Lectures on Dynamical Systems

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PART II

1 Invariant Measures and Ergodic Theorem

Given a transformation $T : X \to X$, we may wonder how often a subset of X is visited by an orbit of T. In the previous sections, we encountered several examples for which some orbits were dense and every nonempty open set was visited infinitely often. To measure the asymptotic fraction of times a set is visited, we may look at the limit points of the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_A(T^j(x)) \tag{1.1}$$

as $n \to \infty$. To have a more tractable situation, let us assume that X is a Polish space (i.e., a complete separable metric space) and that $T : X \to X$ is continuous. It is also more convenient to consider

$$\lim_{n\to\infty} \Phi_n(f)(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x)) \tag{1.2}$$

where f is a bounded continuous function. If the limit of (1.2) exists for every $f \in C_b(X)$, then the limit $\ell_x(f)$ enjoys some obvious properties:

(i) $f \geq 0 \Rightarrow \ell_x(f) \geq 0$, $\ell_x(\mathbb{1}) = 1$.

(ii) $\ell_x(f)$ is linear in f.

(iii) $|\ell_x(f)| \leq \sup_{y\in X} |f(y)|$.

(iv) $\ell_x(f \circ T) = \ell_x(f)$.

If X is also locally compact, then we can use Riesz Representation Theorem to assert that there exists a unique (Radon) probability measure μ such that $\ell_x(f) = \int f d\mu$. Evidently, such a measure $\mu(A)$ measures how often a set A is visited by the orbit $O^+(x)$. Motivated by this, we let $\mathcal{I}_T$ denote the space of probability measures μ such that

$$\int f \circ T d\mu = \int f d\mu, \tag{1.3}$$

for every $f \in C_b(X)$. Such a measure μ is an example of an *invariant measure.*

It seems natural that for analyzing the limit points of (1.1), we should first try to understand the space $\mathcal{I}_T$ of invariant measures. Note that in (1.2), what we have is $\int f d\mu_x^n$ where $\mu_x^n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)}$. We also learned that if (1.2) exists for every f, then μ_x^n has a limit and its limit is an invariant measure. Of course there is a danger that the limit (1.2) does not exist in general. This is very plausible if the orbit is unbounded and some of the mass of the measure μ_x^n is lost as $n \to \infty$ because $T^j(x)$ goes off to infinity. This would not happen if we assume X is compact. To this end, let us review the notion of weak convergence for measures. We say $\mu_n \Rightarrow \mu$ for μ_n, $\mu \in \mathcal{M}(X)$ if

$$\int f d\mu_n \to \int f d\mu, \tag{1.4}$$

for every $f \in C_b(X)$. It turns out that for the weak convergence, we only need to verify (1.4) for $f \in U_b(X)$ where $U_b(X)$ denotes the space of bounded uniformy continuous functions. Since $U_b(X)$ is separable, we can metrize the space of probability measures $\mathcal{M}(X)$. (See for example "Probabilty measures on Metric Spaces" by Parthasarathy.)

Exercise 1.1

(i) Show that the topology associated with (1.4) is metrizable with the metric given by

$$d(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \frac{\left|\int f_n d\mu - \int f_n d\nu\right|}{1 + \left|\int f_n d\mu - \int f_n d\nu\right|},$$

where $\{f_n : n \in N\}$ is a countable dense subset of $U_b(X)$.

(ii) Show that if X is a compact metric space, then $\mathcal{M}(X)$ is compact.

Theorem 1.2 *Suppose X is a compact metric space.*

(i) *(Krylov–Bogobulov) $\mathcal{I}_T \neq \emptyset$*

(ii) *If $\mathcal{I}_T = \{\mu\}$ is singleton, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) = \int f d\mu$$

uniformly for every $f \in C(X)$. In fact $\mu_x^n \Rightarrow \mu$ uniformly in x.

(iii) *If $\left\{\frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j\right\}$ converges uniformly to a constant for functions f in a dense subset of $C(X)$, then $\mathcal{I}_T$ is a singleton.*

Proof.

(i) This is an immediate consequence of Exercise 1.1(ii) and what we had in the beginning of this section. In fact any limit point of $\{\mu_x^n\}$ is in $\mathcal{I}_T$ for every x.

(ii) Let $\{x_n\}$ be any sequence in X and put $\nu_n = \mu^n_{x_n}$. One can readily show that any limit point of $\{\nu_n\}$ is in $\mathcal{I}_T = \{\mu\}$. Hence $\nu_n \Rightarrow \mu$. From this we can readily deduce that in fact $\mu^n_x \Rightarrow \mu$ uniformly.

(iii) We are assuming that $\Phi_n(f)$ converges uniformly to a constant $\hat{f}$ for f in a dense set $\mathcal{A} \subseteq C(X)$. The constant $\hat{f}$ can only be $\int f d\mu$ because for every $\mu \in \mathcal{I}_T$,

$$\int \Phi_n(f) d\mu = \int f d\mu.$$

Let us write $\|f\| = \sup_{x\in X} |f(x)|$. Pick any $g \in C(X)$ and a sequence $f_k \in \mathcal{A}$ such that $\|f_k - g\| \leq k^{-1}$. Since $\|\Phi_n(f)\| \leq \|f\|$ for every f, we learn

$$\limsup_{n\to\infty} \|\Phi_n(g) - \int g d\mu\| \leq \lim_{n\to\infty} \|\Phi_n(f_k) - \int f_k d\mu\| + 2/k \leq 2/k.$$

By sending $k \to \infty$ we deduce that $\lim_n \|\Phi_n(g) - \int g d\mu\| = 0$. Since $\hat{f}$ is a constant independent of $\mu \in \mathcal{I}_T$, we conclude that $\mathcal{I}_T$ is a singleton.

□

From Theorem 5.2 we learn that when $\mathcal{I}_T$ is a singleton, the statistics of the orbits are very simple. However, this does not happen very often. This is a rather rare situation and when it happens, we say that the transformation T is *uniquely ergodic*.

Example 1.3 Consider a translation $T : \mathbb{T}^d \to \mathbb{T}^d$ given by $T(x) = x+\alpha$ with $\alpha = (\alpha_1 \dots \alpha_d)$ and $\alpha_1 \dots \alpha_d, 1$ rationally independent. We claim that $\mathcal{I}_T$ consists of a single measure, namely the Lebesgue measure on $\mathbb{T}^d$, normalized to be a probability measure. One way to see this is by observing that if $\mu \in \mathcal{I}_T$, then

$$\int f(x + n\alpha)\mu(dx) = \int f(x)\mu(dx)$$

for every continuous f and any $n \in \mathbb{N}$. Since $\{n\alpha\}$ is dense, we deduce that μ is translation invariant. It is well-known that the Lebesgue measure is the only translation invariant probability measure. In fact we can use Theorem 1.2(iii) to see this directly. According to this theorem, we need to show that $\Phi_n(f)$ converges uniformly to a constant for f in a dense subset $\mathcal{A}$ of $C(X)$. For $\mathcal{A}$ take the set of trigonometric polynomials $\sum_j c_j e^{2\pi i j\cdot x}$ where the summation is over $j \in \mathbb{Z}^d$ and only finitely many c_j's are nonzero. Evidently, it suffices to verify this for $f(x) = e^{2\pi i j\cdot x}$. When $j \neq 0$,

$$\begin{aligned} |\Phi_n(f)| &= \left| \frac{1}{n} \sum_{\ell=0}^{n-1} e^{2\pi i j\cdot(x+\ell\alpha)} \right| = \left| \frac{1}{n} \sum_{\ell=0}^{n-1} e^{2\pi i \ell j\cdot\alpha} \right| \\ &= \frac{1}{n} \left| \frac{1 - e^{2\pi i n j\cdot\alpha}}{1 - e^{2\pi i j\cdot\alpha}} \right| \to 0 \end{aligned}$$

uniformly as $n \to \infty$. Thus T is uniquely ergodic. We note that the ergodicity of the Lebesgue measure also implies the denseness of the sequence $\{x + n\alpha\}$. $\square$

As we mentioned earlier, in most cases $\mathcal{I}_T$ is not a singleton. There are some obvious properties of the set $\mathcal{I}_T$ which we now state. Note that $\mathcal{I}_T$ is always a convex and closed subset of $\mathcal{M}(X)$. Also, $\mathcal{I}_T$ is compact when X is compact because $\mathcal{M}(X)$ is compact. Let us state a theorem of Choquet that can be used to get a picture of the set $\mathcal{I}_T$. Recall that if $\mathcal{C}$ is a compact convex set then a point $a \in \mathcal{C}$ is *extreme* if $a = \theta b + (1-\theta)c$ for some $\theta \in [0,1]$ and $b, c \in \mathcal{C}$ implies that either $a = b$ or $a = c$. According to *Choquet's theorem*, if $\mathcal{C}$ is convex and compact, then any $\mu \in \mathcal{C}$ can be expressed as an average of the extreme points. More precisely, we can find a probability measure θ on the set of extreme points of $\mathcal{C}$ such that

$$\mu = \int_{C^{ex}} \alpha \theta(d\alpha). \tag{1.5}$$

Let us write $\mathcal{I}_T^{ex}$ for the extreme points of $\mathcal{I}_T$. The extreme points of $\mathcal{I}_T$ are called *ergodic measures*. In view of (1.5), any $\mu \in \mathcal{I}_T$ can be expressed as an average of ergodic ones. Later we give simpler conditions for ergodicity.

Example 1.4 Consider $T : \mathbb{T} \to \mathbb{T}$ with $T(x) = x + \alpha \pmod 1$ with $\alpha = \frac{1}{\ell}$ and ℓ a positive integer. It is not hard to see that $\mathcal{I}_T^{ex} = \{\mu_x : 0 \le x < \alpha\}$ where $\mu_x = \frac{1}{\ell}[\delta_x + \delta_{x+\alpha} + \dots + \delta_{x+(\ell-1)\alpha}]$. Note that if $X(x) = \{x, x+\alpha, \dots, x+(\ell-1)\alpha\}$ then $\mathbb{T} = \bigcup_{x \in [0,\alpha)} X(x)$. Also observe

$$\frac{1}{n}\sum_{0}^{n-1} f \circ T(x) = \frac{1}{\ell}[f(x) + \dots + f(x + (\ell-1)\alpha)] + O\left(\frac{1}{n}\right) \to \int f d\mu_x,$$

as $n \to \infty$. $\square$

Given $\mu \in \mathcal{I}_T^{ex}$, clearly the set

$$X_\mu = \{x : \mu_x^n \Rightarrow \mu \text{ as } n \to \infty\}$$

is invariant under T. That is, if $x \in X_\mu$, then $T(x) \in X_\mu$. Also, if $\mu_1 \neq \mu_2 \in \mathcal{I}_T^{ex}$, then $X_{\mu_1} \cap X_{\mu_2} = \emptyset$. *Ergodic Theorem* below implies that $\mu(X_\mu) = 1$. This confirms the importance of ergodic measures among the invariant measures. Later we find more practical criterion for ergodicity in terms of invariant sets and functions. This will be achieved in two Ergodic Theorems we prove.

Theorem 1.5 (von Neumann) *Let $T : X \to X$ be a Borel measurable transformation and let $\mu \in \mathcal{I}_T$. If $f \in L^2(\mu)$, then $\Phi_n(f) = \frac{1}{n}\sum_0^{n-1} f \circ T^j$ converges in L^2-sense to $\mathcal{P}f$, where Pf is the projection of f onto the space of invariant functions g satisfying $g \circ T = g$.*

Proof. First observe that if $f = g \circ T - g$ for some $g \in L^2$, then $\Phi_n(f) \to 0$ as $n \to \infty$. Let $\mathcal{H}$ denote the linear space of *gradient* type functions $g \circ T - g$. If $f \in \bar{\mathcal{H}}$, then we still have $\lim_{n\to\infty} \Phi_n(f) = 0$. This is because if $f_k \in \mathcal{H}$ converges to f in L^2, then $\|\Phi_n(f)\|_{L^2} \leq \|\Phi_n(f_k)\|_{L^2} + \|f - f_k\|_{L^2}$ because $\|h \circ T^j\|_{L^2} = \|f\|_{L^2}$ by invariance. Since $\|\Phi_n(f_k)\|_{L^2} \to 0$ as $n \to \infty$ and $\|f - f_k\|_{L^2} \to 0$ as $k \to \infty$, we deduce that $\Phi_n(f) \to 0$ as $n \to \infty$.

Given any $f \in L^2(\mu)$, write $f = g+h$ with $g \in \bar{\mathcal{H}}$ and $h \perp \mathcal{H}$. If $h \perp \mathcal{H}$, then $\int h\, \varphi \circ T\, d\mu = \int h\varphi\, d\mu$, for every $\varphi \in L^2(\mu)$. Hence $\int (h \circ T - h)^2\, d\mu = 0$. This means that $h \circ T = h$. As a result, h is invariant and $\Phi_n(f) = \Phi_n(g) + \Phi_n(h) = \Phi_n(g) + h$. Since $\Phi_n(g) \to 0$, we deduce that $\Phi_n(f) \to h$ with $h = Pf$. $\square$

What we have in von Neumann's theorem is an operator $Uf = f \circ T$ that is an isometry of $L^2(\mu)$ and the space of invariant functions $\{\varphi : \varphi \circ T = \varphi\}$ is the eigenspace associated with the eigenvalue one. Hence our theorem simply says $\frac{1}{n}(I + U + \cdots + U^{n-1}) \to P$. Note that if $\lambda = e^{i\theta}$ is an eigenvalue of U and if $\lambda \neq 1$, then $\frac{1}{n}(1 + \lambda + \cdots + \lambda^{n-1}) = \frac{\lambda^n - 1}{n(\lambda - 1)} \to 0$ as $n \to \infty$. The above theorem suggests studying the spectrum of the operator U for a given T. Later we will encounter the notion of mixing dynamical systems. It turns out that the mixing condition implies that discrete spectrum of the operator U consists of the point 1 only.

As our next goal, we would like to have a different type of convergence. In our next theorem we consider an almost everywhere mode of convergence.

To this end let us take a measurable transformation $T : X \to X$ and $\mu \in \mathcal{I}_T$. Let $f \in L^1(\mu)$. First we would like to find a candidate for the limit $\lim_{n\to\infty} \Phi_n(f)$. Theorem 5.5 suggests looking at the projection of f onto the space of conserved (invariant) functions. Motivated by this, let us define

$$\mathcal{F}_T = \{A \in \mathcal{F} : T^{-1}(A) = A\}. \tag{1.6}$$

where T is $\mathcal{F}$-measurable. Note that $\mathcal{F}_T$ is a σ-algebra and consists of sets for which $1\!\!1_A \circ T = 1\!\!1_A$. We may now define Pf as the unique $\mathcal{F}_T$-measurable function such that

$$\int_A Pf d\mu = \int_A f d\mu \tag{1.7}$$

for every $A \in \mathcal{F}_T$. Note that since Pf is $\mathcal{F}_T$-measurable, we have

$$Pf \circ T = Pf,$$

μ-almost everywhere. Also, Pf is uniquely defined as the Radon–Nikodym derivative $f\mu$ with respect to μ, if we restrict it to $\mathcal{F}_T - \sigma$-algebra. More precisely

$$Pf = \frac{d(f\mu|_{\mathcal{F}_T})}{d\mu|_{\mathcal{F}_T}}.$$

We are now ready for the statement of Birkhoff Ergodic Theorem.

Theorem 1.6 *Suppose $\mu \in \mathcal{I}_T$ and $f \in L^1(\mu)$. Let Pf be as above. Then*

$$\mu\left\{x : \lim_{n\to\infty} \frac{1}{n}\sum_0^{n-1} f(T^j(x)) = Pf(x)\right\} = 1.$$

Moreover $\Phi_n(f)$ converges to Pf in L^1 sense.

Proof. Set $g = f - Pf - \epsilon$ for a fixed $\epsilon > 0$. Evidently $Pg \equiv -\epsilon < 0$ and $\Phi_n(f - Pf - \epsilon) = \Phi_n(f) - Pf - \epsilon$. Hence, it suffices to show

$$\limsup_{n\to\infty} \Phi_n(g) \le 0 \quad \mu - \text{a.e.}$$

We expect to have

$$g + g\circ T + \cdots + g\circ T^{n-1} = -\varepsilon n + o(n).$$

From this, it is reasonable to expect that the expression $g+\cdots+g\circ T^{n-1}$ to be bounded above μ-a.e. Because of this, let us define $G_n = \max_{j\le n}\sum_0^{j-1} g\circ T^i$. Set $A = \{x : \lim_{n\to\infty} G_n(x) = +\infty\}$. Without loss of generality, we may assume that g is finite everywhere. Clearly $A \in \mathcal{F}_T$ because $G_{n+1} = g + \max(0, G_n \circ T)$. Note also that if $x \notin A$, then $\limsup_{n\to\infty} \Phi_n(g) \le 0$. To complete the proof, it remains to show that $\mu(A) = 0$. To see this, observe

$$\begin{aligned} 0 &\le \int_A (G_{n+1} - G_n)d\mu = \int_A (G_{n+1} - G_n\circ T)d\mu \\ &= \int_A [g + \max(0, G_n\circ T) - G_n\circ T]d\mu = \int_A (g - \min(0, G_n\circ T))d\mu. \end{aligned}$$

On the set A, $-\min(0, G_n\circ T) \downarrow 0$. Hence by Dominated Convergence Theorem, $0 \le \int_A g d\mu = \int_A Pg d\mu \le -\epsilon\mu(A)$. Thus we must have $\mu(A) = 0$.

It remains to show that $\Phi_n(f)$ converges to Pf in L^1 sense. To show this, let $f_k = f 1\!\!1(f \le k)$ so that

$$\lim_{k\to\infty} \|f_k - f\|_{L^1(\mu)} = 0.$$

Since $\Phi_n(f_k)$ converges to Pf_k almost everywhere and $|\Phi_n(f)|$ is bounded by constant k, we have that $\Phi_n(f_k)$ converges to Pf_k in L^1 sense. Note that

$$\begin{aligned} \|\Phi_n(f) - Pf\|_{L^1(\mu)} &\le \|\Phi_n(f_k) - Pf_k\|_{L^1(\mu)} + \|\Phi_n(f - f_k)\|_{L^1(\mu)} + \|P(f - f_k)\|_{L^1(\mu)} \\ &\le \|\Phi_n(f_k) - Pf_k\|_{L^1(\mu)} + 2\|f - f_k\|_{L^1(\mu)}, \end{aligned}$$

where for the second inequality we used Exercise 1.8(ii). We now send $n \to \infty$ and $k \to \infty$ in this order. $\square$

As a consequence of this theorem, we have the following criterion for ergodicity.

Lemma 2.6 *$\mu \in \mathcal{I}_T^{ex}$ iff $\mu \in \mathcal{I}_T$ and $\mu(A) = 0$ or 1 for every $A \in \mathcal{F}_T$.*

Proof. Suppose $\mu \in \mathcal{I}_T$ and $A \in \mathcal{F}_T$. If $\mu(A) \in (0, 1)$, then

$$\mu_1(B) = \frac{\mu(A \cap B)}{\mu(A)}, \ \mu_2(B) = \frac{\mu(A^c \cap B)}{\mu(A^c)}$$

are well-defined and belong to $\mathcal{I}_T$. Moreover, $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$ for $\alpha = \mu(A)$. Hence if $\exists$ $A \in \mathcal{F}_T$ with $\mu(A) \in (0, 1)$, then $\mu \notin \mathcal{I}_T^{ex}$.

Conversely, suppose $\mu \in \mathcal{I}_T$ and that $\mu(A) = 0$ or 1 if $A \in \mathcal{F}_T$. Note that since Pf is measurable with respect to $\mathcal{F}_T$, we learn that Pf is constant μ-a.e. and the constant can only be $\int f d\mu$. This implies that if

$$B = \left\{ x : \Phi_n(f)(x) \to \int f d\mu \right\},$$

then$\mu(B) = 1$. If $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{I}_T$ and $\alpha \in (0, 1)$, we also have that $\mu_i(A) = 0$ or 1 if $A \in \mathcal{F}_T$ and $i = 1$ or 2. As a result,

$$\mu_i \left\{ x : \Phi_n(f)(x) \to \int f d\mu_i \right\} = 1$$

for $i = 1, 2$. Since $\mu(B) = 1$, we know that $\mu_1(B) = \mu_2(B) = 1$. Now if $\mu \neq \mu_1$, we can find integrable f such that $\int f d\mu \neq \int f d\mu_1$. This contradicts $\mu(B) = \mu_1(B) = 1$. Thus, we must have $\mu = \mu_1$. $\square$

If T is invertible, then we can have an ergodic theorem for T^{-1} as well. Since $\mathcal{F}_T = \mathcal{F}_{T^{-1}}$, it is clear that $P_T f = P_{T^{-1}} f$. As a consequence

Lemma 1.7 *Suppose $T, T^{-1} : X \to X$ are measurable and $\mu \in \mathcal{I}_T = \mathcal{I}_{T^{-1}}$. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_0^{n-1} f \circ T^j = \lim_{n \to \infty} \frac{1}{n} \sum_0^{n-1} f \circ T^{-j} = Pf.$$

$\mu - a.e.$

Exercise 1.8

- (i) Let A be a measurable set with $\mu(A \Delta T^{-1}(A)) = 0$. Show that there exists a set $B \in \mathcal{F}_T$ such that $\mu(A \Delta B) = 0$.

- (ii) Show that $\int Pf d\mu \leq \int |f| d\mu$.

As we mentioned in the introduction, many important ergodic measures enjoy a stronger property known as mixing. A measure $\mu \in \mathcal{I}_T$ is called *mixing* if for any two measurable sets A and B,

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B). \tag{1.8}$$

Mixing implies the ergodicity because if $A \in \mathcal{F}_T$, then $T^{-n}(A) = A$ and $T^{-n}(A) \cap A^c = \emptyset$. As a result, $\mu(A) = \lim_n \mu(T^{-n}(A) \cap A) = \mu(A)\mu(A)$, which implies that either $\mu(A) = 0$ or $\mu(A) = 1$. Also note that if μ is ergodic, then

$$\mu \left\{ x : \frac{1}{n} \sum_0^{n-1} 1\!\!1_A \circ T^j \to \mu(A) \right\} = 1,$$

which in turn implies

$$\lim_{n\to\infty} \int \left(\frac{1}{n} \sum_0^{n-1} 1\!\!1_A \circ T^j \right) 1\!\!1_B \; d\mu = \mu(A)\mu(B).$$

Hence ergodicity means

$$\lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} \mu(T^{-j}(A) \cap B) = \mu(A)\mu(B). \tag{1.9}$$

So, the ergodicity is some type of a weak mixing.

Example 1.9 Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a translation $T(x) = x + \alpha \pmod 1$ with $\alpha = (\alpha_1 \dots \alpha_d)$ and $\alpha_1 \dots \alpha_d, 1$ rationally independent. We now argue that T is not mixing. To see this, take a set A with $\mu(A) > 0$ and assume that A is not dense. Pick $x_0 \notin A$ and let $\delta = \text{dist.}(x_0, A) > 0$. Take any set B open with $\mu(B) > 0$ and $\text{diam}(B) < \delta/2$. By topological transitivity, $x_0 \in T^{-n}(B)$ for infinitely many $n \in \mathbb{N}$. Since $\text{diam}(T^{-n}(B)) = \text{diam}(B)$, we deduce that $T^{-n}(B) \cap A = \emptyset$ for such n's. Clearly $\mu(T^{-n}(B) \cap A) = 0$ does not converge to $\mu(A)\mu(B) \neq 0$ as $n \to \infty$. $\square$

Before discussing examples of mixing systems, let us give an equivalent criterion for mixing.

Lemma 1.10 *A measure μ is mixing iff*

$$\lim_{n\to\infty} \int f \circ T^n \; g \; d\mu = \int f d\mu \int g d\mu \tag{1.10}$$

for f and g in a dense subset of $L^2(\mu)$.

Proof. If μ is mixing, then (1.10) is true for $f = 1\!\!1_A$, $g = 1\!\!1_B$. Hence (1.10) is true if both f and g are simple, i.e., $f = \sum_{j=1}^m c_j 1\!\!1_{A_j}$, $g = \sum_{j=1}^m c'_j 1\!\!1_{B_j}$. We then use the fact that the space of simple functions is dense in $L^2(\mu)$.

For the converse, observe that if $\|f - \hat{f}\|_{L^2}$ and $\|g - \hat{g}\|_{L^2}$ are small, then

$$\left| \int f \circ T^n \; g \; d\mu - \int \hat{f} \circ T^n \; \hat{g} \; d\mu \right|,$$

is small. Indeed,

$$\begin{aligned} \left| \int f \circ T^n \; g \; d\mu - \int \hat{f} \circ T^n \; \hat{g} \; d\mu \right| &\leq \left| \int (f \circ T^n - \hat{f} \circ T^n) g \; d\mu \right| + \left| \int \hat{f} \circ T^n \; (g - \hat{g}) \; d\mu \right| \\ &\leq \|f - \hat{f}\| \|g\| + \|\hat{f}\| \|g - \hat{g}\| \end{aligned}$$

by invariance and Schwartz Inequality. □

Example 1.11 Let $T_m : \mathbb{T} \to \mathbb{T}$ be the expanding map $T_m(x) = mx \pmod 1$ with $m \geq 2$ positive integer. Given any $p = (p_0, \dots, p_{m-1})$ with $p_j \geq 0$ and $p_0 + \dots + p_{m-1} = 1$, we can construct a unique probability measure μ_p such that

$$\mu_p[\cdot a_1 \dots a_k, \cdot a_1 \dots a_k + m^{-k}) = p_{a_1} p_{a_2} \dots p_{a_k}.$$

If $p = (1, 0, \dots, 0)$ then the measure $\mu_p = \delta_0$ corresponds to the fixed point 0. If $p_0 = \dots = p_{m-1} = \frac{1}{m}$, then μ_p is the Lebesgue measure. It is not hard to show that μ_p is an invariant measure for T_m.

****Figure Goes Here****

In fact, if

$$A = \{x : x = \cdot a_1 a_2 \dots a_k * * \dots\},$$

then

$$T^{-1}(A) = \{x : x = \cdot * a_1 a_2 \dots a_k * \dots\}$$

and

$$\mu_p(T^{-1}(A)) = \sum_{b=0}^{m-1} p_b p_{a_1} \dots p_{a_k} = p_{a_1} \dots p_{a_k} = \mu_p(A).$$

To show that each μ_p is mixing observe that if

$$\begin{aligned} A &= \{x : x = \cdot a_1 a_2 \dots a_k * * \dots\}, \\ B &= \{x : x = \cdot b_1 b_2 \dots b_k * * \dots\}, \end{aligned} \tag{1.11}$$

then

$$T^{-n}(A) \cap B = \{x : x = \cdot \underbrace{b_1 b_2 \dots b_k * * \dots *}_{n} a_1 \dots a_k * * \dots\}$$

whenever $n \geq k$, and

$$\mu_p(T^{-n}(A) \cap B) = \mu_p(A)\mu_p(B).$$

This implies the mixing because the set of simple functions $f = \sum_{j=1}^{\ell} c_j 1\!\!1_{A_j}$ with A_j as in (1.11) is dense in $L^2(\mu_p)$ and we can apply Lemma 5.10.

Note also that if x is a periodic point of period ℓ, then $\mu = \frac{1}{\ell}\sum_{j=0}^{\ell-1} \delta_{T^j(x)}$ is an ergodic measure. Such μ is never mixing unless $\ell = 1$. □

Exercise 1.12 Let a be a periodic point for T of period ℓ. Show that $\mu = \frac{1}{\ell}\sum_{j=0}^{\ell-1} \delta_{T^j(x)}$ is not mixing if $\ell > 1$.

Exercise 1.13

- (i)Show that if μ is mixing and $f \circ T = \lambda f$, then either $\lambda = 1$ or $f = 0$.
- (ii) Show that the Lebesgue mesaure λ is ergodic for $T(x, y) = (x + \alpha, x + y) \pmod 1$ iff α is irrational. Show that λ is never mixing.

Example 1.14 Consider a linear transformation on $\mathbb{R}^2$ associated with a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a, b, c, d \in \mathbb{Z}$, then $T(x) = Ax \pmod 1$ defines a transformation on the 2-dimensional torus $\mathbb{T}^2$. Here we are using the fact that if $x = y \pmod 1$, then $Ax = Ay \pmod 1$. If we assume $\det A = 1$, then the Lebesgue measure λ on $\mathbb{T}^2$ is invariant for the transformation T. To have λ mixing, we need to assume that the eigenvalues of T are real and different from 1 and -1. Let us assume that A has eigenvalues α and α^{-1} with $\alpha \in \mathbb{R}$ and $|\alpha| < 1$. By Lemma 5.9, λ is mixing if we can show that for every $n, m \in \mathbb{Z}^2$,

$$\lim_{N\to\infty} \int (\varphi_n \circ T^N)\varphi_m d\lambda = \int \varphi_n d\lambda \int \varphi_m d\lambda \tag{1.12}$$

where $\varphi_n(x) = \exp(2\pi i n \cdot x)$. If $n = 0$, then (1.12) is obvious. If $n \neq 0$, then the right-hand side of (1.12) is zero. We now establish (1.12) for $n \neq 0$ by showing that the left-hand side is zero for sufficiently large N.

Clearly

$$\int \varphi_n \circ T^N \varphi_m d\lambda = \int \varphi_{(A^T)^N n+m}\, d\lambda, \tag{1.13}$$

where A^T denotes the transpose of A. To show that (1.13) is zero for large N, it suffices to show that $(A^T)^N n + m \neq 0$ for large N. For this, it suffices to show that $\lim_{N\to\infty}(A^T)^N n =$

∞. This is certainly true unless n is an eigenvector associated with the eigenvalue α, i.e., $A^T n = \alpha n$. Such an eigenvector can not exist because $\alpha^l n = (A^T)^l n$ would be an integer for all $l \in \mathbb{N}$, which is impossible by $0 < |\alpha| < 1$. $\square$

We end this section with some comments on the ergodicity of continuous dynamical system.

Given a flow $\{\phi_t : t \in \mathbb{R}\}$, let us define

$$\mathcal{I}_\phi = \left\{ \mu : \int f \circ \phi_t d\mu = \int f d\mu \quad \forall\ (f,t) \in C_b(X) \times \mathbb{R} \right\}.$$

Given $\mu \in \mathcal{I}_\phi$ and $f \in L^1(\mu)$, we would like to show

$$\mu \left\{ x : \lim_{t\to\infty} \frac{1}{t} \int_0^t f \circ \phi_\theta(x)\ d\theta \text{ exists} \right\} = 1.$$

To reduce this to the discrete case, let us define $\Omega = \prod_{j\in\mathbb{Z}} \mathbb{R}$ and $\Gamma : X \to \Omega$ by

$$\Gamma(x) = (\omega_j(x) : j \in \mathbb{Z}) = \left(\int_j^{j+1} f \circ \phi_\theta(x)\ d\theta : j \in \mathbb{Z} \right).$$

We then define $T(\omega_j : j \in \mathbb{Z}) = (\omega_{j+1} : j \in \mathbb{Z})$. Clearly $\Gamma \circ \phi_1 = T \circ \Gamma$. Also, if $\mu \in \mathcal{I}_\phi$, then $\tilde{\mu}$ defined by $\tilde{\mu}(A) = \mu(\Gamma^{-1}(A))$ belongs to $\mathcal{I}_T$. Indeed,

$$\begin{aligned} \int g \circ T\ d\tilde{\mu} &= \int g \circ T \circ \Gamma\ d\mu \\ &= \int g \circ \Gamma \circ \phi_1\ d\mu = \int g \circ \Gamma d\ \mu = \int g\ d\tilde{\mu}. \end{aligned}$$

We now apply Theorem 1.6 to assert

$$\tilde{\mu} \left\{ \omega : \lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} \omega_j \text{ exists} \right\} = 1.$$

Hence

$$\mu \left\{ x : \lim_{n\to\infty} \frac{1}{n} \int_0^n f \circ \phi_\theta(x)\ d\theta \text{ exists} \right\} = 1.$$

From this, it is straightforward to deduce

$$\mu \left\{ x : \lim_{t\to\infty} \frac{1}{t} \int_0^t f \circ \phi_\theta\ d\theta \text{ exists} \right\} = 1.$$

To see this, observe

$$\frac{1}{t}\int_0^t f\circ\phi_\theta\ d\theta = \frac{[t]}{t}\frac{1}{[t]}\int_0^{[t]} f\circ\phi_\theta\ d\theta + \frac{1}{t}\int_{[t]}^t f\circ\phi_\theta\ d\theta.$$

Hence it suffices to show

$$\lim_{n\to\infty}\frac{1}{n}\int_n^{n+1}|f\circ\phi_\theta|\ d\theta = 0 \quad \mu-\text{a.e.} \tag{1.14}$$

To prove this, observe

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n |f\circ\phi_\theta|\ d\theta \text{ exists } \mu-\text{a.e.}$$

and this implies

$$\begin{aligned}\frac{1}{n}\int_n^{n+1}|f\circ\phi_\theta|\ d\theta &= \frac{1}{n}\int_0^{n+1}|f\circ\phi_\theta|\ d\theta - \frac{1}{n}\int_0^n|f\circ\phi_\theta|\ d\theta\\ &= \frac{n+1}{n}\frac{1}{n+1}\int_0^{n+1}|f\circ\phi_\theta|\ d\theta - \frac{1}{n}\int_0^n|f\circ\phi_\theta|\ d\theta\end{aligned}$$

converges to 0 $\mu-$a.e., proving (1.14).

