.

§ 7. The groups bP_{2k}

The next two sections will prove that the groups bP_{2k} are finite cyclic for $k \neq 2$. In fact for k odd, the group bP_{2k} has at most two elements. For $k = 2m \neq 2$ we will see in Part II that bP_{4m} is a cyclic group of order[1]

$$\varepsilon_m 2^{2m-2}(2^{2m-1} - 1) \cdot \text{numerator } \frac{4B_m}{m},$$

[1] This expression for the order of bP_{4m} relies on recent results of J. F. Adams [1].

where B_m denotes the m-th Bernoulli number, and ε_m equals 1 or 2.

The proofs will be based on the following.

Lemma 7.1. *Let M be a $(k-1)$-connected manifold of dimension $2k$, $k \geq 3$, and suppose that $H_k M$ is free abelian group with basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$, where*

$$\lambda_i \cdot \lambda_j = 0, \quad \lambda_i \cdot \mu_j = \delta_{ij}$$

for all i, j (where δ_{ij} denotes a Kronecker delta). Suppose further that every imbedded sphere in M which represents a homology class in the subgroup generated by $\lambda_1, \dots, \lambda_r$ has trivial normal bundle. Then $H_k M$ can be killed by a sequence of spherical modifications.

PROOF. According to [17, Lemma 6] or Haefliger [6] any homology class in $H_k M$ can be represented by a differentiably imbedded sphere.

Remark. It is at this point that the hypothesis $k \geq 3$ is necessary. Our methods break down completely for the case $k = 2$ since a homology class in $H_2(M^4)$ need not be representable by a differentiably imbedded sphere (compare Kervaire-Milnor [13]).

Choose an imbedding $\varphi_0 : S^k \to M$ so as to represent the homology class λ_r. Since the normal bundle is trivial, φ_0 can be extended to an imbedding $\varphi : S^k \times D^k \to M$. Let $M' = \chi(M, \varphi)$ denote the modified manifold, and let

$$M_0 = M - \operatorname{Interior} \varphi(S^k \times D^k) = M' - \operatorname{Interior} \varphi'(D^{k+1} \times S^{k-1}).$$

The argument now proceeds just as in [17, p. 54]. There is a diagram

$$\begin{array}{ccccccccc}
 & & Z & & & & & & \\
 & & \downarrow & \searrow \lambda_r & & & & & \\
0 & \to & H_k M_0 & \to & H_k M & \xrightarrow{\cdot \lambda_r} & Z \to H_{k-1} M_0 & \to & 0, \\
 & & \downarrow & & & & & & \\
 & & H_k M' & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where the notation and the proof is similar to that of Lemma 5.6. Since $\mu_r \cdot \lambda_r = 1$ it follows that $H_{k-1} M_0 = 0$. From this fact one easily proves that M_0 and M' are $(k-1)$-connected. The group $H_k M_0$ is isomorphic to the subgroup of $H_k M$ generated by $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_{r-1}\}$. The group $H_k M'$ is isomorphic to a quotient group of $H_k M_0$. It has basis

$\{\lambda'_1, \ldots, \lambda'_{r-1}, \mu'_1, \ldots, \mu'_{r-1}\}$ where each element λ'_i corresponds to a coset

$$\lambda_i + \lambda_r Z \subset H_k M$$

and each μ'_j corresponds to a coset $\mu_j + \lambda_r Z$.

The manifold M' also satisfies the hypothesis of Lemma 7.1. In order to verify that

$$\lambda'_i \cdots \lambda'_j = 0, \quad \lambda'_i \cdot \mu'_j = \delta_{ij},$$

note that each λ'_i or μ'_j can be represented by a sphere imbedded in M_0 and representing the homology class λ_i or μ_j in M. Thus the intersection numbers in M' are the same as those in M. In order to verify that any imbedded sphere with homology class $n_1\lambda'_1 + \cdots + n_{r-1}\lambda'_{r-1}$ has trivial normal bundle note that any such sphere can be pushed off $\varphi'(0 \times S^{k-1})$ and hence can be deformed into M_0. It will represent a homology class

$$(n_1\lambda_1 + \cdots + n_{r-1}\lambda_{r-1}) + n_r\lambda_r \in H_k M$$

and thus will have trivial normal bundle.

Iterating this construction r times, the result will be a k-connected manifold. This completes the proof of Lemma 7.1.

Now consider an s-parallelizable manifold M of dimension $2k$, bounded by a homology sphere. By Theorem 5.5, we can assume that M is $(k-1)$-connected. Using the Poincaré duality theorem it follows that $H_k M$ is free abelian, and the intersection number pairing

$$H_k M \otimes H_k M \to Z$$

has determinant ± 1. The argument now splits up into three cases.

Case 1. Let $k = 3$ or 7 (compare [17, Theorem 4′]). Since k is odd the intersection pairing is skew symmetric. Hence there exists a "symplectic" basis for $H_k M$; that is, a basis $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r\}$ with

$$\lambda_i \cdot \lambda_j = \mu_i \cdot \mu_j = 0, \quad \lambda_i \cdot \mu_j = \delta_{ij}.$$

Since $\pi_{k-1}(SO_k) = 0$ for $k = 3, 7$, any imbedded k-sphere will have trivial normal bundle. Thus Lemma 7.1 implies that $H_k M$ can be killed. Since an analogous result for $k = 1$ is easily obtained, this proves:

Lemma 7.2. *The groups bP_2, bP_6, and bP_{14} are zero.*

Case 2. k is odd, but $k \neq 1, 3, 7$. Again one has a symplectic basis; but the normal bundle of an imbedded sphere is not necessarily trivial. This case will be studied in § 8.

Case 3. k is even, say, $k = 2m$. Then the following is true (compare [17, Theorem 4]).

Lemma 7.3. *Let M be a framed manifold of dimension $4m>4$ bounded by a homology sphere.*[1] *The homotopy groups of M can be killed by a sequence of framed spherical modifications if and only if the signature $\sigma(M)$ is zero.*

Since a proof of Lemma 7.3 is essentially given in [17], we will only give an outline here.

In one direction the lemma follows from the assertion that $\sigma(M)$ is invariant under spherical modifications. (See [17, p. 41]. The fact that M has a boundary does not matter here, since we can adjoin a cone over the boundary, thus obtaining a closed homology manifold with the same signature.)

Conversely suppose that $\sigma(M)=0$. We may assume that M is $(k-1)$-connected. Since the quadratic form $\lambda \to \lambda \cdot \lambda$ has determinant ± 1 and signature zero, it is possible to choose a basis $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r\}$ for $H_k M$ such that $\lambda_i \cdot \lambda_j = 0$, $\lambda_i \cdot \mu_j = \delta_{ij}$. The proof is analogous to that of [17, Lemma 9], but somewhat simpler since we do not put any restriction on $\mu_i \cdot \mu_j$. For any imbedded sphere with homology class $\lambda = n_1\lambda_1 + \cdots + n_r\lambda_r$ the self-intersection number $\lambda \cdot \lambda$ is zero. Therefore, according to [17, Lemma 7], the normal bundle is trivial.

Thus M satisfies the hypothesis of Lemma 7.1. It follows that $H_k M$ can be killed by spherical modifications. Since the homomorphism

$$\pi_k(SO_k) \to \pi_k(SO_{2k+1})$$

is onto for k even, it follows from Lemma 6.2 that we only need to use framed spherical modifications. This completes the proof of Lemma 7.3.

Lemma 7.4. *For each $k=2m$ there exists a parallelizable manifold M_0 whose boundary bM_0 is the ordinary $(4m-1)$-sphere, such that the signature $\sigma(M_0)$ is non-zero.*

PROOF. According to Milnor and Kervaire [18, p. 457] there exists a closed "almost parallelizable" $4m$-manifold whose signature is non-zero. Removing the interior of an imbedded $4m$-disk from this manifold, we obtain the required parallelizable manifold M_0.

Now consider the collection of all $4m$-manifolds M_0 which are s-parallelizable, and are bounded by the $(4m-1)$-sphere. Clearly the corresponding signatures $\sigma(M_0) \in Z$ form a group under addition. Let $\sigma_m > 0$ denote the generator of this group.

[1] This lemma is of course true if bM is vacuous. In this case the signature $\sigma(M)$ is necessarily zero, by Hirzebruch's signature theorem.

Theorem 7.5. *Let Σ_1 and Σ_2 be homotopy spheres of dimension $4m-1$, $m>1$, which bound s-parallelizable manifolds M_1 and M_2 respectively. Then Σ_1 is h-cobordant to Σ_2 if and only if*

$$\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m}.$$

Proof. First suppose that

$$\sigma(M_1) = \sigma(M_2) + \sigma(M_0).$$

Form the connected sum along the boundary

$$(M, bM) = (-M_1, -bM_1)\#(M_2, bM_2)\#(M_0, bM_0)$$

as in § 2; with boundary

$$bM = -\Sigma_1\#\Sigma_2\#S^{4m-1} \approx -\Sigma_1\#\Sigma_2.$$

Since

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2) + \sigma(M_0) = 0,$$

it follows from Lemma 7.3 that $bM = -\Sigma_1\#\Sigma_2$ belongs to the trivial h-cobordism class. Therefore Σ_1 is h-cobordant to Σ_2.

Conversely let W be an h-cobordism between $-\Sigma_1\#\Sigma_2$ and the sphere S^{4m-1}. Pasting W onto $(-M_1, -bM_1)\#(M_2, bM_2)$ along the common boundary $-\Sigma_1\#\Sigma_2$, we obtain a differentiable manifold M bounded by the sphere S^{4m-1}. Since M is clearly s-parallelizable, we have

$$\sigma(M) = 0 \pmod{\sigma_m}.$$

But

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2).$$

Therefore

$$\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m},$$

which completes the proof.

Corollary 7.6. *The group bP_{4m}, $m>1$, is isomorphic to a subgroup of the cyclic group of order σ_m. Hence bP_{4m} is finite cyclic.*

The proof is evident.

Discussion and computation. In part II we will see that bP_{4m} is cyclic of order precisely $\sigma_m/8$. In fact a given integer σ is a signature $\sigma(M)$

for some s-parallelizable manifold M bounded by a homotopy sphere if and only if

$$\sigma \equiv 0 \pmod 8.$$

The following equality is proved in [18, p. 457];

$$\sigma_m = \frac{2^{2m-1}(2^{2m-1}-1)B_m j_m a_m}{m},$$

where B_m denotes the m-th Bernoulli number, j_m denotes the order of the cyclic group

$$J(\pi_{4m-1}(SO)) \subset \Pi_{4m-1}$$

and a_m equals 1 or 2 according as m is even or odd. Thus bP_{4m} is cyclic group of order

$$\frac{\sigma_m}{8} = \frac{2^{2m-4}(2^{2m-1}-1)B_m j_m a_m}{m}. \tag{1}$$

According to recent work of J. F. Adams [1], the integer j_m is precisely equal to the denominator of $B_m/4m$, at least when m is odd. (Compare [18, Theorem 4].) Therefore

$$\frac{B_m j_m a_m}{4m} = a_m \text{ numerator}\left(\frac{B_m}{4m}\right) = \text{numerator}\left(\frac{4B_m}{m}\right),$$

where the last equality holds since the denominator of B_m is divisible by 2, but not 4. Thus bP_{4m} is cyclic of order

$$\frac{\sigma_m}{8} = 2^{2m-2}(2^{2m-1}-1)\cdot \text{numerator}\left(\frac{4B_m}{m}\right), \tag{2}$$

when m is odd.

One can also give a formula for the order of the full group Θ_{4m-1}. In Part II we will see that Θ_{4m-1}/bP_{4m} is isomorphic to $\Pi_{4m-1}/J(\pi_{4m-1}(SO))$ (compare § 4). Together with formula (1) above this implies that:

$$\text{order } \Theta_{4m-1} = \frac{(\text{order } \Pi_{4m-1})2^{2m-4}(2^{2m-1}-1)B_m a_m}{m}.$$

§ 8. A cohomology operation

Let $2 \leq k \leq n-2$ be integers and let (K, L) be a CW-pair satisfying the following

Hypothesis. The cohomology groups $H^i(K, L; G)$ vanish for $k < i < n$ for all coefficient groups G.

Then a cohomology operation

$$\psi : H^k(K, L; Z) \to H^n(K, L; \pi_{n-1}(S^k))$$

is defined as follows.[1] Let $e^0 \in S^k$ denote a base point and let

$$s \in H_k(S^k, e^0; Z)$$

denote a generator. Then $\psi(c)$ will denote the first obstruction to the existence of a map

$$f : (K, L) \to (S^k, e^0),$$

satisfying the condition $f^*(s) = c$.

To be more precise let K^r denote the r-skeleton K. Then given any class

$$x \in H^k(K, L; Z) \simeq H^k(K^{n-1} \cup L, L; Z)$$

it follows from standard obstruction theory that there exists a map

$$f_z : (K^{n-1} \cup L, L) \to (S^k, e^0)$$

with $f_z^* s = x$; and that the restriction

$$f_x|_{(K^{n-2} \cup L, L)}$$

is well defined up to homotopy. The obstruction to extending f_x over $K^n \cup L$ is the required class

$$\psi(x) \in H^n(K, L; \pi_{n-1}(S^k)).$$

Lemma 8.1. *The function*

$$\psi : H^k(K, L; Z) \to H^n(K, L; \pi_{n-1}(S^k))$$

is well defined, and is natural in the following sense. If the CW-pair (K', L') also satisfies the hypothesis above, then for any map

$$g : (K', L') \to (K, L)$$

and any $x \in H^k(K, L; Z)$ the identity

$$g^* \psi(x) = \psi g^*(x)$$

is satisfied.

---

[1] A closely related operation φ_0 has been studied by Kervaire [12]. The operation φ_0 would serve equally well for our purposes.

The proof is straightforward. It follows that ψ does not depend on the particular cell structure of the pair (K, L).

Now let us specialize the case $n = 2k$.

Lemma 8.2. *The operator ψ satisfies the identity*

$$\psi(x + y) = \psi(x) + \psi(y) + [i, i](x \smile y),$$

where the last term stands for the image of the class $x \smile y \in H^{2k}(K, L; Z)$ under the coefficient homomorphism

$$Z \to \pi_{2k-1}(S^k),$$

which carries 1 *into the Whitehead product class* $[i, i]$.

PROOF. Let $U = e^0 \cup e^k \cup \{e_i^{2k}\} \cup \{e_j^{2k+1}\} \cup \cdots$ denote a complex formed from the sphere S^k by adjoining cells of dimensions $\geq 2k$ so as to kill the homotopy groups in dimensions $\geq 2k - 1$. Let

$$u \in H^k(U, e^0; Z)$$

be a standard generator. Evidently the functions

$$\psi : H^k U \to H^{2k}(U; \pi_{2k-1}(S^k))$$

and

$$\psi : H^k(U \times U) \to H^{2k}(U \times U; \pi_{2k-1}(S^k))$$

are defined. We will first evaluate $\psi(u \times 1 + 1 \times u)$.

The $(2k + 1)$-skeleton $U \times U$ consists of the union

$$U^{2k+1} \times e^0 \cup e^0 \times U^{2k+1} \cup e^k \times e^k.$$

Therefore the cohomology class $\psi(u \times 1 + 1 \times u) \in H^{2k}(U \times U; \pi_{2k-1}(S^k))$ can be expressed uniquely in the form

$$a \times 1 + 1 \times b + \gamma(u \times u),$$

with $a, b \in H^{2k}(U; \pi_{2k-1}(S^k))$ and $\gamma \in \pi_{2k-1}(S^k)$. Applying Lemma 8.1 to the inclusion map

$$U \times e^0 \to U \times U,$$

we see that a must be equal to $\psi(u)$. Similarly b is equal to $\psi(u)$. Applying Lemma 8.1 to the inclusion

$$S^k \times S^k \to U \times U,$$

we see that $\psi(s \times 1 + 1 \times s) = \gamma(s \times s)$. But $\psi(s \times 1 + 1 \times s)$ is just the obstruction to the existence of a mapping

$$f : S^k \times S^k \to S^k,$$

satisfying $f(e^0, x) = f(x, e^0) = x$. Therefore γ must be equal to the Whitehead product class $[i, i] \in \pi_{2k-1}(S^k)$. Thus we obtain the identity

$$\begin{aligned}\psi(u \times 1 + 1 \times u) &= \psi(u) \times 1 + 1 \times \psi(u) + [i, i](u \times u) \\ &= \psi(u \times 1) + \psi(1 \times u) + [i, i]((u \times 1) \smile (1 \times u)).\end{aligned}$$

Now consider an arbitrary CW-pair (K, L) and two classes x, $y \in H^k(K, L)$. Choose a map

$$g : (K, L) \to (U \times U, e^0 \times e^0)$$

so that $g^*(u \times 1) = x$, $g^*(1 \times u) = y$. (Such a map can be constructed inductively over the skeletons of K since the obstruction groups $H^i(H, L; \pi_{i-1}(U \times U))$ are all zero.) Then by Lemma 8.1:

$$\begin{aligned}\psi(x + y) &= g^*\psi(u \times 1 + 1 \times u) \\ &= g^*\psi(u \times 1) + g^*(1 \times u) + [i, i]g^*((u \times 1) + (1 \times u)) \\ &= \psi(x) + \psi(y) + [i, i](x \smile y).\end{aligned}$$

This completes the proof of Lemma 8.2.

Now let M be a $2k$-manifold which is $(k-1)$-connected. Then

$$\psi : H^k(M, bM) \to H^{2k}(M, bM; \pi_{2k-1}(S^k)) \simeq \pi_{2k-1}(S^k)$$

is defined.

Lemma 8.3. *Let k be odd*[1] *and let M be s-parallelizable. Then an imbedded k-sphere in M has trivial normal bundle if and only if its dual cohomology class $v \in H^k(M, bM)$ satisfies the condition $\psi(v) = 0$.*

---

[1] This lemma is actually true for even k also.

PROOF. Let N be a closed tubular neighborhood of the imbedded sphere, and let

$$M_0 = M - \text{Interior } N.$$

Then there is a commutative diagram

$$\begin{array}{ccc}
w \in H^k(N, bN) & \xrightarrow{\psi} & H^{2k}(N, bN; \pi_{2k-1}(S^k)) \\
\uparrow \simeq & & \uparrow \simeq \\
H^k(M, M_0) & \xrightarrow{\psi} & H^{2k}(M, M_0; \pi_{2k-1}(S^k)) \\
\downarrow & & \downarrow \simeq \\
v \in H^k(M, bM) & \xrightarrow{\psi} & H^{2k}(M, bM; \pi_{2k-1}(S^k)),
\end{array}$$

where the generator w of the infinite cyclic group $H^k(N, bN)$ corresponds to the cohomology class v under the left-hand vertical arrows. Thus,[1]

$$\psi(v)[M] = \psi(w)[N] \in \pi_{2k-1}(S^k).$$

It is clear that the homotopy class $\psi(w)[N]$ depends only on the normal bundle of the imbedded sphere.

The normal bundle is determined by an element ν of the group $\pi_{k-1}(SO_k)$. Since M is s-parallelizable, ν must belong to the kernel of the homomorphism

$$\pi_{k-1}(SO_k) \to \pi_{k-1}(SO).$$

But this kernel is zero for $k = 1, 3, 7$, and is cyclic of order 2 for other odd values of k. The unique nontrivial element corresponds to the tangent bundle of S^k, or equivalently to the normal bundle of the diagonal in $S^k \times S^k$.

Thus if $\nu \neq 0$ then N can be identified with a neighborhood of the diagonal $S^k \times S^k$. Then

$$\psi(w)[N] = \psi(s \times 1 + 1 \times s)[S^k \times S^k] = [i, i] \neq 0$$

(assuming that $k \neq 1, 3, 7$). On the other hand if $\nu = 0$ then $\psi(w)$ is clearly zero. This completes the proof of Lemma 8.3.

[1]The symbol $[M]$ denotes the homomorphism $H^n(M, bM; G) \to G$ determined by the orientation homology class in $H_n(M, bM; Z)$.

Henceforth we will assume that k is odd and $\neq 1, 3, 7$. The subgroup of $\pi_{2k-1}(S^k)$ generated by $[i, i]$ will be identified with the standard cyclic group Z_2. Thus a function

$$\psi_0(\lambda) = \psi(x)[M],$$

is defined by the formula

$$\psi_0 : H_k M \to Z_2,$$

where $x \in H^k(M, bM)$ denotes the Poincaré dual of the homology class λ. Evidently:

(1) $\psi_0(\lambda + \mu) \equiv \psi_0(\lambda) + \psi_0(\mu) + \lambda \cdot \mu \pmod 2$, and
(2) $\psi_0(\lambda) = 0$ if and only if an imbedded sphere representing the homology class λ has trivial normal bundle.

Now assume that bM has no homology in dimensions k, $k-1$, so that the intersection pairing has determinant ± 1. Then one can choose a symplectic basis for $H_k M$: that is a basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ such that

$$\lambda_i \cdot \lambda_j = 0, \quad \mu_i \cdot \mu_j = 0, \quad \lambda_i \cdot \mu_j = \delta_{ij}.$$

Definition. The *Arf-invariant* $c(M)$ is defined to be the residue class[1]

$$\psi_0(\lambda_1)\psi_0(\mu_1) + \cdots + \psi_0(\lambda_r)\psi_0(\mu_r) \in Z_2$$

(compare [3]). This residue class modulo 2 does not depend on the choice of symplectic basis.

Lemma 8.4. *If $c(M) = 0$ then $H_k M$ can be killed by a sequence of framed spherical modifications.*

The proof will depend on Lemma 7.1. Let $\{\lambda_1, \dots, \lambda_r,\ \mu_1, \dots, \mu_r\}$ be a symplectic basis for $H_k M$. By permuting the λ_i and μ_i we may assume that

$$\begin{aligned} \psi_0(\lambda_i) = \psi_0(\mu_i) &= 1 \quad \text{for } i \leq s, \\ \psi_0(\lambda_i) &= 0 \quad \text{for } i > s, \end{aligned}$$

where s is an integer between 0 and r. The hypothesis

$$c(M) = \sum \psi_0(\lambda_i)\psi_0(\mu_i) = 0$$

implies that $s \equiv 0 \pmod 2$.

[1] This coincides with the invariant $\Phi(M)$ as defined by Kervaire [12].

Construct a new basis $\{\lambda'_1, \ldots, \mu'_r\}$ for $H_k M$ by the substitutions

$$\lambda'_{2i-1} = \lambda_{2i-1} + \lambda_{2i}, \quad \lambda'_{2i} = \mu_{2i-1} - \mu_{2i},$$
$$\mu'_{2i-1} = \mu_{2i-1}, \qquad \mu'_{2i} = \lambda_{2i},$$

for $2i \leq s$,

$$\lambda'_i = \lambda_i, \quad \mu'_i = \mu_i$$

for $i > s$. This basis is again symplectic, and satisfies the condition:

$$\psi_0(\lambda'_1) = \cdots = \psi_0(\lambda'_r) = 0.$$

For any sphere imbedded in M with homology class $\lambda = n_1\lambda'_1 + \cdots + n_r\lambda'_r$ the invariant $\psi_0(\lambda)$ is zero, and hence the normal bundle is trivial. Thus the basis $\{\lambda'_1, \ldots, \mu'_r\}$ satisfies the hypothesis of Lemma 7.1. Thus $H_k M$ can be killed by spherical modifications.

If M is a framed manifold then it is only necessary to use framed modifications for this construction. This follows from Lemma 6.2, since the homomorphism $\pi_k(SO_k) \to \pi_k(SO_{2k+1})$ is onto for $k \neq 1, 3, 7$. This completes the proof of Lemma 8.4.

Theorem 8.5. *For k odd, the group bP_{2k} is either zero or cyclic of order* 2.

According to Lemma 7.2 the groups bP_2, bP_6 and bP_{14} are zero. Thus we may assume that $k \neq 1, 3, 7$.

Let M_1 and M_2 be s-parallelizable and $(k-1)$-connected manifolds of dimension $2k$, bounded by homotopy spheres. If

$$c(M_1) = c(M_2),$$

we will prove that bM_1 is h-cobordant to bM_2. This will clearly prove Theorem 8.5.

Form the connected sum $(M, bM) = (M_1, bM_1)\#(M_2, bM_2)$ along the boundary. Clearly

$$c(M) = c(M_1) + c(M_2) = 0.$$

Therefore, according to Lemma 8.4, it follows that the boundary

$$bM = bM_1 \# bM_2$$

bounds a contractible manifold. Hence, according to Theorem 1.1 the manifold bM_1 is h-cobordant to $-bM_2$. Since a similar argument shows that bM_2 is h-cobordant to $-bM_1$, this completes the proof.

Remark. It seems plausible that $bP_{2k} \simeq Z_2$ for all odd k other than 1, 3, 7; but this is known to be true only for $k = 5$ (compare Kervaire [12]) and $k = 9$.

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3

Homotopically equivalent smooth manifolds[1,2]

S. P. Novikov

Here we introduce a method for the investigation of smooth simply connected manifolds of dimension $n \geq 5$ that permits an exact classification of them up to orientation-preserving diffeomorphisms. This method involves a detailed investigation of the properties of the so-called Thom complexes of normal bundles and is based on a theorem of Smale concerning the equivalence of the concepts of "h-cobordism" and "orientation-preserving diffeomorphism". In the last chapter we work out some simple examples. Appendices are given in which the results of this article are applied to certain other problems.

Introduction

This article is devoted to the study of the following question: What are the invariants that define the property of two smooth oriented manifolds

[1]Translated by V. Poenaru, Izvestiya Akad. Nauk SSSR, ser. matem. **28** (1964), 365–474 (Received March 22, 1963).

[2]The main ideas were first sketched in [14]. This paper contains detailed proofs of all results from [14] plus a number of new results. The paper [14] was named the best mathematical paper of the U.S.S.R. of 1961 by the Academy of Sciences of the U.S.S.R. — S. P. Novikov's remark (2004).

of being diffeomorphic to each other? It is clear that for manifolds to be diffeomorphic it is necessary for them to be homotopically equivalent. A more refined necessary condition is given by the tangent bundle of a manifold. Speaking in modern terms, to a manifold M^n there corresponds an Atiyah–Hirzebruch–Grothendieck functor

$$K_R(M^n) = Z + \widetilde{K}_R(M^n),$$

and by the tangent bundle we mean a certain distinguished element $\tau(M^n) \in \widetilde{K}_R(M^n)$, the "stable tangent bundle" with its degree extracted. Though the ring $\widetilde{K}_R(M^n)$ itself is homotopically invariant, it is well known that the element $\tau(M^n)$ is not homotopically invariant, and what is more, it can have infinitely many values. For two manifolds, M_1^n and M_2^n, to be diffeomorphic it is necessary that there exists a homotopy equivalence $f : M_1^n \to M_2^n$, such that

$$f^*\tau(M_2^n) = \tau(M_1^n),$$

where $f : \widetilde{K}_R(M_2^n) \to \widetilde{K}_R(M_1^n)$. If this latter necessary condition holds, then the direct products $M_1^n \times R^N$ and $M_2^n \times R^N$ are diffeomorphic (Mazur). But this result of Mazur is of little help in determining whether or not M_1^n and M_2^n themselves are diffeomorphic. Even for $n = 3$ there exist nondiffeomorphic manifolds satisfying the necessary conditions indicated above for manifolds to be diffeomorphic (lens spaces). To be sure, these manifolds are not simply connected. For simply connected manifolds the papers of Whitehead on simple homotopy type or the papers by Smale [17, 19] yield a stronger result, namely, that the direct products by a ball $M_1^n \times D^N$ and $M_2^n \times D^N$ are diffeomorphic. Nevertheless examples by Milnor [10] of differentiable structures on spheres show that for simply connected manifolds combinatorially equivalent to a sphere, multiplication by a closed mall actually eliminates the existence of a finer distinction between smooth structures.

In the papers by Milnor [9] and Milnor and Kervaire [6] a more or less complete classification was finally given of homotopy spheres with exactness up to h-homology (J-equivalence) in terms of the standard homotopy groups of spheres.

The foundation for this classification was laid by papers of Smale [17, 19], who demonstrated that, for simply connected manifolds of dimension $n \geq 5$, the concepts "h-homology" and "orientation-preserving diffeomorphism" coincide. In addition, Smale proposed a method that permits this classification and Wall gave a good classification of manifolds in certain simple examples (cf. [18, 27]).

Here we investigate the class of smooth manifolds $\{M_j^n\}$ that are homotopically equivalent among themselves and such that for any pair i, j

there exists a homotopy equivalence $f : M_i^n \to M_j^n$ of degree $+1$, such that

$$f^*\tau(M_j^n) = \tau(M_i^n),$$

where $f^* : \widetilde{K}_R(M_j^n) \to \widetilde{K}_R(M_i^n)$ and $\tau(M^n)$ is the stable tangent bundle. Thus we consider the class of smooth manifolds having the same homotopy type and tangent bundle. The basic problem is to give a classification of manifolds of the class $\{M_i^n\}$ for $n \geq 5$, assuming that $\pi_1(M_i^n) = 0$. The approach used in this paper is connected with a consideration of the Thom complex T_N of the stable normal bundle for the manifold M_0^n belonging to the class $\{M_i^n\}$. The complex $T^N = T_N(M_0^n)$ is obtained by a contraction of the boundary of the ε-neighborhood U_ε^{N+n} of the manifold M_0^n in the space R^{N+n} into a point, i.e.

$$T_N = \frac{U^{N+n}}{\partial U^{N+n}},$$

and it is easily shown that the complex T_N of dimension $n + N$ is a pseudomanifold with fundamental cycle $[T_N]$, belonging to a form of the Hurewicz homomorphism:

$$H : \pi_{n+N}(T_N) \to H_{n+N}(T_N).$$

Consider the finite set $A = H^{-1}([T_N])$. The group $\pi(M_0^n, SO_N)$ acts on this set, and on the set of orbits $A/\pi(M_0^n, SO_N)$ there is an action of the mapping class group $\pi^+(M_0^n, M_0^n)$ for mappings $f : M_0^n \to M_0^n$ of degree $+1$ such that

$$f^*\tau(M_0^n) = \tau(M_0^n).$$

The main goal of Chapter I is to prove the following assertion.

(Classification). There exists a natural mapping of sets $\{M_i^n\} \to (A/\pi(M_0^n, SO_N))/\pi^+(M_0^n, M_0^n)$, possessing the following properties:

(a) if this mapping takes two manifolds M_1^n and M_2^n to the same element, then there exists such a Milnor sphere $\widetilde{S}^n \in \theta^n(\partial\pi)$ that $M_1^n = M_2^n \# \widetilde{S}^n$;
(b) conversely, if $M_1^n = M_2^n \# \widetilde{S}^n$, then these manifolds are mapped to the same element of $(A/\pi(M_0^n, SO_N))/\pi^+(M_0^n, M_0^n)$, where $\widetilde{S}^n \in \theta^n(\partial\pi)$;
(c) if $n \neq 4k + 2$, then this mapping is epimorphic.

From this theorem one can immediately draw certain conclusions. For example, one can easily prove the following.

The homotopy type and the rational Pontrjagin classes determine a smooth simply connected manifold M^n to within a finite number of possibilities for $n \geq 5$. If the groups $H_{4i}(M^n)$ are finite for $0 < 4i < n$, then there exists a finite number of smooth structures on the topological manifold M^n (a result of the finiteness of the set A).

In fact the solution of the problem obtained by the author is much more significant in homotopy terms than in the way it is formulated in the cited Classification Theorem. A number of geometric properties of manifolds admit a natural interpretation in terms of the homotopy properties of the space T_N. These properties studied at the end of Chapter I (Theorems 6.9 and 6.10) and throughout Chapter II, which is also concerned with a development of the methods of numerical calculation. We mention here a number of problems that are studied at the end of Chapter I and in Chapter II.

1. The conditions under which a mapping $f : M^n \to M^n$ of degree $+1$ is homotopic to a diffeomorphism (Theorems 6.9 and 6.10).
2. A study of the action of $\pi^+(M_0^n, M_0^n)$ on the set $A/\pi(M_0^n, SO_N)$ (§ 7).
3. A determination of the obstructions $d_i(M_1^n, M_0^n) \in H_{n-i}(M_1^n, \pi_{N+i}(S^N))$ to the manifold $M_1^n \in \{M_i^n\}$ being diffeomorphic to the manifold M_0^n (§ 8).
4. The connected sum of a manifold with a Milnor sphere and its homotopic meaning (§ 9).
5. The variation of the smooth structure of a π-manifold along a cycle of minimal dimension (§ 9).
6. Variation in smooth structure and Morse's surgery (§ 10).

In Chapter III the results of Chapters I and II are applied to the working out of examples. The result of § 14 was independently obtained by W. Browder [29].

In addition to the main text of the paper there are four appendices, written quite concisely and not very rigorously. The reader can regard these appendices (together with the results of §§ 10 and 12) as annotations of new results, the complete proofs of which will be published in later parts of this article. However, in these appendices and in §§ 10, 12 we have sketched out the proofs with sufficient detail that a specialist might completely analyze them without waiting for the publication of later parts.

In Appendix 1 the results of § 14 are expressed in the language, suitable for calculations, of the Atiyah–Grothendieck–Hirzebruch K- and J-functors, and there is indicated an application of these results to Pontrjagin's theory of classes.

Appendix 2 is devoted to (i) an extension of the results of the paper to combinatorial manifolds; and (ii) an investigation of the relation between smooth and combinatorial manifolds.

Appendix 3 is devoted to a study of the action of the Milnor groups $\theta^{4k-1}(\partial\pi)$ on manifolds and to the problem of singling out the group $\theta^{4k-1}(\partial\pi)$ as a direct summand in the group θ^{4k-1}.

In Appendix 4 we study the problem of determining the Euclidean spaces in which a nontrivial Milnor sphere can be embedded in such a way that its normal bundle there is trivial.

Chapter I

The fundamental construction[1]

§ 1. Morse's surgery

The material of this section is largely borrowed from other papers (for example, from [5, 9]) and is essentially a somewhat generalized account of them in a terminology adapted to our purposes.

Let $M^n \subset R^{n+N}$ be a smooth manifold with or without boundary, smoothly located in a Euclidean space R^{n+N} of sufficiently large dimension. Let $S^i \times D^{n-i}_\varepsilon \subset M^n$ be a smooth embedding of the direct product $S^i \times D^{n-i}_\varepsilon$ in M^n, where D^{n-i}_ε is a ball in the space R^{n-i} (of radius ε) in the natural coordinate system

$$h : \partial D^{i+1} \times D^{n-i}_\varepsilon \to S^i \times D^{n-i}_\varepsilon \subset M$$

such that $h(x,y) = (x, h_x(y))$, where $h_x \in SO_{n-i}$. The set of maps $h_x, x \in S^i$, defines a smooth map $d(h) : S^i \to SO_{n-i}$, which completely defines the diffeomorphism h.

Set

$$\begin{aligned} B^{n+1}(h) &= M^n \times I\left(0, \frac{1}{2}\right) \cup_h D^{i+1} \times D^{n-i}_\varepsilon, \\ M^n(h) &= (M^n \setminus S^i \times D^{n-i}_\varepsilon) \cup_h D^{i+1} \times \partial D^{n-i}_\varepsilon. \end{aligned} \tag{1}$$

The transformation operation from M^n to $M^n(h)$ is called a Morse surgery. It is well known that:

(1) $\partial B^{n+1}(h) = M^n \cup (-M^n(h))$, if M^n is closed;
(2) The manifolds $B^{n+1}(h)$ and $M^n(h)$ can be defined as smooth orientable manifolds.

[1] Chapter I is a detailed account of author's note [14].

(3) The subspace $(M^n \times \frac{1}{2}) \cup_h D^{i+1} \times 0 \subset B^{n+1}(h)$ is a deformation retract of $B^{n+1}(h)$.
(4) The manifold $B^{n+1}(h)$ is defined up to diffeomorphism by the homotopy class $\widetilde{d}(h)$ of the smooth mapping $d(h)\colon S^i \to SO_{n-i}$; $\widetilde{d}(h) \in \pi_i(SO_{n-i})$.
(5) The manifold $B^{n+1}(h)$ can be located in the direct product $R^{n+N} \times I(0,1)$ in such a way that

$$B^{n+1}(h) \cap R^{n+N} \times 1 = M^n(h)$$
$$B^{n+1}(h) \cap R^{n+N} \times 0 = M^n$$

and $B^{n+1}(h)$ approaches the boundary components $R^{n+N} \times 1$ and $R^{n+N} \times 0$ orthogonally.

Assume in the tubular neighborhood $T_{2\varepsilon}$ (of radius 2ε) of the sphere $S^i \subset M^n$, where $T_{2\varepsilon} = S^i \times D^{n-i}_{2\varepsilon}$, there is a vector field τ^N, which is continuous on $T_{2\varepsilon}$ and normal to the manifold in R^{n+N}. We have

Lemma 1.1. *Suppose the inclusion homomorphism $\pi_i(SO_{n-i}) \to \pi_i(SO_{N+n-i})$ is an epimorphism. Then the diffeomorphism*

$$h : \partial D^{i+1} \times D^{n-i}_{\varepsilon} \to S^i \times D^{n-i}_{\varepsilon} \subset M^n$$

may be chosen in such a way that the frame field τ^N, which is normal to $T_{2\varepsilon}$ in R^{N+n}, can be extended to a frame field $\widetilde{\tau}^N$ on $(T_{2\varepsilon} \times I(0,\frac{1}{2})) \cup_h D^{i+1} \times D^{n-i}_{\varepsilon}$, that is normal $B^{n+1}(h)$ in the Cartesian product $R^{n+N} \times I(0,1)$.

Let us choose on $D^{i+1} \times 0 \subset R^{n+N} \times I(0,1)$ some continuous frame field τ_0^{N+n-i}, normal to $D^{i+1} \times 0$ in $R^{n+N} \times I(0,1)$, and let us consider its restriction to the boundary

$$S^i \times 0 \subset M^n \subset R^{n+N} \times 0,$$

which we will also denote by τ_0^{N+n-i}. Since the homomorphism $\pi_i(SO_{n-i}) \to \pi_i(SO_{N+n-i})$ is onto, we can choose on the sphere $S^i \times 0 \subset M^n$ an $(n-i)$-frame field τ^{n-i}, normal to the sphere $S^i \times 0$ in the manifold M^n and such that the combined frame field (τ^N, τ^{n-i}), normal to the sphere $S^i \times 0 \in R^{N+n} \times 0$, is homotopic to the field τ_0^{N+n-i}, which is induced by the $(N+n-i)$-framed field τ_0^{N+n-i} on the ball

$$D^{i+i} \times 0 \subset R^{N+n} \times I(0,1).$$

Hence the field (τ^N, τ^{n-i}) may be extended to the ball

$$D^{i+1} \times 0 \subset R^{N+n} \times I(0,1).$$

We shall denote this extension by $(\widetilde{\tau}^N, \widetilde{\tau}^{n-i})$, where $\widetilde{\tau}^N$ is the extension of the first N-frame and $\widetilde{\tau}^{n-i}$ is the extension of the last $(n-i)$-frame. Let us now "blow-up" the ball

$$D^{i+1} \times 0 \subset R^{N+n} \times (0,1)$$

by the last $n-i$ vectors $\widetilde{\tau}^{n-i}$ of the frame, more exactly, by the linear space of dimension $n-i$, defined by these $n-i$ vectors at each point of the ball. We shall denote this blow-up by Q. The vectors of the frame $\widetilde{\tau}^N$ will be normal to Q and define an extension of the frame τ^N to this blow-up. The frame field τ^{n-i}, which is normal to the sphere $S^i \times 0 \subset M^n$, is different from the original frame field on the sphere $S^i \times 0$ that was defined by the original coordinate system on the Cartesian product $S^i \times D^{n-i}_\varepsilon \subset M^n$. This difference is measured by the "discriminating" map $S^i \to SO_{n-i}$, which also defines the element $\widetilde{d}(h) \in \pi_i(SO_{n-i})$ needed by us and the diffeomorphism

$$h : \partial D^{i+1} \times D^{n-i}_\varepsilon \to M^n.$$

It is easy to see from (1) that

$$B^{n+1}(h) = \left[(M \diagdown T_{2\varepsilon}) \times I\left(0, \frac{1}{2}\right)\right] \cup \left[\left(T_{2\varepsilon} \times I\left(0, \frac{1}{2}\right)\right) \cup_h Q\right]$$

and that the N-frame field is extended onto Q. But

$$Q \approx D^{i+1} \times D^{n-i}_\varepsilon,$$

where $\approx$ means a diffeomorphism.

The lemma is proved.

For convenience in applications of Lemma 1.1 we formulate the following statement.

Lemma 1.2.

(a) *Suppose $i < n-i$. Then the map*

$$\pi_i(SO_{n-i}) \to \pi_i(SO_{N+n-i})$$

is an epimorphism;

(b) *Suppose $i = 2k$ and $i = n-i$. Then the map*

$$\pi_{2k}(SO_{2k}) \to \pi_{2k}(SO_{N+2k})$$

is also an epimorphism;

(c) *Suppose* $i = 2k+1, i = n-i$. *In this case the map is epimorphic if and only if* $i \neq 1, 3, 7$. *If* $i = 1, 3, 7$, *then the quotient group* $\pi_i(SO_{N+n-i})/\pi_i(SO_i)$, $i = n-i$ *contains two elements.*

The proof of (a) and (b) is contained in [20], and that of (c) can be found in [1].

§ 2. Relative π-manifolds

Let M^n be a smooth manifold, either closed or with boundary and let $W^i \subset M^n$ be a submanifold of it. Denote by $\nu^N(M^n)$ the normal bundle of the manifold $M^n \subset R^{N+n}$ and denote by $\nu^{n-i}(W^i, M^n)$ the normal bundle of the manifold W^i in M^n.

Definition 2.1. Let $f : M_1^k \to M_2^n$ be a smooth map. We shall call M_1^k an (f, π)-manifold $\operatorname{mod} M_2^n$, if

$$f^*\nu^N(M_2^n) = \nu^N(M_1^k).$$

Definition 2.2. Suppose a sphere $S^i \subset M_1^k$, is smoothly situated in M_1^k, is such that the map $f|_{S^i \to M_2^n}$ is null-homotopic. Then the bundle $\nu^{k-i}(S^i, M_1^k)$ has the following properties:

(1) for $i < k-i$ the bundle $\nu^{k-i}(S^i, M_1^k)$ is trivial;
(2) for $i = k-i$, $i = 2s$, the bundle $\nu^{k-i}(S^i, M_1^k)$ is trivial if and only if the self-intersection number $S^i \cdot S^i$ is zero;
(3) for $i = k-i$, $i = 1, 3, 7$, the bundle $\nu^{k-i}(S^i, M_1^k)$ is trivial;
(4) for $i = k-i$, $i = 2s+1$, $i \neq 1, 3, 7$, the bundle is completely defined by the value of the invariant $\varphi(S^i) \in Z_2$.

If $x \in \operatorname{Ker} f_* \subset \pi_i(M_1^k)$, where x is the homotopy class of the embedding $S^i \subset M_1^k$ and the group $\pi_1(M_1^k)$ is zero, then φ defines a map

$$\varphi : \operatorname{Ker} f_* \to Z_2$$

and

$$\varphi(x+y) = \varphi(x) + \varphi(y) + [H(x) \cdot H(y)] \operatorname{mod} 2, \tag{2}$$

where $H : \pi_i(M_1^k) \to H_i(M_1^k)$ is the Hurewicz homomorphism.

Let us consider the tubular neighborhood T of the sphere S^i in M_1^k; this neighborhood is the total space of an SO_{k-i}-bundle with base S^i. The map $f_\circ f : T \to M_2^n$ is null-homotopic and, by assumption,

$$j^* f^* \nu^N (M_2^n) = \nu^N (T),$$

where j is an embedding of $S^i \subset M_1^k$. Hence $\nu^N(T)$ is trivial. Since the manifold T is not closed, the triviality of the bundle $\nu^N(T)$ implies that T is parallelizable. Hence the normal bundle of a sphere S^i in a manifold is completely determined by an element $\alpha \in \operatorname{Ker} p$, where

$$p : SO_{k-i} \subset SO_\infty$$

and

$$p_* : \pi_{i-1}(SO_{k-i}) \to \pi_{i-1}(SO_\infty)$$

is a homomorphism of the natural embedding p. For $i < k - i$ the map p is an isomorphism, and this implies property 1.

If $i = k - i, i = 2s$, then

$$\operatorname{Ker} p_* = Z \subset \pi_{2s-1}(SO_{2s}),$$

and, as is well known, the bundles over the sphere S^{2s} defined by the elements $\alpha \in \operatorname{Ker} p_* \subset \pi_{2s-1}(SO_{2s})$, are completely defined by the Euler class $\chi(\alpha)$, where $\chi(\alpha) \equiv 0 \pmod 2$. But the Euler class of a bundle is equal to the self-intersection number $S^i \cdot S^i$, and this implies property 2.

For $i = 1, 3, 7$, $i = k - i$, the kernel $\operatorname{Ker} p_* = 0$, and this implies property 3.

For $i \neq 1, 3, 7$, $i = 2s+1$, we have $\operatorname{Ker} p_* = Z_2$ (see [1]). Thus the normal bundle $\nu^{k-i}(S^i, T)$ is determined by the value of the invariant $\varphi(S^i) \subset Z_2$.

Now let $\pi_1(M_1^k) = 0$. Hence by Whitney's results two spheres $S_1^i, S_2^i \subset M_1^k$, which define one and the same element $x \in \pi_i(M_1^k)$, $i = k - i$, are regularly homotopic (see [25]). Hence

$$\varphi(S_1^i) = \varphi(S_2^i).$$

Thus the map

$$\varphi : \operatorname{Ker} f_* \to Z_2,$$

is defined since each element $x \in \operatorname{Ker} f_*$ can be realized by an embedded smooth sphere $S^i \subset M_i^k$ (see [9]). Let us now prove (2). Let $x, y \in \operatorname{Ker} f_*$ be two cycles. Realize them by spheres $S_1^i, S_2^i \subset M^n$, the number of intersection points of which is equal to the intersection number $|H(x) \cdot H(y)|$ (see [25]).

We form tubular neighborhoods T_1 and T_2 of the spheres S_1^i and S_2^i in M_1^k, respectively. Denote by

$$T(x,y) = T_1 \cup T_2$$

a smooth neighborhood of the union $S_1^i \cup S_2^i$. The manifold $T(x,y)$ is obviously parallelizable, and

$$H_i(T(x,y)) = Z + Z.$$

If the spheres do not intersect, then our statement is obvious. Let us assume $|H(x) \cdot H(y)| = 1$. Then

$$\pi_1(T(x,y)) = 0, \quad H_j(T(x,y)) = 0, \quad j \neq i,$$

and the boundary $\partial T(x,y)$ is a homotopy sphere (see [8]).

Kervaire proved [4] that in the manifold $T(x,y)$ we have

$$\varphi(x+y) = \varphi(x) + \varphi(y) + [H(x) \cdot H(y)] \bmod 2,$$

thus, the same holds in $M_1^k \supset T(x,y)$, since the sphere S^i, realizing the element $x+y$, lies in $T(x,y)$, and φ is an invariant of the normal bundle. If $|H(x) \cdot H(y)| > 1$, then the group

$$\pi_1(T(x,y)) = \pi_1(\partial T(x,y))$$

is free and the number of its generators is equal to $|H(x) \cdot H(y)| - 1$; hence our argument does not go through. But by the Morse surgery described in § 1, it is possible to "paste" the group $\pi_1(T(x,y)) = \pi_1(\partial T(x,y))$ and pass to a simply connected manifold $\widetilde{T}(x,y) \subset M_1^k$ such that

(a) $\widetilde{T}(x,y) = T(x,y) \cup_{h_1} D^2 \times D^{k-2} \cup_{h_2} \cdots \cup_{h_t} D^2 \times D^{k-2}$, where

$$t = |H(x) \cdot H(y)| - 1$$

and

$$h_q : \partial D^2 \times D^{k-2} \to \partial T(x,y);$$

(b) $\widetilde{T}(x,y)$ is parallelizable;
(c) $H_i(\widetilde{T}(x,y)) = Z + Z,\ H_j(\widetilde{T}(x,y)) = 0,\ i \neq j;$
(d) the spheres $S_1^i, S_2^i \subset T(x,y)$ generate the group $H_i(\widetilde{T}(x,y))$.

To do this, we must perform the Morse surgery in the interior of the manifold M_1^k, which is possible for $k \geq 6$. Then we apply Kervaire's results [4] to the manifold $\widetilde{T}(x, y)$ to obtain (2):

$$\varphi(x+y) = \varphi(x) + \varphi(y) + [H(x) \cdot H(y)] \bmod 2.$$

(Concerning Morse's surgery operations cf. papers [2, 9].) Thus the lemma is proved. We note that our description of the behavior of the normal bundle to the sphere in a parallelizable manifold is not original and is contained in papers [4, 9] and others.

Definition 2.3. If the map $f : M_1^n \to M_2^n$ has degree $+1$, we say that the manifold M_1^n is greater than or equal to M_2^n, and write $M_1^n \overset{f}{\geq} M_2^n$.[1]

Lemma 2.4. *If $M_1^n \overset{f}{\geq} M_2^n$, then the map $f : H^*(M_2^n, K) \to H^*(M_1^n, K)$ is a monomorphism for any field K.*

PROOF. Let $x \in H^i(M_2^n, K), x \neq 0$; then there exists $y \in H^{n-i}(M_2^n, K)$ such that $(xy, [M_2^n]) = 1$. Since

$$(f^*(xy), [M_1^n]) = (f^*xf^*y, [M_1^n]) = (xy, f_*[M_1^n]) = (xy, [M_2^n]) = 1,$$

it follows that $f^*xf^*y \neq 0$ and therefore $f^*x \neq 0$.

The lemma is proved.

Lemma 2.5. *If $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$ and $M_1^n \overset{f}{\geq} M_2^n$, $M_2^n \overset{g}{\geq} M_1^n$, then the maps f and g are homotopy equivalences.*

PROOF. The maps $f_\circ g : M_2^n \to M_2^n$ and $g^\circ f : M_1^n \to M_1^n$ are onto of degree $+1$. Hence by Lemma 2.4 they induce an isomorphism of the cohomology groups over an arbitrary field K and hence an isomorphism of the integral cohomology and homology groups. Whitehead's theorem enables us to completes the proof.

Remark 2.6. Lemma 2.5 can also be stated as follows: if $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$, then the homology groups of the manifolds M_1^n and M_2^n are isomorphic and $M_1^n \overset{f}{\geq} M_2^n$, then they are homotopically equivalent.

[1] It is also assumed that M_2^n is an (f, π)-manifold modulo M_1^n.

§ 3. The general construction

Let M^n be a smooth simply connected oriented manifold and let $\nu^N(M^n)$ be its stable normal bundle with fiber closed ball D^N, and suppose that this bundle is oriented, i.e. the structural group is reduced to SO_N. We contract the boundary $\partial\nu^N(M^n)$ to the point and denote by $T_N(M^n)$ the obtained space, which is the Thom space of the bundle (see [7, 22]). We have:

$$T_N(M^n) = \frac{\nu^N(M^n)}{\partial\nu^N(M^n)}. \tag{3}$$

The Thom isomorphism

$$\varphi : H_i(M^n) \to H_{N+i}(T_N(M^n)) \tag{4}$$

is well known.

As usual, we denote by $[M^n]$ the fundamental cycle of the manifold M^n for the selected orientation.

Lemma 3.1. *The homology class $\varphi[M^n]$ belongs to the image of the Hurewicz homomorphism $H : \pi_{N+n}(T_N(M^n)) \to H_{N+n}(T_N(M^n))$.*

PROOF. Let us construct an element $x \in \pi_{N+n}(T_N(M^n))$ such that $H(x) = \varphi[M^n]$. Let the manifold M^n be smoothly situated in the sphere S^{N+n}. Its closed tubular neighborhood $T \subset S^{N+n}$ is diffeomorphic to the total space of the bundle $\nu^N(M^n)$ in a natural way, since T is canonically fibered by normal balls D^N. We effect the natural diffeomorphism $T \to \nu^N(M^n)$ and consider the composition

$$T \to \nu^N(M^n) \to T_N(M^n);$$

the map $T \to T_N(M^n)$ takes the boundary ∂T to a point and is therefore extended to the map $S^{N+n} \to T_N(M^n)$ that takes all of the exterior $S^{N+n} \setminus T$ to the same point. This map obviously represents the desired element $x \in \pi_{N+n}(T_n(M^n))$. The lemma is proved.

In the sequel an important role will be played by the set

$$H^{-1}\varphi[M^n] \subset \pi_{N+n}(T_N(M^n)),$$

which we shall always denote by $A(M^n)$. We consider an arbitrary element $\alpha \in A(M^n)$ and the map

$$\widetilde{f}_\alpha : S^{N+n} \to T_N(M^n)$$

representing it.

From the paper of Thom [22] there easily follows

Lemma 3.2. *There exists a homotopic smooth map*

$$f_\alpha : S^{N+n} \to T_N(M^n)$$

such that

(a) *the inverse image $f_\alpha^{-1}(M^n)$ is a smooth manifold M^n_α, smoothly situated in the sphere S^{N+n};*
(b) *for every point $x \in M^n_\alpha$ the map f_α takes the ε-ball D^N_x, normal to M^n_α in S^{N+n}, to the ε-ball $D^N_{f_\alpha(x)}$, normal to M^n in $T_N(M^n)$, and the map $f_\alpha : D^N_x \to D^N_{f_\alpha(x)}$ is a linear nondegenerate map for all $x \in M^n_\alpha$;*
(c) *the maps $f_\alpha/M^n_\alpha \to M^n$ and $f_\alpha/D^N_x \to D^N_{f_\alpha(x)}$ have degree $+1$ for all $x \in M^n_\alpha$.*

Proof. Points (a) and (b) are taken from Thom's paper [22]. For the proof of point (c) we observe that the map $\widetilde{f}_\alpha : S^{N+n} \to T_N(M^n)$ and hence f_α have degree $+1$ (this makes sense because $T^N(M^n)$ is a pseudomanifold with fundamental cycle $[T_N] = \varphi[M^n]$). Hence the map f_α must have degree $+1$ in the tubular neighborhood of $M^n_\alpha = f_\alpha^{-1}(M^n)$. We reduce the structural group of the bundle $\nu^N(M^n_\alpha)$ to SO_N so that all maps $f_\alpha : D^N_x \to D^N_{f_\alpha(x)}$ have determinants >0. Then on the manifold M^n_α there is a unique orientation which is induced by the orientations of the sphere S^{N+n} and the fiber D^N_x. In this orientation the map $f_\alpha : M^n_\alpha \to M^n$ has degree $+1$ since the degree of the bundle map

$$\nu^N(M^n_\alpha) \to \nu^N(M^n) \to T_N(M^n)$$

is $+1$ and is equal to the product of the degrees of the map for the base M^n_α and for the fiber $D^N_x, x \in M^n_\alpha$; on the fiber D^N_x, as a result of the choice of its orientation, this degree is equal to $+1$, which yields the desired statement. The lemma is proved.

Lemma 3.3. *The manifold $M^n_\alpha \overset{f_\alpha}{\geq} M^n$.*

Proof. The map f_α has degree $+1$ and is clearly such that

$$f_\alpha^* \nu^N(M^n) = \nu^N(M^n_\alpha).$$

Lemma 3.4. *If $\pi_1(M^n_\alpha) = 0$ and $H_i(M^n_\alpha) = H_i(M^n)$, $i = 0, 1, 2, \dots, n$, then the map $f_\alpha : M^n_\alpha \to M^n$ is a homotopy equivalence.*

The proof follows from Corollary 3.3, Lemma 2.5 and Remark 2.6 on page 59.

We denote by $\overline{A}(M^n) \subset A(M^n)$ the subset consisting of those elements $\alpha \in \overline{A}(M^n)$ for which there exist representatives $f_\alpha : S^{N+n} \to T_N(M^n)$ satisfying Lemma 3.2 and such that the inverse image $f_\alpha^{-1}(M^n) = M_\alpha^n$ is a manifold homotopically equivalent to M^n. We are interested in the set $\overline{A}(M^n)$. To study this set, the three following important questions are appropriate:

1. What is the location of the subset $\overline{A}(M^n)$ in $A(M^n)$, i.e. in which classes $\alpha \in A(M^n) \subset \pi_{N+n}(T_N(M^n))$ are there representatives $f_\alpha : S^{N+n} \to T_N(M^n)$, for which the manifold
$$M_\alpha^n = f_\alpha^{-1}(M^n)$$
is homotopically equivalent to M^n (in which classes $\alpha \in A(M^n)$ are there manifolds of the same homotopy type as M^n)?
2. Suppose two manifolds $M_{\alpha,1}^n$ and $M_{\alpha,2}^n$ belong to the same class $\alpha \in \overline{A}(M^n)$ and both are homotopically equivalent to M^n. This means that there are two homotopic maps of the sphere
$$f_{\alpha,i} : S^{N+n} \to T_N(M^n)$$
such that
$$f_{\alpha,i}^{-1}(M^n) = M_{\alpha,i}^n, \quad i = 1, 2.$$
What is the connection between $M_{\alpha,1}^n$ and $M_{\alpha,2}^\alpha$?
3. In which classes $\alpha_i \in \overline{A}(M^n)$ can one find one and the same manifold M_1^n that is homotopy equivalent to M^n?

The following three sections will be devoted to the solution of these questions.

§ 4. Realization of classes

The aim of this section is to study in which classes $\alpha \in A(M^n)$ one can find manifolds homotopically equivalent to M^n. First we prove a number of easy algebraic lemmas. Consider two finite complexes X, Y and a map $f : X \to Y$. Assume that K is an arbitrary field
$$\pi_1(X) = \pi_1(Y) = 0.$$

Lemma 4.1. *Suppose for any K the map $f_* : H_i(X; K) \to H_i(Y; K)$ is epimorphic for $i \leq j+1$ and isomorphic for $i \leq j$. Then $f_* : H_i(X; Z) \to H_i(Y; Z)$ is epimorphic for $i \leq j+1$ and isomorphic for $i \leq j$.*

Proof. We consider the cylinder $C_j = X \times I(0,1) \cup_f Y$, which is homotopy equivalent to Y, and the exact sequence of the pair (C_f, X):

$$H_i(X) \xrightarrow{f_*} H_i(Y) \to H_i(C_f, X) \xrightarrow{\partial} H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \tag{5}$$

for $i \leq j+1$. From (5) it follows that $H_i(C_f, X; K) = 0$ for $i \leq j+1$. Therefore

$$H_i(C_f, X; Z) = 0, \quad i \leq j+1.$$

Returning to the exact sequence (5) (for integral homology) we obtain all the statements of the lemma. The lemma is proved.

Lemma 4.2. *Suppose the map $f : X \to Y$ is such that the map $f_* : H_i(X; Z) \to H_i(Y; Z)$ is an epimorphism for $i \leq j+1$ and an isomorphism for $i \leq j$. Then $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leq j$ and an epimorphism for $i \leq j+1$, and vice versa.*

Proof. Consider the two exact sequences which form the commutative diagram together with the Hurewicz homomorphism

$$\begin{array}{ccccccc}
H_i(X;Z) & \xrightarrow{f^*} & H_i(Y;Z) & \longrightarrow & H_i(C_f,X;Z) & \xrightarrow{\partial} & H_{i-1}(X;Z) \\
{\scriptstyle H}\uparrow & & {\scriptstyle H}\uparrow & & {\scriptstyle H}\uparrow & & {\scriptstyle H}\uparrow \\
\pi_i(X) & \xrightarrow{f^*} & \pi_i(Y) & \longrightarrow & \pi_i(C_f,X) & \xrightarrow{\partial} & \pi_{i-1}(X)
\end{array} \tag{6}$$

for $i \leq j+1$. It is easy to see that

$$H_i(C_f, X; Z) = 0, \quad i \leq j+1.$$

Since $\pi_1(X) = \pi_1(Y) = 0$, we have

$$\pi_1(C_f, X) = 0, \quad i \leq j+1,$$

which yields Lemma 4.2 (the direct statement). The converse statement is proved analogously. The lemma is proved.

For definiteness, in the sequel we shall always denote the homomorphisms $\pi_1(X) \to \pi_i(Y)$ and $H_i(X) \to H_i(Y)$ corresponding to $f : X \to Y$, by $f_*^{(\pi_i)}$ and $f_*^{(H_i)}$.

Lemma 4.3. *Under the same conditions as in Lemma 4.2, the homomorphism*

$$H : \operatorname{Ker} f_*^{(\pi_{j+1})} \to \operatorname{Ker} f_*^{(H_{j+1})}$$

is an epimorphism.

PROOF. The following diagram is commutative and the rows are exact:

$$\begin{tikzcd}
\pi_{j+2}(C_f, X) \arrow[r] \arrow[d, "H", "\wr"'] & \operatorname{Ker} f_*^{(\pi_{j+1})} \arrow[r] \arrow[d, "H"] & 0 \\
H_{j+2}(C_f, X; Z) \arrow[r] & \operatorname{Ker} f_*^{(H_{j+1})} \arrow[r] & 0.
\end{tikzcd} \tag{7}$$

From the proof of Lemma 4.2 we know that

$$H_i(C_f, X) = \pi_i(C_f, X) = 0, \quad i \leq j+1.$$

Therefore

$$\pi_{j+2}(C_f, X) \approx H_{j+2}(C_f, X).$$

The standard argument completes the proof.

Now, let us consider a map $f : M_1^n \to M_2^n$ of degree $+1$. We will be interested in the case when the kernels $\operatorname{Ker} f_*^{(\pi_i)}$ are trivial for $i < [\frac{n}{2}]$. We consider separately the cases of even and odd π. The following two lemmas hold.

Lemma 4.4. *Suppose $n = 2s$ and the groups $\operatorname{Ker} f_*^{(\pi_i)}$ are trivial for $i < s$. Then the group $\operatorname{Ker} f_*^{(H_s)}$ is free abelian, is a direct summand in the group $H_s(M_1^n, Z)$, and the intersection matrix of basic cycles of the group $\operatorname{Ker} f_*^{(H_s)}$ is unimodular.*

Lemma 4.4′. *Suppose $n = 2s + 1$ and the groups $\operatorname{Ker} f_*^{(\pi_i)}$ are trivial for $i < s$. Then the group $\operatorname{Ker} f_*^{(H_{s+1})}$ is free abelian, and both $\operatorname{Ker} f_*^{(H_s)}$ and $\operatorname{Ker} f_*^{(H_{s+1})}$ are direct summands of the groups $H_s(M_1^n, Z)$ and $H_{s+1}(M_1^n, Z)$, respectively. The finite part $\operatorname{Tor} \operatorname{Ker} f_*^{(H_s)}$ of the group $\operatorname{Ker} f_*^{(H_s)}$ is closed under the Alexander duality, i.e. the linking matrix of the generating elements of order p^i is unimodular $\operatorname{mod} p^i$ for some primary decomposition, for fixed values of p, i. The intersection matrix of $\operatorname{Ker} f_*^{(H_{s+1})}$ and $\operatorname{Ker} f_*^{(H_s)}/\operatorname{Tor} \operatorname{Ker} f_*^{(H_s)}$ is unimodular, too.*

We shall prove both lemmas simultaneously, starting from the identity

$$f_*(f^* x \cap y) = x \cap f_* y, \tag{8}$$

which holds for any continuous map f. In our case $f_*[M_1^n] = [M_2^n]$ and the operation $\cap\,[M_1^n]$ coincides with the Poincaré duality isomorphism D. Thus we get:

$$f_* D f^* = D,$$

which yields

$$H_i(M_1^n) = \operatorname{Ker} f_*^{(H_i)} + Df^* H^{n-i}(M_2^n) \tag{9}$$

over any coefficient domain and for any values of i. Consequently, we have proved the statement about direct summands in all cases. The absence of torsion in $\operatorname{Ker} f_*^{(H_s)}$ for $n = 2s$ and $\operatorname{Ker} f_*^{(H_{s+1})}$ for $n = 2s+1$ follows from the fact that the groups $\operatorname{Ker} f_*^{(H_{s-1})}$ are trivial in both cases, and from the Alexander duality for torsions of $H_{s-1}(M_1^n)$ and $H_{n-s}(M_1^n)$ for both values of n. It remains to prove the unimodularity of the correspondent intersection/linking matrices. We show that the groups $\operatorname{Ker} f_*^{(H_i)}$ and $Df^*H^{n-i}(M_2^n)$ are orthogonal to each other with respect to the cycle intersection, for any value of i and over each coefficient domain. Indeed, let $x \in H^{n-i}(M_2^n)$ and $y \in \operatorname{Ker} f_*^{(H_i)}$. Then

$$(f^* \cap [M_1^n]) \cdot y = (f^* x, y) = (x, f_* y) = 0 \tag{10}$$

and any element of the group $Df^*H^{n-i}(M_2^n)$ is of the form

$$f^* x \cap [M_1^n].$$

Thus the groups $\operatorname{Ker} f_*^{(H_i)}$ and $Df^*D^{-1}(M_2^n)$ are orthogonal. Applying this orthogonality, we obtain the unimodularity of the intersection matrices in all cases. The statement concerning linking matrices follows from the fact that the linking can be defined in terms of intersection of cycles modulo p^i. Lemmas 4.4 and 4.4′ are proved.

We note a useful supplement to Lemma 4.4.

Lemma 4.5. *The map* $H : \operatorname{Ker} f_*^{(\pi_s)} \to \operatorname{Ker} f_*^{(H_s)}$ *is an isomorphism for* $n = 2s$ *if the groups* $\operatorname{Ker} f_*^{(\pi_i)} = 0$ *for* $i < s$.

PROOF. As in the proof of Lemma 4.3, consider the commutative diagram:

$$\begin{tikzcd}
H_{s+1}(M_1^n) \arrow[r, "\approx"] & H_{s+1}(M_2^n) \arrow[r] & H_{s+1}(C_f, M_1^n) \arrow[r, "\partial"] & \operatorname{Ker} f_*^{(H_s)} \arrow[r] & 0 \\
\pi_{s+1}(M_1^n) \arrow[r] \arrow[u] & \pi_{s+1}(M_2^n) \arrow[r] \arrow[u] & \pi_{s+1}(C_f, M_1^n) \arrow[r, "\partial"] \arrow[u, "H", "\wr"'] & \operatorname{Ker} f_*^{(\pi_s)} \arrow[r] \arrow[u] & 0.
\end{tikzcd} \tag{11}$$

Since the maps $f_*^{(H_i)}$ for $i < s$ are isomorphisms, the map $f_*^{(H_{s+1})}$ is an isomorphism as well. The exactness of the sequence yields the isomorphism

$$\partial : H_{s+1}(C_f, M_1^n) \to \operatorname{Ker} f_*^{(H_s)}.$$

Thus

$$\partial H = H\partial : \pi_{s+1}(C_f, M_1^n) \to \operatorname{Ker} f_*^{(H_s)}$$

is an isomorphism, and the map

$$H : \operatorname{Ker} f_*^{(\pi_s)} \to \operatorname{Ker} f_*^{(H_s)}$$

is also an isomorphism. The lemma is proved.

