Involution and Symmetry Reductions

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Abstract

After reviewing some notions of the formal theory of differential equations we discuss the completion of a given system to an involutive one. As applications to symmetry theory we study the effects of local solvability and of gauge symmetries, respectively. We consider non-classical symmetry reductions and more general reductions using differential constraints.

1 Introduction

Much of the theory of differential equations is concerned with normal systems, i.e. those who satisfy the assumptions of the Cauchy-Kowalevsky theorem. General systems are more difficult. A priori it is not clear whether they are consistent, as they may generate integrability conditions. Thus the first step in the analysis of such a system must always be its completion.

There exist several approaches to this problem. A geometric approach based on exterior calculus is given by the Cartan-Kähler theory [3]. The Janet-Riquier theory [8] represents an algebraic approach. It can be further extended to so-called Differential Gröbner Bases whose application to symmetry theory is discussed in Ref. [5].

We use the formal theory [13] with the central concept of an involutive system. It is a differential geometric approach containing also some elements of the Janet-Riquier theory. We will apply it in the context of symmetry theory and the reduction of differential equations. The concept of involution has many other applications, especially in differential geometry and physics [14], which we must, however, ignore here.

The article is organized as follows. After a brief review of the basic notions of formal theory, we consider the completion in Section 3. Then we present some simple applications to symmetry theory like the problem of local solvability, "subtraction" of the superposition symmetry for linear systems and of gauge symmetries. Sections 5 and 6 deal with the problem of reducing a given system to ordinary differential equations. Finally, some conclusions are given.

2 Involutive Systems

We cannot give here a detailed introduction into the formal theory or the underlying jet bundle formalism. Our presentation follows Ref. [20]. A general reference is the book of Pommaret [13].

We will always use coordinates, although the whole theory can be expressed in an intrinsic way. The independent variables $x_1, \dots, x_n$ and the dependent variables $u^1, \dots, u^m$ form fiber coordinates for the bundle $\mathcal{E}$. Derivatives are written in the form $p^\alpha_\mu = \partial^{|\mu|} u^\alpha / \partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$ where $\mu = [\mu_1, \dots, \mu_n]$. Adding the derivatives p^α_μ up to order q defines a local coordinate system for the q-th order jet bundle $J_q\mathcal{E}$. A system of differential equations $\mathcal{R}_q$ of order q is a fibred submanifold locally described by

$$\mathcal{R}_q \, : \, \Big\{ \, \Phi^\tau \left(x_i, u^\alpha, p^\alpha_\mu \right) = 0 \, , \quad \tau = 1, \dots, p \, ; \; |\mu| \leq q \, . \tag{1}$$

The *symbol* $\mathcal{M}_q$ of the system (1) is the solution space of the following linear system of (algebraic!) equations in the unknowns v^α_μ

$$\mathcal{M}_q \, : \, \left\{ \sum_{\alpha, |\mu| = q} \left(\frac{\partial \Phi^\tau}{\partial p^\alpha_\mu} \right) v^\alpha_\mu = 0 \, . \right. \tag{2}$$

(We will refer to both the linear system and its solution space as the symbol). The place-holders v^α_μ are coordinates of a finite-dimensional vector space; we introduce one for each derivative of order q. In a linear system the symbol is simply obtained by taking the highest order part and substituting v^α_μ for p^α_μ.

We make a power series ansatz for the general solution of the differential equation $\mathcal{R}_q$ expanding around some point x^0. Substituting into the equations (1) and evaluating at x^0 yields a system of algebraic equations for the Taylor coefficients up to order q.

The *prolonged* systems $\mathcal{R}_{q+r}$ are obtained by differentiating $\mathcal{R}_q$ r times totally with respect to all independent variables. Substituting the power series ansatz into $\mathcal{R}_{q+r}$ and evaluating at x^0 yields an inhomogeneous linear system for the Taylor coefficients of order $q + r$. Its homogeneous part is determined by the prolonged symbol $\mathcal{M}_{q+r}$, i.e. the symbol of $\mathcal{R}_{q+r}$.

This order by order construction fails, if integrability conditions occur. They pose additional conditions on coefficients of lower order and must all be known to pursue the above described procedure. A system containing all its integrability conditions is called *formally integrable.*

The arbitrariness of the general solution is reflected by the dimensions of the prolonged symbols, because at each order $\dim \mathcal{M}_{q+r}$ coefficients are not determined by the differential equations but can be chosen freely [21]. Formal integrability does, however, not suffice to determine these dimensions without explicitly constructing the prolonged symbols.

