Partial Differential Equations
for
Engineers and Scientists

Problem Book I

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Preface

This collection of exercises contains over 150 explicitly solved problems for linear partial differential equations (PDEs) and boundary value problems based on more than 10 years of experience in teaching beginner PDE courses at several North American universities.

A multitude of excellent introductory textbooks are available on PDEs, such as the monographs by Haberman [1], Churchill and Brown [2], Debanth, Mint-U [3], and Asmar [4], among others. Although these books give a concise, detailed, and easily accessible introduction to the theory of linear PDEs, they provide only a limited number of solved examples. When teaching from these textbooks, we are always asked by students for additional problems. Many students in these courses are studying engineering or another science and can be easily confused by abstract constructions. These students tend to benefit from a more drill-like repetition of problems and solutions. Here we address exactly this need. The problems in our textbook are all completely solved and explained in great detail. Any student in a corresponding course should have no trouble understanding the problems.

The final two chapters of this textbook contain four sample midterm exams and four sample final exams. These sample exams are all real exams that were given between 2004 and 2009 at the University of Alberta. They provide students with a useful guideline of what to expect as well as an opportunity to test their abilities.

To help students use this book, we incorporated two special features. First, we rank the problems according to their difficulty. Of course, this is a subjective task. Still, it gives a good indication of the anticipated level of difficulty. We use

rank 0		for very simple problems
rank 1	✗	for simple problems
rank 2	✗✗	for more involved problems
rank 3	✗✗✗	for difficult problems.

To be successful in a classical PDE course, you should be able to solve some of the **rank 2** problems. Candidates for an A grade should be able to solve some of the **rank 3** problems. The other feature of this textbook is a detailed table of contents that lists all of the problems. With this, an instructor or student can keep track of which problems were discussed in class, included in an assignment, or given in an exam. Students can also make notes such as **mastered** or **not yet mastered** or **revisit later** and so forth.

We hope this textbook will provide useful assistance to all interested in learning to solve linear PDEs. We have chosen not to include yet another description of the underlying theory, since several excellent theoretical textbooks are already on the market. Some titles are mentioned above. Nevertheless, we expect these problems will help you understand linear PDEs and enable you to get a glimpse of the beautiful theory behind them.

Ed and Thomas,
Edmonton, November 2009

Part I

Theory

Chapter 1

Introduction

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Many physical, biological and engineering problems can be expressed mathematically by means of partial differential equations (abbreviated PDEs) together with initial and/or boundary conditions. PDEs are used in basically all scientific areas, for example, the Schrödinger equation in quantum mechanics, Maxwells equations in electrodynamics, the reaction-diffusion equations chemistry and in mathematical biology, models for spatial spread of populations, heat conduction problems, and also the Black-Scholes formula for financial markets. A mathematical definition of PDEs is quite simple, since a PDE is an equation which involves partial derivatives. The fascinating aspect of partial differential equations is that they can be classified into three PDE kingdoms which are called **elliptic, parabolic** and **hyperbolic**. Each of these kingdoms has a king, that is, a simple equation which shows all of the properties which are typical for this group. Elliptic equations are represented by the **Laplace equation**, parabolic equations are represented by the **heat equation** and hyperbolic equations are represented by the **wave equation**. The classification is defined for linear second order equation in Section **??**, the properties of these types, however, carry much further and also higher order equations behave "wave-like" or "diffusion-like". The study of the three basic equations which represent the three subgroups is the content of this course. If you learn the methods for the wave equation, you will be able to study fluid flow in a pipeline, and the Schrödinger equation to gain an understand of quantum mechanics. Laplaces equation is a prototype for Maxwells equations in electrostatics, two dimensional fluid flow, and the statics of buildings and bridges. The theory of the heat equation prepares you for the study of reaction-diffusion equations in population biology and for heat flow problems in conducting materials.

In this course we will deal almost exclusively with **linear** partial differential equations (the simplest type) and we will primarily use one technique for solving them. This technique is called **separation of variables**. This technique involves reducing (i.e. simplifying) the PDEs to ordinary differential equations (abbreviated ODEs), which then can be solved using ODE methods.

Generally, a PDE will have infinitely many solutions. To isolate a unique solution, we will introduce side conditions (auxiliary conditions) which typically appear as **initial conditions** and **boundary conditions**. Before we dick into the theory, we recall some basic facts about functions and their partial derivatives. Then we can define the concepts of elliptic, parabolic and hyperbolic PDEs in a proper manner.

1.1 Partial Differential Equations

Let $f : \Omega \to \mathbb{R}$ be a function defined on an open set $\Omega \subset \mathbb{R}^2$. The partial derivatives of $f(x, y)$ are defined as

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ \frac{\partial}{\partial y} f(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \end{aligned}$$

provided the limits exist. Thus, we differentiate with respect to one of the variables while holding the other variable fixed. Alternative notations, which we use in this text include

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f(x,y)}{\partial x} = f_x(x,y), \quad \frac{\partial f}{\partial y}(x,y) = \frac{\partial f(x,y)}{\partial y} = f_y(x,y)$$

or even simpler

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial y} = f_y$$

Further notations, which we will not use here but which can be found in other textbooks include

$$f_x = f_1 = \partial_x f = D_1 f, \quad f_x = f_2 = \partial_y f = D_2 f.$$

As a rule of thumb in mathematics you will observe that the more important a concept is, the more notations it has. Hence partial derivatives are quite important!

In general, a **partial differential equation** for an unknown function $u(x, y)$, $u(x, y, z)$ or $u(x, y, z, t)$, ...etc ...can be written as a general function

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, \cdots) = 0,$$

We now define some terminology. In the above equations u (the unknown function) is referred to as the **dependent** variable with all remaining variables x, y, z, t being called the **independent** variables. A PDE in some unknown function u, is called **linear** if the equation is of first degree in u and the derivatives of u. A PDE is called **homogeneous** if for a solution u each scalar multiple αu is also a solution. The **order** of a PDE refers to the order of the highest derivative that appears in the equation. Finally, the **dimension** of a PDE refers to the number of independent variables present. If there is a clear distinction between time and space variables, then dimension is also used for the spatial part alone.

Exercise 1.1.
Find dimension and order of the following partial differential equations. Which if these are linear and which are homogeneous?

$$\text{heat equation} \qquad u_t = Du_{xx} + f(x) \tag{1.1}$$

$$\text{wave equation} \qquad u_{tt} - c^2 u_{xx} = 0 \tag{1.2}$$

$$\text{Laplace equation} \qquad u_{xx} + u_{yy} = 0 \tag{1.3}$$

$$\text{advection equation} \qquad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{1.4}$$

$$\text{(no name)} \qquad \frac{\partial^2 u}{\partial x^2} + e^y \sin z \, \frac{\partial^2 u}{\partial x \partial z} = u, \tag{1.5}$$

$$\text{(no name)} \qquad \frac{\partial^2 u}{\partial x \partial y} = \sin u, \tag{1.6}$$

$$\text{KdV equation} \qquad u_t + uu_{xx} + u_{xxx} = 1. \tag{1.7}$$

Solution:

- Equation (1.1) is a 2–dimensional, 2nd order, linear, (non-homogeneous for $f \neq 0$) PDE. It is sometimes called one-dimensional heat equation, since the space variable x is one-dimensional.
- Equation (1.2) is a 2–dimensional, 2nd order, linear, homogeneous PDE, which is sometimes called one dimensional wave equation, since the space variable x is one-dimensional.
- Equation (1.3) is a 2–dimensional, 1st order, linear, homogeneous PDE.
- Equation (1.4) is a 2–dimensional, 1st order, linear, homogeneous PDE.
- Equation (1.5) is a 3–dimensional, 2nd order, linear, homogeneous PDE.
- Equation (1.6) is a 2–dimensional, 2nd order, nonlinear, homogeneous PDE.
- Equation (1.7) is a 2–dimensional, 3rd order, nonlinear, non-homogeneous PDE.

1.2 Classification of linear, second order PDEs

The classification into the kingdoms of elliptic, parabolic and hyperbolic can be obtained from the study of linear, second order PDEs. A general homogeneous linear second order PDE can be written as

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0, \tag{1.8}$$

with real coefficients a, b, c, d, e, f (The factor 2 in front of the b-term is just a convention, so that in the end it looks nicer). The type of this equation is defined by its **principal part**, which are the highest order terms

$$au_{xx} + 2bu_{xy} + cu_{yy}.$$

This expression can be written in an abstract matrix notation as

$$\left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} u.$$

Here we just pretend that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are symbols which can be entered as components of a vector. The interpretation is that this vector is applied as derivatives on the function $u(x, y)$:

$$\begin{aligned} \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} u &= \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \\ &= \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right)\begin{pmatrix} au_x + bu_y \\ bu_x + cu_y \end{pmatrix} \\ &= au_{xx} + bu_{yx} + bu_{xy} + cu_{yy} \\ &= au_{xx} + 2bu_{xy} + cu_{yy} \end{aligned}$$

For the last equality we use the assumption that $u(x, y)$ is twice continuously differentiable such that the mixed derivative are identical, $u_{xy} = u_{yx}$. The matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is called **coefficient matrix** of the PDE, or the **symbol** of the PDE. The classification of PDE's is based on the relative sign of the eigenvalues of the symbol. Notice that the symbol is symmetric, hence thei eigenvalues are real (not complex). As in good old linear algebra, the determinant of this matrix gives the product of the eigenvalues:

$$\lambda_1\lambda_2 = \det A = ac - b^2$$

tells us a lot about the type of equation.

Definition 1. *The PDE (1.8) is said to be*

- **elliptic** *if and only if* $ac - b^2 > 0$, *i.e. the eigenvalues of* A *have the same sign and are not zero (both positive or both negative).*
- **parabolic** *if and only if* $ac - b^2 = 0$, *i.e. at least one eigenvalue is* 0.
- **hyperbolic** *if and only if* $ac - b^2 < 0.$, *i.e. the eigenvalues have opposite sign and are non zero.*

Now we have defined our PDE kingdoms. Next we introduce the corresponding rulers of these kingdoms.

- The **Laplace equation** in two dimensions reads $u_{xx} + u_{yy} = 0$. It's symbol is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with determinant $\det A = 1 > 0$, hence the Laplace equation is elliptic.

- The **heat equation** in one (spatial) dimension reads $u_t = k u_{xx}$. It's symbol is $A = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ with determinant $\det A = 0$, hence the heat equation is parabolic.

- The **wave equation** in one (spatial) dimension reads $u_{tt} - c^2 u_{xx} = 0$. It's symbol is $A = \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix}$ with determinant $\det A = -c^2 < 0$, hence the wave equation is hyperbolic.

Exercise 1.2.
Classify the following linear 2nd order PDEs.

1. $u_t + 2u_{tt} + 3u_{xx} = 0$
2. $17u_{yy} + 3u_x + u = 0$
3. $4u_{xy} + 2u_{xx} + u_{yy} = 0$
4. $u_{yy} - u_{xx} - 2u_{xy} = 0$

Solution:

1. Symbol $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\det A = 6 > 0$, elliptic.
2. Symbol $A = \begin{pmatrix} 0 & 0 \\ 0 & 17 \end{pmatrix}$, $\det A = 0$, parabolic.
3. Symbol $A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $\det A = -2 < 0$, hyperbolic.
4. Symbol $A = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$, $\det A = -2 < 0$, hyperbolic.

One would expect that a classification scheme should not depend on the coordinate system in which the PDE is expressed. To see that this is the case, consider a change of independent variable:

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y).$$

The transformation is nonsingular if the Jacobian of the transformation is nonzero, i.e. if

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0.$$

Let us denote the transformed dependent variable as $w(\xi,\eta) = u(x,y)$. Then the PDE (1.8) becomes

$$\alpha w_{\xi\xi} + 2\beta w_{\xi\eta} + \gamma u_{\eta\eta} + \delta u_\xi + \epsilon u_\eta + fu = 0,$$

where

$$\begin{aligned}
\alpha &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\
\beta &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y, \\
\gamma &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\
\delta &= d\xi_x + e\xi_y \\
\epsilon &= d\eta_x + e\eta_y
\end{aligned}$$

Computing the symbol for A for the transformed equation and computing its determinant gives

$$detA = \alpha\gamma - \beta^2 = (ac - b^2)\,(\det J)^2.$$

The sign of $\alpha\gamma - \beta^2$ is the same as the sign of $ac - b^2$, hence the classification of PDEs is invariant under a change of coordinates.

1.3 Side Conditions

Remember from the methods for ODEs, that solving linear ordinary differential equations one usually finds a "general" solution which involves a number of undetermined constants. To find these constants, some side conditions are needed. Quite often initial conditions are used to identify a unique solution. This idea is similar for PDEs. Also here, the PDE alone does not give rise to uniqueness as can be seen for the 2-dimensional Laplace equation:

EXAMPLE 1.1. Take the 2–dimensional Laplace equation:

$$u_{xx} + u_{yy} = 0.$$

This equation is a 2nd order, linear partial differential equation. Solutions of this equation include:

$$\begin{aligned}
&u(x,y) = cxy, && u(x,y) = c\sin(nx)\cosh(ny), \\
&u(x,y) = c(x^2 - y^2), && u(x,y) = c\,e^{-y}\,\cos(x), \\
&u(x,y) = c(x^3 - 3xy^2), && u(x,y) = c\ln(x^2 + y^2), \\
&u(x,y) = c(x^4 - 6x^2y^2 + y^4), && u(x,y) = c\tan^{-1}(y/x), \\
&u(x,y) = c(x^5 - 10x^3y^2 + 5xy^4) && u(x,y), = c\,e^{\sin(x)\cosh(y)}\,\sin(\cos x\,\sinh y),
\end{aligned}$$

where in each case the constant c is arbitrary. The list goes on. Polynomial solutions of any order exist, as do solutions involving various combinations of exponential and trigonometric functions and others. Linear combinations of solutions are again solutions. In fact, we can *not* write down "the" general solution of Laplace's equation without specifying side consitions.

We have just seen that partial differential equations alone often have infinitely many solutions. In order to get a unique solution for a particular problem, additional conditions must be applied. These auxiliary conditions are typically of two types: **initial conditions** and **boundary conditions**.

Initial conditions are typically given at a chosen start time, usually at $t = 0$. Boundary conditions are used to describe what the system does on the domain boundaries. As example, we introduce classical boundary conditions for the heat equation. Let $\Omega \subset \mathbb{R}^n$ be a given piecewise smooth domain. Piecewise smooth means that at all but a finite number of points there exist a unique normal vector to the boundary $\partial\Omega$. The heat equation on Ω reads

$$u_t = k\Delta u.$$

1. **Initial condition**: We prescribe the initial temperature distribution in Ω at some specific time (usually at time $t = 0$). In 3 dimensions this takes the form $u(x, y, z, 0) = u_0(x, y, z)$.

2. **Boundary condition**: give conditions on the boundary $\partial\Omega$ for all time. Boundary conditions are divided into three categories:

   (a) **Dirichlet Condition**: We prescribe u on $\partial\Omega$. This takes the form $u(x, y, z, t) = g(x, y, z, t)$ for $(x, y, z) \in \partial\Omega$. A common example are **homogeneous Dirichlet boundary conditions** $u(x, y, z, t) = 0$ on $\partial\Omega$.

   (b) **Neumann Condition**: We prescribe the heat flow through the boundary $\partial\Omega$. This takes the form $\frac{\partial u}{\partial n} = g$ for $(x, y, z) \in \partial\Omega$, where $\frac{\partial u}{\partial n} = \vec{\nabla} u \cdot \vec{n}$ is a directional derivative ($\vec{n}$ being the outward pointing unit normal to $\partial\Omega$). A common example is no heat flow through the boundary (representing perfect insulation). These are called **homogeneous Neumann boundary conditions**: $\frac{\partial u}{\partial n} = 0$.

   (c) **Robin's Conditions** are a mixture of Dirichlet and Neumann boundary conditions. This takes the form $\alpha u + \beta \frac{\partial u}{\partial n} = g$ for $(x, y, z) \in \partial\Omega$ and $t \geqslant 0$. This type of boundary condition occurs when, for example, Newton's law of cooling is applied. Newton's law of cooling states that the rate at which heat is transferred across a boundary is proportional to the temperature difference across the boundary. If we denote the temperature outside the region Ω by T, then Newton's law of cooling can be written as

$$\kappa u + \nu \frac{\partial u}{\partial n} = \kappa T.$$

A complete problem for a PDE consists of the PDE plus an appropriate number of side conditions. For example a complete problem for a general heat equation is given by

$$\frac{\partial u}{\partial t} = \vec{\nabla}\cdot(K(x)\vec{\nabla}u) + Q(x) \qquad 12 \qquad (x,y,z)\in\Omega, \quad t>0, \tag{1.9}$$

$$u(x,y,z,0) = f(x,y,z) \qquad (x,y,z)\in\Omega, \tag{1.10}$$

$$\alpha u(x,y,z,t) + \beta\frac{\partial u}{\partial n}(x,y,z,t) = g(x,y,z,t) \quad (x,y,z)\in\partial\Omega, \quad t\geqslant 0. \tag{1.11}$$

We see that

$\alpha \neq 0$, $\beta = 0$ corresponds to a Dirichlet condition;
$\alpha = 0$, $\beta \neq 0$ corresponds to a Neumann condition;
$\alpha \neq 0$, $\beta \neq 0$ corresponds to a Robin's condition.

If a PDE is studied on an infinite domain, then typically appropriate decay conditions are used. For example

$$\lim_{x\to\infty} u(x,t) < c_1 e^{-c_2 x^2}$$

with appropriate constants c_1, c_2.

1.3.1 Boundary Conditions on an Interval

Most of this text deals with PDE's on n-dimensional intervals

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

In that case the following rule of thumb can be applied:

> **Hillen's rule of thumb:** To formulate a complete problem for a PDE on an interval the number of side conditions for each of the variables $t, x, y, \ldots$ corresponds to the maximum order of this variable.

For example, the heat equation $u_t = k(u_{xx} + u_{yy})$ on $I = [0,1]\times[0,1]$ needs one initial condition (order of time derivatives is one), two boundary conditions for x and two boundary conditions for y. These could be of Dirichlet form, for example

$$u(x,y,0) = g(x,y), \quad u(0,y,t) = f_1(y,t), \quad u(1,y,t) = f_2(y,t), \quad u(x,0,t) = f_3(x,t), \quad u(x,1,t) = f_4(x,t).$$

The one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ on $[0,1]$ needs two initial conditions, usually for location $u(x,0) = g_0(x)$ and for initial velocity $u_t(x,0) = g_1(x)$ and two boundary conditions, typically at $u(0,t)$ and $u(1,t)$. Hillen's rule of thumb is a nice tool to check if the right number of side conditions is given. This rule can be extended to more general domains (for example circular domains), but one has to be careful to gain the right intuition. We will give many examples later.

1.4 Linear PDEs

In this section we will explore a bit more about linear PDEs. We will find a very important tool called the **superposition principle**. This principle is the very foundation of our solution theory. Without it, we could finish this textbook right here.

Every linear PDE can be written in one of two forms:

$$\begin{aligned} Lu &= 0, \qquad \text{(homogeneous)} \\ Lu &= f, \qquad \text{(nonhomogeneous)} \end{aligned}$$

for some linear differential operator L. What exactly do we mean by a linear differential operator? The definition is analogous to the definition with which you are familiar from your course in linear algebra.

Definition 1 (*Linear Operator*). An operator L with domain of definition $D(L)$ is called *linear* if it satisfies:

1. $L(cu) = cLu$, for any constant $c \in \mathbb{R}$ and $u \in D(L)$
2. $L(u_1 + u_2) = Lu_1 + Lu_2$, for two functions $u_1, u_2 \in D(L)$.

For differential operators we usually take the domain as the set of those functions which are continuously differentiable on the underlying set Ω. A couple of examples.

EXAMPLE 1.2. If $L = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, then the equation $Lu = 0$ is equivalent to $u_x + u_y = 0$. The domain of L is $D(L) = \{\text{set of continuously differentiable functions}\}$. We checked already earlier that the advection equation is linear.

EXAMPLE 1.3. If $L = \frac{\partial^2}{\partial x^2} + e^y \sin(x)\frac{\partial}{\partial y} - 1$, then the equation $Lu = 0$ is equivalent to $u_{xx} + e^y \sin(x) u_y = u$. The domain of L is $D(L) = \{\text{set of twice continuously differentiable functions}\}$. And it is straightforward to check that L is linear.

The main reason that linear PDEs are so much easier to deal with than nonlinear ones is the principle of superposition.

Theorem 2. *(Principle of Superposition)*
If u_1 and u_2 are solutions of a linear, homogeneous PDE $Lu = 0$, then $c_1u_1 + c_2u_2$ is also a solution for arbitrary constants c_1 and c_2.

Proof.
We have $Lu_1 = 0$ and $Lu_2 = 0$ since u_1 and u_2 are solutions. Therefore

$$L(c_1u_1 + c_2u_2) = L(c_1u_1) + L(c_2u_2) = c_1Lu_1 + c_2Lu_2 = c_1(0) + c_2(0) = 0.$$

EXAMPLE 1.4. Consider the 2–dimensional Laplace's equation $Lu = 0$, where $L = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Let $u_n(x, y) = \cos(nx)\sinh(ny)$ for $n = 1, 2, 3, \ldots$. For each n, u_n is a solution to Laplace's equation since

$$\begin{aligned} Lu_n &= \frac{\partial^2}{\partial x^2}(\cos(nx)\sinh(ny)) + \frac{\partial^2}{\partial y^2}(\cos(nx)\sinh(ny)) \\ &= -n^2\cos(nx)\sinh(ny) + n^2\cos(nx)\sinh(ny) = 0. \end{aligned}$$

Hence, by the principle of superposition

$$u(x, y) = \sum_{n=1}^{N} a_n u_n(x, y) = \sum_{n=1}^{N} a_n \cos(nx)\sinh(ny)$$

is also a solution for any integer N and any constants a_n.

What the principle of superposition gives us is a means of constructing new solutions if a few solutions are already known. This does *not* generally hold for nonlinear equations as the following example illustrates.

EXAMPLE 1.5. Consider the 2–dimensional, first order, *nonlinear* PDE

$$u_x + uu_y = 0.$$

The functions

$$u_1(x, y) = 1, \qquad u_2(x, y) = \frac{y}{1 + x},$$

are solutions of the PDE since

$$\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_1}{\partial y} = 0 + (1)(0) = 0, \quad \text{and} \quad \frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{y}{(1+x)^2} + \left(\frac{y}{1+x}\right)\left(\frac{1}{1+x}\right) = 0.$$

But, the sum of the two $u(x, y) = u_1(x, y) + u_2(x, y) = 1 + \frac{y}{1+x}$ is not a solution since

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = -\frac{y}{(1+x)^2} + \left(1 + \frac{y}{1+x}\right)\left(\frac{1}{1+x}\right) = \frac{1}{1+x} \neq 0.$$

The principle of superposition holds for linear, homogeneous equations. For nonhomogeneous equations we have the following result.

Theorem 3. *If*

- u_p *is a solution to the nonhomogeneous PDE* $Lu = f$ *(i.e.* $Lu_p = f$*) and*
- u_h *is a solution to the homogeneous PDE* $Lu = 0$ *(i.e.* $Lu_h = 0$*)*

then $u = cu_h + u_p$ *is a solution to* $Lu = f$ *for any constant* c*.*

Proof.
$Lu = L(cu_h + u_p) = cLu_h + Lu_p = c(0) + f = f$.

What this result says is that if a particular solution to a nonhomogeneous linear PDE is known and a solution to the homogeneous counterpart is also known, then a new solution to the nonhomogeneous PDE may be constructed. In fact, any solution to a nonhomogeneous linear PDE will be of this form, for suppose $\widetilde{u}$ is any solution to the nonhomogeneous PDE $Lu = f$ (i.e. $L\widetilde{u} = f$). Define $u_h = \widetilde{u} - u_p$. Then

$$Lu_h = L(\widetilde{u} - u_p) = L\widetilde{u} - Lu_p = f - f = 0,$$

thus u_h is a solution to the homogeneous equation $Lu = 0$. Hence, *any solution* $\widetilde{u}$ of the nonhomogeneous equation $Lu = f$ can be written as the sum of a solution to the homogeneous equation plus any particular solution:

$$\widetilde{u} = u_h + u_p.$$

1.5 Steady States and Equilibrium Solutions

Definition 4. *A* STEADY STATE *or* EQUILIBRIUM SOLUTION *of an initial-boundary value problem of a PDE is a solution that does not depend on time, i.e.* $u(x,t) = \bar{u}(x)$.

EXAMPLE 1.6. *(Diffusion through a cell membrane)*
We are interested to compute the concentration of a nutrient $u(x,t)$, for example oxygen, through a cell membrane of thickness l. We assume that the oxygen concentration inside and outside of the cell are constant with values c_1 (inside) and c_2 (outside). The corresponding initial-boundary value problem employs the diffusion equation:

$$\begin{aligned} u_t &= Du_{xx} \\ u(0,x) &= f(x) \\ u(t,0) = c_1, \qquad & u(t,l) = c_2. \end{aligned}$$

This is a nonhomogeneous Dirichlet problem for the diffusion equation, where D denotes the diffusion coefficient of oxygen in the cell membrane and $f(x)$ denotes the initial distribution of oxygen. We will learn later how to completely solve this model. Here we are only interested in a steady state distribution. A steady state does not depend on time, hence

$$u_t(t,x) = 0, \qquad \text{i.e.} \quad u(x,t) = \bar{u}(x).$$

Then we obtain

$$\bar{u}_{xx} = 0$$

which leads to $\bar{u}' = a_1$ and $\bar{u}(x) = a_1 x + a_2$ with two unknown constants a_1, a_2. We find these constants from the boundary conditions. At $x = 0$ we have

$$\bar{u}(0) = c_1 = a_2$$

Hence $a_2 = c_1$. At $x = l$ we have

$$\bar{u}(l) = c_2 = a_1 l + c_1.$$

Hence $a_1 = \frac{c_2 - c_1}{l}$ and we find the steady state solution

$$\bar{u}(x) = \frac{c_2 - c_1}{l} x + c_1.$$

To make sure that we found the right solution we test this solution. It is linear, hence indeed $\bar{u}'' = 0$. At $x = 0$ we have $\bar{u}(0) = c_1$ and at $x = l$ we find $\bar{u}(l) = c_2$.

The concentration profile through a membrane is a linear function which interpolates between the two concentration levels c_1 and c_2.

EXAMPLE 1.7. Now we are interested in the steady states of a homogeneous Neumann problems for the heat equation on $[0, l]$.

$$\begin{aligned} u_t(x,t) &= Du_{xx}(x,t) \\ u(0,x) &= f(x) \\ u_x(0,t) = 0 & \qquad u_x(l,t) = 0 \end{aligned}$$

As before, we find that at steady state we have $u(x,t) = \bar{u}(x)$ and $\bar{u}''(x) = 0$. Hence $\bar{u}(x)$ is linear $\bar{u}(x) = a_1 x + a_2$ with two unknown constants a_1 and a_2. Using the boundary conditions we find

$$\bar{u}_x(0) = a_1 = 0 \qquad \text{and} \qquad \bar{u}_x(l) = a_1 = 0.$$

Both boundary conditions require that $a_1 = 0$, hence

$$\bar{u}(x) = a_2$$

is a constant function. But now we have already used both boundary conditions. How can we find the missing constant a_2? To answer this question we need to dig a bit deeper and understand the conservation of mass property of the heat equation with homogeneous Neumann boundary conditions.

The integral

$$M(t) := \int_0^l u(x,t)dx$$

can be understood as the total mass (in case of the diffusion equation) or as total heat (in case of heat equation) in the system. Using the fundamental theorem of calculus we find

$$\begin{aligned}\frac{d}{dt}M(t) &= \frac{d}{dt}\int_0^l u(x,t)dx \\ &= \int_0^l u_t(x,t)dx \\ &= \int_0^l Du_{xx}(x,t)dx \\ &= D\left[u_x(t,l) - u_x(t,0)\right] \\ &= 0.\end{aligned}$$

Hence $M(t)$=const. and the total mass is conserved. Then we expect that the steady state $\bar{u}(x)$ has the same total mass as the initial condition $f(x)$. The initial mass is

$$M_0 = M(0) = \int_0^l f(x)dx$$

Then we require

$$M_0 = \int_0^l \bar{u}(x)dx = \int_0^l a_2 dx = a_2 l.$$

Hence $a_2 = \frac{M_0}{l}$ and the steady state solution for the above problem is

$$\bar{u}(x) = \frac{M_0}{l}.$$

EXAMPLE 1.8. Here we add a source term to the heat equation: Find the steady state of the following PDE on $[0, 2\pi]$:

$$\begin{aligned}u_t &= 3u_{xx} + 9\sin x \\ u(x,0) &= 9\sin x \\ u(0,t) &= 9 \\ u_x(2\pi,t) &= 0\end{aligned}$$

The steady state $\bar{u}(x)$ satisfies

$$3\bar{u}'' + 9\sin x = 0$$

Hence $\bar{u}'' = -3\sin x$ which is solved by

$$\bar{u}(x) = 3\sin x + c_1 x + c_2.$$

The boundary condition at $x = 0$ gives $\bar{u}(0) = 9 = c_2$. For the right boundary, we need the derivative: $\bar{u}_x(x) = 3\cos x + c_1$. The corresponding boundary condition gives

$$u_x(2\pi) = 0 = 3 + c_1,$$

hence $c_1 = -3$. The steady state solution is

$$\bar{u}(x) = 3\sin x - 3x + 9.$$

1.6 First Example for Separation of Variables

Separation of variables is a method that tries to seperate the dependence of the corresponding variables. For example suppose we have a linear PDE for some unknown function $u(x, y)$. Then we look for a solution of the form

$$u(x, y) = X(x)Y(y).$$

So far we do not know if this idea works. Nevertheless, we plug this assumed form of solution into the PDE, perform some manipulations and, hopefully, end up with ODEs for the unknown functions X and Y which we can solve. Let us study our first example for separation:

EXAMPLE 1.9. One dimensional heat equation with Dirichlet boundary conditions on the interval $[0, \ell]$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad x \in \Omega = [0, \ell], \quad t > 0, \tag{1.12}$$

$$u(x, 0) = 3\sin(\frac{2\pi x}{\ell}) =: f(x), \tag{1.13}$$

$$u(0, t) = 0, \tag{1.14}$$

$$u(\ell, t) = 0. \tag{1.15}$$

Here u represents the temperature in a bar at time t at position x. We try the method of separation of variables. Assume a solution of the form

$$u(x, t) = X(x)T(t). \tag{1.16}$$

Then

$$\frac{\partial u}{\partial t}(x, t) = X(x)T'(t), \qquad \frac{\partial^2 u}{\partial x^2}(x, t) = X''(x)T(t).$$

Plug this into Eq. (1.12) to get

$$XT' = kX''T,$$

This can be rewritten

$$\frac{T'}{kT} = \frac{X''}{X}$$

The left side is a function only of time t, whereas the right hand side of the equation is a function of the position x only. Since x and t are *independent* of each other, both sides must equal a constant, which we call $-\lambda$. Hence we get two equations:

$$\frac{T'}{kT} = -\lambda, \qquad \frac{X''}{X} = -\lambda.$$

The value of this constant is, at this point, unknown. Thus, we get two ODEs , one for T and one for X:

$$T' + \lambda kT = 0, \qquad X'' + \lambda X = 0.$$

Now insert Eq. (1.16) into the boundary conditions (1.14) and (1.15):

$$u(0,t) = X(0)T(t) = 0,$$
$$u(\ell,t) = X(\ell)T(t) = 0.$$

But $T(t) = 0$ for all t implies that $u \equiv 0$ which would mean that the initial condition would not be satisfied. Therefore the only conclusion is that

$$X(0) = 0, \qquad X(\ell) = 0.$$

The problems for T and X become:

$$T' + \lambda k T = 0, \quad t > 0, \qquad X'' + \lambda X = 0, \quad x \in [0,\ell], \tag{1.17}$$
$$X(0) = X(\ell) = 0. \tag{1.18}$$

The equation for T is easy to solve:

$$T(t) = ce^{-\lambda k t}, \qquad (c \text{ an arbitrary constant})$$

with no restriction as yet on λ.

