Notes on
Ordinary Differential and Difference Equations

by

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ORDINARY DIFFERENTIAL EQUATIONS

DEFINITIONS

<u>Def</u> A differential equation is an equation that involves derivatives of a dependent variable with respect to one or more independent variables.

Ex. $$y'' + y' = e^x$$

Ex. $$\left(\frac{d^4 s}{dt^4}\right)^2 + 2\left(\frac{d^2 s}{dt^2}\right)^5 + \left(\frac{ds}{dt}\right)^2 = 0$$

<u>Def</u> Any function that is free of derivatives and that satisfies identically a differential equation is a solution of the differential equation.

Ex. $$y'' = -2y(y')^3$$

has

$$y^3 - 3x + 3y = 5$$

(an implicit function of y) as a solution.

<u>Def</u> A differential equation that involves derivatives with respect to a single independent variable is an ordinary differential equation (ODE). One that involves derivatives with respect to more than one independent variable is a partial differential equation (PDE).

<u>Def</u> The order of a differential equation is the order of the highest-order derivative present.

Ex. $$y'' + y' = e^x$$

is of order 2.

Ex. $$\left(\frac{d^4 s}{dt^4}\right)^2 + 2\left(\frac{d^2 s}{dt^2}\right)^5 + \left(\frac{ds}{dt}\right)^2 = 0$$

is of order 4.

<u>Def</u> If a differential equation can be rationalized and cleared of fractions with regard to all derivatives present, the exponent of the highest-order derivative is called the degree of the

equation.
Note: Not every differential equation has a degree.

Ex.

$$\left(\frac{d^4s}{dt^4}\right)^2 + 2\left(\frac{d^2s}{dt^2}\right)^5 + \left(\frac{ds}{dt}\right)^2 = 0$$

is of degree 2.

Ex.

$$y'' + (y')^2 = \ln y''$$

has no degree.

GENERAL REMARKS ON SOLUTIONS

Three important questions arise in attempting to solve a differential equation:

(1) Does a solution exist?

(2) If a solution exists, is it unique?

(3) If solutions exist, how do we find them?

Although most effort concerns (3), questions (1) and (2) are logically prior and must be answered first.

Thm (Existence) A differential equation y' = f(x,y) has at least one solution passing through any given point (x_0, y_0) if f is continuous.

Thm (Uniqueness) A sufficient condition for the solution to the differential equation y' = f(x,y), passing through a given point (x_0,y_0) to be unique is that ∂f/∂y be continuous.

These theorems apply only to first-order equations. However, analagous theorems apply to higher-order equations:

Thm (Existence and Uniqueness) The n-th order equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

has a unique solution passing through the point

$$(x_0, \eta_1, \ldots, \eta_n)$$

if f, f_2, f_3, ..., f_{n+1} all are continuous.

When the last theorem applies, an n-th order differential equation has a solution with n arbitrary constants, called the *general solution*, and this solution is unique.

Ex. $$y''' - 3y'' - 4y' + 12y = 0$$

has the general solution

$$y = Ae^{3x} + Be^{-2x} + Ce^{2x}$$

where the three arbitrary constants are A, B, and C.

Special cases of the general solution are called *particular solutions*; these are determined by the arbitrary initial conditions (x_0, η_1, ..., η_n).

Ex. The equation

$$y = xy' + (y')^2$$

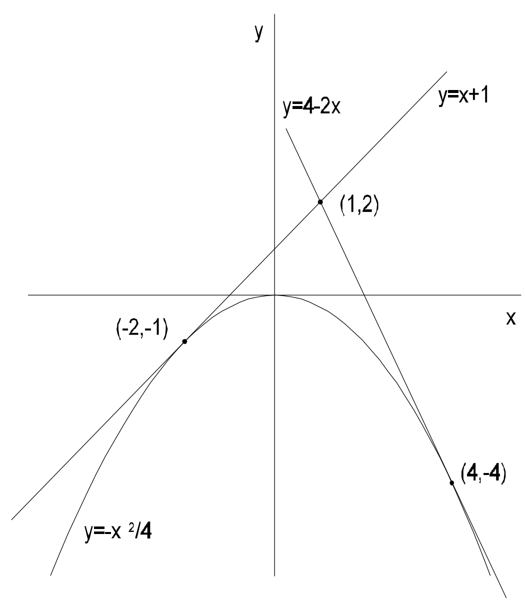
has the general solution

$$y = cx + c^2$$

However, it also has the solution

$$y = -x^2/4$$

which cannot be derived from the general solution and so is not a particular solution. It is called a *singular solution*.



To understand this last example, note that the original differential equation is of degree 2, that is, it is a quadratic equation in the variable y' and therefore can be solved to give

$$y' = \frac{-x \pm \sqrt{x^2 + 4y}}{2}$$

Consider the first of these:

$$y' = \frac{-x + \sqrt{x^2 + 4y}}{2}$$

The partial derivative with respect to y is $(x^2 + 4y)^{-1/2}$, which is discontinuous at $y = -x^2/4$. It therefore does not satisfy the uniqueness conditions. Note that only the region above the parabola $y = -x^2/4$ defines a real equation. For initial points within this region, solutions are unique. [For example, the point (1,2) implies from the general solution $y = cx + c^2$ that $c^2 + c - 2 = 0$, so that c = 1or -2, i.e., that y = x + 1 or y = 4 - 2x. Of these, only the first satisfies the equation

$$y' = \frac{-x + \sqrt{x^2 + 4y}}{2}$$

in agreement with the existence theorem. The second satisfies the other equation

$$y' = \frac{-x - \sqrt{x^2 + 4y}}{2}$$

Note that y = x + 1 and y = 4 - 2x, and in fact $y = cx + x^2$, are tantents to the parabola $y = -x^2/4$, which is the singular solution. The parabola is thus the envbelope of all these tangents, which in turn collect all the possible initial conditions.

Singular solutions, when they occur, often appear on the borders of existence regions.

FIRST-ORDER ODEs

I. Exactness

A first-order, first-degree ODE has the form

$$\frac{dy}{dx} = F(x,y)$$

which may be rewritten as

$$M(x,y)dx + N(x,y)dy = 0$$

$$\Leftrightarrow \quad \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

If this equation satisfies the existence and uniqueness theorems, then the general solution will have one arbitrary constant c, and the solution can be written

$$U(x,y) = c$$

Take the differential of both sides

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial x}dy$$

$$= 0$$

which implies

$$\frac{dy}{dx} = -\frac{\partial U/\partial x}{\partial U/\partial y}$$

so that

$$\frac{\partial U/\partial x}{\partial U/\partial y} = \frac{M}{N}$$

or

$$\frac{\partial U/\partial x}{M} = \frac{\partial U/\partial y}{N}$$

Calling each of these ratios μ, which in general may be a function of x and y, we obtain

$$\frac{\partial U}{\partial x} = \mu M$$

and

$$\frac{\partial U}{\partial y} = \mu N$$

Substituting these back into the equation for dU gives

$$\mu(Mdx + Ndy) = dU = 0$$

an *exact differential equation*. Such an equation can be written

$$U(x,y) = c$$

The function μ enables us to go from the original differential equation to an exact differential and then, by integration, to a function that solves the differential equation. Thus μ is called an *integrating factor*.

If the differential equation

$$Mdx + Ndy = 0$$

already is exact, then by definition there is a function U(x,y) such that

$$Mdx + Ndy = dU$$

But

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial x}dy$$

which implies that

$$\frac{\partial U}{\partial x} = M$$

$$\frac{\partial U}{\partial y} = N$$

For a sufficiently smooth function U,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x}$$

$$= \frac{\partial^2 U}{\partial x \partial y}$$

$$= \frac{\partial N}{\partial y}$$

Thus the relation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

is a necessary condition for exactness. It turns out also to be sufficient. Thus we have

Thm A necessary and sufficient condition for exactness of the differential equation

$$Mdx + Ndy = 0$$

is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex. The differential equation

$$(2xy + 3x^2)dx + x^2dy = 0$$

is exact because

$$\frac{\partial(2xy + 3x^2)}{\partial y} = 2x$$

$$= \frac{\partial x^2}{\partial x}$$

Thus, by the sufficiency part of the theorem, there exists a function U such that

$$(2xy + 3x^2)dx + x^2dy = dU$$

or equivalently such that

$$\frac{\partial U}{\partial x} = 2xy + 3x^2$$

$$\frac{\partial U}{\partial Y} = x^2$$

We use these last two relationships to determine U. By reversing the differentiation (i.e., by integrating), we can recover U. Therefore

$$U = \int(2xy + 3x^2)\partial x + f(y)$$

where f(y) is the constant of integration. So

$$U = x^2 y + x^3 + f(y)$$

To find f(y), use the second relationship above:

$$\frac{\partial U}{\partial y} = x^2 + f'(y)$$

$$= x^2$$

$$\Rightarrow \quad f'(y) = 0$$

$$\Rightarrow \quad f(y) = A \qquad \text{constant}$$

Thus

$$U = x^2 y + x^3 + A$$

II. Solving First-Order Differential Equations

A. Separation of variables.

Ex. To solve

$$\frac{dy}{dx} = \frac{y^2}{x^3}$$

we write the equation in the form

$$\frac{dx}{x^3} = \frac{dy}{y^2}$$

Then integrate

$$\int \frac{dx}{x^3} = \int \frac{dy}{y^2}$$

which gives

$$-\frac{x^{-2}}{2} = -y^{-1} + A$$

or

$$\frac{1}{y} = \frac{1}{2x^2} + A$$

which is an implicit function for y.

This method in fact is a shortcut way of finding an integrating factor. To see this, re-write the original differential equation in the form Mdx + Ndy = 0:

$$y^2 dx - x^3 dy = 0$$

We see that

$$\frac{\partial M}{\partial y} = 2y$$

$$\neq -3x^2$$

$$= \frac{\partial N}{\partial x}$$

so the equation is not exact. However, the function $(x^3y^2)^{-1}$ is an integrating factor; multiplying the differential equation by it gives

$$\frac{dx}{x^3} - \frac{dy}{y^2} = 0$$

for which

$$\frac{\partial M}{\partial y} = 0$$

$$= \frac{\partial N}{\partial x}$$

so that exactness holds.

B. Integrating factors involving one variable.

Ex. The equation

$$(2y^2x - y)dx + x\,dy = 0$$

is not exact and not separable. However, the function y^{-2} is an integrating factor:

$$y^{-2}(2y^2x - y)dx + y^{-2}x\,dy = 0$$

or

$$(2x - y^{-1})dx + xy^{-2}dy = 0$$

or

$$(xy^{-2}dy - y^{-1}dx) + 2x\,dx = 0$$

which is

$$-d\left(\frac{x}{y}\right) + d(x^2) = 0$$

so that we have, upon integrating,

$$-\frac{x}{y} + x^2 = c$$

How do we know y^{-2} is an integrating factor? (Ignore the problem of how we found it in the first place.) To see this, consider the case where $Mdx + Ndy = 0$ is not exact or separable. Multiply by the integrating factor μ (as yet unknown) to obtain the exact equation $\mu Mdx + \mu Ndy = 0$. Then by definition

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

Consider two cases.

(i) μ is a function of x alone.

In this case,

$$\mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{d\mu}{dx}$$

or

$$\frac{d\mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$$

$$= f(x)dx \qquad \text{by hypothesis}$$

$$\Rightarrow \quad \mu = e^{\int f(x)dx}$$

where the arbitrary constant of integration has been set equal to zero. Thus we have the following:

<u>Thm</u> If

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$$

where f(x) is a function of x alone, then $e^{\int f(x)dx}$ is an integrating factor.

(ii) μ is a function of y alone.

Similar reasoning leads to the following:

<u>Thm</u> If

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$$

where g(y) is a function of y alone, then $e^{\int g(y)dy}$ is an integrating factor.

C. Important special case of the previous method - the linear first-order equation.

An equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a linear equation of first order. (Note that it is only required to be linear in y, not x.) This equation has $e^{\int P(x)dx}$ as an integrating factor, so that we may write

$$e^{\int P dx} \frac{dy}{dx} + Pye^{\int P dx} = Qe^{\int P dx}$$

which is equivalent to

$$\frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

which we may integrate to obtain

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + c$$

which is the general solution.

D. Other methods.

Often, equations are not immediately solvable by multiplying by an integrating factor. Numerous specialized methods exist for such equations, usually involving some convenient transformation of variables.

Ex. The equation

$$\frac{dx}{dx} = f(\frac{y}{x})$$

(unfortunately called a homogeneous differential equation) can be solved by the transformation

$$y/x = v$$

$$\Leftrightarrow \quad y = vx$$

Differentiate both sides of y = vx with respect to x:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

so that

$$v + x\frac{dv}{dx} = f(\frac{y}{x})$$

or

$$\frac{dx}{x} = \frac{dv}{f(v)-v}$$

which has variables separated and is easily solved.

III. Line Integrals and Differential Equations

A. Basics.

A line integral generalizes the concept of the integral to include integrals of vector-valued functions on R^n. For simplicity, we will restrict attention n=2.