As before we can readily show that if $\frac{1}{t}\int_0^t f\circ\phi_\theta\ d\theta \to Pf$, then $Pf\circ\phi_t = Pf$ $\mu-$a.e. for every t, and that Pf is the projection of f onto the invariant sets. In particular, if μ is ergodic, then $Pf \equiv \int f d\mu$.

2 Transfer Operator, Liouville Equation

In the previous section we encountered several examples of dynamical systems for which it was rather easy to find "nice" ergodic invariant measures. We also observed in the case of expanding map that the space of invariant measures is rather complex. One may say that the Lebesgue measure is the "nicest" invariant measure for an expanding map. Later in Section 3, we show how the Lebesgue measure stands out as the unique invariant measure of maximum entropy.

In general, it is not easy to find some natural invariant measure for our dynamical system. For example, if we have a system on a manifold with a Riemannian structure with a volume form, we may wonder whether or not such a system has an invariant measure that is absolutely continuous with respect to the volume form. To address and study these sorts of questions in a systematic fashion, let us introduce an operator on measures that would give the evolutions of measures with respect to our dynamical system. This operator is simply the dual of the operator $Uf = f \circ T$. More precisely, define $\mathcal{A} : \mathcal{M}(x) \to \mathcal{M}(x)$ by

$$\int Uf \; d\mu = \int f \circ T \; d\mu = \int f \; d\mathcal{A}\mu$$

for every $f \in C_b(X)$. We certainly have

$$(\mathcal{A}\mu)(A) = \mu(T^{-1}(A)) \tag{2.1}$$

for every measurable A. Even though we have some general results regarding the spectrum of U, the corresponding questions for the operator $\mathcal{A}$ are far more complex. We can now cast the existence of an invariant measure with some properties as the existence of a fixed point of $\mathcal{A}$ with those properties. The operator $\mathcal{A}$ is called *Perron–Frobenious*, *Perron–Frobenious–Ruelle* or *Transfer Operator*, once an expression for it is derived when μ is absolutely continuous with respect to the volume form. We note that an invariant measure μ is mixing iff $\mathcal{A}^n\nu$ converges to μ in high n limit, for every $\nu << \mu$. To get a feel for the operator $\mathcal{A}$, let us examine some examples.

Example 2.1

(i) $T : \mathbb{T}^d \to \mathbb{T}^d$, $T(x) = x + \alpha \pmod 1$. The operator $\mathcal{A}$ simply translates a measure for the amount α. We assume that the numbers $\alpha_1 \dots \alpha_d$, and 1 are rationally independent. We can study the asymptotic behavior of $\mathcal{A}^n\mu$ for a given μ. The sequence $\{\mathcal{A}^n\mu\}$ does not converge to any limit as $n \to \infty$. In facr the set of limit points of the sequence $\{\mathcal{A}^n\mu\}$ consits of all translates of μ. However

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{A}^j\mu = \lambda \tag{2.2}$$

where λ denotes the Lebesgue measure. The proof of (2.1) follows from the unique ergodicity of T that implies

$$\Phi_n(f) \to \int f d\lambda$$

uniformly for every continuous f. This implies

$$\lim_{n\to\infty} \int \Phi_n(f) d\mu = \lim_{n\to\infty} \int f d\left(\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{A}^j \mu\right) = \int f d\lambda,$$

proving (2.2).

(ii) Let (X, d) be a complete metric space and suppose $T : X \to X$ is a *contraction*. In other words, there exists a constant $\alpha \in (0, 1)$ such that $d(T(x), T(y)) \leq \alpha d(x, y)$. In this case T has a unique fix point $\bar{x}$ and $\lim_{n\to+\infty} T^n(x) = \bar{x}$ for every x (the convergence is locally uniform). As a consequence we learn that $\lim_{n\to\infty} \mathcal{A}^n\mu = \delta_{\bar{x}}$ for every measure $\mu \in \mathcal{M}(X)$. For example, if $X = \mathbb{R}$ and $T(x) = \alpha x$ with $\alpha \in (0, 1)$, then $d\mu = \rho dx$ results in a sequence $\mathcal{A}^n\mu = \rho_n dx$ with

$$\rho_n(x) = \alpha^{-n}\rho\left(\frac{x}{\alpha^n}\right).$$

In other words, the measure μ under $\mathcal{A}$ becomes more concentrated about the origin.

(iii) Let $T : \mathbb{T} \to \mathbb{T}$ be the expansion $T(x) = 2x (\mathrm{mod}\, 1)$. If $d\mu = \rho dx$ and $\mathcal{A}^n\mu = \rho_n dx$, then $\rho_1(x) = \frac{1}{2}\left(\rho\left(\frac{x}{2}\right) + \rho\left(\frac{x+1}{2}\right)\right)$ and

$$\rho_n(x) = \frac{1}{2^n}\sum_{j=0}^{2^n-1} \rho\left(\frac{x}{2^n} + \frac{j}{2^n}\right).$$

From this, it is clear that if ρ is continuous, then $\lim_{n\to\infty} \rho_n(x) \equiv 1$. Indeed

$$\begin{aligned}\lim_{n\to\infty}\left|\rho_n(x) - \frac{1}{2^n}\sum_{j=0}^{2^n-1}\rho\left(\frac{j}{2^n}\right)\right| &= \lim_{n\to\infty}\left|\frac{1}{2^n}\sum_{j=0}^{2^n-1}\left(\rho\left(\frac{x}{2^n}+\frac{j}{2^n}\right) - \rho\left(\frac{j}{2^n}\right)\right)\right| \\ &\quad + \lim_{n\to\infty}\frac{1}{2^n}\sum_{j=0}^{2^n-1}\rho\left(\frac{j}{2^n}\right) = \int \rho dx = 1.\end{aligned}$$

This can also be seen by looking at the Fourier expansion of ρ. We now only need to assume that $\rho \in L^2[0, 1]$. If

$$\rho(x) = \sum_n a_n e^{2\pi i n x},$$

then $a_0 = 1$ and

$$\rho_1(x) = \sum_k a_{2k} e^{2\pi i k x},$$

and by induction,

$$\rho_n(x) = \sum_k a_{2^n k} e^{2\pi i k x}.$$

As a result,

$$\int_0^1 |\rho_n(x) - 1|^2 dx = \sum_{k \neq 0} a_{2^n k}^2 \to 0.$$

□

There is a couple of things to learn from Example 2.1. First, when there is a contraction, the operator $\mathcal{A}$ makes measures more concentrated in small regions. Second, if there is an expansion then $\mathcal{A}$ has some smoothing effect. In hyperbolic systems we have both expansion and contraction. In some sense, if we have more contraction than the expansion, then it is plausible that there is a fractal set that attracts the orbits as $n \to \infty$. If this happens, then there exists no invariant measure that is absolutely continuous with respect to the volume measure. Later in this section, we will see an example of such phenomenon. As a result, to have an absolutely continuous invariant measure, we need to make sure that, in some sense, the expansion rates and the contraction rates are balanced out. Let us first derive a formula for $\mathcal{A}\mu$ when μ is absolutely continuous with respect to a volume form. As a warm up, first consider a transformation $T : \mathbb{T}^d \to \mathbb{T}^d$ that is smooth. We also assume that T is invertible with a smooth inverse, i.e., T is a diffeomorphism. We then consider $d\mu = \rho dx$. We have

$$\int_{\mathbb{T}^d} f \circ T \rho \; dx = \int_{\mathbb{T}^d} f \rho \circ T^{-1} |JT^{-1}| \; dy$$

where $JT^{-1} = \det DT^{-1}$. As a result, if $\mathcal{A}\mu = \hat{\rho} dx$, then $\hat{\rho} = |JT^{-1}| \rho \circ T^{-1} = \frac{\rho \circ T^{-1}}{|JT \circ T^{-1}|}$. We abuse the notation to write

$$\mathcal{A}\rho = \frac{\rho \circ T^{-1}}{|JT \circ T^{-1}|}, \tag{2.3}$$

regarding $\mathcal{A}$ as an operator acting on probability density functions. More generally, assume that X is a smooth manifold and T is C^∞. Let ω be a volume form (nondegenerate d-form where d is the dimension of X). Then $T^*\omega$, the pull-back of ω under T, is also a k-form and we define $JT(x)$ to be the unique number such that $T^*\omega_x = JT(x)\omega_{T(x)}$. More precisely, $T^*\omega_x(v_1 \dots v_k) = \omega_{T(x)}(DT(x)v_1, \dots, DT(x)v_k) = JT(x)\omega_{T(x)}(v_1 \dots v_k)$. We then have

$$\int_X (f \circ T)\rho \; \omega = \int_X f(\rho \circ T^{-1})|JT^{-1}| \; \omega.$$

Hence (2.3) holds in general.

If T is not invertible, one can show

$$\mathcal{A}\rho = \sum_{y \in T^{-1}(\{x\})} \frac{\rho(y)}{|JT(y)|}. \tag{2.4}$$

The next proposition demonstrates how the existence of an absolutely continuous invariant measure forces a bound on the Jacobians.

Proposition 2.2 *Let X be a smooth manifold with a volume form ω. Let $T : X \to X$ be a diffeomorphism with $JT > 0$. The following statements are equivalent:*

(i) *There exists $\mu = \rho\omega \in \mathcal{I}_T$ for a bounded uniformly positive ρ.*

(ii) *The set $\{JT^n(x) : x \in X,\ n \in \mathbb{Z}\}$ is uniformly bounded.*

Proof. **(i) $\Rightarrow$ (ii)** Observe

$$\mathcal{A}^2\rho = \frac{\rho \circ T^{-2}}{JT \circ T^{-2}} \frac{1}{JT \circ T^{-1}} = \frac{\rho \circ T^2}{JT^2 \circ T^{-2}}$$

because $JT^2 = (JT \circ T)JT$. By induction,

$$\mathcal{A}^n\rho = \frac{\rho \circ T^{-n}}{JT^n \circ T^{-n}},\ \ n \in \mathbb{N}.$$

Also, $\mathcal{A}^{-1}\rho = (\rho \circ T)JT$, and by induction

$$\begin{aligned} \mathcal{A}^{-n}\rho &= (\rho \circ T^n)JT^n \\ &= (\rho \circ T^n)JT^{-n} \circ T^n;\quad n \in \mathbb{N}. \end{aligned}$$

Hence

$$\mathcal{A}^n\rho = \frac{\rho \circ T^{-n}}{JT^n \circ T^{-n}};\quad n \in \mathbb{Z}. \tag{2.5}$$

If $\rho\omega$ is invariant, then $\mathcal{A}^n\rho = \rho$ for all $n \in \mathbb{Z}$. As a result, $(JT^n \circ T^{-n})\rho = \rho \circ T^{-n}$, or

$$JT^n = \frac{\rho}{\rho \circ T^n};\quad n \in \mathbb{Z}. \tag{2.6}$$

Now it is clear that if ρ is bounded and uniformly positive, then $\{JT^n(x) : n \in \mathbb{Z},\ x \in X\}$ is uniformly bounded.

(ii) $\Rightarrow$ **(i)** Suppose $\{JT^n(x) : n \in \mathbb{Z} \text{ and } x \in X\}$ is bounded and define

$$\rho(x) = \sup_{n \in \mathbb{Z}} JT^n(x).$$

We then have

$$\begin{aligned} JT(x)(\rho \circ T)(x) &= \sup_{n \in \mathbb{Z}} (JT^n) \circ T(x) JT(x) \\ &= \sup_{n \in \mathbb{Z}} J(T^n \circ T)(x) = \rho(x). \end{aligned}$$

Hence $\mathcal{A}\rho = \rho$. Evidently ρ is bounded. Moreover

$$1/\rho = \inf_n [1/JT^n(x)] = \inf_n JT^{-n} \circ T^n = \inf_n JT^n \circ T^{-n}$$

is uniformly bounded by assumption. $\square$

Recall that expansions are harmless and have smoothing effect on $\mathcal{A}\rho$. As a test case, let us consider an expansion of $[0,1]$ given by

$$T(x) = \begin{cases} T_1(x) & x \in [0, \theta_0) = I_1 \\ T_2(x) & x \in [\theta_0, 1] = I_2 \end{cases}$$

with T_1, T_2 smooth functions satisfying $|T_i'(x)| \geq \lambda$ for $x \in I_i$. We assume $\lambda > 1$ and that $T_i(I_i) = [0,1]$.

In this case

$$\mathcal{A}\rho(x) = \frac{\rho_1 \circ T_1^{-1}(x)}{T_1' \circ T_1^{-1}(x)} + \frac{\rho \circ T_2^{-1}(x)}{T_2' \circ T_2^{-1}(x)}. \tag{2.7}$$

Theorem 2.3 *If $T_1, T_2 \in C^2$, then there exists $\mu \in \mathcal{I}_T$ of the form $d\mu = \rho dx$ with ρ of finite variation.*

Proof. Write $S_i = T_i^{-1}$ so that

$$\mathcal{A}\rho = (\rho \circ S_1)S_1' + (\rho \circ S_2)S_2'.$$

We have

$$\begin{aligned}\int_0^1 |(\mathcal{A}\rho)'|dx &\le \lambda^{-1}\int_0^1 \mathcal{A}|\rho'|\ dx \\ &\quad +\beta_0 \int_0^1 \mathcal{A}\rho\ dx,\end{aligned}$$

where $\beta_0 = \max_{x,i\in\{1,2\}} \frac{|S_i''(x)|}{S_i'(x)}$ and here we used $S_i' \le \frac{1}{\lambda}$. Hence

$$\int_0^1 |(\mathcal{A}\rho)'|dx \le \lambda^{-1}\int_0^1 |\rho'|dx + \beta_0.$$

By induction,

$$\int_0^1 |(\mathcal{A}^n\rho)'|dx \le \lambda^{-n}\int_0^1 |\rho'|dx + \beta_0 \frac{1-\lambda^{-n}}{1-\lambda^{-1}}.$$

From this we learn that

$$\sup_n \|\mathcal{A}^n\rho\|_{BV} < \infty.$$

Hence $\mathcal{A}^n\rho$ has convergent subsequences in $L^1[0,1]$. But a limit point may not be an invariant density. To avoid this, let us observe that we also have

$$\sup_n \left\| \frac{1}{n}\sum_0^{n-1} \mathcal{A}^j\rho \right\|_{BV} < \infty.$$

Hence the sequence $\left\{\rho_n = n^{-1}\sum_0^{n-1} \mathcal{A}^j\rho\right\}_n$ has convergent subsequences by Helley Selection Theorem. If $\bar\rho$ is a limit point, then $\mathcal{A}\bar\rho = \bar\rho$ because for every $J \in C([0,1])$,

$$\begin{aligned}\int (J\circ T)\bar\rho\ dx &= \lim_{n\to\infty}\int (J\circ T)\rho_n\ dx \\ &= \lim_{n\to\infty}\int J\mathcal{A}\rho_n\ dx \\ &= \lim_{n\to\infty}\int J\rho_n\ dx = \int J\bar\rho\ dx.\end{aligned}$$

Also, for every periodic $J \in C^1$,

$$\begin{aligned}\left|\int_0^1 J'\bar\rho\ dx\right| &= \lim_{n\to\infty}\left|\int_0^1 J'\rho_n\ dx\right| \le \|J\|_{L^\infty}\sup_n \|\rho_n\|_{BV} \\ &\le \text{const. } \|J\|_{L^\infty}.\end{aligned}$$

Hence $\bar{\rho} \in BV$. $\square$

We now discuss another approach which yields the convergence of $\mathcal{A}^n\rho$. To find a fixed point of $\mathcal{A}$, let us consider the following function space:

$$\mathcal{C}_a = \{e^g : |g(x) - g(y)| \leq a|x-y| \text{ for } x, y \in [0,1]\}. \tag{2.8}$$

We note that $\rho \in \mathcal{C}_a \cup \{0\}$, iff $\rho \geq 0$ and for all $x, y \in [0,1]$,

$$\rho(x) \leq \rho(y)e^{a|x-y|}.$$

Recall that $S_i = T_i^{-1}$ and $\beta_0 = \max_{x,i\in\{1,2\}} \frac{|S_i''(x)|}{S_i'(x)}$.

Lemma 2.4 *We have that $\mathcal{A}\mathcal{C}_a \subseteq \mathcal{C}_{a\sigma}$, whenever $a > \frac{\beta_0}{\sigma - \lambda^{-1}}$ and $\sigma > \lambda^{-1}$.*

Proof. Let $\rho = e^g \in \mathcal{C}_a$. Then

$$\begin{aligned}
\mathcal{A}\rho(x) &= \sum_{i=1}^{2} \rho \circ S_i(x) S_i'(x) \\
&\leq \sum_{i=1}^{2} \rho \circ S_i(y) e^{a|S_i(x) - S_i(y)|} |S_i'(x)| \\
&= \sum_{i=1}^{2} \rho \circ S_i(y) e^{a|S_i(x) - S_i(y)|} |S_i'(y)| e^{\log|S_i'(x)| - \log|S_i'(y)|} \\
&\leq \sum_{i=1}^{2} \rho \circ S_i(y) |S_i'(y)| e^{a\lambda^{-1}|x-y|} e^{\beta_0|x-y|} \\
&= \mathcal{A}\rho(y) e^{(a\lambda^{-1} + \beta_0)|x-y|}.
\end{aligned}$$

As a result, $\mathcal{A}\mathcal{C}_a \subseteq \mathcal{C}_{a\lambda^{-1}+\beta_0} \subseteq \mathcal{C}_{\sigma a}$. $\square$

What we learn from Lemma 6.3 is that if $\sigma \in (\lambda^{-1}, 1]$, then we can find a function space $\mathcal{C}_a$ that is mapped into itself by $\mathcal{A}$. Note that indeed $\mathcal{C}_a$ is a cone in the sense that

$$\begin{cases} \text{if } \rho \in \mathcal{C}_a, & \text{then } \lambda\rho \in \mathcal{C}_a \text{ for } \lambda > 0, \\ \text{if } \rho_1, \rho_2 \in \mathcal{C}_a, & \text{then } \rho_1 + \rho_2 \in \mathcal{C}_a. \end{cases}$$

Define a partial order

$$\rho_1 \preccurlyeq \rho_2 \text{ iff } \rho_2 - \rho_1 \in \mathcal{C}_a \cup \{0\}. \tag{2.9}$$

In other words, $\rho_1 \preccurlyeq \rho_2$ iff $\rho_1 \leq \rho_2$ and

$$\rho_2(x) - \rho_1(x) \leq (\rho_2(y) - \rho_1(y))e^{a|x-y|}, \; x, y \in [0,1]. \tag{2.10}$$

Hilbert metric associated with our cone $\mathcal{C}_a$ is defined as

$$d_a(\rho_1, \rho_2) = \log(\beta_a(\rho_1, \rho_2)\beta_a(\rho_2, \rho_1)), \tag{2.11}$$

where $\beta_a(\rho_1, \rho_2) = \inf\{\lambda \geq 0 : \rho_2 \preccurlyeq \lambda\rho_1\}$. By convention, $\beta_a(\rho_1, \rho_2) = \infty$ if there exists no such λ. We certainly have

$$d_a(\rho_1, \rho_2) = \sup_\alpha \inf_\beta \left\{ \log \frac{\beta}{\alpha} : \alpha\rho_1 \preccurlyeq \rho_2 \preccurlyeq \beta\rho_1 \right\} \geq 0. \tag{2.12}$$

Lemma 2.5 $\beta_a(\rho_1, \rho_2) = \sup\limits_{\substack{x,y \\ x\neq y}} \dfrac{e^{a|x-y|}\rho_2(y) - \rho_2(x)}{e^{a|x-y|}\rho_1(y) - \rho_1(x)} \geq \sup\limits_x \dfrac{\rho_2(x)}{\rho_1(x)}.$

Proof. If $\rho_2 \preccurlyeq \lambda\rho_1$, then $\rho_2 \leq \lambda\rho_1$ and

$$\begin{aligned} -\rho_2(x) + \lambda\rho_1(x) &\leq e^{a|x-y|}(-\rho_2(y) + \lambda\rho_1(y)), \\ -\rho_2(x) + e^{a|x-y|}\rho_2(y) &\leq \lambda(-\rho_1(x) + e^{a|x-y|}\rho_1(y)). \end{aligned}$$

From this we deduce

$$\beta_a(\rho_1, \rho_2) = \max\left\{ \sup_x \frac{\rho_2(x)}{\rho_1(x)}, \; \sup_{x\neq y} \frac{e^{a|x-y|}\rho_2(y) - \rho_2(x)}{e^{a|x-y|}\rho_1(y) - \rho_1(x)} \right\}.$$

Note that if $\sup\limits_x \dfrac{\rho_2(x)}{\rho_1(x)} = \dfrac{\rho_2(\bar{x})}{\rho_1(\bar{x})}$, then

$$\frac{e^{a|x-\bar{x}|}\rho_2(\bar{x}) - \rho_2(x)}{e^{a|x-\bar{x}|}\rho_1(\bar{x}) - \rho_1(x)} = \frac{e^{a|x-\bar{x}|}\rho_1(\bar{x})\frac{\rho_2(\bar{x})}{\rho_1(\bar{x})} - \rho_1(x)\frac{\rho_2(x)}{\rho_1(x)}}{e^{a|x-\bar{x}|}\rho_1(\bar{x}) - \rho_1(x)} \geq \frac{\rho_2(x)}{\rho_1(x)}.$$

This completes the proof of lemma. □

Lemma 2.6

(i) *d_a is a quasimetric with $d_a(\rho_1, \rho_2) = 0$ iff $\rho_1 = \lambda\rho_2$ for some $\lambda > 0$.*

(ii) *If $a_1 \leq a_2$ then $d_{a_1}(\rho_1, \rho_2) \geq d_{a_2}(\rho_1, \rho_2)$ for $\rho_1, \rho_2 \in \mathcal{C}_{a_1}$.*

Proof. **(i)** If $d_a(\rho_1,\rho_2)=1$, and $\beta_a(\rho_1,\rho_2)=\lambda$, then $\beta_a(\rho_2,\rho_1)=\lambda^{-1}$. Hence $\rho_2 \preccurlyeq \lambda\rho_1 \preccurlyeq \rho_2$ which implies that $\lambda\rho_1=\rho_2$. The triangle inequality is a consequence of $\beta_a(\rho_1,\rho_2)\beta_a(\rho_2,\rho_3)\le \beta_a(\rho_1,\rho_3)$. This is a consequence of the fact that if $\rho_2 \preccurlyeq \lambda_1\rho_1$ and $\rho_3 \preccurlyeq \lambda_2\rho_2$, then $\rho_3 \preccurlyeq \lambda_1\lambda_2\rho_1$.

(ii) First observe $\mathcal{C}_{a_1}\subseteq \mathcal{C}_{a_2}$. Hence $\rho_2 \preccurlyeq \lambda\rho_1$ in $\mathcal{C}_{a_1}$ implies the same inequality in $\mathcal{C}_{a_2}$. □

Recall that we are searching for a fixed point for the operator $\mathcal{A}$. By Lemma 6.3, if $\sigma\in(\lambda^{-1},1)$ and $a>\frac{\beta_0}{\sigma-\lambda^{-1}}$, then $\mathcal{A}(\mathcal{C}_a)\subseteq\mathcal{C}_{a\sigma}\subseteq\mathcal{C}_a$. As our next step,, we show that $\mathcal{A}$ is a contraction on $\mathcal{C}_a$. But first let us demonstrate that in fact that the set $\mathcal{C}_{a\sigma}$ is a bounded subset of $\mathcal{C}_a$.

Lemma 2.7 diam $\mathcal{C}_{a\sigma}=\sup_{\rho_1,\rho_2\in\mathcal{C}_{a\sigma}} d_a(\rho_1,\rho_2)\le b:=2\log\dfrac{1+\sigma}{1-\sigma}+2a\sigma.$

Proof. From $\rho_2(x)\le\rho_2(y)e^{-a\sigma|x-y|}$ and $\rho_1(x)\le\rho_1(y)e^{a\sigma|x-y|}$ we deduce

$$\beta_a(\rho_1,\rho_2)\le\sup_{x,y}\frac{e^{a|x-y|}-e^{-a\sigma|x-y|}}{e^{a|x-y|}-e^{a\sigma|x-y|}}\frac{\rho_2(y)}{\rho_1(y)}.$$

To calculate this, set $z=e^{a|x-y|}$. Then $z\ge1$ and $\lim_{z\to1}\frac{z-z^{-\sigma}}{z-z^{\sigma}}=\frac{1+\sigma}{1-\sigma}$. On the other hand, $\frac{z-z^{-\sigma}}{z-z^{\sigma}}\le\frac{1+\sigma}{1-\sigma}$ or equivalently $z^\sigma\le\frac{2\sigma}{1+\sigma}z+\frac{1-\sigma}{1+\sigma}z^{-\sigma}$ which is the consequence of the convexity of the exponential function;

$$e^{\sigma\log z}\le\frac{2\sigma}{1+\sigma}e^{\log z}+\frac{1-\sigma}{1+\sigma}e^{-\sigma\log z}.$$

As a result,

$$\beta_a(\rho_1,\rho_2)\le\frac{1+\sigma}{1-\sigma}\sup_y\frac{\rho_2(y)}{\rho_1(y)}\le\frac{1+\sigma}{1-\sigma}\frac{\rho_2(y_0)e^{a\sigma/2}}{\rho_1(y_0)e^{-a\sigma/2}}=\frac{\rho_2(y_0)}{\rho_1(y_0)}e^{a\sigma}\frac{1+\sigma}{1-\sigma}$$

for $y_0=\frac12$. Hence

$$\beta_a(\rho_1,\rho_2)\beta_a(\rho_2,\rho_1)\le\left(\frac{1+\sigma}{1-\sigma}\right)^2e^{2a\sigma},$$

completing the proof of lemma. □

We are now ready to show that $\mathcal{A}$ is a contraction.

Lemma 2.8 *For every* $\rho_1,\rho_2\in\mathcal{C}_a$,

$$d_a(\mathcal{A}\rho_1,\mathcal{A}\rho_2)\le\tanh\left(\frac{b}{4}\right)d_a(\rho_1,\rho_2).$$

Proof. By Lemma 6.7, $\operatorname{diam}(\mathcal{A}\mathcal{C}_a) \le b$. As a consequence if $\beta\rho_1 \succcurlyeq \rho_2 \succcurlyeq \alpha\rho_1$, then

$$d_a(\mathcal{A}(\rho_2 - \alpha\rho_1),\ \mathcal{A}(\beta\rho_1 - \rho_2)) \le b$$

for every $\rho_1, \rho_2 \in \mathcal{C}_a$ and $\alpha, \beta \ge 0$. This means that we can find two constants $\lambda_1, \lambda_2 \ge 0$ such that $\log \frac{\lambda_1}{\lambda_2} \le b$ and

$$\frac{\beta + \alpha\lambda_1}{1 + \lambda_1}\mathcal{A}\rho_1 \preccurlyeq \mathcal{A}\rho_2 \preccurlyeq \frac{\beta + \alpha\lambda_2}{1 + \lambda_2}\mathcal{A}\rho_1.$$

As a result,

$$d_a(\mathcal{A}\rho_1, \mathcal{A}\rho_2) \le \log \frac{\beta + \alpha\lambda_1}{1 + \lambda_1}\frac{1 + \lambda_2}{\beta + \alpha\lambda_2} = \log \frac{\frac{\beta}{\alpha} + \lambda_1}{\frac{\beta}{\alpha} + \lambda_2} + \log \frac{1 + \lambda_2}{1 + \lambda_1}.$$

Minimizing over α and β yields

$$\begin{aligned} d_a(\mathcal{A}\rho_1, \mathcal{A}\rho_2) &\le \log \frac{e^{d_a(\rho_1,\rho_2)} + \lambda_1}{e^{d_a(\rho_1,\rho_2)} + \lambda_2} + \log \frac{1 + \lambda_2}{1 + \lambda_1} \\ &= \int_0^{d_a(\rho_1,\rho_2)} \frac{e^\theta(\lambda_2 - \lambda_1)}{(e^\theta + \lambda_1)(e^\theta + \lambda_2)} d\theta \le d_a(\rho_1, \rho_2)\frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}} \end{aligned}$$

because $\max\limits_{x \ge 1} \dfrac{x(\lambda_2 - \lambda_1)}{(x + \lambda_1)(x + \lambda_2)} = \dfrac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}}$. We now maximize over $\frac{\lambda_2}{\lambda_1}$ to obtain

$$d_a(\mathcal{A}\rho_1, \mathcal{A}\rho_2) \le d_a(\rho_1, \rho_2)\frac{e^{\frac{1}{2}b} - 1}{e^{\frac{1}{2}b} + 1} = d_a(\rho_1, \rho_2)\tanh\left(\frac{b}{4}\right).$$

□

This evidently gives us a contraction on $\mathcal{C}_a$ for any $a \ge \frac{\beta_0}{\sigma - \lambda^{-1}}$ provided that $\sigma \in (\lambda^{-1}, 1)$, because $\tanh\left(\frac{b}{4}\right) < 1$ always. We may minimize the rate of contraction $\tanh\left(\frac{b}{4}\right)$ by first choosing the best a, namely $a = \frac{\beta_0}{\sigma - \lambda^{-1}}$, and then minimizing b in σ as σ varies in $(\lambda^{-1}, 1)$. Our goal is to show that $\lim_{n\to\infty} \mathcal{A}^n\rho$ converges to a unique invariant density $\bar{\rho}$. For this, let us establish an inequality connecting $d_a(\rho_1, \rho_2)$ to $\|\rho_1 - \rho_2\|_{L^1}$.

Lemma 2.9 *For every $\rho_1, \rho_2 \in \mathcal{C}_a$, with* $\displaystyle\int_0^1 \rho_1\ dx = \int_0^1 \rho_2\ dx = 1$, we have

$$\int_0^1 |\rho_1 - \rho_2|\ dx \le (e^{d_a(\rho_1,\rho_2)} - 1), \quad |\rho_1 - \rho_2| \le (e^{d_a(\rho_1,\rho_2)} - 1)\rho_1.$$

Proof. Let us write $d_a(\rho_1, \rho_2) = \log \frac{\beta}{\alpha}$ with $\alpha\rho_1 \preccurlyeq \rho_2 \preccurlyeq \beta\rho_1$. This in particular implies that $\alpha\rho_1 \leq \rho_2 \leq \beta\rho_1$. Integrating this over $[0,1]$ yields $\alpha \leq 1 \leq \beta$. As a result,

$$\begin{array}{rcl} \rho_2 - \rho_1 & \preccurlyeq & (\beta - 1)\rho_1 \preccurlyeq (\beta - \alpha)\rho_1, \\ \rho_2 - \rho_1 & \succcurlyeq & (\alpha - 1)\rho_1 \succcurlyeq (\alpha - \beta)\rho_1. \end{array}$$

From this we deduce $(\alpha - \beta)\rho_1 \leq \rho_1 - \rho_1 \leq (\beta - \alpha)\rho_1$. As a result, $|\rho_1 - \rho_2| \leq (\beta - \alpha)\rho_1 \leq (\beta/\alpha - 1)\rho_1$ and

$$\int_0^1 |\rho_2 - \rho_1| \; dx \leq (\beta - \alpha) \leq \frac{\beta - \alpha}{\alpha} = \frac{\beta}{\alpha} - 1 = e^{d_a(\rho_1, \rho_2)} - 1.$$

□

We are now ready to state and prove the first main result of this section.

Theorem 2.10 *Let $a = \frac{\beta_0}{\sigma - \lambda^{-1}}$ and $\sigma \in (\lambda^{-1}, 1)$. Then for every $\rho \in \mathcal{C}_a$ with $\int_0^1 \rho = 1$, $\lim_{n\to\infty} \mathcal{A}^n \rho = \bar{\rho}$ exists uniformly and $\bar{\rho}\; dx \in \mathcal{I}_T$ with $\bar{\rho} \in \mathcal{C}_{a\sigma}$. Moreover, there exists a constant $\bar{c}_1$ such that*

$$\left| \int_0^1 f \circ T^n \; g dx - \int_0^1 g dx \int_0^1 f \bar{\rho} dx \right| \leq \bar{c}_1 \hat{\lambda}^n \|f\|_{L^1} (\|g\|_{L^1} + \|g'\|_{L^\infty}) \tag{2.13}$$

where $\hat{\lambda} = \tanh\left(\frac{b}{4}\right)$, $b = 2\log\frac{1+\sigma}{1-\sigma} + 2a\sigma$, $f \in L^1$, and g is Lipschitz.

