Differential Geometry
of
Curves and Surfaces

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Chapter 1

Curves

1.1 Vector Spaces and Normed Spaces

An n-dimensional vector space V^n is a set of elements (called "vectors") with addition $\mathbf{e}_1 + \mathbf{e}_2$ and scalar multiplication $t\mathbf{e}_1$. The addition and scalar multiplication satisfy certain laws such as

$$t(\mathbf{e}_1 + \mathbf{e}_2) = t\mathbf{e}_1 + t\mathbf{e}_2,$$

$$\mathbf{e}_1 + (-1)\mathbf{e}_1 = 0.$$

There is a set of n linearly independent vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ in V^n such that every vector $\mathbf{u}$ in V^n can be *uniquely* expressed in the following form

$$\mathbf{u} = x^1\mathbf{e}_1 + \cdots + x^n\mathbf{e}_n.$$

In this case, we call the set $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ a *basis* for V^n, and denote V^n by

$$\mathrm{V}^n = \mathrm{span}\Big\{\mathbf{e}_1, \cdots, \mathbf{e}_n\Big\}.$$

Note that $\Big\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, \cdots, \mathbf{e}_n\Big\}$ is also a basis for V^n.

Let $\mathbf{e}_1, \cdots, \mathbf{e}_k$ be k linearly independent vectors in V^n. Define

$$\mathrm{W} := \Big\{x^1\mathbf{e}_1 + \cdots + x^k\mathbf{e}_k, \ \Big| \ x^i \text{ real numbers}\Big\}. \tag{1.1}$$

W is a k-dimensional vector space with the induced addition and scalar multiplication from that in V^n. We denote W by

$$\mathrm{W} = \mathrm{span}\Big\{\mathbf{e}_1, \cdots, \mathbf{e}_k\Big\}.$$

Example 1.1.1 Let

$$\mathrm{R}^n := \Big\{ (u^1, \cdots, u^n),\ u^i \ \text{ real numbers } \Big\},$$

with the following addition and scalar multiplication

$$(u^1, \cdots, u^n) + (v^1, \cdots, v^n) := (u^1 + v^1, \cdots, u^n + v^n).$$

$$t(u^1, \cdots, u^n) := (tu^1, \cdots, tu^n).$$

R^n is called the *canonical n-dimensional vector space.* The canonical basis for R^n is

$$\Big\{ \mathbf{e}_1 = (1, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, \cdots, 0, 1) \Big\}.$$

Example 1.1.2 Let

$$\ell := \Big\{ (u, v),\ v = m\, u \Big\}$$

where m is a constant. ℓ is a line in R^2 passing through the origin with slope of m. Note

$$\ell = \mathrm{span}\Big\{ \mathbf{e}_1 \Big\},$$

where $\mathbf{e}_1 = (1, m)$. Thus ℓ is a subspace of R^2.

Example 1.1.3 Let

$$\mathrm{V} := \Big\{ (u, v, w),\ u + 2v = w \Big\}$$

V is a plane in R^3 passing through the origin. There are two linearly independent vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ in V such that

$$\mathrm{V} = \mathrm{span}\Big\{ \mathbf{e}_1, \mathbf{e}_2 \Big\}.$$

For example, we can take $\mathbf{e}_1 = (-1, 1, 1)$ and $\mathbf{e}_2 = (0, 1, 2)$.

An inner product in V^n is a positive definite bilinear symmetric form on V^n, that is a map $\langle , \rangle : \mathrm{V}^n \times \mathrm{V}^n \to \mathrm{R}$ with the following properties

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$;

(b) $\langle \lambda \mathbf{u} + \mu \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{w} \rangle + \mu \langle \mathbf{v}, \mathbf{w} \rangle$;

(c) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and equality holds if and only if $\mathbf{u} = 0$.

Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be a basis for V^n. Then for $\mathbf{u} = \sum_{i=1}^n u^i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i,j=1}^n a_{ij} u^i v^j,$$

where

$$a_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle.$$

The matrix (a_{ij}) is a positive definite symmetric matrix. There is always an *orthonormal* basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ such that

$$a_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Every inner product $\langle , \rangle$ in V^n defines a function on V^n

$$\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

This function $\| \cdot \|$ has the following properties

(a) $\|\mathbf{u}\| \geq 0$ and equality holds if and only if $\mathbf{u} = 0$;

(b) $\|\lambda \mathbf{u}\| = \lambda \|\mathbf{u}\|, \ \forall \lambda > 0$;

(c) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$;

(d) $\| - \mathbf{u}\| = \|\mathbf{u}\|$.

We call $\| \cdot \|$ an *Euclid norm* on V^n and $(\mathrm{V}^n, \| \cdot \|)$ (or $(\mathrm{V}^n, \langle , \rangle)$) an *Euclid space.*

Let R^n be the canonical n-dimensional vector space. The canonical inner product (dot product) $\bullet$ and Euclid norm $| \cdot |$ on R^n are defined by

$$\mathbf{u} \bullet \mathbf{v} = u^1 v^1 + \cdots + u^n v^n,$$

$$|\mathbf{u}| = \sqrt{(u^1)^2 + \cdots + (u^n)^2}.$$

We denote $(\mathrm{R}^n, | \cdot |)$ by $\mathbb{R}^n$ and call $\mathbb{R}^n$ the *canonical Euclid space.*

Every Euclid space $(\mathrm{V}^n, \| \cdot \|)$ is linearly isomorphic to $\mathbb{R}^n$.

There are many interesting functions on V^n with properties similar to that of an Euclid norm. Let $\| \cdot \| : \mathrm{V}^n \to [0, \infty)$ be a function with the following properties

(a) $\|\mathbf{u}\| \geq 0$ and equality holds if and if $\mathbf{u} = 0$;

(b) $\|\lambda \mathbf{b}\| = \lambda \|\mathbf{v}\|, \ \lambda > 0$;

(c) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

$\| \cdot \|$ is called a *norm* on V^n and $(\mathrm{V}^n, \| \cdot \|)$ is called a *normed space.*

When $n = \dim \mathrm{V}^n = 1$, a norm is an Euclid norm if $\| - \mathbf{u}\| = \|\mathbf{u}\|$.

1.2 Curves in Vector Spaces

Let $\mathrm{V}^n = \mathrm{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be a vector space and $I = (a, b)$. Let $c : I \to \mathrm{V}^n$ be a map. Express $c(t)$ in the following form

$$c(t) = c^1(t)\mathbf{e}_1 + \cdots + c^n(t)\mathbf{e}_n.$$

c is said to be *regular* if each component $c^i(t)$ is C^∞ and for any $t \in I$, the derivative

$$\dot{c}(t) := \frac{dc^1}{dt}(t)\mathbf{e}_1 + \cdots + \frac{dc^n}{dt}(t)\mathbf{e}_n \neq 0.$$

A set C in V^n is called a *curve* if it is the image of a regular map $c : I = (a, b) \to \mathrm{V}^n$. In this case, c is called a *coordinate map* (or *parametrization*) of C.

Example 1.2.1 The ellipse

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

can be parametrized by

$$c(t) := \Big(a\cos(t),\ b\sin(t)\Big).$$

Any coordinate map of a curve C determines an orientation of C. There are exactly two orientations for C. If $\bar{c} : \bar{I} = (\bar{a}, \bar{b}) \to C$ is another coordinate map, then there is a function $\bar{t} = \bar{t}(t)$ with $\frac{d\bar{t}}{dt} > 0$ or $\frac{d\bar{t}}{dt} < 0$, such that

$$\bar{c}(\bar{t}) = c(t).$$

If $\frac{d\bar{t}}{dt} > 0$, c and $\bar{c}$ determine the same orientation. Otherwise, they determine the opposite orientations.

Given a curve C in V^n. Let $c : I = (a, b) \to \mathrm{V}^n$ be a coordinate map of a curve C. For $p = c(t_o) \in C$, set

$$T_pC := \mathrm{span}\Big\{\dot{c}(t_o)\Big\}.$$

T_pC is a line in V^2. The line T_pC is independent of the choice of a particular coordinate map of C . We call T_pC the *tangent line* of C at p.

A *length structure* is a family of functions

$$\|\cdot\|_p : T_pC \to [0, \infty), \qquad p \in C,$$

satisfying

(i) $\|\lambda\mathbf{u}\|_p = \lambda\|\mathbf{u}\|_p$, for all $\lambda > 0$ and $\mathbf{u} \in T_pC$;

(ii) for any coordinate map $c : I = (a, b) \to C$, the function

$$\sigma(t) := \|\dot{c}(t)\|_{c(t)}$$

is C^∞.

Given a length structure $\|\cdot\|$ on a curve C in V^n. Let $c: I \to C$ be a coordinate map. The form

$$ds := \sigma(t)dt$$

depends only on the orientation of a coordinate map. We call ds the *length form*. With the length form, we can measure the length of C by

$$L(C) = \int_C ds := \int_a^b \sigma(t)dt.$$

The length $L(C)$ of C depends on the orientation of C.

We can also find a coordinate map $c(s)$ of C such that

$$\|\dot{c}(s)\| = 1.$$

Such a coordinate map is called a *unit speed* coordinate map.

Usually, the length structure is induced by a norm in the ambient space. Let C be a curve in a normed space $(\mathrm{V}^n, \|\cdot\|)$. For a coordinate map $c: I \to C$, define

$$ds := \|\dot{c}(t)\|dt.$$

ds is a length form along C.

Example 1.2.2 $\mathbb{R}^2$ is the canonical vector plane R^2 equipped with the canonical Euclid norm

$$|(u,v)| = \sqrt{u^2+v^2}.$$

Let $y = f(x)$ be a C^∞ function on an interval $I = (a,b)$.

$$C = \Big\{(x,y) \in \mathbb{R}^2,\ y = f(x),\ a < x < b\Big\}.$$

C is a curve in $\mathbb{R}^2$. Take a coordinate map $c: (a,b) \to C$ given by

$$c(t) := (t, f(t)).$$

Define

$$ds = \sqrt{1+[f'(t)]^2}\ dt.$$

$d\ell$ is a length form on C. Then the length of C is given by

$$L(C) = \int_a^b \sqrt{1+[f'(t)]^2}dt.$$

Given a curve C in V^2. Let $c: I = (a,b) \to \mathrm{V}^2$ be a coordinate map of C. Fix a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for V^2. Express $c(t)$ in the following form

$$c(t) = u(t)\mathbf{e}_1 + v(t)\mathbf{e}_2 = \bar{u}(t)\bar{\mathbf{e}}_1 + v(t)\bar{\mathbf{e}}_2.$$

We have

$$\phi(t) := \frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)} = \frac{\bar{u}'(t)\bar{v}''(t) - \bar{u}''(t)\bar{v}'(t)}{\bar{u}(t)\bar{v}'(t) - \bar{u}'(t)\bar{v}(t)}. \tag{1.2}$$

Namely, $\phi(t)$ is independent of the choice of a particular basis. We call $\phi(t)$ the *associated function* of $c(t)$.

Let $t = t(\bar{t})$ be a C^∞ function with $\frac{dt}{d\bar{t}} \neq 0$. Then $\bar{c}(\bar{t}) := c(t(\bar{t}))$ is a coordinate map of C. Let $\bar{\phi}(\bar{t})$ denote the corresponding function of $\bar{c}$. We have

$$\phi(t) = \bar{\phi}(\bar{t})\Big(\frac{d\bar{t}}{dt}\Big)^2. \tag{1.3}$$

Thus the positivity of $\phi(t)$ is independent of a particular coordinate map $c(t)$ of C.

A curve C in V^2 is said to be *strongly convex* if $\phi(t) > 0$ for a coordinate map $c(t)$ of C.

Given a strongly convex curve C in V^2. Define $\|\cdot\| : T_pC \to [0, \infty)$ by

$$\|v\| := \sqrt{\phi(0)}, \qquad \mathbf{v} \in T_pC,$$

where $\phi(t)$ is the associated function of a coordinate map $c(t)$ with $\dot{c}(0) = \mathbf{v}$. By (1.3), we see that $\|\cdot\|$ is independent of the choice of a particular coordinate map. Set

$$d\ell := \|\dot{c}(t)\| dt = \sqrt{\phi(t)} dt$$

$d\ell$ is independent of the choice of a particular coordinate map. We call $d\ell$ the *Landsberg length form*.

There is always a coordinate map $c(\ell)$ such that

$$\|\dot{c}(\ell)\| = \sqrt{\phi(\ell)} = 1.$$

Such a parameter ℓ is called a *Landsberg parameter*.

Example 1.2.3 Let $f = f(t)$ be a positive function satisfying $f''(t) > 0$. Consider the following curve C in R^2:

$$uf\Big(\frac{v}{u}\Big) = 1, \qquad u > 0.$$

Parametrize C by

$$u = \frac{1}{f(t)}, \qquad v = \frac{t}{f(t)}.$$

A simple computation gives

$$\phi(t) = \frac{f''(t)}{f(t)} > 0.$$

The Landsberg length form is given by

$$d\ell = \sqrt{\frac{f''(t)}{f(t)}}\, dt.$$

Example 1.2.4 Let $a > 0$. Consider the curve C in R^2:

$$u\Big(\frac{v}{u}\Big)^{1+a} = 1, \qquad u > 0.$$

Parametrize C by

$$u(t) = t, \qquad v(t) = t^{\frac{a}{1+a}}, \qquad t > 0.$$

A simple computation gives

$$\phi(t) = \frac{a}{(1+a)}\, t^{-2}.$$

The Landsberg length form is given by

$$d\ell = \sqrt{\frac{a}{(1+a)}}\, t^{-1}\, dt.$$

Integrating $d\ell$, we obtain

$$\ell = \sqrt{\frac{a}{(1+a)}}\ln t + \ell_o.$$

ℓ is a Landsberg parameter. With this parameter, we parametrize C by

$$c(\ell) = \Big(\exp\Big[\sqrt{\frac{1+a}{a}}(\ell - \ell_o)\Big],\ \exp\Big[\sqrt{\frac{a}{1+a}}(\ell - \ell_o)\Big]\Big).$$

1.3 Curves in $\mathbb{R}^n$

In this section, we will discuss the geometry of curves in an n-dimensional Euclid space $\mathbb{R}^n$.

Let C be a curve in $\mathbb{R}^n$ and $c : I = (a, b) \to C$ a coordinate map with unit speed, i.e.,

$$|\dot{c}(s)| = 1.$$

Observe that

$$\frac{d}{ds}\Big[|\dot{c}(s)|^2\Big] = \frac{d}{dt}\Big[\dot{c}(s) \bullet \dot{c}(s)\Big] = 2\dot{c}(s) \bullet \ddot{c}(s) = 0. \tag{1.4}$$

Let $\{\dot{c}(s), \mathbf{e}_1(s), \cdots, \mathbf{e}_{n-1}(s)\}$ be a family of positively oriented orthonormal basis along c. We can express

$$\ddot{c}(s) = \kappa_1(s)\mathbf{e}_1(s) + \cdots + \kappa_{n-1}(s)\mathbf{e}_{n-1}(s).$$

We call $\{\kappa_1(s), \cdots, \kappa_{n-1}(s)\}$ the curvatures of c with respect to $\{\mathbf{e}_1(s), \cdots, \mathbf{e}_{n-1}(s)\}$.

I. Curves in $\mathbb{R}^2$: Let $c : I \to C$ be a unit speed coordinate map of a curve C in $\mathbb{R}^2$.

$$c(s) = \Big(x(s), y(s)\Big).$$

Let $\mathbf{t}(s) := \dot{c}(s)$.

$$\mathbf{t}(s) = \Big(\dot{x}(s), \dot{y}(s)\Big).$$

By (1.4), we have

$$\dot{\mathbf{t}}(s) \cdot \mathbf{t}(s) = 0.$$

There is a unique unit vector $\mathbf{n}(s)$ such that $\{\mathbf{t}(s), \mathbf{n}(s)\}$ is a positively oriented orthonormal basis for R^2.

$$\mathbf{n}(s) = \Big(-\dot{y}(s), \dot{x}(s)\Big).$$

There is a function $\kappa(s)$ such that

$$\dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s).$$

$\kappa(s)$ is called the *(signed) curvature.*

For a regular plane curve $c(t) = (x(t), y(t))$ in $\mathbb{R}^2$, the signed curvature $\kappa(t)$ at $c(t)$ is given by

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\Big[x'(t)^2 + y'(t)^2\Big]^{3/2}}.$$

Theorem 1.3.1 *For any unit speed coordinate map of a curve in* $\mathbb{R}^2$, $\{\mathbf{t}(s), \mathbf{n}(s)\}$ *satisfy*

$$\begin{cases} \dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s) \\ \dot{\mathbf{n}}(s) = -\kappa(s)\mathbf{t}(s) \end{cases} . \tag{1.5}$$

Let

$$\mathbf{t}(s) = \Big(u(s), v(s)\Big).$$

Then

$$\mathbf{n}(s) = \Big(-v(s), u(s)\Big).$$

System (1.5) is equivalent to

$$\begin{cases} \dot{u}(s) = -\kappa(s)v(s) \\ \dot{v}(s) = \kappa(s)u(s) \end{cases} . \tag{1.6}$$

We obtain

$$u(s) = \cos\Big(\int \kappa(s)ds + \varphi\Big) \tag{1.7}$$

$$v(s) = \sin\Big(\int \kappa(s)ds + \varphi\Big). \tag{1.8}$$

Since

$$\Big(\dot{x}(s), \dot{y}(s)\Big) = \dot{c}(s) = \mathbf{t}(s) = \Big(u(s), v(s)\Big),$$

we obtain

$$x(s) = \int u(s)ds = \int \cos\Big(\int \kappa(s)ds + \varphi\Big)ds \tag{1.9}$$

$$y(s) = \int v(s)ds = \int \sin\Big(\int \kappa(s)ds + \varphi\Big)ds. \tag{1.10}$$

Theorem 1.3.2 *For a closed oriented curve $C \subset \mathbb{R}^2$, the total curvature*

$$\int_C \kappa(s)\ ds = 2\pi\ \mathrm{I},$$

where I is an integer.

The integer I is called the *index* of C in $\mathbb{R}^2$.

Example 1.3.1 The unit speed plane curve $c(s)$ in $\mathbb{R}^2$ with $\kappa(s) = 2s$ and $c(0) = (0, 0)$ is called *Euler's spiral or the spiral of Cornu.*

II. Curves in $\mathbb{R}^3$: Let $c : I \to C$ be a unit speed coordinate map of a curve C in $\mathbb{R}^3$. Let $\mathbf{t}(s) := \dot{c}(s)$. Assume that $\dot{\mathbf{t}}(s) \neq 0$. By (1.4),

$$\dot{\mathbf{t}}(s) \cdot \mathbf{t}(s) = 0.$$

That is, $\dot{\mathbf{t}}(s)$ is perpendicular to $\mathbf{t}(s)$. Let

$$\mathbf{n}(s) := \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|}.$$

Let

$$\mathbf{b}(s) := \mathbf{t}(s) \times \mathbf{n}(s).$$

We obtain a positively oriented basis $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$. We can express

$$\dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s),$$

where

$$\kappa(s) = |\dot{\mathbf{t}}(s)| > 0.$$

Define

$$\tau(s) := \dot{\mathbf{n}}(s) \bullet \mathbf{b}(s).$$

κ and τ is called the *curvature* and *torsion* of C respectively.

Theorem 1.3.3 (The Frenet formulas) *For any unit speed coordinate map of curve C in $\mathbb{R}^3$, $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ satisfy*

$$\begin{cases} \dot{\mathbf{t}}(s) & = \kappa(s)\mathbf{n}(s) \\ \dot{\mathbf{n}}(s) & = -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s) \\ \dot{\mathbf{b}}(s) & = -\tau(s)\mathbf{n}(s) \end{cases} . \tag{1.11}$$

For any coordinate map $c(t)$ of C, the following formulas hold.

$$\mathbf{t}(t) = \frac{c'(t)}{|c'(t)|}.$$

$$\mathbf{b}(t) = \frac{c'(t) \times c''(t)}{|c'(t) \times c''(t)|}$$

$$\mathbf{n}(t) = \mathbf{b}(t) \times \mathbf{t}(t).$$

$$\kappa(t) = \frac{|c'(t) \times c''(t)|}{|c'(t)|^3}$$

$$\tau(t) = \frac{(c'(t) \times c''(t)) \cdot c'''(t)}{|c'(t) \times c''(t)|^2}$$

Example 1.3.2 Consider the following curve

$$c(s) := \Big(\frac{(1+s)^{3/2}}{3}, \ \frac{(1-s)^{3/2}}{3}, \ \frac{s}{\sqrt{2}}\Big), \qquad -1 < s < 1.$$

(a) $|\dot{c}| = 1$;

(b) $\kappa = 1/\sqrt{8(1-s^2)}$;

(c)

$$\mathbf{n} = \Big(\frac{\sqrt{2(1-s)}}{2}, \ \frac{\sqrt{2(1+s)}}{2}, \ 0\Big), \qquad \mathbf{b} = \Big(\frac{-\sqrt{1-s}}{2}, \ \frac{\sqrt{1-s}}{2}, \ \frac{\sqrt{2}}{2}\Big);$$

(d) $\tau = \kappa$.

The geometric meaning of the torsion lies in the following result:

Theorem 1.3.4 *A curve in $\mathbb{R}^3$ is a plane curve if and only if $\tau = 0$.*

Proof: Suppose that $\tau = 0$. By the Frenet formula, we know that $\mathbf{b}(s) = \mathbf{b}_o$ is a constant vector. Note that

$$\mathbf{b}_o \cdot \dot{c}(s) = \mathbf{b}_o \cdot \mathbf{t}(s) = \mathbf{b}(s) \cdot \mathbf{t}(s) = 0,$$

we obtain

$$\mathbf{b}_o \cdot (c(s) - c(s_o)) = 0.$$

Thus $c(s)$ is contained in the plane passing through $c(s_o)$ and perpendicular to $\mathbf{b}_o$. Q.E.D.

1.4 Exercises

Exercise 1.4.1 Parametrize the circle which is centered at the point (a, b).

Exercise 1.4.2 Let κ with $|\kappa| < 1$. Parametrize the curve C in R^2:

$$\sqrt{u^2 + v^2} + \kappa u = 1.$$

Find the Landsberg length form and set up the integral for the total Landsberg length of C.

