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THE G. F. LAPTEV METHOD: FUNDAMENTAL OBJECTS OF MAPPINGS

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CONTENTS

Introduction

Convention on indices. Latin indices assume values from 1 up to m_1, Greek indices do that from 1 up to m_2; with respect to a repeating index the summation is performed, and under the summation, the differentiation is excluded, for example, if in the writing $X_i x^j$, where the indices i and j are fixed, we differentiate the function x^j with respect to the vector field X_i, then the writing $X_i x^i$ means the linear combination of the vector fields X_i with coefficients x^i. As a rule, the summation index is repeated in the direction $\nearrow$, which is convenient for passing to the matrix writing of formulas.

Let us consider a smooth mapping

$$f : M_1 \to M_2, \qquad \dim M_1 = m_1, \qquad \dim M_2 = m_2. \tag{1}$$

On neighborhoods $U_1 \subset M_1$ and $f(U_1) = U_2 \subset M_2$, let the coordinate functions u^i and v^α be given. In local writing, the mapping f is determined by assignment of the functions f^α on the neighborhood U_1 that are f-connected with the coordinate functions v^α:

$$v^\alpha \circ f = f^\alpha. \tag{2}$$

The functions f^α and their partial derivatives

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial u^i}, \quad f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial u^i \partial u^j}, \quad \dots \tag{3}$$

define jets of the mapping f of all orders. The jets define the differentials of the mapping f, morphisms of k-fold tangent bundles (floors):

$$T^k f : T^k M_1 \to T^k M_2, \quad k = 1, 2, \dots \tag{4}$$

(jet structures and floors, manyfold vector bundles, are studied in [2, 3, 10, 12, 13, 16]).

The partial derivatives (3) are not tensor objects. If we assume that the manifolds M_1 and M_2 are fully parallelizable, and nonholonomic bases (after Schouten [14])

$$(P_i, \omega^j) \quad \text{and} \quad (R_\alpha, \theta^\beta), \tag{5}$$

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 124, Part 2, Geometry, 2010.

are given on them, where the vector fields P_i (resp. R_α) compose a frame field on M_1 (resp. on M_2) and form ω^j (resp. θ^β) is the dual coframe field, then the Laptev method [6] allows us to define the invariant objects

$$\mathcal{F}_i^\alpha, \quad \mathcal{F}_{ij}^\alpha, \quad \mathcal{F}_{ijk}^\alpha, \quad \dots, \tag{6}$$

independent of the local coordinates. On the neighborhoods U_1 and U_2, bases (5) are defined[1] by the matrices A, A^{-1}, B, and B^{-1}:

$$\begin{aligned} A &= (A^i_j), \quad A^{-1} = (\bar{A}^j_i), \qquad B = (B^\alpha_\beta), \quad B^{-1} = (\bar{B}^\beta_\alpha), \\ P_i &= \partial_j \bar{A}^j_i, \quad \omega^i = A^i_j du^j, \qquad R_\alpha = \tilde{\partial}_\beta \bar{B}^\beta_\alpha, \quad \theta^\alpha = B^\alpha_\beta dv^\beta, \end{aligned} \tag{7}$$

and objects (6) are represented in the form

$$\mathcal{F}_i^\alpha = (B^\alpha_\beta \circ f) f^\beta_j \bar{A}^j_i, \tag{8}$$

$$\mathcal{F}_{ij}^\alpha = (B^\alpha_\beta \circ f) F^\beta_{pq} \bar{A}^p_i \bar{A}^q_j, \tag{9}$$

$$\mathcal{F}_{ijk}^\alpha = (B^\alpha_\beta \circ f) F^\beta_{pqr} \bar{A}^p_i \bar{A}^q_j \bar{A}^r_k, \tag{10}$$

$$\dots \qquad \dots$$

where f^β_j, F^β_{pq}, F^β_{pqr}, ... are mixed tensors symmetric in subscripts. In the space T_uM_1 $\forall u \in M_1$, we respectively define the linear, quadratic, and cubic (and higher-order) vector-valued forms with values in T_vM_2, $v = f(u) \in M_2$. Formula (8) describes the transformation of the Jacobi matrix $(f^\alpha_i) \rightsquigarrow (\mathcal{F}^\alpha_i)$, more precisely, the transformation of the matrix of the tangent mapping $T_uf : T_uM_1 \to T_vM_2$ under basis change in the spaces T_uM_1 and T_vM_2. The matter is not so simple for passage (9) from the partial derivatives f^α_{ij} to the invariant object $\mathcal{F}^\alpha_{ij}$:

$$f^\alpha_{ij} \rightsquigarrow F^\alpha_{ij} \rightsquigarrow \mathcal{F}^\alpha_{ij}\,. \tag{11}$$

Our first goal is to show in which way the object $\mathcal{F}^\alpha_{ij}$ appears according to Laptev, in particular, which form the object F^α_{ij} has in the local writing. In what follows, we show how this situation is understood from the viewpoint of connections in general bundles and in which way we can construct higher-order objects (6). In the further development, the crucial role is played by the Lie derivatives.

1. First Prolongations

First, from (2), we deduce two relations[2]:

$$dv^\alpha \circ Tf = f^\alpha_i du^i \;\Leftrightarrow\; \theta^\alpha \circ Tf = \mathcal{F}^\alpha_i \omega^i. \tag{12}$$

The composition $\circ Tf$ is conventionally omitted in such relations. This is not fully correct since we cannot equate functions and forms defined on distinct manifolds, but, according to the tradition, we also do that for simplicity.

By using the Jacobi matrix (f^α_i), the relation $dv^\alpha = f^\alpha_i du^i$ defines the tangent mapping Tf in the cobases du^i and dv^α. The relation $\theta^\alpha = \mathcal{F}^\alpha_i \omega^i$, being understood independently and not related to the previous relation, defines the mapping Tf globally in the cobases ω^i and θ^α.

The relation $\theta^\alpha = \mathcal{F}^\alpha_i \omega^i$ is called the *differential equation of the mapping f*. This equation and the object $\mathcal{F}^\alpha_i$ define the *first prolongation* of the mapping f.

Let us perform the exterior differentiation[3]:

$$\theta^\alpha \circ Tf = \mathcal{F}^\alpha_i \omega^i \rightsquigarrow \mathrm{d}(\theta^\alpha \circ Tf) = d\mathcal{F}^\alpha_i \wedge \omega^i + \mathcal{F}^\alpha_i d\omega^i. \tag{13}$$

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[1]With respect to the bases $\left(\partial_i = \dfrac{\partial}{\partial u^i}, du^j\right)$ on U_1 and $\left(\tilde{\partial}_\alpha = \dfrac{\partial}{\partial v^\alpha}, dv^\beta\right)$ on U_2.

[2]The left relation has a local meaning, and the meaning of the right relation is global. On the neighborhoods U_1 and U_2, they are equivalent.

[3]For a 1-form Φ, we distinguish its differential $d\Phi$ as a scalar-valued function on TM and its exterior differential $\mathrm{d}\Phi$.

The exterior differentials $\mathrm{d}\omega^i$ and $\mathrm{d}\theta^\alpha$ can be represented in the form

$$\mathrm{d}\omega^i = \omega^i_j \wedge \omega^j, \qquad \mathrm{d}\theta^\alpha = \theta^\alpha_\beta \wedge \theta^\beta, \tag{14}$$

where ω^i_j and θ^α_β are fully definite 1-forms on M_1 and M_2 with values in the Lie algebras of the linear groups $GL(M_1, \mathbb{R})$ and $GL(M_2, \mathbb{R})$; see (19) below. The result of the exterior differentiation (13) is written in the form[4]

$$\big(d\mathcal{F}^\alpha_i + \mathcal{F}^\alpha_k \omega^k_i - (\theta^\alpha_\beta \circ Tf)\mathcal{F}^\beta_i\big) \wedge \omega^i = 0, \tag{15}$$

whence, applying the Cartan lemma, we reveal the existence of the invariant object $\mathcal{F}^\alpha_{ij}$ symmetric in subscripts such that the following relation holds:

$$d\mathcal{F}^\alpha_i + \mathcal{F}^\alpha_k \omega^k_i - (\theta^\alpha_\beta \circ Tf)\mathcal{F}^\beta_i = \mathcal{F}^\alpha_{ij}\omega^j. \tag{16}$$

The nonholonomic bases (5) have non-holonomy objects c^k_{ij} and $\tilde{c}^\gamma_{\alpha\beta}$,

$$[P_i, P_j] = P_k c^k_{ij}, \qquad \mathrm{d}\omega^i = -\frac{1}{2}\, c^i_{jk}\, \omega^j \wedge \omega^k, \tag{17}$$

$$[R_\alpha, R_\beta] = R_\gamma \tilde{c}^\gamma_{\alpha\beta}, \qquad \mathrm{d}\theta^\alpha = -\frac{1}{2}\, \tilde{c}^\alpha_{\beta\gamma}\, \theta^\beta \wedge \theta^\gamma. \tag{18}$$

Whence, taking account of the skew-symmetry of the quantities cjk and $\tilde{c}\,^\alpha_{\beta\gamma}$ with respect to subscripts, relation (14) implies

$$\omega^i_j = -c\,^i_{jk}\omega^k, \qquad \theta^\alpha_\beta = -\tilde{c}\,^\alpha_{\beta\gamma}\theta^\gamma. \tag{19}$$

Let us rewrite (16) using the relations

$$d\mathcal{F}^\alpha_i = P_j \mathcal{F}^\alpha_i \omega^j = (P_{(j}\mathcal{F}^\alpha_{i)} + P_{[j}\mathcal{F}^\alpha_{i]})\omega^j,$$

$$\omega^k_i = -c\,^k_{ij}\omega^k, \qquad \theta^\alpha_\beta \circ Tf = -(\tilde{c}\,^\alpha_{\beta\gamma} \circ f)\mathcal{F}^\gamma_j \omega^j.$$

We obtain two relations; one relation for symmetric coefficients and the other for skew-symmetric ones:

$$\mathcal{F}^\alpha_{ij} = P_{(i}\mathcal{F}^\alpha_{j)}, \tag{20}$$

$$P_{[i}\mathcal{F}^\alpha_{j]} + \mathcal{F}^\alpha_k c\,^k_{ij} - (\tilde{c}\,^\alpha_{\beta\gamma} \circ f)\mathcal{F}^\beta_i \mathcal{F}^\gamma_j = 0. \tag{21}$$

Relation (20) defines the object $\mathcal{F}^\alpha_{ij}$ of our interest, and (21) must be omitted. As is seen below, it holds identically.

Let us calculate the derivative $P_i\mathcal{F}^\alpha_j$ in local coordinates, where $P_i = \partial_j \bar{A}^j_i$ and $\mathcal{F}^\alpha_i = (B^\alpha_\beta \circ f) f^\beta_j \bar{A}^j_i$; see (6) and (8). First, from $\theta^\alpha \circ Tf = \mathcal{F}^\alpha_i \omega^i$ (see (12)), it follows that $\theta^\alpha(TfP_i) = \mathcal{F}^\alpha_i$ since $\omega^j(P_i) = \delta^j_i$ is the identity matrix. This means that in the frame R_α, we have $TfP_i = R_\alpha \mathcal{F}^\alpha_i$, i.e., under the mapping Tf, the vectors P_i are translated from the point $u \in M_1$ into the point $v = f(u) \in M_2$ into the vectors $R_\alpha \mathcal{F}^\alpha_i$. Therefore,

$$P_i(B^\alpha_\beta \circ f) = \big((R_\gamma B^\alpha_\beta) \circ f\big)\mathcal{F}^\gamma_i = \big((\tilde{\partial}_\gamma B^\alpha_\beta) \circ f\big) f^\gamma_k \bar{A}^k_i.$$

Furthermore, the differentiation of two more factors (8) yields

$$P_i f^\beta_j = f^\beta_{jk} \bar{A}^k_i,$$

$$P_i \bar{A}^k_j = \partial_p \bar{A}^k_j \bar{A}^p_i = -\bar{A}^k_l \partial_p A^l_q \bar{A}^p_i \bar{A}^q_j.$$

By the Leibnitz rule, from (8), we obtain

$$P_i\mathcal{F}^\alpha_j = (B^\alpha_\beta \circ f)\Big\{f^\beta_{pq} - f^\beta_k \bar{A}^k_l \partial_p A^l_q + \big((\bar{B}^\beta_\sigma \tilde{\partial}_\lambda B^\sigma_\mu) \circ f\big) f^\lambda_p f^\mu_q\Big\}\bar{A}^p_i \bar{A}^q_j. \tag{22}$$

[4]To understand (15) and (16) correctly on M_1, it is essential to keep in mind the composition $\circ Tf$.