Let us turn to the problem (1.17) and (1.18) for X. Notice that $X(x) \equiv 0$ is a solution (called the **trivial solution**) of the problem. However, the trivial solution leads to $u(x,t) \equiv 0$ which does not satisfy the initial condition. Therefore we seek nontrivial solutions to (1.17) and (1.18) for X. As we shall see, nontrivial solutions exist only for certain values of λ, called **eigenvalues**. The corresponding nontrivial solutions X are called **eigenfunctions**. More complicated problems will result in more complicated eigenvalue problems known as "Sturm–Liouville" problems. More on this will appear later.

The solution of (1.17) and (1.18) for X will depend on the sign of λ. There are three cases to consider: λ negative, positive or zero.

- case (i): $(\lambda < 0)$
  Let $\lambda = -\mu^2 \neq 0$. Then

  $$X'' - \mu^2 X = 0 \quad \Longrightarrow \quad X(x) = a\cosh(\mu x) + b\sinh(\mu x).$$

  The left boundary condition gives us

  $$X(0) = 0 \quad \Longrightarrow \quad a = 0 \quad \Longrightarrow \quad X(x) = b\sinh(\mu x).$$

  The right boundary condition now gives us

  $$X(\ell) = 0 \quad \Longrightarrow \quad b\sinh(\mu\ell) = 0 \quad \Longrightarrow \quad b = 0 \qquad 0.5\text{or} \qquad 0.5\sinh(\mu\ell) = 0.$$

  But $\mu\ell \neq 0$, therefore $b = 0$ which means that $X(x) \equiv 0$. There are no nontrivial solutions for $\lambda < 0$.

- case (ii): $(\lambda = 0)$
  Then $X'' = 0$ which yields $X(x) = ax + b$. However, the boundary conditions imply that $a = b = 0$ which again leads to the trivial solution. So there are no nontrivial solutions for the case $\lambda = 0$.

- case (iii): $(\lambda > 0)$
  Let $\lambda = \mu^2 \neq 0$. Then

$$X'' + \mu^2 X = 0 \quad \Longrightarrow \quad X(x) = a\cos(\mu x) + b\sin(\mu x).$$

The left boundary condition gives us

$$X(0) = 0 \quad \Longrightarrow \quad a = 0 \quad \Longrightarrow \quad X(x) = b\sin(\mu x).$$

The right boundary condition now gives us

$$X(\ell) = 0 \quad \Longrightarrow \quad b\sin(\mu\ell) = 0 \quad \Longrightarrow \quad b = 0 \qquad 0.5\text{or} \qquad 0.5\sin(\mu\ell) = 0.$$

A nontrivial solution results only when $\sin(\mu\ell) = 0$. This occurs for $\mu\ell = n\pi$, $n = \pm1, \pm2, \ldots$. Thus, we get nontrivial solutions only for

$$\lambda = \lambda_n = \mu_n^2 = \frac{n^2\pi^2}{\ell^2}, \quad n = 1, 2, 3, \ldots \qquad \text{(eigenvalues)}$$

with corresponding nontrivial solutions

$$X = X_n(x) = \sin(\mu_n x) = \sin(\frac{n\pi x}{\ell}). \qquad \text{(eigenfunctions)}$$

If we now put these eigenvalues into the solution to the T equation, we get

$$T = T_n(t) = e^{-\lambda_n kt}, \quad n = 1, 2, 3, \ldots.$$

For every $n = 1, 2, 3, \ldots$ we have a solution $u_n(x, t) = X_n(x)T_n(t)$ which satisfies the PDE and the boundary conditions. And, since the PDE is linear, a multiple of a solution is also a solution, so

$$u_n(x, t) = a_n e^{-\lambda_n kt}\sin(\frac{n\pi x}{\ell}), \quad n = 1, 2, 3, \ldots$$

are solutions. It remains only to satisfy the initial condition (1.13).

$$u(x, 0) = f(x) \quad \Longrightarrow \quad n = 2, \qquad 0.5a_2 = 3.$$

The solution to the problem is

$$u(x, t) = 3e^{-4\pi^2 kt/\ell^2}\sin(\frac{2\pi x}{\ell}).$$

In the previous example the initial condition was of a very specific form. In fact, the initial condition $f(x)$ is one of the eigenfunctions $\sin(\frac{n\pi x}{\ell})$ for $n = 2$. Now suppose that the initial condition had been $f(x) = x(\ell - x)$. Then the initial condition is not an eigenfunction and hence can not be satisfied for any n. Does this mean that our method doesn't work for this case? But, the PDE is linear and homogeneous (as are the boundary conditions) so the principle of superposition can be applied. Doing so yields

$$u(x,t) = \sum_{n=1}^{N} c_n u_n(x,t),$$

or, more generally

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin(\frac{n\pi x}{\ell}).$$

Now applying the initial condition yields

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \qquad \text{i.e.} \qquad x(\ell - x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{\ell}).$$

At this point a few questions come to mind:

1. Does the infinite series above converge?

2. What class of functions can be represented by an infinite series of the type given above?

3. If a functions f can be represented by such a series, how does one find the constant c_n's?

1.7 Physical basis of the Heat Equation

Consider a region $\Omega \subset \mathbb{R}^3$ (see Figure 1.1), consisting of some substance with boundary given by $\partial\Omega = S$. Let us introduce the following notation:

$$\begin{aligned}
u &\equiv \text{temperature at the point } (x,y,z) \in \Omega \text{ at time } t; \\
K &\equiv \text{thermal conductivity of substance at } (x,y,z); \\
\rho &\equiv \text{density of substance at } (x,y,z); \\
c &\equiv \text{heat capacity per unit mass at } (x,y,z); \\
\vec{J} &\equiv \text{heat flux vector (gives magnitude and direction of heat flow)}; \\
h &\equiv \text{rate of internal heat generation per unit volume at } (x,y,z).
\end{aligned}$$

The above quantities have the following physical dimensions:

$$
\begin{aligned}
[u] &\equiv T; & 8\,T &\equiv \text{ temperature;}\\
[K] &\equiv \frac{H}{LTt}; & H &\equiv \text{ heat;}\\
[\rho] &\equiv \frac{m}{L^3}; & t &\equiv \text{ time;}\\
[c] &\equiv \frac{H}{mT}; & L &\equiv \text{ length;}\\
[h] &\equiv \frac{H}{L^3 t}; & m &\equiv \text{ mass.}
\end{aligned}
$$

Suppose we let

$$
\begin{aligned}
d\mathcal{V} &\equiv \text{ an element of volume;}\\
dS &\equiv \text{ an element of surface area;}\\
\vec{n} &\equiv \text{ outward pointing unit vector of } dS.
\end{aligned}
$$

The vector $\vec{\nabla} u$ points in the direction of the most rapid increase of u, that is $\vec{\nabla} u$ is orthogonal to the surface $u \equiv$ constant. We use **Fourier's law**, which states that heat flows from regions of high temperature to regions of low temperature, where the flux $\vec{J}$ satisfies

$$\vec{J} = -K\vec{\nabla} u. \tag{1.19}$$

We have

$$\left.\begin{matrix}\text{Amt. of heat}\\ \text{in vol. element}\end{matrix}\right\} = \text{mass} \times \text{heat capacity} \times \text{temp.} \ = \rho c u \, d\mathcal{V};$$

$$\left.\begin{matrix}\text{Total amt. of}\\ \text{heat in } \Omega\end{matrix}\right\} = \iiint\limits_{\Omega} \rho c u \, d\mathcal{V};$$

$$\left.\begin{matrix}\text{Rate at which heat}\\ \text{enters through } dS\end{matrix}\right\} = -\vec{J} \cdot \vec{n} \, dS;$$

$$\left.\begin{matrix}\text{Total rate at which heat enters}\\ \text{through the boundary } S\end{matrix}\right\} = -\iint\limits_{S} \vec{J} \cdot \vec{n} \, dS;$$

$$\left.\begin{matrix}\text{Total rate at which heat}\\ \text{is generated internally}\end{matrix}\right\} = -\iiint\limits_{\Omega} h \, d\mathcal{V}.$$

Conservation of energy implies that

$$\begin{Bmatrix}\text{Rate of change}\\ \text{of heat in } \Omega\end{Bmatrix} = \begin{Bmatrix}\text{Rate at which heat}\\ \text{enters through boundary } S\end{Bmatrix} + \begin{Bmatrix}\text{Rate at which heat}\\ \text{is generated internally}\end{Bmatrix},$$

in other words

$$\frac{\partial}{\partial t}\iiint_\Omega \rho c u \, d\mathcal{V} = -\iint_S \vec{J}\cdot\vec{n}\, dS + \iiint_\Omega h\, d\mathcal{V}.$$

Using the divergence theorem from advanced calculus, we can rewrite this as

$$\iiint_\Omega \rho c \frac{\partial u}{\partial t}\, d\mathcal{V} = -\iiint_\Omega \vec{\nabla}\cdot\vec{J}\, d\mathcal{V} + \iiint_\Omega h\, d\mathcal{V},$$

or

$$\iiint_\Omega \left\{\rho c\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{J} - h\right\} d\mathcal{V} = 0.$$

This must hold for an arbitrary region Ω so the integrand must be identically zero. Using the definition for the flux vector in Eq. (1.19), we get

$$\boxed{\frac{\partial u}{\partial t} = \frac{1}{\rho c}\vec{\nabla}\cdot(K\vec{\nabla}u) + \frac{h}{\rho c}}. \qquad 6\text{(heat equation)}$$

It is often the case that ρ, c and K are constant. In that case we define

$$k := \frac{K}{\rho c}, \qquad 6F := \frac{h}{\rho c}$$

The constant k is called the **thermal diffusivity** and F, which is not necessarily constant, is called the **forcing**. The heat equation then becomes:

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u + F}. \qquad 6\text{(heat equation in standard form)} \tag{1.20}$$

Written explicitly in terms of Cartesian coordinates:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F, \qquad 6\text{(3-d heat equation)} \tag{1.21}$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + F. \qquad \text{(1-d heat equation)} \tag{1.22}$$

1.8 Physical basis of the Wave Equation

Here we will only give a derivation for the one–dimensional wave equation. Consider a string stretched over an interval $[a, b]$. Let us introduce the following notation:

$$\begin{aligned}
u &\equiv \text{displacement above the } x\text{-axis},\\
\sigma &\equiv \text{surface tension};\\
\rho &\equiv \text{density per unit length};\\
t &\equiv \text{time};\\
h &\equiv \text{vertical force per unit length};\\
\alpha, \beta &\equiv \text{angles indicated in the figure}.
\end{aligned}$$

The above quantities have the following physical dimensions:

$$\begin{aligned}
[u] &\equiv L; & 8\, L \equiv\ & \text{length}\\
[\sigma] &\equiv F; & F \equiv\ & \text{force};\\
[\rho] &\equiv \frac{m}{L}; & m \equiv\ & \text{mass};\\
[h] &\equiv \frac{F}{L}.
\end{aligned}$$

We shall make two simplifying assumptions: ρ and σ are constant; and $|u|, \alpha, \beta$ are small. The angles α, β being small implies that $\cos\alpha \approx 1$ which in turn implies that $\tan\alpha \approx \sin\alpha$, and similarly for $\tan\beta$. We have

$$\begin{aligned}
\text{mass} &= \rho\Delta x,\\
\frac{\partial u}{\partial x}(x, t) &= \tan\alpha \approx \sin\alpha,\\
\frac{\partial u}{\partial x}(x + \Delta x, t) &= \tan\beta \approx \sin\beta.
\end{aligned}$$

The forces acting on the segment of rope are:

$$\begin{aligned}
&\text{Horizontal force:} \quad & 4F_h &= \sigma(\cos\beta - \cos\alpha) \approx 0;\\
&\text{Vertical force:} & F_v &= \sigma(\sin\beta - \sin\alpha) + h(x^*)\Delta x\\
& & &\approx \sigma\left[\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right] + h(x^*)\Delta x.
\end{aligned}$$

Applying Newton's law of motion: $\vec{F} = m\vec{a}$, we have

$$F_h = 0, \qquad 1(\text{no horizontal motion}) \qquad 4F_v = m\frac{\partial^2 u}{\partial t^2}(x^*, t).$$

Therefore we have

$$\rho\frac{\partial^2 u}{\partial t^2}(x^*,t) = \sigma\frac{\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x} + h(x^*).$$

If we take the limit as $\Delta x \to 0$, then $x^* \to x$ and

$$\rho\frac{\partial^2 u}{\partial t^2}(x,t) = \sigma\frac{\partial^2 u}{\partial x^2}(x,t) + h(x).$$

If we let $c^2 := \frac{\sigma}{\rho}$, and $F := \frac{h(x)}{\rho}$, then we get

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + F}, \qquad 4(\text{1–d wave equation}). \tag{1.23}$$

The higher dimensional wave equations are given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + F, \qquad 6\ (\text{2–d wave equation}) \tag{1.24}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F.(\text{3–d wave equation}) \tag{1.25}$$

In general we have

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + F.$$

The boundary conditions for the wave equation are the same as they are for the heat equation. However, since the wave equation has two time derivatives, two initial conditions are required. Thus, a complete problem for the wave equation consists of the partial differential equation itself plus two initial conditions plus boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + F \qquad 12 \qquad (x,y,z)\in\Omega, \qquad 1t>0, \tag{1.26}$$

$$u(x,y,z,0) = f(x,y,z) \qquad (x,y,z)\in\Omega, \tag{1.27}$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = g(x,y,z) \qquad (x,y,z)\in\Omega, \tag{1.28}$$

$$\alpha u(x,y,z,t) + \beta\frac{\partial u}{\partial n}(x,y,z,t) = h(x,y,z,t)(x,y,z)\in\partial\Omega, \qquad 1t\geqslant 0. \tag{1.29}$$

1.9 Physical basis of the Laplace Equation

From the previous two sections we have

$$u_t = k\nabla^2 u + F, \qquad 6\,(\text{heat equation})$$
$$u_{tt} = c^2\nabla^2 u + F. \qquad 6(\text{wave equation})$$

If we look for a steady state (i.e. time independent) solution to either the heat equation or the wave equation, then we get an equation of the form

$$\nabla^2 u = h. \qquad 6\text{(Poisson's equation)}$$

If $h \equiv 0$, then we have

$$\nabla^2 u = 0. \qquad 6\text{(Laplace's equation)}$$

Laplace's equation, also called the potential equation, occurs in many areas of physics such as hydrodynamics, elasticity, electric field theory, ...etc. A complete problem for Laplace's equation consists of the partial differential equation itself plus boundary conditions:

$$\nabla^2 u = 0, \qquad 18 \qquad (x, y, z) \in \Omega, \tag{1.30}$$

$$\alpha u(x, y, z) + \beta \frac{\partial u}{\partial n}(x, y, z) = f(x, y, z), (x, y, z) \in \partial\Omega. \tag{1.31}$$

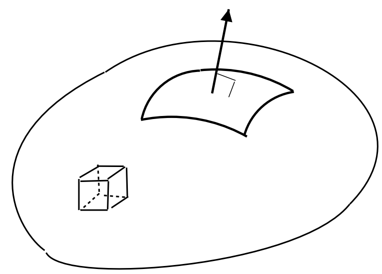


Figure 1.1: Region Ω

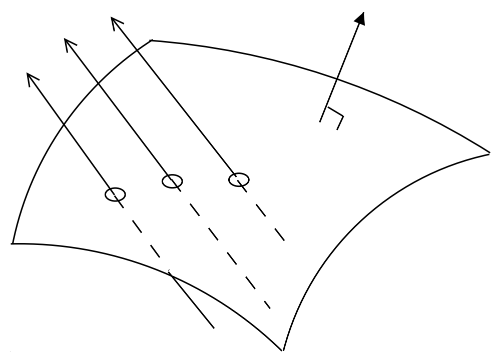


Figure 1.2: Flux J through a surface element dS

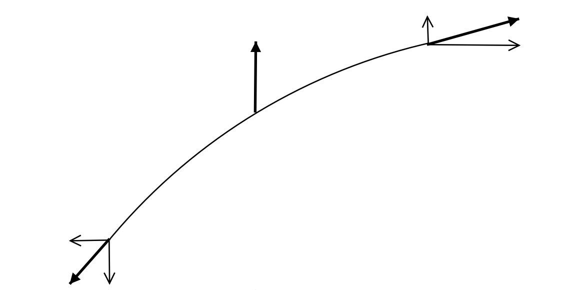


Figure 1.3: Forces acting on a segment of string

Chapter 2

Fourier Series

June 17, 2010

2.1 Piecewise Continuous Functions

In most physical problems we may expect to use "nice" smooth functions. However, it will be convenient to allow for certain types of discontinuous functions. Consider the behaviour of a function f near a point x_0. In particular consider the following limits:

$$f(x_0+) := \lim_{x\to x_0^+} f(x), \qquad f(x_0-) := \lim_{x\to x_0^-} f(x).$$

We make the following definition:

Definition 5. *If*

(i) $f(x_0+) = f(x_0-) = f(x_0)$, *then* f *is continuous at* $x = x_0$*;*
(ii) $f(x_0+) = f(x_0-) \neq f(x_0)$, *then* f *has a removable discontinuity at* $x = x_0$*;*
(iii) $f(x_0+) \neq f(x_0-)$, *then* f *has a jump discontinuity at* $x = x_0$*;*
(iv) $f(x_0+)$ *or* $f(x_0-)$ *does not exist, then* f *has a "bad" discontinuity at* $x = x_0$*.*

Definition 6. *A function* f *is* PIECEWISE CONTINUOUS *(abbreviated p-cts), sometimes called sectionally continuous, on an interval* (a, b) *if*

(i) f *is continuous for* $x \in (a, b)$ *except possibly at a finite number of points;*
(ii) $f(x+)$ *exists for all* $x \in [a, b)$*;*
(iii) $f(x-)$ *exists for all* $x \in (a, b]$*.*

A function f is piecewise continuous on an interval (a, b) if f has at most a finite number of discontinuities, none of which is worse than a jump discontinuity.

EXAMPLE 2.1. The function

$$f(x) = \begin{cases} x & 0 \leqslant x < 1 \\ -1 & 1 < x \leqslant 2 \\ 1 & 2 < x < 3 \end{cases}$$

is piecewise continuous on the interval $(0, 3)$.

EXAMPLE 2.2. The function $f(x) = \frac{1}{x}$ is not piecewise continuous on $(0, 1)$, since

$$f(0+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} \text{ does not exist.}$$

It is convenient to introduce the following notation:

Definition 7. $PC(a, b) := \{$*set of all p-cts functions on* $(a, b)\}$.

Properties of $PC(a, b)$:

(i) if $f, g \in PC(a, b)$, then $\alpha f + \beta g \in PC(a, b)$ for all $\alpha, \beta \in \mathbb{R}$;
(ii) if $f, g \in PC(a, b)$, then $fg \in PC(a, b)$;
(iii) if $f \in PC(a, b)$, then $\int_a^b |f(x)|\, dx$ exists.

¿From property (i) above it follows that $PC(a, b)$ is a vector space.

Definition 8. *A function f is* PIECEWISE SMOOTH *(abbreviated p-smooth) on (a, b) if*

(i) $f \in PC(a, b)$; and
(ii) $f' \in PC(a, b)$.

Definition 9. $PC^1(a, b) := \{$*set of all p-smooth functions on* $(a, b)\}$.

It is clear that $PC^1(a, b) \subset PC(a, b)$.

2.2 Even, Odd and Periodic Functions

We begin with a definition.

Definition 10. *A function f is*

(i) EVEN *if $f(-x) = f(x)$ for all $x \in \mathcal{D}_f$;*
(ii) ODD *if $f(-x) = -f(x)$ for all $x \in \mathcal{D}_f$;*
(iii) PERIODIC *with period p if $f(x + p) = f(x)$ for all $x \in \mathcal{D}_f$.*

EXAMPLE 2.3.

(i) $f(x) = x^n$ is even if n is an even integer.
(ii) $f(x) = x^n$ is odd if n is an odd integer.
(iii) $f(x) = \cos(x)$ is even and 2π-periodic.
(iv) $f(x) = \sin(x)$ is odd and 2π-periodic.
(v) $f(x) = e^x$ is not even, odd or periodic.
(vi) $f(x) = \cosh(x)$ is even.
(vii) $f(x) = \sinh(x)$ is odd.

If f is p-periodic, then

$$\int_a^{a+p} f(x)\, dx = \int_0^p f(x)\, dx \tag{2.1}$$

.

If a function f is defined on an interval (a, b), it is sometimes useful to extend the definition of f to the entire real line.

EXAMPLE 2.4. The PERIODIC EXTENSION of f, denoted $\bar{f}$, is defined as

$$\bar{f}(x) = f(x + np) \quad \text{for} \quad a - np < x < b - np, \qquad 1n \in \mathbb{Z},$$

where $p = b - a$.

Frequently in applications we will encounter a function f defined on an interval $(0, \ell)$. We want to construct periodic extensions of f. Let f be defined on $(0, \ell)$.

Definition 11. *The* ODD EXTENSION *of f on $(-\ell, \ell)$, denoted f_o, is defined as*

$$f_o(x) = \begin{cases} f(x) & 0 < x < \ell \\ -f(-x) & -\ell < x < 0 \end{cases}.$$

Definition 12. *The* EVEN EXTENSION *of f on $(-\ell, \ell)$, denoted f_e, is defined as*

$$f_e(x) = \begin{cases} f(x) & 0 < x < \ell \\ f(-x) & -\ell < x < 0 \end{cases}.$$

We then extend f_e and f_o to get periodic extensions $\bar{f}_e$ and $\bar{f}_o$.

2.3 Orthogonal Functions

Consider vectors in 3–dimensional vector space $\mathbb{R}^3$. A basis for $\mathbb{R}^3$ is given by $\{\vec{i}, \vec{j}, \vec{k}\}$. For convenience we rewrite the basis as

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{\vec{i}, \vec{j}, \vec{k}\}.$$

Consider the vectors:

$$\vec{v} = (a_1, a_2, a_3) = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \sum_{i=1}^{3} a_i\vec{e}_i,$$

$$\vec{w} = (b_1, b_2, b_3) = b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3 = \sum_{i=1}^{3} b_i\vec{e}_i.$$

The dot product of two vectors is given by

$$\vec{v} \cdot \vec{w} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_ib_i.$$

Putting in all of the steps we have:

$$\vec{v} \cdot \vec{w} = \left(\sum_{i=1}^{3} a_i\vec{e}_i\right) \cdot \left(\sum_{j=1}^{3} b_j\vec{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j\vec{e}_i \cdot \vec{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} a_ib_j\delta_{ij} \sum_{i=1}^{3} a_ib_i,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{is the Kronecker delta.}$$

Consider two sets of basis vectors in $\mathbb{R}^2$:

$$\vec{E}_1 = (1,0),\ \vec{E}_2 = (1,1), \qquad \text{and} \qquad \vec{e}_1 = (1,1),\ \vec{e}_2 = (1,-1).$$

Now we want to write the vector $\vec{v} = (1,2)$ in terms of $\{\vec{E}_1, \vec{E}_2\}$ and $\{\vec{e}_1, \vec{e}_2\}$.

If $\vec{v} = A_1\vec{E}_1 + A_2\vec{E}_2$ then

$$\begin{aligned} \vec{v} \cdot \vec{E}_1 &= A_1\vec{E}_1 \cdot \vec{E}_1 + A_2\vec{E}_2 \cdot \vec{E}_1 \\ \vec{v} \cdot \vec{E}_2 &= A_1\vec{E}_1 \cdot \vec{E}_2 + A_2\vec{E}_2 \cdot \vec{E}_2 \end{aligned}$$

which becomes

$$\begin{aligned} 1 &= A_1 + A_2 \\ 3 &= A_1 + 2A_2 \end{aligned}$$

and therefore $A_1 = -1$ and $A_2 = 2$ and

$$\vec{v} = -\vec{E}_1 + 2\vec{E}_2.$$

If $\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2$ then

$$\begin{aligned} \vec{v} \cdot \vec{e}_1 &= a_1\vec{e}_1 \cdot \vec{e}_1 + a_2\vec{e}_2 \cdot \vec{e}_1 \\ \vec{v} \cdot \vec{e}_2 &= a_1\vec{e}_1 \cdot \vec{e}_2 + a_2\vec{e}_2 \cdot \vec{e}_2 \end{aligned}$$

which becomes

$$\begin{aligned} 3 &= 2a_1 \\ -1 &= 2a_2 \end{aligned}$$

and therefore $a_1 = 3/2$ and $a_2 = -1/2$.

$$\vec{v} = \frac{3}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2.$$

While this is a simple 2–dimensional example, it is clear that the calculation on the right is much simpler. The reason for this is the fact that the basis vectors on the right $\{\vec{e}_1, \vec{e}_2\}$ form an orthogonal set. The significance of this becomes more evident in higher dimensional spaces. For example, in $\mathbb{R}^3$ we have

$$\begin{aligned} \vec{v} &= A_1\vec{E}_1 + A_2\vec{E}_2 + A_3\vec{E}_3, \\ \vec{v} &= a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3. \end{aligned}$$

A little manipulation yields:

$$A_1 \vec{E}_1 \cdot \vec{E}_i + A_1 \vec{E}_1 \cdot \vec{E}_i + A_1 \vec{E}_1 \cdot \vec{E}_i = \vec{v} \cdot \vec{E}_i, \quad i = 1, 2, 3.$$

In matrix form this is

$$\begin{bmatrix} \vec{E}_1 \cdot \vec{E}_1 & \vec{E}_2 \cdot \vec{E}_1 & \vec{E}_3 \cdot \vec{E}_1 \\ \vec{E}_1 \cdot \vec{E}_2 & \vec{E}_2 \cdot \vec{E}_2 & \vec{E}_3 \cdot \vec{E}_2 \\ \vec{E}_1 \cdot \vec{E}_3 & \vec{E}_2 \cdot \vec{E}_3 & \vec{E}_3 \cdot \vec{E}_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \vec{v} \cdot \vec{E}_1 \\ \vec{v} \cdot \vec{E}_2 \\ \vec{v} \cdot \vec{E}_3 \end{bmatrix}$$

To solve this equation requires the inversion of a 3×3 matrix.

However, if the second set of basis vectors is orthogonal, i.e. $\vec{e}_i \cdot \vec{e}_j = 0$ for $i \neq j$, then the matrix equation becomes

$$\begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & 0 & 0 \\ 0 & \vec{e}_2 \cdot \vec{e}_2 & 0 \\ 0 & 0 & \vec{e}_3 \cdot \vec{e}_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \vec{v} \cdot \vec{e}_1 \\ \vec{v} \cdot \vec{e}_2 \\ \vec{v} \cdot \vec{e}_3 \end{bmatrix}$$

which is easily solved to get

$$a_1 = \frac{\vec{v} \cdot \vec{e}_1}{\vec{e}_1 \cdot \vec{e}_1}, \quad a_2 = \frac{\vec{v} \cdot \vec{e}_2}{\vec{e}_2 \cdot \vec{e}_2}, \quad a_3 = \frac{\vec{v} \cdot \vec{e}_3}{\vec{e}_3 \cdot \vec{e}_3}.$$

Using summation notation, we have

$$\vec{v} = \sum_{i=1}^{3} a_i \vec{e}_i.$$

$$\vec{v} \cdot \vec{e}_j = \left(\sum_{i=1}^{3} a_i \vec{e}_i \right) \cdot \vec{e}_j = \sum_{i=1}^{3} a_i \vec{e}_i \cdot \vec{e}_j = a_j \vec{e}_j \cdot \vec{e}_j \qquad \text{(since orthogonal)}$$

Therefore

$$a_j = \frac{\vec{v} \cdot \vec{e}_j}{\vec{e}_j \cdot \vec{e}_j}, \quad j = 1, 2, 3.$$

Recall that the **norm** of a vector (or length of a vector) is given by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

If $\vec{v} = \sum_{i=1}^{3} a_i \vec{e}_i$, then

$$\begin{aligned} \|\vec{v}\|^2 = \sqrt{\vec{v} \cdot \vec{v}} &= \left(\sum_{i=1}^{3} a_i \vec{e}_i \right) \cdot \left(\sum_{j=1}^{3} a_j \vec{e}_j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i a_j \vec{e}_i \cdot \vec{e}_j = \sum_{i=1}^{3} a_i^2 \vec{e}_i \cdot \vec{e}_i = \sum_{i=1}^{3} a_i^2 \|\vec{e}_i\|^2. \end{aligned}$$

If the $\vec{e}_i$ are unit vectors (i.e. $\|\vec{e}_i\| = 1$), then

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Definition 13. *An* INNER PRODUCT *is any function* $\langle \cdot, \cdot \rangle$ *that acts on pairs of vectors* $\vec{v}$ *and* $\vec{w}$ *in a vector space* $\mathcal{X}$ *that satisfies the following properties:*

(i) $\left\langle \vec{v}, \vec{w} \right\rangle = \left\langle \vec{w}, \vec{v} \right\rangle$;

(ii) $\left\langle \vec{v}, \vec{w}_1 + \vec{w}_2 \right\rangle = \left\langle \vec{v}, \vec{w}_1 \right\rangle + \left\langle \vec{v}, \vec{w}_2 \right\rangle$;

(iii) $\left\langle c\vec{v}, \vec{w} \right\rangle = c \left\langle \vec{v}, \vec{w} \right\rangle$;

(iv) $\left\langle \vec{v}, \vec{v} \right\rangle \geqslant 0$.