Exercise 1.4.3 Find a unit speed plane curve $c(s)$ in R^2 with $c(0) = (0, 0)$ and

$$\kappa(s) = \frac{1}{1 + s^2}$$

Exercise 1.4.4 Let C be a curve in $\mathbb{R}^2$ and $c : I = (a, b) \to C$ an orientation-preserving coordinate map. Assume that the tangent vector $c'(t)$ is not parallel to the position vector $c(t)$. Show that if the signed curvature $\kappa > 0$, then $\phi(t) > 0$.

Exercise 1.4.5 (The Helix) Consider the helix

$$c(s) := \left(a \cos\left(\frac{s}{c}\right),\ a \sin\left(\frac{s}{c}\right),\ \frac{b}{c} s\right)$$

with $c = \sqrt{a^2 + b^2}$. Show that

$$\kappa(s) = \frac{a}{c^2} = \frac{a}{a^2 + b^2}, \qquad \tau(s) = \frac{a}{c^2} = \frac{b}{a^2 + b^2}.$$

Exercise 1.4.6 Find the curvature κ and the torsion τ of the following curve

$$c(s) = \left(\frac{1}{\sqrt{2}} \cos(s),\ \sin(s),\ \frac{1}{\sqrt{2}} \cos(s)\right).$$

Chapter 2

Minkowski Spaces

2.1 Minkowski Spaces

I. Minkowski Norms: Let $F : \mathrm{V}^n \to [0, \infty)$ be a function. Assume that

(a) F is positively homogeneous of degree one, i.e.,

$$F(\lambda \mathbf{y}) = \lambda F(\mathbf{y}), \qquad \lambda > 0,\ \mathbf{y} \in \mathrm{V}^n.$$

(b) F is C^∞ on $\mathrm{V}^n - \{0\}$.

For a vector $\mathbf{y} \in \mathrm{V}^n - \{0\}$, define a symmetric bilinear form

$$\mathbf{g}_y : \mathrm{V}^n \times \mathrm{V}^n \to \mathrm{R}$$

by

$$\mathbf{g}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \Big[F^2(\mathbf{y} + s\mathbf{u} + t\mathbf{v}) \Big]|_{s=t=0}, \qquad \mathbf{u},\ \mathbf{v} \in \mathrm{V}^n. \tag{2.1}$$

Fix a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ for V^n. Let

$$L(y^1, \cdots, y^n) := \frac{1}{2} F^2\Big(y^1 \mathbf{e}_1 + \cdots + y^n \mathbf{e}_n\Big). \tag{2.2}$$

L is a C^∞ function on $\mathrm{R}^n \setminus \{0\}$. For a vector $\mathbf{y} := \sum_{i=1}^n y^i \mathbf{e}_i \in \mathrm{V}^n - \{0\}$, let

$$g_{ij}(\mathbf{y}) := L_{y^i y^j}\Big(y^1, \cdots, y^n\Big), \qquad i, j = 1, \cdots, n. \tag{2.3}$$

Then

$$\mathbf{g}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^n g_{ij}(\mathbf{y}) u^i v^j \tag{2.4}$$

where $\mathbf{u} = \sum_{i=1}^n u^i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i$.

Definition 2.1.1 F is called a *Minkowski norm* on V^n if

(i) F is positively homogeneous of degree one,

(ii) F is C^∞ on $\mathrm{V}^n - \{0\}$;

(iii) $\mathbf{g_y}$ is positive definite for all $\mathbf{y} \in \mathrm{V}^n - \{0\}$, i.e., the matrix $\Big(g_{ij}(y)\Big)$ is positive definite.

For a Minkowski norm F on V^n,

$$\mathbf{g_y}(\mathbf{y}, \mathbf{y}) = F^2(\mathbf{y}). \tag{2.5}$$

$$\mathbf{g_y}(\mathbf{y}, \mathbf{u}) = \frac{1}{2}\frac{d}{dt}\Big[F^2(\mathbf{y} + t\mathbf{u})\Big]|_{t=0}. \tag{2.6}$$

Example 2.1.1 Any Euclid norm $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$ in a vector space is a Minkowski norm.

Lemma 2.1.1 *Let F be a Minkowski norm on V^n.*

(i) For any $\mathbf{y}, \mathbf{u} \in \mathrm{V}^n - \{0\}$,

$$\mathbf{g_y}(\mathbf{y}, \mathbf{u}) \le F(\mathbf{y})F(\mathbf{u}). \tag{2.7}$$

(ii) For any vectors $\mathbf{u}, \mathbf{v} \in \mathrm{V}^n$

$$F(\mathbf{u} + \mathbf{v}) \le F(\mathbf{u}) + F(\mathbf{v}).$$

Example 2.1.2 (Randers norm) Let $\mathrm{V}^n = \mathrm{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. Let α and β be an Euclid norm and a linear functional on V^n respectively. α can be expressed in the following form

$$\alpha(\mathbf{y}) = \sqrt{\sum_{i,j=1}^{n} a_{ij}\; y^i y^j}, \qquad \mathbf{y} = \sum_{i=1}^{n} y^i \mathbf{e}_i \in \mathrm{V}^n,$$

where (a_{ij}) is a positive definite symmetric $n \times n$ matrix, and β can be expressed by

$$\beta(\mathbf{y}) = \sum_{i=1}^{n} b_i\; y^i, \qquad \mathbf{y} = \sum_{i=1}^{n} y^i \mathbf{e}_i \in \mathrm{V}^n$$

Define

$$F(\mathbf{y}) := \alpha(\mathbf{y}) + \beta(\mathbf{y}).$$

F is a Minkowski norm if and only if the norm of β with respect to α is less than 1,

$$\|\beta\|_\alpha := \sup_{\alpha(\mathbf{y})=1} \beta(\mathbf{y}) < 1.$$

Let (a^{ij}) denote the inverse of (a_{ij}). We have

$$\|\beta\|_\alpha = \sqrt{\sum_{i,j=1}^{n} a^{ij}\, b_i b_j}.$$

A direct computation yields

$$g_{ij} = \frac{F}{\alpha}\Big(a_{ij} - \alpha_{y^i}\alpha_{y^j}\Big) + \Big(\alpha_{y^i} + b_i\Big)\Big(\alpha_{y^j} + b_j\Big).$$

$$\det(g_{ij}) = \Big(\frac{F}{\alpha}\Big) \det(a_{ij}).$$

Definition 2.1.2 Let $\mathbf{b} = (b_1, \cdots, b_n)$ be a vector in $\mathbb{R}^n$ with $|\mathbf{b}| < 1$. Define $|\cdot|_{\mathbf{b}} : \mathrm{R}^n \to [0, \infty)$ by

$$|\mathbf{v}|_{\mathbf{b}} := |\mathbf{v}| + \mathbf{b}\cdot\mathbf{v}, \tag{2.8}$$

$|\cdot|_{\mathbf{b}}$ is a Randers norm on R^n. We denote by $\mathbb{R}^n_{\mathbf{b}} = (\mathrm{R}^n, |\cdot|_{\mathbf{b}})$ and call $\mathbb{R}^n_{\mathbf{b}}$ the *canonical Randers space* associated with $\mathbf{b}$.

II. Minkowski Planes

Let $F : \mathrm{V}^2 \to [0, \infty)$ be a nonnegative function. Assume that

(a) F is positively homogeneous of degree one, i.e.,

$$F(\lambda\mathbf{y}) = \lambda F(\mathbf{y}), \qquad \lambda > 0.$$

(b) F is C^∞ on $\mathrm{V}^2 \setminus \{0\}$.

Fix a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for V^2. Let

$$L(u, v) := \frac{1}{2}F^2\Big(u\mathbf{e}_1 + v\mathbf{e}_2\Big).$$

L is C^∞ on $\mathrm{R}^2 - \{0\}$. Let

$$g_{11} = L_{uu}, \qquad g_{12} = L_{uv} = g_{21}, \qquad g_{22} = L_{vv}.$$

F is a Minkowski norm on V^2 if and only if for any $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in \mathrm{V}^2 - \{0\}$, i.e., $\mathbf{g}_y : \mathrm{V}^2 \times \mathrm{V}^2 \to \mathrm{R}$ is positive definite, i.e.,

$$g_{11} > 0, \qquad g_{22} > 0, \qquad g_{11}g_{22} - g_{12}g_{21} > 0. \tag{2.9}$$

Let $c(t) = u(t)\mathbf{e}_1 + v(t)\mathbf{e}_2$ be a coordinate map for $C = F^{-1}(1)$. Then for $\mathbf{y} = c(t) \in C$,

$$\mathbf{g}_{\mathbf{y}}(\mathbf{v},\mathbf{v}) = a^2 + \frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)} b^2, \qquad \mathbf{v} = a\, c(t) + b\, \dot{c}(t). \tag{2.10}$$

Thus the Landsberg length form of $C = F^{-1}(1)$ is given by

$$d\ell = \sqrt{\mathbf{g}_{c(t)}\Big(\dot{c}(t), \dot{c}(t)\Big)}\, dt.$$

Example 2.1.3 For a number κ with $|\kappa| < 1$, let

$$|(u,v)|_\kappa = \sqrt{u^2+v^2} + \kappa u \tag{2.11}$$

and

$$L(u,v) = \frac{1}{2}\Big(|(u,v)|_\kappa\Big)^2 = \frac{1}{2}(\sqrt{u^2+v^2} + \kappa u)^2.$$

We have

$$\begin{aligned} g_{11} &= 1 + \kappa^2 + \kappa u\Big(2u^2 + 3v^2\Big)\Big(u^2+v^2\Big)^{-3/2} \\ g_{12} &= \kappa v^3\Big(u^2+v^2\Big)^{-3/2} = g_{21} \\ g_{22} &= 1 + \kappa u^3\Big(u^2+v^2\Big)^{-3/2}. \end{aligned}$$

This gives

$$g_{11}g_{22} - g_{12}g_{21} = \Big(\frac{\sqrt{u^2+v^2} + \kappa u}{\sqrt{u^2+v^2}}\Big)^3.$$

Thus $|\cdot|_\kappa$ is a Minkowski norm on R^2.

Example 2.1.4 Let

$$F(u,v) := \Big\{u^4 + 3c\, u^2v^2 + v^4\Big\}^{\frac{1}{4}}.$$

A direct computation yields

$$\begin{aligned} g_{11} &= \frac{2u^6 + 9cu^4v^2 + 6u^2v^4 + 3cv^6}{2\Big(u^4 + 3c\, u^2v^2 + v^4\Big)^{3/2}} \\ g_{12} &= \frac{(9c^2-4)u^3v^3}{2\Big(u^4 + 3c\, u^2v^2 + v^4\Big)^{3/2}} = g_{21} \\ g_{22} &= \frac{3cu^6 + 6u^4v^2 + 9cu^2v^4 + 2v^6}{2\Big(u^4 + 3c\, u^2v^2 + v^4\Big)^{3/2}} \end{aligned}$$

This gives

$$g_{11}g_{22} - g_{12}g_{21} = \frac{3}{4} \cdot \frac{2cu^4 + (4-2c^2)u^2v^2 + 2cv^4}{u^4 + 3c\,u^2v^2 + v^4}.$$

Thus F is a Minkowski norm on R^2 if and only if $0 < c < 2$.

Assume that F is a Minkowski norm on V^2. Since $\mathbf{g_y} : \mathrm{V}^2 \times \mathrm{V}^2 \to \mathrm{R}$ is a positive definite symmetric bilinear form, there exactly two directions perpendicular to $\mathbf{y}$ with respect to $\mathbf{g_y}$. Thus if an orientation is given for V^2, there is a unique vector $\mathbf{y}^\perp$ satisfying

(1) $\{\mathbf{y}, \mathbf{y}^\perp\}$ is positively oriented.

(2) $\mathbf{y}^\perp$ is perpendicular to $\mathbf{y}$,

$$\mathbf{g_y}(\mathbf{y}, \mathbf{y}^\perp) = 0, \tag{2.12}$$

(3) $\mathbf{y}^\perp$ and $\mathbf{y}$ satisfy

$$\mathbf{g_y}(\mathbf{y}^\perp, \mathbf{y}^\perp) = \mathbf{g_y}(\mathbf{y}, \mathbf{y}) = F^2(\mathbf{y}). \tag{2.13}$$

We call $\mathbf{y}^\perp$ the *conjugate vector* to $\mathbf{y}$.

Fix an oriented basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for V^2. Let

$$L(u,v) := \frac{1}{2}F^2\Big(u\mathbf{e}_1 + v\mathbf{e}_2\Big).$$

For a vector $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in \mathrm{V}^2$,

$$\mathbf{y}^\perp := \frac{-L_v\mathbf{e}_1 + L_u\mathbf{e}_2}{\sqrt{L_{uu}L_{vv} - (L_{uv})^2}}. \tag{2.14}$$

$\mathbf{y}^\perp$ is tangent to the indicatrix $C = F^{-1}(1)$. Let $c(t) = u(t)\mathbf{e}_1 + v(t)\mathbf{e}_2$ be a coordinate map for C. For $\mathbf{y} = c(t)$, we have

$$\mathbf{y}^\perp = \frac{\dot{c}(t)}{\sqrt{\phi(t)}}, \tag{2.15}$$

where

$$\phi(t) = \mathbf{g}_{c(t)}\Big(\dot{c}(t), \dot{c}(t)\Big) = \frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)}.$$

Cartan Torsion. Fix an arbitrary basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for a vector plane V^2. Let F be a Minkowski norm on V^2 and

$$L(u,v) := \frac{1}{2}F^2\Big(u\mathbf{e}_1 + v\mathbf{e}_2\Big).$$

L is C^∞ on $\mathrm{R}^2 - \{0\}$.

Definition 2.1.3 For a vector $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in \mathrm{V}^2 - \{0\}$, define

$$\mathbf{C}(\mathbf{y}) = \frac{L_{uuu}(-L_v)^3 + 3L_{uuv}(-L_v)^2 L_u + 3L_{uvv}(-L_v)(L_u)^2 + L_{vvv}(L_u)^3}{4L\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)^{\frac{3}{2}}}.$$

We call $\mathbf{C}(\mathbf{y})$ the *Cartan torsion* of F in the direction $\mathbf{y}$.

Using the homogeneity of L, we have

$$L_u = \frac{2}{u}L - \frac{v}{u}L_v,$$

$$L_{uv} = \frac{1}{u}L_v - \frac{v}{u}L_{vv}, \qquad L_{uu} = \frac{2}{u^2}L - 2\frac{v}{u^2}L_v + \Big(\frac{v}{u}\Big)^2 L_{vv},$$

$$L_{uuv} = -\frac{v}{u}L_{vvv}, \qquad L_{uvv} = \Big(\frac{v}{u}\Big)^2 L_{vvv}, \qquad L_{uuu} = -\Big(\frac{v}{u}\Big)^3 L_{vvv}.$$

By these identities, one can easily simplify the above formula for $\mathbf{C}(\mathbf{y})$ to the following simpler one for $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2$ with $u > 0$.

$$\mathbf{C}(\mathbf{y}) = \frac{2L^2 L_{vvv}}{\Big(2LL_{vv} - L_v L_v\Big)^{\frac{3}{2}}}. \tag{2.16}$$

Clearly, if F is Euclidean, then $\mathbf{C}(\mathbf{y}) = 0$ for any $\mathbf{y}$. The converse is also true.

Proposition 2.1.1 (Cartan) *A Minkowski norm F on V^2 is Euclidean if and only if $\mathbf{C}(\mathbf{y}) = 0$ for any $\mathbf{y} \in \mathrm{V}^2 - \{0\}$.*

Proof: Suppose that $\mathbf{C}(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathrm{V}^2 - \{0\}$. Then

$$L_{vvv} = 0.$$

By the homogeneity of L, we know that

$$L_{uuu} = L_{uuv} = L_{uvv} = L_{vvv} = 0.$$

Thus L is quadratic in (u, v) and F is Euclidean. Q.E.D.

Let $C = F^{-1}(1)$. Parametrize it by $c(t) = (u(t), v(t))$. By (2.10), the Landsberg length form is given by

$$d\ell := \sqrt{\phi(t)}dt = \sqrt{\frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)}}\, dt.$$

A direct computation also yields the following formula for $\mathbf{C}(\mathbf{y})$,

$$\mathbf{C}(\mathbf{y}) = \frac{1}{2}\sqrt{\frac{uv' - u'v}{u'v'' - u''v'}} \left\{ \frac{u'v''' - u'''v'}{u'v'' - u''v'} - 3\frac{uv'' - u''v}{uv' - u'v} \right\}. \tag{2.17}$$

Theorem 2.1.1 *For any Minkowski norm F on V^2, the total Cartan torsion on $C = F^{-1}(1)$ must vanish,*

$$\int_C \mathbf{C}\, d\ell = 0. \tag{2.18}$$

Proof: Let $c(t), a \le t \le b$ be a coordinate map of C with

$$c(a) = c(b), \qquad \dot{c}(a) = \dot{c}(b).$$

Note that

$$\mathbf{C}(\mathbf{y}) d\ell = \Big[\ln \chi(t)\Big]' dt,$$

where

$$\chi(t) = \sqrt{\frac{u'v'' - u''v'}{\Big(uv' - u'v\Big)^3}}.$$

Integrating $\mathbf{C}$ along C gives (2.18). Q.E.D.

Theorem 2.1.2 (**Varga Equation**) *Let C be a strongly convex curve in V^2 which is parametrized by $c(s)$ with a Landsberg parameter ℓ . Then*

$$\ddot{c}(\ell) + \mathbf{C}(\ell)\dot{c}(\ell) + c(\ell) = 0. \tag{2.19}$$

Example 2.1.5 Let

$$|(u, v)|_\kappa := \sqrt{u^2 + v^2} + \kappa u,$$

where κ is a constant with $|\kappa| < 1$. $|\cdot|_\kappa$ is a Randers norm on R^2. The Cartan torsion of $|\cdot|_\kappa$ is given by

$$\mathbf{C}(\mathbf{y}) = -\frac{3}{2}\frac{\kappa v}{(u^2+v^2)^{\frac{1}{4}}\sqrt{\sqrt{u^2+v^2}+\kappa u}}.$$

We have

$$\|\mathbf{C}\| := \max_{\mathbf{y}} |\mathbf{C}(\mathbf{y})| = \frac{3}{\sqrt{2}}\sqrt{1 - \sqrt{1-\kappa^2}} \le \frac{3}{\sqrt{2}}.$$

The absolute maximum is attained at

$$u = -\frac{1 - \sqrt{1-\kappa^2}}{\kappa}, \qquad v = \pm\sqrt{1-u^2}.$$

2.2 Exercises

Exercise 2.2.1 Let κ be a constant $|\kappa| < 1$. Consider the following Randers norm on R^2,

$$|(u,v)|_\kappa = \sqrt{u^2+v^2} + \kappa u.$$

Show that for a vector $\mathbf{y} = (u,v)$, its conjugate vector $\mathbf{y}^\perp$ is given by

$$\mathbf{y}^\perp = \Big(\frac{\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}+\kappa u}\Big)^{\frac{1}{2}}\Big(-v\ ,\ u+\kappa\sqrt{u^2+v^2}\Big).$$

Exercise 2.2.2 Is the following norm Minkowskian on R^2? Why ?

$$F(u,v) = \{u^4+v^4\}^{\frac{1}{4}}.$$

Exercise 2.2.3 Show that for any $\varepsilon \geq 0$, the following norm is a Minkowski norm on R^2.

$$F(u,v) = \sqrt{u^2+v^2+\varepsilon\sqrt{u^4+v^4}}. \tag{2.20}$$

Exercise 2.2.4 Let $\lambda, \mu > 0$ and

$$F(u,v) := \Big\{\Big(\lambda u^2+\mu v^2\Big)\Big(\mu u^2+\lambda v^2\Big)\Big\}^{\frac{1}{4}}.$$

Show that F is a Minkowski norm if and only if

$$3-2\sqrt{2} < \frac{\lambda}{\mu} < 3+2\sqrt{2}.$$

Exercise 2.2.5 Let $F : R^2 \to [0,\infty)$ be a Minkowski norm. In $R^2_+ = \{u > 0\}$, express F in the form

$$F(u,v) = uf\Big(\frac{v}{u}\Big),$$

where $f = f(t)$ is a positive function. Show that at $\mathbf{y} = (u,v) \in R^2_+$,

$$\mathbf{C}(\mathbf{y}) = \frac{3f'(t)f''(t)+f(t)f'''(t)}{2\sqrt{f(t)f''(t)}f''(t)}.$$

where $t = v/u$.

Exercise 2.2.6 Let

$$F(u,v) := \Big\{u^4+3c\,u^2v^2+v^4\Big\}^{1/4},$$

where $0 < c < 2$. F is a Minkowski norm on R^2. Show that for any $\mathbf{y} = (u,v)$ with $u > 0$,

$$\mathbf{C}(\mathbf{y}) = \frac{2\sqrt{3}(9c^2-4)(v^4-u^4)uv}{3\Big(2c\,u^4+(4-3c^2)\,u^2v^2+2c\,v^4\Big)^{3/2}}.$$

Show that for (u, v) with $u/v = \sqrt{2/(3c)}$,

$$\mathbf{C}(\mathbf{y}) = -\frac{(9c^2 - 4)^2}{64c}.$$

Thus

$$\sup_{\mathbf{y}} |\mathbf{C}(\mathbf{y})| \geq \frac{(9c^2 - 4)^2}{64c} \to \infty, \qquad \text{as } c \to 0^+.$$

Chapter 3

Metrics on Surfaces

3.1 Surfaces

Surfaces can be defined as in an abstract way. From differential topology, we know that every surface can be embedded into R^n. Thus we will consider surfaces in a finite-dimensional vector space.