The *class* of a multi-index $\mu = [\mu_1, \dots, \mu_n]$ is the smallest k for which μ_k is different from zero. The columns of the symbol (2) are labeled by the v^α_μ. After ordering them by class, i.e. a column with a multi-index of higher class is always left of one with lower class, we compute a row echelon form.

We denote the number of rows where the pivot is of class k by $\beta_q^{(k)}$ and associate with each such row the *multiplicative variables* $x_1, \dots, x_k$. Prolonging each equation only with respect to its multiplicative variables yields independent equations of order $q+1$, as each has a different leading term. If prolongation with respect to the non-multiplicative variables does not lead to additional independent equations of order $q+1$, the symbol is involutive.

Definition 1 *The symbol $\mathcal{M}_q$ is called* involutive*, if*

$$\operatorname{rank} \mathcal{M}_{q+1} = \sum_{k=1}^{n} k\beta_q^{(k)} . \tag{3}$$

The differential equation $\mathcal{R}_q$ is called involutive*, if it is formally integrable and its symbol is involutive.*

The definition of $\beta_q^{(k)}$ is obviously coordinate dependent. But one can show that almost every coordinate system yields the same values. Such coordinates are called *δ-regular*. Definition 1 applies only to them. (There exist alternative methods to compute the correct values intrinsically [13, 20].)

The prolongation of an involutive symbol is again involutive. Since prolonging an equation with respect to one of its multiplicative variables x_i yields an equation of class i, we get $\beta_{q+1}^{(i)} = \sum_{k=i}^n \beta_q^{(k)}$. Induction allows to compute $\beta_{q+r}^{(k)}$ for any $r > 0$ from the $\beta_q^{(k)}$ and we obtain

$$\operatorname{rank} \mathcal{M}_{q+r} = \sum_{k=1}^{n} \binom{r+k-1}{r} \beta_q^{(k)} . \tag{4}$$

There exists an easily applicable criterion to check whether or not a system is involutive. The problem with formal integrability is to show that no integrability conditions occur at *any* prolongation order, i.e. an infinite number of checks. This can be done in a finite manner for systems with an involutive symbol.

Theorem 2 *Let $\mathcal{R}_q$ be a q-th order differential equation with an involutive symbol $\mathcal{M}_q$. If no integrability conditions arise during the prolongation of $\mathcal{R}_q$ to $\mathcal{R}_{q+1}$, then $\mathcal{R}_q$ is involutive.*

3 Completion to Involution and Arbitrariness

Theorem 3 (Cartan-Kuranishi) *Any differential equation $\mathcal{R}_q$ can be completed to an equivalent involutive one by a finite number of prolongations and projections (i.e. addition of integrability conditions).*

[1]	r $\leftarrow$ 0; s$\leftarrow$ 0	
[2]	compute $\mathcal{R}_{q+1}$	$\{$*prolong*$\}$
[3]	compute $\mathcal{M}_q, \mathcal{M}_{q+1}$	$\{$*extract symbols*$\}$
[4]	`until` $\mathcal{R}^{(s)}_{q+r}$ involutive `repeat`	
[4.1]	`while` $\#multVar(\mathcal{M}^{(s)}_{q+r}) \neq \operatorname{rank} \mathcal{M}^{(s)}_{q+r+1}$ `repeat`	
[4.1.1]	r $\leftarrow$ r+1	$\{$*counter for prolongations*$\}$
[4.1.2]	compute $\mathcal{R}^{(s)}_{q+r+1}$	$\{$*prolong*$\}$
[4.1.3]	compute $\mathcal{M}^{(s)}_{q+r+1}$	$\{$*extract symbol*$\}$
[4.2]	`if` $\dim \mathcal{R}^{(s)}_{q+r+1} - \dim \mathcal{M}^{(s)}_{q+r+1} < \dim \mathcal{R}^{(s)}_{q+r}$ `then`	
[4.2.1]	s $\leftarrow$ s+1	$\{$*counter for projections*$\}$
[4.2.2]	compute $\mathcal{R}^{(s)}_{q+r}$	$\{$*add integrability conditions*$\}$
[4.2.3]	compute $\mathcal{R}^{(s)}_{q+r+1}$	$\{$*prolong*$\}$
[4.2.4]	compute $\mathcal{M}^{(s)}_{q+r}, \mathcal{M}^{(s)}_{q+r+1}$	$\{$*extract symbols*$\}$
[5]	`return` $\mathcal{R}^{(s)}_{q+r}$	

Figure 1: Completion algorithm

Since this theorem depends on some fairly deep results in the formal theory, we only discuss an algorithm to perform the completion. It is based on Theorem 2 and consists of two nested loops. The inner one prolongs the system until its symbol becomes involutive; the outer one checks then for integrability conditions and adds them.