Let

$\xi \equiv (x, y) \in R^2$; $\xi = xi + yj$ where i, j are basis vectors
$\phi: R^2 \to R^2$; $\phi(\xi) = [f(x, y), g(x, y)]$ where f, $g: R^2 \to R^1$
$\rho: R^1 \to R^2$, $\rho(t) = \xi$, with $|\rho'(t)| > 0 \quad \forall \; t \in [a, b]$ and $\rho(t) = [r(t), s(t)]$ where $r,s: R^1 \to R^1$
C = graph of $\rho(t)$ for $t \in [a, b]$, called a directed arc

<u>Def</u> The line integral of ϕ along C, the graph of $\xi = \rho(t)$, is the number

$$\int_C \phi(\xi) \cdot d\xi = \int_C [f(x,y)dx + g(x,y)dy]$$

$$= \int_a^b \{f[r(t), s(t)]r'(t) + g[r(t), s(t)]s'(t)\}dt$$

Ex. Let $\phi(\xi) = (x^2, y^2)$ and C be the graph of $\rho(t) = (5\cos t, 5\sin t)$ for $t \in [-1/2\,\pi, 1/2\pi]$. Then

$$\int_C \phi(\xi) d\xi = \int_{-\pi/2}^{\pi/2} (25\cos^2 t, 25\sin^2 t)\cdot(-5\sin t, 5\cos t)$$

$$= \int_{-\pi/2}^{\pi/2} (-125\cos^2 t\sin t, 125\sin^2 t\cos t)\cdot(-5\sin t, 5\cos t)$$

$$= 125\left(\frac{1}{3}\cos 3t + \frac{1}{3}\sin^3 t\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{250}{3}$$

Many times we integrate around a closed path, which is one that begins and ends at the same point. We must distinguish between the two directions around the path; integrating in one direction yields the opposite sign as integrating in the other direction. By convention, we integrate in the counterclockwise direction. The notation

$$\oint_P \phi(\xi)\cdot d\xi$$

indicates integrating around the closed path P in the counterclockwise direction. (If the direction of integration ever is ambiguous, the notation is altered by putting an arrow on the circle to show the direction of integration.)

Thm (Fundamental Theorem of Calculus for Line Integrals.) Suppose ϕ is the gradient of a scalar-valued function $h:R^2 \to R^1$. Then

$$\int_C \phi(\xi)\cdot d\xi = \int_C \nabla h(\xi)\cdot d\xi$$

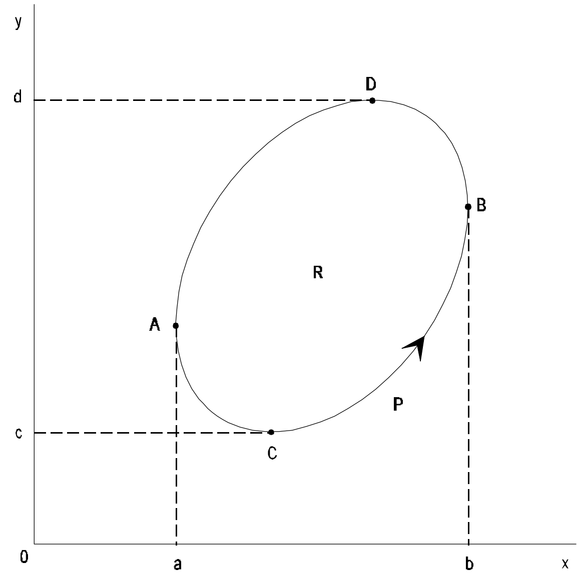
$$= \int_a^b \nabla h[\rho(t)]\cdot\rho'(t)dt$$

$$= \int_a^b D_t h[\rho(t)]dt$$

$$= h[\rho(b)] - h[\rho(a)]$$

B. Green's theorem; functions defined by line integrals.

Let P be the path shown in the graph, and let R be the region bounded by P. Then



$$\oint_P \Phi(\xi)\cdot d\xi = \oint_P [f(x,y)dx + g(x.y)dy]$$

$$= \oint_P f(x,y)dx + \oint_P g(x.y)dy$$

To evaluate these last two integrals, we decompose P into two directed arcs. For the first integral, we consider P as the union of the arc ACB and BDA; for the second, we consider P as the union of CBD and DAC. Thus we have

$$\oint_P f(x,y)dx = \int_{ACB} f(x,y)dx + \int_{BDA} f(x,y)dx$$

$$\oint_P g(x,y)dy = \int_{CBD} g(x,y)dy + \int_{DAC} g(x,y)dy$$

Let ACB be the graph of y = r(x) and ADB be the graph of y = s(x). Then

$$\int_{ACB} f(x,y)dx = \int_a^b f[x,r(x)]dx$$

$$\int_{BDA} f(x,y)dx = -\int_{ADB} f(x,y)dx = -\int_a^b f[x,s(x)]dx$$

so that

$$\oint_P f(x,y)dx = -\int_a^b [f(x,s(x) - f(x,r(x)]dx$$

By the fundamental theorem of calculus, for each x ∊ [a, b] we have

$$f[x,s(x)] - f[x,r(x)] = f(x,y)\Big|_{r(x)}^{s(x)}$$

$$= \int_{r(x)}^{s(x)} f_2(x,y)\,dy$$

so that

$$\oint_P f(x,y)\,dx = -\int_a^b \int_{r(x)}^{s(x)} f_2(x,y)\,dy\,dx$$

$$= -\iint_R f_2(x,y)\,dA$$

Similarly, if the arcs CBD and CAD are the graphs of x = q(y) and x = p(y) for y ∊ [c, d], then

$$\oint g(x,y)\,dy = \int_c^d g[q(y),y]\,dy - \int_c^d g[p(y),y]\,dy$$

$$= \int_c^d [g(q(y),y) - g(p(y),y)]\,dy$$

$$= \int_c^d g(x,y)\Big|_{p(y)}^{q(y)} dy$$

$$\int_c^d \int_{p(y)}^{q(y)} g_1(x,y)\,dy$$

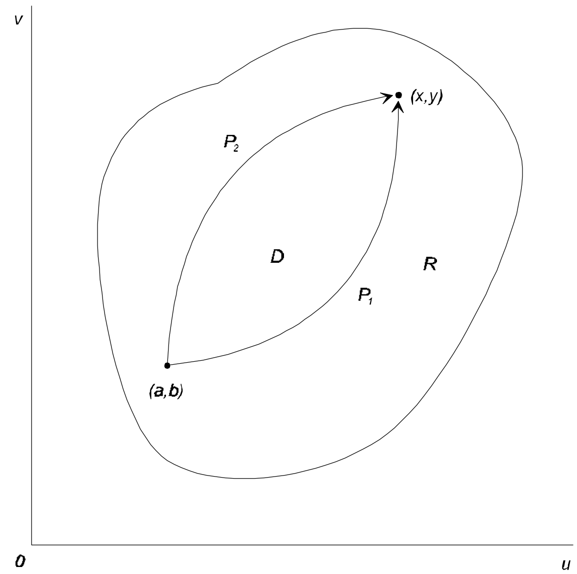
$$= \iint_R g_1(x,y)\,dA$$

Then by substituting into the original integral, we obtain Green's Theorem (also called the fundamental theorem of calculus for double integrals):

<u>Thm</u> $$\oint_P \Phi(\xi)\cdot d\xi = \iint_R [g_1(x,y) - f_2(x,y)]\,dA$$

Green's theorem is important because it allows sensible definitions of functions in terms of line integrals. To define a function $F:R^2 \to R^1$ with a line integral, choose a point $\alpha = (a, b)$ and define $F(\xi)$, for any $\xi = (x, y)$,

$$F(\xi) = \int_{\alpha}^{\xi} \phi(\gamma)\cdot d\gamma$$



In general, this definition is inadequate in that the value of F will depend on the path chosen from α to ξ. To have a useful definition, we onlyn use line integrals to define functions when the integral is independent of path. We use Green's theorem to identify path-independent integrals.

Let P_1 and P_2 be two directed paths in R that connect the points (a, b) and (x, y). We want F to be path-independent, so require:

$$\int_{P_1} \phi(\gamma)\cdot d\gamma = \int_{P_2} \phi(g)\cdot d\gamma$$

$$\Leftrightarrow \quad \oint_{P} \phi(\gamma)\cdot d\gamma = 0$$

where $P = P_1 \cup P_2$. Suppose the region D bounded by P is simply connected (essentially, has no holes). Then by Green's theorem

$$\oint \phi(\gamma)\cdot d\gamma = \iint_{D} [g_1(u,v) - f_2(u,v)]dA$$

where $(u, v) \equiv \gamma$. The double integral is zero if, for each point of D, $g_1 = f_2$. Thus we have:

<u>Thm</u> If R is simply connected and $f_2 = g_1 \ \forall \ \delta \in R$, then the equation

$$F(\xi) = \int_{\alpha}^{\xi} \phi(\gamma)\cdot d\gamma$$

defines a function $F:R^2 \to R^1$.

We can compute the partial derivatives for this function F as follows:

<u>Thm</u> For F defined by

$$F(x,y) = \int_{(a,b)}^{(x,y)} [f(u,v)du + g(u,v)dv]$$

where f and g are continuous and satisfy $f_2 = g_1$, we have

$$D_x F(x,y) = f(x,y)$$

$$D_y F(x,y) = g(x,y)$$

Proof: Write

$$F(x,y) = \int_{(a,b)}^{(c,y)} \phi(\gamma)\cdot d\gamma + \int_{(c,y)}^{(x,y)} \phi(\gamma)\cdot d\gamma$$

The first of these integrals is independent of x so that

$$F_1(x,y) = D_x \int_{(a,b)}^{(c,y)} \phi(\gamma)\cdot d\gamma + D_x \int_{(c,y)}^{(x,y)} \phi(\gamma)\cdot d\gamma$$

$$= D_x \int_{(c,y)}^{(x,y)} \phi(\gamma)\cdot d\gamma$$

$$= D_x \int_{(c,y)}^{(x,y)} [\ (u,v)du + g(u,v)dv]$$

We may use any path to evaluate this integral; it is convenient to use the horizontal segment joining (c, y) and (x, y), along which v is constant at the value y:

$u = r(t) = t,\ t \in [c, x]$
$v = s(t) = y$

Then we have

$$\int_{(c,y)}^{(x,y)} [\ (u,v)du + g(u,v)dv] \ = \ \int_c^x f[r(t),s(t)]r'(t)dt + \int_y^y g[r(t),s(t)]s'(t)dt$$

$$= \int_c^x f(t,y)dt$$

Consequently,

$$F_1 \ = \ D_x \int_c^x f(t,y)dt$$

$$= f(x,y)$$

The proof that $D_y F = g(x,y)$ is similar. <u>QED</u>

The Fundamental Theorem of Calculus for Line Integrals tells us that the line integral of a gradient is independent of path. The converse is also true:

<u>Thm</u> If line integrals of a continuous function ϕ are independent of path in some region R, then ϕ is the gradient of a scalar-valued function.

Proof: We can take as this scalar-valued function the function F defined by

$$F(\xi) \ = \ \int_\alpha^\xi \phi(\gamma) \cdot d\gamma$$

for which

$$\nabla F(\xi) \ = \ [F_1(x,y), F_2(x,y)]$$

$$= \ [f(x,y), g(x,y)]$$

$$= \ \phi(\xi)$$

<u>QED</u>

C. Application to Differential Equations

Let f(x,y) and g(x,y) satisfy $f_2 = g_1$, and let

$$F(x,y) = \int_{(a,b)}^{(x,y)} [f(u,v), g(u,v)] \cdot d[u,v]$$

For any c in the range of F, the set of points S satisfying F(x,y) = c constitutes a level curve (contour) of F. Suppose r is a differentiable function whose graph is a subset of this level curve, i.e.,

$$F(s, r(x)) = c \qquad \forall x \in I \subset S$$

Then

$$\begin{aligned} D_x F(x, r(x)) &= F_1(x, r(x)) + F_2(x, r(x)) r'(x) \\ &= f(x, r(x)) + g(x, r(x)) r'(x) \\ &= D_x c \\ &= 0 \end{aligned}$$

Therefore, in the interval I, the function y = r(x) satisfies the exact differential equation

$$f(x,y) + g(x,y) y' = 0$$

which in our earlier notation can be written

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

or

$$M(x,y)dx + N(x,y)dy = 0$$

So we conclude that

(1) level curves of functions defined by line integrals provide solutions to exact differential equations;

(2) the condition for exactness of the differential equation is the same as that for path independence of the corresponding line integral.

Ex. The differential equation

$$6x + y^3 + 3xy^2 y' = 0$$

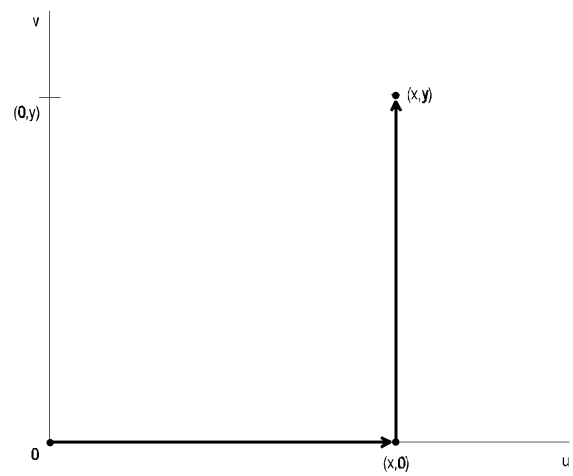
is exact because

$$f_2 = 3y^2$$

$$= g_1$$

Therefore, we can set

$$F(x,y) = \int_{(0,0)}^{(x,y)} [(6u + v^3)du + 3uv^2 dv]$$



and use the path shown to find that

$$F(x,y) = xy^3 + 3x^2$$

Level curves of this function are graphs of the form

$$xy^3 + 3x^2 = c$$

so the function

$$y = \left(\frac{c - 3x^2}{x} \right)^{1/3}$$

satisfies the differential equation. The value of c is determined by initial conditions. For example, if the initial condition is that y = 3 when x = 1 [that is, $(x_0, y_0) = (1,3)$], then c = 30.