It remains to symmetrize expression (22) in order to obtain (23) and to alternate it in order to verify identity (21).

The symmetrization of expression (22) yields

$$\mathcal{F}^\alpha_{ij} = (B^\alpha_\beta \circ f)\Big\{f^\beta_{pq} - f^\beta_k \Gamma^k_{pq} + (\Lambda^\beta_{\lambda\mu} \circ f) f^\lambda_p f^\mu_q\Big\} \bar{A}^p_i \bar{A}^q_j\,, \tag{23}$$

where Γ^k_{pq} and $\Lambda^\beta_{\lambda\mu}$ are coefficients of affine torsion-free connections on the neighborhoods U_1 and U_2:

$$\Gamma^k_{pq} = \bar{A}^k_l \partial_{(p} A^l_{q)}, \qquad \Lambda^\beta_{\lambda\mu} = \bar{B}^\beta_\sigma \tilde{\partial}_{(\lambda} B^\sigma_{\mu)}\,. \tag{24}$$

In short, expression (23) reduces to the formula

$$\mathcal{F}^\alpha_{ij} = (B^\alpha_\beta \circ f) F^\beta_{pq} \bar{A}^p_i \bar{A}^q_j$$

(see (9)), where

$$\boxed{F^\beta_{pq} = f^\beta_{pq} - f^\beta_k \Gamma^k_{pq} + (\Lambda^\beta_{\lambda\mu} \circ f) f^\lambda_p f^\mu_q\,.} \tag{25}$$

If, as was said above, the matrix $(\mathcal{F}^\alpha_i)$ or, in local form, the Jacobi matrix (f^β_j) (see (8)) defines the *first differential prolongation* of the mapping f, then the objects $(\mathcal{F}^\alpha_i, \mathcal{F}^\alpha_{ij})$ or, in local form, the objects $(f^\beta_j, F^\beta_{pq})$ (see (23) and (25)) with coefficients (24) define the *second differential prolongation* of the mapping f according to Laptev [6].

Alternating expression (22) and taking into account the formulas for the nonholonomy objects[5] of bases (5) on U_1 and U_2:

$$c^{\,k}_{pq} = 2\partial_{\,[s} A_{l]} \bar{A}^l_p \bar{A}^s_q, \qquad \tilde{c}^{\,\beta}_{\lambda\mu} = 2\tilde{\partial}_{\,[\sigma} B_{\nu]} \bar{B}^\nu_\lambda \bar{B}^\sigma_\mu, \tag{26}$$

we verify that relation (21) indeed turns to an identity.

Hence the coefficients $\mathcal{F}^\alpha_{ij}$ in (16) whose existence is revealed in applying the Cartan lemma to relation (13) are uniquely defined by formula (23) or formula (9) taking account of (25).

Conclusion. The second differential prolongation of the mapping f according to the Laptev method reduces to the following actions:

the differential equation $\theta^\alpha \circ Tf = \mathcal{F}^\alpha_i \omega^i$ (see (12)) is subjected to the exterior differentiation (see (13));
the Cartan lemma is applied to the result (15) (see (16));
on the manifolds M_1 and M_2, the objects $\mathcal{F}^\alpha_i$ and $\mathcal{F}^\alpha_{ij}$, which are independent of local coordinates, are defined;
on the neighborhoods U_1 and U_2, the object $\mathcal{F}^\alpha_{ij}$ is represented in form (23) or in form (9) with coefficients (25).

Note that objects (6) are defined with accuracy up to fixing bases (5) on M_1 and M_2. When bases (5) are related to some structures[6] on $M_1$and M_2 in a certain way, we can speak of *absolute invariants* of the mapping f.

[5]The formulas for the nonholonomy objects (26) are deduced by a direct calculation of the brackets $[P_i, P_j]$ and $[R_\alpha, R_\beta]$; see also (7), (17), and (18).

[6]Numerous ways in which the bases can be related to various structures are proposed by the Cartan movable frame method [4].

2. Morphisms of Bundles with Connections

2.1. Bases. Let us consider a bundle with n-dimensional base and r-dimensional fibers:

$$\pi : \mathcal{M} \to M, \qquad \dim \mathcal{M} = n + r, \quad \dim M = n. \tag{27}$$

A *connection* in such a bundle is defined by assigning the structure $\Delta_h \oplus \Delta_v$ on the manifold $\mathcal{M}$, where Δ_v is the r-dimensional *vertical* distribution tangent to the fibers, and Δ_h is the n-dimensional *horizontal* distribution complement to Δ_v. In this case, $T\pi(\Delta_v) = 0$ and $T\pi(\Delta_h) = TM$.

On the base manifold M, let a nonholonomic basis $(\bar{R}_i, \bar{\theta}^j)$ with nonholonomy object $\bar{c}^{\,i}_{jk}$ be given. On the manifold $\mathcal{M}$, let us define the basis[7]

$$(R_i, R_\alpha; \theta^j, \theta^\beta), \tag{28}$$

but specialize it in such a way that the vector fields R_i and R_α respectively belong to the distributions Δ_h and Δ_v and the fields R_i are π-projectable on M, more precisely they are lifts of the fields $\bar{R}_i$ to the distribution Δ_h, i.e., $T\pi R_i = \bar{R}_i$, $\theta^j = \bar{\theta}^j \circ T\pi$. Such a specialization simplifies the lift of any vector fields from the manifold M to the distribution Δ_h:

$$\bar{X} = \bar{R}_i \bar{x}^i \rightsquigarrow X = R_i x^i, \quad \text{where} \quad x^i = \bar{x}^i \circ \pi.$$

In the structure $\Delta_h \oplus \Delta_v$, the nonholonomy object of basis (28) is divided into subobjects

$$c^{\,i}_{jk},\ c^{\,i}_{\beta k},\ c^{\,i}_{\beta\gamma},\ c^{\alpha}_{jk},\ c^{\alpha}_{\beta k},\ c^{\alpha}_{\beta\gamma}. \tag{29}$$

Because of the basis specialization, two subobjects vanish: $c^{\,i}_{\beta\gamma} = 0$ since the distribution Δ_v is integrable, and $c^{\,i}_{\beta k} = 0$ since the fields R_k are π-projectable. Moreover, $c^{\,i}_{jk} = \bar{c}^{i}_{jk} \circ \pi$. The subobject $c^{\alpha}_{\beta\gamma}$ tells us about the basis nonholonomy[8] in the distribution Δ_v. The subobjects $c^{\,\alpha}_{jk}$ and $c^{\,\alpha}_{\beta k}$ are most important. They are respectively called the *curvature object* and the *connection object*.[9]

On the local neighborhood $U \subset \mathcal{M}$, the *base* and *fiber* coordinate functions (u^i, u^α) are concordant with the bundle in such a way that $u^i = \bar{u}^i \circ \pi$, where $\bar{u}^i$ are coordinate functions on the neighborhood $\pi(U) = \bar{U} \subset M$. On U, we define the *natural basis*

$$\left(\partial_i = \frac{\partial}{\partial u_i},\ \partial_\alpha = \frac{\partial}{\partial u_\alpha};\ du^j,\ du^\beta\right).$$

In the natural basis, we define the *adapted basis* of the structure $\Delta_h \oplus \Delta_v$:

$$\begin{aligned} (X_i\ X_\alpha) &= \left(\frac{\partial}{\partial u^j}\ \frac{\partial}{\partial u^\beta}\right) \cdot \begin{pmatrix} \delta^j_i & 0 \\ \Gamma^\beta_i & \delta^\beta_\alpha \end{pmatrix}, \\ \begin{pmatrix} \omega^i \\ \omega^\alpha \end{pmatrix} &= \begin{pmatrix} \delta^i_j & 0 \\ -\Gamma^\alpha_j & \delta^\alpha_\beta \end{pmatrix} \cdot \begin{pmatrix} du^j \\ du^\beta \end{pmatrix}. \end{aligned} \tag{30}$$

We define the vector fields X_i composing a basis of the distribution Δ_h and the forms ω^j that are annihilated by the distribution Δ_h:

$$X_i = \partial_i + \Gamma^\alpha_i \partial_\alpha, \qquad \omega^\alpha = du^\alpha - \Gamma^\alpha_i du^i. \tag{31}$$

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[7]The Latin indices assume the values in the range from 1 up to n, and the Greek ones from $n+1$ up to $n+r$.

[8]In the case of the principal bundle where the fibers are orbits of the structural group, the quantities $c^\alpha_{\beta\gamma}$ are the structural constants of this group.

[9]The relation $c^\alpha_{jk} = 0$ means that the distribution Δ_h is integrable. The matrix $c^\alpha_{\beta k}$ (see Greek indices) defines an infinitesimal turn of the distribution Δ_v under a displacement in direction R_k.

In the general case, the quantities Γ^α_i depend on the base coordinates, as well as on fiber ones.[10] The Pfaffian system $\omega^\alpha = 0$ is equivalent to the system of differential equations,

$$\omega^\alpha = 0 \quad \Leftrightarrow \quad \frac{\partial u^\alpha}{\partial u^i} = \Gamma^\alpha_i, \tag{32}$$

where u^i are independent variables and u^α are the unknown functions. In this way we provide the passage from the structure $\Delta_h \oplus \Delta_v$ to differental equations and vice versa.

In the case where bundle (27) is a vector bundle whose fibers are r-dimensional vector spaces, it is necessary that the translation of fibers along any path on the base M be linear. This holds iff the quantities Γ^α_i depend on fiber coordinates in a linear and homogeneous way:[11]

$$\Gamma^\alpha_i = \Gamma^\alpha_{\beta i} u^\beta \tag{33}$$

with coefficients $\Gamma^\alpha_{\beta i}$ depending only on base coordinates. Such a connection in vector bundle (27) is called a *linear connection.*[12]

The adapted basis (30) is specialized in the structure $\Delta_h \oplus \Delta_v$ in the sense of (28). It is defined only on the local neighborhood U and, as is shown, is adapted to the coordinates (u^i, u^α). For this bases, among six subobjects (29), only the following two objects remain: the curvature and connection objects. They are expressed through the quantities Γ^α_i:

$$c^{\,\alpha}_{ij} = X_i \Gamma^\alpha_j - X_j \Gamma^\alpha_i = \partial_i \Gamma^\alpha_j - \partial_j \Gamma^\alpha_i + \partial_\beta \Gamma^\alpha_j \Gamma^\beta_i - \partial_\beta \Gamma^\alpha_i \Gamma^\beta_j, \tag{34}$$

$$c^{\,\alpha}_{\beta i} = \partial_\beta \Gamma^\alpha_i. \tag{35}$$

In the case where the connection is linear (see (33)), we have

$$c^\alpha_{ij} = K^\alpha_{ij\beta} u^\beta, \quad \text{where} \quad K^\alpha_{ij\beta} = \partial_i \Gamma^\alpha_{\beta j} - \partial_j \Gamma^\alpha_{\beta i} + \Gamma^\alpha_{\gamma j} \Gamma^\gamma_{\beta_i} - \Gamma^\alpha_{\gamma i} \Gamma^\gamma_{\beta_j}, \tag{36}$$

$$c^\alpha_{\beta i} = \Gamma^\alpha_{\beta i}. \tag{37}$$

In particular, the tangent bundle to the manifold M is the vector bundle with fibers $T_u M\ \forall u \in M$, and in this case, all that was done in bundle (27) remains valid if we set $\mathcal{M} = TM$ and $n = r$. The Greek indices become Latin.