The dot product is but one example of an innerproduct. What we want to do is generalize these concepts of inner product and norms to p-cts functions.

Here we will give an heuristic motivation for how to define an inner product for p-cts functions. Let

$$f, g, w \in PC(a, b) \quad \text{with} \quad w \geqslant 0.$$

Recall from elementary calculus the definition of the integral as the limit of a Riemann sum:

$$\int_a^b f(x)g(x)w(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)g(x_i)w(x_i)\Delta x_i,$$

where $a = x_0 < x_1 < \cdots < x_N = b$ is a paartition of the interval (a, b) and $\Delta x_i = x_i - x_{i-1}$. If we define

$$a_i = f(x_i)\sqrt{w(x_i)\Delta x_i}, \quad b_i = g(x_i)\sqrt{w(x_i)\Delta x_i},$$

then

$$\int_a^b f(x)g(x)w(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} a_i b_i.$$

The sum on the right looks like an inner product in N–dimensions. This leads us to make the following definition.

Definition 14. *Let* $f, g, w \in PC(a, b)$ *with* $w(x) \geqslant 0$. *The* INNER PRODUCT *of* f *and* g *with weight* w *is defined as*

$$\langle f, g \rangle := \int_a^b f(x)g(x)w(x)\,dx.$$

The function $w(x)$ *is called the* WEIGHT *function. Quite often we use* $w(x) = 1$.

It is easily verified that this definition satisfies all the properties given in the definition of inner product. Recall that we saw earlier that $PC(a, b)$ is a vector space. One can think of this definition as an inner product defined on an "infinite dimensional" vector space $PC(a, b)$.

We give a couple more definitions.

Definition 15. *The* NORM *of $f \in PC(a,b)$ with weight w is given by $\|f\| := \sqrt{\langle f, f\rangle}$.*

Definition 16. *Let $f, g, w \in PC(a,b)$ with $w(x) \geqslant 0$. Then f and g are said to be* ORTHOGONAL *on (a,b) relative to the weight w if $\langle f, g\rangle = 0$.*

We will illustrate these concepts with the following important example.

EXAMPLE 2.5. Consider the functions $c_n, s_n \in PC(a,b)$ for $n = 0, 1, 2, 3, \ldots$ defined by

$$c_n(x) := \cos(\frac{n\pi x}{\ell}), \quad s_n(x) := \sin(\frac{n\pi x}{\ell}), \quad \ell := \frac{b-a}{2}, \tag{2.2}$$

with inner product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,dx. \qquad \text{(weight } w(x) \equiv 1\text{)}$$

We see that c_n is 2ℓ-periodic, since

$$c_n(x+2\ell) = \cos(\frac{n\pi(x+2\ell)}{\ell}) = \cos(\frac{n\pi x}{\ell} + 2n\pi) = \cos(\frac{n\pi x}{\ell}) = c_n(x).$$

Similarly, s_n is 2ℓ-periodic. Thus, for $n = 0$, we have

$$\langle c_0, c_0\rangle = \int_a^b c_0^2(x)\,dx = \int_a^b dx = b - a = 2\ell.$$

For $n \neq 0$ we have

$$\begin{aligned}
\langle c_n, c_m\rangle &= \int_a^b c_n(x)c_m(x)\,dx = \int_0^{2\ell} c_n(x)c_m(x)\,dx \qquad \text{(using property (2.1) with } p = 2\ell\text{)} \\
&= \int_0^{2\ell} \cos(\frac{n\pi x}{\ell})\cos(\frac{m\pi x}{\ell})\,dx = \frac{\ell}{\pi}\int_0^{2\pi} \cos(n\xi)\cos(m\xi)\,d\xi \quad \text{(using } \xi = \frac{\pi x}{\ell}\text{)} \\
&= \frac{\ell}{\pi}(\delta_{nm}\pi) = \delta_{nm}\ell.
\end{aligned}$$

A similar calculation yields

$$\langle s_n, s_m\rangle = \delta_{nm}\ell, \qquad \langle s_n, c_m\rangle = 0.$$

Therefore the set $\{c_0, c_1, s_1, c_2, s_2, \ldots\} = \{1, \cos(\frac{\pi x}{\ell}), \sin(\frac{\pi x}{\ell}), \cos(\frac{2\pi x}{\ell}), \sin(\frac{2\pi x}{\ell}), \ldots\}$ is an orthogonal set of functions on $[a,b]$ relative to the above inner product.

2.4 Fourier Series

If f is periodic with period p, then one might attempt to represent f with an infinite series of p-periodic functions. Notice that the set

$$\{1, \cos(\frac{\pi x}{\ell}), \sin(\frac{\pi x}{\ell}), \cos(\frac{2\pi x}{\ell}), \sin(\frac{2\pi x}{\ell}), \ldots\} \tag{2.3}$$

consists of functions having common period $p = 2\ell$. Suppose f is 2ℓ-periodic. We could try to represent f as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}).$$

Each term in the series has period 2ℓ, so the sum, if it exists, we would expect would also be 2ℓ-periodic. Two questions come to mind:

1. How does one find the constants a_0, a_n, b_n?
2. Once the constants are found, does the series converge to f?

In example 2.5 we saw that the elements of the set (2.3) are orthogonal. This makes calculation of the constants in the series easy. Writing f in the more compact notation

$$f(x) = \frac{a_0}{2} c_0 + \sum_{n=1}^{\infty} a_n c_n(x) + b_n s_n(x)$$

and using the inner product of example 2.5 we get

$$\begin{aligned}
\langle f, c_0 \rangle &= \left\langle \frac{a_0}{2} c_0 + \sum_{n=1}^{\infty} (a_n c_n + b_n s_n), c_0 \right\rangle \\
&= \frac{a_0}{2} \langle c_0, c_0 \rangle + \sum_{n=1}^{\infty} (a_n \langle c_n, c_0 \rangle + b_n \langle s_n, c_0 \rangle) \\
&= \frac{a_0}{2} (2\ell) + \sum_{n=1}^{\infty} (a_n \delta_{n0} + 0) = a_0 \ell.
\end{aligned}$$

Therefore

$$a_0 = \frac{1}{\ell} \langle f, c_0 \rangle .$$

For $m \neq 0$ we have

$$\begin{aligned}
\langle f, c_m \rangle &= \left\langle \frac{a_0}{2} c_m + \sum_{n=1}^{\infty} (a_n c_n + b_n s_n), c_m \right\rangle \\
&= \frac{a_0}{2} \langle c_0, c_m \rangle + \sum_{n=1}^{\infty} (a_n \langle c_n, c_m \rangle + b_n \langle s_n, c_m \rangle) \\
&= 0 + \sum_{n=1}^{\infty} (a_n \ell \delta_{nm} + 0) = a_m \ell.
\end{aligned}$$

Therefore

$$a_m = \frac{1}{\ell}\langle f, c_m\rangle\,,$$

and similarly

$$b_m = \frac{1}{\ell}\langle f, s_m\rangle\,.$$

This prompts the following definition:

Definition 17. *The series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}),$$

where

$$a_n = \frac{1}{\ell}\langle f, c_m\rangle = \frac{1}{\ell}\int_a^b f(x)\cos(\frac{n\pi x}{\ell})\,dx, \qquad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\ell}\langle f, s_m\rangle = \frac{1}{\ell}\int_a^b f(x)\sin(\frac{n\pi x}{\ell})\,dx, \qquad n = 1, 2, \ldots,$$

is called the FOURIER SERIES *for f, and a_n and b_n are called the* FOURIER COEFFICIENTS.

Note that if $f \in PC(a,b)$, then all of the above integrals exist. We have answered question 1 above: we have simple formulas for the Fourier coefficients. Again, we reiterate that the reason for the simplicity of these formulas is due to the orthogonality of the sine and cosine functions on the given interval relative to the given inner product. It remains to determine whether this infinite series converges, and if so, to what function does it converge? Until convergence can be determined we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}).$$

EXAMPLE 2.6. Find the Fourier series for the 2π-periodic function f defined

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}, \quad f(x + 2\pi) = f(x).$$

Solution
We take the interval to be $(a,b) = (-\pi, \pi)$. Then $\ell = \pi$, $c_n(x) = \cos(nx)$ and $s_n(x) =$

$\sin(nx)$. We get

$$a_0 = \frac{1}{\ell}\langle f, c_0\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2},$$
$$a_n = \frac{1}{\ell}\langle f, c_n\rangle = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{(-1)^n - 1}{\pi n^2},$$
$$b_n = \frac{1}{\ell}\langle f, s_n\rangle = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{(-1)^{n+1}}{n}.$$

Therefore, the Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)] \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{(-1)^n + 1}{n}\sin(nx)]$$
$$\sim \frac{\pi}{4} - \frac{2}{\pi}\cos(x) + \sin(x) - \frac{1}{2}\sin(2x) - \frac{2}{9\pi}\cos(3x) + \frac{1}{3}\sin(3x) - \ldots.$$

Another example.

EXAMPLE 2.7. Find the Fourier series for the 2ℓ-periodic function f defined

$$f(x) = x, \quad -\ell < x < \ell, \quad f(x + 2\ell) = f(x).$$

Solution
In this case we have $c_n(x) = \cos(\frac{n\pi x}{\ell})$ and $s_n(x) = \sin(\frac{n\pi x}{\ell})$. We get

$$a_n = \frac{1}{\ell}\langle f, c_n\rangle = \frac{1}{\ell}\int_{-\ell}^{\ell} x\cos(\frac{n\pi x}{\ell})\,dx = 0,$$
$$b_n = \frac{1}{\ell}\langle f, s_n\rangle = \frac{1}{\ell}\int_{-\ell}^{\ell} x\sin(\frac{n\pi x}{\ell})\,dx = \frac{2\ell}{n\pi}(-1)^{n+1}.$$

Therefore, the Fourier series is given by

$$x \sim \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi x}{\ell}) \quad \text{for } -\ell < x < \ell. \tag{2.4}$$

Note that in this last example at $x = \ell$ the *L.H.S.* of (2.4) equals ℓ whereas the *R.H.S.* equals 0. This indicates that at least at one point, the Fourier series does not converge to the function. Another observation concerning this last example: the function f is an odd function and the Fourier series is a sum of odd terms. This should not be surprising. In general,

if f is odd, then $f(x) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{\ell})$, called a Fourier sine series;

if f is even, then $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \cos(\frac{n\pi x}{\ell})$, called a Fourier cosine series.

2.5 Convergence of Fourier Series

Definition 18. *A function f is* PIECEWISE SMOOTH *on (a,b) if*

(i) f is piecewise continuous (i.e. $f \in PC(a,b)$);

(ii) f' is piecewise continuous (i.e. $f' \in PC(a,b)$).

Let $PC^1(a,b) := \{\text{all piecewise smooth functions on } (a,b)\}$. Then $PC^1(a,b) \subset PC(a,b)$.

EXAMPLE 2.8. If

$$f(x) = \begin{cases} x & -1 < x < 0 \\ x^2 & 0 < x < 1 \end{cases} \quad \text{then} \quad f'(x) = \begin{cases} 1 & -1 < x < 0 \\ 2x & 0 < x < 1 \end{cases}.$$

Therefore $f \in PC^1(a,b)$.

EXAMPLE 2.9. If $f(x) = x^{2/3}$ 1on $(-1,1)$, then $f'(x) = \frac{2}{3x^{1/3}}$. We have $f \in PC(-1,1)$ but $f' \notin PC(-1,1)$. Therefore $f \notin PC^1(-1,1)$.

Theorem 19. *If*

(i) $f \in PC^1(a,b)$;

(ii) $\bar{f}$ is the $(b-a)$-periodic extension of f,

then the Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}), \quad \textit{where } \ell = \frac{b-a}{2},$$

converges to

$$\frac{\bar{f}(x+) + \bar{f}(x-)}{2}.$$

This says that if $\bar{f}$ is continuous at x (i.e. $\bar{f}(x+) = \bar{f}(x-) = \bar{f}(x)$), then the Fourier Series converges to $\bar{f}(x)$ and if $\bar{f}$ has a jump discontinuity at x (i.e. $\bar{f}(x+) \neq \bar{f}(x-)$), then the Fourier Series converges to the point midway between the limiting values.

EXAMPLE 2.10. If we go back to Example 2.7, we found the Fourier Series for the function

$$f(x) = x, \quad -\ell < x < \ell, \quad f(x+2\pi) = f(x),$$

to be

$$f(x) \sim \frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(\frac{n\pi x}{\ell}).$$

But $\bar{f}(\ell+) = -\ell$ and $\bar{f}(\ell-) = +\ell$, therefore

$$\frac{\bar{f}(\ell+) + \bar{f}(\ell-)}{2} = \frac{-\ell + \ell}{2} = 0.$$

The Fourier Series evaluated at $x = \ell$:

$$\frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi) = 0.$$

EXAMPLE 2.11. Again we go back to Example 2.7 but this time we evaluate the Fourier Series at $x = \ell/2$. Since f is continuous at $x = \ell/2$ we get

$$\frac{\ell}{2} = \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi}{2}) = \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{2n}}{2n-1}\sin(\frac{(2n-1)\pi}{2}) + \frac{(-1)^{2n+1}}{2n}\sin(n\pi)\right\}.$$

Rearranging this expression gives:

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(\frac{(2n-1)\pi}{2}) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

2.6 Operations on Fourier Series

Here we will examine what kind of manipulation may be legitimately be carried out on Fourier Series. The next few results will be stated without proof.

Theorem 20. *If f is periodic and p-cts, then its Fourier Series is unique.*

Next we show that the operation of finding the Fourier series is a linear operation.

Theorem 21. *If*

(i) $\widehat{f}(x) \sim \frac{\widehat{a}_0}{2} + \sum_{n=1}^{\infty}\widehat{a}_n\cos(\frac{n\pi x}{\ell}) + \widehat{b}_n\sin(\frac{n\pi x}{\ell})$*; and*

(ii) $\widetilde{f}(x) \sim \frac{\widetilde{a}_0}{2} + \sum_{n=1}^{\infty}\widetilde{a}_n\cos(\frac{n\pi x}{\ell}) + \widetilde{b}_n\sin(\frac{n\pi x}{\ell})$,

then

$$c_1\widehat{f} + c_2\widetilde{f} \sim c_1\frac{\widehat{a}_0}{2} + c_2\frac{\widetilde{a}_0}{2} + \sum_{n=1}^{\infty}[(c_1\widehat{a}_n + c_2\widetilde{a}_n)\cos(\frac{n\pi x}{\ell}) + (c_1\widehat{b}_n + c_2\widetilde{b}_n)\sin(\frac{n\pi x}{\ell})].$$

Theorem 22. *If*

(i) $f \in PC(\mathbb{R})$*, 2ℓ-periodic with* $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{\ell}) + b_n\sin(\frac{n\pi x}{\ell})$*; and*

(ii) $g \in PC(\alpha,\beta)$,

then

$$\int_\alpha^\beta f(x)g(x)\,dx = \int_\alpha^\beta \frac{a_0}{2}g(x)\,dx + \sum_{n=1}^{\infty}\int_\alpha^\beta[a_n\cos(\frac{n\pi x}{\ell}) + b_n\sin(\frac{n\pi x}{\ell})]g(x)\,dx.$$

EXAMPLE 2.12. Going back to Eq. (2.4) of Example 2.7, using $g(x) = 1$, we get

$$\frac{\ell^2}{2} = \int_0^\ell x\,dx = \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^\ell\sin(\frac{n\pi x}{\ell})\,dx = \frac{2\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[(-1)^n - 1].$$

After a little manipulation we get

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

The following theorem gives us the conditions under which term-by-term differentiation is justified.

Theorem 23. *If f is 2ℓ-periodic, continuous and p-smooth for all x, then the Fourier Series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell})$$

is differentiable for all x at which f'' exists and

$$f'(x) = \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n\pi x}{\ell}) + \beta_n \sin(\frac{n\pi x}{\ell})$$

where $\alpha_n = \frac{n\pi b_n}{\ell}$ and $\beta_n = -\frac{n\pi a_n}{\ell}$.

EXAMPLE 2.13. Again consider Eq. (2.4) of Example 2.7

$$f(x) = x \sim \frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(\frac{n\pi x}{\ell}) \quad \text{for } -\ell < x < \ell.$$

We have $f'(x) = 1$, $f'(0) = 1$ and the differentiated Fourier Series looks like

$$\frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{n\pi}{\ell} \cos(\frac{n\pi x}{\ell}) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos(\frac{n\pi x}{\ell}).$$

At $x = 0$: $2 \sum_{n=1}^{\infty} (-1)^{n+1} = 2\{1 - 1 + 1 - 1 + 1 - 1 + \cdots\}$ does not converge. However, this does not contradict the theorem since f is not continuous.

One last result.

Theorem 24. *If*

(i) f is periodic with Fourier coefficients a_n, b_n; and

(ii) $\sum_{n=1}^{\infty} (|n^k a_n| + |n^k b_n|)$ converges for some integer $k \geqslant 1$,

then f has continuous derivatives f', f'', ..., $f^{(k)}$ whose Fourier Series are the differentiated series of f.

2.7 Mean Error

While some functions can be represented by an infinite Fourier Series, in practice we can only evaluate a finite series. The question that arises is: how good an approximation is a truncated Fourier Series? Before we can answer this we need a way to measure the "distance" between two functions.

Suppose we have a function f together with two approximations to f given by g_1 and g_2. The question is: which is a better approximation t f? We use norms to measure the "distance" between functions. We say that g_1 is a better approximation to f than g_2 if

$$\|f - g_1\| < \|f - g_2\|.$$

For $f \in PC(a, b)$ there is more than one way to define a norm. Two possible norms are:

$$\|f\|_\infty := \sup_{a \leqslant x \leqslant b} |f(x)|, \quad \text{and} \quad \|f\|_{L^2} := (\langle f, f\rangle)^{1/2} = \left[\int_a^b f^2(x)\, dx\right]^{1/2}.$$

Which is the "better" norm? The answer is not at all obvious, as the following illustrates.

Consider $f, g_1, g_2 \in PC(0, \frac{1}{\varepsilon^2})$ defined as follows:

$$f(x) = 1, \qquad g_1(x) = 1 + \varepsilon, \qquad g_2(x) = \begin{cases} 2 & 0 < x \leqslant \varepsilon^2 \\ 1 & \varepsilon^2 < x < \frac{1}{\varepsilon^2} \end{cases}.$$

Then

$$|f(x) - g_1(x)| = \varepsilon, \qquad \text{and} \qquad |f(x) - g_2(x)| = \begin{cases} 1 & 0 < x \leqslant \varepsilon^2 \\ 0 & \varepsilon^2 < x < \frac{1}{\varepsilon^2} \end{cases}$$

so that

$$\begin{aligned}
\|f - g_1\|_\infty &= \sup_{0 \leqslant x \leqslant \frac{1}{\varepsilon^2}} |f(x) - g_1(x)| = \varepsilon, \\
\|f - g_2\|_\infty &= \sup_{0 \leqslant x \leqslant \frac{1}{\varepsilon^2}} |f(x) - g_2(x)| = 1, \\
\|f - g_1\|_{L^2} &= \left\{\int_0^{1/\varepsilon^2} |f(x) - g_1(x)|^2\, dx\right\}^{1/2} = \left\{\int_0^{1/\varepsilon^2} \varepsilon^2\, dx\right\}^{1/2} = 1, \\
\|f - g_2\|_{L^2} &= \left\{\int_0^{1/\varepsilon^2} |f(x) - g_2(x)|^2\, dx\right\}^{1/2} = \left\{\int_0^{1/\varepsilon^2} 1^2\, dx\right\}^{1/2} = \varepsilon.
\end{aligned}$$

Hence, relative to the sup–norm, g_1 is a better approximation to f than is g_2, whereas, relative to the L^2–norm, g_2 is the better approximation to f. Because the L^2–norm is related to an inner product (see definition 15), we will find it convenient to adopt the L^2–norm.

Now that we have a way to measure the distance between functions, we return to the original question, namely how good an approximation is a truncated Fourier Series?

Suppose $f \in PC(a, b)$ has Fourier Series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}), \quad \text{where } \ell = \frac{b-a}{2}.$$

Define

$$f_N(x) := \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}), \qquad (f_N \text{ is the truncated Fourier Series of } f)$$

$$g_N(x) := \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(\frac{n\pi x}{\ell}) + B_n \sin(\frac{n\pi x}{\ell}).$$

We now try to find constants A_n, and B_n so that g_N is a better approximation to f than is f_N relative to the L^2–norm.

Define:

$$E_N := \|f - g_N\|.$$

Now find A_n, and B_n which minimize E_N. We have

$$E_N^2 = \|f - g_N\|^2 = \langle f - g_N, f - g_N\rangle = \langle f, f\rangle - 2\langle f, g_N\rangle + \langle g_N, g_M\rangle \tag{2.5}$$

Examine each term in (2.5) individually. The first term is easy: $\langle f, f\rangle = \|f\|^2$. Using the notation introduced in (2.2), the second term becomes

$$\begin{aligned}
\langle f, g_N\rangle &= \left\langle f, \frac{A_0}{2}c_0(x) + \sum_{n=1}^{N} A_n c_n(x) + B_n s_n(x)\right\rangle \\
&= \frac{A_0}{2}\langle f, c_0\rangle + \sum_{n=1}^{N}(A_N\langle f, c_n\rangle + B_n\langle f, s_n\rangle) \\
&= \frac{A_0}{2}\ell a_0 + \sum_{n=1}^{N}(A_N \ell a_n + B_n \ell b_n) \\
&= \ell\left\{\frac{1}{2}a_0 A_0 \sum_{n=1}^{N}(a_N A_n + b_n B_n)\right\}.
\end{aligned}$$

The third term becomes

$$\begin{aligned}
\langle g_N, g_N\rangle &= \left\langle \frac{A_0}{2}c_0(x) + \sum_{n=1}^{N} A_n c_n(x) + B_n s_n(x), \frac{A_0}{2}c_0(x) + \sum_{m=1}^{N} A_m c_m(x) + B_m s_m(x)\right\rangle \\
&= \frac{A_0^2}{4}\langle c_0, c_0\rangle + 2\sum_{n=1}^{N}\left(\frac{A_0}{2}A_N\langle c_0, c_n\rangle + \frac{A_0}{2}B_n\langle c_0, s_n\rangle\right) \\
&\quad + \sum_{n=1}^{N}\sum_{m=1}^{N}(A_n A_m\langle c_n, c_m\rangle + A_n B_m\langle c_n, s_m\rangle + A_m B_n\langle c_m, s_n\rangle + B_n B_m\langle s_n, s_m\rangle) \\
&= \frac{A_0^2}{4}(2\ell) + \sum_{n=1}^{N}\sum_{m=1}^{N}(A_n A_m \ell\delta_{nm} + B_n B_m \ell\delta_{nm}) \\
&= \ell\left\{\frac{A_0^2}{2} + \sum_{n=1}^{N}(A_n^2 + b_n^2)\right\}.
\end{aligned}$$

Plug these expressions into (2.5) to get

$$\begin{aligned}
E_n^2 &= \|f\|^2 - 2\langle f, g_N\rangle + \langle g_N, g_M\rangle \\
&= \|f\|^2 - 2\ell\left\{\frac{1}{2}a_0A_0\sum_{n=1}^{N}(a_NA_n + b_nB_n)\right\} + \ell\left\{\frac{A_0^2}{2} + \sum_{n=1}^{N}(A_n^2 + b_n^2)\right\} \\
&= \|f\|^2 + \ell\left\{\frac{A_0^2}{2} - a_0A_0 + \sum_{n=1}^{N}(A_n^2 - 2a_nA_n + B_n^2 - 2b_nB_n)\right\} \\
&= \|f\|^2 + \ell\left\{\frac{1}{2}(A_0 - a_0)^2 - \frac{a_0^2}{2} + \sum_{n=1}^{N}[(A_n - a_n)^2 - a_n^2 + (B_n - b_n)^2 - b_n^2]\right\} \\
&= \|f\|^2 + \ell\left\{\frac{1}{2}(A_0 - a_0)^2 + \sum_{n=1}^{N}[(A_n - a_n)^2 + (B_n - b_n)^2]\right\} - \ell\left\{\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right\}.
\end{aligned}$$

The first and last term are fixed; only the middle term contains A_n's and B_n's. The middle term, being the sum of squares, is non-negative and so is minimized by setting $A_n = a_n$ and $B_n = b_n$ for all n. Thus, the best g_N to approximate f is $g_N = f_N$. In this case the minimum error is

$$E_N^2 = \|f\|^2 - \ell\left\{\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right\}.$$

Using $E_N^2 \geqslant 0$ gives Bessel's inequality:

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leqslant \frac{1}{\ell}\|f\|^2 = \frac{1}{\ell}\int_a^b f^2(x)\,dx.$$

Taking the limit $N \to \infty$ yields

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leqslant \frac{1}{\ell}\|f\|^2 = \frac{1}{\ell}\int_a^b f^2(x)\,dx.$$

Using the fact that the Fourier Series converges and the previous theorem on term-by-term integration, this condition can be strenghtened to what is called Parseval's equation

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\ell}\|f\|^2.$$

See Figure 2.1 for several approximations to the function defined in Example 2.7 with $\ell = \pi$.

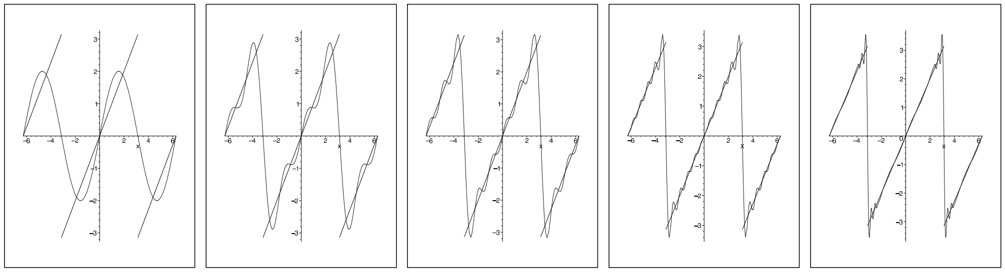


Figure 2.1: Truncated Fourier Series f_N for $N = 1, 3, 5, 10, 20$.

2.8 Complex Fourier Series

A well known formula in complex variables is

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad 8\text{(Euler's formula)}$$

¿From this we can deduce

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh i\theta, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -i\sinh i\theta.$$

We can make use of these formulas to write a Fourier Series in complex form. Suppose $f \in PC(a, b)$ has Fourier Series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell}), \quad \text{where } \ell = \frac{b-a}{2}.$$

Then

$$\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \left\{a_n(e^{i\frac{n\pi x}{\ell}} + e^{-i\frac{n\pi x}{\ell}}) - ib_n(e^{i\frac{n\pi x}{\ell}} - e^{-i\frac{n\pi x}{\ell}})\right\} \\
&\sim \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \left\{(a_n - ib_n)e^{i\frac{n\pi x}{\ell}} + (a_n + ib_n)e^{-i\frac{n\pi x}{\ell}}\right\} \\
&\sim c_0 + \sum_{n=1}^{\infty} (c_n e^{i\frac{n\pi x}{\ell}} + c_{-n} e^{-i\frac{n\pi x}{\ell}}),
\end{aligned}$$

where

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \qquad i = 1, 2, 3, \ldots.$$

The Fourier Series may be conveniently rewritten

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}.$$

Recall the formulas for the Fourier coefficients:

$$a_n = \frac{1}{\ell}\int_a^b f(x)\cos(\frac{n\pi x}{\ell})\,dx, \qquad b_n = \frac{1}{\ell}\int_a^b f(x)\sin(\frac{n\pi x}{\ell})\,dx.$$

These lead to

$$\begin{aligned}
c_0 &= \frac{a_0}{2} = \frac{1}{2\ell}\int_a^b f(x)\,dx,\\
c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2\ell}\int_a^b f(x)[\cos(\frac{n\pi x}{\ell}) - i\sin(\frac{n\pi x}{\ell})]\,dx = \frac{1}{2\ell}\int_a^b f(x)e^{-i\frac{n\pi x}{\ell}}\,dx,\\
c_{-n} &= \frac{1}{2}(a_n + ib_n) = \frac{1}{2\ell}\int_a^b f(x)[\cos(\frac{n\pi x}{\ell}) + i\sin(\frac{n\pi x}{\ell})]\,dx = \frac{1}{2\ell}\int_a^b f(x)e^{i\frac{n\pi x}{\ell}}\,dx.
\end{aligned}$$

To summarize, we have the following:

Theorem 25. *The complex Fourier Series for $f \in PC(a,b)$ is*

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}},$$

where

$$c_n = \frac{1}{2\ell}\int_a^b f(x)e^{-i\frac{n\pi x}{\ell}}\,dx, \qquad \ell = \frac{b-a}{2}.$$

We finish this section with an example.