Let us now investigate an arbitrary element $\alpha \in A(M^n)$. We have the following:

Lemma 4.6. *For every element $\alpha \in A(M^n)$, there exists a map $f_\alpha : S^{N+n} \to T_N(M^n)$ satisfying Lemma* 3.2, *such that the inverse image $M_\alpha^n = f_\alpha^{-1}(M^n) \subset S^{N+n}$ possesses the following properties*:

(a) $\pi_1(M_\alpha^n) = 0$;
(b) *the maps $f_*^{(H_s)} : H_s(M_\alpha^n) \to H_s(M^n)$ are isomorphisms for $s < [\frac{n}{2}]$.*

PROOF. We will inductively construct the maps

$${}_s f_\alpha : S^{N+n} \to T_N(M^n),$$

satisfying Lemma 3.2, for which the groups

$$H_i(M_{\alpha,s}^n), \quad M_{\alpha,s}^n = {}_\alpha f_s^{-1}(M^n)$$

are isomorphic to the groups $H_i(M^n), i < s$. Since the maps ${}_s f_\alpha$: $M_{\alpha,s}^n \to M^n$ have degree $+1$, this isomorphism is established by the map ${}_s f_{\alpha *}^{(H_i)}$. From Lemmas 4.1–4.3 it follows that the map ${}_s f_\alpha^{(H_s)}$ is an epimorphism, and all of the basic cycles $x_1, \ldots, x_l \in \operatorname{Ker} {}_s f_\alpha^{(H_s)}$ can be realized by a system of smoothly embedded disjoint spheres $S_1^s, \ldots, S_l^s \subset M_{\alpha,s}^n$, on which the map ${}_s f_\alpha / S_j^s$ is null-homotopic. We assume that the maps ${}_i f_\alpha$ are already constructed for $i \leq s$; let us construct the map ${}_{s+1} f_\alpha$ by reconstructing ${}_s f_\alpha$.

Step 1. We deform the map ${}_s f_\alpha$ to a map ${}_s \widetilde{f}_\alpha$, such that

$$ {}_s \widetilde{f}_\alpha (T(S_1^s)) = g_0 \in M^n, $$

where g_0 is a point in M^n. The deformation is assumed to be smooth, and $T(S_1^s) \subset M_{\alpha,s}^n$ denotes a smooth tubular neighborhood of the sphere $S_1^s \subset M_{\alpha,s}^n$. In the fiber $D_{g_0}^N \subset \nu^N(M^n)$, we take the frame τ_0^N, defining the orientation of the fiber $D_{g_0}^N$. The inverse image ${}_s \widetilde{f}_\alpha^* \tau_0^N$ is a continuous N-frame field τ^N on $T(S_1^s)$ that is normal to $T(S_1^s) \subset S^{N+n}$, since the map ${}_s \widetilde{f}_\alpha$ satisfies Lemma 3.2. The arbitrariness in the choice of the frame τ_0^N is immaterial for our purposes.

Step 2. According to Lemma 2.1 the tube $T(S_1^s)$ is diffeomorphic to $S_1^s \times D_\varepsilon^{n-s}$, where $\varepsilon > 0$ is a small number.

We assign in $T(S_1^s)$ the coordinates $(x, y), x \in S_1^s, y \in D_\varepsilon^{n-s}$. As a result of Step 1, on the tube $T(S_1^s)$ we have a field τ^N. Consider the Cartesian product $S^{N+1} \times I(0,1)$. We shall assume that

$$ {}_s \widetilde{f}_\alpha : S^{N+n} \times 0 \to T_N(M^n), \quad M_{\alpha,s}^n \subset S^{N+n} \times 0. $$

Let us construct a membrane $B^{n+1}(h) \in S^{N+n} \times I(0,1)$ orthogonally approaching to the boundaries, such that the field τ^N can be extended to a certain field $\widetilde{\tau}^N$, that is normal to

$$ B^{N+1}(h) \Big\backslash \left[(M_{\alpha,s}^n \diagdown T(S_1^s)) \times I \left(0, \frac{1}{2} \right) \right] $$

in the Cartesian product $S^{N+n} \times I(0,1)$, where

$$ \begin{gathered} h : \partial D^{s+1} \times D_\varepsilon^{n-s} \to T(S_1^s), \quad h(x,y) = (x, d(h)_x(y)), \\ d(h) : S_1^s \to SO_{n-s}. \end{gathered} $$

Such a membrane $B^{n+1}(h)$ can be chosen according to Lemmas 1.1 and 1.2.

Step 3. We extend the map ${}_s \widetilde{f}_\alpha : M_{\alpha,s}^n \to M^n$ to a smooth map ${}_s F_\alpha :$ $B^{n+1}(h) \to M^n$ by setting

$$ \begin{gathered} {}_s F_\alpha = {}_s f_\alpha \diagup B^{n+1}(h) \cap S^{N+n} \times 0, \\ {}_s F_\alpha (D^{s+1} \times D_\varepsilon^{n-s}) = g_0 = {}_s f_\alpha (T(S_1^s)). \end{gathered} \tag{12} $$

Extend

$$ {}_sF_\alpha : B^{n+1}(h) \to M^n $$

to

$$ {}_aF_\alpha : T(B^{n+1}(h)) \to T_N(M^n), $$

where $T(B^{n+1}(h))$ is the tubular neighborhood of $B^{n+1}(h)$ in $S^{N+1} \times I(0,1)$, according to the frame field $\widetilde{\tau}^N$ that is normal to the part $B^{n+1}(h)$ in $S^{N+n} \times I(0,1)$, the latter being diffeomorphic to $D^{s+1} \times D_\varepsilon^{n-s} \subset B^{n+1}(h)$. On the remaining part

$$ B^{n+1}(h) \diagdown D^{s+1} \times D_\varepsilon^{n-s} = M_{\alpha,s}^n \times I\left(0, \frac{1}{2}\right) $$

the extension of the map is trivial. In their intersection

$$ M_{\alpha,s}^n \times I\left(0, \frac{1}{2}\right) \cap D^{s+1} \times D_\varepsilon^{n-s} = T(S_1^s) $$

these extensions are compatible with the frame field τ^N. Furthermore, by using Thom's method, we extend the map ${}_sF_\alpha$ to the whole $S^{N+n} \times I(0,1)$.

Now we put

$$ {}_sf_\alpha^{(1)} = \frac{{}_sF_\alpha}{S^{N+n} \times 1}. $$

Clearly, the map ${}_sf_\alpha^{(1)}$ satisfies Lemma 3.2 and

$$ {}_sf_\alpha^{(1)-1}(M^n) = M_{\alpha,s}^n(h). $$

Since $2s+1 < n$, we conclude that

$$ \operatorname{Ker} {}_sf_{\alpha*}^{(1)} = \frac{\operatorname{Ker} {}_sf_{\alpha*}}{(x_1)}. $$

Iterating the construction, we put

$$ {}_{s+1}f_\alpha = {}_sf_\alpha^{(l)}, $$

which yields the statement of the lemma.

The analysis of the case $s = [\frac{n}{2}]$ is more difficult. We shall subdivide it into the following cases:

(1) $n = 4k, \quad s = 2k, \quad k \geq 2;$
(2) $n = 4k + 2, \quad s = 2k + 1, \quad k \geq 1, \quad k \neq 1, 3;$
(3) $n = 4k + 2, \quad s = 2k + 1, \quad k = 1, 3;$
(4) $n = 4k + 1, \quad s = 2k, \quad k \geq 1;$
(5) $n = 4k + 3, \quad s = 2k + 1, \quad k \geq 1.$

Lemma 4.7. *Let $n = 4k$. For every element $\alpha \in A(M^n)$ there exists a map $f_\alpha : S^{N+n} \to T_N(M^n)$ satisfying Lemma 3.2 such that the inverse image $M^n_\alpha = f_\alpha^{-1}(M^n)$ is homotopically equivalent to M^n.*

PROOF. Applying Lemma 4.6, we can construct a map ${}_{2k}f_\alpha : S^{N+n} \to T_N(M^n)$, such that

$$\operatorname{Ker} {}_{2k}f^{(H_i)}_{\alpha *} = 0, \quad i < 2k,$$

where

$${}_{ak}f_\alpha : M^n_{\alpha,k} = {}_{2k}f_\alpha^{-1}(M^n) \to M^n.$$

According to Lemma 4.4, the group

$$\operatorname{Ker} {}_{2k}f^{(H_{2k})}_{\alpha *} = L_{2k} \subset H_{2k}(M^n_{\alpha,2k})$$

is free Abelian; it is a direct summand of the group $H_{2k}(M^n_{\alpha,2k})$, and the intersection matrix for basic cycles $l_1, \ldots, l_m \subset L_{2k}$ is unimodular. In the group $H_{2k}(M^n_{\alpha,2k})/\mathrm{Tor}$, choose a basis $l_1, \ldots, l_m$, $q_1, \ldots, q_p$ such that

$$q_i \circ l_j = 0, \quad i = 1, \ldots, p, \quad j = 1, \ldots, m;$$

this can be done because of the unimodularity of

$$(l_j \circ l_t), \quad j, t = 1, \ldots, m.$$

The matrix $(q_i \circ q_j)$ is equivalent to the intersection matrix for basic cycles of the group $H_{2k}(M^n)/\mathrm{Tor}$; moreover,

$$({}_{2k}f_{\alpha *}q_i) \circ ({}_{2k}f_{\alpha *}q_j) = q_i \circ q_j.$$

Since

$${}_{2k}f^*_\alpha \nu^N(M^n) = \nu^N(M^n_{\alpha,2k})$$

and the degree of ${}_{2k}f_\alpha$ is $+1$, the Hirzebruch formula [3] yields that the indices (signatures) of the manifolds $M^n_{\alpha,2k}$ and M^n are equal to each other. Thus the signature of the matrix $(l_i \circ l_j)$, $i, j = 1, \ldots, m$, is equal to zero (the intersection matrix of $M^n_{\alpha,2k}$ thus splits into two matrices, one of which is identical to the intersection matrix for the manifold M^n, and the other one is $(l_i \circ l_j), i, j = 1, \dots, m)$. On the other hand, the self-intersection indices $l_i \circ l_i$ are even. To prove that $l_i \circ l_i$ are even, let us realize the cycle l_i by a smooth sphere $S^{2k}_i \subset M^n_{\alpha,2k}$ according to Whitney [25] and Lemma 4.3. Then we consider the tubular neighborhood of the sphere, $T(S^{2k}_i) \subset M^n_{\alpha,2k}$, which is a parallelizable manifold (cf. Proof of Lemma 2.2, point 1). The self-intersection index of a sphere in a parallelizable manifold is always even, which yields the desired statement. Thus the signature of the matrix $(l_i \circ l_j)$ is zero and

$$l_i \circ l_i \equiv 0 \pmod 2.$$

According to [9], one can find a basis $l'_1, \dots, l'_m, m = 2m'$ such that

(a) $l'_i \circ l'_i = 0, \quad 1 \le i \le m;$
(b) $l'_{2i+1} \circ l'_{2i+2} = 1$, $i = 0, 1, \dots, m' - 1$,
(c) $l'_k \circ l'_j = 0$ otherwise,

i.e. the matrix can be reduced to the form

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \text{\Large 0} \\ \cdots & \cdots \\ \cdots & \cdots \\ \text{\Large 0} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}. \tag{13}$$

We realize the cycles $l_i, i = 1, \dots, m$, by smoothly embedded spheres $S^{2k}_i \subset M^n_{\alpha,2k}$ in such a way that their geometric intersections correspond to the algebraic intersection indices (the number of intersection points $S^{2k}_i \cap S^{2k}_j$ is equal to the index $|S^{2k}_i \circ S^{2k}_j|$; this can be done for $k > 1$; cf. [9, 26]). According to Lemma 2.2, the normal bundles $\nu^{2k}(S^{2k}_i, M^n_{\alpha,2k})$ are trivial. Then, we exactly repeat Steps 1, 2, 3 of the proof of Lemma 4.6, using Lemma 1.2. As a result of Morse surgery, the manifold $M^n_{\alpha,2k}$ is simplified (one Morse surgery over the sphere S^{2k}_i kills the square $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; cf. [9]). Iterating the operation, we obtain the map

$$f_\alpha; S^{N+n} \to T_N(M^n)$$

such that $\operatorname{Ker} f^{(H_i)}_{\alpha *} = 0, i \le 2k$, and $\pi(M^n_\alpha) = 0$. By Poincaré duality,

$$\operatorname{Ker} f^{(H_j)}_{2*} = 0, \quad j > 2k,$$

and the groups $H_i(M^n_\alpha)$ and $H_i(M^n)$ are isomorphic. By Lemma 2.4 and Remark 2.6 on page 59, the manifold M^n_α is homotopically equivalent to the manifold M^n. The lemma is proved.

Now let $n = 4k+2, k \neq 1, 3, k > 1$.

Lemma 4.8. *For every element $\alpha \in A(M^n)$ there exists a map $f_\alpha : S^{N+n} \to T_N(M^n)$ satisfying Lemma* 3.2, *such that the inverse image $M^n_\alpha = f_\alpha^{-1}(M^n)$ possesses the following properties*:

(a) $\pi_1(M^n_\alpha) = 0$;

(b) $H_i(M^n_\alpha) = H_i(M^n), \quad i \neq 2k+1$;

(c) $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z$ *or* 0;

(d) *denote the base cycles of the group* $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})}$ *by $x, y, x \circ y = 1$ if* $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z$. *Then $\varphi(x) = \varphi(y) = 1$.*

PROOF. By using the results of Lemma 4.6, we consider the map

$$ {}_{2k+1}f_\alpha : S^{N+n} \to T_N(M^n), $$

satisfying Lemma 3.2 and such that

$$ \begin{aligned} H_i(M^n_{\alpha,2k+1}) &= H_i(M^n), \quad i < 2k+1, \\ \operatorname{Ker} {}_{2k+1}f_{\alpha *}^{(H_{2k+1})} &= Z + \cdots + Z; \end{aligned} $$

the intersection matrix for the base cycles of the group $\operatorname{Ker} {}_{2k+1}f_{\alpha *}^{(H_{2k+1})}$ is skew-symmetric and unimodular. It can therefore be reduced to the basis $x_1, \dots, x_{2l} \in \operatorname{Ker} {}_{2k+1}f_{\alpha *}^{(H_{2k+1})}$, for which the intersection matrix is of the form

$$ \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ \dots & \dots & \dots & \dots \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}. \tag{14} $$

Thus we determine the invariant $\varphi(x) \in Z_2$, $x \in \operatorname{Ker} {}_{2k+1}f_{\alpha *}^{(H_{2k+1})}$, such that

$$ \varphi(x+y) = \varphi(x) + \varphi(y) + (x \circ y) \mod 2 $$

by Lemmas 2.2 and 4.4. Set

$$ \varphi({}_{2k+1}f_\alpha) = \sum_{i=1}^{l} \varphi(x_{2i-1})\varphi(x_{2i}). $$

If $\varphi({}_{2k+1}f_\alpha) = 0$, then it is possible to choose a basis $x'_1, \dots, x'_{2l}$, such that

$$\varphi(x'_i) = 0, \quad i = 1, \dots, 2l.$$

If $\varphi({}_{2k+1}f_\alpha) = 1$, then one can find a basis $x'_1, \dots, x'_{2l}$, such that

$$\varphi(x'_1) = \varphi(x'_2) = 1$$

and

$$\varphi(x'_i) = 0, \quad i > 2$$

(cf. [4]). Let us realize the cycles by smoothly embedded spheres $S_i^{2k+1} \subset M^n_{\alpha,2k+1}$, that intersect each other if and only if the corresponding intersection index is nonzero, and there is at most one intersection between any two spheres (cf. [9, 25]). If $\varphi({}_{2k+1}f_\alpha) = 0$ then the normal bundles $\nu^{2k+1}(S_i^{2k+1}, M^n_{\alpha,2k+1})$ are trivial. If $\varphi({}_{2k+1}f_\alpha) = 1$ then the bundles $\nu^{2k+1}(S_i^{2k+1}, M^n_{\alpha,2k+1})$ are trivial only for $i > 2$. Repeating Steps 1, 2, 3 of Lemma 4.6 and using Lemmas 1.2 and 4.7, we employ the Morse surgery to paste the spheres S^{2k+1}_{2i-1}, $i \geq 2$, every time killing the square $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $\varphi({}_{2k+1}f_\alpha) = 0$, then we paste the sphere S_1^{2k+1} as well, because its normal bundle in the manifold $M^n_{\alpha,2k+1}$ is trivial. As a result we get the map

$$f_\alpha : S^{N+n} \to T_N(M^n),$$

which possesses properties (a)–(d).

Thus the lemma is proved.

We now investigate the case $n = 6,\ 14 = 4k+2,\ k = 1, 3$.

Lemma 4.8′. *For every element $\alpha \in A(M^n)$ there exists a map $f_\alpha : S^{N+n} \to T_N(M^n)$ such that*

(1) $\pi_1(M^n_\alpha) = 0$;

(2) $H_i(M^n_\alpha) = H_i(M^n), \quad i \neq 2k+1$;

(3) $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z$ *or* 0.

Though the formulations of Lemmas 4.8 and 4.8′ are analogous, we shall see from the proof that these cases are essentially distinct. As above, we construct the map

$${}_{2k+1}f_\alpha : S^{N+n} \to T_N(M^n).$$

We have:

$$\mathrm{Ker}\ {}_{2k+1}f_{\alpha *}^{(H_i)} = 0, \quad i < 2k+1,$$

and the group $\mathrm{Ker}\ {}_{2k+1}f_{\alpha *}^{(H_{2k+1})}$ is free abelian; in the latter group we choose basic cycles $x_1, \dots, x_{2l}$, for which the intersection matrix is of the form (14). Let us realize these cycles by the spheres $S_i^{2k+1} \subset M_{\alpha,2k+1}^n$. The map ${}_{2k+1}f_\alpha$ can be thought of such that

$${}_{2k+1}f_\alpha(S_{2i-1}^{2k+1}) = {}_{2k+1}f_\alpha(S_{2i}^{2k+1}) = g_0 \in M^n, \quad i = 1, \dots, l,$$

where g_0 is a point in M^n. On the spheres S_{2i-1}^{2k+1} and S_{2i}^{2k+1}, there are the frame fields τ_{2i-1}^N and τ_{2i}^N, which are normal to $M_{\alpha,2k+1}^n$. The maps

$$f_* : \pi_3(SO_3) \to \pi_3(SO_{N+3})$$

and

$$j_* : \pi_7(SO_7) \to \pi_7(SO_{N+7})$$

are not epimorphic. In fact,

$$\mathrm{Coker} f_* = Z_2.$$

We select arbitrary frame fields τ_{2i-1}^{2k+1}, τ_{2i}^{2k+1} that are normal to S_{2i-1}^{2k+1} and S_{2i}^{2k+1} in $M_{\alpha,2k+1}^n$ (we recall that in this case the normal bundles $\nu^{2k+1}(S_{2i-1}^{2k+1}, M_{\alpha,2k+1}^n)$ and $\nu^{2k+1}(S_{2i}^{2k+1}, M_{\alpha,2k+1}^n)$ are trivial).

If we vary the fields τ_{2i-1}^{2k+1} and τ_{2i}^{2k+1} arbitrarily, the total frame fields $(\tau_{2i-1}^N, \tau_{2i-1}^{2k+1})$ and $(\tau_{2i}^N, \tau_{2i}^{2k+1})$, which are normal to the spheres S_{2i-1}^{2k+1} and S_{2i}^{2k+1} in S^{N+n}, τ_{2i-1}^{2k+1} and τ_{2i}^{2k+1} form the elements $\psi_{2i-1}, \psi_{2i} \in \mathrm{Coker} j_*$. If $\psi_{2i-1} \neq 0$ and $\psi_{2i} \neq 0$, then the framing cannot be extended to the balls $D_{2i-1}^{2k+2}, D_{2i}^{2k+2} \subset S^{N+n} \times I(0,1)$. This yields an obstruction to transmission of framings τ_{2i-1}^N and τ_{2i}^N under the Morse surgery (depending on the field τ_{2i-1}^{2k+1} or τ_{2i}^{2k+1}) valued in the group $\mathrm{Coker} j_*$, and equal to

$$\psi_{2i-1} = \psi_{2i-1}(S_{2i-1}^{2k+1})$$

and

$$\psi_{2i} = \psi_{2i}(S_{2i}^{2k+1}).$$

It is easy to see that the invariants ψ depend only on the cycle $x_s \in \mathrm{Ker}\ {}_{2k+1}f_{\alpha *}^{(H_{2k+1})}$ and do not depend on the sphere S_s^{2k+1} realizing the cycle x_s because

$$\mathrm{Ker}\ {}_{2k+1}f_{\alpha *}^{(H_{2k+1})} = \mathrm{Ker}\ {}_{2k+1}f_{\alpha *}^{(\pi_{2k+1})}$$

according to Lemma 4.5, and the homotopic spheres of dimension $2k+1$ in $M^n_{2k+1,\alpha}$ are regularly homotopic (see [25]). Thus, we determine the invariant

$$\psi(x) \in Z_2, \quad x \in \operatorname{Ker}\, {}_{2k+1}f^{(H_{2k+1})}_{\alpha *}.$$

We note further that analogously to the construction of φ, one may find a basis $x'_1, \dots, x'_{2l}$ such that $\psi(x'_s) = 0, s > 2$ (see [15]). It is therefore possible, following the previous proofs, to paste the cycles $x'_s, s \geq 3$ by a Morse surgery. If $\psi(x'_1) \neq 0$ and $\psi(x'_2) \neq 0$, the further pastings are impossible (the framing transmission obstruction is nonzero). If $\psi(x'_s) = 0$, $s = 1$ or $s = 2$, then the cycle x'_s can be repasted and therefore delete the whole square $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As a result, we obtain the claim of the lemma. The lemma is proved.

Remark 4.9. For a detailed analysis of the invariant ψ and Morse surgery (for $k = 0$) see L. S. Pontrjagin [15].

It remains for us to investigate the case of odd values of n. First of all we note that in this case there is no obstruction for the Morse surgery to transmit the framing; however, it is not clear whether the manifold can be simplified as a result of Morse surgery (just this question had a trivial solution in the remaining cases). If $n = 2i + 1$, then the Morse surgery over a cycle (sphere) of dimension i yields a new cycle of the same dimension i, which would be null-homotopic in any other case. Consider an arbitrary closed simply connected manifold Q^n. Assume the group $H_i(Q^n)$ has a torsion $\operatorname{Tor} H_i(Q^n) \neq 0$. Choose in $\operatorname{Tor} H_i(Q^n)$ a minimal system of generators $x_1, \dots, x_l$ of orders $q_1, \dots, q_l$. As it is well known, for any two cycles $x, y \in \operatorname{Tor} H_i(Q^n)$ there is a "linking coefficient" $Lk(x, y) \in Z_{d(q,q')}$, where q and q' are orders of the elements x and y and $d(q, q')$ is their greatest common divisor. Namely,

$$Lk(x, y) = \partial^{-1}(qx) \circ y \equiv x \circ \partial^{-1}(q'y) \bmod d(q, q'). \tag{15}$$

We formulate the Poincaré–Alexander duality[1]:

Suppose $x_1, \dots, x_l \in \operatorname{Tor} H_i(Q^n)$ is a minimal system of p-primary generators of orders $q_1, \dots, q_l$, respectively. Then there exists a minimal generator system $y_1, \dots, y_l \in \operatorname{Tor} H_i(Q^n)$ of orders $q_1, \dots, q_l$, such that

$$Lk(x_m, y_l) = \delta_{ml} \bmod d(q_m, q_l). \tag{16}$$

[1]Linking coefficients and duality are well defined not only for a system of p-generators.

Assume that the cycle x_1 is realized by a sphere $S_1^i \subset Q^n$ in such a way that the bundle $\nu^{i+1}(S_1^i, Q^n)$ is trivial.

The tubular neighborhood $T(S_1^i)$ of the sphere S_1^i in Q^n is diffeomorphic to $S_1^i \times D_\varepsilon^{i+1}$, $\varepsilon > 0$ being a small number.

We decompose the Morse surgery into two steps:

Step 1: $Q^n \to Q^n \setminus S_1^i \times D_\varepsilon^{i+1} = \overline{Q}^n$,
Step 2: $\overline{Q}^n \to \overline{Q}^n \cup_h D^{i+1} \times S_\varepsilon^i = Q^n(h)$, where $h : \partial D^{i+1} \times D_\varepsilon^{i+1} \to Q^n$ (see § 1).

Consider the cycle

$$b(x_1) = g_0 \times \partial D_\varepsilon^{i+1} \subset \overline{Q}^n, \quad g_0 \in S_1^i.$$

Lemma 4.10. *$H_s(\overline{Q}^n) = H_s(Q^n)$ for $s < i$. There is an epimorphism $x_{1*} : H_i(\overline{Q}^n) \to H_i(Q^n)$, the kernel of which is generated by the cycle $b(x_1)$. In the group $H_i(\overline{Q}^n)$ it is possible to choose generators $\widetilde{y}_j = x_{1*}^{-1} y_j$, $j = 1, \ldots, l$ such that*

$$b(x_1) = q_1 \widetilde{y}_1. \tag{17}$$

Proof. $H_s(\overline{Q}^n) = H_s(Q^n)$, $s < i$, as long as $n = 2i + 1 > 2s + 1$, and therefore all s-dimensional cycles and $(s+1)$-dimensional membranes can be assumed to be nonintersecting with S_1^i. For $s = i$ we can assume that the s-dimensional cycles do not intersect S_1^i. Therefore an embedding induces the epimorphism

$$x_{1*} : H_i(\overline{Q}^n) \to H_i(Q^n).$$

But the membranes have dimension $i+1$ and intersect $\overline{S}_1^i$ at isolated points. Consequently, two cycles which are homologous in Q^n, are homologous in $\overline{Q}^n$ modulo $b(x_1)$. Consequently,

$$\operatorname{Ker} x_{1*} = (b(x_1)).$$

In the homology class $y_1 \in H_i(Q^n, Z)$ one can find a cycle $\overline{y}_1$ and a membrane $\partial^{-1}(g\overline{y}_1)$ such that the intersection index

$$\partial^{-1}(q\overline{y}_1) \circ x_1 = 1,$$

which yields the cycle $b(x_1)$ is homologous to $q\widetilde{y}_1$. Thus, the lemma is proved.

It is well known that the linking coefficients $Lk(x, y)$ are bilinear, symmetric for odd i and antisymmetric for even i. In the group

$\operatorname{Tor} H_i(Q^n, Z)$, choose some p-primary subgroup system

$$H(p, s_p) \supset H(p, s_p - 1) \supset \cdots \supset H(p, 1),$$

where

$$\operatorname{Tor} H_i(Q^n, Z) = \sum_{p,k} H(p, k)/H(p, k-1).[1]$$

Thus, the group $H(p, s_p)$ contains all elements of the group of orders p^j, and the group $H(p,k)/H(p,k-1)$ contains all p-primary generators of orders p^k, and $H(p,k)/H(p,k-1)$ is a subgroup spanned by these elements: $\widetilde{H}(p,k) \subset H_i(Q^n, Z)$.

Lemma 4.11.

(a) *The direct sum decomposition of* $\operatorname{Tor} H_i(\overline{Q}^n, Z)$ *as* $H(p,k)/H(p,k-1)$ *can be made* (*for an appropriate choice of* p*-primary generators*) *such that* $Lk(x,y) = 0$, *if* $x \in \widetilde{H}(p,k_1), y \in \widetilde{H}(p,k_2),\ k_1 \neq k_2$;

(b) *In each group* $\widetilde{H}(p,k)$ *one can choose a system of* p*-primary generators* $x_1, \ldots, x_l, y_1, \ldots, y_{2m} \in \widetilde{H}(p,k)$ *such that*:

$$\left.\begin{aligned} Lk(x_s, y_l) &= 0, \quad 1 \le s \le t, \quad 1 \le l \le 2m,\\ Lk(x_{s_1}, x_{s_2}) &= 0, \quad s_1 \neq s_2,\\ Lk(y_{l_1}, y_{l_2}) &= 0, \quad |l_1 - l_2| > 1,\\ Lk(y_{l_1}, y_{l_2}) &= 0, \quad l_1 + l_2 \equiv 1 \,(\mathrm{mod}\ 4), \end{aligned}\right\} \tag{18}$$

$$Lk(x_s, x_s) \not\equiv 0 \,(\mathrm{mod}\ p), \qquad 1 \le s \le t, \tag{19}$$

$$\left.\begin{aligned} Lk(y_l, y_l) &\equiv 0 \,(\mathrm{mod}\ p), \qquad 1 \le l \le 2m,\\ Lk(y_{2l-1}, y_{2l}) &\equiv 1 \,(\mathrm{mod}\ p^k), \quad 1 \le l \le m. \end{aligned}\right\} \tag{20}$$

PROOF. It can be easily seen that for any choice of a system of p-primary generators in the group $H(p, s_p)$, the linking coefficient matrix for generating elements of order p^{s_p} (considered mod p^{s_p}) has a determinant which is relatively prime to p. We put $k = s_p$ and consider the subgroup $H(p, s_p - 1)$ such that

$$Lk(x, y) = 0,$$

where $x \in H(p, s_p - 1)$ and y is a generator of order p^{s_p}. Now one can choose a new system of p-primary generators in which all generators of orders

[1] The choice is such that $H(p,k) = H(p,k)/H(p,k-1) + H(p,k-1)$, $\widetilde{H}(p,k) = H(p,k)/H(p,k-1)$.

less than p^{s_p}, belong to the subgroup $H(p, s_p - 1)$. Assume by induction hypothesis that in the group $H(p, s_p)$ there are fixed subgroups $H(p, k)$ together with a system of p-primary generators such that:

(a) all generators of order greater than p^k belong to $H(p, k)$;
(b) $Lk(x, y) = 0, x \in H(p, k)$, y is a generator of order $>p^k$.

Let us construct the group $H(p, k-1)$. Consider the subgroup $H(p, k)$ and suppose that $H(p, k-1)$ consists of all elements $x \in H(p, k-1)$ such that

$$Lk(x, y) = 0,$$

where $x \in H(p, k-1)$, y is a generator of order p^k. Since the linking coefficient matrix for basic cycles of orders p^k for $H(p, k)$ (here coefficients are considered $\bmod p^k$) has a determinant coprime to p, then the group $H(p, k-1)$, we have constructed, possesses all necessary properties. Thus, we have decomposed the group $H(p, s_p)$ as a direct sum

$$\widetilde{H}(p, k) = \frac{H(p, k)}{H(p, k-1)},$$

so that

$$Lk(\widetilde{H}(p, k_1), \widetilde{H}(p, k_2)) = 0, \quad k_1 \neq k_2.$$

Point (a) of the lemma is completely proved. For the proof of point (b) we note that each group $\widetilde{H}(p, k)$ represents a linear space over the ring Z_{p^k} with scalar product $Lk(x, y)$, having determinant coprime with p. Consequently, either

(1) in the original basis there is a generator $\widetilde{x}_1$ such that $Lk(\widetilde{x}_1, \widetilde{x}_1) \not\equiv 0 \pmod p$, or
(2) there is a pair of generators $\widetilde{y}_1, \widetilde{y}_2$ such that

$$Lk(\widetilde{y}_1, \widetilde{y}_1) \equiv 0 \pmod p, \quad Lk(\widetilde{y}_2, \widetilde{y}_2) \equiv 0 \pmod p,$$
$$Lk(\widetilde{y}_1, \widetilde{y}_2) \not\equiv 0 \pmod p.$$

If case (1) holds, then one must select a basis $(\widetilde{x}_1, x_2, \dots, x_l, y_1, \dots, y_s)$ such that

$$Lk(x_j, \widetilde{x}_1) = Lk(y_j, \widetilde{x}_1) = 0, \quad j \geq 2.$$

If case (2) holds, then

$$\begin{vmatrix} Lk(\widetilde{y}_1, \widetilde{y}_2), & Lk(\widetilde{y}_1, \widetilde{y}_2) \\ \pm Lk(\widetilde{y}_1, \widetilde{y}_2), & Lk(\widetilde{y}_2, \widetilde{y}_2) \end{vmatrix} = \begin{vmatrix} pa_{11}, & a_{12} \\ \pm a_{12}, & pa_{22} \end{vmatrix} \not\equiv 0 \pmod p;$$

we select a new basis $\{x_j, \widetilde{y}_\varepsilon, y_l\}, l \geq 2$, such that

$$Lk(x_j, \widetilde{y}_\varepsilon) = Lk(y_l, \widetilde{y}_\varepsilon) = 0, \quad \varepsilon = 1, 2.$$

In the second case we put

$$y_1 = \widetilde{y}_1, \quad y_2 = \frac{1}{Lk(\widetilde{y}_1, \widetilde{y}_2)}\widetilde{y}_2.$$

Then in both cases we select the other required generators in subgroups orthogonal to $\widetilde{x}_1$ (in the first case) or subgroups orthogonal to $\widetilde{y}_1, \widetilde{y}_2$ (in the second case) in such a way that the relations (18)–(20) hold. The lemma is proved.

In the sequel we will always compose a minimal system of generators of the group $\operatorname{Tor} H_i(Q^n, z)$ by taking p-primary generators constructed in Lemma 4.11. We shall select a minimal system (with respect to the number of generators), and the generating element x of order

$$q = \prod_{p \in I} p^{k_p}$$

will be canonically represented as a sum of primary generators $x = \sum_p x(p)$ of orders p^{k_p}. We split the set of indices J into two parts: for the first part J_1 we take all p for which the elements $x(p)$ satisfy (19), and for the second part J_2 we take all p, for which $x(p)$ satisfies (20). Setting

$$\overline{x} = \sum_{p \in J_1} x(p), \quad \overline{\overline{x}} = \sum_{p \in J_2} x(p),$$

we see that for $\overline{\overline{x}}$ there is a basic element $\overline{\overline{y}}$, independent with $\overline{\overline{x}}$, such that the linking number $Lk(\overline{\overline{x}}, \overline{\overline{y}})$ is relatively prime to the order of $\overline{\overline{x}}$ (the latter being equal to $\overline{\overline{y}}$).

Lemma 4.12. *If $n = 2i+1$ and i is even, then the order of the element $\overline{x}$ is equal to* 2 *(if $\overline{x} \neq 0$).*

It is evident that the proof of the lemma immediately follows from the antisymmetry $Lk(\overline{x}, \overline{x}) = -Lk(\overline{x}, \overline{x})$ that must be relatively prime to the order of $\overline{x}$. The lemma is proved.

Suppose the cycle $\overline{x}_1$ is realized by the sphere $S_1^i \subset Q^n$, and i is even. According to Lemma 4.10, with the element $\overline{x}_1 \in H_i(Q^n)$ we associate the element $\widetilde{x}_1 \in H_i(Q^n)$ such that $b(\overline{x}_1) = 2\widetilde{\overline{x}}_1$. One can assume that $\widetilde{\overline{x}}_1$ lies on the boundary of the tubular neighborhood

$$T(S^i) \subset Q^n, \quad T(S^i) = S^i \times D^{i+1}.$$

Lemma 4.13. *The kernel of the homomorphism*

$$H_i(\overline{Q}^n) \to H_i(Q^n(h))$$

for any $h : \partial D^{i+1} \times D_\varepsilon^{n-i} \to T(S_1^i)$ *such that*

$$h(x, y) = (x, h_x(y)), \quad h_x \in SO_{i+1},$$

is generated by the element $(1 + 2\lambda(h))\widetilde{\overline{x}}_1$, *where* $\lambda(h)$ *is an integer.*

Proof. Consider the map $d(h) : S_1^i \to SO_{i+1}$, defining the Morse surgery, and denote by $y(h)$ the homology class of the cycle $\widetilde{y}(h) \subset \partial T(S_1^i)$, the latter defined by the first vector of the frame field $d(h)$, normal to S_1^i in $Q^n, y(h) \in H_i(\overline{Q}^n)$. There is a number $\lambda(h)$ such that

$$y(h) = \widetilde{\overline{x}}_1 + \lambda(h) b(\overline{x}_1)$$

or

$$y(h) = (1 + 2\lambda(h))\widetilde{\overline{x}}_1.$$

Evidently, with respect to the inclusion homomorphism $H_i(\overline{Q}^n) \to H_i(Q^n(h))$, the kernel is generated by the element

$$y(h) = (1 + 2\lambda(h))\widetilde{\overline{x}}_1$$

The lemma is proved.

Thus, we have eliminated the element $\overline{x}_1$ of order 2. Therefore the group $H_i(Q^n(h))$ of generators not satisfying (20), will be one element less (for even i) since all such generators are of order 2 according to Lemma 4.12.

Let i be arbitrary (odd or even) and let $\overline{\overline{x}}_1$ be a generating cycle $\overline{\overline{x}}_1 \in H_i(Q^n)$, satisfying (20) and realized by a sphere $S_1^i \subset Q^n$ with trivial normal bundle $\nu^{i+1}(S_1^i, Q^n)$. Suppose also that the cycle $\overline{\overline{x}}_2 \in H_i(Q^n)$ is such that $Lk(\overline{\overline{x}}_1, \overline{\overline{x}}_2) = 1$. We denote, as in Lemma 4.10, the generators corresponding to them by $\widetilde{\overline{\overline{x}}}_1, \widetilde{\overline{\overline{x}}}_2 \in H_i(\overline{Q}^n)$, where $b\overline{\overline{x}}_1 = q_1\widetilde{\overline{\overline{x}}}_2$, q_1 is the order of generators $\overline{\overline{x}}_1, \overline{\overline{x}}_2 \in H_i(Q^n)$ and $\widetilde{\overline{\overline{x}}}_1$ is the homology class in $H_i(\overline{Q}^n)$ of the cycle $\overline{\overline{x}}_1(h)$ defined by the first vector of the framed

$i+1$-field $h : S^i \to SO_{i+1}$ on the boundary $\partial T(S^i_1)$ for some fixed h. Then we have

Lemma 4.14. *The kernel of the inclusion homomorphism $H_i(\overline{Q}^n) \to H_i(Q^n(h))$ is generated by the element $\widetilde{\overline{\overline{x}}}_1$, and the group $H_i(Q^n(h))$ has one generator less than the group $H_i(Q^n)$.*

The proof of the lemma follows from the definition of the Morse surgery and the relation $b(\overline{\overline{x}}_1) = q_1\widetilde{\overline{\overline{x}}}_2$.

Proof. The element $\widetilde{\overline{\overline{x}}}_2 \in H_i(Q^n(h))$ has order λq_1, where

$$\lambda \equiv Lk(\overline{\overline{x}}_1, \overline{\overline{x}}_1) \bmod q_1,$$

and the number $Lk(\widetilde{\overline{\overline{x}}}_2, \widetilde{\overline{\overline{x}}}_2)$ is relatively prime to λq_1, if $\lambda \neq 0$ (i.e. the element $\widetilde{\overline{\overline{x}}}_2$ satisfies (19) in the manifold $(Q^n(h))$).

Assume i is odd. Consider the element $\overline{x}_1 \in H_i(Q^n)$ realized by the sphere $S^i_1 \subset Q^n$ with trivial normal bundle $\nu^{i+1}(S^i_1, Q^n)$. The linking coefficient

$$Lk(\overline{x}_1, \overline{x}_1) = \lambda\,(\mathrm{mod}\ q),$$

where q is the order of $\overline{x}_1$ and λ is relatively prime to q. From Lemma 4.10 it follows that on the boundary $\partial T(S^i_1)$ one can find a cycle $\widetilde{\overline{x}}_1$ such that in the homology group $H_i(\overline{Q}^n)$ the relation

$$\lambda b(\overline{x}_1) = q\widetilde{\overline{x}}_1$$

holds.

Consider the map $h : S^i_1 \to SO_{i+1}$ and the kernel of the inclusion

$$j_* : \pi_i(SO_{i+1}) \to \pi_i(SO_\infty),$$

which is isomorphic to Z for odd i, $\operatorname{Ker} j_* = Z$.

Denote by $y(h)$ the homology class in $\overline{Q}^n$ of the cycle defined on $\partial T(S^i_1) = S^i_1 \times S^i_\varepsilon(b(\overline{x}_1)) = g_0 \times S^i_\varepsilon, g_0 \in S^i_1$, by the first vector of the framed field h. Let $\mu \in \operatorname{Ker} j_* = Z$ (μ is a number).

Lemma 4.15. *The kernel of the inclusion homomorphism*

$$H_i(\overline{Q}^n) \to H_i(Q^n(h))$$

is generated by the element $y(h) = \tilde{\overline{x}}_1 + \gamma b(\overline{x}_1)$. *The kernel of the inclusion homomorphism*

$$H_i(\overline{Q}^n) \to H_i(Q^n(h+\mu)), \quad \mu \in \operatorname{Ker} j_* = Z,$$

is generated by the element $y(h+\mu) = y(h) + 2\mu b(\overline{x}_1)$.

The proof of Lemma 4.15 immediately follows from the definition of Morse surgery and the structure of the homomorphism $\operatorname{Ker} j_* \to H_i(S^i)$ generated by the mapping $SO_{i+1} \to S^i$ (projection), where the generator of the group $\operatorname{Ker} j_*$ is taken to the cycle $2[S^i]$. Therefore

$$y(h+\mu) = y(h) + 2\mu b(\overline{x}_1).$$

Let us prove that

$$y(h) = \tilde{\overline{x}}_1 + \gamma b(\overline{x}_1).$$

To do this, we consider the intersection index

$$[\partial^{-1} q_1 y(h)] \cdot \overline{x}_1 = \lambda \bmod q_1 = \lambda + \gamma q_1.$$

On the other hand,

$$[\partial^{-1} b(\overline{x}_1)] \cdot \overline{x}_1 = 1.$$

Therefore

$$[\partial^{-1}(q_1 y(h) - q_1 \gamma b(\overline{x}_1))] \cdot \overline{x}_1 = \lambda,$$

from which it follows that one can put $\tilde{\overline{x}}_1 = y(h) - \gamma b(\check{x})$. The lemma is proved.

Lemma 4.16. *There exists a number* μ *such that in the group* $H_i(Q^n(h+\mu))$ *we have*:

(a) $\tilde{\overline{x}}_1 = 0$, $\lambda b(\overline{x}_1) = 0$ (γ *is even*),

(b) $\tilde{\overline{x}}_1 = b(\overline{x}_1)$, $(\lambda_1 - q_1)\tilde{\overline{x}}_1 = 0$ (γ *is odd*), *where in both cases the order of the "new" element* $b(\overline{x}_1)$ *is less than* q_1; *the number* $Lk(b(\overline{x}_1), b(\overline{x}_1))$ *is relatively prime to the order of* $b(\overline{x}_1)$.

PROOF. Since $\lambda b(\overline{x}_1) = q_1\widetilde{\overline{x}}_1$ in $\overline{Q}^n$ and $\widetilde{\overline{x}}_1 = y(h) - \gamma b(\overline{x}_1)$, then

$$y(h+\mu) = y(h) + 2\mu b(\overline{x}_1) = \widetilde{\overline{x}} + \gamma b(\overline{x}_1) + 2\mu b(\overline{x}_1).$$

Passing to $Q^n(h+\mu)$, we will get the relation $y(h+\mu) = 0$. Thus

$$\begin{aligned} \widetilde{\overline{x}}_1 &= -(\gamma + 2\mu) b(\overline{x}_1) && (\text{in } Q^n(h+\mu)), \\ \lambda b(\overline{x}_1) &= q_1 \widetilde{\overline{x}}_1 && (\text{in } \overline{Q}^n), \end{aligned}$$

which yields a possibility to choose μ ($\mu = -\frac{\gamma}{2}$ for γ and $2\mu - 1 = \gamma$ for odd γ).

Evidently, by Lemma 4.11, the element $b(\overline{x}_1)$ is not linked with other basis elements. The assertion is proved.

Now we apply the proved lemmas to study the maps

$$f_\alpha : S^{N+n} \to T_N(M^n),$$

where $n = 2i + 1$.

Lemma 4.17. *Let $\alpha \in A(M^n)$. There exists a map*

$$f_\alpha : S^{N+n} \to T_N(M^n),$$

such that the inverse image $f_\alpha^{-1}(M^n) = M_\alpha^n$ is homotopically equivalent to M^n.

PROOF. As above, consider the map

$${}_if_\alpha : S^{N+n} \to T^n(M^n),$$

constructed according to Lemma 4.6, and the inverse image

$$M_{\alpha,i}^n = {}_if_\alpha^{-1}(M^n),$$

for which the groups $H_s(M_{\alpha,i}^n)$ are isomorphic to $H_s(M^n)$ for $s < i$ and $\pi_1(M_{\alpha,i}^n) = 0$. The group $\operatorname{Ker} {}_if_{\alpha*}^{(H_i)}$ is a direct summand in $H_i(M_{\alpha,i}^n)$, $n = 2i+1$, according to Lemma 4.4′. The group $\operatorname{Ker} {}_if_{\alpha*}^{(H_{i+1})}$ is free abelian by Lemma 4.5. First, let us use the Morse surgery, and try to kill the group $\operatorname{Tor}\operatorname{Ker} {}_if_{\alpha*}^{(H_i)}$, by using the Poincaré–Alexander duality. If i is odd, then Lemmas 4.12 and 4.13 allow us to kill all elements not satisfying (20) without increasing the number of generators. Then by Lemma 4.14, we kill all elements satisfying (20), where each Morse surgery decreases the number of generators by 1. If i is odd, then consequent Morse surgeries will let us kill all generators satisfying (20), each time decreasing the number of generators by 1 (according to Lemma 4.14), and then, according to Lemmas 4.15 and 4.16, we shall start decreasing the order of some generator satisfying (19) without increasing the number of generators and decreasing the order each time (we vary the surgery $\operatorname{mod} \operatorname{Ker} j_* \subset \pi_i(SO_{i+1})$), which preserves the possibility to transmit the frame fields (cf. proofs of Lemmas 1.1 and 1.2). Thus, as a result, we kill the group $\operatorname{Tor} \operatorname{Ker}_i f_{\alpha *}^{(H_i)}$. Then, according to [4], we easily kill the elements of infinite order and get the desired manifold M_α^n and mapping

$$f_\alpha : S^{N+n} \to T_N(M^n)$$

analogously to Theorems 4.7–4.9. The lemma is proved.

We collect the results of the lemmas in the following:

Lemma 4.18. *If $n = 4k, k \geq 2$ or $n = 2k+1$, then each element*

$$\alpha \in A(M^n) \subset \pi_{N+n}(T_N(M^n)), \quad A(M^n) = H^{-1}\varphi[M^n],$$

is represented by a map $f_\alpha : S^{N+n} \to T_N(M^n)$, which is t-regular and such that

$$\pi_1(M_\alpha^n) = 0, \quad H_i(M_\alpha^n) = H_i(M^n)$$

for $i = 2, \dots, n-2$, where $M_\alpha^n = f_\alpha^{-1}(M^n)$. Thus the manifold M_α^n is homotopically equivalent to M^n with degree $+1$, and $\nu^N(M_\alpha^n) = f_\alpha^*(M^n)$. If $n = 4k+2, k \geq 1$, then for any element $\alpha \in A(M^n)$ one can choose a map $f_\alpha : S^{N+n} \to T_N(M^n)$ of homotopy class α such that

$$\pi_1(M_\alpha^n) = 0, \quad H_i(M_\alpha^n) = H_i(M^n)$$

for $i \leq 2k$, where $M_\alpha^n = f_\alpha^{-1}(M^n)$; moreover,

$$\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z,$$

and there are well-defined invariants $\varphi(\alpha) \in Z_2$ for $n = 4k+2$, $k \neq 1, 3$, and $\psi(\alpha) \in Z_2$ for $n = 6, 14$; if these invariants are zero, then it is possible to repaste the groups $\operatorname{Ker} f_{\alpha *}^{(H_{2k+1})} = Z + Z$ by a sequence of Morse surgeries.

This theorem is a formal unification of the following lemmas.

§ 5. The manifolds in one class

Definition 5.1. For any element $\alpha \in \overline{A}(M^n) \subset A(M^n)$ the map representing it

$$f_\alpha : S^{N+n} \to T_N(M^n)$$

is called admissible if it satisfies Lemma 3.2 and the inverse image

$$f_\alpha^{-1}(M^n) = M_\alpha^n$$

is homotopically equivalent to M^n.

Theorem 5.2. *Let $f_{\alpha,i} : S^{N+n} \to T_N(M^n), i = 1, 2,$ be two admissible homotopic maps and let $M^n_{\alpha,i} = f^{-1}_{\alpha,i}(M^n)$. If n is even, then manifolds $M^n_{\alpha,i}$ are diffeomorphic of degree $+1$. If n is even, then there exists a Milnor sphere $\widetilde{S}^n \in \theta^n(\partial\pi)$, which is a boundary of a π-manifold such that the manifolds $M^n_{\alpha,1}$ and $M^n_{\alpha,2}\#\widetilde{S}^n$ are diffeomorphic of degree $+1$.*

Proof. Consider the homotopy

$$F : S^{N+n} \times I \to T_N(M^N),$$

where $F/S^{N+n} \times 0 = f_{\alpha,1}$ and $F/S^{N+n} \times 1 = f_{\alpha,2}$. We split the proof into several steps.

Step 1. Let us make the homotopy F t-regular. After this, consider the inverse image

$$F^{-1}(M^n) \subset S^{N+n} \times I(0, 1),$$

which is a manifold N^{n+1} with boundary

$$\partial N^{n+1} = M^n_{\alpha,1} \cup (-M^n_{\alpha,2}),$$

such that

$$\nu^N(N^{N+1}) = F * \nu^N(M^n).$$

Thus, we have a well-defined map $F/N^{n+1} \to M^n$, which is homotopy equivalence of degree $+1$ for each boundary component. The manifold N^{n+1} is an (F, π)-manifold mod M^n.

Step 2. Consider the following direct sum decompositions

$$\left.\begin{aligned} H_j(N^{n+1}) &= H_j(M^n_{\alpha,i}) + \operatorname{Ker} F_*^{(H_j)}, \quad i=1,2,\\ \pi_j(N^{n+1}) &= \pi_j(M^n_{\alpha,i}) + \operatorname{Ker} F_*^{(\pi_j)}, \quad i=1,2,\\ H^j(N^{n+1}) &= H^j(M^n_{\alpha,i}) + \operatorname{Coker} F^*, \quad i=1,2, \end{aligned}\right\} \tag{21}$$

that arise from natural retractions of the membrane to the boundary components:

$$(f_{\alpha,i})^{-1}\cdot F : N^{n+1} \to M^n_{\alpha,i}, \tag{22}$$

where the maps $f(_{\alpha,i})^{-1} f_{\alpha,i} : M^n_{\alpha,i} \to M^n_{\alpha,i}$ are homotopic to the identity.

It is evident that

$$\left.\begin{aligned} H_j(N^{n+1}, M^n_{\alpha,i}) &= \operatorname{Ker} F_*^{(H_j)}, \quad i=1,2,\\ \pi_j(N^{n+1}, M^n_{\alpha,i}) &= \operatorname{Ker} F_*^{(\pi_j)}, \quad i=1,2,\\ H^j(N^{n+1}, M^n_{\alpha,i}) &= \operatorname{Coker} F^*, \quad i=1,2. \end{aligned}\right\} \tag{23}$$

We have the following:

Lemma 5.3. *Between the groups* $\operatorname{Ker} F_*^{(H_j)}/\operatorname{Tor}$ *and* $\operatorname{Ker} F_*^{(H_{n+1-j})}/\operatorname{Tor}$ *there is a non-degenerate unimodular scalar product determined by the intersection index. Between the groups* $\operatorname{Tor}\operatorname{Ker} F_*^{(H_j)}$ *and* $\operatorname{Tor}\operatorname{Ker} F_*^{(H_{n-j})}$, *there is the Alexander duality: for every minimal generator system* $x_1,\ldots,x_l \in \operatorname{Tor}\operatorname{Ker} F_*^{(H_j)}$ *there exists a minimal generator system* $y_1,\ldots,y_l \in \operatorname{Tor}\operatorname{Ker} F_*^{(H_{n-j})}$ *such that the order of* y_i *is equal to the order of* x_i, $i=1,\ldots,l$ *and* $Lk(x_i,y_j)=\delta_{ij}$.

Proof. Lemma 5.3 is an immediate consequence from the decompositions (21), isomorphisms (23) between relative groups $\operatorname{mod} M^n_{\alpha,i}$ and groups $\operatorname{Ker} F_*^{(H_j)}$, and the Poincaré–Alexander duality D:

$$\begin{aligned} &D : H_j(N^{n+1}, M^n_{\alpha,1}) \xrightarrow{\approx} H^{n+1-j}(N^{n+1}, M^n_{\alpha,2}),\\ &\operatorname{Tor} H_j(N^{n+1}, M^n_{\alpha,1}) \approx \operatorname{Tor} H_{n-j}(N^{n+1}, M^n_{\alpha,2}). \end{aligned} \tag{24}$$

The lemma is proved.

Step 3. By means of the Morse decomposition, we consequently kill the groups $\pi_1(N^{n+1}), \operatorname{Ker} F_*^{(H_2)}, \ldots$ and so on, by modifying F to the reconstructed membrane and using all of the techniques proved in § 4.

Case 1. If n is even, then $n+1$ is odd and the successive reconstructions of the groups $\operatorname{Ker} F_*^{(H_j)}$ up to $j=\frac{n}{2}$ have no obstructions. If $\operatorname{Ker} F_*^{(H_j)} = 0$

for $j \leq \frac{n}{2}$, then, by Lemma 5.3, $\operatorname{Ker} F_*^{(H_{n+1-j})} = 0$ (and $\pi_1 = 0$). Thus the membrane N^{n+1} is contractible to any of its boundary components, which yields the J-equivalence (h-cobordism) of the boundary components. According to the Smale Theorem [19] the manifolds $M^n_{\alpha,1}$ and $M^n_{\alpha,2}$ are diffeomorphic.

Case 2. If $n = 4k - 1$, then $n + 1 = 4k$. Analogously to the preceding case, one can obtain the result that $\operatorname{Ker} F_*^{(H_j)} = 0$ for $j < 2k$ and $\operatorname{Ker} F_*^{(H_j)} = 0$ for $j > 2k$. The intersection matrix of the free abelian group $\operatorname{Ker} F_*^{(H_{2k})}$ will be unimodular and will have even numbers on its diagonal (exactly analogously to Lemma 4.7), however, the signature of this matrix, should not be zero, unlike the situation in Lemma 4.7, since the Hirzebruch formula [3] is acceptable only for closed manifold. Denote the intersection matrix by $B = (b_{ij})$, where $b_{ij} = x_i \cdot x_j, x_1, \dots, x_s$ is the basis of the group $\operatorname{Ker} F_*^{(H_{2k})}$. Denote the signature of B by $\tau(B)$. It is known (see [8]) that $\tau(B) \equiv 0 \pmod 8$, because $\det B = \pm 1$ and $b_{ii} \equiv 0 \pmod 2$.

Let us construct, following Milnor [8], a π-manifold $M^{n+1}(B)$ such that:

(a) $\pi_1(M^{n+1}(B)) = 0$;
(b) $H_j(M^{n+1}(B)) = 0, j \neq 0, 2k$;
(c) $\partial M^{n+1}(B)$ is a homotopy sphere

$$\widetilde{S}^n = \partial M^{n+1}(B) \in \theta^n(\partial\pi);$$

(d) the intersection matrix of the basic cycles of $H_{2k}(M^{n+1}(B))$ is such that its signature

$$\tau(M^{n+1}(B)) = -\tau(B).$$

Now, let us consider the manifold

$$N^{n+1} \cup_{f_0} D^n_\varepsilon \times I(0,1) \cup_{f_1} M^{n+1}(B) = N^{n+1}(B), \tag{25}$$

where

$$\begin{aligned} f_0 &: D^n_\varepsilon \times 0 \to M^n_{\alpha,2}, \\ f_1 &: D^n_\varepsilon \times 1 \to \partial M^{n+1}(B) \end{aligned}$$

(f_0, f_1 are diffeomorphisms of the desired degree ∓ 1). Clearly,

$$\partial N^{n+1}(B) = M^n_{\alpha,1} \cup (-M^n_{\alpha,2} \# \widetilde{S}^n).$$

In addition, there are the following retractions

$$\begin{aligned} F_1 &: N^{n+1}(B) \to M^n_{\alpha,1}, \\ F_2 &: N^{n+1}(B) \to M^n_{\alpha,2} \# \widetilde{S}^n, \end{aligned} \tag{26}$$

induced by the retractions $(f_{\alpha,1})^{-1} \cdot F$ and $(f_{\alpha,2})^{-1} \cdot F$. Since $M^{n+1}(B)$ is a π-manifold, it is easy to see that

$$F_i^* \nu^N(M^n_{\alpha,i}) = \nu^N(N^{n+1}(B)), \quad i = 1, 2.$$

By construction, the signature of the intersection matrix of basic cycles of $\operatorname{Ker} F_{i*}^{(H_{2k})}$, $i = 1, 2$, is equal to the sum of signatures

$$\tau(B) + \tau(M^{n+1}(B)) = 0.$$

Further, we repeat completely the arguments of Lemma 4.7, we reconstruct the group $\operatorname{Ker} F_{i*}^{(H_{2k})}$, $i = 1, 2$ by using the same method, and kill this group, and apply the Smale theorem (cf. Case 1). Thus, Case 2 is investigated.

Case 3. $n = 4k + 1, n = 1 = 4k + 2$. Analogously to Cases 1 and 2 and proofs of Lemmas 4.8 and 4.9 we assume that the membrane N^{n+1} is such that:

(a) $\operatorname{Ker} f_*^{(H_j)} = 0$, $j < 2k + 1$,
(b) $\pi_1(N^{n+1}) = 0$,
(c) $\operatorname{Ker} F_*^{(H_{2k+1})} = Z + Z$ or 0 depending on the values of the invariants φ (for $k \neq 1, 3$) or ψ (for $k = 1, 3$), being obstructions for the Morse surgery.

First, the invariant ψ (for the cases $k = 1, 3$) defines an obstruction not to Morse surgery, but for a transmission of frame fields (cf. Lemma 4.9), which plays no role for us. Thus, we perform the Morse surgery (without being concerned about the fields) and get

$$\operatorname{Ker} F_*^{(H_{2k+1})} = 0, \quad k = 1, 3.$$

Thus the membrane contracts onto each of its boundaries and therefore (cf. [19]) it is diffeomorphic to $M^n_{\alpha,1} \times I$.

If $k \neq 1, 3$ then on the basic cycles $x, y \in \operatorname{Ker} F_*^{(H_{2k+1})}$ there exists a well-defined invariant $\varphi(x), \varphi(y)$.

If $\varphi(x) = 0$ or $\varphi(y) = 0$, we perform the Morse surgery recalling the sense of φ (the invariant of normal bundle for an embedded sphere $S^{2k+1} \subset N^{4k+2}$). Let $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$. We construct, according to Kervaire [4], a π-manifold $M^{4k+2}(\varphi)$ such that:

(a) the boundary $\partial M^{4k+2}(\varphi)$ is a homotopy sphere;
(b) $\pi_1(M^{4k+1}(\varphi)) = H_j(M^{4k+2}(\varphi)) = 0,\ j \neq 0, 2k+1$;
(c) $H_{2k+1}(M^{4k+2}(\varphi)) = Z + Z$; and denoting the basic cycles by $\overline{x}, \overline{y}$.
(d) $\varphi(\overline{x}) = \varphi(\overline{y}) = 1$.

KAs in Case 2, we set:

$$N^{4k+2}(\varphi) = N^{4k+2} \cup_{f_0} D_\varepsilon^{4k+1} \times I(0,1) \cup f_1 M^{4k+2}(\varphi), \tag{27}$$

where

$$f_0 : D_\varepsilon^{4k+1} \times 0 \to M_{\alpha,1}^{4k+1},$$
$$f_1 : D_\varepsilon^{4k+1} \times 1 \to M_{\alpha,2}^{4k+2}$$

are diffeomorphisms of the desired degree $+1$. Then

$$\partial N^{4k+2}(\varphi) = M_{\alpha,1}^{4k+1} \cup (-M_{\alpha,2}^{4k+1} \# \partial M^{4k+2}(\varphi)).$$

Using next the relation

$$\varphi(z+t) = \varphi(z) + \varphi(t) + z \cdot t|_{\text{mod } 2},$$

we find a new basis $x_1, x_2, x_3, x_4 \in \operatorname{Ker} F_{1*}^{(H_{2k+1})}$, where

$$F_1 : N^{4k+2}(\varphi) \to M_{\alpha,1}^{4k+1},$$

is a natural retraction (here $\varphi(x_i) = 0, i = 1,2,3,4$), and we paste the cycles by using Morse surgery. Then we apply Smale's Theorem again (cf. Case 1). The theorem is proved.

§ 6. One manifold in different classes

We shall consider only maps

$$f_\alpha : S^{N+n} \to T^N(M^n)$$

which are admissible in the sense of § 5.

Lemma 6.1. *The homotopy class of an admissible map*

$$f_\alpha : S^{N+n} \to T^N(M^n)$$

is completely defined by:

(a) *a manifold M_α^n that is homotopically equivalent to the manifold M^n with degree $+1$ and such that $M_\alpha^n \geq M^n$;*

(b) *some (arbitrary) embedding* $M^n_\alpha \subset S^{N+n}$;
(c) *some (arbitrary up to homotopy) smooth map* $\widetilde{f}_\alpha : M^n_\alpha \to M^n$ *of degree* $+1$, *for which* $\widetilde{f}^*_\alpha \nu^N(M^n) = \nu^n(M^n_\alpha)$;
(d) *some (arbitrary up to homotopy) smooth map of* SO_N*-bundles*

$$\widetilde{\widetilde{f}}_\alpha : \nu^N(M^n_\alpha) \to \nu^N(M^n)$$

that covers the smooth map $\widetilde{f}_\alpha : M^n_\alpha \to M^n$.

Proof. If we are given a manifold M^n_α, an embedding $M^n_\alpha \subset S^{N+n}$, a map $\widetilde{f}_\alpha : M^n_\alpha \to M^n$ and a covering map of bundles $\widetilde{\widetilde{f}}_\alpha : \nu^N(M^n_\alpha) \to \nu^N(M^n)$, then the map f_α is completely defined for the tubular neighborhood $T(M^n_\alpha) \subset S^{N+n}$ because the tube $T(M^n_\alpha)$ is the total space of the normal bundle $\nu^N(M^n_\alpha)$. By construction of the Thom complex $T_N(M^n)$, the extension of the map f_α to the remaining part of the sphere S^{N+n} is trivial (in the neighborhood of the singular point of the Thom complex) and in a unique way up to homotopy. Now assume that we perform an isotopy to the embedding $M^n \subset S^{N+n}$, and we perform a homotopy $\widetilde{f}_\alpha$ for $\widetilde{\widetilde{f}}_\alpha$ such that all isotopies and homotopies are smooth, and the homotopy of the map $\widetilde{\widetilde{f}}_\alpha$ is a homotopy of SO_N-bundles which covers the homotopy $\widetilde{f}_\alpha$. These isotopies and homotopies together define an embedding of

$$M^n_\alpha \times I(0,1) \subset S^{N+n} \times I(0,1),$$

and a map F of the tubular neighborhood

$$T(M^n_\alpha \times I(0,1)) \subset S^{N+n} \times I$$

$T(M^n_\alpha \times I(0,1))$ is diffeomorphic $\nu^N(M^n_\alpha) \times I(0,1)$ into the space $T_N(M^n)$, where $F(M^n_\alpha \times I) \subset M^n$. Furthermore, the map

$$F : T(M^n_\alpha \times I) \to T_N(M^n)$$

is extended in a well-known manner to the map

$$F : S^{N+n} \times I \to T_N(M^n),$$

where $F/S^{N+n} \times 0 = f_\alpha$. Consequently, the homotopy class α of the map f_α does not depend on the arbitrariness in the choice of embedding (all embeddings are isotopic for $N > n$), neither does it depend on the maps $\widetilde{f}_\alpha, \widetilde{\widetilde{f}}$ chosen in their homotopy classes.

The lemma is proved.

Thus, for a fixed manifold M^n_α the homotopy class of an admissible map f_α,

$$f_\alpha : S^{N+n} \to T_N(M^n),$$

is completely defined by the homotopy class of a map $\widetilde{f}_\alpha : M^n_\alpha \to M^n$ of degree $+1$ such that

$$\nu^N(M^n_\alpha) = \widetilde{f}^*_\alpha \nu^N(M^n),$$

and by the homotopy class of a map of SO_N-bundles:

$$\widetilde{\widetilde{f}}_\alpha : \nu^N(M^n_\alpha) \to \nu(M^n)$$

that covers $\widetilde{f}_\alpha$ (in the sequel, it will be assumed without further comment that the embedding $M^n_\alpha \subset S^{N+n}$ is fixed).

Lemma 6.2. *If both manifolds $M^n_{\alpha,i} \geq M^n$, $i = 1, 2$, which are homotopy equivalent to M^n, belong to one class $\alpha \in \overline{A}(M^n) \subset A(M^n)$, then for every class α_1, for which there exists an admissible map*

$$f_{\alpha_1,1} : S^{N+n} \to T_N(M^n)$$

such that $f^{-1}_{\alpha_1,1}(M^n) = M^n_{\alpha,1}$, there also exists another admissible map

$$f_{\alpha_1,2} : S^{N+n} \to T_N(M^n),$$

for which $f^{-1}_{\alpha_1,2}(M^n) = M^n_{\alpha,2}$.

PROOF. Consider the t-regular homotopy

$$F : S^{N+n} \times I(0,1) - T_N(M^n),$$

where $F/S^{N+n} \times 0 = f_{\alpha,1}$ and $f/S^{N+n} \times 1 = f_{\alpha,2}$. We put

$$N^{n+1} = F^{-1}(M^n) \subset S^{N+n} \times I,$$

where

$$\nu^N(N^{n+1}) = F^*\nu^N(M^n).$$

Since the map F restricted to the boundary components, represents homotopy equivalences $\widetilde{f}_{\alpha,1}$ and $\widetilde{f}_{\alpha,2}$, the membrane N^{n+1} naturally retracts to each of the boundary components. Denote these retractions by

$$F_i = (f_{\alpha,i})^{-1} \cdot F, \quad i = 1, 2.$$

By Lemma 6.1, the element α_1 can be obtained as follows: on the boundary of $M^n_{\alpha,1} \subset \partial N^{n+1}$ we change the map $\widetilde{f}_{\alpha,1}$ to $\widetilde{f}_{\alpha_1,1}$ and, analogously, we change the map of bundles $\widetilde{\widetilde{f}}_{\alpha,1}$ to the map $\widetilde{\widetilde{f}}_{\alpha_1,1}$. Since the membrane

N^{n+1} retracts to the boundary and

$$\nu^N(N^{n+1}) = F_1^* \nu^N(M^n_{\alpha,1}),$$

we may extend the maps $\widetilde{f}_{\alpha,1}, \widetilde{\widetilde{f}}_{\alpha_1,1}$ to the maps

$$\widetilde{F} : N^{n+1} \to M^n$$

and

$$\widetilde{\widetilde{F}} : \nu^N(N^{n+1}) \to T_N(M^n).$$

Then we extend this map $\widetilde{\widetilde{F}}$ from the tubular neighborhood $T(N^{n+1}) \subset S^{N+n} \times I$ to the whole Cartesian product $S^{N+n} \times I$ by using Thom's method, and denote this extension by

$$\overline{F} : S^{N+n} \times I \to T_N(M^n).$$

Clearly,

$$\frac{\overline{F}}{S^{N+n}} \times 0 = f_{\alpha_1,1}.$$

Putting

$$f_{\alpha_1,2} = \frac{\overline{F}}{S^{N+n}} \times 1,$$

if the extension $\overline{F}$ is smooth on $T(N^{n+1})$, which can always be attained. This completes the proof of the lemma.