Let $\mathrm{V}^n = \mathrm{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be an n-dimensional vector space ($n \geq 2$). Let $D \subset \mathrm{R}^2$ be an open subset and $\varphi : D \to \mathrm{V}^n$ a map. φ can be expressed in the following form

$$\varphi(x, y) = \varphi^1(x, y)\mathbf{e}_1 + \cdots + \varphi^n(x, y)\mathbf{e}_n.$$

At a point $\mathbf{x} = (x, y) \in D$, define

$$d\varphi_{\mathbf{x}} : \mathrm{R}^2 \to \mathrm{V}^n$$

by

$$d\varphi_{\mathbf{x}}(\mathbf{u}) := \frac{d}{dt}\Big[\varphi(\mathbf{x} + t\mathbf{u})\Big]|_{t=0}, \qquad \mathbf{u} \in \mathrm{R}^2.$$

We have

$$d\varphi_{\mathbf{x}}(\mathbf{u}) = \mu\varphi_x(\mathbf{x}) + \nu\varphi_y(\mathbf{x}), \qquad \mathbf{u} = (\mu, \nu) \in \mathrm{R}^2,$$

where

$$\varphi_x(\mathbf{x}) := \frac{\partial\varphi}{\partial x}(\mathbf{x}), \qquad \varphi_y(\mathbf{x}) := \frac{\partial\varphi}{\partial y}(\mathbf{x}).$$

Thus the following subset

$$\Big\{d\varphi_{\mathbf{x}}(\mathbf{u}) \in \mathrm{V}^n \;\Big|\; \mathbf{u} \in \mathrm{R}^2\Big\} = \mathrm{span}\Big\{\varphi_x(\mathbf{x}), \varphi_y(\mathbf{x})\Big\}$$

is a subspace of V^n. φ is called a *regular map* if for any $\mathbf{x} = (x, y) \in D$, $\varphi_x(\mathbf{x})$ and $\varphi_y(\mathbf{x})$ are linearly independent.

A subset S in V^n is called a *surface* if at every point $p \in S$, there is an open subset $\mathcal{U}_p$ of p in V^n such that $S \cap \mathcal{U}_p$ is the image of a regular map $\varphi : D \to \mathrm{R}^n$, i.e.,

$$\varphi(D) = S \cap \mathcal{U}_p.$$

φ is called a *coordinate map* of S at p and (D, φ) is called a *coordinate system* at p.

For $p \in S$, the subspace,

$$T_pS := \text{span}\Big\{\varphi_x(\mathbf{x}),\ \varphi_y(\mathbf{x})\Big\},$$

is independent the choice of a particular coordinate map $\varphi : D \to \mathrm{V}^n$ with $p = \varphi(\mathbf{x})$.

Definition 3.1.1 T_pS is called the *tangent plane* of S at p. Vectors in T_pS are called the *tangent vectors* of S at p.

Example 3.1.1 Let $z = f(x, y)$ be a smooth function on an open subset $D \subset \mathrm{R}^2$. Define $\varphi : D \to \mathrm{R}^3$ by

$$\varphi(x, y) = \Big(x, y, f(x, y)\Big). \tag{3.1}$$

φ is a regular map. The surface $S = \varphi(D)$ is called the graph of f.

Example 3.1.2 Let

$$\varphi(x, y) := \Big(\sin x,\ xy,\ y\cos x\Big), \qquad (x, y) \in D = (0, \pi/2) \times \mathrm{R}.$$

φ is a regular map. Thus the image $S = \varphi(D)$ is a surface in R^3.

Example 3.1.3 Let

$$\varphi(\theta, \phi) := \Big(\sin\phi\cos\theta,\ \sin\phi\sin\theta, \cos\phi\Big), \qquad (\theta, \phi) \in D = (0, 2\pi) \times (0, \pi).$$

φ is a coordinate map of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

Let $\mathcal{U}$ be an open subset in R^3 and $\Phi : \mathcal{U} \to \mathrm{R}$ be a smooth function. Define

$$\nabla\Phi(\mathbf{x}) := \Big(\Phi_u(\mathbf{x}),\ \Phi_v(\mathbf{x}),\ \Phi_w(\mathbf{x})\Big),$$

where $\mathbf{x} = (u, v, w) \in \mathcal{U}$. $\nabla\Phi(\mathbf{x})$ is called the *gradient* of Φ at $\mathbf{x}$. Consider the level set

$$\Phi^{-1}(c) := \Big\{\mathbf{x} \in \mathrm{R}^3 \ \Big|\ \Phi(\mathbf{x}) = c\Big\}.$$

Theorem 3.1.1 (Implicit Function Theorem) *Assume that at some point* $\mathbf{x} \in \Phi^{-1}(c)$,

$$\nabla\Phi(\mathbf{x}) \neq 0.$$

Then there is an open subset $\mathcal{U}_{\mathbf{x}}$ *of* $\mathbf{x}$ *in* R^3 *such that* $\Phi^{-1}(c) \cap \mathcal{U}_{\mathbf{x}}$ *is a surface.*

Example 3.1.4 Consider the set S in R^3 defined by

$$S := \left\{ (u, v, w) \in \mathrm{R}^3 \;\Big|\; \frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} = 1 \right\}.$$

where $a, b, c > 0$. S is a closed surface called an *ellipsoid*.

3.2 Riemann, Randers and Finsler Metrics

Let X be a set. A function $d : X \times X \to \mathrm{R}$ is called a *distance function* if it has the following properties

(i) For any $p, q \in X$,

$$d(p,q) \geq 0 \tag{3.2}$$

and equality holds if and only if $p = q$.

(ii) For any $p, q, r \in X$,

$$d(p,r) \leq d(p,q) + d(q,r). \tag{3.3}$$

The number $d(p,q)$ is called the *distance* from p to q. In general, the distance from p to q is not equal to that from q to p. The distance d is said to be *reversible* if for any $p, q \in X$,

$$d(p,q) = d(q,p). \tag{3.4}$$

Example 3.2.1 (Funk) Let $\mathcal{U}$ be a strictly convex domain in $\mathbb{R}^n = (\mathrm{R}^n, |\cdot|)$. For a pair of points $p, q \in \mathcal{U}$, let $z_{pq} \in \partial\mathcal{U}$ be the intersection point of the line with $\partial\mathcal{U}$ which passes through p and q in the order p, q, z_{pq}. Define

$$d(p,q) := \ln \frac{|z_{pq} - p|}{|z_{pq} - q|}. \tag{3.5}$$

d is a non-reversible distance on $\mathcal{U}$, i.e.,

$$d(p,q) \leq d(p,r) + d(r,q).$$

d is called the *Funk distance*.

Example 3.2.2 (Klein) Let d denote the Funk distance on a strictly convex domain $\mathcal{U} \subset \mathrm{R}^n$. Define $\tilde{d} : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ by

$$\tilde{d}(p,q) := \frac{1}{2}\Big\{ d(p,q) + d(q,p) \Big\}, \tag{3.6}$$

Observe that for $p, q, r \in \Omega$,

$$\begin{aligned} \tilde{d}(p,q) &= \frac{1}{2}\Big\{ d(p,q) + d(q,p) \Big\} \\ &\leq \frac{1}{2}\Big\{ d(p,r) + d(r,q) \Big\} + \frac{1}{2}\Big\{ d(q,r) + d(r,p) \Big\} \\ &= \frac{1}{2}\Big\{ d(p,r) + d(r,p) \Big\} + \frac{1}{2}\Big\{ d(q,r) + d(r,q) \Big\} \\ &= \tilde{d}(p,r) + \tilde{d}(r,q). \end{aligned}$$

Thus $\tilde{d}$ is a distance too. We call $\tilde{d}$ the *Klein distance.*

Now we take a look at the special case when $\mathcal{U} = \mathbb{B}^n$ is the standard unit ball in the Euclid space $\mathbb{R}^n$. For a pair of points $p, q \in \mathbb{B}^n$, let

$$z_{pq} = p + \lambda(q - p) \in \partial\mathbb{B}^n, \qquad \lambda > 1.$$

From the equation $|z|^2 = 1$, we obtain

$$\lambda = \frac{\sqrt{\langle p, q-p\rangle^2 + |p-q|^2(1-|p|^2)} - \langle p, q-p\rangle}{|q-p|^2}.$$

By (3.5), we obtain a formula for the Funk distance on $\mathbb{B}^n$

$$d(p,q) = \ln\frac{\lambda}{\lambda - 1} = \ln\frac{\sqrt{\langle p, q-p\rangle^2 + |p-q|^2(1-|p|^2)} - \langle p, q-p\rangle}{\sqrt{\langle p, q-p\rangle^2 + |p-q|^2(1-|p|^2)} - \langle q, q-p\rangle}.$$

Note that

$$\lim_{q \to \partial\mathrm{B}^n} d(0,q) = \infty, \qquad \lim_{p \in \partial\mathrm{B}^n} d(p,0) = \ln 2.$$

Example 3.2.3 Let $(\mathrm{V}^n, \|\cdot\|)$ be an n-dimensional normed space. The norm $\|\cdot\|$ on V^n induces a distance $d : \mathrm{V}^n \times \mathrm{V}^n \to \mathrm{R}$ by

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{y} - \mathbf{x}\|.$$

Let S be a surface in V^n. The norm $\|\cdot\|$ also induces a family of norms F_p on T_pS, $p \in S$, by

$$F_p(\mathbf{y}) := \|\mathbf{y}\|, \qquad \mathbf{y} \in T_pS \subset \mathrm{V}^n.$$

With this family of norms $F = \{F_p\}_{p \in S}$, we can measure the length of a curve on S.

Let C be a curve on S and let $c : I = [a, b] \to C \subset S$ be a coordinate map for the whole curve C. The length form along C is

$$ds := F\Big(\dot{c}(t)\Big)dt = \|\dot{c}(t)\|dt.$$

The length of C is defined by

$$L(C) = \int_C ds = \int_a^b F\Big(\dot{c}(t)\Big)dt.$$

Let $\mathcal{C}$ denote the set of all curves on S. Then L is a function on $\mathcal{C}$. L is called the *length structure* defined by F. The length structure induces a distance function $\tilde{d}$ on S by

$$\tilde{d}(p, q) := \inf_{c \in C} L(C),$$

where the infimum is taken over all curves from p to q. Note that

$$d(p, q) \leq \tilde{d}(p, q).$$

Example 3.2.4 (Induced Riemannian metrics) Consider a surface in the Euclidean space $\mathbb{R}^n$. Let $\varphi : D \to S$ be a coordinate map of S. The induced Riemannian metric F is given by

$$F_p(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2}, \tag{3.7}$$

where $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$ and

$$a = \varphi_x \cdot \varphi_x, \quad b = \varphi_x \cdot \varphi_y, \quad c = \varphi_y \cdot \varphi_y.$$

Example 3.2.5 (Induced Randers metrics) Let $\mathbf{b} = (b_1, \cdots, b_n)$ be a vector with $|\mathbf{b}| < 1$. Consider a Randers space $\mathbb{R}^n_{\mathbf{b}} = (\mathrm{R}^n, |\cdot|_{\mathbf{b}})$, where $|\cdot|_{\mathbf{b}}$ is defined by

$$|\mathbf{v}|_{\mathbf{b}} := |\mathbf{v}| + \mathbf{b} \cdot \mathbf{v}, \qquad \mathbf{v} \in \mathrm{R}^n. \tag{3.8}$$

Let $\varphi : D \to S$ be a coordinate map of S. The induced Randers metric on S is given by

$$F_p(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2} + \lambda u + \mu v, \tag{3.9}$$

where $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$ and

$$a = \varphi_x \cdot \varphi_x, \quad b = \varphi_x \cdot \varphi_y, \quad c = \varphi_y \cdot \varphi_y,$$

$$\lambda = \mathbf{b} \cdot \varphi_x, \quad \mu = \mathbf{b} \cdot \varphi_y.$$

The above discussion leads to the following

Definition 3.2.1 Let S be a surface in V^n. A family of functions $F = \{F_p\}_{p \in S}$, where $F_p : T_pS \to \mathrm{R}$, is called a *Finsler metric* if it has the following properties:

(i) for any coordinate map $\varphi : D \to S$, the function

$$F := F\Big(u\varphi_x(x,y) + v\varphi_y(x,y)\Big),$$

is a C^∞ function on $D \times (\mathrm{R}^2 - \{0\})$.

(ii) F_p is a Minkowski norm on T_pS.

F is said to be *reversible* if each F_p is reversible, i.e., $F_p(-y) = F_p(y)$, $y \in T_pS$.

Given a Finsler metric F on a surface S. For a curve C on S, we obtain a length form along C,

$$ds = F\Big(\dot{c}(t)\Big)\,dt,$$

where $c : [a,b] \to C$ is a coordinate map. The length of C is defined by

$$L(C) := \int_a^b ds = \int_a^b F\Big(\dot{c}(t)\Big)\,dt.$$

With the length structure, we obtain a distance d on S defined by

$$d(p,q) := \inf_C L(C), \tag{3.10}$$

where the infimum is taken over all curves on S from p to q. d and F are related by

$$F\Big(\dot{c}(t)\Big) = \lim_{\varepsilon \to 0^+} \frac{d\Big(c(t), c(t+\varepsilon)\Big)}{\varepsilon}. \tag{3.11}$$

Thus any distance on a surface is either induced by a unique Finsler metric or not induced by any Finsler metric (d is said to be singular in this case). Given a distance on a surface, we can use (3.11) to find the Finsler metric that induces it.

Let $S := D$ be an open domain in R^2. S is a special surface in R^2. The canonical coordinate map is given by

$$\varphi(x,y) = (x,y).$$

For a point $p = (x,y) \in S$, the tangent space T_pS can be identified with R^2. Namely, any vector $\mathbf{y} = (u,v) \in \mathrm{R}^2$ is a tangent vector to S at p. A Finsler metric F on S is a function $F(\mathbf{y})$ of tangent vectors $\mathbf{y} = (u,v) \in T_{(x,y)}S$. Thus $F = F(\mathbf{y})$ is a function of $(x,y,u,v) \in D \times \mathrm{R}^2$.

Let S be a surface in V^n and $\varphi : D \to S$ a coordinate map for S. For a Finsler metric F on S, we obtain a function of $(x,y,u,v) \in D \times \mathrm{R}^2$,

$$F := F\Big(u\varphi_x(x,y) + v\varphi_y(x,y)\Big).$$

F is a function of $(x,y,u,v) \in D^2 \times \mathrm{R}^2$. Thus it can be viewed a Finsler metric on D.

Example 3.2.6 Let $f = f(x,y)$ and $h = h(x,y)$ be positive C^∞ functions on an open subset $\mathcal{U} \subset \mathrm{R}^2$. Suppose that f satisfies

$$3 - 2\sqrt{2} < f(x,y) < 3 + 2\sqrt{2}.$$

Let

$$F := h(x,y)\Big\{ \Big(f(x,y)\, u^2 + v^2 \Big) \Big(u^2 + f(x,y)\, v^2 \Big) \Big\}^{\frac{1}{4}}.$$

F is a Finsler metric on $\mathcal{U}$.

Example 3.2.7 (Funk and Klein metrics) Let $\mathcal{U} \subset \mathrm{R}^2$ be a strongly convex domain, that is, there is a Minkowski norm φ on R^2 and a point $\mathbf{x}_o \in \mathcal{U}$ such that

$$\mathcal{U} = \Big\{ \mathbf{y} \in \mathrm{R}^2,\ \varphi(\mathbf{y} - \mathbf{x}_o) < 1 \Big\}.$$

For a point $\mathbf{x} = (x,y) \in \mathcal{U}$ and a vector $\mathbf{y} = (u,v) \in \mathrm{R}^2 - \{0\}$, define $F_{\mathbf{x}}(\mathbf{y}) > 0$ by

$$\mathbf{x} + \frac{\mathbf{y}}{F_{\mathbf{x}}(\mathbf{y})} \in \partial\mathcal{U}. \tag{3.12}$$

Set $F_{\mathbf{x}}(0) = 0$. $F_{\mathbf{x}}$ is a Minkowski norm on $T_{\mathbf{x}}\mathcal{U} = \mathrm{R}^2$. Thus F is a Finsler metric on $\mathcal{U}$. F is a function of $(x,y,u,v) \in \mathcal{U} \times \mathrm{R}^2$, which satisfies

$$F_x = F F_u, \qquad F_y = F F_v. \tag{3.13}$$

Define another function $\tilde{F}$ by

$$\tilde{F}_{\mathbf{x}}(\mathbf{y}) := \frac{1}{2}\Big\{ F_{\mathbf{x}}(\mathbf{y}) + F_{\mathbf{x}}(-\mathbf{y}) \Big\}. \tag{3.14}$$

$\tilde{F}$ is also a Finsler metric on $\mathcal{U}$.

Let d and $\tilde{d}$ denote the Funk distance and Klein distance on $\mathcal{U}$. See (3.5) and (3.6). One can easily verify that

$$\begin{aligned} F_{\mathbf{x}}(\mathbf{y}) &= \lim_{\varepsilon \to 0^+} \frac{d\Big(\mathbf{x}, \mathbf{x} + \varepsilon\mathbf{y}\Big)}{\varepsilon}, \\ \tilde{F}_{\mathbf{x}}(\mathbf{y}) &= \lim_{\varepsilon \to 0^+} \frac{\tilde{d}\Big(\mathbf{x}, \mathbf{x} + \varepsilon\mathbf{y}\Big)}{\varepsilon}. \end{aligned}$$

Thus if d and $\tilde{d}$ are defined by some Finsler metrics on $\mathcal{U}$, they must be F and $\tilde{F}$. By additional work, one can show that the distance functions defined by F and $\tilde{F}$ are indeed d and $\tilde{d}$. Thus we call F and $\tilde{F}$ the *Funk metric* and *Klein metric*, respectively.

Let $\mathcal{U} = \mathbb{D}^2$ be the unit disk in R^2. The Funk metric and Klein metric are given by

$$F = \frac{\sqrt{(xu+yv)^2 + (u^2+v^2)(1-x^2-y^2)} + xu + yv}{1-x^2-y^2} \tag{3.15}$$

$$\tilde{F} = \frac{\sqrt{(xu+yv)^2 + (u^2+v^2)(1-x^2-y^2)}}{1-x^2-y^2}. \tag{3.16}$$

Example 3.2.8 Let (S, α) be a Riemannian surface and $\mathbf{x}$ be a vector field on S with $\alpha(\mathbf{x}) < 1$. Let g denote the family of inner products on tangent planes, which are determined by α,

$$\alpha(\mathbf{y}) = \sqrt{g(\mathbf{y}, \mathbf{y})}, \qquad \mathbf{y} \in T_pS.$$

Suppose that a machine with full power can move around on the surface S at constant speed 1 if there are no other force (such as wind) pushing it. Now consider $\mathbf{x}$ as wind on S. Its speed $\alpha(\mathbf{x})$ in the direction $\mathbf{x}$ is not a constant in general. Let the machine move along a curve C on S. Due to the wind, at a point $p \in C$, the machine moves at the speed $\alpha_p(\mathbf{v})$ in the direction

$$\mathbf{v} := \mathbf{x} + \mathbf{u} \in T_pC,$$

where $\mathbf{u} \in T_pS$ is a unit vector which is the force of the machine (the resulting force is $\mathbf{v}$). Fix a unit tangent vector $\mathbf{y} \in T_pC$ and express $\mathbf{v} = \lambda(p)\mathbf{y}$ for some $\lambda(p) > 0$. Then $\mathbf{u} = \lambda(p)\mathbf{y} - \mathbf{x}$. Observe that

$$1 = \alpha(\mathbf{u})^2 = \lambda(p)^2 - 2\lambda(p)g(\mathbf{x}, \mathbf{y}) + \alpha(\mathbf{x})^2.$$

We obtain

$$\lambda(p) = \sqrt{g(\mathbf{x}, \mathbf{y})^2 + 1 - \alpha(\mathbf{x})^2} + g(\mathbf{x}, \mathbf{y}). \tag{3.17}$$

$\lambda = \lambda(p)$ is a function along C. There is a coordinate map $c : [0, T] \to C$ which coveres the whole C such that

$$\alpha(\dot{c}(\ell)) = \lambda(c(\ell)), \qquad 0 \le \ell \le T. \tag{3.18}$$

T is the time that it takes for the machine to complete trip along C. The *time-length* of C is defined as the time

$$\mathcal{L}(C) := T.$$

$\mathcal{L}$ is a length structure which is different from the original Riemannian length structure defined by α.

The length structure $\mathcal{L}$ is induced by a Finsler metric. To determine it, we take an arbitrary curve C on S. Let $p = c(\ell_o) \in C$, where c is the coodinate map of C defined as above. There is an increasing function on $[0, \varepsilon)$, $\ell = \ell(t)$, with $\ell(0) = \ell_o$ such that $\gamma(t) := c(\ell(t))$ is a unit speed coordinate map with respect to α. We have

$$\ell(t) = \int_0^t F\Big(\dot{\gamma}(\tau)\Big) d\tau.$$

Let $\mathbf{y} = \dot{\gamma}(0) \in T_pC$. Then

$$F(\mathbf{y}) = \ell'(0).$$

Note that by (3.18),

$$1 = \alpha\Big(\dot{\gamma}(0)\Big) = \alpha\Big(\dot{c}(\ell_o)\ell'(0)\Big) = \lambda(p)\ell'(0).$$

We obtain

$$\ell'(0) = \frac{1}{\lambda(p)}.$$

By (3.17) we obtain

$$\begin{aligned} F(\mathbf{y}) &= \frac{1}{\sqrt{g(\mathbf{x},\mathbf{y})^2 + 1 - \alpha(\mathbf{x})^2} - g(\mathbf{x},\mathbf{y})} \\ &= \frac{\sqrt{g(\mathbf{x},\mathbf{y})^2 + 1 - \alpha(\mathbf{x})^2} - g(\mathbf{x},\mathbf{y})}{1 - \alpha(\mathbf{x})^2}. \end{aligned}$$

Note that the above vector $\mathbf{y}$ satisfies $\alpha(\mathbf{y}) = 1$. If $\mathbf{y}$ is an arbitrary vector, then F is given by

$$F(\mathbf{y}) := \frac{\sqrt{g(\mathbf{x},\mathbf{y})^2 + \alpha(\mathbf{y})^2(1 - \alpha(\mathbf{x})^2)} - g(\mathbf{x},\mathbf{y})}{1 - \alpha(\mathbf{x})^2}. \tag{3.19}$$

F is a Randers metric of Funk type. See the following example.