Involution of a symbol can be checked using Definition 1, if the coordinate system is δ-regular what we will assume in the sequel. This requires only linear algebra. Whether or not integrability conditions arise during a prolongation can be deduced from a dimensional argument. Denote the projection of $\mathcal{R}_{q+1}$ into $J_q\mathcal{E}$ by $\mathcal{R}^{(1)}_q$. Then

$$\dim \mathcal{R}^{(1)}_q = \dim \mathcal{R}_{q+1} - \dim \mathcal{M}_{q+1}\,, \tag{5}$$

since integrability conditions are connected with rank defects in the symbol. None has appeared during the prolongation from $\mathcal{R}_q$ to $\mathcal{R}_{q+1}$, if and only if this dimension is equal to $\dim \mathcal{R}_q$.

Fig. 1 shows the algorithm in a more formal language. $\mathcal{R}^{(s)}_{q+r}$ denotes the system obtained after $r+s$ prolongations and s projections and $\mathcal{M}^{(s)}_{q+r}$ the corresponding symbol. Determining the dimensions of the submanifolds $\mathcal{R}^{(s)}_{q+r}$ poses the main problem, especially for non-linear systems. The other calculations require only differentiations or linear algebra. Refs. [19, 22] describe an implementation in the computer algebra system AXIOM.

To conclude this section we recall some results of Ref. [21] concerning the arbitrariness of the general solution. (4) yields only the ranks of the prolonged symbols, but their dimensions are more interesting. They can be expressed in a similar way, if we introduce the *Cartan characters* $\alpha_q^{(k)}$ of a differential equation

$$\alpha_q^{(k)} = m\binom{q+n-k-1}{q-1} - \beta_q^{(k)}, \qquad k = 1, \dots, n\,. \tag{6}$$

They form a descending sequence $\alpha_q^{(1)} \geq \alpha_q^{(2)} \geq \cdots \geq \alpha_q^{(n)} \geq 0$.

Now we can introduce the *Hilbert polynomial* of $\mathcal{R}_q$

$$H_q(r) = \dim \mathcal{M}_{q+r} = \sum_{k=1}^{n} \alpha_{q+r}^{(k)} = \sum_{k=1}^{n} \binom{r+k-1}{r} \alpha_q^{(k)}\,. \tag{7}$$

(it can be written explicitly as a polynomial in r). Analyzing the number of arbitrary Taylor coefficients in the power series expansion of the general solution and comparing with these dimensions yields the following result.

Theorem 4 *The general solution of a first-order system of differential equations $\mathcal{R}_1$ contains f_k arbitrary functions depending on k arguments with*

$$f_n = \alpha_1^{(n)} = m - \beta_1^{(n)}\,, \qquad f_k = \alpha_1^{(k)} - \alpha_1^{(k+1)} = \beta_1^{(k+1)} - \beta_1^{(k)}\,. \tag{8}$$

Obviously the f_k are always non-negative. Note that Theorem 4 refers to algebraic representations of the general solution, i.e. no integrals or derivatives of the arbitrary functions may occur.[1] One can derive more general results covering also higher-order equations and other representations of the solution [21].

If all Cartan characters $\alpha_q^{(k)}$ of a differential equation $\mathcal{R}_q$ vanish, then it follows from Theorem 4 that its general solution does not contain any arbitrary functions and hence its solution space is finite-dimensional. Such a system is said to be of *finite type*. Since its Hilbert polynomial $H_q(r)$ vanishes, too, all the arbitrary Taylor coefficients are of order less than q.

4 Symmetry Theory

Symmetry theory [2, 10, 25] represents one of the most important approaches to non-linear differential equations. Besides some special techniques for completely integrable systems it provides more or less the only systematic way to construct solutions. These reduction methods are the topic of the next section.

A somewhat subtle and often overlooked point is the *local solvability* of a differential equation. Essentially there are two different notions of a symmetry. The most general definition says it is a transformation mapping solutions into solutions. The geometric definition calls a transformation of $\mathcal{E}$ a symmetry

[1] A typical example is the general solution of the wave equation in the form $f(x+t) + g(x-t)$. A non-algebraic representation is given by the d'Alembert form of the solution $[f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(\tau)d\tau]/2$.

of the differential equation $\mathcal{R}_q$, if its prolongation to $J_q\mathcal{E}$ leaves $\mathcal{R}_q$ invariant. For simplicity we will consider here only point symmetries, i.e. we restrict the allowed transformations to diffeomorphisms of $\mathcal{E}$.