EXAMPLE 2.14. Calculate the complex Fourier Series for

$$f(x) = x, \quad -\pi < x < \pi, \quad f(x+2\pi) = f(x).$$

Solution
The complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

We get

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0,\\
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx = \frac{1}{2\pi}\left\{\frac{2\pi i}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi\right\} = \frac{i}{n}(-1)^n, \quad n \neq 0.
\end{aligned}$$

Therefore

$$f(x) \sim i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{inx}.$$

Chapter 3

Separation of Variables

June 17, 2010

3.1 Homogeneous Equations

In Example 1.9 we successfully applied separation of variables to a linear homogeneous PDE The PDE was the 1-d heat equation $u_t = ku_{xx}$, where k was assumed constant. Looking for a solution of the form $u(x,t) = X(x)T(t)$ we got:

$$\frac{T'}{kT} = \frac{X''}{X}.$$

The left side of the equation is a function of time t only whereas the right side is a function of x only. This means that both sides must equal some constant ($-\lambda$ say):

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

This lead to ODEs for T and X:

$$T' + \lambda kT = 0, \qquad X'' + \lambda X = 0.$$

Let us try this method for a Neuman problems of the heat equation

Exercise 3.1. ✗✗

Consider the Neumann problem for the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad 0 < x < L,\ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < L.$$

(a) Give a short physical interpretation of this problem.

(b) Use the method of separation of variables to solve this problem. First show that there are no separated solutions which grow exponentially in time.

Hint: The solution can be written as

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t} \cos\frac{n\pi x}{L}.$$

Find the λ_n .

(c) Show that the initial condition, $u(x,0) = f(x)$, leads to the Fourier cosine series:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

(d) Solve for A_n by using

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} \frac{L}{2} & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

for $n,\, m \geqslant 0$,.

(e) Consider the limit $\lim_{t\to\infty} u(x,t)$. Find the steady state solution and show that $u(x,t)$ approaches this steady-state solution as $t \to \infty$.

Solution:

(a) The partial differential equation describes heat flow in a one-dimensional rod where

- all thermal properties are constant in any cross-section,
- there is no heat flow through the lateral sides, that is, the lateral sides are perfectly insulated,
- there are no internal heat sources in the rod,
- the boundary conditions imply that there is no heat flow through the ends of the rod, and
- the initial temperature distribution in the rod is $u(x, 0) = f(x)$ for $0 < x < L$.

(b) In order to show that there are no separated solutions which grow exponentially in time, we have to show that there are no negative eigenvalues.

Assuming a solution of the form

$$u(x, t) = \phi(x) \cdot G(t)$$

and separating variables, we get

$$\frac{\phi''(x)}{\phi(x)} = \frac{G'(t)}{k\,G(t)} = -\lambda$$

where λ is a constant. The partial differential equation is reduced to two ordinary differential equations:

Spatial Equation:

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, \quad 0 \leqslant x \leqslant L \\ \phi'(0) = 0 \\ \phi'(L) = 0. \end{aligned}$$

Time Equation:

$$G'(t) + k\lambda G(t) = 0, \quad t \geqslant 0.$$

Case 1: We consider the spatial equation first, suppose that $\lambda = -\mu^2 < 0$, where $\mu \neq 0$, the general solution in this case is

$$\phi(x) = A\cosh\mu x + B\sinh\mu x.$$

Applying the boundary conditions, since

$$\phi'(x) = \mu A\sinh\mu x + \mu B\cosh\mu x,$$

we get $\phi'(0) = \mu B = 0$, and $B = 0$. Also, $\phi'(L) = \mu A\sinh\mu L = 0$, so that $A = 0$. Thus, the only solution in this case is the trivial solution $\phi(x) \equiv 0$, and there are no separated solutions which grow exponentially in time.

Case 2: $\lambda = \mu^2 > 0$, where $\mu \neq 0$.

The general solution in this case is

$$\phi(x) = A\cos\mu x + B\sin\mu x.$$

Since

$$\phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x,$$

applying the first boundary condition, we have

$$\phi'(0) = \mu B \cdot 1 = 0,$$

so that $B = 0$. Applying the second boundary condition, we have

$$\phi'(L) = -A\mu\sin\mu L = 0,$$

and if $A = 0$ we get only the trivial solution again. Therefore, we get nontrivial solutions if and only if $\sin\mu L = 0$, that is, when

$$\mu = \frac{n\pi}{L}$$

where $n \geqslant 1$. The eigenvalues for this problem are

$$\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2},$$

with corresponding eigenfunctions

$$\phi_n(x) = \cos\tfrac{n\pi x}{L}$$

for $n = 1, 2, 3, \ldots$.

Case 3: $\lambda = 0$.

The general solution in this case is

$$\phi(x) = Ax + B,$$

Applying the boundary conditions, we get $\phi'(0) = A = 0$, and $\phi'(L) = A = 0$, and the only solution in this case is the constant solution $\phi_0(x) = B$. The eigenvalue for this problem is $\lambda_0 = 0$, with corresponding eigenfunction $\phi_0(x) = 1$.

The solutions to the time equation corresponding to these nontrivial solutions are

$$G_n(t) = e^{-\frac{kn^2\pi^2 t}{L^2}}$$

for $n = 0, 1, 2, 3, \ldots$.

For $n \geqslant 0$, the functions

$$u_n(x,t) = \phi_n(x) \cdot G_n(t) = \cos \tfrac{n\pi x}{L} e^{-\frac{kn^2\pi^2 t}{L^2}}$$

are also solutions to the partial differential equation satisfying the boundary conditions, and since the partial differential equation and the boundary conditions are homogeneous, by the superposition principle, the solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L} \cdot e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

Note that the constant A_0 corresponds to the $n = 0$ term.

(c) The initial condition is

$$u(x,0) = f(x)$$

for $0 < x < L$, so that if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L}$$

for $0 < x < L$, then the initial condition will be satisfied.

(d) In order to determine the coefficients A_n we use the fact that $\cos \frac{n\pi x}{L}$ and $\cos \frac{m\pi x}{L}$ are orthogonal on the interval $[0, L]$ in the sense that

$$\int_0^L \cos \tfrac{n\pi x}{L} \cos \tfrac{m\pi x}{L}\, dx = \begin{cases} 0 & \text{for} \quad n, m \geqslant 0,\ n \neq m \\ \dfrac{L}{2} & \text{for} \quad n = m \neq 0 \\ L & \text{for} \quad n = m = 0. \end{cases}$$

Starting from the initial condition

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L} = \sum_{n=0}^{\infty} A_n \cos \tfrac{n\pi x}{L}$$

for $0 < x < L$, we multiply both sides of this equation by $\cos \frac{m\pi x}{L}$, and integrate over the interval $[0, L]$ to get

$$\int_0^L f(x) \cos \tfrac{m\pi x}{L}\, dx = \sum_{n=0}^{\infty} \int_0^L A_n \cos \tfrac{n\pi x}{L} \cos \tfrac{m\pi x}{L}\, dx,$$

and from the orthogonality of the eigenfunctions on the interval $[0, L]$, we have

$$\int_0^L f(x) \cos \tfrac{m\pi x}{L}\, dx = \begin{cases} \displaystyle\int_0^L A_0\, dx = LA_0 & \text{for} \quad m = 0 \\ \dfrac{LA_m}{2} & \text{otherwise.} \end{cases}$$

Therefore

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\tfrac{n\pi x}{L}\,dx, \quad n \geqslant 1.$$

(e) The solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\tfrac{n\pi x}{L}\cdot e^{-\frac{kn^2\pi^2 t}{L^2}},$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\tfrac{n\pi x}{L}\,dx, \quad n \geqslant 1.$$

Taking the limit as $t \to \infty$, we have

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left(A_0 + \sum_{n=1}^{\infty} A_n \cos\tfrac{n\pi x}{L}\cdot e^{-\frac{kn^2\pi^2 t}{L^2}}\right)\\ &= A_0 + \lim_{t\to\infty}\left(\sum_{n=1}^{\infty} A_n \cos\tfrac{n\pi x}{L}\cdot e^{-\frac{kn^2\pi^2 t}{L^2}}\right)\\ &= A_0,\end{aligned}$$

and

$$\lim_{t\to\infty} u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx.$$

Let v denote the steady-state solution, then v satisfies the boundary value problem

$$\begin{aligned}&\frac{d^2v}{dx^2} = 0, \quad 0 < x < L\\ &\frac{dv}{dx}(0) = 0\\ &\frac{dv}{dx}(L) = 0\end{aligned}$$

The general solution is

$$v(x) = Ax + B,$$

and

$$\frac{dv}{dx}(x) = A.$$

Applying the boundary conditions, we have

$$A = \frac{dv}{dx}(0) = \frac{dv}{dx}(L) = 0,$$

and the steady-state solution is

$$v(x) = B, \quad 0 < x < L.$$

To evaluate the constant B, we note that since the total heat energy in the bar is constant, then

$$\int_0^L B\,dx = \int_0^L v(x)\,dx = \int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx,$$

so that

$$B \cdot L = \int_0^L f(x)\,dx,$$

that is,

$$v(x) = B = \frac{1}{L}\int_0^L f(x)\,dx$$

for $0 < x < L$.

3.1.1 General Linear Homogeneous Equation

Now suppose we consider a slightly more general linear homogeneous PDE:

$$a_1 u_{tt} + a_2 u_t = k_1 u_{xx} + k_2 u_x + k_3 u.$$

If we try separation of variables $u(x,t) = X(x)T(t)$ we get

$$a_1 \frac{T''}{T} + a_2 \frac{T'}{T} = k_1 \frac{X''}{X} + k_2 \frac{X'}{X} + k_3.$$

If the coefficients are constant, then the left side of the equation is a function of time t only and the right side of the equation is a function of the spatial variable x only. This means that both sides must equal some constant ($-\lambda$ say):

$$a_1 \frac{T''}{T} + a_2 \frac{T'}{T} = k_1 \frac{X''}{X} + k_2 \frac{X'}{X} + k_3 = -\lambda,$$

and so we get ODEs for T and X:

$$a_1 T'' + a_2 T' + \lambda T = 0, \qquad k_1 X'' + k_2 X' + (k_3 + \lambda) X = 0.$$

So, once again, separation of variables seems to work.

But what happens if the coefficients are not constant? For example consider the PDE $u_t = xtu_{xx}$. In this case, separation of variables leads to

$$\frac{T'}{tT} = \frac{xX''}{X},$$

which leads to ODEs in T and X. But for $u_t = (x+t)u_{xx}$, separation of variables leads to

$$\frac{T'}{T} = (x+t)\frac{X''}{X}.$$

In this case it doesn't work.

The success of the method of separation of variables in Example 1.9 depended not only on the fact that the PDE was linear and homogeneous, but also on the fact that the boundary conditions were linear and homogeneous. Inserting $u(x,t) = X(x)T(t)$ into the homogeneous boundary conditions lead to

$$X(0) = X(\ell) = 0.$$

More generally, the homogeneous Robin's conditions

$$\alpha u(0,t) + \beta \frac{\partial u}{\partial x}(0,t) = 0$$

leads to

$$\alpha X(0) + \beta X'(0) = 0.$$

What happens when either the PDE or the boundary conditions are no longer homogeneous? For a nonhomogeneous PDE like

$$u_t = k u_{xx} + \gamma,$$

using $u(x,t) = X(x)T(t)$ leads to

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{\gamma}{kXT}.$$

so the method doesn't work. For the nonhomogeneous boundary conditions

$$u(0,t) = c_1, \qquad u(\ell,t) = c_2$$

we get the inconsistency

$$T(t) = \frac{c_1}{X(0)} = \frac{c_2}{X(\ell)}.$$

Again, the method doesn't work.

In summary, what we conclude is that if we have a linear homogeneous PDE with linear homogeneous boundary conditions, then separation of variables *may* work. But the method may not work. It depends on the particular problem. For nonhomogeneous problems, the method does not work; at least not directly. In the latter case, however, all is not lost, as we shall see in the next section.

Exercise 3.2. ✗✗✗

The one-dimensional wave equation in presence of a damping term, where in the simplest case the resistance can be assumed to be proportional to the velocity, is called the **damped one-dimensional wave equation**:

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad 0 < x < L,\ t > 0.$$

Solve this equation subject to the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for all} \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{for} \quad 0 < x < L.$$

Solution:

(a) Assuming a product solution of the form $u(x,t) = X(x)T(t)$, and substituting this into the equation we have

$$XT'' + 2kXT' = c^2X''T.$$

Dividing by c^2XT and separating variables, we obtain

$$\frac{T''}{c^2T} + \frac{2kT'}{c^2T} = \frac{X''}{X}.$$

Since x and t are independent variables and the left-hand-side depends only on t, while the right hand side depends only on x, then both sides must be constant, and so

$$\frac{T''}{c^2T} + \frac{2kT'}{c^2T} = -\lambda \qquad \text{and} \qquad \frac{X''}{X} = -\lambda,$$

where λ is the separation constant, and the functions X and T satisfy the following indexOrdinary differential equationordinary differential equations

$$\begin{aligned} X'' + \lambda X &= 0 \\ T'' + 2kT' + \lambda c^2 T &= 0. \end{aligned}$$

Now, we can satisfy the boundary conditions by requiring that $X(0) = X(L) = 0$, and so X satisfies the boundary value problem

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X(L) &= 0. \end{aligned}$$

As in the previous problems, we only get a nontrivial solution if the separation constant λ is positive, say $\lambda = \mu^2$ where $\mu \neq 0$, and in this case, the equations for X and T become

$$X'' + \mu^2 X = 0, \quad X(0) = 0, \quad X(L) = 0,$$
$$T'' + 2kT' + (\mu c)^2 T = 0,$$

where $\mu \neq 0$.

(b) The general solution to the equation

$$X'' + \mu^2 X = 0$$

is given by

$$X(x) = A\cos\mu x + B\sin\mu x,$$

where the constants are determined from the boundary conditions. Since $X(0) = 0$, then we must have $A = 0$; and since $X(L) = 0$, the only nontrivial solutions arise when $\sin\mu L = 0$, and this happens if and only if $\mu L = n\pi$, where n is an integer.

Therefore, the only nontrivial solutions to the boundary value problem for X occur for

$$\mu = \mu_n = \tfrac{n\pi}{L}$$

and the solutions are

$$X = X_n = \sin\tfrac{n\pi x}{L}$$

for $n = 1, 2, \ldots$.

(c) For each integer $n \geqslant 1$, the corresponding equation for T is

$$T'' + 2kT' + \left(\tfrac{n\pi c}{L}\right)^2 T = 0,$$

a second order, linear, homogeneous, constant coefficient equation which we know how to solve. Assuming a solution of the form $T(t) = e^{\alpha t}$, and plugging this into the differential equation we get the characteristic equation

$$\alpha^2 + 2k\alpha + \tfrac{n^2\pi^2c^2}{L^2} = 0,$$

and the roots of this quadratic equation are

$$\alpha_{n,1} = -k + \sqrt{k^2 - \tfrac{n^2\pi^2c^2}{L^2}} \qquad \text{and} \qquad \alpha_{n,2} = -k - \sqrt{k^2 - \tfrac{n^2\pi^2c^2}{L^2}}.$$

In order to find the corresponding solutions $T_n(t)$, we need to consider three cases, according to whether $k^2 - \frac{n^2\pi^2c^2}{L^2}$ is zero, positive or negative.

Case 1: $k^2 - \frac{n^2\pi^2c^2}{L^2} = 0$. In this case, we have equal real roots, and the solution is

$$T_n(t) = e^{-kt}\left(a_n + b_n t\right)$$

where $k = \frac{n\pi c}{L} > 0$.

Case 2: $k^2 - \frac{n^2\pi^2c^2}{L^2} > 0$. In this case, we have two distinct real roots, and the solution is

$$T_n(t) = e^{-kt}\left(a_n \cosh \alpha_n t + b_n \sinh \alpha_n t\right)$$

where $\alpha_n = \sqrt{k^2 - \frac{n^2\pi^2c^2}{L^2}}$.

Case 3: $k^2 - \frac{n^2\pi^2c^2}{L^2} < 0$. In this case, we have two distinct imaginary roots, and the solution is

$$T_n(t) = e^{-kt}\left(a_n \cos \alpha_n t + b_n \sin \alpha_n t\right)$$

where $\alpha_n = \sqrt{\frac{n^2\pi^2c^2}{L^2} - k^2}$.

(d) Since the partial differential equation and the boundary conditions are linear and homogeneous, we use the superposition principle to write the solution as a linear combination of the solutions that we found in part (c)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} X_n(x) \cdot T_n(t).$$

If $\frac{kL}{\pi c}$ is not a positive integer, then

$$k^2 - \tfrac{n^2\pi^2c^2}{L^2} \neq 0,$$

and either $1 \leqslant n < \frac{kL}{\pi c}$, or $n > \frac{kL}{\pi c}$, so we are in Case 2 or Case 3, and the solution is

$$\begin{aligned} u(x,t) = e^{-kt} & \sum_{1 \leqslant n < kL/\pi c} \sin \tfrac{n\pi x}{L}\left(a_n \cosh \alpha_n t + b_n \sinh \alpha_n t\right) \\ + e^{-kt} & \sum_{kL/\pi c < n < \infty} \sin \tfrac{n\pi x}{L}\left(a_n \cos \alpha_n t + b_n \sin \alpha_n t\right) \end{aligned}$$

where these sums run over nonnegative integers only, and $\alpha_n = \sqrt{\left|k^2 - \left(\frac{n\pi c}{L}\right)^2\right|}$.

Also, to satisfy the initial conditions, the a_n are the Fourier sine coefficients for the odd periodic extension of $f(x)$, that is,

$$a_n = \frac{2}{L}\int_0^L f(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n = 1, 2, \ldots$.

If we differentiate this expression for $u(x,t)$ with respect to t, and set $t=0$, then we see that $-ka_n+\alpha_n b_n$ are just the Fourier sine coefficients of the odd periodic extension of $g(x)$, that is,

$$-ka_n + \alpha_n b_n = \frac{2}{L}\int_0^L g(x)\sin\tfrac{n\pi x}{L}\,dx$$

for $n = 1, 2, \ldots$.

(e) If $\frac{kL}{\pi c}$ is a positive integer, then we have to add the corresponding term in the sum when the $n = \frac{kL}{\pi c}$.

In this case, if $n_0 = \frac{kL}{\pi c}$, the solution is as in (d) with the one additional term

$$\sin\tfrac{kx}{c}\left(a_{kL/\pi c}e^{-kt} + b_{kL/\pi c}te^{-kt}\right)$$

with a_n and b_n as in (d), except that $b_{kL/\pi c}$ is determined from the equation

$$-ka_{kL/\pi c} + b_{kL/\pi c} = \frac{2}{L}\int_0^L g(x)\sin\tfrac{kx}{c}\,dx.$$

3.2 Nonhomogeneous Equations

We saw in the previous section that the method of separation of variables can not be directly applied to problems where either the PDE or the boundary conditions are nonhomogeneous. To handle problems of this type, where either the PDE or the boundary conditions are nonhomogeneous, we split the problem into two parts: one part is an ODE to which the nonhomogeneities are attached, and another part consisting of the homogeneous counterpart of the original PDE This is best illustrated by means of an example.

EXAMPLE 3.1. Consider the following nonhomogeneous, 1–dimensional heat equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \gamma, \qquad 0 \leqslant x \leqslant 1,\ t > 0,\\
u(0,t) &= c_L \qquad (\gamma, c_L, c_R \text{ are positive constants})\\
u(1,t) &= c_R\\
u(x,0) &= 0.
\end{aligned}$$

This problem governs the temperature inside a one–dimensional rod of unit length in which heat is generated internally. Notice that neither the PDE nor the boundary conditions are homogeneous.

If we try the usual separation of variables form of solution $u(x,t) = X(x)T(t)$ in the PDE we get

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{\gamma}{kXT},$$

which is not separable, and if we try this in the boundary conditions we get the inconsistency

$$T(t) = \frac{c_1}{X(0)} = \frac{c_2}{X(\ell)}.$$

What we will do is split the solution into two pieces

$$u(x,t) = v(x) + w(x,t). \tag{3.1}$$

Inserting this into the original PDE yields

$$w_t = w_{xx} + v'' + \gamma.$$

We have a single PDE in two unknown functions: v and w. Since we have two unknowns, we need two equations. We impose the following condition:

$$v'' + \gamma = 0.$$

If we now insert (3.1) into the boundary conditions we get

$$\begin{aligned} v(0) + w(0,t) &= c_L, \\ v(1) + w(1,t) &= c_R. \end{aligned}$$

Impose the conditions $v(0) = c_L$ and $v(1) = c_R$. The initial condition becomes

$$v(x) + w(x,0) = 0 \qquad \text{or} \qquad w(x.0) = -v(x).$$

Putting everything together we get that v satisfies

$$\begin{aligned} v'' &= -\gamma, \qquad 0 \leqslant x \leqslant 1, \\ v(0) &= c_L, \qquad 1v(1) = c_R, \end{aligned}$$

and w satisfies

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} \qquad 0 \leqslant x \leqslant 1, \qquad 1t > 0, \\ w(0,t) &= 0, \\ w(1,t) &= 0, \\ w(x,0) &= -v(x). \end{aligned}$$

The nonhomogeneities of the original problem have been shifted over to the ODE for v. The problem for w is a linear, homogeneous PDE with homogeneous boundary conditions.

Solving the problem for v we get

$$v(x) = -\frac{1}{2}\gamma x^2 + (\frac{1}{2}\gamma - c_L + c_R)x + c_L.$$

To solve the problem for w we apply separation of variables:

$$w(x,t) = X(x)T(t).$$

This leads to

Problem for T:	Problem for X:
$T' + \lambda T = 0,$	$X'' + \lambda X = 0,$
	$X(0) = X(1) = 0.$

Solving the problem for X, we get a nontrivial solution only for $\lambda > 0$:

$$\lambda = \lambda_n = n^2\pi^2, \quad X = X_n(x) = \sin(n\pi x), \qquad n = 1, 2, \ldots.$$

This means that $T = T_n(t) = e^{-n^2\pi^2 t}$ and the family of solutions we get for w is

$$w_n(x,t) = X_n(x)T_n(t) = e^{-n^2\pi^2 t}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

For each $n = 1, 2, \ldots$ w_n is a solution to the PDE for w which also satisfies the homogeneous boundary conditions. We construct more general solutions by applying the principle of superposition:

$$w(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin(n\pi x).$$

We now apply the initial condition $w(x,0) = -v(x)$ to get

$$-v(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x). \qquad \text{(Fourier sine series)}$$

Since we have a Fourier sine series, we let v_o be the odd, 2ℓ-periodic extension of v (with $\ell = 1$) so we get

$$\begin{aligned} b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} -v_o(x)\sin(\frac{n\pi x}{\ell})\,dx = -2\int_0^1 v(x)\sin(n\pi x)\,dx \\ &= -2\int_0^1 [-\frac{1}{2}\gamma x^2 + (\frac{1}{2}\gamma - c_L + c_R)x + c_L]\sin(n\pi x)\,dx \\ &= \frac{2\gamma}{n^3\pi^3}[(-1)^n - 1] + \frac{2}{n\pi}[c_L - (-1)^n c_R]. \end{aligned} \tag{3.2}$$

The final solution is

$$\begin{aligned} u(x,t) &= v(x) + w(x,t) = v(x) + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin(n\pi x) \\ &= -\frac{1}{2}\gamma x^2 + (\frac{1}{2}\gamma - c_L + c_R)x + c_L + 2\sum_{n=1}^{\infty}\left\{\frac{2\gamma}{n^3\pi^3}[(-1)^n - 1] + \frac{2}{n\pi}[c_L - (-1)^n c_R]\right\} e^{-n^2\pi^2 t}\sin(n\pi x \end{aligned}$$

Notice that $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\{v(x)+w(x,t)\} = v(x)$. For this reason, w is sometimes called the **transient solution**, while $v(x)$ is called the **equilibrium solution**, or the **steady state solution**. For the special case $\gamma = 0$, the solution reduces to

$$u(x,t) = (c_R - c_L)x + c_L + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[c_L - (-1)^n c_R]e^{-n^2\pi^2 t}\sin(n\pi x).$$

Here we plot some results with the following parameter values: $\gamma = 15$, $c_L = 1$, $c_R = 2$. Figure 3.1 contains a plot of the steady state solution $v(x)$. Figure 3.2 shows approximations to $u(x,t)$ by using a truncated series with $N = 5, 50$, at early times. In Figure 3.3 we show the evolution of $u(x,t)$ as time $t \to 0.6$. Notice that convergence to the limiting solution takes place very quickly.

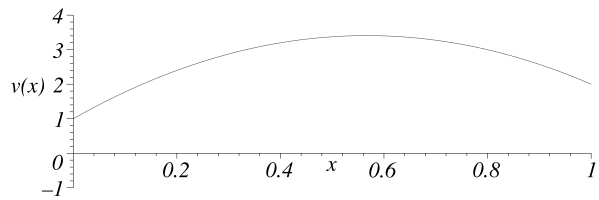


Figure 3.1: Steady state solution $v(x)$

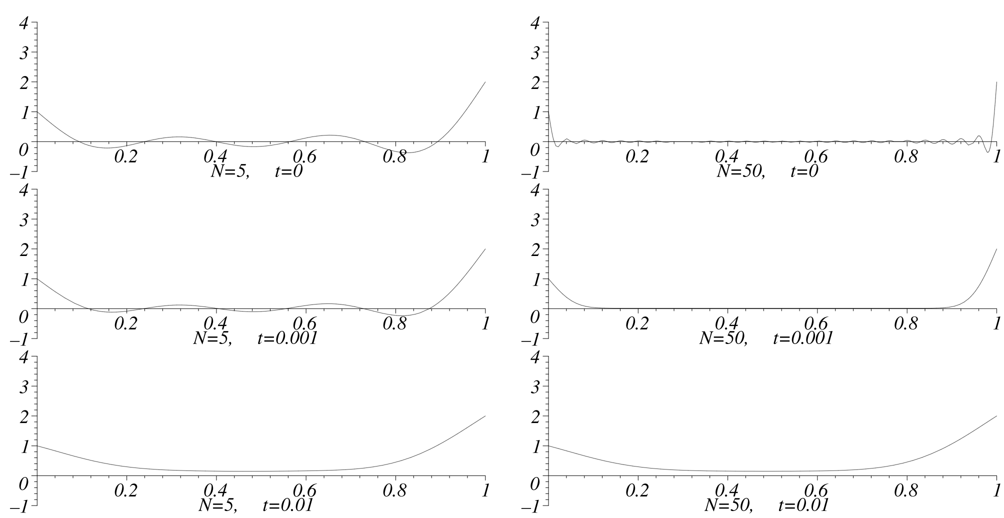


Figure 3.2: Different approximations for $u(x,t)$ with $N = 5, 50$

The previous example dealt with a nonhomogeneous problem in which the nonhomogeneities, both in the PDE and in the boundary conditions, were constant. In the next example we consider a problem with non–constant nonhomogeneities.

EXAMPLE 3.2. Consider the following nonhomogeneous, 1–dimensional heat equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \gamma(x,t), \qquad 0 \leqslant x \leqslant 1,\ t > 0, \\ &\quad u(0,t) = c_L(t) \qquad (\gamma, c_L, c_R \text{ are positive constants}) \\ &\quad u(1,t) = c_R(t) \\ &\quad u(x,0) = g(x). \end{aligned}$$

The technique we employed in the previous example, namely writing $u(x,t) = v(x) + w(x,t)$, won't work in this case since nonhomogeneous terms depend on the time t. In order to solve this problem we proceed in two steps. First we remove the nonhomogeneous terms from the boundary conditions (actually move them up into the PDE itself) and then we solve the nonhomogeneous PDE with homogeneous boundary conditions.