Let $\mathbf{x}$ denote the radial vector field on the unit disk $\mathbb{D}^2$, which is given by

$$\mathbf{x}_p = -(x,y), \qquad p = (x,y) \in \mathbb{D}^2.$$

The Finsler metric defined as above is given by

$$F = \frac{\sqrt{(xu + yv)^2 + (u^2 + v^2)(1 - x^2 - y^2)} + (xu + yv)}{1 - x^2 - y^2}.$$

This is just the Funk metric on $\mathbb{D}^2$. See (3.15). If a machines travels along a ray issuing from the center, it takes infinite time to reach the boundary of the disk, and finite time to reach the center, because the flow points toward to the center and its speed near the boundary is almost equal to that of the machine. This can be verified directly. For any unit vector $\mathbf{y}$, let $\gamma(t) = t\mathbf{y}$, $0 \le t < 1$. For any ε with $0 < \varepsilon < 1$,

$$\int_0^\varepsilon F\Big(\dot{\gamma}(t)\Big) dt < \infty, \qquad \int_\varepsilon^1 F\Big(\dot{\gamma}(t)\Big) dt = \infty.$$

Below is another specific model. A boat travels across the river of equal width. Take a xy-coordinate system such that the x-axis is the shore of the river. The water flow is parallel to the shore. Then the flow vector takes the following form

$$\mathbf{x} = \Big(h(y), 0\Big), \qquad 0 \leq y \leq h_o,$$

where h_o denotes the width of the river. The resulting Randers metric is given by

$$F = \frac{\sqrt{h(y)^2 u^2 + (u^2 + v^2)(1 - h(y)^2)} - h(y)u}{1 - h(y)^2}.$$

Assume that the speed of the boat in still water is a constant $= 1$. The time for which the boat travels along a curve is the length of the curve with respect to F.

3.3 Exercises

Exercise 3.3.1 Show that the set S in R^3 defined by the following equation is a surface, where

$$S := \left\{ (u, v, w) \in \mathrm{R}^3 \;\Big|\; \frac{u^2}{a^2} + \frac{v^2}{b^2} - \frac{w^2}{c^2} = 1 \right\},$$

where $a, b, c > 0$.

Exercise 3.3.2 Let

$$\mathrm{S} := \left\{ (u, v, w) \neq (0,0,0) \;\Big|\; w^2 = u^2 + v^2 \right\}.$$

Show that S is a surface in R^3.

Exercise 3.3.3 Let

$$\varphi(x, y) := \Big(x, y, x^2 + y^2, x^2 - y^2 \Big), \qquad (x, y) \in \mathrm{R}^2.$$

Show that φ is a regular map, hence $S = \varphi(\mathrm{R}^2)$ is a surface in $\mathbb{R}^4$.

Exercise 3.3.4 Let

$$\mathcal{U} := \left\{ (x, y) \in \mathrm{R}^2 \;\Big|\; y > x^2 \right\}.$$

For $\mathbf{x} = (x, y) \in \mathcal{U}$ and $\mathbf{y} = (u, v)$ with $u \neq 0$, define $F = F_{\mathbf{x}}(\mathbf{y}) > 0$ by

$$\mathbf{x} + \frac{\mathbf{y}}{F_{\mathbf{x}}(\mathbf{y})} \in \partial \mathcal{U}.$$

Find a formula for F. F is a function of $(x, y, u, v) \in \mathcal{U} \times (\mathrm{R}^2 \setminus \{(0, v),\ v > 0\})$. Verify that $F = F_{\mathbf{x}}(\mathbf{y})$ satisfies

$$F_x = F F_u, \qquad F_y = F F_v.$$

Exercise 3.3.5 Let

$$F = \frac{\sqrt{(u^2 + v^2) - (xv - yu)^2} + (xu + yv)}{1 - x^2 - y^2}.$$

Verify directly that F satisfies

$$F_x = F F_u, \qquad F_y = F F_v.$$

Exercise 3.3.6 Let a spherical metric on R^2,

$$\alpha := \frac{\sqrt{u^2 + v^2 + (xv - yu)^2}}{1 + x^2 + y^2},$$

and a radial vector field on R^2,

$$\mathbf{x}_p := -\sqrt{1 + x^2 + y^2} \cdot (x, y), \qquad p = (x, y) \in \mathrm{R}^2.$$

Using (3.19), find the Randers metric of Funk type for the pair $\{\alpha, \mathbf{x}\}$.

Chapter 4

Areas on Metric Surfaces

4.1 Area Forms of Metrics

Area is a concept in geometry that represents the amount of space a geometrical shape will occupy. Area is always expressed by a number value that is measured in square units (like square feet or square inches).

Long before the Romans, both the Babylonians and the Egyptians had very practical reasons for learning how to calculate how much space was contained in a plot of land that had certain, fixed-shape boundaries. They first measured the area of a rectangular lot by counting how many same-sized squares could fit inside the rectangle. After doing this many times, someone made the generalization that multiplying the number of squares on each adjoining side was exactly the same operation as counting all the squares inside. For a rectangle, the rule then became length times width. The ancients eventually developed rules for finding the area of triangles and circles and for less symmetrical shapes such as parallelograms and trapezoids. The standard technique for dealing with odd-shaped figures was to subdivide the shape into sections or shapes for which the area could easily be calculated. Thus if they are broken up into rectangles, triangles, or even circles and their individual areas added together, the area of an irregular figure can be determined.

The ancient ideas can be generalized as follows.

Example 4.1.1 The standard Euclidean area form on $\mathbb{R}^2$ is given by

$$dA = dxdy.$$

For a bounded region D in $\mathbb{R}^2$, define the Euclidean area $\mathcal{A}(D)$ of D by

$$\mathcal{A}(D) = \int_D dxdy.$$

For the unit disk $\mathbb{D}^2$,

$$\mathcal{A}(\mathbb{D}^2) = \pi.$$

If

$$(x,y) = \Big(x(\bar{x},\bar{y}), y(\bar{x},\bar{y})\Big) : \bar{D} \to D,$$

then

$$\int_D dxdy = \int_{\bar{D}} \Big|\frac{\partial(x,y)}{\partial(\bar{x},\bar{y})}\Big| \, d\bar{x}d\bar{y},$$

where

$$\Big|\frac{\partial(x,y)}{\partial(\bar{x},\bar{y})}\Big| := \Big|\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}} - \frac{\partial y}{\partial \bar{x}}\frac{\partial x}{\partial \bar{y}}\Big|.$$

Consider the following more general region

$$D := \Big\{(x,y) \in \mathrm{R}^2 \;\Big|\; \sqrt{ax^2 + 2bxy + cy^2} + \lambda x + \mu y < 1\Big\}$$

where

$$a >, \quad c > 0, \quad ac - b^2 > 0 \tag{4.1}$$

$$\kappa := \sqrt{\frac{c\lambda^2 - 2b\lambda\mu + a\mu^2}{ac - b^2}} < 1. \tag{4.2}$$

From linear algebra, there is a linear transformation

$$x = p_{11}\bar{x} + p_{12}\bar{y}, \qquad y = p_{21}\bar{x} + p_{22}\bar{y},$$

such that

$$\Big|\frac{\partial(x,y)}{\partial(\bar{x},\bar{y})}\Big| = p_{11}p_{22} - p_{12}p_{21} = \frac{1}{\sqrt{ac-b^2}}$$

and D is the image of the following domain

$$\bar{D} = \Big\{(\bar{x},\bar{y}) \in \mathrm{R}^2 \;\Big|\; \sqrt{\bar{x}^2 + \bar{y}^2} + \kappa\bar{x} < 1\Big\}.$$

Thus

$$\mathcal{A}(D) = \frac{\mathcal{A}(\bar{D})}{\sqrt{ac-b^2}}.$$

Observe that the equation

$$\sqrt{\bar{x}^2 + \bar{y}^2} + \kappa\bar{x} = 1$$

is equivalent to the equation,

$$(1-\kappa^2)^2\Big(\bar{x} + \frac{\kappa}{1-\kappa^2}\Big)^2 + (1-\kappa^2)\bar{y}^2 = 1.$$

Thus $\bar{D}$ is enclosed by an elliptic curve. Take another transformation

$$\bar{x} = \frac{\tilde{x}}{1-\kappa^2} - \frac{\kappa}{1-\kappa^2}, \qquad \bar{y} = \frac{\tilde{y}}{\sqrt{1-\kappa^2}}.$$

Then $\bar{D}$ is the image of the unit disk

$$\mathbb{D}^2 = \Big\{(\tilde{x},\tilde{y}) \in \mathrm{R}^2 \;\Big|\; \tilde{x}^2 + \tilde{y}^2 < 1\Big\}.$$

Note that

$$\Big|\frac{\partial(\bar{x},\bar{y})}{\partial(\tilde{x},\tilde{y})}\Big| = \frac{1}{\Big(1-\kappa^2\Big)^{3/2}}.$$

We obtain

$$\mathcal{A}(\bar{D}) = \int_{\mathbb{D}^2} \frac{1}{\Big(1-\kappa^2\Big)^{3/2}} d\tilde{x}d\tilde{y} = \frac{\pi}{(1-\kappa^2\Big)^{3/2}}.$$

Finally, we obtain the Euclidean area of D

$$\mathcal{A}(D) = \frac{\pi}{\sqrt{ac-b^2}\Big(1-\kappa^2\Big)^{3/2}} = \frac{(ac-b^2)\pi}{\Big(ac-b^2-c\lambda^2+2b\lambda\mu-a\mu^2\Big)^{\frac{3}{2}}}. \tag{4.3}$$

Let S be a surface and F be a Finsler metric on S. Let $\varphi : D \to S$ be a coordinate map. For a point $(x,y) \in D$, let

$$D_{(x,y)} := \Big\{(u,v) \in \mathrm{R}^2 \;\Big|\; F\Big(u\varphi_x(x,y) + v\varphi_y(x,y)\Big) < 1\Big\}.$$

$D_{(x,y)}$ is a bounded convex region in R^2. Let

$$\sigma(x,y) := \frac{\pi}{\mathcal{A}\Big(D_{(x,y)}\Big)}. \tag{4.4}$$

Define

$$dA := \sigma(x,y)dxdy.$$

Let $\bar{\varphi} : \bar{D} \to S$ be another coordinate map and

$$\bar{D}_{(\bar{x},\bar{y})} := \Big\{(\bar{u},\bar{v}) \in \mathrm{R}^2 \;\Big|\; F(\bar{x},\bar{y},\bar{u},\bar{v}) < 1\Big\}.$$

Assume that $\mathcal{U} = \varphi(D) \cap \bar{\varphi}(\bar{D}) \neq \emptyset$, Then for any point $p = \varphi(x,y) = \bar{\varphi}(\bar{x},\bar{y}) \in S$,

$$\bar{\sigma}(\bar{x},\bar{y})\; d\bar{x}d\bar{y} = \sigma(x,y)\; dxdy. \tag{4.5}$$

Since

$$dxdy = \Big|\frac{\partial(x,y)}{\partial(\bar{x},\bar{y})}\Big| d\bar{x}d\bar{y},$$

(4.5) is equivalent to the following

$$\bar{\sigma}(\bar{x},\bar{y}) = \sigma(x,y)\Big|\frac{\partial(x,y)}{\partial(\bar{x},\bar{y})}\Big|. \tag{4.6}$$

Thus $dA = \sigma(x,y)dxdy$ is well-defined.

Let $f : S \to \mathrm{R}$ be a continuous function on a Finsler surface (S,F). Let

$$\mathrm{supp}(f) := \Big\{p \in S \;\Big|\; f(p) \neq 0.\Big\}.$$

Suppose that supp(f) is contained in a coordinate neighborhood $\varphi(D)$. Define

$$\int_{\varphi(D)} f dA := \int_D f(x,y)\, \sigma(x,y) dxdy,$$

where $f(x,y) = f \circ \varphi(x,y)$ is a function on D. By (4.6), we see that $\int_{\varphi(D)} f dA$ is well-defined.

The number

$$\mathcal{A}(S) := \int_S dA$$

is called the *area* of S. In particular, the area of $S_o = \varphi(D)$ is given by

$$\mathcal{A}(S_o) = \int_{\varphi(D)} dA = \int_D \sigma(x,y)\; dxdy.$$

I. Area Forms on Riemann Surfaces: Let S be a surface and $\varphi : D \to S$ be a coordinate map . Let F be a Riemannian metric on S in the following form

$$F_p(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2}, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$$

where $a = a(x,y), b = b(x,y), c = c(x,y)$ are functions of $(x,y) \in D$. For the domain

$$D = \Big\{ (u,v) \in \mathrm{R}^2 \;\Big|\; \sqrt{au^2 + 2buv + cv^2} < 1 \Big\},$$

according to Example 4.1.1,

$$\mathcal{A}(D) = \frac{\pi}{\sqrt{ac - b^2}}.$$

Thus the area form $dA = \sigma(x,y)dxdy$ is given by

$$\sigma(x,y) = \sqrt{ac - b^2}. \tag{4.7}$$

Area Forms on surfaces in $\mathbb{R}^n$: Let S be a surface in $\mathbb{R}^n$ and $\varphi : D \to \mathbb{R}^n$ be a coordinate map for S. The induced Riemannian metric F on S is given by

$$F_p(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2}, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS,$$

where

$$a = \varphi_x \cdot \varphi_x, \qquad b = \varphi_x \cdot \varphi_y, \qquad c = \varphi_y \cdot \varphi_y.$$

By (4.7), we obtain the area form $dA = \sigma(x,y)dxdy$ of F, where $\sigma(x,y)$ is given by

$$\sigma(x,y) = \sqrt{ac - b^2}.$$

Example 4.1.2 Let S be a graph in $\mathbb{R}^3$ defined by

$$z = f(x,y), \qquad (x,y) \in D.$$

The natural coordinate map for S is given by

$$\varphi(x,y) = \Big(x, y, f(x,y)\Big), \qquad (x,y) \in D.$$

We have

$$\begin{aligned} \varphi_x &= \Big(1, 0, f_x\Big) \\ \varphi_y &= \Big(0, 1, f_y\Big). \end{aligned}$$

Thus

$$(\varphi_x \cdot \varphi_x)(\varphi_y \cdot \varphi_y) - (\varphi_x \cdot \varphi_y)^2 = 1 + [f_x]^2 + [f_y]^2.$$

The area form $dA = \sigma(x,y)dxdy$ under φ is given by

$$\sigma(x,y) = \sqrt{1 + [f_x]^2 + [f_y]^2}.$$

The area of S is given by the following integral

$$\mathcal{A}(S) = \int_D \sqrt{1 + [f_x]^2 + [f_y]^2}\, dxdy.$$

II. Area Forms on Randers Surfaces: Let S be a surface and $\varphi : D \to S$ be a coordinate map . Let F be a Randers metric on S in the following form

$$F_p(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2} + \lambda u + \mu v, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS,$$

where $a = a(x,y), b = b(x,y), c = c(x,y), \lambda = \lambda(x,y), \mu = \mu(x,y)$ are functions of $(x,y) \in D$ satisfying (4.1) and (4.2). Then the area form $dA = \sigma(x,y)dxdy$ is given by

$$\begin{aligned} \sigma(x,y) &= \frac{\Big(ac - b^2 - c\lambda^2 + 2b\lambda\mu - a\mu^2\Big)^{\frac{3}{2}}}{ac - b^2} \\ &= \sqrt{ac - b^2}\Big(1 - \frac{c\lambda^2 - 2b\lambda\mu + a\mu^2}{ac - b^2}\Big)^{\frac{3}{2}}. \end{aligned}$$

The surface area of $S_o = \varphi(D)$ is given by

$$\begin{aligned} \mathcal{A}(S_o) &= \int_D \sqrt{ac - b^2}\Big(1 - \frac{c\lambda^2 - 2b\lambda\mu + a\mu^2}{ac - b^2}\Big)^{\frac{3}{2}} dxdy \\ &\le \int_D \sqrt{ac - b^2}\, dxdy. \end{aligned}$$

Area Forms on surfaces in a Randers space $\mathbb{R}^n_{\mathbf{b}}$: Let $\mathbb{R}^n_{\mathbf{b}} = (\mathbb{R}^n, |\cdot|_{\mathbf{b}})$ be a Randers space, where

$$|\mathbf{v}|_{\mathbf{b}} = |\mathbf{v}| + \mathbf{b}\cdot\mathbf{v} \qquad \mathbf{v} \in \mathrm{R}^n.$$

Let S be a surface in $\mathbb{R}^n_{\mathbf{b}}$. For a coordinate map $\varphi : D \to S$, the induced Randers metric on S is given by

$$F_p(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2} + \lambda u + \mu v, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS,$$

where

$$a = \varphi_x \cdot \varphi_x, \quad b = \varphi_x \cdot \varphi_y, \quad c = \varphi_y \cdot \varphi_y,$$
$$\lambda = \mathbf{b}\cdot\varphi_x, \quad \mu = \mathbf{b}\cdot\varphi_y.$$

The area form $dA = \sigma(x,y)dxdy$ is given by

$$\sigma(x,y) = \sqrt{ac-b^2}\Big(1 - \frac{c\lambda^2 - 2b\lambda\mu + a\mu^2}{ac-b^2}\Big)^{\frac{3}{2}}.$$

4.2 Exercises

Exercise 4.2.1 Find the area of the graph S in $\mathbb{R}^3$ defined by

$$z = \sqrt{x^2 + y^2}, \qquad x^2 + y^2 < 1.$$

Exercise 4.2.2 Let

$$\varphi(x, y) = \Big(x,\ y,\ x^2 + y^2,\ x^2 - y^2\Big).$$

Find an area form $dA = \sigma(x, y)dxdy$ for the surface $S = \varphi(\mathrm{R}^2)$ in $\mathbb{R}^4$. Find the integral formula for the area of $S_o = \varphi(D)$ for the square $D = (0, 1) \times (0, 1)$.

Exercise 4.2.3 Let $(\mathrm{R}^3, |\cdot|_\kappa)$ be a canonical Randers space, where $|\cdot|_\kappa$ is given by

$$|(u, v, w)|_\kappa = \sqrt{u^2 + v^2 + w^2} + \kappa w.$$

Consider a graph S in R^3 given by

$$z = f(x, y).$$

Verify that the area form $dA = \sigma(x, y)dxdy$ of F on S is given by

$$\sigma(x, y) = \frac{\Big[1 + (1 - \kappa^2)([f_x]^2 + [f_y]^2)\Big]^{\frac{3}{2}}}{1 + [f_x]^2 + [f_y]^2}.$$

Exercise 4.2.4 Let S denote the graph of f in $(\mathrm{R}^3, |\cdot|_\kappa)$, where

$$f(x, y) = \frac{-\kappa + \sqrt{1 - (1 - \kappa^2)(x^2 + y^2)}}{1 - \kappa^2},$$

where

$$(x, y) \in D := \Big\{(x, y) \in \mathrm{R}^2 \ \Big|\ x^2 + y^2 < \frac{1}{1 - \kappa^2}\Big\}.$$

Using Exercise 4.2.3, find the area form $dA = \sigma(x, y)dxdy$ of S. Verify that for $\kappa \neq 0$,

$$\mathrm{Area}(S) = \frac{\pi}{\kappa} \ln \frac{1 + \kappa}{1 - \kappa}.$$

Exercise 4.2.5 Let

$$D = (0, 1) \times (0, 1).$$

Let S be a surface and $\varphi : D \to S$ be a coordinate map . Let F be a Randers metric on S given by

$$F_p(\mathbf{y}) = \sqrt{u^2 + \sinh^2(x)v^2} + \tanh(x)\, u, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS.$$

Find the area of $S' = \varphi(D)$.

Chapter 5

Geodesics on Metric Surfaces

5.1 Shortest Paths and Geodesics

Let (S, F) be a Finsler surface. For a curve C on S issuing from p to q, the length of C is given by

$$L(C) := \int_a^b F\Big(\dot{c}(t)\Big) dt,$$

where $c : [a, b] \to S$ is a coordinate map of C with $c(a) = p$ and $c(b) = q$. Assume that C is a shortest curve among those nearby C and issuing from p to q. Take an arbitrary smooth family of curves C_s, $|s| < \varepsilon$, on S issuing from p to q such that $C_0 = C$. Let

$$\mathcal{L}(s) := L(C_s).$$

$\mathcal{L}(s)$ is the length of the curve C_s. By assumption,

$$\mathcal{L}(s) \geq \mathcal{L}(0).$$

This implies that

$$\mathcal{L}'(0) = 0.$$

Let $c : [a, b] \to C$ be a constant speed coordinate map of C. Assume that C is covered by a coordinate map $\varphi : D \to S$. We can express $c(t)$ by

$$c(t) = \varphi\Big(x(t), y(t)\Big).$$

Put

$$L(x, y, u, v) := \frac{1}{2} F^2(\mathbf{y}), \qquad \mathbf{y} = u\varphi_x + v\varphi_y.$$

By the method in the calculus of variations, we know that the functions $x(t)$ and $y(t)$ satisfy the following second order ordinary equations:

$$x''(t) + 2G\Big(x(t), y(t), x'(t), y'(t)\Big) = 0 \tag{5.1}$$

$$y''(t) + 2H\Big(x(t), y(t), x'(t), y'(t)\Big) = 0 \tag{5.2}$$

where $G = G(x, y, u, v)$ and $H = H(x, y, u, v)$ are given by

$$G := \frac{(L_x L_{vv} - L_y L_{uv}) - (L_{xv} - L_{yu})L_v}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{5.3}$$

$$H := \frac{(-L_x L_{uv} + L_y L_{uu}) + (L_{xv} - L_{yu})L_u}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{5.4}$$

We call G and H the *geodesic coefficients* of F.

Definition 5.1.1 A *geodesic* on a Finsler surface (S, F) is a regular map $c : I \to S$ such that its coordinates $(x(t), y(t))$ in a coordinate system (D, φ) satisfy (5.1) and (5.2). The image of a geodesic on S is called a *path*.