An immediate consequence of Theorem 2.10 is the mixing property of $\bar{\rho}$ because we may choose $g = h\bar{\rho} / \int h\bar{\rho}$ to deduce

$$\lim_{n\to\infty} \int_0^1 f \circ T^n \; h\bar{\rho} \; dx = \int_0^1 f\bar{\rho} dx \int_0^1 h\bar{\rho} dx.$$

Proof of Theorem 2.10. We first show that if $\rho \in \mathcal{C}_a$, then $\mathcal{A}^n\rho$ converges to a function $\bar{\rho} \in \mathcal{C}_a$ in L^1-sense. Indeed

$$\begin{array}{rcl} \|\mathcal{A}^{n+m}\rho - \mathcal{A}^n\rho\|_{L^1} & \leq & \exp(d_a(\mathcal{A}^{n+m}\rho, \mathcal{A}^n\rho)) - 1 \\ & \leq & \exp(\hat{\lambda}^{n-1} d_a(\mathcal{A}^{m+1}\rho, \mathcal{A}\rho)) - 1 \\ & \leq & e^{\hat{\lambda}^{n-1}b} - 1 \leq \hat{\lambda}^{n-1} b e^{\hat{\lambda}^{n-1}b} \leq c_0 \hat{\lambda}^{n-1} \end{array}$$

for a constant c_0 that depends on b only. This implies that $\mathcal{A}^n\rho$ is Cauchy in L^1. Let $\bar{\rho} = \lim \rho_n$ where $\rho_n = \mathcal{A}^n\rho$. Since $\rho_n(x) \leq \rho_n(y)e^{a\sigma|x-y|}$ and $\rho_{n_k} \to \bar{\rho}$ a.e. for a subsequence, we deduce that $\bar{\rho}(x) \leq \bar{\rho}(y)e^{a\sigma|x-y|}$ for a.e. x and $y \in [0,1]$. By modifying $\bar{\rho}$ on a set of zero

Lebesgue measure if necessary, we deduce that $\bar\rho \in \mathcal{C}_a$. Note that $\bar\rho$ is never zero, because if $\bar\rho(x_0) = 0$ for some x_0, then $\bar\rho(x) \le \bar\rho(x_0)e^{a\sigma|x_0-x|}$ implies that $\bar\rho(x) = 0$ for every x. But $\int_0^1 \rho dx = 1$ implies that $\int_0^1 \bar\rho dx = 1$. So $\bar\rho > 0$, completing the proof of $\bar\rho \in \mathcal{C}_a$.

We now show that $\mathcal{A}^n\rho \to \bar\rho$ uniformly. Indeed from $\mathcal{A}^n\rho \to \bar\rho$ in L^1 we deduce that $\int f \circ T^n \rho dx \to \int f\bar\rho dx$ for every bounded f, which implies that $\mathcal{A}\bar\rho = \bar\rho$. Moreover

$$\begin{aligned} |\mathcal{A}^n\rho - \bar\rho| &= |\mathcal{A}^n\rho - \mathcal{A}^n\bar\rho| \le (e^{d_a(\mathcal{A}^n\rho,\mathcal{A}^n\bar\rho)} - 1)\mathcal{A}^n\bar\rho \\ &\le (e^{\hat\lambda^{n-1}d_a(\mathcal{A}\rho,\mathcal{A}\bar\rho)} - 1)\bar\rho \le (e^{\hat\lambda^{n-1}b} - 1)\bar\rho \\ &\le \hat\lambda^{n-1}be^{\hat\lambda^{n-1}b}\bar\rho \le c_0\hat\lambda^n\bar\rho \end{aligned}$$

with c_0 depending on b only. From this we learn that

$$\|\mathcal{A}^n\rho - \bar\rho\|_{L^\infty} \le c_0\hat\lambda^n\|\bar\rho\|_{L^\infty},$$

for every $\rho \in \mathcal{C}_a$ with $\int_0^1 \rho dx = 1$.

We now turn to the proof of (2.13). Without loss of generality, we may assume that $g \ge 0$. Given such a function g, find $l > 0$ large enough so that $\rho = g + l\bar\rho \in \mathcal{C}_a$. Indeed, for $y > x$, we have that $\rho(y) \le g(y) + l\bar\rho(x)\exp(a\sigma(y-x)) =: \exp(h(y))$. On the other hand

$$h'(y) = \frac{g'(y) + la\sigma\bar\rho(x)e^{a\sigma(y-x)}}{g(y) + l\sigma\bar\rho(x)e^{a\sigma(y-x)}} \le \frac{\|g'\|_{L^\infty}}{l\bar\rho(x)} + \frac{la\sigma\bar\rho(x)e^{a\sigma(y-x)}}{l\bar\rho(x)e^{a\sigma(y-x)}} \le \frac{\|g'\|_{L^\infty}}{\inf\bar\rho}\frac{1}{l} + a\sigma.$$

This is at most a if we choose

$$l = \frac{\|g'\|_{L^\infty}}{(1-\sigma)\inf\bar\rho}.$$

Hence

$$\left\|\mathcal{A}^n\frac{g + l\bar\rho}{Z} - \bar\rho\right\|_{L^\infty} \le c_0\hat\lambda^n\|\bar\rho\|_{L^\infty}$$

where $Z = \int_0^1 (g + l\bar\rho)dx$. Since $\mathcal{A}\bar\rho = \bar\rho$, we deduce

$$\begin{aligned} \left\|\frac{\mathcal{A}^n g}{Z} + \frac{l}{Z}\bar\rho - \bar\rho\right\|_{L^\infty} &\le c_0\hat\lambda^n\|\bar\rho\|_{L^\infty}, \\ \|\mathcal{A}^n g - (Z-l)\bar\rho\|_{L^\infty} &\le c_0\hat\lambda^n\|\bar\rho\|_{L^\infty}Z. \end{aligned}$$

Hence

$$\left\|\mathcal{A}^n g - \bar\rho\int_0^1 g\,dx\right\|_{L^\infty} \le c_1\hat\lambda^n\left[\int g\,dx + l\right] \le c_2\hat\lambda^n\left[\int g\,dx + \|g'\|_{L^\infty}\right].$$

From this, we can readily deduce (2.13). □

As our next scenario, let us study an example of a 2-dimensional system that has expanding and contracting direction but there is no absolutely continuous invariant measure. As a toy model for such a phenomenon, we consider a (generalized) *baker's transformation*:

$$T:\mathbb{T}^2\to\mathbb{T}^2,\ T(x_1,x_2)=\begin{cases}\left(\frac{x_1}{\alpha},\beta x_2\right) & \text{if } 0\le x_1\le\alpha,\\ \left(\frac{x_1-\alpha}{\beta},\beta+\alpha x_2\right) & \text{if } \alpha<x_1\le 1.\end{cases}$$

with $\alpha,\beta>0$ and $\alpha+\beta=1$.

Note

$$|JT(x_1,x_2)|=\begin{cases}\frac{\beta}{\alpha} & \text{if } 0\le x\le\alpha,\\ \frac{\alpha}{\beta} & \text{if } \alpha<x\le 1.\end{cases}$$

As we will see later, the transformation T does not have an absolutely continuous invariant measure unless $\alpha=\beta=\frac{1}{2}$. To analyze Perron–Frobenious operator, let us define $F_\mu(x_1,x_2)=\mu([0,x_1]\times[0,x_2])$. If $\hat{F}=F_{\mathcal{A}\mu}$, then

$$\hat{F}(x_1,x_2)=\begin{cases}F(\alpha x_1,x_2/\beta) & \text{if } 0\le x_2\le\beta,\\ F(\alpha x_1,1)+F\left(\beta x_1+\alpha,\frac{x_2-\beta}{\alpha}\right)-F\left(\alpha,\frac{x_2-\beta}{\alpha}\right) & \text{if } \beta<x_2\le 1.\end{cases} \tag{2.14}$$

To see this, recall that $\hat{F}(x_1,x_2)=\mu(T^{-1}([0,x_1]\times[0,x_2]))$. Also

$$T^{-1}(x_1,x_2)=\begin{cases}\left(\alpha x_1,\frac{x_2}{\beta}\right) & \text{if } 0\le x_2\le\beta,\\ \left(\alpha+\beta x_1,\frac{x_2-\beta}{\alpha}\right) & \text{if } \beta<x_2\le 1.\end{cases} \tag{2.15}$$

Now if $0\le x_2\le\beta$, then $T^{-1}([0,x_1]\times[0,x_2])=[0,\alpha x_1]\times\left[0,\frac{x_2}{\beta}\right]$ which implies that $\hat{F}(x_1,x_2)=F\left(\alpha x_1,\frac{x_2}{\beta}\right)$ in this case. On the other hand, if $\beta<x_2\le 1$, then

$$\begin{aligned}T^{-1}([0,x_1]\times[0,x_2]) &= T^{-1}([0,x_1]\times[0,\beta])\cup T^{-1}([0,x_1]\times[\beta,x_2]),\\ T^{-1}([0,x_1)\times[0,\beta]) &= [0,\alpha x_1]\times[0,1],\\ T^{-1}([0,x_1]\times(\beta,x_2]) &= [\alpha,\alpha+\beta x_1]\times\left(0,\frac{x_1-\beta}{\alpha}\right].\end{aligned}$$

Clearly $\mu([0,\alpha x_1]\times[0,1]) = F(\alpha x_1, 1)$.

Moreover,

$$\begin{aligned}\mu\left([\alpha,\alpha+\beta x_1]\times\left(0,\frac{x_2-\beta}{\alpha}\right]\right) &= F\left(\alpha+\beta x_1,\frac{x_2-\beta}{\alpha}\right)\\ &\quad -\mu\left([0,\alpha)\times\left(0,\frac{x_2-\beta}{\alpha}\right)\right)\\ &= F\left(\alpha+\beta x_1,\frac{x_2-\beta}{\alpha}\right)-F\left(\alpha,\frac{x_2-\beta}{\alpha}\right),\end{aligned}$$

completing the proof of (2.14).

Since the expanding and contracting directions are the x, y-axis, we may separate variable to solve the equation $\hat{\mathcal{A}}F := \hat{F} = F$. In other words, we search for a function $F(x_1,x_2) = F_1(x_1)F_2(x_2)$ such that $\hat{\mathcal{A}}F = F$. Since for pure expansion in dimension one the Lebesgue measure is invariant, we may try $F_1(x_1) = x_1$. Substituting this in $\hat{\mathcal{A}}F$ yields $\hat{\mathcal{A}}F(x_1,x_2) = x_1\hat{F}_2(x_2)$ where

$$\mathcal{B}F_2 := \hat{F}_2(x_2) = \begin{cases}\alpha F_2\left(\frac{x_2}{\beta}\right) & 0\le x_2\le\beta,\\ \alpha+\beta F_2\left(\frac{x_2-\beta}{\alpha}\right) & \beta < x_2 \le 1.\end{cases}$$

Here we are using $F_2(1) = 1$. We are now searching for F_2 such that $\mathcal{B}F_2 = F_2$. It turns out that this equation has a unique solution F_2 that has zero derivative almost everywhere. Hence our invariant measure $\bar{\mu} = \lambda_1\times\lambda_2$ with λ_1 the Lebesgue measure and λ_2 a singular measure. One can show that the support of the measure λ_2 is of Hausdorff dimension $\frac{\alpha\log\alpha+\beta\log\beta}{\alpha\log\beta+\beta\log\alpha} =: \Delta$. To explain this heuristically, we show that if A denotes the set of points x such that there exists a sequence of intervals $I_n(x)$ with $x\in I_n(x)$, $\cap_n I_n(x) = \{x\}$, and

$$\lim_{n\to\infty}\frac{\log\lambda_2(I_n(x))}{\log\lambda_1(I_n(x))} = \Delta,$$

then $\lambda_2(A) = 1$. To construct I_n, let us first define a family of intervals $I_{a_1,\dots,a_n}$, with $a_1,\dots,a_n\in\{0,1\}$, so that $I_0 = [0,\beta)$, $I_1 = [\beta,1)$, and if $I_{a_1,\dots,a_n} = [p,q)$, then $I_{a_1,\dots,a_n,0} =$

$[p, p+\beta(q-p))$, and $I_{a_1,\dots,a_n,1} = [p+\beta(q-p), q)$. It is not hard to show

$$\lambda_2(I_{a_1,\dots,a_n}) = \alpha^{L_n}\beta^{R_n}, \qquad \lambda_1(I_{a_1,\dots,a_n}) = \beta^{L_n}\alpha^{R_n}, \tag{2.16}$$

where L_n and R_n denote the number of 0 and 1 in the sequence $a_1, \dots, a_n$, respectively. Given x, we can find a sequence $\omega(x) = (a_1, \dots, a_n, \dots) \in \Omega = \{0,1\}^{\mathbb{N}}$, such that $x \in I_{a_1,\dots,a_n}$ for every n. The transformation $x \mapsto \omega(x)$ pushes forward the measure λ_2 to the product measure λ_2' such that each a_n is 0 with probability α. If $L_n(x)$ and $R_n(x)$ denote the number of 0 and 1 in $a_1, \dots, a_n$ with $\omega(x) = (a_1, \dots, a_n, \dots)$, then by Birkhoff Ergodic Theorem

$$\lambda_2\left\{x : \lim_n \frac{L_n(x)}{n} = \alpha, \ \ \lim_n \frac{R_n(x)}{n} = \beta\right\} = 1.$$

From this and (2.16) we can readily deduce that $\lambda_2(A) = 1$.

Note that the support of $\bar{\mu}$ is of dimension $1 + \Delta$. Evidently $\Delta < 1$ unless $\alpha = \beta = \frac{1}{2}$.

What we have constructed is the *Sinai–Ruelle–Bowen* (SRB) measure $\bar{\mu}$ of our baker's transformation T. Note that this measure is absolutely continuous with respect to the expanding direction x-axis. A remarkable result of Sinai–Ruelle–Bowen asserts

$$\lim_{n\to\infty} \frac{1}{n}\sum_{0}^{n-1} f(T^j(x)) = \int f d\bar{\mu}$$

for almost all x with respect to the Lebesgue measure. This is different from Birkoff's ergodic theorem because Birkhoff's ergodic theorem only gives us convergence for $\bar{\mu}$-a.e. and $\bar{\mu}$ is singular with respect to Lebesgue measure.

Exercise 2.11

(i) Show that the baker's transformation is reversible in the following sense: If $\Phi(x,y) = (1-x, 1-y)$ then $\Phi^2 =$ identity and $T^{-1} = \Phi T \Phi$.

(ii) Show that if $\mu \in \mathcal{I}_T$ then $\mu\Phi \in \mathcal{I}_{T^{-1}}$ where $\mu\Phi$ is defined by $\int f d(\mu\Phi) = \int f \circ \Phi d\mu$.

Exercise 2.12 Let $T : (0,1] \to (0,1]$ by $T(x) = \left\{\frac{1}{x}\right\}$ where $\{\cdot\}$ means the fractional part. Derive the corresponding Perron–Frobenious equation. Show that $\rho(x) = \frac{1}{\log 2}\frac{1}{1+x}$ is a fixed point for the corresponding Perron–Frobenious operator.

We end this section with a discussion regarding the flow-analog of Perron–Frobenious equation. Given a flow ϕ_t associated with the ODE $\frac{dx}{dt} = f(x)$, let us define

$$T_t g = g \circ \phi_t.$$

This defines a group of transformations on the space of real-valued functions g. The dual of T_t acts on measures. More precisely, $T_t^*\mu$ is defined by

$$\int T_t f d\mu = \int f dT_t^* \mu,$$

or equivalently $T_t^*\mu(A) = \mu(\phi_t^{-1}A) = \mu(\phi_{-t}(A))$. The following theorem of *Liouville* gives an infinitesimal description of $T_t^*\mu$ when μ is absolutely continuous with respect to Lebesgue measure.

Theorem 2.13 *Suppose that there exists a differentiable function $\rho(x,t)$ such that $d(T_t^*\mu) = \rho(x,t)dx$. Then ρ satisfies the Liouville's equation*

$$\rho_t + \operatorname{div}(f\rho) = 0.$$

Proof. Let g be a differentiable function of compact support. We have

$$\begin{aligned}
\int g(y)\rho(y,t+h)dy &= \int g(\phi_{t+h}(x))\rho(x,0)dx \\
&= \int g(\phi_h(\phi_t(x)))\rho(x,0)dx \\
&= \int g(\phi_h(y))\rho(y,t)dy \\
&= \int g(y+hf(y)+o(h))\rho(y,t)dy \\
&= \int g(y)\rho(y,t)dy + h\int \nabla\ g(y)\cdot f(y)\rho(y,h)dy \\
&\quad +o(h).
\end{aligned}$$

This implies that $\frac{d}{dt}\int g(y)\rho(y,t)dy = \int f(y)\cdot\nabla g(y)\rho(y,t)dy$. After an integration by parts,

$$\frac{d}{dt}\int g(y)\rho(y,t)dy = \int g(\rho_t + \operatorname{div}(f\rho))dy.$$

Since g is arbitrary, we are done. $\square$

Exercise 2.14

(i) Let $u(x,t) = T_t g(x) = g(\phi_t(x))$. Show that u satisfies $u_t = \mathcal{L}u$ where $\mathcal{L}u = f(x)\cdot\frac{\partial u}{\partial x}$.

(ii) Show that $\mu \in \mathcal{I}_\phi$ iff $\int \mathcal{L}g d\mu = 0$ for every $g \in C^1$ of compact support.

In particular a measure ρdx is invariant if

$$\operatorname{div}(f\rho) = 0,$$

or equivalently $\rho\nabla\, f + \rho \text{ div } f = 0$. The generalization of this to manifolds is straightforward. If $\mathcal{L}_f$ denotes the *Lie derivative* and f is the velocity of the flow, then $\rho\omega$ is invariant if and only if

$$\mathcal{L}_f \rho + \rho \text{ div } f = 0.$$

Example 2.15 Let

$$T(x) = \begin{cases} \frac{x}{1-x} & \text{for } x \in \left[0, \frac{1}{2}\right), \\ 2x - 1 & \text{for } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Note that for this example, the condition $|T'(x)| > 1$ is violated at a single point $x = 0$. It turns out T has no invariant measure which is absolutely continuous with respect to Lebesgue measure. We omit the proof and refer the reader to [LaYo].

Notes The proof of Theorem 2.10 was taken from [Li].

3 Entropy

Roughly speaking, the entropy measures the exponential rate of increase in dynamical complexity as a system evolves in time. We will discuss two notions of entropy in this section, the *topological entropy* and (Kolmogorov–Sinai) *metric entropy*. We define the topological entropy first even though chronologically metric entropy was defined first.

Let (X,d) be a compact metric space and $T : X \to X$ be a continuous transformation. Define

$$\begin{aligned} d_n(x,y) &= \max\{d(x,y), d(T(x),T(y)),\dots,d(T^{n-1}(x),T^{n-1}(y))\}, \\ B^n(x,r) &= B^n_{T,d}(x,r) = \{y : d_n(x,y) < r\}. \end{aligned}$$

We then define two numbers. First $S^n_{T,d}(r)$ is defined as the smallest number k for which we can find a set A of cardinality k such that $X = \bigcup_{x\in A} B^n_{T,d}(x,r)$. We also define $N^n_{T,d}(r)$ to be the maximal number of points in X with pairwise d_n-distances at least r. Set

$$\begin{aligned} h_{\text{top}}(T;d) &= h_{\text{top}}(T) = \lim_{r\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S^n_{T,d}(r), \\ \bar{h}_{\text{top}}(T;d) &= \bar{h}_{\text{top}}(T) = \lim_{r\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N^n_{T,d}(r). \end{aligned}$$

We will see below that $\bar{h}_{\text{top}} = h_{\text{top}}$ and we call $h_{\text{top}}(T)$, the *topological entropy* of T. We will see that $h_{\text{top}}(T;d)$ is independent of the choice of the metric and depends on the topology of the underlying space. In some sense, "higher entropy" means "more orbits". But the number of orbits is usually uncountably infinite. Hence we fix a "resolution" r, so that we do not distinguish points that are of distance less than r. Hence $N^n(r)$ represents the number of distinguishable orbits of length n, and this number grows like $e^{nh_{\text{top}}(T)}$. Here are some properties of the topological entropy.

Proposition 3.1

(i) *If the metrics d and d' induce the same topology, then $h_{\text{top}}(T;d) = h_{\text{top}}(T;d')$.*

(ii) *If $F : X \to Y$ is a homeomorphism, $T : X \to X$, $S : Y \to Y$, $S \circ F = F \circ T$, then $h_{\text{top}}(T) = h_{\text{top}}(S)$.*

(iii) *$h_{\text{top}}(T^n) = nh_{\text{top}}(T)$. Moreover, if T is a homeomorphism, then $h_{\text{top}}(T) = h_{\text{top}}(T^{-1})$.*

(iv) *$h_{\text{top}}(T) = \bar{h}_{\text{top}}(T)$.*

Proof.

(i) Set $\eta(\epsilon) = \min\{d'(x,y) : d(x,y) \geq \epsilon\}$. Then

$$d'(x,y) < \eta(\epsilon) \Rightarrow d(x,y) < \epsilon.$$

As a result, $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$ and $B^n_{T,d'}(x, \eta(\epsilon)) \subseteq B^n_{T,d}(x, \epsilon)$. Hence $S^n_{T,d'}(\eta(\epsilon)) \geq S^n_{T,d}(\epsilon)$. Thus $h_{\text{top}}(T,d) \leq h_{\text{top}}(T,d')$.

(ii) Given a metric d on X, define a metric d' on Y by $d'(x,y) = d(F^{-1}(x), F^{-1}(y))$. Evidently $h_{\text{top}}(T;d) = h_{\text{top}}(S;d')$.

(iii) Evidently $B^{nk}_{T,d}(x,r) \subseteq B^k_{T^n,d}(x,r)$. Hence

$$S^{nk}_{T,d}(r) \geq S^k_{T^n,d}(r),\ \ h_{\text{top}}(T^n) \leq n h_{\text{top}}(T).$$

For the converse, find a function $\eta : (0,\infty) \to (0,\infty)$ such that $\lim_{r \to 0} \eta(r) = 0$ and $B_d(x, \eta(r)) \subset B^n_{T,d}(x,r)$. Then $B^k_{T^n,d}(x, \eta(r)) \subset B^{kn}_{T,d}(x,r)$. This implies that $S^k_{T^n,d}(\eta(r)) \geq S^{kn}_{T,d}(r)$, which in turn implies

$$\frac{1}{k} \log S^k_{T^n,d}(\eta(r)) \geq n \frac{k-1}{k} \max_{(k-1)n \leq \ell \leq kn} \frac{1}{\ell} \log S^\ell_{T,d}(r).$$

From this, it is not hard to deduce that $h_{\text{top}}(T^n) \geq n h_{\text{top}}(T)$.

For $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$, observe $T^{n-1}(B^n_{T,d}(x,r)) = B^n_{T^{-1},d}(T^{n-1}(x), r)$. Hence $X = \bigcup_{j=1}^k B^n_{T,d}(x_j, r)$ implies that $X = \bigcup_{j=1}^k B^n_{T^{-1},d}(T^{n-1}(x_j), r)$. From this we deduce $S^n_{T^{-1},d}(r) \leq S^n_{T,d}(r)$. This implies that $h_{\text{top}}(T^{-1}) \leq h_{\text{top}}(T)$ which in turn implies that $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$.

(iv) This is an immediate consequence of the following straightforward inequalities:

$$N^n_{T,d}(2r) \leq S^n_{T,d}(r) \leq N^n_{T,d}(r).$$

The first inequality follows from the fact that if $N^n(r) = L$ and $\{x_1, \ldots, x_L\}$ is a maximal set, then $X = \bigcup_{j=1}^L B_{d_n}(x_j, r)$. The second inequality follows from the fact that no d_n-ball of radius r can contain two points that are $2r$-apart. □

Exercise 3.2 Let (X_1, d_1), (X_2, d_2) be two compact metric spaces and let $T_i : X_i \to X_i$, $i = 1,2$ be two continuous functions. show that $h_{\text{top}}(T_1 \times T_2) = h_{\text{top}}(T_1) + h_{\text{top}}(T_2)$.

Hint: For $T = T_1 \times T_2$ and a suitable choice of a metric d for $X_1 \times X_2$, show that

$$S^n_{T,d}(r) \leq S^n_{T_1,d_1}(r) S^n_{T_2,d_2}(r), \quad N^n_{T,d}(r) \geq N^n_{t,d_1}(r_1) N^n_{T,d_2}(r_2).$$

□

Example 3.3 Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a translation. Since T is an isometry, $d_n(x, y) = d(x, y)$ for $d(x, y) = |x - y|$. Thus $S^n(r)$ is independent of n and $h_{\text{top}}(T) = 0$. $\square$

Example 3.4 Let $X = \{0, 1, \dots, N-1\}^{\mathbb{Z}}$. Given $\omega = (\omega(j) : j \in \mathbb{Z}) \in X$, define $(T\omega)(j) = \omega(j+1)$. Consider the metric

$$d(\omega, \omega') = \sum_{j \in \mathbb{Z}} \lambda^{-|j|} |\omega(j) - \omega'(j)|,$$

with $\lambda > 1$. Fix $\alpha \in X$ and take any $\omega \in X$. Evidently

$$\sum_{|j|>m} \lambda^{-|j|} |\alpha(j) - \omega(j)| \leq 2(N-1) \sum_{m+1}^{\infty} \lambda^{-\ell} = \frac{2(N-1)}{\lambda^m(\lambda - 1)}.$$

Also, if $\omega(j) \neq \alpha(j)$ for some $j \in \{-m, \dots, m\}$, then

$$\sum_{|j| \leq m} \lambda^{-|j|} |\alpha(j) - \omega(j)| \geq \lambda^{-m}.$$

Evidently d induces the product topology on X no matter what $\lambda \in (1, \infty)$ we pick. Choose λ large enough so that $\frac{2(N-1)}{\lambda - 1} < 1$. For such a choice of λ,

$$B_d\left(\alpha, \lambda^{-m}\right) = \{\omega : \omega(j) = \alpha(j) \text{ for } j \in \{-m, \dots, m\}\}.$$

Since

$$\left\{\omega : d(T^i(\omega), T^i(\alpha)) < \lambda^{-m}\right\} = \{\omega : \omega(j+i) = \alpha(j+i) \text{ for } j \in \{-m, \dots, m\}\},$$

we deduce

$$B_{d_n}\left(\alpha, \lambda^{-m}\right) = \{\omega : \omega(j) = \alpha(j) \text{ for } j \in \{-m, \dots, m+n-1\}\}.$$

Evidently every two d_n-balls of radius λ^{-m} are either identical or disjoint. As a result, $S^n_{T,d}\left(\lambda^{-m}\right) = N^{2m+n}$. Thus

$$h_{\text{top}}(T) = \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log N^{2m+n} = \log N.$$

$\square$

Example 3.5 Let (X, T) be as in the previous example and let $A = [a_{ij}]$ be an $N \times N$ matrix with $a_{ij} \in \{0, 1\}$ for all $i, j \in \{0, 1, \dots, N-1\}$. Set

$$X_A = \{\omega \in X : a_{\omega(i), \omega(i+1)} = 1 \text{ for all } i \in \mathbb{Z}\}.$$

Evidently X_A is an invariant set and the restriction of T to X_A gives a dynamical system. To have a irreducible situation, we assume that each row of A contains at least one 1 (if for example $a_{0j} = 0$ for all j, we may replace X with $\{1, 2, \ldots, N-1\}^{\mathbb{Z}}$). For such A,

$$\begin{aligned}
S^n_{T,d}\left(\lambda^{-m}\right) &= \#\text{ of balls of radius } \lambda^{-m} \text{ with nonempty intersection with } X_A \\
&= \#\text{ of } (\alpha_{-m}, \ldots, \alpha_{m+n-1}) \text{ with } a_{\alpha_i,\alpha_{i+1}} = 1 \text{ for } -m \le i < m+n+1 \\
&= \sum_{r,s=0}^{N-1} \#\{(\alpha_{-m}, \ldots, \alpha_{m+n-1}) : a_{\alpha_i,\alpha_{i+1}} = 1 \text{ for } -m \le i < m+n-1 \\
&\qquad \text{and } \alpha_{-m} = r,\ \alpha_{m+n-1} = s\} \\
&= \sum_{r,s=0}^{N-1} a^{2n+m-1}_{r,s} = \|A^{2m+n-1}\|
\end{aligned}$$

where $a^k_{r,s}$ is the (r, s) entry of the matrix A^k, and $\|A\|$ denotes the norm of A, i.e., $\|A\| = \sum_{r,s} |a_{r,s}|$. We now claim

$$h_{\text{top}}(T) = \lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \|A^{2m+n-1}\| = \log r(A),$$

where $r(A) = \max\{|\lambda| : \lambda \text{ an eigenvalue of } A\}$. To see this, first observe that if $Av = \lambda v$, then $A^k v = \lambda^k v$. Hence

$$|\lambda|^k \max_j |v_j| \le |\lambda|^k \sum_j |v_j| \le \sum_{i,j} |a^k_{i,j}||v_i| \le \|A^k\| \max_j |v_j|.$$

As a result, $\|A^k\| \ge |\lambda|^k$. This shows that $h_{\text{top}}(T) \ge \log r(A)$. For the converse, we choose a basis so that the off-diagonal entries in Jordan normal form of A become small (see Theorem ?? of Part I). Using this we can show that $|Av| \le (r(A) + \delta)|v|$ which in turn implies that $|A^k v| \le (r(A)+\delta)^k|v|$. From this we deduce that $h_{\text{top}}(T) \le \log(r(A)+\delta)$. Finally send $\delta \to 0$ to deduce that $h_{\text{top}}(T) \le \log r(A)$. This completes the proof of $h_{\text{top}}(T) = \log r(A)$.

$\square$

Example 3.6 Let $X = \mathbb{T}^2$ and $T : X \to X$ is given by $T(x) = A_0 X \pmod 1$ where A_0 is an integer-valued matrix with eigenvalues λ_1, λ_2 satisfying $|\lambda_2| < 1 < |\lambda_1| = |\lambda_2|^{-1}$. For the sake of definiteness, let us take $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ with eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$ and eigenvectors $v_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ \frac{-\sqrt{5}-1}{2} \end{bmatrix}$. T is a contraction along v_2 and an expansion along v_1. We now draw the eigen lines from the origin and let them intersect several times to separate torus into disjoint rectangles. Let us write R_1 and R_2 for these rectangles. We now study $T(R_1)$ and $T(R_2)$. We set

$$\begin{matrix} T(R_1)\cap R_1 = Z_0\cup Z_1 \\ T(R_1)\cap R_2 = Z_3 \end{matrix} \quad , \quad R_1 = Z_0\cup Z_1\cup Z_2.$$

We then define Z_4 so that $R_2 = Z_3\cup Z_4$. One can then show that $T(R_2) = Z_2\cup Z_4$. We now define $Y = \{0,1,2,3,4\}^{\mathbb{Z}}$ and $h : Y_C \to \mathbb{T}^2 = X$ with

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = [c_{ij}]$$

where $h(\omega) = x$ for $\{x\} = \bigcap_{n\in\mathbb{Z}} T^{-n}(Z_{\omega(n)})$. If $\hat{T}$ denotes the shift on Y_C, then we have $T\circ h = h\circ\hat{T}$. Here we are using the fact that if $x\in Z_i$ and $T(x)\in Z_j$, then $c_{ij} = 1$. Also, since T is contracting in v_2-direction and T^{-1} is contracting in v_1-direction, then $\bigcap_{n\in\mathbb{Z}} T^{-n}(Z_{\omega(n)})$ has at most one point. To show that the intersection is nonempty, first we verify that indeed whenever $c_{ij} = 1$, then $T(Z_i)\cap Z_j \neq \emptyset$. Using this, it is not hard to deduce that $\bigcap_{-N}^{N} T^{-n}(Z_{\omega(n)}) \neq \emptyset$ whenever $\omega\in Y_C$. This and the compactness of the space imply that $\bigcap_{n\in\mathbb{Z}} T^{-n}(Z_{\omega(n)}) \neq \emptyset$.