On the neighborhood $U \subset M$, the coordinate functions u^i are defined, and on the neighborhood $TU = \pi^{-1}(U) \subset TM$, they define the coordinate functions $(u^i \circ \pi, du^i)$. The differentials $(d(u^i \circ \pi), d^{\,2} u^i)$ of these functions define a natural coframe on TU. Under the passage $U \subset \mathcal{M} \rightsquigarrow TU \subset TM$, we introduce another notation:

$$\begin{gathered} (u^i, u^\alpha, du^i, du^\alpha) \rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}), \\ \text{where} \quad u^i_1 = du^i, \quad u^i_2 = d(u^i \circ \pi), \quad u^i_{12} = d^{\,2} u^i. \end{gathered} \tag{38}$$

In the classical terminology [8], a linear connection in the tangent bundle TM is equivalent to assigning an *affine connection* on the manifold M. In tensor analysis (see [8, p. 128]), one defines the *curvature tensor*:

$$K^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{is} \Gamma^s_{js} - \Gamma^l_{os} \Gamma^s_{is}. \tag{39}$$

To put in concordance these formulas with those of tensor analysis and in particular, to ensure that formula (39) directly follows from (36), we change the sign in the passage (we refer to the notation):

$$\Gamma^\alpha_{\beta j} \ \rightsquigarrow \ -\Gamma^k_{ij}, \quad K^\alpha_{ij\beta} \ \rightsquigarrow \ -K^l_{ijk}.$$

[10] There exist nr quantities Γ^α_i. Precisely with this arbitrariness, we define an n-dimensional subspace in the $(n+r)$-dimensional vector space $T_u\mathcal{M}$, $u \in \mathcal{M}$.

[11] In what follows, when speaking of linear functions on fibers, we assume that they are linear and homogeneous.

[12] Various authors define connections, including linear connections, in different ways. The proposed structure is simplest, and such a terminology is completely justified.

Then the forms $\omega^\alpha = du^\alpha - \Gamma^\alpha_{\beta i} u^\beta du^i$ (see (31) and (33)) on the neighborhood TU take the form

$$U^i_{12} = \Gamma^i_{jk} u^j_1 u^k_2 + u^i_{12}. \tag{40}$$

Therefore, if in the neighborhood $T^2U \subset T^2M$, the coordinates $(u^i, u^i_1, u^i_2, u^i_{12})$ are defined, then in these coordinates, the "bad" second differentials u^i_{12} can be replaced by the vector quantities U^i_{12}. As for vector-valued quantities u^i_1 and u^i_2, they cannot be identified in general since their differentials du^i and $d(u^i \circ \pi)$ are respectively defined on U and TU, and, therefore, $du^i \circ \pi_1 \neq d(u^i \circ \pi)$. The quadruple $(u^i, u^i_1, u^i_2, U^i_{12})$ defines the *adapted coordinates* in the neighborhood T^2U.

2.2. Fundamental objects. A morphism of two bundles (see π_1 and π_2) is defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{f} & \mathcal{M}_2 \\ \downarrow\pi_1 & & \downarrow\pi_2 \\ M_1 & \xrightarrow{\bar f} & M_2 \end{array}. \tag{41}$$

The dimensions of the manifolds are

$$\dim \mathcal{M}_1 = n_1 + r_1, \quad \dim M_1 = n_1,$$

$$\dim \mathcal{M}_2 = n_2 + r_2, \quad \dim M_2 = n_2.$$

Let us consider the situation on local neighborhoods. On the neighborhoods (see the scheme below to the left), the coordinate functions are concordant with the bundles (see the scheme to the right):

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow\pi_1 & & \downarrow\pi_2 \\ \bar U & \xrightarrow{\bar f} & \bar V \end{array}, \qquad \begin{array}{ccc} (u^i, u^\alpha) & \xrightarrow{f} & (v^{i'}, v^{\alpha'}) \\ \downarrow\pi_1 & & \downarrow\pi_2 \\ (\bar u^i) & \xrightarrow{\bar f} & (\bar v^{i'}) \end{array}, \tag{42}$$

and moreover,

$$\bar U = \pi_1(U), \qquad V = f(U), \qquad \bar V = \bar f(\bar U) = \pi_2(V);$$

$$u^i = \bar u^i \circ \pi_1, \qquad v^{i'} = \bar v^{i'} \circ \pi_2.$$

The mapping $\bar f$ is defined by the functions$\bar f^{i'}$ on $\bar U$, which are $\bar f$-related to the coordinate functions $\bar v^{i'}$ on $\bar V$:

$$\bar v^{i'} \circ \bar f = \bar f^{i'}.$$

The mapping f is defined by the functions $(f^{i'}, f^{\alpha'})$ on U, which are f-related to the coordinate functions $(v^{i'}, v^{\alpha'})$ on V:

$$\begin{cases} v^{i'} \circ f & = f^{i'}, \\ v^{\alpha'} \circ f & = f^{\alpha'}. \end{cases} \tag{43}$$

The functions $f^{\alpha'}$ depend on all coordinates (u^i, u^α), whereas $f^{i'}$ depend only on the base ones (u^i) since they are π_1-related to the functions $\bar f^{i'}$. Indeed,

$$v^{i'} \circ f = (\bar v^{i'} \circ \pi_2) \circ f = (\bar v^{i'} \circ \bar f) \circ \pi_1 = \bar f^{i'} \circ \pi_1,$$

and hence $f^{i'} = \bar f^{i'} \circ \pi_1$. By using the Jacobi matrix, the tangent mapping Tf translates the natural frame forms, i.e., the differentials of the coordinate functions from V to U:

$$\begin{pmatrix} dv^{i'} \\ dv^{\alpha'} \end{pmatrix} \circ Tf = \begin{pmatrix} f^{i'}_j & 0 \\ f^{\alpha'}_j & f^{\alpha'}_\beta \end{pmatrix} \cdot \begin{pmatrix} du^j \\ du^\beta \end{pmatrix}. \tag{44}$$

On the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, let connections

$$\Delta_h \oplus \Delta_v \quad \text{and} \quad \Delta'_h \oplus \Delta'_v$$

be given.[13] Then on the neighborhoods U and V, the adapted bases are defined. Let us write only the coframes (see (30)):

$$\begin{pmatrix}\omega^i\\ \omega^\alpha\end{pmatrix} = \begin{pmatrix}\delta^i_j & 0\\ -\Gamma^\alpha_j & \delta^\alpha_\beta\end{pmatrix}\cdot\begin{pmatrix}du^j\\ du^\beta\end{pmatrix},$$
$$\begin{pmatrix}\theta^{i'}\\ \theta^{\alpha'}\end{pmatrix} = \begin{pmatrix}\delta^{i'}_{j'} & 0\\ -\Lambda^{\alpha'}_{j'} & \delta^{\alpha'}_{\beta'}\end{pmatrix}\cdot\begin{pmatrix}du^{j'}\\ du^{\beta'}\end{pmatrix}. \tag{45}$$

The tangent mapping Tf translates the adapted coframe forms from V to U, and, similarly to (44), we have the formula

$$\begin{pmatrix}\theta^{i'}\\ \theta^{\alpha'}\end{pmatrix}\circ Tf = \begin{pmatrix}f^{i'}_j & 0\\ F^{\alpha'}_j & f^{\alpha'}_\beta\end{pmatrix}\cdot\begin{pmatrix}\omega^j\\ \omega^\beta\end{pmatrix}, \tag{46}$$

but with one difference that here, the non-invariant left lower block of the Jacobi matrix is replaced by the invariant block

$$\boxed{F^{\alpha'}_i = f^{\alpha'}_i + f^{\alpha'}_\beta\Gamma^\beta_i - (\Lambda^{\alpha'}_{j'}\circ f)f^{j'}_i.} \tag{47}$$

Formulas (44) and (46) define the same mapping Tf, but in different cobases. Block (47) appears under the matrix multiplication:

$$\begin{pmatrix}\delta^{i'}_{j'} & 0\\ -\Lambda^{\alpha'}_{j'}\circ f & \delta^{\alpha'}_{\beta'}\end{pmatrix}\cdot\begin{pmatrix}f^{j'}_k & 0\\ f^{\beta'}_k & f^{\beta'}_\gamma\end{pmatrix}\cdot\begin{pmatrix}\delta^k_i & 0\\ -\Gamma^\gamma_i & \delta^\gamma_\alpha\end{pmatrix}.$$

When speaking of a morphism of vector bundles with linear connections, the functions $f^{\alpha'}$ in (43) and the connection coefficients Γ^α_i and $\Lambda^{\alpha'}_{i'}$ are linear functions of the fiber coordinates (see (33)):

$$f^{\alpha'} = f^{\alpha'}_\beta u^\beta, \quad \Gamma^\alpha_i = \Gamma^\alpha_{\beta i}u^\beta,$$
$$\Lambda^{\alpha'}_{i'} = \Lambda^{\alpha'}_{\gamma' i'}v^{\gamma'}, \quad \Lambda^{\alpha'}_{i'}\circ f = (\Lambda^{\alpha'}_{\gamma' i'}\circ f)f^{\gamma'}_\beta u^\beta.$$

Quantities (47) on U are also linear functions of fiber coordinates with coefficients depending only on base coordinates:

$$F^{\alpha'}_i = F^{\alpha'}_{\beta i}u^\beta, \quad \boxed{F^{\alpha'}_{\beta i} = \partial_i f^{\alpha'}_\beta + f^{\alpha'}_\gamma\Gamma^\gamma_{\beta i} - (\Lambda^{\alpha'}_{\gamma' j'}\circ f)f^{\gamma'}_\beta f^{j'}_i.} \tag{48}$$

In particular, the differential of a mapping $f : M_1 \to M_2$ defines the tangent bundle morphism

$$\begin{array}{ccc} TM_1 & \xrightarrow{Tf} & TM_2\\ \downarrow\pi_1 & & \downarrow\pi_2\\ M_1 & \xrightarrow{f} & M_2\end{array}. \tag{49}$$

In this case, the mapping Tf is defined by the system (see (43))

$$\begin{cases} v^{i'}\circ f = f^{i'},\\ d(v^{i'}\circ f) = df^{i'},\end{cases} \qquad \begin{pmatrix}f^{i'}_i & 0\\ df^{i'}_i & f^{i'}_i\end{pmatrix}. \tag{50}$$

To the right, the Jacobi matrix of the mapping Tf is wtitten, and moreover,

$$df^{i'} = f^{i'}_i u^i_1, \quad df^{i'}_i = f^{i'}_{ij}u^j_1, \quad \text{and} \quad d(v^{i'}\circ f) = dv^{i'}_1\circ Tf.$$

Furthermore, let the manifolds M_1 and M_2 be equipped with affine connections. Let us include the connection coefficients

$$\Gamma^\alpha_{\beta j} \rightsquigarrow -\Gamma^k_{ij}, \qquad \Lambda^{\alpha'}_{\gamma' j'} \rightsquigarrow -\Lambda^{k'}_{i'j'}$$

[13] In such a case, we of about a *morphism of bundles with connections*.

in our formulas and give the following form to the object (48):

$$F_i^{i'} = F_{ij}^{i'} u_1^j, \qquad \boxed{F_{ij}^{i'} = f_{ij}^{i'} - f_k^{i'} \Gamma_{ij}^k + (\Lambda_{j'k'}^{i'} \circ f) f_i^{i'} f_j^{j'}.} \tag{51}$$

We obtain an object of the same form as (25) but with the difference that here, the connection coefficients are specialized, as in case (24).