For any geodesic $c(t)$ of a Finsler metric F,

$$F(\dot{c}(t)) = constant.$$

The geodesic functions G and H have the following properties

(i) G and H are C^∞ on $D \times (\mathrm{R}^2 - \{0\})$;

(ii) G and H are positively homogeneous of degree two in (u, v), that is, for any $\lambda > 0$,

$$\begin{aligned} G(x, y, \lambda u, \lambda v) &= \lambda^2 G(x, y, u, v), \\ H(x, y, \lambda u, \lambda v) &= \lambda^2 H(x, y, u, v). \end{aligned}$$

(iii) If $\bar{\varphi} : \bar{D} \to S$ is another coordinate map with $\bar{\varphi}(\bar{D}) \cap \varphi(D) \neq \emptyset$, then (G, H) and $(\bar{G}, \bar{H})$ are related by

$$\begin{aligned} 2\bar{G} &= 2G\frac{\partial \bar{x}}{\partial x} + 2H\frac{\partial \bar{x}}{\partial y} - \frac{\partial^2 \bar{x}}{\partial x^2}u^2 - 2\frac{\partial^2 \bar{x}}{\partial x \partial y}uv - \frac{\partial^2 \bar{x}}{\partial y^2}v^2 \\ 2\bar{H} &= 2G\frac{\partial \bar{y}}{\partial x} + 2H\frac{\partial \bar{y}}{\partial y} - \frac{\partial^2 \bar{y}}{\partial x^2}u^2 - 2\frac{\partial^2 \bar{y}}{\partial x \partial y}uv - \frac{\partial^2 \bar{y}}{\partial y^2}v^2. \end{aligned}$$

Let $c(t) = \varphi\Big(x(t), y(t)\Big)$ be a geodesic with $x'(t) > 0$. Then $x(t)$ is an increasing function so that the inverse function $t = t(x)$ exists and $y = y(t) =$

$y(t(x))$ becomes a function of x. By the chain rule, we have

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}, \tag{5.5}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\Big(y'(t)\Big)}{x'(t)} = \frac{y''(t)x'(t) - x''(t)y'(t)}{x'(t)^3}. \tag{5.6}$$

By (5.1) and (5.2), we obtain

$$\frac{d^2y}{dx^2} = \frac{-2H\Big(x(t), y(t), x'(t), y'(t)\Big)x'(t) + 2G\Big(x(t), y(t), x'(t), y'(t)\Big)y'(t)}{x'(t)^3}.$$

Let

$$\Phi := 2\frac{v}{u}G(x, y, u, v) - 2H(x, y, u, v).$$

Φ satisfies

$$\Phi(x, y, \lambda u, \lambda v) = \lambda^2 \Phi(x, y, u, v), \qquad \lambda > 0.$$

Now two ODEs (5.1) and (5.2) are combined into the following ODE after we eliminate the parameter t,

$$\frac{d^2y}{dx^2} = \Phi\Big(x, y, 1, \frac{dy}{dx}\Big). \tag{5.7}$$

If $x'(t) < 0$, we can still view $y(t)$ as a function of x. In this case, two ODEs (5.1) and (5.2) are combined into the following ODE:

$$\frac{d^2y}{dx^2} = \Phi\Big(x, y, -1, -\frac{dy}{dx}\Big). \tag{5.8}$$

Proposition 5.1.1 *Let (S, F) be a Finsler surface and $\varphi : D \to S$ a coordinate map such that the geodesic coefficients G and H are in the following forms*

$$G = P(x, y, u, v)\, u, \qquad H = P(x, y, u, v)\, v.$$

Then any path on the surface must be the image of a straight line under φ.

Proof: Let $c(t) = \varphi\Big(x(t), y(t)\Big)$ be a geodesic. Since

$$\Phi = 2\frac{v}{u}Pu - 2Pv = 0,$$

$y = y(x)$ satisfies

$$\frac{d^2y}{dx^2} = 0.$$

The general solution of the above ODE is that $y = mx + b$. Q.E.D.

Example 5.1.1 Let F be the Funk metric on a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$. Hence it satisfies

$$F_x = F F_u, \qquad F_y = F F_v. \tag{5.9}$$

A direct computation yields

$$G = \frac{1}{2} F\, u, \qquad H = \frac{1}{2} F\, v. \tag{5.10}$$

Thus paths are straight lines in $\mathcal{U}$.

A Finsler surface (S, F) is said to be *positively complete* if every geodesic $c : (a, b) \to S$ can be extended to a geodesic defined on (a, ∞).

Let (S, F) be a positively complete Finsler surface. For a point $p \in S$, define

$$\exp_p : T_p S \to S$$

by

$$\exp_p(\mathbf{y}) = c_{\mathbf{y}}(1),$$

where $c_{\mathbf{y}} : [0, \infty) \to S$ denotes the geodesic with $\dot{c}_{\mathbf{y}}(0) = \mathbf{y}$. We have

$$\exp_p(t\mathbf{y}) = c_{\mathbf{y}}(t), \qquad t \geq 0.$$

We call $\exp_p$ the *exponential map* at p.

Theorem 5.1.1 *Let F be a positively complete Finsler metric on a surface. For any point $p \in S$,*

$$\exp_p : T_p S \to S$$

is an onto map.

I. Geodesics on Riemann Surfaces:

Consider a Riemannian metric F on a surface S,

$$F(\mathbf{y}) = \sqrt{au^2 + 2buv + cv^2}, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_p S,$$

where $a = a(x, y), b = b(x, y)$ and $c = c(x, y)$ are C^∞ functions on D satisfying

$$a > 0, \quad c > 0, \quad ac - b^2 > 0.$$

The geodesic coefficients G and H are given by

$$\begin{aligned} G &= \frac{\Big(ca_x + ba_y - 2bb_x\Big)u^2 + 2\Big(ca_y - bc_x\Big)uv + \Big(2cb_y - cc_x - bc_y\Big)v^2}{4\Big(ac - b^2\Big)} \\ H &= \frac{\Big(2ab_x - ba_x - aa_y\Big)u^2 + 2\Big(ac_x - ba_y\Big)uv + \Big(bc_x + ac_y - 2bb_y\Big)v^2}{4\Big(ac - b^2\Big)}. \end{aligned}$$

Thus G and H are quadratic functions of $(u, v) \in \mathrm{R}^2$.

Example 5.1.2 Consider the following Riemannian metric on a domain $\mathcal{U}$ in R^2,

$$F = e^{\eta/2}\sqrt{u^2+v^2}, \tag{5.11}$$

where $\eta = \eta(x,y)$ is a C^∞ function on $\mathcal{U}$. The geodesic coefficients G and H are given by

$$\begin{aligned} G &= (u^2-v^2)\,\eta_x + 2uv\,\eta_y \\ H &= 2uv\,\eta_x - (u^2-v^2)\,\eta_y. \end{aligned}$$

Example 5.1.3 Let

$$F = \sqrt{\frac{u^2+v^2+(xv-yu)^2}{1+x^2+y^2}}.$$

F is a Riemannian metric on R^2. The geodesic coefficients G and H are given by

$$\begin{aligned} G &= -\frac{1}{2}F^2\,x \\ H &= -\frac{1}{2}F^2\,y. \end{aligned}$$

Thus any geodesic $c(t) = (x(t), y(t))$ with $\lambda = F(\dot{c}(t)) > 0$ must be given by

$$\begin{aligned} x(t) &= a\cosh(\lambda t) + b\sinh(\lambda t) \\ y(t) &= c\cosh(\lambda t) + d\sinh(\lambda t). \end{aligned}$$

Example 5.1.4 Consider the following Riemannian metric on R^2,

$$F = \frac{\sqrt{u^2+v^2+(xv-yu)^2}}{1+x^2+y^2}.$$

The geodesic coefficients G and H are given by

$$G = P\,u, \qquad H = P\,v,$$

where

$$P = -\frac{xu+yv}{1+x^2+y^2}.$$

Thus paths are straight lines.

Example 5.1.5 Let S be the graph in $\mathbb{R}^3$ which is defined by

$$z = f(x, y), \qquad (x, y) \in D.$$

The standard coordinate map of S is given by

$$\varphi(x, y) = \Big(x, y, f(x, y)\Big).$$

The induced Riemannian metric is given by

$$F(\mathbf{y}) := \sqrt{(1 + [f_x]^2)u^2 + 2f_x f_y uv + (1 + [f_y]^2)v^2},$$

where $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$. The geodesic coefficients G and H are given by

$$G = \frac{f_x}{2(1 + [f_x]^2 + [f_y]^2)}\Big(f_{xx}u^2 + 2f_{xy}uv + f_{yy}v^2\Big) \tag{5.12}$$

$$H = \frac{f_y}{2(1 + [f_x]^2 + [f_y]^2)}\Big(f_{xx}u^2 + 2f_{xy}uv + f_{yy}v^2\Big). \tag{5.13}$$

If

$$z = f(x, y) = a(y) + b(y)x,$$

where $a(y)$ and $b(y)$ are functions of y, then

$$H(x, y, u, 0) = 0.$$

Thus the straight lines C_m on the surface S,

$$C_m := \Big\{\Big(x,\ m,\ a(m) + b(m)x\Big)\Big\},$$

are paths.

II. Geodesics on Randers Surfaces: Consider a Randers metric $F = \alpha + \beta$ on a surface S, where

$$\begin{aligned} \alpha(\mathbf{y}) : &= \sqrt{a(x, y)u^2 + 2b(x, y)uv + c(x, y)v^2} \\ \beta(\mathbf{y}) : &= \lambda(x, y)u + \mu(x, y)v \end{aligned}$$

where $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$.

Let

$$\begin{aligned} A &= \lambda_x - \frac{ca_x + ba_y - 2bb_x}{2(ac - b^2)}\lambda - \frac{2ab_x - ba_x - ac_y}{2(ac - b^2)}\mu \\ B &= \frac{1}{2}\Big(\lambda_y + \mu_x\Big) - \frac{ca_y - bc_x}{2(ac - b^2)}\lambda - \frac{ac_x - ba_y}{2(ac - b^2)}\mu \\ C &= \mu_y - \frac{2cb_y - cc_x - bc_y}{2(ac - b^2)}\lambda - \frac{bc_x + ac_y - 2bb_y}{2(ac - b^2)}\mu \\ D &= \frac{1}{2}\Big(\lambda_y - \mu_x\Big). \end{aligned}$$

Let G, H denote the geodesic functions of the Riemannian metric α and $\tilde{G}, \tilde{H}$ denote the geodesic functions of the Randers metric $F = \alpha + \beta$. Then

$$\begin{aligned}
\tilde{G} &= G + \frac{u}{2F}\Big\{Au^2 + 2Buv + Cv^2\Big\} + \frac{D\alpha^2}{(ac-b^2)F}\Big(bu + cv + \alpha\mu\Big) \\
\tilde{H} &= H + \frac{v}{2F}\Big\{Au^2 + 2Buv + Cv^2\Big\} - \frac{D\alpha^2}{(ac-b^2)F}\Big(au + bv + \alpha\lambda\Big).
\end{aligned}$$

Let

$$\tilde{\Phi} := 2\frac{v}{u}\tilde{G}(x,y,u,v) - 2\tilde{H}(x,y,u,v).$$
$$\Phi := 2\frac{v}{u}G(x,y,u,v) - 2H(x,y,u,v).$$

Observe that

$$\tilde{\Phi} = \Phi + \frac{2D\alpha^3}{(ac-b^2)u}. \tag{5.14}$$

Then the paths $y = y(x)$ of F satisfies

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \Phi\Big(x, y, \pm 1, \pm\frac{dy}{dx}\Big) \\
&\pm \frac{\lambda_y(x,y) - \mu_x(x,y)}{a(x,y)c(x,y) - b(x,y)^2}\Big\{a(x,y) + 2b(x,y)\frac{dy}{dx} + c(x,y)\Big(\frac{dy}{dx}\Big)^2\Big\}^{3/2}.
\end{aligned}$$

The sign in the above equation is equal to that of $x'(t)$. We conclude that if

$$\lambda_y = \mu_x,$$

then the paths of $F = \alpha + \beta$ coincide with that of α.

Example 5.1.6 Let $|\cdot|_\kappa$ be the Randers norm on R^3 given by

$$|(u,v,w)|_\kappa = \sqrt{u^2 + v^2 + w^2} + \kappa w.$$

Let S be a graph in $(\mathrm{R}^3, |\cdot|_\kappa)$ given by

$$\varphi(x,y) = \Big(x,\ y,\ f(x,y)\Big).$$

The induced Finsler metric F is given by

$$F_\kappa = \sqrt{a\,u^2 + 2b\,uv + c\,v^2} + \lambda\,u + \mu\,v,$$

where

$$a = 1 + [f_x]^2, \quad b = f_x f_y, \quad c = 1 + [f_y]^2$$
$$\lambda = \kappa f_x, \quad \mu = \kappa f_y.$$

Note that

$$\lambda_y = \mu_x.$$

Thus the paths of F_κ remains unchanged when κ changes.

Example 5.1.7 Consider the following special Randers metric on S

$$F(\mathbf{y}) = \sqrt{u^2 + \phi^2(x)v^2} + \psi(x)u, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS.$$

We have

$$\begin{aligned} G &= -\frac{1}{2}\phi(x)\phi'(x)v^2 + \frac{\psi'(x)u^2 + \psi(x)\phi(x)\phi'(x)v^2}{2\Big(\sqrt{u^2+\phi^2(x)v^2} + \psi(x)u\Big)}\, u \\ H &= \frac{\phi'(x)}{\phi(x)}uv + \frac{\psi'(x)u^2 + \psi(x)\phi(x)\phi'(x)v^2}{2\Big(\sqrt{u^2+\phi^2(x)v^2} + \psi(x)u\Big)}\, v. \end{aligned}$$

5.2 Exercises

Exercise 5.2.1 Let F be a Finsler metric on a domain $D \subset \mathrm{R}^2$. Express a Finsler metric F in the form

$$F = u\, f\Big(x, y, \frac{v}{u}\Big), \qquad u > 0, \tag{5.15}$$

where $f = f(x, y, \xi)$ is a function of $(x, y, \xi) \in D \times \mathrm{R}$. Express the geodesic coefficients G and H of F in the form

$$\begin{aligned} G &= \frac{1}{2}\Theta\Big(x, y, \frac{v}{u}\Big)\; u^2 \\ H &= \frac{1}{2}\Theta\Big(x, y, \frac{v}{u}\Big)\; uv - \frac{1}{2}\Phi\Big(x, y, \frac{v}{u}\Big)\; u^2, \end{aligned}$$

where $\Theta = \Theta(x, y, \xi)$ and $\Phi = \Phi(x, y, \xi)$ are functions of (x, y, ξ). Show that

$$\Phi = \frac{f_y - f_{x\xi} - \xi f_{y\xi}}{f_{\xi\xi}}, \tag{5.16}$$

$$\Theta = \frac{f_y - f_{x\xi} - \xi f_{y\xi}}{f_{\xi\xi}} \cdot \frac{f_\xi}{f} + \frac{f_x + \xi f_y}{f}. \tag{5.17}$$

Exercise 5.2.2 Consider the following Riemannian metric on the right half plane of R^2.

$$F = \frac{1}{x}\sqrt{u^2 + v^2}. \tag{5.18}$$

(a) Let $c(t) = (x(t), y(t))$ be a geodesic of F. Show that $x(t)$ and $y(t)$ satisfy

$$\begin{aligned} x''(t) - \frac{1}{x(t)}\Big(x'(t)^2 - y'(t)^2\Big) &= 0 \\ y''(t) - 2\frac{1}{x(t)}x'(t)y'(t) &= 0. \end{aligned}$$

(b) View a geodesic $c(t) = \Big(x(t), y(t)\Big)$ as a graph of $y = f(x)$. Show that it satisfies

$$\frac{d^2y}{dx^2} = \frac{1}{x}\Big\{\frac{dy}{dx} + \Big(\frac{dy}{dx}\Big)^3\Big\}.$$

(c) Verify that semi-circles

$$x^2 + (y - b)^2 = a^2, \qquad x > 0$$

are paths.

Exercise 5.2.3 Let S be a graph in $\mathbb{R}^4$ given by

$$\varphi(x, y) = \Big(x,\ y,\ x^2 + y^2,\ x^2 - y^2\Big).$$

Find G and H of the induced Riemannian metric F.

Exercise 5.2.4 Consider the following Randers metric $F = \alpha + \beta$ on a domain $\Omega = \{(x, y),\ x^2 + y^2 < 1\}$ in R^2,

$$F = \frac{\sqrt{u^2 + v^2 - (xv - yu)^2} + xu + yv}{1 - x^2 - y^2}.$$

Find the geodesic coefficients G and H of F. Describe the paths of F in Ω ?

Exercise 5.2.5 Let $|\kappa| < 1$. Consider the following Randers metric on a domain $\mathcal{U}$ in R^2.

$$F = \sqrt{\frac{1 + \kappa^2 x^2}{1 - (1 - \kappa^2)x^2} u^2 + x^2 v^2} + \frac{\kappa x u}{\sqrt{1 - (1 - \kappa^2)x^2}}.$$

Show that

$$H(x, y, u, 0) = 0.$$

Explain why any horizontal line (after an appropriate parametrization) is a paths of F.

Exercise 5.2.6 Let

$$F = \sqrt{\frac{u^2 + v^2 - (xv - yu)^2}{1 - x^2 - y^2}}.$$

F is a Riemannian metric on the unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$.

(a) Verify that the geodesics of F are given by

$$\begin{aligned} G &= \frac{1}{2} F^2\, x \\ H &= \frac{1}{2} F^2\, y. \end{aligned}$$

(b) Show that any geodesic $c(t) = (x(t), y(t))$ with $\lambda = F(\dot{c}(t)) > 0$ is given by

$$\begin{aligned} x(t) &= a \cos(\lambda t) + b \sin(\lambda t), \\ y(t) &= c \cos(\lambda t) + d \sin(\lambda t), \end{aligned}$$

where a, b, c and d are constants. Explain why the paths of F are either elliptic curves with center at the origin or straight lines passing through the origin.

Chapter 6

Geometry of Surfaces in R^n

6.1 Normal Curvature, Mean Curvature and Gauss Curvature

In this section, we shall discuss the extrinsic geometry of surfaces in an Euclid space $\mathbb{R}^n$. Let S be a surface in the n-dimensional Euclid space $\mathbb{R}^n$. Let $\varphi : D \to S$ be a coordinate map of S. The induced Riemannian metric is given by

$$F_p(\mathbf{y}) = \sqrt{a\, u^2 + 2b\, uv + c\, v^2},$$

where $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$ and

$$a = \varphi_x \cdot \varphi_x, \qquad b = \varphi_x \cdot \varphi_y, \qquad c = \varphi_y \cdot \varphi_y.$$

This gives rise to an inner product in T_pS given by

$$\mathbf{g}_p(\mathbf{y}, \mathbf{u}) := a\, us + b\,(ut + vs) + c\, vt,$$

where $\mathbf{y} = u\varphi_x + v\varphi_y,\ \mathbf{u} = s\varphi_x + t\varphi_y \in T_pS$.

Definition 6.1.1 For a tangent vector $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$, define

$$\mathbf{A}_p(\mathbf{y}) := \varphi_{xx}u^2 + 2\varphi_{xy}uv + \varphi_{yy}v^2 - 2G\varphi_x - 2H\varphi_y, \tag{6.1}$$

where $G = G(x, y, u, v)$ and $H = H(x, y, u, v)$ are the geodesic coefficients of F. We call $\mathbf{A}_p(\mathbf{y})$ the *normal curvature* of S in the direction $\mathbf{y} \in T_pS$.

To compute the normal curvature, we need the formulas for the geodesic coefficients G and H of F,

$$G \quad = \quad \frac{\Big(ca_x + ba_y - 2bb_x\Big)u^2 + 2\Big(ca_y - bc_x\Big)uv + \Big(2cb_y - cc_x - bc_y\Big)v^2}{4\Big(ac - b^2\Big)}$$

$$H = \frac{\Big(2ab_x - ba_x - aa_y\Big)u^2 + 2\Big(ac_x - ba_y\Big)uv + \Big(bc_x + ac_y - 2bb_y\Big)v^2}{4\Big(ac - b^2\Big)}.$$

Plugging them into (6.1), we obtain

$$\mathbf{A}_p(\mathbf{y}) = \mathbf{a}\, u^2 + 2\mathbf{b}\, uv + \mathbf{c}\, v^2, \tag{6.2}$$

where

$$\begin{aligned}
\mathbf{a} &= \varphi_{xx} - \frac{ca_x + ba_y - 2bb_x}{2\Big(ac - b^2\Big)}\varphi_x - \frac{2ab_x - ba_x - aa_y}{2\Big(ac - b^2\Big)}\varphi_y \\
\mathbf{b} &= \varphi_{xy} - \frac{ca_y - bc_x}{2\Big(ac - b^2\Big)}\varphi_x - \frac{ac_x - ba_y}{2\Big(ac - b^2\Big)}\varphi_y \\
\mathbf{c} &= \varphi_{yy} - \frac{2cb_y - cc_x - bc_y}{2\Big(ac - b^2\Big)}\varphi_x - \frac{bc_x + ac_y - 2bb_y}{2\Big(ac - b^2\Big)}\varphi_y.
\end{aligned}$$

By the above formula, we see that the normal curvature $\mathbf{A}_p(\mathbf{y})$ is quadratic in $\mathbf{y} \in T_pS$.

Remark 6.1.1 If S is a surface in a Minkowski space $(\mathrm{R}^n, \|\cdot\|)$, then the normal curvature $\mathbf{A}_p(\mathbf{y})$ is defined by the same formula as (6.1). In general, the normal curvature $\mathbf{A}_p(\mathbf{y})$ is not quadratic in $\mathbf{y} \in T_pS$, unless the geodesic coefficients $G = G(x, y, u, v)$ and $H = H(x, y, u, v)$ are quadratic in $(u, v) \in \mathrm{R}^2$ for each (x, y).

Lemma 6.1.1 *Assume that S is a surface in an Euclid space $\mathbb{R}^n$. Let $\varphi : D \to S$ be a coordinate map of S. For any $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$, $\mathbf{A}_p(\mathbf{y})$ is perpendicular to T_pS,*

$$\mathbf{A}_p(\mathbf{y}) \cdot \mathbf{v} = 0, \qquad \forall \mathbf{v} \in T_pS. \tag{6.3}$$

Thus the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are perpendicular to T_pS.

Take an arbitrary basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for T_pS (e.g., $\mathbf{e}_1 = \varphi_x, \mathbf{e}_2 = \varphi_y$). Express

$$\begin{aligned}
F_p(\mathbf{y}) &= \sqrt{a\, u^2 + 2b\, uv + c\, v^2}, \\
\mathbf{A}_p(\mathbf{y}) &= \mathbf{a}\, u^2 + 2\mathbf{b}\, uv + \mathbf{c}\, v^2,
\end{aligned}$$

where $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in T_pS$. Define

$$\mathbf{H}_p = \frac{c\mathbf{a} - 2b\mathbf{b} + a\mathbf{c}}{ac - b^2} \tag{6.4}$$

$$\mathbf{K}_p = \frac{\mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{b}}{ac - b^2}. \tag{6.5}$$

$\mathbf{H}_p$ and $\mathbf{K}_p$ are independent of the choice of $\{\mathbf{e}_1, \mathbf{e}_2\}$. But they might depend on the shape of S in $\mathbb{R}^n$. We call $\mathbf{H}_p$ and $\mathbf{K}_p$ the *mean curvature* and the *Gauss curvature*, respectively.