The transformation h is onto because for each x we can find $\omega\in Y_C$ such that $T^n(x)\in Z_{\omega(n)}$. However, h is not one-to-one. For example if $\bar{\alpha}$ denotes $\bar{\alpha} = (\omega(n) : n\in\mathbb{Z})$ with $\omega(n) = \alpha$ for all n, then $\bar{0},\bar{1},\bar{4}\in Y_C$ (but not $\bar{2}$ and $\bar{3}$). Moreover $\hat{T}(\bar{0}) = \bar{0}$, $\hat{T}(\bar{1}) = \bar{1}$, $\hat{T}(\bar{4}) = \bar{4}$. On the other hand the only x with $T(x) = x$ is $x = 0$. In fact $h(\bar{0}) = h(\bar{1}) = h(\bar{4})$ is equal to the origin. From $T\circ h = h\circ\hat{T}$ and Example 3.5 we conclude that $h_{\text{top}}(T) \leq h_{\text{top}}(\hat{T}) = \log r(C)$. See Exercise 3.7. A straightforward calculation yields $r(C) = \lambda_1 = \frac{3+\sqrt{5}}{2}$. Later we discuss the metric entropy, and using the metric entropy of T we will show in Example 3.15 below that indeed $h_{\text{top}}(T) = \log\frac{3+\sqrt{5}}{2}$. □

Exercise 3.7

(i) Let $F : X\to Y$ be a continuous function with $F(X) = Y$. Let $T : X\to X$, $T' : Y\to Y$ be continuous and $F\circ T = T'\circ F$. show that $h_{\text{top}}(T') \leq h_{\text{top}}(T)$.

(ii) Let C be as in Example 3.6. show that $r(C) = \frac{3+\sqrt{5}}{2}$. □

The metric entropy is the measure-theoretic version of the topological entropy. Let $T : X\to X$ be a measurable transformation and take $\mu\in\mathcal{I}_T$. A countable collection ξ

of measurable subsets of X is called a *μ-partition* if $\mu(C \cap D) = 0$ for every two distinct $A, B \in \xi$, and $\mu\left(X - \bigcup_{A\in\xi} A\right) = 0$. If ξ and η are two μ-partition, then their *common refinement* $\xi \vee \eta$ is the partition

$$\xi \vee \eta = \{A \cap B : A \in \xi,\ B \in \eta,\ \mu(A \cap B) > 0\}.$$

Also, if ξ is a μ-partition, then we set

$$T^{-1}\xi = \{T^{-1}(A) : A \in \xi\},$$

which is also a μ-partition because $\mu \in \mathcal{I}_T$. We also define

$$\xi_{-n}^T = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi.$$

As we discussed in the introduction, the metric entropy measures the exponential gain in the information. Imagine that we can distinguish two points x and y only if x and y belong to different elements of the partition ξ. Now if the orbits up to time $n-1$ are known, we can use them to distinguish more points. The partition ξ_{-n}^T represents the accumulated information gained up to time $n-1$. Except for a set of zero μ-measure, each x belongs to a unique element $C_n(x) \in \xi_{-n}^T$. Let's have an example.

Example 3.8 Let $T(x) = mx \pmod 1$, $T : \mathbb{T} \to \mathbb{T}$ with $m \in \mathbb{Z}$, $m \geq 2$. Let $\xi = \left\{\left[\frac{j}{m}, \frac{j+1}{m}\right) : j = 0, \ldots, m-1\right\}$. Then

$$\eta_n = \xi_{-n}^T = \{[\cdot a_1 \ldots a_n, \cdot a_1 \ldots a_n + m^{-n}) : a_1 \ldots a_n \in \{0, 1, \ldots, m-1\}\}.$$

Given x, let $\cdot a_1 a_2 \ldots a_n * * \ldots$ denote its base m expansion. Note that for points on the boundary of the intervals in η_n, we may have two distinct expansions. Since we have chosen closed-open intervals in ξ, we dismiss expansions which end with infinitely many m. In other words, between $.a_1 \ldots a_k mm \ldots$, with $a_k < m$ and $.a_1 \ldots a_k' 00 \ldots$ for $a_k' = a_k + 1$, we choose the latter. we have

$$C_{\eta_n}(x) = [\cdot a_1 \ldots a_n, \cdot a_1 \ldots a_n + m^{-n}).$$

If $\mu_p \in \mathcal{I}_T$ with $p = (p_0, \ldots, p_{m-1})$, $p_j \geq 0$, $\sum_j p_j = 1$, $\mu_p([\cdot a_1 \ldots a_n, \cdot a_1 \ldots a_n + m^{-n})) = p_{a_1} p_{a_2} \ldots p_{a_n}$, then $\mu_p(C_{\eta_n}(x)) = p_{a_1} \ldots p_{a_n}$ and

$$\frac{1}{n} \log \mu_p(C_{\eta_n}(x)) = \frac{1}{n} \sum_1^n \log p_{a_j} = \frac{1}{n} \sum_0^{n-1} \log f(T^j(x))$$

where $f(\cdot a_1 a_2 \ldots) = p_{a_1}$. By ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_p(C_{\eta_n}(x)) = \sum_1^m p_j \log p_j.$$

$\square$

In general, if we are interested in the amount of information the partition $\eta_n = \xi^T_{-n}$ carries out, perhaps we should look at $\mu(C_n(x))$ where $C_n(x) = C_{\eta_n}(x)$. This is typically exponentially small in n. Motivated by Example 3.8, we define

$$
\begin{aligned}
I_\xi(x) &= -\log \mu(C_\xi(x)),\\
H_\mu(\xi) &= \int I_\xi(x)\mu(dx) = -\sum_{C\in\xi} \mu(C)\log\mu(C),\\
h_\mu(T,\xi) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi^T_{-n}).
\end{aligned}
$$

Theorem 3.9 *The limit in the definition $h_\mu(T,\xi)$ exists. Moreover, if $C_n(x) = C_{\xi^T_{-n}}(x)$, then*

$$
\lim_{n\to\infty} \int \left| \frac{1}{n}\log\mu(C_n(x)) + h_\mu(T,\xi)\right| d\mu = 0, \tag{3.1}
$$

provided that μ is ergodic. (Shannon–McMillan–Breiman Theorem)

We do not give the full proof of (3.1) that involves some results from the probability theory. The proof of the existence of the limit is an immediate consequence of Lemmas 3.10 and 3.11 below. To this end let us define,

$$
\begin{aligned}
I_{\xi|\eta}(x) &= -\log\mu(C_\xi(x) \mid C_\eta(x)) = -\log \frac{\mu(C_\xi(x)\cap C_\eta(x))}{\mu(C_\eta(x))},\\
H_\mu(\xi\mid\eta) &= \int I_{\xi|\eta}\,d\mu = -\sum_{A\in\xi,B\in\eta} \mu(A\cap B)\log\frac{\mu(A\cap B)}{\mu(B)},
\end{aligned}
$$

where η and ξ are two μ-partitions.

Lemma 3.10 We have

$$
H_\mu(\xi\vee\eta) = H_\mu(\eta) + H_\mu(\xi\mid\eta), \quad H_\mu(\xi\vee\eta) \le H_\mu(\xi)+H_\mu(\eta), \quad H_\mu(T^{-1}\xi) = H_\mu(\xi). \tag{3.2}
$$

Lemma 3.11 *Let a_n be a sequence of numbers such that $a_{n+m} \le a_n + a_m$. Then $\lim_{n\to\infty}\frac{1}{n}a_n = \inf_n \frac{a_n}{n}$.*

Proof of Lemma 3.10. We certainly have $I_{\xi\vee\eta} = I_\eta + I_{\xi|\eta}$. From this we deduce the first equality in (3.2). For the inequality $H_\mu(\eta\vee\xi) \le H_\mu(\xi) + H_\mu(\eta)$, it suffices to show that $H_\mu(\xi|\eta) \le H_\mu(\xi)$. Set $\varphi(x) = x\log x$ and note that φ is convex. Then

$$\begin{aligned}\varphi(\mu(B)) &= \varphi\left(\sum_{A\in\eta}\mu(A)\frac{\mu(A\cap B)}{\mu(A)}\right) \le \sum_{A\in\eta}\mu(A)\varphi\left(\frac{\mu(A\cap B)}{\mu(A)}\right)\\ &= \sum_{A\in\eta}\mu(A\cap B)\log\frac{\mu(A\cap B)}{\mu(A)}.\end{aligned}$$

This completes the proof of $H_\mu(\eta\vee\xi) \le H_\mu(\xi) + H_\mu(\eta)$. The statement $H_\mu(T^{-1}\xi) = H_\mu(\xi)$ is obvious because $\mu(T^{-1}(A)) = \mu(A)$ for every $A \in \xi$. $\square$

Proof of Lemma 3.11. Evidently $\liminf_{n\to\infty}\frac{a_n}{n} \ge \inf_n \frac{a_n}{n}$. On the other hand, if $n = \ell m + r$ with $m, \ell \in \mathbb{N}$, $r \in [0, m)$, then

$$\begin{aligned} a_n = a_{\ell m + r} &\le a_{\ell m} + a_r \le \ell a_m + a_r,\\ \frac{a_n}{n} &\le \frac{\ell m}{n}\frac{a_m}{m} + \frac{a_r}{n}.\end{aligned}$$

After sending $n \to \infty$, we obtain,

$$\limsup_{n\to\infty}\frac{a_n}{n} \le \frac{a_m}{m}$$

for every $m \in \mathbb{Z}^+$. This completes the proof. $\square$

Proof of Theorem 3.9. Let us define $\xi(n,m) = T^{-n}\xi \vee T^{-n-1}\xi \vee \cdots \vee T^{-m}\xi$ whenever $n < m$. We have

$$\begin{aligned} I_{\xi^T_{-n-1}} &= I_{\xi(0,n)} = I_{\xi\vee T^{-1}\xi(0.n-1)} (= I_{\xi\vee\xi(1,n)})\\ &= I_{T^{-1}\xi(0,n-1)} + I_{\xi|T^{-1}\xi(0,n-1)}.\end{aligned}$$

Since $C_{T^{-1}\eta}(x) = C_\eta(T(x))$, we deduce

$$I_{\xi^T_{-n}} = I_{\xi(0,n-1)}\circ T + I_{\xi|\xi(1,n)}.$$

Applying this repeatedly, we obtain

$$\text{(3.3)}\quad \begin{aligned} I_{\xi^T_{-n-1}} &= I_{\xi|\xi(1,n)} + I_{\xi|\xi(1,n-1)}\circ T + \cdots + I_{\xi|\xi(1,2)}\circ T^{n-2} + I_{\xi|T^{-1}\xi}\circ T^{n-1} + I_\xi\circ T^n,\\ \frac{1}{n}I_{\xi^T_{-n-1}} &= \frac{1}{n}\sum_{j=0}^{n-1} I_{\xi|\xi(1,n-j)}\circ T^j + \frac{1}{n}I_\xi\circ T^{n-1}.\end{aligned}$$

If it were not for the dependence of $I_{\xi|\xi(1,n-j)}$ on $n-j$, we could have used the Ergodic Theorem to finish the proof. However, if we can show that $\lim_{m\to\infty} I_{\xi|\xi(1,m)} = \hat{I}$ exists, say in $L^1(\mu)$-sense, then we are almost done because we can replace $I_{\xi|\xi(1,n-j)}$ with $\hat{I}$ in (3.3) with an error that is small in L^1-sense. We then apply the ergodic theorem to assert

$$\lim_{n\to\infty} \frac{1}{n} I_{\xi^T_{-n}} = \int \hat{I} d\mu.$$

Note that if we write $\mathcal{F}_\eta$ for the σ–algebra generated by η, then $\mu(C_\xi(x)|C_\eta(x))$ is nothing other than

$$\mu(C_\xi \mid \mathcal{F}_\eta)(x) = \sum_{A\in\xi} \mu(A \mid \mathcal{F}_\eta)(x) 1\!\!1_A(x),$$

i.e. the conditional expectation of the indicator function of the set C_ξ, given the σ-field $\mathcal{F}_\eta$. Hence, we simply have

$$\hat{I}(x) = -\log\left\{\lim_{n\to\infty} \sum_{A\in\xi} \mu(A \mid \xi(1,n))(x) 1\!\!1_A(x)\right\} = -\sum_{A\in\xi} \log\left\{\lim_{n\to\infty} \mu(A \mid \xi(1,n))(x)\right\} 1\!\!1_A(x).$$

This suggests studying $\lim_{n\to\infty} \mu(A \mid \xi(1,n))$. The existence and interpretation of the limit involve some probabilistic ideas. We may define $\mathcal{F}_{1,n}$ to be the σ-algebra generated by the partition $\xi(1,n)$. We then have $\mathcal{F}_{1,2} \subseteq \mathcal{F}_{1,3} \subseteq \dots$ and if $\mathcal{F}_{1,\infty}$ is the σ-algebra generated by all $\xi(1,n)$'s, then

$$\lim_{n\to\infty} \mu(A \mid \xi(1,n)) = \mu(A \mid \mathcal{F}_{1,\infty}),$$

μ-almost surely and in $L^1(\mu)$–sense. The right-hand side is the conditional measure of A given the σ-algebra $\mathcal{F}_{1,\infty}$. The proof of convergence follows the celebrated martingale convergence theorem. We only provide a proof for the $L^1(\mu)$–convergence and refer the reader to any textbook on martingales for the almost sure convergence.

Write $f = \mu(A \mid \mathcal{F}_{1,\infty})$ so that

$$\mu(A \mid \mathcal{F}_{1,n}) = E^\mu(f \mid \mathcal{F}_{1,n}),$$

where the right-hand side denotes the μ–conditional expectation of f given the σ–algebra $\mathcal{F}_{1,n}$. Hence the martingale convergence theorem would follow if we can show

$$\lim_{n\to\infty} E^\mu(f \mid \mathcal{F}_{1,n}) = f, \tag{3.4}$$

for every $\mathcal{F}_{1,\infty}$-measurable function f. Given such a function f and $\delta > 0$, we can find a positive integer k and $\mathcal{F}_{1,k}$-measurable function g such that $\|f-g\|_{L^1(\mu)} \le \delta$. We certainly have

$$\lim_{n\to\infty} E^\mu(g \mid \mathcal{F}_{1,n}) = g, \quad \|E^\mu(f \mid \mathcal{F}_{1,n}) - E^\mu(g \mid \mathcal{F}_{1,n})\|_{L^1(\mu)} \le \delta.$$

We use this and send δ to 0 to deduce (3.4). For our purposes, we need something stronger, namely

$$\lim_{n\to\infty} \log \mu(A \mid \mathcal{F}_{1,n}) = \log \mu(A \mid \mathcal{F}_{1,\infty}). \tag{3.5}$$

This would follow from (3.4) provided that we can show

$$\int_A \left(\sup_n (-\log \mu(A \mid \mathcal{F}_{1,n})) \right) d\mu \le -\mu(A) \log \mu(A) + \mu(A). \tag{3.6}$$

Indeed if we pick $\ell > 0$ and define

$$A_n = \left\{ x : \mu\left(A \mid \mathcal{F}_{1,n}\right)(x) < e^{-\ell}, \;\; \mu\left(A \mid \mathcal{F}_{1,k}\right)(x) \ge e^{-\ell} \;\text{ for } k = 1, 2, \dots, n-1 \right\},$$

then $A_n \in \mathcal{F}_{1,n}$ and we can write

$$\begin{aligned}
&\mu\left\{ x \in A : \sup_n (-\log \mu(A \mid \mathcal{F}_{1,n})(x)) > \ell \right\} = \mu\left(A \cap \cup_{n=1}^{\infty} A_n\right) = \sum_1^{\infty} \mu(A \cap A_n) \\
&= \sum_1^{\infty} \int_{A_n} \mu(A \mid \mathcal{F}_{1,n}) d\mu \\
&\le \sum_1^{\infty} \int_{A_n} e^{-\ell} d\mu = e^{-\ell} \sum_1^{\infty} \mu(A_n) \le e^{-\ell}.
\end{aligned}$$

From this we deduce

$$\begin{aligned}
&\int_A \left(\sup_n (-\log \mu(A \mid \mathcal{F}_{1,n}))(x) \right) d\mu = \int_0^{\infty} \mu\left\{ x \in A : \sup_n (-\log \mu(A \mid \mathcal{F}_{1,n})(x)) > \ell \right\} d\ell \\
&\le \int_0^{\infty} \min\{\mu(A), e^{-\ell}\} d\ell = -\mu(A) \log \mu(A) + \mu(A).
\end{aligned}$$

This completes the proof of (3.6). □

The proof of Theorem 3.9 suggests an alternative formula for the entropy. In some sense $h_\mu(T, \xi)$ is the entropy of the "presence" ξ relative to its "past" $\xi(1, \infty)$. To make this rigorous, first observe that by (3.3),

$$H_\mu(\xi_{-n}^T) = H_\mu(\xi(0, n-1)) = \sum_{j=1}^{n-1} H_\mu(\xi \mid \xi(1, j)) \tag{3.7}$$

where $H_\mu(\xi \mid \xi(1, 1))$ means $H_\mu(\xi)$. In fact we have

Proposition 3.12 $h_\mu(T,\xi) = \inf_n H_\mu(\xi \mid T^{-1}\xi \vee \cdots \vee T^{-n}\xi)$ *and the sequence* $H_\mu(\xi \mid T^{-1}\xi \vee \cdots \vee T^n\xi)$ *is nondecreasing.*

Proof. The monotonicity of the sequence $a_n = H_\mu(\xi \mid T^{-1}\xi \vee \cdots \vee T^n\xi)$ follows from Lemma 3.13 below. We then use (3.7) to assert

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}H_\mu(\xi_{-n}^T) &= \lim_{n\to\infty}\frac{1}{n}\sum_1^{n-1}H_\mu(\xi \mid \xi(1,j)) \\ &= \lim_{n\to\infty}H_\mu(\xi \mid \xi(1,n)) = \inf_n H_\mu(\xi \mid \xi(1,n)).\end{aligned}$$

$\square$

It remains to show the monotonicity of the sequence a_n. Let us write $\alpha \le \beta$ when β is a *refinement* of α. This means that for every $B \in \beta$, there exists a set $A \in \alpha$ such that $\mu(B - A) = 0$. Evidently $\xi(1,1) \le \xi(1,2) \le \cdots \le \xi(1,n)$. Let us write $X = Y (\operatorname{mod} 0)$ if $\mu(X\Delta Y) = 0$. If $\alpha \le \beta$, then for every $A \in \alpha$, $A = \cup\{B \in \beta : \mu(B - A) = 0\} (\operatorname{mod} 0)$. For the monotonicity of a_n, it suffices to show this:

Lemma 3.13 *If* $\alpha \le \beta$*, then* $H_\mu(\xi \mid \alpha) \ge H_\mu(\xi \mid \beta)$*.*

Proof. Recall $\varphi(z) = z\log z$. We have

$$H_\mu(\xi \mid \alpha) = -\sum_{A,C}\mu(A\cap C)\log\frac{\mu(A\cap C)}{\mu(A)} = -\sum_{A,C}\mu(A)\varphi\left(\frac{\mu(A\cap C)}{\mu(A)}\right),\ A \in \alpha,\ C \in \xi.$$

Fix A and write $A = \cup\{B : B \in J\} (\operatorname{mod} 0)$ for a family J, so that $\{B : B \in J\} \subseteq \beta$. Hence

$$\varphi\left(\frac{\mu(A\cap C)}{\mu(A)}\right) = \varphi\left(\sum_{B\in J}\frac{\mu(B)}{\mu(A)}\frac{\mu(C\cap B)}{\mu(B)}\right) \le \sum_{B\in J}\frac{\mu(B)}{\mu(A)}\varphi\left(\frac{\mu(C\cap B)}{\mu(B)}\right).$$

From this we deduce $H_\mu(\xi \mid \alpha) \ge H_\mu(\xi \mid \beta)$. $\square$

We finally define the entropy of a transformation by

$$h_\mu(T) = \sup\{h_\mu(T,\xi) : H_\mu(\xi) < \infty,\ \xi \text{ a partition}\}.$$

Exercise 3.14

(i) If ξ has m elements, then $0 \le H_\mu(\xi) \le \log m$.

(ii) If $\mu_1, \mu_2 \in \mathcal{I}_T$ and $\alpha \in [0,1]$, then

$$
\begin{aligned}
H_{\alpha\mu_1+(1-\alpha)\mu_2}(\xi) &\geq \alpha H_{\mu_1}(\xi) + (1-\alpha)H_{\mu_2}(\xi) \\
h_{\alpha\mu_1+(1-\alpha)\mu_2}(T,\xi) &\geq \alpha h_{\mu_1}(T,\xi) + (1-\alpha)h_{\mu_2}(T,\xi) \\
h_{\alpha\mu_1+(1-\alpha)\mu_2}(T) &\geq \alpha h_{\mu_1}(T) + (1-\alpha)h_{\mu_2}(T).
\end{aligned}
$$

(iii) If $\alpha \leq \beta$, then $H_\mu(\alpha) \leq H_\mu(\beta)$ and $h_\mu(T,\alpha) \leq h_\mu(T,\beta)$. □

We continue with some basic properties of the entropy.

Proposition 3.15

(i) $h_\mu(T,\xi) \leq h_\mu(T,\eta) + H_\mu(\xi \mid \eta)$.

(ii) $h_\mu(T^k) = kh_\mu(T)$ *and if* T *is invertible, then* $h_\mu(T) = h_\mu(T^{-1})$.

(iii) *If* $\mu \perp \nu$ *and* $\mu, \nu \in \mathcal{I}_T$, *then* $h_{\alpha\mu+(1-\alpha)\nu}(T) = \alpha h_\mu(T) + (1-\alpha)h_\nu(T)$.

Proof.

(i) Recall $\xi(m,n) = T^{-m}\xi \vee \cdots \vee T^{-n}\xi$. We certainly have

$$H_\mu(\xi(0,n-1)) \leq H_\mu(\eta(0,n-1)) + H_\mu(\xi(0,n-1) \mid \eta(0,n-1)).$$

It suffices to show that $H_\mu(\xi(0,n-1) \mid \eta(0,n-1)) \leq nH_\mu(\xi \mid \eta)$. To show this, first observe that in general,

$$H_\mu(\alpha \vee \beta \mid \gamma) = H_\mu(\alpha \mid \gamma) + H_\mu(\beta \mid \alpha \vee \gamma),$$

which follows from

$$
\begin{aligned}
I_{(\alpha\vee\beta)|\gamma}(x) &= -\log \frac{\mu(C_{\alpha\vee\beta}(x) \cap C_\gamma(x))}{\mu(C_\gamma(x))} \\
&= -\log \frac{\mu(C_\alpha(x) \cap C_\beta(x) \cap C_\gamma(x))}{\mu(C_\gamma(x))} \\
&= -\log \frac{\mu(C_\alpha(x) \cap C_\gamma(x))}{\mu(C_\gamma(x))} - \log \frac{\mu(C_{\alpha\vee\beta}(x) \cap C_\gamma(x))}{\mu(C_\alpha(x) \cap C_\gamma(x))} \\
&= I_{\alpha|\gamma}(x) + I_{\beta|(\alpha\vee\gamma)}(x).
\end{aligned}
$$

Using this we write,

$$
\begin{aligned}
H_\mu(\xi(0,n-1) \mid \eta(0,n-1)) &\leq H_\mu(\xi \mid \eta(0,n-1)) + H_\mu(\xi(1,n-1) \mid \eta(0,n-1) \vee \xi) \\
&\leq H_\mu(\xi \mid \eta) + H_\mu(\xi(1,n-2) \mid \eta(1,n-1)) \\
&\leq H_\mu(\xi \mid \eta) + H_\mu(T^{-1}\xi(0,n-2) \mid T^{-1}\eta(0,n-2)) \\
&= H_\mu(\xi \mid \eta) + H_\mu(\xi(0,n-2) \mid \eta(0,n-2)) \\
&\qquad \cdots \\
&\leq nH_\mu(\xi \mid \eta).
\end{aligned}
$$

(ii) We have

$$\frac{k}{nk} H_\mu \left(\bigvee_0^{nk-1} T^{-r}\xi \right) = \frac{1}{n} H_\mu \left(\bigvee_{j=0}^{n-1} (T^k)^{-j} (\xi \vee T^{-1}\xi \vee \cdots \vee T^{-k+1}\xi) \right).$$

Hence $kh_\mu(T,\xi) = h_\mu(T^k, \eta)$ where $\eta = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-k+1}\xi$. Since $\eta \geq \xi$, we deduce that $kh_\mu(T) = h_\mu(T^k)$.

The claim $h_\mu(T^{-1}) = h_\mu(T)$ follows from the invariance of μ and the fact

$$\xi(0, n-1) = \xi \vee \cdots \vee T^{-n+1}\xi = T^{-n+1}(\xi \vee \cdots \vee T^{n-1}\xi).$$

(iii) Let A be such that $\mu(A) = 1$, $\nu(A) = 0$. Set $B = \bigcup_{m=1}^\infty \bigcap_{n \geq m} T^{-n}(A)$. We can readily show that $T^{-1}B = B$ and that $\mu(B) = 1$, $\nu(B) = 0$. Set $\beta = \{B, X - B\}$ and given a partition ξ, define $\hat{\xi} = \xi \vee \beta$. If $\gamma = \alpha\mu + (1-\alpha)\nu$, then

$$H_\gamma(\eta_n) = \alpha H_\mu(\xi_n) + (1-\alpha) H_\nu(\xi_n) + \alpha \log \alpha + (1-\alpha)\log(1-\alpha), \tag{3.8}$$

where $\eta_n = \hat{\xi} \vee \cdots \vee T^{-n+1}\hat{\xi}$ and $\xi_n = \xi \vee \cdots \vee T^{-n+1}\xi$. To see this, observe that if $C \in \eta_n$ and $\varphi(z) = z \log z$, then

$$\varphi(\gamma(C)) = \begin{cases} \alpha\mu(C)\log(\alpha\mu(C)) & \text{if } C \subseteq B, \\ (1-\alpha)\nu(C)\log((1-\alpha)\nu(C)) & \text{if } C \subseteq X - B. \end{cases}$$

This clearly implies (3.8). Hence,

$$h_\gamma(T, \hat{\xi}) = \alpha h_\mu(T,\xi) + (1-\alpha) h_\nu(T,\xi).$$

From this we deduce

$$h_\gamma(T) \leq \alpha h_\mu(T) + (1-\alpha) h_\nu(T).$$

This and Exercise 3.8(ii) complete the proof. □

Exercise 3.16 (Rokhlin Metric) Define $d(\eta, \xi) = H_\mu(\eta \mid \xi) + H_\mu(\xi \mid \eta)$. Show that d is a metric on the space of μ-partition. □

In practice, we would like to know whether $h_\mu(T) = h_\mu(T,\xi)$ for a partition ξ. In the next theorem, we provide a sufficient condition for this.

Theorem 3.17 *Let ξ be a finite μ-partition and assume that the σ-algebra consisting of $T^{-n}(C)$, $n \in \mathbb{N}$, $C \in \xi$, equals to the Borel σ-algebra. Then $h_\mu(T) = h_\mu(T,\xi)$.*

Proof. For a given partition η, we apply Proposition 3.9 to assert

$$h_\mu(T,\eta) \le h_\mu(T, \xi \vee \cdots \vee T^{-n+1}\xi) + H_\mu(\eta \mid \xi \vee \cdots \vee T^{-n+1}\xi). \tag{3.9}$$

From the definition, it is not hard to see that indeed $h_\mu(T, \xi \vee \cdots \vee T^{-n+1}\xi) = h_\mu(T,\xi)$. From this and (3.9), it suffices to show that for every partition η,

$$\lim_{n\to\infty} H_\mu(\eta \mid \xi \vee \cdots \vee T^{-n+1}\xi) = 0. \tag{3.10}$$

To believe this, observe that if $\eta \le \alpha$, then $H_\mu(\eta \mid \alpha) = 0$ because

$$I_{\eta|\alpha}(x) = -\log \frac{\mu(C_\eta(x) \cap C_\alpha(x))}{\mu(C_\alpha(x))} = -\log \frac{\mu(C_\alpha(x))}{\mu(C_\alpha(x))} = 0.$$

Now if the σ-algebra generated by all $\xi_n = \xi \vee \cdots \vee T^{-n+1}\xi$, $n \in \mathbb{N}^*$ is the full σ-algebra, then $\eta \le \xi_n$ at least asymptotically. We may prove this by the Martingale Convergence Theorem. In fact if $\mathcal{F}_n$ is the σ-algebra generated by ξ_n, then

$$\begin{aligned} \mu(C_\eta(x) \mid C_{\xi_n}(x)) &= \sum_{A\in\eta} 1\!\!1_A(x)\mu(A \mid \mathcal{F}_n)(x) \\ &\to \sum_{A\in\eta} 1\!\!1_A(x)\mu(A \mid \mathcal{F}_\infty)(x) = \sum_{A\in\eta} 1\!\!1_A(x) 1\!\!1_A(x) = 1. \end{aligned}$$

This and (3.6) imply that $H_\mu(\eta \mid \xi_n) = -\int \log \mu(C_\eta(x) \mid C_{\xi_n}(x))\mu(dx) \to 0$. $\square$

Example 3.18 Consider the dynamical system of Example 3.2. Let ξ be as in Example 3.2. The condition of Theorem 3.17 is satisfied for such ξ and we deduce

$$h_{\mu_p}(T) = -\sum_0^{m-1} p_j \log p_j.$$

$\square$

Example 3.19 Consider a translation $T(x) = x + \alpha \pmod 1$ in dimension 1. If $\alpha \in \mathbb{Q}$, then $T^m =$ identity for some $m \in \mathbb{N}^*$. This implies that $h_\mu(T) = \frac{1}{m} h_\mu(T^m) = 0$ where μ is the Lebesgue measure. If α is irrational, then set $\xi = \{[0,1/2),[1/2,1)\}$. By the denseness of $\{T^{-n}(1/2) : n \in \mathbb{N}\}$, we deduce that ξ satisfies the condition of Theorem 3.17. As a result, $h_\mu(T) = h_\mu(T,\xi)$. On the other hand $\xi \vee \cdots \vee T^{-n+1}\xi$ consists of $2n$ elements. From this and Exercise 3.8(i), $H_\mu(\xi \vee \cdots \vee T^{-n+1}\xi) \le \log(2n)$. This in turn implies $\lim_{n\to\infty} \frac{1}{n} H_\mu(\xi \vee \cdots \vee T^{-n+1}\xi) = 0$. Thus $h_\mu(T) = 0$. $\square$

In fact we can show that the entropy of a translation is zero using the fact that the topological entropy of a translation zero. More generally we always have

$$\sup_{\mu\in\mathcal{I}_T} h_\mu(T) = h_{\text{top}}(T). \tag{3.11}$$

To prepare for the proof of (3.11), let us make some definitions. Given $r, \delta > 0$, we define $S^n_{T,d}(r,\delta)$ to be the smallest k such that there exists a set E with $\#E = k$ and $\mu\left(\bigcup_{x\in E} B^n_{T,d}(x,r)\right) > 1-\delta$. We then define

$$\hat{h}_\mu(T) = \lim_{\delta\to 0}\lim_{r\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S^n_{T,d}(r,\delta).$$

Evidently

$$\hat{h}_\mu(T) \le h_{\text{top}}(T). \tag{3.12}$$

Moreover, a theorem of Katok asserts:

Theorem 3.20 *For every ergodic $\mu\in\mathcal{I}_T$, we have $h_\mu(T)\le\hat{h}_\mu(T)$.*

Proof. Let $\xi = \{C_1,\dots,C_\ell\}$ be a μ-partition. Choose compact sets $K_1,\dots,K_\ell$ with $K_j\subseteq C_j$ such that $\mu(C_j - K_j)\le\epsilon$ for $j = 1,\dots,\ell$. Let $K_0 = X - K_1\cup\dots\cup K_\ell$ and put $\eta = \{K_0, K_1,\dots,K_\ell\}$. Evidently η is a partition and

$$\begin{aligned} H_\mu(\xi\mid\eta) &= -\sum_{i,j}\mu(C_i\cap K_j)\log\frac{\mu(C_i\cap K_j)}{\mu(K_j)} \\ &= -\sum_{i}\mu(C_i\cap K_0)\log\frac{\mu(C_i\cap K_0)}{\mu(K_0)} \\ &= -\mu(K_0)\sum_i\frac{\mu(C_i\cap K_0)}{\mu(K_0)}\log\frac{\mu(C_i\cap K_0)}{\mu(K_0)} \\ &\le \mu(K_0)\log\ell \le \epsilon\ell\log\ell \end{aligned}$$

by Exercise 3.14(i). From this and Proposition 3.15(i) we deduce,

$$h_\mu(T,\xi)\le h_\mu(T,\eta) + \varepsilon\ell\log\ell. \tag{3.13}$$

Set $\eta_n = \eta\vee\dots\vee T^{-n+1}\eta$. Recall that by Theorem 3.9,

$$\lim_{n\to\infty}\frac{1}{n}\log\mu(C_n(x)) = -h_\mu(T,\eta)$$

in L^1-sense, when $C_n(x) = C_{\eta_n}(x)$. Choose a subsequence $n_j \to \infty$ so that

$$\lim_{n_j \to \infty} \frac{1}{n_j} \log \mu(C_{n_j}(x)) = -h_\mu(T, \eta),$$

μ-almost everywhere. Pick $\varepsilon' > 0$ and set

$$X_N = \left\{ x \in X : \frac{1}{n_j} \log \mu(C_{n_j}(x)) \leq -h_\mu(T, \eta) + 1 \text{ for } n_j > N \right\}.$$

Since $\mu(X_N) \to 1$ as $N \to \infty$, for every $\delta > 0$, there exists N such that $\mu(X_N) > 1 - \delta$. Let

$$r = \frac{1}{2} \min\{\operatorname{dist}(K_i, K_j) : i \neq j,\ i, j \in \{1, \dots, \ell\}\}.$$

Clearly a ball $B_d(x, r)$ intersects at most two elements of η, one K_j with $j \in \{1, \dots, n\}$ and perhaps K_0. We now argue that $B_{d_n}(x, r)$ intersects at most 2^n elements of η_n. To see this, observe

$$B_{d_n}(x, r) = B_d(x, r) \cap T^{-1}(B_d(T(x), r)) \cap \cdots \cap T^{-n+1}(B_d(T^{n-1}(x), r)).$$

Also, if $A \in \eta_n$, then $A = A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n+1}(A_{n-1})$ with $A_j \in \eta$. Now if $B^n_{T,d} \cap A \neq \emptyset$, then $T^{-j}(B_d(T^j(x), r)) \cap T^{-j}(A_j) \neq \emptyset$ for $j = 0, \dots, n-1$. Hence $B_d(T^j(x), r) \cap A_j \neq \emptyset$ for $j = 0, \dots, n-1$. As a result, there are at most 2^n-many choices for A. Now assume that $\mu\left(\bigcup_{x \in E} B_{d_n}(x, r)\right) > 1 - \delta$. We would like to bound $\#E$ from below. First observe

$$\begin{aligned} 1 - 2\delta &\leq \mu\left(\bigcup_{x \in E} B_{d_n}(x, r) \cap X_N\right) \leq \sum_{x \in E} \mu(B_{d_n}(x, r) \cap X_N) \\ &= \sum_{x \in E} \sum_{A \in \eta_n} \mu(B_{d_n}(x, r) \cap X_N \cap A). \end{aligned}$$

But if $B_{d_n}(x, r) \cap X_N \cap A \neq \emptyset$ for $n = n_j > N$, then

$$\mu(B_{d_n}(x, r) \cap X_N \cap A) \leq \mu(A) \leq e^{-n(h_\mu(T, \eta) + \varepsilon')}.$$

As a result,

$$1 - 2\delta \leq 2^{n_j} e^{-n(h_\mu(T, \eta) - \varepsilon')} (\#E).$$

Hence

$$h_\mu(T, \eta) \leq \lim_{n_j \to \infty} \frac{1}{n_j} \log S^{n_j}_{T,d}(r, \delta) + \varepsilon' + \log 2.$$

From this we deduce that $h_\mu(T,\eta) \le \hat{h}_\mu(T) + \varepsilon' + \log 2$. From this and (3.13) we learn that $h_\mu(T,\xi) \le \hat{h}_\mu(T) + \varepsilon\ell\log\ell + \varepsilon' + \log 2$. By sending $\varepsilon, \varepsilon' \to 0$ and taking supremum over ξ we deduce $h_\mu(T) \le \hat{h}_\mu(T) + \log 2$. Since this is true no matter what T is, we learn

$$h_\mu(T) = \frac{1}{m} h_\mu(T^m) \le \frac{1}{m}\hat{h}_\mu(T^m) + \frac{\log 2}{m}.$$

A repetition of the proof of Proposition 3.1(iii) yields $\frac{1}{m}\hat{h}_\mu(T^m) = \hat{h}_\mu(T)$. We then pass to the limit $m \to \infty$ to complete the proof of Theorem. □

Example 3.21 Consider $T : \mathbb{T}^2 \to \mathbb{T}^2$, $Tx = Ax \pmod 1$ with A an integer matrix with $\det A = 1$. We assume that A is symmetric and its eigenvalues $\lambda_1, \lambda_2 = \lambda_1^{-1}$ satisfy $|\lambda_1| > 1 > |\lambda_2|$. We claim that if μ is the Lebesgue measure, then $h_\mu(T) \ge \log|\lambda_1|$. In case of $T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, we can use our result $h_{\text{top}}(T) \le \log|\lambda_1|$ from Example 3.6 to conclude that in fact $h_\mu(T) = h_{\text{top}}(T) = \log|\lambda_1|$.