Conclusion. First of all, the fundamental object appears in the general bundle, but in passing to vector and tangent bundles, it is specialized. More precisely,

under a morphism of bundles with connections (41), there arises the object $F_i^{\alpha'}$, (47), which shows the deviation of the image $Tf\Delta_h$ from the distribution Δ'_h; for $F_i^{\alpha'} = 0$, we have the embedding $Tf\Delta_h \subset \Delta'_h$;

if bundles (41) are vector and the connections are linear, the object $F_i^{\alpha'}$ becomes linear on fibers with coefficients $F_{\beta i}^{\alpha'}$, (48);

under floor morphism (49), we obtain the object $F_{ij}^{i'}$, (51), which is characteristic for mappings of one affine connection space into another;

finally, if on the manifolds M_1 and M_2, global bases (5) are given, then connections (24) are defined on them and in the course of the prolongation of differential equations (12), there arises the object $\mathcal{F}_{ij}^{\alpha}$, (23), with local components F_{ij}^{α}, (25).

Under a general bundle morphism, we have the following equivalent writings:[14]

$$\boxed{dv^{\alpha'} \circ Tf = f_j^{\alpha'} du^j + f_\beta^{\alpha'} du^\beta} \;\leftrightsquigarrow\; \boxed{\theta^{\alpha'} \circ Tf = F_j^{\alpha'} du^j + f_\beta^{\alpha'} \omega^\beta.} \tag{52}$$

In the right writing, we have the sum of two invariant summands, whereas in the left writing, these summands are not invariant. Under the floor morphism (49), where the adapted coordinates[15] (see (40))

$$U_{12}^i = \Gamma_{jk}^i u_1^j u_2^k + u_{12}^i, \qquad V_{12}^\alpha = \Lambda_{\beta\gamma}^\alpha v_1^\beta v_2^\gamma + v_{12}^\alpha \tag{53}$$

are given, we have the same writings where to the right there is the sum of two invariant summands, which do not exist in the left sum:

$$\boxed{v_{12}^\alpha \circ T^2 f = f_{ij}^\alpha u_1^i u_2^j + f_i^\alpha u_{12}^i} \;\leftrightsquigarrow\; \boxed{V_{12}^\alpha \circ T^2 f = F_{ij}^\alpha u_1^i u_2^j + f_i^\alpha U_{12}^i.} \tag{54}$$

The object

$$\boxed{F_{ij}^\alpha = f_{ij}^\alpha - f_k^\alpha \Gamma_{ij}^k + (\Lambda_{\beta\gamma}^\alpha \circ f) f_i^\beta f_j^\gamma} \tag{55}$$

in formula (54) plays a fundamental role.[16]

If in the neighborhood U (resp. in the neighborhood V), we make the basis change using matrices A and A^{-1} (resp. B and B^{-1}), for example, in passing to global bases (5), then the vector components transform

$$(u_1^i, u_2^i, U_{12}^i) \;\rightsquigarrow\; (\tilde{u}_1^i = A_j^i u_1^j,\ \tilde{u}_2^i = A_j^i u_1^j,\ \tilde{U}_{12}^i = A_j^i U_{12}^j),$$

$$(v_1^\alpha, v_2^\alpha, V_{12}^\alpha) \;\rightsquigarrow\; (\tilde{v}_1^\alpha = B_\beta^\alpha v_1^\beta,\ \tilde{v}_2^\alpha = B_\beta^\alpha v_2^\beta,\ \tilde{V}_{12}^\alpha = B_\beta^\alpha V_{12}^\beta),$$

and the right relation in (54) is written in the form

$$\boxed{\tilde{V}_{12}^\alpha \circ T^2 f = \mathcal{F}_{ij}^\alpha \tilde{u}_1^i \tilde{u}_2^j + \mathcal{F}_i^\alpha \tilde{U}_{12}^i,} \tag{56}$$

where (see (8) and (9))

$$\mathcal{F}_i^\alpha = (B_\beta^\alpha \circ f) F_j^\alpha \bar{A}_i^j, \qquad \mathcal{F}_{ij}^\alpha = (B_\beta^\alpha \circ f) F_{pq}^\alpha \bar{A}_i^p \bar{A}_j^q. \tag{57}$$

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[14]See (44) and (46).

[15]We return to the initial indices; see Comment 1, p. 675.

[16]Not only in geometry; see Sec. 2.3 below, but in applications, e.g., in Lagrangian mechanics; see [15, p. 12.].

In this way, the global objects $\mathcal{F}_i^\alpha$ and $\mathcal{F}_{ij}^\alpha$ are expressed through the tensor quantities F_i^α and F_{ij}^α on the local neighborhoods. The higher objects $\mathcal{F}_{ijk}^\alpha$, ... are not considered. They are constructed in floors analogously; see [10, Sec. 14, p. 168]. A connection in double bundle

$$T^2M \underset{T\pi_1}{\overset{\pi_2}{\rightrightarrows}} TM$$

is defined, e.g., as the structure on T^3M (or on T^2M):

$$\Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12},$$

and moreover,

$$T^2\pi_1(\Delta_1 \oplus \Delta_{12}) = T\pi_2(\Delta_2 \oplus \Delta_{12}) = 0,$$

$$T^2\pi_1(\Delta \oplus \Delta_2) = T\pi_2(\Delta \oplus \Delta_1) = T^2M,$$

and the object F_{ijk}^α, as F_{ij}^α in (54), plays a fundamental role in the formula

$$V_{123}^\alpha \circ T^3 f = F_{ijk}^\alpha u_1^i u_2^j u_3^k + F_{ij}^\alpha (u_1^i U_{23}^j + u_2^i U_{13}^j + u_3^i U_{12}^j) + f_i^\alpha U_{123}^i.$$

2.3. Various theories.

2.3.1. Covariant differentiation. In the structure $\Delta_h \oplus \Delta_v$, a tensor field of type (p, q) is decomposed into 2^{p+q} invariant blocks. In this case, the field components in the adapted basis are expressed through the components in the natural basis, and vice versa. Formulas (58), (59), and (60) (see below) show how this happens in the cases $(1, 0)$, the case of a vector field X, $(0, 1)$, the case of a 1-form Φ and $(1, 1)$, and the case of an affinor field $\mathcal{A}$:

$$X = \partial_i \tilde{x}^i + \partial_\alpha \tilde{x}^\alpha = X_i x^i + \partial_\alpha x^\alpha, \quad x^i = \tilde{x}_i,\ x^\alpha = \tilde{x}^\alpha - \Gamma_i^\alpha \tilde{x}^i, \tag{58}$$

$$\begin{gathered} \Phi = \tilde{\varphi}_i du^i + \tilde{\varphi}_\alpha du^\alpha = \varphi_i du^i + \varphi_\alpha \omega^\alpha, \\ \varphi_i = \tilde{\varphi}_i + \tilde{\varphi}_\alpha \Gamma_i^\alpha, \quad \varphi_\alpha = \tilde{\varphi}_\alpha, \end{gathered} \tag{59}$$

$$\begin{gathered} \mathcal{A} = (\partial_i\ \partial_\alpha) \cdot \begin{pmatrix} \tilde{a}_j^i & \tilde{a}_\beta^i \\ \tilde{a}_j^\alpha & \tilde{a}_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} du^j \\ du^\beta \end{pmatrix} = (X_i\ \partial_\alpha) \cdot \begin{pmatrix} a_j^i & a_\beta^i \\ a_j^\alpha & a_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} du^j \\ \omega^\beta \end{pmatrix}, \\ \begin{pmatrix} a_j^i & a_\beta^i \\ a_j^\alpha & a_\beta^\alpha \end{pmatrix} = \begin{pmatrix} \tilde{a}_j^i + \tilde{a}_\gamma^k \Gamma_j^\gamma & \tilde{a}_\beta^i \\ \tilde{a}_j^\alpha + \tilde{a}\,_\gamma^\alpha \Gamma_j^\gamma - \Gamma_k^\alpha \tilde{a}_j^k - \Gamma_k^\alpha \tilde{a}_\gamma^k \Gamma_j^\gamma & \tilde{a}\,_\beta^\alpha - \Gamma_k^\alpha \tilde{a}_\beta^k \end{pmatrix}. \end{gathered} \tag{60}$$

We especially highlight the case where the right upper block of matrix (60) is equal to zero

$$\tilde{a}_\beta^i = 0 \quad \Rightarrow \quad \begin{pmatrix} a_j^i & a_\beta^i \\ a_j^\alpha & a_\beta^\alpha \end{pmatrix} = \begin{pmatrix} \tilde{a}_j^i & 0 \\ \tilde{a}_j^\alpha + \tilde{a}_\gamma^\alpha \Gamma_j^\gamma - \Gamma_k^\alpha \tilde{a}_j^k & \tilde{a}_\beta^\alpha \end{pmatrix}. \tag{61}$$

When the bundle is vector, and the connection in it is linear, it is natural to assume that not only the quantities $\Gamma_i^\alpha = \Gamma_{i\beta}^\alpha u^\beta$, but the quantities $\tilde{x}^\alpha, \tilde{\varphi}_i, \tilde{a}_i^\alpha$, together with quantities $x^\alpha, \varphi_i, a_i^\alpha$, are linear on fibers:

$$\tilde{x}^\alpha = \tilde{x}_\beta^\alpha u^\beta \quad \Rightarrow \quad x^\alpha = x_\beta^\alpha u^\beta, \quad x_\beta^\alpha = \tilde{x}_\beta^\alpha - \Gamma_{\beta i}^\alpha \tilde{x}^i, \tag{62}$$

$$\tilde{\varphi}_i = \tilde{\varphi}_{\beta i} u^\beta \quad \Rightarrow \quad \varphi_i = \varphi_{\beta i} u^\beta, \quad \varphi_{\beta i} = \tilde{\varphi}_{\beta i} + \tilde{\varphi}_\alpha \Gamma_{\beta i}^\alpha, \tag{63}$$

$$\tilde{a}\,_i^\alpha = \tilde{a}\,_{\beta i}^\alpha u^\beta \quad \Rightarrow \quad a_i^\alpha = a_{\beta i}^\alpha u^\beta, \quad a_{\beta i}^\alpha = \tilde{a}\,_{\beta i}^\alpha + \tilde{a}\,_\gamma^\alpha \Gamma_{\beta i}^\gamma - \Gamma_{\beta k}^\alpha \tilde{a}_i^k. \tag{64}$$

The tensor-type coefficients appearing here (see x_β^α, $\varphi_{\beta i}$, and $a_{\beta i}^\alpha$) are prototypes of covariant derivatives.

The meaning of the covariant derivatives themselves is revealed in the tangent bundle TM. A vector field $\tilde{X}$ is lifted from the manifold M to a vector field X on TM:[17]

$$a_t = \exp t\tilde{X} \;\rightsquigarrow\; Ta_t = \exp tX.$$

The covariant differential and the covariant derivatives of the components $\tilde{x}^i$ in the classical sense (here, the lifted from M to TM) appear in the adapted frame:

$$\tilde{X} = \bar{\partial}_i \tilde{x}^i \quad \rightsquigarrow \quad X = \partial_i x^i + \partial_i^1 dx^i = X_i x^i + \partial_i^1 \nabla x^i,$$

$$x^i = \tilde{x}^i \circ \pi, \;\; dx^i = \partial_j x^i u_1^j, \qquad \boxed{\nabla x^i = \nabla_j x^i u_1^j, \;\; \nabla_j x^i = \partial_j x^i + \Gamma^i_{jk} x^k.}$$

The relation $\nabla x^i = 0$ means that the field X belongs to the distribution Δ_h.