In the previous example we saw, that for $\gamma = 0$, that v was a linear function, i.e. v was of the form $v(x) = a + bx$. This suggests that we try writing u as follows:

$$u(x,t) = A(t) + xB(t) + w(x,t), \tag{3.3}$$

where A and B are chosen to eliminate the nonhomogeneous terms in the boundary conditions. Plugging this expression into the boundary conditions yields

$$\left.\begin{aligned} u(0,t) = c_L(t) \\ u(1,t) = c_R(t) \end{aligned}\right\} \quad \Longrightarrow \quad \left\{\begin{aligned} A(t) + w(0,t) = c_L(t) \\ A(t) + B(t) + w(1,t) = c_R(t) \end{aligned}\right. .$$

We want homogeneous boundary conditions for w, so we choose

$$A(t) := c_L(t), \qquad B(t) := c_R(t) - c_L(t).$$

This leads to $w(0,t) = w(1,t) = 0$. Now plugging (3.3) into the PDE gives:

$$A'(t) + xB'(t) + \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \gamma(x,t),$$

and the initial condition becomes $A(0) + xB(0) + w(x,0) = g(x)$. The problem for w is:

$$\begin{aligned} &\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + F(x,t), \\ &w(0,t) = 0, \qquad 1w(1,t) = 0, \\ &w(x,0) = f(x), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} F(x,t) &:= \gamma(x,t) - A'(t) - xB'(t), \\ f(x) &:= -A(0) - xB(0) + g(x), \\ A(t) &:= c_L(t), \\ B(t) &:= c_R(t) - c_L(t). \end{aligned}$$

Equation (3.4) is still nonhomogeneous but the boundary conditions associated with it are homogeneous. Before solving (3.4), we consider the homogeneous problem associated with it, namely

$$\begin{aligned}&\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2},\\&v(0,t) = 0, \qquad 1v(1,t) = 0,\\&v(x,0) = f(x).\end{aligned} \tag{3.5}$$

This problem is now in the form of the problem in Example 1.9. Applying separation of variables $v(x,t) = X(x)T(t)$ leads to

$$X = X_n(x) = \sin(n\pi x), \quad T = T_n(t) = e^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, \ldots.$$

We know that any piecewise continuous function can be expanded in a Fourier series, in this case a Fourier sine series. In particular

$$\begin{aligned}f(x) &= \sum_{n=1}^{\infty} f_n \sin(n\pi x),\\F(x,t) &= \sum_{n=1}^{\infty} F_n(t)\sin(n\pi x),\end{aligned}$$

where

$$\begin{aligned}f_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx,\\F_n(t) &= 2\int_0^1 F(x,t)\sin(n\pi x)\,dx.\end{aligned}$$

The homogeneous problem sets the stage for this problem. We use the eigenfucntions of the homogeneous problem to exoand the solution as

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)X_n(x) = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x), \tag{3.6}$$

which is called **method of eigenfunction expansion**. Plugging this into (3.4) gives

$$\sum_{n=1}^{\infty} w_n'(t)\sin(n\pi x) = -\sum_{n=1}^{\infty} n^2\pi^2 w_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} F_n(t)\sin(n\pi x)$$

which, upon rearrangement gives

$$\sum_{n=1}^{\infty}\left[w_n'(t) + n^2\pi^2 w_n(t) - F_n(t)\right]\sin(n\pi x) = 0.$$

If we plug (3.6) into the initial condition associated with (3.4), we get

$$\sum_{n=1}^{\infty} w_n(0)\sin(n\pi x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x).$$

This leads to the following ODEs for the w_n's:

$$\begin{aligned} &w_n'(t) + n^2\pi^2 w_n(t) = F_n(t), \quad n = 1, 2, 3, \ldots, \\ &w_n(0) = f_n. \end{aligned}$$

This is a first order, homogeneous linear ODE , which can always be solved by means of an integrating factor. The solution is

$$w_n(t) = e^{-n^2\pi^2 t}\left[f_n + \int_0^t e^{n^2\pi^2 s} F_n(s)\,ds\right].$$

Thus, the solution to the original problem is:

$$u(x,t) = A(t) + xB(t) + \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x),$$

where

$$\begin{aligned} w_n(t) &= e^{-n^2\pi^2 t}\left[f_n + \int_0^t e^{n^2\pi^2 s} F_n(s)\,ds\right], \\ F_n(t) &= 2\int_0^1 F(x,t)\sin(n\pi x)\,dx, \\ f_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx, \\ F(x,t) &:= \gamma(x,t) - A'(t) - xB'(t), \\ f(x) &:= -A(0) - xB(0) + g(x), \\ A(t) &:= c_L(t), \\ B(t) &:= c_R(t) - c_L(t). \end{aligned}$$

Speial cases:

1. ($\gamma \equiv c_L \equiv c_R \equiv 0$). (i.e. homogeneous case)
   Then $A(t) \equiv B(t) \equiv F(x,t) \equiv 0$, $f(x) = g(x)$, $w_n(t) = g_n e^{-n^2\pi^2 t}$ and

$$u(x,t) = \sum_{n=1}^{\infty} g_n e^{-n^2\pi^2 t}\sin(n\pi x).$$

2. (γ, c_L, c_R constant, $g(x) \equiv 0$). (i.e. Example 3.1)
Then $A = c_L$, $B = c_R - c_L$, $A' \equiv B' \equiv 0$, $F(x,t) = \gamma$, $f(x) = -c_l - (c_R - c_L)x$,

$$f_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = \frac{2}{n\pi}[c_l - (-1)^n c_R],$$
$$F_n(t) = 2\int_0^1 \gamma\sin(n\pi x)\,dx = \frac{2\gamma}{n\pi}[1 - (-1)^n].$$

This gives

$$\begin{aligned} w_n(t) &= e^{-n^2\pi^2 t}\left[f_n + F_n\int_0^t e^{n^2\pi^2 s}\,ds\right] \\ &= e^{-n^2\pi^2 t}\left[f_n + \frac{F_n}{n^2\pi^2}(e^{n^2\pi^2 t} - 1)\right] = \left(f_n - \frac{F_n}{n^2\pi^2}\right)e^{n^2\pi^2 t} + \frac{F_n}{n^2\pi^2}. \end{aligned}$$

Therefore

$$\begin{aligned} u(x,t) &= A(t) + xB(t) + \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) = -f(x) + \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) \\ &= \sum_{n=1}^{\infty}[w_n(t) - f_n]\sin(n\pi x) = \sum_{n=1}^{\infty}\left[\left(f_n - \frac{F_n}{n^2\pi^2}\right)e^{n^2\pi^2 t} + \frac{F_n}{n^2\pi^2} - f_n\right]\sin(n\pi x) \\ &= \sum_{n=1}^{\infty}\left(f_n - \frac{F_n}{n^2\pi^2}\right)\left(e^{n^2\pi^2 t} - 1\right)\sin(n\pi x). \end{aligned}$$

Does this agree with the result of Example 3.1? To see that it does, recall that

$$-v(x) = \sum_{n=1}^{\infty} b_n\sin(n\pi x),$$

where the b_n's are given by (3.2). Thus, we have

$$f_n - \frac{F_n}{n^2\pi^2} = \frac{2}{n\pi}[c_l - (-1)^n c_R] - \frac{2\gamma}{n^3\pi^3}[1 - (-1)^n] = b_n.$$

Therefore

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n\left(e^{n^2\pi^2 t} - 1\right)\sin(n\pi x) \\ &= -\sum_{n=1}^{\infty} b_n\sin(n\pi x) + \sum_{n=1}^{\infty} b_n e^{n^2\pi^2 t}\sin(n\pi x) = v(x) + \sum_{n=1}^{\infty} b_n e^{n^2\pi^2 t}\sin(n\pi x). \end{aligned}$$

We now have a relatively systematic way to handle nonhomogeneities that may arise in our PDEs and/or boundary conditions. The crucial part is being able to solve the corresponding homogeneous problem. In the examples we have considered thus far, the homogeneous PDEs have been relatively easy to solve. Suppose we now consider some slightly more complicated homogeneous PDEs. Recall the general heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{\rho c}\vec{\nabla} \cdot (K\vec{\nabla} u) + \frac{h}{\rho c},$$

where ρ is density, c is heat capacity, K is thermal conductivity, and h is the rate of internal heat generation. Let us consider the one–dimensional case with $h = h_1 u$:

$$\rho(x)c(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(x)\frac{\partial u}{\partial x}\right] + h_1(x)u$$
$$1+ \text{ auxiliary conditions}$$

If we try separation of variables $u(x, t) = X(x)T(t)$ on this equation we get

$$\rho c X T' = \frac{d}{dx}(KX')T + h_1 XT$$

which separates to

$$\frac{T'}{T} = \frac{(KX')'}{\rho c X} + \frac{h_1}{\rho c} = -\lambda.$$

The equation for X is $(KX')' + (h_1 + \lambda\rho c)X = 0$. This second order ODE with nonconstant coefficients is, of course, much more difficult to solve than the $X'' + \lambda X = 0$ equation we had before. In general, we need to consider ODEs of the following type:

$$(r(x)\phi')' + [q(x) + \lambda p(x)]\phi = 0, \quad x \in (a, b),$$
$$1+ \text{ boundary conditions.}$$

Notice that for the case $r(x) \equiv p(x) \equiv 1$ and $q(x) \equiv 0$ this ODE reduces to $\phi'' + \lambda\phi = 0$, which is the simple ODE that what we had before. The question is: for what values of λ will nontrivial solutions exist? These *eigenvalue* problems are called *Sturm–Liouville* problems and will be the subject of study in the next chapter.

Exercise 3.3. ✗✗
Solve the nonhomogeneous heat equation with time-dependent source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$u(x,0) = f(x)$$

and nonhomogeneous boundary conditions

$$u(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = B(t).$$

Hint: Reduce the problem to one with homogeneous boundary conditions by writing $u(x,t) = w(x,t) + v(x,t)$ and assuming that v satisfies just the boundary conditions (and nothing else), then use the method of eigenfunction expansions to solve for $w(x,t)$.

Solution: If $u(x,t)$ is a solution to the problem $(*)$, we reduce the problem to one with homogeneous boundary conditions by writing

$$u(x,t) = v(x,t) + w(x,t)$$

where $v(x,t)$ satisfies only the boundary conditions

$$v(0,t) = A(t) \qquad (**)$$
$$\frac{\partial v}{\partial x}(L,t) = B(t)$$

for $t \geqslant 0$. We take the simplest possible such function, namely,

$$v(x,t) = B(t)\,x + A(t),$$

then

$$u(x,t) = w(x,t) + B(t)\,x + A(t),$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} + \frac{dB(t)}{dt}\,x + \frac{dA(t)}{dt},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}.$$

Therefore

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} - \frac{dB(t)}{dt}\,x - \frac{dA(t)}{dt} + Q(x,t).$$

Also,

$$A(t) = u(0,t) = w(0,t) + A(t) \qquad \text{so that} \qquad w(0,t) = 0,$$

while

$$B(t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial w}{\partial x}(L,t) + B(t) \qquad \text{so that} \qquad \frac{\partial w}{\partial x}(L,t) = 0.$$

Therefore, $w(x,t)$ satisfies the problem with homogeneous boundary conditions given by

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} - \frac{dB(t)}{dt}\,x - \frac{dA(t)}{dt} + Q(x,t), \quad 0 \leqslant x \leqslant L,\ t \geqslant 0 \qquad (***)$$

$$w(0,t) = 0, \quad t \geqslant 0$$

$$\frac{\partial w}{\partial x}(L,t) = 0, \quad t \geqslant 0$$

$$w(x,0) = f(x) - B(0)\,x - A(0), \quad 0 \leqslant x \leqslant L.$$

The initial value–boundary value problem for $w(x,t)$ now consists of a nonhomogeneous partial differential equation, but with homogeneous boundary conditions. As is usual with nonhomogeneous equations, we first find the solution to the homogeneous problem

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}$$

$$w(0,t) = 0$$

$$\frac{\partial w}{\partial x}(L,t) = 0$$

using separation of variables. Assuming a solution of the form $w(x,t) = \phi(x) \cdot T(t)$, we get two ordinary differential equations:

$$\phi''(x) + \lambda\phi(x) = 0, \quad 0 \leqslant x \leqslant L, \qquad T'(t) + \lambda k\,T(t) = 0, \quad t \geqslant 0,$$

$$\phi(0) = 0$$

$$\phi'(L) = 0$$

The eigenvalues are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin\tfrac{(2n-1)\pi x}{2L}$$

for $n \geqslant 1$.

Now, we are not solving the T equation and finding the general solution to the homogeneous problem, instead we use the method of eigenfunction expansions to write the solution $w(x,t)$ to $(***)$, the nonhomogeneous problem, in terms of the eigenfunctions of the homogeneous problem:

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \tfrac{(2n-1)\pi x}{2L}, \tag{†}$$

where indexVariation of parameters(similar to the method of variation of parameters) the coefficients $a_n(t)$ depend on t.

Next, we force this to be a solution to the equation $(***)$ by requiring that each $a_n(t)$ satisfies a first-order ordinary differential equation together with an initial condition. We look at the initial conditions first, when $t = 0$ we want

$$w(x,0) = f(x) - B(0)\,x - A(0) = \sum_{n=1}^{\infty} a_n(0) \sin \tfrac{(2n-1)\pi x}{2L},$$

and from the orthogonality of the eigenfunctions on the interval $[0, L]$, we find the coefficients

$$a_n(0) = \frac{2}{L} \int_0^L [f(x) - B(0)\,x - A(0)] \sin \tfrac{(2n-1)\pi x}{2L}\, dx$$

for $n \geqslant 1$.

Now from (†) we have

$$\frac{\partial w}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n(t)}{dt} \sin \tfrac{(2n-1)\pi}{2L} x \qquad \text{and} \qquad \frac{\partial^2 w}{\partial x^2} = -\sum_{n=1}^{\infty} a_n(t) \left(\tfrac{(2n-1)\pi}{2L}\right)^2 \sin \tfrac{(2n-1)\pi x}{2L},$$

and substituting these expressions into the equation $(***)$, after some simplification, we obtain

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} + k\lambda_n\, a_n\right] \sin \tfrac{(2n-1)\pi x}{2L} = -\frac{dB(t)}{dt}\, x - \frac{dA(t)}{dt} + Q(x,t).$$

The left-hand side of this equation is just the generalized Fourier series of the function

$$g(x,t) = -\frac{dB(t)}{dt}\, x - \frac{dA(t)}{dt} + Q(x,t),$$

so that

$$\frac{da_n}{dt} + k\lambda_n\, a_n = \frac{2}{L} \int_0^L g(x,t) \sin \tfrac{(2n-1)\pi x}{2L}\, dx = G_n(t), \tag{††}$$

and $a_n(t)$ satisfies the initial-value problem

$$\frac{da_n(t)}{dt} + k\lambda_n\, a_n(t) = G_n(t), \quad t \geqslant 0$$

$$a_n(0) = \frac{2}{L} \int_0^L [f(x) - B(0)\,x - A(0)] \sin \tfrac{(2n-1)\pi x}{2L}\, dx.$$

Multiplying by the integrating factor $e^{\lambda_n kt}$, we can solve this first-order linear equation to get

$$a_n(t) = a_n(0)e^{-\lambda_n kt} + e^{-\lambda_n kt}\int_0^t G_n(s)e^{\lambda_n ks}\,ds, \quad t \geqslant 0 \tag{$\dagger\dagger\dagger$}$$

for $n \geqslant 1$.

The solution to the original equation is

$$u(x,t) = B(t)\,x + A(t) + \sum_{n=1}^{\infty} a_n(t)\sin\sqrt{\lambda_n}\,x$$

for $0 \leqslant x \leqslant L,\ t \geqslant 0$, where

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

and $a_n(t)$ is given by $(\dagger\dagger\dagger)$ for $n \geqslant 1$.

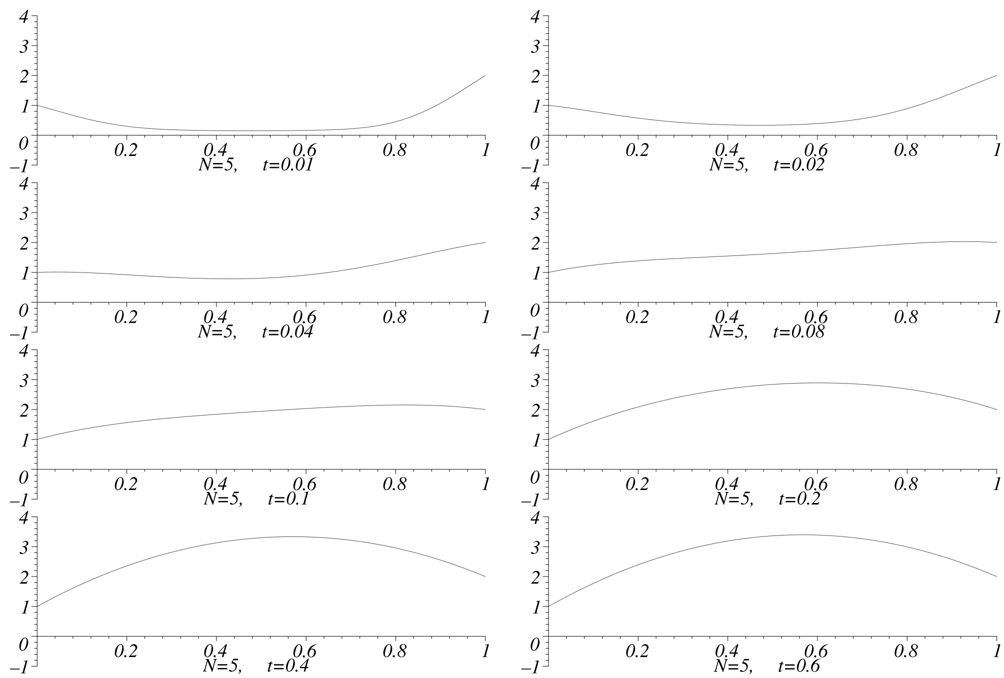


Figure 3.3: Time evolution of $u(x, t)$ with $N = 5$

Chapter 4

Sturm-Liouville Problems

June 17, 2010

4.1 Formulaton

Consider the following ODE for an unknown function $\phi(x)$ together with boundary conditions:

$$\begin{aligned} &(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \qquad a \leqslant x \leqslant b, \\ &\alpha_1\phi(a) - \beta_1\phi'(a) = 0, \\ &\alpha_2\phi(b) + \beta_2\phi'(b) = 0. \end{aligned} \tag{4.1}$$

The function $\phi(x) \equiv 0$ is always a solution, called the trivial solution. What we wish to determine is for what values of the constant λ, if any, do nontrivial solutions exist. Such problems are generally called eigenvalue problems.

Definition 26. *A* REGULAR STURM–LIOUVILLE EIGENVALUE PROBLEM *denotes the problem to find a pair (ϕ, λ) of eigenfucntion and eigenvalue which solve (4.1), where*

(i) p, p', q, σ are continuous for $a \leqslant x \leqslant b$;
(ii) $p(x) > 0$ and $\sigma(x) > 0$ for $a \leqslant x \leqslant b$;
(iii) $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.

Definition 27. *A Sturm–Liouville problem is called* SINGULAR *if at least one of the conditions in the above definition fails.*

The most common singular Sturm–Liouville problem one encounters is one where $p(x) > 0$ for $a < x < b$, but either $p(a) = 0$ or $p(b) = 0$.

EXAMPLE 4.1. Consider the following boundary value problem that we have solved several times before:

$$\begin{aligned} &\phi'' + \lambda\phi = 0, \qquad 0 \leqslant x \leqslant \ell, \\ &\phi(0) = \phi(\ell) = 0. \end{aligned}$$

This is a regular Sturm–Liouville problem with $p(x) \equiv \sigma(x) \equiv 1$, $q(x) \equiv 0$, and $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$. Nontrivial solutions exist only for

$$\lambda = \lambda_n = \frac{n^2\pi^2}{\ell^2}, \quad n = 1, 2, 3, \ldots, \qquad \text{(eigenvalues)}$$
$$\phi_n(x) = \sin(\frac{n\pi x}{\ell}). \qquad \text{(eigenfunctions)}$$

Clearly $\lambda_n \geqslant 0$ for all n and $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{n^2\pi^2}{\ell^2} = \infty$.

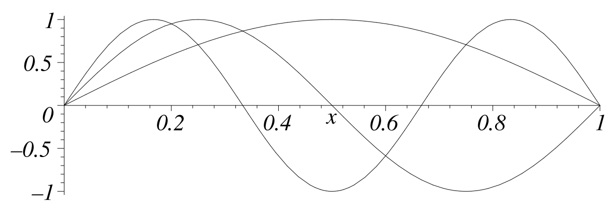


Figure 4.1: Eigenfunctions $\phi_1(x)$, $\phi_2(x)$ $\phi_3(x)$

Definition 28. *The values of λ for which nontrivial solutions to (4.1) exist are called* EIGENVALUES. *The set of all eigenvalues is the called* SPECTRUM. *The nontrivial solution corresponding to an eigenvalue is called an* EIGENFUNCTION.

4.2 Properties of Sturm–Liouville Problems

Theorem 29. *A regular Sturm–Liouville problem has an infinite spectrum.*

Theorem 30. *If λ_m and λ_n are distinct eigenvalues to a regular Sturm–Liouville problem (i.e. $\lambda_m \neq \lambda_n$), then the corresponding eigenfunctions ϕ_m and ϕ_n are orthogonal relative to the inner product*

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\,dx.$$

Proof.
Since ϕ_m and ϕ_n are solutions to the Sturm–Liouville problem, they satisfy the boundary conditions:

$$\alpha_1\phi_m(a) - \beta_1\phi_m'(a) = 0, \qquad \alpha_2\phi_m(b) + \beta_2\phi_m'(b) = 0,$$
$$\alpha_1\phi_n(a) - \beta_1\phi_n'(a) = 0, \qquad \alpha_2\phi_n(b) + \beta_2\phi_n'(b) = 0.$$

The boundary condition at $x = a$ can be written in matrix form

$$\begin{bmatrix} \phi_m(a) & \phi_m'(a) \\ \phi_n(a) & \phi_n'(a) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ -\beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\alpha_1^2 + \beta_1^2 \neq 0$, the determinant of the coefficient matrix must be singular

$$\begin{vmatrix} \phi_m(a) & \phi_m'(a) \\ \phi_n(a) & \phi_n'(a) \end{vmatrix} = 0.$$

Therefore

$$\phi_m(a)\phi_n'(a) - \phi_m'(a)\phi_n(a) = 0. \tag{4.2}$$

A similar argument at the right boundary $x = b$ yields

$$\phi_m(b)\phi_n'(b) - \phi_m'(b)\phi_n(b) = 0. \tag{4.3}$$

Since ϕ_m and ϕ_n also satisfy the ODE we have

$$\begin{aligned} &(p(x)\phi_m')' + [q(x) + \lambda_m \sigma(x)]\phi_m = 0, \\ &(p(x)\phi_n')' + [q(x) + \lambda_n \sigma(x)]\phi_n = 0. \end{aligned}$$

Multiplying the first by ϕ_n and the second by ϕ_m and subtracting yields

$$\phi_n(p\phi_m')' - \phi_m(p\phi_n')' + (\lambda_m - \lambda_n)\sigma\, \phi_m\phi_n = 0.$$

Integrating over the interval gives

$$\int_a^b [\phi_n(p\phi_m')' - \phi_m(p\phi_n')']\, dx = (\lambda_n - \lambda_m) \int_a^b \phi_m\phi_n\, \sigma\, dx = 0.$$

Further manipulation results in

$$\begin{aligned} (\lambda_n - \lambda_m)\, \langle \phi_m, \phi_n \rangle &= \int_a^b [\phi_n(p\phi_m')' - \phi_m(p\phi_n')']\, dx \\ &= p\phi_n\phi_m' \Big|_a^b - \int_a^b p\phi_m'\phi_n'\, dx - p\phi_m\phi_n' \Big|_a^b + \int_a^b p\phi_m'\phi_n'\, dx \\ &= p(\phi_n\phi_m' - \phi_m\phi_n') \Big|_a^b - \int_a^b p(\phi_m'\phi_n' - \phi_m'\phi_n')\, dx \\ &= p(b)[\phi_n(b)\phi_m'(b) - \phi_m(b)\phi_n'(b)] - p(a)[\phi_n(a)\phi_m'(a) - \phi_m(a)\phi_n'(a)] \\ &= 0. \end{aligned}$$

But, since the eigenvalues are distinct, we have $\langle \phi_m, \phi_n \rangle = 0$. Thus, the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

EXAMPLE 4.2. (Cauchy – Euler equation)

$$\begin{aligned} &(x\phi')' + \frac{\lambda}{x}\phi = 0, \quad 1 \leqslant x \leqslant \ell, \\ &\phi(1) = 0, \quad \phi(\ell) = 0. \end{aligned}$$

This is a regular Sturm–Liouville problem with $p(x) = x$, $\sigma(x) = 1/x$, $q(x) = 0$, and $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$.

Making the transformation

$$x = e^s, \quad u(s) = \phi(x),$$

leads to

$$u'' + \lambda u = 0.$$

This is easily solved (with $\lambda = \mu^2$) to give

$$u(s) = c_1 \cos \mu s + c_2 \sin \mu s.$$

Therefore

$$\phi(x) = c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x).$$

Applying the boundary conditions yields

$$\lambda_n = \frac{n^2\pi^2}{(\ln \ell)^2}, \quad n = 1, 2, \ldots, \qquad \text{(eigenvalues)}$$

$$\phi_n(x) = \sin(n\pi \frac{\ln x}{\ln \ell}). \qquad \text{(eigenfunctions)}$$

For $m \neq n$ we have

$$\langle \phi_m, \phi_n \rangle = \int_1^\ell \frac{\sin(m\pi \frac{\ln x}{\ln \ell}) \sin(n\pi \frac{\ln x}{\ln \ell})}{x}\, dx = \frac{\ln \ell}{\pi} \int_0^\pi \sin(mt) \sin(nt)\, dt \qquad 6(\text{using } t = \frac{\pi \ln x}{\ln \ell})$$

$$= \frac{\ln \ell}{2\pi} \int_0^\pi [\cos(n-m)t - \cos(n+m)t]\, dt = \frac{\ln \ell}{2\pi} \left[\frac{\sin(n-m)t}{n-m} - \frac{\sin(n+m)t}{n+m} \right]_0^\pi = 0.$$

Therefore eigenfunctions are orthogonal.

Theorem 31.

1. *The regular Sturm–Liouville problem has an infinite spectrum with $\lambda_n \to \infty$ as $n \to \infty$.*
2. *If the eigenvalues are ordered $\lambda_1 < \lambda_2 < \cdots$, then the eigenfunction corresponding to λ_n has exactly $(n-1)$ zeros in the interval $a < x < b$.*
3. *If $q(x) \leqslant 0$, $\alpha_1\beta_1 \geqslant 0$ and $\alpha_2\beta_2 \geqslant 0$, then $\lambda_n \geqslant 0$ for all n.*

Proof.
(part (3) only)
The inner product we use is one with weight function $\sigma(x)$:

$$\langle \square, \triangle \rangle := \int_a^b \square \triangle \sigma(x)\, dx.$$

Let λ_n be an eigenvalue with corresponding eigenfunction ϕ_n. Then

$$(p(x)\phi_n')' + [q(x) + \lambda_n \sigma(x)]\phi_n = 0.$$

Manipulating, we get

$$\begin{aligned}
\lambda_n \|\phi_n\|^2 = \lambda_n \langle \phi_n, \phi_n \rangle = \lambda_n \int_a^b \phi_n^2(x)\sigma(x)\,dx \\
&= -\int_a^b q(x)\phi_n^2(x)\,dx - \int_a^b [p(x)\phi_n'(x)]'\phi_n(x), dx \\
&=: A_n + B_n.
\end{aligned}$$

We have

$$\begin{aligned}
A_n &= -\int_a^b q(x)\phi_n^2(x)\,dx \geqslant 0, \qquad \text{(since, by assumption } q(x) \leqslant 0) \\
B_n &= -\int_a^b [p(x)\phi_n'(x)]'\phi_n(x), dx = -p(x)\phi_n'(x)\phi_n(x)\Big|_a^b + \int_a^b p(x)\phi_n'(x)\,dx =: C_n + D_n.
\end{aligned}$$

Clearly

$$D_n = \int_a^b p(x)\phi_n'(x)\,dx \geqslant 0, \qquad \text{(since, by assumption } p(x) \geqslant 0)$$

and

$$C_n = F_n(a) - F_n(b), \quad \text{where} \quad F_n(x) := p(x)\phi_n(x)\phi_n'(x).$$

We first examine $F_n(a)$. There are two possibilities: either $\beta_1 = 0$ or $\beta_1 \neq 0$. We have

$$\begin{aligned}
\beta_1 = 0 \quad &\Longrightarrow \quad \phi_n(a) = 0 \quad \Longrightarrow \quad F_n(a) = 0 \\
\beta_1 \neq 0 \quad &\Longrightarrow \quad \phi_n'(a) = \frac{\alpha_1}{\beta_1}\phi_n(a) \quad \Longrightarrow \quad F_n(a) = \frac{\alpha_1}{\beta_1}p(a)\phi_n^2(a) \geqslant 0.
\end{aligned}$$

Either way, $F_n(a) \geqslant 0$. Similarly for $F_n(b)$. There are two possibilities: either $\beta_2 = 0$ or $\beta_2 \neq 0$. We have

$$\begin{aligned}
\beta_2 = 0 \quad &\Longrightarrow \quad \phi_n(b) = 0 \quad \Longrightarrow \quad F_n(b) = 0 \\
\beta_2 \neq 0 \quad &\Longrightarrow \quad \phi_n'(b) = -\frac{\alpha_2}{\beta_2}\phi_n(b) \quad \Longrightarrow \quad F_n(b) = -\frac{\alpha_2}{\beta_2}p(b)\phi_n^2(b) \leqslant 0.
\end{aligned}$$

Either way, $F_n(b) \leqslant 0$. Therefore $C_n = F_n(a) - F_n(b) \geqslant 0$. Finally

$$\lambda_n \|\phi_n\|^2 = A_n + C_n + D_n \geqslant 0.$$

But $\|\phi_n\|^2 \neq 0$, therefore $\lambda_n \geqslant 0$.

4.3 Eigenfunction Expansions

Consider the regular Sturm–Liouville problem

$$
\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \qquad a \leqslant x \leqslant b, \\
&\alpha_1\phi(a) - \beta_1\phi'(a) = 0, \\
&\alpha_2\phi(a) + \beta_2\phi'(a) = 0.
\end{aligned}
\tag{4.4}
$$

Denote the eigenvalues by λ_n and the corresponding eigenfunctions by ϕ_n. Define the inner product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\sigma(x)\,dx.$$

We know that $\langle \phi_m, \phi_n\rangle = 0$ for $m \neq n$.
Question: Can we represent a function f defined on (a, b) by a series of eigenfunctions?

$$\text{i.e. } f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x).$$

Assume for the moment that this is the case. Then the coefficients of the series are calculated as follows:

$$
\begin{aligned}
\langle f, \phi_m\rangle &= \left\langle \sum_{n=1}^{\infty} c_n\phi_n, \phi_m \right\rangle = \sum_{n=1}^{\infty} c_n \langle \phi_n, \phi_m\rangle \\
&= c_1 \langle \phi_1, \phi_m\rangle + c_2 \langle \phi_2, \phi_m\rangle + \cdots + c_m \langle \phi_m, \phi_m\rangle + \cdots \\
&= 0 + 0 + \cdots + 0 + c_m\|\phi_m\|^2 + 0 + \cdots
\end{aligned}
$$

Therefore

$$c_m = \frac{\langle f, \phi_m\rangle}{\|\phi_m\|^2}.$$

Definition 32. *Let f be a piecewise continuous function on $[a, b]$. The eigenfucntion expansion*

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x)$$

with coefficients

$$c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2}$$

where the inner product is based on the weight function $\sigma(x)$, is called a GENERALIZED FOURIER SERIES *of f .*

Theorem 33. *If f is piecewise smooth on (a, b), then the generalized Fourier series*

$$\sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{where} \quad c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2},$$

converges to

$$\frac{f(x+)+f(x-)}{2}.$$

This says that if f is continuous at x (i.e. $f(x+)=f(x-)=f(x)$), then the eigenfunction expansion converges to $f(x)$ and if f has a jump discontinuity at x (i.e. $f(x+)\neq f(x-)$), then the eigenfunction expansion converges to the point midway between the limiting values.

EXAMPLE 4.3. Consider the regular Sturm–Liouville problem

$$\begin{aligned}&\phi''+\lambda\phi=0, \qquad 0<x<1,\\&\phi(0)=0,\\&2\phi(1)-\phi'(1)=0.\end{aligned}$$

Note that in this case $\alpha_2\beta_2=-2<0$, hence Theorem 31 does *not* guarantee non-negative eigenvalues.