LINEAR DIFFERENTIAL EQUATIONS

I. General

Def A linear differential equation of order n has the form

$$a_0(x)\frac{d_n y}{dx^n} + a_1(x)\frac{d_{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x)$$

where $a_i(x)$ and F(x) do not depend on y.

It is often convenient to express a linear differential equation in terms of differential operators:

$$a_0(x)D^n y + a_1(x)D^{n-1}y + \ldots + a_{n-1}(x)Dy + a_n(x)y = F(x)$$

or

$$[a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)D + a_n(x)]y = F(x)$$

or

$$\phi(D)y = F$$

Clearly,

$$D^n(u+v) = D^n u + D^n v$$

$$D^n(au) = aD^n u$$

Any operator satisfying these properties is called a *linear operator*. It is easy to show that ϕ(D) is a linear operator.

We want to discover how to find the general solution to a linear differential equation. To do so, we will examine the *homogeneous* (or complementary) *equation*

$$\phi(D)y = 0$$

Its solution provides a crucial step in finding the solution to the original equation.

Thm If y = u(x) is any solution of the given equation and y = v(x) is any solution of the homogeneous equation, then y = u(x) + v(x) is a solution of the given equation.

Proof: We have

$$\phi(D)u = F(x)$$

$$\phi(D)v = 0$$

Adding, we have

$$\phi(D)(u+v) = F(x)$$

<u>Def</u> The general solution of the homogeneous equation is called the *homogeneous* (or complementary) *solution*. A solution of the given equation in which all constants have specified values is called a *particular solution*.

<u>Thm</u> The general solution of $\phi(D)y = F(x)$ may be obtained by finding a particular solution y_p and adding it to the homogeneous solution y_c.

Ex. The homogeneous equation for

$$(D^2 - 5D + 6)y = 3x$$

is

$$(D^2 - 5D + 6)y = 0$$

It may be verified that the homogeneous solution is

$$y_c = c_1 e^{3x} + c_2 e^{2x}$$

It also may be verified that the particular solution is

$$y_p = \frac{2}{2}x + \frac{5}{12}$$

Therefore, the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{3x} + c_2 e^{2x} + \frac{1}{2}x + \frac{5}{12} \end{aligned}$$

Obviously, two important questions must be answered:

(1) How do we find the homogeneous solution?

(2) How do we find the particular solution?

II. Existence and Uniqueness

Thm Let $a_i(x)$ and F(x) be continuous on the interval [a,b] and suppose $a_0(x) \neq 0 \; \forall \; x \in [a,b]$. Then there exists a unique solution y(x) satisfying the differential equation and also the initial conditions

$$y^{(i)}(c) = p_i \qquad i = 0, 1, \ldots, n-1 \quad \text{and} \quad c \in [a,b], \text{and } p_i \text{ constants}$$

III. Obtaining a Homogeneous Solution

Suppose we want to solve

$$(D-2)y = 0$$

which is the same as

$$dy/dx - 2y = 0$$

This is easily solved by the methods discussed earlier:

$$y = ce^{2x}$$

This solution could also be found by lucky guessing. Assume a solution of the form $y = e^{mx}$ with m to be determined. For this form to be a solution, it must satisfy $(m-2)\, e^{mx} = 0$, i.e., m=2 because e^{mx} is never zero. This solution can be generalized to ce^{2x}.

Would the same guessing method work on the equation

$$(D^2 - 3D + 2)y = 0$$

Letting $y = e^{mx}$, we obtain

$$(m^2 - 3m + 2)e^{mx} = 0$$

$$\Rightarrow \qquad m^2 - 3m + 2 = 0$$

$$\Rightarrow \qquad m = 1 \text{ or } 2$$

$\Rightarrow$ e^x and e^{2x} are solutions

What is the general solution? We know a general solution must have two arbitrary constants, so let us propose

$$y = c_1 e^{x} + c_2 e^{2x}$$

This function is easily shown to satisfy the original equation and therefore must be the general solution.

Thm If $y_1, y_2, ..., y_p$ are solutions to $\phi(D)y = 0$, then $c_1y_1 + c_2y_2 + ... + c_py_p$ also is a solution.

The equation used above for determination of m is called the *characteristic* (or auxiliary) *equation* and has the form $\phi(m) = 0$. Note that this equation applies only to differential equations with constant coefficients, that is, with $a_i(x) = a_i$. For the most part, we will be concerned only with equations that have constant coefficients.

Thm If the differential equation $\phi(D) = 0$ has constant coefficients and if the roots of the characteristic equation are distinct, then the general solution is

$$y = \sum_{i=1}^{n} c_i e^{m_i x}$$

where m_i are the roots of the characteristic equation.

Thm If the characteristic equation has repeated roots, then the general solution is

$$y = \sum_{i=1}^{d} \left(\sum_{j=1}^{n_i} c_j x^{j-1} \right) e^{m_i x}$$

where d = number of distinct roots, n_i = multiplicity of the ith root with $n_1 + ... + n_d = n$ = the order of the equation.

Ex. The differential equation

$$(D^2 - 6D + 9)y = 0$$

has one root equal to 3 with multiplicity 2. Therefore the general solution is

$$(c_1 + c_2 x)e^{3x}$$

The foregoing theorem is a consequence of the following:

Thm If $y = y_1$ is one solution of the nth-order differential equation $\phi(D) = F(x)$ [with variable or constant coefficients], then the substitution $y = y_1 v$ will transform the given equation into one of the (n-1)st order in dv/dx.

Ex. One solution to

$$(D^2 - 6D + 9)y = 0$$

is e^{3x}. According to the theorem, let $y = ve^{3x}$. Then

$$Dy = v'e^{3x} + 3ve^{3x}$$

$$d^2y = v''e^{3x} + 6v'e^{3x} + 9ve^{3x}$$

so that we obtain

$$v''e^{3x} = 0$$

which implies $v'' = 0$ or $v = c_1 + c_2 x$. Consequently,

$$y = (c_1 + c_2 x)e^{3x}$$

IV. Linear Independence and Wronskians

Suppose we have the differential equation

$$(D^3 - 6D^2 + 11D - 6)y = 0$$

and somehow arrive at the three solutions

$$e^{2x} + 2e^{x}$$

$$5e^{2x} + 4e^{x}$$

$$e^{x} - e^{2x}$$

Is the general solution given by

$$y = A(e^{2x} + 2e^{x}) + B(5e^{2x} + 4e^{x}) + C(e^{x} - e^{2x})$$

The answer is no because we can rearrange terms to get

$$y = (2A + 4B + C)e^{x} + (A + 5B - C)e^{2x}$$

$$= c_1 e^{x} + c_2 e^{2x}$$

which has only two arbitrary constants.

A basic principle in determining general solutions is the following

Thm If $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of the nth order linear differential equation

$$\phi(D)y = [a_0(x)D^{n} + a_1(x)D^{n-1} + \ldots + a_n]y$$

$$= 0$$

then all solutions have the form

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

which is the general solution.

The problem in the foregoing example is that the three solutions are not linearly independent. A test for linear independence is given by the following:

Thm The set of functions $y_1, y_2, \ldots, y_n$ is linearly independent if and only if the Wronskian

$$W(y_1, y_2, \ldots, y_n) \equiv \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is not identically zero.

Ex. The functions x^2 and x^3 have the Wronskian

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^4$$

which is not identically zero. Note that W = 0 at x = 0, but that is different from W being identically zero.

V. Behavior of Solutions to Second-Order Equations

Suppose the differential equation

$$\phi(D)y = (a_0 D^2 + a_1 D + a_2)y$$

$$= 0$$

has constant coefficients. Then the behavior of the solution depends on the values of the roots of the characteristic equation.

A few examples will be mentioned here; detailed discussion is deferred until systems of equations have been discussed.

(A) Distinct negative roots

The solution has the form

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

with m_1, $m_2 < 0$. Therefore, $y \to 0$ as $x \to \infty$. In this case, y is asymptotically stable.

(B) Distinct positive roots

$y \to \infty$ as $x \to \infty$

so y is unstable.

(C) Distinct imaginary roots

For the equation

$$(D^2 + 1)y = 0$$

we have the characteristic equation

$$m^2 + 1 = 0$$

which has the conjugate pair of imaginary roots ±i. The general solution is

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

What does this solution mean?

The functions e^{ix} and e^{-ix} are defined to be

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

which are derived from the series expansion of e^u

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots$$

with u replaced by ix or -ix.

Therefore

$$\begin{aligned} c_1 e^{ix} + c_2 e^{-ix} &= c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x) \\ &= (c_1 + c_2)\cos x + (c_1 - c_2) i \sin x \\ &= A \cos x + B \sin x \\ &= y \end{aligned}$$

Because y must be real, both A and B must be real. Therefore

$$c_1 + c_2 = A \qquad \text{real}$$

$$(c_1 - c_2) i = B \qquad \text{real}$$

$$\Rightarrow \qquad c_1 = \frac{B}{i} + c_2$$

$$\Rightarrow \qquad 2c_2 + \frac{B}{i} = A$$

$$\Rightarrow \qquad c_2 = -\frac{B}{2i} + \frac{A}{2}$$

$$\Rightarrow \qquad c_1 = \frac{B}{2i} + \frac{A}{2}$$

In this particular example, the solution

$$y = A\cos x + B\sin x$$

produces endless non-damping and non-explosive cycles as x → ∞.

In general, imaginary roots yield cyclic solutions. The solutions may be damped, endless, or explosive, depending on the values of the real parts of the roots.

Ex. The equation

$$(D^2 + 2D + 5)y = 0$$

has roots m = -1 ± 2i and the general solution

$$y = c_1 e^{(-1+2i)x} + c_2 e^{(-1-2i)x}$$

$$= e^{-x}(c_1 e^{2ix} + c_2 e^{-2ix})$$

$$= e^{-x}(A\cos 2x + B\sin 2x)$$

which leads to damped cycles as x → ∞.

VI. Obtaining a Particular Solution

Our next step is to discover how to find a particular solution to the equation ϕ(D)y = F(x). We briefly examine two methods.

A. <u>Method of Undetermined Coefficients</u>

This method applies when F(x) consists of a polynomial in x, terms of the form sin(px), cos(px), and e^{px} where p = constant, or combinations of sums and products of these. When products of these terms are not present, the rules for using the method are:

(1) Write the homogeneous solution y_H.

(2) Assume a particular solution with the same kinds of terms constituting F(x):

(a) For a polynomial of degree n, assume a polynomial of degree n.

(b) For terms sin(px), cos(px), or sums and differences of such terms, assume a sin(px) + b cos(px).

(c) For terms e^{px}, assume ae^{px}.

(3) If any of the assumed terms already appear in the homogeneous solution, they must be multiplied by a power of x sufficiently high (but no higher) that they do not occur in the homogeneous solution.

(4) Evaluate the coefficients of the assumed particular solution, thus obtaining y_P.

(5) Add y_H and y_P to obtain the general solution.

When products of the basic terms are present, one first must differentiate the right side indefinitely, keeping track of all essentially different terms that arise. If these are finite in number, then the method of undetermined coefficients is applicable. The assumed particular solution is formed by multiplying each of the terms that appeared in the differentiation by an undetermined constant and adding the results.

Ex. (step 2.b) The homogeneous solution of

$$(D^2 + 4D + 4)y = 6\sin 3x$$

is

$$y_H = (c_1 + c_2 x)e^{-2x}$$

To find a particular solution, we ask what functions differentiated once or twice yield sin3x or constant multiples of it. The answer is sin3x or cos3x. Therefore, we try

$$y_P = a\sin 3x + b\cos 3x$$

Substituting in the given equation gives

$$(D^2 + 4D + 4)y = (-5a - 12b)\sin 3x + (12a - 5b)\cos 3x$$

$$= 6\sin 3x$$

The second equality will be satisfied if and only if

-5a - 12b = 6
12a - 5b = 0

⇒ a = -30/169, b = -72/169

⇒ $y = y_C + y_P$

$$= (c_1 + c_2 x)e^{-2x} - \frac{30}{169}\sin 3x - \frac{72}{169}\cos 3x$$

Ex. (need to differentiate RHS) Suppose we have

$$(D^2 + 1)y = x^2\cos 5x$$

The homogeneous solution is

$$y_C = c_1\cos x + c_2\sin x$$

To find a particular solution, begin differentiating the functions $x^2\cos 5x$:

1st round: yields terms of the form $x^2\sin 5x$ and $x\cos 5x$

2nd round: yields $x^2\cos 5x$, $x\sin 5x$, and $\cos 5x$

Continuing in this manner, we find that no terms other than the following arise (where we ignore numerical constants):

$x^2\cos 5x$
$x^2\sin 5x$
$x\cos 5x$
$x\sin 5x$
$\cos 5x$
$\sin 5x$

Therefore, we assume as a particular solution

$$y_P = ax^2\cos5x + bx^2\sin5x + cx\cos5x + fx\sin5x + g\cos5x + h\sin5x$$

If the RHS of the given equation were lnx, successive differentiations would give 1/x, $1/x^2$, ... which are infinite in number. In this case, the method of undetermined coefficients is inapplicable and some other method must be used.