For a form $\tilde{\Phi}$ on the manifold M as a scalar-valued function on TM, we define the differential $d\tilde{\Phi}$. The covariant differential and the covariant derivatives of the components $\tilde{\varphi}_i$ (lifted from M to TM) appear in the adapted coframe:

$$\tilde{\Phi} = \tilde{\varphi}_i d\bar{u}^i \quad \rightsquigarrow \quad \Phi \doteq d\tilde{\Phi} = d\varphi_i u_2^i + \varphi_i u_{12}^i = \nabla\varphi_i u_2^i + \varphi_i U_{12}^i\,,$$

$$\varphi_i = \tilde{\varphi}_i \circ \pi, \;\; d\varphi_i = \partial_j \varphi_i u_1^j, \qquad \boxed{\nabla\varphi_i = \nabla_j \varphi_i u_1^j, \;\; \nabla_j \varphi_i = \partial_j \varphi_i - \varphi_k \Gamma^k_{ij}.}$$

The relation $\nabla\varphi_i = 0$ means that the differential $d\tilde{\Phi}$ is annihilated by the distribution Δ_h. Diffeomorphisms of the manifold M induce the action of the linear group $GL(n,\mathbb{R})$ on the tangent space $T_u M$, and the differentials of these diffeomorphisms induce the action of the tangent group $T\big(GL(n,\mathbb{R})\big)$ on the tangent space[18] $T_{(u,u_1)}(TM)$. We have the representations of the groups $GL(n,\mathbb{R})$ and $T\big(GL(n,\mathbb{R})\big)$ in floors TM and T^2M. Moreover, if the action of $GL(n,\mathbb{R})$ on TM is represented by the matrix $\tilde{A}$ (affinor field on M), then the action of $T\big(GL(n,\mathbb{R})\big)$ on T^2M is defined by the block-matrix[19]

$$\begin{pmatrix} \tilde{A} & 0 \\ d\tilde{A} & \tilde{A} \end{pmatrix}, \quad \text{where} \quad \tilde{A} = (\tilde{a}^i_j), \quad d\tilde{A} = (\partial_k \tilde{a}^i_j u_1^k).$$

This block-matrix defines the affinor field $\mathcal{A}$ on TM, but in the natural basis. In the adapted basis, the field $\mathcal{A}$ is decomposed into invariant blocks (see (61)):

$$\begin{pmatrix} \tilde{A} & 0 \\ \nabla\tilde{A} & \tilde{A} \end{pmatrix},$$

where

$$\boxed{\nabla\tilde{A} = (\nabla_k \tilde{a}^i_j u_1^k), \;\; \nabla_k \tilde{a}^i_j = \partial_k \tilde{a}^i_j - \tilde{a}^i_l \Gamma^l_{jk} + \Gamma^i_{jl} \tilde{a}^l_k.}$$

For $\nabla\tilde{A} = 0$, one says that the affinor $\mathcal{A}$ *respects* the structure $\Delta_h \oplus \Delta_v$.

2.3.2. Symmetries of the distribution Δ_h. We consider Cartan's theory and Sophus Lie's theory as one unified theory, the *Lie–Cartan calculus* being a duality theory in which the roles of the Lie derivatives and that of Cartan exterior forms are of the same importance.

If on the manifold M, a nonholonomic basis (P_i, ω^j), having the nonholonomy object c^k_{ij} is given (see (17)), then this basis naturally moves in the flow of the vector field $X = P_i x^i$, and its infinitesimal transformation is determined by the *derivation formulas*

$$\boxed{P' = -PC, \quad \omega' = C\omega.} \tag{65}$$

Here, the prime denotes the Lie derivatives with respect to X, and the matrix C has the form

$$C = (c^i_{jk} x^k + P_i x^j). \tag{66}$$

[17]The field X is called the *complete lift* of the field $\tilde{X}$ on TM; see [7].

[18]We assume that $u \in M$ and $(u, u_1) \in TM$.

[19]The structure $\Delta_h \oplus \Delta_v$, the adopted basis, and the action of $T\big(GL(n,\mathbb{R})\big)$ in T^2M, all this refers to gauge theory; see [2, Pt. 2], [5, Chap. 9], and Sec. 3.3.4 below.

Formulas (65) simplify the calculations.[20] For example, the Lie derivatives of a vector field Y, a 1-form Φ and an affinor field $\mathcal{A}$ in the flow of X are calculated according to the scheme

$$\begin{aligned} Y &= Py & &\rightsquigarrow & Y' &= P(y' - Cy), \\ \Phi &= \varphi\omega & &\rightsquigarrow & \Phi' &= (\varphi' + \varphi C)\omega, \\ \mathcal{A} &= PA\,\omega & &\rightsquigarrow & \mathcal{A}' &= P(A' - CA + AC)\omega. \end{aligned}$$

In the case of a holonomic basis (∂_i, du^j), we have from this the familiar formulas in coordinates, where $C = (\partial_i x^j)$. In the case where the coefficients c^i_{jk} are constant, it is convenient to apply formulas (65) in Lie group theory, in particular, in group representation theory.

In bundle (27), there exists the adapted basis (30). The derivation formulas of this basis in the flow of the vector field $X = X_i x^i + \partial_\alpha x^\alpha$ are defined by the block-matrix:[21]

$$C = \begin{pmatrix} X_j x^i & \partial_\beta x^i \\ X_j x^\alpha - \partial_\beta \Gamma^\alpha_j x^\beta + (X_i \Gamma^\alpha_j - X_j \Gamma^\alpha_i) x^j & \partial_\beta x^\alpha + \partial_\beta \Gamma^\alpha_k x^k \end{pmatrix}. \tag{67}$$

In matrix (67), a special role is played by its left lower block (see (34) and (35))

$$X_j x^\alpha - c^\alpha_{\beta j} x^\beta - c^\alpha_{ij} x^i. \tag{68}$$

This block defines the infinitesimal turn of the distribution Δ_h in the flow of the field X. If

$$X_j x^\alpha - c^\alpha_{\beta j} x^\beta - c^\alpha_{ij} x^i = 0, \tag{69}$$

then in the flow of X, Δ_h does not change, and the field X is an *infinitesimal symmetry* of the distribution Δ_h. If, moreover, $x^\alpha = 0$, then the field X is horizontal, and under the condition $c^\alpha_{ij} x^j = 0$, it is the *characteristic vector field* of the distribution[22] Δ_h.

In the case of a vector bundle and a linear connection where we specialize the quantities $c^\alpha_{ij} = K^\alpha_{ij\beta} u^\beta$, $c^\alpha_{\beta i} = \Gamma^\alpha_{\beta i}$, and $x^\alpha = x^\alpha_\beta u^\beta$ (see (36) and (37)), block (68) takes the form

$$(\partial_j x^\alpha_\beta + x^\alpha_\gamma \Gamma^\gamma_{\beta j} - \Gamma^\alpha_{\gamma j} x^\gamma_\beta - K^\alpha_{ij\beta} x^i) u^\beta. \tag{70}$$

In such a case, the vertical field X $(x^i = 0)$ is an infinitesimal symmetry of the distribution Δ_h iff (see (64))

$$\partial_j x^\alpha_\beta + x^\alpha_\gamma \Gamma^\gamma_{\beta j} - \Gamma^\alpha_{\gamma j} x^\gamma_\beta = 0.$$

In the classical theory, one knows the concept of *Lie derivative of affine connection coefficients* Γ^i_{jk} with respect to the vector field $\tilde{X} = \partial_i x^i$ on the neighborhood $U \subset M$; see [17, p. 41],

$$\mathcal{L}\Gamma^i_{jk} = \frac{\partial^2 x^i}{\partial u^j \partial u^k} + \partial_l \Gamma^i_{jk} x^l - \partial_l x^i \Gamma^l_{jk} + \Gamma^i_{jl} \partial_k x^l + \Gamma^i_{lk} \partial_j x^l. \tag{71}$$

If we pass to the tangent bundle with affine connection using the change

$$x^\alpha \rightsquigarrow dx^i = \partial_j x^i du^j, \quad x^\alpha_\beta \rightsquigarrow \partial_j x^i, \quad \Gamma^\alpha_{\beta j} \rightsquigarrow -\Gamma^k_{ij},$$

then the expression in the parentheses in (70) takes exactly form (71). If $\mathcal{L}\Gamma^i_{jk} = 0$, then the vector field $\tilde{X}$ on the manifold M is called an *affine collineation*; in Riemannian geometry, this is the *Killing*

[20] Formulas (65), as well as subsequent formulas are written in the matrix form.

[21] Matrix (67) is the same as in $\omega' = C\omega$ (65). To calculate the blocks of this matrix, we need the differentials of the functions in basis (30). They are easily written by using the identity affinor:

$$d = X_i \otimes \omega^i + \partial_\alpha \otimes du^\alpha \quad \rightsquigarrow \quad df = X_i f\,\omega^i + \partial_\alpha f\,du^\alpha.$$

[22] The characteristic vector field allows us to reduce the dimension of the distribution considered. So, for example, a two-dimensional distribution admitting the characteristic vector field is projected on the isocline field, which is easily integrated. In this case, the distribution itself is integrable.

vector field. Now we add the following: when $\mathcal{L}\Gamma^i_{jk} = 0$, the complete lift of the field $\tilde{X}$ to TM is an infinitesimal symmetry for the distribution Δ_h.

2.4. Automorphisms of bundles with connection. An automorphism of a bundle π is defined by a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{a} & \mathcal{M} \\ \downarrow\pi & & \downarrow\pi \\ M & \xrightarrow{\tilde{a}} & M \end{array}, \tag{72}$$

where $\tilde{a}$ and a are diffeomorphisms of manifolds M and $\mathcal{M}$ respectively. Situation (41) presupposes that in general, on the manifold $\mathcal{M}$, two structures $\Delta_h \oplus \Delta_v$ and $\Delta'_h \oplus \Delta_v$ are given, i.e., it is assumed that in the bundle π two horizontal distributions Δ_h and Δ'_h are given. Under the diffeomorphism Ta, we have a distribution transformation $\Delta_h \rightsquigarrow Ta\Delta_h$, and the deviation $Ta\Delta_h$ from Δ'_h on $U \subset \mathcal{M}$ is determined by the quantity[23]

$$\boxed{A^\alpha_i = a^\alpha_i + a^\alpha_\beta \Gamma^\beta_i - (\Lambda^\alpha_j \circ a) a^j_i.} \tag{73}$$

The relation $A^\alpha_i = 0$ means that we have a transformation $\Delta_h \rightsquigarrow \Delta'_h$, i.e., $Ta\Delta_h = \Delta'_h$. In the case of an affine connection, from this, we arrive at connection transformation theory and the concept of *deformation tensor* A^α_i; see [8, p. 131].

In particular, when $\Delta_h = \Delta'_h$, formula (73) takes the form

$$\boxed{A^\alpha_i = a^\alpha_i + a^\alpha_\beta \Gamma^\beta_i - (\Gamma^\alpha_j \circ a) a^j_i,} \tag{74}$$

and the relation $A^\alpha_i = 0$ means that the diffeomorphism a is a *symmetry* of the distribution Δ_h. We obtain the theory of motions in spaces with connection.