We show now with a very simple example that the two definitions are equivalent only for involutive systems:

$$\mathcal{R}_1 \,:\, \left\{ \begin{array}{l} u_z + yu_x = 0\,, \\ u_y = 0\,. \end{array} \right. \tag{9}$$

Since y appears explicitly, $\mathcal{R}_1$ is not invariant under y-translations. Cross-differentiation yields the integrability condition $u_x = 0$. Hence the general solution is given by $u(x, y, z) = const$ and y-translations are symmetries in the sense of the first definition.

The geometric approach "looses" this symmetry, because it requires that $\mathcal{R}_1$ should remain invariant and not its submanifold $\mathcal{R}_1^{(1)}$. But the prolongation of any solution lies in $\mathcal{R}_1^{(1)}$. Thus the second definition imposed a stronger condition than the first one. We call a differential equation $\mathcal{R}_q$ *locally solvable*, if for every point $p \in \mathcal{R}_q$ there exists a solution f such that $p \in j_q(f)$. Then both definitions are equivalent.

Besides the occurance of integrability conditions Lewy type effects can disturb the local solvability [10]. If we ignore these by considering only analytic differential equation, we can conclude from the Cartan-Kähler theorem that every involutive differential equation is locally solvable [20].

One always hopes that the determining system can be solved explicitly, but sometimes this is not possible. Nevertheless one can extract information about the symmetry algebra [17]. Here we are only interested in its size which can be computed by a straightforward application of the results of Section 3 to the determining system. This method is not restricted to Lie point symmetries but can also be applied to generalized or non-classical symmetries.

One interesting feature of this approach is the possibility to formally subtract some effects. We will discuss two problems of this kind. The first one is fairly trivial and concerns the superposition symmetry. Since all linear equations have it, one always finds an infinite-dimensional symmetry algebra for them. But usually the other symmetries are of much more interest. Thus one would like to know the size of the remaining algebra.

We illustrate the procedure at the heat equation, although it is trivial to compute its symmetries explicitly. Its determining system is of second order and becomes involutive after five prolongations and four projections, i.e. completion leads to a third-order equation [20]. Its Hilbert polynomial is constant, $H_3(r) = 2$. But this is also the Hilbert polynomial of the heat equation itself. Thus the infiniteness stems solely from the superposition symmetry.

The involutive system describes a 13-dimensional manifold in a third-order jet bundle. Hence the general solution of the determining system contains 13 arbitrary Taylor coefficients of order less than four. One prolongation of the heat equation yields a seven-dimensional manifold in another third-order jet bundle. Thus its general solution depends on seven arbitrary Taylor coefficients of order less than four.

To obtain the number of coefficients connected to other symmetries than the superposition we must subtract these seven. This yields the well-known result that besides superposition the heat equation possesses a six-dimensional symmetry group.

A more interesting application concerns the size of the physical solution space of a gauge theory. In such theories one identifies solutions related by symmetry transformations. The size of the reduced solution space can be determined as in Section 3, if one uses *gauge corrected* Cartan characters.

We define mathematically a gauge symmetry as a fiber-preserving transformation of the bundle $\mathcal{E}$ depending on some arbitrary functions of all independent variables which maps solutions into solutions. Let us assume that the transformations can be written in the form

$$\begin{aligned} \bar{x}_i &= \Omega_i(x_j)\,, \\ \bar{u}^\alpha &= \Lambda^\alpha\big(x_i, u^\beta, \lambda_a^{(0)}(x), \partial\lambda_a^{(1)}(x), \dots, \partial^p\lambda_a^{(p)}(x)\big) \end{aligned} \tag{10}$$

where γ_0 gauge functions $\lambda_a^{(0)}$ enter algebraically, γ_1 gauge functions $\lambda_a^{(1)}$ enter with their first derivatives etc. Ref. [20] shows how to handle more general cases using a pseudogroup approach.