B. Method of Variation of Constants

This method is generally applicable but can be difficult because it requires integration of functions to determine the general solution. The method is illustrated by finding the solution to the following equation:

$$y'' + y = \tan x$$

The homogeneous solution is

$$y_H = A\cos x + B\sin y$$

The solution is the general solution to the equation

$$y'' + y = 0$$

A particular solution to the homogeneous equation therefore is

$$y_{HP} = \cos x + \sin x$$

We therefore can regard the homogeneous solution as the product of the vectors (cosx, sinx) and (A, B).

To find a general solution to the original equation, we try generalizing y_H by letting A and B be functions of x rather than constants (hence the name of the method). If we are to determine two functions A(x) and B(x), we need to have two conditions to indentify the two functions. One of these is that

$$y = A(x)\cos x + B(x)\sin x$$

satisfies the original equation. The other we are free to choose as we need.

We proceed by differentiating the function

$$y = A(x)\cos x + B(x)\sin x$$

to obtain

$$y' = -A(x)\sin x + B(x)\cos x + A'(x)\cos x + B'(x)\sin x$$

Clearly, further differentiation will introduce more terms, so we seek a condition we may impose that will simplify this last equation. The following is what we seek:

$$A'(x)\cos x + B'(x)\sin x = 0$$

which then gives

$$y' = -A(x)\sin x + B(x)\cos x$$

Therefore

$$y'' = -A(x)\cos x - B(x)\sin x - A'(x)\sin x + B'(x)\cos x$$

Substituting these expressions for y' and y" into the original equation gives

$$-A(x)\sin x + B'(x)\cos x = \tan x$$

This, together with the earlier condition that

$$A'(x)\cos x + B'(x)\sin x = 0$$

gives the following expressions for A'(x) and B'(x):

$$A'(x) = \frac{-\sin^2 x}{\cos x}$$

$$B'(x) = \sin x$$

Therefore, by integration,

$$A(x) = \int -\frac{\sin^2 x}{\cos x}dx$$

$$= \int \frac{\cos^2 x - 1}{\cos x}dx$$

$$= \int (\cos x - \sec x)dx$$

$$= \sin x - \ln(\sec x + \tan x) + c_1$$

$$B(x) = \int \sin x\, dx$$

$$= -\cos x + c_2$$

which gives

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x)$$

as the general solution to the original equation.

C. Operator Methods

Suppose we "solve" the differential equation

$$\phi(D)y = F(x)$$

by writing

$$y = \frac{1}{\phi(D)}F(x)$$

where $\phi^{-1}(D)$ represents an operation to be performed on F(x). What is this operation?

To gain insight, consider the simpler equation Dy = x. Then we would have

$$y = \frac{1}{D}x$$

However, we can solve Dy = x by the usual methods:

$$y = \int x\,dx$$

so it seems natural to define

$$\frac{1}{D}x = \int x\,dx$$

It is then straightforward to show that, analogously,

$$D^{-n}x = \int\int\int\ldots\int x\,dx\,dx\,dx\ldots dx \qquad \text{n-fold}$$

Now consider

$$(d-p)y = f(x) \qquad \text{p constant}$$

Formally, we have

$$y = \frac{1}{D-p}f(y)$$

Solving the differential equation by usual methods gives

$$y = e^{px}\int e^{-px}f(x)dx$$

so it is natural to define

$$(D-p)^{-1}f(x) = e^{px}\int e^{-px}f(x)dx$$

Note that this reduces to our earlier definition for p = 0.

Now consider

$$(D-p_1)(D-p_2)y = f(x)$$

The operator $(D-p_1)(D-p_2)$ has several important properties:

(1) $(D-p)y = \frac{dy}{dx} - py$, so that

$$(D-p_1)(D-p_2)y = (\frac{d}{dx}-p_1)(\frac{d}{dx}-p_2)y$$

$$= (\frac{d}{dx}-p_1)(\frac{dy}{dx}-p_2 y)$$

$$= \frac{d}{dx}(\frac{dy}{dx}-p_2 y)-p_1(\frac{dy}{dx}-p_2 y)$$

$$= \frac{d^2y}{dx^2}-(p_1+p_2)\frac{dy}{dx}+p_1p_2 y$$

$$= [D^2--(p_1+p_2)D+p_1p_2]y$$

The reasoning used here establishes that D operators may be multiplied or factored like algebraic quantities, as long as the p_i are constants.

(2) One may show that the factorization

$$a_0D^n+a_1D^{n-1}+\ldots+a_n \equiv a_0(D-p_1)(D-p_2)\ldots(D-p_n)$$

is always possible and unique when the a_i are constants.

(3) D operators obey the commutative, associative, and distributive laws.

This last fact allows us to solve our original differential equation by writing

$$y = \frac{1}{(D-p_1)(D-p_2)}f(x)$$

By applying our earlier results, we have

$$\frac{1}{(D-p_1)(D-p_2)}f(x) \equiv e^{p_1x}\int e^{-p_1x}\left[e^{p_2x}\int e^{-p_2x}f(x)\right]dx$$

In a similar manner, we may write

$$[(D-p_1)(D-p_2)\ldots(D-p_n)]^{-1}f(x) \equiv e^{p_1x}\int e^{-p_1x}e^{p_2x}\int e^{-p_2x}\ldots e^{p_nx}\int e^{-p_nx}f(x)dx^n$$

Expressions like the left side of the last equation are often easier tohandle if resoved into partial fractions. We briefly review partial fractions.

Def The ratio of two polynomials is called a rational expression.

Thm Every rational expression can be written as a sum of the forms

$$q_r x^r + q_{r-1} x^{r-1} + \ldots + q_0$$

$$\frac{A}{(x-a)^r}$$

$$\frac{Bx+C}{(x^2+bx+c)^r} \qquad \text{where} \quad b^2 - 4x < 0$$

To apply the method of partial fractions, first examine the rational expression. If the degree of the numerator is less than that of the denominator, we can proceed. Otherwise, we use long division (synthetic division) to write

$$F(x) = Q(x) + \frac{R(x)}{D(x)}$$

where deg R(x) < deg D(x).

Ex. If we have

$$\frac{(x^3+1)}{(x-2)}$$

we perform long division

$$\begin{array}{r|l} & x^2 + 2x + 4 \\ \hline x-2 & x^3 + 1 \\ & \underline{x^3 - 2x^2} \\ & \quad 2x^2 + 1 \\ & \quad \underline{2x^2 - 4x} \\ & \qquad 4x + 1 \\ & \qquad \underline{4x + 1} \\ & \qquad\quad 9 \end{array}$$

to obtain

$$\frac{x^3+1}{x-2} = x^3+2x+4+\frac{9}{x-2}$$

Once we have a rational expression with only the degree of the numerator less than that of the denominator, we apply the method, according to which

(a) each factor $(x\text{-}a)^r$ in the denominator leads to a sum of the form

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_r}{(x-a)^r}$$

(b) each factor $(x^2 + bx + c)^r$ in the denominator leads to a sum of the form

$$\frac{B_1x+C_1}{(x^2+bx+c)}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\ldots+\frac{B_rx+C_r}{(x^2+bx+c)^r}$$

Ex. We use partial fractions to write

$$\frac{6x^3+5x^2+21x+12}{x(x+1)(x^2+4)} = \frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2+4}$$

which can be rewritten as

$$6x^3+5x^2+21x+12 = A(x+1)(x^2+4)+Bx(x^2+4)+Cx+Dx(x+1)$$

$$= (A+B+C)x^3+(A+C+D)x^2+(4A+4B+D)x+4A$$

Therefore, equating coefficients,

$$A+B+C = 6$$

$$A+C+D = 5$$

$$4A+4B+D = 21$$

$$4A = 12$$

which we can solve for A, B, C, and D to obtain

$$A = 3$$

$$B = 2$$

$$C = 1$$

$$D = 1$$

With these results on partial fractions, we can re-write expressions such as $[(D-p_1)(D-p_2)...(D-p_n)]^{-1}$ in the form

$$\frac{A_1}{D-p_1} + \frac{A_2}{D-p_2} + \ldots + \frac{A_n}{D-p_n}$$

Therefore, we have

$$\frac{1}{(D-p_1)(D-p_2)\ldots(D-p_n)}f(x) = \frac{A_1}{D-p_1}f(x) + \frac{A_2}{D-p_2}f(x) + \ldots + \frac{A_n}{D-p_n}f(x)$$

$$= A_1 e^{p_1 x}\int e^{-p_1 x} f(x)dx + \ldots + A_n e^{p_n x}\int e^{-p_n x} f(x)dx$$

which involves only single integrations.

Ex. To find the general solution of $(D^2 - 1)y = e^{-x}$, we write

$$y = \frac{1}{(D-1)(D-2)}e^{-x}$$

$$= \left(\frac{1/2}{D-1} - \frac{1/2}{D+1}\right)e^{-x}$$

$$= \frac{1}{2}e^{x}\int e^{-2x}dx - \frac{1}{2}e^{-x}\int dx$$

$$= c_1 e^{x} + c_2 e^{-x} - \frac{1}{4}e^{-x} - \frac{1}{2}xe^{-x}$$

$$= Ae^{x} + Be^{-x} - \frac{1}{2}xe^{-x}$$

Note that if we set the constants of integration c_1 and c_2 equal to zero, we obtain a particular solution.

Ex. To find a particular solution of

$$(D^2 - D + 1)y = x^3 - 3x^2 + 1$$

we write

$$y = \frac{1}{1 - D + D^2}(x^3 - 3x^2 + 1)$$

By long division in ascending powers of D:

$$\frac{1}{1 - D + D^2} = 1 + D - D^3 - D^4 + \ldots$$

Therefore, formally,

$$y = (1 + D - D^3 - D^4 + \ldots)(x^3 - 3x^2 + 1)$$

$$= (x^3 - 3x^2 + 1) + D(x^3 - 3x^2 + 1) - D^3(x^3 - 3x^2 + 1) - D^4(x^3 - 3x^2 + 1) + \ldots$$

$$= x^3 - 6x - 5$$

which can be verified as a particular solution.

SYSTEMS OF EQUATIONS

I. General

We wish to analyze systems of the following type:

$$y_1' = f_1(t, y_1, y_2, \ldots, y_n)$$

$$y_2' = f_2(t, y_1, y_2, \ldots, y_n)$$

$$\vdots \qquad \vdots$$

$$y_n' = f_n(t, y_1, y_2, \ldots, y_n)$$

where f_i are functions defined in some region D of R^{n+1}. To proceed more compactly, we define

$$y \equiv (y_1, y_2, \ldots, y_n)$$

$$f(t,y) \equiv [f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)]$$

$$y' \equiv (y_1', y_2', \ldots, y_n')$$

Thus we can write our original equation system in vector form:

$$y' = f(t,y)$$

To *solve* this system means to find a real interval I and a vector $\phi(t)$ defined on I such that

(i) $\phi'(t)$ exists for each t in I

(ii) the point $(t, \phi(t))$ lies in D for each t in I

(iii) $\phi'(t) = f(t, \phi(t))$ for every t in I

II. Existence and Uniqueness

Thm If f and $\partial f/\partial y_k$ $\forall k$ are continuous in D, then given any point (t_0, η) there exists a unique solution ϕ of the system

$$y' = f(t,y)$$

satisfying the initial condition $\phi(t_0) = \eta$. The solution exists on any interval I containing t_0 for which the pints $(t, \phi(t))$ lie in D. Furthermore, the solution is a continuous function of the triple (t, t_0, η).

III. Linear Systems

A. General

A linear system is one which is linear in y, so that

$$f(t,y) = A(t)y + g(t)$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_1n(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_2n(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

$$g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

In what follows, we will need a definition of the norm (length) of a matrix.

Def The absolute value norm of A is

$$|A| \equiv \sum_{i,\,j=1}^{n} |a_{ij}|$$

Other definitions of norm are possible. In general, a norm is any function N:D → R satisfying the following three conditions:

(i) $N(a) \geq 0$ and $= 0$ if and only if $a = 0$

(ii) $N(ca) = |c|\, N(a)$ for constant c

(iii) $N(a+b) \leq N(A) = N(B)$

Def The sequence of matrices $\{A_k\}$ converges to the matrix A if and only if the sequence of real numbers $\{|A-A_k|\}$ converges to zero.

Because of the way the matrix norm is defined, this definition of matrix convergence means that $A_k \to A$ if and only if $a_{ij}(k) \to a_{ij}$ as $k \to \infty$ $\forall$ i, j.

B. Existence and Uniqueness

Thm If A(t) and g(t) are continuous on an interval I, $t_0 \in I$, and $|y| < \infty$, then the equation

$$y' = A(t)y + g(t)$$

has a unique solution $\phi(t)$ satisfying the initial condition $\phi(t_0) = \eta$ and existing on I.

C. <u>Linear Homogeneous Systems</u>

The linear homogeneous system is of the form

$$y' = A(t)y$$

Note that, given the point $(t_0, 0)$, the homogeneous system has the unique solution $\phi = 0$. [By inspection, this is a solution; by the existence and uniqueness theorem, it is unique.]