For $h_\mu(T) \ge \log|\lambda_1|$ we use an idea of Hopf. First observe that by the invariance of μ with respect to T, $H_\mu(T^{-n}\xi \vee \cdots \vee T^n\xi) = H_\mu(\xi \vee \cdots \vee T^{-2n}\xi)$. Hence it suffices to study $\lim_{n\to\infty} \frac{1}{2n} H_\mu(T^{-n}\xi \vee \cdots \vee T^n\xi)$. For estimating this, we show that the area of each $C \in \eta_n = T^{-n}\xi \vee \cdots \vee T^n\xi$ is exponentially small. This is achieved by showing that $\text{diam}(C) = O(|\lambda_1|^{-n})$. There is a natural metric on $\mathbb{T}^2$ that is closely related to the Euclidean distance. Given two points $a = (a_1, a_2)$, $b = (b_1, b_2)$, we define $d(a,b) = (\bar{d}(a_1,b_1)^2 + \bar{d}(a_2,b_2)^2)^{1/2}$ where $\bar{d}(x,y)$ is the length of shortest arc connecting x to y.

Pick $C \in \eta_n$. To estimate $\text{diam}(C)$, we pick two points $x, y \in C$. Let v_1, v_2 be the eigenvectors corresponding to λ_1 and λ_2. We draw a line through x in direction v_1 and a line through y in direction v_2. Assume that these lines intersect at z. We also assume that $\text{diam}(A) < \frac{1}{20}$ for every $A \in \eta$. Hence the same is true for $A \in \eta_n$. To estimate $d(x,y)$, it suffices to estimate $d(x,z)$ and $d(y,z)$. Let us start with $d(x,z)$. Suppose that we have $|T^n(x) - T^n(z)| < \frac{1}{10}$. Then $d(x,z) \le |x - z| = |T^{-n}(T^n(x)) - T^{-n}(T^n(z))| = |\lambda_1|^{-n}|T^n(x) - T^n(z)|$ because T^{-1} contracts in v_1 direction with rate $|\lambda_1|^{-1} = |\lambda_2|$. This would imply that $d(x,z) \le |\lambda_1|^{-n}/10$. To show that $|T^n(x) - T^n(z)| < \frac{1}{10}$, first observe that $d(T^n(z), T^n(y)) \le |\lambda_1|^{-n} d(y,z) \le |\lambda_1|^{-n}/20 < \frac{1}{20}$ and $d(T^n(x), T^n(y)) \le \frac{1}{20}$ because $T^n(x), T^n(y)$ belongs to a member of ξ. As a result, $d(T^n(x), T^n(z)) < \frac{1}{10}$. We actually need $|T^n(x) - T^n(z)| < \frac{1}{10}$. To prove this, first note that the above argument can be used to show that indeed $d(T^k(x), T^k(y)) < \frac{1}{10}$ for $k = 0, 1, \ldots, n$. We now use induction to show that $|T^k(x) - T^k(y)| < \frac{1}{10}$ for $k = 0, 1, \ldots, n$. To see this, observe that if $u = x - y$, then $|u| < \frac{1}{10}$, and since $T(u) = T(x) - T(y)$, we also have $|T(u)| < \frac{1}{10}$. Indeed $|T(u)| = |\lambda_1||u|$ and since $|u| < \frac{1}{10}$, $|\lambda_1||u| < \frac{1}{2}$, which means that $d(T(x)T(y)) < \frac{1}{10}$ does imply $|T(x) - T(y)| < \frac{1}{10}$. Note that we are using the fact that $d(T(x), T(y)) = |T(x) - T(y) + a|$ for some $a \in \mathbb{Z}^2$ and since $|\lambda_1||u| < \frac{1}{2}$, we must have $a = 0$. By induction we can extend this to all $k \le n$. In the same way we prove $d(y,z) < \frac{1}{10}|\lambda_1|^{-n}$. Hence $d(x,y) < \frac{1}{5}|\lambda_1|^n$ for $n \in \mathbb{N}$. This implies that

$\mu(C) \leq$ constant $\times|\lambda_1|^{-2n}$ for $C \in \eta_n$. This evidently implies that $\frac{1}{2n} H_\mu(\eta_n) \geq \log|\lambda_1| + o(1)$, and as $n \to \infty$ we deduce that $h_\mu(T) \geq \log|\lambda_1|$. $\square$

We end this section by establishing (3.11). Half of (3.11) is a consequence of Theorem 3.20 and (3.12). It remains to show this:

Theorem 3.22 $h_{\text{top}}(T) \leq \sup_{\mu \in \mathcal{I}_T} h_\mu(T)$.

Proof. For each n, pick a set E_n such that $\#E_n = N^n_{T,d}(r) = N^n(r)$. In other words, E_n is a maximal set satisfying $d_n(x,y) \geq r$ for $x, y \in E_n$ with $x \neq y$. Define $\mu_n = \frac{1}{N^n(r)} \sum_{x \in E_n} \delta_x$ and

$$\hat{\mu}_n = \frac{1}{n} \sum_0^{n-1} T^{-j} \mu_n = \frac{1}{n} \sum_0^{n-1} \mathcal{A}^j \mu_n.$$

This means that

$$\int f d\hat{\mu}_n = \frac{1}{n} \sum_0^{n-1} \int f(T^j(x)) \mu_n(dx) = \frac{1}{n} \sum_0^{n-1} \frac{1}{N^n(r)} \sum_{x \in E_n} f(T^j(x)).$$

Let $\bar{\mu}$ be a limit point of the sequence $\{\hat{\mu}_n\}$. It is not hard to show that $\bar{\mu} \in \mathcal{I}_T$ because

$$\lim_{n\to\infty} T^{-1}\bar{\mu} - \bar{\mu} = (T^{-n}\mu_n - \mu_n)/n \to 0,$$

as $n \to \infty$. It remains to show

$$\limsup_{n\to\infty} \frac{1}{n} \log N^n(r) \leq h_{\bar{\mu}}(T) \leq \sup_\mu h_\mu(T). \tag{3.14}$$

For (3.14), it suffices to find a partition ξ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log N^n(r) \leq h_{\bar{\mu}}(T, \xi). \tag{3.15}$$

Fix $\delta > 0$. We first would like to find a partition $\xi = \{C_1 \dots C_\ell\}$ such that $\text{diam}(C_j) \leq \delta$ for $j = 1, \dots, \ell$, and $\bar{\mu}(\partial C_j) = 0$ where ∂C_j denotes the boundary of C_j. The construction of such a partition ξ is straightforward. First, if $B_d(x,a)$ is a ball of radius a, then we consider

$$\bigcup \{\partial B_d(x, a') : a - \epsilon \leq a' \leq a\},$$

to observe that there exists $a' \in (a - \epsilon, a)$ such that $\mu(\partial B_d(x, a')) = 0$. From this, we learn that we can cover X by finitely many balls B_j, $j = 1, \dots, \ell$ of radius at most $\frac{\delta}{2}$ such that $\bar{\mu}(\partial B_j) = 0$ for $j = 1, \dots, \ell$. We finally define $\xi = \{C_1 \dots C_\ell\}$ by $C_1 = \bar{B}_1$, $C_2 = \bar{B}_2 - \bar{B}_1, \dots, C_n = \bar{B}_n - \bigcup_{j=1}^{n-1} \bar{B}_j$. Since $\partial C_j \subseteq \bigcup_{k=1}^{\ell} \partial B_k$, we are done. We now argue that the partition $\xi_n = \xi \vee \cdots \vee T^{-n+1}\xi$ enjoys the same property; $\bar{\mu}(\partial C) = 0$ if $C \in \xi_n$. This is because $\partial C \subseteq \bigcup_{A\in\xi} \bigcup_{k=0}^{n-1} T^{-j}(\partial A)$ and by invariance, $\bar{\mu}(T^{-j}(\partial A)) = \bar{\mu}(\partial A) = 0$. Such a partition is advantageous for our purposes because if η is a partition with $\alpha(\partial A) = 0$ for $A \in \eta$ and every n, and if $\alpha_n \Rightarrow \alpha$, then $\alpha_n(A) \to \alpha(A)$ for every $A \in \eta$, and, as a result, $H_{\alpha_n}(\eta) \to H_\alpha(\eta)$.

First observe that $H_{\mu_n}(\xi_n) = \log N^n(r)$ provided that $\mathrm{diam}(C) < r$ for every $C \in \xi$. Indeed $\mathrm{diam}_n(A) < r$ for every $A \in \xi_n$ if $\mathrm{diam}_n(A)$ denotes the diameter of A with respect to d_n. This in turn implies that $\mu_n(A) = 0$ or $\frac{1}{N^n(r)}$ for every $A \in \xi_n$ simply because each such A contains at most one element of E_n. As a result $H_{\mu_n}(\xi_n) = N^n(r)\left(-\log \frac{1}{N^n(r)}\right)\frac{1}{N^n(r)} = \log N^n(r)$. Using this, we would like to estimate from below $H_{\hat{\mu}_n}(\xi_m)$. Recall that only a subsequence of $\hat{\mu}_n$ converges, say $\lim_{j\to\infty} \hat{\mu}_{n_j} = \mu$. Let $0 \le k < m < n$. Set $a(k) = \left[\frac{n-k}{m}\right]$ so that we can write

$$\{0, 1, \ldots, n-1\} = \{k + tm + i : 0 \le t < a(k),\ 0 \le i < m\} \cup R$$

with $R = \{0, 1, \ldots, k-1\} \cup \{k + ma(k),\ k + ma(k) + 1, \ldots, n-1\} =: R_1 \cup R_2$. Clearly $\#R_1 \le m$, $\#R_2 \le m$. We then write

$$\xi_n = \bigvee_{t=0}^{a(k)-1} T^{-(tm+k)}(\xi \vee \cdots \vee T^{-m+1}\xi) \vee \bigvee_{i\in R} T^{-i}\xi.$$

Using $H(\alpha \vee \beta) \le H(\alpha) + H(\beta)$ we learn,

$$\begin{aligned}
\log N^n(r) = H_{\mu_n}(\xi_n) &\le \sum_{t=0}^{a(k)-1} H_{\mu_n}(T^{-(tm+k)}\xi_m) + \sum_{i\in R} H_{\mu_n}(T^{-i}\xi) \\
&= \sum_{t=0}^{a(k)-1} H_{\mathcal{A}^{tm+k}\mu_n}(\xi_m) + \sum_{i\in R} H_{\mu_n}(T^{-i}\xi) \\
&\le \sum_{t=0}^{a(k)-1} H_{\mathcal{A}^{tm+k}\mu_n}(\xi_m) + 2m\log(\#\xi).
\end{aligned}$$

This is true for every k. Hence

$$\begin{aligned}
m \log N^n(r) &\le \sum_{k=0}^{m-1} \sum_{t=0}^{a(k)-1} H_{\mathcal{A}^{tm+k}\mu_n}(\xi_m) + 2m^2\log(\#\xi) \\
&\le \sum_{j=0}^{n-1} H_{\mathcal{A}^j\mu_n}(\xi_m) + 2m^2\log(\#\xi) \\
&\le nH_{\hat{\mu}_n}(\xi_m) + 2m^2\log(\#\xi),
\end{aligned}$$

where for the last inequality we used Exercise 3.14(ii). As a result,

$$\frac{1}{n}\log N^n(r) \le \frac{1}{m}H_{\hat{\mu}_n}(\xi_m) + 2\frac{m}{n}\log(\#\xi).$$

Choose any sequence of $\{n_j\}$ such that the limit of ${n_j}^{-1}\log N^{n_j}(r)$ exists and choose a subsequence of it $\{n_j'\}$ so that $\hat{\mu}_{n_j'} \Rightarrow \bar{\mu}$ for some $\bar{\mu} \in \mathcal{I}_T$. We then have

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{n_j}\log N^{n_j}(r) &= \lim_{n_j'\to\infty}\frac{1}{n_j'}\log N^{n_j'}(r) \le \lim_{n_j'\to\infty}\frac{1}{m}H_{\hat{\mu}_{n_j'}}(\xi_m)\\ &= \frac{1}{m}H_{\bar{\mu}}(\xi_m).\end{aligned}$$

We now send m to infinity to deduce

$$\lim_{j\to\infty}\frac{1}{n_j}\log N^{n_j}(r) \le h_{\bar{\mu}}(T,\xi) \le \sup_{\mu\in\mathcal{I}_T} h_\mu(T).$$

Since $\{n_j\}$ can be chosen any sequence for which the limit exists, we conclude

$$\limsup_{n\to\infty}\frac{1}{n}\log N^n(r) \le \sup_{\mu\in\mathcal{I}_T} h_\mu(T),$$

as desired. □

Exercise 3.23 Let $\alpha_n \Rightarrow \alpha$ and $\alpha(\partial A) = 0$. Deduce that $\alpha_n(A) \to \alpha(A)$. (*Hint:* For such A, we can approximate the indicator function of A with continuous functions.)

Theorem 3.8 provides us with a rather local recipe for calculating the entropy. It turns out that there is another local recipe for calculating the entropy that is related to $\hat{h}_\mu(T)$. A theorem of Brin and Katok[BK] asserts that if $\mu \in \mathcal{I}_T$ is ergodic, then $\frac{1}{n}\log\mu(B_{d_n}(x,r))$ approximates $h_\mu(T)$. More precisely,

$$h_\mu(T) = \lim_{r\to 0}\limsup_{n\to\infty}\left[-\frac{1}{n}\log\mu(B_{d_n}(x,r))\right]$$

for μ-almost all x.

4 Lyapunov Exponents

In section 3 we learned that if $\mu \in \mathcal{I}_T$ with $T : X \to X$ continuous and X a compact metric space, then $h_\mu(T) \le h_{\text{top}}(T)$. It turns out that for a nice hyperbolic system a lot more can be said. For example, if X is a manifold with a volume measure m, then there exists a unique $\bar\mu = \mu_{SRB} \in \mathcal{I}_T$ such that $h_{\text{top}}(T) = h_{\bar\mu}(T)$, and if $I(\mu) = h_{\text{top}}(T) - h_\mu(T) = h_{\bar\mu}(T) - h_\mu(T)$, then, we roughly have

$$m\left\{x : \frac{1}{n}\sum_{0}^{n-1} \delta_{T^j(x)} \text{ is near } \mu\right\} \approx e^{-nI(\mu)}.$$

This is known in probability theory as a large deviation principle. Recall that the entropy $h_\mu(T)$ is affine in μ. Hence I is affine, and its convex conjugate, the pressure, is defined by

$$\Phi(F) = \sup_\mu \left(\int F d\mu - I(\mu)\right)$$

satisfies

$$\Phi(F) = \lim_{n\to\infty} \frac{1}{n} \log \int \exp\left(\sum_{0}^{n-1} F(T^j(x))\right) m(dx).$$

Also, Pesin's formula asserts that $h_{\text{top}}(T) = h_{\bar\mu}(T) = \sum_i n_i l_i^+(\bar\mu)$ where l_i's are the logarithm of the rate of expansions and n_i is the multiplicity of l_i.

For general $\mu \in \mathcal{I}_T$, we have Ruelle's inequality $h_\mu(T) \le \sum_i n_i l_i^+(\mu)$. In the case of $T(x) = Ax (\operatorname{mod} 1)$, we simply have $l_i = \log|\lambda_i|$ where λ_i's are the eigenvalues of A. In this section we define the *Lyapunov exponents* l_i's and establish the Ruelle's inequality.

Consider a transformation $T : M \to M$ where M is a compact C^1 manifold and T is a C^1 transformation. To study the rate of expansion and contraction of T, we may study $D_xT^n : T_xM \to T_{T^n(x)}M$. We certainly have

$$D_xT^n = D_{T^{n-1}(x)}T \dots D_{T(x)}TD_xT. \tag{4.1}$$

If we write $A(x) = D_xT$, then (4.1) can be written as

$$A_n(x) := D_xT^n = A(T^{n-1}(x)) \dots A(T(x))A(x). \tag{4.2}$$

Here we are interested in the long time behavior of the dynamical system associated with $F : TM \to TM$ that is defined by $F(x, v) = (T(x), (D_xT)(v)) = (T(x), A(x)v)$. Let us assume that M is a Riemannian manifold. This means that for each x there exists an inner product $\langle\cdot,\cdot\rangle_x$ and (an associated norm $|\ |_x$) that varies continuously with x. The formula

(4.1) suggests an exponential growth rate for D_xT^n. For example, if we take the norm of both sides of (4.2) we obtain

$$\|A_n(x)\| \leq \prod_0^{n-1} \|A(T^j(x))\|.$$

Set $S_n(x) = \log \|A_n(x)\|$. We then have that $S_0 = 0$ and

$$S_{n+m}(x) \leq S_n(x) + S_m(T^n(x)). \tag{4.3}$$

Theorem 4.1 *Let T be a diffeomorphism and assume that $\mu \in \mathcal{I}_T^{ex}$. Then there exists $\lambda \in \mathbb{R}$ such that*

$$\mu\left\{x : \frac{1}{n} \log \|D_xT^n\| \to \lambda\right\} = 1.$$

This theorem is an immediate consequence of *Kingman's subadditive ergodic theorem*:

Theorem 4.2 *Let $\mu \in \mathcal{I}_T^{ex}$ and suppose that $\{S_n(\cdot) : n \to \mathbb{N}\}$ is a sequence of $L^1(\mu)$ functions satisfying* (4.3)*. Then*

$$\mu\left\{x : \frac{1}{n} S_n(x) \to \lambda\right\} = 1$$

for $\lambda = \inf_n \left\{\frac{1}{n}\int S_n d\mu\right\} \in [-\infty, +\infty)$.

Proof of Theorem 4.1. On account of Theorem 8.2, we only need to show $\lambda \neq -\infty$. From $id = D_{T^n(x)}T^{-n}\ D_xT^n$, we learn that $1 \leq \|D_{T^n(x)}T^{-n}\| \|D_xT^n\|$. Let us write α for $\sup_x \|D_xT^{-1}\|$. Then

$$\|D_{T^n(x)}T^{-n}\| = \| \dots D_{T^{n-1}(x)}T^{-1} D_{T^n(x)}T^{-1}\| \leq \alpha^n.$$

Hence $\|D_xT^n\| \geq \alpha^{-n}$ which implies that $\lambda \geq -\log \alpha$. □

To prepare for the proof of Theorem 8.2, let us state a useful fact regarding the precompactness of a set of measures.

Exercise 4.3 Let X be a Polish (separable metric complete) space. Suppose $\{\mu_N\}$ is a sequence of probability measures on X. Assume that for every $\delta > 0$ there exists a compact set K_δ such that $\sup_N \mu_N(K_\delta^c) \leq \delta$. Show that $\{\mu_N\}$ has a convergent subsequence.

Proof of Theorem 4.2. Fix $N > 0$. Any $n > N$ can be written as $n = kN + r$ for some $k \in \mathbb{N}^*$ and $r \in \{0, 1, \dots, N-1\}$. As a result, if $i \in \{1, \dots, N\}$, then $n = i + lN + m$ with $l = l(i) = \begin{cases} k & \text{if } r \geq i \\ k-1 & \text{if } r < i \end{cases}$, $m = m(i) = \begin{cases} r - i & \text{if } r > i \\ r - i + N & \text{if } r < i. \end{cases}$ By subadditivity,

$$\begin{aligned} S_n(x) &\leq S_i(x) + S_{lN}(T^i(x)) + S_m(T^{i+lN}(x)) \\ &\leq S_i(x) + \sum_{j=0}^{l-1} S_N(T^{i+jN}(x)) + S_m(T^{i+lN}(x)). \end{aligned}$$

We now some over i to obtain

$$S_n(x) \leq \frac{1}{N}\sum_1^N S_i(x) + \sum_1^{l(i)N} \frac{S_N}{N}(T^i(x)) + \frac{1}{N}\sum_1^N S_{m(i)}(T^{i+lN}(x)).$$

Hence

$$\frac{1}{n}S_n(x) \leq \frac{1}{n}\sum_1^n \frac{S_N}{N}(T^i(x)) + R_{n,N}(x),$$

where $\|R_{n,N}\|_{L^1} \leq \text{constant} \times \frac{N}{n}$, because $\int |S_l(T^r)|d\mu = \int |S_l| d\mu$. By the Ergodic Theorem,

$$\limsup_{n\to\infty} \frac{1}{n}S_n(x) \leq \int \frac{S_N}{N} d\mu.$$

Since N is arbitrary,

$$\limsup_{n\to\infty} \frac{1}{n}S_n(x) \leq \lambda, \tag{4.4}$$

almost everywhere and in L^1-sense. For the converse, we only need to consider the case $\lambda > -\infty$.

For the reverse inequality, let us take a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ that is nondecreasing in each of its arguments. We certainly have

$$\begin{aligned} \int \varphi(S_1, \dots, S_n) d\mu &= \int \varphi(S_1 \circ T^k, \dots, S_n \circ T^k) d\mu \\ &\geq \int \varphi(S_{k+1} - S_k, S_{k+1} - S_k, \dots, S_{k+n} - S_k) d\mu \end{aligned}$$

for every k. Hence

$$\begin{aligned} \int \varphi(S_1, \dots, S_n) d\mu &\geq \frac{1}{N}\sum_0^{N-1} \int \varphi(S_{k+1} - S_k, \dots, S_{k+n} - S_k) d\mu \\ &= \int \varphi(S_{k+1} - S_k, \dots, S_{k+n} - S_k) d\mu \nu_N(dk) \end{aligned} \tag{4.5}$$

where $\nu_N = \frac{1}{N}\sum_0^{N-1}\delta_j$. We think of k as a random number that is chosen uniformly from 0 to $N-1$. To this end let us define $\Omega = \mathbb{R}^{\mathbb{Z}^+} = \{w : \mathbb{Z}^+ \to \mathbb{R}\}$ and $T : M \times \mathbb{N} \to \Omega$ such that $T(x,k) = w$ with $w(j) = S_{k+j}(x) - S_{k+j-1}(x)$. We then define a measure μ_N on Ω by $\mu_N(A) = (\mu \times \nu_N)(T^{-1}(A))$. Using this we can rewrite (4.5) as

$$\int \varphi(S_1,\dots,S)d\mu \geq \int \varphi(w(1), w(1)+w(2),\dots,w(1)+\cdots+w(n))\mu_N(dw). \tag{4.6}$$

We want to pass to the limit $N \to \infty$. Note that Ω is not a compact space. To show that μ_N has a convergent subsequence, observe

$$\begin{aligned}
\int w(j)^+\mu_N(dw) &= \int (S_{k+j}(x) - S_{k+j-1}(x))^+\mu(dx)\nu_N(dx) \\
&= \frac{1}{N}\sum_0^{N-1}\int (S_{k+j}(x) - S_{k+j-1}(x))^+\mu(dx) \\
&\leq \frac{1}{N}\sum_0^{N-1}\int (S_1(T^{k+j-1}(x)))^+\mu(dx) = \int S_1^+ d\mu, \\
\int w(j)\mu_N(dw) &= \frac{1}{N}\sum_0^{N-1}\int (S_{k+j}(x) - S_{k+j-1}(x))\mu(dx) \\
&= \frac{1}{N}\int (S_{j+N-1} - S_{j-1})d\mu \geq \lambda\frac{j+N-1}{N} - \frac{1}{N}\int S_{j-1}d\mu \\
&> -\infty,
\end{aligned}$$

uniformly in N. As a result $\int w(j)^-\mu_N(dw)$ is uniformly bounded. Hence

$$\sup_N \int |w_j| d\mu_N = \beta_j < \infty$$

for every j. We now define

$$K_\delta = \left\{ w : |w_j| \leq \frac{2^{j+1}\beta_j}{\delta} \right\}.$$

The set K_δ is compact and

$$\mu_N(K_\delta^c) \leq \frac{1}{2}\sum_j 2^{-j}\beta_j^{-1}\delta\beta_j = \delta.$$

From this and Exercise 4.3 we deduce that μ_N has a convergent subsequence. Let $\bar\mu$ be a limit point and set $\bar S_j = w(1) + \cdots + w(j)$. By (4.6),

$$\int \varphi(S_1,\dots,S)d\mu \geq \int \varphi(\bar S_1,\dots,\bar S_n)d\bar\mu, \tag{4.7}$$

for every continuous monotonically decreasing φ. We now define $\tau : \Omega \to \Omega$ by $(\tau w)(j) = w(j+1)$. It is not hard to see $\bar{\mu} \in \mathcal{I}_\tau$. By Ergodic Theorem, $\frac{1}{n}\bar{S}_n \to Z$ for almost all w. Moreover, $\int Z d\bar{\mu} = \int w(1)\bar{\mu}(dw) = \lim_{N\to\infty} \int \frac{1}{N}(S_N - S_0)d\mu = \lambda$. We use (**??**) to assert that for every bounded continuous increasing ψ,

$$\int \psi\left(\min_{k\le n\le l} \frac{S_n}{n}\right) d\mu \ge \int \psi\left(\min_{k\le n\le l} \frac{\bar{S}_n}{n}\right) d\bar{\mu}.$$

We now apply the bounded convergence theorem to deduce

$$\int \psi(\underline{S})d\mu \ge \int \psi(Z)d\bar{\mu}$$

where $\underline{S} = \liminf_{n\to\infty} \frac{S_n}{n}$. Choose $\psi(z) = \psi^{r,l}(z) = (zv(-l)) \wedge r$, $\psi_l(z) = zv(-l)$. We then have

$$\int \psi_l(\underline{S})d\mu \ge \int \psi^{r,l}(\underline{S})d\mu \ge \int \psi^{r,l}(Z)d\bar{\mu}.$$

After sending $r \to \infty$, we deduce

$$\int \psi_l(\underline{S})d\mu \ge \int Z d\bar{\mu} = \lambda, \text{ or} \tag{4.8}$$

$$\int (\psi_l(\underline{S}) - \lambda)d\mu \ge 0.$$

Recall $\underline{S} \le \limsup \frac{S_n}{n} \le \lambda$. But (4.8) means

$$\int_{\underline{S}\ge -l} (\underline{S} - \lambda)d\mu + (-l-\lambda)\mu\{\underline{S} \le -l\} \ge 0.$$

Since $\lambda > -\infty$, we can choose l large enough to have $-l - \lambda < 0$. For such l, $\underline{S} - \lambda = 0$ on the set $\{\underline{S} \ge -l\}$. By sending $l \to +\infty$ we deduce $\underline{S} = \lambda$ almost everywhere, and this completes the proof. $\square$

We now state *Oseledets Theorem* regarding the existence of *Lyapunov exponents.*

Theorem 4.4 *Let $T : M \to M$ be a C^1-diffeomorphism with $\dim M = m$ and let $\mu \in \mathcal{I}_T$. Let A be a measurable function such that $A(x) : T_xM \to T_{T(x)}M$ is linear for each x and $\log^+ \|A(x)\| \in L^1(\mu)$. Define $A_n(x) = A(T^{n-1}(x)) \dots A(T(x))A(x)$. Then there exists a set $X \subseteq M$ with $\mu(X) = 1$, numbers $l_1 < l_2 < \dots < l_k$ and $n_1, \dots, n_k \in \mathbb{N}^*$ with $n_1 + \dots + n_k = m$, and a linear decomposition $T_xM = E_x^1 \oplus \dots \oplus E_x^k$ with $x \mapsto (E_x^1, \dots, E_x^k)$ measurable such that*

$$\lim_{n\to\infty} \frac{1}{n} \log |A_N(x)v| = l_j$$

for $x \in X$ *and* $v \in F_x^j := E_x^1 \oplus \cdots \oplus E_x^j - E_x^1 \oplus \cdots \oplus E_x^{j-1}$.