In addition we are now able to consider only one fixed manifold $M^n_\alpha \geq M^n$, $M^n \geq M^n_\alpha$ and study the problem of determining the set of classes $\alpha_i \in \overline{A}(M^n) \subset A(M^n)$ this manifold may belong to $B(M^n_\alpha)$ the set of classes $\alpha_i \in \overline{A}(M^n)$ for which there exist admissible maps

$$f_{\alpha_i} : S^{N+n} \to T_N(M^n)$$

such that

$$f^{-1}_{\alpha_i}(M^n) = M^n_\alpha.$$

We denote by $\pi^+(M^n_\alpha, M^n)$ the set of homotopy classes of maps $f : M^n_\alpha \to M^n$ of degree $+1$ such that

$$f^* \nu^N(M^n) = \nu^N(M^n_\alpha).$$

We denote by $\pi(X, Y)$ the set of homotopy classes of maps $X \to Y$ for any complexes X, Y. In particular, the sets $\pi^+(M^n, M^n)$ and $\pi(M^n, SO_N)$ are groups, moreover, the group $\pi(M^n, SO_N)$ is abelian, and the group $\pi^+(M^n, M^n)$ acts transitively without fixed points on $\pi^+(M^n_\alpha, M^n)$.

Lemma 6.3. *The set $B(M^n_\alpha) \subset \overline{A}(M^n)$ splits into a union of disjoint sets*

$$B(M^n_\alpha) = \cup_f B_f(M^n_\alpha),$$

where $f \in \pi^+(M^n_\alpha, M^n)$ and $B_f(M^n_\alpha)$ is the subset of the set $B(M^n_\alpha)$ that consists of classes $\alpha \in \overline{A}(M^n)$ for which there exists an admissible map

$$f_\alpha : S^{N+n} \to T_N(M^n),$$

such that $f_\alpha^{-1}(M^n) = M^n_\alpha$ and such that the restriction of f_α to M^n_α, is of homotopy class $f \in \pi^+(M^n_\alpha, M^n)$.

Proof. It has already been established that the set $B_f(M^n_\alpha)$ is well defined, i.e. to homotopic maps $M^n_\alpha \to M^n$ we associate identical sets of homotopy classes. Let us prove that if two sets $B_{f_1}(M^n_\alpha)$ and $B_{f_2}(M^n_\alpha)$ are intersecting, then they coincide. Analogously to the proof of Lemma 6.2, let us consider the element

$$\alpha_0 \in B_{f_1}(M^n_\alpha) \cap B_{f_2}(M^n_\alpha)$$

and the two corresponding admissible maps

$$f_{\alpha_0,i} : S^{N+n} \to T_N(M^n), \quad i = 1, 2,$$

such that $f_{\alpha_0,1}/M^n_\alpha \to M^n$ and $f_{\alpha_0,2}/M^n_\alpha \to M^n$ have homotopy classes f_1, f_2.

Consider their t-regular homotopy

$$F : S^{N+n} \times I(0,1) \to T_N(M^n)$$

and the membrane

$$N^{n+1} = F^{-1}(M^n) \subset S^{N+n} \times I(0,1),$$

which retracts onto each of its boundaries. By analogy with Lemma 6.2, on the lower boundary we change the bundle map

$$\nu^N(M^n_\alpha) \to \nu^N(M^n),$$

keeping the map $f_{\alpha_0,1}/M^n_\alpha \to M^n$ fixed. We can extend this variation of a bundle map to a variation of the bundle map

$$\nu^N(N^{n+1}) \to \nu^N(M^n),$$

keeping it fixed on N^{n+1}, which can be achieved, starting from a retraction of the membrane to the boundary $M^n_\alpha \subset S^{N+n} \times 0$. Then, by means of a well-known method, we extend the map varied in a tubular neighborhood onto the whole Cartesian product $S^{N+n} \times I(0,1)$. According to Lemma 6.1, in this way we can get from α_0 any other element $\alpha_1 \in B_{f_1}(M^n_\alpha)$. Therefore

$$B_{f_1}(M^n_\alpha) \supset B_{f_2}(M^n_\alpha).$$

By symmetry,

$$B_{f_1}(M^n_\alpha) = B_{f_2}(M^n_\alpha).$$

The lemma is proved.

Lemma 6.4. *The group $\pi(M^n\alpha, SO_N)$ acts transitively on each set $B_f(M^n_\alpha)$.*

Proof. Suppose there exist two classes $\alpha_i \in B_f(M^n_\alpha)$, $i = 1, 2$, and representing them, admissible maps

$$f_{\alpha_i} : S^{N+n} \to T_N(M^n)$$

such that

$$f^{-1}_{\alpha_i}(M^n) = M^n_\alpha, \quad i = 1, 2,$$

and the maps $f_{\alpha_i}/M^n_\alpha \to M^n$ are homotopic. By means of the homotopy constructed in Lemma 6.1, we change the map f_{α_2} to an admissible map $f^{(1)}_{\alpha_2}$ that is homotopic to it and such that

$$f^{(1)}_{\alpha_2} = \frac{f_{\alpha_1}}{M^n_\alpha}.$$

Then the bundle maps $f^{(1)}_{\alpha_2}$ and $f_{\alpha_1} : \nu^N(M^n_\alpha) \to \nu^N(M^n)$ differ in each fiber D^N_x over a point $x \in M^n_\alpha$ by a discriminating orthogonal transformation $h_x \in SO_N$, which depends smoothly on the point $x \in M^n_\alpha$. This yields a smooth map

$$h : M^n_\alpha \to SO_N,$$

discriminating the maps $f^{(1)}_{\alpha_2}$ and f_{α_1} in a neighborhood $T(M^n_\alpha) \subset S^{N+n}$ of the manifold M^n_α. According to Lemma 6.1, if $h : M^n_\alpha \to SO_N$ is null-homotopic, then the elements α_1 and α_2 are equal to each other. Thus, the discriminator h is defined up to homotopy, and the map f_{α_1}, "twisted" in each fiber D^N_x over $x \in M^n_\alpha$ by $h_x \in SO_N$, coincides with $f^{(1)}_{\alpha_2}$. On the set of classes, $B_f(M^n_\alpha)$, there is a transitive action of the group $\pi(M^n_\alpha, SO_N)$. The lemma is proved.

These lemmas combine into the following:

Theorem 6.5. *On the set*

$$\overline{A}(M^n) \subset A(M^n) = H^{-1}\varphi[M^n] \subset \pi_{N+n}(T_N(M^n))$$

there is an action of the group $\pi(M^n_\alpha, SO_N)$*. On the set of orbits,*

$$\frac{\overline{A}(M^n)}{\pi(M^n, SO_N)}$$

there is an action of $\pi^+(M^n, M^n)$*. The elements of the orbit set*

$$B = \frac{\left[\frac{\overline{A}(M^n)}{\pi(M^n, SO_N)}\right]}{\pi^+(M^n, M^n)}$$

are in one-to-one correspondence with classes of manifolds $M^n_\alpha \geq M^n$, $M^n \geq M^n_\alpha$ *with respect to diffeomorphism of degree* $+1$ modulo $\theta^n(\partial\pi)$ *for odd* n, *and with respect to diffeomorphism of degree* $+1$ *for even* n.

PROOF. According to Lemmas 6.3 and 6.4, to the manifold M^n there corresponds a set

$$B(M^n_\alpha) = \cup_{f \in \pi^+(M^n_\alpha, M^n)} B_f(M^n_\alpha),$$

and for each set $B_f(M^n_\alpha)$ there is a transitive action of the group $\pi(M^n_\alpha, SO_n)$. However the groups $\pi(M^n_\alpha, SO_n)$ and $\pi(M^n, SO_n)$ are isomorphic, and if the homotopy class $f \in \pi^+(M^n_\alpha, M^n)$ is given, then we have the corresponding isomorphism

$$f^* : \pi(M^n, SO_N) \to \pi(M^n_\alpha, SO_N).$$

Thus, for each set $B_f(M^n_\alpha)$ we have a natural action of the group $\pi(M^n, SO_n)$; here

$$h(\alpha) = f^* h(\alpha), \quad \alpha \in B_f(M^n_\alpha), \quad h \in \pi(M^n, SO_N).$$

On the other hand, on the set of classes $f \in \pi^+(M^n_\alpha, M^n)$ we have a fixed-point free action of $\pi^+(M^n, M^n)$ (this action is transitive). Thus, on the quotient set $B(M^n_\alpha)/\pi(M^n, SO_N)$ we have a transitive action $\pi^+(M^n, M^n)$, i.e. the quotient set

$$\frac{\left[\frac{B(M^n_\alpha)}{\pi(M^n, SO_N)}\right]}{\pi^+(M^n, M^n)}$$

consists of one element. By using the actions of groups $\pi(M^n, SO_N)$ and $\pi^+(M^n, M^n)$ on each of the sets $B(M^n_\alpha)$ for all manifolds M^n_α where

$$M^n_\alpha \geq M^n, \quad M^n \geq M^n_\alpha,$$

we get an action of these groups on the set $\overline{A}(M^n)$, such that the quotient set with respect to the action of both groups is in the natural one-to-one corresponence with the set of manifolds, which are identified if and only if they belong at least once (thus, always, by Lemma 6.2) to the same class $\alpha \in A(M^n)$. Applying Theorem 5.2, we obtain the desired statement. The theorem is proved.

For subsequent applications it will be convenient to note the following:

Lemma 6.6. *To an automorphism of the SO_N-bundle*

$$h : \nu^N(M^n) \to \nu^N(M^n),$$

fixed on the base M^n (or equivalently, to

$$h : M^n \to SO_N),$$

there corresponds a map

$$Th : T_N(M^n) \to T_N(M^n);$$

for homotopic maps $h_i : M^n \to SO_N$, $i = 0, 1$, the corresponding maps Th_i are homotopic; moreover, while performing the homotopy process Th_t, $0 \le t \le 1$, the manifold $M^n \subset T_N(M^n)$ remains fixed, and the normal ball D^N_x, $x \in M^n \subset T_N(M^n)$ is deformed by means of maps $h_t(x) \in SO_N$, $0 \le t \le 1$. If $h \in \pi(M^n, SO_N)$ and $\alpha \in \pi_{N+n}(T_N(M^n))$, where $\alpha \in \overline{A}(M^n)$ then

$$h(\alpha) = Th_*(\alpha),$$

where $\pi(M^n, SO_N)$ acts on $\overline{A}(M^n)$ according to Theorem 6.5.

Proof. The definition of the map

$$T : \pi(M^n, SO_N) \to \pi(T_N(M^n), T_N(M^n))$$

follows easily from the definition of the Thom space for the bundle $\nu^N(M^n)$.

Let us prove the formula

$$h(\alpha) = Th_*(\alpha).$$

We recall how we defined the action of the group $\pi(M^n, SO_N)$ on the set $\overline{A}(M^n)$: suppose f_α is an admissible map $S^{N+n} \to T_N(M^n)$, $f_\alpha^{-1}(M^n) = M^n_\alpha$ and f_α / M^n_α has homotopy class $f \in \pi^+(M^n_\alpha, M^n)$. There is a naturally defined action of the group $\pi(M^n_\alpha, SO_N)$ and an isomorphism

$$\widetilde{f} : \pi(M^n_\alpha, SO_N) \to \pi(M^n, SO_N).$$

Let $h \in \pi(M^n, SO_N)$ and $\widetilde{f}^{-1}h \in \pi(M^n_\alpha, SO_N)$. Then for an element h there is a corresponding "twisting" of the bundle $\nu^N(M^n)$ in each fiber D^N_x by the element $h_x \in SO_N$, $x \in M^n$. To this twisting, in turn, we associate a twisting f_x^* in the fiber $D^N_{f_\alpha^{-1}(x)}$ by the same element $h_x \in SO_N$ at each point of $f_\alpha^{-1}(x)$. This defines the map

$$f_\alpha^* = \widetilde{f}^{-1} : \pi(M^n, SO_N) \to \pi(M^n_\alpha, SO_N).$$

One can define the action of the group $\pi(M^n_\alpha, SO_N)$ on the set $B_f(M^n_\alpha)$ only in such a way that it looks like $f_\alpha^*(h)$, since when passing to homotopy classes, there is no more distinction in definitions, because f_α/M^n_α is a homotopy equivalence, and $\widetilde{f}_\alpha = f_\alpha^{*-1}$ is an isomorphism.

The lemma is proved.

Lemma 6.7. *With each map $f : M^n \to M^n$ of degree $+1$ such that $f^*\nu^N(M^n) = \nu^N(M^n)$, there corresponds a non-empty set of maps*

$$(\overline{T}f) : T_N(M^n) \to T_N(M^n).$$

Two maps $\overline{T}_1, \overline{T}_2 \in (\overline{T}f)$ differ by an automorphism Th for some $h : M^n \to SO_N$.

For homotopic maps $f_1, f_2 : M^n \to M^n$, we have mod $T(\pi(M^n, SO_N))$-*homotopic maps $\overline{T}f_1$ and $\overline{T}f_2 : T_N(M^n) \to T_N(M^n)$.*

With the product $f_1 \circ f_2$ we associate the product

$$\overline{T}f_1 \circ \overline{T}f_2 = \overline{T}f_1 \circ f_2 \mod \operatorname{Im} T.$$

Suppose $f \in \pi^+(M^n, M^n)$ and $\alpha \in \overline{A}(M^n)/\pi(M^n, SO_N)$. Then

$$f(\alpha) = \overline{T}f_*(\alpha),$$

where

$$\overline{T}f_* : \pi_{N+n}(T_N(M^n)) \to \pi_{N+n}(T_N(M^n)).$$

The proof of this lemma is analogous to the Proof of Lemma 6.6, and it readily follows from the well-known definition of the action of $\pi^+(M^n, M^n)$ on $\pi^+(M^n_\alpha, M^n)$ and the dependence of $\alpha \in \overline{A}(M^n)/\pi(M^n, SO_N)$ from the map $M^n_\alpha \to M^n$ of degree $+1$ (an element of $\pi^+(M^n, M^n)$) (cf. Lemmas 6.1, 6.3, Theorem 6.5 and their proofs).

Now, let us consider the particular case when M^n is a π-manifold. In this case the bundle $\nu^N(M^n)$ is trivial. We define a frame field τ^N_x that is smoothly dependent on a point $x \in M^n$ and normal to M^n in $T_N(M^n)$. Following [15], we call the pair (τ^N, M^n), a "framed manifold". Then it is easy to see that for any element $\alpha \in \overline{A}(M^n)$ and any admissible map

$$f_\alpha : S^{N+n} \to T_N(M^n)$$

the manifold

$$M^n_\alpha = f_\alpha^{-1}(M^n)$$

obtains a natural framing $f_\alpha^* \tau^N$ and becomes a framed manifold.

In this case we have the following:

Lemma 6.8. *There is a one-to-one homomorphism*

$$T_0 : \pi^+(M^n, M^n) \to \pi(T_N(M^n), T_N(M^n))$$

such that for any $h \in \pi(M^n, SO_N)$, $f \in \pi^+(M^n, M^n)$ *the following relations hold*:

(a) $Th \cdot \overline{T}_0 f = \overline{T}_0 f \cdot T f^* h$, *where* $f^* \pi(M^n, SO_N) \to \pi(M^n, SO_N)$;
(b) $T_0 = \overline{T} \bmod \operatorname{Im} T$.

Proof. Let us construct the homomorphism $\overline{T}_0$; to do this, consider the automorphism

$$f : M^n \to M^n,$$

$f \in \pi^+(M^n, M^n)$, and cover it to get a map

$$\nu^N(M^n) \to \nu^N(M^n),$$

assuming that the vector with coordinates

$$(\lambda_1, \dots, \lambda_N) \in D^N_x, \quad x \in M^n,$$

defined by a frame τ^N_x in the fiber normal to a point x, is mapped to the vector with coordinates $(\lambda_1, \dots, \lambda_N)$ at the point $f(x)$. Since the field τ^N is smooth, we obtain a (smooth, if f is smooth) map

$$\nu^N(M^n) \to \nu^N(M^n),$$

which gives the desired map

$$\overline{T}_0 f : T_N(M^n) \to T_N(M^n).$$

We have proved point (a) of the lemma.

We consider a map $h : M^n \to SO_N$ and the composition

$$h \cdot \overline{T}_0 f : \nu^N \to \nu^N(M^n),$$

covering the map $f : (M^n) \to M^n$. The maps $h \cdot \overline{T}_0 f$ and $\overline{T}_0 f$ differ at each point $x \in M^n$ by $h_x \in SO_N$ and at each point $f^{-1}(x) \in M^n$ by

$$f^* h_{f^{-1}(x)} \in SO_N, \quad h_x = f^* h_{f^{-1}(x)}.$$

Thus

$$h \cdot \overline{T}_0 f = \overline{T}_0 f \cdot f^* h$$

(f^* is the automorphism $f^* : \pi(M^n, SO_N) \to \pi(M^n, SO_N)$ induced by f).

Further, we have:

$$Th \cdot \overline{T}_0 f = \overline{T}_0 f \cdot T f^* h.$$

Point (b) readily follows from the construction of $\overline{T}_0$.

The lemma is proved.

We consider the set $\pi^+(M^n_\alpha, M^n)$ defined above. It admits a left action of the group $\pi^+(M^n, M^n)$ and a right action of the group $\pi^+(M^n_\alpha, M^n_\alpha)$, where

$$M^n_\alpha \geq M^n, \quad M^n \geq M^n_\alpha.$$

In other words, for every

$$f \in \pi^+(M^n, M^n), \quad g \in \pi^+(M^n_\alpha, M^n), \quad f_1 \in \pi^+(M^n_\alpha, M^n_\alpha)$$

there is a well-defined composition

$$f \cdot g \cdot f_1 \in \pi^+(M^n_\alpha, M^n).$$

Moreover, for every $f \in \pi^+(M^n, M^n)$, $g \in \pi^+(M^n_\alpha, M^n)$ the following formula holds:

$$f \cdot g = g \cdot (g^* f),$$

where $g^* : \pi^+(M^n, M^n) \to \pi^+(M^n_\alpha, M^n_\alpha)$ is an isomorphism defined by the element $g \in \pi^+(M^n_\alpha, M^n)$.

We introduce the following notation: by

$$D^+(M^n_\alpha) \subset \pi^+(M^n_\alpha, M^n_\alpha)$$

we denote the subgroup consisting of all those homotopy classes of maps for which there is a representative

$$h : M^n_\alpha \to M^n_\alpha,$$

being a diffeomorphism by

$$\widetilde{D}^+ \subset \pi^+(M^n_\alpha, M^n_\alpha)$$

we denote the analogous subgroup, where a certain representative

$$\widetilde{h} : M^n_\alpha \to M^n_\alpha$$

is a diffeomorphism everywhere except a spherical neighborhood of one point, and the obstruction to an extension of the diffeomorphism to that point belongs to the group $\theta^n(\partial\pi)$. In view of the canonical Smale isomorphism $\theta^n = \Gamma^n$, one may assume that $\theta^n(\partial\pi) \subset \Gamma^n$ for $n \geq 5$. Denote by

$$\Lambda^n(M^n_\alpha) \subset \theta^n(\partial\pi)$$

the subgroup such that for every element $\gamma \in \Lambda^n(M^n_\alpha)$ of it there exists a map

$$\widetilde{h}_\gamma : M^n_\alpha \to M^n_\alpha,$$

homotopic to the identity, and being a diffeomorphism everywhere except a spherical neighborhood of one point, and the obstruction to an extension of the diffeomorphism at this point is equal to γ.

Theorem 6.9. *The group $D^+(M^n_\alpha)$ is a normal subgroup of $\widetilde{D}^+(M^n_\alpha)$. The quotient group $\widetilde{D}^+(M^n_\alpha)/D^+(M^n_\alpha)$ is isomorphically embedded into $\theta^n(\partial\pi)/\Lambda^n(M^n_\alpha)$. If n is even, then $D^+(M^n_\alpha) = \widetilde{D}^+(M^n_\alpha)$; if n is odd then the quotient group $\widetilde{D}^+(M^n_\alpha)/D^+(M^n_\alpha)$ is a finite cyclic group.*

Proof. With a representative $\widetilde{h} : M^n_\alpha \to M^n_\alpha$ of an element from $\widetilde{D}^+$ we associate the obstruction for extension of the diffeomorphism to the point. It is easy to see that the non-uniqueness of the obstruction belongs to the group $\Lambda^n(M^n_\alpha)$, and the group $D^+(M^n_\alpha)$ is mapped to zero. Thus the embedding

$$\frac{\widetilde{D}^+(M^n_\alpha)}{D^+(M^n_\alpha)} \subset \frac{\theta^n(\partial\pi)}{\Lambda^n(M^n_\alpha)}$$

is constructed. The remaining statements follow from the results of [6], [8] concerning the groups $\theta^n(\partial\pi)$. The theorem is proved.

Theorem 6.10. *The element $g^*f \in \pi^+(M^n_\alpha, M^n_\alpha)$ belongs to the subgroup $\widetilde{D}^+(M^n_\alpha)$, if and only if $\overline{T}f_*(\alpha) = \alpha$, where $\alpha \in \overline{A}(M^n)/\pi(M^n, SO_N)$.*

We note certain consequences of Theorems 6.9 and 6.10. If $M^n_\alpha = M^n$ then $g^*f = gfg^{-1}$, thus, Theorem 6.10 yields

Lemma 6.11. *The subgroup $\widetilde{D}^+(M^n_\alpha)$ is normal in $\pi^+(M^n, M^n)$; the quotient group $\pi^+(M^n, M^n)/\widetilde{D}^+(M^n)$ is finite (it is not known whether it is abelian or not).*

Lemma 6.12. *The group $D^+(M^n)$ is of finite index in $\pi^+(M^n, M^n)$.*

Proof of Theorem 6.10. By definition, the manifold M^n_α is constructed as follows: a map

$$f_\alpha : S^{N+n} \to T_N(M^n),$$

representing an element $\widetilde{\alpha}$ from the class α; this map is admissible if $f_\alpha^{-1}(M^n) \geq M^n$ and $M^n \geq f_\alpha^{-1}(M^n)$, where $f_\alpha^{-1}(M^n)$ is the inverse image of M^n under the map satisfying Lemma 3.2. Then we set

$$M^n_\alpha = f_\alpha^{-1}(M^n).$$

Suppose $f_\alpha/M^n_\alpha \to M^n$ is of the homotopy class $g \in \pi^+(M^n_\alpha, M^n)$, and let f be an element of the group $\pi^+(M^n, M^n)$ such that

$$\overline{T}f_*(\widetilde{\alpha}) \equiv \widetilde{\alpha} \bmod \operatorname{Im} T.$$

Since all our objects are defined up to a degree +1 diffeomorphism, the fact that g^*f is homotopic to a diffeomorphism of degree +1 implies that the sets

$$B_g(M^n_\alpha) \subset \overline{A}(M^n)$$

and

$$B_{g\cdot g^* f}(M^n_\alpha) = B_{f\cdot g}(M^n_\alpha)$$

are identical, which yields one of the statements of the theorem. Now, let us show that if

$$\overline{T} f_*(\alpha) = \alpha, \quad \alpha \in \frac{\overline{A}(M^n)}{\pi(M^n, SO_N)},$$

then the map $g^* f$ is homotopic to a diffeomorphism (of degree $+1$). We split the proof into several steps.

Step 1. We consider homotopic admissible maps $f^{(')}_\alpha$ and $f^{('')}_\alpha : S^{N+n} \to T_N(M^n)$ such that

(a) $f^{(')-1}_\alpha(M^n) = f^{('')-1}_\alpha(M^n) = M^n_\alpha$,
(b) $f^{(')}_\alpha/M^n_\alpha = g$, $f^{('')}_\alpha/M^n_\alpha = g \cdot g^* f = f \cdot g$.

We construct a homotopy $F : S^{N+n} \times I(0,1) \to T_N(M^n)$, which is t-regular and such that $F/S^{N+n} \times 0 = f^{(')}_\alpha$.

Step 2. We define the membrane $N^{n+1} = F^{-1}(M^n) \subset S^{N+n} \times I$; it is evident that

$$F^* \nu^N(M^n) = \nu^N(N^{n+1})$$

and

$$\partial N^{n+1} = M^n_\alpha \cup (-M^n_\alpha).$$

By using Morse surgery, we kill the groups

$$\pi_1(N^{n+1}), \quad \operatorname{Ker} F^{(H_2)}_*, \dots, \operatorname{Ker} F^{(H_i)}_*, \quad i < \left[\frac{n}{2}\right],$$

and simultaneously take to the “new membrane” N^{n+1} the map F and the “framing” (analogously to §§ 4 and 5). Thus, we may assume that

$$\pi_1(N^{n+1}) = 0$$

and

$$\operatorname{Ker} F^{(H_i)}_* = 0, \quad i < \left[\frac{n}{2}\right].$$

Step 3. *Case 1.* If $n+1$ is odd, then, following § 4, we reconstruct the group $\operatorname{Ker} F^{(H_{[n/2]})}_*$. Then (see § 5, Case 1) we shall have a membrane which is diffeomorphic to $M^n_\alpha \times I(0,1)$, according to Smale [19]. The theorem is proved.

Case 2. If $n+1$ is even ($n+1 = 4k+2$ or $n+1 = 4k$), then it is necessary to use the fact that the boundary components of the manifold N^{n+1} are already diffeomorphic. Then, analogously to Cases 2 and 3 from § 5, we have to construct the membranes $\overline{M}^{n+1}(B)$ and $\overline{M}^{n+1}(\varphi)$, in order to kill the obstructs to Morse surgery, and then consider the unions

$$\overline{N}^{n+1}(B) = N^{n+1} \cup_{f_0} D^n \times I(0,1) \cup_{f_1} \overline{M}^{n+1}(B),$$

$$N^{n+1}(\varphi) = N^{n+1} \cup_{f_0} D^n \times I(0,1) \cup_{f_1} \overline{M}^{n+1}(\varphi),$$

as in § 5, Cases 2 and 3 (here B is the intersection matrix for the membrane N^{n+1} and φ is the Kervaire invariant). The maps

$$F : N^{n+1} \to M^n$$

define, in a natural way, the maps

$$F(B) : \overline{N}^{n+1}(B) \to M^n$$

and

$$F(\varphi) : \overline{N}^{n+1}(\varphi) \to M^n$$

in such a way that

$$F(B)^* \nu^N(M^n) = \nu^N(N^{n+1}(B))$$

and

$$F(\varphi)^* \nu^N(M^n) = \nu^N(\overline{N}^{n+1}(\varphi)).$$

It is easy to see that

$$\partial \overline{N}^{n+1}(B) = [M^n_\alpha \# \widetilde{S}^n(B)] \cup (-M^n_\alpha)$$

and

$$\partial \overline{N}^{n+1}(\varphi) = [M^n_\alpha \# \widetilde{S}^n(\varphi)] \cup (-M^n_\alpha).$$

We reconstruct by a Morse surgery the manifolds $\overline{N}^{n+1}(B)$ and $\overline{N}^{n+1}(\varphi)$; the resulting manifolds $\overline{\overline{N}}^{n+1}(B)$ and $\overline{\overline{N}}^{n+1}(\varphi)$ will define a J-equivalence (diffeomorphism) of degree $+1$ between manifolds M^n_α and $M^n_\alpha \# \widetilde{S}^n(B)$, M^n_α and $M^n_\alpha \# \widetilde{S}^n(\varphi)$, where $\widetilde{S}^n(B),\ \widetilde{S}^n(\varphi) \in \theta^n(\partial\pi)$. Denote the maps $F(B)$, $F(\varphi)$, reconstructed to the membranes $\overline{\overline{N}}^{n+1}(B)$ and $\overline{\overline{N}}^{n+1}(\varphi)$, by $\overline{F}(B)$, $\overline{F}(\varphi)$. Moreover, $\overline{\overline{N}}(B)$ is diffeomorphic to $M^n_\alpha \times I$ ($n = 4k-1$), $\overline{\overline{N}}(\varphi)$ is diffeomorphic to $M^n_\alpha \times I$ ($n = 4k+1$) and $\overline{F}(B) = F/M^n_\alpha \times 1$ ($n = 4k-1$), $\overline{F}(\varphi) = F/M^n_\alpha \times 1$ ($n = 4k+1$).

The map

$$\overline{F}(B) : M^n_\alpha \times 0 \to M^n$$

is homotopic to the composition

$$F_1(B)g(B) : M^n_\alpha \xrightarrow{g(B)} M^n_\alpha \# \widetilde{S}^n(B) \xrightarrow{F_1(B)} M^n, \quad n = 4k-1,$$

and the map

$$F(\varphi) : M^n_\alpha \times 0 \to M^n$$

is homotopic to the composition

$$M^n_\alpha \xrightarrow{g(\varphi)} M^n_\alpha \# \widetilde{S}^n(\varphi) \xrightarrow{F_1(\varphi)} M^n, \quad n = 4k+1,$$

where $g(B)$ and $g(\varphi)$ are diffeomorphisms of degree $+1$, induced by some direct product decomposition

$$\overline{\overline{N}}(B) = M^n_\alpha \times I, \quad \overline{\overline{N}}(\varphi) = M^n_\alpha \times I.$$

The maps $F_1(B)$ and $F_1(\varphi)$ are homotopic to the maps $F/M^n_\alpha \times 1$ ($n = 4k-1$ and $n = k+1$), respectively, which yields the desired statement.[1] The theorem is proved.

Chapter II

Processing the results

§ 7. The Thom space of a normal bundle. Its homotopy structure

In order to understand and apply the results of §§ 1–6, we shall study the homotopy structure of the Thom complex $T_N(M^n)$, where M^n is a simply connected manifold, $n \geq 4$.

In the manifold M^n, we select the $n-2$-frame K^{n-2} in such a way that

$$H_i(K^{n-2}) = H_i(M^n), \quad i < n.$$

[1] It remains to add that the diffeomorphism $g(B) : M^n_\alpha \to M^n_\alpha \# \widetilde{S}^n(B)$ must be thought of as a diffeomorphism modulo point: $M^n_\alpha \to M^n_\alpha$. An analogous statement holds for $g(\varphi)$.

Then the manifold $M^n \setminus x_0$, $x_0 \in M^n$ is contractible to K^{n-2}. The embedding $K^{n-2} \subset M^n$ induces the bundle $j^*\nu^N(M^n)$ on K^{n-2}; we denote the Thom space of this bundle by T_N^{n-2}. There is a natural embedding $T_N^{n-2} \subset T^N(M^n)$. Analogously, one can select frames of lower dimension:

$$K^0 = K^1 \subset K^2 \subset \cdots \subset K^{n-2}$$

and construct the Thom complexes

$$T_N^0 = S^N \subset T_N^2 \subset \cdots \subset T_N^{n-2}.$$

We may think that the complex T_N^i is the $N+i$-dimensional skeleton of the complex $T_N(M^n)$, $i = 0, 2, \ldots, n-2$.

Lemma 7.1. *The Thom complex $T_N(M^n)$ is homotopically equivalent to the wedge $S^{N+n} \vee T_N^{n-2}$.*

Proof. Lemma 7.1 is an immediate consequence of Lemma 3.1 on the sphericity of the cycle

$$\varphi[M^n] \in H_{N+n}(T_N(M^n)).$$

We consider the group $\pi_n(M^n)$ and select the subgroup $\widetilde{\pi}_n(M^n) \subset \pi_n(M^n)$ consisting of those elements $\gamma \in \widetilde{\pi}_n(M^n)$ such that $H(\gamma) = 0$. In the group $\widetilde{\pi}_n(M^n)$ we select the even smaller subgroup $\pi_n^\nu(M^n)$, consisting of elements $\gamma \in \pi_n^\nu(M^n)$ such that, for any map $g_\gamma : S^n \to M^n$, representing the element γ, the bundle $g_\gamma^*\nu^N(M^n)$ over the sphere S^n, is trivial.

Now let L^i be an arbitrary i-dimensional complex, over which a vector SO_N-bundle ν^N is given. Denote the Thom complex of this bundle by $T_N(\nu^N)$. Suppose $\gamma \in \pi_n(L^i)$, and the bundle $\gamma^*\nu^N$ over the sphere S^n is trivial. We shall say that $\gamma \in \pi_n(L^i, \nu^N)$. For $L^i = M^n$ and $\nu^N = \nu^N(M^n)$ we have already defined such a group. Clearly, there is a well-defined epimorphism

$$\pi_n(K^{n-2}, \nu^N(M^n)) \to \pi_n^\nu(M^n).$$

There is a well-defined embedding $\kappa : S^N \subset T_N(\nu^N)$, corresponding to the embedding of the point $x_0 = L^0 \subset L^i$. We have:

Lemma 7.2. *There is a well-defined natural homomorphism*

$$T^N : \pi_n(L^i, \nu^N) \to \frac{\pi_{n+N}(T_N(\nu^N))}{\operatorname{Im} \kappa}. \tag{29}$$

If there exist two bundles ν_1^N, ν_2^N over complexes L_1^i, L_2^i, respectively, and a map $F : \nu_1^N \to \nu_2^N$ is given, then there is a well-defined map

$$T(F) : T_N(\nu_1^N) \to T_N(\nu_2^N)$$

such that the diagram

$$\begin{array}{ccc} \pi_n(L_1^{i_1}, \nu_1^N) & \xrightarrow{\widetilde{F}_*} & \pi_n(L_2^{i_2}, \nu_2^N) \\ \Big\downarrow{\scriptstyle T^N} & & \Big\downarrow{\scriptstyle T^N} \\ \dfrac{\pi_{n+N}(T_N(\nu_1^N))}{\operatorname{Im}\kappa} & \xrightarrow{T(F)_*} & \dfrac{\pi_{n+N}(T_N(\nu_2^N))}{\operatorname{Im}\kappa} \end{array} \tag{30}$$

is commutative.

Proof. It is easy to see that the bundle map F corresponds to a map

$$\widetilde{F}_* : \pi_n(L_1^{i_1}, \nu_1^N) \to \pi_n(L_2^{i_2}, \nu_2^N).$$

Namely, let the map F defined on the bases spaces $L_1^{i_1} \to L_2^{i_2}$ be denoted by $\widetilde{F}$. Then, clearly,

$$\widetilde{F}_*(\pi_n(L_1^{i_1}, \nu_1^N)) \subset \pi_n(L_2^{i_2}, \nu_2^N)$$

by the definition of the induced bundle. In this way the upper row of the diagram is constructed. We shall denote the constructed natural map

$$\pi_n(L_1^{i_1}, \nu_1^N) \to \pi_n(L_2^{i_2}, \nu_2^N)$$

by $\widetilde{F}_*$. The construction of the lower row is evident. Now, let us construct the homomorphisms T^N. For this sake, consider the element $\gamma_s \in \pi_n(L_s^{i_s}, \nu_s^N)$, $s = 1, 2$, and consider the map

$$\widetilde{\gamma}_s : S^n \to L_s^{i_s},$$

representing γ_s. The bundle $\widetilde{\gamma}_s^* \nu_s^N$ over S^n is trivial. Thus the maps

$$\mu : S^{N+n} \to T_N(S^n, \widetilde{\gamma}_s \nu_s^N),$$

$$T\widetilde{\gamma}_s : T_N(S^n, \widetilde{\gamma}_s \nu_s^N) \to T_N(L_s^{i_s}, \nu_s^N),$$

are well-defined, where $T\widetilde{\gamma}_s$ is a natural map between Thom complexes corresponding to the bundle map $\widetilde{\gamma}_s \nu_s^N \to \nu_s^N$, and the map μ is such that

$$\mu_*[S^{N+n}] = \varphi[S^n],$$

where $\varphi : H_n(S^n) \to H_{n+N}(T_N(S^n, \widetilde{\gamma}_s^*, \nu_s^N))$ is the Thom isomorphism. The cycle $\varphi[S^n]$ is spherical by Lemma 3.1 since the sphere is a π-manifold. By Lemma 7.1, the space $T_N(S^n, \widetilde{\gamma}_s^*, \nu_s^N)$ is homotopically equivalent to

$S^{N+n} \vee S^N$, thus the homotopy class of μ is well-defined mod $\pi_{n+N}(S^N) = \operatorname{Im}\kappa_*$. Then the composition

$$T\widetilde{\gamma}_s\mu : S^{N+n} \to T_N(L_s^{i_s}, \nu_s^N)$$

determines the element to be denoted $T : T(\gamma_s)$; this element is well defined modulo mod $\operatorname{Im}\kappa_*$. After we have given the definition, its naturality (commutativity of the diagram in Lemma 7.2) is evident.

The lemma is proved.

Proof. We call the groups $\pi_n(L^i, \nu^N)$ the homotopy groups of the bundle ν^N, and we call T^N the suspension homomorphism. This notation agrees with the following lemma.

Lemma 7.3. *If the bundle ν^N over the complex L^i is trivial then*:

(a) $\pi_n(L^i, \nu^N) = \pi_n(L^i)$ *for all* n;
(b) *the space $T_N(L^i, \nu^N)$ is homotopically equivalent to the wedge $S^N \vee E^N L^i$, where E^N is the N-multiple suspension*;
(c) *the homomorphism T^N coincides with the N-th iteration of the suspension homomorphism*

$$E^N : \pi_n(L^i) \to \pi_{n+N}(E^N L^i) = \frac{\pi_{n+N}(T_N(L^i, \nu^N))}{\operatorname{Im}\kappa_*}$$

for $N > n+1$.

Proof. The Thom space of the trivial bundle for closed balls D^N, $\nu^N = L^i \times D^N$ is, obviously, homotopically equivalent to the suspension for $N > 1$:

$$\begin{aligned} T_N(L^i, \nu^N) &= \frac{L^i \times D^N}{L^i \times \partial D^N} \\ &= ET_{N-1}(L^i, \nu^{N-1}) = E\left(\frac{L^i \times D^{N-1}}{L^i \times \partial D^{N-1}}\right). \end{aligned}$$

Furthermore, for $N = 1$ we have:

$$T_1(L^i, \nu^1) = \frac{L^i \times I(0,1)}{L^i \times \partial I(0,1)} = E(L^i \cup x_0),$$

where $L^i \cup x_0$ denotes the union of L^i with x_0. Since the space $E(L^i \cup x_0)$ is homotopically equivalent to $S^1 \vee EL^i$, we see that the space $T_N(L^i, \nu^N)$ is homotopically equivalent to the suspension

$$E^{N-1}(S^1 \vee EL^i) = S^N \vee E^N L^i.$$

The second part of the lemma follows trivially from the definition of a suspension homomorphism and is actually a definition of it. The lemma is proved.

Suppose M^n is a smooth simply connected oriented manifold, let $\nu^N(M^n)$ be its normal bundle

$$T_N(M^n) = T_N(M^n, \nu^N(M^n)),$$

$j : K^{n-2} \subset M^n$ be its $n-2$-skeleton

$$\pi_n^\nu(M^n) = \frac{\pi_n(K^{n-2}, j^*\nu^N(M^n))}{\operatorname{Ker} j_*}.$$

According to Lemma 7.1,

$$\pi_{n+N}(T_N(M^n)) = Z + \pi_{n+N}(T_N^{n-2}). \tag{31}$$

The generator of the group $Z = \pi_{n+N}(S^{N+n})$ depends on the wedge decomposition

$$T_N(M^n) = S^{N+n} \vee T_N^{n-2}.$$

We shall choose this decomposition in such a way that the generator of the direct summand $Z = \pi_{n+N}(S^{N+n})$ is a generator constructed in the proof of Lemma 3.1. Denote this generator by

$$1_{N+n} \in \pi_{n+N}(S^{N+n}) \subset T_N(M^n).$$

We have the following:

Lemma 7.4. *For any element $\gamma \in \pi_n^\nu(M^n)$ there exists a degree $+1$ map $g_\gamma : M^n \to M^n$ such that:*

(a) $g_\gamma^*\nu^N(M^n) = \nu^N(M^n)$,
(b) g_γ *fixes the frame K^{n-2},*
(c) *the discriminator between g_γ and the identical map is nonzero only on one simplex $\sigma^n \subset M^n$, and it is equal to $\gamma \in \pi_n^\nu(M^n)$ on this simplex.*

PROOF. We consider the identical map and change it on a simplex $\sigma^n \subset M^n$ by the element $\gamma \in \pi_n^\nu(M^n)$. We denote the resulting map by g_γ since the degree of $\widetilde{\gamma} : S^n \to M^n$ representing γ is zero by definition of the group, $\pi_n^\nu(M^n)$, the degree of $g_\gamma : M^n \to M^n$ is $+1$. Consider the bundles $g_\nu^*\nu^N(M^n)$ and $\nu^N(M^n)$, which we identify, as usual, with homotopy classes of $\nu : M^n \to B_{SO_N}$ (for the bundle $\nu^N(M^n)$) and the map $\nu \cdot g_\gamma : M^n \to M^n \to B_{SO_N}$ (for the bundle $g_\gamma^*\nu^N(M^n)$). The discriminator between ν and $\nu \cdot g_\gamma$ is supported on the same simplex $\sigma^n \subset M^n$, as the supporter of g_γ and the identical map, and it is easy to see that it is equal to the element

$$\nu_*(\gamma) \in \pi_n(B_{SO_N}), \quad \nu_* : \pi_n(M^n) \to \pi_n(B_{SO_N}).$$

The bundle $\gamma^*\nu^N(M^n)$ over S^n is trivial by definition of the group $\pi_n^\nu(M^n)$; it is defined by the composition

$$\nu \cdot \widetilde{\gamma} : S^n \to M^n \to B_{SO_N};$$

its triviality is equivalent to the condition

$$\nu_*(\gamma) = 0.$$

Therefore the discriminator between $\nu : M^n \to B_{SO_N}$ and $\nu \cdot g_\gamma : M^n \to B_{SO_N}$ is zero, and these maps are homotopic. The lemma is proved.

Lemma 7.4 yields:

Lemma 7.5. *There is a well-defined homomorphism $g_* : \pi_n^\nu(M^n) \to \pi^+(M^n, M^n)$ which is a map onto the set of all elements of $\pi^+(M^n, M^n)$ whose representatives fix the frame $K^{n-2} \subset M^n$.*

Proof. The map g_* has already been constructed in Lemma 7.4; namely, with an element $\gamma \in \pi_n^\nu(M^n)$ we associate the homotopy class of the map $g_\gamma : M^n \to M^n$. The fact that it is a homeomorphism is evident. Let us calculate the image

$$\operatorname{Im} g_* \subset \pi^+(M^n, M^n).$$

We consider any map $f : M^n \to M^n$ of degree $+1$ representing some element of the group $\pi^+(M^n, M^n)$ and fixed on the skeleton K^{n-2}.

The discriminator between it and the identity map is the cocycle

$$\lambda(f) \in H^n(M^n, \pi_n(M^n)),$$

where one can assume that the cochain $\lambda(f)$ is nonzero only on one simplex $\sigma^n \in M^n$. Then

$$\lambda(f)[\sigma^n] \in \pi_n(M^n).$$

Since the map f is of degree $+1$, the degree of the map of $S^n \to M^n$, representing the element

$$\lambda(f)[\sigma^n] \in \pi_n(M^n),$$

is zero. Since

$$f^*\nu^N(M^n) = \nu^N(M^n),$$

the discriminator between the maps

$$\nu : M^n \to B_{SO_N}$$

and

$$\nu \cdot f : M^n \to B_{SO_N},$$

defining the bundles $\nu^n(M^n)$ and $f^*\nu^N(M^n)$ is equal to

$$\nu_*\lambda(f)[\sigma^n] \in \pi_n(B_{SO_N})$$

and

$$\nu_*\lambda(f)[\sigma^n] = 0,$$

since $f^*\nu^N(M^n) = \nu^N(M^n)$. Therefore

$$\lambda(f)[\sigma^n] \in \pi_n^\nu(M^n).$$

The lemma is proved.

We recall that in § 6 we defined a map

$$\overline{T} : \pi^+(M^n, M^n) \to \pi(T_N(M^n), T_N(M^n)),$$

homeomorphic and single-valued modulo the action of the group $\pi(M^n, SO_N)$, i.e. modulo the image of the homomorphism

$$T : \pi(M^n, SO_N) \to \pi(T_N(M^n), T_N(M^n)).$$

Lemma 7.6. *The formula*

$$\overline{T} g_*\gamma(1_{N+n} + \alpha) \equiv 1_{N+n} + \alpha + T^N\gamma \ (\operatorname{mod} \operatorname{Im} T \cup \operatorname{Im} \kappa_*) \tag{32}$$

is valid for all $\gamma \in \pi_n^\nu(M^n)$, where 1_{N+n} is the generator chosen above and α is an element of the group $\pi_{N+n}(T_N^{n-2}) \subset \pi_{N+n}(T_N(M^n))$.

PROOF. The map $g_*\gamma$ fixes K^{n-2}, thus $\overline{T}g_*\gamma$ can be chosen to be fixed on $T_N^{n-2} \subset T_N(M^n)$. Therefore, the map

$$\overline{T}g_*\gamma : T_N(M^n) \to T_N(M^n)$$

is completely defined by the map

$$\frac{\overline{T}g_*\gamma}{S^{N+n}} \to T_N(M^n)$$

and

$$[\overline{T}g_*\gamma] \cdot (\alpha) = \alpha$$

for all

$$\alpha \in \pi_{N+n}(T_N^{n-2}) \subset \pi_{N+n}(T_N(M^n)).$$

Let us investigate the image $[\overline{T}g_*\gamma]_*(1_{N+n})$. The discriminator between g_γ and $1 : M^n \to M^n$ is supported on the simplex $\sigma^n \subset M^n$ and it is equal to γ; the complex $M^n \diagdown \sigma^n$ contracts onto K^{n-2}. Therefore, the discriminator between

$$\overline{T}g_\gamma : T_N(M^n) \to T_N(M^n)$$

and the identical map

$$1 : T_N(M^n) \to T_N(M^n)$$

can initially be regarded as maps from the Thom complex $T_N(S^N, \nu^N)$ (ν^N is the trivial bundle) to the Thom complex $T_N(M^N)$, where on $S^N \subset T_N(S^n, \nu^N)$ the maps are homotopic (equal). Therefore the discriminator of the maps $\overline{T}g_*\gamma$ and 1 is $T^N\gamma$ by definition of the homomorphism T^N. The non-uniqueness in the formula from Lemma 7.6 arises because of non-uniqueness in the definition of the homomorphisms T^N and $\overline{T}$. The lemma is proved.

Remark 7.7. For π-manifolds the definition of the homomorphism $\overline{T}^N$ coincides with E^N and is therefore unique; the homomorphism $\overline{T}$ in this case is also well defined according to Lemma 6.8, and the formula from Lemma 7.6 has the meaning of an exact equality, not a congruence.

We shall not prove the assertion made in the remark since we shall not make use of it.

§ 8. Obstructions to a diffeomorphism of manifolds having the same homotopy type and a stable normal bundle

Let us consider the filtration

$$T_N(M^n) \supset T_N^{n-2} \supset \cdots \supset T_N^2 \supset S^N,$$

where T_N^i is the Thom space of the i-dimensional frame K^i of the manifold M^n in minimal cell decomposition (the number of i-cells is equal to $\max rkH^i(M^n, K)$ over all fields K). We denote the numbers $\max rkH^i(M^n, K)$ by $b^i_{\max}$. By $T_N^{(i,j)}$ we denote

$$T_N^{(i,j)} = \frac{T_N^i}{T_N^j} \quad j < i.$$

In particular,

$$T_N^{(i,i-j)} = \bigvee_{k=1}^{b^i_{\max}} S_k^{N+i}.$$

Clearly,

$$H_{N+i}(T_N^i, T_N^{i-1}) = H_{N+i}\left(\bigvee_{k=1}^{b^i_{\max}} S_k^{N+i}\right) = \underbrace{Z + \cdots + Z}_{b^i_{\max}\text{ factors}}.$$

The homomorphism

$$\partial : H_{N+i}(T_N^i, T_N^{i-1}) \to H_{N+i-1}(T_N^{i-1}) \to H_{N+i-1}(T_N^{i-1}, T_N^{i-2})$$

defines a boundary operator in the complex $T_N(M^n)$ together with its homology and cohomology groups. We shall have in mind precisely this interpretation of boundary homomorphism.

Proof of the Obstruction to a Diffeomorphism. We shall identify the manifolds $M^n_\alpha \geq M^n$, $M^N \geq M^n_\alpha$ modulo $\theta^n(\partial\pi)$ with orbits of the groups $\pi(M^N, SO_N)$ and $\pi^+(M^n, M^n)$ in the set $\overline{A}(M^n)$ according to the results of §§ 1–6. With M^n_α we associate the orbit $B(M^n_\alpha) \subset \overline{A}(M^n)$. Suppose we are given two manifolds M^n_α and M^n_β, $\alpha \in B(M^n_\alpha), \beta \in B(M^n_\beta)$.

According to Lemma 7.1 the elements α, β are of the form:

$$\alpha = 1_{N+n} + \overline{\alpha}, \quad \overline{\alpha} \in \pi_{N+n}(T_N^{n-2}),$$
$$\beta = 1_{N+n} + \overline{\beta}, \quad \overline{\beta} \in \pi_{N+n}(T_N^{n-2}).$$

The exact sequences (for the pairs $T_N^i, T_N^j, j < i$)

$$\cdots \to \pi_{N+n}(T_N^j) \to \pi_{N+n}(T_N^i) \to \pi_{N+n}(T_N^{i,j}) \xrightarrow{\partial} \pi_{N+n-1}(T_N^j) \to \cdots \quad (33)$$

are defined.

In particular, we have:

$$\begin{aligned}
&\cdots \to \pi_{N+n}(S^N) \xrightarrow{j_{0,2}} \pi_{N+n}(T_N^2) \xrightarrow{\Lambda_2} \pi_{N+n}\left(\bigvee_{k_2=1}^{b_{\max}^2} S_{k_2}^{N+2}\right) \to \cdots \\
&\cdots \to \pi_{N+n}(T_N^i) \xrightarrow{j_{i,i+1}} \pi_{N+n}(T_N^{i+1}) \xrightarrow{\lambda_{i+1}} \pi_{N+n}\left(\bigvee_{k_{i+1}=1}^{b_{\max}^{i+1}} S_{k_{i+1}}^{N+i+1}\right) \to \cdots \\
&\cdots \to \pi_{N+n}(T_N^{n-3}) \xrightarrow{j_{n-3,n-2}} \pi_{N+n}(T_N^{n-2}) \xrightarrow{\Lambda_{n-2}} \pi_{N+n}\left(\bigvee_{k_{n-2}=1}^{b_{\max}^{n-2}} S_{k_{n-2}}^{N+n-2}\right) \dots .
\end{aligned} \quad (34)$$

Consider the difference $\overline{\alpha} - \overline{\beta} \in \pi_{N+n}(T_N^{n-2})$. Then we have:

$$\Lambda_{n-2}(\overline{\alpha} - \overline{\beta}) \in \sum_{k_{n-2}=1}^{b_{\max}^{n-2}} \pi_{N+n}(S_{k_{n-2}}^{N+n-2}).$$

Thus with each sphere $S_{k_{n-2}}^{N+n-2}$ we associate an element $d_{n-2}(\overline{\alpha}, \overline{\beta}$ $k_{n-2}) \in \pi_{N+n}(S_{k_{n-2}}^{N+n-2})$ (the direct summand of the element $\Lambda_{n-2}(\overline{\alpha} - \overline{\beta})$ corresponding to the number k_{n-2}). The spheres $S_{k_{n-2}}^{N+n-2}$ are in a natural one-to-one correspondence with cells of dimension $N+n-2$ of the complex $T_N(M^n)$, and, consequently, with cells of dimension $n-2$ of the complex M^n. Therefore $d_{n-2}(\overline{\alpha}, \overline{\beta}, k_{n-2})$ (under variation of k_{n-2}) runs over the chain $d_{n-2}(\overline{\alpha}, \overline{\beta})$ of the complex $T_N(M^n)$ valued in $\pi_{N+n}(S^{N+n-2})$. If the chain $d_{n-2}(\overline{\alpha}, \overline{\beta}) = 0$, we put:

$$d_{n-3}(\overline{\alpha}, \overline{\beta}, k_{n-3}) = \Lambda_{n-3} \cdot j^{-1}_{n-3,n-2}(\overline{\alpha} - \overline{\beta}) \quad \text{(on the sphere } S^{N+n-3}_{k_{n-3}});$$

if $d_{n-(i-1)}(\overline{\alpha}, \overline{\beta}) = 0$, then we set:

$$d_{n-i}(\overline{\alpha}, \overline{\beta}) = \Lambda_{n-i} \cdot j^{-1}_{n-i,n-(i-1)} \cdot \dots \cdot j^{-1}_{n-3,n-2}(\overline{\alpha} - \overline{\beta})$$

(on the sphere $S^{N+n-i}_{k_{n-i}}$ the value of the chain $d_{n-i}(\overline{\alpha}, \overline{\beta})$ is equal to the corresponding direct summand of the element $\Lambda_{n-i} \cdot j^{-1}_{n-i,n-(i-1)} \cdot \dots \cdot j^{-1}_{n-3,n-2}(\overline{\alpha} - \overline{\beta})$).

Clearly, the chain $d_{n-1}(\overline{\alpha}, \overline{\beta})$ is ambiguously defined up to

$$\Lambda_{n-i} \cdot \operatorname{Ker}(j_{n-3,n-2} \cdot \dots \cdot j_{n-i,n-(i-1)}) = Q_{n-i}.$$

Lemma 8.1. *The chain $d_{n-i}(\overline{\alpha}, \overline{\beta})$ is well defined if $d_{n-j}(\overline{\alpha}, \overline{\beta}) = 0$, $j < i$, and this chain is a cycle with coefficients in $\pi_{N+n}(S^{N+n-i})$.*

Proof. Let us prove that $d_{n-i}(\overline{\alpha}, \overline{\beta})$ is a cycle. According to the definition of a boundary operator in our complex $T_N(M^n)$ for the selected decomposition (cf. above) it suffices to consider some element

$$j^{-1}_{n-i,n-(i-1)} \cdot \dots \cdot j^{-1}_{n-3,n-2}(\overline{\alpha} - \overline{\beta}) \in \pi_{N+n}(T^{n-i}_N)$$

and the boundary homomorphism

$$\partial : H_{N+n-i}(T^{n-i,n-i-1}_N) \to H_{N+n-i-1}(T^{n-i-1,n-i-2}_N).$$

Consider the homomorphisms

$$\begin{array}{ccccc} \overline{\partial} : \pi_{N+n}(T^{n-i,n-i-1}_N) & \xrightarrow{\partial} & \pi_{N+n-1}(T^{n-i-1}_N) & \to & \pi_{N+n-1}(T^{n-i-1,n-i-2}_N) \\ & & \wr\wr & & \wr\wr \\ & \displaystyle\sum_{k_{n-i}} \pi_{n+N}(S^{N+n-i}_{k_{n-i}}) & \xrightarrow{\overline{\partial}} & \displaystyle\sum_{k_{n-i-1}} \pi_{n+N-1}(S^{N+n-i-1}_{k_{n-i-1}}). & \end{array}$$

Then we consider the chain $d_{n-i}(\overline{\alpha}, \overline{\beta})$. Since

$$d_{n-i}(\overline{\alpha}, \overline{\beta}) = \Lambda_{n-i} \cdot j^{-1}_{n-i,n-(i-1)} \cdot \dots \cdot j^{-1}_{n-3,n-2}(\overline{\alpha} - \overline{\beta})$$

it follows from the exact sequences on page 112 that $\operatorname{Im} \Lambda_{n-i} \subset \operatorname{Ker} \overline{\partial}$, and hence

$$\overline{\partial} d_{n-i}(\overline{\alpha}, \overline{\beta}) = 0.$$

The lemma is proved.

In this way,

$$d_{n-i}(\overline{\alpha},\overline{\beta}) \in H_{N+n-i}(T_N(M^n),\pi_{N+n}(S^{N+n-i})),$$

or, by the Thom isomorphism φ, we obtain the element

$$\widetilde{d}_{n-i}(\overline{\alpha},\overline{\beta}) = \varphi^{-1}d_{n-i}(\overline{\alpha},\overline{\beta}) \in H_{n-i}(M^n,\pi_{N+n}(S^{N+n-i})),$$

defined with a large degree of ambiguity.

PROOF OF THE MINIMAL DISCRIMINATOR. We commence to vary arbitrarily the elements $\alpha \in B(M^n_\alpha)$ and $\beta \in B(M^n_\beta)$ within the sets $B(M^n_\alpha)$ and $B(M^n_\beta)$, corresponding to the manifolds M^n_α and M^n_β in such a way that the difference

$$\overline{\alpha} - \overline{\beta} \in \pi_{N+n}(T_N^{n-2})$$

belongs to

$$\operatorname{Im} j_{n-3,n-2}\cdot\dots\cdot j_{n-i,n-(i-1)}(\pi_{N+n}(T_N^{n-i}))$$

for

$$i = \max_{\alpha,\beta} i[\alpha \in B(M^n_\alpha),\beta \in B(M^n_\beta)]$$

and only then we define the ("minimal") discriminator

$$d_{n-i}(M^n_\alpha, M^n_\beta) = d_{n-i}(\overline{\alpha}_0,\overline{\beta}_0),$$

where $\alpha_0 \in B(M^n_\alpha)$ and $\beta_0 \in B(M^n_\beta)$ are such elements that the difference $\overline{\alpha}_0 - \overline{\beta}_0$ belongs to

$$\operatorname{Im} j_{n-3,n-2}\cdot\dots\cdot j_{n-i,n-i+1}$$

for i maximal possible. It is evident that:

(1) the homology class $d_{n-i}(M^n_\alpha, M^n_\beta)$ is defined with ambiguity:
(2) its ambiguity has two reasons:
   (a) generally speaking, the non-triviality of the group

$$\operatorname{Ker}(j_{n-3,n-2}\cdot\dots\cdot j_{n-i,n-i+1})$$

   and
   (b) the ambiguity in the choice of elements α_0,β_0 in orbits $B(M^n_\alpha)$ and $B(M^n_\beta)$.

We shall explain the situation more precisely in the Appendices at the end of the paper by analyzing examples.

§ 9. Variation of a smooth structure keeping triangulation preserved

We start by recalling the results of Milnor, Smale, Kervaire (see [4, 6, 8, 9, 10, 17, 18]). Milnor [8] defined a group of smooth structures on a sphere of dimension n, denoted by θ^n, and introduced in it the filtration

$$\theta^n \supset \theta^n(\pi) \supset \theta^n(\partial\pi).$$

Any element of the group θ^n is a smooth oriented manifold having the homotopy type of sphere. It was shown that:

(1) $\theta^n/\theta^n(\pi) = 0,\ n \neq 8k+1,\ 8k+2,\ k \geq 2,$[1] $\theta^n/\theta^n(\pi) = Z_2$ or 0 for $\pi = 8k+1, 8k+2,\ k \geq 2$;

(2) there is an inclusion homomorphism

$$\frac{\theta^n(\pi)}{\theta^n(\partial\pi)} \subset \frac{\pi_{N+n}(S^N)}{J\pi_n(SO_N)},$$

is an epimorphism for $n \neq 4k+2$ and for $n = 10$;

(3) for $n = 4k+2$ the subgroup $\theta^n(\pi)/\theta^n(\partial\pi)$ has index 2 or 1 in the group $\pi_{N+n}(S^N)/J\pi_n(SO_N)$, moreover, for $n = 2, 6, 14$ it has index 2;

(4) the group $\theta^n(\partial\pi)$ is trivial for even n and for $n \leq 6\,(n \neq 3)$, $n = 13$; the group $\theta^{2k+1}(\partial\pi)$ is always cyclic; for even k it contains at most two elements and $\theta^9(\partial\pi) = Z_2$, and for odd k the order of this group grows rapidly, and it is nontrivial for $k = 2s-1$, $s \geq 2(\theta^7(\partial\pi) = Z_{28}$, $\theta^{11}(\partial\pi) = Z_{992,\dots})$.

As already stated before, an element of the group θ^n, $n \geq 5$ is a smooth oriented manifold of homotopy type S^n, the inverse element is the same manifold with the opposite orientation, and the group operation is the “connected sum” of oriented manifolds (see [10]), which makes sense, generally speaking, for arbitrary manifolds (however the connected sum of topological spheres is a topological sphere). We shall denote the elements of θ^n by $\widetilde{S}^n_i$, thus determining their topological structure. Our first goal is the study of the connected sum $M^n\#\widetilde{S}^n$, where M^n is an arbitrary simply connected manifold $n \geq 5$. Evidently, the manifolds M^n and $M^n\#\widetilde{S}^n$ are homeomorphic for $n \geq 5$, moreover, they are then combinatorially-equivalent (cf. (17)), though possibly, they are not diffeomorphic if the smooth structure on the sphere $\widetilde{S}^n$ is not standard (if $\widetilde{S}^n \neq 0$ in the group θ^n).

[1] Adams [36] showed that $\theta^n/\theta^n(\pi) = 0$ for all n.

In the sequel, we shall denote the stable group $\pi_{N+n}(S^n)$ by $G(n)$ for $N > n+1$. According to Milnor,

$$\frac{\theta^n(\pi)}{\theta^n(\partial\pi)} \subset \frac{\pi_{N+n}(S^n)}{\operatorname{Im} J},$$

to each element $\widetilde{S}^n \in \theta^n(\pi)$ there corresponds a set $\widetilde{B}(\widetilde{S}^n) \subset G(n)$ such that

$$\widetilde{B}(\widetilde{S}^n_1 \# \widetilde{S}^n_2) = \widetilde{B}(\widetilde{S}^n_1) + \widetilde{B}(\widetilde{S}^n_2)$$

and

$$\widetilde{B}(\widetilde{S}^n) = \operatorname{Im} J,$$

if $\widetilde{S}^n \in \theta^n(\partial\pi)$. We recall that in the preceding sections, with every manifold $M^n_1 \geq M^n,\ M^n \geq M^n_1$ we canonically associated the sets

$$B(M^n_1) \subset \overline{A}(M^n) \subset A(M^n) \subset \pi_{N+n}(T_N(M^n)).$$

In addition, there is a natural embedding

$$\kappa : S^N \subset T_N(M^n),$$

where $S^N = T^0_N$ (cf. § 6).

This leads to a homomorphism

$$\kappa_* : G(n) \to \pi_{N+n}(T_N(M^n)).$$

We have the following:

Lemma 9.1. $B(M^n_1 \# \widetilde{S}^n) = B(M^n_1) + \kappa_* \widetilde{B}(\widetilde{S}^n)$.

Proof. Let us show that

$$B(M^n_1 \# \widetilde{S}^n) \supset B(M^n_1) + \kappa_* \widetilde{B}(\widetilde{S}^n).$$

Suppose $\alpha \in B(M^n_1)$, $\gamma \in \widetilde{B}(\widetilde{S}^n)$ and

$$f_\alpha : S^{N+n} \to T_N(M^n), \quad f_\gamma : S^{N+n} \to S^N$$

are some maps representing the elements α and γ, respectively, which are t-regular in the sense of Pontrjagin–Thom, where

$$f^{-1}_\alpha(M^n) = M^n_1$$

and

$$f^{-1}_\gamma(x_0) = \widetilde{S}^n, \quad x_0 \in S^N.$$

We assume that the sphere S^N lies in the Thom complex $T_N(M^n)$ in the standard manner and that

$$f_\gamma : S^{N+n} \to T_N(M^n), \quad f_\gamma(S^{N+n}) \in \kappa S^N, \quad f_\gamma^{-1}(M^n) = f_\gamma^{-1}(x_0).$$

Then there is a well-defined "mapping connected sum" (cf. [8, 10, 15])

$$f_{\alpha+\gamma} : S^{N+n} \to T_N(M^n)$$

such that

$$f_{\alpha+\gamma}^{-1}(M^n) = M_1^n \# \widetilde{S}^n$$

and the map $f_{\alpha+\gamma}$, represents, by definition, the element $\alpha + \kappa_* \gamma$. Let us show that

$$B(M_1^n \# \widetilde{S}^n) \subset B(M_1^n) + \kappa_* \widetilde{B}(\widetilde{S}^n).$$

Suppose $\beta \in B(M_1^n \# \widetilde{S}^n)$ and the map

$$f_\beta : S^{N+n} \to T_N(M^n)$$

represents the element β, satisfies Lemma 3.2 and is such that

$$f_\beta^{-1}(M^n) = M_1^n \# \widetilde{S}^n \subset S^{N+n}.$$

By definition of the connected sum #, in M_1^n there exists a sphere $S_0^{n-1} \subset M_1^n \# \widetilde{S}^n$ such that

$$(M_1^n \# \widetilde{S}^n) \setminus S_0^{n-1} = (M_1^n \setminus D_\varepsilon^n) \cup (\widetilde{S}^n \setminus D_\varepsilon^n),$$

where $D_\varepsilon^n \subset M_1^n$ and $D_\varepsilon^n \subset \widetilde{S}^n$ are balls of radius ε, given in some local coordinate system by a canonical equation, and $\varepsilon > 0$ is a small number. Since $\widetilde{S}^n$ is a π-manifold ($\widetilde{S}^n \subset \theta^n(\pi)$), it follows that every frame field τ^N, that is normal to $\widetilde{S}^n \subset S^{N+n}$ and is defined everywhere except $D_\varepsilon^n \subset S^{N+n}$, can be extended onto the ball D_ε^n. We deform smoothly the map f_β to a map

$$\widetilde{f}_\beta : S^{N+n} \to T_N(M^n),$$

such that

$$\widetilde{f}_\beta^{-1}(x_0) \supset \widetilde{S}^n \setminus D_\varepsilon^n \subset M_1^n \# \widetilde{S}^n, \quad x_0 \in M^n \subset T_N(M^n)$$

(the map $\widetilde{f}_\beta$ is assumed to be t-regular). We consider a frame $\tau^N_{x_*}$ that is normal to the manifold $M^n \subset T_N(M^n)$ at x_0. The inverse image of the frame under a t-regular map $\widetilde{f}_\beta$ (cf. [15, 22]) generates a frame field

$$\tau^N = \widetilde{f}_\beta^{-1}(\tau^N_{x_*}),$$

that is normal to $\widetilde{S}^N \diagdown D^n_\varepsilon$ in S^{N+n}. We now "cut" the manifold $M^n_1 \# \widetilde{S}^n$ along the sphere S^{n-1}_0 into two parts and extend the frame field τ^N from the sphere

$$S^{n-1}_0 = (\widetilde{S}^n \diagdown D^n_\varepsilon) \cap (M^n_1 \diagdown D^n_\varepsilon)$$

to the ball D^n_ε. More rigorously, we consider the membrane

$$B^{n+1}(h) = (M^n_1 \# \widetilde{S}^n) \times I\left(0, \frac{1}{2}\right) \cup_h D^n_\varepsilon \times D^1,$$

where

$$h : \partial D^n_\varepsilon \times D^1 \to S^{n-1}_0 \times D^1 \subset M^n_1 \# \widetilde{S}^n,$$
$$h(x, y) = (x, y).$$

Clearly,

$$\partial B^{n+1}(h) = (M^n_1 \# \widetilde{S}^n) \cup (-M^n_1 \cup -\widetilde{S}^n).$$

Further, as in § 1, we embed in the usual way the membrane $B^{n+1}(h)$ to the direct product $S^{N+n} \times I(0, 1)$, where

$$B^{n+1}(h) \cap S^{N+n} \times 0 = M^n_1 \# \widetilde{S}^n,$$

and extend the map $f_\beta / S^{N+n} \times 0$ to the map

$$F : S^{N+n} \times I \to T_N(M^n),$$

where

$$F^{-1}(M^n) = B^{n+1}(h),$$

making use of the possibility to extend the field τ^N from the sphere $S^{n-1}_0 \subset S^{N+n} \times 0$ to the ball $D^n_\varepsilon \subset S^{N+n} \times I(0, 1)$. This extension can be chosen in such a way that

$$F^{-1}(M^n) \cap S^{N+n} \times 1 = \widetilde{S}^n \cup M_1^n, \quad \widetilde{S}^n \subset F^{-1}(x_0).$$

Since

$$F^{-1}(M^n) \cap S^{N+n} \times 1 = \widetilde{S}^n \cup M_1^n,$$

it follows that the map $F/S^{N+n} \times 1$ is decomposed into a sum of maps $f_\beta^{(')}$ and $f_\beta^{('')}$, representing elements of type $\beta_1 \in B(M_1^n)$ and $\beta_2 \in \kappa_* \widetilde{B}(\widetilde{S}^n)$, respectively.