According to Lemma 6.1.1, the mean curvature $\mathbf{H} = \mathbf{H}_p$ is always perpendicular to T_pS, while the Gauss curvature $\mathbf{K} = \mathbf{K}_p$ is a function on S.

Theorem 6.1.1 (Gauss Theorem) *For any surface S in an Euclid space $\mathbb{R}^n$, $\mathbf{K}$ depends only on the induced Riemannian metric F.*

Proof: Let $\varphi : D \to S$ be an arbitrary coordinate map. Observe that

$$\begin{aligned}
\varphi_{xx}\cdot\varphi_x &= \frac{1}{2}\Big(\varphi_x\cdot\varphi_x\Big)_x = \frac{1}{2}a_x\\
\varphi_{xx}\cdot\varphi_y &= \Big(\varphi_x\cdot\varphi_y)_x - \frac{1}{2}\Big(\varphi_x\cdot\varphi_x\Big)_y = b_x - \frac{1}{2}a_y\\
\varphi_{xy}\cdot\varphi_x &= \frac{1}{2}\Big(\varphi_x\cdot\varphi_x\Big)_y = \frac{1}{2}a_y\\
\varphi_{xy}\cdot\varphi_y &= \frac{1}{2}\Big(\varphi_y\cdot\varphi_y\Big)_x = \frac{1}{2}c_x\\
\varphi_{yy}\cdot\varphi_x &= \Big(\varphi_y\cdot\varphi_x\Big)_y - \frac{1}{2}\Big(\varphi_y\cdot\varphi_y\Big)_x = b_y - \frac{1}{2}c_x\\
\varphi_{yy}\cdot\varphi_y &= \frac{1}{2}\Big(\varphi_y\cdot\varphi_y\Big)_y = \frac{1}{2}c_y
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{xx}\cdot\varphi_{yy} - \varphi_{xy}\cdot\varphi_{xy} &= \Big(\varphi_x\cdot\varphi_{yy}\Big)_x - \Big(\varphi_x\cdot\varphi_{xy}\Big)_y\\
&= \Big(b_y - \frac{1}{2}c_x\Big)_x - \Big(\frac{1}{2}b_y\Big)_y\\
&= -\frac{1}{2}\Big(a_{yy} + c_{xx} - 2b_{xy}\Big).
\end{aligned}$$

By (6.5) and the above identities, we obtain

$$\begin{aligned}
\mathbf{K}(\mathbf{y}) &= -\frac{a_{yy} + c_{xx} - 2b_{xy}}{2\Big(ac - b^2\Big)}\\
&\quad + \frac{a_y c_y + c_x c_x - 2b_x c_y}{4\Big(ac - b^2\Big)^2}\, a\\
&\quad + \frac{a_x c_y - a_y c_x + 2b_x b_y - 2a_y b_y - 2b_x c_x}{4\Big(ac - b^2\Big)^2}\, b\\
&\quad + \frac{a_x c_x + a_y a_y - 2a_x b_y}{4\Big(ac - b^2\Big)^2}\, c. \qquad (6.6)
\end{aligned}$$

Thus the Gauss curvature $\mathbf{K}$ defined in (6.5) can be expressed in terms of $a = \varphi_x \cdot \varphi_x$, $b = \varphi_x \cdot \varphi_y$, $c = \varphi_y \cdot \varphi_y$, and their (first and second order) partial derivatives with respect to x and y. This proves the Gauss Theorem. Q.E.D.

By (6.6), the Gauss curvature can be defined for all Riemannian metrics. J. Nash has proved that every Riemannian metric F on a surface S is induced by an Euclid norm. More precisely, (S, F) is isometric to a surface S' in an Euclid space $\mathbb{R}^n$ with the induced Riemannian metric F'.

Theorem 6.1.2 (Gauss-Bonnet) *Let (S, F) be a closed oriented Riemann surface. Then the total Gauss curvature*

$$\int_S \mathbf{K}\, dA = 2\pi\, \chi,$$

where χ is an integer.

The integer χ is called the *Euler number* of S. It is a topological invariant of S, independent of metrics on S. For example,

$$\chi(\mathbb{S}^2) = 2, \qquad \chi(\mathbb{T}^2) = 0, \qquad \cdots$$

Consider a surface S in $\mathbb{R}^3$. Take a normal vector $\mathbf{n}$ at p, i.e., $\mathbf{n}$ is a unit vector such that

$$\mathbf{n} \cdot \mathbf{v} = 0, \qquad \mathbf{v} \in T_pS.$$

Let

$$\Lambda_{\mathbf{n}}(\mathbf{y}) := \mathbf{A}_p(\mathbf{y}) \cdot \mathbf{n}.$$

Then

$$\mathbf{A}_p(\mathbf{y}) = \Lambda_{\mathbf{n}}(\mathbf{y})\, \mathbf{n}.$$

From the definition, we see that $\Lambda_{\mathbf{n}}$ is a quadratic form on T_pS. By linear algebra, there is a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for T_pS such that

$$\begin{aligned} F_p(\mathbf{y}) &= \sqrt{u^2 + v^2} \\ \Lambda_{\mathbf{n}}(\mathbf{y}) &= \kappa_1 u^2 + \kappa_2 v^2, \end{aligned}$$

where $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in T_pS$. κ_1 and κ_2 are called the *principle curvatures* at p with respect to $\mathbf{n}$. We have

$$\mathbf{A}_p(\mathbf{y}) = \Big(\kappa_1 u^2 + \kappa_2 v^2\Big)\, \mathbf{n} = (\kappa_1 \mathbf{n})\, u^2 + (\kappa_2 \mathbf{n})\, v^2.$$

By (6.4) and (6.5), we obtain

$$\mathbf{H}_p = (\kappa_1 + \kappa_2)\mathbf{n}, \qquad \mathbf{K}_p = \kappa_1 \kappa_2.$$

Example 6.1.1 Consider a graph S in $\mathbb{R}^3$ given by

$$z = f(x, y).$$

Take the standard coordinate map

$$\varphi(x, y) = (x, y, f(x, y)).$$

Let

$$\mathbf{y} = u\varphi_x + v\varphi_y = \Big(u,\ v,\ uf_x(x, y) + vf_y(x, y)\Big) \in T_pS.$$

The induced Riemannian metric is given by

$$F(\mathbf{y}) = \sqrt{(1 + [f_x]^2)u^2 + 2f_x f_y uv + (1 + [f_y]^2)v^2}. \tag{6.7}$$

We have

$$\begin{aligned} G &= \frac{f_x}{2\Big(1 + [f_x]^2 + [f_y]^2\Big)}\Big(f_{xx}u^2 + 2f_{xy}uv + f_{yy}v^2\Big) \\ H &= \frac{f_y}{2\Big(1 + [f_x]^2 + [f_y]^2\Big)}\Big(f_{xx}u^2 + 2f_{xy}uv + f_{yy}v^2\Big). \end{aligned}$$

Finally, we get

$$\mathbf{A}_p(\mathbf{y}) = \frac{f_{xx}u^2 + 2f_{xy}uv + f_{yy}v^2}{\sqrt{1 + [f_x]^2 + [f_y]^2}}\ \mathbf{n}, \tag{6.8}$$

where

$$\mathbf{n} := \frac{1}{\sqrt{1 + [f_x]^2 + [f_y]^2}}\Big(-f_x,\ -f_y,\ 1\Big)$$

is a unit vector field perpendicular to T_pS. The mean curvature is given by

$$\mathbf{H} = \frac{(1 + [f_y]^2)f_{xx} - 2f_x f_y f_{xy} + (1 + [f_x]^2)f_{yy}}{\Big(1 + [f_x]^2 + [f_y]^2\Big)^{3/2}}\mathbf{n}. \tag{6.9}$$

The Gauss curvature is given by

$$\mathbf{K} = \frac{f_{xx}f_{yy} - f_{xy}^2}{\Big(1 + [f_x]^2 + [f_y]^2\Big)^2}. \tag{6.10}$$

Consider the sphere $\mathbb{S}^2(r)$ of radius r in $\mathbb{R}^3$. The upper semisphere is the graph of the following function

$$f(x, y) = \sqrt{r^2 - x^2 - y^2}.$$

By the above formulas, we obtain the mean curvature $\mathbf{H}_p$ and the Gauss curvature $\mathbf{K}_p$ at $p = (x, y, z) \in \mathbb{S}^2(r)$,

$$\mathbf{H}_p = \frac{2}{r}\mathbf{n}, \qquad \mathbf{K}_p = \frac{1}{r^2}.$$

To understand the geometric meaning of the principle curvatures, we consider the following graph S in $\mathbb{R}^3$,

$$f(x,y) = \frac{1}{2}\Big(c_1\, x^2 + c_2\, y^2\Big),$$

where c_1, c_2 are constants. At the point $p = (0,0,0) \in S$, $\mathbf{n} = (0,0,1)$ is a normal vector to S. The tangent plane $T_pS = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1 = (1,0,0)$ and $\mathbf{e}_2 = (0,1,0)$. For a vector vector $\mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in T_pS$,

$$\begin{aligned} F_p(\mathbf{y}) &= \sqrt{u^2 + v^2} \\ \mathbf{A}_p(\mathbf{y}) &= \Big(c_1\, u^2 + c_2\, v^2\Big)\mathbf{n}. \end{aligned}$$

Thus the principle curvatures $\kappa_1 = c_1$ and $\kappa_2 = c_2$. We obtain

$$\mathbf{H}_p = (c_1 + c_2)\,\mathbf{n}, \qquad \mathbf{K}_p = c_1 c_2.$$

Example 6.1.2 Consider a surface S in R^4 given by

$$\varphi(x,y) = \Big(x, y, f(x), g(y)\Big).$$

Let

$$\mathbf{y} = u\varphi_x + v\varphi_y = \Big(u, v, uf'(x), vg'(y)\Big).$$

The induced Riemannian metric is

$$F(\mathbf{y}) = \sqrt{(1 + f'(x)^2)u^2 + (1 + g'(y)^2)v^2}.$$

The geodesic coefficients G and H are given by

$$\begin{aligned} G &= \frac{f'(x)f''(x)}{2(1 + f'(x)^2)}u^2 \\ H &= \frac{g'(y)g''(y)}{2(1 + g'(y)^2)}v^2. \end{aligned}$$

The normal curvature is given by

$$\mathbf{A}_p(\mathbf{y}) = \frac{f''(x)}{\sqrt{1 + f'(x)^2}}\mathbf{n}\, u^2 + \frac{g''(y)}{\sqrt{1 + g'(y)^2}}\mathbf{m}\, v^2,$$

where

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{1 + f'(x)^2}}\Big[-f'(x), \;\; 0, \;\; 1, \;\; 0 \Big] \\ \mathbf{m} &= \frac{1}{\sqrt{1 + g'(y)^2}}\Big[\; 0, \;\; -g'(y), \;\; 0, \;\; 1\Big] \end{aligned}$$

are unit vectors that are perpendicular to T_pS. By the above formulas, we obtain the mean curvature and the Gauss curvature

$$\mathbf{H}_p = \frac{1}{\left(1 + f'(x)^2\right)^{3/2}}\ \mathbf{n} + \frac{1}{\left(1 + g'(y)^2\right)^{3/2}}\ \mathbf{m}.$$

$$\mathbf{K}_p = 0.$$

6.2 Exercises

Exercise 6.2.1 Let S be a graph in $\mathbb{R}^3$ defined by

$$z = x^2 - 2y^2.$$

Find the normal curvature $\mathbf{A}_p(\mathbf{y})$ for the vector $\mathbf{y} = (u, v, 2xu - 4yv) \in T_pS$, where $p = (x, y, x^2 - 2y^2)$.

Exercise 6.2.2 Let S be a graph in $\mathbb{R}^3$ defined by

$$z = x^2 + y^2.$$

Find the Gauss curvature $\mathbf{K}$. Show that the Gauss curvature is positive and approaches zero as (x, y) approaches infinity

$$\lim_{x^2+y^2\to\infty} \mathbf{K} = 0.$$

Exercise 6.2.3 Let S be a graph in $\mathbb{R}^3$ defined by

$$z = \sqrt{x^2 + y^2}.$$

Describe the surface. Find the mean curvature $\mathbf{H}$.

Exercise 6.2.4 Let $0 < r < R$. Consider a surface S in $\mathbb{R}^3$ which is obtained by rotating the following circle around the third coordinate line.

$$C = \Big\{(x, 0, z) \;\Big|\; x^2 + z^2 = r^2\Big\}.$$

We can parametrize S by

$$\varphi(x, y) = \Big((R + r\cos x)\cos y,\ (R + r\cos x)\sin y,\ r\sin x\Big),$$

where $(x, y) \in (0, 2\pi) \times (0, 2\pi)$. Find the Gauss curvature $\mathbf{K}$. The surface is almost covered by φ. Verify that

$$\frac{1}{2\pi}\int_S \mathbf{K}\, dA = 0.$$

.

Exercise 6.2.5 Let S be the surface in $\mathbb{R}^4$, which is parametrized by

$$\varphi(x, y) = \Big(x,\ y,\ x^2 - y^2,\ x^2 + y^2\Big).$$

(a) Find the induced Riemannian metric F on S;

(b) Find the normal curvature $\mathbf{A}_p$;

(c) Find the mean curvature $\mathbf{H}_p$;

(d) Find the Gauss curvature $\mathbf{K}_p$.

Chapter 7

Gauss Curvature

7.1 Gauss Curvature of Metrics

Now we are going to define the Gauss curvature for Finsler metrics.

Let (S,F) be a Finsler surface. Let $\varphi : D \to S$ be a coordinate map and

$$L(x,y,u,v) := \frac{1}{2}F^2(\mathbf{y}), \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS.$$

The geodesic coefficients $G = G(x,y,u,v)$ and $H = H(x,y,u,v)$ are given by

$$G : \;=\; \frac{(L_x L_{vv} - L_y L_{uv}) - (L_{xv} - L_{yu})L_v}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{7.1}$$

$$H : \;=\; \frac{(-L_x L_{uv} + L_y L_{uu}) + (L_{xv} - L_{yu})L_u}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{7.2}$$

Now we are ready to introduce the Gauss curvature.

$$\begin{aligned}\mathbf{K}(\mathbf{y}) : \;=\; & \frac{1}{F^2(\mathbf{y})}\Big\{2G_x + 2H_y + 2G_uH_v - 2H_uG_v \\ & -Q^2 - Q_xu - Q_yv + 2GQ_u + 2HQ_v\Big\},\end{aligned} \tag{7.3}$$

where $Q := G_u + H_v$.

Consider a Riemannian metric F on a surface S,

$$F(\mathbf{y}) = \sqrt{a(x,y)u^2 + 2b(x,y)uv + c(x,y)v^2}, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS.$$

The geodesic coefficients G and H of F are given by

$$G \;=\; \frac{\Big(ca_x + ba_y - 2bb_x\Big)u^2 + 2\Big(ca_y - bc_x\Big)uv + \Big(2cb_y - cc_x - bc_y\Big)v^2}{4\Big(ac - b^2\Big)}$$

$$H = \frac{\Big(2ab_x - ba_x - aa_y\Big)u^2 + 2\Big(ac_x - ba_y\Big)uv + \Big(bc_x + ac_y - 2bb_y\Big)v^2}{4\Big(ac - b^2\Big)}.$$

Plugging them into (7.3), we obtain a formula for the Gauss curvature $\mathbf{K}$,

$$\begin{aligned} \mathbf{K}(\mathbf{y}) &= -\frac{a_{yy} + c_{xx} - 2b_{xy}}{2\Big(ac - b^2\Big)} \\ &\quad + \frac{a_y c_y + c_x c_x - 2b_x c_y}{4\Big(ac - b^2\Big)^2}\, a \\ &\quad + \frac{a_x c_y - a_y c_x + 2b_x b_y - 2a_y b_y - 2b_x c_x}{4\Big(ac - b^2\Big)^2}\, b \\ &\quad + \frac{a_x c_x + a_y a_y - 2a_x b_y}{4\Big(ac - b^2\Big)^2}\, c. \end{aligned} \tag{7.4}$$

Proposition 7.1.1 *For a Riemannian metric F on a surface S in an Euclid space $\mathbb{R}^n$, the Gauss curvature $\mathbf{K}(\mathbf{y})$ defined in (6.6) coincides with that defined in (7.3). Moreover, $\mathbf{K}(\mathbf{y})$ can be expressed in terms of a, b, c and their partial derivatives. Thus at any point $p \in S$, the Gauss curvature $\mathbf{K}(\mathbf{y}) = \mathbf{K}_p$ is independent of $\mathbf{y} \in T_pS - \{0\}$.*

Example 7.1.1 Let $z = f(x, y)$ be a C^∞ function on an open subset $\mathcal{U} \subset \mathrm{R}^2$. Let S denote the graph of f in the Eiclid space $\mathbb{R}^3$. The induced Riemannian metric on S is given by

$$F(\mathbf{y}) := \sqrt{(1 + [f_x]^2)u^2 + 2f_x f_y uv + (1 + [f_y]^2)v^2}, \tag{7.5}$$

where $\mathbf{y} = (u, v, uf_x + vf_y) \in T_pS$. By (7.4), we obtain a formula for the Gauss curvature of F,

$$\mathbf{K} = \frac{f_{xx}f_{yy} - f_{xy}f_{xy}}{\Big(1 + [f_x]^2 + [f_y]^2\Big)^2}. \tag{7.6}$$

This is exactly same as the formula in (6.10).

Take

$$f = \sqrt{1 - x^2 - y^2}.$$

Plugging it into (7.5) yields

$$F = \sqrt{\frac{(1 - x^2 - y^2)(u^2 + v^2) + (xu + yv)^2}{1 - x^2 - y^2}}.$$

By (7.6), we obtain

$$\mathbf{K} = 1.$$

Example 7.1.2 Consider the following Riemannian metric F on a surface S. Let $\varphi : D \to S$ be a coordinate map and express

$$F(\mathbf{y}) := \sqrt{a\,u^2 + c\,v^2}, \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS \tag{7.7}$$

where $a = a(x,y)$ and $b = b(x,y)$ are functions of $(x,y) \in D$. An easy computation yields

$$G = \frac{a_x\,u^2 + 2a_y\,uv - c_x\,v^2}{4a} \tag{7.8}$$

$$H = \frac{-c_y\,u^2 + 2c_x\,uv + c_y\,v^2}{4c} \tag{7.9}$$

Using either formula in (7.3), we obtain

$$\mathbf{K} = -\frac{1}{2\sqrt{ac}}\Big[\Big(\frac{c_x}{\sqrt{ac}}\Big)_x + \Big(\frac{a_y}{\sqrt{ac}}\Big)_y\Big]. \tag{7.10}$$

If

$$F = e^{\eta/2}\sqrt{u^2 + v^2}, \tag{7.11}$$

where $\eta = \eta(x,y)$ is a C^∞ function on D. Then

$$\mathbf{K} = -\frac{1}{2}e^{-\eta}\Big(\eta_{xx} + \eta_{yy}\Big). \tag{7.12}$$

Example 7.1.3 Consider the following Riemannian metric F on a domain $\mathcal{U} \subset \mathrm{R}^2$,

$$F = \sqrt{u^2 + \phi^2(x)v^2}.$$

By (7.3), we obtain

$$\mathbf{K} = -\frac{\phi''(x)}{\phi(x)}.$$

Let ϕ be a solution of the following equation

$$\phi''(x) + \lambda\phi(x) = 0, \tag{7.13}$$

where λ is a constant. Then

$$\mathbf{K} = \lambda.$$

Equation (7.13) can be solved. The general solution is given by

$$\phi(x) = a\mathbf{s}_\lambda(x) + b\mathbf{s}'_\lambda(x),$$

where

$$\mathbf{s}_\lambda(x) = \begin{cases} \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} & \text{if } \lambda > 0 \\ x & \text{if } \lambda = 0 \\ \frac{\sinh(\sqrt{-\lambda}x)}{\sqrt{-\lambda}} & \text{if } \lambda < 0 \end{cases} \tag{7.14}$$

Example 7.1.4 (Poincare Disk) Let

$$F := \frac{2\sqrt{u^2+v^2}}{1-x^2-y^2} \tag{7.15}$$

F is a Riemannian metric on the unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$. This Riemannian metric has the following properties:

(i) all rays from the origin have infinite length,

(ii) $\mathbf{K} = -1$.

Example 7.1.5 Let

$$F = \sqrt{\frac{1}{1-x^2}u^2 + x^2v^2}.$$

The Gauss curvature of F satisfies

$$\mathbf{K} = 1.$$

The following proposition is very useful in computing the Gauss curvature.

Proposition 7.1.2 If the geodesic coefficients G and H of a Finsler metric F are in the following form

$$G = Pu, \qquad H = Pv,$$

then

$$\mathbf{K}(\mathbf{y}) = \frac{1}{F^2(\mathbf{y})}\Big\{P^2 - P_x u - P_y v\Big\}. \tag{7.16}$$

Proof: Plugging $G = Pu$ and $H = Pv$ into (7.3) and using $P_u u + P_v v = P$ yield (7.16). We give more details below. First, observe that

$$G_u = P_u u + P, \qquad G_v = P_v u$$

$$H_u = P_u v, \qquad H_v = P_v v + P.$$

This gives

$$Q := G_u + H_v = P_u u + P_v v + 2P = 3P.$$

Pluging them into (7.3), we obtain

$$\begin{aligned}\mathbf{K}(\mathbf{y}) &= \frac{1}{F^2(\mathbf{y})}\Big\{2P_x u + 2P_y v + 2(P_u u + P)(P_v v + P) - 2P_u P_v uv \\ &\quad -9P^2 - 3P_x u - 3P_y v + 6PP_u u + 6PP_v v\Big\} \\ &= \frac{1}{F^2(\mathbf{y})}\Big\{ - P_x u - P_y v - 9P^2 + 2P^2 + 8P(P_u u + P_v v)\Big\} \\ &= \frac{1}{F^2(\mathbf{y})}\Big\{P^2 - P_x u - P_y v\Big\}.\end{aligned}$$

Q.E.D.