Remark If $a \in M$ is a periodic point of period N, then $\mu = N^{-1}\sum_{j=0}^{N-1}\delta_{T^j(a)}$ is an ergodic invariant measure. In this case the Oseledets Theorem can be readily extablished. Indeed if $\lambda_1, \dots, \lambda_m$ denote the eigenvalues of $R = A(T^{N-1}(a))\dots A(T(a))A(a)$, then $\ell_1 < \cdots < \ell_k$ are chosen so that $\{\ell_1, \dots, \ell_k\} = \{N^{-1}\log|\lambda_1|, \dots, N^{-1}\log|\lambda_m|\}$ and $E_a^j = \oplus_i\{V_i : N^{-1}\log|\lambda_i| = \ell_j$ where $V_i = \{v \in T_aM; (A(a) - \lambda_i)^r v = 0$ for some $r\}$ is the generalized eigenspace associated with λ_i.

Note that when $m = 1$, Theorem 4.4 is an immediate consequence of the Ergodic Theorem and the only Lyapunov exponent $l_1 = \int \log|A(x)|\mu(dx)$. We only prove Theorem 4.4 when $m = 2$ and $A(x) = D_xT$. The proof of general case is similar in spirit but more technical.

Proof of Theorem 4.4 for $m = 2$***,*** $A(x) = D_xT$***.*** By Theorem 4.1, there exist numbers l_1 and l_2 such that if

$$X_0 = \left\{x : \lim_{n\to\infty}\frac{1}{n}\log\|D_xT^n\| = l_2,\ \lim_{n\to\infty}\frac{1}{n}\log\|D_xT^{-n}\| = -l_1\right\}$$

then $\mu(x_0) = 1$. Evidently $|A_nv|^2 = \langle A_n^*A_nv, v\rangle = |B_nv|^2$ where $B_n = (A_n^*A_n)^{1/2}$. Clearly $A_n^*A_n \geq 0$ and B_n is well-deTined. Since $B_n \geq 0$, we can Tind numbers $\mu_2^n(x) \geq \mu_1^n(x) \geq 0$ and vectors $a_1^n(x), a_2^n(x)$ such that $|a_1^n| = |a_2^n| = 1$, $\langle a_1^n, a_2^n\rangle_x = 0$ and $B_na_j^n = \mu_j^na_j^n$ Tor $j = 1, 2$.

Note that since $\|A_n(x)\| = \|B_n(x)\|$,

$$l_2 = \lim_{n\to\infty}\frac{1}{n}\log\mu_2^n. \tag{4.9}$$

To obtain a similar formula Tor l_1, Tirst observe that $D_{T^{-n}(x)}T^nD_xT^{-n} = id$ implies that $A_{-n}(x) = D_xT^{-n} = (A_n(T^{-n}(x)))^{-1}$. If we set $S_{-n}(x) = \log\|A_{-n}(x)\|$ and $R_n(x) = \log\|A_n(x)^{-1}\|$ then both $\{S_{-n}(x) : n \in \mathbb{N}\}$ and $\{R_n(x) : n \in \mathbb{N}\}$ are subadditive; $S_{-n-m} \leq S_{-n}\circ T^{-n} + S_{-m}$, $R_{n+m} \leq R_n \circ T^m + R_m$. Clearly, $-l_1 = \lim_{n\to\infty}\frac{1}{n}S_{-n}$ by definition. So, $-l_1 = \inf_n \frac{1}{n}\int S_{-n}d\mu$. On the other hand $\hat{l} = \lim_{n\to\infty}\frac{1}{n}R_n = \inf_n\frac{1}{n}\int R_nd\mu$. Since $S_{-n} = R_n\circ T^{-n}$, we have $\int R_nd\mu = \int S_{-n}d\mu$. This in turn implies that $\hat{l} = -l_1$. As a result,

$$-l_1 = \lim_{n\to\infty}\frac{1}{n}\log\|A_n^{-1}\| = \lim_{n\to\infty}\frac{1}{n}\log\|A_n^{*-1}\|.$$

(Recall that $\|A\| = \|A^*\|$. We then have

$$\begin{aligned} -l_1 &= \lim_{n\to\infty}\frac{1}{n}\log\|(A_n^*A_n)^{-1/2}\| = \lim_{n\to\infty}\frac{1}{n}\log\|B_n^{-1}\| \\ &= -\lim_{n\to\infty}\frac{1}{n}\log(\mu_1^n \wedge \mu_2^n) = -\lim_{n\to\infty}\frac{1}{n}\log\mu_1^n. \end{aligned} \tag{4.10}$$

Naturally we expect E^2_x to be the limit of the lines $\{ta^n_2 : t \in \mathbb{R}\}$ as $n \to \infty$. For this, let us estimate $|a^{n+1}_2(x) - a^n_2(x)|$. If necessary, replace a^n_2 with $-a^n_2$ so that we always have $\langle a^{n+1}_2, a^n_2\rangle_x \geq 0$. We certainly have

$$\begin{aligned} |a^{n+1}_2 - a^n_2|^2 &= 2 - 2\langle a^{n+1}_2, a^n_2\rangle, \\ 1 &= |a^{n+1}_2|^2 = \langle a^{n+1}_2, a^n_1\rangle^2 + \langle a^{n+1}_2, a^n_2\rangle^2. \end{aligned}$$

We now use the elementary inequality $1 - z^2 \leq \sqrt{1-z^2}$ to assert

$$\begin{aligned} |a^{n+1}_2 - a^n_2|^2 &= 2 - 2(1 - \langle a^{n+1}_2, a^n_1\rangle^2)^{1/2} \leq 2\langle a^{n+1}_2, a^n_1\rangle^2 \\ &= 2\langle B_{n+1}a^{n+1}_2/\mu^{n+1}_2, a^n_1\rangle^2 \\ &= 2(\mu^{n+1}_2)^{-2}\langle a^{n+1}_2, B_{n+1}a^n_1\rangle^2 \\ &\leq 2(\mu^{n+1}_2)^{-2}|B_{n+1}a^n_1|^2 = 2(\mu^{n+1}_2)^{-2}|A_{n+1}a^n_1|^2 \\ &= 2(\mu^{n+1}_2)^{-2}|A(T^n(x))A_n(x)a^n_1(x)|^2 \\ &\leq 2(\mu^{n+1}_2)^{-2}c_0|A_n(x)a^n_1(x)|^2 \\ &= 2(\mu^{n+1}_2)^{-2}c_0|B_na^n_1|^2 \\ &= 2c_0(\mu^{n+1}_2/\mu^n_1)^{-2} \end{aligned}$$

for $c_0 = \max_x \|A(x)\|$. From this, (4.9) and (4.10) we deduce

$$\limsup_{n\to\infty} \frac{1}{n} \log |a^{n+1}_2 - a^n_2| \leq -(l_2 - l_1).$$

Let us now assume that $l_2 - l_1 > \delta > 0$. We then have that for constants c_1, c_2,

$$|a^{n+1}_2 - a^n_2| \leq c_1 e^{-\delta n}, \ \ |a^{n+r}_2 - a^n_2| \leq c_2 e^{-\delta n}$$

for all positive n and r. As a result, $\lim_{n\to\infty} a^n_2 = b_2$ exists for $x \in X$ and

$$|a^n_2 - b_2| \leq c_2 e^{-\delta n}$$

for all n. We now define $E^x_2 = \{tb_2(x) : t \in \mathbb{R}\}$. To show that $\lim_{n\to\infty} \frac{1}{n} \log |A_n(x)b_2(x)| = l_2$, observe

$$\begin{aligned} |A_nb_2| &\leq |A_na^n_2| + |A_n(a^n_2 - b_2)| \\ &\leq |B_na^n_2| + \|A_n\||a^n_2 - b_2| \\ &\leq \mu^n_2 + \|A_n\|c_2e^{-\delta n}. \end{aligned}$$

As a result,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log |A_nb_2| &\leq \max\left(\limsup \frac{1}{n} \log \mu^n_2, \limsup \frac{1}{n} \log(\|A_n\|e^{-\delta n})\right) \\ &= \max(l_2, l_2 - \delta) = l_2. \end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned} |A_n b_2| &\geq \mu_2^n - \|A_n\| c_2 e^{-\delta n}, \\ l_2 = \lim_{n\to\infty} \frac{1}{n} \log \mu_2^n &\leq \liminf_{n\to\infty} \max\left(\frac{1}{n}\log|A_n b_2|, \frac{1}{n}\log\|A_n\|e^{-\delta n}\right) \\ &\leq \liminf_{n\to\infty} \max\left(\frac{1}{n}\log|A_n b_2|, l_2 - \delta\right). \end{aligned}$$

From this we can readily deduce that $l_2 \leq \liminf_{n\to\infty} \frac{1}{n}\log|A_n b_2|$. From this and (4.11) we conclude

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n(x) b_2| = l_2$$

for $x \in X$.

To find E_1^x, replace f with T^{-1} in the above argument. This completes the proof when $l_1 \neq l_2$.

It remains to treat the case $l_1 = l_2$. We certainly have

$$|A_n v|^2 = |B_n v|^2 = \langle v, a_1^n\rangle^2 (\mu_1^n)^2 + \langle v, a_2^n\rangle^2 (\mu_2^n)^2.$$

Hence

$$\mu_1^n |v| \leq |A_n v| \leq \mu_2^n |v|.$$

We are done because $\lim \frac{1}{n}\log\mu_2^n = \lim\frac{1}{n}\log\mu_1^n = l_1 = l_2$. □

Example 4.5

(i) Let $T : \mathbb{T}^m \to \mathbb{T}^m$ be a translation. Then $D_x T^n = id$ and the only Lyapunov exponent is zero.

(ii) Let $T : \mathbb{T}^m \to \mathbb{T}^m$ be given by $T(x) = Ax(\operatorname{mod} 1)$ with A a matrix of integer entries. Let $\lambda_1, \dots, \lambda_r$ denote the eigenvalues of A. Let $l_1 < l_2 < \cdots < l_k$ be numbers with $\{l_1, \dots, l_k\} = \{\log|\lambda_1|, \dots, \log|\lambda_r|\}$. We also write n_j for the sum of the multiplicities of eigenvalues λ_i with $\log|\lambda_i| = l_j$. The space spanned by the corresponding generalized eigenvectors is denoted by E_j. We certainly have that if $v \in E_j$ then $\lim_{n\to\infty} \frac{1}{n}\log|A^n v| = l_j$. □

Some comments on Oseledets Theorem is in order. First the identity $A_n(T(x))A(x)v = A_{n+1}(x)v$ implies that for $j = 1, \dots, k$

$$A(x) F_x^j = F_{T(x)}^j, \tag{4.12}$$

where $F_x^j = E_x^1 \oplus \cdots \oplus E_x^j$. Also, we have

$$\frac{1}{n} \log |\det A_n(x)| = \frac{1}{n} \sum_0^{n-1} |\det A(T^j(x))| \to \int \log |\det D_x T| d,$$

by Ergodic Theorem. On the other hand, if $B_n = (A_n^* A_n)^{1/2}$ then $(\det B_n)^2 = (\det A_n)^2$, or $\det B_n = |\det A_n|$. It turns out that if $\mu_1^n \geq \cdots \geq \mu_k^n$ are the eigenvalues of B_n, then $\frac{1}{n} \log \mu_j^n \to \hat{l}_j$, where $\{l_1, \dots, l_k\} = \{\hat{l}_1, \dots, \hat{l}_m\}$. This in turn implies that $\frac{1}{n} \log \det B_n \to \sum_1^k \hat{l}_j$ because $\det B_n = \mu_1^n \dots \mu_m^n$. In summary

$$\int \log |\det D_x T| d\mu = \sum_1^k n_j l_j. \tag{4.13}$$

It turns out that the most challenging part of Theorem 4.4 is the existence of the limit. Indeed if we define

$$l(x,v) = \limsup_{n \to \infty} \frac{1}{n} \log |A_n(x) v|, \tag{4.14}$$

then we can show that as in Theorem 4.4 there exists a splitting $T_x M = E_x^1 \oplus \cdots \oplus E_x^k$ with $l(x,v) = l_j$ for $v \in F_j(x)$.

Exercise 4.6 Verify the following properties of $l(x,v)$ without using Theorem 4.4:

(i) $l(x, \alpha v_1) = l(x, v_1)$, $l(x, v_1 + v_2) \leq \max(l(x, v_1), l(x, v_2))$ for every x, v_1, and v_2 and scalar $\alpha \neq 0$.

(ii) $l(T(x), A(x)v) = l(x,v)$

(iii) We have $\mu\{x : l(x,v) \in [-\infty, +\infty)\} = 1$ for every $v \in \mathbb{R}^m$ and ergodic $\mu \in \mathcal{I}_T$.

(iv) The space $\{v : l(x,v) \leq t\} = V_x(t)$ is linear and that $V_x(s) \subseteq V_x(t)$ for $s \leq t$, $A(x) V_x(t) \subseteq V_{T(x)}(t)$.

(v) There exists $k(x) \in \mathbb{N}$, numbers $l_1(x) < l_2(x) < \cdots < l_{k(x)}(x)$ and splitting $T_x M = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$ such that if $v \in E_x^1 \oplus \cdots \oplus E_x^j - E_x^1 \oplus \cdots \oplus E_x^{j-1}$ then $l(x,v) = l_j$. Indeed $E_x^1 \oplus \cdots \oplus E_x^j = V_x(l_j)$.

We now state and prove an inequality of Ruelle.

Theorem 4.7 *Let $T : M \to M$ be C^1 and $\mu \in \mathcal{I}_T$ be ergodic. Then*

$$h_\mu(T) \leq \sum_1^k n_j l_j^+.$$

Proof. We only present the proof when $\dim M = m = 2$. First we would like to divide M into "small squares". For this we take a triangulation of M; $M = \cup_i \Delta_i$ where each Δ_i is a diffeomorphic copy of a triangle in $\mathbb{R}^2$ and $\Delta_i \cap \Delta_j$ is either empty, or a common vertex, or a common side. We then divide each triangle into squares of side length ε and possibly triangles of side length at most ε (we need these triangles near the boundary of Δ_i's).

The result is a covering of M that is denoted by ξ^ε. Note that we may choose members of ξ^ε such that $\mu(\delta A) = 0$ for $A \in \xi^\varepsilon$. (If not move each element of ξ^ε by small amount and use the fact that for some translation of boundary side we get zero measure. Otherwise we have $\sum_\alpha a_\alpha < \infty$ with $a_\alpha > \delta > 0$ for an infinite sum.) As a result, ξ^ε is a μ-partition. It is not hard to show

$$h_\mu(T) = \lim_{\varepsilon \to 0} h_\mu(T, \xi^\varepsilon). \tag{4.15}$$

Recall that $h_\mu(T, \xi^\varepsilon) = \lim_{k\to\infty} \int I_{\xi^\varepsilon|\xi^{\varepsilon,k}} d\mu$ where $\xi^{\varepsilon,k} = T^{-1}(\xi^\varepsilon) \vee T^{-2}(\xi^\varepsilon) \vee \cdots \vee T^{-k}(\xi^\varepsilon)$ and

$$I_{\xi^\varepsilon|\xi^{\varepsilon,k}} = -\sum_{A\in\xi^\varepsilon} \sum_{B\in\xi^{\varepsilon,k}} \frac{\mu(A\cap B)}{\mu(B)} \log \frac{\mu(A\cap B)}{\mu(B)} \mathbb{1}_B.$$

Given x, let $B = B_{\varepsilon,k}(x)$ be the unique element of $\xi^{\varepsilon,k}$ such that $x \in B$. Such B is of the form $T^{-1}(C_1) \cap \cdots \cap T^{-k}(C_k)$ with $C_1 \dots C_k \in \xi^\varepsilon$, where $C_i = C_{\xi^\varepsilon}(T^j(x))$. Let us write simply write $C_1(x)$ for $C_{\xi^\varepsilon}(T^1(x))$. We have

$$\begin{aligned} I_{\xi^\varepsilon|\xi^{\varepsilon,k}}(x) &\le \log \#\{A \in \xi^\varepsilon : A \cap B_{\varepsilon,k}(x) \neq \emptyset\} \\ &\le \log \#\{A \in \xi^\varepsilon : A \cap T^{-1}(C_1(x)) \neq \emptyset\}. \end{aligned} \tag{4.16}$$

Each $C_1(x)$ is a regular set; either a diffeomorphic image of a small square or a triangle. Since the volume of C is of order $O(\varepsilon^2)$, we have

$$\operatorname{vol}(T^{-1}(C)) \le c_1 \varepsilon^2 \max_{z\in C} |\det D_z T^{-1}|,$$

for a constant c_1. If $A \cap T^{-1}(C) \neq \emptyset$, then for a constant α_0,

$$A \subseteq \{y : |y - x_0| \leq \alpha_0 \varepsilon \text{ for some } x_0 \in T^{-1}(C)\} =: D.$$

We now want to bound $\mathrm{vol}(D)$. The boundary of $T^{-1}(C)$ is a regular curve. Hence its length is comparable to the diameter of $T^{-1}(C)$, and this is bounded above by a multiple of the norm of DT^{-1}. In other words we have a bound of the form

$$\text{const. } \varepsilon \max_{z \in C} \|D_z T^{-1}\|.$$

Using this we obtain

$$\mathrm{vol}(D) \leq c_2 \max_{z \in C} (1 + \|D_x T^{-1}\| + |\det D_z T^{-1}) \varepsilon^2. \tag{4.17}$$

for a constant c_2. (We could have bounded $\mathrm{vol}(A)$ by $(\|D_x T^{-1}\| \varepsilon)^2$ but (4.17) is a better bound.)

We now use (4.17) to obtain an upper bound for the right-hand side (4.16). Indeed

$$\#\{A : A \cap T^{-1}(C^{\varepsilon,k}(x)) \neq \emptyset\} \leq c_3 \max_{z \in C} (1 + \|D_z T^{-1}\| + |\det D_z T^{-1}|) \tag{4.18}$$

for a constant c_3. This is because the union of such A's is a subset of D, for two distinct A, B, we have $\mu(A \cap B) = 0$, and for each $A \in \xi^\varepsilon$ we have that $\mathrm{vol}(A) \geq c_4 \varepsilon^2$ for some positive constant c_4. From (4.18) and (4.16) we learn

$$I_{\xi^\varepsilon | \xi^{\varepsilon,k}}(x) \leq c_5 + \log \max_{z \in C} (\|D_z T^{-1}\| + |\det D_z T^{-1}| + 1)$$

for $C = C_1(x)$. By sending $k \to \infty$ we deduce

$$h_\mu(T, \xi^\varepsilon) \leq c_5 + \int \log \max_{z \in C_{\xi^e}(T(x))} (1 + \|D_z T^{-1}\| + |\det D_z T^{-1}|) d\mu. \tag{4.19}$$

By the invariance of μ,

$$h_\mu(T, \xi^\varepsilon) \leq c_5 + \int \log \max_{z \in C_{\xi^\varepsilon}(x)} (1 + \|D_z T^{-1}\| + |\det D_z T^{-1}|) \mu(dx).$$

Send $\varepsilon \to 0$ to yield

$$h_\mu(T) \leq c_5 + \int \log(1 + \|D_x T^{-1}\| + |\det D_x T^{-1}|) \mu(dx).$$

The constant c_5 is independent of f. This allows us to replace T with T^{-n} to have

$$n h_\mu(T) \leq c_5 + \int \log(1 + \|D_x T^n\| + |\det D_x T^n|) \mu(dx).$$

First assume that there are two Lyapunov exponents. Since $\frac{1}{n}\log\|D_xT^n\| \to l_2$ and $\frac{1}{n}\log|\det D_xT^n| \to l_1+l_2$, we deduce

$$h_\mu(T) \leq \max(0, l_2, l_1+l_2) \leq l_1^+ + l_2^+. \tag{4.20}$$

In the same way we treat the case of one Lyapunov exponent. □

The bound (4.20) may appear surprising because $h_\mu(T) \geq 0$ would rule out the case $l_1, l_2 < 0$. In fact we can not have $l_1, l_2 < 0$ because we are assuming T is invertible. An invertible transformation can not be a pure contraction. Moreover if $h_\mu(T) > 0$ we must have a hyperbolic transformation in the following sense:

Corollary 4.8 *If* $\dim M \geq 2$ *and* $h_\mu(T) > 0$*, then there exists a pair of Lyapunov exponents* α, β *such that* $\alpha > 0$*,* $\beta < 0$*. In particular, if* $\dim M = 2$ *and* $h_\mu(T) > 0$*, then* $l_1 < 0 < l_2$*.*

Proof. Observe that if $l_1 < \cdots < l_k$ are Lyapunov exponents of T, then $-l_k < \cdots < -l_1$ are the Lyapunov exponents of T^{-1}. Simply because if $A_n(x) = D_xT^n$, then $A_{-n} \circ T^n = A_n^{-1}$. Now by Theorem 4.7,

$$\begin{aligned} h_\mu(T) = h_\mu(T^{-1}) &\leq \sum_i n_i(-l_i)^+ = \sum_i n_i l_i^-, \\ h_\mu(T) &\leq \sum_i n_i l_i^+. \end{aligned}$$

From these we deduce that $\sum_i l_i^- < 0 < \sum_i l_i^+$ whenever $h_\mu(T) > 0$. □

Pesin's theorem below gives a sufficient condition for having equality in Theorem 4.7. We omit the proof of Pesin's formula.

Theorem 4.9 *Let* M *be a* C^1*-manifold and assume* $T : M \to M$ *is a* C^1 *diffeomorphism. Assume* DT *is Hölder continuous. Let* $\mu \in \mathcal{I}_T$ *be an ergodic measure that is absolutely continuous with respect to the volume measure of* M*. Then*

$$h_\mu(T) = \sum_i n_i l_i^+.$$

In the context of Theorem 4.7, it is natural to define

$$E_x^s = \bigoplus_{l_i<0} E_x^i, \ E_x^u = \bigoplus_{l_i>0} E_x^i.$$

If there is no zero Lyapunov exponent, we have $T_xM = E^s_x \oplus E^u_x$ μ-almost everywhere. If we write $l^\pm = \min_i l^\pm_i$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \log |(D_x T^{-n})v| \le -l^+$$

for $v \in E^u_x - \{0\}$, and

$$\lim_{n\to\infty} \frac{1}{n} \log |(D_x T^{n})v| \le -l^-$$

for $v \in E^s_x - \{0\}$, μ-almost everywhere. If this happens in a uniform fashion, then we say that μ is an *Anosov measure*. More precisely, we say a that the measure $\mu \in \mathcal{I}^{ex}_T$ is *Anosov* if there exists a decomposition $T_xM = E^u_x \oplus E^s_x$ and constants $K > 0$ and $\alpha \in (0,1)$ such that

$$\begin{aligned}
&(D_xT)E^u_x = E^u_{T(x)}, \ (D_xT)E^s_x = E^s_{T(x)}, \\
&|(D_xT^n)v| \le K\alpha^n|v| \text{ for } v \in E^s_x, \\
&|(D_xT^{-n})v| \le K\alpha^n|v| \text{ for } v \in E^u_x.
\end{aligned}$$

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$$\begin{aligned}
W^s(x) &= \left\{ y : \lim_{n\to\infty} d(T^n(x), T^n(y)) = 0 \right\} \\
W^u(x) &= \left\{ y : \lim_{n\to\infty} d(T^{-n}(x), T^{-n}(y)) = 0 \right\}
\end{aligned}$$

with d a metric on M, then we have a nice foliation of M. In fact

$$\begin{aligned}
&W^s(x) \cap W^s(y) \ne \emptyset \Rightarrow W^s(x) = W^s(y), \\
&W^u(x) \cap W^u(y) \ne \emptyset \Rightarrow W^u(x) = W^u(y), \\
&E^u_x = T_xW^u(x), \quad E^s_x = T_xW^s(x).
\end{aligned}$$

We also have a simple formula for the topological entropy:

$$\begin{aligned}
h_{\text{top}}(T) &= \int \log |\det D_xT|_{E^u_x}| \mu(dx) \\
&= \sum_i n_i l^+_i.
\end{aligned}$$

An obvious example of an Anosov transformation is the Arnold cat transformation.

In the continuous case the Lyapunov exponents are defined likewise. Consider a group of C^1-transformations $\{\phi_t : t \in \mathbb{R}\}$. Here each ϕ_t is from an m-dimensional manifold M onto itself. We then pick an ergodic measure $\mu \in \mathcal{I}_\phi$ and find a splitting $T_xM = E^1_x \oplus \cdots \oplus E^k_x$ such that for $v \in E^1_x \oplus \cdots \oplus E^j_x - E^1_x \oplus \cdots \oplus E^{j-1}_x$,

$$\lim_{t\to\infty} \frac{1}{t} \log |(D_x\phi_t)v| = l_j.$$

It turns out that we always have a zero Lyapunov exponent associated with the flow direction. More precisely, if $\frac{d}{dt}\phi_t(x)|_{t=0} = \xi(x)$, then

$$\lim_{t\to\infty} \frac{1}{t} \log |(D_x\phi_t)\xi(x)| = 0.$$

Intuitively this is obvious because two phase points that lie close to each other on the same trajectory do not separate exponentially.

In the next section we study the Lyapunov exponents for Hamiltonian systems. As a prelude, we show that the Lyapunov exponents for a Hamiltonian flow come in a pair of numbers of opposite signs.

In the case of a Hamiltonian system, we have a *symplectic transformation* $T : M \to M$. This means that M is equipped with a *symplectic form* ω and if $A(x) = D_xT$, then

$$\omega_x(a, b) = \omega_{T(x)}(A(x)a, A(x)b). \tag{4.21}$$

By a symplectic form we mean a C^1 map $x \mapsto \omega_x$, $\omega_x : T_xM \times T_xM \to \mathbb{R}$ such that ω_x is bilinear, $\omega_x(a, b) = -\omega_x(b, a)$, and if $\omega_x(a, b) = 0$ for every $b \in T_xM$, then $a = 0$. Indeed one can find a basis for T_xM such that with respect to this basis, $\omega_x(a, b) = \bar{\omega}(a, b)$ with $\bar{\omega}(a, b) = Ja \cdot b$, and

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $d \times d$ identity matrix and $\dim M = 2d$. Use this basis for T_xM and $T_{T(x)}M$ yields

$$\bar{\omega}(a, b) = \bar{\omega}(A(x)a, A(x)b). \tag{4.22}$$

Equivalently,

$$A(x)^t JA(x) = J. \tag{4.23}$$

As is well-known, this in particular implies that $\det A(x) = 1$. Of course we already know this for Hamiltonian systems by Liouville's theorem, namely the volume is invariant under a Hamiltonian flow.

Theorem 4.10 *The Lyapunov exponents $l_1 < l_2 < \cdots < l_k$ satisfy $l_j + l_{k-j+1} = 0$ and $n_j = n_{2r-j+1}$ for $j = 1, 2, \ldots, k$. Moreover the space $\hat{E}_x^{j-1} := \bigoplus_{i=1}^{j-1} E_x^j$ is ω–orthogonal complement of $\hat{E}_x^{2d-j+1}$.*

Proof. Write $l(x, v) = \lim_{n\to\infty} \frac{1}{n} \log |A_n(x)v|$ where $A_n(x) = D_xT^n$ and $v \in T_xM$. Note that since M is compact, we can Tind a constant c_0 such that

$$|\omega_x(a, b)| \le c_0|a||b|$$

Tor all $a, b \in T_x M$ and all $x \in M$. As a result,

$$|\omega_x(a,b)| = |\omega_{T^n(x)}(A_n(x)a, A_n(x)b)| \le c_0 |A_n(x)a||A_n(x)b|,$$

and if $\omega_x(a,b) \neq 0$, then

$$l(x,a) + l(x,b) \ge 0. \tag{4.24}$$

By Theorem 4.4, there exist numbers $\beta_1 \le \beta_2 \le \cdots \le \beta_{2d}$ and spaces

$$\{0\} = V_0 \subseteq V_1(x) \subseteq \cdots \subseteq V_{2d-1}(x) \subseteq V_{2d}(x) = T_x M$$

such that $\dim V_j(x) = j$ and if $v \in V_{j+1}(x) - V_j(x)$, then $l(x,v) = \beta_j$. Of course $l_1 < \cdots < l_k$ are related to $\beta_1 \le \cdots \le \beta_{2d}$ by $\{l_1, \ldots, l_k\} = \{\beta_1, \ldots, \beta_{2d}\}$ and $n_j = \#\{s : \beta_s = l_j\}$. Note that if W is a linear subspace of $T_x M$ and

$$W^{bot} = \{b \in T_x M : \omega(a,b) = 0 \text{ for all } a \in W\},$$

then one can readily show that $\dim W + \dim W^{\perp} = 2d$. As a result, we can use $\dim V_j + \dim V_{2d-j+1} = 2d+1$ to deduce that there exist $a \in V_j$ and $b \in V_{2d-j+1}$ such that $\omega(a,b) \neq 0$. Indeed the set

$$\Lambda = \{(a,b) \in (T_x M)^2 : a \in V_j,\ b \in V_{2d-j+1},\ \omega_x(a,b) \neq 0\}$$

is a nonempty open subset of $V_j \times V_{2d-j+1}$. Hence

$$\tilde{\Lambda} = \{(a,b) \in (T_x M)^2 : a \in V_j - V_{j-1},\ b \in V_{2d-j+1} - V_{2d-j},\ \omega_x(a,b) \neq 0\}$$

is also nonempty. As a result, we can use (4.24) to assert

$$\beta_j + \beta_{2d-j+1} \ge 0, \tag{4.25}$$

for $j \in \{1, 2, \ldots, d\}$. On the other hand

$$\sum_{j=1}^{d} (\beta_j + \beta_{2d-j+1}) = \sum_i n_i l_i = 0$$

by (4.14) because $\det D_x T^n = 1$. From this and (4.25) we deduce that

$$\beta_j + \beta_{2d-j+1} = 0.$$

From this we can readily deduce that $l_j + l_{k-j+1} = 0$ and $n_j = n_{k-j+1}$.

For the last claim, observe that since $l_j + l_{k-j+1} = 0$, we have $l_j + l_i < 0$ whenever $i + j \le k$. From this and (4.25) we learn that if $i + j \le k$ and $(a,b) \in E_x^i \times E_x^j$, then $\omega_x(a,b) = 0$. Hence $\hat{E}_x^{j-1} \subseteq (\hat{E}_x^{k-j+1})^{\perp}$. Since

$$n_1 + \cdots + n_{k-j+1} + n_1 + \cdots + n_{j-1} = n_1 + \cdots + n_{k-j+1} + n_k + \cdots + n_{k-j+2} = 2d,$$

we deduce that

$$\dim \hat{E}_x^{j-1} = \dim (\hat{E}_x^{k-j+1})^{\perp}.$$

This in turn implies that $\hat{E}_x^{j-1} = (\hat{E}_x^{k-j+1})^{\perp}$. $\square$

5 Ergodicity of Hyperbolic Systems

Lyapunov exponents can be used to measure the hyperbolicity of dynamical systems. Anosov measures (systems) are examples of uniformly or strongly hyperbolic systems which exhibit chaotic and stochastic behavior. In reality, dynamical systems are rarely strongly hyperbolic and those coming from Hamiltonian systems are only weakly (or even partially) hyperbolic.

An argument of Hopf shows that hyperbolicity implies ergodicity. We examin this argument for two models in this sections; Example 5.1 and Example 5.2. To explain Hopf's argument, let us choose the simplest hyperbolic model with expansion and contraction, namely Arnold cat transformation, and use this argument to prove its ergodicity. In fact in Example 1.14 we showed the mixing of Arnold cat transformation which in particular implies the ergodicity. But our goal is presenting a second proof of ergodicity which is the key idea in proving ergodicity for examples coming from Hamiltonian systems.