Thus, it is established that

$$B(M_1^n \# \widetilde{S}^n) \supset B(M_1^n) + \kappa_* \widetilde{B}(\widetilde{S}^n),$$
$$B(M_1^n \# \widetilde{S}^n) \subset B(M_1^n) + \kappa_* \widetilde{B}(\widetilde{S}^n).$$

The lemma is proved.

We now investigate a more complicated operation for the variation of a smooth structure. Suppose the manifold M^n is $k-1$-connected, where $k \leq [\frac{n}{2}]$. Clearly,

$$H_k(M^n) = \pi_k(M^n).$$

We consider an element $z \in H_k(M^n)$ and a smooth sphere $S^k \subset M^n$ realizing it. The tubular neighborhood $T(S^k) \subset M^n$ of the sphere represents the SO_{n-k}-bundle of balls D^{n-k} over the sphere S^k. Assume this bundle is trivial. Consider a map

$$g : S^k \to \operatorname{diff} S^{n-k-1},$$

taking the whole sphere S^k into one point $g(S^k) \in \operatorname{diff} S^{n-k-1}$ (we note that according to [23], [17], [8], there exists a natural isomorphism $\operatorname{diff} S^{n-k-1}/j \operatorname{diff} D^{n-k} \approx \theta^{n-k}$, $n-k \neq 3, 4$). Therefore to the map g there corresponds a smooth sphere $\widetilde{S}^{n-k}(g) \in \theta^{n-k}$. We shall consider only those maps

$$g : S^k \to \operatorname{diff} S^{n-k-1},$$

for which $\widetilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$.

Consider the automorphism[1]

$$\widetilde{g} : \partial T(S^k) \to \partial T(S^k),$$

induced by the map

$$g(S^k) : S^{n-k-1} \to S^{n-k-1}.$$

Namely, in each fiber of the bundle of $(n-k-1)$-dimensional spheres $\partial T(S^k)$ over S^k we define an automorphism $g(S^k)$. We set

$$M^n(S^k, g) = (M^n \diagdown T(S^k)) \cup_{\widetilde{g}} T(S^k).$$

From the paper [17] and the fact that $\widetilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$ we get the following:

Lemma 9.2. *The manifolds M^n and $M^n(S^k, g)$ are combinatorially-equivalent. The combinatorial equivalence*

$$f(g) : M^n(S^k, g) \to M^n$$

can be chosen in such a way that:

(a) $f(g)^* \nu^N(M^n) = \nu^N(M^n(S^k, g)$,
(b) $f(g) \diagup M^n(S^k, g) \diagdown T(S^k)$ *is the identity*,
(c) $f(g) \diagup S^k$ *is the identity*,
(d) $f(g)/T(S^k) \subset M^n(S^k, g)$ *fiberwise.*

PROOF. The diffeomorphism $g(S^k) : \partial D^{n-k} \to \partial D^{n-k}$ extends to a combinatorial equivalence $G : D^{n-k} \to D^{n-k}$, which is a diffeomorphism everywhere except the origin $O \in D^{n-k}$. Let us define a map

$$f(g) : M^n(S^k, g) \to M^n$$

as follows:

$$\begin{aligned} f(g) &= 1 \quad \text{on } M^n(S^k, g) \diagdown T(S^k) = M^n \diagdown T(S^k), \\ f(g) &= 1 \quad \text{on } S^k \subset M^n(S^k, g), \end{aligned}$$

$f(g) = G$ on the fiber D_x^{n-k} at each point $x \in S^k$, where by 1 we denote the identity map.

For such a map $f(g)$, the properties (b)–(d) are evident. To prove (a), it is necessary to make use of the fact that $\widetilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$. Namely, it

[1]Here we assume that the tube $T(S^k)$ is endowed with a coordinate system, i.e. a normal field of $n-k$ frames on S^k.

turns out that the discriminator between the "classifying" maps $\nu_1 \cdot f(g)$ and ν_2 in

$$M^n(S^k, g) \xrightarrow{f(g)} M^n \xrightarrow{\nu_1} B_{SO_N},$$

$$M^n(S^k, g) \xrightarrow{\nu_2} B_{SO_N}$$

of the bundles $f(g)^*\nu^N(M^n)$ and $\nu^N(M^n(S^k, g))$, respectively, is valued in the group

$$H^{n-k}\left(M^n(S^k, g), \frac{\theta^{n-k}}{\theta^{n-k}(\pi)}\right),$$

where

$$\frac{\theta^{n-k}}{\theta^{n-k}(\pi)} \subset \pi_{n-k-1}(SO_N) = \pi_{n-k}(B_{SO_N})$$

(cf. [8]), and if this discriminator is equal to zero then the maps $\nu_1 \cdot f(g)$ and ν_2 are homotopic. Moreover, if $\widetilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$, then the discriminator is zero. From the definition of $f(g)$ it immediately follows that the discriminator is

$$z(g) \in H^{n-k}(M^n(S^k, g), \pi_{n-k}(B_{SO_N}))$$

and the fact that it vanishes is sufficient for $\nu_1 \cdot f(g)$ and ν_2 to be homotopic. The element $z(g)$ is represented by a cocycle $\overline{z}(g)$, having the same value on each fiber D^{n-k}_x, $x \in S^k \subset M^n(S^k, g)$. This value (on a given fiber D^{n-k}_x) is by definition (cf. [8]) an element of the group $\pi_{n-k}(B_{SO_N})$ defining the normal bundle of the smooth sphere $S^{n-k}(g)$, i.e. an element of the group $\theta^n/\theta^n(\pi)$ that is equal to zero if $\widetilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$.

Thus all assertions of the lemma are proved.

Now let $M^n = S^k \times S^{n-k}$. In this case we get the following:

Lemma 9.3. *The manifold $M^n(S^k, g)$ is of degree $+1$ diffeomorphic to the manifold $S^k \times \widetilde{S}^{n-k}(g)$.*

Proof. Clearly,

$$M^n(S^k, g) = (S^k \times D^{n-k}) \cup_{\widetilde{g}} (S^k \times D^{n-k}).$$

The diffeomorphism

$$\widetilde{g} : S^k \times S^{n-k-1} \to S^k \times S^{n-k-1},$$

constructed above is such that

$$\widetilde{g}(x, y) = (x, g(S^k)y).$$

At the same time the diffeomorphism of

$$S^{n-k}(g) = D^{n-k} \cup_{g(S^k)} D^{n-k}, \quad g(S^k) : S^{n-k-1} \to S^{n-k-1},$$

holds by definition. Thus the diffeomorphism $\widetilde{g}$ is a fiberwise diffeomorphism that introduces a new direct product structure on $S^k \times S^{n-k-1}$. As a result of pasting

$$M^n(S^k, g) = S^k \times D^{n-k} \cup_{\widetilde{g}} S^k \times D^{n-k}$$

we obtain the direct product

$$S^k \times \left(D^{n-k} \cup_{g(S^k)} D^{n-k}\right) = S^k \times \widetilde{S}^{n-k}(g).$$

The lemma is proved.

We now define the "sum of manifolds along a cycle" operation. Suppose M_1^n and M_2^n are manifolds and $S_i^k \subset M_i^n$, $i = 1, 2$, are smoothly located in k-dimensional spheres, having trivial normal bundles $\nu^{n-k}(S_i^k, M_i^n)$, $i = 1, 2$, In the tubular neighborhoods,

$$T(S_i^k) \subset M_i^n, \quad i = 1, 2,$$

we introduce the direct product coordinates

$$T(S_i^k) = S_i^k \times D_\varepsilon^{n-k},$$

by using geodesic ε-balls D_ε^{n-k}, which are normal to the spheres $S_i^k \subset M_i^n$ in a certain Riemannian metric. Then we put

$$[M_1^n \diagdown T(S_1^k)] \cup_h [M_2^n \diagdown T(S_2^k)] = M^n(S_1^k, S_2^k, h),$$

where

$$h : S_1^k \times D_\varepsilon^{n-k} \to S_2^k \times D_\varepsilon^{n-k},$$
$$h(x, y) = (x, h_x(y)), \quad h_x \in SO_{n-k},$$
$$d(h) : S_1^k \to SO_{n-k}.$$

Lemma 9.4. *If $k < [\frac{n}{2}]$ and $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$, then the manifold $M^n(S_1^k, S_2^k, h)$ depends only on the homotopy classes α_i of embeddings of $S_i^k \subset M_i^n$, $i = 1, 2$, and the homotopy class $\widetilde{d}$ of mapping $d(h) : S_1^k \to SO_{n-k}$.*

Proof. If two spheres $S^k_{i,1}, S^k_{i,2}$, $i = 1, 2$, are smoothly situated in a manifold M^n_i and are homotopic for $k < [\frac{n}{2}]$ then they are diffeotopic. From this fact and results of [16] it follows that two embeddings

$$f_{i,j} : S^k_{i,j} \times D^{n-k}_\varepsilon \to M^n_i, \quad i, j = 1, 2,$$

are defined up to diffeotopy by the pair $(\alpha_i, \widetilde{d}_i)$, where $\alpha_i \in \pi_k(M^n_i)$ and $d_i \in \pi_k(SO_{n-k})$. From the fact that $M^n(S^k_1, S^k_2, h)$ is defined by diffeotopy classes of embeddings

$$f_{i,j} : S^k_{i,j} \times D^{n-k}_\varepsilon \to M^n_i, \quad i, j = 1, 2,$$

it immediately follows that it depends only on the quadruple

$$(\alpha, \widetilde{d}_1, \alpha_2, \widetilde{d}_2) \quad \alpha_i \in \pi_k(M^n_i), \quad \widetilde{d}_i \in \pi_k(SO_{n-k}).$$

Clearly, the quadruples $(\alpha, \widetilde{d}_1, \alpha_2, \widetilde{d}_2)$ and $(\alpha_1, 0, \alpha_2, \widetilde{d}_2 - \widetilde{d}_1)$ define the same manifold. The lemma is proved.

Below we will denote the manifold $M^n(S^k_1, S^k_2, h)$ by $M^n(\alpha_1, \alpha_2, \widetilde{d})$, where $\alpha_i \in \pi_k(M^n_i)$, $i = 1, 2$, and $\widetilde{d}_i \in \pi_k(SO_{n-k})$.

Remark. According to our definitions the bundles $\nu^{n-k}(S^k_i, M^n_i)$ must be trivial; as a result, for $2k < n$ we have $\alpha_i \in \pi_k(M^n, \nu^N(M^n))$ (cf. § 7).

The following lemma is a consequence of the definition of a connected sum along a cycle and Lemma 9.3.

Lemma 9.5. *Let $M^n_1 = S^k \times \widetilde{S}^{n-k}(g)$ and let M^n_2 be a $k-1$-connected manifold, $\alpha \in \pi_k(M^n_2, \nu^N(M^n_2))$, $\beta \in \pi_k(M^n_1)$, $\widetilde{d} \in \pi_k(SO_{n-k})$, where β is the generator. Then the manifold $M^n(\alpha, \beta, d)$ is diffeomorphic with degree $+1$ to the manifold $M^n_2(\alpha, g)$ $(\operatorname{mod} \theta^n)$ for any element $\widetilde{d} \in \pi_k(SO_{n-k})$.*

Proof. The element $\widetilde{d} \in \pi_k(SO_{n-k})$ defines a diffeomorphism

$$h(\widetilde{d}) : S^k \times D^{n-k} \to S^k \times D^{n-k}$$

such that

$$h(\widetilde{d})(x, y) = (x, h(\widetilde{d})_x y), \quad h(\widetilde{d})_x \in SO_{n-k},$$

where $h(\widetilde{d}) : S^k \to SO_{n-k}$ is a representative of $\widetilde{d}$. The diffeomorphism $h(\widetilde{d})$ is extended to a diffeomorphism

$$\overline{h}(\widetilde{d}) : S^k \times \widetilde{S}^{n-k}(g) \to S^k \times \widetilde{S}^{n-k}(g)$$

(everywhere except a point), since

$$S^k \times \widetilde{S}^{n-k}(g) = (S^k \times D^{n-k}) \cup_{\widetilde{g}} (S^k \times D^{n-k}),$$

where $\widetilde{S}^{n-k}(g) \in \theta^{n-k}$. Therefore the result of the gluing

$$M^n(\alpha, \beta, \widetilde{d}) = (M_1^n \diagdown S^k \times D^{n-k}) \cup_{h(\widetilde{d})} (M_2^n \diagdown S^k \times D^{n-k})$$

does not depend (up to an element from θ^n) on the diffeomorphism $h(\widetilde{d})$. But if we put $\widetilde{d} = 0$, then the equality

$$M^n(\alpha, \beta, 0) = M_2^n(\alpha, g)$$

is a tautology. The lemma is proved.

We now examine the Thom complex $T_N(S^k \times S^{n-k})$ and the subset

$$A(S^k \times S^{n-k}) \subset \pi_{N+n}(T_N(S^k \times S^{n-k})).$$

The manifold $S^k \times \widetilde{S}^{n-k}(g)$ is a π-manifold, if $\widetilde{S}^{n-k}(g) \in \theta^{n-k}(\pi)$, and is combinatorially equivalent to the manifold $S^k \times S^{n-k}$. There is therefore (cf. §§ 1–6) defined by the subset

$$B(S^k \times \widetilde{S}^{n-k}(g)) \subset A(S^k \times S^{n-k}).$$

In addition, with the smooth sphere $\widetilde{S}^{n-k}(g)$ one associates the subset

$$\widetilde{B}(\widetilde{S}^{n-k}(g)) \subset G(n-k), \quad k < n-k.$$

Lemma 9.6. *The Thom complex $T_N(S^k \times S^{n-k})$ is homotopically equivalent to*

$$S^{N+n} \vee S^{N+n-k} \vee S^{N+k} \vee S^N.$$

The group

$$\pi_{N+n}(T_N(S^k \times S^{n-k}))$$

is isomorphic to the direct sum

$$Z + G(k) + G(n-k) + G(n).$$

The set $A(S^k \times S^{n-k})$ consists of all elements of the form

$$1_{N+n} + \gamma, \quad 1_{N+n} \in Z, \quad \gamma \in G(k) + G(n-k) + G(n),$$

where the element $1_{N+n} + 0 \in B(S^k \times S^{n-k})$.

The direct decomposition

$$\pi_{N+n}(T_N(S^k \times S^{n-k})) = Z + G(k) + G(n-k) + G(n)$$

can be chosen in such a way that:

(a) $G(n) = \operatorname{Im} \kappa_*$;
(b) the subgroup $G(n-k)$ belongs to the image of the inclusion homomorphism $f_* : \pi_{N+n}(T_N^k) \to \pi_{N+n}(T_N(S^k \times S^{n-k}))$,

$$f : T_N^k \subset T_N(S^k \times S^{n-k})$$

is the embedding constructed in § 7, and $T_N^k = S^{N+k} \vee S^N$; the subgroup $G(n-k)$ is defined uniquely $\operatorname{mod} G(n)$;
(c) $B(S^k \times \widetilde{S}^{n-k}(g) \# \theta^n(\pi)) \supset 1_{N+n} + j_* \widetilde{B}(\widetilde{S}^{n-k}(g)) \operatorname{mod} \operatorname{Im} \kappa_*$, where $j : T_N^k \subset T_N(S^k \times S^{n-k})$ is the embedding.

Proof. The decomposition of the Thom space into a wedge union of spheres follows from

$$E(S^i \times S^i) = S^{i+1} \vee S^{j+1} \vee S^{i+j+1},$$

and Lemma 7.3. All assertions of the lemma, except the last one, are trivial and follow immediately from the natural decomposition of the Thom complex into a wedge union of spheres. Furthermore, from Lemma 9.1 it follows that

$$B(S^k \times \widetilde{S}^{n-k}(g) \# \widetilde{S}^n) = B(S^k \times \widetilde{S}^{n-k}(g)) + \kappa_* \widetilde{B}(\widetilde{S}^n),$$

where $\widetilde{S}^n \in \theta^n(\pi)$. Therefore, for the proof of the lemma, it is sufficient to show that

$$B(S^k \times \widetilde{S}^{n-k}(g)) \supset 1_{N+n} + j_* \widetilde{B}(\widetilde{S}^{n-k}(g)) \operatorname{mod} \operatorname{Im} \kappa_*.$$

We consider the “auxiliary Thom complex”

$$T_N(S^k) = S^{N+k} \vee S^N \subset T_N(S^k \times S^{n-k}), \quad T_N^k = T_N(S^k), \quad k < n-k.$$

We also consider a map

$$f : S^k \times \widetilde{S}^{n-k}(g) \to S^k,$$

where

$$f(x,y) = x, \quad x \in S^k, \quad y \in \widetilde{S}^{n-k}(g).$$

We extend the map f to a map

$$\widetilde{F} : S^k \times \widetilde{S}^{n-k}(g) \times D^n \to S^k \times D^N,$$

by putting $\widetilde{F} = f \times 1$. We extend the map $\widetilde{F}$ to a map $F : S^{N+n} \to T_N(S^k)$ in the usual way, so that

$$\frac{F}{T(S^k \times \widetilde{S}^{n-k}(g))} = \widetilde{F},$$

since the usual tubular neighborhood $T(S^k \times \widetilde{S}^{n-k}(g)) \subset S^{N+n}$ is diffeomorphic to $S^k \times \widetilde{S}^{n-k}(g) \times D^N$ by virtue of the fact that $S^k \times \widetilde{S}^{n-k}(g)$ is a π-manifold. The map $\widetilde{F}$ factors into a composition of maps

$$\widetilde{F} = 1 \circ \widetilde{F} : S^k \times \widetilde{S}^{n-k}(g) \times D^N \to S^k \times D^N \to S^k \times D^N,$$

where $\widetilde{F}^{-1}(x_0) = \widetilde{S}^{n-k}(g),\ x_0 \in S^k$, and the maps are t-regular. Therefore the induced map

$$F : S^{N+n} \to T_N(S^k)$$

factors into a composition of maps

$$F = F_2 \circ F_1 : S^{N+n} \to S^{N+k} \to T_N(S^k),$$

where $F_2^{-1}(S^k) = S^k$, $F_2/S^k = 1$ and $F_1^{-1}(x_0) = \widetilde{S}^{n-k}(g),\ x_0 \in S^k$.

By definition (cf. Lemma 3.1), the map F_2 represents a generating element of the group

$$\pi_{N+k}(S^{N+k}) \subset \pi_{N+k}(T_N^k) = \pi_{N+k}(T_N(S^k)) = \pi_{N+k}(S^{N+k} \vee S^N).$$

The map F_1 represents an arbitrary element of the set

$$\widetilde{B}(\widetilde{S}^{n-k}(g)) \subset \pi_{N+n}(S^{N+n-k}) = G(n-k).$$

We now consider the sum

$$1_{N+n} + j_* \widetilde{B}(\widetilde{S}^{n-k}(g)) \subset \pi_{N+n}(T_N(S^k \times S^{n-k})).$$

Let the map

$$g : S^{N+n} \to T_N(S^k \times S^{n-k})$$

represent the element

$$1_{N+n} \in \pi_{N+n}(S^{N+n}) \subset \pi_{N+n}(T_N(S^k \times S^{n-k}))$$

and let the map

$$F : S^{N+n} \to T_N^k \subset T_N(S^k \times S^{n-k})$$

represent an element of the set $j_*\widetilde{B}(\widetilde{S}^{n-k}(g))$ (the map F is constructed above). We consider the "sum" of maps

$$(g+F) : S^{N+n} \to T_N(S^k \times S^{n-k}),$$

where

$$\begin{aligned}(g+F)^{-1}(S^k \times S^{n-k}) &= g^{-1}(S^k \times S^{n-k}) \cup F^{-1}(S^k \times S^{n-k}) \\ &= S^k \times S^{n-k} \cup S^k \times \widetilde{S}^{n-k}(g).\end{aligned}$$

We consider the product $S^k \times D_\varepsilon^{n-k} \times I(0,1)$ and form the membrane $B^{n+1} \subset S^{n+N} \times I(0,1)$. We have:

$$B^{n+1} = [S^k \times S^{n-k} \cup S^k \times \widetilde{S}^{n-k}(g)] \times I\left(0, \frac{1}{2}\right) \cup_{h_1,h_2} S^k \times D_\varepsilon^{n-k} \times I(0,1),$$

where

$$\begin{aligned}h_1 &: S^k \times D_\varepsilon^{n-k} \times 0 \to S^k \times D_\varepsilon^{n-k} \subset S^k \times S^{n-k}, \\ h_2 &: S^k \times D_\varepsilon^{n-k} \times 1 \to S^k \times D_\varepsilon^{n-k} \subset S^k \times \widetilde{S}^{n-k}(g),\end{aligned}$$

and

$$h_i(x,y) = (x, h_{ix}(y)), \quad h_{ix} \in SO_{n-k}, \quad i = 1,2.$$

Clearly,

$$\partial B^{n+1} = [S^k \times S^{n-k} \cup S^k \times \widetilde{S}^{n-k}(g)] \cup S^k \times \widetilde{S}^{n-k}(g).$$

In addition, on the manifold

$$[S^k \times S^{n-k} \cup S^k \times \widetilde{S}^{n-k}(g)] = \partial B^{n+1} \cap S^{N+n} \times 0$$

a framed N-field is given, which is normal to this manifold at the sphere S^{N+n}, and it is induced by the map $(g+F)$ from some *a priori* given and fixed frame N-field, which is normal to the submanifold $S^k \times S^{n-k}$ in $T_N(S^k \times S^{n-k})$ (cf. §§ 1–6). We shall place the membrane B^{n+1} in the Cartesian product $S^{N+n} \times I(0,1)$ smoothly and we shall assume, as in §§ 1–6, that $S^{N+n} \times 0$ admits a map $(g+F)$ and

$$\begin{aligned}&B^{n+1} \cap S^{N+n} \times 0 = \partial B^{n+1} \cap S^{N+n} \times 0 = (g+F)^{-1}(S^k \times S^{n-k}) \\ &B^{n+1} \cap S^{N+n} \times 1 = S^k \times \widetilde{S}^{n-k}(g),\end{aligned}$$

where the membrane B^{n+1} orthogonally approaches the boundary components of the Cartesian product $S^{N+n} \times I(0,1)$. Since the difference between the cycles $S^k \times x_0,\ x_0 \in S^{n-k}$ and $S^k \times x_1,\ x_1 \in \widetilde{S}^{n-k}(g)$, belongs to the kernel

$$\operatorname{Ker}(g+F)_*^{(H_k)} \subset H_k(S^k \times (S^{n-k} \cup \widetilde{S}^{n-k}(g)),$$

it is possible to extend the map of a submanifold

$$B^{n+1} \cap S^{N+n} \times 0$$

to the map

$$\widetilde{(g+F)} : B^{n+1} \to S^k \times S^{N-k} \subset T_N(S^k \times S^{n-k}).$$

In addition, it is always possible to choose maps h_1, h_2 in such a way that the map $\widetilde{(g+F)}$ is extendable to a map

$$\widetilde{\widetilde{(g+F)}} : T(B^{n+1}) \to T_N(S^k \times S^{n-k}),$$

where $T(B^{n+1})$ is a tubular neighborhood of the manifold $B^{n+1} \subset S^{N+n} \times I$, as in §§ 1–6 (or, what is the same thing, an N-frame field normal to the manifold $B^{n+1} \cap S^{N+n} \times 0$, can be extended to an N-framed field normal to the whole of membrane B^{n+1} in $S^{N+n} \times I(0,1)$). Then we extend the map $\widetilde{\widetilde{(g+F)}}$ from the tube $T(B^{n+1})$ to the direct product $S^{N+n} \times I(0,1)$ in the usual way. As a result, we arrive at a certain map

$$\frac{\widetilde{\widetilde{(g+F)}}}{S^{N+n}} \times 1 \to T_N(S^k \times S^{n-k}),$$

that is homotopic to $(g+F)$ and such that

$$\widetilde{\widetilde{(g+F)}}^{-1}(S^k \times S^{n-k}) \cap S^{N+n} \times 1 = S^k \times \widetilde{S}^{n-k}(g).$$

We have thus proved that in any homotopy class of the manifold $1_{N+n} + j_*\widetilde{B}(\widetilde{S}^{n-k}(g))$ there exists a representative

$$\widetilde{\widetilde{(g+F)}} : S^{N+n} \times 1 \to T_N(S^k \times S^{n-k}),$$

such that

$$\widetilde{\widetilde{(g+F)}}^{-1}(S^k \times S^{n-k}) = S^k \times \widetilde{S}^{n-k}(g).$$

Consequently,

$$1_{N+n} + j_*\widetilde{B}(\widetilde{S}^{n-k}(g)) \subset B(S^k \times \widetilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*.$$

Comparing our results with Lemma 7.3, we obtain the desired statement. The lemma is proved.

From Lemma 9.6 we get an immediate result:

Lemma 9.7.

$$B(S^k \times \widetilde{S}^{n-k}(g)\#\theta^n(\pi)) \supset B(S^k \times S^{n-k}) + j_*\widetilde{B}(\widetilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*.$$

PROOF. The proof formally follows from Lemma 9.6. It is only necessary to note that, accroding to Lemma 9.6,

$$B(S^k \times \widetilde{S}^{n-k}(g)\#\theta^n(\pi)) \supset 1_{N+n} + \widetilde{B}(\widetilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*,$$

where $1_{N+n} \in \pi_{N+n}(S^{N+n}) \subset \pi_{N+n}(T_N(S^k \times S^{n-k}))$; although the decomposition

$$T_N(S^k \times S^{n-k}) = S^{N+n} \vee S^{N+n-k} \vee S^{N+k} \vee S^N$$

is chosen ambiguously. Namely, if we take another element of the set $B(S^k \times S^{n-k})$ as a new generator

$$1'_{N+n} \in \pi_{N+n}(S^{N+n})$$

and choose, according to the choice of this new generator, a new decomposition of the Thom complex into a union, then, by replacing 1_{N+n} with $1'_{N+n}$, all the arguments of Lemma 9.6 remain true and we get

$$B(S^k \times \widetilde{S}^{n-k}(g)\#\theta^n(\pi)) \supset 1'_{N+n} + j_*\widetilde{B}(\widetilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*$$

for any element $1'_{N+n} \in B(S^k \times S^{n-k})$.

The lemma is proved.

Combining the results of the preceding lemmas, we can state that there have been introduced two elementary operations for changing the smoothness which preserve the triangulation: the connected sum with a Milnor sphere from $\theta^n(\pi)$ and the "connected sum along a cycle" $S^k \subset M^n$, $k < [\frac{n}{2}]$ (if the normal bundle $\nu^{n-k}(S^k, M^n)$ is trivial), of the manifolds M^n and $S^k \times \widetilde{S}^{n-k}$, where $\widetilde{S}^{n-k} \in \theta^{n-k}(\pi)$. The homotopy meaning of these operations for the case $M^n = S^k \times S^{n-k}$ was found in Lemmas 9.1–9.7.

Denote by $B_{\gamma,\delta}(M_1^n) \subset B(M_1^n)$ the subset consisting of those elements

$$\alpha_i \in B_{\gamma,\delta}(M_1^n) \subset B(M_1^n) \subset A(M^n) \subset \pi_{N+n}(T_N(M^n)),$$

for which there are representatives $f_{\alpha_i} : S^{N+n} \to T_N(M^n)$ satisfying Lemma 3.2 and possessing the following properties:

(a) the manifolds $f_{\alpha_i}^{-1}(M^n)$ are diffeomorphic to M_1^n, though the map f_{α_i}/M_1^n need not be a diffeomorphism;
(b) $f_{\alpha_i *} \cdot (\delta) = \gamma$, where $\gamma \in \pi_k(M^n)$, $\delta \in \pi_k(M_1^n)$.

Lemma 9.8. *If there exists a diffeomorphism $h : M_1^n \to M_1^n$ of degree $+1$ such that $h_*(\delta_1) = \delta_2$, $\delta_1, \delta_2 \in \pi_k(M_1^n)$, then the sets $B_{\gamma,\delta_1}(M_1^n)$ and $B_{\gamma,\delta_2}(M_1^n)$ coincide.*

The proof of the lemma follows immediately from the fact that we distinguish all our objects only up to an equivalence induced by diffeomorphisms of the manifold M_1^n onto itself of degree $+1$. The lemma is proved.

Below we shall always denote a "connected sum along a cycle" of two manifolds M_1^n and M_2^n in the following standard notation:

$$M^n(\gamma_1, \gamma_2, d) = M_1^n \#_{\gamma_1,\gamma_2}^d M_2^n,$$

where $\gamma_i \in \pi_k(M_i^n, \nu^N(M_i^n))$, $d \in \pi_k(SO_{n-k})$. In the case when $M_2^n = S^k \times \widetilde{S}^{n-k}$, $\gamma \in \pi_k(M_1^n, \nu^N(M_1^n))$ and $\beta \in \pi_k(S^k \times \widetilde{S}^{n-k})$ is a generating element, we then, taking into account Lemma 9.5, use the notation:

$$M_1^n \#_{\gamma,\beta}^d S^k \times \widetilde{S}^{n-k} = M_1^n \#_\gamma S^k \times \widetilde{S}^{n-k} \bmod \theta^n.$$

Lemma 9.9. *Suppose M^n is a $(k-1)$-connected manifold and $\gamma, \delta \in \pi_k(M^n, \nu^N(M^n))$, $k < n-k$. Then in the Thom complex $T_N(M^n)$ the relation*

$$B_{\gamma,\delta}(M^n) + \widetilde{B}(\widetilde{S}^{n-k}(g)) \cdot T^N\gamma \subset B(M^n \#_\delta S^k \times \widetilde{S}^{n-k}(g)) \bmod \operatorname{Im} \kappa_*, \quad (33)$$

is valid, where $\widetilde{B}(\widetilde{S}^{n-k}(g)) \subset G(n-k)$ and

$$T^N : \pi_k(M^n, \nu^N(M^n)) \to \frac{\pi_{N+k}(T_N(M^n))}{\operatorname{Im} \kappa_*}$$

is the homomorphism constructed in § 7.

PROOF. We realize the element $\gamma \in \pi_k(M^n, \nu^N(M^n))$ by a smoothly embedded sphere $\widetilde{\gamma} : S^k \subset M^n$, which has a trivial normal bundle $\nu^{n-k}(S^k, M^n)$ in the manifold M^n, since the bundle $\widetilde{\gamma}^* \nu^N(M^n)$ (by condition) and the bundle

$$\nu^{n-k}(S^k, M^n) \oplus \widetilde{\gamma}^* \nu^N(M^n) = \nu^{N+n-k}(S^k)$$

are trivial and $k < n-k$. The embedding $\widetilde{\gamma} : S^k \subset M^n$ determines in a natural way the embedding

$$T^N \widetilde{\gamma} : T_N(S^k, \widetilde{\gamma}^* \nu^N(M^n)) \subset T_N(M^n).$$

By analogy with the proof of Lemmas 9.6 and 9.7 we consider the two maps:

$$\begin{aligned} f &: S^{N+n} \to T_N(M^n), \quad \widetilde{f} \in B_{\gamma,\delta}(M^n), \\ F &: S^{N+n} \to T_N(S^k, \widetilde{\gamma}^* \nu^N(M^n)) \subset T_N(M^n), \end{aligned}$$

having the following properties:

$$\widetilde{F} \in \widetilde{B}(\widetilde{S}^{n-k}(g)) \circ B(S^k), \quad B(S^k) \subset \pi_{N+k}(T_N(S^k, \widetilde{\gamma}^* \nu^N(M^n))$$
$$T_N(S^k, \widetilde{\gamma}^* \nu^N(M^n)) = T_N(S^k)$$

($\widetilde{f}$ and $\widetilde{F}$ respectively denote the homotopy classes of the maps f and F).

It is easy to see that $f^{-1}(M^n) = M^n$ and $F^{-1}(S^k) = S^k \times \widetilde{S}^{n-k}(g)$.

Further, we consider the map

$$(f+F) : S^{N+n} \to T_N(M^n),$$

representing the element $\widetilde{f} + T^N \widetilde{\gamma} \widetilde{F} \in \pi_{N+n}(T_N(M^n))$. Clearly,

$$(f+F)^{-1}(M^n) = M^n \cup S^k \times \widetilde{S}^{n-k}(g) \subset S^{N+n},$$

the element $f_*^{-1}(\gamma) - F_*^{-1}(\gamma)$ belongs to the kernel $\mathrm{Ker}(f+F)_*$, $\delta = f_*^{-1}(\gamma)$. By analogy with the proof of Lemma 6.9 we construct a membrane $B^{n+1} \subset S^{N+1} \times I(0,1)$ such that:

(a) $B^{n+1} \cap S^{N+n} \times 0 = (f+F)^{-1}(M^n)$,
(b) $B^{n+1} \cap S^{N+n} \times 1 = M^n \#_\delta S^k \times \widetilde{S}^{n-k}$,
(c) $B^{n+1} = (f+F)^{-1}(M^n) \times I\left(0, \frac{1}{2}\right) \cup_{h_1,h_2} S^k \times D_\varepsilon^{n-k} \times I(0,1)$,
(d) $h_1 : S^k \times D_\varepsilon^{n-k} \times 0 \to M^n \times \frac{1}{2}$,
(e) $h_2 : S^k \times D_\varepsilon^{n-k} \times 1 \to S^k \times \widetilde{S}^{n-k} \times \frac{1}{2}$,
(f) $h_i(x, y, i-1) = (x, h_{ix}(y))$, where $i = 1, 2$, $h_{ix} \in SO(n-k)$, $x \in S^k$, $y \in D_\varepsilon^{N-k}$.

The membrane is chosen in such a way that the map

$$\frac{(f+F)}{S^{N+n}} \times 0$$

could be extended to a map

$$F_1 : S^{N+n} \times I(0,1) \to T_N(M^n)$$

such that

$$F_1^{-1}(M^n) = B^{n+1}.$$

The choice of the membrane is made according to the choice of $h_{ix},\ i = 1, 2$, as in Lemma 9.6, and it is always possible for $k < \underset{\sim\sim\sim}{n - k}$. On the upper boundary, the map $F_1/S^{N+n} \times 1$ will define a map $(q + F)$, such that

$$(f + F)^{-1}(M^n) = M^n \#_\delta S^k \times \widetilde{S}^{n-k}(g).$$

Thus we have shown that the sum $\widetilde{f} + T^N \widetilde{\gamma} \widetilde{F}$ belongs to the set

$$B(M^n \#_\delta S^k \times \widetilde{S}^{n-k}(g)),$$

whence $\widetilde{f} \in B_{\gamma,\delta}(M^n)$ and

$$\widetilde{F} \in \widetilde{B}(\widetilde{S}^{n-k}(g)) \circ B(S^k), \quad T^N \widetilde{\gamma} \widetilde{F} \in \widetilde{B}(\widetilde{S}^{n-k}(g)) \cdot T^N \gamma.$$

By definition of homomorphism,

$$T^N : \pi_k(M^n, \nu^N(M^n)) \to \frac{\pi_{N+n}(T_N(M^n))}{\operatorname{Im} \kappa_*}.$$

The theorem is proved.

§ 10. Varying smooth structure and keeping the triangulation preserved. Morse surgery[1]

Assume the manifold M^n is $k-2$-connected and it is a π-manifold for $k < n - k - 1$, $k - 2 \geq 1$. In the group

$$H_{k-1}(M^n) = \pi_{k-1}(M^n) = \pi_{k-1}(M^n, \nu^N(M^n)),$$

let us choose some element γ and let us realize it by a sphere $S^{k-1} \subset M^n$, which, by $k-1$-parallelizability of the manifold M^n, has trivial normal

[1] The main theorem of this section, Theorem 10.2, is not completely proved. The reader may omit this section, since the results given here are not used in the sequel. A detailed proof of Theorem 10.2 will be given in the second part of the work.

bundle $\nu^{n-k+1}(S^{k-1}, M^n)$, and let us form the manifold

$$B^{n+1}(h) = M^n \times I\left(0, \frac{1}{2}\right) \cup_h D^k \times D_\varepsilon^{n-k+1},$$

where

$$h : \partial D^k \times D_\varepsilon^{n-k+1} \to T(S^{k-1}) = S^{k-1} \times D_\varepsilon^{n-k+1},$$
$$h(x, y) = (x, h_x(y)), \quad h_x \in SO_{n-k+1}.$$

We choose the diffeomorphism h in such a way that the manifold $B^{n+1}(h)$ is a π-manifold, which is possible (see §§ 1–2 or § 9). Obviously,

$$\partial B^{n+1}(h) = M^n \cup (-M^n(h))$$

and

$$H_k(B^{n+1}(h), M^n) = H_{n+1-k}(B^{n+1}(h), M^n(h)) = Z,$$
$$H_i(B^{n+1}(h), M^n) = H_{n+1-i}(B^{n+1}(h), M^n(h)) = 0, \quad i \neq k.$$

Let us change the smooth structure on the manifold $M^n(h)$, and preserve the normal bundle $\nu^N(M^n(h))$ and triangulation. Denote the obtained manifold by $M_1^n(h)$. This change of smooth structure is associated, by § 8, with the set of elements $(\alpha_i) \in \pi_{N+n}(T_N^{n-2})$, which is the set of all differences

$$B(M^n(h)) - B(M_1^n(h)), \quad T_N^{n-2} \subset T_N(M^n(h)).$$

Denote by $q : M_1^n(h) \to M^n(h)$ the standard combinatorial equivalence. In the set $B(M_1^n(h))$ we choose a subset $B^{(q)}(M_1^n(h))$ consisting of all elements $\alpha \in B^{(q)}(M_1^n(h))$ having t-regular representatives

$$f_2 : S^{N+n} \to T_N(M^n(h))$$

such that

$$f_2^{-1}(M^n(h)) = M_1^n(h)$$

and

$$\frac{f_2}{M_1^n(h)} = q.$$

Let us fix a standard element $1_{N+n} \in B(M^n(h))$ constructed in the proof of Lemma 3.1, and consider the set of differences of the type

$$\{1_{N+n} - B^{(q)}(M^n(h))\} \in \pi_{N+n}(T_N^{n-2}), \quad T_N^{n-2} \subset T_N(M^n(h)).$$

We extend the smooth structure from the manifold $M_1^n(h)$ to the whole membrane $B^{n+1}(h)$. This leads to the obstruction

$$\varphi^s \in H^s(B^{n+1}(h), M^n(h), \theta^{n-s}), \quad \Gamma^{n-s} \subset \theta^{n-s}$$

with coefficients in Milnor's groups (see [12, 23]). But since

$$H^s(B^{n+1}(h), M^n(h)) = 0, \quad s \neq n+1-k,$$

we get exactly one obstruction

$$\varphi^{n+1-k} \in H^{n+1-k}(B^{n+1}(h), M^n(h), \theta^{n-k}) = \theta^{n-k}.$$

Thus, with each manifold $M_1^n(h)$, which is combinatorially equivalent to $M^n(h)$ there corresponds an element $\varphi^{n+1-k} \in \theta^{n-k}$. According to Munkres [12], if $\varphi^{n+1-k} = 0$ then the change of smooth structure can be extended to $B^{n+1}(h)$ from the boundary $M^n(h)$ without changing the triangulation.

In the group

$$H_{k-1}(M^n) = \pi_{k-1}(M^n)$$

let us choose a minimal system of generators $\gamma_1, \ldots, \gamma_l$, and let us realize them by smoothly embedded pairwise disjoint spheres $S_1^{k-1}, \ldots, S_l^{k-1} \subset M^n$. For each of these spheres the bundles $\nu^{n-k+1}(S_i^{k-1}, M^n)$, $i = 1, \ldots, l$ are trivial. Let us construct the manifold

$$B_l^{n+1}(h) = M^n \times I\left(0, \frac{1}{2}\right) \cup_{h_1,\ldots,h_l} [(D_1^k \times D_\varepsilon^{n-k+1}) \cup \cdots \cup (D_l^k \times D_\varepsilon^{n-k+1})],$$

where

$$h_i : \partial D_i^k \times D_\varepsilon^{n-k+1} \to S_i^{k-1} \times D_\varepsilon^{n-k+1} \subset M^n, \quad i = 1, \ldots, l,$$

such that

$$h_i(x, y) = (x, h_{ix}(y)), \quad x \in S_i^{k-1}, \quad y \in D_\varepsilon^{n-k+1}, \quad h_{ix} \in SO_{n-k+1}.$$

According to §§ 1–2, we choose the diffeomorphism h_i in such a way that the manifolds

$$M_l^n(h) = \left(M^n \setminus \cup_i T(S_i^{k-1})\right) \cup_{h_1,\ldots,h_l} \left[\cup_i D_i^k \times S_\varepsilon^{n-k}\right]$$

and $B_l^{n+1}(h)$ are π-manifolds, which is possible for $k < n-k$. Evidently,

$$\partial B_l^{n+1}(h) = M^n \cup (-M_l^n(h))$$

and

$$H^s(B_l^{n+1}(h), M^n) = H^{n+l-s}(B_l^{n+1}(h), M_l^n(h)) = 0, \quad s \neq k.$$

Since $k < n-k-1$, the manifold $M_l^n(h)$ is $k-1$-connected. By analogy with the above argument, each change of smooth structure on $M_l^n(h)$ without change of the triangulation generates an element

$$\varphi^{n+1-k} \in H^{n+1-k}(B_l^{n+1}(h), M_l^n(h), \theta^{n-k}) = \underbrace{\theta_{(1)}^{n-k} + \cdots + \theta_{(l)}^{n-k}}_{l\,\text{factor}}.$$

Let

$$H_i(M_l^n(h)) = 0, \quad i < k+p \quad (p \geq 0)$$

and

$$H_{k+p}(M_l^n(h)) = \pi_{k+p}(M_l^n(h)) \neq 0,$$

where $k+p < n-k-p-1$. We change the smooth structure on $M_l^n(h)$ by using the results from § 9, namely, in the group $\pi_{k+p}(M_l^n(h))$ we choose a basis $\delta_1, \ldots, \delta_m$ and consider the sum

$$M_l^n(h)\#_{\delta_1} S^{k+p} \times \widetilde{S}_1^{n-k-p} \#_{\delta_2} \cdots \#_{\delta_m} S^{k+p} \times \widetilde{S}_m^{n-k-p},$$

where $\widetilde{S}_i^{n-k-p} \in \theta^{n-k-p}(\pi)$. Let us try to "pull" the new smooth structure along the membrane $B_l^{n+1}(h)$ to M^n. This leads to an obstruction

$$\varphi^{n+1-k} \in \theta^{n-k} + \cdots + \theta^{n-k} \quad (l \text{ copies});$$

this obstruction defines a map

$$\varphi^{n+1-k} : \sum_{i=1}^{m} \theta_i^{n-k-p} \to \sum_{j=1}^{l} \theta_j^{n-k} \tag{35}$$

(with a change of smooth structure of $M_l^n(h)$ by an element $\theta \in \sum_{i=1}^m \theta_i^{n-k-p}$ one associates the obstruction $\varphi^{n+1-k}(\theta) \in \sum_{j=1}^l \theta_j^{n-k}$). If $\theta \in \operatorname{Ker} \varphi^{n+1-k}$ then the change of smooth structure by θ can be "pulled". Now let us study the homotopy nature of the constructed map φ^{n+1-k} in terms of Thom's complex. To do that, recall the filtration of the Thom complex

$$T_N(M^n) \supset T_N^{n-2} \supset \cdots \supset T_N^2 \supset S^N = T_N^0.$$

If the manifold M^n is $k-2$-connected, then

$$T_N^2 = T_N^s = \cdots = T_N^{k-2} = T_N^0 = S^n$$

and

$$T_N^{n-2} = \cdots = T_N^{n-k+1}.$$

Generally, we shall always choose the filtration

$$T_N^i = T_N(K^i, j^*\nu^N(M^n)),$$

where K^i is the i-frame in the minimal cell decomposition and $j : K^i \subset M^n$ (the number of cells $\sigma^i \subset M^n$ is equal to max $rk\, H^i(M^n, k)$ over all fields k). With each cell $\sigma^i \subset M^n$ one associates a cell

$$T_N\gamma^i \subset T_N^i \subset T_N(M^n),$$

in such a way that the boundary operators in complexes M^n and $T_N(M^n)$ are identical:

$$\partial T_N(\gamma^i) = T_N(\partial\sigma^i).$$

In § 7 it was proved that if M^n is a π-manifold then the space $T_N(M^n)$ is homotopically equivalent to a wedge of spheres

$$E^N(M^n) \vee S^N = E^N(M^n \cup x_0),$$

where x_0 is point. In this case we may assume that

$$E^N(K^i \cup x_0) = E^N K^i \vee S^N = T_N^i,$$

in such a way that

$$T_N(M^n) = S^{N+n} \vee E^N K^{n-2} \vee S^N.$$

Now, consider the Thom complex $T_N(B_l^{n+1}(h))$, which is a pseudomanifold with boundary

$$\partial T_N(B_l^{n+1}(h)) = T_N(M^n) \vee T_N(M_l^n(h)).$$

As it is well known (see § 1), the space $B_l^{n+1}(h)$ is contracted to its part

$$M^n \times \frac{1}{2} \cup_{h_1,\ldots,h_l} (D_1^k \times 0 \cup \cdots \cup D_l^k \times 0).$$

The homotopy type of the Thom complex depends only on the homotopy type of the base. Thus the Thom complex $T_N(B_l^{n+1}(h))$ is homotopically equivalent to the Thom complex

$$T_N(M^n) \cup_{T_N h_1, \ldots, T_N h_l} (D_1^{N+k} \cup \cdots \cup D_l^{N+k}),$$

where

$$T_N h_i : \partial D_i^{N+k} \to E^N K^{k-1} \subset T_N^{k-1}, \quad i = 1, \ldots, l.$$

It is evident now that

$$K^{k-1} = S_1^{k-1} \vee \cdots \vee S_l^{k-1}$$

and

$$T_N h_i : \partial D_i^{N+k} \to E^N S_i^k = S_i^{N+k-1},$$

if the spheres $S_i^{k-1} \subset K^{k-1}$ are chosen according to the (previously chosen) system of generators $\gamma_1, \ldots, \gamma_l$ of the group

$$H_{k-1}(M^n) = \pi_{k-1}(M^n)$$

when defining the manifold

$$B_l^{n+1}(h), \quad h = (h_1, \ldots, h_l).$$

Now let us investigate the Thom complex $T_N(M_l^n(h))$. If the element γ_s is of infinite order then when passing from M^n to $M_l^n(h)$, from the cycle $\widetilde{\gamma}_s \in H_{n-k+1}(M^n)$ such that $\widetilde{\gamma}_s \cdot \gamma_s = 1$, we remove a neighborhood of a point (this neighborhood being orthogonal to the sphere $S_s^{k-1} \subset M^n$). If for all generating cycles of infinite order

$$\gamma_{i_1}, \ldots, \gamma_{i_s} \in \pi_{k-1}(M^n)$$

one finds the dual system of generators

$$\widetilde{\gamma}_{i_1}, \ldots, \widetilde{\gamma}_{i_s} \in H_{n-k+1}(M^n)$$

such that

$$\widetilde{\gamma}_{i_j} \cdot \gamma_{i_t} = \delta_{jt},$$

and every element $\widetilde{\gamma}_{i_j}$ is defined by exactly one cell $\sigma_j^{n-k+1} \subset M^n$ then when passing from M^n to $M_l^N(h)$ from the interior of each cell σ_j^{n-k+1} we remove a small ball neighborhood of a point, and the complement can be contracted to K^{n-k}. If the element γ_t is of finite order q_t then there is an element

$$\widetilde{\gamma}_t \in H^{n-k+1}(M^n, Z_{q_t})$$

such that

$$\gamma_t \cdot \widetilde{\gamma}_t = 1 \pmod{q_t};$$

if γ_t is also defined by one cell

$$\sigma_t^{n-k+1} \in K^{n-k+1} \subset M^n$$

(which can always be assumed if $n-k+1 \neq k-1$), then when passing from M^n to $M_l^n(h)$, we remove from this cell a ball neighborhood of the intersection point of σ_t^{n-k+1} and S_1^{k-1}, and after this operation the complement can be contracted to the boundary $\partial\sigma_t^{n-k+1} \subset K^{n-k}$. Besides that, the whole group $\pi_{k-1}(M^n)$ is mapped to zero when passing from M^n to $M_l^n(h)$ (each sphere S_i^{k-1}, $i = 1, \dots, l$ moved to the boundary of the tubular neighborhood $\partial T(S_i^{k-1}) \subset M^n$, is spanned by a ball D_i^k). This leads to the following statement.

Lemma 10.1. *The complex $T_N(B_l^{n+1}(h))$ is homotopy equivalent to the Thom complex*

$$T_N(M^n) = S^{N+n} \vee E^N K^{n-k+1} \vee S^N$$

with a cone spanning the $N+k-1$-dimensional subcomplex

$$E^N K^{k-1} = S_1^{N+k-1} \vee \dots \vee S_l^{N+k-1} \subset E^N K^{n+k-1} \subset T_N(M^n).$$

If $k-1 < n-(k-1)-1$, then the Thom complex $T_N(M_l^n(h))$ is a subcomplex of the complex $T_N(B_l^{n+1}(h))$, and it is contracted along itself to the subcomplex

$$\frac{(S^{N+n} \vee E^N K^{n-k} \vee S^N)}{E^N K^{k-1}}$$

of the complex

$$T_N(B_l^{n+1}(h)) = \frac{(S^{N+n} \vee E^N K^{n-k+1} \vee S^N)}{E^N K^{k-1}}.$$

A proof of Lemma 10.1 follows from the arguments given before the formulations, and passing to the Thom complexes.

The lemma is proved.

In § 8, we have already considered the exact sequences (33) and (34) of the form

$$\begin{aligned} \dots \to \pi_{N+i}(T_N^{k-1}) \to \pi_{N+i}(T_N^{k+p}) \\ \to \pi_{N+i}\left(\frac{T_N^{k+p}}{T_N^{k-1}}\right) \xrightarrow{\partial} \pi_{N+i-1}(T_N^{k-1}) \to \cdots \end{aligned}$$

for $i = n, p \geq 0$. In our case

$$T_N^i = E^N K^i \vee S^N,$$
$$T_N^{k-1} = S_1^{N+k-1} \vee \cdots \vee S_l^{N+k-1} \vee S^N = E^N K^{k-1} \vee S^N.$$

Assume now $i = n$. Consider the exact sequence

$$\begin{aligned}\pi_{N+n}(E^N K^{k-1}) &\to \pi_{N+n}(E^N K^{k+p}) \\ &\to \pi_{N+n}\left(\frac{E^N K^{k+p}}{E^N K^{k-1}}\right) \\ &\xrightarrow{\partial} \pi_{N+n-1}(E^N K^{k-1}) \\ &\to \pi_{N+n-1}(E^N K^{k+p}), \quad p \geq 0, \end{aligned} \tag{36}$$

corresponding to the exact sequences (33) and (34), since

$$T_N^m = E^N K^m \vee S^N.$$

To emphasize the dependence of the manifold, we shall write:

$$T_n^m = T_N^m(M^n) \subset T_N(M^n), \quad T_N^m(M_l^n(h)) \subset T_N(M_l^n(h)),$$
$$T_N^m(B_l^{n+1}(h)) \subset T_N(B_l^{n+1}(h)).$$

From Lemma 10.1, it follows that

$$T_N^m(M_l^n(h)) = T_N^m(B_l^{n+1}(h)) = \left(\frac{E^N K^m}{E^N K^{k-1}}\right) \vee S^N$$

for $m \leq n-k$ and

$$T_N^{n-k+1}(B_l^{n+1}(h)) = \left(\frac{E^N K^{n-k+1}}{E^N K^{k-1}}\right) \vee S^N.$$

We shall also write:

$$K^m = K^m(M^n) \subset M^n,$$
$$K^m(M_l^n(h)) \subset M_l^n(h), \quad K^m(B_l^{n+1}(h)) \subset B_l^{n+1}(h),$$

denoting the frames of dimension m of the corresponding manifolds M^n, $M_l^n(h)$ or $B_l^{n+1}(h)$ by symbols depending on the manifold. Note that

$$\pi_{N+n-1}(E^N K^{k-1}) = G(n-k) + \cdots + G(n-k) \quad (l \text{ summands}).$$

Let us rewrite the exact sequence (36) as

$$\sum_{i=1}^{l} G_i(n-k+1) \to \pi_{N+n}(E^N K^{k+p}(M^n)) \xrightarrow{\Lambda} \pi_{n+N}(E^N K^{k+p}(B_l^{n+1}(h)))$$
$$\xrightarrow{\partial} \sum_{i=1}^{l} G_i(n-k) \to \pi_{N+n-1}(E^N K^{k+p}(M^n)), \qquad (37)$$

moreover if $k+p \le n-k$ then

$$E^N K^{k+p}(B_l^{n+1}(h)) = E^N K^{k+p}(M_l^n(h)).$$

According to the notation of § 8 from Lemma 10.1, we get:

$$T_N^{k+p,k-1}(M^n) = T_N^{k+p}(M_l^n(h)) = T_N^{k+p}(B_l^{n+1}(h)),$$
$$k-1 < n-k-2, \quad p \ge 0, \quad k+p \le n-k.$$

Now, let us consider the "framed" smooth spheres $\widetilde{S}^i \subset S^{N+1}$ in Pontrjagin's sense [15]. In this case the sphere $\widetilde{S}^i$ with a normal frame field τ^N ("framing") in S^{N+i} defines an element of the group $G(i)$. The "connected sum along a cycle" operation defined in § 9, will always be performed for "framed" π-manifolds M_1^n, $M_2^N \subset S^{N+n}$, in such a way that the manifolds

$$M_1^n \#_{\gamma_1,\gamma_2}^d M_2^n$$

gets a natural framing for a suitable choice of d. Since the "framed" smooth sphere $\widetilde{S}^i$ defines exactly one element $\alpha \in (\widetilde{S}^i, \tau^N) \subset G(i)$, Theorem 9.9 can be reformulated as follows:

Each element $\beta \in B_{\gamma,\delta}(M_1^n) \subset A(M^n)$ is: (a) a "framed" manifold M_1^n plus (b) fixed (up to a homotopy) map $f : M_1^n \to M^n$ of degree $+1$ such that

$$f_*\delta = \gamma, \quad \gamma \in \pi_k(M^n, \nu^N(M^n)), \quad \delta \in \pi_k(M_1^n, \nu^N);$$

on $M_1^n \#_\delta S^k \times \widetilde{S}^{n-k}$ there is a natural framing and a natural mapping

$$\widetilde{f} : M_1^n \#_\delta S^k \times \widetilde{S}^{n-k} \to M^n;$$

these framing and mapping $\widetilde{f}$ jointly define an element

$$\beta + \alpha(\widetilde{S}^{n-k}, \tau^N) \circ T^N\gamma \in B_{\gamma,\delta}(M_1^n \#_\delta S^k \times \widetilde{S}^{n-k}),$$

where

$$\alpha(\widetilde{S}^{n-k}, \tau^N) \in \widetilde{B}(\widetilde{S}^{n-k}), \quad \beta \in B_{\gamma,\delta}(M_1^n).$$

This new formulation is somewhat stronger than the former one, but this was, in fact, proved when proving Theorem 9.9. We shall call this (stronger) statement Theorem 9.9′. Moreover, when "pulling" the smooth structure along the membrane $B_l^{n+1}(h)$, we shall always try to pull the new framing obtained by changing the boundary of $M_l^n(h)$ by a framed smooth sphere $\widetilde{S}^{n-k-p}$, $p \geq 0$ (the manifold M^n is $k-2$-connected, and the manifold $M_l^n(h)$ is $k+p-1$-connected). Recall that the manifold M^n is "framed" and, according to § 2, we defined the membrane $B_l^{n+1}(h)$ in such a way that the framing on M^n can be extended to the framing of the membrane

$$B_l^{n+1}(h) \subset S^{N+n} \times I(0,1), \quad M^n \subset S^{n+N} \times 0,$$
$$M_l^n(h) \subset S^{N+n} \times 1.$$

In this case the obstruction to "pulling" of the new framing (together with the smooth structure) from the boundary $M_l^n(h)$ to the membrane $B_l^{n+1}(h)$ is a homology class

$$\widetilde{\varphi}^{n+1-k} \in H^{n+1-k}(B_l^{n+1}(h), M_l^n(h); G(n-k))$$
$$= G(n-k) + \cdots + G(n-k) \quad (l \text{ copies}).$$

This obstruction to extending the smooth structure and framing from the boundary to the membrane is split into two parts:

(1) on the boundary $\partial\sigma^{n+1-k} = S^{n-k}$ of each simplex

$$\sigma^{n+1-k} \in B_l^{n+1}(h)$$

one defines a new smooth structure

$$\widetilde{S}^{n-k}(\sigma^{n+1-k}) \in \theta^{n-k}$$

(see [12, 23]);

(2) on the boundary $\partial\sigma^{n+1-k}$ there is a frame field τ^N, which is normal to the whole membrane $B_l^{n+1}(h) \subset S^{N+n} \times I(0,1)$; the latter makes sense since the new smooth structure is defined on the neighborhood of the $n-k$-dimensional frame together with the new "framing" of the membrane $B_l^{n+1}(h)$.

(3) On $\partial\sigma^{n+1-k}$, we have a frame field normal to $\partial\sigma^{n+1-k}$ in $B_l^{n+1}(h)$ (in the new smooth structure). We denote the latter frame field by τ^k; it will make sense in the new smooth structure.

(4) The smooth structure $\widetilde{S}^{n-k}$ on $\partial\sigma^{n+1-k}$ and the fields (τ^N, τ^k) together define an element

$$\alpha(\sigma^{n+1-k}) \in G(n-k);$$

if the smoothness and the framing (τ^N, τ^k) extend from the neighborhood of a boundary of the simplex σ^{n+1-k} and define a "smoothness with a framing" on a neighborhood of the $n-k$-dimensional framing plus the neighborhood of simplex (see [12, 23]), then

$$\alpha(\sigma^{n+1-k}) = 0.$$

According to the previous results, we can change the smooth structure and the framing on a $k+p$-frame of the manifold $M_l^n(h)$ by an element

$$\alpha \in \sum_{i=1}^{m} G_i(n-k-p),^{1}$$

where m is the number of generators of the group

$$H_{k+p}(M_l^n(h)) = \pi_{k+p}(M_l^n(h)).$$

With the element $\alpha \in \sum_{i=1}^{m} G_i(n-k-p)$ one associates the element

$$\widetilde{\varphi}^{n+1-k}(\alpha) \in \sum_{j=1}^{l} G_j(n-k) = H^{n+1-k}(B_l^{n+1}(h), M_l^n(h); G(n-k)).$$

On the other hand, we have constructed an exact sequence (37)

$$\begin{aligned} \cdots \to \pi_{N+n}(E^N K^{k+p}(M^n)) \xrightarrow{\Lambda} \pi_{N+n}(E^N K^{k+p}(M_l^n(h)) \\ \xrightarrow{\partial} \sum_{i=1}^{l} G_i(n-k) \to \cdots, \end{aligned}$$

such that

$$\pi_{N+n}(E^N K^{k+p}(M_l^n(h)) = \sum_{j=1}^{m} G_j(n-k-p),$$

thus

$$\partial : \sum G_i(n-k-p) \to \sum G_i(n-k).$$

The following theorem holds:

Theorem 10.2. *The homomorphism $\partial : \sum_{j=1}^{m} G_i(n-k-p) \to \sum_{i=1}^{l} G_i(n-k)$ of the exact sequence* (37) *coincides with the map $\widetilde{\varphi}^{n+1-k}$ for those values where both are defined.*

SKETCH OF THE PROOF. The definition of ∂ is algebraic, and the map $\widetilde{\varphi}^{n+1-k}$ was defined geometrically. Consequently, to establish the connection between them, it is necessary to translate the definition of $\widetilde{\varphi}^{n+1-k}$ into algebraic language.

[1] It is important to note that framed smooth spheres do not generate the whole group $G(i)$ for $i = 4q+2$, thus $\widetilde{\varphi}^{n+1-k}$ is not always well defined.

Consider the manifold

$$\left(M^n \Big\backslash \bigcup_{i=1}^{l} T(S_i^{k-1})\right) = B^n,$$

where

$$\partial B^n = \bigcup_{i=1}^{l} S_i^{n-k} \times S_i^{k-1}.$$

Evidently,

$$M_l^n(h) = B^n \cup_{h_1,\dots,h_l} \left[\bigcup_{i=1}^{l} S_i^{n-k} \times D_i^k\right]$$

and

$$M^n = B^n \cup \left[\bigcup_{i=1}^{l} D_i^{n-k+1} \times S_i^{k-1}\right].$$

Let us change the smooth structure of the manifold $M_l^n(h)$ as descibed above (together with the framing, if the latter is defined) by the element

$$\alpha \in \sum_{j=1}^{m} G_j(n-k-p), \quad \alpha = \sum_j \alpha_j, \quad \alpha_j \in G_j(n-k-p).$$

Thus, the smooth structure and the framing are changed only in the neighborhood of the cycles $S_j^{k+p} \subset M_l^n(h)$. The intersection

$$S_j^{k+p} \cdot S_i^{n-k} = M_{ij}^p$$

is a smooth submanifold $M_{ij}^p \subset S_i^{n-k}$ which is framed in the sphere S_i^{n-k} by a framed field induced by a coordinate system in a neighborhood of the sphere S_j^{k+p}; we assume that the spheres S_j^{k+p} and S_i^{n-k} are orthogonal to each other in their common points. The obtained framed manifold defines an element $\beta_{ij} \in \pi_{n-k}(S^{n-k-p})$; when changing the smooth structure of the manifold $M_l^n(h)$ in the neighborhood of the cycle S_j^{k+p} by the sphere $\widetilde{S}_j^{n-k-p}(\alpha_j) \in \theta^{n-k-p}(\pi)$ the smooth structure on the sphere $S_i^{n-k} \subset M_l^n(h)$ is changed in the tubular neighborhood of the manifold

$M_{ij}^p \subset S_i^{n-k}$: namely,

$$T(M_{ij}^p) \subset S_i^{n-k},$$
$$T(M_{ij}^p) = M_{ij}^p \times D_\varepsilon^{n-k-p},$$
$$\partial T(M_{ij}^p) = M_{ij}^p \times S_\varepsilon^{n-k-p-1}.$$

Consider the map

$$\widetilde{g} : M_{ij}^p \to \operatorname{diff} S_\varepsilon^{n-k-p-1},$$

induced by the map

$$g : S_j^{k+p} \to \operatorname{diff} S_\varepsilon^{n-k-p-1},$$

which takes the sphere S_j^{k+p} to the point $g(S_j^{k+p})$, where

$$\widetilde{S}^{n-k-p}(g(S_j^{k+p})) = \widetilde{S}^{n-k-p}(\alpha_j).$$

Furthermore, we set

$$\widetilde{S}_i^{n-k}(\alpha_j) = [S_j^{n-k} \diagdown T(M_{ij}^p)] \cup_{\widetilde{g}} T(M_{ij}^p), \tag{38}$$

where

$$\widetilde{g} : \partial T(M_{ij}^p) \to \partial T(M_{ij}^p),$$
$$\widetilde{g}(x,y) = (x, \widetilde{g}(M_{ij}^p) \circ y), \quad x \in M_{ij}^p, \quad y \in S_\varepsilon^{n-k-p-1}.$$

The following lemma clarifies the sense of the elements $\beta_{ij} \in G(p)$.

Lemma 10.3. *The complex $T_N^{k+p}(M^n)$ is homotopically equivalent to the wedge*

$$S^N \bigvee \left[\left(\bigcup_{j=1}^m D_i^{k+p+N} \right) \cup_{\beta_{ij}} \bigvee_{i=1}^l S_i^{N+k-1} \right],$$

where $\beta_{ij} \in \pi_{N+k+p-1}\big(S_i^{N+k-1}\big) = G(p)$.