In the case of an automorphism of a vector bundle with linear connection, object (74) is specialized (see (48)):

$$A^\alpha_i = A^\alpha_{\beta i} u^\beta, \qquad \boxed{A^\alpha_{\beta i} = \partial_i a^\alpha_\beta + a^\alpha_\gamma \Gamma^\gamma_{\beta i} - (\Gamma^\alpha_{\gamma j} \circ a) a^\gamma_\beta a^j_i.} \tag{75}$$

The differential of the diffeomorphism $a : M \to M$ is understood as an automorphism of the floor TM:

$$\begin{array}{ccc} TM & \xrightarrow{Ta} & TM \\ \downarrow\pi & & \downarrow\pi \\ M & \xrightarrow{a} & M \end{array}. \tag{76}$$

If M is an affine connection space here, then object (75) looks as follows[24] (see (55)):

$$\boxed{A^k_{ij} = a^k_{ij} - a^k_l \Gamma^l_{pq} + (\Gamma^k_{pq} \circ a) a^p_i a^q_j.} \tag{77}$$

If, on the manifold M, the vector field $\tilde{X} = \partial_i x^i$ induces the flow $a_t = \exp t\tilde{X}$, where a_t for a fixed parameter t is a diffeomorphism, then in diagram (72) and in expression (77), we set $a \rightsquigarrow a_t$, and the object A^k_{ij} depending on t can be differentiated in t as $t \to 0$. There arise the derivatives

$$(\Gamma^k_{ij} \circ a, a^i_j, a^i_{jk}) \rightsquigarrow (\tilde{X}\Gamma^k_{ij}, \partial_j x^i, \partial^2_{jk} x^i),$$

and, as a result, we obtain

$$(A^k_{ij})'_{t=0} = \mathcal{L}\Gamma^k_{ij}$$

[23]Formula (73) is a particular case of (47), and it requires no additional explanation.

[24]The dual situation: on the one hand, $A^k_{ij} = 0$ means that $\Delta_h = \Delta_h$, and, on the other hand, this relation can be treated as the connection coefficient transformation $\Gamma^i_{jk} \circ a \rightsquigarrow \Gamma^i_{jk}$ under the coordinate transformation a^{-1}; see [8, p. 125].

(see (71)). If for each fixing of t, the diffeomorphism a_t is a symmetry of the distribution Δ_h, then the vector field $\tilde{X}$ is an infinitesimal symmetry of the distribution Δ_h,

$$A^k_{ij} = 0 \;\Rightarrow\; \mathcal{L}\Gamma^k_{ij} = 0.$$

Conclusion. As was shown above, under a morphism of general, vector, and tangent bundles with connections, there respectively arise the objects

$$F^{\alpha'}_i \ (47) \;\to\; F^{\alpha'}_{\beta i} \ (48) \;\to\; F^{\alpha}_{ij} \ (55).$$

Now, under an automorphism of general, vector, and tangent bundles with connections, there arise the objects

$$A^{\alpha}_i \ (73) \;\to\; A^{\alpha}_{\beta i} \ (75) \;\to\; A^k_{ij} \ (77) \;\to\; \mathcal{L}\Gamma^k_{ij} \ (71).$$

From the symmetries of the distribution Δ_h, we arrive at *motion theory* in connection spaces. The following derivation formulas and objects yield an infinitesimal treatment of motions in the structure $\Delta_h \oplus \Delta_v$:

$$(65) \to (67) \to (68) \to (70) \to (71).$$

We arrive at *geodesic mapping theory* starting from the implication

$$F^{\alpha}_{ij} = 0 \quad\Rightarrow\quad \{\, U^i_{12} = 0 \;\Rightarrow\; V^{\alpha}_{12} = 0 \,\}. \tag{78}$$

Then $Tf\Delta_h \subset \Delta'_h$, and geodesics of the manifold M_1 are mapped[25] into geodesics of the manifold M_2; see (54).

2.5. Immersions and submersions. It is known that an injective mapping is left invertible, and a surjective mapping is right invertible. This also refers to an immersion, a submersion, and their Jacobi matrices. So, if f^{α}_i is the Jacobi matrix of the mapping f, then in the case where f is an immersion (resp. a submersion), there exists a matrix φ^j_{α} (resp. ψ^i_{β}) such that $\varphi^j_{\alpha} f^{\alpha}_i = \delta^j_i$ (resp. $f^{\alpha}_i \psi^i_{\beta} = \delta^{\alpha}_{\beta}$). Moreover, in the case where f is an immersion (resp. a submersion), the columns (resp. rows) of the matrix f^{α}_i are linearly independent, and the Gramian matrix $g_{ij} = f^{\alpha}_i f^{\alpha}_j$ (resp. $g^{\alpha\beta} = f^{\alpha}_i f^{\beta}_i$) is positive-definite. This means that the matrix g_{ij} (resp. $g^{\alpha\beta}$) admits the inverse matrix g^{ij} (resp. $g_{\alpha\beta}$), and the matrix φ^{β}_i (resp. ψ^i_{β}) can be defined as follows:

$$\varphi^j_{\alpha} = f^{\alpha}_k g^{kj}$$

(resp. $\psi^i_{\beta} = g_{\beta\gamma} f^{\gamma}_i$).

We are interested in the object F^{α}_{ij}, (55), in the following two cases:

(1) when f is an immersion, and the n-dimensional manifold M_1 is immersed in the $(n+r)$-dimensional manifold M_2, thus defining an n-dimensional surface $f(M_1) \subset M_2$ there;

(2) when f is a submersion, and the $(n+r)$-dimensional manifold M_1 is projected on the n-dimensional manifold M_2, thus defining a bundle with n-dimensional base and r-dimensional leaves.

The object F^{α}_{ij} depends on the choice of the connections Γ^k_{ij} and $\Lambda^{\alpha}_{\beta\gamma}$ on the manifolds M_1 and M_2. The Laptev method presupposes the assignment of bases (5) on M_1 and M_2; then the quantities Γ^k_{ij} and $\Lambda^{\alpha}_{\beta\gamma}$ are defined by formulas (24). In general, in cases (1) and (2) the following fixings are possible:

$$1)\ \varphi^k_{\alpha} F^{\alpha}_{ij} = 0 \qquad\Rightarrow\qquad \Gamma^k_{ij} = \varphi^k_{\alpha}\big(f^{\alpha}_{ij} + (\Lambda^{\alpha}_{\beta\gamma} \circ f) f^{\beta}_i f^{\gamma}_j\big), \tag{79}$$

$$2)\ F^{\alpha}_{ij} \psi^i_{\beta} \psi^j_{\gamma} = 0 \qquad\Rightarrow\qquad \Lambda^{\alpha}_{\beta\gamma} \circ f = -\,(f^{\alpha}_{ij} - f^{\alpha}_k \Gamma^k_{ij}) \psi^i_{\beta} \psi^j_{\gamma}. \tag{80}$$

From formula (79), *Riemannian geometry* starts. Let M_2 be an Euclidean space. On the surface $f(M_1)$, we define the metric tensor g_{ij}. Since zero connection in M_2 has an invariant meaning, we set

[25] If on a curve $\gamma(t) \subset M_1$, the functions U^i_{12} vanish, $\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0$, then this curve is a geodesic, but then $V^{\alpha}_{12} = 0$, and its image, the curve $f(\gamma(t)) \subset M_2$, is also a geodesic.

$\Lambda^{\alpha}_{\beta\gamma} = 0$. If we assume that the object F^{α}_{ij} for fixed i and j lies in the plane normal to $f(M_1)$, then the connection coefficients Γ^k_{ij} are uniquely defined on M_1 and on $f(M_1)$; these are the Christoffel symbols

$$f^{\alpha}_k F^{\alpha}_{ij} = 0 \quad \Rightarrow \quad \Gamma^k_{ij} = f^{\alpha}_l f^{\alpha}_{ij} g^{lk}. \tag{81}$$

From (55), we deduce the following *Gauss formula*, where F^{α}_{ij} are coefficients of the quadratic form with values in the plane normal to $f(M_1)$:

$$f^{\alpha}_{ij} = f^{\alpha}_k \Gamma^k_{ij} + F^{\alpha}_{ij}.$$

In case (2), we have the dual situation. It is called *co-Riemannian geometry*. Let M_1 be an Euclidean space foliated into r-dimensional fibers. The base M_2 is n-dimensional. In M_1, we can set $\Gamma^k_{ij} = 0$. If we require that the relation $F^{\alpha}_{ij} f^{\beta}_i f^{\gamma}_j = 0$ holds, then, according to (80),

$$F^{\alpha}_{ij} f^{\beta}_i f^{\gamma}_j = 0 \quad \Rightarrow \quad \Lambda^{\alpha}_{\beta\gamma} \circ f = -g_{\beta\lambda} g_{\gamma\mu} f^{\alpha}_{ij} f^{\lambda}_i f^{\mu}_j. \tag{82}$$

The second partial derivatives f^{α}_{ij} are replaced by the tensor object:

$$F^{\alpha}_{ij} = f^{\alpha}_{ij} + (\Lambda^{\alpha}_{\beta\gamma} \circ f) f^{\beta}_i f^{\gamma}_j. \tag{83}$$

Co-Riemannian geometry is elaborated a little, but it promises to be not less rich in content than Riemannian geometry.

In general, the scheme (79)–(80) covers not only Riemannian and co-Riemannian geometries, but the general theory of immersed manifolds, including surface theory, on the one hand, and bundle theory, in particular the theory of scalar fields with Laplacians, etc., on the other hand.[26]

3. Sector-Bundles

The further development is related to the floors.[27] Only in floors, we can formulate higher-order motion laws.

3.1. Iterations of the tangent functor. The floors are defined in the course of successive iterations of the tangent functor T. Under the kth iteration T^k, to a smooth n-dimensional manifold M, we put in correspondence its kth floor $T^k M$, being a $2^k n$-dimensional manifold, and, at the same time, a k-fold vector bundle, and to a smooth mapping f, we put in correspondence its kth-order differential $T^k f$, a morphism of kth floors. Elements in floors are defined as follows:

$$u \in M, \quad (u, u_1) \in TM, \quad (u, u_1, u_2, u_{12}) \in T^2 M,$$
$$(u, u_1, u_2, u_{12}, u_3, u_{13}, u_{23}, u_{123}) \in T^3 M, \ \ldots .$$

A point of the floor $T^k M$ is the pair of a point of the floor $T^{k-1}M$ and the vector at this point tangent to $T^{k-1}M$. Moreover, *a point of the floor $T^{k-1}M$ is denoted by $k-1$ symbols, and the vector tangent to $T^{k-1}M$ at this point is denoted by the same symbols but equipped with an additional index k* ($k =$ 1,2,…, $T^0 M = M$). We see below that such an indexation is very convenient.

The natural projections

$$\pi_l : T^l M \to T^{l-1} M, \quad l = 1, 2, \ldots, k,$$

define k distinct projections from the floor $Y^k M$ on the floor $T^{k-1}M$:

$$T^{k-1}\pi_1, \ T^{k-2}\pi_2, \ \ldots, \ T\pi_{k-1}, \ \pi_k ; \tag{84}$$

note that *under the lth projection of the kth floor on the $(k-1)$th floor, the symbols with index l are deleted*. So, to go down from the third floor to the second, we have three projections, and our rule applies:

[26] See, e.g.,[9].