The gauge correction term $\Delta\alpha_q^{(k)}$ which must be subtracted from $\alpha_q^{(k)}$ to adjust for the effect of the symmetry can be computed recursively through

$$\Delta\alpha_q^{(k)} = \frac{(k-1)!}{(n-1)!}\sum_{l=0}^{p}\gamma_l s_{n-k-1}^{(n-1)}(q+l) - \sum_{i=k+1}^{n}\frac{(k-1)!}{(i-1)!}\,\Delta\alpha_q^{(i)} s_{i-k}^{(i-1)}(0)\,, \tag{11}$$

where the $s_k^{(n)}(q)$ denote some combinatorial factors [20, 21]. With these gauge corrected Cartan characters one can intrinsically define the number of degrees of freedom in a gauge theory as $\alpha_q^{(n-1)} - \Delta\alpha_q^{(n-1)}$ using the identification of the Dirac algorithm for systems with constraints with our completion algorithm [24].

5 Symmetry Reduction

The problem of local solvability might appear fairly artificial and of not much interest in concrete applications. But the situation changes as soon as we start to consider the *non-classical method* of Bluman and Cole [1].

Proposition 5 *The integrability conditions of the augmented system consisting of the original differential equations plus the invariant surface condition are identically satisfied, if and only if the corresponding vector field generates a non-classical symmetry in the sense of Bluman and Cole.*

Proof. We consider the involutive system of differential equations

$$\Delta^\tau(x^i, u^\alpha, p_i^\alpha) = 0\,, \qquad \tau = 1, \dots, p\,. \tag{12}$$

Without loss of generality we assume that the x^n-component of the vector field does not vanish and write the invariant surface condition as (from now on we use the Einstein summation convention)

$$p_n^\alpha + \xi^k p_k^\alpha - \eta^\alpha = 0\,. \tag{13}$$

We now analyze when the augmented system (12,13) does not generate integrability conditions. (13) is an equation of class n and can be used to eliminate p_n^α in the other equations which are thus of lower class. Since (12) was assumed to be involutive, the only possibility for integrability conditions arise from its prolongation with respect to the non-multiplicative variable x^n.

In these equations we eliminate all derivatives of the form p_{nj}^α with the prolongations of (13). The remaining second order derivatives are eliminated using the multiplicative prolongations of (12). This yields

$$\frac{\partial \Delta^\tau}{\partial x^n} + \xi^k \frac{\partial \Delta^\tau}{\partial x^k} + \eta^\alpha \frac{\partial \Delta^\tau}{\partial u^\alpha} + (D_k \eta^\alpha - D_k \xi^l p_l^\alpha) \frac{\partial \Delta^\tau}{\partial p_k^\alpha} = 0 \tag{14}$$

where D_k denotes the total derivative with respect to x^k. Thus these equations must vanish identically on the manifold defined by (12,13).

Now we apply the non-classical method. Prolonging once the vector field $\vec{v} = \partial_{x^n} + \xi^k \partial_{x^k} + \eta^\alpha \partial_{u^\alpha}$ yields $pr(\vec{v}) = \vec{v} + \eta_i^\alpha \partial_{p_i^\alpha}$ where $\eta_i^\alpha = D_i \eta^\alpha - D_i \xi^k p_k^\alpha$. Applying it to the differential equation (12) leads again to (14). We require that it vanishes modulo (12,13). Thus the determining equations are exactly the conditions from the involution analysis. •

If a function is invariant under a one-parameter group of Lie symmetries, it satisfies the corresponding invariant surface condition. Bluman and Cole require that this condition is *compatible* with the original system in the sense that the appearing integrability conditions are satisfied. But for the existence of invariant solutions it suffices, if the invariant surface condition is *consistent* with the original system, i.e. the augmented system has a solution.

If the augmented system is not locally solvable, we can generalize the method of Bluman and Cole. Then (14) does not vanish as a consequence of the other equations and we have to add it. This may lead to further integrability conditions and again we can either require that they are satisfied automatically or add them, too. This process leads to a finite tree of different systems and each can lead to new symmetry reductions.

Pucci and Saccomandi [15] proved Proposition 5 for the case of one dependent variable based on some results by Darboux. They also noted that the next integrability conditions can be easily constructed by iteratively applying the proposition to the system obtained by adding the previous integrability conditions. In each case one must simply apply the prolonged vector field to the equations of the system.