Note that, for any two solutions ϕ_1 and ϕ_2 and constants c_1 and c_2:

$$(c_1\phi_1 + c_2\phi_2)' = c_1\phi_1' + c_2\phi_2'$$

$$= c_1A\phi_1 + c_2A\phi_2$$

$$= A(c_1\phi_1 + c_2\phi_2)$$

so that $(c_1\phi_1 + c_2\phi_2)$ also is a solution. Consequently, the solutions to the homogeneous equation form a vector space, denoted V.

<u>Thm</u> If A(t) is continuous on I, then the solutions to the homogeneous system form a vector space of dimension n.

Proof: We already know the solutions form a vector space. We need only discover its dimension..

Let t_0 be any point of I and let $\delta_1, \delta_2, ..., \delta_n$ be n linearly independent points in the space; the elementary vectors $\delta_i = (0, 0, ..., 0, 1, 0, ..., 0)$ are obviously such. By the existence and uniqueness theorem, the homogeneous system has n solutions ϕ_i satisfying

$$\phi_i(t_0) = \delta_i$$

To see that the ϕ_i are linearly independent, consider

$$a_1\phi_1(t) + a_2\phi_2(t) + ... + a_n\phi_n(t) = 0$$

Setting $t = t_0$, this becomes

$$a_1\delta_1 + a_2\delta_2 + ... + a_n\delta_n = 0$$

which can only occur if the a_i all are zero because of the linear independence of the δ_i. Therefore the ϕ_i are linearly independent.

To see that the ϕ_i span V, consider any element of V, say $\psi(t)$. Let $\psi(t_0) = \delta \in R^n$. The δ_i form a basis of R^n (because they are linearly independent), so:

$$\delta = c_1\delta_1 + c_2\delta_2 + \ldots + c_n\delta_n$$

Now consider

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t)$$

This is a solution because the solutions are a vector space. Also,

$$\phi(t_0) = c_1\delta_1 + c_2\delta_2 + \ldots + c_n\delta_n$$

$$= \delta$$

Therefore $\phi(t)$ and $\psi(t)$ are both solutions satisfying the same initial condition. By uniqueness,

$$\psi(t) = \phi(t)$$

$$= c_1\phi_1(t) + \ldots + c_n\phi_n(t)$$

So the ϕ_i span V, which therefore has dimension n. **QED.**

Def An n-dimensional set of linearly independent solutions is called a fundamental set of solutions.

Def An nxn matrix whose coumns are solutions of the homogeneous system is called a solution matrix.

Def An nxn solution matrix whose columns are linearly independent (i.e., whose rank is n) is called a fundamental matrix.

We denote the fundamental matrix formed from solutions ϕ_i, ϕ_2, ..., ϕ_n by Φ. Then the foregoing theorem's implication that every solution ψ is the linear combination

$$\psi(t) = \sum c_i\phi_i$$

for some unique choice of constants c_i can be restated as

$$\psi(t) = \Phi(t)c$$

where c is the column vector $(c_i, ..., c_n)'$.

Thm (Abel's formula) If M is a solution matrix of the homogeneous system on I and if $t_0 \in I$, then $\forall\, t \in I$

$$\det M(t) = \det M(t_0) \exp\left[\int_{t_0}^{t} \sum_{j=1}^{n} a_{jj}(s)\,ds\right]$$

Proof: Let the columns of M be ϕ_j, each having the components $(\phi_{1j}, \phi_{2j}, \ldots, \phi_{nj})$. Because ϕ_j is a solution,

$$\phi_{ij}' = \sum_{k=1}^{n} a_{ik}\phi_{kj} \qquad \text{for } i,j = 1,\ldots,n$$

It is a fact from linear algebra that

$$(\det M)' = \begin{vmatrix} \phi_{11}' & \cdots & \phi_{1n}' \\ \phi_{21} & \cdots & \phi_{2n} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \phi_{21}' & \cdots & \phi_{2n}' \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{n-1,1} & \cdots & \phi_{n-1,n} \\ \phi_{n1}' & \cdots & \phi_{nn}' \end{vmatrix}$$

so that here

$$(\det M)' = \begin{vmatrix} \sum a_{ik}\phi_{k1} & \cdots & \sum a_{ik}\phi_{k1} \\ \phi_{21} & \cdots & \phi_{2n} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{n-1,1} & \cdots & \phi_{n-1,n} \\ \sum a_{nk}\phi_{k1} & \cdots & \sum a_{nk}\phi_{kn} \end{vmatrix}$$

Each determinant on the RHS can be evaluated by elementary row operations. For example, in the first determinant we multiply the second row by a_{12}, the third by a_{13}, and so on, add these n-1 rows and then subtract the result from the first row. The resulting first row will be $(a_{11}\phi_{11}, a_{11}\phi_{12}, \ldots, a_{11}\phi_{1n})$, so that the value of the first determinant is $a_{11}(\det M)$. Proceeding similarly with the other determinants, we obtain

$$
\begin{aligned}
(\det M)' &= a_{11}\det M + a_{22}\det M + \ldots + a_{nn}\det M \\
&= \left[\sum_{k=1}^{n} a_{kk}(t)\right]\det M \\
&= [tr A(t)]\det M
\end{aligned}
$$

This is a first-order *scalar* equation for detM, whose solution is

$$
\det M(t) = \det M(t_0)\exp\left[\int_{t_0}^{t}\sum_{j=1}^{n} a_{jj}(s)\,ds\right] \qquad \textbf{QED}
$$

Cor Either detM(t) $\neq 0\ \forall t \in I$ or detM(t) $= 0\ \forall t \in I$.

Thm A solution matrix M is fundamental if and only if detM(t) $\neq 0\ \forall t$ on I.

Proof: (A) If detM(t) $\neq 0\ \forall t \in I$, then the ϕ_i are linearly independent and M is fundamental.
(B) If M is fundamental, then every solution has the form $\phi(t) = M(t)c$ for some constant vector c. Then $\forall t_0 \in I$ and $\phi(t_0)$, the system $\phi(t_0) = M(t_0)c$ has a unique solution for a given c because M is fundamental. Therefore $\det M(t_0) \neq 0$, because any system Ax = b has a unique solution if and only if detA $\neq 0$ (with the solution given by $x = A^{-1}b$). So by Abel's formula, detM(t) $\neq 0\ \forall t \in I$.   **QED**

Note that in general a matrix may have linearly independent columns but have its determinant identically zero. For example,

$$
M(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & 2 & t \\ 0 & 0 & 0 \end{bmatrix}
$$

According to the last theorem, this cannot happen for solutions to homogeneous systems.

Note that, by Abel's formula, it is sufficient to test whether a matrix is fundamental by evaluating it at one point. This point often can be chosen to make the calculation easy.

Thm If Φ is fundamental and C is a nonsingular constant matrix, then ΦC also is fundamental. Every fundamental matrix is of the form ΦC for some nonsingular C.

Proof: Let Φ and Ψ be fundamental. The jth column ψ_j of Ψ can be written $\psi_j = \Phi c_j$ for some c_j. Define C as the matrix whose columns are c_j. Then $\Psi = \Phi C$. Because $\det\Phi \neq 0$ and $\det\Psi \neq 0$, we also have $\det C \neq 0$. Reversing the argument gives the converse. **QED**

D. Linear Nonhomogeneous Systems

The nonhomogenous system is

$$y' = A(t)y + g(t)$$

where g(t) is usually called the forcing function.

Suppose Φ is fundamental for the homogeneous system. Also suppose that ϕ_1 and ϕ_2 are solutions to the nonhomogeneous system. Then ϕ_1-ϕ_2 is a solution to the homogeneous system, in which case $\exists$ c such that

$$\phi_1 - \phi_2 = \Phi c = \phi_n$$

Thus, to find any solution to the nonhomogeneous system, we need only know one, because every other solution differs from the known one by some solution to the homogeneous system:

$$\begin{aligned}\phi_1 &= \phi_n + \phi_2 \\ &= \Phi c + \phi_2\end{aligned}$$

We will now see how to construct solutions to the nonhomogeneous system by using the method of variation of constants.

Let Φ be a fundamental matrix of the associated homogeneous system. Suppose we attempt to construct a solution to the nonhomogeneous system of the form

$$\Psi(t) = \Phi(t)v(t)$$

where v is to be determined. Suppose such a solution exists. Then

$$\begin{aligned}\psi'(t) &= \Phi'(t)v(t) + \Phi(t)v'(t) && \text{by differentiation} \\ &= A(t)\Phi(t)v(t) + g(t) && \text{by supposition}\end{aligned}$$

Because Φ is fundamental,

$$\Phi'(t) = A(t)\Phi(t)$$

so that we have

$$\Phi(t)v'(t) = g(t)$$

$$\Rightarrow \quad v'(t) = \Phi^{-1}(t)g(t)$$

$$\Rightarrow \quad v(t) = \int_{t_0}^{t} \Phi^{-1}(s)g(s)ds$$

Therefore

$$\psi(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds$$

Conversely, if we define $\psi(t)$ by

$$\psi(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds$$

then

$$\psi'(t) = \Phi'(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds + \Phi(t)\Phi^{-1}(t)g(t)$$

$$= A(t)\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds + g(t)$$

$$= A(t)\psi(t) + g(t)$$

Obviously, $\psi(t_0) = 0$.

We therefore have proven the variation of constants formula:

Thm If Φ is a fundamental matrix of $y' = A(t)y$ on I, then the function

$$\psi(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds$$

is the unique solution of $y' = A(t)y + g(t)$ satisfying the initial condition $\psi(t_0) = 0$.

We see, then, that every solution of the nonhomogeneous equation has the form

$$\phi(t) = \phi_h(t) + \psi(t)$$

where

ψ	=	variation of constants function
ϕ_h	=	solution of homogeneous system satisfying the same initial condition at ϕ, for example, $\phi_h(t_0) = \eta$.

Note that, in the variation of constants formula, Φ^{-1} acts like the integrating factor in our earlier single equation problems. Moreover, the entire theory of linear equations developed earlier is a special case of the present development, for any such equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_{n-1}(t)y' + a_n(t) = b(t)$$

or

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & \cdots & \cdots & \cdots & -a_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}$$

or

$$y' = A(t)y + g(t)$$

Note that the eigenvalues of A equal the roots of the characteristic equation.

E. Linear Systems with Constant Coefficients

In general, it is not easy to find the fundamental matrix Φ. However, when A(t) is constant, Φ is easy to determine.

Def A series $\sum_{k=0}^{\infty} U_k$ of matrices converges if and only if the sequence $\sum_{k=0}^{m} U_k$ of partial sums converges, where convergence of a sequence of matrices is as defined earlier in terms of the absolute value norm.

Lem A sequence $\{A_k\}$ of matrices converges if and only if, given a number $\epsilon \geq 0$, there exists an integer $N(\epsilon)>0$ such that $|A_m-A_p| < \epsilon$ whenever m, p > $N(\epsilon)$.

Def The matrix e^M is defined as

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots$$

This definition is sensible only if the series on the RHS converges. To see that it does, define

$$S_k = I + M + \frac{M^2}{2!} + \ldots + \frac{M^k}{k!}$$

Note that

$$|S_k| = \left| \sum_{j=0}^{k} \frac{M^j}{j!} \right|$$

Then, for m > p,

$$|S_m - S_p| = \left| \sum_{k=p+1}^{m} \frac{M^k}{j!} \right|$$

$$\leq \sum_{k=p+1}^{\infty} \frac{|M|^k}{j!}$$

$$\leq \sum_{k=0}^{\infty} \frac{|M|^k}{j!}$$

$$\leq e^{|M|} \in \mathbb{R}^1$$

$$\leq \infty$$

Thus $|S_m-S_p|$ is less than the tail of the infinite series defining $e^{|M|}$. Because $e^{|M|}$ is finite, the series that defines it converges, which means its tail vanishes as the lower index of summation is raised. Therefore, given $\epsilon>0$, there exists N>0 such that $|S_m-S_n| < \epsilon$ for m,p > N, so that $\{S_k\}$ and therefore the series defining $e^{|M|}$ converges.

Thm If M and P commute (i.e., MP = PM), then

$$e^{M}e^{P} = e^{M+P}$$

Thm If T is a non-singular nxn matrix, then

$$T^{-1}e^{M}T = e^{T^{-1}MT}$$

Thm The matrix

$$\Phi(t) = e^{At}$$

is fundamental for the system

$$y' = Ay$$

with Φ(0) = I on t ∊ (-∞, ∞).

Proof: Φ(0) = I is obvious. We have by differentiation

$$\left(e^{At}\right)' = A + \frac{A^2 t}{1!} + \frac{A^3 t}{2!} + \ldots$$

$$= Ae^{At}$$

for t ∊ (-∞, ∞). Therefore e^{At} is a solution matrix. Furthermore,

$$\det \Phi(0) = \det I$$

$$= 1$$

so by Abel's formula

$$\det\left(e^{At}\right) = \det\left(e^{A \cdot 0}\right)\exp\left[\left(\sum_{kl=0}^{n} a_{kk}\right)t\right]$$

$$\neq 0 \qquad\qquad \text{QED}$$

Ex. To find the fundamental matrix of y' = Ax if

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

note that

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

These matrices commute, so

$$e^{At} = \exp \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t \cdot \exp \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t$$

$$= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} \left\{ I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 \frac{t^2}{2!} + \ldots \right\}$$

But

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^j = 0 \quad \text{for } j \geq 2$$

$$\Rightarrow \quad e^{At} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} + \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t$$

$$= e^{3t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

We next develop a general method for determining the form of e^{At} when A is an arbitrary matrix.