Exercise 5.1 Let $A = \begin{bmatrix} 1+\alpha^2 & \alpha \\ a & 1 \end{bmatrix}$ with $\alpha \in \mathbb{Z}$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the projection $\pi(a) = a(\operatorname{mod} 1)$ and define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by $T \circ \pi = \pi \circ \hat{T}$ where $\hat{T}(a) = Aa$. Since $\alpha \in \mathbb{Z}$ and $\det A = 1$, we know that T is continuous and that the normalized Lebesgue measure μ on $\mathbb{T}^2$ is invariant for T. The eigenvalues of A are

$$\lambda_1 = \lambda(\alpha) = \frac{1}{2}[2 + \alpha^2 - \alpha\sqrt{4+\alpha^2}] < 1 < \lambda_2 = (\lambda(\alpha))^{-1},$$

provided that $\alpha > 0$. The corresponding eigenvectors are denoted by v_1 and v_2. Define

$$\hat{W}^s(a) = \{a + tv_1 : t \in \mathbb{R}\},\ \hat{W}^u(a) = \{a + tv_2 : t \in \mathbb{R}\}.$$

We then have that $W^s(x)$ and $W^u(x)$ defined by

$$W^s(\pi(a)) = \pi(\hat{W}^s(a)),\ W^u(\pi(a)) = \pi(\hat{W}^a(a))$$

are the stable and unstable manifolds. Take a continuous periodic $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$. This induces a continuous $f : \mathbb{T}^2 \to \mathbb{R}$ such that $f \circ \pi = \hat{f}$. We have that $f \circ T^n \circ \pi = \hat{f} \circ \hat{T}^n$. Define $\hat{X}^{\pm}$ to be the set of points a such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} \hat{f}(\hat{T}^{\pm j}(a)) =: \hat{f}^{\pm}(a)$$

exists. Then $\pi(\hat{X}^{\pm}) = X^{\pm}$ with $X^{\pm}$ consisting of points x such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} f(T^{\pm j}(x)) =: f^{\pm}(x)$$

exists with $f^{\pm} = \hat{f}^{\pm} \circ \pi$. Evidently $f^{\pm} \circ T = f^{\pm}$ on $X^{\pm}$ and $\hat{f}^{\pm} \circ \hat{T} = \hat{f}^{\pm}$ on $\hat{X}^{\pm}$. From definition, we see that if $b \in \hat{W}^s(a)$ (resp. $b \in \hat{W}^u(a)$), then

$$|\hat{T}^n(b) - \hat{T}^n(a)| = \lambda^n |a - b|,$$

$$(\text{resp. } |\hat{T}^{-n}(b) - \hat{T}^{-n}(a)| = \lambda^n |a - b|).$$

for $n \in \mathbb{N}$. Hence $a \in \hat{X}^+$ (resp. $\hat{X}^-$) implies that $\hat{W}^s(a) \subseteq \hat{X}^+$ (resp. $\hat{W}^u(a) \subseteq \hat{X}^-$). Let $d(\cdot, \cdot)$ be the standard distance on the torus. More precisely,

$$d(x, y) = \min\{|a - b| : \pi(a) = x,\ \pi(b) = y\}.$$

Again if $y \in W^s(x)$ (resp. $y \in W^u(x)$), then

$$d(T^n(x), T^n(y)) = \lambda^n d(x, y),$$

$$(\text{resp. } d(T^{-n}(x), T^{-n}(y)) = \lambda^n d(x, y))$$

for $n \in \mathbb{N}$. Similarly $x \in X^+$ (resp. X^-) implies that $W^s(x) \subseteq X^+$ (resp. $W^u(x) \subseteq X^-$). Let Y denote the set of points $x \in X^- \cap X^+$ such that $f^+(x) = f^-(x)$. By Lemma 1.7, $\mu(Y) = 1$. Choose a point x_0 such that $\hat{W}^u(x_0) - Y$ is a set of 0 length. The function $\hat{f}^-$ is constant on $\hat{W}^u(x_0)$. The function $\hat{f}^+$ is constant on $\hat{W}^s(y)$ for every $y \in \hat{W}^u(x_0) \cap Y$ and this constant coincides with the value $\hat{f}^-$ at y. Hence $\hat{f}^+ = \hat{f}^-$ is a constant on the set

$$\bigcup_{y \in \hat{W}^u(x_0) \cap Y} \hat{W}^s(y).$$

But this set is of full measure. So $\hat{f}^+ = \hat{f}^-$ is constant a.e. and this implies that $f^+ = f^-$ is constant a.e. □

Let us call a discrete dynamical system *hyperbolic* if its Lyapunov exponents are nonzero. According to a result of *Pesin*, a hyperbolic diffeomorphism with a smooth invariant measure has at most countably many ergodic components. Pesin's theory also proves the existence of stable and unstable manifolds for hyperbolic systems.

Sinai studied the issue of ergodicity and hyperbolicity for a system of colliding balls in the late 60's. These systems can be regarded as hyperbolic systems with discontinuities. To get a feel for Sinai's method, we follow a work of Liverani and Wojtkowski [LiW] by studying a toral transformation as in Example 9.1 but now we assume that the entry $\hat{a} \notin \mathbb{Z}$ so that the induced transformation is no longer continuous. As we will see below, the discontinuity of the transformation destroys the uniform hyperbolicity of Example 9.1 and, in some sense our system is only weakly hyperbolic.

Exercise 5.2 As in Example 5.1, let us write $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ for the $(\bmod 1)$ projection onto the torus and consider $A = \begin{bmatrix} 1+\alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}$ and $\hat{T}(a) = Aa$ which induces $T : \mathbb{T}^2 \to \mathbb{T}^2$ by $T \circ \pi = \pi \circ \hat{T}$. If $0 < \alpha < 1$, then T is discontinuous. However the Lebesgue measure μ is still invariant for T. To understand T, let us express $\hat{T} = \hat{T}_2 \circ \hat{T}_1$, $T = T_2 \circ T_1$, $\hat{T}_i(a) = A_i a$ for $i = 1, 2$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

If we regard $\mathbb{T}$ as $[0, 1]$ with $0 = 1$, then

$$\begin{aligned} T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1 \\ \alpha x_1 + x_2 (\bmod 1) \end{bmatrix}, \\ T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1 + \alpha x_2 (\bmod 1) \\ x_2 \end{bmatrix} \end{aligned}$$

with $x_1, x_2 \in [0, 1]$. Note that T_i is discontinuous on the circle $x_i \in \{0, 1\}$. As a result, T is discontinuous on the circle $x_2 \in \{0, 1\}$ and on the curve $x_1 + \alpha x_2 \in \mathbb{Z}$. One way to portray this is by introducing the sets

$$\begin{aligned} \Gamma^+ &= \{(x_1, x_2) : 0 \le x_2 + \alpha x_1 \le 1, \ 0 \le x_1 \le 1\} \\ \Gamma^- &= \{(x_1, x_2) : 0 \le x_2 \le 1, \ \alpha x_2 \le x_1 \le \alpha x_2 + 1\} \end{aligned}$$

and observing that $\hat{T}$ maps Γ^+ onto Γ^- but T is discontinuous along $S^+ = \partial \Gamma^+$. Moreover $\hat{T}^{-1} = \hat{T}_2^{-1} \circ \hat{T}_1^{-1}$ with $\hat{T}_i^{-1}(a) = A_i^{-1} a$ for $i = 1, 2$, where

$$A_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}, \ A_2^{-1} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}.$$

Since T_1^{-1} is discontinuous on the circle $x_2 \in \{0, 1\}$ and T_2^{-1} is discontinuous on the circle $x_1 \in \{0, 1\}$, we deduce that T^{-1} is discontinuous on $S^- = \partial \Gamma^-$.

Note that the line $x_2 = 0$ is mapped onto the line $x_2 = ax_1$ and the line $x_2 = 1$ is mapped onto the line $x_2 = ax_1 + 1$. Also note that distinct points on S^+ which correspond to a single point on $\mathbb{T}^2$ are mapped to distinct points on $\mathbb{T}^2$.

We now examine the stable and unstable manifolds. For the unstable manifold, we need to have that if $y \in W^u(x)$, then $d(T^{-n}(x), T^{-n}(y)) \to 0$ as $n \to +\infty$. We may try

$$W_0^n(x) = \{\pi(a + v_2 t) : t \in \mathbb{R}\}$$

where a is chosen so that $\pi(a) = x$ and v_2 is the expanding direction. This would not do the job because of the discontinuity. Indeed the discontinuity set S^- cut the set $W_0^u(x)$ into pieces.

Let us write $W_1^u(x)$ for the connected component of $W_0^u(x)$ inside Γ^-. Since crossing S^- causes a jump discontinuity for T^{-1}, we have that $d(T^{-n}(x), T^{-n}(y)) \not\to 0$ if $y \in W_0^u(x) - W_1^u(x)$. However note that if $y \in W_1^u(x)$, then $d(T^{-1}(x), T^{-1}(y)) = \lambda d(x, y)$. As a result, $d(T^{-1}(x), T^{-1}(y))$ gets smaller than $d(x, y)$ by a fator of size λ. To have $d(T^{-n}(x), T^{-n}(y)) = \lambda^n d(x, y)$, we need to make sure that the segment joining $T^{-n}(x)$ to $T^{-n}(y)$ is not cut into pieces by S^-. That is, the segment xy does not intersect $T^n(S^-)$. Motivated by this, let us pick $x \in \mathbb{T}^2 - \bigcup_{i=0}^{\infty} T^i(S^-)$ and define $W_j^u(x)$ to be the component of $W_0^u(x)$ which avoids $\bigcup_{i=0}^{j} T^i(S^-)$. We now claim that for μ-almost all points, $W^u(x) = \bigcap_{j=0}^{\infty} W_j^u(x)$ is still a nontrivial segment. (This would be our unstable manifold.) More precisely, we show that for μ-almost all x, there exists a finite $N(x)$ such that $W^u(x) = \bigcap_{j=0}^{\infty} W_j^u(x) = \bigcap_{j=0}^{N(x)} W_j^u(x)$. To see this, let us observe that for example

$$W_2^u(x) = T(T^{-1}W_1^u(x) \cap W_1^u(T^{-1}(x))).$$

In other words, we take $W_1^u(x)$ which is a line segment with endpoints in S^-. We apply T^{-1} on it to get a line segment $T^{-1}W_1^u(x)$ with $T^{-1}(x)$ on it. This line segment is shorter than $W_1^u(x)$; its length is λ times the length of $W_1^u(x)$. If this line segment is not cut by S^-, we set $W_2^u(x) = W_1^u(x)$; otherwise we take the connected component of $T^{-1}W_1^u(x)$ which lies inside S^- and has $T^{-1}(x)$ on it. This connected component lies on $W_1^u(T^{-1}(x))$. We then map this back by T. Note that $W_2^u(x) \neq W_1^u(x)$ only if $d(T^{-1}(x), S^-) =$ distance of $T^{-1}(x)$ from S^- is less than

$$\text{length}(T^{-1}W_1^u(x)) = \lambda^{-1}\ \text{length}(W_1^u(x)).$$

More generally,

$$W_{i+1}^u(x) = T^i(T^{-i}W_i^u(x) \cap W_1^u(T^{-i}(x)),$$

and $W^u_{i+1}(x) \neq W^u_i(x)$ only if

$$d(T^{-i}(x), S^-) < \lambda^{-i} \text{ length } (W^u_i(x)).$$

Since length $(W^u_i(x)) \leq$ length $(W^u_1(x)) =: c_0$, we learn that if $W^u(x) = \{x\}$, then

$$d(T^{-i}(x), S^-) < c_0 \lambda^i,$$

for infinitely many i. Set $S^-_\delta = \{x \in \Gamma^- : d(x, S^-) < \delta\}$. We can write

$$\begin{aligned}
\{x : W^u(x) = \{x\}\} &\subseteq \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^i(S^-_{c_0\lambda^i}), \\
\mu(\{x : W^u(x) = \{x\}\}) &\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(T^i(S^-_{c_0\lambda^i})) \\
&= \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(S^-_{c_0\lambda^i}) \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} c_1 c_0 \lambda^i = 0
\end{aligned}$$

for some constant c_1. From this we deduce that for μ-almost all points x, the set $W^u(x)$ is an interval of positive length with endpoints in $\bigcup_{i=0}^{\infty} T^i(S^-)$. Moreover, if $y \in W^u(x)$, then

$$d(T^{-n}(y), T^{-n}(x)) = \lambda^n d(x, y) \to 0$$

as $n \to \infty$. In the same fashion, we construct $W^s(x)$.

We now apply the Hopf's argument. To this end, let us take a dense subset $\mathcal{A}$ of $C(\mathbb{T}^2)$ and for $f \in C(\mathbb{T}^2)$ define $f^\pm$ as in Example 5.1. Set

$$\begin{aligned}
X_f &= \{x \in \mathbb{T}^2 : f^\pm(x), W^s(x), W^u(x) \text{ are well-defined and } f^+(x) = f^-(x)\} \\
X &= \bigcap_{f \in \mathcal{A}} X_f.
\end{aligned}$$

So far we know that $\mu(X) = 1$. Regarding $\mathbb{T}^2$ as $[0,1]^2$ with $0 = 1$ and slicing $\mathbb{T}^2$ into line segments parallel to v_i for $i = 0, 1$, we learn that each stable or unstable leaf intersects X on a set of full length, except for a family of leaves of total μ-measure 0. Let us pick a leaf $W^s(x_0)$ which is not one of the exceptional leaf and define

$$Z_0 = \bigcup \{W^u(y) : y \in W^s(x_0) \text{ and } y \in X\}.$$

Since $W^u(y)$ is of positive length, for each $y \in W^s(x)$, we deduce that $\mu(Z_0) > 0$. On the other hand f^+ is constant on $W^s(x_0)$ and f^- is constant on each $W^u(y)$, $y \in W^s(x_0) \cap X$. Since $f^+ = f^-$ on $W^s(x_0)$, we deduce that $f^+ = f^-$ is constant on Z_0 for every $f \in \mathcal{A}$.

With the aid of Hopf's argument, we managed to show that $f^\pm$ is constant on a set of positive μ-measure. But for ergodicity of μ, we really need to show this on a set of μ-full measure. This is where Hopf's argument breaks down, however it does show that μ has at most countably many ergodic components. Indeed if we define

$$Z(x_0) = \{x : f^\pm(x) \text{ exist and } f^\pm(x) = f^\pm(x_0)\},$$

then $\mu(Z(x_0)) > 0$ because $Z(x_0) \supseteq Z_0$. Since this is true for μ-almost all x_0, we deduce that μ can only have countably many ergodic components.

We now explain how Sinai's method can be used to prove the ergodicity of μ. To this end, let us take a box B with boundary lines parallel to v_1 and v_2 and define

$$\begin{aligned} W^u(B) \;=\; & \{y \in B \cap Y : W^u(y) \cap Y \text{ is of full length and } W^u(y) \\ & \text{reaches the boundary of } B \text{ on both ends}\} \end{aligned}$$

where

$$Y = \{y : f^+(y) \text{ and } f^-(y) \text{ are defined and } f^+(y) = f^-(y)\}.$$

In the same fashion we define $W^s(B)$. We now claim that f^+ is constant on $W^s(B)$, f^- is constant on $W^u(B)$, and these constants coincide. To see this, we fix $W^u(y) \subseteq W^u(B)$ and take all $z \in W^u(y) \cap Y$. We have that f^- is constant on $W^u(y)$ and that $f^-(z) = f^+(z)$ for such $z \in W^u(y) \cap Y$. Since f^+ is constant on each $W^s(z)$, we deduce that f^+ is constant on $\bigcup_{z \in W^u(y) \cap Y} (W^s(z) \cap Y)$ and this constant coincides with $f^-(y)$. By varying $y \in W^u(B)$, we obtain the desired result. (Here we are using the fact that if $W^u(y) \subseteq W^u(B)$ and $W^s(z) \subseteq W^s(B)$, then $W^u(y)$ and $W^s(z)$ intersect.)

We now take two boxes which overlap. For example, imagine that $B_1 = I_1 \times J_1$, $B_2 = I_2 \times J_2$ in the (v_1, v_2) coordinates, where either $J_1 = J_2$ and $I_1 \cap I_2 \neq \emptyset$, or $I_1 = I_2$ and $J_1 \cap J_2 \neq \emptyset$.

We wish to have that the constant $f^{\pm}$ of $W^{u(s)}(B_1)$ equaling the constant $f^{\pm}$ of $W^{u(s)}(B_2)$. We know that f^+ is constant on $W^s(B_1) \cup W^s(B_2)$ and that f^- is constant on $W^u(B_1) \cup W^u(B_2)$. We also know that $f^+ = f^-$ in Y. Clearly if $J_1 = J_2$, $I_1 \cap I_2 \neq \emptyset$ and $W^s(B_1) \cap W^s(B_2) \neq \emptyset$ (respect. $I_1 = I_2$, $J_1 \cap J_2 \neq \emptyset$ and $W^u(B_1) \cap W^u(B_2) \neq \emptyset$), then the constant f^+ (respect. f^-) for $W^s(B_1)$ (respect. $W^u(B_1)$) conincides with the constant f^+ (respect. f^-) for $W^s(B_2)$ (respect. $W^u(B_2)$). Let us identify a scenario for which $\mu(W^s(B_1) \cap W^s(B_2)) > 0$. Given $\beta > 0$, let us call a box B *β-uconnected* if the set

$$\begin{aligned} B^u \;\; = \;\; & \{x \in B : W^u(x) \text{ is defined and reaches} \\ & \text{the boundary of } B \text{ on both ends}\} \end{aligned}$$

satisfies $\mu(B^u) > \beta\mu(B)$. The set B^s is defined in a similar way and we say that B is β-sconnected if $\mu(B^s) > \beta\mu(B)$. Note that if $\mu(B^{u(s)}) > \beta\mu(B)$, then $\mu(W^{u(s)}(B)) > \beta\mu(B)$ because Y is of full-measure. (Here we are using Fubini's theorem to write the measures of Y as an integral of the lengths of v_1 or v_2 slices of Y.) Now if both B_1 and B_2 are β-uconnected (respect. sconnected), B_2 is to the right of B_1 (respect. B_2 is on the top of B_1) and $\mu(B_1 \cap B_2) \geq (1-\beta)\max(\mu(B_1), \mu(B_2))$, then for sure $\mu(W^s(B_1) \cap W^s(B_2)) > 0$ (respect. $\mu(W^u(B_1) \cap W^u(B_2)) > 0$).

Based on this observation, let us take a box $\bar{B}$ and cover it by overlapping small boxes. Pick $\beta \in (0, 1/2)$ and take a grid

$$\left\{ \frac{\beta}{n} i \in \bar{B} : i \in \mathbb{Z}^2 \right\}$$

and use the points of this grid as the center of squares of side length $\frac{1}{n}$. Each such square has area $\frac{1}{n^2}$, and two adjacent squares overlap on a set of area $(1-\beta)\frac{1}{n^2}$.

Let us write $\mathcal{B}_n^{\beta}(\bar{B})$ for the collection of such overlapping squares. We now state a key result of Sinai regarding the α-u(s)connected boxes.

Theorem 5.3 *There exists $\alpha_0 < 1$ such that for every $\beta \in (0, \alpha_0)$,*

$$\lim_{n\to\infty} n\mu \left(\cup \{ B \in \mathcal{B}_n^{\beta}(\bar{B}) : B \textit{ is not either } \beta\textit{-uconnected or } \beta\textit{-sconnected } \} \right) = 0.$$

We now demonstrate how Theorem 9.3 can be used to show that f^+ and f^- are constant almost everywhere in $\bar{B}$. We choose $\beta < \alpha < \alpha_0$ and would like to show that if $y, z \in X_f \cap \bar{B}$, then $f^-(y) = f^+(z)$.

To prove this, we first claim that there exists a full column of boxes in $\mathcal{B}_n^\beta(\bar{B})$ such that each box B in this column is α-uconnected and $W^u(y)$ reaches two boundary sides of a box in the column provided that n is sufficiently large.

Here y is fixed and since $W^u(y)$ is a nontrivial interval, it crosses $c_1 n$ many columns of total area $c_2 n^2$. If each such column has a box which is not α-uconnected, then

$$\mu(\cup\{B \in \mathcal{B}_n^\beta(\bar{B}) : B \text{ is not } \alpha\text{-uconnected}\}) \geq c_3 n \cdot \frac{1}{n^2}$$

for some $c_3 > 0$ (note that a point x belongs to at most $\left(\frac{1}{2\beta} + 1\right)^2$ many boxes). This contradicts Theorem 2.2 for large n. Hence such a column exists. Similarly, we show that there exists a full row of boxes in $B_n^\beta(\bar{B})$ such that each box is α-sconnected and at least one box in this row is fully crossed by $W^s(z)$. Since $\beta < \alpha$, we now that f^- is constant (with the same constant) on $\cup W^s(B)$ with the union over the boxes B on that row, and that f^+ is constant on $\cup W^u(B)$ with union over the boxes B on that column. Since the row and the column intersect on a box, we deduce that $f^+(y) = f^-(z)$. This completes the proof of $f^+ = f^- =$ constant a.e. in $\bar{B}$.

We now turn to the proof of Theorem 5.3.

Proof of Theorem 5.3***.*** First we define a sector

$$\mathcal{C} = \{(a, b) \in \mathbb{R}^2 : |a| \leq \gamma |b|\}$$

which is symmetric about the unstable line v_2 and contains the two directions of sides of Γ^-. We use the explicit value of the slope of v_2 to see that in fact γ can be chosen in $(0,1)$. We now argue that all the line segments in $\bigcup_0^\infty T^i(S^-)$ have directions in the sector $\mathcal{C}$. This is because $\mathcal{C}$ already has the directions of S^-. On the other hand, since the sides of S^- are not parallel to v_1, T^i pushes these lines toward v_2.

Now let us measure the set of points not in $W^u(B)$ for a box in $\mathcal{B}_n^\beta(B)$. Note that if a point $x \in B$ is not in $W^u(B)$, it means that $W^u(x)$ is cut by one of $T^i(S^-)$, $i \in \mathbb{N}^*$ inside B. Let us first consider the case when B is intersected by precisely one line segment of $\bigcup_i T^i(S^-)$. Since this line segment is in sector $\mathcal{C}$, we learn that $\mu(B - W^u(B)) \leq \frac{\gamma}{n^2}$.

This means

$$\mu(W^u(B)) \geq (1-\gamma)\mu(B).$$

Let us choose $\alpha_0 = \frac{1}{2}(1-\gamma)$ so that if $\beta < \alpha_0$ and B is not β-uconnected, then B must intersect at least two segments in $\bigcup_i T^i(S^-)$. (This would be true even when $\beta < 1-\gamma$ but we need a smaller β later in the proof.) We now look at $R_L = \bigcup_{i=0}^{L-1} T^i(S^-)$ and study those boxes which intersect at least two line segments in R_L. Note that each box B is of length $1/n$ and the line segments in R_L are distinct. So, a box $B \in \mathcal{B}_n^\beta$ intersects at least two lines in R_L only if it is sufficiently close to an intersection point of two lines in R_L.

More precisely, we can find a constant $c_1(L)$ such that such a box is in a $\frac{c_1(L)}{n}$ neighborhood of an intersection point. (In fact $c_1(L)$ can be chosen to be a constant multiple of $L^2 e^{c_0 L}$ because there are at most $4L(4L-1)$ intersection points and the smallest possible angle between two line segment in R_L is bounded below by $e^{-c_0 L}$ for some constant c_0.) Hence the total area of such boxes is $c_1(L)n^{-2}$. Now we turn to those boxes which intersect at most one line in R_L and at least one line in $R'_L = \bigcup_{i=L}^{\infty} T^i(S^-)$. Let us write $\mathcal{D}_L$ for the set of such boxes. Let us write $B - W^u(B) = B'_L \cup B''_L$, where

$$\begin{aligned} B'_L &= \{x \in B : W^u(x) \cap B \cap R_L \neq \emptyset\} \\ B''_L &= \{x \in B : W^u(x) \cap B \cap R'_L \neq \emptyset\}. \end{aligned}$$

If $B \in \mathcal{D}_L$, then B can intersect at most one line segment in R_L. Hence $\mu(B'_L) \leq \gamma\mu(B) \leq (1-2\beta)\mu(B)$. If $B \in \mathcal{D}_L$ is not β-uconneted, then

$$(1-\beta)\mu(B) \leq \mu(B - W^u(B)) \leq (1-2\beta)\mu(B) + \mu(B''_L).$$

From this we deduce

$$\begin{aligned} \mu(\cup\{B \in \mathcal{D}_L : B \text{ is not } \alpha\text{-uconnected}\}) &\leq \sum\{\mu(B) \in \mathcal{D}_L : B \text{ is not } \alpha\text{-uconnected}\} \\ &\leq \beta^{-1}\sum\{\mu(B''_L) \in \mathcal{D}_L : B \text{ is not } \alpha\text{-uconnected}\} \\ &\leq \frac{c(\beta)}{\beta}\mu\left(\cup\{B''_L \in \mathcal{D}_L : B \text{ is not } \alpha\text{-uconnected}\}\right), \end{aligned}$$

where for the last inequlity we have used the fact that each point belongs to at most $c(\beta) = (1/(2\beta)+1)^2$ many boxes in $\mathcal{B}_n^\beta$. Let $x \in B''_L$ for some $B \in \mathcal{D}_L$. This means that $W^u(x) \cap B$ intersects $T^i(S^-)$ for some $i \geq L$. Hence $T^{-i}(W^u(x) \cap B) \cap S^- \neq \emptyset$. Note that $T^{-i}(W^u(x) \cap B)$ is a line segment of length at most $\lambda^{-i}n^{-1}$. As a result, $T^{-i}x$ must be within $\lambda^{-i}n^{-1}$-distance of S^-. That is, $x \in T^i(S^-_{\lambda^i n^{-1}})$. So,

$$\begin{aligned} \mu(\cup\{B''_L : B \in \mathcal{D}_L\}) &\leq \mu\left(\bigcup_{i=L}^{\infty} T^i(S^-{}_{\lambda^{-i}n^{-1}})\right) \\ &\leq \sum_{i=L}^{\infty} \mu(T^i(S^-{}_{\lambda^{-i}n^{-1}})) \\ &= \sum_{i=L}^{\infty} \mu(S^-{}_{\lambda^{-i}n^{-1}}) \\ &\leq c_2 \sum_{i=L}^{\infty} n^{-1}\lambda^i \leq c_3 n^{-1}\lambda^{-L}. \end{aligned}$$

This yields

$$\mu(\cup\{B \in \mathcal{B}_n^\beta(\bar{B}) : B \text{ is not } \alpha\text{-usconnected}\} \leq c_1(L)n^{-2} + c_4(\beta)n^{-1}\lambda^{-L}$$

for every n and L. This completes the proof of Theorem 5.3. □

6 Lorentz Gases

So far we have discussed various statistical notions such as ergodicity, entropy and Lyapunov exponents, for dynamical systems. We have examined these notions for a rather limited number of examples, namely toral automorphisms, translations (or free motions) and one-dimensional expansions. In this section we study examples coming from classical mechanics. A *Lorentz gas* is an example of a gas in which heavy molecules are assumed to be immobile and light particles are moving under the influence of forces coming from heavy particles. The dynamics of a light particle with position $q(t)$ is governed by the Newton's law

$$\frac{d^2q}{dt^2} = -\nabla V(q), \tag{6.1}$$

where $V(q) = \sum_j W(|q - q_j|)$ with q_j denoting the center of immobile particles and $W(|z|)$ represents a central potential function. For simplicity we set the mass of the light particle to be zero. We may rewrite (6.1) as

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -\nabla V(q). \tag{6.2}$$

Recall that the total energy $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ is conserved. Because of this, we may wish to study the ergodicity of our system restricted to an energy shell

$$\{(q,p) : H(q,p) = E\}.$$

When W is of compact support, we may simplify the model by taking

$$W(|z|) = \begin{cases} 0 & \text{if } |z| > \varepsilon, \\ \infty & \text{if } |z| \le \varepsilon. \end{cases} \tag{6.3}$$

To interpret (6.2) for W given by (6.3), let us first assume that the support of $W(|q - q_i|)$, $i \in \mathbb{Z}$ are nonoverlapping. Assume a particle is about to enter the support of $W(|q-q_i|)$. For such a scenario, we may forget about other heavy particles and assume that the potential energy is simply given by $W(|q - q_i|)$. For such a potential we have two conservation laws:

$$\text{conservation of energy: } \frac{d}{dt}\left(\frac{1}{2}|p|^2 + V(|q - q_i|)\right) = 0$$

$$\text{conservation of angular momentum: } \frac{d}{dt} p \times (q - q_i) = 0.$$

Let us assume that a particle enters the support at a position q with velocity p and exits the support at a position q' with velocity p'. For a support choose a ball of center q_i and diameter ε. If $n = \frac{q-q_i}{|q-q_i|}$ and $n' = \frac{q'-q_i}{|q'-q_i|}$, then we can use the above conservation laws to conclude that $|p'| = |p|$ and the angle between (p, n) is the negation of the angle between (p', n').

The same conservation laws hold for the case (6.3). We are now ready for interpretation of dynamics when W is given by (6.3). Draw a ball of diameter ε and center q_i for each i. Then the phase space is

$$\begin{aligned} X &= \{(q,p) : |q - q_i| \geq \varepsilon \text{ for all } i, \text{ and } p \in \mathbb{R}^d\} \\ &= \left(\mathbb{R}^d - \bigcup_i B_{\varepsilon/2}(q_i)\right) \times \mathbb{R}^d. \end{aligned}$$

For $q \notin \partial X$ we simply have $\frac{dq}{dt} = p$. When $|q - q_i| = \varepsilon$ then the dynamics experiences a jump discontinuity in p-component. More precisely

$$|q(t) - q_i| = \varepsilon \text{ implies } p(t_+) = p(t_-) - 2p(t_-) \cdot n_i(t) n_i(t), \tag{6.4}$$

where $n_i(t) = \frac{q(t)-q_i}{|q(t)-q_i|}$. As our state, we may consider

$$\begin{aligned} M &= \{q : |q - q_i| \gamma \varepsilon \text{ for all } i\} \times \{p : |p| = 1\} \\ &=: Y_\varepsilon \times \mathbb{S}^{d-1}. \end{aligned}$$

Classically two possibilities for the configurations of q_i's are considered. As the first possibility, imagine that the q_i's are distributed periodically with period 1. Two cases occur. Either $\varepsilon < 1$ which corresponds to an *infinite horizon* because a light particle can go off to infinity. Or $\varepsilon \geq 1$ which corresponds to a finite horizon.

As our second possibility we distribute q_i's randomly according to a Poissonian probability distribution.

In this section we will study Lorentz gases on tori. In the periodic case of an infinite horizon, we simply have a dynamical system with phase space

$$M = (\mathbb{T}^d - B_\varepsilon) \times \mathbb{S}^{d-1} = Y_\varepsilon \times \mathbb{S}^{d-1},$$

where $\mathbb{T}^d - B_\varepsilon$ represents a torus from which a ball of radius $\varepsilon/2$ is removed. In the case of finite horizon our $M = Y_\varepsilon \times \mathbb{S}^{d-1}$ but now Y_ε is a region confined by 4 concave arcs. In the random case we may still restrict the dynamics to a torus. For example, we select N points $q_1, \dots, q_j$ randomly and uniformly from the set

$$X_\varepsilon = \{(q_1, \dots, q_N) : |q_i - q_j| > \varepsilon \text{ for } i \neq j\},$$

and then we set

$$Y_\varepsilon = \{q : |q - q_i| \geq \varepsilon \text{ for } i = 1, \dots, N\}.$$

Next we find an invariant measure for the dynamical system $(q(t), p(t))$. We write x for (q, p) and denote its flow by $\phi_t(x)$. Recall that the phase space is $M = Y_\varepsilon \times \mathbb{S}^{d-1} = Y \times S$. This is a manifold of dimension $2d-1 =: m$. We have $\partial M = \partial Y \times S$ with $\partial M = \bigcup_j (\Gamma_j^+ \cup \Gamma_j^-)$ where $\Gamma_j^\pm = \{(q,p) : |q - q_i| = \varepsilon,\ p \in S,\ \pm n_i(q) \cdot p \geq 0\}$ where $n_i(q) = \frac{q-q_i}{|q-q_i|}$. If $(q,p) \in \Gamma_j^-$, then we have a pre-collisional scenario and (q,p) corresponds to a post-collisional scenario. For an invariant measure we take a normalized Lebesgue measure $\frac{1}{Z} dq\, dp = \frac{1}{Z} dx$ where Z is a normalizing constant. To prove this, let us take a smooth test function $J : M \to \mathbb{R}$ such that $J(q, p') = J(q, p)$ whenever $(q,p) \in \partial M$ and $p' = p - 2p \cdot n\, n$ with $n = n_j(q)$ in the case of $(q,p) \in \Gamma_j^-$. Such a test function produces

$$(T_t J)(x) = u(x,t) = J(\phi_t(x)),$$

that is continuous in (x,t). In fact u satisfies a Liouville-type equation with boundary conditions:

$$\begin{cases} u_t = p \cdot u_q, & x \in M - \partial M; \\ u(q, p', t) = u(q, p, t), & t \geq 0,\ (q,p) \in \partial M. \end{cases} \tag{6.5}$$

We expect (6.5) to be true weakly; if K is a smooth function, then

$$\begin{aligned} \frac{d}{dt} \int u(x,t) K(x) dx = & - \int u(x,t) v \cdot K_x(x) dx \\ & - \varepsilon^{d-1} \sum_j \int_{|p|=1} dp \int_{|n|=1} u(q_j + \varepsilon n, p) K(q_j + \varepsilon n, p)(p \cdot n) dn. \end{aligned} \tag{6.6}$$

Let us verify (6.6) when the horizon is infinite. Under such an assumption, we find a sequence of functions

$$\tau_0(x) = 0 < \tau_1(x) < \tau_2(x) < \dots$$

for almost all x, such that $\phi_t(x) \in M - \partial M$ for $t \in (\tau_j(x), \tau_{j+1}(x))$,

$\phi_{\tau_j(x)}(x) \in \partial M$ if $j > 0$, and each finite interval $[0,T]$ can have only finitely many τ_i's. Let us explain this further. Note that if $v = (v^1, \dots, v^d)$ with $v^1, \dots, v^d$ rationally independent, then $x + vt$ would enter any open set eventually. This proves the existence of τ_1 for such v. Since the set of such v is of full measure, we have the existence of $\tau_1(x)$ for almost all x. Similarly we can prove the existence of τ_j's inductively for almost all x.