6 General Reductions

The goal of reductions is usually to obtain ordinary differential equations. One may ask whether there are other ways to reach it than group theory. An example is the direct method of Clarkson and Kruskal [4] later extended by Galaktionov [7]. Although it is equivalent to the non-classical method restricted to fibre-preserving groups [11], it was originally designed without any group theory. The same holds for Rubel's method of quasi-solutions [16, 18]

Lie [9] showed that the solution of every system of finite type can be constructed by solving only ordinary differential equations. Thus one should study when the addition of differential constraints leads to such a system. For most of this section we concentrate on the addition of one differential constraint to a single second-order equation

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0\,. \tag{15}$$

Proposition 6 *Adding a consistent first-order differential constraint to (15) leads to an equation of finite type, if (15) is not a differential consequence of it.*

PROOF. Without loss of generality we solve the constraint for u_x

$$u_x - \phi(x,t,u,u_t) = 0\,. \tag{16}$$

Hence all x-derivatives in Eq. (15) can be eliminated and we obtain

$$\tilde{\Delta}(x,t,u,u_t,u_{tt}) = 0\,. \tag{17}$$

We distinguish between a degenerate case when $\tilde{\Delta}_{u_{tt}} = 0$ and the generic case where (17) is still a second order equation. This distinction can be characterized intrinsically by the rank of a symbol.

Generic case: We have to investigate, when the equation

$$\mathcal{R}_2\ :\ \left\{\begin{array}{l} u_x - \phi(x,t,u,u_t) = 0\,,\\ u_{xx} - \phi_{u_t}u_{xt} - \phi_u u_x - \phi_x = 0\,,\\ u_{xt} - \phi_{u_t}u_{tt} - \phi_u u_t - \phi_t = 0\,,\\ \tilde{\Delta}(x,t,u,u_t,u_{tt}) = 0 \end{array}\right. \tag{18}$$

is involutive. In the degenerate case its symbol $\mathcal{M}_2$ has not full rank, which happens if and only if

$$\tilde{\Delta}_{u_{tt}} = \Delta_{u_{xx}}\phi_{u_t}^2 + \Delta_{u_{xt}}\phi_{u_t} + \Delta_{u_{tt}} = 0\,. \tag{19}$$

Thus $\dim \mathcal{M}_2 = 0$ in the generic case and the symbol is trivially involutive. Since a system with a vanishing symbol is always of finite type, our claim is proven. One can now continue to analyze the integrability conditions of $\mathcal{R}_2$ in order to determine the size of the solution space, but we omit this here.

Degenerate case: In this case we are left with a first order equation

$$\mathcal{R}_1 : \begin{cases} u_x - \phi(x,t,u,u_t) = 0\,, \\ \tilde{\Delta}(x,t,u,u_t) = 0\,. \end{cases} \tag{20}$$

If $\tilde{\Delta}_{u_t} \neq 0$, the symbol vanishes and $\mathcal{R}_1$ is of finite type. Otherwise $\tilde{\Delta}$ is algebraic and can be consider as an implicit solution, as under the assumption of the proposition it does not vanish. •

It may not always be possible to find constraints such that the degeneracy condition (19) is satisfied, especially for non-linear equations. For quasi-linear systems we rediscover the distinction into hyperbolic, parabolic and elliptic equations.

Proposition 7 *Adding a consistent second-order differential constraint to (15) does not lead to an equation of finite type, if and only if the combined system is involutive and the symbol of its prolongation has rank 3.*

PROOF. Without loss of generality we solve again the constraint

$$u_{xx} - \psi(x,t,u,u_x,u_t,u_{xt},u_{tt}) = 0\,. \tag{21}$$

Eliminating u_{xx} from (15) leads to three different cases:

$$\begin{array}{ll} \text{A)} & u_{tt} - \tilde{\Delta}(x,t,u,u_x,u_t,u_{xt}) = 0\,, \\ \text{B)} & u_{xt} - \tilde{\Delta}(x,t,u,u_x,u_t) = 0\,, \\ \text{C)} & \tilde{\Delta}(x,t,u,u_x,u_t) = 0\,. \end{array} \tag{22}$$

Case C is trivial: one considers the reduced equation as a first order constraint for (21) and recovers the situation of Proposition 6. The distinction between the three cases is induced by the rank of symbols. We are interested in the equation

$$\mathcal{R}_2 : \begin{cases} u_{xx} - \psi(x,t,u,u_x,u_t,u_{xt},u_{tt}) = 0\,, \\ \Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0\,. \end{cases} \tag{23}$$

In Case C the symbol $\mathcal{M}_2$ has not maximal rank. Cases A and B differ in the rank of the prolonged symbol $\mathcal{M}_3$.