Example 7.1.6 (Spherical Plane) Let

$$F = \frac{\sqrt{u^2 + v^2 + (xv - yu)^2}}{1 + x^2 + y^2}.$$

F is a Riemannian metic on R^2. The geodesic coefficients G and H are given by

$$G = P\,u, \qquad H = P\,v,$$

where

$$P = -\frac{xu + yv}{1 + x^2 + y^2}.$$

Thus all paths of F are straight lines in R^2. By (7.16), we obtain

$$\mathbf{K} = 1.$$

Example 7.1.7 (Funk Metric) Let $F = F(x, y, u, v)$ be the Funk metric on a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$. It satisfies

$$F_x = F F_u, \qquad F_y = F F_v.$$

Thus the geodesic coefficients G and H are given by

$$G = \frac{1}{2} F\ u, \qquad H = \frac{1}{2} F\ v.$$

By Proposition 7.1.2, we obtain

$$\begin{aligned} \mathbf{K} &= \frac{1}{F^2}\Big\{\frac{1}{4}F^2 - \frac{1}{2}F_x u - \frac{1}{2}F_y v\Big\} \\ &= \frac{1}{F^2}\Big\{\frac{1}{4}F^2 - \frac{1}{2}F F_u u - \frac{1}{2}F F_v v\Big\} \\ &= \frac{1}{F^2}\Big\{\frac{1}{4}F^2 - \frac{1}{2}F^2\Big\} \\ &= -\frac{1}{4}. \end{aligned}$$

Thus the Gauss curvature $\mathbf{K} = -1/4$.

The Funk metric F on the unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$ is given by

$$F := \frac{\sqrt{(u^2 + v^2) - (xv - yu)^2} + (xu + yv)}{1 - x^2 - y^2}.$$

By the above argument, we know that the geodesics are straight lines and the Gauss curvature of F is equal to $-1/4$.

Example 7.1.8 (Klein Metric) Let $\varphi = \varphi(x, y, u, v)$ be the Funk metric on the strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$ and $\bar{\varphi} := \varphi(x, y, -u, -v)$. φ and $\bar{\varphi}$ satisfy

$$\varphi_x = \varphi\varphi_u, \qquad \varphi_y = \varphi\varphi_v$$

$$\bar{\varphi}_x = -\bar{\varphi}\bar{\varphi}_u, \qquad \bar{\varphi}_y = -\bar{\varphi}\bar{\varphi}_v.$$

The Klein metric F on $\mathcal{U}$ is given by

$$F := \frac{1}{2}\Big(\varphi + \bar{\varphi}\Big) = \frac{1}{2}\Big(\varphi(x, y, u, v) + \varphi(x, y, -u, -v)\Big).$$

The geodesic coefficients G and H of F are in the following form

$$G = P\,u, \qquad H = P\,v,$$

where

$$P = \frac{1}{2}(\varphi - \bar{\varphi}).$$

By Proposition 7.1.2, we obtain

$$\begin{aligned}
\mathbf{K} &= \frac{1}{4F^2}\Big\{(\varphi - \bar{\varphi})^2 - 2(\varphi\varphi_u + \bar{\varphi}\bar{\varphi}_u)u - 2(\varphi\varphi_v + \bar{\varphi}\bar{\varphi}_v)v\Big\} \\
&= \frac{1}{4F^2}\Big\{\varphi^2 - 2\varphi\bar{\varphi} + \bar{\varphi}^2 - 2\varphi^2 - 2\bar{\varphi}^2\Big\} \\
&= -\frac{1}{4F^2}(\varphi + \bar{\varphi})^2 = -1.
\end{aligned}$$

The Klein metric on the unit disk $\mathbb{D}^2 \subset \mathrm{R}^2$ is given by

$$F := \frac{\sqrt{(u^2 + v^2) - (xv - yu)^2}}{1 - x^2 - y^2}. \tag{7.17}$$

By the above argument, the geodesics of F are straight lines and the Gauss curvature of F satisfies

$$\mathbf{K} = -1.$$

Example 7.1.9 Let $\varphi = \varphi(x, y, u, v)$ be the Funk metric on a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$. If a Finsler metric $F = F(x, y, u, v)$ on $\mathcal{U}$ satisfies

$$F_x = (\varphi F)_u, \qquad F_y = (\varphi F)_v.$$

The geodesic coefficients G and H are given by

$$G = \varphi\,u, \qquad H = \varphi\,v.$$

By Proposition 7.1.2, we obtain

$$\begin{aligned}\mathbf{K} &= \frac{1}{F^2}\Big\{\varphi^2 - \varphi_x u - \varphi_y v\Big\}\\ &= \frac{1}{F^2}\Big\{\varphi^2 - \varphi\varphi_u u - \varphi\varphi_v v\Big\}\\ &= \frac{1}{F^2}\Big\{\varphi^2 - \varphi^2\Big\} = 0.\end{aligned}$$

Thus the geodesics of F are straight lines and the Gauss curvature of F satisfies $\mathbf{K} = 0$.

Example 7.1.10 (R. Bryant, 1995) Let

$$\begin{aligned}\alpha &= \frac{\Big(ax^4 + 2x^2 + a\Big)\, u^2 + x^2\Big(x^2 + a\Big)\Big(x^4 + 2ax^2 + 1\Big)\, v^2}{\Big(x^4 + 2ax^2 + 1\Big)^2}\\ \beta &= \frac{u^4 + 2x^2\Big(1 + ax^2\Big)\, u^2v^2 + x^4\Big(x^4 + 2ax^2 + 1\Big)\, v^4}{\Big(x^4 + 2ax^2 + 1\Big)^2}\end{aligned}$$

Define

$$F_a = \sqrt{\frac{\alpha + \sqrt{\beta}}{2}} + \frac{\sqrt{1 - a^2}\, x}{x^4 + 2ax^2 + 1}\, u.$$

F_a is a Finsler metric for $0 < a \leq 1$. Note that when $a = 1$,

$$F_1 = \frac{\sqrt{u^2 + x^2(x^2 + 1)v^2}}{x^2 + 1}$$

is a Riemannian metric , while F_{-1} is not a Finsler metric. For any $0 < a \leq 1$, the Gauss curvature of F_a is always equal to 1, i.e.,

$$\mathbf{K} = 1.$$

The Gauss curvature of $\mathbb{R}^2$ satisfies

$$\mathbf{K} = 0.$$

The meanings of the signs of $\mathbf{K}$ lies in the following comparison theorem with $\mathbb{R}^2$.

Theorem 7.1.1 (Cartan-Hadamard) *Let F be a positively complete Finsler metric on a surface S. Suppose that the Gauss curvature satisfies the bound:*

$$\mathbf{K} \leq 0.$$

Then $\exp_p : T_pS \to S$ *is a non-singular map.*

The Gauss curvature of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ satisfies

$$\mathbf{K} = 1.$$

The diameter of $\mathbb{S}^2$ satisfies

$$\mathrm{Diam}(\mathbb{S}^2) = \pi.$$

Below is a comparison theorem with $\mathbb{S}^2$ for Finsler surfaces satisfying $\mathbf{K} \geq 1$.

Theorem 7.1.2 (Bonnet-Myers) *Let F be a Finsler metric on a closed surface S. Suppose bthat the Gauss curvature satisfies the bound:*

$$\mathbf{K} \geq 1.$$

Then the diameter of S satisfies

$$\mathrm{Diam}(S) := \sup_{p,q \in S} d(p,q) \leq \pi.$$

7.2 Exercises

Exercise 7.2.1 Let S be a surface in $\mathbb{R}^3$ parametrized by

$$\varphi(x,y) = \Big(\cos x,\ 2\sin x,\ y\Big).$$

Verify that the Gauss curvature of the induced Riemann metric vanishes, $\mathbf{K} = 0$.

Exercise 7.2.2 Let S denote the graph of f in $\mathbb{R}^3$, where

$$f(x,y) = k(x^2 - y^2), \qquad (x,y) \in \mathrm{R}^2.$$

Find the Gauss curvature $\mathbf{K}$ of S and evaluate the following integral

$$\chi := \frac{1}{2\pi}\int_S \mathbf{K}\, dA.$$

Exercise 7.2.3 (Beltrami Half-Plane) Let

$$F = \frac{\sqrt{u^2+v^2}}{x}.$$

Show that the Gauss curvature is a constant $\mathbf{K} = -1$.

Exercise 7.2.4 Let

$$F = \sqrt{\frac{u^2+v^2+(xv-yu)^2}{1+x^2+y^2}}.$$

Show that the Gauss curvature of F is a constant $\mathbf{K} = -1$.

Exercise 7.2.5 Let

$$F = \sqrt{u^2+\sinh^2(x)v^2} + \tanh(x)u.$$

Show that the Gauss curvature of F is a constant, $\mathbf{K} = -1/4$.

Exercise 7.2.6 Let

$$F = \frac{\sqrt{u^2+x^2(1-x^2)v^2}+xu}{1-x^2}.$$

Show that the Gauss curvature of F is a constant, $\mathbf{K} = -1/4$.

Exercise 7.2.7 Let

$$F = \frac{\sqrt{(r^2-(x^2+y^2))(u^2+v^2)+r^2(xu+yv)^2} - \sqrt{r^2-1}(xu+yv)}{\sqrt{r^2-x^2-y^2}},$$

where r is a constant with $r \geq 1$. Verify that the Gauss curvature $\mathbf{K}$ of F at the center of the disk, $(x,y) = (0,0)$ is given by

$$\mathbf{K}(\mathbf{y}) = 1 + \frac{3(r^2-1)}{4r^2} \geq 1.$$

Chapter 8

Non-Riemannian Curvatures

8.1 Non-Riemannian Curvatures of Metrics

In this section, we will introduce and discuss several geometric quantities of Finsler metrics. These quantities vanish when the metric is Riemannian. Thus they are called non-Riemannian curvature.

Let (S, F) be a Finsler surface. The Finsler metric F is a family of Minkowski norms F_p on tangent planes T_pS. Thus the Cartan torsion $\mathbf{C}(\mathbf{y})$ is defined for each F_p on T_pS. This is our first non-Euclidean quantity. From the definition, one can see that the Cartan torsion at a point $p \in S$ depends only on the Minkowski norm F_p on T_pS.

Let $\varphi : D \to S$ be a coordinate map and

$$L(x, y, u, v) := \frac{1}{2} F^2(\mathbf{y}), \qquad \mathbf{y} = u\varphi_x + v\varphi_y \in T_pS.$$

The Cartan torsion is given by

$$\mathbf{C}(\mathbf{y}) = \frac{L_{uuu}(-L_v)^3 + 3L_{uuv}(-L_v)^2 L_u + 3L_{uvv}(-L_v)(L_u)^2 + L_{vvv}(L_u)^3}{4L\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)^{\frac{3}{2}}},$$

where $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$. By the homogeneity of L, we can simplify the above formula to the following one.

$$\mathbf{C}(\mathbf{y}) = \frac{2L^2 L_{vvv}}{\Big(2LL_{vv} - (L_v)^2\Big)^{3/2}}, \tag{8.1}$$

where $\mathbf{y} = u\varphi_x + v\varphi_y$ with $u > 0$. We see that F is a Riemannian metric if and only if $\mathbf{C}(\mathbf{y}) = 0$ for all $\mathbf{y} \neq 0$.

In general, given two points p, q on S, the Minkowski planes (T_pS, F_p) and (T_qS, F_q) are not linearly isometric to each other. Namely, the geometric properties of (T_pS, F_p) change over the surface. To study the rate of change of F, we define several new quantities.

First, recall that the geodesic coefficients G and H of F are given by

$$G := \frac{(L_xL_{vv} - L_yL_{uv}) - (L_{xv} - L_{yu})L_v}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{8.2}$$

$$H := \frac{(-L_xL_{uv} + L_yL_{uu}) + (L_{xv} - L_{yu})L_u}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{8.3}$$

The Cartan torsion $\mathbf{C}(\mathbf{y})$ is a function on non-zero tangent vectors $\mathbf{y} \in T_pS$. For a coordinate map $\varphi : D \to S$,

$$\mathrm{C}(x, y, u, v) := \mathbf{C}\Big(u\varphi_x + v\varphi_y\Big).$$

$\mathrm{C} = \mathrm{C}(x, y, u, v)$ is a C^∞ function on $D \times (\mathrm{R}^2 \setminus \{0\})$.

Definition 8.1.1 For a vector $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$, define

$$\mathbf{L}(\mathbf{y}) := \frac{1}{F(\mathbf{y})}\Big\{\mathrm{C}_x u + \mathrm{C}_y v - 2\mathrm{C}_u G - 2\mathrm{C}_v H\Big\}. \tag{8.4}$$

We call $\mathbf{L}(\mathbf{y})$ the *Landsberg curvature* (the L-curvature for short) in the direction $\mathbf{y}$.

Let $c(t)$ be a geodesic on S. Express it by

$$c(t) = \varphi\Big(x(t), y(t)\Big).$$

Then

$$\dot{c}(t) = \varphi_x\Big(x(t), y(t)\Big)x'(t) + \varphi_y\Big(x(t), y(t)\Big)y'(t).$$

Observe that

$$\begin{aligned}
\frac{d}{dt}\Big[\mathbf{C}(\dot{c})\Big] &= \mathrm{C}_x x' + \mathrm{C}_y y' + \mathrm{C}_u x'' + \mathrm{C}_y y'' \\
&= \mathrm{C}_x x' + \mathrm{C}_y y' - 2\mathrm{C}_u G - 2\mathrm{C}_v H \\
&= F(\dot{c})\mathbf{L}(\dot{c}).
\end{aligned}$$

This proves the following

Proposition 8.1.1 *Let $c(t)$ be an arbitrary geodesic on a Finsler surface.*

$$\frac{d}{dt}\Big[\mathbf{C}(\dot{c})\Big] = F(\dot{c})\mathbf{L}(\dot{c}).$$

From Proposition 8.1.1, we see that if a Finsler metric satisfies $\mathbf{L} = 0$, then the Cartan torsion $\mathbf{C}(\dot{c}(t)) = constant$ along any geodesic $c(t)$. Thus the geometry of (T_pS, F_p) does not change too much over the surface.

Below is a useful formula to compute the L-curvature.

Lemma 8.1.1 *The L-curvature can be expressed by the formula*

$$\mathbf{L}(\mathbf{y}) = -\frac{F^3(\mathbf{y})}{2} \cdot \frac{G_{vvv}L_u + H_{vvv}L_v}{\Big(2LL_{vv} - L_vL_v\Big)^{\frac{3}{2}}}, \tag{8.5}$$

where $\mathbf{y} = u\varphi_x + v\varphi_y$ *with* $u > 0$.

There is another simple non-Riemannian quantity which is closely related to the area form. Let F be a Finsler metric on a surface S. Let $\varphi : D \to S$ be a coordinate map. Express the area form by

$$dA = \sigma(x,y)dxdy.$$

Definition 8.1.2 For a vector $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS$, define

$$\mathbf{S}(\mathbf{y}) := \frac{1}{F(\mathbf{y})}\Big\{G_u + H_v - \frac{u}{\sigma}\sigma_x - \frac{v}{\sigma}\sigma_y\Big\}. \tag{8.6}$$

We call $\mathbf{S}(\mathbf{y})$ the S-curvature in the direction $\mathbf{y} \in T_pS$.

Let

$$\tilde{\mathrm{S}}(x,y,u,v) := G_u + H_v - \frac{u}{\sigma}\sigma_x - \frac{v}{\sigma}\sigma_y.$$

$\tilde{\mathrm{S}} = \tilde{\mathrm{S}}(x,y,u,v)$ is a C^∞ function on $D \times (\mathrm{R}^2 \setminus \{0\})$. Define

$$\mathbf{E}(\mathbf{y}) := \frac{1}{2F(\mathbf{y})}\frac{\tilde{\mathrm{S}}_{uu}(-L_v)^2 + 2\tilde{\mathrm{S}}_{uv}(-L_v)(L_u) + \tilde{\mathrm{S}}_{vv}(L_u)^2}{L_{uu}L_{vv} - L_{uv}L_{uv}}. \tag{8.7}$$

We call $\mathbf{E}(\mathbf{y})$ the *E-curvature* in the direction $\mathbf{y}$.

Proposition 8.1.2 *By the S-curvature, we define another quantity.*

$$\mathbf{E}(\mathbf{y}) := \frac{1}{2}\Big\{\mathbf{S}(\mathbf{y}) + \frac{d^2}{dt^2}\Big[\mathbf{S}(\mathbf{y} + t\mathbf{y}^\perp)\Big]|_{t=0}\Big\}. \tag{8.8}$$

Proof: Be definition,

$$\mathbf{E}(\mathbf{y}) = \frac{1}{2F(\mathbf{y})}\frac{d^2}{dt^2}\Big[F(\mathbf{y} + t\mathbf{y}^\perp)\mathbf{S}(\mathbf{y} + t\mathbf{y}^\perp)\Big]|_{t=0}.$$

Note that

$$\frac{d}{dt}\Big[F(\mathbf{y} + t\mathbf{y}^\perp)\Big]|_{t=0} = 0.$$

$$\frac{d^2}{dt^2}\Big[F(\mathbf{y} + t\mathbf{y}^\perp)\Big]|_{t=0} = F(\mathbf{y}).$$

Thus

$$\frac{d^2}{dt^2}\Big[F(\mathbf{y}+t\mathbf{y}^\perp)\mathbf{S}(\mathbf{y}+t\mathbf{y}^\perp)\Big]|_{t=0} = F(\mathbf{y})\Big\{\mathbf{S}(\mathbf{y}) + \frac{d^2}{dt^2}\Big[\mathbf{S}(\mathbf{y}+t\mathbf{y}^\perp)\Big]|_{t=0}\Big\}.$$

This gives (8.8). Q.E.D.

Below is a useful formula to compute the E-curvature.

Lemma 8.1.2

$$\mathbf{E}(\mathbf{y}) = -\frac{F^3(\mathbf{y})}{2}\cdot\frac{G_{vvv}v/u - H_{vvv}}{2LL_{vv} - L_vL_v}, \tag{8.9}$$

where $\mathbf{y} = u\varphi_x + v\varphi_y$ *with* $u > 0$.

Proof: For a vector $\mathbf{y} = u\varphi_x + v\varphi_y$, the conjugate vector is given by

$$\mathbf{y}^\perp = \frac{-L_v\varphi_x + L_u\varphi_y}{\sqrt{L_{uu}L_{vv} - L_{uv}L_{uv}}}.$$

Let $Q = G_u + H_v$. By the homogeneity of G and H, we obtain

$$Q_{uu}(-L_v)^2 + 2Q_{uv}(-L_v)(L_u) + Q_{vv}(L_u)^2 = -\frac{F^4(\mathbf{y})}{u^2}\Big(G_{vvv}v/u - H_{vvv}\Big).$$

Thus

$$\begin{aligned}\mathbf{E}(\mathbf{y}) &= \frac{1}{2F(\mathbf{y})}\frac{d^2}{dt^2}\Big[F(\mathbf{y}+t\mathbf{y}^\perp)\mathbf{S}(\mathbf{y}+t\mathbf{y}^\perp)\Big]|_{t=0}\\ &= \frac{Q_{uu}(-L_v)^2 + 2Q_{uv}(-L_v)(L_u) + Q_{vv}(L_u)^2}{2F(\mathbf{y})\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)}\\ &= -\frac{F^3(\mathbf{y})}{2}\cdot\frac{G_{vvv}v/u - H_{vvv}}{2LL_{vv} - L_vL_v}.\end{aligned}$$

This gives (8.9). Q.E.D.

We know that for any Riemannian metric on a surface, the geodesic coefficients G and H are quadratic in $(u, v) \in \mathrm{R}^2$ in any coordinate system. From (8.5) and (8.9), one can see that $\mathbf{L}(\mathbf{y}) = 0$ and $\mathbf{E}(\mathbf{y}) = 0$ for any vector $\mathbf{y}$ if and only if G and H are quadratic in $(u, v) \in \mathrm{R}^2$. A Finsler metric F satisfying $\mathbf{L} = 0$ and $\mathbf{E} = 0$ is called a *Berwald metric*. As matter of fact, Berwald metrics are either Riemannian or locally Minkowskian, i.e., there is a coordinate map $\varphi : D \to S$ such that

$$F = F\Big(u\varphi_x + v\varphi_y\Big)$$

is independent of $(x, y) \in D$. This fact is due to Z. I. Szabó (1981). However, this fact is not true if the Finsler metric is singular. For example, the following (singular) Finsler metrics on an open subset $\mathcal{U} \subset \mathrm{R}^2$ are Berwald metrics:

$$\begin{aligned} F &= e^{\eta/2} \exp\left[Q \arctan\left(\frac{v}{u}\right)\right]\left(u^2+v^2\right) \\ F &= e^{\eta/2} \frac{v^{1+a}}{u^a} \\ F &= e^{\eta/2} u \exp\left(Q\frac{v}{u}\right), \end{aligned}$$

where $a > 0$ is a constant and $\eta = \eta(x, y)$ is an arbitrary C^∞ on $\mathcal{U}$. F is a C^∞ function on $\mathcal{U} \times \mathrm{R}^2_+$, where $\mathrm{R}^2_+ := \{(u, v) \in \mathrm{R}^2 \mid u > 0\}$. All of these examples are constructed by L. Berwald in 1941.

Example 8.1.1 Let (S, α) be a Riemannian surface and $\mathbf{x}$ be a vector field on S with $\alpha(\mathbf{x}) < 1$. Let g denote the family of inner products on tangent planes, which are determined by α,

$$\alpha(\mathbf{y}) = \sqrt{g(\mathbf{y}, \mathbf{y})}, \qquad \mathbf{y} \in T_pS.$$

Define F by

$$F(\mathbf{y}) := \frac{\sqrt{g(\mathbf{x}, \mathbf{y})^2 + \alpha(\mathbf{y})^2(1-\alpha(\mathbf{x})^2)} - g(\mathbf{x}, \mathbf{y})}{1-\alpha(\mathbf{x})^2}. \tag{8.10}$$

F is a Randers metric of Funk type.

Let α denote the standard Euclidean metric on $\mathbb{R}^2$. Take a vector field $\mathbf{x}$ on the unit disk $\mathbb{D}^2$ given by

$$\mathbf{x}_p = (-y, x), \qquad p = (x, y) \in \mathbb{D}^2.$$

The Randers metric of Funk type on $\mathbb{D}^2$ is given by

$$F = \frac{\sqrt{(-yu+xv)^2 + (u^2+v^2)(1-x^2-y^2)} - (-yu+xv)}{1-x^2-y^2}.$$

At the origin $(0, 0)$, the L-curvature and the E-curvature are given by

$$\mathbf{L}(\mathbf{y}) = -\frac{3}{2}, \qquad \mathbf{E}(\mathbf{y}) = 0,$$

where $\mathbf{y}$ is an arbitrary tangent vector at origin $(0, 0)$. Thus L-curvature and the E-curvature are independent of the directions at the center. The above result suggests that for any tangent vector at any point inside $\mathbb{D}^2$,

$$\mathbf{E}(\mathbf{y}) = 0.$$

By a direct computation, we see that this is actually true.