- case (i): $(\lambda\leqslant 0)$
  Let $\lambda=-\mu^2$. Then

  $$\phi''-\mu^2\phi=0 \quad\Longrightarrow\quad \phi(x)=a\cosh(\mu x)+b\sinh(\mu x).$$

  The left boundary condition gives us

  $$\phi(0)=0 \quad\Longrightarrow\quad a=0 \quad\Longrightarrow\quad \phi(x)=b\sinh(\mu x).$$

  The right boundary condition now gives us

  $$2\phi(1)-\phi'(1)=0 \quad\Longrightarrow\quad 2b\sinh(\mu)-b\mu\cosh(\mu)=0 \quad\Longrightarrow\quad \tanh(\mu)=\frac{\mu}{2}.$$

  This equation has a solution at $\mu=0,\pm\mu_0$. Only μ_0 yields a nontrivial solution

  $$\phi_0(x)=\sinh(\mu_0 x).$$

- case (ii): $(\lambda>0)$
  Let $\lambda=\mu^2\neq 0$. Then

  $$\phi''+\mu^2\phi=0 \quad\Longrightarrow\quad \phi(x)=a\cos(\mu x)+b\sin(\mu x).$$

  The left boundary condition gives us

  $$\phi(0)=0 \quad\Longrightarrow\quad a=0 \quad\Longrightarrow\quad \phi(x)=b\sin(\mu x).$$

  The right boundary condition now gives us

  $$2\phi(1)-\phi'(1)=0 \quad\Longrightarrow\quad 2b\sin(\mu)-b\mu\cos(\mu)=0 \quad\Longrightarrow\quad \tan(\mu)=\frac{\mu}{2}.$$

  This equation has infinitely many solutions $\mu_1,\mu_2,\ldots$ with $\phi_n(x)=\sin(\mu_m x)$.

To summarize, we have an infinite set of eigenvalues:

$$-\mu_0^2, \mu_1^2, \mu_2^2, \ldots, \quad \text{where} \quad \tanh\mu_0 = \frac{\mu_0}{2},\ \tan\mu_n = \frac{\mu_n}{2},\ n = 1, 2, 3, \cdots,$$

with corresponding eigenfunctions

$$\phi_0(x) = \sinh(\mu_0 x), \quad \phi_n(x) = \sin(\mu_n x), \quad n = 1, 2, 3, \cdots.$$

The norms of the eigenfunctions are given by

$$n = 0: \quad \|\phi_0\|^2 = \langle\phi_0, \phi_0\rangle = \int_0^1 \sinh^2(\mu_0 x)dx = \frac{1}{2}\left(1 - \frac{\sinh(2\mu_0)}{2\mu_0}\right),$$
$$n \neq 0: \quad \|\phi_n\|^2 = \langle\phi_n, \phi_n\rangle = \int_0^1 \sin^2(\mu_n x)dx = \frac{1}{2}\left(1 - \frac{\sin(2\mu_n)}{2\mu_n}\right),$$

An eigenfunction expansion for a function $f \in PC(0,1)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} c_n\phi_n(x), \quad \text{where} \quad c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2}.$$

Exercise 4.1. ✗

Given the boundary value problem

$$y'' + \left(\frac{1+\lambda x}{x}\right) y = 0$$
$$y(1) = 0$$
$$y(2) = 0,$$

on the interval $[1, 2]$. Put the equation in Sturm-Liouville form and decide whether the problem is regular or singular.

Solution: An equation is in Sturm-Liouville form if it has the form

$$(p(x)\, y')' + q(x)\, y + \lambda\sigma(x)\, y = 0.$$

We can rewrite the boundary value problem above in the form

$$(1 \cdot y')' + \frac{1}{x}\, y + \lambda\, y = 0$$
$$y(1) = 0$$
$$y(2) = 0$$

and here $p(x) = 1,\ p'(x) = 0,\ q(x) = \dfrac{1}{x},\ \sigma(x) = 1$ are all continuous on the interval $[1, 2]$, with $p(x) > 0$ and $r(x) > 0$ for all $x \in [1, 2]$.

The boundary conditions are of the form

$$\begin{aligned} c_1 y(1) + c_2 y'(1) &= 0 \\ d_1 y(2) + d_2 y'(2) &= 0 \end{aligned}$$

where $c_1 = d_1 = 1$ and $c_2 = d_2 = 0$, and so are Sturm-Liouville type boundary conditions.

Therefore, this is a **regular** Sturm-Liouville problem on the interval $[1, 2]$.

Exercise 4.2. ✗✗

Consider the regular Sturm-Liouville problem

$$\phi'' + \lambda^2 \phi = 0 \qquad 0 \leqslant x \leqslant \pi$$
$$\phi'(0) = 0$$
$$\phi(\pi) = 0$$

(a) Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

(b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Given the function $f(x) = \dfrac{\pi^2 - x^2}{2}, \quad 0 < x < \pi$, find the eigenfunction expansion for f.

(d) Show that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots$$

Solution:

(a) case(i): $\lambda = 0$

The general solution to the equation $\phi'' + \lambda^2 \phi = 0$ in this case is

$$\phi(x) = c_1\, x + c_2,$$

and differentiating, $\phi'(x) = c_1$, and the condition $\phi'(0) = 0$ implies that $c_1 = 0$. The condition $\phi(\pi) = 0$ implies that $c_2 = 0$, so there are no nontrivial solutions in this case.

case(ii): $\lambda \neq 0$ The general solution to the equation $\phi'' + \lambda^2 \phi = 0$ in this case is

$$\phi(x) = c_1 \cos \lambda x + c_2 \sin \lambda x,$$

and differentiating, we get

$$\phi'(x) = -c_1\lambda \sin \lambda x + c_2\lambda \cos \lambda x.$$

The condition $\phi'(0) = 0$ implies that $c_2\,\lambda = 0$, and so $c_2 = 0$. The solution is then

$$\phi(x) = c_1 \cos \lambda x,$$

and the condition $\phi(\pi) = 0$ implies that $\cos \lambda\pi = 0$, and therefore the eigenvalues are

$$\lambda^2 = \lambda_n^2 = \left(\tfrac{2n-1}{2}\right)^2,$$

for $n \geqslant 1$. The corresponding eigenfunctions are

$$\phi_n(x) = \cos \tfrac{(2n-1)}{2} x,$$

for $n \geqslant 1$.

(b) Let $\lambda_n = \frac{2n-1}{2}$ for n=1,2,3,..., then for $m \neq n$, we have

$$\begin{aligned}
\int_0^\pi \phi_m(x)\phi_n(x)\,dx &= \int_0^\pi \cos \lambda_m x\, \cos \lambda_n x\, dx \\
&= \frac{1}{2}\int_0^\pi \{\cos(\lambda_m + \lambda_n)x + \cos(\lambda_m - \lambda_n)x\}\, dx \\
&= \frac{1}{2(\lambda_m + \lambda_n)} \sin(\lambda_m + \lambda_n)x\Big|_0^\pi + \frac{1}{2(\lambda_m - \lambda_n)} \sin(\lambda_m - \lambda_n)x\Big|_0^\pi \\
&= \frac{1}{2(\lambda_m + \lambda_n)} \sin(\lambda_m + \lambda_n)\pi + \frac{1}{2(\lambda_m - \lambda_n)} \sin(\lambda_m - \lambda_n)\pi \\
&= 0
\end{aligned}$$

since $(\lambda_m + \lambda_n)\pi = (m + n - 1)\pi$ and $(\lambda_m - \lambda_n)\pi = (m - n)\pi$.

(c) Writing

$$f(x) = \frac{\pi^2 - x^2}{2} \sim \sum_{n=1}^{\infty} c_n\phi_n(x),$$

the coefficients c_n in the eigenfunction expansion are found using the orthogonality of the eigenfunctions on $[0, \pi]$.

$$\begin{aligned}
c_n &= \frac{2}{\pi}\int_0^\pi \left(\frac{\pi^2 - x^2}{2}\right) \cos \lambda_n x\, dx \\
&= \frac{2}{\pi}\frac{\sin \lambda_n\pi}{\lambda_n^3} = \frac{16}{\pi(2n-1)^3} \sin \tfrac{(2n-1)}{2}\pi \\
&= \frac{16(-1)^{n+1}}{\pi(2n-1)^3},
\end{aligned}$$

where the integral was evaluated by repeated integration by parts.

Therefore, the eigenfunction expansion of f is given by

$$\frac{\pi^2 - x^2}{2} \sim \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \cos \tfrac{(2n-1)}{2} x.$$

(d) In this particular problem, the eigenfunction expansion is actually the Fourier cosine series for f. Since the function f is piecewise smooth on the interval $[0, \pi]$ and since the even extension of f to $[-\pi, \pi]$ is continuous at $x = 0$, then by Dirichlet's theorem the series converges to $f(0) = \dfrac{\pi^2}{2}$ when $x = 0$, and therefore

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots$$

EXAMPLE 4.4. Here we summarize the four most frequent Sturm Liouville problems. The underlying computations can be found in Part II within the problems on Sturm Liouville problems. These SL-problems should be memorized:

Model type	SL-problem	Spectrum	eigenfunctions
homogeneous Dirichlet b.c.	$\phi''(x) = -\lambda\phi(x)$ $\phi(0) = \phi(L) = 0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$	$\phi_n = \sin \frac{n\pi x}{L}$
homogeneous Neumann b.c.	$\phi''(x) = -\lambda\phi(x)$ $\phi'(0) = \phi'(L) = 0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, \ldots$	$\phi_n = \cos \frac{n\pi x}{L}$
mixed b.c. I	$\phi''(x) = -\lambda\phi(x)$ $\phi(0) = 0, \phi'(L) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$, $n = 1, 2, \ldots$	$\phi_n = \sin \frac{(2n-1)\pi x}{2L}$
mixed b.c. II	$\phi''(x) = -\lambda\phi(x)$ $\phi'(0) = 0, \phi(L) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$, $n = 1, 2, \ldots$	$\phi_n = \cos \frac{(2n-1)\pi x}{2L}$

4.3.1 Rayleigh Quotient

It is often important to estimate the first (LEADING) eigenvalue. If the leading eigenvalue is positive, then the system is stable, in the sense that small perturbations are damped and the system converges to the equilibrium steady state. If the leading eigenvalue is negative, then the system is unstable and small perturbations are amplified. This can have adverse consequences to the system at hand (crashing bridges or towers, for example).

The Rayleigh quotient is a simple and elegant methods to estimate the leading eigenvalue.

Theorem 34. *If (λ_n, ϕ_n) is a solution of a regular SL problem, then λ_n can be calculated by the* RAYLEIGH QUOTIENT

$$\lambda_n = \frac{-p(x)\phi_n(x)\phi_n'(x)|_0^L + \int_0^L (p(x)\phi_n'(x)^2 - q(x)\phi_n(x)^2)dx}{\int_0^L \phi_n(x)^2\sigma(x)dx} \tag{4.5}$$

<u>*Proof.*</u>
Multiplication of the SL problem by ϕ_n gives

$$\phi_n \frac{d}{dx}(p\phi_n') + q\phi_n^2 + \lambda_n \sigma \phi_n^2 = 0.$$

Integration and integrating py parts gives

$$\begin{aligned} 0 &= \int_0^L \phi_n \frac{d}{dx}(p\phi_n')dx + \int_0^L q\phi_n^2 dx + \int_0^L \lambda_n \sigma \phi_n^2 dx \\ &= \phi_n(p\phi_n')|_0^L - \int_0^L \phi_n' p \phi_n' dx + \int_0^L q\phi_n^2 dx + \int_0^L \lambda_n \sigma \phi_n^2 dx \end{aligned}$$

Hence (4.5) follows after rearrangement.

Lemma 35. *If $-p\phi_n\phi_n'|_0^L \geq 0$ and $q \leq 0$ and $0 \leq x \leq L$, then $\lambda_n > 0$!*

We can write down the Rayleigh quotient for **any** function (it does not have to be an eigenfunction). We define

$$\mathcal{R}(u) = \frac{-puu'|_0^L + \int_0^L (pu'^2 - qu^2)dx}{\int_0^L u^2 \sigma dx}$$

We find that the leading eigenvalue is the smallest of all those Rayleigh quotients, for functions that satisfy the correct boundary conditions:

Theorem 36. *Let $D(L)$ denote the set of all continuous functions that satisfy the boundary conditions*

$$\alpha_1 u(0) - \beta_1 u'(0) = 0, \qquad \alpha_2 u(L) + \beta_2 u'(L) = 0,$$

then

$$\lambda_1 = \min_{u \in D(L)} \mathcal{R}(u)$$

is the leading eigenvalue.

EXAMPLE 4.5. Find a good upper and lower estimate for the leading order eigenvalue of the SL problem

$$2\phi''(x) + \lambda\phi(x) = 0, \qquad \phi'(0) = \phi'(L) = 0.$$

Here we have $p = 2, q = 0, \sigma = 1$ and the boundary term in $\mathcal{R}$ is

$$-p\phi\phi'|_0^L = -2\phi(L)\phi'(L) - 2\phi(0)\phi'(0) = 0.$$

Hence the Rayleigh quotient reads

$$\mathcal{R}(\phi) = \frac{\int_0^L 2\phi'(x)dx}{\int_0^L \phi^2(x)dx} \geq 0$$

and a lower estimate is

$$\lambda_1 \geq 0.$$

To find an upper estimate, we choose a test function which satisfies the boundary conditions (but possibly not the equation). For example $w(x) = 5$ satisfies the boundary conditions. For this function we find

$$\mathcal{R}(w) = \frac{\int_0^L 2 \cdot 0 \, dx}{\int_0^L 25 dx} = 0$$

hence

$$\lambda_1 \leq 0.$$

Together with the previous estimate we find

$$\lambda_1 = 0.$$

We can compare the exact solution of this SL problem. With the transformation of $\lambda/2 = \mu$ the above SL problem becomes

$$\phi'' + \mu\phi = 0, \qquad \phi'(0) = \phi'(L) = 0,$$

which is one of our standard problems. The eigenvalues are $\lambda_n = (n\pi/L)^2$ for $n = 0, 1, 2, \ldots$. Hence the leading eigenvalue is $\lambda_0 = 0$. Notice that here the leading eigenvalue is called λ_0 just for convenience, since the index n starts at 0.

EXAMPLE 4.6. Find good upper and lower bounds for the leading eigenvalue of

$$\phi'' - x\phi + \lambda\phi = 0, \qquad \phi'(0) = 0, \quad \phi'(1) + 2\phi(i) = 0.$$

We have $p = 1, q = -x, \sigma = 1$ and the boundary term reads

$$-p\phi\phi'|_0^L = -\phi(1)\phi'(1) + \phi(0)\phi'(0) = 2\phi(1)^2 \geq 0$$

Since $q = -x \leq 0$ we have $\mathcal{R}(\phi) \geq 0$, hence

$$\lambda_1 \geq 0.$$

For a lower estimate we choose a test function which satisfies the boundary conditions. Here it is not so easy to guess a function, so we make the Ansatz:

$$w(x) = Ax^2 + Bx + C.$$

Substituting this function into the boundary conditions we get

$$0 = B, \qquad 2A + 2A + 2C = 0$$

Hence $B = 0$ and $C = -2A$. We choose $A = 1$, then $f(x) = x^2 - 2$ is a test function which satisfies the boundary conditions. We compute

$$\mathcal{R}(f) = \frac{-ff'|_0^1 + \int_0^1 (f'^2 + xf^2)dx}{\int_0^1 f^2 dx}$$

The first term is

$$-ff'|_0^1 = 2.$$

The integrals are

$$\int_0^1 f'^2 + xf^2 dx = \int_0^1 4x^2 + x(x^4 - 4x^2 + 4)dx = \frac{5}{2}$$

and

$$\int_0^1 f^2 dx = \int_0^1 (x^4 - 4x^2 + 4)dx = \frac{43}{15}.$$

Together we get

$$\mathcal{R}(f) = \frac{135}{86} \approx 1.57.$$

Hence we find

$$0 \le \lambda_1 \le 1.57$$

Exercise 4.3. Find the general Fourier series solution for the following homogeneous Neumann problem for the wave equation. Use the Rayleigh quotient to show that $\lambda_1 > 0$.

$$\begin{aligned} \alpha(x)u_{tt} &= (\tau(x)u_x)_x - \beta(x)u \qquad\qquad 0 \le x \le L \\ u_x(0,t) &= 0, \quad u_x(L,t) = 0 \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x) \end{aligned}$$

where $\alpha(x), \tau(x), \beta(x) > 0$.

Solution Separation of variables with $u(x,t) = T(t)X(x)$ leads to the time problem

$$T''(t) = -\lambda T(t)$$

and the spatial problem

$$\begin{aligned} (\tau(x)X'(x))' - \beta(x)X(x) + \lambda\alpha(x)X(x) &= 0 \\ X'(0) = 0, \quad X'(L) &= 0 \end{aligned}$$

which is a regular SL problem with

$$p(x) = \tau(x), q(x) = -\beta(x) < 0, \quad \text{and} \quad \sigma(x) = \alpha(x).$$

From the general theory it follows that we have a complete set of eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunctions $\phi_1, \phi_2, \ldots$.

We study the Rayleigh quotient. The boundary term reads

$$-p(x)\phi(x)\phi'(x)|_0^L = \tau(L)\phi(L)\phi'(L) - \tau(0)\phi(0)\phi'(0) = 0$$

Since $q(x) < 0$ we find $\lambda_1 > 0$. This is the information which we needed for the time problem. $T'' = -\lambda_n T$ for $\lambda_n > 0$ is solved by

$$T_n(t) = a_n \cos(\sqrt{\lambda_n} t) + b_n \sin(\sqrt{\lambda_n} t).$$

Hence, after using superposition, we find the solution in the form of a generalized Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(\sqrt{\lambda_n} t) + b_n \sin(\sqrt{\lambda_n} t))\ \phi_n(x).$$

Finally, we adapt the initial conditions. At $t = 0$ we get for the initial displacement

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

which is the generalized Fourier series of $f(x)$ with coefficients

$$a_n = \frac{\langle f(x), \phi_n(x) \rangle}{\|\phi_n\|}$$

Similarly, we get for the initial velocity

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} b_n \phi_n(x).$$

Hence the coefficients of the generalized Fourier series of $g(x)$ are

$$\sqrt{\lambda_n} b_n = \frac{\langle g(x), \phi_n(x) \rangle}{\|\phi_n\|}$$

which gives

$$b_n = \frac{1}{\sqrt{\lambda_n}} \frac{\langle g(x), \phi_n(x) \rangle}{\|\phi_n\|}.$$

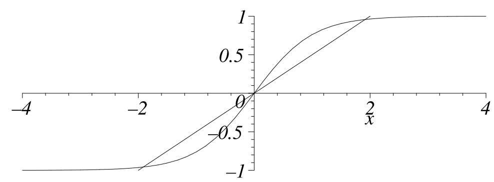


Figure 4.2: Intersection of $y = \tanh(\mu)$ and $y = \mu/2$.

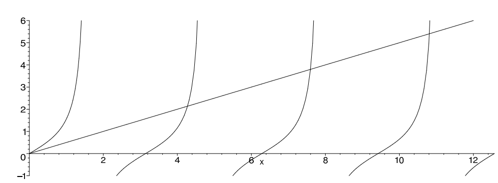


Figure 4.3: Intersection of $y = \tan(\mu)$ and $y = \mu/2$.

Chapter 5

Problems in Cartesian Coordinates

June 17, 2010

5.1 Heat Equation

EXAMPLE 5.1. Consider the following 1-dimensional heat equation:

$$\begin{aligned} u_t &= k(u_{xx} + \gamma u), \qquad 0 < x < \ell,\ t > 0, \\ &u_x(0,t) = 0, \\ &\kappa u(\ell,t) + u_x(\ell,t) = 0, \\ &u(x,0) = f(x). \end{aligned}$$

This is a homogeneous PDE with homogeneous boundary conditions. Try a separation of variables solution $u(x,t) = X(x)T(t)$. Plug into the PDE to get:

$$\frac{T'}{kT} - \gamma = \frac{X''}{X} = -\lambda.$$

This gives the following problems for X and T:

$$T' + (\lambda - \gamma)kT = 0, \qquad \begin{aligned} &X'' + \lambda X = 0, \\ &X'(0) = 0, \\ &\kappa X(\ell) + X'(\ell) = 0. \end{aligned}$$

The problem for X is a regular Sturm–Liouville problem with $q(x) = 0$, $\alpha_1\beta_1 = 0$ and $\alpha_2\beta_2 \geqslant 0$. According to Theorem 31 (part 3) all the eigenvalues are non-negative. So, letting $\lambda = \mu^2$ we get

$$\begin{aligned} X(x) &= a\cos(\mu x) + b\sin(\mu x), \\ X'(x) &= -a\mu\sin(\mu x) + b\mu\cos(\mu x). \end{aligned}$$

At the left boundary $x = 0$ we have

$$X'(0) = 0 \quad \Longrightarrow \quad b = 0 \quad \Longrightarrow \quad X(x) = a\cos(\mu x),\ X'(x) = -a\mu\sin(\mu x).$$

At the right boundary $x = \ell$ we have

$$\kappa X(\ell) + X'(\ell) = 0 \quad \Longrightarrow \quad a\kappa\cos(\mu\ell) - a\mu\sin(\mu\ell) = 0 \quad \Longrightarrow \quad \tan(\mu\ell) = \frac{\kappa}{\mu}.$$

Thus we get

$$\begin{aligned}\lambda_n &= \mu_n^2, \quad n = 1, 2, \cdots, \quad \text{where } \tan(\mu_n\ell) = \frac{\kappa}{\mu_n},\\ X_n(x) &= \cos(\mu_n x),\\ T_n(t) &= e^{-(\mu_n^2-\gamma)kt}.\end{aligned}$$

The norm of the eigenfunctions, relative to the appropriate inner product, is

$$\|X_n\|^2 = \langle X_n, X_n\rangle = \int_0^\ell X_n^2(x)\,dx = \int_0^\ell \cos^2(\mu_n x)\,dx = \frac{1}{2}\left(\ell + \frac{\sin(2\mu_n\ell)}{2\mu_n}\right).$$

Apply the principle of superposition to get

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x)T_n(t) = \sum_{n=1}^{\infty} c_n e^{-(\mu_n^2-\gamma)kt}\cos(\mu_n x).$$

Now apply the initial condition: $u(x,0) = f(x)$ to get

$$f(x) = \sum_{n=1}^{\infty} c_n \cos(\mu_n x).$$

Thus, the problem is solved provided we can expand f in an eigenfunction expansion. We know that we can, if $f \in PC(0,\ell)$, with the coefficients given by

$$c_n = \frac{\langle f, X_n\rangle}{\|X_n\|^2} = \frac{2}{\ell + \frac{\sin(2\mu_n\ell)}{2\mu_n}} \int_0^\ell f(x)\cos(\mu_n x)\,dx.$$

Let us interpret the long time behaviour of the results. We have

$$0 \leqslant \mu_1 \leqslant \frac{\pi}{2\ell}, \quad \frac{\pi}{\ell} \leqslant \mu_2 \leqslant \frac{3\pi}{2\ell}, \quad \cdots \quad (n-1)\frac{\pi}{\ell} \leqslant \mu_n \leqslant (2n-1)\frac{\pi}{2\ell}.$$

Clearly μ_n depends on κ so we can write

$$(n-1)\frac{\pi}{\ell} \leqslant \mu_n(\kappa) \leqslant (2n-1)\frac{\pi}{2\ell}.$$

We consider two cases.

- case (i): (perfect insulation $\kappa \to 0$)
  We have $\lim_{\kappa\to 0} \mu_n(\kappa) = (n-1)\dfrac{\pi}{\ell}$ and, in particular, $\lim_{\kappa\to 0} \mu_1(\kappa) = 0$. Other quantities simplify as follows:

$$\|X_n\|^2 = \frac{1}{2}\left(\ell + \frac{\sin(2\mu_n\ell)}{2\mu_n}\right) = \frac{1}{2}\left(\ell + \frac{\sin 2(n-1)\pi}{2(n-1)\pi/\ell}\right) = \begin{cases} \frac{\ell}{2} & n \neq 1 \\ \ell & n = 1 \end{cases},$$

$$c_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{(n-1)\pi x}{\ell}\right) dx,$$

$$u(x,t) = e^{\gamma kt}\left\{c_1 + \sum_{n=2}^{\infty} c_n e^{-\mu_n^2 kt}\cos(\mu_n x)\right\}.$$

  Taking the limit as $t \to \infty$ we get

$$\lim_{t\to\infty} u(x,t) = +\infty. \qquad \text{(assuming } c_1, \gamma > 0\text{)}$$

  Does this result make sense? Yes it does. Remember that $\gamma > 0$ represents internal heat generation. With perfect insulation, no heat can escape so the temperature must continue to rise indefinitely.

- case (ii): (imperfect insulation $\kappa > 0$)
  The solution is

$$u(x,t) = c_1 e^{(\gamma-\mu_1^2)kt}\cos(\mu_1 x) + \sum_{n=2}^{\infty} c_n e^{(\gamma-\mu_n^2)kt}\cos(\mu_n x).$$

  Taking the limit as $t \to \infty$ we get

$$\lim_{t\to\infty} u(x,t) = \begin{cases} +\infty & \gamma > \mu_1^2, \\ c_1\cos(\mu_1 x) & \gamma = \mu_1^2, \\ 0 & \gamma < \mu_1^2. \end{cases}$$

  With imperfect insulation there is heat loss. What this result is saying is that if the internal heat generation is sufficiently large (i.e. $\gamma > \mu_1^2$), then heat is generated at a rate faster than it can escape through the boundary. Therefore the temperature rises indefinitely. On the other hand, if internal heat generation is very low, then heat escapes faster than it is generated internally, and the temperature eventually goes to zero. But, if the rate of internal heat generation is just right, (i.e. $\gamma = \mu_1^2$), then there is a balance between internal heat generation and heat loss through the boundary and an equilibrium temperature distribution is reached. The critical value of internal heat generation is precisely the value of the smallest eigenvalue λ_1.

5.2 Wave Equation

Recall the general wave equation derived earlier

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + F.$$

The one–dimensional, homogeneous version is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

The one–dimensional wave equation is one of the rare PDEs for which the "general" solution can actually be found. How this is done is not immediately obvious, so we will apply separation of variables.

EXAMPLE 5.2. Consider the following one-dimensional wave equation:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \ell,\ t > 0, \quad (c \equiv \text{ constant}) \\ &u(0,t) = 0, \quad u(\ell,t) = 0, \\ &u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \end{aligned}$$

This problem governs the vertical displacement of a string with its end points fixed at $x = 0$ and $x = \ell$. At time $t = 0$ we give the string an initial displacement $f(x)$, and an initial velocity $g(x)$, and then the solution to this problem governs the subsequent motion of the string.

The boundary conditions are homogeneous so we try separation of variables $u(x,t) = X(x)T(t)$. Plug this into the equation to get

$$XT'' = c^2 X'' T \quad \Longrightarrow \quad \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda. \quad \text{(constant)}$$

The boundary conditions imply that $X(0) = X(\ell) = 0$, so we get the following problems for X and T:

$$T'' + \lambda c^2 T = 0 \qquad\qquad \begin{aligned} &X'' + \lambda X = 0. \\ &X(0) = X(\ell) = 0. \end{aligned}$$

For the problem in X, Theorem 31 implies that all the eigenvalues are non-negative. Letting $\lambda = \mu^2$ we get

$$\begin{aligned} \lambda_n = \mu_n^2 &= \frac{n^2\pi^2}{\ell^2}, \quad n = 1, 2, \ldots, \\ X_n(x) &= \sin(\mu_n x). \end{aligned}$$

The equation for T becomes

$$T'' + c^2\mu_n^2 T = 0,$$

which is easily solved:

$$T_n(t) = \alpha_n \cos(c\mu_n t) + \beta_n \sin(c\mu_n t).$$

Since the equation and the boundary conditions are homogeneous we may apply the principle of superposition to get

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \sin(\mu_n x)[\alpha_n \cos(c\mu_n t) + \beta_n \sin(c\mu_n t)],$$
$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} c\mu_n \sin(c\mu_n x)[-\alpha_n \sin(c\mu_n t) + \beta_n \cos(c\mu_n t)]$$

Apply the initial conditions:

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\} \implies \begin{cases} f(x) = \sum_{n=1}^{\infty} \alpha_n \sin(\mu_n x), & \text{(Fourier sine series)} \\ g(x) = \sum_{n=1}^{\infty} c\beta_n\mu_n \sin(\mu_n x). & \text{(Fourier sine series)} \end{cases}$$

Let $\bar{f}_o$ and $\bar{g}_o$ be the odd, 2ℓ-periodic extensions of f and g respectively. Then

$$\bar{f}_o(x) = \sum_{n=1}^{\infty} \alpha_n \sin(\mu_n x) \qquad \text{where} \qquad \alpha_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\mu_n x)\,dx,$$
$$\bar{g}_o(x) = \sum_{n=1}^{\infty} \alpha_n \sin(\mu_n x) \qquad \text{where} \qquad \beta_n = \frac{2}{c\mu_n \ell}\int_0^{\ell} g(x)\sin(\mu_n x)\,dx.$$

For convenience we define

$$G(x) := \int \bar{g}_o(x)\,dx = -\sum_{n=1}^{\infty} c\beta_n \cos(\mu_n x).$$

Note that G is even and 2ℓ-periodic. Looking more closely at the solution $u(x,t)$ yields

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \sin(\mu_n x)[\alpha_n \cos(c\mu_n t) + \beta_n \sin(c\mu_n t)] \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\{\alpha_n[\sin\mu_n(x-ct) + \sin\mu_n(x+ct)] + \beta_n[\cos\mu_n(x-ct) + \cos\mu_n(x+ct)]\} \\ &= \frac{1}{2}\{\bar{f}_o(x-ct) + \bar{f}_o(x+ct) - \frac{1}{c}[G(x-ct) - G(x+ct)]\} \\ &= \frac{1}{2}[\bar{f}_o(x-ct) + \bar{f}_o(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_o(\xi)\,d\xi. \end{aligned}$$

We have the solution in closed form! In other words, the series has been summed.

We now attempt to give a physical interpretation to the solution. For simplicity we consider the case $g(x) \equiv 0$ (zero initial velocity):

$$u(x,t) = \frac{1}{2}[\bar{f}_o(x - ct) + \bar{f}_o(x + ct)].$$

Define $\xi := x - ct$, called a *phase*. If ξ is constant, then $d\xi/dt = 0$, that is $dx/dt = c$. Therefore the function $\bar{f}_o(\xi) = \bar{f}_o(x - ct)$ represents a wave travelling to the right with speed c. Similarly, if we define another phase $\eta := x + ct$, then $\eta =$ constant implies that $dx/dt = -c$. That is, $\bar{f}_o(x + ct)$ represents a wave travelling to the left with speed c.

Suppose f is defined as the "hat" function as given in Figure 5.2.