Generic case (A): $\mathcal{M}_2$ has three multiplicative variables (one equation of class 2 and one of class 1). To check whether or not it is involutive, we must compute the rank of its prolongation

$$\mathcal{M}_3 : \begin{cases} v_{xxx} - \psi_{u_{xt}} v_{xxt} - \psi_{u_{tt}} v_{xtt} = 0\,, \\ v_{xxt} - \psi_{u_{xt}} v_{xtt} - \psi_{u_{tt}} v_{ttt} = 0\,, \\ v_{xtt} - \tilde{\Delta}_{u_{xt}} v_{xxt} = 0\,, \\ v_{ttt} - \tilde{\Delta}_{u_{xt}} v_{xtt} = 0\,. \end{cases} \tag{24}$$

Since it is four, $\mathcal{M}_2$ is not involutive but $\mathcal{M}_3$, as it vanishes. If no integrability condition arises in the prolongation from $\mathcal{R}_3$ to $\mathcal{R}_4$, $\mathcal{R}_3$ is an involutive finite type equation with a four-dimensional solution space.

One can compute the general form of the possible integrability condition, but it is a rather lengthy expression. It is at most of second order, since $\mathcal{M}_3$ vanishes. There arise many case distinctions depending on its order and on whether the projected system is involutive. We omit these and conclude that we have in any case a finite type equation with an at most three-dimensional solution space.

Degenerate case (B): We write the symbol $\mathcal{M}_3$ in the following form

$$\mathcal{M}_3 : \begin{cases} v_{xxx} - \psi_{u_{xt}} v_{xxt} - \psi_{u_{tt}} v_{xtt} = 0\,, \\ v_{xxt} - \psi_{u_{xt}} v_{xtt} - \psi_{u_{tt}} v_{ttt} = 0\,, \\ v_{xxt} = 0\,, \\ v_{xtt} = 0\,. \end{cases} \tag{25}$$

Its rank depends on $\psi_{u_{tt}}$. If it is not zero, the rank of the symbol is still four and we are in Case A. Otherwise, $\operatorname{rank} \mathcal{M}_3 = 3$ and the symbol $\mathcal{M}_2$ is involutive.

During the prolongation from $\mathcal{R}_2$ to $\mathcal{R}_3$ one integrability condition may arise. If it vanishes identically, $\mathcal{R}_2$ is involutive but not of finite type, as $\dim \mathcal{M}_2 = 1$. Otherwise, the projected equation $\mathcal{R}_2^{(1)}$ is of finite type. Again we get many case distinctions depending on the order the integrability condition. The solution space is at most three-dimensional. •

These propositions can be extended to equations of higher than second order. We will restrict ourselves to the generic cases. Many more case distinctions arise, but they affect only the dimension of the solution space.

The case of first order constraints runs completely analogously to Proposition 6. All derivatives with respect to x can be replaced by t-derivatives using the constraint; the reduced equation is a parametrized ordinary differential equation. The degeneracy condition (19) is replaced by

$$\Delta_{u_{[q,0]}} \phi_{u_t}^q + \Delta_{u_{[q-1,1]}} \phi_{u_t}^{q-1} + \cdots + \Delta_{u_{[0,q]}} = 0 \tag{26}$$

where q denotes the order of the differential equation and where we have used the notation $u_{[r,s]} = \partial^{r+s} u / \partial x^r \partial t^s$.

Prolongation of the constraint (16) to order q yields q independent equations. One can easily see, that the symbol $\mathcal{M}_q$ of the equation $\mathcal{R}_q$ which arises by adjoining these to the original equation always vanishes, if condition (26) does not hold. If no integrability conditions occur, $\mathcal{R}_q$ is an involutive finite type equation with a q-dimensional solution space.

Proposition 7 is generalized by considering a constraint of order q. For simplicity we assume again that it is in solved form

$$u_{[q,0]} = \psi(x, t, u, u_x, u_t, \ldots, u_{[0,q-1]}, u_{[q-1,1]}, \ldots, u_{[0,q]})\,. \tag{27}$$

Eliminating $u_{[q,0]}$ yields the equation

$$\tilde{\Delta}(x,t,u,u_x,u_t,\dots,u_{[0,q-1]},u_{[q-1,1]},\dots,u_{[0,q]}) = 0\,. \qquad (28)$$

We must now analyze the system $\mathcal{R}_q$ comprising (27) and (28).