[27] The floor structure is described in the books [2, 3], and [16] in sufficiently more detail. According to White, the floors are called the *sector-bundles*. This theory is still rather new, and, as is seen, its applications are only sketched.

$$\begin{array}{ccc} & (u, u_1, u_2, u_{12}, u_3, u_{13}, u_{23}, u_{123}) & \\ \swarrow_{T^2\pi_1} & \downarrow_{T\pi_2} & \searrow_{\pi_3} \\ (u, u_2, u_3, u_{23}) & (u, u_1, u_3, u_{13}) & (u, u_1, u_2, u_{12}). \end{array}$$

In general, to go down from the floor T^kM to the manifold M, there exist $k!$ different ways.[28]

If to each smooth function Φ defined on the floor $T^{k-1}M$, we put in correspondence the pair $(\Phi \circ \pi_k, d\Phi)$ on the floor T^kM, then according to this principle, to each smooth function f on the manifold M, we put in correspondence 2^k distinct functions

$$f \rightsquigarrow (f \circ \pi_1, df) \rightsquigarrow (f \circ \pi_1\pi_2, df \circ \pi_2, d(f \circ \pi_1), d^2 f) \rightsquigarrow \dots.$$

defined on the floor T^kM. Using the same indexation as for the floor elements, let us introduce the following notation for this functions:

$$f \rightsquigarrow (f, f_1) \rightsquigarrow (f, f_1, f_2, f_{12}) \rightsquigarrow (f, f_1, f_2, f_{12}, f_3, f_{13}, f_{23}, f_{123}) \rightsquigarrow \dots.$$

Here, the differentials f, df, d^2f, d^3f, $\dots$ have a broader meaning than in the ordinary analysis.[29]

3.2. Sector forms. From the neighborhood $U \subset M$, on the neighborhoods $T^kU \subset T^kM$, the coordinate functions induce the coordinate functions

$$u^i \rightsquigarrow (u^i, u^i_1) \rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}) \rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}, u^i_3, u^i_{13}, u^i_{23}, u^i_{123}) \rightsquigarrow \dots.$$

The coordinates with subscript l are fiber coordinates for lth projection (84), and the others are basis coordinates, $l = 1, 2, \dots, k$. A point of the kth floor is defined by 2^kn coordinates.[30]

The differentials defined by the function f on the floors T^kM are expressed in the coordinates as follows:

$$\begin{aligned} &f_1 = f_i u^i_1, \quad f_2 = f_i u^i_2, \quad f_3 = f_i u^i_3, \ \dots, \\ &f_{12} = f_{ij} u^i_1 u^j_2 + f_k u^k_{12}, \quad f_{13} = f_{ij} u^i_1 u^j_3 + f_k u^k_{13}, \\ &f_{23} = f_{ij} u^i_2 u^j_3 + f_k u^k_{23}, \ \dots, \\ &f_{123} = f_{ijk} u^i_1 u^j_2 u^k_3 + f_{ij}(u^i_1 u^j_{23} + u^i_2 u^j_{13} + u^i_3 u^j_{12}) + f_k u^k_{123}, \ \dots\ , \end{aligned}$$

where

$$f_i = \frac{\partial f}{\partial u^i}, \qquad f_{ij} = \frac{\partial^2 f}{\partial u^i \partial u^j}, \ \dots.$$

All these differentials are linear functions on fibers of the corresponding bundles. So, for example, the function f_{123} is decomposed into the sum of five summands, and the index 1 (resp. 2 and 3) enters each of the summands only one time. This means that the function f_{123} is linear on fibers of three bundles $T^2\pi_1$, $T\pi_2$, and π_3.

If in these formulas, the partial derivatives f_i, f_{ij}, f_{ijk}, $\dots$ are replaced by arbitrary scalar coefficients, we obtain the general case. A scalar-valued function on the floor T^kM, linear on all fibers of all bundles (84), is called a *k-sector form* in the sense of White [16].

The White sector form theory is much more general than the Cartan exterior form theory, which is included in it as a particular case. For example, a 1-form Φ on the manifold M as a scalar-valued function on the floor TM has the differential $d\Phi$. In coordinates on T^2U, we have

$$\Phi = \varphi_i u^i_1 \quad \rightsquigarrow \quad d\Phi = \partial_j \varphi_i \, u^i_1 u^j_2 + \varphi_k u^k_{12}.$$

[28] The structure of T^kM is described by a commutative diagram in the form of the k-dimensional cube and composition projections from T^kM to M in it; indeed, there are $k!$ such projections.

[29] According to the principle of u^i, du^i, d^2u^i, $\dots$, d^ku^i in the Miron–Atanasiu theory, the so-called *osculating bundle* $\mathrm{Osc}^k M$ is constructed; it is a $(k+1)$-dimensional subbundle in T^kM defined by the equality of projections (84); see [1, 2]. This recalls the situation in the case $n = 1$, $k = 2$ where on the paraboloid $z = xy$, by the relation $x = y$, a parabola is cut out. In such a case, we cannot speak of the multi-fold vector bundle.

[30] Whence $\dim T^kM = 2^k \dim M$.

If we alternate the derivatives $\partial_j\varphi_i$ in indices and symmetrize them $\partial_j\varphi_i = \partial_{[j}\varphi_{i]} + \partial_{(j}\varphi_{i)}$, then in the expression $d\Phi$, the exterior differential $\mathrm{d}\,\Phi = \partial_{[j}\varphi_{i]}\varphi_i\, u_1^i u_2^j$ of this form is isolated. In the structure of sector-bundles (floors), on the sector-forms, one treats any operations with exterior forms.

3.3. Tangent groups. Lie groups and their representations are well developed by the Cartan method. However, the tangent functor and Lie derivatives give a special beauty to this theory.

3.3.1. Generalized Leibnitz formula. A smooth mapping λ,

$$\lambda : \big((u, u_1), (v, v_1)\big) \mapsto (w, w_1),$$

where U, V, and W are smooth manifolds, admits the tangent mapping $T\lambda$,

$$\lambda : U \times V \to W : (u, v) \mapsto w, w_1.$$

To a pair of points $u \in U$ and $v \in V$, in W, we put in correspondence the point w denoted by $u \cdot v$. Let us define the mappings

$$l_u : V \to W : v \mapsto u \cdot v, \qquad r_v : U \to W : u \mapsto u \cdot v.$$

To tangent vectors u_1 and v_1 to U and V at the points u and v, we put in correspondence the vector w_1 tangent to W at the point w, which is the sum of images[31] $Tr_v(u_1)$ and $Tl_u(v_1)$, which are respectively denoted by $u_1 \cdot v$ and $u \cdot v_1$. We obtain convenient formulas in which, to the right, there is the *generalized Leibnitz formula*[32]:

$$\boxed{w = u \cdot v, \qquad w_1 = u_1 \cdot v + u \cdot v_1\,.} \tag{85}$$

Let us pass to group theory. Let G be a Lie group with multiplication law[33]:

$$\gamma : G \times G \to G : (a, b) \mapsto c = ab.$$

The first floor TG becomes a Lie group[34] with the multiplication law $T\gamma$. The situation on TG is described by the formulas

$$\boxed{c = ab, \qquad c_1 = a_1 b + a b_1\,.} \tag{86}$$

By the right shift r_b and, respectively, by the left shift l_a, more precisely, by their differentials, two vectors $a_1 \in T_aG$ and $b_1 \in T_bG$ are translated from points $a \in G$ and $b \in G$ to T_cG, where the sum of their images a_1b and ab_1 composes the vector c_1 at the point $c \in G$. In this way[35], the vectors a_1 and b_1 are multiplied in the group TG. The identity in the group TG is a zero vector in T_eG, where e is the identity of the group G. The inversion of elements in the group TG is executed according to the rule

$$\boxed{(a, a_1) \ \rightsquigarrow\ (a, a_1)^{-1} = (a^{-1}, -a^{-1}a_1a^{-1})\,.} \tag{87}$$

---

[31]This is easily proved in coordinates:

$$w^\alpha = \lambda^\alpha(u, v) \quad \rightsquigarrow \quad dw^\alpha = \frac{\partial\lambda^\alpha}{\partial u^i}du^i + \frac{\partial\lambda^\alpha}{\partial v^\sigma}dv^\sigma, \ \ldots.$$

[32]Using this formula, we can deduce all basic calculation formulas for Lie derivatives. So, from $(Yf)' = Y'f + Yf'$, where the prime denotes differentiation with respect to a vector field X, we deduce $Y' = XY - YX$, from $(\Phi(Y))' = \Phi'(Y) + \Phi(Y')$, we obtain Φ', and so on, in essence resolving the equation $w_1 = u_1 \cdot v + u \cdot v_1$ with respect to u_1. Exercise: Basing on $\omega(P) = E$, prove the equivalence of derivation formulas (65).

[33]Between the group elements, the dot is not written: $c = ab$.

[34]This holds automatically for the kth floor T^kG, which becomes a Lie group with multiplication law $T^k\gamma$. For example, for T^2G, to formulas (86), we add the formulas $c_2 = a_2b + ab_2$, $c_{12} = a_{12}b + a_1b_2 + a_2b_1 + ab_{12}$, etc.

[35]If formula (86) is represented in the form $c_1 = a(a^{-1}a_1 + b_1b^{-1})b$, then we argue in another way: the vectors a_1 and b_1 are translated in T_eG into the vectors $a^{-1}a_1$ and b_1b^{-1}, where they are added, and then their sum is translated into T_cG, as the formula shows.

An arbitrary vector $e_1 \in T_eM$ is translated in T_aG into two vectors ae_1 and e_1a. On the group G, we define the vector fields ae_1 and e_1a $\forall a \in G$, the left-invariant field ae_1, and the right-invariant field e_1a. Under inversion (87), these fields change their roles: the left-invariant field becomes right-invariant and, vice versa, the right-invariant field becomes left-invariant:

$$(ae_1)^{-1} = -e_1a^{-1}, \qquad (e_1a)^{-1} = -a^{-1}e_1.$$

As is seen below, the difference of fields $e_1a - ae_1$ defines the *adjoint representation operator*.

3.3.2. Group representations. Let the manifold M be the representation space of the group G. The mappings λ and $T\lambda$ define the action of the group G on M and the action of the tangent group TG on TM:

$$\lambda : M \times G \to M : (u,a) \mapsto v = u \cdot a,$$
$$T\lambda : \big((u,u_1),(a,a_1)\big) \mapsto (v,v_1), \qquad \boxed{v_1 = u_1 \cdot a + u \cdot a_1.}$$

For each $a \in G$, the mapping $\lambda_a : M \to M : u \mapsto v = u \cdot a$ is a diffeomorphism, and moreover, $a \mapsto \lambda_a$ is a homomorphism of the group G into the diffeomorphism group of the manifold M; moreover, $\lambda_{ab} = \lambda_b \circ \lambda_a$. The mapping

$$\lambda_u : G \to M : a \mapsto v = u \cdot a$$

defines the orbit $\lambda_u(G) \subset M$ of a point $u \in M$ in M, and the mapping $T_u\lambda$ translates all vectors in TG into this orbit:

$$T\lambda_u : TG \to TM : (a,a_1) \mapsto (v,v_1), \quad \text{where} \quad v_1 = u \cdot a_1 = v \cdot a^{-1}a_1.$$

A fixed vector $a^{-1}a_1 \in T_eG$ is translated into TM composing a vector field on M tangent to orbits, the *operator of the group* G. An arbitrary vector basis (frame) is translated from T_eG on the manifold M into a system of operators. The distribution, being their linear span, is integrable, with integral surfaces that are the orbits of points $u \in M$. The equation $v_1 = v \cdot a^{-1}a_1$ is called the *defining relation*[36] of this representation.

Example. A group G acts on itself by inner automorphisms (*adjoint representation*):

$$\mathrm{Ad}_a : G \;\to\; G : \; b \;\mapsto\; \tilde{b} = aba^{-1}.$$

The general *adjoint representation operator* is defined as the difference of the left-invariant and right-invariant vector fields on G:

$$\tilde{b}_1 = a_1ba^{-1} + ab(-a^{-1}a_1a^{-1}) = (a_1a^{-1})\tilde{b} - \tilde{b}(a_1a^{-1}),$$
$$a_1a^{-1} \in T_eG \quad \forall \tilde{b} \in G.$$

3.3.3. Groups $GL(2,\mathbb{R})$ and $T^k\big(GL(2,\mathbb{R})\big)$. Let us consider the linear group $GL(2,\mathbb{R})$, and together with it, its tangent group $T\big(GL(2,\mathbb{R})\big)$. An element of $GL(2,\mathbb{R})$ is a regular matrix A, and let the tangent vector to $GL(2,\mathbb{R})$ at the point A be defined by the matrix C, so that $(A,C) \in T\big(GL(2,\mathbb{R})\big)$,

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}.$$

In $GL(2,\mathbb{R})$, we define the coordinates and the natural basis (∂_i, da^j), where $\partial_i = \dfrac{\partial}{\partial a_i}$, $i,j = 1,2,3,4$. The basis at the identity $E \in GL(2,\mathbb{R})$ spreads on the whole group into two bases[37]: the left-invariant

[36]In coordinates, it is written in the form of the system $dv^\alpha = \xi^\alpha_i \omega^i$, where ξ^α_i are components of the group operators of this representation on the orbit and ω^i are left-invariant forms on the group G.