Figure 5.3 shows how a string, with $y = f(x)$ as its initial displacement, evolves with time. Specifically, the figure exhibits snapshots of the string's displacement at various times. The left side of Figure 5.3 shows the evolution of $\bar{f}_o$, the odd, 2ℓ-periodic extension of f. The right side of Figure 5.3 shows the evolution of the actual physical string itself.

Since the solution to the wave equation in the previous example represents two waves: one travelling to the left and one travelling to the right, we consider going back to the beginning and making a change of coordinates:

$$\xi = x + ct, \quad \eta = x - ct, \quad \text{and} \quad u(x,t) = w(\xi, \eta).$$

By the chain rule we get

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 w}{\partial \xi^2} + 2\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2}, \\
\frac{\partial u}{\partial t} &= c\left(\frac{\partial w}{\partial \xi} - \frac{\partial w}{\partial \eta}\right), \\
\frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 w}{\partial \xi^2} - 2\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2}\right).
\end{aligned}$$

Plug into the wave equation to get

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = c^2\left(\frac{\partial^2 w}{\partial \xi^2} - 2\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2}\right) - c^2\left(\frac{\partial^2 w}{\partial \xi^2} + 2\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2}\right) = -4c^2\frac{\partial^2 w}{\partial \xi \partial \eta}.$$

Therefore

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = 0.$$

This is easily solved:

$$\frac{\partial}{\partial \eta}\left(\frac{\partial w}{\partial \xi}\right) = 0 \quad \Longrightarrow \quad \frac{\partial w}{\partial \xi} = \phi'(\xi) \quad \Longrightarrow \quad w(\xi, \eta) = \phi(\xi) + \psi(\eta).$$

Therefore

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

This is the general solution to the 1-dimensional wave equation. It is known as *d'Alembert's solution*.

We now apply the auxiliary conditions. Taking $\partial/\partial t$ we get

$$\frac{\partial u}{\partial t}(x,t) = c[\phi'(x+ct) - \psi'(x-ct)].$$

At $t = 0$ we have

$$\left.\begin{array}{r} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{array}\right\} \implies \left\{\begin{array}{c} \phi(x) + \psi(x) = f(x) \\ c[\phi'(x) - \psi'(x)] = g(x) \end{array}\right\} \implies \left\{\begin{array}{c} \phi(x) + \psi(x) = f(x) \\ \phi'(x) - \psi'(x) = \frac{1}{2}G(x) \end{array}\right. .$$

Therefore

$$\phi(x) = \frac{1}{2}[f(x) + \frac{1}{c}G(x)], \qquad \psi(x) = \frac{1}{2}[f(x) - \frac{1}{c}G(x)].$$

The functions ϕ and ψ are only defined for $0 < x < \ell$. But $x - ct$ and $x + ct$ go beyond the interval $(0, \ell)$. Let $\widetilde{f}$ and $\widetilde{G}$ be extensions of f and G to the entire real line. Then we have

$$\phi(x) = \frac{1}{2}[\widetilde{f}(x) + \frac{1}{c}\widetilde{G}(x)], \qquad \psi(x) = \frac{1}{2}[\widetilde{f}(x) - \frac{1}{c}\widetilde{G}(x)], \qquad \forall x.$$

We now apply the boundary conditions. At $x = 0$ we have

$$u(0,t) = 0 \implies \phi(ct) + \psi(-ct) = 0 \implies \widetilde{f}(ct) + \widetilde{f}(-ct) + \frac{1}{c}[\widetilde{G}(ct) - \widetilde{G}(-ct)] = 0.$$

But f and g are independent functions, so we must have

$$\widetilde{f}(ct) + \widetilde{f}(-ct) = 0 \quad \text{and} \quad \widetilde{G}(ct) - \widetilde{G}(-ct) = 0.$$

This means that

$$\begin{aligned} \widetilde{f}(ct) &= -\widetilde{f}(-ct) \quad \text{for all } t, \text{ therefore } \widetilde{f} \text{ is odd,} \\ \widetilde{G}(ct) &= \widetilde{G}(-ct) \quad \text{for all } t, \text{ therefore } \widetilde{G} \text{ is even.} \end{aligned}$$

At the right boundary $x = \ell$ we have $u(\ell, t) = 0$ which implies

$$\phi(\ell+ct) + \psi(\ell-ct) = 0 \implies \widetilde{f}(\ell+ct) + \widetilde{f}(\ell-ct) + \frac{1}{c}[\widetilde{G}(\ell+ct) - \widetilde{G}(\ell-ct)] = 0.$$

Again, since f and g are independent functions, we must have

$$\widetilde{f}(\ell+ct) = -\widetilde{f}(\ell-ct) \quad \text{and} \quad \widetilde{G}(\ell+ct) = \widetilde{G}(\ell-ct).$$

Manipulating further

$$\begin{aligned} \widetilde{f} \text{ odd} &\implies \widetilde{f}(\ell+ct) = \widetilde{f}(-\ell+ct) = \widetilde{f}(\ell+ct-2\ell) \\ \widetilde{G} \text{ even} &\implies \widetilde{G}(\ell+ct) = \widetilde{G}(-\ell+ct) = \widetilde{G}(\ell+ct-2\ell) \end{aligned}$$

Therefore $\widetilde{f}$ and $\widetilde{G}$ are 2ℓ-periodic. Hence

$$\widetilde{f} = \bar{f}_o, \quad \text{and} \quad \widetilde{G} = \int \bar{g}_o(x)\, dx = G.$$

Therefore

$$u(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}[\bar{f}_o(x-ct) + \bar{f}_o(x+ct)] + \frac{1}{2c}[G(x+ct) - G(x-ct)].$$

In the next example we consider a 1–dimensional wave equation on an infinite domain with time dependent boundary conditions.

EXAMPLE 5.3. Consider the following:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x>0,\ t>0, \quad (c \equiv \text{ constant}) \\ &u(0,t) = h(t), \\ &u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0. \end{aligned}$$

This problem governs the vertical displacement of a "semi–infinite" string with the motion at its (one and only) boundary $x = 0$ precribed. The general solution to the equation is

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

Differentiating with respect to t gives

$$\frac{\partial u}{\partial t}(x,t) = c[\phi'(x+ct) - \psi'(x-ct)].$$

Applying the initial conditions at $t = 0$ gives

$$\phi(x) + \phi(x) = 0, \quad \phi'(x) - \psi'(x) = 0, \quad \text{for } x > 0.$$

We have

$$\psi(x) = -\phi(x) \quad \implies \quad \phi'(x) = 0 \quad \implies \quad \phi(x) = A, \quad \psi(x) = -A, \quad \text{for } x > 0,$$

where A is some constant. At the boundary $x = 0$ we have

$$\phi(ct) + \psi(-ct) = h(t) \quad \implies \quad A + \psi(-ct) = h(t) \quad \implies \quad \psi(-ct) = h(t) - A, \quad \text{for } t > 0.$$

Thus we have

$$\psi(\xi) = \begin{cases} -A & \text{for } \xi > 0, \\ h(-\xi/c) - A & \text{for } \xi < 0. \end{cases}$$

The solution becomes

$$u(x,t) = \phi(x+ct) + \psi(x-ct) = A + \psi(x-ct) =: \widetilde{\psi}(x-ct),$$

where

$$\widetilde{\psi}(\xi) = A + \psi(\xi) = \begin{cases} 0 & \text{for } \xi > 0, \\ h(-\xi/c) & \text{for } \xi < 0. \end{cases}$$

Therefore

$$u(x,t) = \widetilde{\psi}(x-ct) = \begin{cases} 0 & \text{for } x > ct, \\ h(t - x/c) & \text{for } x < ct. \end{cases}$$

This represents a rightward travelling wave moving at speed "c". A physical interpretation is given in Figure 5.4. Here we have the displacement given as a function of time at various positions:

$$u(0,t) = \begin{cases} 0 & \text{for } t < 0 \\ h(t) & \text{for } t > 0 \end{cases}, \quad u(c,t) = \begin{cases} 0 & \text{for } t < 1 \\ h(t-1) & \text{for } t > 1 \end{cases}, \quad u(2c,t) = \begin{cases} 0 & \text{for } t < 2 \\ h(t-2) & \text{for } t > 2 \end{cases},$$

and here we give snapshots of the string at various time intervals:

$$u(x,0) = 0, \quad u(x,1) = \begin{cases} 0 & \text{for } x > c \\ h(1 - x/c) & \text{for } x < c \end{cases}, \quad u(x,2) = \begin{cases} 0 & \text{for } x > 2c \\ h(2 - x/c) & \text{for } x < 2c \end{cases}.$$

5.3 Laplace's Equation

Laplace's equation, also called the potential equation, may be compactly written as

$$\nabla^2 u = 0.$$

Written out explicitly, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{(2-d Laplace equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \text{(3-d Laplace equation)}$$

Time independent solutions of either the heat equation or the wave equation satisfy Laplace's equation. Solutions of Laplace's equation are called *harmonic functions*. A complete well–posed problem consists of Laplace's equation together with boundary conditions:

$$\nabla^2 u = 0, \quad \text{in } \Omega,$$

$$\alpha u + \beta \frac{\partial u}{\partial n} = f, \quad \text{on } \partial\Omega,$$

where
$$\frac{\partial u}{\partial n} = \vec{\nabla} u \cdot \vec{n}, \quad \text{with} \quad \vec{n} \text{ beint the unit outward pointing normal of } \Omega.$$

EXAMPLE 5.4. (Potential in a rectangle)
Consider Laplace's equation defined on a rectangle:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 < x < a, \quad 0 < y < b, \\ u(x,0) = f_1(x), &\quad u(0,y) = 0, \\ u(x,b) = f_2(x), &\quad u(a,y) = 0. \end{aligned}$$

Notice that according to Hillen's rule of thumb, we have four boundary conditions. Two are homogeneous boundary conditions on one pair of opposite sides. Try separation of variables: $u(x,y) = X(x)Y(y)$. Plug into the equation to get:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \quad \text{(constant)}$$

Using the homogeneous boundary conditions, we get

$$\left.\begin{matrix} u(0,y) = 0 \\ u(a,y) = 0 \end{matrix}\right\} \implies \left\{\begin{matrix} X(0)Y(y) = 0 \\ X(a)Y(y) = 0 \end{matrix}\right\} \implies \left\{\begin{matrix} X(0) = 0 \\ X(a) = 0 \end{matrix}\right. .$$

We get the following ODEs for X and Y:

$$\begin{aligned} X'' + \lambda X &= 0, \qquad\qquad & Y'' - \lambda Y = 0, \\ X(0) = X(a) &= 0. \end{aligned}$$

Clearly, the qualitative nature of the solutions of the X equation will be different than those of the Y equation. The problem fo X is a regular Sturm–Liouville problem. From Theorem 31, the eigenvalues for the X problem are all non-negative. Therefore set $\lambda = \mu^2$. We get

$$\left.\begin{matrix} X'' + \mu^2 X = 0 \\ X(0) = X(a) = 0 \end{matrix}\right\} \implies \left\{\begin{matrix} \lambda_n = \mu_n^2 = \frac{n^2\pi^2}{a^2} \\ X_n(x) = \sin(\frac{n\pi x}{a}) \end{matrix}\right. \quad n = 1, 2, \ldots .$$

The equation for Y becomes
$$Y'' - \mu_n^2 Y = 0,$$

which is easily solved to give

$$Y_n(y) = \alpha_n \cosh(\mu_n y) + \beta_n \sinh(\mu_n y), \qquad n = 1, 2, \ldots .$$

For each $n = 1, 2, \ldots$ we have $u_n(x,y) = X_n(x)Y_n(y)$ which is a solution to Laplace's equation which also satisfies the homogeneous boundary conditions. However, none of these can individually be made to satisfy the nonhomogeneous boundary conditions. Since Laplace's equation is linear and homogeneous we can apply the principle of superposition:

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty}[\alpha_n \cosh(\mu_n y) + \beta_n \sinh(\mu_n y)] \sin(\mu_n x).$$

We now apply the remaining boundary conditions. At $y = 0$ we have

$$u(x,0) = f_1(x) \quad \Longrightarrow \quad f_1(x) = \sum_{n=1}^{\infty} \alpha_n \sin(\mu_n x).$$

Therefore the α_n's must be the Fourier sine coefficients

$$\alpha_n = \frac{2}{a}\int_0^a f_1(x)\sin(\frac{n\pi x}{a})\,dx.$$

At $y = b$ we get

$$u(x,b) = f_2(x) \quad \Longrightarrow \quad f_2(x) = \sum_{n=1}^{\infty}[\alpha_n \cosh(\mu_n b) + \beta_n \sinh(\mu_n b)] \sin(\mu_n x),$$

from which we conclude

$$\alpha_n \cosh(\mu_n b) + \beta_n \sinh(\mu_n b) = \frac{2}{a}\int_0^a f_2(x)\sin(\frac{n\pi x}{a})\,dx =: \gamma_n,$$

or

$$\beta_n = \frac{\gamma_n - \alpha_n \cosh(\mu_n b)}{\sinh(\mu_n b)}.$$

Hence we get

$$\begin{aligned}
Y_n(y) &= \alpha_n \cosh(\mu_n y) + \beta_n \sinh(\mu_n y) = \alpha_n \cosh(\mu_n y) + \frac{\gamma_n - \alpha_n \cosh(\mu_n b)}{\sinh(\mu_n b)} \sinh(\mu_n y) \\
&= \frac{\alpha_n}{\sinh(\mu_n b)}[\sinh(\mu_n b)\cosh(\mu_n y) - \cosh(\mu_n b)\sinh(\mu_n y)] + \frac{\gamma_n}{\sinh(\mu_n b)}\sinh(\mu_n y) \\
&= \frac{1}{\sinh(\mu_n b)}[\alpha_n \sinh(\mu_n(b-y)) + \gamma_n \sinh(\mu_n y)].
\end{aligned}$$

The final solution is:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{1}{\sinh(\mu_n b)}[\alpha_n \sinh(\mu_n(b-y)) + \gamma_n \sinh(\mu_n y)] \sin(\mu_n x),$$

where

$$\begin{aligned}
\alpha_n &= \frac{2}{a}\int_0^a f_1(x)\sin(\frac{n\pi x}{a})\,dx, \qquad \mu_n = \frac{n\pi}{a}, \\
\gamma_n &= \frac{2}{a}\int_0^a f_2(x)\sin(\frac{n\pi x}{a})\,dx.
\end{aligned}$$

Consider the following more general problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$$
$$u(x,0) = f_1(x), \quad u(0,y) = g_1(y),$$
$$u(x,b) = f_2(x), \quad u(a,y) = g_2(y).$$

None of the boundary conditions is homogeneous. To solve this problem we split it into two, each of which is in a form similar to the one in the preceding example. Consider the following problems for v and w:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$
$$v(x,0) = f_1(x), \quad v(0,y) = 0, \qquad w(x,0) = 0, \quad w(0,y) = g_1(y),$$
$$v(x,b) = f_2(x), \quad v(a,y) = 0, \qquad w(x,b) = 0, \quad w(a,y) = g_2(y).$$

Then $u(x,y) = v(x,y) + w(x,y)$ is the solution to the original problem.

5.4 Maximum Principle

Consider a closed, bounded region $R \in \mathbb{R}^2$ with boundary ∂R.

Theorem 37. *Let $u(x,y)$ be any continuous solution of*

$$\nabla^2 u = F(x,y). \qquad \textit{(Poisson's equation)}$$

Then

1. *The maximum of u in R occurs on the boundary ∂R if $F > 0$ in R.*
2. *The minimum of u in R occurs on the boundary ∂R if $F < 0$ in R.*

Proof.
(by contradiction)
We do only part 1 with $F(x,y) > 0$ for all $(x,y) \in R$. Proof of the second part is similar. Since u is a continuous solution in R (a closed, bounded set) it has a maximum at some point $(x_0,y_0) \in R$. Suppose that $(x_0,y_0) \in R^o$ (the interior of R), (i.e. $(x_0,y_0) \notin \partial R$). It follows that

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial u}{\partial y}(x_0,y_0) = 0.$$

Since u has a maximum at (x_0,y_0) we have

$$\frac{\partial^2 u}{\partial x^2}(x_0,y_0) \leqslant 0, \quad \frac{\partial^2 u}{\partial y^2}(x_0,y_0) \leqslant 0. \qquad \text{(i.e. concave down)}$$

Summing these yields

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \leqslant 0,$$

which contradicts $u_{xx} + u_{yy} = F > 0$. Hence, it cannot be that the point (x_0, y_0) at which u attains its maximum in the interior of R. Therefore this point lies on the boundary.

Theorem 38. *If u is a continuous solution of $\nabla^2 u = 0$ in the closed, bounded region R, then the maximum and minimum of u occur on the boundary of R.*

Proof.
Let M be the maximum of u on ∂R, which exists since ∂R is a closed, bounded set. In other words

$$|u(x,y)| \leqslant M, \quad \forall (x,y) \in \partial R.$$

What we want to show is that this inequality holds, not only for $(x,y) \in \partial R$, but for all $(x,y) \in R$. To this end, consider a square of length 2ℓ, where ℓ is large enough so that the square contains all of R. Then

$$(x,y) \in R \quad \Longrightarrow \quad |x| \leqslant \ell.$$

Let $\varepsilon > 0$ be arbitrary. Define

$$v(x,y) := u(x,y) + \varepsilon x^2.$$

For $(x,y) \in \partial R$ we have

$$v(x,y) = u(x,y) + \varepsilon x^2 \leqslant M + \varepsilon \ell^2.$$

Differentating yields

$$\nabla^2 v = \nabla^2 u + \varepsilon \nabla^2 (x^2) = 2\varepsilon.$$

Since $\nabla^2 v = 2\varepsilon > 0$, the previous theorem implies that v attains its maximum on ∂R. Therefore

$$v(x,y) \leqslant M + \varepsilon \ell^2, \quad \forall (x,y) \in R.$$

For u we have

$$u(x,y) = v(x,y) - \varepsilon x^2 \leqslant v(x,y) \leqslant M + \varepsilon \ell^2, \quad \forall (x,y) \in R.$$

But ε was arbitrary, so letting $\varepsilon \to 0$ we get

$$u(x,y) \leqslant M, \quad \forall (x,y) \in R.$$

Thus M, the global maximum for u, occurs on the boundary ∂R. The proof for the case of the minimum value is similar.

5.5 Wave Equation (2–d)

Recall te general wave equation derived earlier:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + f.$$

We now consider the 2–d wave equation on a rectangle.

EXAMPLE 5.5. Consider the following 2–dimensional, homogeneous wave equation defined on a rectangle;

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u, \qquad 0 < x < a,\ 0 < y < b,\ t > 0,\\ u(0,y,t) &= 0, \quad u(x,0,t) = 0,\\ u(a,y,t) &= 0, \quad u(x,b,t) = 0,\\ u(x,y,0) &= f(x,y),\\ u_t(x,y,0) &= 0. \end{aligned}$$

Notice that we need six side conditions, according to our rule of thumb. This problem governs the vertical displacement u of a membrane (think of a rectangular drum) stretched over a rectangle and fastened at the edges. Since the PDE itself and the boundary conditions are homogeneous, we try separation of variables: $u(x,y,t) = \phi(x,y)T(t)$. Here we look for a solution whereby the spatial variables can be separated from the time variable. Plug into the equation to get:

$$\phi T'' = c^2(\nabla^2\phi)T \quad \Longrightarrow \quad \frac{T''}{c^2T} = \frac{\nabla^2\phi}{\phi} = -\lambda. \quad \text{(const.)}$$

Plug $u(x,y,t) = \phi(x,y)T(t)$ into the homogeneous boundary conditions to get boundary conditions for ϕ. We end up with the following problems for ϕ and T:

$$\begin{aligned} \nabla^2\phi &= -\lambda\phi, \quad 0 < x < a,\ 0 < y < b, \\ \phi(0,y) &= 0,\ \phi(x,0) = 0,\\ \phi(a,y) &= 0,\ \phi(x,b) = 0. \end{aligned} \qquad\qquad \begin{aligned} T'' + \lambda c^2 T &= 0, \quad t > 0\\ T'(0) &= 0. \end{aligned}$$

The equation for ϕ is a linear, homogeneous PDE with homogeneous boundary conditions. We try separation of variables: $\phi(x,y) = X(x)Y(y)$. The ϕ equation becomes:

$$X''Y + XY'' = -\lambda XY \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\gamma. \quad \text{(const.)}$$

Plug $\phi(x,y) = X(x)Y(y)$ into the boundary conditions and we get the following problems for X and Y:

$$\begin{aligned} X'' + \gamma X &= 0, \quad 0 < x < a,\\ X(0) &= X(a) = 0. \end{aligned} \qquad\qquad \begin{aligned} Y'' + (\lambda - \gamma)Y &= 0, \quad 0 < y < b,\\ Y(0) &= Y(b) = 0. \end{aligned}$$

The problem for X is a regular Sturm–Liouville problem. By Theorem 31, the eigenvalues are all non-negative. So setting $\gamma = \mu^2$ we get:

$$\gamma_n = \mu_n^2 = \frac{n^2\pi^2}{a^2}, \quad n = 1, 2, \ldots \qquad \text{(eigenvalues)}$$
$$X_n(x) = \sin(\mu_n x). \qquad \text{(eigenfunctions)}$$

The problem for Y is a regular Sturm–Liouville problem. By Theorem 31, the eigenvalues are all non-negative. So setting $\lambda - \gamma = \omega^2$ we get:

$$\omega_m = \frac{m\pi}{b}, \quad m = 1, 2, \ldots$$
$$Y_m(y) = \sin(\omega_m y).$$

For ϕ we can write

$$\phi_{mn}(x, y) = X_n(x)Y_m(y) = \sin(\mu_n x)\sin(\omega_m y).$$

But $\lambda = \gamma + \omega^2 = \mu^2 + \omega^2$, therefore

$$\lambda_{mn} = \mu_n^2 + \omega_m^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) =: \alpha_{mn}^2.$$

The equation for T becomes

$$T'' + c^2\alpha_{mn}^2 T = 0,$$

which is easily solved to give

$$T_{mn}(t) = A_{mn}\cos(c\alpha_{mn}t) + B_{mn}\sin(c\alpha_{mn}t).$$

Applying the initial condition $T'_{mn}(0) = 0$ implies that $B_{mn} = 0$, so we end up with

$$T_{mn}(t) = A_{mn}\cos(c\alpha_{mn}t).$$

We now apply the principle of superposition:

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(x, y)T_{mn}(t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin(\mu_n x)\sin(\omega_m y)\cos(c\alpha_{mn}t).$$

It now remains to satisfy the last initial condition: $u(x, y, 0) = f(x, y)$.

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin(\mu_n x)\sin(\omega_m y) = \sum_{m=1}^{\infty}F_m(x)\sin(\omega_m y), \tag{5.1}$$

where we have defined

$$F_m(x) := \sum_{n=1}^{\infty}A_{mn}\sin(\mu_n x). \tag{5.2}$$

¿From (5.1) it is clear that the F_m's are just the Fourier sine coefficients of f, and from (5.2) it is clear that the A_{mn}'s are just the Fourier sine coefficients of the F_m's. Hence we have

$$F_m(x) = \frac{2}{b}\int_0^b f(x,y)\sin(\omega_m y)\,dy,$$
$$A_{mn} = \frac{2}{a}\int_0^a F_m(x)\sin(\mu_n x)\,dx.$$

Combining results the final solution to the problem is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(\mu_n x)\sin(\omega_m y)\cos(c\alpha_{mn}t),$$

where

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\mu_n x)\sin(\omega_m y)\,dy\,dx.$$

For the one dimensional problems, the coefficients in the series solution were obtained by means of inner products. Can the same be done in this case? Can the formula for the A_{mn}'s be expressed in terms of inner products? If we examine the formula for A_{mn}'s more closely we see that

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\phi_{mn}(x,y)\,dy\,dx \quad \text{which resembles} \quad (\text{const.})\cdot\langle f, \phi_{mn}\rangle\,,$$

for some "appropriate" inner product. In the next section we shall see how this can be done in a systematic way.

5.6 Eigenfunctions in Two Dimensions

Consider the homogeneous version of either the two-dimensional heat equation or the two-dimensional wave equation:

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \qquad\qquad \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u,$$
$$+\,\text{b.c.'s}, \qquad\qquad +\,\text{b.c.'s},$$
$$+\,\text{i.c.'s}. \qquad\qquad +\,\text{i.c.'s}.$$

If we apply separation of variables $u(x,y,t) = \phi(x,y)T(t)$, we get

$$\frac{T'}{kT} = \frac{\nabla^2\phi}{\phi} = -\lambda, \qquad \text{or} \qquad \frac{T''}{c^2T} = \frac{\nabla^2\phi}{\phi} = -\lambda.$$

Either way, we end up with a two-dimensional eigenvalue problem for ϕ

$$\nabla^2\phi = -\lambda\phi,$$
$$+\,\text{b.c.'s}.$$

Typically, problems of this type will have an infinite, discrete (i.e. countable) spectrum. In other words, nontrivial solutions exist for an infinite, but discrete, set of values of λ. The main result of this section will be to show that, with the appropriate inner product, distinct eigenfunctions are orthogonal.

To accomplish these results, we will need Green's Theorem, along with what are known as Green's identities.

Theorem 39 (Green's Theorem)**.** *If*

(i) Ω is a domain in the xy-plane with boundary $\partial\Omega$;

(ii) P and Q are continuous with continuous partial derivatives in Ω;

then

$$\oint_{\partial\Omega} P(x,y)\,dx + Q(x,y)\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

We will demonstrate the following:

$$\left.\begin{matrix}\text{Green's}\\ \text{theorem}\end{matrix}\right\} \implies \left\{\begin{matrix}\text{Green's } 1^{st}\\ \text{identity}\end{matrix}\right\} \implies \left\{\begin{matrix}\text{Green's } 2^{nd}\\ \text{identity}\end{matrix}\right\} \implies \left\{\begin{matrix}\text{eigenfunctions}\\ \text{are orthogonal}\end{matrix}\right.$$

Theorem 40 (Green's 1^{st} identity)**.**
If u and v are twice continuously differentialble in the region $\Omega \subset \mathbb{R}^2$, then

$$\iint_{\Omega} (u\nabla^2 v + \vec{\nabla} u \cdot \vec{\nabla} v)\,dA = \oint_{\partial\Omega} u\frac{\partial v}{\partial n}\,ds.$$

In the above equation, $\frac{\partial v}{\partial n}$ refers to the directional derivative of v in the direction of $\vec{n}$, the unit outward pointing normal of Ω, and s refers to arc-length.

Proof.
Parameterize the boundary $\partial\Omega$ in terms of arc-length as follows:

$$\partial\Omega : \begin{cases} x = x(s), \\ y = y(s). \end{cases}$$

Let $\vec{r}$ be the position vector to a point $(x,y) \in \partial\Omega$, $\vec{T}$ be the unit tangent vector at (x,y), and $\vec{n}$ be the unit, outward pointing normal vector. Then

$$\vec{r} = (x,y), \quad \vec{T} = \frac{d\vec{r}}{ds} = (\frac{dx}{ds}, \frac{dy}{ds}), \quad \vec{n} = (\frac{dy}{ds}, -\frac{dx}{ds}).$$

Let

$$P = u\frac{\partial v}{\partial y}, \qquad Q = -u\frac{\partial v}{\partial x}.$$

Then
$$\frac{\partial P}{\partial y} = u\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \qquad \frac{\partial Q}{\partial x} = -u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}.$$

Now we get

$$\begin{aligned}
\iint\limits_{\Omega} (u\nabla^2 v + \vec{\nabla} u\cdot\vec{\nabla} v)\,dA &= \iint\limits_{\Omega} \left[u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + (\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})\cdot(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y})\right] dA \\
&= \iint\limits_{\Omega} \left[\left(u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) + \left(u\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\right] dA \\
&= \iint\limits_{\Omega} \left(-\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dA = -\oint\limits_{\partial\Omega} P(x,y)\,dx + Q(x,y)\,dy \\
&= -\oint\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial y}\,dx - u\frac{\partial v}{\partial x}\,dy\right) = \oint\limits_{\partial\Omega} u\left(-\frac{\partial v}{\partial y}\frac{dx}{ds} + \frac{\partial v}{\partial x}\frac{dy}{ds}\right) ds \\
&= \oint\limits_{\partial\Omega} u(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y})\cdot(\frac{dy}{ds},-\frac{dx}{ds})\,ds = \oint\limits_{\partial\Omega} u(\vec{\nabla} v\cdot\vec{n})\,ds = \oint\limits_{\partial\Omega} u\frac{\partial v}{\partial n}\,ds.
\end{aligned}$$

The next result is much easier to prove.

Theorem 41 (Green's 2^{nd} identity)**.**
If u and v are twice continuously differentialble in the region $\Omega\subset\mathbb{R}^2$, then

$$\iint\limits_{\Omega} (u\nabla^2 v - v\nabla^2 u)\,dA = \oint\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds.$$

Proof.
¿From Green's 1^{st} identity we have

$$\iint\limits_{\Omega} (u\nabla^2 v + \vec{\nabla} u\cdot\vec{\nabla} v)\,dA = \oint\limits_{\partial\Omega} u\frac{\partial v}{\partial n}\,ds.$$

Re writing this with u and v reversed yields

$$\iint\limits_{\Omega} (v\nabla^2 u + \vec{\nabla} v\cdot\vec{\nabla} u)\,dA = \oint\limits_{\partial\Omega} v\frac{\partial u}{\partial n}\,ds.$$

Subtraction gives the result.

We now come to one of the two main results of this section.