Omitting the analogue of Case C above, $\operatorname{rank}\mathcal{M}_q = 2$ and there are three multiplicative variables. The prolonged symbols $\mathcal{M}_{q+r}$ are given by

$$\mathcal{M}_{q+r}\ :\ \begin{cases} v_{[q+i,r-i]} - \displaystyle\sum_{j=1}^{q} \psi_{u_{[q-j,j]}} v_{[q-j+i,r+j-i]} = 0 \\ \displaystyle\sum_{j=1}^{q} \tilde{\Delta}_{u_{[q-j,j]}} v_{[q-j+i,r+j-i]} = 0 \end{cases} \qquad i = 0,\dots,r\,. \qquad (29)$$

As long as these symbols have maximal rank, we get $\operatorname{rank}\mathcal{M}_{q+r} = 2r+2$ and $\mathcal{M}_{q+r}$ has $2r+3$ multiplicative variables. Since this symbol is defined in a vector space of the dimension $q+r+1$ (the number of derivatives of order $q+r$), we reach a vanishing and thus involutive symbol for $r = q-1$. The corresponding system $\mathcal{R}_{2q-1}$ comprises $q(q+1)$ equations and its solution space has dimension q^2.

The number of multiplicative variables of a symbol provides a lower bound for the rank of its prolongation. Thus if one of the symbols $\mathcal{M}_{q+r}$ has not maximal rank, its rank is $2r+1$ and the preceding symbol is involutive. If no integrability conditions arise, the corresponding system is involutive but not of finite type. This never happens, if $\psi_{[0,q]} \neq 0$, as one can see from (29).

The analysis becomes much more involved with more than two independent variables. In general, one constraint is not sufficient to obtain an equation of finite type. We can extend Proposition 6 by considering $n-1$ constraints

$$p_k = \phi(x^i, u, p_j)\,, \qquad k = 1,\dots,n-1, \quad j \neq k\,. \qquad (30)$$

The only way to be sure that q-th order constraints lead to an equation of finite type is to add so many constraints that the symbol $\mathcal{M}_q$ vanishes. But sometimes just one constraint may suffice, if it generates enough integrability conditions. In the case of the equation $u_{zz} + yu_{xx} = 0$, it is well-known from a famous example of Janet that addition of the constraint $u_{yy} = 0$ suffices to obtain an equation of finite type.

7 Conclusion

Olver and Rosenau wrote in Ref. [12]: "The most important conclusion to be drawn from this approach is that the unifying theme behind finding special solutions to partial differential equations is *not*, as is commonly supposed, group theory, but rather the more analytic subject of over-determined systems of partial differential equations".

The importance of group theory is not that it provides the most general framework for the reduction of differential equations. But it provides techniques which are applicable in concrete computations. In contrast, simply requiring that one obtains ordinary differential equations yields hardly any restrictions on the constraints and in practice one does not know what constraints are useful.

We do not think that the "generalized non-classical method" indicated in Section 5 has great practical importance, although Pucci and Saccomandi [15] applied it to some systems of interest. It is rarely possible to solve the arising determining systems due to their nonlinearity and at each step they become more complicated.

Of more interest seems to be the approach by Duzhin and Lychagin [6] who combined the idea of reduction to a finite type system with Lie's integration theory for ordinary differential equations. They try to determine the constraints in such a way that the resulting ordinary differential equations possess enough symmetries to be integrable by quadratures. Although the method is not completely algorithmically, one obtains a clear criterion for *useful* constraints. Especially for systems with more than two independent variables there are probably not many alternatives.

Olver and Rosenau also mention in Ref. [12] that, in principle, one can give a group-theoretic explanation to arbitrary differential constraints. If we consider the constraint as the characteristic of a generalized symmetry [10], one could speak of conditional generalized symmetries. But this point of view seems artificial and without real implications in applications.

Another application of formal methods in symmetry theory consists of determining the loss of arbitrariness during a reduction. It is well-known that opposed to the situation for ordinary differential equations one cannot reconstruct the general solution of system of partial differential equations from a group invariant solution. The lost generality can be easily quantified using the techniques presented in Section 3. We do not go into details here but refer to Ref. [23] where some examples are considered.

Finally, we want to comment on the importance of computer algebra. The completion of a given system to an involutive one is in general a rather complicated process despite the apparent simplicity of the algorithm presented in Section 3. As soon as there are more than two independent variables one might need fairly complicated linear combinations to exhibit the integrability conditions. If in addition the system contains equations of different order, the completion becomes very tedious. One needs therefore powerful computer algebra tools to perform such calculations, if one goes beyond trivial examples.

Most of the steps in our algorithm do not consume much computing time. The main obstacle, especially for non-linear systems, is the determination of the dimensions of the submanifolds. In the case of polynomial nonlinearities this can be done using Gröbner bases. A detailed discussion of this and other problems can be found in Ref. [22].

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