<u>Thm</u> Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of the nxn matrix A with respective multiplicities $n_1, \ldots, n_k$ (note that $\Sigma n_j = n$). Then

(1) there exist k subspaces X_j of E^n such that $E^n = X_1 \oplus X_2 \oplus \ldots \oplus X_k$, i.e., such that $x = x_1 + x_2 + \ldots + x_k$, i.e., E^n is the direct sum of the X_j

(2) X_j is invariant under A, i.e., $Ax_j \in X_j \ \forall \ x_j \in X_j$

(3) $A - \lambda_j I$ is nilpotent un X_j of index at most n_j, i.e., $\exists \ q_j \leq n_j$ such that, $\forall \ x_j \in X_j$,

$$(A - \lambda_j I)^{q_j} x_j = 0$$

Our goal is to find that solution $\phi(t)$ to the system y' = Ay that satisfies the initial condition $\phi(0) = \eta$. We know that $\phi(t) = e^{At}\eta$, so we willknow $\phi(t)$ once we find e^{At}. But we know that

$$\eta = v_1 + \ldots + v_k$$

where v_j is some (as yet unknown) vector in the subspace X_j. We also know that v_j satisfies

$$(A - \lambda_j I)^{q_j} v_j = 0$$

Therefore,

$$e^{At} v_j = e^{\lambda_j t} e^{(A-\lambda_j I)t} v_j$$

$$= e^{\lambda_j t}\left[I + (A-\lambda_j I)t + (A-\lambda_j I)^2 \frac{t^2}{2!} + \ldots + (A-\lambda_j I)^{n_j - 1} \frac{t^{n_j - 1}}{(n_j - 1)!}\right] v_j$$

[Note that

$$e^{xI} = I + xI + \frac{x^2 I^2}{2!} + \ldots$$

$$= I(1 + x + \frac{x^2}{2!} + \ldots)$$

$$= I e^x$$

$$\Rightarrow \quad e^{\lambda_j I t} + e^{\lambda_j t} I$$

]

So we can write

$$
\begin{aligned}
\phi(t) &= e^{At}\eta \\
&= e^{At}\sum_{j=1}^{k} v_j \\
&= \sum_{j=1}^{k} e^{At} v_j \\
&= \sum_{j=1}^{k} e^{\lambda_j t}\left[I + (A-\lambda_j I)t + \ldots + (A-\lambda_j I)^{n_j-1}\frac{t^{n_j-1}}{(n_j-1)!}\right]v_j \\
&= \sum_{j=1}^{k} e^{\lambda_j t}\left[\sum_{i=0}^{n_j-1}\frac{t^i}{i!}(A-\lambda_j I)^I\right]v_j
\end{aligned}
$$

To find e^{At}, let η successively equal the basis vectors e_1, e_2, ..., e_n because

$$
\begin{aligned}
e^{At} &= e^{At} I \\
&= \left[e^{At}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e^{At}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e^{At}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right]
\end{aligned}
$$

Ex. Let

$$
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
$$

so that we have a general 2-dimensional system. The eigenvalues of A are found from

$$
\begin{aligned}
|A - \lambda I| &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\
&= 0
\end{aligned}
$$

$$\Rightarrow \quad \lambda = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

In this example, we consider the case where the two roots λ_1 and λ_2 are distinct. We have

$$\begin{aligned}
\phi(t) &= e^{At}\eta \\
&= e^{\lambda_1 t}v_1 + e^{\lambda_2 t}v_2 \\
&= e^{\lambda_1 t}\begin{bmatrix} v_{11} \\ \\ v_{12} \end{bmatrix} + e^{\lambda_2 t}\begin{bmatrix} v_{21} \\ \\ v_{22} \end{bmatrix} \\
&= \begin{bmatrix} e^{\lambda_1 t}v_{11} + e^{\lambda_2 t}v_{21} \\ \\ e^{\lambda_1 t}v_{12} + e^{\lambda_2 t}v_{22} \end{bmatrix}
\end{aligned}$$

which establishes the usual result that each part of the solution vector $\phi(t)$ depends on both roots and two arbitrary constants.

As yet, the vectors v_1 and v_2 are unknown. They are determined by the initial conditions $\phi(0) = \eta$. Once we determine v_1 and v_2, we can determine e^{At}, which we would need if we wanted the general form of $\phi(t) = e^{At}\eta$ or if we wanted to solve a non-homogeneous equation for which $y'(t) = Ay(t)$ was the associated homogeneous equation.

To determine v_1 and v_2, we use the conditions

$$(A - \lambda_1 I)x = 0 \qquad \forall\ x \in X_1$$

$$(A - \lambda_2 I)x = 0 \qquad \forall\ x \in X_2$$

$$(A-\lambda_1 I)x = \begin{bmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}-\lambda_1)x_1 + a_{12}x_2 \\ a_{21}x_1 + (a_{22}-\lambda_1)x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving the first equation for x_2 gives

$$x_2 = \frac{\lambda_1 - a_{11}}{a_{12}} x_1$$

Substituting into the second equation gives

$$0 = a_{21}x_1 + (a_{22}-\lambda_1)\frac{\lambda_1 - a_{11}}{a_{12}} x_1$$

$$= [\lambda_1^2 - (a_{11}+a_{22})\lambda_1 + (a_{11}a_{22} - a_{12}a_{21})x_1$$

$$\equiv 0 \quad \text{because this is the characteristic equation for A evaluated at a root of that equation}$$

$$\Rightarrow \begin{cases} x_1 & \text{arbitrary} \\ x_2 & \dfrac{\lambda_1 - a_{11}}{a_{12}} x_1 \end{cases}$$

[Note that if we had solved the equations of $(A-\lambda_1 I)x = 0$ in the other order, we would have obtained

$$x_1 \quad \text{arbitrary}$$

$$x_2 = \frac{\lambda_1 - a_{11}}{a_{12}} x_1$$

which seems different but is not because

$$\frac{\lambda_1 - a_{11}}{a_{12}} = \frac{a_{21}}{\lambda_1 - a_{22}}$$

This last equality can be seen by cross-multiplying to obtain

$$\lambda_1^2 - (a_{11} + a_{22})\lambda_1 + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

which is the characteristic equation for A evaluated at the root λ_1 and so necessarily true.]

Similarly, the condition

$$(A - \lambda_2 I)x = 0$$

$$\Rightarrow \begin{cases} x_1 \quad \text{arbitrary} \\ x_2 = \dfrac{\lambda_2 - a_{11}}{a_{12}} x_1 \end{cases}$$

So v_1 and v_2 have the forms

$$v_1 = \begin{bmatrix} \alpha \\ \dfrac{\lambda_1 - a_{11}}{a_{12}} \alpha \end{bmatrix}, \quad v_2 = \begin{bmatrix} \beta \\ \dfrac{\lambda_2 - a_{11}}{a_{12}} \beta \end{bmatrix}$$

Therefore

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \dfrac{\lambda_1 - a_{11}}{a_{12}} \alpha \end{bmatrix} + \begin{bmatrix} \beta \\ \dfrac{\lambda_2 - a_{11}}{a_{12}} \beta \end{bmatrix}$$

or

$$\eta_1 = \alpha + \beta$$

$$\eta_2 = \frac{\lambda_1 - a_{11}}{a_{12}}\alpha + \frac{\lambda_2 - a_{11}}{a_{12}}\beta$$

$$\Rightarrow \begin{cases} \alpha = \dfrac{(\lambda_2 - a_{11})\eta_1 - a_{12}\eta_2}{\lambda_2 - \lambda_2} \\ \beta = \dfrac{(\lambda_1 - a_{11})\eta_1 - a_{12}\eta_2}{\lambda_2 - \lambda_2} \end{cases}$$

Substitute these results into the expressions for v_1 and v_2 to obtain

$$v_1 = \begin{bmatrix} \dfrac{(\lambda_2 - a_{11})\eta_1 - a_{12}\eta_2}{\lambda_2 - \lambda_1} \\ \left(\dfrac{\lambda_1 - a_{11}}{a_{12}}\right)\left\{\dfrac{(\lambda_2 - a_{11})\eta_1 - a_{12}\eta_2}{\lambda_2 - \lambda_1}\right\} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \dfrac{-(\lambda_1 - a_{11})\eta_1 + a_{12}\eta_2}{\lambda_2 - \lambda_1} \\ \left(\dfrac{\lambda_2 - a_{11}}{a_{12}}\right)\left\{\dfrac{-(\lambda_1 - a_{11})\eta_1 + a_{12}\eta_2}{\lambda_2 - \lambda_1}\right\} \end{bmatrix}$$

We know that

$$\phi(t) = e^{\lambda_1 t}v_1 + e^{\lambda_2 t}v_2$$

$$= e^{\lambda_1 t}\begin{bmatrix} \dfrac{(\lambda_2 - a_{11})\eta_1 - a_{12}\eta_2}{\lambda_2 - \lambda_1} \\ \left(\dfrac{\lambda_1 - a_{11}}{a_{12}}\right)\left\{\dfrac{(\lambda_2 - a_{11})\eta_1 - a_{12}1\eta_2}{\lambda_2 - \lambda_1}\right\} \end{bmatrix} + e^{\lambda_2 t}\begin{bmatrix} \dfrac{-(\lambda_1 - a_{11})\eta_1 + a_{12}\eta_2}{\lambda_2 - \lambda_1} \\ \left(\dfrac{\lambda_2 - a_{11}}{a_{12}}\right)\left\{\dfrac{-(\lambda_1 - a_{11})\eta_1 + a_{12}1\eta_2}{\lambda_2 - \lambda_1}\right\} \end{bmatrix}$$

We let η successively equal the basis vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$:

(a)

$$\eta = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \quad \phi(t) = e^{\lambda_1 t}\begin{bmatrix}\dfrac{(\lambda_2 - a_{11})}{\lambda_2 - \lambda_1}\\ \left(\dfrac{\lambda_1 - a_{11}}{a_{12}}\right)\left(\dfrac{(\lambda_2 - a_{11})}{\lambda_2 - \lambda_1}\right)\end{bmatrix} + e^{\lambda_2 t}\begin{bmatrix}\dfrac{-(\lambda_1 - a_{11})}{\lambda_2 - \lambda_1}\\ -\left(\dfrac{\lambda_2 - a_{11}}{a_{12}}\right)\left(\dfrac{(\lambda_1 - a_{11})}{\lambda_2 - \lambda_1}\right)\end{bmatrix}$$

(b)

$$\eta = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow \quad \phi(t) = e^{\lambda_1 t}\begin{bmatrix}\dfrac{-a_{12}}{\lambda_2 - \lambda_1}\\ -\left(\dfrac{\lambda_1 - a_{11}}{\lambda_2 - \lambda_1}\right)\end{bmatrix} + e^{\lambda_2 t}\begin{bmatrix}\dfrac{a_{12}}{\lambda_2 - \lambda_1}\\ \left(\dfrac{\lambda_2 - a_{11}}{a_{12}}\right)\end{bmatrix}$$

So from the relation

$$e^{At} = e^{At}I = \left[e^{At}\begin{pmatrix}1\\0\end{pmatrix}, e^{At}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

we get

$$e^{At} = \begin{bmatrix} \left(\frac{\lambda_2 - a_{11}}{\lambda_2 - \lambda_1}\right) e^{\lambda_1 t} - \left(\frac{\lambda_1 - a_{11}}{\lambda_2 - \lambda_1}\right) e^{\lambda_2 t} & -\left(\frac{a_{12}}{\lambda_2 - \lambda_1}\right) e^{\lambda_1 t} + \left(\frac{a_{12}}{\lambda_2 - \lambda_1}\right) e^{\lambda_2 t} \\ \left(\frac{\lambda_1 - a_{11}}{a_{12}}\right)\left(\frac{\lambda_2 - a_{11}}{\lambda_2 - \lambda_1}\right) e^{\lambda_1 t} - \left(\frac{\lambda_2 - a_{11}}{a_{12}}\right)\left(\frac{\lambda_1 - a_{11}}{\lambda_2 - \lambda_1}\right) e^{\lambda_2 t} & -\left(\frac{\lambda_1 - a_{11}}{\lambda_2 - \lambda_1}\right) e^{\lambda_1 t} + \left(\frac{\lambda_2 - a_{11}}{\lambda_2 - \lambda_1}\right) e^{\lambda_2 t} \end{bmatrix}$$

Ex. Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Then $\lambda_1 = 3$, k = 1, and $n_1 = 2$. $X_1 = E^2$.

$$A - 3I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$(A - 3I)^2 = 0$$

Therefore

$$\phi(t) = e^{3t}[I + (A - 3I)t]\eta$$

$$= e^{3t}\left(I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \right) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= e^{3t} \begin{bmatrix} \eta_1 + (-\eta_1 + \eta_2)t \\ \eta_2 + (-\eta_1 + \eta_2)t \end{bmatrix}$$

is the solution with $\phi(0) = \eta$. To construct e^{At}, successively let $\eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$e^{At}\begin{pmatrix}1\\0\end{pmatrix} = e^{3t}\begin{bmatrix}1-t\\-t\end{bmatrix}$$

$$e^{At}\begin{pmatrix}0\\1\end{pmatrix} = e^{3t}\begin{bmatrix}t\\1+t\end{bmatrix}$$

so that

$$e^{At} = e^{At}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \left[e^{At}\begin{pmatrix}1\\0\end{pmatrix}, e^{At}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$= e^{3t}\begin{bmatrix}1-t & t\\-t & 1+t\end{bmatrix}$$

Ex. Let

$$A = \begin{bmatrix}3 & -1 & 1\\2 & 0 & 1\\1 & -1 & 2\end{bmatrix}$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda-1)(\lambda-2)^2$$

so that the eigenvalues are

$\lambda_1 = 1$, $n_1 = 1$

$\lambda_2 = 2$, $n_2 = 2$

$k = 2$, $X_1 + X_2 = E^3$

We use the equations (A-I)x = 0 and $(A\text{-}2I)^2x = 0$ to determine X_1 and X_2.

$$(A-I)x = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} x = 0$$

$$\Leftrightarrow \begin{cases} 2x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases}$$

so that X_2 is spanned by the vectors $v_2 = (x_1, x_2, x_3)'$ with $x_1 = x_2$ and x_3 arbitrary.