PROOF. Evidently,

$$M^n = B^n \cup \left[\cup_i D_i^{n-k+1} \times S_i^{k+1}\right],$$

where

$$B^n = M_l^n(h) \diagdown \left(\cup_i S_i^{n-k} \times D_i^k\right);$$

the manifold $M_{ij}^p \subset S_i^{n-k}$ is the intersection

$$S_i^{n-k} \cdot S_j^{k+p} \subset M_l^n(h), \quad i = 1, \ldots, l, \quad j = 1, \ldots, m.$$

We shall assume, unreservedly, that the spheres S_i^{n-k} and S_j^{k+p} intersect orthogonally at each point of M_{ij}^p. Consider the tubular neighborhood $\overline{T}(M_{ij}^p) \subset S_j^{k+p}$ of the manifold M_{ij}^p in S_i^{k+p}. Evidently,

$$T(M_{ij}^p) = M_{ij}^p \times D_{\varepsilon i}^k$$

and

$$\partial T(M_{ij}^p) = M_{ij}^p \times S_\varepsilon^{k-1} \subset S_j^{k+p}.$$

Note that on M_{ij}^p there exists a framing, which is normal to M_{ij}^p inside S_i^{n-k}, on the whole manifold $M_{ij}^p \times S_{\varepsilon i}^{k-1}$ there is a framing normal to $M_{ij}^p \times S_{\varepsilon i}^{k-1}$ in the manifold

$$\partial T(S_i^{n-k}) = S_i^{n-k} \times S_{\varepsilon i}^{k-1},$$

and on $M_{ij}^p \times S_{\varepsilon i}^{k-1}$ there is an N-frame field, normal to the manifold $M_l^n(h)$ in the sphere S^{N+n}. Consider the Thom complex $T_N(S_{\varepsilon i}^{k-1})$ and note that the sphere $S_{\varepsilon i}^{k-i} \subset B^n$ defines a (generally, nontrivial) cycle in homology $H_k(M^n)$, so that the group $H_k(M^n)$ is generated by cycles $S_{\varepsilon i}^{k-1} \subset B^n$, which appear when passing from $M_l^n(h)$ to $B^n \subset M^n$ by removing tubular neighborhoods $T(S_i^{n-k}) \subset M_l^n(h)$. The pair of framings on the manifold

$$M_{ij}^p \times S_i^{k-1} \subset M_l^n(h) \subset S^{N+n},$$

given above, together with the natural projection

$$p : M_{ij}^p \times S_i^{k-1} \to S_i^{k-1}$$

defines a map of the sphere

$$F(\beta_{ij}) : S^{N+k+p-1} \to T_N(S_i^{k-1}) = S^N \vee S^{N+k-1},$$

satisfying Lemma 3.2 such that

$$F(\beta_{ij})^{-1}(S_i^{k-1}) = M_{ij}^p \times S_i^{k-1}, \quad F(\beta_{ij}) = \frac{p}{M_{ij}^p} \times S_i^{k-1},$$

and the map $F(\beta_{ij})$ on a tubular neighborhood of the manifold $M_{ij}^p \times S_i^{k-1}$ is defined by the pair of framings constructed above, these framings should be normal to $M_{ij}^p \times S_i^{k-1} \subset S_i^{n-k} \times S_i^{k-1}$ and to $M_l^n(h) \subset S^{N+n}$. It is easy to see that the map

$$F(\beta_{ij}) : S^{N+n} \to T_N(S^{k-1})$$

has homotopy class $\beta_{ij} \circ T_{N\gamma i}$, where γi is the generating element of the group $\pi_{k-1}(S_i^{k-1})$. Recall that the framing normal to $M_l^n(h)$, was given on the membrane

$$B_l^{n+1}(h) \subset S^{N+n} \times I(0,1)$$

and, consequently, on the manifold $M^n \subset S^{N+n} \times 0$, where

$$M_l^n(h) \subset S^{N+n} \times 1.$$

Thus the constructed mapping

$$\sum_i F(\beta_{ij}) : S^{N+n} \to T_N \left(\vee_i S_i^{k-1}\right)$$

is null-homotopic in the complex $T_N(M^n)$, since the framing to

$$\cup_i M_{ij}^p \times S_i^{k-1} \subset B^n \subset M^n$$

has already been extended to the membrane

$$\left(S_j^{k+p} \backslash \cup_i \left(M_{ij}^p \times D_{\varepsilon i}^k\right)\right) \subset B^n$$

by definition of this framing, and the framing normal to the whole manifold $M_l^n(h)$, has been extended to the membrane $B_l^{n+1}(h)$. Thus the element

$$\sum_i \beta_{ij} \circ T^N \gamma_i \in \pi_{N+k+p-1}(T_N^{k+p}(M^n))$$

is equal to zero. It is easy to see that every element

$$\beta \in \pi_{N+k+p-1}(T_N^{k-1}(M^n)),$$

belonging to the kernel of the inclusion homomorphism

$$T_N^{k-1}(M^n) \subset T_N^{k+p}(M^n),$$

is a linear combination of elements $\sum_i \beta_{ij} \circ T_{\gamma i}^N$, which yields the desired statement. The lemma is proved.

Remark. If $p = 0$ then the manifold M_{ij}^p is a collection of points, and there is a well-defined intersection index

$$\beta_{ij} = S_j^{k+p} \cdot S_i^{n-k}, \quad i = 1, \dots, l, \quad j = 1, \dots, m.$$

The proof of Lemma 10.3 is trivial in this case, and in terms of intersection indices $S_j^{k+p} \cdot S_i^{n-k}$ one can express the boundary operator in the complex $T_N^{k+p}(M^n)$ (the elements $\beta_{ij} \in G(0) = Z$ are integer numbers).

We study the behavior of smooth structure on the spheres $S_i^{n-k} \supset M_{ij}^p$ when varying the smooth structure in the tubular neighborhood

$$T(M_{ij}^p) = M_{ij}^p \times D^{n-k-p},$$

described above. Namely,

$$\widetilde{S}^{n-k} = (S^{n-k} \setminus T(M_{ij}^p)) \cup_g T(M_{ij}^p),^1$$

$$g : M_{ij}^p \to \operatorname{diff} S^{n-k-p-1},$$

$g(M_{ij}^p)$ consists of one point (one diffeomorphism), corresponding to the sphere $\widetilde{S}^{n-k-p}(g) \in \theta^{n-k-p}(\pi)$. Let us separately consider the manifold

$$M_{ij}^p \times \widetilde{S}^{n-k-p}(g)$$

and define a framing τ^N on it inside the sphere S^{N+n-k} in such a way that the framed manifold

$$M_{ij}^p \times \widetilde{S}^{n-k-p}(g) \subset S^{N+n-k}$$

defines an element from the set

$$\beta_{ij} \circ \widetilde{B}(\widetilde{S}^{n-k-p}(g)) \in G(n-k).$$

[1]The operation of changing the smooth structure seriously depends on the choice of the map $M_{ij}^p \to SO_{n-k-p}$, defining normal coordinates.

On the sphere $S^{n-k} \subset S^{N+n}$ we first define the zero framing τ_0^N. Consider the framed manifold

$$M^{n-k} = (S^{n-k} \cup M_{ij}^p \times \widetilde{S}^{n-k-p}(g))$$

in the sphere $S^{N+n-k} \times 0$ and the membrane

$$N_q^{n-k+1} = M^{n-k} \cup_q D^{n-k-p} \times I(0,1) \times M_{ij}^p,$$

where

$$\begin{gathered} q = (q_0, q_1), \\ q_0 : D^{n-k-p} \times M_{ij}^p \times 0 \to S^{n-k} \\ q_1 : D^{n-k-p} \times M_{ij}^p \times 1 \to M_{ij}^p \times \widetilde{S}^{n-k-p}(g), \end{gathered}$$

so that

$$q_i(x, y, i) = (q_{iy}(x), y, i), \quad i = 0, 1, \quad q_{iy} \in SO_{n-k-p}.$$

We shall assume that

$$N_q^{n-k+1} \subset S^{N+n-k} \times I(0,1),$$

and it is evident that

$$\begin{gathered} N_q^{n-k+1} \cap S^{N+n-k} \times 0 = M^{n-k} \\ N_q^{n-k+1} \cap S^{N+n-k} \times 1 = \widetilde{S}^{n-k}, \end{gathered}$$

and the membrane N_q^{n-k+1} touches the boundary orthogonally.

Lemma 10.4. *The mappings $q_i : M_{ij}^p \to SO_{n-k-p}$, $i = 0, 1$, can be chosen in such a way that the framing $\tau^N \cup \tau_0^N$, given on the manifold $M^{n-k} \subset S^{N+n-k} \times 0$, can be extended to the membrane $N_q^{n-k+1} \subset S^{N+n-k} \times I(0,1)$.*

PROOF. Since, by assumption, p is smaller than $n - k - p$, the natural inclusion homomorphism

$$\pi(M_{ij}^p, SO_{n-k-p}) \to \pi(M_{ij}^p, SO_N)$$

is an epimorphism. Thus for the fixed map

$$q_0 : M_{ij}^p \to SO_{n-k-p}$$

we may find a map q_1

$$q_1 : M_{ij}^p \to SO_{n-k-p},$$

such that the framing $\tau^N \cup \tau_0^N$ extends from M^{n-k} to the membrane N_q^{n-k+1}, $q = (q_0, q_1)$ because the membrane N_q^{n-k+1} is always contractible to the subcomplex

$$M^{n-k} \cup_q 0 \times M_{ij}^p \times I(0,1),$$

and it suffices to extend the framing to this subcomplex, which is done completely analogously to the proof of Lemma 2.1.

The lemma is proved.

Thus, Lemma 10.4 gives information about new smooth structures and framings on spheres S_i^{n-k}, $i = 1, \dots, l$ when we deform the smooth structure and the framing on the initial manifold $M_l^n(h)$. Namely, when changing the smooth structure (and framing) on j-th basic cycle of the group

$$H_{k+p}(M_l^n(h)) = \pi_{k+p}(M_i^n(h))$$

by a Milnor sphere $\widetilde{S}^{n-k-p}(\alpha_j) \in \theta^{n-k-p}(\pi)$ together with the framing (an element of α_j-group $G(n-k-p)$ on the sphere S_i^{n-k}) the smooth structure and the framing define an element

$$\sum_j \beta_{ij} \circ \alpha_j \in G(n-k).$$

Since the homomorphism

$$\partial : \pi_{N+n}(T_N^{k+p}(M_l^n(h))) \to \pi_{N+n-1}(T_N^{k-1}(M^n)),$$

constructed above, is defined (as known in homotopy topology) in such a way that

$$\alpha \to \sum_{i,j} \alpha_j \circ \beta_{ij},$$

where $\alpha = \sum \alpha_j$ for all

$$\alpha \in \sum_{j=1}^m G_j(n-k-p) = \pi_{N+n}(T_N^{k+p}(M_l^n(h))),$$

and the elements

$$\beta_{ij} \in \sum_{i=1}^{l} G_i(p) \subset \pi_{N+k+p-1}(T_N^{k-1}(M^n))$$

possess the properties from Lemma 10.3, our theorem is proved.

Summarizing the results of Chapter II, we may say that we have partially studied the homotopy structure of Thom's complexes, the action of the group $\pi^+(M^n, M^n)$, the connected sum operation of a manifold with Milnor's sphere and the variation of smooth structure along a cycle of minimal nonzero dimension (for the case of π-manifolds). Besides, we have kept track of how the homotopy structure of Thom's complex varies when performing Morse's surgery, and, finally, we studied the connection between changing the smooth structure in the manifold operated on by a surgery and a homomorphism in some exact sequence closely connected to the homotopy structure of Thom's complex. The study of the latter connection was performed only for elementary operations changing smooth structure, however, in a sequel of this work, the author will give a more systematic treatment of changing the smooth structure and their connection to homomorphisms of type ∂.

In the next chapter, we shall extract corollaries from the general theorem established above and analyze examples.

Chapter III

Corollaries and applications

§ 11. Smooth structures on Cartesian product of spheres

We shall apply the results of the previous sections to the following example:

$$M^n = S^k \times S^{n-k}, \quad n - k > k.$$

From § 7 it follows that

$$T_N(M^n) = S^{N+n} \vee S^{N+n-k} \vee S^{N+k} \vee S^N$$

and

$$\pi_{N+n}(T_N) = Z + G(k) + G(n-k) + G(n).$$

The set $A(M^n)$ consists of all elements of type

$$1_{N+n} + \alpha, \quad 1_{N+n} \in Z, \quad \alpha \in G(k) + G(n-k) + G(n),$$

so that $1_{N+n} + 0 \in B(S^k \times S^{n-k})$.

Let us investigate the action of the group $\pi(M^n, SO_N)$ on the set $A(M^n)$. It is easy to see that the sequence

$$\pi_n(SO_N) \to \pi(M^n, SO_N) \xrightarrow{p} \pi_{n-k}(SO_N) + \pi_k(SO_N) \to 0 \tag{39}$$

is exact.

Lemma 11.1. *If $b \in \pi_n(SO_N) \subset \pi(M^n, SO_N)$, then for each element $1_{N+n} + \alpha \in A(M^n)$ we have*:

$$b(1_{N+n} + \alpha) = 1_{N+n} + \alpha + J(b). \tag{40}$$

Proof. Consider the following two maps

$$f_i : S^{N+n} \to T_N(M^n), \quad i = 1, 2,$$

representing elements $1_{N+n} + \alpha$ and $b(1_{N+n} + \alpha)$, respectively, so that

$$f_1^{-1}(M^n) = f_2^{-1}(M^n) = M_\alpha^n$$

and

$$\frac{f_1}{M_\alpha^n} = \frac{f_2}{M_\alpha^n}.$$

But in the tubular neighborhood $T(M_\alpha^n)$, the maps f_1 and f_2 differ by

$$b \in \pi_n(SO_N) \subset \pi(M^n, SO_N),$$

and this difference is supported near one point $x_0 \in M_\alpha^n$. We may say it as follows: the manifold M_α^n is framed in two different ways τ_i^N, $i = 1, 2$, and these framings differ only in a neighborhood of x_0 by element $b \in \pi_n(SO_N)$ In this case on the sphere S^n there exists a framing τ^N corresponding to the element b such that for the framed manifold (τ_1^N, M_α^n), (τ_2^N, M_α^n), (τ^N, S^n) we have:

$$(\tau_1^N, M_\alpha^n)\#(\tau^N, S^n) = (\tau_2^N, M_\alpha^n).$$

Thus the framings τ_i^N, $i = 1, 2$ on the manifold M_α^n differ by a framed sphere S^n, and in the homotopy groups we have $\pi_{n+N}(T_N)$

$$b(1_{N+n} + \alpha) = 1_{N+n} + \alpha + J(b).$$

The lemma is proved.

Lemma 11.2. *If $a \in \pi(M^n, SO_N)$ and $p(a) \in \pi_{n-k}(SO_N)$ then for every element $1_{N+n} + \alpha \in B(S^k \times \widetilde{S}^{n-k})$*

$$a(1_{N+n} + \alpha) = 1_{N+n} + \alpha + J(p(a)) \pmod{\operatorname{Im} \kappa_* \in G(n)}. \tag{41}$$

Proof. Let, as above,

$$f_i : S^{N+n} \to T_N(M^n), \quad i = 1, 2,$$

represent the elements $a(1_{N+n} + \alpha)$ and $1_{N+n} + \alpha$, in such a way that the manifold

$$M_\alpha^n = f_i^{-1}(M^n), \quad i = 1, 2,$$

is diffeomorphic to the manifold $S^k \times \widetilde{S}^{n-k}$ and framed in two different ways. These framings τ_i^N, $i = 1, 2$ differ on the basic cycle $\widetilde{S}^{n-k} \subset M_\alpha^n$, and, besides,

$$\frac{f_1}{M_\alpha^n} = \frac{f_2}{M_\alpha^n}.$$

Let us choose a standard framing τ_0^k, tangent to S^k at $x_0 \in S^k$, and choose the frame fields (τ_i^N, τ_0^k) on the sphere

$$x_0 \times \widetilde{S}^{n-k} \subset M_\alpha^n,$$

which differ by $j_* p(a)$, where

$$j_* : \pi_{n-k}(SO_N) \xrightarrow{\approx} \pi_{n-k}(SO_{N+k}).$$

Separately, let us consider the manifold

$$S^k \times S^{n-k} \subset S^{N+n}$$

and for the cycle $x_0 \times S^{n-k}$ let us define a framing τ^{N+k} defined by the element $j_* p(a)$, where the last k vectors are tangent to the factor S^k, and the first N vectors are normal to $S^k \times S^{n-k}$ (and defined for $x_0 \times S^{n-k}$).

We extend the vector field τ^N, defined by the first N vectors of τ^{N+k}, to the whole manifold

$$S^k \times S^{n-k} \subset S^{N+n},$$

which is possible; let us now define a map

$$F : S^k \times S^{n-k} \to S^k \times y_0 \subset S^k \times S^{n-k},$$

by setting $F(x, y) = x$. Consider the element β of the group $\pi_{N+n}(T_N^k(M^n))$, defined by this extended framing and by F, which is evidently represented by the map

$$f_\beta : S^{N+n} \to T_N^k$$

such that

$$f_\beta^{-1}(S^k) = S^k \times S^{n-k}, \quad \frac{f_\beta}{S^k \times S^{n-k}} = F.$$

It is easy to see that the sum $1_{N+n} + \alpha + \beta$ is represented by

$$(f_\alpha + f_\beta) : S^{N+n} \to T_N,$$

where

$$\begin{aligned}(f_\alpha + f_\beta)^{-1}(S^k \times S^{n-k}) &= (S^k \times \widetilde{S}^{n-k}) \cup (S^k \times S^{n-k}) \\ &= f_\alpha^{-1}(S^k \times S^{n-k}) \cup f_\beta^{-1}(S^k \times S^{n-k}).\end{aligned}$$

Analogously to § 10, using the “connected sum

$$S^k \times S^{n-k} \#_\gamma S^k \times S^{n-k}$$

along the cycle” $\gamma = S^k$ for framed manifolds $S^k \times \widetilde{S}^{n-k}$ and $S^k \times S^{n-k}$, construct a map

$$\widetilde{(f_\alpha + f_\beta)} : S^{N+n} \to T_N$$

of homotopy class $1_{N+n} + \alpha + \beta$ such that

$$\begin{aligned}\widetilde{(f_\alpha + f_\beta)}^{-1}(S^k \times S^{n-k}) &= (S^k \times S^{n-k} \#_\gamma S^k \times \widetilde{S}^{n-k}) \\ &= S^k \times \widetilde{S}^{n-k} \bmod \theta^n.\end{aligned}$$

The map $\widetilde{(f_\alpha + f_\beta)}$, considered on $S^k \times \widetilde{S}^{n-k}$, coincides with both f_1 and f_2 on $S^k \times \widetilde{S}^{n-k}$, and in the tubular neighborhood differs from f_1 only in a neighborhood of a point (the difference is nontrivial only for the dimension n since we have killed the difference $p(a)$ on the $n-k$-frame). Thus we conclude that

$$1_{N+n} + \alpha + \beta = a(1+\alpha) \operatorname{mod} \operatorname{Im} \kappa_* \subset G(n)$$

by Lemma 1. By virtue of Theorem 9.9 (or its modification Theorem 9.9′ given in § 10)

$$\beta = Jp(a) \circ E^N_\gamma \quad \operatorname{mod} \operatorname{Im} \kappa_*,$$

where γ is the fundamental class of the sphere S^k.

The lemma is proved.

Now let us study the action of the group $\pi^+(M^n, M^n)$ on the set $A(M^n)$, according to the results of § 7.

It is easy to see that

$$\pi^\nu_n(S^k \times S^{n-k}) = \pi_n(S^k \times S^{n-k}) = \pi_n(S^k) + \pi_n(S^{n-k})$$

and that the sequence

$$0 \to \pi_n(S^k) + \pi_n(S^{n-k}) \to \pi^+(S^k \times S^{n-k})$$

is exact. Since $n-k > k$, the homomorphism

$$T^N = E^N : \pi_n(S^{n-k}) \to G(k) \subset \frac{\pi_{n+N}(T^N(M^n))}{G(n)},$$

constructed in § 7, is an epimorphism. Applying Lemma 7.6, we get the following statement.

Lemma 11.3. *The set $B(M^n_\alpha)$ contains all elements of type*

$$1_{N+n} + \alpha + \beta \pmod{G}(n),$$

where $\beta \in G(k),\ \alpha \in G(k) + G(n-k) + G(n)$.

Proof. Let $\gamma \in \pi_n(S^{n-k}) \subset \pi^+(M^n, M^n)$. According to § 6, the group $\pi^+ + (M^n, M^n)$ acts on the set

$$B(1_{N+n} + \alpha) \subset A(M^n),$$

and, according to § 7 (Lemma 7.6), we have:

$$\gamma(1_{N+n} + \alpha) = E^N\gamma + 1_{N+n} + \alpha[\operatorname{mod} G(n)];$$

but the homomorphism E^N is an epimorphism, which yields the desired statement. The lemma is proved.

Comparing Lemmas 11.2 and 11.3 with the results of § 10, we get the following.

Lemma 11.4. *For each smooth sphere* $\widetilde{S}^{n-k} \subset \theta^{n-k}(\pi)$ *the set*

$$B(S^k \times \widetilde{S}^{n-k}) \subset A(M^n)$$

contains all elements of the type

$$1_{N+n} + \widetilde{B}(\widetilde{S}^{n-k}) + G(k) \quad [\text{mod } G(n)],$$

where with the element $1_{N+n} + 0$ *one associates the manifold*

$$M^n = S^k \times S^{n-k},$$

so that the set $\widetilde{B}(\widetilde{S}^{n-k})$ *represents the residue class* $\operatorname{mod} \operatorname{Im} J$ *in the group* $G(n-k)$.

The proof of the lemma follows from a formal combination of previous lemmas.

Lemma 11.5. (1) *If* $n - k \not\equiv 2 \mod 4$, *then each element of the set* $A(M^n) \operatorname{mod} G(n)$ *belongs to one of the sets* $B(S^k \times \widetilde{S}^{n-k})$, $\widetilde{S}^{n-k} \in \theta^{n-k}(\pi)$, *so that the following inclusion holds*:

$$B(S^k \times \widetilde{S}^{n-k}) \supset 1_{N+n} + \widetilde{B}(\widetilde{S}^{n-k}) + G(k) \operatorname{mod} G(n). \tag{42}$$

For every pair $\widetilde{S}^k \in \theta^k(\pi)$, $\widetilde{S}^{n-k} \in \theta^{n-k}(\pi)$ *there exists a smooth sphere* $\widetilde{S}_1^{n-k} \in \theta^{n-k}(\pi)$ *such that*

$$B(\widetilde{S}^k \times \widetilde{S}^{n-k}) = B(S^k \times \widetilde{S}_1^{n-k}) \operatorname{mod} G(n). \tag{43}$$

(2) *If* M_1^n *is such that*

$$B(M_1^n) \neq B(\widetilde{S}^k \times \widetilde{S}^{n-k}) \operatorname{mod} G(n)$$

for any $\widetilde{S}^k \in \theta^k$, $\widetilde{S}^{n-k} \in \theta^{n-k}$, *then the manifold* M_1^n *is combinatorially equivalent to the manifold* $M^n = S^k \times S^{n-k}$.

(3) *If* $B(M_1^n) = B(M_2^n) \operatorname{mod} G(n)$ *then the manifolds* M_1^n *and* M_2^n *are diffeomorphic mod point.*[1]

PROOF. If $n - k \not\equiv 2 \operatorname{mod} 4$, then $\widetilde{\theta}(n-k) = G(n-k)$ and, by Lemma 11.4, every element of the set $A(M^n)$ belongs to one of the sets

$$B(S^k \times \widetilde{S}_1^{n-k}) \operatorname{mod} G(n),$$

which yields Statement (1).

[1] $\widetilde{\theta}(n-k) \subset G(n-k)$ consists of framed Milnor's spheres.

If $n-k \equiv 2 \bmod 4$ and $G(n-k)/\widetilde{\theta}(n-k) = Z_2$ (see [6]), then it is possible that

$$B(M_1^n) \neq B(\widetilde{S}^k \times \widetilde{S}^{n-k}) \bmod G(n)$$

for any $\widetilde{S}^k$, $\widetilde{S}^{n-k}$ such that $\widetilde{S}^k \times \widetilde{S}^{n-k}$ is a π-manifold. In the latter case assume the contrary: M_1^n is combinatorially equivalent to $S^k \times S^{n-k}$ and some map

$$f : M_1^n \to S^k \times S^{n-k}$$

realizes this combinatorial equivalence. By [11], there is a first obstruction

$$p^k(f) \in H^{n-k}(M_1^n, \theta^k) = \theta^k,$$

i.e. $p^k(f) \in \theta^k$ and with the element $p^k(f)$ one associates the sphere $\widetilde{S}^k \in \theta^k$.

Consider the standard combinatorial equivalence

$$f_0 : S^k \times S^{n-k} \to \widetilde{S}_1^k \times S^{n-k}, \quad \widetilde{S}_1^k = -p^k(f),$$

such that

$$p^k(f_0) = -p^k(f) = \widetilde{S}_1^k \in \theta^k.$$

Evidently,

$$p^k(f_0 \circ f) = p^k(f) + p^k(f_0) = 0.$$

Consider the second obstruction

$$p^{n-k}(f_0 \cdot f) \in H^k(M_1^n, \theta^{n-k}) = \theta^{n-k},$$

the sphere

$$\widetilde{S}_1^{n-k} = -p^{n-k}(f_0 \cdot f)$$

and the map

$$f_1 : \widetilde{S}_1^k \times S^{n-k} \to \widetilde{S}_1^k \times \widetilde{S}_1^{n-k}.$$

Obviously,

$$p^{n-k}(f_1 \circ f_0 \cdot f) = p^{n-k}(f_1) + p^{n-k}(f_0 \cdot f) = 0.$$

According to the results of [9, 11, 17], the manifolds M_1^n and $\widetilde{S}_1^k \times \widetilde{S}_1^{n-k}$ are diffeomorphic mod point, and from § 9, we get:

$$B(M_1^n) \equiv B(\widetilde{S}_1^k \times \widetilde{S}_1^{n-k}) \bmod G(n).$$

Thus we obtain a contradiction with our assumption, thus Statement (2) is proved. As for Statement (3), it was essentially proved in § 9 (see Lemma 9.1). The theorem is proved.[1]

Remark. Since the sphere $\widetilde{S}^{n-k} \subset \theta^{n-k}(\partial\pi)$ can always be smoothly realized in R^n for $k \geq 2$, it follows from Smale [19] that $\widetilde{S}^{n-k} \times D^{k+1}$ is diffeomorphic to $S^{n-k} \times D^{k+1}$, $k \geq 2$. Thus $\widetilde{S}^{n-k} \times S^K$ is diffeomorphic to $S^{n-k} \times S^k$.

Lemma 11.6. *If* $n-k \not\equiv 2 \bmod 4$ *then any direct product* $\widetilde{S}^k \times \widetilde{S}_1^{n-k}$ *is diffeomorphic* mod *point to the direct product* $\widetilde{S}^k \times \widetilde{S}_2^{n-k}$ *for some sphere* $\widetilde{S}_2^{n-k}$, *where*

$$\widetilde{S}^k \in \theta^k(\pi), \quad \widetilde{S}_i^{n-k} \in \theta^{n-k}(\pi), \quad i=1,2, \quad k \geq 2, \quad n-k > k.$$

This fact immediately follows from Theorem 11.5 and Lemma 9.1.

Example 1. Let $M^n = S^2 \times S^6$. Then $\pi(M^n, SO_N) = Z_2$, and the sequence

$$0 \to \pi_8(S^2) + \pi_8(S^6) \to \pi^+(S^2 \times S^6, S^2 \times S^6) \xrightarrow{q} \pi_6(S^2) + Z_2 \to 0$$

is exact. Furthermore,

$$T_N(S^2 \times S^6) = S^{N+8} \vee S^{N+6} \vee S^{N+2} \vee S^N,$$

the set $A(M^n) = \widetilde{A}(M^n)$ consists of all elements of type

$$1_{N+n} + G(2) + G(6) + G(8)$$

and

$$B(S^2 \times S^6) \supset 1_{N+n} + 0.$$

How does the group $\pi^+(M^n, M^n)$ act? If $a \in \pi_8(S^2)$ and $b \in \pi_8(S^6)$, then, according to § 7, we have:

$$(b+a)(1_{N+n} + \alpha) \equiv 1_{N+n} + \alpha + E^N a + E^N b \bmod G(8). \tag{44}$$

Consider the subgroup $Z_2 \in \pi^+(M^n, M^n)$, generated by the diffeomorphism

$$f : S^2 \times S^6 \to S^2 \times S^6$$

such that $f(x, y) = (-x, -y)$.

[1] In part II we shall prove that if the quotient group $G(n)/\widetilde{\theta}(n) = Z_2$, then for all M^n the set $\widetilde{A}(M^n)$ contains half (exactly half) of the set $A(M^n)$, $n = 4k+2$.

According to § 6 we have:

$$T^N f(1_{N+n} + \alpha) = 1_{N+n} - \alpha \bmod G(8). \tag{45}$$

We know that $\pi_6(S^2) = Z_{12}$; let η be the generator of the group $\pi_6(S^2) = Z_{12}$ and $\widetilde{\eta} \in q^{-1}(\eta)$. Assume also $\alpha \in G(2) + G(6)$. Let us show that

$$\widetilde{\eta}(1_{N+n} + \alpha) = 1_{N+n} + \alpha \bmod G(2) + G(8).$$

By virtue of § 6, the map

$$f_{\widetilde{\eta}} : S^2 \times S^6 \to S^2 \times S^6,$$

representing the element $\widetilde{\eta} \in \pi^+(S^2 \times S^6, S^2 \times S^6)$, induces the map

$$E^N f_{\widetilde{\eta}} : E^N(S^2 \times S^6) \to E^N(S^2 \times S^6)$$

and, since $T_N(S^2 \times S^6) = S^N \vee E^N(S^2 \times S^6)$, it follows from § 6 that

$$E^N f_{\widetilde{\eta}_*}(1_{N+n} + \alpha) = \widetilde{\eta}(1_{N+n} + \alpha) \bmod G(8).$$

Consider the following map

$$f_{\widetilde{\eta}} : E^N(S^2 \times S^6) \to E^N(S^2 \times S^6).$$

Note that the space $E(S^2 \times S^6)$ is homotopically equivalent to the complex $S^3 \vee S^7 \vee S^9$ and that

$$\pi_9(E(S^2 \times S^6)) = \pi_9(S^3) + \pi_9(S^7) + \pi_9(S^9) + \operatorname{Ker} E^{N-1},$$

where

$$\pi_9(S^3) = Z_3, \quad \pi_9(S^7) = Z_2, \quad \pi_9(S_9) = Z.$$

It is evident that

$$E f_{\widetilde{\eta}}(\lambda_9) = \lambda_9 + \mu_9^{(1)} + \mu_9^{(2)} \pmod{\operatorname{Ker} E^{N-1}},$$

where

$$\mu_9^{(1)} \in \pi_9(S^3), \quad \mu_9^{(2)} \in \pi_9(S^7), \quad \lambda_9 \in \pi_9(E(S^2 \times S^6)).$$

Since

$$E^N f_{\widetilde{\eta}}(1_{N+n} + \alpha) = 1_{N+n} + \alpha + E^{N-1}(\mu_9^{(1)} + \mu_9^{(2)})$$

and

$$E^{N-1}(\mu_9^{(1)}) = 0, \quad E^{N-1}(\mu_9^{(2)}) \in G(2),$$

we get

$$\widetilde{\eta}(1_{N+n}+\alpha) \equiv 1_{N+n}+\alpha \pmod{G(2)+G(8)}.$$

Thus we have proved that the set $A(S^2\times S^6)$ is split into the following sets:

(a) $\cup_{\widetilde{S}^8\in\theta^8} B(S^2\times S^6\#\widetilde{S}^8) = 1_{N+n}+G(2)+G(8)$.

(b) Since $G(6)=Z_2$ and $G(6)\neq \operatorname{Im} E^{N-1}\pi_8(S^2)$, the set

$$A(S^2\times S^6)\Big\backslash \bigcup_{\widetilde{S}^8\in\theta^8} B(S^2\times S^6\#\widetilde{S}^8)$$

is non-empty. There is a π-manifold M_1^n of homotopy type $S^2\times S^6$, which is not diffeomorphic to $S^2\times S^6 \bmod \theta^8$.

(c) Since $\theta^2=\theta^6=0$, we see that the manifold M_1^n is not combinatorially-equivalent to $S^2\times S^6$.

Lemma 11.7. *There exist simply connected and combinatorially non-equivalent π-manifolds of homotopy type $S^2\times S^6$.*

§ 12. Low-dimensional manifolds. Cases $n=4,5,6,7$[1]

Let M^n be a simply connected manifold of dimension n. Consider the Thom complex $T_N(M^n)$ and the Thom isomorphism

$$\varphi : H^i(M^n)\to H^{N+i}(T_N(M^n)),\quad i\geq 0.$$

By $u_N\in H^N(T_N)$ we denote, as usual, the fundamental class of the Thom complex. Let $\overline{w}_i\in H^i(M^n,Z_2)$ be the normal Stiefel–Whitney classes. The following lemma is well known.

Lemma 12.1. *The formula*

$$\varphi(\overline{w}_i)=Sq^iu_N \tag{46}$$

holds.

The proof of this lemma (in the case of the tangent bundle and its Thom complex) belongs to Thom [21] and Wu [26], and analogously for Thom's complexes of any bundle (in our case, the normal bundle).

---

[1] A detailed proof of theorems from this section will be given in the next part of this work.

If we denote by $p_1 \in H(M^n, Z_3)$ the modulo 3 reduction of the Pontrjagin class of normal bundle, then (for $n \geq 6$) we have an analogous formula

$$\varphi(p_1) = P^1 u_N, \tag{46$'$}$$

where

$$P^1 : H^k(x, Z_3) \to H^{4+k}(x, Z_3)$$

is the Steenrod cube. For $n = 4$ the Pontrjagin class is equal to $\frac{1}{3}\tau$, where τ is the signature of M^n (see [3, 16]), and for $n = 5$ the class p_1 is zero because the manifold M^5 is simply connected.

Assume $n = 4$. Then the following lemma holds.

Lemma 12.2. *The group $\pi(M^4, SO_N)$ is trivial for any simply connected manifold M^4.*

The proof follows from the fact that

$$\pi_2(SO_N) = \pi_4(SO_N) = 0.$$

One can also easily prove the following:

Lemma 12.3. *The map*

$$T^N : \pi_4(M^4, \nu^N(M^4)) \to \pi_{N+4}(T_N^2(M^4))$$

is an epimorphism for any simply connected manifold M^4; the group $\operatorname{Im} \kappa_*(\pi_{N+4}(S^N))$ is zero.

PROOF. Since the group $G(4)$ is zero, the image $\operatorname{Im} \kappa_*$ is trivial. Since the suspension homomorphism

$$E^N : \pi_4(S^2) \to G(2)$$

is an epimorphism, the map

$$T^N : \pi_4(K^2(M^4), \nu^N) \to T_N^2(M^4),$$

(which can be easily reduced to the suspension homomorphism) is also an epimorphism (note that $\pi_4(K^2(M^4), \nu^N) = \pi_4(K^2(M^4)))$. Since the natural map

$$\pi_4(K^2(M^4), \nu^N) \to \pi_4(M^4, \nu^N(M^n))$$

is an epimorphism, the lemma is proved.

Taking into account that

$$T_N(M^4) = T_N^2(M^n) \vee S^{N+4},$$

we obtain the desired statement.

Lemma 12.4. *The set $B(M^4) \subset A(M^4) \subset \pi_{N+4}(T_N)$ coincides with the whole set $A(M^4)$. Thus*

$$\widetilde{A}(M^4) = A(M^4) = B(M^4),$$

and any two simply connected homotopy equivalent 4-manifolds are J-equivalent.

The proof immediately follows from Lemma 12.3 and the results from § 7.

Lemma 12.5. (1) *If $n = 5, 6$, then there is a well-defined canonical epimorphism*

$$H^3(M^n, Z) \to \pi(M^n, SO_N).$$

(2) *If $n = 7$ then the sequence*

$$Z = \pi_7(SO_N) \to (M^7, SO_N) \to H^3(M^7, Z) \to 0$$

is exact.

Proof. Because

$$\pi_7(SO_N) = \pi_3(SO_N) = Z$$

and

$$\pi_2(SO_N) = \pi_4(SO_N) = \pi_5(SO_N) = \pi_8(SO_N) = 0, \quad \pi_1(M^n) = 0,$$

the lemma follows trivially from the obstruction theory for homotopy of mapping.

Let us study the action of $\pi(M^n, SO_N)$ on the set

$$\widetilde{A}(M^n) \subset \pi_{N+n}(T_N(M^n)).$$

Note that the filtration

$$T_N \supset T_N^{n-2} \supset \cdots \supset T_N^2 \supset S^N$$

for $n \leq 7$ contains no more than 6 levels. Taking into account that $G(4) = G(5) = 0$, we get the following well-defined exact sequences

$$\pi_{N+n}(T_N^{n-3}) \xrightarrow{\Delta^{(2)}} \pi_{N+n}(T_N^{n-2}) \xrightarrow{\Lambda} \sum_{i=1}^{l} G_i(2),$$

$$\pi_{N+n}(T_N^{n-4}) \xrightarrow{\Delta^{(3)}} \pi_{N+n}(T_N^{n-3}) \to \sum_{j=1}^{m} G_j(3),$$

$$l = rkH_2(M^n, Z_2), \quad m = rkH_3(M^n, Z_{24}),$$

$$G(n) \to \pi_{N+n}(T_N^{n-4}) \to 0,$$

$$G(n) \to \pi_{N+n}(T_N^{n-3}) \to \sum G_j(3)$$

for $n = 7$. These exact sequences are induced by exact sequences (33)–(34). Note that

$$G(2) = Z_2, \quad G(3) = Z_{24} = \operatorname{Im} J,$$
$$G(6) = Z_2, \quad G(7) = Z_{240} = \operatorname{Im} J.$$

One can easily prove the following:

Lemma 12.6. *For $n = 6$ the cardinality of the set $\widetilde{A}(M^n)$ is half the cardinality of $A(M^n)$.*

If $\alpha \in \widetilde{A}(M^n)$ and $\beta \in G(6)$, $\beta \neq 0$, then $\alpha + \beta \in A(M^n)$, but $\alpha + \beta \notin \widetilde{A}(M^n)$.

Proof. Consider an admissible map

$$f_\alpha : S^{N+6} \to T_N(M^6)$$

such that the manifold $M_\alpha^6 = f_\alpha^{-1}(M^6)$ is homotopically equivalent to M^6. Besides, let us consider a map

$$F_\beta : S^{N+6} \to S^N$$

such that

$$F_\beta^{-1}(x_0) = S^3 \times S^3,$$

where $x_0 \in S^N$. The inverse image

$$F_\beta^{-1}(x_0) = S^3 \times S^3 \subset S^{N+6}$$

is a framed manifold, and for the cycles

$$S^3 \times x \subset S^3 \times S^3$$

and

$$y \times S^3 \subset S^3 \times S^3$$

there is a well-defined invariant $\psi \in Z_2$, which is an obstruction for pulling the framing for a Morse surgery (see §§ 2,4). The sum of maps

$$(F_\beta + f_\alpha) : S^{N+6} \to T_N(M^6)$$

represents an element $\alpha + \beta$ and

$$(F_\beta + f_\alpha)^{-1}(M^6) = S^3 \times S^3 \cup M^6_\alpha.$$

By a Morse surgery, one may transform $(F_\beta + f_\alpha)$ in a way such that the manifolds M^6 for our new map $(f_\alpha + F_\beta)$ homotopic to $(F_\beta + f_\alpha)$, is a framed connected sum

$$M_1^6 = M_\alpha^6 \# S^3 \times S^3$$

analogously to § 4 and § 9. For the cycles $y \times S^3$ and $S^3 \times x \subset M_1^6$, there is a well-defined invariant $\psi \in Z_2$ which is an obstruction to a Morse surgery. We have a well-defined invariant $\psi(\alpha + \beta) \neq 0$ giving an obstruction for simplification of the inverse image of M_1^6 by Morse surgery (because of the obstruction ψ to pulling frame fields). It is easy to see that the invariant ψ is well-defined and the class $\alpha + \beta \notin \widetilde{A}(M^6)$.

The lemma is proved.

Since $G(3) = \operatorname{Im} J$ and $G(7) = \operatorname{Im} J$, then from Lemma 12.5 and the definition of J we easily obtain the following:

Lemma 12.7. *For every element $\alpha \in \widetilde{A}(M^n)$ the orbit $\pi(M^n, SO_N) \circ \alpha$ for $n \leq 7$ contains all elements of the form $\alpha + \beta$, where*

$$\beta \in \Delta_*^{(2)} \pi_{N+n}(T_N^{n-3}) \subset \pi_{N+n}(T_N^{n-2}) \subset \pi_{N+n}(T_N(M^n))$$

(*here Δ_* is the inclusion homomorphism $\Delta : T_N^{n-3} \subset T_N^{n-2}$ in the exact sequence* (33)).

The proof follows from the fact that the sequence

$$G(n) \to \pi_{N+n}(T_N^{n-3}) \to \sum_j G_j(3)$$

is exact for $n \leq 7$, and from Lemma 12.6 (for the case $n = 6$).

Lemma 12.8. *The image of the homomorphism composition*

$$\Lambda \cdot T^N : \pi_n^\nu(M^n) \to \sum_{i=1}^{l} G_i(2)$$

coincides with the image of Λ.

The proof of the lemma easily follows from the form of non-stable homotopy groups of spheres in low dimensions (≤ 7), the structure of the suspension homomorphism E^N and the definition of the homomorphism T^N, having all properties analogous to the properties of the suspension homomorphism (see § 7).

Comparing lemmas and results of §§ 1–7, we get the following:

Theorem 12.9. *For* $n \leq 7$, *the sets* $\widetilde{A}(M^n)$ *and* $B(M^n) \subset \widetilde{A}(M^n)$ *coincide.*

PROOF. In a sequel of this work we shall study the properties of T^N and the connection of J with the action of $\pi(M^n, SO_N)$ in more details.

§ 13. Connected sum of a manifold with Milnor's sphere

Using § 9, let us study the question when manifolds M^n and $M^n \# \widetilde{S}^n$ are diffeomorphic of degree $+1$ $(\operatorname{mod} \theta^n(\partial\pi))$.

By Lemma 9.1, to perform this deed we should understand the structure of the homomorphism $\kappa_* : G(n) \to \pi_{N+n}(T_N(M^n))$, where $\kappa : S^N \subset T_N(M^n)$ is the natural embedding of the fiber

$$D_x^N \subset \nu^N(M^n), \quad x \in M^n,$$

where the boundary ∂D_x^N is contracted to a point when passing to $T_N(M^n)$. By Lemma 9.1 we have:

$$B(M^n \# \widetilde{S}^n) = B(M^n) + \kappa_* \widetilde{B}(\widetilde{S}^n),$$

moreover, $\widetilde{B}(\widetilde{S}^n) \subset G(n)$ is a residue class $\operatorname{mod} \operatorname{Im} J$. The following lemma holds:

Lemma 13.1. *If in the set* $\widetilde{B}(\widetilde{S}^n)$ *there is an element* $\beta \in \widetilde{B}(\widetilde{S}^n) \subset G(n)$ *such that* $\kappa_*\beta = 0$ *then the manifolds* M^n *and* $M^n \# \widetilde{S}^n$ *are diffeomorphic* $\operatorname{mod} \theta^n(\partial\pi)$; *in this case there is a sphere* $\widetilde{S}_1^n \in \theta^n(\partial\pi)$ *such that the manifolds* M^n *and* $M^n \# (\widetilde{S}^n \# \widetilde{S}_1^n)$ *are diffeomorphic of degree* 1.

PROOF. Let $\kappa_* \beta = 0$, where $\beta \in \widetilde{B}(\widetilde{S}^n)$. Then the intersection $B(M^n) \cap B(M^n \# \widetilde{S}^n)$ is non-empty, hence

$$B(M^n) = B(M^n \# \widetilde{S}^n).$$

Applying § 6, we get the first claim of the lemma. The second claim follows from the associativity of the operation #. The lemma is proved.

Now let us try to find examples of manifolds M^n for which the kernel of κ_* is nontrivial.

Consider an SO-bundle ν with fiber S^m and base S^l, where $m \geq l + 1$. The bundle ν is defined by a certain element $h \in \pi_{l-1}(SO_{m+1})$. Denote by M^n the total space of the bundle ν, $n = m + l$. We have the following:

Lemma 13.2. *The complex $T^l_N(M^n)$ is homotopy equivalent to the complex $D^{N+l} \cup_{Jh} S^N$, where $Jh \in G(l-1)$.*

PROOF. Consider the bundle $j^* \nu^N(M^n)$, which is a restriction of the normal bundle to the frame

$$K^l(M^n) = S^l \overset{j}{\subset} M^n$$

of dimension l. It is easy to see that the normal bundle $j^* \nu^N(M^n)$ is defined by the invariant

$$\pm h \in \pi_{l-1}(SO_N) \approx \pi_{l-1}(SO_{m+1}),$$

since $m \geq l + 1$. Clearly, the complexes $T^l_N(M^n)$ and $T_N(S^l, j^* \nu^N(M^n))$ coincide, and, by Milnor's definition [7] of the J-homomorphism, we get the desired statement. The lemma is proved.

Lemma 13.3. *Let, as above, $h \in \pi_{l-1}(SO_{m+1})$, $m \geq l + 1$, and $\alpha \in G(m+1)$, so that $\alpha \cdot Jh \notin \operatorname{Im} J$. Then there is a Milnor sphere $\widetilde{S}^{m+l}$, such that $\alpha \cdot Jh \in \widetilde{B}(\widetilde{S}^{m+l})$ and the manifolds M^n and $M^n \# \widetilde{S}^{m+l}$, $n = m + l$, are diffeomorphic of degree* $+1$ modulo $\theta^n(\partial\pi)$.[1]

PROOF. Evidently, the element $\alpha \cdot Jh$ belongs to the kernel $\operatorname{Ker} \kappa_*$. If $n \not\equiv 2 \bmod 4$, then the lemma follows from the previous arguments and the results of Milnor [see [6, 8]]. If $m \not\equiv 1 \bmod 4$ then the element $\alpha \in G(m+1)$ can be thought of as a framed smooth sphere $\widetilde{S}^{m+1}_\alpha$, and the element $\alpha \cdot Jh$ is a framed direct product $\widetilde{S}^{m+1} \times S^{l-1}$; by a single Morse surgery we may kill the cycles of dimensions $l-1$ and $m+1$, and after that the element $\alpha \cdot Jh$ is realizable by a homotopy sphere. The lemma is proved. If $m + 1 \equiv 2 \bmod 4$

[1] Cf. also [32].

and $m + l \equiv 2 \bmod 4$, then the element α can be realized by a framed manifold Q^{m+1} such that

$$\pi_1(Q^{m+1}) = 1, \quad H_i(Q^{m+1}) = 0, \quad i \neq 0, \frac{m+1}{2}, m+1,$$

and the group

$$H_{\frac{m+1}{2}}(Q^{m+1}) = Z + Z,$$

moreover, for the basic cycles $Z_1, Z_2 \in H_{\frac{m+1}{2}}$ the Kervaire invariant

$$\varphi(Q^{m+1}) = \varphi(\alpha) \in Z_2$$

(or $\psi(\alpha) \in Z_2$, if $m+1 = 6, 14$) is defined. The element $\alpha \cdot Jh$ can be realized by a direct product $Q^{m+1} \times S^{l-1}$. By using Morse surgery, let us paste the cycle

$$Z_i \otimes 1 \in H_{\frac{m+1}{2}}(Q^{m+1} \times S^{l-1}),$$

and then the cycle of dimension $l-1 < m+1$. Since homology groups have no torsion, this would not lead to new cycles; Morse surgery and pulling the framings are possible because $\frac{m+1}{2} < \left[\frac{n}{2}\right]$ and $l-1 < [\frac{n}{2}]$. The element $\alpha \cdot Jh$ will hence be realized by a smooth framed sphere. The lemma is proved.

In [13] there is a multiplication table for homotopy groups of spheres. In particular,

$$G(1) = \operatorname{Im} J = Z_2, \quad G(8) = Z_2 + Z_2 \supset \operatorname{Im} J = Z_2,$$
$$G(9) = Z_2 + Z_2 + Z_2 \supset \operatorname{Im} J = Z_2, \quad G(10) = Z_2 + Z_3 \supset \operatorname{Im} J = 0.$$

The products $G(1) \cdot G(8) \subset G(9)$ and $G(1) \cdot G(9) \subset G(10)$, moreover

$$G(1) \cdot G(8) = Z_2 + Z_2, \quad G(1) \cdot G(9) = Z_2.$$

Analogously, $G(13) = Z_3$ and $G(3) = Z_{24} + \operatorname{Im} J$, so that

$$G(13) = G(3) \cdot G(10), \quad G(13) \supset \operatorname{Im} J = 0.$$

Comparing the information above in the groups $G(i)$ and $\operatorname{Im} J \subset G(i)$ with the previous statements, we get the following:

Theorem 13.4. (a) *There exist manifolds M^n of dimension $n = 9$ and $n = 10$ such that* (1) $w_2(M^n) \neq 0$; (2) *There is a Milnor sphere $\widetilde{S}^n \subset \theta^n(\pi)$ such that $M^n = M^n \# \widetilde{S}^n$;*

(b) *There is a manifold* M^{13} *such that* (1) $p_1(M^{13}) \not\equiv 0 \pmod 3$, (2) *For every Milnor's sphere* $\widetilde{S}^{13} \subset \theta^{13}(\pi) = Z_3$ *the manifolds* M^{13} *and* $M^{13}\#\widetilde{S}^{13}$ *are diffeomorphic of degree* $+1$.

Remark. Theorem 13.4 holds for every manifold M^9 (or M^{10}) such that $w_2 \neq 0, \pi_1 = 0$; analogously for dimension 13.

PROOF. For $M^9(M^{10})$ we should take the total space of the bundle ν of spheres of dimensions 7 (or 8) over the sphere S^2 with $w_2(\nu) \neq 0$. Comparing Lemma 13.3 with the information about the groups $G(i), \operatorname{Im} J$ given above, we obtain the desired statement.

For dimension 13 the proof is analogous. The theorem is proved.

To conclude, the author states the conjecture that for π-manifolds (and all manifolds homotopically equivalent to them) the connected sum with a Milnor sphere always change the smooth structure modulo $\theta(\partial\pi)$.

§ 14. Normal bundles of smooth manifolds[1]

Completely analogously to the proofs of Theorems from § 4 on realizability of cycles from $A(M^n) \subset \pi_{N+n}(T_N(M^n))$ we may prove the following three statements.

Theorem 14.1. *Let* M^{2k+1} *be a smooth simply connected manifold. In order for an* SO_N*-bundle* ν *over* M^{2k+1} *to be a normal bundle of some smooth manifold* $\widetilde{M}^{2k+1}$ *which is homotopy equivalent to* M^{2k+1}, *it is necessary and sufficient that the Thom complex* $T_N(M^{2k+1}, \nu)$ *possesses the following property: the cycle* $\varphi[M^{2k+1}]$ *is spherical.*

Theorem 14.2. *Let* $M^{4k}, k > 1$ *be a smooth simply connected manifold. For the* SO_N*-bundle* ν *to be normal bundle of some manifold* $\widetilde{M}^{4k}$ *homotopically equivalent to* M^{4k}, *it is necessary and sufficient that the Thom complex* $T_N(M^{4k}, \nu)$ *possesses the following properties:*

(1) *the cycle* $\varphi[M^n]$ *is spherical;*
(2) *if* $p(\nu^N) = 1 + p_1(\nu^N) + \cdots + p_k(\nu^N)$ *and*

$$\overline{p}(\nu^N) = p(\nu^N)^{-1} = 1 + \overline{p}_1 + \cdots + \overline{p}_k,$$

then the Hirzebruch polynomial $L_k(\overline{p}_1, \ldots, \overline{p}_k)$ *is equal to the signature* $\tau(M^n)$.

[1]The result of this section is independently obtained by Browder [29].

Theorem 14.3. *Let $n = 4k + 2$, let M^n be a smooth manifold, $\pi_1(M^n) = 0$, ν^N be a vector SO_N-bundle, and let $T_N(M^n, \nu^N)$ be its Thom's complex. If the cycle $\varphi[M^n]$ is spherical, then there exists a manifold M_1^n with boundary $\partial M_1^n = \widetilde{S}^{n-1} \in \theta^{n-1}(\partial\pi)$ such that there is a mapping of pairs*

$$f : (M_1^n, \partial M_1^n) \to (M^n, x_0), \quad x_0 \in M^n,$$

for which

$$f_* : \pi_i(M_1^n, \partial M_1^n) \to \pi_i(M^n, x_0)$$

is an isomorphism for $i \leq n$ and $f^*\nu^N = \nu^N(M_1^n)$.

The proofs of the above three theorems are analogous to the proofs of theorems from § 4, and they use properties of degree 1 maps and properties of Thom's complexes.

Remark. Theorems 14.1–14.3 can be given combinatorial meaning (in their formulation we need not require smoothness of the manifold M^n): namely, if M^n is a combinatorial manifold in the sense of Brower–Whitehead, then Thom's t-regularity notion generalizes for the combinatorial case and the inverse images $f^{-1}(M^n) \subset S^{N+n}$ of the map

$$f : S^{N+n} \to T_N(M^n, \nu^N)$$

will be combinatorial sumanifolds of the sphere S^{N+n} located in this sphere with a transverse field in Whitehead's sense [25]. Thus on the manifold $f^{-1}(M^n) \subset S^{N+n}$ there appears a smooth structure such that

$$\nu^N(f^{-1}(M^n)) = f^*\nu^N.$$

Then we may apply the argument of §§ 1–4. Thus, Theorems 14.1–14.3 can be considered as theorems of finding a smooth homotopy equivalent analog for a combinatorial manifold.

Appendix 1. Homotopy type and Pontrjagin classes

a. There are plenty of relations for homotopy invariance of classes considered modulo something (Thom, Wu), i.e. congruence-type relations. Moreover, for manifolds of dimension $4k$ the Thom–Rokhlin–Hirzebruch formula expresses the index in terms of Pontrjagin numbers and thus gives one invariance relation for rational classes. A sequence of negative examples due to Dold, Milnor, Thom shows that Pontrjagin numbers and Pontrjagin classes are, "generally", not invariant. Moreover, in a private conversation, J. Milnor communicated to the author several examples of manifold, which show that among the Pontrjagin numbers, the linear subspace of homotopically invariant numbers has dimension presumably less than or equal to half of the total dimension for $k \geq 2, n = 4k$.

b. One should especially consider the class $p_1(M^5)$, or, more generally, the class $L_k(p_1, \dots, p_k)(M^{4k+1})$ as rational classes. Rokhlin [35] proved the topological invariance of these classes. However, the homotopy invariance is neither proved nor disproved. The author can show that these classes are not determined by any cohomology invariants. Nothing else is known.

c. In § 14 we gave a condition for an SO-bundle, which is necessary and sufficient for this bundle to be normal for some homotopy equivalent manifold for $n > 4, n \neq 4k+2$ ($n = 6$ and $n = 14$ are admitted). Translating this result into the language of Atiyah–Hirzebruch papers (see [37]), we obtain a manifold M_0^n, for which the Atiyah–Hirzebruch–Grothendieck functors

$$K_R(M_0^n) = Z + \widetilde{K}_R(M_0^n)$$

and

$$J_R(M_0^n) = Z + \widetilde{J}_R(M_0^n)$$

and the natural epimorphism $J_R : \widetilde{K}_R \to \widetilde{J}_R$.

Denote by $\alpha \in \widetilde{K}_R(M_0^n)$ the normal bundle to M_0^n itself minus its dimension. Our theorem says: an element $\beta \in \widetilde{K}_R(M_0^n)$ corresponds to a normal bundle of some M_1^N of homotopy type M_0^n for $n \neq 4k, 4k+2$ or $n = 6, n = 14$ if $J(\beta) = J(\alpha)$ (Atiyah proved that Thom's complex $T_N(\beta)$ of the bundle $\beta + N$ is reducible if and only if $J(\beta) = J(\alpha)$, where $\alpha + N$ is the normal bundle); for $n = 4k$ one should also add the Rokhlin–Thom–Hirzebruch condition for the Pontrjagin classes of the element β. For concrete calculations it is recommended here to use Adams' methods, his operations Φ_R^k and "generalized characteristic classes", which in certain cases lead to exact computation of J_R (see [28, 36]).

d. Let X be a finite complex and let

$$\widetilde{H}^*_{(4)}(X) = \sum_{i \geq 0} \widetilde{H}^{4i}(X, Z),$$

where

$$\widetilde{H}^{4i}(X, Z) = \frac{H^{4i}(X, Z)}{2\text{-torsion}}.$$

In the ring $\widetilde{H}^*_{(4)}(X)$, let us consider elements of the form

$$1 + x_1 + \cdots + x_i + \cdots ,$$

where $x_i \in \widetilde{H}^{4i}(X, Z)$. The set of such elements forms a group $\Lambda(X)$ with respect to multiplication. There is a well-defined group homomorphism

$$P : \widetilde{K}_R(X) \to \Lambda(X),$$

which takes a stable SO-bundle (we consider the homomorphism P only on elements with $w_1 = 0$) to its Pontrjagin polynomial.

It is easy to show that the group $\operatorname{Im} P$ is of finite index in the group $\Lambda(X)$. Bott's paper allow us to calculate the image $\operatorname{Im} P$ in the group $\Lambda(X)$.

e. Let $X = M_0^n$ and let α be, as above, the element in $\widetilde{K}_R$, corresponding to the normal SO-bundle to M_0^n. The kernel $\operatorname{Ker} J$ consists of SO-bundles. It is easy to see that the group $\operatorname{Im} P(\operatorname{Ker} J)$ has finite index in $\Lambda(X)$. Let us denote this group by $\Lambda'(X) = P(\operatorname{Ker} J)$. From the above we get the following:

Theorem. *If n is odd or equal to $n = 6, 14$, then the Pontrjagin polynomials of normal bundles to manifolds of homotopy type M_0^n altogether constitute the residue of the element $P(\alpha) \in \Lambda(X)$ by the subgroup $\Lambda'(X)$ of finite index in $\Lambda(X)$. For $n = 4k$ they form not the whole residue class of the element of $P(\alpha)$, but only its part satisfying the Thom–Rokhlin–Hirzebruch condition.*

From this theorem by considering many examples one may conclude that for simply connected manifolds of dimension $n \geq 6, n \neq 4k+2$ be polynomial in Pontrjagin classes except $L_k(M^{4k})$, is not a homotopy invariant.

f. The case $n = 4k+2, n \neq 6, 14$ is more complicated. However, with some homological restriction on the manifold M_0^n, e.g., if the group

$$H^{2k+1}(M_0^{4k+2}, Z) \otimes Z_2$$

is trivial, this case can be considered. If $n = 4k+2$, with each element $\beta \in \widetilde{K}_R(M_0^n)$ such that $J(\beta) = J(\alpha)$, one associates the invariant $\varphi(\beta) \in Z_2$, so that $\varphi(\beta) = 0$ if there exists a manifold M_1^{4k+2} of homotopy type M_0^{4k+2} with normal bundle $\beta + N$, and $\varphi(\beta) = 1$ in the opposite case. We set $\beta = \alpha + \gamma$, where $\gamma \in \operatorname{Ker} J$. Analogously to the author's work [33] one shows that

$$\varphi(\alpha + \gamma_1 + \gamma_2) = \varphi(\alpha) + \varphi(\alpha + \gamma_1) + \varphi(\alpha + \gamma_2),$$

where $\gamma_1, \gamma_2 \in \operatorname{Ker} J$. Since $\varphi(\alpha) = 0$, we define a homomorphism $\overline{\varphi} : \operatorname{Ker} J \to Z_2$, where $\overline{\varphi}(\gamma) = \varphi(\alpha + \gamma), \gamma \in \operatorname{Ker} J$ (we assume that

$H^{2k+1}(M_0^{4k+2}, Z) \otimes Z_2 = 0)$. Thus we have either

$$\text{Ker } \overline{\varphi} = \text{Ker } J$$

or

$$\text{Ker } \overline{\varphi} = \frac{1}{2} \text{Ker } J.$$

In the formulation of part e one should replace the group $\Lambda'(X)$ with the group $P(\text{Ker } \overline{\varphi})$ coinciding with $\Lambda'(X)$ or having index 2 in the latter.

Appendix 2. Combinatorial equivalence and Milnor's microbundle theory

Is it possible to present, for the class of combinatorial manifolds, any analogue of the construction given by the author in the present work for detecting diffeomorphism of smooth manifolds (with the same dimension restrictions and provided that the manifold is simply connected)?

a. First of all, we need the notion of stable bundle. For the sake of smoothing combinatorial manifold, Milnor suggested to consider "combinatorial microbundles" over complexes (see [31, 34]). Roughly speaking, a microbundle is a bundle over a complex, whose fiber is the Euclidean space R^n and whose structure group is the group of "micro-automorphisms", i.e. piecewise-linear automorphisms with a common fixed point, which are identified if they coincide in a neighborhood of this point. Besides, the definition includes the combinatorial structure of the bundle space (the description of a microbundle given here is not quite exact). Milnor proved that there is a uniquely defined stable normal microbundle, though the normal bundle itself does not always exist.

b. Thus, it is worth considering the class of simply connected combinatorial manifolds $\{M_i^n\}$ for $n \geq 5$ of the same homotopy type and with the same stable normal microbundle (as in the smooth case). As before, we may consider the Thom complex T_N of the normal bundle for one manifold $M_0^n \in \{M_i^n\}$. Further analogy requires the notion of t-regularity in the combinatorial case. This notion is rather local, and since the transversality notion makes sense for combinatorial manifolds, then the notion of t-regularity naturally generalizes. The cycle

$$\varphi[M_0^n] \in H_{N+n}(T_N)$$

is spherical as in the case of smooth M_0^n, thus the preimages

$$f^{-1}(M_0^n) \subset S^{N+n}$$

for a t-regular $f : S^{N+n} \to T_N$ will possess good properties. An analogous result holds for preimages under homotopy

$$F : S^{N+n} \times I \to T_N.$$

c. We have to consider Morse surgery in the new situation, when we want to kill the kernels of the maps

$$M^n_f \to M^n_0,$$

where $M^n_f = f^{-1}(M^n_0)$, or

$$W^{n+1}_f \to M^n_0,$$

where $F : S^{N+n} \times I \to M^n_0$. Here we meet the following difficulties:

(1) the sphere $S^i \subset M^n_f$ or $S^i \subset W^{n+1}_F$, generally, has no normal microbundle in a manifold;

(2) if the sphere $S^i \subset M^n_f, S^i \subset W^{n+1}_F$ and it has normal microbundle, it need not be trivial;

(3) is it possible to "pull" the framings, even if the Morse surgery is possible? Note that for solving (2) and (3) we were seriously used the fast stabilization of the embeddings $SO_k \subset SO_{k+1} \subset \cdots$ and the results of Bott, having no combinatorial analog. To avoid these difficulties we should introduce "local smooth structures" and framings in the neighborhood of the cycle in question. Recall that a neighborhood of this cycle can be considered as the preimage of one point $x_0 \in M^n_0$. Thus in this neighborhood we may set up the smooth structure and the framing. The cycle we are investigating will be a smooth sphere in this smooth structure. The last remark annihilates all difficulties caused by Morse surgery.

d. Thus all the results work. In all formulations one should replace SO_N with PL and remove the group $\theta^n(\partial\pi)$ from our formulation: this group consists of spheres which are combinatorially standard. Accordingly, one should change the group $\pi^+(M^n, M^n)$.

e. If M^n_0 is smooth, we may apply either combinatorial or smooth construction to it. As a result, we will be able to study the relation between smooth and combinatorial method of Thom's complexes.

f. To apply the combinatorial theory it is important to know the homotopy group $\pi_i(SO)$, $\pi_i(PL)$ and the inclusion

$$\pi_i(SO) \to \pi_i(PL).$$

Recently, Mazur (see [31]) showed that

$$\pi_i(PL, SO) = \Gamma^i$$

(the Milnor–Thom group)[1] As it is well known (see [17]), $\Gamma^i = \theta^i$ for $i \neq 3, 4$, $\Gamma^3 = 0$ and the group Γ^4 is unknown. Since the inclusion $\pi_i(SO) \to \pi_i(PL)$ is monomorphic in all dimensions (Bott [1], Thom, Rokhlin–Schwarz, Adams), we have:

$$\Gamma^i = \frac{\pi_i(PL)}{\pi_i(SO)}.$$

Let us give a table of groups $\pi_i(PL)$ and inclusions $\pi_i \subset \pi_i(PL)$ for $i \leq 14$[2]:

$i=0$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi_i(PL)$	0	Z_2	0	Z	Γ^4	0	0	$Z+Z_4$	Z_2+Z_2	$Z_2+Z_2+Q_4$	Z_6	$Z+Z_8$	0	Z_3	Z_2

The inclusion homomorphism $\pi_i \subset \pi_i(PL)$ for $i \leq 14$ is trivially obtained by using the theorem on monomorphism of the inclusion and the structure of the groups Γ^i (see [6]), except $i = 7, 11$. Here we have:

$$\pi_7(SO) = Z, \quad \pi_7(PL) = Z + Z_4,$$

so that $u_{SO} = 7u_{PL} + v_{PL}$, where u_{PL} is an infinite order generator and v_{PL} is a generator of order 4;

$$\pi_{11}(SO) = Z, \quad \pi_{11}(PL) = Z + Z_8,$$

moreover, $u_{SO} = 124u_{PL} + v_{PL}$, where v_{PL} is a generator of order 8.

g. The Whitehead homomorphism $J_{PL} : \pi_i(PL) \to \pi_{N+i}(S^n)$[3] is an epimorphism for $i \neq 4k + 2$ or $i = 10$, and the quotient group $\pi_{N+i}(S^N)/\operatorname{Im} J_{PL}$ contains two elements for $i = 2, 6, 14$ and no more than two elements, otherwise. Note that for $i = 9$

$$\operatorname{Ker} J_{PL} = Z_2\theta^9(\partial\pi).$$

Conjecture. *For $i = 4k - 1$ the group $\pi_i(PL)$ looks like*

$$\pi_i(PL) = Z + Z_{\lambda_k} + \frac{\pi_{N+i}(S^N)}{\operatorname{Im} J_{SO}},$$

where λ_k is, possibly, a power of two.

[1] This result was independently obtained by Hirsch [38].

[2] $Q_4 = Z_4$ or $Z_2 + Z_2$.

[3] The definition of J_{PL} was not given above, but it can be given analogously to the usual J-homomorphism.

It might be possible that this conjecture can be proved by an arithmetic argument and comparison of coefficients of the L-genus for almost parallelizable Milnor's manifolds M_0^{4k} with index 8, Bott's results on divisibility of Pontrjagin's classes of SO-bundles over the sphere and Adams' result on stable J-homomorphism, in particular, in representing the image of $\operatorname{Im} J_{SO}$ as a direct summand in $\pi_{N+4k-1}(S^N)$. We assume that

$$J_{PL}(Z + Z_{\lambda_k}) = \operatorname{Im} J_{SO}$$

and also

$$\pi_{N+4k-1}(S^N) = J_{PL}(Z + Z_{\lambda_k}) + \frac{\pi_{N+4k-1}(S^N)}{\operatorname{Im} J_{SO}}.$$

This would yield that the group $\theta^{4k-1}(\partial\pi) \subset \theta^{4k-1}$ is a direct summand. Besides, the group

$$\pi_{4k-1}(SO) = Z \subset \pi_{4k-1}(PL)$$

should be included as follows:

$$u_{SO} = \delta_k u_{PL} + v_{PL},$$

where u_{PL} is an infinite order generator and v_{PL} is a generator of order λ_k. Then the order of $\theta^{4k-1}(\partial\pi)$ is equal to $\delta_k\lambda_k$. If the conjecture is true, we may assume Bott's theorem for the combinatorial case.

Let $a_k = 1$ for k even, and let $a_k = 2$ for k odd; let

$$L_k(p_1, \ldots, p_k) = \frac{t_k}{s_k p_k} + \cdots,$$

where t_k, s_k are relatively prime. Since $L_k(M_0^{4k}) = 8$, we have:

$$p_k(M_0^{4k}) = 8\frac{s_k}{t_k}.$$

For SO-bundles over the sphere the class p_k is divisible by $a_k(2k-1)!$ Let us find the common denominator $\widetilde{t}_k$ for $\frac{8s_k}{t_k}$ and $a_k(2k-1)!$, where $\widetilde{t}_k$ is a divisor of t_k (and it is equal to $t_k, t_k/2, t_k/4$ or $t_k/8$, if t_k is divisible by the corresponding powers of two). After that, let us find the greatest common divisor d_k of the numerators of the corresponding irreducible fractions.