[37]The left-invariant coframe is defined by the formula $\omega = A^{-1}dA$, and the right-invariant one by the formula $\omega = dAA^{-1}$, and the dual frames by inverse matrices.

basis (X_i, ω^j) and the right-invariant basis $(\tilde{X}_i, \tilde{\omega}^j)$:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix},$$

$$\begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} da_1 & da_2 \\ da_3 & da_4 \end{pmatrix},$$

$$\begin{pmatrix} \tilde{X}_1 & \tilde{X}_2 \\ \tilde{X}_3 & \tilde{X}_4 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix},$$

$$\begin{pmatrix} \tilde{\omega}^1 & \tilde{\omega}^2 \\ \tilde{\omega}^3 & \tilde{\omega}^4 \end{pmatrix} = \begin{pmatrix} da_1 & da_2 \\ da_3 & da_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1}.$$

In the left-invariant frame, a general left-invariant field X has constant coefficients C, and in exactly the same way, in the right-invariant frame, a general right-invariant field $\tilde{X}$ is represented:

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4,$$
$$\tilde{X} = c_1 \tilde{X}_1 + c_2 \tilde{X}_2 + c_3 \tilde{X}_3 + c_4 \tilde{X}_4.$$

The flows $a_t = \exp tX$ and $\tilde{a}_t = \exp t\tilde{X}$ are defined by the same one-parameter subgroup e^{Ct} of the group $GL(2, R)$, and moreover, the trajectories of the field X are left cosets, whereas the trajectories of the field $\tilde{X}$ are right cosets:

$$a_t \rightsquigarrow A' = A\,C \quad \Rightarrow \quad A_t = A\, e^{Ct},$$
$$\tilde{a}_t \rightsquigarrow A' = CA \quad \Rightarrow \quad A_t = e^{Ct} A.$$

Derivation formulas (65) for the left-invariant basis (X_i, ω^j) in the flow of the field X are written in the form

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}' = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} - \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix},$$

$$\begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix}' = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix}.$$

From this, the meaning of the group operator commutator table is clear: the derivative $X_i' = [X, X_i]$ defines the ith row of this table.[38] The right-invariant basis $(\tilde{X}_i, \tilde{\omega}^j)$ is invariant in the flow of the field X since the left and right shifts on $GL(2, R)$ commute and left-invariant operators commute with right-invariant operators.

The adjoint representation operators are defined as the differences $Y_i = \tilde{X}_i - X_i$:

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} - \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}.$$

The operators Y_i are linearly dependent

$$Y_1 + Y_4 = 0, \quad a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 = 0,$$

and on $GL(2, R)$, their linear span $\langle Y_i \rangle$ defines the two-dimensional integrable distribution, being the annihilator of the forms

$$\theta^1 = a_1 \omega^1 + a_3 \omega^2 + a_2 \omega^3 + a_4 \omega^4 = a_1 \tilde{\omega}^1 + a_3 \tilde{\omega}^2 + a_2 \tilde{\omega}^3 + a_4 \tilde{\omega}^4,$$
$$\theta^2 = \omega^1 + \omega^4 = \tilde{\omega}^1 + \tilde{\omega}^4.$$

Moreover, $d(\mathrm{tr} A) = \theta^1$ and $d(\det A) = (\det A)\theta^2$.

[38]More precisely, by the coefficient c_j we have the jth element of the ith row of the table.

The general adjoint representation operator is represented here as the linear combination of the operators Y_i with coefficients C:

$$Y = c_1Y_1 + c_2Y_2 + c_3Y_3 + c_4Y_4.$$

The flow of the operator Y is defined by the 1-parameter subgroup of the inner automorphism group:

$$A' = CA - AC \quad \Rightarrow \quad A_t = e^{Ct}Ae^{-Ct}.$$

The invariants of the operator Y and the integrals of the distribution $\langle Y_i \rangle$ are the trace and the determinant of the matrix A:

$$\mathrm{tr}A = a_1 + a_4, \qquad \det A = a_1a_4 - a_2a_3.$$

The operators Y_i admit an infinitesimal symmetry P; see below. In the flow of the field P, the integrals of the distribution $\langle Y_i \rangle$, the quantities $(\mathrm{tr}\, A, \det A)$, transform but remain integrals.

The common invariant of the operators Y_i and the field P is the discriminant Δ:

$$\begin{aligned}
P = \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_1}, \qquad & \exp tP : A \rightsquigarrow A_t = A + tE, \\
& (\mathrm{tr}\, A)_t = \mathrm{tr}\, A + 2t, \\
& (\det A)_t = \det A + \mathrm{tr}\, A \cdot t + t^2, \\
\Delta = \mathrm{tr}^2 A - 4 \det A, \qquad & \Delta_t = \Delta.
\end{aligned}$$

There arises the *syzygy*, precisely, the discriminant Δ connects the invariants $(\mathrm{tr}\, A, \det A)$ of the operators Y_i with the invariants $(a_1 - a_4, a_2, a_3)$ of the field P:

$$\Delta = (a_1 + a_2)^2 - 4(a_1a_4 - a_2a_3) = (a_1 - a_4)^2 + 4a_2a_3.$$

The field P can be projected on the plane uu':

$$\zeta : A \rightsquigarrow (u, u'), \qquad \begin{cases} u \circ \zeta &= \det A, \\ u' \circ \zeta &= \mathrm{tr}\, A, \end{cases}$$

$$P \ \rightsquigarrow \ T\zeta P = u'\frac{\partial}{\partial u} + 2\frac{\partial}{\partial u'}\,, \qquad \begin{cases} u_t &= u + u't + t^2, \\ u'_t &= u' + 2t. \end{cases}$$

The trajectories of the vector field $T\zeta P$ are parabolas, and the discriminant $\tilde{\Delta}$ of the quadratic function u_t coincides with the discriminant Δ:

$$\tilde{\Delta} = (u')^2 - 4u \quad \rightsquigarrow \quad \Delta = \tilde{\Delta} \circ \zeta\,.$$

Conclusion. The general situation is as follows. In a continuous medium, a certain system is subjected to perturbations. First, the system is moved in a certain flow. The flow is linearized by the Jacobian matrix A similar to the tangent vector to the trajectory of a moving point. Under one perturbation, the matrix transforms: $A \rightsquigarrow UAU^{-1}$ (the operators Y_i act), and under the other, its diagonal changes: $A \rightsquigarrow A + tE$ (the operator P acts). Under each of the perturbations, invariants are isolated. Both operations commute: $U(A + tE)U^{-1} = UAU^{-1} + tE$ (P is a symmetry of the operators Y_i), which tells us about the existence of a common invariant (discriminant Δ). One and other invariants are connected by a syzygy.

The same situation holds with a matrix of order n. In this case, coefficients of the corresponding polynomials and their discriminants exactly coincide with the initial and central moments of probability theory; see [11]. Hence we speak of the stability as a whole.

3.3.4. Gauge groups. The tangent group of the linear group $GL(n,\mathbb{R})$ is monomorphically embedded in the linear group $GL(2n,\mathbb{R})$:

$$T\big(GL(n,\mathbb{R})\big) \hookrightarrow GL(2n,\mathbb{R}): \quad (A,A_1) \rightsquigarrow \begin{pmatrix} A & 0 \\ A_1 & A \end{pmatrix}. \tag{88}$$

Rules (86) and (87) are satisfied[39]:

to the product of elements (A,A_1) and (B,B_1), we put in correspondence the product of matrices:

$$\begin{pmatrix} A & 0 \\ A_1 & A \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ B_1 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ A_1B + AB_1 & AB \end{pmatrix}$$

$$= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} E & 0 \\ A^{-1}A_1 + B_1B^{-1} & E \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix};$$

to the inversion of an element $(A,A_1)^{-1}$, we put in correspondence the matrix inversion:

$$\begin{pmatrix} A & 0 \\ A_1 & A \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -A^{-1}A_1A^{-1} & A^{-1} \end{pmatrix}.$$

This is the first step of the iteration process

$$T^k\big(GL(n,\mathbb{R})\big) \hookrightarrow GL(2^kn,\mathbb{R}), \quad k = 1, 2, \ \dots. \tag{89}$$

In the second step, the tangent group $T^2\big(GL(n,\mathbb{R})\big)$ is embedded in the group $GL(4n,\mathbb{R})$, and the step is repeated:

$$(A, A_1, A_2, A_{12}) \rightsquigarrow \begin{pmatrix} A & 0 & 0 & 0 \\ A_1 & A & 0 & 0 \\ A_2 & 0 & A & 0 \\ A_{12} & A_2 & A_1 & A \end{pmatrix}. \tag{90}$$

Rules (86) and (87) are naturally satisfied here[40]:

$$\begin{pmatrix} A & 0 & 0 & 0 \\ A_1 & A & 0 & 0 \\ A_2 & 0 & A & 0 \\ A_{12} & A_2 & A_1 & A \end{pmatrix} \cdot \begin{pmatrix} B & 0 & 0 & 0 \\ B_1 & B & 0 & 0 \\ B_2 & 0 & B & 0 \\ B_{12} & B_2 & B_1 & B \end{pmatrix} = \begin{pmatrix} C & 0 & 0 & 0 \\ C_1 & C & 0 & 0 \\ C_2 & 0 & C & 0 \\ C_{12} & C_2 & C_1 & C \end{pmatrix},$$

$$\begin{aligned} C &= AB, \\ C_1 &= A_1B + AB_1, \\ C_2 &= A_2B + AB_2, \\ C_{12} &= A_{12}B + A_2B_1 + A_1B_2 + AB_{12}, \end{aligned}$$

and if the result of the multiplication is the identity matrix, then in $T^2\big(GL(n,\mathbb{R})\big)$, we define the inversion of elements[41]:

$$B = A^{-1},$$

[39]See footnote 34, p. 691.

[40]See footnote 33, p. 691.

[41]The following formulas are generalized: $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$, $\left(\frac{1}{u}\right)'' = \frac{-u''u + 2(u')^2}{u^3}$.

$$B_1 = -A^{-1}A_1A^{-1},$$
$$B_2 = -A^{-1}A_2A^{-1},$$
$$B_{12} = -A^{-1}(A_{12} - A_1A^{-1}A_2 - A_2A^{-1}A_1)A^{-1}.$$

As a result of the next embedding, we obtain a sequence of the so-called *gauge groups*.

In gauge theory, the central problem is the isolation of the stationary subgroup in the transformation group for a point of representation space. In the floors T^kM, the diffeomorphism group $\mathcal{G}$ of the manifold M acts jetwise. The Jacobian matrices of diffeomorphisms a, Ta, T^2a, ... are inductively constructed:

$$A \rightsquigarrow \begin{pmatrix} A & 0 \\ A_1 & A \end{pmatrix} \rightsquigarrow \begin{pmatrix} A & 0 & 0 & 0 \\ A_1 & A & 0 & 0 \\ A_2 & 0 & A & 0 \\ A_{12} & A_2 & A_1 & A \end{pmatrix} \rightsquigarrow \dots,$$

$$A = (a^i_j), \quad A_1 = (a^i_{jk}u^i_1), \quad A_2 = (a^i_{jk}u^i_2),$$
$$A_{12} = (a^i_{jkl}u^k_1u^l_2 + a^i_{jk}u^k_{12}), \dots .$$

The group of invertible jets of order k is homomorphically mapped into the group $GL(2^kn, \mathbb{R})$, and moreover, the kernel of this homomorphism is the stationary subgroup of an element of the floor T^kM. The floor T^kM is a homogeneous space with action of the group $\mathcal{G}$.

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