Now,

$$\eta = v_1 + v_2$$

or

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ b \end{bmatrix} + \begin{bmatrix} c \\ c \\ d \end{bmatrix}$$

which leads to

$$v_1 = \begin{bmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \eta_1 \\ \eta_1 \\ \eta_3 - \eta_2 + \eta_1 \end{bmatrix}$$

Therefore

$$\phi(t) = e^{t} v_1 + e^{2t}[I + (A-2I)t]v_2$$

$$= e^{t} \begin{bmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1+t & -t & t \\ 2t & 1-2t & t \\ t & -t & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_1 \\ \eta_3 - \eta_2 + \eta_1 \end{bmatrix}$$

The fundamental matrix is

$$e^{At} = \begin{bmatrix} (1+t)e^{2t} & -te^{2t} & te^{2t} \\ -e^{t} + (1+t)e^{2t} & e^{t} - te^{2t} & te^{2t} \\ -e^{t} + e^{2t} & e^{t} - e^{2t} & e^{2t} \end{bmatrix}$$

Finally, we may now use the variation of constants formula to solve the nonhomogeneous equation

$$y' = Ay + g(t)$$

to get

$$\phi(t) = e^{A(t-t_0)}\eta + \int_{t_0}^{t} e^{A(t-s)} g(s) ds$$

Two useful properties of solutions to linear systems with constant coefficients are:

Thm If all eigenvalues of A have real parts negative, then every solution ϕ(t) of y' = Ay approaches zero as t → ∞. More precisely, ∃ $\tilde{K}$ > 0, σ > 0 such that

$$|\phi(t)| < \tilde{K} e^{-\sigma t} \quad \forall\ t \in (0, \infty)$$

Thm If ∃ M > 0, T > 0, and *a* such that

$$|g(t)| \leq M e^{at} \quad \forall\ t \geq T$$

then ∃ K > 0 and *b* such that every solution ϕ(t) of y' = Ay + g(t) satisfies

$$|\phi(t)| \leq K e^{bt} \quad \forall\ t \geq T$$

and ∃ H > 0 and *c* such that

$$|\phi'(t)| \leq H e^{ct} \quad \forall\ t \geq T$$

Notice that our earlier treatment of nth-order linear equations is a special case of the theory of linear systems. In particular, when the coefficients are constant, the fundamental matrix has the form e^{At}, which is a generalization of the terms in the nth-order equation's solutions of the form $cx^j e^{\lambda_j t}$, where λ_i was a root of the characteristic equation. Also note that such roots are the same as the eigenvalues in the system representation. Finally, the integrating factor in the 1st-order linear equation was e^{px} for the case of constant coefficients, which is a special case of the system result that the fundamental matrix e^{At} is the integrating factor in the variation of constants formula.

F. Linearization

Results for linear systems are far easier to obtain than for non-linear systems. Consequently, it often is advantageous to linearize a non-linear system by application of Taylor's formula. The expansion is performed about whatever point is of interest, usually a stable equilibrium pint when one exists.

Taylor's formula for the multivariate case is

$$f(x) = f(x^0) + \sum_{i=1}^{n} f_i(x^0)(x_1 - x_i^0) + \frac{1}{2!}\sum_{i,j=1}^{n} f_{ij}(x^0)(x_i - x_i^0)(x_j - x_j^0) + \ldots$$

BOUNDARY VALUE PROBLEMS

These are problems that involve finding solutions of a differential equation that satisfy prescribed conditions at two given points, called the boundary conditions. Methods of solving such problems are not discussed here.

DIFFERENCE EQUATIONS

Difference equations are the discrete-time analog of differential equations.

Ex. $y_t = a + by_{t-1} + e_t$

Aside from details, the theory behind difference equations is the same as that for differential equations. Our goal is to learn how to solve equations of the form

$$y_t = a_0 + a_1 y_{t-1} + \dots + a_n y_{t-n} + x_t$$

As with differential equations, the parameters a_i may be functions of time, but here we will concentrate on the constant coefficients case because most of the results in the literature pertain to that case.

I. <u>Basics</u>

We begin with the simplest difference equation

$$y_t = a y_{t-1}$$

or

$$y_t - y_{t-1} = 0$$

which is a first-order homogeneous equation. By successive substitution, we obtain

$$\begin{aligned} y_t &= a y_{t-1} \\ &= a(a y_{t-2}) \\ &= a[a(a y_{t-3})] \\ &\vdots \\ &= a^t y_0 \end{aligned}$$

where y_0 is the value of y at time initial time t=0. Thus a general solution to this equation is

$$y_t = c a^t$$

where *c* is an arbitrary constant to be determined byinitial conditions. In this example, the initial condition is y_{t0} = y0 given, so that c = y_0.

We now consider the nth-order equation

$$y_t + a_1 y_{t-1} + \dots + a_n y_{t-n} = 0$$

Recall that, with linear differential equations, we found that ce^{mt} was a solution of the first-order equation and then searched for the set of values for m that would make ce^{mt} a solution to the nth-order equation. We then used the principle of superposition to conclude that the general solution was $\sum c_i e^{m_i t}$. We proceed analogously here. We know that ca^t solves the first-order equation. Would functions of the form $y_t = r^t$ solve the nth-order equation? If so, for what values of r?

Let us try $y_t = r^t$ in the equation

$$r^t + a_1 r^{t-1} + \dots + a_n r^{t-n} = 0$$

$$\Rightarrow \quad r^t (1 + a_1 r^{-1} + \dots + a_n r^{-n}) = 0$$

$\Rightarrow$ either r = 0 or r is a root of

$$1 + a_1 r^{-1} + \dots + a_n r^{-n} = 0$$

$$\Leftrightarrow \quad r^n + a_1 r^{n-1} + \dots + a_n = 0$$

Thus, if this *characteristic equation* has n distinct roots, we have n solutions to the homogeneous equation of the form $y_t = r_i^t$. To obtain a general solution, we use the following

Thm If the function $y_i(t)$ satisfy the homogeneous equation, then so does

$$y(t) = c_1 y_1(t) + \dots + c_n y_n(t)$$

where the c_i are arbitrary constants.

Thus our general solution to the homogeneous equation is

$$y_t = c_1 r_1^t + \dots + c_n r_n^t$$

to deal with the case of repeated roots, we appeal to the following

<u>Thm</u> If r_i, i = 1, ..., p, are the distinct roots of the characteristic equation, with respective multiplicities n_i ($\Sigma n_i = n$), then the general solution of the homogeneous equation is

$$y_t = \sum_{j=1}^{p} \left(\sum_{i=0}^{n_j-1} c_i t^i \right) r_j^t$$

When complex roots occur, they always occur in conjugate pairs (a ± bi). Thus the expression for y_t will contain terms of the form

$$\sum_{k=0}^{n_j-1} c_k t^k (a+bi)^t + \sum_{k=0}^{n_j-1} d_k t^k (a-bi)^t$$

These terms can be rewritten as

$$\sum_{k=0}^{n_j-1} \tilde{c}_k t^k \rho^t \cos\phi t + \sum_{k=0}^{n_j-1} \tilde{d}_k t^k \rho^t \sin\phi t$$

where

$$\tilde{c}_k = c_k + d_k$$

$$\tilde{d}_k = (c_k - d_k) i$$

$$\rho = (a^2 + b^2)^{1/2}$$

$$\phi = \tan^{-1}(b/a)$$

The ability to re-write the complex root terms this way follows from three facts:

(1) $a \pm bi = \rho e^{i\phi}$

$= \rho(\cos\phi \pm i \sin\phi)$

(2) $(a \pm bi)^k = \rho^k(\cos\phi k \pm i \sin\phi k)$

(3) $c(a+bi)^k + d(a-bi)^k = \rho^k[(c+d)\cos\phi k + i(c-d)\sin\phi k]$

Ex. The second-order equation

$$y_t - 2ay_{t-1} + y_{t-2} = 0$$

has the characteristic equation

$$r^2 - 2ar + 1 = 0$$

If |a| < 1, this equation has roots

$$r-1, r_2 = a \pm \sqrt{a^2 - 1}$$

so that the general solution is

$$y_t = c_1\left(a + \sqrt{a^2-1}\right)^t + c_2\left(a - \sqrt{a^2-1}\right)^t$$

$$= \tilde{c}_1 \cos\phi t + \tilde{c}_2 \sin\phi t$$

with

$$\rho = [a^2 + (1-a^2)]^{1/2}$$

$$= 1$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-a^2}}{a}\right)$$

Ex. The same equation has repeated roots at 1 if a = 1, in which case the solution is

$$y_t = c_1 + c_2 t$$

II. <u>Lag Operators</u>

The lag operator is defined by

$$L^n X_t = x_{t-n}, \quad n \in Z$$

Formally, the opeator L^n operates on one sequence $\{X_t\}_{t=-\infty}^{\infty}$ to give a new sequence $\{Y_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}$.

The polynomial A(L) in the lag operator L is defined by

$$A(L) = a_0 + a_1 L + a_2 L^2 + \ldots$$

$$= \sum_{j=0}^{\infty} a_j L^j$$

where the a_j are constant. Note that A(L) is itself an operator; applying it to X_t gives

$$A(L)X_t = a_0 X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots$$

$$= \sum_{j=0}^{\infty} a_j X_{t-j}$$

We assume henceforth that A(L) is rational, in which case it can be expressed as

$$A(L) = B(L)/C(L)$$

The simplest example of a rational polynomial in L is

$$A(L) = \frac{1}{1 - \lambda L}$$

By long division (synthetic division), we can rewrite this as

$$\frac{1}{1-\lambda L} = 1 + \lambda L + \lambda^2 L^2 + \ldots$$

$$= \sum_{j=0}^{\infty} \lambda^j L^j$$

If we apply this particular A(L) to X_t, we obtain

$$A(L)X_t = \sum_{j=0}^{\infty} \lambda^j X_{t-j}$$

which is sometimes called the backward expansion of A(L)X_t. Note that if $|\lambda| \geq 1$, then if $X_t = \bar{X}$ a constant, we have

$$A(L)X_t = \sum_{j=0}^{\infty} \lambda^j X_{t-j}$$

$$= \bar{X} \sum_{j=0}^{\infty} \lambda^j$$

$$= \infty$$

Thus sometimes it is important to restrict $|\lambda| < 1$.

An alternative expansion is available:

$$\frac{1}{1-\lambda L} = \frac{-(\lambda L)^{-1}}{1-(\lambda L)^{-1}}$$

$$= -\frac{1}{\lambda L}\left[1 + \frac{1}{\lambda}L^{-1} + \left(\frac{1}{\lambda}\right)^2 L^{-2} + \ldots\right]$$

$$= -\frac{1}{\lambda}L^{-1} - \left(\frac{1}{\lambda}\right)^2 L^{-2} - \ldots$$

This is the forward expansion. Thus we can write

$$A(L)X(t) = -\frac{1}{\lambda}X_{t+1} - \left(\frac{1}{\lambda}\right)^2 X_{t+1} - \ldots$$

$$= -\sum_{j=1}^{\infty}\left(\frac{1}{\lambda}\right)^j X_{t+j}$$

If $|\lambda| < 1$, this forward expansion equals $+\infty$; if $|\lambda| > 1$, then $|1/\lambda| < 1$ and the forward expansion converges. Thus the forward expansion sometimes is useful when $|\lambda| > 1$; indeed, it often has a natural interpretation in terms of current expectations of future values.

III. <u>Nonhomogeneous Equations</u>

We will examine two ways of solving nonhomogeneous difference equations:

(1) undetermined coefficients
(2) inverse operators

A. <u>Undetermined coefficients</u>

This method can be used whenever the forcing sequence is itself the solution to some linear difference equation with constant coefficients. Given a difference equation

$$\phi(L)y_t = x_t$$

we seek an operator $\theta(L)$, called an *annihilator operator*, such that

$$\theta(L)x_t = 0$$

This operator is then applied to both sides of the original equation, giving

$$\theta(L)\phi(L)y_t = 0$$

which is a homogeneous equation that can be solved as previously discussed. The solution that emerges then is substituted into the original equation to evaluate the undetermined coefficients arising from $\theta(L)$.