Conjecture. *The Pontrjagin class of the stable microbundle over the sphere S^{4k} is a multiple of $d_k/\widetilde{t}_k$, and there is a microbundle with such class.*

In particular, for $k = 2, 3$ this conjecture is proved by the author:

$$\frac{d_2}{\widetilde{t}_2} = \frac{6}{7}, \quad \frac{d_3}{\widetilde{t}_3} = \frac{2 \cdot 5!}{124}.$$

Thus we have proved the following:

Corollary. *The Pontrjagin classes of microbundles over the spheres* S^8 *and* S^{12} *are multiples of* $\frac{6}{7}$ *and* $2 \cdot 5!/124$, *respectively, and there exist microbundles with such classes.*

Remark. The results of § 11 are naturally attached to part (e) of the present appendix, which deals with the relation between smooth and combinatorial manifolds (provided that normal bundles coincide). It is especially important to understand the example $S^2 \times S^6$, showing the non-triviality of the combinatorial theory. This follows from $G(6)/\operatorname{Im} J_{PL} = Z_2$.

Conjecture. *If simply connected manifolds* M_1^n *and* M_2^n, $n > 7$ *having the same homotopy type and normal bundle are such that* $H_{4k+2}(M_i^n, Z_2) = 0$, $2 \leq 4k+2 < n$ *then they are combinatorially equivalent* (*possibly, it is sufficient to require this condition only for* $k = 1, 3$).

Appendix 3. On groups $\theta^{4k-1}(\partial\pi)$

a. Starting from the Hirzebruch formula and Milnor–Kervaire results [6] we see that the order of the group $\theta^{4k-1}(\partial\pi)$ can be expressed in terms of the image of Whiteheads homomorphism

$$J_{SO} : \pi_{4k-1}(SO_N) \to \pi_{N+4k-1}(S^N).$$

In recent Adams' works the image $\operatorname{Im} J_{SO}$ is completely calculated for even k, and up to a factor 1 or 2 for even $k+1$, and in all known cases this factor is equal to 1. Moreover, it follows from Adams' works that the order of the image $\operatorname{Im} J_{SO}$ is completely defined by the integrality property of the Borel–Hirzebruch A-genus [30] (up to a constant factor). From comparison of papers by Milnor–Kervaire [5] and Adams [28] one can see that an odd order factor of $\operatorname{Im} J_{SO}$ is completely defined by the Hirzebruch L-genus. Combining these results, we get the following statement:

Theorem 1. *The odd part of the group* $\theta^{4k-1}(\partial\pi) \subset \theta^{4k-1}$ *is a direct summand in* θ^{4k-1}.

To prove the statement, one should construct a homomorphism

$$h : \theta^{4k-1} \to \overline{\theta}^{4k-1}(\partial\pi),$$

where $\overline{\theta}^{4k-1}(\partial\pi)$ is the odd part of the group $\theta^{4k-1}(\partial\pi)$. The homomorphism h can be constructed quite easily. To do that, we should span the sphere $\widetilde{S}^{4k-1} \subset \theta^{4k-1}$ by a membrane W^{4k}, fill in the boundary

$\partial W^{4k} = \widetilde{S}^{4k-1}$ by a ball, and for the obtained manifold W_0^{4k} take the value of the combinatorial class $p_k(W_0^{4k}) \bmod 1$. If

$$\widetilde{S}^{4k-1} \subset \theta^{4k-1}(\partial\pi),$$

then the constructed homomorphism can identify only those elements which differ by an order element of type $2s$: these follow from Adams' results.

b. The study of the even part of $\theta_2^{4k-1}(\partial\pi) \subset \theta^{4k-1}(\partial\pi)$ is more complicated. To do it, let us consider the homomorphism

$$p \circ q : \theta^{4k-1} \xrightarrow{q} \frac{\pi_{N+4k-1}(S^N)}{\operatorname{Im} J_{SO}} \xrightarrow{p} V_{\text{spin}}^{4k-1},$$

where q is the Milnor homomorphism and p is the "forgetting of framing" homomorphism of sphere homotopy groups to "spinor cobordisms", made out of simply connected manifolds satisfying $W_2 = 0$. It is evident that

$$\theta^{4k-1}(\partial\pi) \subset \operatorname{Ker}(p \circ q).$$

Applying Adams' results, we obtain the following Statement:

Theorem 2. *If k is even then the subgroup $\theta_{(2)}^{4k-1}(\partial\pi) \subset \operatorname{Ker}(p \circ q)$ is represented as a direct summand. If k is odd then either*

$$\theta_2^{4k-1}(\partial\pi) \subset \operatorname{Ker}(p \circ q)$$

is a direct summand or

$$\frac{\theta_2^{4k-1}(\partial\pi)}{Z_2} \subset \frac{\operatorname{Ker}(p \circ q)}{Z_2}$$

is a direct summand.

The proof is similar to that of Theorem 1, but the membrane should be spanned for $W_2 = 0$ and instead of the class p_k one should take the A-genus for k even and the $\frac{1}{2}A$ for k odd (modulo 1). Note that for dimensions 9 and 10 (and also 17, 18) the image of the homomorphism $p \circ q$ is nontrivial [see [33]].

Conjecture. *For dimensions $4k-1$, the homomorphism $p \circ q$ is trivial.*

c. The study of action of $\theta^{4k-1}(\partial\pi)$ on manifolds is a difficult problem which cannot be solved by using only our methods. Let us show some relatively simple cases, where we are able to shed light on this question. Let the manifold M^{4k-1} (not necessarily simply connected) be such that the groups $H^{4l}(M^{4k-1}, Q)$ are trivial ($l = 1, 2, \ldots, Q$ is the field of rational numbers).

Theorem 3.[1] *If the order of the sphere* $\widetilde{S}^{4k-1} \in \theta^{4k-1}(\partial\pi)$ *in the group* $\theta^{4k-1}(\partial\pi)$ *is odd, then the manifolds* M^{4k-1} *and* $M^{4k-1}\#\widetilde{S}^{4k-1}$ *are not diffeomorphic of degree* $+1$.

To prove this theorem, we shall use the following scheme.

1. We construct a membrane W^{4k}, $\partial W^{4k}=(-M^{4k-1})\cup(M^{4k-1}\#\widetilde{S}^{4k-1})$ such that

$$H_i(W^{4k}, M^{4k-1}) = 0, \quad i \neq 2k,$$

for which a retraction $F : W^{4k} \to M^{4k-1}$ is given such that

$$F^*\nu^N(M^{4k-1}) = \nu^N(W^{4k}),$$

where $\nu^N(M)$ is the normal bundle of the manifold M.

2. Given a diffeomorphism

$$h : M^{4k-1} \to M^{4k-1}\#\widetilde{S}^{4k-1}$$

of degree $+1$. Let us identify the boundaries of the membrane W^{4k-1} by using h. Denote the obtained orientable closed manifold by V^{4k}.

3. One can show that the groups $H^{4l}(V^{4k}, Q) = 0, l = 1, \dots, k-1, l \neq \frac{k}{2}$, but for $l = \frac{k}{2}$ the group

$$H^{2k}(V^{4k}, Q) = H^{2k}(W^{4k}, M^{4k-1}, Q) + B, \quad I(B) = 0.$$

4. If the sphere $\widetilde{S}^{4k-1} \subset \theta^{4k-1}(\partial\pi)$ is of odd order, then the class $p_k(V^{4k})$ is fractional analogously to Theorem 1. The contradiction proves the theorem.

If, in addition, $H_1(M^{4k-1}) = 0$ and $W_2(M^{4k-1}) = 0$, then for $\widetilde{S}^{4k-1} \subset \theta^{4k-1}(\partial\pi)$ we may, analogously to Theorems 2 and 3, prove, by using the Hirzebruch A-genus and Adams' theorems that the spehre $\widetilde{S}^{4k-1}$ changes the smooth structure after addition of M^{4k-1} (one should note that $W_2(W^{4k}) = 0$ and $W_2(V^{4k}) = 0$, and instead of the class p_k one should take $A_k[V^{4k}]$ if k is even, and $\frac{1}{2}A_k[V^{4k}]$ if k is odd).

d. For $n = 4k+1$, as shown above, the image of

$$p \circ q : \theta^{4k-1} \to V^{4k-1}_{\text{spin}}$$

[1] If $H^4(M^7, Q) \neq 0$ and $p_1 \neq 0$, then Theorem 3 is inapplicable, as one example due to the author shows.

may be nontrivial. For example, for $k = 2$ the image $\operatorname{Im} p \circ q = Z_2$. Moreover (see Appendix 2), the group $\pi_9(PL) = Z_2 + Z_2 + Q_4$, where $Q_4 = Z_4$ or $Z_2 + Z_2$.

By analogous arguments we may show that the group

$$\operatorname{Ker} p \circ q = Z_2 + Z_2 \subset \theta^9,$$

and the group $\theta^9(\partial\pi) \subset \operatorname{Ker}(p \circ q)$ is a direct summand. Moreover, we may show that

$$\operatorname{Im} J_{PL} = G(9),$$

where $G(i) = \pi_{N+i}(S^N)$, so that

$$J_{PL}(Z_2 + Z_2) = G(1)G(8) = Z_2 + Z_2$$

and

$$J_{PL}(Q_4) = \frac{G(9)}{G(1)G(8)} = Z_2,$$

$$\operatorname{Im} J_{PL} = Z_2 = \theta^9(\partial\pi) \subset \pi_9(PL)$$

(namely, $J_{PL}^{-1} J_{SO} = Z_2 + Z_2$ and $\theta^9(\partial\pi) = J_{PL}^{-1} J_{SO}/\pi_9(SO)$). Since

$$\frac{\theta^9}{\operatorname{Ker}(p \circ q)} = Z_2,$$

we cannot prove that the group $\theta^9(\partial\pi)$ is a direct summand.

Conjecture. $\theta^9 = Z_2 + Z_4$ *and* $\pi_9(PL) = Z_2 + Z_2 + Z_4$.

Appendix 4. Embedding of homotopy spheres into Euclidean space and the suspension stable homomorphism

It is well known that the usual sphere S^n can be in the standard way embedded in R^{n+1}. Moreover, it follows from Smale's works that a homotopy sphere $\widetilde{S}^n$ for $n \neq 3, 4$ is diffeomorphic to the standard sphere S^n if and only if it can be smoothly embedded into R^{n+1}. It follows from Milnor, Kervaire and Hirsch [6, 19] that a homotopy sphere $\widetilde{S}^n$ is a boundary of a π-manifold if and only if it can be embedded into the Euclidean space R^{n+2}. On the other hand, Haefliger showed that any homotopy sphere $\widetilde{S}^n$ is embeddable into R^{n+j} approximately for $j > \frac{n}{2} + 1$.[1] We shall consider only

[1]The order of the normal bundle $\alpha \in \pi_{n-1}(SO_j)$ is 2^h for $j > \frac{n}{2} + 1$.

embeddings of homotopy spheres $\widetilde{S}^n \subset R^{n+k}$ for $2 \leq k \leq n-1$ with trivial normal bundle, the "π-embeddings". It is easy to extract some necessary conditions for the existence of a π-embeddings $\widetilde{S}^n \subset R^{n+k}$ from homotopy groups of spheres.

Consider the set $\widetilde{B}(\widetilde{S}^n) \subset \pi_{N+n}(S^N)$ which is a residue class modulo $J\pi_n(SO_N)$.

Lemma 1. *If there is a π-embedding $\widetilde{S}^n \subset R^{n+k}$ then there exists an element $\alpha \in \widetilde{B}(\widetilde{S}^n)$ such that $\alpha \in E^{N-k}(\pi_{n+k}(S^k))$ where E is the suspension.*

The proof of the lemma trivially follows from the interpretation of the suspension homomorphism in terms of framed manifolds. As for sufficiency of the condition of Lemma 1, we have the following:

Theorem 1. *If there is an element $\alpha \in \widetilde{B}(\widetilde{S}^n)$ such that $\alpha \in \operatorname{Im} E^{N-k}$, then there exists a π-embedding $S^n \subset R^{n+k+1}$.*

The proof of the theorem uses the results of § 11 on differentiable structures on the Cartesian product of spheres and follows from Lemmas 1, 2 and 3.

Lemma 2. *With the assumptions of Theorem* 1, *the sets $B(S^n \times S^k)$ and*

$$B(\widetilde{S}^n \times S^k) \subset \widetilde{A} \subset \pi_{N+n}(T_n(S^N \times S^k))$$

coincide up to $\operatorname{Im} \kappa_*$, *where $\kappa : S^n \subset T_N$.*

Lemma 3. *If the sets $B(M_1^m)$ and $B(M^m) \subset \widetilde{A}$ coincide* modulo $\operatorname{Im} \kappa_*$, *then the manifolds M_1^m and M^m are diffeomorphic* modulo $\theta^m(\pi)$.

The proof of Lemma 3 is given in § 9 for all cases except $m \equiv 2 \pmod 4$. For a proof of Lemma 3 for $m \equiv 2 \pmod 4$, see [33].

Lemma 4. *If a manifold M^{n+k} is diffeomorphic to $S^n \times S^k$* $\operatorname{mod} \theta^{n+k}$, *where $M^{n+k} = \widetilde{S}^n \times S^k$, then the homotopy sphere $\widetilde{S}^n$ admits a π-embedding into R^{n+k+1}.*

The proof of Lemma 4 is obvious.

Consider the special case $k = 3$. The following lemma holds:

Lemma 5.[1] *If the sphere S^n is π-embedded into S^{n+3}, then it bounds a manifold $W^{n+1} \subset S^{n+3}$, whose normal bundle is an SO_2-bundle with Chern class $c_1 \in H^2(W^{n+1})$ such that $c_1^2 = 0$.*

[1]The idea of the proof of Lemma 5 is borrowed from V. A. Rokhlin's works.

PROOF. For the sphere $\widetilde{S}^n$, construct a frame field τ_3, normal to the sphere in S^{n+3}, and let us take the small sphere to the boundary $S^2 \times \widetilde{S}^n$ of the tubular neighborhood by using the first vector of the frame field. The obtained manifold $\widetilde{S}^n \subset S^2 \times \widetilde{S}^n$ is null-homologous in the complement

$$S^{n+3} \diagdown \operatorname{Int} D^3 \times \widetilde{S}^n,$$

in such a way that the membrane spanning can be thought of as a manifold W^{n+1} with boundary $\widetilde{S}^n \subset S^2 \times \widetilde{S}^n = \partial(S^{n+3} \diagdown \operatorname{Int} D^3 \times \widetilde{S}^n)$. By the way, it trivially follows from Smale [19] that

$$S^{n+3} \diagdown \operatorname{Int} D^3 \times \widetilde{S}^n$$

is diffeomorphic to $S^2 \times D^{n+1}$. The membrane W^{n+1} realizes the basic cycle of the group

$$H_{n+1}(S^2 \times D^{n+1}, \partial(S^2 \times D^{n+1})) = Z.$$

The normal bundle to the membrane W^{n+1} in S^{n+3} is an SO_2-bundle, and it is defined by the Chern class $c_1 \in H^2(W^{n+1})$. Let us show that $c_1^2 = 0$. We shall assume $n > 3$. Then

$$H_{n-1}(S^2 \times D^{n+1}) = 0.$$

The self-intersection

$$W^{n+1} \cdot W^{n+1} \subset S^{n+3} \diagdown \operatorname{Int} D^3 \times \widetilde{S}^n$$

defines an $(n-1)$-dimensional cycle modulo boundary, and it is a submanifold $V^{n-1} \subset W^{n+1}$. Since

$$\partial W^{n+1} = \widetilde{S}^n \subset S^2 \times \widetilde{S}^n,$$

we may assume that V^{n-1} lies strictly inside W^{n+1} and it is closed (it is easy to see that in dimension $n-1$ we have: $H_{n-1}(S^2 \times D^{n+1}) = H_{n-1}(S^2 \times D^{n+1}, \partial(S^2 \times D^{n+1})) = 0$).

Denote by

$$D_M : H_j(M, \partial M) \to H^{l-j}(M)$$

the Poincaré duality isomorphism, and denote by i the embedding

$$W^{n+1} \subset S^{n+3} \diagdown \operatorname{Int} D^3 \times \widetilde{S}^n.$$

Then

$$\begin{aligned} c_1^2 &= i^*\{D_M i_*[W^{n+1}]\}^2 = i^* D_M\{i_*[W^{n+1}] \cdot i_*[W^{n+1}]\} \\ &= i^* D_M i_*[V^{n-1}] = 0, \end{aligned}$$

where $M = S^{n+3} \diagdown \operatorname{Int} D^3 \times \widetilde{S}^n$.

The lemma is proved.

From lemma it immediately follows that the connected submanifold

$$V^{n-1} = W^{n+1} \cdot W^{n+1},$$

with $V^{n-1} \subset W^{n+1}$, has a trivial normal bundle in W^{n+1}. Moreover, if for the boundary $\widetilde{S}^n \subset S^2 \times \widetilde{S}^n$ we define a 2-frame field τ_2, which is normal to $\widetilde{S}^n$ in $S^2 \times \widetilde{S}^n$, and extend it to the interior of W^{n+1}, then for an appropriate choice of the field and its extension (which will also be denoted by τ_2) the singularity manifold of the field τ_2 inside W^{n+1} coincides with the manifold $V^{n-1} \subset W^{n+1}$. The tubular neighborhood $D \times V^{n-1}$ of V^{n-1} in W^{n+1} has boundary $S^1 \times V^{n-1}$ on which the field τ_2 is defined and degenerate. Let us add to $\tau_2/S^1 \times V^{n-1}$ the radius-vector directed inside the ball D^2 normally to the boundary $S^1 = \partial D^2$ at each point. Thus we get a 3-field $\widetilde{\tau}_3$ on $S^1 \times V^{n-1}$.

The following lemma is evident.

Lemma 6. *Framed manifolds $(\widetilde{S}^n, \tau_3)$ and $(S^1 \times V^{n+1}, \widetilde{\tau}_3)$ define the same element of the group $\pi_{n+3}(S^3)$ (we should take $W^{n+1} \setminus \operatorname{Int} D^2 \times V^{n-1}$ to be the membrane connecting these framed manifolds).*

Conjecture. *If the sphere $\widetilde{S}^n$ is π-embedded into S^{n+3}, then there is a normal framed field τ_3 on this sphere such that the framed manifold $(\widetilde{S}^n, \tau_3)$ defines an element of $\pi_{n+3}(S^3)$ which factors as $\beta \circ \alpha$, where $\alpha \in \pi_{n+3}(S^4)$ and $\beta \in \pi_4(S^3) = Z_2$.*

Proof. In the group G_n the set $\widetilde{B}(\widetilde{S}^n) \subset G_n$ contains $\alpha\beta$, where $\alpha \in G_{n-1}$, $\beta \in G_1$ (thus the element $\alpha\beta$ has order not greater than two), if $\widetilde{S}^n$ is π-embeddable into S^{n+3}.

Since the image of the higher suspension of the groups $\pi_{n+3}(S^3)$ contains elements of odd order p not belonging to the group $J\pi_n(SO_N)$, then for $k = 2$ and $k = 3$ in Theorem 1 we cannot get rid of the difference by one in the necessary condition (Lemma 1) and the sufficient condition (Theorem 1).

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4

Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds[1]

S. P. Novikov

In a number of special cases it is proved that the rational Pontrjagin–Hirzebruch classes may be computed in terms of cohomology invariants of various infinitely-sheeted coverings. This proves their homotopy invariance for the cases in question (Theorems 1 and 2). The methods are applied to the problem of topological invariance of the indicated classes (Theorem 3). From the results there follow various homeomorphism and homotopy types of closed simply connected manifolds, which yields a solution to the problem of Hurewicz for the first time in dimension larger than three (Theorem 4). We note that in the paper [3] the author completed the proof of the topological invariance of all the rational Pontrjagin classes by using quite a different method.[2]

[1]Translated by J. M. Danskin (edited by V. O. Manturov), Izvestiya Akademii Nauk SSSR, ser. matem., 1965, T. 29, ss. 1373–1388 (Received April 3, 1965).

[2]Many years ago, M. Gromov completely realized this plan of purely homological proof of topological invariance for rational Pontrjagin classes in all dimensions. — S. P. Novikov's remark (2004).

Introduction

As it is well known, already for three-dimensional manifolds homeomorphism is distinct from the homotopy type in the sense that there exist closed manifolds which are homotopically equivalent but not homeomorphic. They are distinguished by the Reidemeister invariant called "torsion". It is natural to expect that in dimension $n > 3$ homeomorphism will not coincide with homotopy type, either. For example, they are distinguished by the torsion invariant in higher dimensions as well, if one proves that torsion is topological invariant. Another widely known invariant, not a homotopy invariant, but, conjecturally, a topological invariant, is the Pontrjagin class, considered as rational. However, in dimension $n > 3$ no invariant has been established as topological unless it is also obviously homotopic. It is interesting that for $n = 3$ the torsion invariant, as a means of distinguishing combinatorial lens spaces, has been known since the 1930s, and its topological invariance was obtained only in 1950s in the form of a consequence of the "Hauptvermutung" (Moise). The situation is that in three dimensions a continuous homeomorphism may be approximated by a piecewise linear one. This can hardly be true in higher dimensions, and even if it is true, at the present time there are no means in sight for the proof of this fact.

In the present paper we study the rational Pontrjagin classes as topological and homotopy invariants. It is known that for simply connected manifolds there are no "rational relations" of homotopy invariance of classes other than the signature theorem:

$$(L_k(p_1, \dots, p_k), [M^{4k}]) = \tau(M^{4k}),$$

where $\tau(M^{4k})$ denotes the signature of the quadratic form $(x^2, [M^{4k}])$, $x \in H^{2k}(M^{4k}, R)$ and L_k are the Hirzebruch polynomials for the Pontrjagin classes. In what follows we shall speak about the classes $L_k = L_k(p_1, \dots, p_k)$ along with the classes p_k for manifolds, since the former are convenient in the investigation of invariance. This is shown by the signature theorem presented above and the combinatorial results of Thom, Rokhlin, and Schwarz (see [4–6]). The only "gap" in the theory of Pontrjagin classes, from the point of view of the problems posed, was the theorem of Rokhlin, proved in 1957, establishing that the class $L_k(M^{4k+1})$ is a topological invariant, but here it was not known whether the indicated class was a homotopy invariant (see [4]). Though Rokhlin's proof does not formally involve the fundamental group, one should note that this theorem is empty for simply connected manifolds since $H_{4k}(M^{4k+1}) = 0$.

In the present paper we establish for certain cases the algebraic connection of the classes with the fundamental group. From the resulting relation, it follows that the class $L_k(M^{4k+1})$ is in essence a homotopy invariant. The formulas (see § 3) found by the author may be (up to some extent) considered as generalizations of Hirzebruch's formulas. Their connection to coverings was rather unexpected, since in the characteristic class theory the fundamental group had hitherto played no role at all.

It was possible to apply these formulas to the question of topological invariance of Pontrjagin's classes. Under certain conditions we were able to prove that the scalar products (L_k, x), where $x \in H_{4k}(M^n)$, $n = 4k+2$, are topological invariants. Already for $M^n = S^2 \times S^{4k}$ this fact allows one to solve affirmatively the distinction question between homeomorphism and homotopy type for all dimensions of the form $4k + 2, k \geq 1$, and in the class of simply connected manifolds, for which the "simple" homotopy type coincides with the ordinary one.

The basic results of this paper were sketched in [1].

We take this opportunity to express my gratitude to V. A. Rokhlin for useful discussions on this work.

§ 1. Signature of a cycle and its properties

In this section we collect a number of simple algebraic facts on quadratic forms to be used in the sequel.

We assume that we are given a real linear space P, possibly of infinite dimension, and that on P there is a symmetric bilinear form $\langle x, y\rangle$-valued in R. We shall be interested only in the case when P can be represented as $P = P_1 + P_2$, where P_1 is finite-dimensional and $\langle x, y\rangle = 0$, $y \in P_2$, $x \in P$, i.e. the entire form is concentrated on a finite-dimensional subspace $P_1 \subset P$; certainly the choice of the latter is non-unique. In this case we shall say that the form is of finite type. The quadratic form $\langle x, x\rangle$ is concentrated, essentially, on P_1, and one can consider its signature, which we shall use as the signature of $\langle x, x\rangle$ on P. The signature does not depend on the choice of P_1. Evidently, every subspace $P' \subset P$ is such that the form $\langle x, x\rangle$ for $x \in P$ is of finite type, too, and has a signature in the same sense: one can easily find a decomposition $P' = P_1' + P_2'$, where $\langle x, y\rangle = 0$, $y \in P_2'$ and P_2' is finite-dimensional.

The following facts on the signature easily follow from the analogous facts for forms on finite-dimensional spaces.

a) Given two subspaces $P' \subset P$ and $P'' \subset P$ such that every element of P is a sum $x_1 + x_2$, $x_1 \in P'$, $x_2 \in P''$. If the form $\langle x, y\rangle$ vanishes identically on P' and on P'', then the signature of $\langle x, x\rangle$ is zero on P. If now the forms on P' and P'' are nontrivial but P' and P'' are as $P' = (P' \bigcap P'') + P_1'$

and $P'' = (P' \bigcap P'') + P_1''$ so that $\langle x, y\rangle = 0$, for $y \in P_1'$, $x \in P'$, $y \in P_1''$, $x \in P''$, then the signature of $\langle x, x\rangle$ on P coincides with the signature of $\langle x, x\rangle$ on $P' \bigcap P''$.

b) Given a subspace $P' \subset P$ such that $\langle x, y\rangle = 0$ implies that for all $x \in P'$ we have $\langle y, y\rangle = 0$, then the signature of $\langle x, x\rangle$ on P' coincides with the signature of $\langle x, x\rangle$ on P.

Suppose that K is any locally finite complex and $z \in H_{4k}(K, Z)/\,\mathrm{Torsion}$. Consider the group $H^{2k}(K, R) = P$ and the bilinear form $\langle x, y\rangle = (xy, z)$, $x, y \in P$. It is easy to prove:

Lemma 1.1. *The bilinear form $\langle x, y\rangle$ has finite type on the group $P = H^{2k}(K, R)$.*

Proof. One can find a finite subcomplex $K_1 \overset{i}{\subset} K$ such that in K_1 there is an element $z_1 \in H_{4k}(K_1)$ and $z = i_* z_1$. The group $H^{2k}(K_1, R)$ is finite-dimensional. The homomorphism $i^*\colon P \to H^{2k}(K_1, R)$ is defined. Since

$$((i^*x)(i^*y), z_1) = (xy, z) = \langle x, y\rangle,$$

the kernel $\operatorname{Ker} i^* \subset P$ consists only of those elements $y \in \operatorname{Ker} i^*$ for which $\langle x, y\rangle = 0$. The image $\operatorname{Im} i^*$ is finite-dimensional, and therefore the form $\langle x, y\rangle$ has finite type on P. Thus lemma is proved.

Therefore the signature of the form $P = H^{2k}(K, R)$ is determined.

By nondegenerate part of a form of finite type on a linear space P we shall mean a subspace $P_1 \subset P$ such that the form is nondegenerate on P_1 and is trivial on the orthogonal complement to P_1. It is natural to consider P_1 as a factor of P. Evidently the signature is defined by the nondegenerate part of the quadratic form, the latter being uniquely defined.

Lemma 1.2. *Suppose that $K_1 \subset K_2 \subset \cdots \subset K$ is an increasing sequence of locally finite complexes and $K = \bigcup_j K_j$. Denote the inclusion $K_1 \subset K_j$ by i_j and denote the inclusion $K_1 \subset K$ by i. Given an element $K_1 \in H_{4k}(K_1, Z)/\,\mathrm{Torsion}$ such that $i_{j*} z_1 \neq 0$, $i_* z_1 \neq 0$. Consider the elements $i_{j*} z_1 = z_j$ and forms on the spaces $P_j = H^{2k}(K_j, R)$. Then the nondegenerate part of a quadratic form on P_j is one and the same for all sufficiently large indices and it coincides with the nondegenerate part of the quadratic form on $P = H^{2k}(K, R)$.*

Proof. Consider the homomorphisms $i_j^*\colon P_j \to P_1$ and $i^*\colon P \to P_1$. In P_1, select a finite-dimensional nondegenerate part $P_1' \subset P_1$; then we may suppose that the images of all nondegenerate parts $P_j' \subset P$ under i_j^* belong to $P_1' \subset P_1$.[1] But the image

$$\operatorname{Im} i^* = \bigcap_j \operatorname{Im} i^*_{j+1};$$

because of finite-dimensionality of P_1' and the inclusion

$$\operatorname{Im} i^* \supset \operatorname{Im} i^*_{j+1}$$

for all j, we obtain a stabilization of the images $i_j^* P_j' \subset P_1'$. Now, since the kernel $\operatorname{Ker} i_j^*$ consists only of the purely degenerate part, it follows that the forms coincide on P_j' and on $i_j^* P_j'$. The lemma is proved.

In the sequel the signature of the natural form on $P = H^{2k}(K, R)$ for a given element $z \in H_{4k}(K, Z)/\mathrm{Tor}$ will be called the signature *of* z; it is denoted by $\tau(z)$. If $K = M^{4k}$ and $z = [M^{4k}]$, then $\tau(z) = \tau(M^{4k})$.

Evidently $\tau(-z) = -\tau(z)$ and $\tau(\lambda z) = \tau(z)$, if $\lambda > 0$.

§ 2. The basic lemma

Assume W^n is an open manifold and assume V^{n-1} is a submanifold separating W^n into W_1 and W_2 in such a way that $W_1 \bigcup W_2 = W^n$ and $W_1 \bigcap W_2 = V^{n-1}$. We assume that V and W are smooth (or PL) manifolds and the embedding $i \colon V^{n-1} \subset W^n$ is smooth or PL. Now, given a continuous (not necessarily smooth or PL) mapping $T \colon W^n \to W^n$ such that the intersection $TV^{n-1} \bigcap V^{n-1}$ is empty, while V^{n-1} and TV^{n-1} cobound a connected piece of the manifold W^n. Moreover, require that the mapping $W^n \to W^n/T$ is a covering, so that the intersection $TN \bigcap N$ coincides with TV^{n-1} and so that W^n is as

$$W^n = \bigcup_l T^l N.$$

Under the conditions above, the following lemma holds.

Basic lemma. *For any element $z \in H_{4k}(V^{n-1}, Z)/\mathrm{Tor}$ such that $i_* z \neq 0 \bmod \mathrm{Tor}$, $T_* i_* z = i_* z$ and the membrane between z and Tz lies in N, then*

$$\tau(z) = \tau(i_* z)$$

provided that either of the following holds:

a) $n = 4k+1, V^{n-1}$ *is compact*, $z = [V^{n-1}]$;

[1]To prove the stabilization, it is convenient here to select in K_j finite subcomplexes $\overline{K}_j \subset K_j$ such that $\overline{K}_j \subset \overline{K}_{j+1}$ and $\bigcup_j \overline{K} = K$, and carry out the argument for $\overline{K}_j$.

b) *n is arbitrary, but the group* $H_{2k+1}(W^n, R)$ *has no T-free elements.* (*This means that for any* $\alpha \in H_{2k+1}(W^n, R)$ *there is an index* $q = q(\alpha)$ *such that*

$$\alpha = \sum_{l=1}^{q} \lambda_l T_*^l \alpha.$$

For example, this is satisfied if the group $H_{2k+1}(W^n, R)$ *is finite-dimensional.*)

PROOF. Denote by i_1 and i_2, the embeddings $V^{n-1} \subset W_1$ and $V^{n-1} \subset W_2$, respectively, and denote by $J_l \subset H^{2k}(V^{n-1}, R)$ the image $H^{2k}(W_l, R)$. On J_l, the form $(x^2, z) = \langle x, x\rangle$ is defined. The signature of this form coincides with the signature of the cycle $i_{l*}z \in H_{4k}(W_l)$, as shown in § 1. We have:

Lemma 2.1. $\tau(i_{l*}z) = \tau(i_*z)$, $l = 1, 2$.

The proof of Lemma 2.1 follows from Lemma 1.2. Indeed, for the proof of the equation $\tau(i_{l*}z) = \tau(i_*z)$ one should set

$$K_1 = N \cup T^{-1}N, \dots, K_i = K_{i-1} \cup T^{i-1}N \cup T^{-i}N, \dots, K = W^n,$$

and analogously decompose

$$W_2 = \bigcup_j K_j', \quad K_i' = T^{-i}K_i, \quad W_2 = K',$$

take into account that T^q homeomorphically maps K_q' onto K_q, and apply Lemma 1.2 from § 1.[1]

From the proof of Lemma 2.1 we have:

Lemma 2.1′. *Let* J *be the image* $i^*H^{2k}(W^n, R)$. *Then the nondegenerate part of the form on* J_l, $l = 1, 2$, *can be chosen with support on* $J = J_1 \cap J_2$.

In order to finish the proof of the basic lemma, we need to establish that the signature of the quadratic form on J coincides with the signature of the quadratic form on the entire group $P = H^{2k}(V^{n-1}, R)$.

1. Assume first $n = 4k + 1$ and $z = [V^{n-1}]$. Suppose that $\alpha \in P$ and $\langle \alpha, x\rangle = 0$, $x \in J_1$. Then the element $\alpha \cap [V^{n-1}] = \beta \in H_{2k}(V^{n-1}, R)$,

[1] Certainly, the main role is played by the T-invariance of i_*z, and the condition on the film cobounding z and Tz, where $z \in H_{4k}(V), Tz \in H_{4k}(TV)$.

obviously satisfies $(\beta, x) = 0$, $x \in J_1$. This means that $i_{1*}\beta = 0$. Since $i_{1*}\beta = 0$, the self-intersection index $\beta \circ \beta = 0$. Thus

$$(\alpha^2, [V^{n-1}]) = \beta \circ \beta = 0.$$

From the algebraic properties of the signature (see § 1, b)) we conclude that the signature of the form on J_1 coincides with the signature of the form on P, the latter signature being equal $\tau(z) = \tau(V^{n-1})$.

The theorem is proved for $n = 4k + 1$.

2. Now assume $n > 4k + 1$. It follows from Lemmas 2.1 and 2.1′ and properties of the signature (see § 1, a)) that the signature $\tau(i_* z)$, which coincides with the signature of the form on $J \subset P$, is equal to the signature of the form on the space P', the latter defining the linear envelope of J_1 and J_2.

Now suppose that $\alpha \in P$ and $\langle \alpha, x \rangle = 0$, $x \in P'$. Consider the element $\beta = \alpha \cap z \in H_{2k}(V^{n-1}, R)$. Since $(\beta, x) = 0$, $x \in P'$, we have $i_{1*}\beta = i_{2*}\beta = 0$. The two membranes ∂_1 and ∂_2, spanning the cycle representing the element in W_1 and W_2, respectively, define a cycle $\delta = \partial_1 - \partial_2$, to be considered as an element $\delta \in H_{2k+1}(W^n, R)$. Since by assumption

$$\delta = \sum_{l=1}^{q(\delta)} \lambda_l T_*^l \delta,$$

there exists a $2k+2$-chain c_0 in W^n, whose boundary defines this relation. Set

$$c = c^0 + \sum_{l=1}^{q(\delta)} \lambda_l T^l c^0 + \cdots + \sum_{l_1, \ldots, l_m} \lambda_{l_1} \cdot \lambda_{l_2} \cdots \lambda_{1_m} T^{l_1 + \cdots + l_n} c_0 + \cdots .$$

Though c is a noncompact chain, its compact boundary in δ, and the intersection $c \cap V^{n-1}$ is compact. However, the boundary of the intersection $\partial(c \cap V^{n-1})$ is exactly β. Therefore

$$\beta = \alpha \cap z = 0 \quad \text{and} \quad (\alpha^2, z) = 0.$$

The basic lemma is proved.

Remark. As V. A. Rokhlin pointed to me, in the part of the basic lemma related to $n = 4k + 1$, it is essentially proved that if M^{4k} is one of the boundary components of any (say, open) manifold W_1^{4k+1}, then

$$\tau(M^{4k}) = \tau(i_{1*}[M^{4k}]);$$

the conclusion concerning the signature $\tau(i_* z)$ in the union $W = W_1 \cup W_2$ therefore proved by using the transformation $T\colon W \to W$. One can avoid this to prove an analogue of the lemma for the case when W is an open manifold and M^{4k} is a separating compact cycle, thus the transformation T essentially does not play a great role here. However, for $n = 4k+2$ this argument not using T, has not been successfully applied in the homotopy theorem.

§ 3. Theorems on homotopy invariance. Generalized signature theorem

Consider a closed manifold M^n where $n = 4k+m$. Given an element $z \in H_{4k}(M^n, Z)/\mathrm{Tor}$ whose dual $D_z \in H^m(M^n, Z)$ is a product of indivisible elements $D_z = y_1 \cdots y_m \bmod \mathrm{Tor}, y_i \in H^1(M^n, Z)$. We define the covering $p\colon \widehat{M} \to M^n$, under which a path $\gamma \subset M^n$ is closed if and only if $(\gamma y_j) = 0$, $j = 1, \dots, m$. Evidently, we have an action of the monodromy group on $\widehat{M}$, this monodromy group is generated by mutually commuting transformations $T_1, \dots, T_m\colon \widehat{M} \to \widehat{M}$.

Lemma 3.1. *There exists an element $\widehat{z} \in H_{4k}(\widehat{M}, Z)$ such that $T_j \widehat{z} = \widehat{z}$, $j = 1, \dots, m$, and $p_* \widehat{z} = z$.*

PROOF. Let us realize the cycles $Dy_j \in H_{n-1}(M^n, Z)$ by submanifolds $M_i^{n-1} \subset M^n$, and realize z by the intersection

$$M^{4k} = M_1^{n-1} \cap \cdots \cap M_m^{n-1}.$$

It is easy to see that all paths lying in M^{4k} are covered by closed loops. Thus there is a well-defined covering embedding $M^{4k} \subset \widehat{M}$, which delivers the required element $\widehat{z}$. The lemma is proved.

Now consider the Serre fibration

$$q\colon M^n \xrightarrow{\widehat{M}} T^m,$$

where the base space has the homotopy type of T^m, the total space is of type M^n and the fiber is of type $\widehat{M}$. This fibration is dual to the covering. It is defined in a homotopically invariant manner. Evidently the term $E_2^{m,4k}$ of the homology spectral sequence is isomorphic to the subgroup $H_{4k}^{\mathrm{inv}} \subset H_{4k}(\widehat{M}, Z)$, consisting of elements which are invariant under the action of the monodromy group. We have a group $E_\infty^{m,4k} \subset E_2^{m,4k}$, consisting of cycles of all differentials of the spectral sequence of the covering.

Lemma 3.2. *The subgroup $E_\infty^{m,4k}$ is infinite cyclic. It is precisely the group $H_n(M^n) = Z$, and*

$$E_\infty^{m-1,4k+1} = \cdots = E_\infty^{1,4k+m-1} = 0.$$

PROOF. The fact that $E_\infty^{m,4k}$ is a quotient of $H_n(M^n)$, is a consequence of the definition of filtration in the homological spectral sequence. Therefore it is a cyclic group. We note that $E_\infty^{m,4k}$ is infinite and the corresponding element was constructed in Lemma 3.1. Therefore $E_\infty^{m-s,4k+s}$ is trivial for $s > 0$. The lemma is proved.

As a corollary of Lemmas 3.1 and 3.2 we get:

Lemma 3.3. *There exists a unique element $\widehat{z} \in H_{4k}(\widehat{M}, Z)$ such that $T_*\widehat{z} = \widehat{z}$, and in terms $E_\infty^{m,4k}$ of the covering spectral sequence the element $\widehat{z} \bigotimes [T^m]$ belongs to the group $E_\infty^{m,4k} = Z$, i.e. $\widehat{z} \bigotimes [T^m]$ is a cycle of all the differentials; $[T^m]$ is the fundamental cycle of the torus.*

Lemma 3.3 is a unification of Lemmas 3.1 and 3.2 with the additional observation that in Lemma 3.1 an element of $E_\infty^{m,4k}$ was explicitly constructed. The element $\widehat{z}$ indicated in Lemma 3.3, will be called canonical.

Theorem 3.4. *For $m = 1$ and $m = 2$ with the additional condition that the group $H_{2k+1}(\widehat{M}, R)$ is finite-dimensional, we have the formula for indivisible $z \in H_{4k}(\widehat{M}^n, Z)$, $Dz = y_1, \ldots, y_m$:*

$$(L_k(M^n), z) = \tau(\widehat{z}),$$

where z is a canonical element. In particular, this scalar product is a homotopy invariant.

Corollary 3.5. *The rational class $L_k(M^{4k+1})$ is a homotopy invariant.*

We note for example that if $\pi_1(M^5) = Z$ and $p_1(M^5) \neq 0$, then the group $\pi_2(M^5)$ is infinite, although this may not be seen in homology. The resulting formula makes it possible to define $L_k(M^{4k+1})$ for all homology manifolds.

Corollary 3.6. *The class $L_k(M^{4k+2})$ of a manifold of T^{4k+2} homotopy type is trivial. The scalar product of $L_k(M^{4k} \times T^2)$ with the cycle $z = [M^4k] \times 0$ is homotopically invariant and is equal to $\tau(M^{4k})$.*

It would be interesting to deal with the question as to whether there exist invariant relations on the stable tangent bundle other than those which are given by the J-functor and Theorem 1 for $n = 4k + 1$ under the assumption that the group π_1 is commutative and $H^{4k}(M^n) = 0$, $i < k$.

PROOF OF THEOREM 1. First we consider the case $m = 1$, $n = 1 + 4k$. In this case the elements z and $\widehat{z}$ are indivisible. From the fundamental lemma proved in § 2, we have

$$\tau(\widehat{z}) = \tau(M^{4k}),$$

where $M^{4k} \overset{i}{\subset} \widehat{M}$ and $\widehat{z} = i_*[M^{4k}]$. On the other hand, $z = p_*\widehat{z}$ and

$$L_k(\widehat{M}) = p^* L_k(M^{4k+1}).$$

Therefore

$$(L_k(\widehat{M}), \widehat{z}) = \tau(M^{4k}) = \tau(\widehat{z}) = (L_k(M^{4k+1}), z).$$

For $m = 1$ the theorem is proved.

Now we turn to the case $m = 2$. We recall first that the element z is indivisible, where $Dz = y_1 y_2$. The indivisible elements Dy_1, Dy_2 are realized by submanifolds M_1^{n-1} and M_2^{n-1}, and the element z is realized by their intersection

$$M^{4k} = M_1^{n-1} \cap M_2^{n-1}.$$

Consider the covering $p\colon \widehat{M} \to M^n$ defined above. The manifold $M^{4k} \subset M_1^{n-1}$ defines an indivisible element $z_1 \in H_{4k}(M_1^{n-1})$. By the previous lemma for $m = 1$ we conclude that on $i\colon \widehat{M}_1^{n-1} \subset \widehat{M}$, covering M_1^{n-1}, there is one cycle $\widehat{z}_1$ such that

$$\tau(\widehat{z}_1) = (L_k(M_1^{n-1}), z).$$

The mapping $T_2\colon \widehat{M} \to \widehat{M}$ is such that the basic lemma can be applied to the ball $\widehat{M} \supset \widehat{M}_1^{n-1}$ and to the elements $\widehat{z}_1$, $i_*\widehat{z}_1$. Thus,

$$\tau(\widehat{z}_1) = \tau(i_*\widehat{z}_1).$$

Accordingly

$$\tau(i_*\widehat{z}_1) = (L_k(M_1^{n-1}), z_1) = \tau(M^{4k}).$$

But $i_*\widehat{z}_1 = \widehat{z}$ and $\tau(M^{4k}) = (L_k(M^n), z)$, which implies Theorem 3.4 for the indivisible cycle z. The theorem is proved.

Now suppose that $z = \lambda z'$ and $Dz = y_1 y_2$, where y_1, y_2 are indivisible elements of the group $H^1(M^n, z)$. As before, suppose that

$M^{4k} = M_1^{n-1} \cap M_2^{n-1}$ and that on M_1^{n-1} and M_2^{n-1} the manifold M^{4k} realizes z_1 and z_2, respectively. If at least one of z_1 or z_2 is indivisible, then the former argument works. Moreover, if $z_1 = \lambda_1 z_1'$ and $z_2 = \lambda_2 z_2'$, then for $\widehat{M}_1^{n-1}$ and $\widehat{M}_2^{n-1}$ we have:

$$(L_k(M_l^{n-1}), z_l) = \lambda_l (L_k(M_l^{n-1}), z_l') = \lambda_l \tau(\widehat{z}_l') = \lambda_l \tau(i_{l*}\widehat{z}_l') = \lambda_l \tau(\widehat{z}),$$

since $\tau(\mu\widehat{z}) = \tau(\widehat{z})$ for $\mu > 0$, $l = 1, 2$, $\lambda_l > 0$. Thus $\lambda_1 = \lambda_2$, if $\tau(\widehat{z}) \neq 0$. Thus, the cycles z_1 and z_2 are divisible by one and the same number $\mu = \lambda_1 = \lambda_2$.

Remark. M^{4k} separates each of M_1^{n-1} and M_2^{n-1} into exactly μ pieces, $a_1, \ldots, a_\mu$ and $b_1, \ldots, b_\mu$, respectively, where

$$M_1^{n-1} = \bigcup_j a_j,$$
$$M_2^{n-1} = \bigcup_j b_j.$$

The pieces a_j and b_j are cyclically ordered. Therefore the boundary of each of those pieces is split into two parts ∂_j' and ∂_j'' for a_j and δ_j' and δ_j'' and b_j, passing one after another in the cyclic order.

The preceding argument yields:

Theorem 3.7. *If the element $z \in H_{4k}(M^n, Z)$ is divisible by λ, where $Dz = y_1 y_2$ and y_1, y_2 are indivisible elements of the group $H^1(M^n, Z)$, then the scalar product $(L_k(M^n), z)$ is equal to $\mu\tau(\widehat{z})$, where $\widehat{z}$ is a canonical element and μ is a divisor of λ.*

Corollary 3.8. *If $\tau(\widehat{z}) = 0$, then the scalar product $(L_k(M^n), z)$ is homotopically invariant and is equal to zero. Since z/λ is an integral indivisible class, then*

$$\left(L_k(M^n), \frac{z}{\lambda}\right) = \mu(\lambda)\tau(\widehat{z}).$$

If $\tau(\widehat{z})$ and λ are coprime, then $\mu = \lambda$. The scalar product $(L_k(M^n), z)$ may have only finitely many values $\mu_i\tau(\widehat{z})$, where μ_i are divisors of λ.

Remark. It was shown here that if we have two indivisible cycles $M_1^{n-1}, M_2^{n-1} \subset M^n$, $n = 4k + 2$, and their intersection is divisible by λ, and is not equal to zero, then in each of them the intersection with z is divisible by one and the same number μ provided that $\tau(\widehat{z}) \neq 0$. Moreover,

$$\mu = \frac{(L_k(M^n), z)}{\tau(\widehat{z})},$$

and therefore μ is topologically invariant (see the next section). Is it possible to prove that μ is always equal to λ?

Example 3.9. Theorem 1 states a fair question: why is the formula $(L_k(M^n), z) = \tau(z)$ not true, rather than the formula $(L_k(M^n), z) = \tau(\widehat{z})$? *A priori* it would be natural to expect just such a formula.

Concerning that, I wish to show on the simplest examples that such a formula is "generally" false. We say that a manifold M_1^{n-1} has homology type of M_0^n if there exists a mapping $f\colon M_1^n \to M_0^n$, inducing an isomorphism of all the homology groups.

We consider $M_0^n = S^1 \times S^{4k}$ and show that there exist infinitely many manifolds M_i^n of "homology type" of $S^1 \times S^{4k}$ with different Pontrjagin class $p_k(M_i^n)$ such that $\pi_1(M_i^n) = Z$ and all $\pi_l(M_i^n) = 0$, $1 < l < 2k$. Moreover, for $k \geq 2$, among the manifolds M_i^n there are those for which the class $p_k(M_i^n)$ is fractional and therefore they are homotopically nonequivalent to smooth manifolds.

Consider the functor $J_{PL}(M_0^n)$ and take a stable microbundle η_{PL} which is J-equivalent to the trivial one. We form the Thom complex T_N. Since the fundamental cycle for it is spherical, we may apply a customary method to reconstruct the preimages of $S^{N+n} \to T_N$, by pasting together the kernel of the mapping π_1, all groups π_l of this preimage up to $l = 2k-1$ and the kernel of the map in dimension $l = 2k$, but only in homology. We get a preimage of M_i^{4k+1} with the given "normal" microbundle. Since the functor J_{PL}^0 is finite, we get the desired result: the class p_k can be varied. By Poincaré duality, the homology type of M_i^{4k+1} is as desired.

Example 3.10. In an analogous way we now show that in the part of Theorem 1, devoted to codimension 2, it is impossible to remove the restriction on finite-dimensionality of the group $H_{2k+1}(\widehat{M}, R)$.

Consider the direct product of $T^2 \times S^{4k}$ and its J-functor. We again select a J-trivial bundle over $T^2 \times S^{4k}$ and denote its Thom complex by T_N. Take an element $\alpha \in H^{-1}[T_N]$ and a representative $f_\alpha : S^{N+n} \to T_N$ of α. By Morse surgery over

$$M_\alpha^n = f_\alpha^{-1}(T^2 \times S^{4k})$$

we may get that

$$\pi_1(M_\alpha^n) = Z + Z$$

and

$$\pi_i(M^n_\alpha) = 0, \quad i \le 2k.$$

However if we choose a J-trivial bundle such that $p_k \neq 0$, we would have:

$$p_k(M^n_\alpha) \neq 0;$$

and at the same time $\tau(\widehat{z}) = 0$, since

$$H_{2k}(\widehat{M}) = \pi_{2k}(\widehat{M}) = 0,$$

where $\widehat{M}$ is the universal cover of M^n_α. Therefore we can deduce that $\pi_{2k+1}(\widehat{M}) = H_{2k+1}(\widehat{M})$ is of infinite type if $p_k(M^n_\alpha) \neq 0$.

§ 4. The topological invariance theorem

We consider a cycle $x \in H_{4k}(M^n, Z)$ for $n = 4k+2$ such that $(Dx)^2 = 0$ mod Tor. Under these conditions we have:

Theorem 4.1. *The scalar product $(L_k(M^n), x)$ is a topological invariant. Here we may assume that M^n is complex which is a homology manifold over Q.*

PROOF. We find an integer λ such that $(D(\lambda x))^2 = 0$. Realize λx by a submanifold $M^{4k} \subset M^n$. It is known that the normal bundle to M^{4k} in M^n is trivial. There is a well-defined embedding $M^{4k} \times R^2 \subset M^n$, which represents an open neighborhood $U = M^{4k} \times R^2$ of M^{4k}. Evidently,

$$(L_k(M^n), x) = \frac{1}{\lambda}\tau(M^{4k}).$$

Now we choose on M^n another smooth (or PL) structure. We denote the class in this smooth (or PL) structure by $L'_k(M^n)$. Let us prove that

$$(L'_k(M^n), \lambda x) = \tau(M^{4k}).$$

The new smooth (PL) structure induces a structure on the neighborhood $U = M^{4k} \times R^2$ and the neighborhood $W = U \backslash (M^{4k} \times 0)$, since U and W are both open. W is homeomorphic to $M^{4k} \times S^1 \times R$. Denote the coordinate along S^1 by φ, and the coordinate along R by t. The coordinate system (m, φ, t) is not smooth in the new smooth structure, $m \in M^{4k}$. Evidently, $H_{4k+1}(W) = Z$ is generated by the cycle $M^{4k} \times S^1 \times 0$. Let us realize this cycle by a smooth submanifold $V^{4k+1} \subset W$ in the new smooth structure. There is a well-defined projection of degree $+1$:

$$f\colon V^{4k+1} \to M^{4k} \times S^1, \quad \widehat{f}\colon \widehat{V} \to M^{4k} \times R.$$

Thus on V^{4k+1} there is a $4k$-dimensional cycle $z \in H_{4k}(V^{4k+1})$ such that $z = [M_*^{4k} \times 0]$, however, it might not be unique. But the scalar product $(L_k(V^{4k+1}), z)$ does not depend on the choice of such cycle z.

Consider the covering $p\colon \widehat{W} \to W$, which lifts all closed loops $M^{4k} \times 0$ to closed loops. Evidently $\widehat{W}$ is homeomorphic to $M^{4k} \times R \times R$. The full preimage $\widehat{V} = p^{-1}(V^{4k+1})$ covers V^{4k+1} with the same monodromy group. There is an invariant cycle $\widehat{z} \in H_{4k}(\widehat{V})$ such that[1]

$$f_* p_* \widehat{z} = [M^{4k} \times 0], \quad \widehat{z} = D\widehat{f}^* D[M^{4k}].$$

From Theorem 3.4 in § 3 we conclude that

$$\tau(\widehat{z}) = (L_k(V^{4k+1}), p_*\widehat{z}) = (L_k'(M^n), \lambda x).$$

Since $V = V^{4k}$ is compact, we may assume that $\widehat{V}$ lies between the levels $t = 0$ and $t = 1$ in $\widehat{W}$.

Consider the (nonsmooth) transformation $T'\colon \widehat{W} \to \widehat{W}$ such that

$$T'(m, \varphi, t) = (m, \varphi, t + 1).$$

Denote the inclusion $\widehat{V} \subset \widehat{W}$ by i. Obviously, $T_*' i_* \widehat{z} = i_* \widehat{z}$ and the group $H_{2k+1}(\widehat{W}) = H_{2k+1}(\widehat{M}^{4k})$ is finite-dimensional. By the basic lemma in § 2 we conclude that

$$\tau(\widehat{z}) = \tau(i_* \widehat{z}).$$

However, $i_* \widehat{z}$ realizes the cycle $M^{4k} \times 0 \times 0$ on $\widehat{W} = M^{4k} \times R$. Thus

$$\tau(i_* \widehat{z}) = \tau(M^{4k}).$$

Since $\tau(\widehat{z}) = (L_k'(M^n), \lambda x)$, we get that

$$(L_k'(M^n), \lambda x) = \tau(M^{4k}).$$

The theorem is proved.

[1]The cycle $z = p_* \widehat{z} \in H_{4k}(V)$ is obtained by intersection $(M^{4k} \times 0 \times R) \cap V$ and V from the homological point of view. The same is true for $\widehat{z}$ on $\widehat{V}$.

Remark. Rokhlin drew my attention to the fact that for the manifold $V = V^{4k+1} \overset{i}{\subset} W$, constructed in the proof of Theorem 4.1, there is a cycle $z \in H_{4k}(V, Z)$ such that

$$\tau(i_* z) = \tau(z) = \tau(M^{4k}).$$

This shows that $\tau(z) = \tau(\widehat{z})$ for the case at hand, which, generally speaking, is not true for arbitrary $4k+1$-dimensional manifolds, as shown in § 3 for simple examples. It is interesting, however, that anyway we have to use coverings, since the formula from § 3 for $L_k(V)$, refers to the cycle $\widehat{z}$, and we use it in the proof.

§ 5. Consequences of the topological invariance theorem

We collect in this section some consequences of Theorem 4.1. Obviously one has the following:

Corollary 5.1. *The class $L_k(M^{4k+2})$ is topologically invariant on the subgroup $H \subset H_{4k}(M^{4k+2})/\mathrm{Tor}$, which admits a basis $x_1, \dots, x_s \in H$ such that $Dx_j^2 = 0 \bmod \mathrm{Tor}$. Here M^{4k+2} is a smooth (or PL)-manifold. For example, for an M^{4k+2}, which is a direct product of any collection of spheres, this is always so.*

Now suppose that M^{4k+2} is any simply connected manifold for which the subgroup $H \subset H_{4k}(M_i^{4k+2})/\mathrm{Tor}$ is nontrivial. Since the functor $J^0_{PL}(M^{4k+2})$ is always finite, we may apply the "realization theorem" for tangent bundles and obtain an infinite collection of PL-manifolds M_i with distinct values of the class $L_k(M_i^{4k+2})$ on the subgroup H, so that there does not exist any mapping $M_i^{4k+2} \to M_j^{4k+2}$ which takes the class into the class. If we wish to obtain smooth manifolds, then we must use the functor $J^0 = J^0_{S_0}$. Here, however, for $k \neq 1, 3$, we are obstructed by the Arf-invariant of Kervaire (for these results see [2], § 14, Appendices I and II). This may be avoided if instead of M^{4k+2} one chooses the homotopy type $M^{4k+2} \# M^{4k+2}$ (in the class of PL-manifolds the Arf-invariant does not obstruct the construction of such manifolds). Thus one obtains the following:

Theorem 5.2. *If the subgroup $H \subset H_{4k}(M^{4k+2}, Z)/\mathrm{Tor}$ for a simply connected manifold M^{4k+2} is nontrivial, then there exists an infinite family of PL-manifolds of homotopy type M^{4k+2} which are not homeomorphic to each other. If $n = 6, 14$, this is also true in the class of smooth manifolds.*

In the class of smooth manifolds there exists an infinite collection of pairwise nonhomeomorphic manifolds of homotopy type $M^{4k+2}\#M^{4k+2}$.

If, for instance, $M^{4k+2} = S^2 \times S^{4k}$, then for $k \geq 2$ one may indicate among these manifolds those which will have a fractional Pontrjagin class $P_k(S^2 \times S^{4k})$ and, accordingly, will be nonhomeomorphic to smooth manifolds, although their homotopy type is $S^2 \times S^{4k}$.

Remark. For $S^2 \times S^{4k}$ such manifolds may be obtained by using Morse surgery over diffferent Haefliger knots $S^3 \subset S^6$. If we choose these manifolds for the type $S^2 \times S^{4k}$ and perform the Morse surgery over S^2, then for equal values of the class p_k we will get distinct nodes $S^{4k-1} \subset S^{4k+2}$.

We define the notion of "topological knot with trivial microbundle". This is an embedding

$$S^n \times R^k \subset S^{n+k},$$

where the equivalence is a homeomorphism preserving the fiberwise structure around $S^n \times 0$. From our results it follows that the knots

$$S^{4k-1} \times R^3 \subset S^{4k+2}, \quad k \geq 1,$$

distinguished by the class p_k of the reconstructed manifold of topological type $S^2 \times S^{4k}$, are not equivalent as topological knots with microbundle.

We note finally that for certain manifolds, for example, for the homotopy type $S^2 \times S^{4k}$ and their sums connected with one another, the "Hauptvermutung" follows. Here the point is that from the results of Appendix II of [2] one may extract the fact that the rational Pontrjagin class in this case is a complete combinatorial invariant. Since it is topologically invariant, we also find by using a simple comparison of invariants that from the existence of a continuous homeomorphism there follows the existence of a piecewise linear homeomorphism. However, no such approximation theorems are proved here. From the homeomorphism, we have used for the proof of the theorem only the fact that sets which are open with respect to both smooth structures, are smooth open manifolds with the same set of cycles. Moreover, our method makes it possible to define the classes L_k of the topological manifold M^{4k+2}. In essence the proof is only that for an arbitrarily introduced smooth structure in the scalar product of the class L_k with a cycle is the same. But it is necessary to use a smooth structure, because it makes it possible to discover a large collection of submanifolds realizing cycles. This is hardly the case for purely topological manifolds.

Appendix (V. A. Rokhlin[1]). Diffeomorphisms of the manifold $S^2 \times S^3$

I want to indicate our further application of the theorem on the topological invariance of the class L_k in codimension two[2]: there exist diffeomorphisms of smooth manifolds, for example, diffeomorphisms of the manifold

$$V = S^2 \times S^3,$$

which are homotopic but not topologically isotopic.

The following elementary arguments are necessary for the proof. To each mapping $f\colon V \to V$ there corresponds a composite mapping

$$S^3 \to V \xrightarrow{f} V \to S^2,$$

where the first arrow denotes the natural mapping of the sphere S^3 onto some fiber $a \times S^3$ of the product $S^2 \times S^3$, and the third arrow is the projection of this product onto the first component. The absolute value of the Hopf invariant of this composite mapping is defined by the homotopy class of the mapping f and will be denoted by $\gamma(f)$. The number of homotopy classes of mappings $f\colon V \to V$ with a given value of $\gamma(f)$ is infinite, but it becomes finite if we restruct ourselves to classes consisting of homotopy equivalences. In particular, there exists only a finite number of pairwise nonhomotopic diffeomorphisms $f\colon V \to V$ with a given value of $\gamma(f)$.

Now consider the manifold $S^2 \times D^4$ with boundary V and denote by M_f the smooth manifold obtained from two copies of W by pasting them by a diffeomorphism $f\colon V \to V$. The homology groups of M_f do not depend on f, i.e. they are the same as those of the product $S^2 \times S^4$ (which corresponds to the identity diffeomorphism $V \to V$), and the multiplicative structure of the integer-valued homology ring is defined by the formula

$$u_2^2 = \pm\gamma(f) u_4,$$

where u_2 and u_4 are the generators of the groups $H^2(M_f; Z)$ and $H^4(M_f; Z)$. In particular, $\gamma(f)$ is a homotopy invariant of the manifold M_f.

Denote by K the class of all manifolds diffeomorphic to the manifolds M_f, and denote by K_0 the class of smooth six-dimensional manifolds topologically equivalent to the product $S^2 \times S^4$.

[1]From a letter of January 20, 1965, from V. A. Rokhlin to the author. The letter was a reply to my note [1] sent to V. A. Rokhlin, and was published with permission (this and the footnotes which follow are due to S. P. Novikov).

[2]I.e. Theorem 1 of the present paper.

Lemma. $K_0 \subset K$.

Proof. Suppose $M \subset K_0$. Then the generator of the group $H_2(M)$ is realized by a smooth embedding of a sphere, and the normal bundle of this sphere, having the invariant homotopy type of the manifold M, is trivial. Accordingly, a tubular neighborhood of this sphere is diffeomorphic to W. If one diffeotopically carries this sphere beyond the limits of this tubular neighborhood, the normal bundle of the sphere will remain trivial, and, as shown by standard calculations, its embedding into the closed complement of the tubular neighborhood will be homotopic to an equivalence. From Smale's theorem it therefore follows that this closed complement is diffeomorphic to W, hence $M \in K$.

Proof of the Theorem. Let $M_1, M_2, \dots$ be pairwise nonhomeomorphic manifolds lying in K_0.[1] From the lemma there exist diffeomorphisms $f_n \colon V \to V$, such that M_n and M_{f_n} are diffeomorphic. Since $\gamma(f)$ is a homotopy invariant of the manifold M_f, we have $\gamma(f_n) = 0$, and since there are only a finite number of pairwise nonhomotopic diffeomorphisms

$$f \colon V \to V \text{ with } \gamma(f) = 0,$$

it follows that there exist indices k, l such that the diffeomorphisms f_k and f_l are homotopic. They are not isotopic, and moreover the diffeomorphism $f_k f_l^{-1} \colon V \to V$ does not extend to a homeomorphism of the manifold W, since otherwise the manifolds M_{f_k} and M_{f_l} would be homeomorphic.

This proof can be made more effective, replacing the rough finiteness arguments by a precise homotopy classification of diffeomorphisms of the manifold V.

One can also give a complete homotopy and differential classification of manifolds of the class K. As for the topological classification, it coincides with the differential one (as holds for manifolds of the class K_0) if the class $p_1(M_f)$ is topologically invariant. The obvious generalization of the preceding lemma shows that the class K contains all the smooth six-dimensional manifolds homotopically equivalent to the total manifolds of orthogonal bundles with basis S^4 and fiber S^2.

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5

On manifolds with free abelian fundamental group and their applications (Pontrjagin classes, smooth structures, high-dimensional knots)[1]

S. P. Novikov

In this paper we establish topological invariance of rational Pontrjagin classes on smooth and piecewise-linear manifolds and give several corollaries. These methods can be applied to other problems.

Introduction

As shown in previous papers by the author [10–13], the question of topological invariance of rational Pontrjagin classes is closely connected to some problems of homotopy and differential topology of non-simply connected manifolds and their covering spaces, such that the fundamental group of the non-simply connected manifold is free abelian. The reduction of the invariance problem to homotopical problems in these series of papers has one common idea. The idea is based on the fact that one can make reasonable conclusion from the notion "continuous homeomorphism" by

[1]Izvestiya AN SSSR, ser. matem., 1966, vol. 30, c. 207–246 (received August 25, 1965). — Translated by V.O. Manturov.

using special non-simply connected open subsets to be studied later by means of purely smooth topology, using non-simply connectedness, though the fundamental group has no relation to the initial problems. Thus, in the initial work [10, 13], this problem was solved by analogues of the Hirzebruch formula for covering spaces, which yielded the difference of homeomorphism and homotopy type. The direct development of this "signature" method led the author to the proof of the topological invariance of the Pontrjagin–Hirzebruch class $L_k(M^n)$ for $n \leq 4k+3$. This intermediate argument is given in the appendix; it was found before the general result [11] appeared, and it generally lost its initial importance because the author could find a general proof of the invariance of classes (short publication see in [11]), with no "signature" arguments and analogues of the Hirzebruch formula.

Here, a solution to the problem of classes is given by using a generalization of the technique of [3, 14] to the non-simply connected case for studying smooth structures on manifolds of type $M^n \times R$, $\pi_1(M^n) = Z + \cdots + Z$, though the reduction of the problem to such problem of differential topology is similar to that from author's work [10] on topological invariance of classes. At this moment, it was useful for us to perform this work, to receive the manuscript of W. Browder (soon published in [4]), where the problem of smooth structures on manifolds of the type $M \times R$ was solved for the simply connected case $\pi_1(M) = 0$. Some ideas from [4] helped the author to perform this work, and the author expresses his gratitude to W. Browder.

The results are formulated in § 1. The central result is Theorem 1, which establishes topological invariance of rational Pontrjagin classes for smooth and piecewise-linear manifolds.

§ 2 is very important in our work: it contains the reduction of Theorem 1 to Theorem 3, and the connection to the other results. It is the place where we use the fact that the manifolds M_1 and M_2 from Theorem 1 are homeomorphc.

In §§ 3–8 we prove Theorem 3. § 5 is of a special interest: these results can be easily generalized for a larger class of groups.

In § 9 we prove Theorem 6 concerning knot theory.

In § 10 we formulate (without proof) one generalization of Theorem 5.

From Theorem 1 of the present work jointly with some previously known results we deduce several corollaries.

Some corollaries from the invariance of classes:

1. The number of smooth structures on a simply connected topological manifold M^n, $n \neq 4$ is finite, and does not exceed the constant $c(M^n)$, where

$$c(M^n) < e^{q_n + \sum_{i=2}^{n} b_{n-i} \ln c_i + \sum_{4k} d_{4k}},$$

whence

$$\begin{aligned} q_n &= \ln|\theta^n(\partial\pi)|, & d_i &= \ln|\operatorname{Tor} H_i(M^n)|, \\ b_j &= \max\nolimits_{p\geq 2} rkH_j(M^n,\, Z_p), & c_j &= a_j\,|\,\pi_{N+j}(S^N)|, \end{aligned}$$

$a_j = 1$ for $j \not\equiv 1, 2 \pmod 8$ and $a_j = 2$ for $j \equiv 1, 2 \pmod 8$. This corollary follow from comparing Theorem 1 with Boot periodicity and author's results concerning the diffeomorphism problem (see [14]).

An analogous finiteness result and estimate (with other universal constants c_i) hold for the number of combinatorial structures on M^n with the same restrictions. Here one should use the result of Surf that $\pi_0(\operatorname{diff} S^3) = 0$. This yields the Hauptvermuting up to a finite number of possible PL-structures for given restrictions. These results follow from [14, see Appendix 2].