Let α denote the standard spherical metric on R^2, which is given by

$$\alpha := \frac{\sqrt{(u^2+v^2)(1+x^2+y^2)-(xu+yv)^2}}{1+x^2+y^2}.$$

Take a vector field $\mathbf{x}$ on R^2 given by

$$\mathbf{x}_p = \sqrt{1+x^2+y^2}\cdot(x,y), \qquad p=(x,y)\in\mathrm{R}^2.$$

The Randers metric of Funk type on R^2 is given by

$$F = \sqrt{u^2+v^2} - \frac{xu+yv}{\sqrt{1+x^2+y^2}}.$$

At the origin $(0,0)$, the L-curvature and the E-curvature are given by

$$\mathbf{E}(\mathbf{y}) = -\frac{3}{4}, \qquad \mathbf{L}(\mathbf{y}) = 0,$$

where $\mathbf{y}$ is an arbitrary tangent vector at the origin $(0,0)$. One could expect that $\mathbf{L}(\mathbf{y})=0$ for any tangent vector at any point. However, this is not the case.

Open Problem: Is there a Finsler metric on an open subset $\mathcal{U}\subset R^2$ satisfying

$$\mathbf{L}=0, \qquad \mathbf{E}\neq 0.$$

When the geodesic coefficients take a simple form, we can simplify the formulas for $\mathbf{L}(\mathbf{y})$ and $\mathbf{E}(\mathbf{y})$.

Proposition 8.1.3 *Assume that the geodesic coefficients G and H are in the following form*

$$G = Pu, \qquad H = Pv.$$

Then $\mathbf{L}(\mathbf{y})$ and $\mathbf{E}(\mathbf{y})$ are given by

$$\mathbf{L}(\mathbf{y}) = -\frac{2\sqrt{2}L\Big(L^{3/2}P_{vv}\Big)_v}{\Big(2LL_{vv}-(L_v)^2\Big)^{3/2}} \tag{8.11}$$

$$\mathbf{E}(\mathbf{y}) = \frac{3\sqrt{2}L^{3/2}P_{vv}}{2LL_{vv}-(L_v)^2}, \tag{8.12}$$

where $\mathbf{y}=u\varphi_x+v\varphi_y$ with $u>0$. Thus $\mathbf{E}=0$ implies that $\mathbf{L}=0$.

Proof: Observe that

$$G_{vvv} = P_{vvv}u, \qquad H_{vvv} = P_{vvv}v + 3P_{vv}.$$

Thus

$$\begin{aligned}
G_{vvv}L_u + H_{vvv}L_v &= P_{vvv}u\Big(\frac{2}{u}L - \frac{v}{u}L_v\Big) + \Big(P_{vvv}v + 3P_{vv}\Big)L_v \\
&= P_{vvv}\Big(2L - L_v v\Big) + \Big(P_{vvv}v + 3P_{vv}\Big)L_v \\
&= 2P_{vv}L + 3P_{vv}L_v \\
&= 2L^{-1/2}\Big(L^{3/2}P_{vv}\Big)_v \\
G_{vvv}v - H_{vvv}u &= P_{vvv}uv - \Big(P_{vvv}v + 3P_{vv}\Big)u = -3P_{vv}u.
\end{aligned}$$

Plugging them into (8.5) and (8.9) give (8.11) and (8.12). Q.E.D.

Example 8.1.2 Consider the Funk metric F on a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$. The geodesic coefficients G and H of F are given by

$$G = \frac{1}{2}F\ u, \qquad H = \frac{1}{2}F\ v.$$

We have

$$P = \frac{1}{2}F = \frac{1}{\sqrt{2}}L^{1/2}.$$

A direct computation yields

$$\begin{aligned}
P_v &= \frac{1}{2\sqrt{2}}L^{-1/2}L_v \\
P_{vv} &= -\frac{1}{4\sqrt{2}}L^{-3/2}(L_v)^2 + \frac{1}{2\sqrt{2}}L^{-1/2}L_{vv} \\
P_{vvv} &= \frac{3}{8\sqrt{2}}L^{-5/2}(L_v)^3 - \frac{3}{4\sqrt{2}}L^{-3/2}L_vL_{vv} + \frac{1}{2\sqrt{2}}L_{vvv}.
\end{aligned}$$

Rewrite the formula for P_{vv} as follows

$$P_{vv} = \frac{L^{-3/2}}{4\sqrt{2}}\Big(2LL_{vv} - (L_v)^2\Big). \tag{8.13}$$

Plugging (8.13) into (8.12), we obtain

$$\mathbf{E}(\mathbf{y}) = \frac{3}{4}.$$

By the above identities, we obtain

$$2LP_{vvv} + 3L_vP_{vv} = \frac{L^{1/2}}{\sqrt{2}}L_{vvv}. \tag{8.14}$$

Plugging (8.14) into (8.11) gives

$$\mathbf{L}(\mathbf{y}) = -\frac{\sqrt{2}L^{3/2}L^{1/2}L_{vvv}}{\sqrt{2}\Big(2LL_{vv} - (L_v)^2\Big)^{3/2}} = -\frac{L^2L_{vvv}}{\Big(2LL_{vv} - (L_v)^2\Big)^{3/2}}.$$

Comparing it with the formula for $\mathbf{C}(\mathbf{y})$ in (8.1), we obtain the relation between the Landsberg curvature and the Cartan torsion of the Funk metric.

$$\mathbf{L}(\mathbf{y}) = -\frac{1}{2}\mathbf{C}(\mathbf{y}).$$

Problem: Is there a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$ such that the Funk metric on $\mathcal{U}$ satisfies

$$\sup_{\mathbf{y}} \mathbf{C}(\mathbf{y}) = \infty \ ?$$

Example 8.1.3 Let $F = \frac{1}{2}(\varphi + \bar{\varphi})$ be the Klein metric on a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$, where φ is the Funk metric on $\mathcal{U}$ and $\bar{\varphi}$ is the reverse of φ, i.e.,

$$\bar{\varphi}(x, y, u, v) := \varphi(x, y, -u, -v).$$

The geodesic coefficients G and H are in the following form

$$G = P\, u, \qquad H = P\, v,$$

where

$$P = \frac{1}{2}\Big(\varphi - \bar{\varphi}\Big).$$

First, by (8.1), we have

$$\mathbf{C}(\mathbf{y}) = \frac{(\varphi + \bar{\varphi})(\varphi_{vvv} + \bar{\varphi}_{vvv}) + 3(\varphi_v + \bar{\varphi}_v)(\varphi_{vv} + \bar{\varphi}_{vv})}{2(\varphi + \bar{\varphi})^{1/2}(\varphi_{vv} + \bar{\varphi}_{vv})^{3/2}}. \tag{8.15}$$

Plugging P into (8.12) we obtain

$$\mathbf{E}(\mathbf{y}) = \frac{3}{2} \cdot \frac{\varphi_{vv} - \bar{\varphi}_{vv}}{\varphi_{vv} + \bar{\varphi}_{vv}}. \tag{8.16}$$

Plugging P into (8.11) we obtain

$$\mathbf{L}(\mathbf{y}) = \frac{(\varphi + \bar{\varphi})(\varphi_{vvv} - \bar{\varphi}_{vvv}) + 3(\varphi_v + \bar{\varphi}_v)(\varphi_{vv} - \bar{\varphi}_{vv})}{2(\varphi + \bar{\varphi})^{1/2}(\varphi_{vv} + \bar{\varphi}_{vv})^{3/2}}. \tag{8.17}$$

Open Problem: Is there a Finsler metric on an open domain $\mathcal{U} \subset \mathrm{R}^2$ satisfying

$$\mathbf{L} = 0, \qquad \mathbf{E} \neq 0.$$

Example 8.1.4 Let F be the Funk metric on a strongly convex domain $\mathcal{U} \subset \mathrm{R}^2$. The geodesic coefficients G and H are given by

$$G = \frac{1}{2} F\ u, \qquad H = \frac{1}{2} F\ v.$$

For any point $(x,y) \in \mathcal{U}$, the convex domain in R^2,

$$\Big\{ (u,v) \in \mathrm{R}^2 \ \Big| \ F(x,y,u,v) < 1 \Big\} = \mathcal{U} \setminus \Big\{ (x,y) \Big\} \quad \text{(shifted } \mathcal{U}).$$

Thus

$$\sigma(x,y) = \frac{\pi}{\mathcal{A}\Big\{ (u,v) \in \mathrm{R}^2 \ \Big| \ F(x,y,u,v) < 1 \Big\}} = \frac{\pi}{\mathcal{A}(\mathcal{U})}.$$

We obtain a simple area form $dA = \frac{\pi}{\mathcal{A}(\mathcal{U})}\ dxdy$. Thus

$$\begin{aligned} \mathbf{S}(\mathbf{y}) &= \frac{1}{F} \Big\{ G_u + H_v - \frac{u}{\sigma}\sigma_x - \frac{v}{\sigma}\sigma_y \Big\} \\ &= \frac{1}{2F} \Big\{ \frac{\partial}{\partial u}\Big(Fu \Big) + \frac{1}{2}\frac{\partial}{\partial v}\Big(Fv \Big) \Big\} \\ &= \frac{3}{2}. \end{aligned}$$

That is,

$$\mathbf{S}(\mathbf{y}) = \frac{3}{2}. \tag{8.18}$$

By (8.8) and (8.18), we obtain

$$\mathbf{E}(\mathbf{y}) = \frac{3}{4}. \tag{8.19}$$

Let

$$F := \frac{\sqrt{u^2 + v^2 - (xv - yu)^2} + xu + yv}{1 - x^2 - y^2}.$$

F is the Funk metric on the unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$. By (8.18) and (8.18), we obtain that $\mathbf{S}(\mathbf{y}) = \frac{3}{2}$ and $\mathbf{E}(\mathbf{y}) = \frac{3}{4}$.

Theorem 8.1.1 (Area Comparison Theorem) *Let (S, F) be a positively complete Finsler surface. Let $\mathcal{A}(B(p,r))$ denote the area of the metric ball $B(p,r)$ of radius r around p. We have*

$$\lim_{r \to 0^+} \frac{\mathcal{A}\Big(B(p,r) \Big)}{2\pi \int_0^r e^{\delta t} \mathrm{s}_\lambda(t) dt} = 1.$$

Assume that the Gauss curvature $\mathbf{K}$ *and the S-curvature* $\mathbf{S}$ *satisfy the following bounds*

$$\mathbf{K} \geq \lambda, \qquad \mathbf{S} \geq -\delta.$$

The ratio

$$\frac{\mathcal{A}\Big(B(p,r)\Big)}{2\pi\int_0^r e^{\delta t}\mathrm{s}_\lambda(t)dt}$$

is non-increasing. In particular,

$$\mathcal{A}\Big(B(p,r)\Big) \le 2\pi\int_0^r e^{\delta t}\mathrm{s}_\lambda(t)dt. \tag{8.20}$$

Here $\mathrm{s}_\lambda(t)$ *denotes the solution of the following ODE*

$$y''(t)+\lambda y(t)=0, \qquad y(0)=0,\ y'(0)=1.$$

At the end, let us discuss two more non-Riemannian curvatures from projective geometry. Let G and H be the geodesic coefficients of a Finsler metric in a coordinate system. Define

$$\Phi(x,y,u,v) := 2\frac{v}{u}G(x,y,u,v) - 2H(x,y,u,v).$$

We call Φ the projection function of F. This function Φ is not an invariant. It depends on the choice of a coordinate map. Nevertheless, the paths of F are determined by

$$\frac{d^2y}{dx^2} = \Phi\Big(x,y,\pm 1,\pm\frac{dy}{dx}\Big).$$

For $\mathbf{y} = u\varphi_x + v\varphi_y \in T_pS - \{0\}$, define

$$\mathbf{D}_{\mathbf{y}}(\mathbf{v}) = \frac{1}{6}\Phi_{vvvv}\Big(\mu - \frac{v}{u}\lambda\Big)^3, \qquad \mathbf{v} = \lambda\varphi_x + \mu\varphi_y \in T_pS. \tag{8.21}$$

$\mathbf{D}_{\mathbf{y}} : T_pS \to \mathrm{R}$ is a homogeneous polynomial of degree three on T_pS for any $\mathbf{y} \in T_pS - \{0\}$. We call it the *Douglas curvature.*

Define

$$\mathbf{W}_{\mathbf{y}}(\mathbf{v}) = W\cdot\Big(\mu - \frac{v}{u}\lambda\Big), \qquad \mathbf{v} = \lambda\varphi_x + \mu\varphi_y \in T_pS$$

by

$$\begin{aligned} W \;=\; & \frac{1}{6}u^2\Phi_{xxvv} + \frac{1}{3}uv\Phi_{xyvv} + \frac{1}{6}u\Phi_x\Phi_{vvv} + \frac{1}{3}u\Phi\Phi_{xvvv} \\ & +\frac{1}{6}v^2\Phi_{yyvv} + \frac{1}{6}v\Phi_y\Phi_{vvv} + \frac{1}{3}v\Phi\Phi_{yvvv} + \frac{1}{6}\Phi^2\Phi_{vvvv} \\ & -\frac{1}{2}\Phi\Phi_{yvv} - \frac{1}{6}u\Phi_v\Phi_{xvv} - \frac{1}{6}v\Phi_v\Phi_{yvv} + \frac{2}{3}\Phi_v\Phi_{yv} \\ & -\frac{2}{3}v\Phi_{yyv} - \frac{2}{3}u\Phi_{xyv} - \frac{1}{2}\Phi_y\Phi_{vv} + \Phi_{yy}. \end{aligned} \tag{8.22}$$

$\mathbf{W_y} : T_pS \to \mathrm{R}$ is a linear function on T_pS for any $\mathbf{y} \in T_pS - \{0\}$. We call it the *Berwald-Weyl curvature.*

If two Finsler metrics $\tilde{F}$ and F on a surface S have common paths, or equivalently, their geodesic coefficients are related by

$$\tilde{G} = G + P\,u, \qquad \tilde{H} = H + P\,v,$$

then

$$\tilde{\Phi} = 2\frac{v}{u}\tilde{G} - 2\tilde{H} = 2\frac{v}{u}G - 2H = \Phi.$$

Thus

$$\widetilde{\mathbf{D}}_{\mathbf{y}} = \mathbf{D_y}, \qquad \widetilde{\mathbf{W}}_{\mathbf{y}} = \mathbf{W_y}.$$

That is, the Douglas curvatures of $\tilde{F}$ and F are equal and the Berwald-Weyl curvatures of $\tilde{F}$ and F are equal.

Suppose that the geodesic coefficients G and H are in the following form

$$G = P\,u, \qquad H = P\,v,$$

then

$$\Phi = 2\frac{v}{u}G - 2H = 2\frac{v}{u}Pu - 2Pv = 0.$$

Thus

$$\mathbf{D_y} = 0, \qquad \mathbf{W_y} = 0.$$

The converse is also true. This is due to L. Berwald, J. Douglas and H. Weyl.

Proposition 8.1.4 (Berwald-Douglas-Weyl) *If a Finsler metric F on a surface S satisfies*

$$\mathbf{D_y} = 0, \qquad \mathbf{W_y} = 0, \qquad \mathbf{y} \in T_pS - \{0\},$$

then every point $p \in S$ has a neighborhood covered by a coordinate map for which the geodesic coefficients G and H are in the following form

$$G = P\,u, \qquad H = P\,v.$$

Proposition 8.1.5 *Let $F = \alpha + \beta$ be a Randers metric on an open domain $\mathcal{U} \subset \mathrm{R}^2$, where*

$$\begin{aligned} \alpha &= \sqrt{a(x,y)u^2 + 2b(x,y)uv + c(x,y)v^2}, \\ \beta &= \lambda(x,y)u + \mu(x,y)v. \end{aligned}$$

Then $\mathbf{D_y} = 0$ if and only if

$$\lambda_y = \mu_x. \tag{8.23}$$

If λ and μ satisfy (8.23), then the Berwald-Weyl curvatures of F and α are equal.

Proof: Let $\tilde{\Phi}$ and Φ denote the projective functions of F and α respectively. According to (5.14),

$$\tilde{\Phi} = \Phi + \frac{(\lambda_y - \mu_x)\alpha^3}{(ac - b^2)u}.$$

Since α is a Riemannian metric, $\Phi_{vvvv} = 0$. This implies that $\tilde{\Phi}_{vvvv} = 0$ if and only if (8.23) is satisfied.

Assume that λ and μ satisfy (8.23). Then

$$\tilde{\Phi} = \Phi.$$

Thus

$$\widetilde{\mathbf{W}}_{\mathbf{y}} = \mathbf{W}_{\mathbf{y}}.$$

Q.E.D.

Example 8.1.5 Let

$$F = \frac{\sqrt{u^2 + x^2(1 - x^2)v^2} - xu}{1 - x^2}.$$

We have

$$\Phi = -\Big(\frac{2}{x}u^2 + xv^2\Big)\frac{v}{u}.$$

By a direct computation, we obtain

$$\Phi_{vvvv} = 0, \qquad W = 0.$$

Thus both the Douglas curvature and the Weyl curvature vanish,

$$\mathbf{D}_{\mathbf{y}} = 0, \qquad \mathbf{W}_{\mathbf{y}} = 0.$$

Example 8.1.6 Let

$$F = \frac{\sqrt{u^2 + x^2v^2} - xu}{1 - x^2}.$$

We have

$$\Phi = -\frac{1}{1 - x^2}\Big(\frac{2}{x}u^2 + x(1 + x^2)v^2\Big)\frac{v}{u}.$$

By a direct computation, we obtain

$$\Phi_{vvvv} = 0, \qquad W = 0.$$

Thus both the Douglas curvature and the Weyl curvature vanish,

$$\mathbf{D}_{\mathbf{y}} = 0, \qquad \mathbf{W}_{\mathbf{y}} = 0.$$

The Gauss curvature is not a constant.

(a) If

$$x = 0.9, \qquad u = 1.1, \qquad v = 3.2,$$

then

$$\mathbf{K} \approx -2.69347348$$

(b) If

$$x = 0.9, \qquad u = 0.5, \qquad v = -1.0,$$

then

$$\mathbf{K} \approx -3.925811359$$

8.2 Exercises

Exercise 8.2.1 Let

$$F = \Big\{ u^4 + 3c\, u^2 v^2 + v^4 \Big\}^{\frac{1}{4}},$$

where $c = c(x,y)$ is a C^∞ function on a domain $\mathcal{U} \subset \mathrm{R}^2$ satisfying $0 < c < 2$. Thus F is a Finsler metric on $\mathcal{U}$. Find the Landsberg curvature of F and show that $\mathbf{L} = 0$ if and only if $c = constant$.

Exercise 8.2.2 Consider the following Funk metric on the unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$,

$$F = \frac{\sqrt{u^2 + v^2 - (xv - yu)^2} - xu - yv}{1 - x^2 - y^2}.$$

Compute $\mathbf{C}(\mathbf{y})$, $\mathbf{E}(\mathbf{y})$ and $\mathbf{L}(\mathbf{y})$. What is the relationbship between $\mathbf{C}(\mathbf{y})$ and $\mathbf{L}(\mathbf{y})$?

Exercise 8.2.3 Consider the following Randers metric on $(0, \infty) \times \mathrm{R}$,

$$F = \sqrt{u^2 + \sinh^2(x) v^2} + \tanh^2(x) u.$$

(a) Find the geodesic coefficients G and H;

(b) Compute $\mathbf{C}(\mathbf{y})$, $\mathbf{E}(\mathbf{y})$ and $\mathbf{L}(\mathbf{y})$;

(c) Verify that

$$\mathbf{E}(\mathbf{y}) = \frac{3}{4}, \qquad \mathbf{L}(\mathbf{y}) = -\frac{1}{2}\mathbf{C}(\mathbf{y}).$$

Exercise 8.2.4 Let

$$F = \frac{\sqrt{u^2 + x^2(1 - x^2)v^2} + xu}{1 - x^2}.$$

F is a Randers metric on the strip $\mathcal{U} \subset \mathrm{R}^2$,

$$\mathcal{U} := \Big\{ (x,y) \in \mathrm{R}^2 \ \Big|\ |x| < 1 \Big\}.$$

(a) Find the geodesic coefficients G and H;

(b) Show that for each (x,y) with $|x| < 1$, the region $D_{(x,y)}$ enclosed by the indicatrix, $F = 1$, is the region enclosed by the following ellipse

$$C = \Big\{ (u,v) \in \mathrm{R}^2 \ \Big|\ \Big(u + x\Big)^2 + x^2 v^2 = 1 \Big\}.$$

Fins the Euclidean area $\mathcal{A}(D_{(x,y)})$ of $D_{(x,y)}$ and the area form $dA = \sigma(x,y)\, dxdy$ of F on $\mathcal{U}$;

(c) Verify that

$$\mathbf{S}(\mathbf{y}) = \frac{3}{2}.$$

Exercise 8.2.5 Let

$$F = \sqrt{u^2 + \sinh^2(x)v^2} - \tanh(x)u.$$

Show that $\mathbf{D_y} = 0$ and $\mathbf{W_y} = 0$.

Exercise 8.2.6 Let

$$F = \frac{\sqrt{(r^2 - (x^2+y^2))(u^2+v^2) + r^2(xu+yv)^2} - \sqrt{r^2-1}(xu+yv)}{\sqrt{r^2 - x^2 - y^2}},$$

where r is a constant with $r \geq 1$. F is a Randers metric on the disk of radius r around the origin. Verify that at the origin $(0,0)$,

$$\mathbf{E}(\mathbf{y}) = -\frac{3\sqrt{r^2-1}}{4r}, \qquad \mathbf{L}(\mathbf{y}) = 0,$$

where $\mathbf{y}$ is an arbitrary tangent vector at $(0,0)$.

Exercise 8.2.7 Let

$$F = \sqrt{u^2+v^2} - \frac{-yu+xv}{\sqrt{r^2+x^2+y^2}},$$

where $r > 0$. F is a Randers metric on R^2. Verify that at the origin $(0,0)$,

$$\mathbf{E}(\mathbf{y}) = 0, \qquad \mathbf{L}(\mathbf{y}) = -\frac{3}{2r},$$

where $\mathbf{y}$ is an arbitrary tangent vector at $(0,0)$.