Ex. Consider

$$(1 - \frac{5}{6}L + \frac{1}{6}L^2)y_t = 3^t$$

The forcing function is

$$x_t = 3^t$$

Because this particular function is the solution to the homogeneous equation

$$(1 - 3L)y_t = 0$$

we try

$$\theta(L) \equiv 1 - 3L$$

as the annihilator. In the case at hand,

$$\theta(L)x_t = (1-3L)3^t$$

$$= 3^t - 3\cdot 3^{t-1}$$

$$= 3^t - 3^t$$

$$= 0$$

so indeed this θ(L) is an annihilator for 3^t. Therefore, we apply θ(L) to the original equation:

$$(1-3L)(1-\frac{5}{6}L+\frac{1}{6}L^2)y_t = (1-3L)3^t$$

$$= 0$$

The auxiliary equation is

$$(r-3)(r^2-\frac{5}{6}r+\frac{1}{6}) = 0$$

which has roots

$$r_1 = 1/2$$

$$r_2 = 1/3$$

$$r_3 = 3$$

the last of which arises from θ(L). The general solution would be of the form

$$y_t = c_1(1/2)^t + c_2(1/3)^t + c_3(3)^t$$

from which we can determine the particular solution

$$y_{pt} = c_3(3)^t$$

by letting $c_1 = c_2 = 0$. Substituting this particular solution into the original solution gives

$$3^t = c_3 3^t - \frac{5}{6} c_3 3^{t-1} + \frac{1}{6} c_3 3^{t-2}$$

$$= 3^t (c_3 - \frac{5}{6} c_3 3^{-1} + \frac{1}{6} c_3 3^{-2})$$

$$\Rightarrow \quad (c_3 - \frac{5}{6} c_3 3^{-1} + \frac{1}{6} c_3 3^{-2}) = 1$$

$$\Rightarrow \quad c_3 = 27/20$$

Therefore the general solution is

$$y_t = c_1 \left(\frac{1}{2} \right)^t + c_2 \left(\frac{1}{3} \right)^t + \frac{27}{20} 3^t$$

Some general notes in applying this method:

(1) If the forcing functions a sum of other functions

$$x_t = x_{1t} + x_{2t}$$

then the annihilator is the product fo the annihilators $\theta_1(L)$ and $\theta_2(L)$ of x_1 and x_2:

$$\theta(L) = \theta_1(L) \theta_2(L)$$

(2) If the additional roots of the augmented characteristic equation that arise from the annihilator repeat any of the roots fo the original characteristic equation, we must follow the rules for repeated roots, as usual.

(3) All that has been said applies when the forcing function is of the form $\pi(L)x_t$. The annihilator is the same for x_t alone; consequently the solution function is of the same form as well, with only the constants differing.

(4) Gabel and Roberts has a table fo some annihilators.

B. Inverse Operators

An inverse operator is an operator that is the inverse of some other operator.

Ex. The inverse operator corresponding to

$$A(L) = 1 - aL$$

is

$$A^{-1} = (1 - aL)^{-1}$$

In general, for a linear operator $\phi(L) = a_0 + a_1L + ... + a_nL^n$, the corresponding inverse operator is

$$\phi^{-1}(L) = (a_0 + a_1L + \ldots + a_nL^n)^{-1}$$

$$= (1 - \lambda_1L)(1 - \lambda_2L)\ldots(1 - \lambda_nL) \quad \text{by application of partial fractions}$$

The method of partial fractions was discussed earlier in the section on differential equations. We can use inverse operators to solve difference equations. If the given equation is

$$\phi(L)y_t = x_t$$

Then the solution is

$$y_t = \phi^{-1}(L)x_t$$

To get some insight into how to express this solution in explicit terms, consider the simple equation

$$(1 - aL)y_t = 0$$

We know the solution is

$$y_t = ca^t$$

But, by inverse operators,

$$y_t = \frac{1}{1-aL}(0)$$

Therefore, we define

$$\frac{1}{1-aL}(0) \equiv ca^t$$

In essence, inverse operation in linear difference equations is similar to inverse operation in linear differential equations. There, the solution method was

(a) multiply both sides by the integrating factor
(b) integrate
(c) divide out any function multiplying the dependent variable

Ex. $$dy/dt + py = f(t)$$

To solve, we multiply by the integrating factor e^{pt}:

$$e^{pt}\frac{dy}{dt} + e^{pt}py = e^{pt}f(t)$$

We then integrate to obtain

$$ye^{pt} = \int e^{pt}f(t) + c$$

(Note the constant of integration.) Finally, we isolate y:

$$y = e^{-pt}\int e^{pt}f(t)\,dt + ce^{-pt}$$

Just as a constant of integration multiplied by e^{-pt} appears in the continuous case, so does a "constant of summation" multiplied by a^t appear in the discrete case:

$$\frac{1}{1-aL}(0) = ca^t$$

So now consider a slightly more complicated case:

$$(1-aL)y_t = x_t$$

The solution is

$$y_t = \frac{1}{1-aL}x_t + ca^t$$

The first term on the RHS is the discrete analog of the integral of the forcing function that appears in the continuous case; the second term is the constant of summation. As before,

$$\frac{1}{1-aL}x_t = (1 + aL + a^2L^2 + \ldots)x_t$$

$$= \sum_{j=0}^{\infty} a^j x_{t-j}$$

In general, this is as far as we can go; but if x_t has some special form, we may be able to write the solution in a more informative way. In particular, if x_t is a constant x, then the solution is

$$y_t = x\sum_{j=0}^{\infty} a^j + ca^t$$

If |a| < 1, then y_t is finite for all t:

$$y_t = x\left(\frac{1}{1-a}\right) + ca^t$$

$$\rightarrow \frac{x}{1-a} \quad \text{as } t \rightarrow \infty$$

Note that, if |a| > 1, we may want to use the forward expansion of $(1\text{-aL})^{-1}$:

$$y_t = \frac{-a^{-1}L^{-1}}{1-a^{-1}L^{-1}}x_t + ca^t$$

$$= -x\sum_{j=0}^{\infty}\left(\frac{1}{a}\right)^j + ca^t$$

which is finite if c = 0 but → ∞ otherwise.

IV. Systems of Equations

A system of difference equations has the form

$$y_t = A(t)y_{t-1} + B(t)x_{te}$$

where y and x are nx1 vectors and A and B are nxn matrices. We will be concerned with the case where A and B are constant.

One method of finding y_t is iteration:

$$
\begin{aligned}
y_1 &= Ay_0 + Bx_0 \\
y_2 &= Ay_1 + Bx_1 \\
&= A^2 y_0 + ABx_0 + Bx_1 \\
\vdots & \qquad \vdots \\
y_t &= A^t y_0 + \sum_{m=0}^{t-1} A^{t-1-m} B x_m
\end{aligned}
$$

or, if initial time is t_0,

$$
\begin{aligned}
y_t &= A^{t-t_0} y_{t_0} + \sum_{m=t_0}^{t-1} A^{t-1-m} B x_m \\
&= A^{t-t_0} y_{t_0} + \sum_{m=0}^{t-t_0-1} A^{t-t_0-1-m} B x_{t_0+m}
\end{aligned}
$$

The problem with this type of solution is that the matrices in each term change with each iteration in a way that is difficult to foresee in general. Little insight into the structure of the solution is provided. To get around this problem, we need a closed-form solution for A^k. The method for obtaining one that is discussed here is based on the spectral decomposition of a matrix.

<u>Thm</u> Any nxn matrix A has a representation of the form

$$
A = \sum_{i=1}^{p} (\lambda_i E_i + N_i)
$$

where p is the number of distinct eigenvalues, λ_i are the eigenvalues, each having multiplicity r_i, N_i is a matrix such that $N_i^{r_i} = 0$, and the matrices E_i and N_i satisfy the following properties:

$$E_i E_j = \begin{cases} 0 & i \neq j \\ E_i & i = j \end{cases}$$

$$E_i N_j = N_j E_i = \begin{cases} 0 & i \neq j \\ N_i & i = j \end{cases}$$

$$\sum_{i=1}^{p} E_i = I$$

<u>Thm</u> Any function f of the matrix A has the representation

$$f(A) = \sum_{i=1}^{p} \left[f(\lambda_i) E_i + \sum_{k=2}^{r_i} \frac{f^{(k-1)}(\lambda_i) N_i^{k-1}}{(k-1)!} \right]$$

Ex. Suppose

$$A = \begin{bmatrix} 0 & 1 \\ -1/8 & 3/4 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = 1/2$, $\lambda_2 = 1/4$. Therefore

$$A^k = \lambda_1^k E_1 + \lambda_2^k E_2$$

$$= (1/2)^k E_1 + (1/4)^k E_2$$

Now,

$$E_1 + E_2 = I$$

and

$$A = \lambda_1 E_1 + \lambda_2 E_2$$

so

$$A - \lambda I = \lambda_1 E_1 + \lambda_2 E_2 - (\lambda_1 E_1 + \lambda_1 E_2)$$

$$= (\lambda_2 - \lambda_1) E_2$$

$$\Rightarrow \quad E_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$$

and similarly

$$E_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$$

Therefore,

$$E_1 = \begin{bmatrix} -1 & 4 \\ & \\ -1/2 & 2 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} -2 & -4 \\ & \\ -1/2 & -1 \end{bmatrix}$$

so

$$A^k = \left(\frac{1}{2}\right)^k \begin{bmatrix} -1 & 4 \\ -1/2 & 2 \end{bmatrix} + \left(\frac{1}{4}\right)^k \begin{bmatrix} 2 & -4 \\ 1/2 & -1 \end{bmatrix}$$

Ex. Suppose

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = \lambda_2 = 1/2$. Therefore

$$A = \frac{1}{2} E_1 + N_1$$

and

$$A^k = (1/2)^k E_1 + k(1/2)^{k-1} N_1$$

From this last equation, we have

$$A^0 = I = E_1$$

$$A = \frac{1}{2} E_1 + N_1$$

$$\Rightarrow \quad N = A - \frac{1}{2} E_1$$

$$= A - \frac{1}{2} I$$

so

$$A^k = (1/2)^k I + k(1/2)^{k-1} [A - (1/2)I]$$

$$= \begin{bmatrix} (1/2)^k & 0 \\ \\ k(1/2)^k & (1/2)^k \end{bmatrix}$$

<u>Thm</u> There exists a nonsingular matrix T such that

$$T^{-1}AT = J$$

$$= \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_p \end{bmatrix}$$

where p is the number of distinct eigenvalues of A and

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

is of dimension $r_i x r_i$ with r_i the multiplicity of λ_i.

The importance fo this theorem is that

$$A = TJT^{-1}$$

so that

$$A^k = (TJT^{-1})(TJT^{-1})\ldots(TJT^{-1})$$

$$= TJ^kT$$

The terms of J^k are powers of the λ_i multiplied by numbers determined by k, so that the elements of A^k comprise linear combinations fo powers of the λ_i. As a result, y_t also comprises such linear combinations so that the asymptotic behavior of y_t depends entirely on the values of these λ_i. In particular, if $|\lambda_i| < 1 \ \forall$ i, then y_t is asymptotically stable (if the forcing function x_t is bounded).

Thm Suppose the eigenvalues of A are distinct. Then

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$J^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

$$A^k = TJ^kT^{-1}$$

Finally, note that any nth order difference equation can be put into system form. The usual kind of equation we will deal with has the form

$$y_t + a_1 y_{t-1} + \ldots + a_n y_{t-n} = u_t$$

which can be put into system form by defining

$$\begin{aligned}
y_1(t) &= y_2(t) \\
y_2(t) &= y_3(t) \\
\vdots \quad & \quad \vdots \\
y_{n-1}(t) &= y_n(t) \\
y_n(t) &= -a_1 y_n(t_1) - a_2 y_{n-1}(t-1) - \ldots - a_n y_1(t-1) + u_t
\end{aligned}$$

or

$$y_t = A y_{t-1} + B u_t$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The more general form of the nth order equation

$$y_t + b_1 y_{t-1} + \ldots + b_n y_{t-n} = a_0 u_t + a_1 u_{t-1} + \ldots + a_m u_{t-m}$$

is only slightly more difficult to handle. For the moment, assume m $\leq$ n. Then re-write the difference equation as

$$y_t = \frac{\phi(L)}{\theta(L)} u_t$$

$$= \phi(L)[\theta^{-1}(L) u_t]$$

Define

$$\theta^{-1}(L) u_t \equiv z(t+1), \quad \text{next period's state}$$

Then

$$\theta(L) z(t+1) = u_t$$

or

$$z(t+1) + b_1 z(t) + \ldots + b_n z(t-n+1) = u_t$$

If we now define

$$x_n(t) = z(t)$$

$$x_{n-1}(t) = z(t-1)$$

$$\vdots \qquad \vdots$$

$$x_1(t) = z(t-n+1)$$

we can write the state-space system

$$x_n(t+1) = U_t - b_1 x_n(t) - b_2 x_{n-1}(t) - \ldots - b_n x_1(t)$$

$$x_{n-1}(t+1) = x_n(t)$$

$$\vdots \qquad \vdots$$

$$x_1(t) = x_2(t)$$

or

$$x_{t+1} = Ax_t + Bu_t$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & \cdots & -b_2 & -b_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\begin{aligned} y_t &= \phi(L) z(t+1) \\ &= \phi(L) x(t+1) \\ &= a_0 x_n(t+1) + a_1 x_n(t) + \ldots + a_m x_n(t-m+1) \\ &= a_0 [u_t - b_1 x_n(t) - \ldots - b_n x_1(t)] + a_1 x_n(t) + \ldots + a_m x_{n-m-1}(t) \end{aligned}$$

which is called the output equation.

In the case where m > n, simply define enough extra “early” x_i to bring n up to equality with m. then the first m-n elements of the nth row of A will be zeroes, and the foregoing analysis applies.