2. As already shown in [10], for dimensions $4k + 2$ the invariance of Pontrjagin classes and Browder's results [see [3], [14], Appendix 1] yield the difference between homeomorphism and homotopy type of closed simply connected manifolds. From Theorem 1 it follows that for any simply connected manifold M^n, $n \geq 6$, for which at least for one $k \neq 0$, $\frac{n}{4}$ the homology group $H_{4k}(M^n)$ is infinite, there is an infinite number of smooth (possibly, outside a point) pairwise non-homeomorphic manifolds M^n having the same homotopy type of M^n. If the homology condition given above fails, the number of such manifolds is presumably finite, as it follows from [14].

3. On odd-dimensional spheres S^{2n+1}, $n \geq 3$, there exists a finite number of smooth (or smooth outside a point) actions of the circle S^1 without fixed points, which are pairwise distinct (not homotopically equivalent). This fact follows from the result of previous paragraph, applied to the quotient space S^{2n+1}/S^1 of homotopy type CP^n because topologically equivalent actions generate homeomorphic quotient spaces.

4. Since Pontrjagin numbers are topologically invariant, two smooth manifolds belonging to different classes of orientable cobordisms Ω_{SO}, are never homeomorphic.

5. All piecewise-linear manifolds with fractional Pontrjagin classes are not homeomorphic to smooth ones. In each dimension $n \geq 8$ many examples of such manifolds are known and many of them (though, not all) are homotopy equivalent to smooth ones.

6. The spaces of SO_n-fibrations with base S^{4k} and fiber R^n, disk D^n or S^{n-1} for $n > 4k + 1$ are completely classified from the topological point of view, by the Pontrjagin class of the bundle. This is true for many other examples. It is known for a long time (Dold) that there are only finitely many pairwise distinct homotopy types.

7. If for a smooth manifold M^n we have an elliptic integrodifferential operator A which takes the section of F_1 over M^n to the section of F_2 over M^n, then it defines the "symbol" $\sigma(A)$. This symbol defines an isomorphism of bundles F_1 and F_2, extended to $\tau(M^n)$ and then restricted to the subspace

$$\tau(M^n) \setminus M^n \subset \tau(M^n),$$

where $\tau(M^n)$ is the tangent bundle space for M^n with fiber R^n and $M^n \subset \tau(M^n)$ is the zero section. Since neither the space $\tau(M^n)$ nor $M^n \subset \tau(M^n)$ depend on the smoothness of M^n, the "symbol" σ is a topologically invariant notion; however, for different smooth structures on M^n the same symbol σ defines operators A_1, A_2 acting in different spaces, but such that $\sigma(A_1) = \sigma(A_2)$ (these operators are defined with ambiguity, but up to some quite continuous addition for each smooth structures). The well-known Atiyah–Singer operator expresses the index of the operator in terms of invariants of the triple (F_1, F_2, σ) independent of the smooth structure and Pontrjagin classes of the manifold M^n. From Theorem 1 we see that the index of the operator is defined only by the symbol independently from the smoothness on the manifold M^n; this index is the same for operators with the same (homotopic) symbols defined for different smooth structures.

8. The natural mapping $\pi_i(BSO) \to \pi_i(B\mathrm{Top})$ is monomorphic, and $H^*(B\mathrm{Top}, Q) \to H^*(BSO, Q)$ is epimorphic.

9. The mapping class group of diffeomorphisms for a simply connected manifold of dimension at least five is of finite index in the analogous subgroup for homeomorphisms [see [14], Theorems 6.9 and 6.10].

Finally, I express my gratitude to V. A. Rokhlin for various fruitful discussions and advices. Note that the invariance proof for rational Pontrjagin classes found by the author is in its major part a natural sequel of the papers by Rokhlin and Thom [15, 19] devoted to this problem. I also express my gratitude to S. P. Demushkin, I. R. Shafarevich, Yu. I. Manin for their help in algebraic questions which arose while performing this work, and to A. V. Chernavsky for questions related to Theorem 6.

§ 1. Formulation of results

From the application point of view, the main theorem of this chapter is the following:

Theorem 1. *Let M_1 and M_2 be two smooth (or PL)- manifolds and let $h\colon M_1 \to M_2$ be a continuous homeomorphism. Then*

$$h^* p_i(M_2) = p_i(M_1),$$

where $p_i(M_q)$, $q = 1, 2$, are rational Pontrjagin classes of M_1 and M_2.

Theorem 2. *Let M^{4k} be a closed manifold, and let W^{m+4k} be a smooth closed manifold of homotopy type $M^{4k} \times T^m$, where T^m is the m-dimensional torus, $\pi_1(M^{4k}) = Z + \cdots + Z$, $h\colon W^{m+4k} \to M^{4k} \times T^m$ is some homotopy equivalence. Then the following formula holds*:

$$(L_k(W^{m+4k}),\, h^*[M^{4k}] \otimes 1) = \tau(M^{4k}),$$

where L_k are the Hirzebruch polynomials, and τ is the signature of a manifold.

The condition $\pi_1 = Z + \cdots + Z$ in Theorem 2 can, certainly, be removed, but we are not going to do it here.

Now let W be an open smooth manifold of dimension $n+1$ having homotopy type of a closed n-dimensional manifold, for which a discrete action (possibly, non-smooth) $T\colon W \to W$ is given such that the quotient W/T is compact. Under these assumptions, the following theorem holds:

Theorem 3. *If $n \geq 5$ and $\pi_1(W)$ is isomorphic to a free abelian group then there exists a closed manifold V such that W is diffeomorphic to $V \times R$.*

This theorem is proved in §§ 3–8, and it yields Theorems 1 and 2 for the smooth case (see § 2). The case of PL-manifolds is completely analogous, and it requires only a combinatorial analog of Theorem 3, the latter is proved with no changes by using remarks from the author's paper [14] [see [14], Appendix 2, on combinatorial Morse surgery].

Among the remaining results, we indicate the following.

Theorem 4. *Let M^n be a smooth manifold, such that $\pi_1(M^n)$ is the free abelian group of rank k. Then the smoothness for a direct product $M^n \times R^q$ for $q > n$ is defined by the tangent bundle, and it may have only finite number of values.*

Theorem 5. *Let M^n be a smooth closed manifold, let $\pi_1(M^n) = \pi$ be the free abelian group of rank k and M^n has homotopy type of skew product with T^l as base and M^{n-l} as fiber, where M^{n-l} is a closed topological manifold. If $l \leq n-5$, then the covering space $\widehat{M}$ over M^n having homotopy type M^{n-l} is diffeomorphc to the direct product $M_1^{n-l} \times R^l$ where M_1^{n-l} is a closed smooth manifold.*

Theorem 5 follows directly from Theorem 3.

An indirect argument shows that Theorem 3 (or its analog) yields:

Theorem 6. *Let $S^n \subset S^{n+2}$, $n \geq 5$, be a topological locally-flat embedding. Then this embedding is topologically equivalent to a smooth embedding $S^n \subset S^{n+2}$ for some smooth structure on S^n. In particular, this embedding is globally flat.*

The reduction of Theorem 6 from previous results will be given at the end of the paper. Unlike Theorems 1, 2, 4, 5, this needs one supplementary argument (see § 9).

In § 10, we indicate (without proof) one generalization of Theorem 5.

§ 2. The proof scheme of main theorems

1. The main theorems will be proved according to the following plan.

a) First, we prove Theorem 3 (see §§ 3–8).
b) From Theorem 3 we will deduce: Theorem 1 for the simply connected case, Theorems 2, 4, 5 (see § 2). It is known that the general statement of Theorem 1 follows from the simply connected case. Moreover, it suffices to prove Lemma 2.1, (see ahead) only for spheres S^{4k}.
c) At the end, we shall give a separate proof of Theorem 6 based on Theorem 3 and its generalizations (see §§ 9,10).

2. The main part of the work is devoted to the proof of Theorem 3. Here we shall indicate the scheme of obtaining the statement of Theorem 1 for the simply connected case as well as the statement of Theorem 2, from Theorem 3.

The following lemma is, in principle, contained in papers [15], [16], [19]. It was communicated to the author by V. A. Rokhlin quite long ago.

Lemma 2.1. *Let W be an arbitrary smooth manifold homeomorphic to $M^{4k}\times R^m$, where M^{4k} is a simply connected closed manifold. If the formula*

$$(L_k(W),\,[M^{4k}])=\tau(M^{4k}),$$

always holds, then the rational Pontrjagin classes of simply connected smooth manifolds are topologically invariant.

Here L_k are the Hirzebruch polynomials, and τ is the signature of a manifold. We shall not give a proof of this lemma, assuming this is well known after the papers by Thom, Rokhlin, Schwarz, (see [15–17]), where it is essentially used, however, only for piecewise-linear homeomorphisms.

Our aim is to prove the following statement.

Lemma 2.2. *With the assumptions of Lemma* 2.1 *the following formula holds*:

$$(L_k(W),\,[M^{4k}])=\tau(M^{4k}).$$

Moreover, this formula holds for piecewise-linear manifolds and "combinatorial" *Pontrjagin classes.*

The argument to deduce Lemma 2.2 from Theorem 3, is, from the ideal point of view, the crux of the paper, since here we use the fact that two manifolds are homeomorphic. Indeed, Theorem 3 itself is unrelated to the invariance problem of Pontrjagin classes.

Let us give this deduction below.

We use the topological structure of the manifold W in the following way: the usual torus T^{m-1} can be smoothly realized in the Euclidean space $R^m \supset T^{m-1} \times R$; consider the open submanifold $i\colon W_1 \subset W$, where $W_1 \approx M^{4k} \times T^{m-1} \times R$, so that the embedding $i\colon W_1 \subset W$ is defined according to the homeomorphism $W \approx M^{4k} \times R^m$ and the embedding $T^{m-1} \times R \subset R^m$. Obviously, $i^* L_k(W) = L_k(W_1)$ and $i_*\colon H_{4k}(W_1) \to H_{4k}(W)$ is an epimorphism. Thus, instead of $L_k(W)$ we may study the class $L_k(W_1)$. Since W_1 is homeomorphic to $(M^{4k} \times T^{m-1}) \times R$ and $\pi_1(M^{4k}) = 0$, we may apply Theorem 3 to W_1 if $k > 1$ or $k = 1$, but $m > 1$.

The following arguments are "periodic". Let us give the explicit construction of the first period.

a) Based on Theorem 3, we can find a closed submanifold $V_1 \subset W_1$ such that W_1 is diffeomorphic to $V_1 \times R$; thus $L_k(W_1) = L_k(V_1)$.

b) Consider the covering space over the torus $T^{m-2} \times R \to T^{m-1}$ and, according to this covering, let us construct a covering over V_1, where V_1 has homotopy type $M^{4k} \times T^{m-1}$,

$$\widehat{V}_1 \underset{p_1}{\to} V_1,$$

where $\widehat{V}_1$ has homotopy type $M^{4k} \times T^{m-2} \times R$, and the mapping class group is Z. Evidently, $L_k(\widehat{V}_1) = p_1^* L_k(V_1)$, and the mapping

$$p_{1*}\colon H_{4k}(\widehat{V}_1) \to H_{4k}(V_1)$$

is such that $H_{4k}(V_1) = \operatorname{Im} p_{1*} + A$, where $L_k/A = 0$ for an appropriate choice of A.

c) Now denote $\widehat{V}_1$ by W_2 and note that to the manifold W_2 we may again apply Theorem 3 if $k > 1$ or $m - 1 > 1$. Thus we obtain the following "period":

$$\underbrace{W_1 \supset V_1 \underset{p_1}{\leftarrow} \widehat{V}_1} = \underbrace{W_2 \supset V_2 \underset{p_2}{\leftarrow} \widehat{V}_2} = W_3.$$

It is important that $\dim W_2 = \dim W_1 - 1$ and the class L_k actually remains unchanged.

Furthermore, starting from the manifold W_2 we again search (as in the first period) the manifolds $V_2 \subset W_2$ and $W_3 = \widehat{V}_2$, and so on, until we reach a simply connected manifold W_m of dimension $4k+1$ and homotopy type M^{4k}.

If $4k > 4$ we again apply Theorem 3 to $W_m = V_m \times R$ and note the following:

$$(L_k(V_m), [V_m]) = (L_k(W), [M^{4k}])$$

by construction;

$$(L_k(V_m), [V_m]) = \tau(M^{4k})$$

by Hirzebruch formula, because V_m is of homotopy type M^{4k} (by construction) and is closed. This yields Lemma 2.2 for the case $4k > 4$.

If $4k = 4$, we may note that the manifold V_{m-1} is of homotopy type $M^{4k} \times S^1$. Thus from Theorem 1 of the author's paper [13] it follows that $(L_k(V_{m-1}), [M^{4k}]) = \tau(M^{4k})$, and we again obtain Lemma 2.2 for $k = 1$.

Analogously, from Theorem 3 we may deduce Theorem 2.

3. Now let us show that Theorem 3 yields Theorem 4. Consider a smooth manifold W which is homeomorphic to $M^n \times R^m$ for a large m. Let us embed $M^n \subset W$ smoothly (see [5]). The neighborhood of M^n in W is the SO-bundle total space for the bundle β such that

$$\beta \oplus \alpha(M) = \alpha(W),$$

where $\alpha(X)$ is the tangent bundle of the smooth manifold X.

Denote by $V = V^{n+m-1}$ the total space of the SO-bundle β with fiber S^{m-1} over M^n. Remove from W a closed neighborhood of the manifold M^n in W, which is homeomorphic to $M^n \times D^m$. The remainder is homeomorphic to

$$M^n \times S^{m-1} \times R = W_1.$$

By Theorem 3, W_1 is diffeomorphic to $V_1 \times R$, where V_1 is a smooth closed manifold of homotopy type $M^n \times S^{m-1}$. However, V_1 is h-homological to the manifold V — the total space of the spherical bundle β. Since $\pi_1 = Z + \cdots + Z$, we see that V_1 is diffeomorphic to V, and the whole manifold W is diffeomorphic to the space β with fiber R^m over M^n. The theorem is proved.

Note that for $M^n = S^1$ the tangent bundles $\alpha(S^1)$ and $\alpha(W)$ are always trivial. Thus $W = S^1 \times R^m$.

4. Note that Theorem 5 formally follows from Theorem 3 for the case when the dimension of the torus is equal to 1: to see this, one should

consider the manifold W, being the covering space over M^n with mapping class group Z. The general case follows by a subsequent use of Theorem 3 to this situation.

§ 3. A geometrical lemma

The aim of this section is to prove a standard-type lemma from a smooth embedding theory. Its only difference from the usual case is that it is necessary in the non-simply connected situation, though it causes no serious changes in the proof.

Lemma 3.1.[1] *Let W^{n+1}, V^n be a smooth manifold $W^{n+1} = W$, with one boundary component $V^n = V$; W^{n+1} is, possibly, open. If the inclusion $\pi_1(V) \to \pi_1(W)$ is an isomorphism, the group $\pi_1(V)$ has no 2-torsion, and all the groups $\pi_i(W, V)$ are zeros for $i \le s$, then any map of pairs $f\colon (D^{l+1}, S^l) \to (W, V)$ is homotopic to a smooth embedding if $3l + 3 < 2n$ and $2l - n + 1 < s$.*

Moreover, under the same dimension restrictions, any finite collection of maps $f_i\colon (D^{l+1}, S^l) \to (W, V)$, $i = 1, \ldots, q$, is homotopic to a system of pairwise non-intersecting smooth embeddings.

Proof. Let us first consider the first part of the lemma about mappings of one object.

Let $f\colon (D^{l+1}, S^l) \to (W, V)$ be an arbitrary mapping of pairs. Consider the universal covering spaces $(\widehat{W}, \widehat{V})$ and the covering mapping of pairs $\hat{f}\colon (D^{l+1}, S^l) \to (\widehat{W}, \widehat{V})$. Since the pair $(\widehat{W}, \widehat{V})$ is simply connected, then we may assume that the mapping $\hat{f}$ is a smooth embedding (see [23]). Moreover, the mapping f has only double intersection points (by genericity assumptions). These intersection points form a submanifold $M^t \subset D^{l+1}$, which, in general, has a boundary, here $t = 2l - n + 1$. The map $f/M^t \to W$ is a two-fold covering. Let us show that this covering is trivial, i.e.

$$M^t = M_1^t \cup M_2^t$$

and

$$f(M_1^t) = f(M_2^t).$$

[1] The author is not sure that this lemma cannot be deduced directly from the works of Haefliger [5] or J. Levine. This lemma will be applied only for the case $n = 2l + 1$ and $n = 2l$ (§ 8), thus the reader should not pay much attention to it.

Indeed, if there were one connected component $M_0^t \subset M^t$, where the map f is two-fold, then the image $\hat{f}(M_0^t) \subset \widehat{W}$ would be such that there exists an element $\alpha \in \pi_1(V) = \pi_1(W)$ such that

$$\alpha(M_0^t) = M_0^t,$$

where $\alpha\colon \widehat{W} \to \widehat{W}$ and $\alpha^2/M_0^t = 1$; which would yield $\alpha^2 = 1$, that contradicts the assumption of the lemma.

Thus, $M^t = M_1^t \cup M_2^t$ and $f(M_1^t) = f(M_2^t)$.

On the manifold M_1^t let us construct a Morse function g which is equal to zero on the boundary $\partial M_1^t \subset S^l$. After passing the first critical point $g = x_0$, the topology of the "large value region" changes. Let us show, analogously to Haefliger [5], that one can accordingly change the map

$$f\colon (D^{l+1}, S^l) \to (W, V)$$

in such a way that instead of

$$M_1^t = \{g \geq 0\} = \{g \geq x_0 - \varepsilon\}$$

we shall have the intersection manifold

$$\overline{M}_1^t = \{g \geq x_0 + \varepsilon\}, \quad \varepsilon > 0,$$

for the new map

$$\overline{f}\colon (D^{l+1}, S^l) \to (W, V),$$

homotopic to f.

Consider the region $G = \{g \leq x_0 + \varepsilon\}$. Denote the index of $(g = x_0, \operatorname{grad} g = 0)$ by k. Then

$$G = \partial M_1^t \times I(0, 1) \bigcup_h D^k \times D^{t-k},$$

$$h\colon \partial D^k \times D^{t-k} \to \partial M_1^t \times 1.$$

Let

$$S^{k-1} = h(\partial D^k \times 0) \subset \partial M_1^t,$$

$$D_0^k = h(D^k \times 0) \subset D^{l+1}.$$

Consider the disk $D^{k+1} \subset D^{l+1}$, where $\partial D^{k+1} = D_0^k \cup D^k$, such that $D^{k+1} \cap M_2^t = \emptyset$,

$$D^{k+1} \cap M_1^t = D_0^k$$

(in general position) and

$$D^{k+1} \cap \partial D^{l+1} = D_1^k$$

(in general position). Let T be the neighborhood of the disk $f(D^{k+1})$ in W and let $\operatorname{Int} T$ be the interior of T. Set

$$W' = W \backslash \operatorname{Int} T.$$

Obviously, W' is diffeomorphic to W: we "pushed" the interior of T away from the boundary $\partial W = V$. Preserving the initial notation, denote W' by W, and denote $\partial W'$ by V.

Consider the abstract disk D^{l+1} and submanifolds M_1^t, M_2^t in it. Delete from D^{l+1} the set $D^{k+1} \subset D^{l+1}$ together with its "hull" $f^{-1}f(D^{k+1})$, in such a way that we also delete the neighborhood of the disk

$$f^{-1}f(D_0^k) \cap M_2^t = \overline{D}_0^k$$

from D^{l+1}. The topological effect of this operation is as follows: from D^{l+1} we delete the neighborhood of the disk $\overline{D}_0^k$ such that

$$\partial \overline{D}_0^k = \overline{D}_0^k \cap \partial D^{l+1}.$$

Thus the boundary of this new body is $S^k \times S^{l-k}$, and the body itself is $D^{k+1} \times S^{l-k}$. We have:

$$D' = D^{l+1} \backslash f^{-1}f(D^{k+1}) = D^{k+1} \times S^{l-k},$$
$$D' \cap \partial W' = S^k \times S^{l-k}.$$

The disk $D^{k+1} \times 0 \subset D'$ defines an element of the group

$$\pi_{k+1}(W', \partial W') = \pi_{k+1}(W, \partial W) = 0, \quad k+1 \leq s.$$

Consider a disk $D^{k+2} \subset W' = W$ such that:

$$\partial D^{k+2} = D_0^{k+1} \cup D_1^{k+1},$$
$$D^{k+2} \cap \partial W' = D_0^{k+2},$$
$$D^{k+2} \cap f(D') = D_1^{k+1} = f(D^{k+1} \times 0)$$

(all intersections are transverse). Let us perform surgery of the manifold D' along D^{k+2}, under which the boundary is operated on by a Morse surgery over the basic cycle $S^k \times 0$. After the surgery, we again obtain a map of the disk $\bar{f}\colon D^{l+1} \to W = W'$, and the singular manifold will "lose" one critical point of the function $g\colon M_1^t \to R$.

More precisely, we have a mapping $f'\colon D' \to W'$ induced by $f\colon D^{l+1} \to W$, such that

$$f'\colon \partial D' \to \partial W', \quad \partial D' = S^k \times S^{l-k}, \quad D' = D^{k+1} \times S^{l-k},$$

and the singular manifold for f' is diffeomorphic to the region $\{g \geq x_0 + \varepsilon\}$ on M_1^t. On the disk $D^{k+1} \times 0$, the mapping f' is a homeomorphism, and there exists a disk $D^{k+2} \subset W'$ such that

$$\partial D^{k+2} = f'(D^{k+1} \times 0) \cup D_0^{k+1}$$

and

$$D_0^{k+1} \subset \partial W', \quad D^{k+2} \cap f'(D') = f'(D^{k+1} \times 0).$$

Consider an abstract disk $D^{k+2} \times D_0^{l-k}$, where $\partial D^{k+2} = D_0^{k+1} \cup D_1^{k+1}$, and paste it to D' as follows:

$$A = D' \bigcup_h D^{k+2} \times D^{l-k}, \quad h\colon D_1^{k+1} \times D^{l-k} \to D' = D^{k+1} \times S^{l-k},$$

where $h(D_1^{k+1} \times 0) = D^{k+1} \times 0 \subset D'$; let

$$B = A \backslash [D^{k+2} \times \operatorname{Int} D^{l-k}].$$

The result of pasting is homeomorphic to the disk $B = D^{l+1}$. This naturally leads to a mapping $f\colon D^{l+1} \to W'$,

$$D^{l+1} = B = A \backslash [D^{k+2} \times \operatorname{Int} D^{l-k}], \quad A = D' \bigcup_h D^{k+2} \times D^{l-k},$$

constructed from $f'\colon D' \to W'$ and the embedded disk $D^{k+2} \subset W'$.

It is easy to see that the mapping of pairs

$$\bar{f}\colon (D^{l+1}, S^l) \to (W', \partial W') = (W, \partial W)$$

is homotopic to

$$f\colon (D^{l+1}, S^l) \to (W, \partial W)$$

and has a "one point less" intersection set (we lose one critical point of g). Reiterating the process, we will get a map without self-intersections, which proves the first part of the lemma.

Analogously one can remove intersections of pairs of mappings

$$f_1, f_2\colon (D^{l+1}, S^l) \to (W, \partial W).$$

This proves the second part of the lemma.

The lemma is proved.

§ 4. An analog of the Hurewicz theorem

Let $f\colon X \to Y$ be a map of complexes such that

$$f_*\colon \pi_1(X) \to \pi_1(Y)$$

is an isomorphism, let the mapping f itself as well as the corresponding covering maps $\hat{f}\colon \widehat{X} \to \widehat{Y}$ on universal covering spaces $\widehat{X}, \widehat{Y}$ induce epimorphisms in all dimensions:

$$H_i(\widehat{X}) \xrightarrow{\hat{f}_*} H_i(\widehat{Y}) \to 0,$$
$$H_i^*(X) \xrightarrow{f_*} H_i(Y) \to 0.$$

Under these conditions we get the following:

Lemma 4.1. *If $f_*\colon \pi_j(X) \to \pi_j(Y)$ is a monomorphism in all dimensions $j < k$ then it is an isomorphism in dimensions $j < k$, it is an epimorphism in dimension k, and for the kernels we have the following* "Hurewicz theorem":

(a) $\operatorname{Ker} f_*^{(\pi_k)} = \operatorname{Ker} \hat{f}_*^{(H_k)} = M_k$,

(b) $M_k/Z_0(\pi)M_k = \operatorname{Ker} f_*^{(H_k)}$,

where $\pi = \pi_1(X) = \pi_1(Y)$, $Z_0(\pi)$ is the kernel of the augmentation $\varepsilon\colon Z(\pi) \to Z$ of the integral group ring, and the kernel in homology for M_k is considered as a $Z(\pi)$-module.

Before proving this lemma, let us list those situations where it can be applied.

1. Let $f\colon M_1^n \to M_2^n$ be a map of closed manifolds of degree $+1$ such that $\pi_1(M_1^n) = \pi_1(M_2^n)$. Then the map $\hat{f}\colon \widehat{M_1} \to \widehat{M_2}$ of universal (and any other) covering spaces has degee $+1$ as a proper map. Thus $\hat{f}$ induces a map $\hat{f}_*$ of open homology groups, and that of $\hat{f}^*$-compact homology groups, for which the formula

$$\hat{f}_* D \hat{f}^* D(x) = x, \quad x \in H_q(\widehat{M_2^n})$$

holds. Consequently,

$$H_q(\widehat{M_1}) = \operatorname{Ker} \hat{f}_*^{(H_q)} + D\hat{f}^* D H_q(\widehat{M_2}).$$

In this case, one can evidently apply Lemma 4.1.

2. Let W be a smooth manifold as in Theorem 3 (see § 1) and let $i\colon V_1 \subset W$ be a smooth submanifold splitting W into two parts and realizing the basic cycle of the group $H_n(W) = Z$ in such a way that $\pi_1(V_1) = \pi_1(W)$. Denote by A and B, respectively, the "right" and "left" sides of W with respect to V_1, where

$$A \cup B = W, \quad A \cap B = V_1.$$

Then the following statements (a) and (b) hold.

a) The embeddings $i_1\colon V_1 \subset A$, $i_2\colon V_1 \subset B$ and $i\colon V_1 \subset W$ satisfy the conditions of Lemma 4.1.
b) For all covering spaces we have a direct sum decomposition:

$$\operatorname{Ker} \hat{\imath}_*^{(H_k)} = \operatorname{Ker} \hat{\imath}_{1*}^{(H_k)} + \operatorname{Ker} \hat{\imath}_{2*}^{(H_k)},$$

and the maps

$$\hat{\imath}_{2*}\colon \operatorname{Ker} \hat{\imath}_{1*}^{(H_k)} \to H_k(\widehat{B}),$$
$$\hat{\imath}_{1*}\colon \operatorname{Ker} \hat{\imath}_{2*}^{(H_k)} \to H_k(\widehat{A}),$$

are monomorphic, and the images $\hat{\imath}_{2*} \operatorname{Ker} \hat{\imath}_{1*}^{(H_k)}$ and $\hat{\imath}_{1*} \operatorname{Ker} \hat{\imath}_{2*}^{(H_k)}$ coincide with the kernels of the embeddings $H_k(\widehat{A}) \to H_k(\widehat{W})$ and $H_k(\widehat{B}) \to H_k(\widehat{W})$.

Let us prove a). Since

$$\pi_1(W) = \pi_1(A) *_{\pi_1(V_1)} \pi_1(B)$$

and $\pi_1(W) = \pi_1(V_1)$, we see that $\pi_1(A) = \pi_1(V_1)$ and $\pi_1(B) = \pi_1(V_1)$.

Consider the basis $x_1, \dots, x_s \in H_k(W)$; let us realize it by cycles $z_1, \dots, z_s \subset W$. Then there is a large N such that all $T^N z_1, \dots, T^N z_s$ lie in $B \subset W$. Since T is an epimorphism, these cycles form a basis of the group $H_k(W)$. Let $x \in H_k(A)$, and let $z \subset A$ be the cycle representing it. Then z is homologous in W to the linear combination $\sum a_i T^N z_i$ by means of a membrane $c \subset W$. The intersection $c \cap V_1$ is a cycle $\overline{z} \subset V_1$, realizing the homology class $\overline{x} \in H_k(V_1)$ such that $x = i_{1*}\overline{x}$. The arguments for B and for the whole W are identical.

Now, let us consider the covering spaces $\widehat{A}$, $\widehat{B}$, $\widehat{V}_1$, $\widehat{W}$ and the covering embeddings $\hat{\imath}$, $\hat{\imath}_1$, $\hat{\imath}_2$. Note that the homology groups $H_k(\widehat{V}_1)$, $H_k(\widehat{A})$, $H_k(\widehat{B})$, $H_k(\widehat{W})$ are finitely generated $Z(\pi_1)$-modules, because π_1 is a Noetherian group ($\pi_1 = Z + \dots + Z$). The following argument is analogous, but instead of the basis of the group one should take a π_1-basis of the module.

The same is true for all intermediate covering spaces. Thus, we may apply Lemma 4.1 here.

Let us prove statement b). If the intersection

$$\operatorname{Ker}\, \hat{\imath}_{1*}^{(H_k)} \cap \operatorname{Ker}\, \hat{\imath}_{2*}^{(H_k)}$$

is non-empty, then there exists a cycle $z \subset \widehat{V}_1$ which is null-homologous in both $\widehat{A}$ and $\widehat{B}$. The membranes define a cycle c in $\widehat{W}$ of dimension $k+1$. This cycle c, according to the arguments above, is homologous in $\widehat{W}$ to a cycle $\overline{c} \subset \widehat{W}$ such that $\overline{c} \cap \widehat{V}_1 = \emptyset$, by means of a membrane $d \subset \widehat{W}$. The intersection $d \cap \widehat{V}_1$ is such that

$$\partial(d \cap \widehat{V}_1) = c \cap \widehat{V}_1 = z,$$

and z is null-homologous. Thus,

$$\operatorname{Ker}\, \hat{\imath}_{1*}^{(H)} \cap \operatorname{Ker}\, \hat{\imath}_{2*}^{(H)} = 0$$

for all covering spaces.

Now let us consider the kernel of the embedding $H_k(\widehat{A}) \to H_k(\widehat{W})$. Let z be a cycle in $\widehat{A}$, which is null-homologous in $\widehat{W}$, by means of a membrane c. Then $z_1 = c \cap \widehat{V}_1$ is such that $z = \hat{\imath}_{1*} z_1$ and $z_1 \in \operatorname{Ker}\, \hat{\imath}_{2*}^{(H)}$. The statement is proved.

PROOF OF LEMMA 4.1. Let us first consider the "simply connected" case of the map $\hat{f}\colon \widehat{X} \to \widehat{Y}$. Denote by C the mapping cylinder for $\hat{f}$, which is contractible to $\widehat{Y}$. Let us write the exact sequences:

$$\begin{array}{ccccccc}
H_i(\widehat{X}) & \longrightarrow & H_i(\widehat{Y}) & \longrightarrow & H_i(C,\, \widehat{X}) & \xrightarrow{\partial} & H_{i-1}(\widehat{X}) \\
\uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} \\
\pi_i(\widehat{X}) & \longrightarrow & \pi_i(\widehat{Y}) & \longrightarrow & \pi_i(C,\, \widehat{X}) & \xrightarrow{\partial} & \pi_{i-1}(\widehat{X}).
\end{array}$$

Since $\hat{f}_*\colon \pi_{i-1}(\widehat{X}) \to \pi_{i-1}(\widehat{Y})$ are all monomorphisms for $i \leq k$, the mapping

$$\partial\colon \pi_i(C,\, \widehat{X}) \to \pi_{i-1}(\widehat{X})$$

is trivial. Since $\hat{f}_*\colon H_i(\widehat{X}) \to H_j(\widehat{Y})$ are epimorphisms, then $\partial\colon H_i(C,\, \widehat{X}) \to H_{i-1}(\widehat{X})$ are monomorphisms onto the kernel $\operatorname{Ker} \hat{f}_*^{(H_{i-1})}$. Because

$H\partial = \partial H$, we have for the first i for which $\pi_i(C, \widehat{X}) \neq 0$:

$$\pi_i(C, \widehat{X}) = H_i(C, \widehat{X}),$$

and ∂H is an isomorphism:

$$\pi_i(C, \widehat{X}) \approx \operatorname{Ker} \hat{f}_*^{(H_{i-1})}.$$

But this is possible only for $i \geq k+1$; otherwise $H\partial = 0$. For $i = k+1$ we have

$$\operatorname{Ker} \hat{f}_*^{(\pi_k)} = \operatorname{Ker} \hat{f}_*^{(H_k)},$$

and for $i \leq k+1$ the mapping $\hat{f}_*\colon \pi_{i-1}(\widehat{X}) \to \pi_{i-1}(\widehat{Y})$ is an epimorphism.

Following Serre, let us transform the map $\hat{f}\colon \widehat{X} \to \widehat{Y}$ into a fibration $\widetilde{f}\colon X_1 \xrightarrow{F} Y_1$, where X_1, Y_1 are of homotopy type $\widehat{X}$, $\widehat{Y}$, and $\widehat{f}$ has homotopy type $\hat{f}$. According to the exact sequence of this fibration, we see that

$$\pi_k(F) = H_k(F) = \operatorname{Ker} \hat{f}_*^{(\pi_k)}$$

according to the previous results.

Let us consider the mapping $f\colon X \to Y$, and transform it into a fibration; the fiber F' has the homotopy type of F, and

$$\pi_k(F) = \operatorname{Ker} \hat{f}_*^{(\pi_k)} = \operatorname{Ker} \hat{f}_*^{(H_k)} = M_k,$$

moreover, $\pi_i(F) = 0$, $i < k$.

Consider the spectral sequence of this fibration. Evidently, $E_2^{0,k} = M_k/Z_0(\pi)M_k$ and $E_2^{q,i} = 0$ for $0 < i < k$.

Since $f_*\colon H_{k+1}(X) \to H_{k+1}(Y)$ is an epimorphism, the differential

$$d_{k+1}\colon E_2^{k+1,0} \to E_2^{0,k}, \quad E_2^{k+1,0} = H_{k+1}(Y),$$

is trivial. Thus,

$$E_\infty^{0,k} = \frac{M_k}{Z_0(\pi)M_k}.$$

Obviously,

$$E_\infty^{0,k} = \operatorname{Ker} f_*^{(H_k)} = \frac{M_k}{Z_0(\pi)M_k}.$$

All statements of the lemma have been proved.

§ 5. The functor $P = \mathrm{Hom}_c$ and its application to the study of homology properties of degree one maps

Let π be a Noetherian group, K a ring or a field, $K(\pi)$ be a group ring with coefficients in K, $\varepsilon\colon K(\pi) \to K$ be the augmentation, $K_0(\pi) = \mathrm{Ker}\ \varepsilon$. We shall assume that K is either Z or a field.

Let M be a finitely generated $K(\pi)$-module.

Definition 5.1. We define the module $PM = \mathrm{Hom}_c(M, K)$ as the submodule $PM \subset \mathrm{Hom}(M, K)$, consisting of linear forms $h\colon M \to K$ such that for any element $x \in M$ the function on the group $f_{h,x}(a) = (h, ax)$ is finite, $a \in \pi$.

Here we note several simple properties of the functor $P = \mathrm{Hom}_c$:

1. For a free module F the module PF is free;
2. For a projective module there is a natural isomorphism $P^2\colon M \to P^2 M$;
3. There is always a natural map $P^2\colon M \to P^2 M$, which is possibly non-monomorphic and non-epimorphic. Denote the kernel of this map by $M_\infty \subset M$. Then we have

$$0 \to M_\infty \to M \to P^2 M \to \mathrm{Coker}\, P^2 \to 0.$$

Example 1. Let $p\colon \widehat{M} \to M^n$ be a regular covering with mapping class group $\pi\colon \widehat{M} \to \widehat{M}$. The homology groups $H_i(\widehat{M}, K) = N_i$ are $K(\pi)$-modules, which are finitely generated if π is Noetherian and M^n is a compact manifold. There is a homomorphism:

$$\frac{N_i}{N_{i\infty}} \to PN_{n-i},$$

defined by the intersection index.

Example 2. Let $f\colon M_1^n \to M_2^n$ be a degree $+1$ map, and let

$$\pi_1(M_1^n) = \pi_1(M_2^n).$$

Denote by $\hat{f}\colon \widehat{M}_1^n \to \widehat{M}_2^n$ the map of covering spaces $\widehat{M}_1 \to M_1^n$ and $\widehat{M}_2 \to M_2^n$ with mapping class group π. Set

$$M_i = \mathrm{Ker}\, \hat{f}_*^{(H_i)} \subset H_i(\widehat{M}_1).$$

Analogously to Example 1, we have:

$$\frac{M_i}{M_{i\infty}} \xrightarrow{h} PM_{n-i},$$
$$(hx, y) = x \circ y.$$

Now, let us consider the derived functors of the functor $P = \mathrm{Hom}_c$. We shall denote them by Ext^i_c, $i \geq 0$. Note that unlike the usual Hom, the functor $P = \mathrm{Hom}_c$ is not exact even for the field K. Thus, it is possible that

$$\mathrm{Ext}^i_c(M,\, K) \neq 0, \quad i > 0.$$

Example 3. Let M_0 be a module on one generator u and $au = u$ for all $a \in \pi$. If $\pi = Z + \cdots + Z$ is a free abelian group on n generators, then

$$\mathrm{Ext}^n_c(M_0,\, K) = M_0$$

and

$$\mathrm{Ext}^i_c(M_0,\, K) = 0, \quad i < n.$$

Let us prove this fact. Consider the triangulated torus T^n and a covering $R^n \to T^n$ with group $\pi = Z + \cdots + Z$. Denote by F_i the free $Z(\pi)$-module of i-dimensional chains on R^n. We have:

$$0 \to F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} \cdots \to F_1 \xrightarrow{\partial} F_0 \xrightarrow{\varepsilon} M_0 \to 0,$$

and the sequence is exact because

$$H_i(R^n) = 0, \quad i > 0, \quad H_0(R^n) = M_0.$$

Let us apply the function P to the resolvent:

$$PF_n \leftarrow PF_{n-1} \leftarrow \cdots \leftarrow PF_1 \leftarrow PF_0,$$

but $PM_0 = 0$ and the resulting complex is the complex of compact cochains for R^n. Thus

$$H^n_c(R^n,\, K) = M_0$$

and

$$H^i_c(R^n,\, K) = 0, \quad i < n,$$

so that

$$H^k_c(R^n,\, K) = \mathrm{Ext}^k_c(M_0,\, K).$$

The following simple lemma holds.

Lemma 5.1. *If M is such that $\mathrm{Ext}^i_c(M,\, K) = 0$, $i > 0$, and $\pi = Z + \cdots + Z$, then the module PM is stably free, i.e. there is a free module F such that $PM + F$ is a free module.*

Proof. Since $\pi = Z + \cdots + Z$, there is such an acyclic resolvent of finite length

$$0 \to F_l \to F_{l-1} \to \cdots \to F_0 \to M \to 0.$$

By the assumption of the lemma, the sequence

$$0 \leftarrow PF_l \leftarrow PF_{l-1} \leftarrow \cdots \leftarrow PF_0 \leftarrow PM$$

is exact. The functor P possesses the property that the modules PF_i are free. Besides, the functor P is "half-exact from the right": it maps an epimorphism to a monomorphism. Thus, the kernel of $PF_0 \to PF_1$ is exactly PM. According to the properties of a free module we can prove the following equality in the usual way:

$$\cdots PF_4 + PF_2 + PF_0 = \cdots + PF_3 + PF_1 + PM,$$

so that all PF_i are free. The lemma is proved.

Let C be a complex of free or projective modules:

$$C = \{\cdots \to F_l \xrightarrow{\partial} F_{l-1} \xrightarrow{\partial} \cdots \to F_1 \xrightarrow{\partial} F_0\}.$$

Then the groups $H_i(C) = N_i$ are π-modules. Consider the complex PC:

$$\{\leftarrow PF_l \xleftarrow{\delta} PF_{l-1} \xleftarrow{\delta} \cdots \xleftarrow{\delta} PF_0\}, \quad \delta = P\partial,$$

and denote its homology groups by $H^i_c(C)$, since they are the "compact support homology groups".

There is a well-known fact: there exists a spectral sequence $\{E_r,\, d_r\}$,

$$E_r = \sum_{p\geq 0,\, q\geq 0} E_r^{p,q}, \quad E_2^{p,q} = \operatorname{Ext}^p_c(N_q,\, K)$$

and the module

$$\sum_{p+q=l} E_\infty^{p,q}$$

is adjoint to $H^l_c(C)$.

This fact is a "universal coefficient formula".

As seen from the examples, the functor P is such that the modules $H_i(C) = N_i$ do not affect $H_l^{i+k}(C)$ for k large enough (see Example 3). We shall be interested in those complexes which are, in some sense, manifolds and admit a certain geometric realization.

A necessary and sufficient condition for realizability of the complex

$$C = \{F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_0\}$$

as a covering space over a finite complex with π as mapping class group, are the following:

a) it is free: all F_i's are free modules;
b) $H_0(C) = M_0$ (see Example 3).

A necessary "geometric" requirement for morphisms of complexes $f\colon C_1 \to C_2$ is that

$$f_*\colon H_0(C_1) \to H_0(C_2)$$

is an isomorphism.

Later we shall need manifolds with maps of degree 1. For realizability as a homological manifold we certainly need that the complexes of modules

$$C = \{F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_0\}$$

and

$$PC = \{PF_n \xleftarrow{\delta} PF_{n-1} \xleftarrow{\delta} \cdots \xleftarrow{\delta} PF_0\},$$

(with $\delta = P\partial$), are in the algebraic sense "homotopy equivalent" (the meaning of this phrase is well known). This will lead us to the Poincaré duality laws:

$$D\colon H_i(C) \approx H_c^{n-i}(C), \quad i \geq 0.$$

Furthermore, if we want to have the duality in the form connected with cohomological multiplication and the section operation, we should require that the complex C be a coalgebra, etc. We shall not dwell on an exact formalization of all necessary notions. Note that for all algebraic complexes obtained from triangulations of manifolds, we get the following: for mappings of degree $\lambda f\colon C_n^1 \to C_n^2$ we may define an operator $Df^*D\colon C_2^n \to C_1^n$ such that

$$f_*Df^*D\colon C_2^n \to C_2^n$$

is the multiplication by λ; if, $\lambda = 1$, then

$$C_1^n = \operatorname{Ker} f + Df^*DC_2^n.$$

Here we get the complex $\operatorname{Ker} f$ made of projective modules and such that the complex $P(\operatorname{Ker} f)$ is algebraically homotopic to it. Consequently, the following duality law holds:

$$D\colon H_i(\operatorname{Ker} f) = H_c^{n-i}(\operatorname{Ker} f),$$

where

$$H_i(\operatorname{Ker} f) = \operatorname{Ker} f_*^{(H_i)}$$

and

$$H_c^{n-i}(\operatorname{Ker} f) = \operatorname{Coker} f^{*(H_c^{n-i})},$$

and the kernels and cokernels are taken for maps of the complexes $f\colon C_1^n \to C_2^n$. Thus to the kernels $\operatorname{Ker} f_*^{(H_i)}$ and cokernels $\operatorname{Coker} f^*(H_c^i)$ we may apply the Poincaré duality law and the "universal coefficient formula", which, could be certainly foreseen before.

We have the following:

Theorem 5.1. *If $f\colon M_1^n \to M_2^n$ is a mapping of degree $+1$ of closed manifolds, $n = 2k$, and $\hat{f}\colon \widehat{M_1} \to \widehat{M_2}$ is a covering mapping such that $\widehat{M}_j$ are regular covering over M_j^n with group $\pi = Z + \cdots + Z$, and the kernels $M_s = \operatorname{Ker} \hat{f}_*^{(H_s)} = 0$, $s < k$, then the kernel $M_k = \operatorname{Ker} \hat{f}_*^{(H_k)}$ is a stably free $Z(\pi)$-module.*

Proof. Since all $M_s = 0$ for $s < k$, we have

$$\operatorname{Ext}_c^i(M_s, Z) = 0, \quad s < k,$$

thus, according to the "universal coefficient formula" given above as a spectral sequence, we get

$$\operatorname{Coker} \hat{f}^{*(H_c^s)} = 0, \quad s < k.$$

Since

$$\operatorname{Coker} \hat{f}^{*(H_c^s)} = \operatorname{Ker} \hat{f}_*^{(H_{n-s})} = 0, \quad s < k,$$

we see that all $M_{n-s} = 0$ for $s < k$, $n = 2k$, and all $M_q = 0$, except $q = k$.

Consequently, according to the "universal coefficient formula",

$$\operatorname{Coker} \hat{f}^{*(H_c^{k+q})} = \operatorname{Ext}_c^q(M_k, K).$$

But

$$\operatorname{Coker} \hat{f}^{*(H_c^{k+q})} = M_{k-q} = 0, \quad q > 0.$$

Thus $\operatorname{Ext}_c^q(M_k, Z) = 0$ for all $q > 0$. According to Lemma 1, the module PM_k is stably free, $PM_k = M_k$. The theorem is proved.

In the case of odd $n = 2k+1$ we again have $f\colon M_1^n \to M_2^n$ of degree $+1$, $f\colon \widehat{M}_1 \to \widehat{M}_2$ is a map of regular coverings with a Noetherian mapping class group π.

Theorem 5.2. *If* $M_s = \operatorname{Ker} \hat{f}_*^{(H_s)} = 0$, $s < k$, *then the following relations hold*:

a) $PM_k = M_{k+1}$;
b) $\operatorname{Ext}_c^i(PM_k, Z) = \operatorname{Ext}_c^{i+2}(M_k, Z), \quad i \geq 1$;
c) *The sequence*

$$0 \to \operatorname{Ext}_c^1(M_k, Z) \to M_k \xrightarrow{P^2} P^2 M_k \to \operatorname{Ext}_c^2(M_k, Z) \to 0$$

$$(M_{k\infty} = \operatorname{Ext}_c^1(M_k, Z), \quad \operatorname{Coker} P^2 = \operatorname{Ext}_c^2(M_k, Z))$$

is exact.

If $\operatorname{Ext}_c^i(M_k, Z) = 0$, $i \geq 3$, *then the module* $PM_{k+1} = P^2 M_k$ *is stably free* $(\pi = Z + \cdots + Z)$.

The proof of this theorem can be easily obtained from the Poincaré duality law:

$$D\colon M_k = \operatorname{Coker} \hat{f}^{*(H_c^{k+1})},$$
$$D\colon M_{k+1} = \operatorname{Coker} \hat{f}^{*(H_c^{k})},$$

and the universal coefficient formulas as a spectral sequence.

Indeed, since $M_j = 0$, $j < k$, we have

$$\operatorname{Coker} \hat{f}^{*(H_c^k)} = PM_k = M_{k+1},$$

which yields a).

The isomorphism from b) is established by the differential d_2, where

$$\begin{array}{ccc} d_2\colon E_2^{i,k+1} & \to & E_2^{i+2,k} \\ \| & & \| \\ \operatorname{Ext}_c^i(M_{k+1}, Z) & \to & \operatorname{Ext}_c^{i+2}(M_k, Z), \quad i \geq 1, \end{array}$$

since $M_{k+j} = 0$, $j \geq 2$, and $\operatorname{Coker} \hat{f}^{*(H_c^{k+j})} = 0$, $j \geq 2$. The statement of c) is also obtained from the spectral sequence of the universal coefficient formula because

$$P^2 M_k = PM_{k+1} = E_2^{k+1,0},$$

the map

$$P^2 M_k \to \operatorname{Ext}_c^2(M_k, Z)$$

is d_2, and the module

$$\operatorname{Ker} d_2 + \operatorname{Ext}_c^1(M_k,\, Z)$$

is adjoint to

$$M_k = \operatorname{Coker} \hat{f}^{*(H_c^{k+1})}.$$

The stable freeness of the module $P^2 M_k = PM_{k+1}$ follows from a) to c), and Lemma 5.1 if

$$\operatorname{Ext}_c^i(PM_k,\, Z) = \operatorname{Ext}_c^{i+2}(M_k,\, Z) = 0, \quad i \geq 1.$$

Theorem 5.2 is proved.

Remark. For degree $+1$ maps $f\colon M_1^n \to M_2^n$ of the covering spaces $\widehat{f}\colon \widehat{M_1} \to \widehat{M_2}$ the following formula always takes place:

$$\operatorname{Coker} \hat{f}^{*(H_c^k)} = \operatorname{Hom}_c(M_k,\, Z),$$

if $M_j = 0$, $j < k$, for any n and k.

Corollary 5.1. *If, under the assumptions of Theorem* 5.2 *we have* $\pi = Z + \cdots + Z$, *then the module* $PM_{k+1} = P^2 M_k$ *is stably free.* (*For the case* $\pi = Z$ *this fact is true, but it is trivial.*)

Proof. If $\pi = Z + Z$ then $\operatorname{Ext}_c^i(M_k,\, Z) = 0$ for $i \geq 3$ for any module M_k. By virtue of Theorem 5.2 we get the desired statement.

§ 6. Stably freeness of kernel modules under the assumptions of Theorem 3

Let $V_1 \overset{i}{\subset} W$ be a connected submanifold separating W into two parts A, B, where

$$A \cap B = V_1, \quad A \cup B = W.$$

Denote the embeddings $V_1 \subset A$ and $V_1 \subset B$, as in § 4, by i_1, i_2, and denote the embedding of universal covering spaces over V_1, W, A, B by $\hat{\imath}\colon \widehat{V_1} \subset \widehat{W}$, $\hat{\imath}_1\colon \widehat{V_1} \subset \widehat{A}$, $\hat{\imath}_2\colon \widehat{V_1} \subset \widehat{B}$. Here W is an $n+1$-dimensional manifold having homotopy type of closed manifold M^n, the group $\pi = \pi_1(W)$ is Noetherian, and on W a discrete transformation T is given such that

$$\pi_1(V_1) = \pi_1(A) = \pi_1(B) = \pi_1(W)$$

and the quotient space W/T is compact. Then we get the following:

Lemma 6.1. *If $\pi = Z + \dots + Z$ and the kernels $M_j = \operatorname{Ker} i_*^{(\pi_i)}$ are trivial for $j < k$ then for $n = 2k$ the modules*

$$M_k' = \operatorname{Ker} i_{1*}^{(\pi_k)}, \quad M_k'' = \operatorname{Ker} i_{2*}^{(\pi_k)}$$

are stably free. If $n = 2k + 1$ and

$$M_j' = \operatorname{Ker} i_{1*}^{(\pi_j)} = 0, \quad j < k,$$

$$M_j'' = \operatorname{Ker} i_{2*}^{(\pi_j)} = 0, \quad j < k + 1,$$

then the kernels $M_k' = \operatorname{Ker} i_{1*}^{(\pi_k)}$, $M_{k+1}'' = \operatorname{Ker} i_{2*}^{(\pi_{k+1})}$ are stably free. In both cases under the assumptions of the lemma there is a natural isomorphism, established by the intersection index of the cycles $M_k' = PM_{n-k}''$.

Proof. Let $n = 2k$. According to Theorem 5.1 under the asumptions of Lemma 6.1, the module $M_k = M_k' + M_k''$ (see § 4) is stably free. Thus both modules M_k' and M_k'' are projective ones, and since $\pi = Z + \dots + Z$, we see that M_k' and M_k'' are stably free. As we know, M_k is the kernel

$$\operatorname{Ker} \hat{\imath}_*^{(H_k)} = \operatorname{Ker} i_*^{(\pi_k)}.$$

Since

$$\operatorname{Ker} \hat{\imath}_*^{(H_k)} = \operatorname{Coker} \hat{\imath}_*^{(H_c^k)} = PM_k$$

(see § 5) and both modules M_k' and M_k'' have nonzero intersection index each, then $M_k' = PM_k''$ and $M_k'' = PM_k'$, which yields the lemma for even $n = 2k$.

Now let $n = 2k + 1$. Let us first prove that under the assumptions of the lemma the kernel

$$M_{k+1}' = \operatorname{Ker} \hat{\imath}_{1*}^{(H_{k+1})}$$

is trivial. Since

$$\operatorname{Ker} \hat{\imath}_*^{(H_{k+1})} = M_{k+1} = M_{k+1}' + M_{k+1}'' = \operatorname{Coker} \hat{\imath}^{*(H_c^k)} = PM_k = PM_k',$$

we have

$$PM_k' \overset{h}{\approx} M_{k+1}' + M_{k+1}'',$$

because $(hx, y) = x \circ y$, where $x \in M_{k+1}' + M_{k+1}''$, $y \in M_k' = M_k$ and $x \circ y$ is the intersection index. But the intersection index $M_k' \circ M_{k+1}'$ is identically zero. Thus $M_{k+1}' = 0$.

Consider a sufficiently large integer s. Then the intersection $T^s V_1 \cap V_1$ is empty. Denote the region between V_1 and $T^s V_1$ by Q and denote $T^s V_1$ by V', $\partial Q = V_1 \cup V_1'$. Here we assume that $T^s V_1 \subset A$.

Consider the embeddings $j\colon \widehat{V}_1 \subset \widehat{Q}$, $j'\colon \widehat{V}' \subset \widehat{Q}$ for the universal covering space $\widehat{W}$. We have (for s large enough):

$$\operatorname{Ker} j_*^{(H_q)} = \begin{cases} 0, & q \neq k, \\ M_k', & q = k, \end{cases}$$

$$\operatorname{Ker} j_*'^{(H_q)} = \begin{cases} 0, & q \neq k+1, \\ M_{k+1}'', & q = k+1, \end{cases}$$

$$\operatorname{Coker} j_*^{(H_q)} \approx \begin{cases} 0, & q \neq k, \\ i_{2*}, & M_k' \approx M_k'', q = k, \end{cases}$$

$$\operatorname{Coker} j_*'^{(H_q)} \approx \begin{cases} 0, & q \neq k+1, \\ i_{1*}, & M_{k+1}' \approx M_{k+1}'', q = k+1. \end{cases}$$

From the equalities listed above we get:

$$H_q(\widehat{Q}, \widehat{V}_1) = 0, \quad q \neq k, k+1,$$

$$H_k(\widehat{Q}, \widehat{V}_1) \approx H_{k+1}(\widehat{Q}, \widehat{V}_1) \approx M_k',$$

$$H_q(\widehat{Q}, \widehat{V}') = 0, \quad q \neq k+1, k+2,$$

$$H_{k+1}(\widehat{Q}, \widehat{V}') \approx H_{k+2}(\widehat{Q}, \widehat{V}_1') \approx M_{k+1}''.$$

Thus

$$H_c^q(\widehat{Q}, \widehat{V}_1) \approx \begin{cases} 0, & q \neq k, k+1, \\ M_{k+1}'', & q = k, k+1, \end{cases}$$

$$H_c^q(\widehat{Q}, \widehat{V}') \approx \begin{cases} 0, & q \neq k+1, k+2, \\ M_k', & q = k+1, k+2. \end{cases}$$

By virtue of the universal coefficient formulas for $H_c^q(\widehat{Q}, \widehat{V}_1)$,

$$M_{k+1}'' = PM_k' = H_c^k(\widehat{Q}, \widehat{V}_1),$$

$$d_2\colon \operatorname{Ext}_c^i(M_k', Z) \to \operatorname{Ext}_c^{i+2}(M_k', Z)$$

is an epimorphism for $i = 0$ and an isomorphism for $i > 0$. Recall that

$$M_k' \approx H_k(\widehat{Q}, \widehat{V}_1) \approx H_{k+1}(\widehat{Q}, \widehat{V}_1)$$

and

$$E_2^{p,q} = \operatorname{Ext}_c^p(H_q, Z), \quad d_2\colon E_2^{p,q} \to E_2^{p+2,q-1}.$$

Since $\pi = Z + \cdots + Z$, for $p > rk\pi$ we have $\operatorname{Ext}_c^p = 0$. Thus

$$\operatorname{Ext}_c^i(M_k', Z) = 0, \quad i > 0.$$

By Lemma 5.1, the module PM'_k is stably free. Since $M'_k = PM''_{k+1}$, the same is true for M'_k. The lemma is proved.

Remark. When proving the acyclicity of the module M'_k we used the fact that $\mathrm{Ext}_c^i = \mathrm{Ext}_c^{i+2}$ and $\mathrm{Ext}_c^p = 0$ for $p > rk\pi$. Actually, the triviality of modules $\mathrm{Ext}_c^i(M'_k, Z)$ for $i > 0$ can be proved in an alternative way for any Noetherian group π under the assumptions of Lemma 6.1.

§ 7. The homology effect of a Morse surgery

Let W be the same as in Theorem 3 (§ 1), $V_1 \overset{i}{\subset} W$, $W = A \cup B$, $A \cap B = V_1$, and the embeddings

$$i_1 \colon V_1 \subset A, \quad i_2 \colon V_1 \subset B$$

are such that

$$\pi_1(V_1) = \pi_1(A) = \pi_1(B) = \pi_1(W)$$

and

$$\mathrm{Ker}\ i_{1*}^{(\pi_k)} = 0, \quad k < p, \quad \mathrm{Ker}\ i_{2*}^{(\pi_k)} = 0, \quad k < n-p.$$

Set

$$\mathrm{Ker}\ i_{1*}^{(\pi_p)} = M'_p, \quad \mathrm{Ker}\ i_{2*}^{(\pi_{n-p})} = M''_{n-p}.$$

Both modules M''_{n-p} and M'_p are $Z(\pi)$-modules. According to Lemma 4.1 we have

$$M'_p = \mathrm{Ker}\ \hat{\imath}_{1*}^{(H_p)}, \quad M''_{n-p} = \mathrm{Ker}\ \hat{\imath}_{2*}^{(H_{n-p})}.$$

On the universal covering space $\widehat{V}_1$ between M'_p and M''_{n-p} there is a scalar product, which is integer-valued and π-invarant; it is generated by the intersection index of cycles.

By virtue of Lemma 4.1 we have

$$\mathrm{Ker}\ i_{1*}^{(H_p)} = \frac{M'_p}{Z_0(\pi)M'_p}$$

and

$$\mathrm{Ker}\ i_{2*}^{(H_{n-p})} = \frac{M''_{n-p}}{Z_0(\pi)M''_{n-p}}.$$

Choose a π-basis $\alpha_1, \dots, \alpha_q$ in M'_p. Let p satisfy the conditions of Lemma 3.1. Let us find disks $D_1^{p+1}, \dots, D_q^{p+1} \subset A$ such that their boundaries $\partial Dj^{p+1} \subset \partial A = V_1$ realize the elements $\alpha_1, \dots, \alpha_q \in M'_p$, and let us paste the handles

$$\begin{aligned} B' &= B \cup T_1 \cup \dots \cup T_q, \\ A' &= A \backslash (\operatorname{Int} T_1 \cup \dots \cup \operatorname{Int} T_q), \end{aligned}$$

where T_i are neighborhoods of disks D_j^{p+1} in A. Then it is easy to see that for $V'_1 = \partial B' = \partial A'$ the kernels

$$\operatorname{Ker}\ \hat{\imath}_1^{\prime(H_j)} = \widetilde{M}'_j$$

and

$$\operatorname{Ker}\ \hat{\imath}_{2*}^{\prime(H_j)} = \widetilde{M}''_j$$

will look like:

$$\begin{gathered} \widetilde{M}'_j = 0, \quad j \le p, \quad \widetilde{M}'_j = M'_j, \quad j > p+1, \\ \widetilde{M}''_j = 0, \quad j < n-p-1, \quad \widetilde{M}''_j = M''_j, \quad j \ge n-p. \end{gathered}$$

Denote the scalar product between the modules M'_p and M''_{n-p} by (,). Let $\beta_1, \dots, \beta_t$ be the π-generators of the module M''_{n-p}. Then the following lemma holds.

Lemma 7.1. *The module $\widetilde{M}''_{n-p-1}$ can be described as follows: its generators $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_q$ are in one-to-one correspondence with generators of M'_p, and the relators are given by the generators of the module M''_{n-p} as follows:*

$$\sum_{\substack{\alpha \in \pi \\ m=1, \dots, q}} (a^{-1}\beta_j, \alpha_m) a\, \widetilde{\alpha}_m = 0.$$

Proof. The geometrical sense of the generators $\widetilde{\alpha}_m$ is the following: these are spheres $S_m^{n-p-1} \subset V'_1$, which are linked to the spheres $\partial D_m^{p+1} \subset V_1$ deleted from V_1. Obviously, the elements $\widetilde{\alpha}_m$ are π-generators in $\widetilde{M}''_{n-p-1}$ because $M''_{n-p-1} = 0$.

Let us now consider the geometrical picture for the universal covering $\widehat{W}$. The geometrical sense of the above relations is evident because on $\widehat{W} \supset \widehat{V}_1$ the cycle β_j has intersection indices with the cycles $a\alpha_m$, $a \in \pi$, and after removing neighborhoods of the cycles α_m from V_1, the cycle β_j yields the desired relation.

The fact that this gives a complete relation system in our case follows from the fact that we have a complete relation system in the module

$$\hat{\imath}'_{1*}\widetilde{M}''_{n-p-1} \subset H_{n-p-1}(\widehat{A}').$$

Indeed, homotopically A' is obtained from A by a simple removal of disks D^{p+1}_m. It is easy to see that

$$\pi_{p+1}(A,\, V_1) = H_{p+1}(\widehat{A},\, \widehat{V}_1).$$

Since the relation in $\hat{\imath}'\widetilde{M}''_{n-p-1}$ appears because of the intersection of cycles from $H_{n-p}(\widehat{A})$ with covering disks $\widehat{D}^{p+1}_m \subset \widehat{A}$ and since $H_{n-p}(\widehat{V}_1) \to H_{n-p}(\widehat{A})$ is epimorphic, the system or relations in the lemma is complete. The lemma is proved.

§ 8. Proof of Theorem 3

Let $n \geq 5$. We preserve the notation for $V_1 \subset W$, A, B, i_1, i_2, i, $\hat{\imath}_1$, $\hat{\imath}_2$, $\hat{\imath}$, M'_t, M''_t, etc.

The proof will consist of three steps.

Step 1. We make $V_1 \subset W$ connected such that

$$\pi_1(V) = \pi_1(W).$$

Here we do not use any restrictions for $\pi_1(W)$ except that it is finitely-generated.

Step 2. By Morse surgery we paste the homotopy kernels of the embedding $V_1 \subset W$ in dimensions $k < [n/2]$, and for odd $n = 2t + 1$ we also paste the kernels

$$\operatorname{Ker}\, i^{(\pi_t)}_{2*} = M''_t$$

by using Lemma 3.1. Here we use the fact that the fundamental group is Noetherian.

Step 3. Pasting handles on one side $V_1 \to V_1 \# S^t \times S^{n-t}$ to the manifold $V_1 \subset W$ we "stabilize" the module $M'_t \to M'_t + F$ for $n = 2t$ or $n = 2t + 1$ in such a way that the kernel of M'_t becomes a free module over $Z(\pi)$. Here we use the results of Theorem 5.2. Then, applying Lemma 3.1, and using surgery over the π-free basis from M'_t, we kill M'_t and M''_{t+1} for $n = 2t + 1$ and M'_t and M''_t for $n = 2t$. By using Lemma 7.1, the kernels in other dimensions (in particular, M''_{n-t-1}) remain trivial. As a result of surgery, we get a closed submanifold $V \subset W$ which is a deformation retract. At this point, Theorem 3 follows trivially: there is a number k such that $T^kV \cap V = \emptyset$. The neighborhood of the manifold T^kV in W is homeomorphic to $V \times R$. According to the above, in this neighborhood there is a smooth $V' \subset W$ near T^kV, having homotopy type W. Between V and V' there is a smooth h-cobordism. Thus, this domain is $V \times I(0, 1)$ and $V' = V$ because $Wh(\pi) = 0$, $\pi = Z + \cdots + Z$ (see [1, 2, 9]). Considering such domains for all k we see that $W = V \times R$.

The theorem is proved.

Remark. If in Step 3 we perform a surgery not over a free π-basis in M'_t, but over any other one according to the projection $F \to M'_t$, then after the surgery we would get a module of relations R, $0 \to R \to F \to M'_t \to 0$, where $R = \widetilde{M}'_{t+1}$ (see § 7) for the manifold after the surgery. By virtue of Lemma 7.1 for this manifold, we would have

$$M''_{n-t-1} = PM'_{t+1} = PR.$$

§ 9. Proof of Theorem 6

Let $S^n \subset S^{n+2}$ be a topological locally flat embedding and let $n \geq 5$. Note that the difference $G = S^{n+2} \backslash S^n$ is a smooth open manifold with "homotopy type at the infinity" $S^n \times S^1$. We shall construct a smooth closed manifold $V \subset G$ of homotopy type $S^n \times S^1$, which bounds in S^{n+2} a manifold D of homotopy type S^n, and contains the "knot" $S^n \subset D \subset S^{n+2}$.

In the case when we know that the knot $S^n \subset S^{n+2}$ is globally flat, i.e. it has a neighborhood $U \supset S^n$ homeomorphic to $S^n \times R^2$, this problem can be easily solved by Theorem 3: namely, we set $W = U \backslash S^n$. Then W is homeomorphic to $S^n \times S^1 \times R$ and it is smooth. By Theorem 3 there is a smooth $V \subset W$ such that W is diffeomorphic to $V \times R$. Evidently, V bounds in $U \supset W \supset V$ a manifold D of homotopy type $S^n \subset D$, $n \geq 5$.

If the global plane is not known, let us consider the decreasing sequence of smooth manifolds with boundary

$$U_1 \supset U_2 \supset \cdots \supset U_i \supset \cdots$$

such that $U_j \supset S^n$ and $\cap_j U_j = S^n$.

Set $W_j = U_j \backslash S^n$. Obviously, the group $H_{n+1}(W_j) \neq 0$, and for j_1 large enough in comparison with $j_0 \gg 1$, the image

$$H_{n+1}(W_{j_1}) \to H_{n+1}(W_{j_0})$$

is isomorphic to Z.

If j_0, j_1 are large enough, one can realize the basic cycle of the image inside W_{j_1} by a submanifold $V_1 \subset W_{j_1}$; it is easy to see that for large

$j_1 \gg j_0 \gg 1$ the map $V_1 \subset W_{j_0}$ is "similar" to the map $V_1 \to S^n \times S^1$. More exactly, this means the following: for j large enough one may find a natural map $W_j \xrightarrow{q_j} S^n \times S^1$ (which, in the case when a global plane exists may be thought of as a projection to $S^n \times S^1$), which induces the map $g_{j_1} \colon W_{j_1} \to S^n \times S^1$ for $j_1 \geq j$. The composition of the embedding $V_1 \subset W_j$ and $g_j \colon W_j \to S^n \times S^1$ is a degree $+1$ map $f_j \colon V_1 \to S^n \times S^1$.

One can easily make V_1 connected and such that $\pi_1(V_1) = Z$, as before. Then V_1 separates W_{j_1} into two parts A and B, and homotopy kernels of embeddings $i_1 \colon V_1 \subset A$ and $i_2 \colon V_1 \subset B$ possess the same properties as the kernels discussed in Theorem 3 (see §§ 4–8), though here we cannot assume that the embeddings $V_1 \subset A$ and $V_1 \subset B$ are homologously epimorphic, unlike Theorem 3. However, note that this epimorphism takes place related to the "interior" part $A \subset W_{j_1}$ such that its closure in S^{n+2} contains S^n. As before, denote the embeddings $V_1 \subset A$ and $V_1 \subset B$, by i_1 and i_2, respectively.