Note that $u(x,t) = J(\phi_t(x))$ is as smooth as J in (x,t) provided $\phi_t(x) \notin \partial M$. This means that u is as smooth as J with $u_t = p \cdot u_q$, provided $(x,t) \in M \times (0,\infty) - \bigcup_j S_j$, where

$$S_j = \{(x,t) : \tau_j(x) = t\}.$$

Note that when t is restricted to a finite interval $[0,T]$, then finitely many S_j's are relevant, each S_j is of codimension 1 in $M \times (0,T)$, and different S_j's are well-separated. It is a general fact that if u is continuous and $u_t = p \cdot u_q$ off $\bigcup_j S_j$, then $u_t = p \cdot u_q$ weakly in M. To see this, take a test function $R(x,t)$ with support in an open set U such that exactly one of the S_j's bisect U into U^+ and U^-. We then have $\int u(R_t - p \cdot R_q)dx\ dt = \int_{U^+} + \int_{U^-}$ and that if we integrate by parts on each $U^\pm$ we get two contributions. One contribution comes from carrying out the differentiation on u, i.e., $\int_{U^\pm}(-u_t + p \cdot u_q)R\ dx\ dt$, which is 0 because $u_t = p \cdot u_q$ in $U^\pm$. The other contribution comes from the boundary of $U^\pm$, and they cancel each other out by continuity of u. In a similar fashion we can verify (6.6). In the periodic case of infinite horizon, we only have one heavy particle per period. This means that in (6.6) the summation has one term.

As a consequence of (6.5) we have that the Lebesgue measure $dq\ dp$ is invariant. In fact if initially x is selected according to a probability measure $d\mu = f^0(x)dx$, then at later times $x(t)$ is distributed according to $d\mu_t = f(x,t)dx$ where $f(x,t) = f^0(\phi_{-t}(x))$. To see this observe that if we choose $K \equiv 1$ in (6.6) we yield

$$\frac{d}{dt}\int J(\phi_t(x))dx = -\varepsilon^{d-1}\int_{|p|=1} dp \int_{|n|=1} u(\bar{q}+\varepsilon n, p)p \cdot n\ dn \tag{6.7}$$

where $\bar{q}$ denotes the center of the only existing ball in the unit square. If we integrate over p first and make a change of variable $p \mapsto p' = p - 2p \cdot n\ n$, then u does not change and $p \cdot n$ becomes $p' \cdot n = -p \cdot n$. Also the Jacobian of such a transformation is 1. As a result, the right-hand side of (6.7) is equal to its negation. This implies

$$\int J(\phi_t(x))dx = \int J(x)dx, \tag{6.8}$$

for every t and every J continuous with $J(q,p') = J(q,p)$ on ∂M. If K and f^0 have the same property and we choose

$$J(x) = f^0(\phi_{-t}(x))K(x),$$

then we deduce

$$\int K(x)f^0(\phi_{-t}(x))dx = \int K(\phi_t(x))f^0(x)dx.$$

From this we conclude

$$f(x,t) = f^0(\phi_{-t}(x)), \tag{6.9}$$

as was claimed before.

Our dynamical system is a simple free motion between collision times. Perhaps we should free out system from the free motion part by focusing on the collisions. For example, let us define

$$\Gamma = \{(n,p) : |p| = |n| = 1,\ p \cdot n \geq 0\} \subseteq \mathbb{T}^2,$$

and $T : \Gamma \to \Gamma$ by $T(n,p) = (n',p')$ where $\phi_{\tau_1(\bar{q}+\varepsilon n,p)+}(\bar{q}+\varepsilon n,p) = (\bar{q}+\varepsilon n',p')$ and $p' = p - 2n' \cdot p\, n'$. In other words, if $(\bar{q}+\varepsilon n,p)$ is a post-collisional pair then at the next collision we get $(\bar{q}+\varepsilon n',p)$, and after the collision the result is $(\bar{q}+\varepsilon n',p')$. Here $\tau_1(x)$ is the first collision time of the point x. Again for a set of full measure, the transformation T is well-defined. Let us write m for the Lebesgue measure on M. This invariant measure induces an invariant measure on Γ. For this let us define $\hat{\Gamma} = \{(y,t) : y = (\bar{q}+\varepsilon n,p),\ 0 \le t < \tau_1(y)\}$ and $F : \hat{\Gamma} \to M$ by $F(y,t) = \phi_t(y)$. It is not hard to see that F is invertible. In fact F is an automorphism between the measure spaces (M,dm) and $(\hat{\Gamma}, |n \cdot p| d\sigma(y)dt)$ where $d\sigma(y) = \varepsilon^{d-1} dn\, dp$ denotes the surface measure on Γ. This simply follows from the fact that the Jacobian of the transformation

$$(\bar{q}+\varepsilon n, t) \mapsto \bar{q}+\varepsilon n + pt = q$$

equals $\varepsilon^{d-1}|n \cdot p|$. In other words $dq = \varepsilon^{d-1}|n \cdot p| dn\, dt$. The transformation F provides us with a useful representation of points in M. Using this representation we can also represent our dynamical system in a special form that is known as *special flow representation.* Let us study $F^{-1} \circ \phi_\theta \circ F$. Let us write $T(\bar{y}+\varepsilon n,p) = (\bar{y}+\varepsilon n',p')$ where $T(n,p) = (n',p')$

$$\hat{\phi}_\theta := F^{-1} \circ \phi_\theta \circ F(y,t) = \begin{cases} (y, \theta + t) & \theta + t < \tau_1(y) \\ (T(y), \theta + t - \tau_1(y)) & \theta + t - \tau_1(y) < \tau_1(T(y)) \\ \vdots \end{cases} \tag{6.10}$$

The measure $\varepsilon^{d-1}|n \cdot p| d\sigma(y)dt$ is an invariant measure for the flow $\hat{\phi}_\theta$. We now claim that if

$$d\mu = \varepsilon^{d-1}|n \cdot p| d\sigma(y) \tag{6.11}$$

then μ is an invariant measure for T. To see this take a subset of $\bar{\Gamma}$. We choose A sufficiently small in diameter so that we can find θ_1, θ_2 and θ_3 with the following property:

$$t \in [\theta_1, \theta_2] \Rightarrow \tau_1(y) < \theta_3 + t < \tau_1(T(y))$$

for every $y \in A$. This means

$$\hat{\phi}_\theta(A \times [\theta_1, \theta_2] = \{(T(y), \theta_3 + t - \tau_1(y) : y \in A,\ t \in [\theta_1, \theta_2]\}.$$

Since $\hat{\phi}_\theta$ has $d\mu dt$ for an invariant measure,

$$(\theta_2 - \theta_1)\mu(A) = (\theta_2 - \theta_1)\mu(T(A)).$$

Since T is invariant, we deduce that μ is invariant.

There are various questions we would like to ask concerning the ergodicity of the dynamical system (ϕ_t, m).

For example, we would like to know whether m is an ergodic invariant measure. Does ϕ_t have nonzero Lyapunov exponents? Can we calculate $h_m(\phi)$? For the last two questions, we need to study $D_x\phi_t$. Recall that if ϕ_t is the flow associated with an ODE of the form $\frac{dx}{dt} = f(x)$, then the matrix-valued function $A(x,t) = D_x\phi_t$ solves

$$\frac{dA}{dt} = (D_{\phi_t(x)}f)A. \tag{6.12}$$

This means that for small δ,

$$\phi_t(x + \delta\hat{x}) - \phi_t(x) \approx \delta A(x,t)\hat{x},$$

with A solving (6.12). Hence, $\hat{x}(t) = A(x,t)\hat{x}$ solves the equation

$$\frac{d\hat{x}}{dt} = B(x,t)\hat{x},$$

where $B(x,t) = (D_{\phi_t(x)}f)$. In the case of a Hamiltonian flow of the form (6.2), we have $f(q,p) = (p, -\nabla V(q))$ and we simply that $\hat{x} = (\hat{q},\hat{p})$ solves

$$\frac{d\hat{q}}{dt} = \hat{p}, \quad \frac{d\hat{p}}{dt} = -D_q^2 V\ \hat{q}.$$

But for our Lorentz gas model associated with (6.3), some care is needed because $\phi_t(x)$ is not differentiable. Let us examine the evolution of $\hat{x}$ for a billiard in a domain Y. That is, a particle of position q travel according to its velocity p, and the velocity p changes to the new velocity $p' = p - 2p\cdot n\ n$ after a collision with the boundary. Here n denotes the inner unit normal at the point of the collision.

To this end, let us take the bounded domain Y where ∂Y is piecewise smooth and study the flow of a billiard inside Y. For this, we compare two trajectories $x(t)$ and $x^*(t)$ where $x^*(0) = x + \delta\hat{x}$, $x(0) = x$, with $\delta \ll 1$. Then at later times we would have $x^*(t) = x(t) + \delta\hat{x}(t) + o(\delta)$ and we would like to derive an equation for the evolution of $\hat{x}(t)$. In between collisions, we simply have $\frac{d\hat{q}}{dt} = \hat{p}$, $\frac{d\hat{p}}{dt} = 0$. To figure out how $(\hat{q},\hat{p})$ changes at a collision, assume that a collision for x occurs at time 0 and a collision at time $\bar{t} = \delta\tau + o(\delta)$ occurs for x^*. Without loss of generality, we may assume that $\tau \geq 0$. Assume that at this collision, the coordinates are $(q,p,\hat{q},\hat{p})$ and right after collision we have the coordinates $(q,p',\hat{q}',\hat{p}')$. Collision for x and x^* occur at $a = x$ and a^* on ∂Y. Let us assume that near a, the boundary $\partial\Gamma$ is represented by $g(y) = 0$ for a smooth function g. We write $a^* = a + \delta\hat{a} + o(\delta)$ and $n^* = n + \delta\hat{n} + o(\delta)$ where n and n^* are normal vectors at a and a^* respectively. We know

$$a^* = a + \delta\hat{a} + o(\delta) = q^* + \bar{t}p^* = a + \delta(\hat{q} + \tau p) + o(\delta),$$

which means that $\hat{a} = \hat{q} + \tau p$. Since $g(a^*) = g(a + \delta\hat{a} + o(\delta)) == 0$, we deduce that $n\cdot\hat{a} = 0$. Hence

$$\tau = -\frac{\hat{q}\cdot n}{p\cdot n}, \quad \hat{a} = V\hat{q} := \left(I - \frac{p\otimes n}{p\cdot n}\right)\hat{q}. \tag{6.13}$$

The operator V is the p-projection onto $n^\perp$. That is $(I-V)\hat{q}$ is parallel to p and $V\hat{q}\cdot n=0$ always. Since $\nu(a^*)=n^*$, $\nu(a)=n$, for $\nu(y)=\frac{\nabla g(y)}{|\nabla g(y)|}$, we deduce

$$\hat{n}=(D\nu(a))\hat{a}=D\nu(a)V(\hat{q}). \tag{6.14}$$

The operator $D\nu(a)$ is known as the shape operator of ∂Y at a. To figure out what $\hat{q}'$ is, we calculate

$$\begin{aligned} q^*(\bar{t})-q(\bar{t})&=a^*-(q+\bar{t}p')=\delta(\hat{q}+\tau(p-p'))+o(\delta)\\ &=\delta(\hat{q}-2\hat{q}\cdot nn)+o(\delta), \end{aligned}$$

and for $t>\hat{t}$,

$$\begin{aligned} q^*(t)-q(t)&=\delta(\hat{q}-2\hat{q}\cdot nn)+(p^{*\prime}-p')(t-\bar{t})+o(\delta)\\ &=\delta(\hat{q}-2\hat{q}\cdot nn)+(p^{*\prime}-p')t+o(\delta). \end{aligned}$$

From this we deduce

$$\hat{q}'=R\hat{q}=(I-2n\otimes n)\hat{q} \tag{6.15}$$

with R denoting the reflection with respect to n. Moreover

$$\begin{aligned} p^{*\prime}-p'&=p^*-2p^*\cdot n^*n^*-p+2p\cdot nn\\ &=p^*-p-2(p^*-p)\cdot nn-2p^*\cdot n^*n^*+2p^*\cdot nn\\ &=\delta(\hat{p}-2\hat{p}\cdot nn)-2p^*\cdot(n+\delta\hat{n})(n+\delta\hat{n})+2p^*\cdot nn+o(\delta)\\ &=\delta[\hat{p}-2\hat{p}\cdot nn-2(p^*\cdot n)\hat{n}-2(p^*-\hat{n})n]+o(\delta)\\ &=\delta[\hat{p}-2\hat{p}\cdot nn-2(p\cdot n)\hat{n}-2(p\cdot\hat{n})n]+o(\delta). \end{aligned}$$

As a result, $p^{*\prime}=p'+\delta\hat{p}'+o(\delta)$ where

$$\hat{p}'=R\hat{p}-2A\hat{q}, \tag{6.16}$$

with

$$\begin{aligned} A\hat{q}&=(p\cdot n)\hat{n}+(p\cdot\hat{n})n=(p\cdot n)\hat{n}+(n\otimes p)\hat{n}\\ &=(p\cdot n)\hat{V}D\nu(a)V\hat{q}, \end{aligned} \tag{6.17}$$

where $\tilde{V}=I+\frac{n\otimes p}{p\cdot n}$. Note that $|\nu|=1$ implies that $nD\nu(a)=0$, or $D\nu(a)$ map $n^\perp$ onto $n^\perp$. Also the range of V is $n^\perp$ and $V:p^\perp\to n^\perp$ is an isomorphism. Moreover, $\tilde{V}$ restricted to

$n^\perp$ equals $I - \frac{n\otimes p'}{p'\cdot n}$, and that $\tilde V : n^\perp \to p'^\perp$ is an isomorphism, which simply n-projects onto $p'^\perp$. Also, for $w, w' \in n^\perp$,

$$\begin{aligned} R\tilde{V} &= (I - 2n\otimes n)\left(I + \frac{n\otimes p}{n\cdot p}\right) \\ &= R + \frac{n\otimes p}{n\cdot p} - 2\frac{n\otimes p}{n\cdot p} = R - \frac{n\otimes p}{n\cdot p}, \\ w\cdot(R\tilde{V})w' &= w\cdot\left(I - \frac{n\otimes p}{n\cdot p}\right)w' = w\cdot w' - \frac{(p\cdot w')(n\cdot w)}{n\cdot p} \\ &= (Vw)\cdot w', \end{aligned}$$

so $R\tilde{V} = V^t$ is the transpose of V. As a result,

$$A = (p\cdot n)RV^tD\nu(a)V\hat{q}. \tag{6.18}$$

One way to explore the dispersive behavior of a dispersive billiard is to study the evolution of the quadratic form $Q(\hat q,\hat p) = \hat q\cdot\hat p$. If we write $Q(t) = Q(\hat q(t),\hat p(t))$, then in between collisions, $\frac{dQ}{dt} = |\hat p|^2$ and at a collision,

$$\begin{aligned} Q(t+) &= \hat q'\cdot\hat p' = R\hat q\cdot(R\hat p - 2A\hat q) \\ &= Q(t-) - 2\hat q\cdot(RA\hat q) = Q(t-) - 2(p\cdot n)(\hat q\cdot V^tD\nu(a)V\hat q) \\ &= Q(t-) - 2(p\cdot n)V\hat q\cdot D\nu(a)V\hat q. \end{aligned}$$

Note that if $V\hat q \neq 0$ and $D\nu(a) > 0$, then $Q(t+) > Q(t-)$ because $p\cdot n < 0$. Also $V\hat q \neq 0$ if $\hat q \neq 0$ and $\hat q \in p^\perp$. The condition $D\nu(a) > 0$ means that the boundary is concave and this is exactly what we mean by a *dispersive billiard*. For such a billiard we expect to have $d-1$ positive Lyapunov exponent, and since we have a Hamiltonian flow, then by Theorem **??**, we have $d-1$ negative Lyapunov exponents also. The remaining Lyapunov exponent is 0. This has to do with the fact that in the flow direction, the Lyapunov exponent is 0. To avoid the vanishing Lyapunov directions, we assume that initially $\hat p\cdot p = 0$ (conservation of $\frac12|p|^2$) and that $\hat q\cdot p = 0$ (i.e., $(\hat q,\hat p)$ is orthogonal to the flow direction $(p,0)$). This suggests that we restrict $(\hat q,\hat p)$ to $W(x) = \{(\hat q,\hat p) : \hat q\cdot p = \hat p\cdot p = 0\} = p^\perp$ for $x = (q,p)$. Note that if $(\hat q,\hat p) \in W(x)$ initially, then $(\hat q(t),\hat p(t)) \in W(\phi_t(x))$ at later times. This is obvious in between collisions and at a collision use the fact that the range of $\tilde V$ is $p'^\perp$.

Once $(\hat q,\hat p)$ is chosen in $W(x)$ initially, then we can say that $Q(t)$ is strictly increasing for a dispersive billiard. To take advantage of this, let us define a sector

$$C(x) = \{(\hat q,\hat p) \in W(x) : \hat q\cdot\hat p > 0\}.$$

What we have learned so far is that

$$\hat D_x\phi_t(C(x) - \{0\}) \subsetneq C(\phi_t(x)) \tag{6.19}$$

where $\hat{D}_x\phi_t$ is a short hand for the flow of $\hat{x}$, so that when ϕ_t is differentiable, then $\hat{D}_x$ is the same as D_x. The property (6.19) is promising because by iteration, we get a slimmer and slimmer sector and in the limit, we expect to get $E^+(x)$ associated with the positive Lyapunov exponents. To see how this works in principle, let us examine an example.

Example 6.1. Consider a matrix-valued function $A(x)$, $x \in \mathbb{T}^2$ such that for almost all x, A has positive entries and $\det A(x) = 1$. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be invariant with respect to the Lebesgue measure μ and define $l(x,v) = \lim_{n\to\infty} \frac{1}{n} \log |A_n(x)v|$, where $A_n(x) = A(T^{n-1}(x))A(T^{n-2}(x)) \cdots A(T(x))A(x)$. Define the sector $C(x) \equiv C = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} : v_1 v_2 > 0 \right\}$. Note that if $\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = A(x) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $A(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$, then

$$\begin{aligned} Q(v_1', v_2') = v_1'v_2' &= (av_1 + bv_2)(cv_1 + dv_2) \\ &\geq v_1v_2 + acv_1^2 + bdv_2^2 \\ &\geq v_1v_2 + 2bc\sqrt{\frac{ad}{bc}} v_1 v_2 \\ &> (1 + 2bc)Q(v_1, v_2). \end{aligned}$$

Hence A maps C onto a sector which lies strictly inside C. If $A_n(x)v^n = (v_1^n, v_2^n)$, then

$$|A_n(x)v|^2 \geq 2v_1^n v_2^n \geq v_1 v_2 \prod_{i=0}^{n-1} (1 + 2b(T^i(x))c(T^i(x))),$$

$$\liminf_{n\to\infty} \frac{1}{n} \log |A_n(x)v| \geq \frac{1}{2} \int \log(1 + 2b(x)c(x))\mu(dx) =: \bar{l} > 0,$$

whenever $v_1v_2 > 0$. In particular, by choosing any v with $v_1^2 + v_2^2 = 1$, $v_1v_2 > 0$, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_n(x)\| \geq \liminf_{n\to\infty} \frac{1}{n} |A_n(x)v| \geq \bar{l},$$

or $l_2 > 0$. Since $\det A_n \equiv 1$, we know that $l_1 + l_2 = 0$. So $l_1 < 0 < l_2$. □

From this example, we learn that perhaps we should try to get a lower bound on $Q(\hat{q}', \hat{p}')/Q(\hat{q}, \hat{p})$. Note that $\hat{q}$ is gaining in size in between collisions. However the gain in the $\hat{p}$ is occuring only at collisionss. If we have a reasonable lower bound on the ratio of $Q(\hat{q}', \hat{p}')$ and $Q(\hat{q}, \hat{p})$, then the gain is exponential as a function of time.

Let us consider $T : \Gamma \to \Gamma$ where

$$\Gamma = \{(q, p') : q \in \partial Y \text{ and } p' \cdot \nu(q) > 0\}$$

with $\nu(q)$ the inner unit normal to the boundary ∂Y, and $T(q,p') = (\tilde{q}, \tilde{p})$ where $\tilde{q}$ denotes the location of the next collision and $\tilde{p}$ denotes the post-collisional velocity after such a collision. We also write $\tau : \Gamma \to (0, \infty)$ for the time between the collision at (q,p) and the next collision at $(\tilde{q}, \tilde{p})$. Now the total gain in Q from the time of a previous collision, till right after a collision at (q,p) is given by

$$\Delta Q = \tau(T^{-1}(q,p'))|\hat{p}|^2 + 2(p \cdot n)^+ V\hat{q} \cdot D\nu(a)V\hat{q}.$$

If we assume that ∂Y is uniformly concave, i.e., $D\nu(a) \geq \delta I$, then

$$\Delta Q \geq \tau(T^{-1}(q,p'))|\hat{p}|^2 + 2\delta|p \cdot n| \; |V\hat{q}|^2.$$

Note that for $\hat{q} \in p^\perp$,

$$|V\hat{q}|^2 = |\hat{q}|^2 + \left(\frac{n \cdot \hat{q}}{n \cdot p}\right)^2 |p|^2.$$

If initially we start from the sector C, then $(\hat{q}, \hat{p})$ stays in C for all times and for such $(\hat{q}, \hat{p})$,

$$Q(\hat{q}, \hat{p}) = \hat{q} \cdot \hat{p} \geq \frac{1}{2}(|\hat{q}|^2 + |\hat{p}|^2).$$

As a result,

$$\begin{aligned}\frac{\Delta Q}{Q} &\geq 2\frac{\tau|\hat{p}|^2 + 2\delta|p \cdot n| \left[|\hat{q}|^2 + \left(\frac{n\cdot\hat{q}}{n\cdot p}\right)^2 |\hat{p}|^2\right]}{|\hat{q}|^2 + |\hat{p}|^2} \\ &\geq 2\min\left(\tau + 2\delta\frac{(n \cdot \hat{q})^2}{|n \cdot p|}, 2\delta|p \cdot n|\right) \\ &\geq 2\min(\tau, 2\delta|p \cdot n|).\end{aligned} \tag{6.20}$$

From this we deduce that if t_n is the time of the n-th collision, then

$$\liminf_{n\to\infty} \frac{1}{n} \log Q(t_n) \geq \int \log(1 + \min(2\tau, 4\delta|p \cdot n|))d\mu > 0 \tag{6.21}$$

where $d\mu = |p \cdot n|dn$ with dn the surface measure on ∂Y.

As in Example 6.1, we can use (6.21) to deduce that there are two Lyapunov exponents l^+, l^- with $l^+ + l^- = 0$, $l^- < 0 < l^+$ when $d = 2$. Also the sector C can be used to construct the corresponding Osledect's directions,

$$E^+(x) = \bigcap_{n>0} D_{T^{-n}(x)}T^nC^+(T^{-n}(x)), \; E^-(x) = \bigcap_{n>0} D_{T^n(x)}T^{-n}C^-(T^n(x))$$

where $C^\pm = \{(\hat{q}, \hat{p}) : \pm\hat{q} \cdot \hat{p} > 0\}$.

There is a simple geometric interpretation for $\hat{x}(t)$. Assume that γ is a curve with $\gamma(0)=x$, $\dot{\gamma}(0)=\hat{x}$. This means that $\gamma(\delta)=x+\delta\hat{x}+o(\delta)$, $\phi_t(\gamma(t))=x(t)+\delta\hat{x}(t)+o(\delta)$, with $x(t)=\phi_t(x)$. In analogy with Riemannian geometry, we may regard $\hat{x}(t)$ on the Jacobi field associated with $x(t)$, and (6.15), (6.16) are the corresponding Jacobi's equations at a collision.

If we take a surface Λ of codimension one in $M=\bar{Y}\times\mathbb{R}^d$, then $T\Lambda\subseteq TM$ evolves to $T\phi_t(\Lambda)$. In this case, it is easier to study the evolution of the unit normal vectors. If $z(t)=(a(t),b(t))\in TM$ is normal to $T(\phi_t(\Lambda))$ at all times, then we would like to derive an evolution equation for it. The vector (a,b) is chosen so that for every t,

$$a(t)\cdot\hat{q}(t)+b(t)\cdot\hat{p}(t)=0,$$

where $(\hat{q}(t),\hat{p}(t))\in T_{x(t)}\Lambda(t)$ with $\Lambda(t)=\phi_t(\Lambda)$. In between collisions, $\hat{x}(t)=(\hat{q}+t\hat{p},\hat{p})$ and $a(t)\cdot(\hat{q}+t\hat{p})+b(t)\cdot\hat{p}=0$, or $a(t)\cdot\hat{q}+(ta(t)+b(t))\cdot\hat{p}=0$. Hence if initially $(a(0),b(0))=(a,b)$, then $a(t)=a$ and $b(t)=b-ta$. So in between collisions we simply have $\frac{da}{dt}=0$, $\frac{db}{dt}=-a$. At a collision (a,b) experiences a jump discontinuity. If after a collision the normal vector is given by (a',b'), then

$$\begin{aligned}&a'\cdot(R\hat{q})+b'\cdot(R\hat{p}-2A\hat{q})=0,\\&(Ra')\cdot\hat{q}+(Rb')\cdot\hat{p}-2(A^tb')\cdot\hat{q}=0.\end{aligned}$$

This suggests

$$\begin{cases}b'=Rb\\a'=Ra+2RA^tRb=:Ra+2Bb.\end{cases}\tag{6.22}$$

Note that if $Q(t)=a(t)\cdot b(t)$, then in between collisions,

$$\frac{dQ}{dt}=-|a|^2,$$

and at a collision

$$\begin{aligned}Q(t+)=a'\cdot b'&=(Ra+2RA^tRb)\cdot Rb\\&=Q(t-)+2A^tRb\cdot b\\&=Q(t-)+2b\cdot RAb\\&=Q(t-)+2(p\cdot n)D\nu(a)(Vb)\cdot V(b),\end{aligned}$$

and in the case of a dispersive billiard,

$$Q(t+)-Q(t-)\le 2\delta(p\cdot n)|Vb|^2<0.$$

Hence $Q(t)$ is decreasing.

As an example of a submanifold Λ of codimension 1, take a function $f^0 : \bar{Y} \times \mathbb{R}^d \to \mathbb{R}$ and set

$$\Lambda = \{(q,p) : f^0(q,p) = c\}$$

for a regular value c. If $f(q,p,t) = f^0(\phi_{-t}(q,p))$, then $\phi_t(\Lambda) = \{(q,p) : f(q,p,t) = c\}$ and for $z = (a,b)$ we may choose $z = (f_q, f_p)$. We know

$$\begin{cases} f_t + p \cdot f_q = 0 & \text{inside } Y \times \mathbb{R}^d, \\ f(q,p,t-) = f(q,p',t+) & \text{on } \partial Y \times \mathbb{R}^d, \end{cases} \tag{6.23}$$

where $q \in \partial Y$ and t is collision time. Setting a $(q,p,t) = f_q(q,p,t)$, $b(q,p,t) = f_p(q,p,t)$, we then have

$$\begin{cases} a_t + pD_q a = 0, \\ b_t + pD_p a = -a, \end{cases} \tag{6.24}$$

which is consistent with $\frac{da}{dt} = 0$, $\frac{db}{dt} = -a$ in between collisions. The formula (6.24) provides a relationship between $z(q,p,t)$ on $\partial Y \times \mathbb{R}^d$. In the case of smooth potential (6.1), if $f(x,t) = f^0(\phi_{-t}(x))$, then f solves the Liouville's equation

$$f_t + p \cdot f_q - \nabla V(q) \cdot f_p = 0. \tag{6.25}$$

If $\alpha = f_q$ and $\beta = f_p$, then after differentiating (6.25) we obtain

$$\begin{cases} \alpha_t + \alpha_q p - \alpha_p \nabla V(q) = D^2 V(q)\beta, \\ \beta_t + \beta_q p - \beta_p \nabla V(q) = -\alpha. \end{cases}$$

This is consistent with (6.22) if we interpret the hard-sphere model as a Hamiltonian system with potential $V(q) = \begin{cases} \infty & \text{if } q \notin Y, \\ 0 & \text{if } q \in Y \end{cases}$. In fact, in some sense, $D^2 V(q)\beta = 2B\beta$ of (6.27) when V is the above "concave" function. We note that if $\bar{\alpha}(x,t) = \alpha(\phi_t(x),t)$, then

$$\frac{d\bar{\alpha}}{dt} = D^2 V(q(t))\bar{\beta}$$
$$\frac{d\bar{\beta}}{dt} = -\bar{\alpha}$$

where $(q(t),p(t)) = \phi_t(x)$. Here $(\bar{\alpha},\bar{\beta})$ is the normal vector to the level sets of f as we mentioned earlier. Hence our method of showing the hyperbolicity of dispersive billiards should be applicable to general V if V is uniformly concave. Indeed, if

$$Q(x,t) = f_q(x,t) \cdot f_p(x,t) = \alpha(x,t) \cdot \beta(x,t),$$

then

$$Q_t + p \cdot Q_q - \nabla V(q) \cdot Q_p = D^2V(q)\beta \cdot \beta - |\alpha|^2$$

or equivalently

$$\bar{Q}_t = D^2V(q(t))\bar{\beta} \cdot \bar{\beta} - |\bar{\alpha}|^2$$

for $\bar{Q}(x,t) = Q(\phi_t(x),t)$, and if for some $\delta \in (0,1)$, $-D^2V(q) \geq \delta I$, then

$$\bar{Q}_t \leq -\delta(|\bar{\alpha}|^2 + |\bar{\beta}|^2) \leq -2\delta\bar{Q}$$

which implies that $|\bar{Q}(t)| \geq e^{2\delta t}|\bar{Q}(0)|$. However, in the case of a billiard, we only have $-D^2V(q) \geq \delta I$ only at collisions, which makes the proof of hyporbolicity much harder.

A particularly nice example of Λ is a normal bundle of a q-surface. More precisely, suppose Θ is a surface of codimension one in Y and set

$$\Lambda = \{(q,p) : q \in \Theta,\ p \text{ is the normal vector at } q\}.$$

Here we are assuming that Λ is orientable and a normal vector p at each $q \in \Theta$ is specified. In this case $(q,p,\hat{q},\hat{p}) \in T\Lambda$ means that $\hat{q} \in T_q\Theta$ and that $\hat{p} = C(q)\hat{q}$ for a suitable matrix $C(q)$ which is known as the *curvature matrix*. (If $p = p(q)$ is the normal vector, then $C(q) = Dp(q)$.) At later times, $(q(t),p(t),\hat{q}(t),C(q,t)\hat{q}(t)) \in T\phi_t(\Lambda)$. In between collisions, $\hat{p}(t)$ stays put, so

$$\frac{d}{dt}(C\hat{q}) = \left(\frac{d}{dt}C\right)\hat{q} + C\left(\frac{d}{dt}\hat{q}\right) = 0.$$

But $\frac{d}{dt}\hat{q} = \hat{p} = C\hat{q}$, so

$$\frac{d}{dt}C(t,q) = -C^2(t,q). \tag{6.26}$$

At a collision C changes to C' with $\hat{p}' = C'\hat{q}'$. Using (6.15) and (6.16), we deduce

$$C' = RCR - 2AR. \tag{6.27}$$

Recall that by our choice, p is the unit normal vector of Θ. Hence $\hat{q} \cdot p = 0$ and $\hat{p} \cdot p = 0$. This means that $\hat{q}, \hat{p} \in p^\perp$. As a result, we only need a matrix $C^\perp$ which is acting on $p^\perp$. Since the same is true after collision, we have $C^\perp$ is the restriction of C to $p^\perp$ and maps $p^\perp$ onto $p^\perp$. The same is true for $C^{\perp\prime}$. Hence

$$\begin{cases} \frac{dC^\perp}{dt} = -(C^\perp)^2 & \text{in between collisions,} \\ C^{\perp\prime} = RC^\perp R - 2AR & \text{at a collision.} \end{cases} \tag{6.28}$$

Note that $C^{\perp\prime} : (p')^\perp \to (p')^\perp$. Indeed if $v \in (p')^\perp = (Rp)^\perp$, then $Rv \in p^\perp$, and A maps p^2 onto $(p')^\perp$. Moreover $C^\perp : p^\perp \to p^\perp$ and R maps $p^\perp$ onto $(Rp)^\perp$.

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