By locally flatness of the knot $S^n \subset S^{n+2}$, the manifold G possesses the following property: there exists $\varepsilon > 0$ such that any map $h \colon P \to G$ of any complex P is homotopic in G to a map $\overline{h} \colon P \to G$, whose image is at a distance $>\varepsilon$ from S^n in S^{n+2}. We assume all W_j lie in the ε-neighborhood of the knot $S^n \subset S^{n+2}$, i.e. j is large enough. But this means that we may apply Lemma 4.1 to the interior part of A (with respect to V_1). Obviously, Lemma 4.1 is applicable to the map $f_j \colon V_1 \to S^n \times S^1$ as well.

We start by gluing handles (as in Theorem 3) to V_1 inside G to eliminate the kernels $\mathrm{Ker}\ i_{1*}^{(\pi_q)}$ and $\mathrm{Ker}\ i_{2*}^{(\pi_q)}$ for $q \leq [n/2]$, and for odd $n+1$ we also kill $\mathrm{Ker}\ i_{1*}^{(\pi_q)}$, $2q+1 = n+1$ (here the dimension of V_1 is $n+1$).

Furthermore, note that

$$\mathrm{Ker}\ i_{1*}^{(\pi_q)} = \mathrm{Ker}\ \hat{\imath}_{1*}^{(H_q)}$$

and

$$\mathrm{Ker}\ f_{j*}^{(\pi_q)} = \mathrm{Ker}\ \hat{f}_{j*}^{(H_q)},$$

and also

$$\mathrm{Ker}\ \hat{f}_{j*}^{(H_q)} = \mathrm{Ker}\ \hat{\imath}_{1*}^{(H_q)} + \mathrm{Ker}\ \hat{\imath}_{2*}^{(H_q)},$$

which yields that one may apply the "Hurewicz" theorem from § 4 to $\mathrm{Ker}\ \hat{\imath}_{2*}^{(H_q)}$.

Now, as in the proof of Theorem 3, we reconstruct $\mathrm{Ker}\ i_{2*}^{(\pi_q)}$, by applying Lemma 7.1 for $n = 2q+1$. The case $n = 2q$ is analogous to Theorem 3 also by virtue of the Remark that the "Hurewicz theorem" (Lemma 4.1) can be applied to the kernel $\mathrm{Ker}\ i_{2*}^{(\pi_q)}$.

Thus, we have proved the following:

Theorem 9.1. *Under assumptions of Theorem* 6 *there exists a submanifold* $V \subset S^{n+2}\backslash S^n$ *of homotopy type* $S^n \times S^1$ *such that the domain* $\overline{A} \subset S^{n+2}$ *restricted by* V *has homotopy type* S^n.

This is the analog of Theorem 3 for the case we consider.

Note that by virtue of the Browder–Levine theorem [see [20], § 5], the manifold V is a skew product with fiber $\widetilde{S}^n \in \theta^n(\partial\pi)$ and has S^1. For even n we have $\theta^n(\partial\pi) = 0$. However, in any case $\widetilde{S}^n$ is PL-homeomorphic to S^n, and V is PL-homeomorphic to $S^n \times S^1$, since the group of PL-automorphisms of the sphere S^n is connected. From now on, we shall work in terms of PL-manifolds.

For a domain $\overline{A}$, $\partial\overline{A} = V$ we take its "dual domain" which is PL-homeomorphic to $D^{n+1} \times S^1$, and paste $\overline{A} \cup_h D^{n+1} \times S^1$, where $h\colon \partial D^{n+1} \times S^1 \to V$ is a PL-homeomorphism. As it is well known, in these conditions $\overline{A} \cup_h D^{n+1} \times S^1$ is PL-homeomorphic to S^{n+2}. The initial sphere S^n lies in $\overline{A}$, and the complement $\overline{A}\backslash S^n$ is contractible to $V = \partial\overline{A}$. Thus the pair $(\overline{A} \cup_h D^{n+1} \times S', S^n)$ satisfies the Stallings theorem [18]. Without loss of generality, we might assume that the embedding $S^n \subset S^{n+2}$ is linear on a small simplex. From the methods of [18], the result below easily follows.

There exists a homeomorphism (PL-homeomorphism everywhere except a small neighborhood of S^n), which takes S^n to the standard sphere. Consequently, on A there is a new PL-structure such that:

a) it coincides with the old structure on $\partial\overline{A}$;
b) $\partial\overline{A} = S^n \times S^1$ is h-cobordant to the boundary of the tubular neighborhood $T(S^n) \subset \overline{A}$.

Thus in the new PL-structure we see that $\overline{A}$ is PL-homeomorphic to $S^n \times D^2$ (see [17]).

This evidently yields that the knot $S^n \subset S^{n+2}$ is globally flat.

Let us prove the remaining part of Theorem 6.

There is a PL-homeomorphism everywhere except in a neighborhood of $S^n \subset \overline{A}$:

$$d\colon \overline{A} \to S^n \times D^2,$$
$$d(S^n) = S^n \times 0.$$

We glue to $S^n \times D^2$ a closed complement $Q = (\overline{S^{n+2}\backslash\overline{A}})$ with respect to the identification $d/\partial\overline{A} = \partial Q$. Then

$$M = S^n \times D^2 \bigcup_d Q,$$

where $d\colon \partial Q \to S^n \times S^1$, and $d/\partial Q$ is a PL-homeomorphism. It is easy to see that M is a homotopy sphere of dimension $n+2$. Thus we have a simultaneous transformation $d'\colon M \to S^{n+2}$, where $d' = d/A$ and $d' = 1/Q$ which takes “knot” to a PL-knot, with a direct product $S^n \times D^2 \subset M$. The PL-knot can be smoothed in such situation, and on $S^n \subset M$ there is a smooth structure from $\theta^n(\partial\pi)$ ($= bP^{n+1}$ see [7]).

Theorem 6 is proved.

§ 10. One generalization of Theorem 5

Let K be a finite “Browder complex”. For the simply connected case this means that there is a “fundamental cycle” of dimension n: $\mu \in H_n(K)$ such that the map $D\colon Z \to Z \cap \mu$ is an isomorphism $H^j(K) \to H_{n-j}(K)$. If the complex K is not simply connected and $p\colon K' \to K$ is a finite-fold covering with m sheets, we have to require that $H_n(K') = Z$ and that there is an element

$$\mu' \in H_n(K'), \quad p\mu' = m\mu,$$

such that the map $D\colon Z \to Z \cap \mu'$ is an isomorphism. If the group $\pi_1(K)$ is finite then this gives a definition of the Browder complex. When $\pi_1(K)$ is infinite, this is not sufficient. Let $K' \to K$ be the covering with subgroup $\pi' \subset \pi = \pi_1(K)$ and fiber $F = \pi/\pi'$ on which π acts by left shifts. Assume $\alpha \cdot f$, $\alpha \in \pi$, $f \in F$, and the groups $H^0(F)$, $H_0(F)$, $H^0_c(F)$, $H^{(0)}_0(F)$, are defined, where the group π acts (here $H^0_c(F)$ are finitely supported functions on F valued in Z, $H^0_0(F)$ are infinite linear combinations $\sum a_i f_i$, $a_i \in Z$, $f_i \in F$). Then we have:

$$H^*(K') = H^*(K, H^0(F)), \quad H^*_c(K') = H^*(K, H^0_c(F)),$$
$$H_*(K') = H_*(K, H_0(F)), \quad H^{(0)}_*(K') = H_*(K, H^{(0)}_0(F)),$$

and all homology groups are assumed to have local coefficients.

Consider the generating element

$$g = \sum_i f_i \in H^{(0)}_0(F).$$

Then the correspondence $Z \to Z \otimes g$ takes $H_i(K)$ to

$$H_i(K, H^{(0)}_0(F)) = H^{(0)}_i(K').$$

If F consists of m elements, then the composition $p_*(Z \otimes g)$ is the multiplication by $m\colon Z \to mZ$.

We require that the maps $D\colon Z \to Z \cap (\mu \otimes g)$, $\mu \in H_n(K)$ are isomorphisms:

$$D\colon H^i_c(K') \to H_{n-i}(K'), \quad Z \in H^i_c(K'),$$
$$D\colon H^i(K') \to H^{(0)}_{n-i}(K'), \quad Z \in H^i(K').$$

The element $\mu \in H_n(K)$, is, as before, the fundamental cycle in K, and $\mu \otimes g$ is the fundamental cycle in K'.

In this case we call K the *Browder complex*.

The following lemma holds.

Lemma 10.1. *If W is an open smooth $(n+1)$-dimensional manifold having homotopy type of a finite complex and there is a (possibly, non-smooth) action on W of the discrete transformation $T\colon W \to W$ such that the quotient space is compact and $H_n(W) = Z$ then W is a Browder complex with respect to the fundamental cycle of dimension n.*

We leave this lemma without proof.[1] We note that the condition on T (on its existence) can be replaced by a simple condition on "homotopy type at the infinity" for W.

From Lemma 10.1, and by virtue of Theorem 3, where the condition about the homotopy type of a closed manifold is replaced by Lemma 10.1, we easily get:

Theorem 10.1. *Let M^n be a smooth closed manifold, $\pi_1(M^n) = \pi = Z + \dots + Z$, and there is a decomposition $\pi = \pi' + \pi''$. Then the covering space M with fundamental group $\pi' \subset \pi$ is diffeomorphic to $M^{n-l} \times R^l$, where $l = rk\pi''$, and M^{n-l} is a closed smooth manifold $n - l \geq 5$.*

For the case $\pi' = Z$, $\pi'' = 0$ this theorem was proved by Browder and Levine (see [21]).

Appendix 1. On the signature formula

As in [10], [13], consider a manifold M^n, $n = m + 4k$, and an indivisible element $z \in H_{4k}(M^n, Z)$ such that $Dz = y_1 \cdots y_m$, $y_i \in H^1(M^n, Z)$, $j = 1, \dots, m$. As it is shown in [10, 13], there is one canonical element $\hat{z} \in H_{4k}(\widehat{M}^n, Z)$, where $\widehat{M}$ is a covering over M^n with group $Z + \dots + Z$ (m summands), such that exactly those paths $\gamma \subset M^n$ are closed for which

$$(\gamma, y_1) = \dots = (\gamma, y_m) = 0.$$

[1]Note that the proof uses homology with special properties on the support, introduced by Rokhlin in yet unpublished paper.

We shall not recall the algebraic definition of $\hat{z} \in H_{4k}(\widehat{M}^n, Z)$ by z. Geometrically, it is defined as follows: we realize the cycles Dy_j by submanifolds $M_j^{n-1} \subset M^n$ and realize z by their intersection

$$M^{4k} = M_1^{n-1} \cap \dots \cap M_m^{n-1}.$$

In this case the manifold M^{4k} is covered by a closed manifold in $\widehat{M}$ and defines a cycle $\hat{z}$.

For $m = 2$ the following theorem holds.

Theorem. *If the intersection index for the group $H_{2k+1}(\widehat{M})$ is identically zero then the following formula holds*:

$$(L_k(M^n), Z) = \tau(\hat{z}).$$

Note that if $H_{2k+1}(\widehat{M}, R)$ is finite-dimensional, then the conditions of our lemma hold. Thus, this theorem is a generalization of Theorem 2 from [10].

Proof of the Theorem. Consider the covering space $\widehat{M}$ constructed above, where we have the inverse images of the manifolds M_1^{n-1} and M_2^{n-1} with respect to the projection $p\colon \widehat{M} \to M^{4k+2}$, $n = 4k+2$. Denote the basic transformation of the mapping class group $Z + Z$ of $\widehat{M}$, by $T_1, T_2\colon \widehat{M} \to \widehat{M}$. Then the inverse image of the manifold M_1^{n-1} splits into a connected sum $\cup_j M_j^{(1)}$, and the inverse image $p^{-1}(M_2^{n-1})$ splits into $\cup_q M_q^{(2)}$ in such a way that $M_s^{(\varepsilon)}$, where $\varepsilon = 1, 2$, $-\infty < s < +\infty$, separates $\widehat{M}$ into two parts: $A_s^{(\varepsilon)}$ and $B_s^{(\varepsilon)}$, where

$$A_s^{(\varepsilon)} \cup B_s^{(\varepsilon)} = \widehat{M}, \quad A_s^{(\varepsilon)} \cap B_s^{(\varepsilon)} = M_s^{(\varepsilon)}.$$

Besides, the notation is chosen in such a way that

$$T_1 M_s^{(1)} = M_s^{(1)}, \quad T_2 M_s^{(1)} = M_{s+1}^{(1)},$$
$$T_2 M_s^{(2)} = M_s^{(2)}, \quad T_1 M_s^{(2)} = M_{s+1}^{(2)},$$

and $M_s^{(\varepsilon)}$ for any s are Z-covering spaces over M_ε^{n-1}. The inverse image of the manifold $M^{4k} = M_1^{n-1} \cap M_2^{n-1}$ can be represented as follows:

$$p^{-1}(M^{4k}) = \bigcup_{j,q} (M_j^{(1)} \cap M_q^{(2)}) = \bigcup_{j,q} M_{j,q}^{4k},$$

so that all $M_{j,q}^{4k}$ are diffeomorphic to the initial $M^{4k} = M_1^{n-1} \cap M_2^{n-1}$. Denote the cycle defined by $M_{j,q}^{4k}$ in $M_j^{(1)}$, by $t_j \in H_{4k}(M_j^{(1)})$, $T_{1*} t_j = t_j$, and denote the embedding $M_j^{(1)} \subset \widehat{M}$ by λ_j. Evidently,

$$\lambda_{j*} t_j = \hat{z}.$$

By virtue of Theorem 1 from [10] (or Theorem 2 from [13]) we get the formula:

$$\tau(t_j) = \tau(M^{4k}).$$

Let us prove the following equality

$$\tau(t_j) = \tau(\widehat{z}).$$

Let $j = 0$, $t_0 \in H_{4k}(M_0^{(1)})$. Denote $M_0^{(1)}$ just by M, t_0 by t, $A_0^{(1)}$ by A and $B_0^{(1)}$ by B. Then

$$B \cap A = M, \quad B \cup A = \widehat{M}.$$

Denote the manifold $M_0^{(2)}$ by N, then

$$M \cap N = M_{0,0}^{4k} = M^{4k}.$$

Now recall a result from [13]. If for each $\alpha \in H^{2k}(M, R)$ such that the cycle $\beta = \alpha \cap t \in H_{2k}(M)$ is null-homologous in A and in B, the equality $(\alpha^2, t) = 0$ holds, then the following formula is true:

$$\tau(t) = \tau(\hat{z}).$$

Note that the cycle $\beta = \alpha \cap t$ geometrically lies in $M^{4k} = M \cap N$, and the self-intersection index $\beta \circ \beta$ (on M^{4k}) is equal to (α^2, t) in M. Besides, the cycle β is cut out from M^{4k} by an open cycle $D\alpha \in H_{2k+1}^{(0)}(M)$.

The cycle β is spanned by membranes $\delta_1 \subset A$ and $\delta_2 \subset B$ such that $\partial\delta_1 = \partial\delta_2 = \beta$. Furthermore, the pair M, N separates $\widehat{M}$ into four parts: W_1, W_2, W_3, W_4, where

$$\bigcap_i W_i = M^{4k},$$

$$\bigcap_i W_i = \widehat{M}, \quad (W_1 \cup W_2) \cap (W_3 \cup W_4) = M,$$

$$(W_1 \cup W_4) \cap (W_2 \cup W_3) = N.$$

Denote by $J_i \subset H^{2k+1}(M, R) \cap t \subset H_{2k}(M, R)$ the subgroups consisting of elements with zero-homologous representatives in W_i, $i = 1, 2, 3, 4$. Analogously, let us define the subgroups $J_{(\varepsilon)} \subset H^{2k+1}(M, R) \cap t$, for $\varepsilon = 1, 2$ consisting of elements zero-homologous in A for $\varepsilon = 1$ or in B for $\varepsilon = 2$.

Clearly,

$$J_1 \cup J_2 = J_{(1)}, \quad J_3 \cup J_4 = J_{(2)}.$$

Denote the group $H^{2k+1}(M, R) \cap t$ by H. Let us define the operator $P\colon H \to H$ by setting

$$P(\alpha \cap t) = (T_1 \cdot \alpha) \cap t.$$

Since $T_{1*}t = t$, then P is an isomorphism. Note that H is a finite dimensional space over R.

The following relations hold:

$$P^k J_{(1)} \subset J_2, \quad P^{-k} J_{(1)} \subset J_1,$$
$$P^k J_{(2)} \subset J_3, \quad P^{-k} J_{(2)} \subset J_4,$$

for k sufficiently large because of finite dimensionality of H, J_i, $J_{(\varepsilon)}$. Thus (again, by virtue of finite dimensionality) we have:

$$J_{(1)} = J_1 = J_2, \quad J_{(2)} = J_3 = J_4.$$

Now, let us return to the element $\beta = \alpha \cap t$ which is null-homologous in A and in B, belongs to M^{4k} and is represented by a cycle $\overline{\beta} \subset M^{4k}$. Since $\beta \in J_{(1)} \cap J_{(2)}$, the cycle $\overline{\overline{\beta}}$ on $T^{-2k}M^{4k}$ (for large k), representing $P^{-2k}\beta$, becomes null-homologous in the manifolds $T^{-2k}W_1$ and $T^{-2k}W_4$, if we add to this cycle the cycle $h \subset T^{-2k}M^{4k}$ null-homologous in M. From finite dimensionality of the group $H_{2k}(M^{4k})$ it follows that k can be chosen so large that the membrane $\partial^{-1}(h)$ can be chosen not to intersect $T^{-k}M^{4k}$. Then the cycle $\overline{\overline{\beta}} \subset T^{-2k}M^{4k}$ is null-homologous in $T^{-k}W_1$ and $T^{-k}W_4$.

Denote the corresponding membranes by δ_3 and δ_4:

$$\delta_3 \subset T^{-k}W_1, \quad \delta_4 \subset T^{-k}W_4, \quad \partial\delta_3 = \partial\delta_4 = \overline{\overline{\beta}}.$$

Since $\alpha \in H^{2k}(M, R)$, we have $D\alpha \in H^{(0)}_{2k+1}(M, R)$, and $D\alpha$ is represented by an open cycle in M, whose intersection with M^{4k} is $\overline{\beta}$, and whose intersection with $T^{-2k}M^{4k}$ is $\overline{\overline{\beta}}$. Denote the segment of this open cycle from $\overline{\beta}$ to $\overline{\overline{\beta}}$ by d, $\partial\alpha = \overline{\beta} - \overline{\overline{\beta}}$.

Set

$$g_1 = \delta_3 - d + \delta_1,$$
$$g_2 = \delta_4 - d + \delta_2,$$

where g_1 and g_2 are $(2k+1)$-dimensional cycles in $\widehat{M}$. The cycle

$$\overline{\overline{\overline{\beta}}} = d \cap T^{-k}M^{4k}$$

is such that it is null-homologous in $T^{-k}W_1$, $T^{-k}W_2$, $T^{-k}W_3$, $T^{-k}W_4$ and its self-intersection index in $T^{-k}M^{4k}$ is equal to

$$\overline{\overline{\overline{\beta}}} \circ \overline{\overline{\overline{\beta}}} = (\alpha^2,\, t) = \overline{\beta} \circ \overline{\beta}.$$

But, it is easy to see that

$$g_1 \circ g_2 = \overline{\overline{\overline{\beta}}} \circ \overline{\overline{\overline{\beta}}}$$

and

$$g_1 \circ g_2 = 0$$

by the assumption of the theorem. Thus, we conclude that the condition

$$(\alpha J_{(1)},\, t) = (\alpha J_{(2)},\, t) = 0$$

yields

$$(\alpha^2,\, t) = 0.$$

By using analogously to [10, 13], we conclude the proof of the theorem.

Now let us make some conclusions from the theorem.

1. It is easy to show that if the condition $N/Z_0(\pi)N = 0$ holds then

$$N_{2k+1} = N_\infty \supset N_{2k+1}^{\perp}.$$

As a matter of fact, every element $\sigma \in N$ satisfies some polynomial relation

$$Q(T_1,\, T_2)\sigma = 0,$$

where T_1, T_2 are generators of π and $\varepsilon Q = 1$, $\varepsilon\colon Z(\pi) \to Z$. Indeed, if $\sigma_1, \dots, \sigma_s$ are generators of N over $Z(\pi)$ and $N/Z_0(\pi)N = 0$, then there exists a matrix $P = (P_{ij})$ with coefficients in $Z(\pi)$ such that $\varepsilon P = E$ and $\sum_i P_{ij}\sigma_i = 0$. But then

$$(\det P)\sigma_j = 0$$

and

$$Q = \det P, \quad \varepsilon Q = 1.$$

We may assume that

$$Q = [1 + P_0(T_2)] + T_1 P_1(T_2) + \cdots + T_1^n P_n(T_2),$$

where P_0 depends only on positive powers of T_2 and $P_0(0) = 0$. Thus the polynomial Q has an inverse in formal series on T_1^j and T_2^s, where $j \geq 0$, $s \geq f(j) > -\infty$. Consequently, the element σ is zero-homologous in open homology and orthogonal to N in the sense of intersection index.

The condition sufficient to apply the theorem ($N/Z_0(\pi)N = 0$) holds, for example, if the image

$$p_*\colon H_{2k+1}(\widehat{M},\, R) \to H_{2k+1}(M^n,\, R)$$

is trivial and the differential

$$d_2\colon E_2^{2,2k} \to E_2^{0,2k+1} = \frac{N}{Z_0(\pi)N},$$
$$E_2^{2,2k} = H_{2k}^{\text{invariant}} \subset H_{2k}(\widehat{M}),$$

is trivial in the Cartan spectral sequence for the covering $p\colon \widehat{M} \to M^n$.

2. Let us give another proof of the invariance of $L_k(M^n)$ for $n \le 4k+3$ and $\pi_1(M^n) = 0$. Indeed, if M^n is homeomorphic to $M^{4k} \times R^3$, where M^{4k} is simply connected and closed, we can, as in § 2, select the submanifold $W = M^{4k} \times T^2 \times R$ and realize the cycle $[M^{4k} \times T^2]$ by a smooth $V \overset{i}{\subset} W$ such that the inclusion homomorphism $i_*\colon \pi_q(V) \to \pi_q(W)$ is an isomorphism $q \le 2k$, which is trivial. Then V splits W into two parts A and B, $A \cap B = V$, and $i_1\colon V \subset A$, $i_2\colon V \subset B$. Set

$$M'_{2k+1} = \operatorname{Ker}\, i_{1*}^{(H_{2k+1})}.$$

Since the intersection index on $M'_{2k+1} = \operatorname{Ker}\, \hat{\imath}_{1*}^{(H_{2k+1})}$ is trivial, where $\hat{\imath}_1$: $\widehat{V} \subset \widehat{A}$ (universal coverings), we can, following Whitney, realize the $Z(\pi)$-basis in M'_{2k+1} by embedded spheres and apply Morse surgery along these spheres (the possibility of such realization is proved identically to the proof of Whitney, for more details see [6]). The surgery can be performed without changing the Pontrjagin classes; after a surgery we get a manifold V_1, to which the theorem from this appendix can be applied. These surgeries, evidently, do not change the “cycle signature” for the covering space over V and V_1. Comparing the above arguments with the basic lemma of [13], applied to the embedding $\widehat{V} \subset \widehat{W}$, with the theorem above, and the equality of “cycle signatures” for $\widehat{V}$ and $\widehat{V}_1$, we obtain the statement in the same vein as in [10, 13].

Appendix 2. Unsolved questions concerning characteristic class theory

Below we give several problems which are directly connected to the results of the author [10–12] and Rokhlin; these problems are mainly concerned with Pontrjagin classes.

I. *Topological problems*

1[1]. Does there exist a number $n = n(k)$ depending only on k such that for all prime $p > n(k)$ the Pontrjagin classes p_k modulo p^h are topologically invariant. This would follow from the fact that the groups $\pi_i(\mathrm{BTop})$ are finitely-generated for all $i \leq 4k$. However, for the solution it seems to be more convenient to use some generalization of the method of the present work or the author's work [13]. Such a result would have nice applications, say, for classical lens spaces of dimensions ≥ 5. For example, $p \neq 7$ for $k = 2$ (see [8]).

2. Are the rational Pontrjagin classes of complexes and rational homological manifolds topological invariants? Positive results in this direction are obtained only for $L_k(M^n)$, $n \leq 4k + 2$ (see [10, 12]).

3. Is it possible to define rational Pontrjagin classes $p_i \in H^{4i}(\mathrm{BTop}, Q)$ for Milnor's topological microbundles to satisfy the following axioms:

a) for O and PL-microbundles they coincide with the usual Pontrjagin classes;
b) the Whitney formula for the sum;
c) the Hirzebruch formula for $L_k(M^{4k})$ and the formulas due to the author for $L_k(M^{4k+1})$ and sometimes for $L_k(M^{4k+m})$, $m > 1$ [see [10, 13] and Theorem 2 of the present paper].

II. *Homotopy problems*

1. Let $x \in H_{4k}(M^n)$ be such an element that $Dz = y_1 \cdots y_m$, $m = n - 4k$, $y_i \in H^1(M^n)$. Is the scalar product $(L_k(M^n), Z)$ a homotopy invariant? The problem is solved by the author for $m = 1$, and partially for $m = 2$ [see [10, 12] and Appendix 1 of the present work] and sometimes for $m > 2$ [see Theorem 2 of the present work]. For $m = 2$ the final solution is obtained by Rokhlin.

2. In those cases when the previous problem has a positive solution, the problem of calculation of L_k classes in terms of homotopy invariants arises. This problem is not solved even in the situation of Rokhlin's theorem for codimension $m = 2$. Important partial cases of this problem will be discussed in the next section devoted to differential-topological questions.

III. *Stably-algebraic problems*

Before discussing problems, let us first give an algebraic introduction. Let π be a Noetherian group and let M be a finitely-generated $Z(\pi)$-module.

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[1] Added when reading the proofs. Problem 1 was recently published in a yet unpublished collection of works by the author and V. A. Rokhlin.

By a scalar product we mean a homomorphism of modules $h\colon M \to PM$, where $PM = \mathrm{Hom}_c(M, Z)$, $(x, y) = hx(y)$. Certainly, a symmetrical and a skew-symmetrical case arise.

We call a scalar product unimodular if h is an isomorphism.

If $\pi' \subset \pi$, then on $N = M(Z_0(\pi')M)$ there naturally arises a bilinear form $(px, py) = \sum_{a\in\pi'}(x, ay)$, which is a scalar product in the same sense, if π' is a normal subgroup. Here $p\colon M \to N$ is the natural projection. We call this bilinear form *the induced scalar product.*

We call a symmetric scalar product even if (x, x) and (x, ax) are divisible by 2 for all $a \in \pi$, $a^2 = 1$.

For subgroups π' of finite index in π and a symmetrical case it makes sense to speak about the signature of the scalar product (induced) on $N = M(Z_0(\pi')M)$, and the signature of a form on N is defined as a function of a subgroup $\pi' \subset \pi$, $\tau = \tau(\pi')$, if the index of π' in π is finite. Set $\tau(M) = \tau(\pi)$, where $I(\pi')$ is the index of π'. Then we require that $\tau(\pi') = \tau(M)I(\pi')$.

Assume the scalar product is skew-symmetrical. By Arf-invariant we mean the map $\varphi\colon M \to Z_2$ such that $\varphi(ax) = \varphi(x)$, $a \in \pi$, and

$$\varphi(x + y) = \varphi(x) + \varphi(y) + (x, y) \bmod 2.$$

Let $\pi' \subset \pi$ and $N = M/Z_0(\pi')M$, $p\colon M \to N$. By induced Arf-invariant we mean the map $\varphi_{\pi'}\colon N \to Z_2$ such that

$$\varphi_{\pi'}(px) = \varphi(x) + \sum_{a\in\frac{\pi'}{2}} (x, ax) \bmod 2,$$

where $\frac{\pi'}{2} \subset \pi'$ denotes the subset in π', which for every pair of elements $a, a^{-1} \in \pi$, contains exactly one. The case $a = a^{-1}$ is inessential because in this case $(x, ax) = (a^{-1}x, x) = -(x, ax) = 0$. For $\varphi_{\pi'}$, it is easy to check the correctedness and the identity for Arf. If π' is of finite index $I(\pi')$ in π, then for $M/Z_0(\pi')M$ there is a well-defined "total" Arf-invariant $\Phi(\pi') \in Z_2$. Set $\varphi(M) = \Phi(\pi)$. Then let $\Phi(\pi') = \Phi(M)I(\pi')$.

Now let π be a finite or abelian group. We say that a module M with symmetric or skew-symmetric scalar product has a Poincaré duality if for all subgroups $\pi' \subset \pi$ the induced scalar products are unimodular.

Let F_1 be a free module on two generators $x, y \in F_1$, such that $(x, ax) = (y, ay) = 0$ for all $a \in \pi$, $(x, ay) = 0$ for $a \neq 1$ and $(x, y) = 1$. We assume the scalar product to be symmetric or skew-symmetric. In the latter case we also require $\varphi(x) = \varphi(y) = 0$, i.e. in the module there is an Arf-invariant of special type. We call such a module one-dimensional free module.

By a free module we mean a sum $F = F_1 + \cdots + F_1$ with respect to the scalar product and Arf-invariant (for the skew-symmetrical case).

We consider the isomorphisms of modules, the direct sum, etc., with respect to all structures preserved.

Admissible classes of modules:

C_1: projective modules with symmetric even scalar product and Poincaré duality.
$C_1^0 \subset C_1$: modules with zero signature $\tau(M) = 0$;
C_2: projective modules with skew-symmetric scalar product, Poincaré duality and Arf-invariant;
$C_2^0 \subset C_2$: modules with zero Arf-invariant $\Phi(M) = 0$;
C_2': as in C_2, but without Arf-invariant taken into account;
$\overline{C}_i \subset C_i$, $i = 1, 2$: invertible modules $M \subset \overline{C}_i$, for which there exists a module M' such that $M + M' = F$ with respect to all structures, where F is as above.

Analogously, one defines the class $\overline{C}_2' \subset C_2$ without Arf-invariant.

Denote the subclasses $C_i^0 \cap \overline{C}_i$ by D_i.

With each class $C_1, C_2, C_2', C_1^0, C_2^0$ we naturally associate the "Grothendieck group":

$$A(\pi) = \widetilde{K}^0(C_1), \quad B(\pi) = \widetilde{K}^0(C_2),$$

$$C(\pi) = \widetilde{K}^0(C_2'), \quad D(\pi) = \widetilde{K}^0(C_1^0),$$

$$E(\pi) = \widetilde{K}^0(C_2).$$

There is a well-defined homomorphism $B(\pi) \to C(\pi)$. The subclasses $\overline{G}_1, \overline{G}_2, \overline{G}_2', D_1, D_2$ define the subgroups of "really invertible" elements.

Algebraic problem: calculate the groups $A(\pi), B(\pi), C(\pi), D(\pi), E(\pi)$. It would be interesting to find these groups for $\pi = Z + \cdots + Z$ and $\pi = Z_p$. For $\pi = Z_p$ this is related with the arithmetics of the number p, because here for "bad" p even the usual functor $\widetilde{K}^0(Z(Z_p))$ without scalar products can be nontrivial.

For $\pi = Z + Z$, the usual $\widetilde{K}^0(\pi)$ is trivial, but $B(\pi)$ and $C(\pi)$ are nontrivial, as Example 2 from § 3 of [13] shows. As we shall see from the further topological problems, for $\pi = Z + \cdots + Z$ all A, B, C, D, E can be nontrivial.

In the case $\pi = Z + \cdots + Z$ we may assume that we always deal with scalar products on algebraically free modules, since projective modules are stably free.

IV. *Differential-topological problems*

Our question will be related to the following situations:

a) There is a commutative diagram of degree +1 maps and (regular) coverings

$$\begin{array}{ccc} M_1^{2n} & \xrightarrow{f} & M_2^{2n} \\ {\scriptstyle p_1}\uparrow & & {\scriptstyle p_1}\uparrow \\ \widehat{M}_1 & \xrightarrow{\hat{f}} & \widehat{M}_2, \end{array}$$

where the monodromy group of the coverings is π and we have an element $\alpha \in K_R^0(M_2^{2n})$ such that $f^*\alpha \in K_R^0(M_1^{2n})$ is the "stable tangent bundle". We assume that the homology kernels of $\hat{f}$ are trivial in dimensions $<n$. Then the kernel $M = \operatorname{Ker} \hat{f}_*^{(H_m)}$ is a π-module, and it defines an element from $A(\pi)$ for $n = 2k$ or $B(\pi)$ for $n = 2k + 1$. For $n = 3$, 7 we need only the image of $B(\pi) \to C(\pi)$.

b) There is a membrane W^{2n} with two boundary components M_1^{2n-1}, M_2^{2n-1} and retractions $r_i\colon W^{2n} \to M_i^{2n-1}$ which are tangential maps. We impose to r_i restrictions analogous to those imposed on f in example a) for coverings $\widehat{W} \to W^{2n}$, $\widehat{M}_i \to M_i^{2n-1}$. Then the kernel $M = \operatorname{Ker} \hat{r}_i^{(H_n)}$ defines an element from $A(\pi)$, $n = 2k$, or from $B(\pi)$, $n = 2k + 1$, moreover, here it is easy to reduce these elements to $D(\pi)$ for $n = 2k$ or to $E(\pi)$ for $n = 2k + 1$.

Problems

1. Realizability of elements $x \in A(\pi)$, $B(\pi)$, $C(\pi)$, $D(\pi)$, $E(\pi)$ in the situations of Examples a) and b).
2. It is interesting to study the case of the previous problem when in a) the element $\alpha \in K_R^0(M_2^{2n})$ is the "stable tangent bundle" to M_2^{2n}.
3. Rational Pontrjagin classes: if in a), the manifold M_2^{2n} is the torus T^{2n}, then $\alpha \in \operatorname{Ker} J$, and the Pontrjagin classes

$$f^* p_i(\alpha) = p_i(M_2),$$

are defined, so that $\pi = Z + \cdots + Z$. As the author has shown, the stable tangent bundle to manifolds of homotopy type T^q is always trivial (it easily follows from Theorem 2 of the present work, Bott periodicity for BO, Adams' result about the $J \otimes Z_2$-homomorphism and the fact that the suspension over the torus T^q is of homotopy type wedge of spheres). Thus for $\alpha \neq 0$ the classes $p_i(\alpha) \in H^*(M_2^{2n})$ are nontrivial, and there is a (possibly, not uniquely defined) invariant $x(\alpha) \in A(\pi)$ for $n = 2k$ and $x(\alpha) \in C(\pi)$ for $n = 2k+1$. The equality $x(\alpha) = 0$ yields $\alpha = 0$ by the author's theorem. The classes p_i are linear forms in exterior powers:

$$p_i(\alpha)\colon \Lambda^{4i}\pi \to Z,$$
$$\pi = Z + \cdots + Z, \quad \mathrm{Hom}(\Lambda^{4i}\pi, Z) = \Lambda^{2n-4i}\pi.$$

Generally speaking, one should assume that $p_i(\alpha) \in \Lambda^{2n-4i}\pi$ for $\pi = Z + \cdots + Z$ ($2n$ copies).

The problem is to calculate $p_i(\alpha) \in \Lambda^{2n-4i}\pi$ as functions of $x(\alpha) \in A(\pi)$ or $C(\pi)$. The above argument shows that there does exist a connection between $p_i(\alpha)$ and $x(\alpha)$.

Certainly, in this problem one can take instead of torus $T^{2n} = M_2^{2n}$ the direct product $S^{4k} \times T^{2n-4k}$, then we shall get a number. This question is closely connected to Problem 2 ("homotopy problems").

4. The situation with non-Noetherian fundamental groups is not clear to the author; there are many geometrical examples of "finite-dimensional groups" here, and the corresponding theory would have a series of applications. Certainly, the functor $P = \mathrm{Hom}_c$ can be defined by using "locally-finite" classes of bases, which always exist geometrically. However, in the applications we need that the modules of kernels are finite-dimensional over $Z(\pi)$. These questions, are however, unrelated to characteristic classes, and they have not been studied by the author.
5. Consider the odd-dimensional case $q = 2k+1$. The restrictions on the module given by Theorem 5.2 of the present paper, are quite insufficient.

Later on, we shall denote $\widetilde{K}^0(Z(\pi))$ by $\widetilde{K}^0(\pi)$.

In addition, we note that the usual $K^0(\pi)$, consisting of stable classes of projective modules, is embeddable in $D(\pi)$ and $E(\pi)$ as follows:

If $\alpha \in K^0(\pi)$, then $P\alpha \in K^0(\pi)$, and for the module $\alpha + P\alpha$ there is a natural scalar product. We get the following inclusions:

$$K^0(\pi) \subset D(\pi) \subset A(\pi),$$
$$K^0(\pi) \subset E(\pi) \subset B(\pi),$$

assuming in the $E(\pi)$ case that the Arf-invariant on $\alpha \subset \alpha + P\alpha$ and $P\alpha \subset \alpha + P\alpha$ is trivial.

By using other functors except $P = \mathrm{Hom}_c(\ , Z)$, the universal coefficient formula, and the Poincaré duality, it is easy to prove the following:

Theorem.[1] *If in Theorem* 3 § 2 *we replace the group* $\pi_\perp = Z + \cdots + Z$ *by any* (*Noetherian*) *group* $\pi = \pi_1$, *then the obstruction to existence of the manifold* $V^n \subset W^{n+1}$, *being a deformation retract in* W^{n+1}, *lies in the Grothendieck group* $K^0(\pi)$, *and the condition that this obstruction is zero is sufficient for the deformation retract to exist* $V^n \subset W^{n+1}$.

Remark. The uniqueness question for such $V^n \subset W^{n+1}$ is reduced to the h-cobordism problem, and hence to $K^1(\pi)$, more exactly, the quotient group $Wh(\pi)$ (see [8]). Thus we get the following picture.

A. Problems like Theorem 3 and § 2 are related only with $K^0(\pi)$ (or to its image in $A(\pi)$ and $B(\pi)$) and to $K^1(\pi) \to Wh(\pi)$. As one can see from the proof of Theorem 6 (see § 9) and the paper of Browder–Levine–Livesay (see [21]), these questions are analogous to the question of finding the boundary of an open manifold.

B. The diffeomorphism problem is subdivided into the following:

1. the J-functor, K_R-functor and normal bundles of compact manifolds: here for $n = 2k$ $A(\pi)$ and $B(\pi)$ play their roles (see [3, 22, 14], Appendix 1).
2. Realization of classes in the Thom complex for $n = 2k$ (see Problem 2). Here the torsions tor $A(\pi)$ and tor $B(\pi)$ are important (see Theorem 1 of [22] for $\pi_1 = 0$).
3. The relations between the h-cobordism and homotopy class in the Thom complex (see Theorem 2 of [22]). Here one should consider invertible elements from $D(\pi)$, $E(\pi)$ for $n = 2k - 1$.
4. For $n = 2k - 1$ in 1. and 2. and $n = 2k$ in 3. Ext^i_c come into play; their role is not known. They generalize the torsion for $\pi_1 = 0$.
5. The relation between h-cobordism and diffeomorphism for $n \geq 5$ is well known and it is connected only to $Wh(\pi) = K^1(\pi)/(\pi \cup -\pi)$.

Appendix 3. Algebraic remarks about the functor $P = \mathrm{Hom}_c$

Here we discuss the following questions:

1. the connection between $\mathrm{Ext}^i_c(M, Z)$ and $\mathrm{Ext}^i_c(PM, Z)$;
2. the notion of "reflexive" module: $P^2M = M$;
3. the functor Q for open homology groups.

[1] Added when reading proofs. This theorem was independently discovered by Siebenmann.

Let us first address the following question. Let M be an admissible π-module. Consider an acyclic (projective) free resolvent

$$C = \{\cdots \to F_n \to \cdots \to F_0 \xrightarrow{\varepsilon} M \to 0\}$$

and apply the functor P:

$$PC = \{0 \to PM \xrightarrow{P_\varepsilon} PF_0 \to \cdots \to PF_n \to \cdots\}.$$

We obtain a sequence which is exact for the term PF_0.

Now consider a resolvent of PM

$$C' = \{\cdots \to F'_n \to \cdots \to F'_0 \xrightarrow{\varepsilon'} PM \to 0\}.$$

Let us paste the complexes C and C':

$$\begin{tikzcd}
C'' = \{\cdots \arrow[r] & F'_n \arrow[r] & \cdots \arrow[r] & F'_0 \arrow[rr, "\delta"] \arrow[dr, "\varepsilon'"'] & & PF_0 \arrow[r] & \cdots \arrow[r] & PF_n \arrow[r] & \cdots\} \\
& & & & PM \arrow[ur, "P\varepsilon"'] \arrow[dr] & & & & \\
& & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

in such a way that $\delta = (P\varepsilon) \circ \varepsilon'$.

Set $F''_n = F'_n$, $F''_{-n-1} = PF_n$, $n \geq 0$. Evidently, we have:

$$H_i(C'') = 0, \quad i \geq -1, \quad H_i(C'') = \operatorname{Ext}_c^{-i-1}(M, Z), \quad i \leq -2.$$

Moreover, for the complex PC''

$$\begin{aligned}
H_c^i(C'') = {} & H_i(PC'') = \operatorname{Ext}_c^i(PM, Z), \quad i > 0, \\
H_c^0(C'') = {} & H_0(PC'') = \operatorname{Coker} P^2 = \frac{P^2 M}{\operatorname{Im} P^2}, \\
H_c^{-1}(C'') = {} & H_{-1}(PC'') = \operatorname{Ker} P^2 \subset M, \\
H_c^{-i}(C'') = {} & H_{-i}(PC'') = 0, \quad i \geq +2.
\end{aligned}$$

All these equality follow from the fact that for projective modules P^2 there exist a natural isomorphism. Thus, $\operatorname{Ker} P^2$ and $\operatorname{Coker} P^2$ obtain a geometrical meaning.

Since $H_i(C'')$ and $H_c^i(C'')$ are connected by the Cartan–Eilenberg–Grothendieck spectral sequences, the following conclusions are in order:

A. Let the homological dimension of the group π be equal to n (for example, $\pi = Z + \cdots + Z$). Then we see that

$$\operatorname{Ext}_c^n(PM, Z) = \operatorname{Ext}_c^{n-1}(PM, Z) = 0.$$

B. If $\mathrm{Ext}_c^i(\mathrm{Ext}_c^i(M, Z), Z) = 0$, $i > 0$, then $\mathrm{Ker}\, P^2 = 0$.
C. If $\mathrm{Ext}_c^{i+1}(\mathrm{Ext}_c^i(M, Z), Z) = 0$, $i > 0$, then $\mathrm{Coker}\, P^2 = 0$.

We call modules M for which $P^2M = M$, reflexive and those M' for which $PM' \approx M'$, will be called self-adjoint. Every reflexive module is a direct summand in a self-adjoint module, and vice versa, because in this case $P(M + PM) = M + PM$ and P is an additive functor.

Corollaries:

1. *if* $\mathrm{Ext}_c^i(M, Z) = 0$, $i > 0$, *and* $\pi = Z + \cdots + Z$, *then* M *is stably free because* PM *is stably free according to Lemma* 5.1 *and* $P^2M = M$;

2. *if* $\pi = Z + Z$, *then for every module* M *the module* PM *is stably free because* $\mathrm{Ext}_c^1(PM, Z) = \mathrm{Ext}_c^2(PM, Z) = 0$.

Note that for $\pi = Z + Z + Z$ this is no longer true because there exists a module $M \neq 0$ which is reflexive and such that

$$\begin{aligned} \mathrm{Ext}_c^2(M, Z) &= \mathrm{Ext}_c^3(M, Z) = 0, \\ \mathrm{Ext}_c^1(\mathrm{Ext}_c^1(M, Z), Z) &= \mathrm{Ext}_c^2(\mathrm{Ext}_c^1(M, Z), Z) = 0, \\ \mathrm{Ext}_c^1(M, Z) &= \mathrm{Ext}_c^3(\mathrm{Ext}_c^1(M, Z), Z) \neq 0. \end{aligned}$$

Let us construct such a module. Let M_0 be a one-dimensional module with generator $u \in M_0$ such that $Z_0(\pi) \circ u = 0$. The resolvent of M_0 (see § 5, Example 1) is three-dimensional,

$$0 \to F_3 \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} M_0 \to 0,$$

moreover,

$$\mathrm{Ext}_c^i(M_0, Z) = 0, \quad 0 \le i \le 2, \quad \mathrm{Ext}_c^3(M_0, Z) = M_0.$$

Let $M = F_2 / \mathrm{Im}\, d$. We have:

$$0 \to F_3 \xrightarrow{d} F_2 \xrightarrow{\varepsilon} M \to 0.$$

Thus $\mathrm{Ext}_c^i(M, Z) = 0$, $i > 1$, and $\mathrm{Ext}_c^1(M, Z) = \mathrm{Ext}_c^3(M_0, Z) = M_0$. This module M is the desired example of reflexive but not projective module for $\pi = Z + Z + Z$.

Let us introduce topology for $Z(\pi)$: namely, for the base system of neighborhoods of zero we take all linear spaces over Z generated by the elements $\alpha \in \pi \backslash A_i$, where A_i is any finite set in π.

For a finitely generated module we define the topology as follows: if $x_1, \ldots, x_k \in M$ are π-generators and $A_1, \ldots, A_k$ are any finite sets in π, we take for neighborhoods of zero in M all $x \in M$ such that $\lambda x = \sum_{i,j} \lambda_{ij} \alpha_{ij} x_j$, $\lambda \neq 0$, where $\alpha_{ij} \in \pi \backslash A_j$, $\lambda, \lambda_{ij} \in Z$. Such neighborhoods generate a system of neighborhoods of zero in M. In this topology, points, are in general, non-separable.

We have: PM are continuous characters of the continuous group M in Z (in discrete topology), so that PM is a topological $Z(\pi)$-module. $\operatorname{Ker} P^2$ are points of M, which are infinitely close to zero.

Let us define the completion $Q\colon M \to \widehat{M}$, where $\widehat{M}$ is the compactification of M, and we equate $\operatorname{Ker} P^2$ to zero in $\widehat{M}$. The derived functors of the functor Q correspond to open homology, thus for the field K we have:

$$Q = \operatorname{Hom}(\operatorname{Hom}_c(M, K), K),$$
$$\operatorname{Tor}^i_Q(M, K) = \operatorname{Hom}(\operatorname{Ext}^i_c(M, K), K), \quad i \geq 0.$$

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6

Stable homeomorphisms and the annulus conjecture[1]

R. Kirby

A homeomorphism h of R^n to R^n is stable if it can be written as a finite composition of homeomorphisms, each of which coincides with the identity in some domain, that is $h = h_1, h_2, \ldots, h_r$, and $h_i|_{U_i}$ = identity for each i where U_i is open subset in R^n.

Stable Homeomorphism Conjecture, SHC_n: All orientation preserving homeomorphisms of R^n are stable.

Stable homeomorphisms are particularly interesting because (see [3]) $\mathrm{SHC}_n \Rightarrow \mathrm{AC}_n$, and AC_k for all $k \leq n \Rightarrow \mathrm{SHC}_n$ where AC_n is the Annulus Conjecture: Let $f, g : S^{n-1} \to R^n$ be disjoint, locally flat imbeddings with $f(S^{n-1})$ inside the bounded component of $R^n - g(S^{n-1})$ (complement to $g(S^{n-1})$). Then the closed region A bounded by $f(S^{n-1})$ and $g(S^{n-1})$ is homeomorphic to $S^{n-1} \times [0,1]$.

Numerous attempts on these conjectures have been made; for example, it is known that an orientation preserving homeomorphism is stable if it is differentiable at one point [10, 12], if it can be approximated by a PL homeomorphism [6], or if it is $(n-2)$-stable [4]. "Stable" versions of AC_n are known: $A \times [0,1)$ is homeomorphic to $S^{n-1} \times I \times [0,1)$, $A \times R$ is $S^{n-1} \times I \times R$, and $A \times S^k = S^{n-1} \times I \times S^k$ if k is odd (see [7, 13]). A counter-example to AC_n would provide a non-triangulable n-manifold [3].

[1] Annals of Math., **89** (1969), 575–582 (received October 29, 1968).

Here we reduce these conjectures to the following problem in PL theory. Let T^n be the Cartesian product of n circles.

Hauptvermutung for tori, HT_n: Let T^n and τ^n be homeomorphic PL n-manifolds. Then T^n and τ^n are PL homeomorphic.

Theorem 1. *If* $n \geq 6$, *then* $\mathrm{HT}_n \Rightarrow \mathrm{SHC}_n$.

(Added December 1, 1968). It can now be shown that SHC_n is true for $n \neq 4$. If $n = 3$, this is a classical result. Theorem 1 also holds for $n = 5$, since Wall [19, p. 67] has shown that an end which is homeomorphic to $S^4 \times R$ is also PL homeomorphic to $S^4 \times R$.

In the proof of Theorem 1, a homeomorphism $f : T^n \to \tau^n$ is constructed. If $\widetilde{f} : \widetilde{T}^n \to \widetilde{\tau}^n$ is any covering of f, then clearly f is stable if and only if $\widetilde{f}$ is stable. Using only the fact that f is a simple homotopy equivalence, Wall's non-simply connected surgery techniques [15] provide an "obstruction" in $H^3(T^n; Z_2)$ to finding a PL homeomorphism between T^n and τ^n. It is Siebenmann's idea to investigate the behavior of this obstruction under lifting $f: T^n \to \tau^n$ to a 2^n-fold cover; he suggested that the obstruction would become zero. Wall [16] and Hsiang and Shaneson [17] have proved this is the case; that is, if $\widetilde{\tau}^n$ is the 2^n-fold cover of a homotopy torus τ^n, $n \geq 5$, then $\widetilde{\tau}^n$ is PL homeomorphic to T^n ($=\widetilde{T}^n$). Therefore, following the proof of Theorem 1, $\widetilde{f} : \widetilde{T}^n \to \widetilde{\tau}^n$ is stable, so f is stable, and thus SHC_n holds for $n \neq 4$. Hence the annulus conjecture AC_n holds for $n \neq 4$.)

(Added April 15, 1969 Siebenmann has found a beautiful and surprising counter-example which leads to non-existence and non-uniqueness of triangulation of manifolds. In particular HT_n is false for $n \geq 5$, so it is necessary to take the 2^n-fold covers, as above. One may then use the fact that $\widetilde{f} : \widetilde{T}^n \to \widetilde{\tau}^n$ is homotopic to a PL homeomorphism to show that $f : T^n \to \tau^n$ was actually isotopic to a PL homeomorphism. Thus, although there are homeomorphisms between T^n and another PL manifold which are not even homotopic to PL homeomorphisms, they cannot be constructed as in Theorem 1. Details will appear in a forthcoming paper by Siebenmann and the author. See also R. C. Kirby and L. C. Siebenmann, On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc., to appear in Bull. Amer. Math. Soc.)

Let $\mathcal{H}(M^n)$ denote the space (with the compact-open topology) of orientation preserving homeomorphisms of an oriented stable n-manifold M, and let $\mathcal{SH}(M^n)$ denote the subspace of stable homeomorphisms.

Theorem 2. $\mathcal{SH}(R^n)$ *is both open and closed in* $\mathcal{H}(R^n)$.

Since a stable homeomorphism of R^n is isotopic to the identity, we have the:

Corollary. $\mathcal{SH}(R^n)$ *is exactly the component of the identity in* $\mathcal{H}(R^n)$.

Corollary. *A homeomorphism of* R^n *is stable if and only if it is isotopic to the identity.*

Theorem 3. *If* M^n *is a stable manifold, then* $\mathcal{SH}(M^n)$ *contains the identity component of* $\mathcal{H}(M^n)$.

In general this does not imply that the identity component is arcwise connected (as it does for $M^n = R^n$ or S^n), but arcwise connectivity does follow from the remarkable result of Chernavskii [5] that $\mathcal{H}(M^n)$ is locally contractible if M^n is compact and closed or $M^n = R^n$. From the techniques in this paper, we have an easy proof of the last case.

Theorem 4. $\mathcal{H}(R^n)$ *is locally contractible.*

We now give some definitions, then a few elementary propositions, the crucial lemma, and finally the proofs of Theorems 1–4 in succession.

The following definitions may be found in Brown and Gluck [3], a good source for material on stable homeomorphisms. A homeomorphism h between open subsets U and V of R^n is called stable if each point $x \in U$ has a neighborhood $W_x \subset U$ such that $h|_{W_x}$ extends to a stable homeomorphism of R^n. Then we may define stable manifolds and stable homeomorphisms between stable manifolds in the same way as is usually done in the PL and differential categories. Whenever it makes sense, we assume that a stable structure on a manifold is inherited from the PL or differentiable structure. Homeomorphisms will always be assumed to preserve orientation.

Proposition 1. *A homeomorphism of* R^n *is stable if it agrees with a stable homeomorphism on some open set.*

Proposition 2. *Let* $h \in \mathcal{H}(R^n)$ *and suppose there exists a constant* $M > 0$ *so that* $|h(x) - x| < M$ *for all* $x \in R^n$. *Then* h *is stable.*

Proof. This is Lemma 5 of [6].

Letting rB^n be the n-ball of radius r, we may consider $rD^n = i(rB^n)$ as a subset of T^n, via some fixed differentiable imbedding $i : rB^n \to T^n$.

Proposition 3. *There exists an immersion* $\alpha : T^n - D^n \to R^n$.

Proof. Since $T^n - D^n$ is open and has a trivial tangent bundle, this follows from [8, Theorem 4.7].

Proposition 4. *If* A *is an* $n \times n$ *matrix of integers with determinant one, then there exists a diffeomorphism* $f : T^n \to T^n$ *such that* $f_* = A$ *where* $f_* : \pi_1(T^n, t_0) \to \pi_1(T^n, t_0)$.

PROOF. A can be written as a product of elementary matrices with integer entries, and these can be represented by diffeomorphisms.

Proposition 5. *A homeomorphism of a connected stable manifold is stable if its restriction to some open set is stable.*

Proposition 6. *Let $f : S^{n-1} \times [-1,1] \to R^n$ be an imbedding which contains S^{n-1} in its interior. Then $f|_{S^{n-1}\times 0}$ extends canonically to an imbedding of B^n in R^n.*

PROOF. This is shown in [9]. However, there is a simple proof; one just re-proves the necessary part of [2] in a canonical way. This sort of canonical construction is done carefully in the proof of Theorem 1 of [11].

The key to the paper is the following observation.

Lemma. *Every homeomorphism of T^n is stable.*

PROOF. Let $e : R^n \to T^n$ be the usual covering map defined by

$$e(x_1, \ldots, x_n) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}),$$

and let $t_0 = (1, \ldots, 1) = e(0, \ldots, 0)$. e fixes a differential and hence stable structure on T^n.

Let h be a homeomorphism of T^n, and assume at first that $h(t_0) = t_0$ and $h_* : \pi_1(T^n, t_0) \to \pi_1(T^n, t_0)$ is the identity matrix. h lifts to a homeomorphism $\widehat{h} : R^n \to R^n$ so that the following diagram commutes.

$$\begin{array}{ccc} R^n & \xrightarrow{\widehat{h}} & R^n \\ \Big\downarrow e & & \Big\downarrow e \\ T^n & \xrightarrow{h} & T^n. \end{array}$$

Since $I^n = [0,1] \times \cdots \times [0,1]$ is compact,

$$M = \sup\{|\widehat{h}(x) - x| \,|x \in I^n|\}$$

exists. The condition $h_* =$ identity implies that $\widehat{h}$ fixes all lattice points with integer coordinates $Z^n \subset R^n$. Thus $\widehat{h}$ moves any other unit n-cube with vertices in this lattice in the "same" way it moves I^n; in particular $|\widehat{h}(x) - x| \leq M$ for all $x \in R^n$. By Proposition 2, $\widehat{h}$ is stable. e provides the coordinate patches on T^n, so h is stable because $e^{-1}he|_{e^{-1}(U)}$(patch) extends to the stable homeomorphism $(\widehat{h})$ for all patches.

Given any homeomorphism h of T^n, we may compose with a diffeomorphism g so that $gh(t_0) = t_0$. If $A = (gh)_*^{-1}$, then Proposition 4 provides a diffeomorphism f with $f_* = A = (gh)_*^{-1}$, so $(fgh)_*$ is the product of stable homeomorphisms and therefore stable.

Proof of Theorem 1. Let g be a homeomorphism of R^n. $g\alpha$ (see Proposition 3) induces a new differentiable structure on $T^n - D^n$, and we call this differential manifold $\widetilde{T^n - D^n}$. We have the following commutative diagram,

$$\begin{tikzcd} T^n - D^n \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha"] & \widetilde{T^n - D^n} \arrow[d, "g\alpha"] \\ R^n \arrow[r, "g"] & R^n. \end{tikzcd}$$

α and $g\alpha$ are differentiable and therefore stable, so g is stable if and only if the identity is stable (use Proposition 1).

Since $\widetilde{T^n - D^n}$ has one end, which is homeomorphic to $S^{n-1} \times R$, and $n \geq 6$, there is no difficulty in adding a differentiable boundary [1]. Since the boundary is clearly a homotopy $(n-1)$-sphere, we can take a C^1-triangulation and use the PL h-cobordism theorem to see that the boundary is a PL $(n-1)$-sphere. To be precise, there is a proper PL imbedding $\beta : S^{n-1} \times [0,1) \to \widetilde{T^n - D^n}$, and we add the boundary by taking the union $\widetilde{T^n - D^n} \cup_\beta S^{n-1} \times [0,1]$ over the map β.

Finally we add B^n to this union, via the identity map on the boundaries, to obtain a closed PL manifold τ^n.

We can assume that $\partial 2D^n$ lies in $\beta(S^{n-1} \times [0,1))$. Thus $\partial 2D^n$ lies in an n-ball in τ^n and, since it is locally flat, bounds an n-ball by the topological Schoenflies theorem [2]. Now, we may extend $\mathrm{id}|_{T^n - 2D^n}$, by coning on $\partial 2D^n$, to a homeomorphism $f : T^n \to \tau^n$.

Using HT_n we have a PL (hence stable) homeomorphism $h : T^n \to \tau^n$. By the lemma, $h^{-1}f : T^n \to T^n$ is stable, so $f = h(h^{-1}f)$ is stable, $f|_{T^n-2D^n} =$ identity is stable, and finally g is stable.

Note that it is only necessary that HT_n gives a stable homeomorphism h.

Proof of Theorem 2. We shall show that a neighborhood of the identity consists of stable homeomorphisms. But then by translation in the topological group $\mathcal{H}(R^n)$, any stable homeomorphism has a neighborhood of stable homeomorphisms, so $\mathcal{SH}(R^n)$ is open. Now it is well known that an open subgroup is also closed (for a coset of $\mathcal{SH}(R^n)$ in $\mathcal{H}(R^n)$ is open, so the union of all cosets of $\mathcal{SH}(R^n)$ is open and is also the complement of $\mathcal{SH}(R^n)$, which is therefore closed.

If C is a compact subset of R^n and $\varepsilon > 0$, then it is easily verified that $N(C, \varepsilon) = \{|h \in \mathcal{H}(R^n)|\, |h(x) - x| < \varepsilon$ for all $x \in C\}$ is an open set in the CO-topology. Let C be a compact set containing $\alpha(T^n - D^n)$. If $\varepsilon > 0$ is chosen small enough, then

$$\overline{h\alpha(T^n - 5D^n)} \subset \alpha(T^n - 4D^n) \subset h\alpha(T^n - 3D^n)$$
$$\subset \overline{h\alpha(T^n - 2D^n)} \subset \alpha(T^n - D^n)$$

for any $h \in N(C, \varepsilon)$. There exists an imbedding $\widehat{h}$, which "lifts" h so that the following diagram commutes.

$$\begin{array}{ccc} T^n - 2D^n & \xrightarrow{\widehat{h}} & T^n - D^n \\ \downarrow \alpha & & \downarrow \alpha \\ R^n & \xrightarrow{h} & R^n. \end{array}$$

To define $\widehat{h}$, first we cover C with finitely many open sets $\{U_i\}$, $i = 1, \ldots, k$, so that α is an imbedding on each component of $\alpha^{-1}(U_i)$, $i = 1, \ldots, k$. Let $\{V_i\}$, $i = 1, \ldots, k$, be a refinement of $\{U_i\}$. If ε was chosen small enough, then $h(V_i) \subset U_i$. Let $W_i = U_i \cap \alpha(T^n - D^n)$ and $X_i = V_i \cap \alpha(T^n - 2D^n)$. Since $h\alpha(T^n - 2D^n) \subset \alpha(T^n - D^n)$, we have $h(X_i) \subset W_i$, $i = 1, \ldots, k$. Let $W_{i,j}$, $j = 1, \ldots, w_i$ be the components of $\alpha^{-1}(W_i)$, let $X_{i,j} = W_{i,j} \cap (T^n - 2D^n)$, $\alpha_{i,j} = \alpha|_{W_{i,j}}$ for all i, j. Now we can define $\widehat{h}$ by

$$\widehat{h}|X_{i,j} = (\alpha_{i,j})^{-1} h\alpha|_{X_{i,j}}$$

for all i, j. Clearly $\widehat{h}$ is an imbedding.

$\alpha(T^n - 4D^n) \subset h\alpha(T^n - 3D^n)$ implies that $\alpha(4D^n - D^n) \supset h\alpha(\partial 3D^n)$, so $\widehat{h}(\partial 3D^n) \subset 4D^n$, and hence $\widehat{h}(\partial 3D^n)$ bounds an n-ball in $4D^n$. By coning, we extend $\widehat{h}|_{(T^n - 3D^n)}$ to a homeomorphism $H : T^n \to T^n$. H is stable by the lemma, so $\widehat{h}$ is stable and h is stable. Hence $N(C, \varepsilon)$ is a neighborhood of the identity consisting of stable homeomorphisms, finishing the proof of Theorem 2.

PROOF OF THEOREM 3. As in the proof of Theorem 2, it suffices to show that a neighborhood of the identity consists of stable homeomorphisms; then $\mathcal{SH}(M^n)$ is both open and closed and therefore contains the identity component.

Let $j : R^n \to M$ be a coordinate patch. Let $\varepsilon > 0$ and $r > 0$ be chosen so that $N(rB^n, \varepsilon) \subset \mathcal{H}(R^n)$ consists of stable homeomorphisms. Then there exists a $\delta > 0$ such that if $h \in N(j(rB^n), \delta) \subset \mathcal{H}(M^n)$, then $hj(2rB^n) \subset j(R^n)$, $j^{-1}hj|_{2rB^n} \in N(rB^n, \varepsilon)$. We may isotope $j^{-1}hj|_{2rB^n}$ to a homeomorphism H of R^n with $H = j^{-1}hj$ on rB^n and therefore $H \in N(rB^n, \varepsilon) \subset \mathcal{H}(R^n)$. Thus H is stable and so $j^{-1}hj|_{2rB^n}$ is stable.

By Proposition 5, h is stable, and hence $N(j(rB^n), \delta)$ is our required neighborhood of the identity.

Proof of Theorem 4. We will observe that Theorem 2 can be proved in a "canonical" fashion; that is, if h varies continuously in $\mathcal{H}(R^n)$, then H varies continuously in $\mathcal{H}(T^n)$. First note that $\mathcal{H}(R^n)$ may be contracted onto $\mathcal{H}_0(R^n)$, the homeomorphisms fixing the origin. The immersion $\alpha : T^n - D^n \to R^n$ can be chosen so that $\alpha e = \mathrm{id}$ on $(1/4)B^n$. Pick a compact set C and $\varepsilon > 0$ as in the proof of Theorem 2 and let $h \in N(C, \varepsilon)$. h lifts canonically to $\widehat{h} : T^n - 2D^n \to T^n - D^n$. Since $\widehat{h}(\mathrm{int} 5\, D^n - 2D^n)$ contains $\partial 4D^n$, it follows from Proposition 6 that $\widehat{h}(\partial 3D^n)$ bounds a canonical n-ball in $4D^n$. Then $\widehat{h}|_{(T^n - 3D^n)}$ extends by coning to $H : T^n \to T^n$.

Clearly $H(t_0) = t_0$ and $H_* =$ identity so H lifts uniquely to a homeomorphism $g : R^n \to R^n$, with $|g(x) - x| <$ constant for all $x \in$ (see lemma). We have the commutative diagram

$$\begin{tikzcd}
R^n \arrow[r, "g"] \arrow[d, "e"'] & R^n \arrow[d, "e"] \\
T^n \arrow[r, "H"] & T^n \\
T^n - 3D^n \arrow[u, phantom, "\cup" description] \arrow[r, "\widehat{h}"] \arrow[d, "\alpha"'] & T^n - 2D^n \arrow[u, phantom, "\cup" description] \arrow[d, "\alpha"] \\
R^n \arrow[r, "h"] & R^n.
\end{tikzcd}$$

Since $e((1/4)B^n) \cap 4D^n$ is empty and $\alpha e = \mathrm{id}$ on $(1/4)B^n$, it follows that $g = h$ on $(1/4)B^n$. The construction of g being canonical means that the map $\psi : \mathcal{H}_0(R^n) \to \mathcal{H}_0(R^n)$, defined by $\psi(h) = g$, is continuous.

Let $P_t : R^n \to R^n$, $t \in [0, 1]$, be the isotopy with $P_0 = h$ and $P_1 = g$ defined by

$$P_t(x) = g\left\{\frac{1}{1-t} \cdot \left[g^{-1}h((1-t)x)\right]\right\}$$

if $t < 1$ and $P_1 = g$. Let $Q_t : R^n \to R^n$ be the isotopy with $Q_0 = g$ and $Q_1 = \mathrm{id}$ defined by

$$Q_t(x) = (1-t) \cdot g\left(\frac{1}{1-t} \cdot x\right)$$

if $t < 1$ and $Q_1 = \text{id}$. Now let $h_t : R^n \to R^n$, $t \in [0,1]$ be defined by

$$h_t(x) = \begin{cases} P_{2t}(x) & \text{if } 0 \le t \le \dfrac{1}{2} \\ Q_{2t-1}(x) & \text{if } \dfrac{1}{2} \le t \le 1. \end{cases}$$

It can be verified that h_t is an isotopy of h to the identity which varies continuously with respect to h. Then $H_t : N(C, \varepsilon) \to \mathcal{H}_0(R^n)$, $t \in [0,1]$ defined by $H_t(h) = h_t$ is a contraction of $N(C, \varepsilon)$ to the identity where $H_t(\text{id}) = \text{id}$ for all $t \in [0,1]$.

The proof can be easily modified to show that if a neighborhood V of the identity in $\mathcal{H}_0(R^n)$ is given, then C and ε can be chosen so that $N(C, \varepsilon)$ contracts to the identity and the contraction takes place in V. To see this, pick $r > 0$ and δ so that $N(rB^n, \delta) \subset V$. Then we may re-define α and e so that $\alpha e = $ identity on rB^n. If $h \in N(rB^n, \delta)$, then $P_t \in N(rB^n, \delta)$, and if ε is chosen small enough (with respect to δ), then $h \in N(rB^n, \varepsilon)$ implies that $Q_t \in N(rB^n, \delta)$. Therefore $N(rB^n, \varepsilon)$ contracts in V.

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