Theorem 42 (main result 1)**.**
Eigenfunctions corresponding to distinct eigenvalues of either problem

$$I: \begin{cases} \nabla^2\phi = -\lambda\phi & in\ \Omega \\ \phi = 0 & on\ \partial\Omega \end{cases} \qquad 3or \qquad 3II: \begin{cases} \nabla^2\phi = -\lambda\phi & in\ \Omega \\ \frac{\partial\phi}{\partial n} = 0 & on\ \partial\Omega \end{cases}$$

are orthogonal relative to the following inner product:

$$\langle f, g\rangle = \iint\limits_{\Omega} f(x,y)g(x,y)\,dA.$$

<u>*Proof*</u>.
Let $\widetilde{\lambda}$ and $\widehat{\lambda}$ be distinct eigenvalues to one of the above problems (i.e. $\widetilde{\lambda} \neq \widehat{\lambda}$), with corresponding eigenfunctions $\widetilde{\phi}$ and $\widehat{\phi}$ respectively. Then we have

$$\nabla^2\widetilde{\phi} = -\widetilde{\lambda}\widetilde{\phi}, \qquad \text{and} \qquad \nabla^2\widehat{\phi} = -\widehat{\lambda}\widehat{\phi}.$$

Therefore

$$\iint\limits_{\Omega} (\widetilde{\phi}\nabla^2\widehat{\phi} - \widehat{\phi}\nabla^2\widetilde{\phi})\,dA = \iint\limits_{\Omega} [\widetilde{\phi}(-\widehat{\lambda}\widehat{\phi}) - \widehat{\phi}(-\widetilde{\lambda}\widetilde{\phi})]\,dA = (\widetilde{\lambda} - \widehat{\lambda})\iint\limits_{\Omega} \widetilde{\phi}\widehat{\phi}\,dA = (\widetilde{\lambda} - \widehat{\lambda})\left\langle \widetilde{\phi}, \widehat{\phi}\right\rangle.$$

Using Green's 2^{nd} identity we get

$$(\widetilde{\lambda} - \widehat{\lambda})\left\langle \widetilde{\phi}, \widehat{\phi}\right\rangle = \iint\limits_{\Omega} (\widetilde{\phi}\nabla^2\widehat{\phi} - \widehat{\phi}\nabla^2\widetilde{\phi})\,dA = \oint\limits_{\partial\Omega} \left(\widetilde{\phi}\frac{\partial\widehat{\phi}}{\partial n} - \widehat{\phi}\frac{\partial\widetilde{\phi}}{\partial n}\right) ds = 0.$$

Since $\widetilde{\lambda} \neq \widehat{\lambda}$, it follows that $\left\langle \widetilde{\phi}, \widehat{\phi}\right\rangle = 0$.

The final result.

Theorem 43 (main result 2)**.**
The eigenvalues of either of the problems

$$I: \begin{cases} \nabla^2\phi = -\lambda\phi & in\ \Omega \\ \phi = 0 & on\ \partial\Omega \end{cases} \qquad 3or \qquad 3II: \begin{cases} \nabla^2\phi = -\lambda\phi & in\ \Omega \\ \frac{\partial\phi}{\partial n} = 0 & on\ \partial\Omega \end{cases}$$

are non-negative.

<u>*Proof*</u>.
Let λ be an eigenvalue to either problem I or II above with corresponding eigenfunction ϕ.

Then we have

$$
\begin{aligned}
\lambda\|\phi\|^2 &= \lambda\left\langle \phi,\phi\right\rangle = \left\langle \lambda\phi,\phi\right\rangle = \left\langle -\nabla^2\phi,\phi\right\rangle = -\iint\limits_{\Omega} \phi\nabla^2\phi\,dA \\
&= -\left\{\oint\limits_{\partial\Omega} \phi\frac{\partial\phi}{\partial n}\,ds - \iint\limits_{\Omega} \vec{\nabla}\phi\cdot\vec{\nabla}\phi\,dA\right\} \qquad \text{(using Green's } 1^{st} \text{ identity with } u=v=\phi) \\
&= 0 + \iint\limits_{\Omega} |\vec{\nabla}\phi|^2\,dA \geqslant 0.
\end{aligned}
$$

It follows that $\lambda \geqslant 0$.

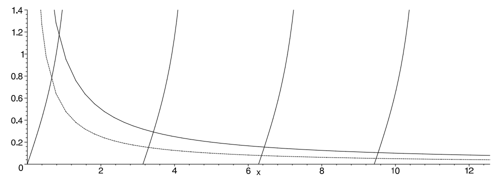


Figure 5.1: Eigenvalues

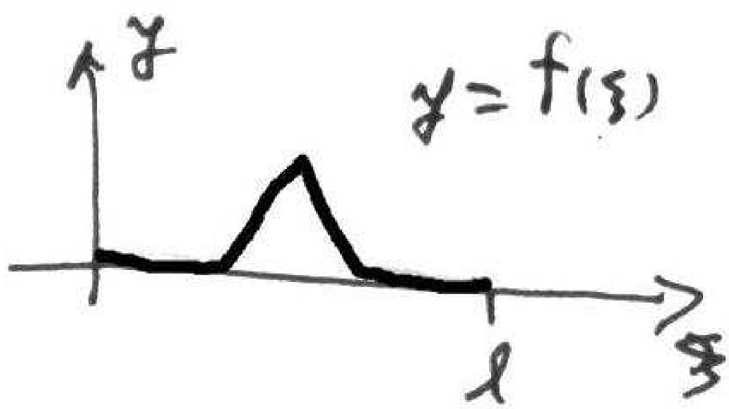


Figure 5.2: The hat function

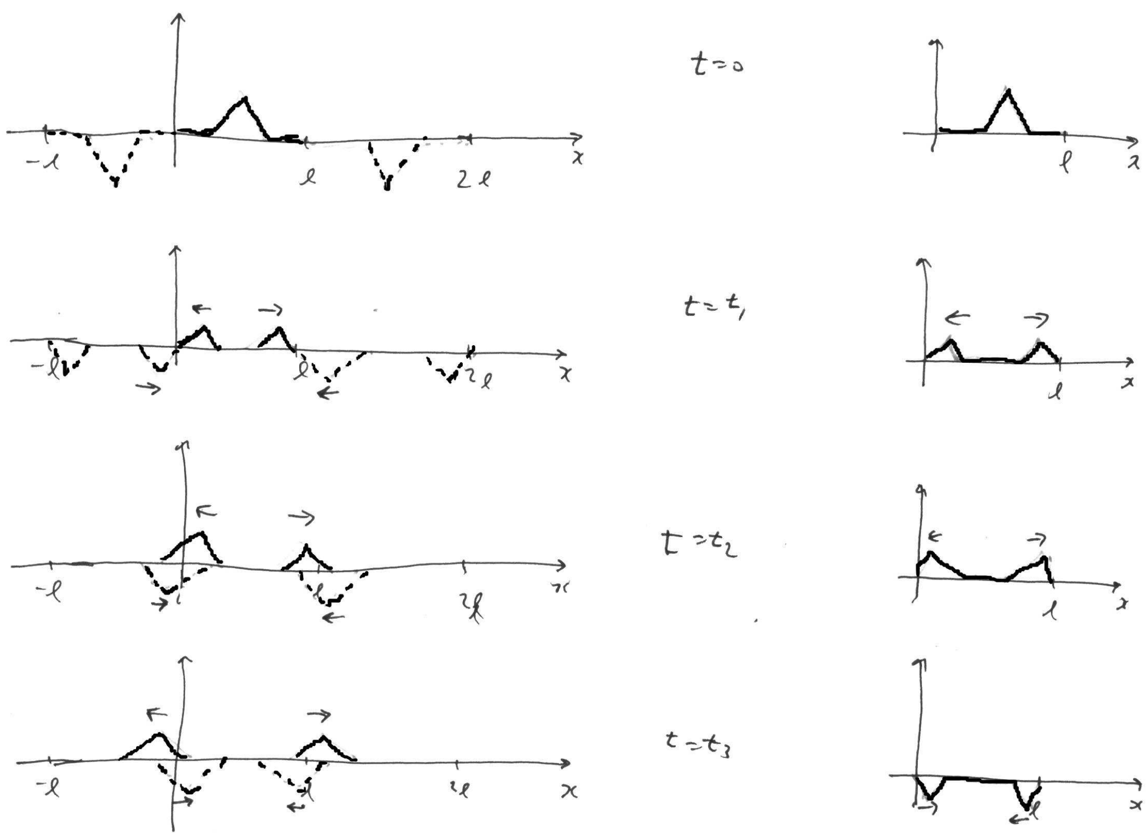


Figure 5.3: Snapshots of the string at various times.

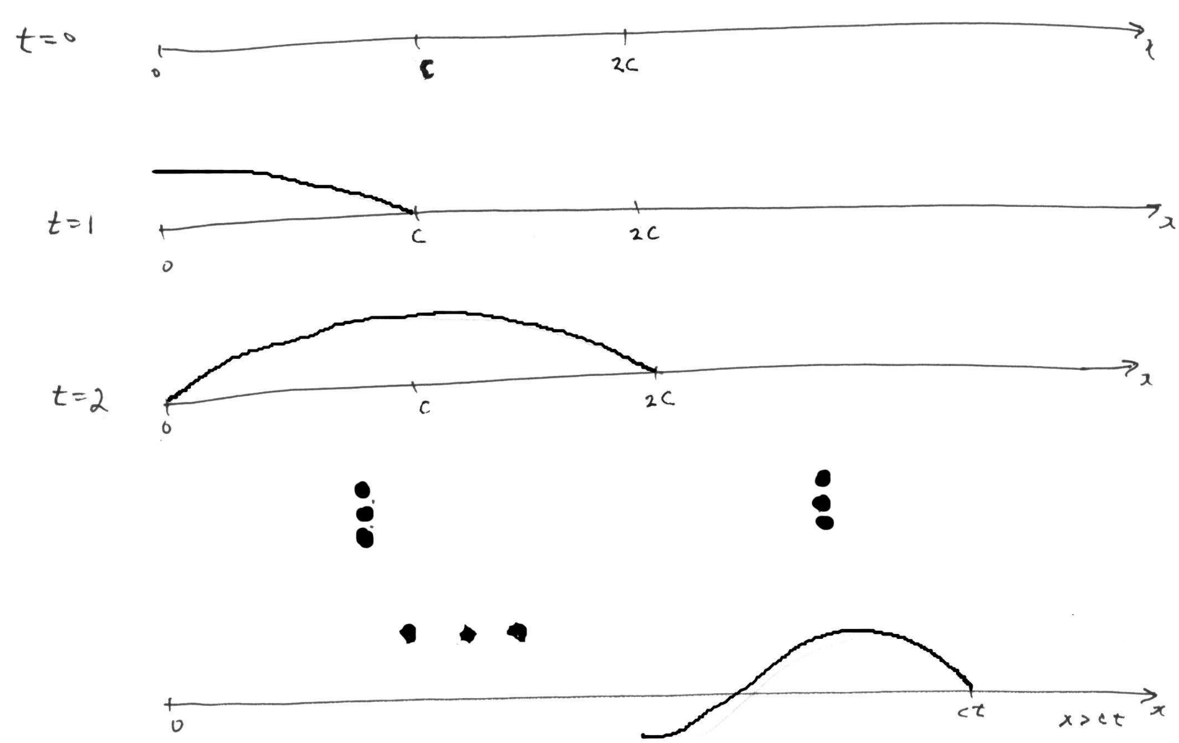


Figure 5.4: Snapshots of the wave front at various times.

Chapter 6

Problems in Cylindrical Coordinates

June 17, 2010

6.1 Polar Coordinates

There are certain geometries, such as those shown in Figure 6.1, for which polar coordinates are more useful than Cartesian coordinates.

Polar coordinates are defines as follows

$$x = \rho\cos\phi, \qquad \rho^2 = x^2 + y^2,$$
$$y = \rho\sin\phi, \qquad \phi = \tan^{-1}\frac{y}{x}.$$

The Jacobian determinant for the transformation is

$$\frac{\partial(x,y)}{\partial(\rho,\phi)} = \begin{vmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\sin\phi \end{vmatrix} = \rho$$

which indicates that the transformation is singular at $\rho = 0$ (i.e. at the origin). The Laplacian in polar coordinates is:

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}.$$

We shall begin by considering Laplace'e equation defined in a circular region.

EXAMPLE 6.1. (potential in a disk)
Consider Laplace's equation in a circular disk:

$$\nabla^2 u = 0, \qquad 0 < \rho < a,\ -\pi < \phi < \pi,$$
$$u(a,\phi) = f(\phi).$$

When we consider the domain in the $\rho\phi$-plane (see Figure 6.2), it appears as a rectangle. Three of the boundaries of this rectangle are not "real physical" boundaries. Nevertheless, if we treat the region as a rectangle in the $\rho\phi$-plane, we need to impose boundary conditions on these "unphysical boundaries". To this end, we make the following additional assumptions:

A1: $u(\rho,\pi) = u(\rho,-\pi)$ and $\frac{\partial u}{\partial\phi}(\rho,\pi) = \frac{\partial u}{\partial\phi}(\rho,-\pi)$;

A2: $|u(\rho,\phi)|$ bounded as $\rho \to 0$.

The first assumption is justified since the line $\phi = \pi$ and $\phi = -\pi$ in the $\rho\phi$-plane actually represent the same line in the physical plane. The second assumption is to preclude the occurrence of singular solutions that may arise due to the singular nature of the transformation to polar coordinates. While these boundary conditions are quite different than those we have experienced before, and are not homogeneous, they do allow us to use separation of variables.

Look for a solution of the form $u(\rho,\phi) = R(\rho)S(\phi)$. Then

$$\nabla^2 u = 0 \quad \Longrightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho R')S + \frac{1}{\rho^2}RS'' = 0 \quad \Longrightarrow \quad \frac{\rho}{R}(\rho R')' = -\frac{S''}{S} = \lambda \qquad 1(\text{constant}).$$

Assumption (A1) implies that $S(\pi) = S(-\pi)$, and likwise for S', and assumption (A2) implies that $|R(\rho)|$ is bounded as $\rho \to 0$. We get the following problems for R and S:

$$\begin{aligned} \rho(\rho R')' - \lambda R &= 0 \\ |R(\rho)| \text{ is bdd as } \rho &\to 0 \end{aligned} \qquad\qquad \begin{aligned} S'' + \lambda S &= 0, \\ S(\pi) &= S(-\pi), \\ S'(\pi) &= S'(-\pi). \end{aligned}$$

The problem for S is not a standard Sturm–Liouville problem as we described such problems earlier. Each boundary condition for the S equation involves both boundary points, which differs from the standard Sturm Liouville problem. Nevertheless, it is easily verified that one gets nontrivial solutions for S equation only for $\lambda \geqslant 0$. If we set $\lambda = \mu^2$, then

$$S(\phi) = a\cos(\mu\phi) + b\sin(\mu\phi) \quad \text{and} \quad S'(\phi) = \mu[-a\sin(\mu\phi) + b\cos(\mu\phi)].$$

¿From the boundary conditions for S we get

$$\left.\begin{aligned} a\cos\mu\pi + b\sin\mu\pi &= a\cos\mu\pi - b\sin\mu\pi \\ \mu[-a\sin\mu\pi + b\cos\mu\pi] &= \mu[a\sin\mu\pi + b\cos\mu\pi] \end{aligned}\right\} \quad \Longrightarrow \quad \left\{\begin{aligned} b\sin\mu\pi &= 0 \\ \mu a\sin\mu\pi &= 0 \end{aligned}\right\} \quad \Longrightarrow \quad \mu = \mu_n = n.$$

Therefore we get

$$\begin{aligned} \lambda_n &= \mu_n^2 = n^2, && \text{(eigenvalues)} \\ S_n(\phi) &= a_n\cos(n\phi) + b_n\sin(n\phi). && \text{(eigenfunctions)} \end{aligned}$$

The equation for R becomes

$$\rho(\rho R')' - \mu_n^2 R = 0, \quad \text{or} \quad \rho^2 R'' + \rho R' - n^2 R = 0. \quad \text{(Cauchy–Euler equation)}$$

Looking for a solution of the form $R(\rho) = \rho^m$, we get

$$[m(m-1) + m - n^2]\rho^m = 0 \quad \Longrightarrow \quad m = \pm n \quad \Longrightarrow \quad R_n(\rho) = \alpha_n\rho^n + \beta_n\rho^{-n}.$$

But (A2) implies that $\beta_n = 0$ for all n. Therefore

$$R_n(\rho) = \alpha_n \rho^n \quad \Longrightarrow \quad u_n(\rho,\phi) = R_n(\rho)S_n(\phi) = \rho^n[A_n\cos(n\phi) + B_n\sin(n\phi)].$$

Applying the principle of superposition we get

$$u(\rho,\phi) = \sum_{n=0}^{\infty} R_n(\rho)S_n(\phi) = \sum_{n=0}^{\infty} \rho^n[A_n\cos(n\phi) + B_n\sin(n\phi)].$$

It remains to satisfy the last boundary condition in the $\rho\phi$-plane, which corresponds to the boundary condition at the actual physical boundary: $u(a,\phi) = f(\phi)$, we get

$$f(\phi) = \sum_{n=0}^{\infty} a^n[A_n\cos(n\phi) + B_n\sin(n\phi)] = A_0 + \sum_{n=1}^{\infty} a^n[A_n\cos(n\phi) + B_n\sin(n\phi)].$$

Hence the A_n's and B_n's are the Fourier coefficients of f. The final solution is

$$u(\rho,\phi) = A_0 + \sum_{n=1}^{\infty} \rho^n[A_n\cos(n\phi) + B_n\sin(n\phi)], \tag{6.1}$$

where

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi, \\ A_n &= \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,d\phi, \quad n = 1,2,\ldots, \\ B_n &= \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,d\phi, \quad n = 1,2,\ldots. \end{aligned}$$

A couple of observations:

- At $\rho = 0$ we have

$$u(0,\phi) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a,\phi)\,d\phi$$

  so u at the centre of the disk is just the average of u over the boundary.

- If $f(\phi) = f_0$ is a constant, then $A_0 = f_0$, and $A_n = B_n = 0$, so that $u(\rho,\phi) \equiv f_0$ (i.e. u const. on the boundary means that u is const. everywhere). This should not be surprising, since the maximum principle implies that the maximum and minimum of u occur on the boundary. So if u is constant on the boundary, then u must be constant everywhere.

We have the solution u in the form of an infinite series. It turns out that we can express u as an integral over the boundary. To see how to do this, we examine the terms in (6.1):

$$\rho^n A_n \cos(n\phi) = \frac{\rho^n}{a^n \pi} \int_{-\pi}^{\pi} f(\xi) \cos(n\xi) \cos(n\phi)\, d\xi, \quad \rho^n B_n \sin(n\phi) = \frac{\rho^n}{a^n \pi} \int_{-\pi}^{\pi} f(\xi) \sin(n\xi) \sin(n\phi)\, d\xi.$$

Adding these yields

$$\begin{aligned} \rho^n [A_n \cos(n\phi) + B_n \sin(n\phi)] &= \frac{\rho^n}{a^n \pi} \int_{-\pi}^{\pi} f(\xi)[\cos(n\xi)\cos(n\phi) + \sin(n\xi)\sin(n\phi)]\, d\xi \\ &= \frac{\rho^n}{a^n \pi} \int_{-\pi}^{\pi} f(\xi) \cos[n(\phi - \xi)]\, d\xi. \end{aligned}$$

The solution (6.1) becomes

$$\begin{aligned} u(\rho,\phi) &= A_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\rho^n}{a^n} \int_{-\pi}^{\pi} f(\xi) \cos[n(\phi-\xi)]\, d\xi \\ &= A_0 + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \left(\sum_{n=1}^{\infty} \frac{\rho^n}{a^n} \cos[n(\phi-\xi)] \right) d\xi \\ &= A_0 + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \sum_{n=1}^{\infty} \Re \left\{ \frac{\rho^n}{a^n} e^{in(\phi-\xi)} \right\} d\xi \\ &= A_0 + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi)\, \Re \left\{ \sum_{n=1}^{\infty} \left(\frac{\rho}{a} e^{i(\phi-\xi)} \right)^n \right\} d\xi \\ &= A_0 + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi)\, \Re \left\{ \frac{\frac{\rho}{a} e^{i(\phi-\xi)}}{1 - \frac{\rho}{a} e^{i(\phi-\xi)}} \right\} d\xi \qquad 5(\text{since } \sum_{n=1}^{\infty} \mu^n = \frac{\mu}{1-\mu} \text{ for } |\mu| < 1) \\ &= A_0 + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi)\, \Re \left\{ \frac{\rho[\cos(\phi-\xi) + i\sin(\phi-\xi)]}{a - \rho[\cos(\phi-\xi) + i\sin(\phi-\xi)]} \right\} d\xi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \frac{a\rho\cos(\phi-\xi) - \rho^2}{a^2 - 2a\rho\cos(\phi-\xi) + \rho^2}\, d\xi \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \left\{ \frac{1}{2} + \frac{a\rho\cos(\phi-\xi) - \rho^2}{a^2 - 2a\rho\cos(\phi-\xi) + \rho^2} \right\} d\xi. \end{aligned}$$

Thus we get

$$u(\rho,\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) \frac{a^2 - \rho^2}{a^2 - 2a\rho\cos(\phi-\xi) + \rho^2}\, d\xi. \qquad \text{(Poisson's integral formula)}$$

6.2 Bessel Functions

6.2.1 Series Solution of Bessel's Equation

The general second order linear homogeneous ODE can be written in the form

$$u'' + P(x)u' + Q(x)u = 0. \tag{6.2}$$

Definition 44. *A point x_0 is called an ordinary point of (6.2) if P and Q are analytic at $x = x_0$ (i.e. P and Q can be expanded in a Taylor series at $x = x_0$). Otherwise the point $x = x_0$ is called a singular point.*

If x_0 is an ordinary point of Eq. (6.2), then two linearly independent solutions of the form

$$u(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

can be found. What happens if x_0 is a singular point? We can get an idea as to what happens if we re–examine the Cauchy–Euler equation:

$$x^2 u'' + pxu' + qu = 0, \quad (p, q \equiv \text{ const.}). \tag{6.3}$$

If we look for a solution of the form $u = x^r$, this leads to

$$r^2 + (p-1)r + q = 0. \tag{6.4}$$

Thus, $u = x^r$ is a solution to Eq. (6.3) only if r is a root of the quadratic equation.

EXAMPLE 6.2. For the equation $3x^2u'' + 11xu' - 3u = 0$ the quadratic (6.4) becomes $r^2 + (8/3)r - 1 = 0$ which leads to two linearly independent solutions $u_1(x) = x^{1/3}$ and $u_2(x) = 1/x^3$.

In general, the solution to the Cauchy–Euler equation (6.3) is

$$u(x) = \begin{cases} c_1 x^{r_1} + c_2 x^{r_2}, & \text{if } r_1 \neq r_2, \\ c_1 x^r + c_2 x^r \ln x, & \text{if } r_1 = r_2 = r, \\ x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)], & \text{if } r_1, r_2 = \alpha \pm i\beta. \end{cases} \tag{6.5}$$

So how does this help? When written in standard form

$$u'' + \frac{p}{x} u' + \frac{q}{x^2} u = 0, \tag{6.6}$$

it is clear that $x = 0$ is a singular point of the equation. The solutions will also usually be singular at $x = 0$ as was the case in Example 6.2. This prompts us to make the following definition.

Definition 45. *Suppose x_0 is a singular point of Eq. (6.2). If $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$, then x_0 is called a regular singular point of (6.2), otherwise it is called an irregular singular point.*

If x_0 is a regular singular point of Eq. (6.2), then P and Q can be written as

$$P(x) = \frac{A(x)}{x - x_0}, \qquad Q(x) = \frac{B(x)}{(x - x_0)^2},$$

where A and B are analytic at x_0. Equation (6.2) now becomes

$$u'' + \frac{A(x)}{x - x_0}u' + \frac{B(x)}{(x - x_0)^2}u = 0,$$

or alternatively

$$(x - x_0)^2u'' + A(x)(x - x_0)u' + B(x)u = 0. \tag{6.7}$$

Equation (6.7) resembles the Cauchy–Euler equation so we look for a solution of the form

$$u(x) = (x - x_0)^rU(x),$$

where U is analytic at x_0. Since U is assumed to be analytic, it can be expanded in a Taylor series so the form of solution we seek is

$$u(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}, \qquad 3a_0 \neq 0. \tag{6.8}$$

The series in (6.8) is called a *Frobenius* series and the exponent r is called the indicial exponent. In fact, if x_0 is a regular singular point of (6.2), then a solution in the form of a Frobenius series always exists and a second linearly independent solution will be of the form

$$u(x) = (x - x_0)^sV(x) \quad \text{or} \quad u(x) = (x - x_0)^rU(x)\ln(x - x_0) + (x - x_0)^sV(x),$$

where V is analytic at $x = x_0$.

Now consider Bessel's equation:

$$x^2u'' + xu' + (x^2 - \lambda^2)u = 0. \qquad (\lambda \equiv \text{ const.}) \tag{6.9}$$

Written in standard form:

$$u'' + \frac{1}{x}u' + \frac{x^2 - \lambda^2}{x^2}u = 0, \quad \text{with} \quad P(x) = \frac{1}{x},\ Q(x) = \frac{x^2 - \lambda^2}{x^2}.$$

Clearly $x = x_0 = 0$ is a singular point of Bessel's equation. It is, in fact, a regular singular point since

$$xP(x) = 1, \qquad \text{which is analytic at } x = 0,$$
$$xQ(x) = x^2 - \lambda^2, \qquad \text{which is analytic at } x = 0.$$

So we look for a solution to Bessel's equation in the form of a Frobenius series:

$$u(x) = x^r \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{n+r}, \qquad a_0 \neq 0.$$

Plug this into Eq. (6.9) to get

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)a_nx^{n+r} + \sum_{n=0}^{\infty}(n + r)a_nx^{n+r} + \sum_{n=0}^{\infty} a_n(x^{n+r+2} - \lambda^2x^{n+2}) = 0.$$

After simplification this becomes

$$(r^2-\lambda^2)a_0x^r+[(1+r)^2-\lambda^2]a_1x^{r+1}+\sum_{n=2}^{\infty}\{[(n+r)^2-\lambda^2]a_n+a_{n-2}\}x^{n+r}=0.$$

Since this must be identically zero, we set the coefficients of x^{n+r} to zero, yielding

$$n=0: \quad (r^2-\lambda^2)a_0=0, \tag{6.10}$$
$$n=1: \quad [(1+r)^2-\lambda^2]a_1=0, \tag{6.11}$$
$$n\geqslant 2: \quad [(n+r)^2-\lambda^2]a_n+a_{n-2}=0. \tag{6.12}$$

We have

$$(6.10) \implies r=\pm\lambda \qquad 1(\text{since } a_0\neq 0). \qquad \text{For now we consider } r=+\lambda.$$
$$(6.11) \implies (2\lambda+1)a_1=0 \implies a_1=0. \quad (\text{unless } \lambda=-\frac{1}{2})$$
$$(6.12) \implies a_n=\frac{-a_{n-2}}{(n+\lambda)^2-\lambda^2}=\frac{-a_{n-2}}{n(n+2\lambda)}, \quad n=2,3,4,\dots.$$

It is clear that

$$a_1=0 \implies a_3=0 \implies a_5=0 \implies \cdots \implies a_{2n+1}=0,$$

and

$$a_2=\frac{-a_0}{2(2+2\lambda)}=-\frac{a_0}{2^2(1+\lambda)},$$
$$a_4=\frac{-a_2}{4(4+2\lambda)}=\frac{-a_2}{2^2\cdot 2(2+\lambda)}=\frac{a_0}{2^4\cdot 2(1+\lambda)(2+\lambda)},$$
$$a_6=\frac{-a_4}{6(6+2\lambda)}=\frac{-a_4}{2^2\cdot 3(3+\lambda)}=\frac{-a_0}{2^6\cdot 2\cdot 3(1+\lambda)(2+\lambda)(3+\lambda)},$$
$$a_8=\frac{-a_6}{8(8+2\lambda)}=\frac{-a_6}{2^2\cdot 4(3+\lambda)}=\frac{a_0}{2^8\cdot 2\cdot 3\cdot 4(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)},$$
$$\vdots$$
$$a_{2n}=\frac{-a_{2(n-1)}}{2^2\cdot n(n+\lambda)}=\frac{(-1)^n a_0}{2^{2n}\,n!(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)\cdots(n+\lambda)}.$$

For convenience we set $a_0=\frac{1}{2^\lambda\Gamma(1+\lambda)}$. Then we get

$$a_{2n}=\frac{(-1)^n}{2^{\lambda+2n}\,n!\;\Gamma(1+\lambda)(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)\cdots(n+\lambda)}=\frac{(-1)^n}{2^{\lambda+2n}\,n!\;\Gamma(n+\lambda+1)}.$$

One solution of Bessel's equation is

$$J_\lambda(x):=\sum_{n=0}^{\infty}a_nx^{n+\lambda}=\sum_{n=0}^{\infty}\frac{(-1)^nx^{\lambda+2n}}{2^{\lambda+2n}n!\;\Gamma(n+\lambda+1)}.$$

This is known as Bessel's function of the first kind of order λ. If λ is not an integer, then a second linearly independent solution (recall that the indicial exponents are $r = \pm\lambda$) is given by $J_{-\lambda}$. In this case the general solution is

$$u(x) = c_1 J_\lambda(x) + c_2 J_{-\lambda}(x).$$

If $\lambda = n \in \mathbb{Z}$, then J_n and J_{-n} are linearly independent (since indicial exponents differ by an integer $r_1 - r_2 = 2n$). In this case a second linearly independent solution can be obtained as follows:

$$Y_\lambda(x) := \frac{J_\lambda(x)\cos(\lambda\pi) - J_{-\lambda}(x)}{\sin(\lambda\pi)}, \qquad 8\lambda \notin \mathbb{Z},$$
$$Y_n(x) := \lim_{\lambda\to n} Y_\lambda(x) = \frac{1}{\pi}\left[\frac{\partial J_\lambda(x)}{\partial\lambda} - (-1)^n \frac{\partial J_{-\lambda}(x)}{\partial\lambda}\right]_{\lambda=n}.$$

It can be shown that J_n and Y_n are linearly independent. The function Y_λ is called a Bessel function of the second kind of order λ. The general solution in this case is

$$u(x) = c_1 J_n(x) + c_2 Y_n(x).$$

6.2.2 Properties of Bessel Functions

We look at Bessel functions of integer order. For $\lambda = n \in \mathbb{Z}$ we have

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k}\, k!\,(n+k)!} = \frac{x^n}{2^n}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}\, k!\,(n+k)!} = \frac{x^n}{2^n}\left[\frac{1}{n!} - \frac{x^2}{2^2\,(n+1)!} + \frac{x^4}{2^4\,2!\,(n+2)!} - \cdots\right].$$

In particular we have

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots,$$
$$J_1(x) = \frac{x}{2}\left[1 - \frac{x^2}{8} + \frac{x^4}{194} - \cdots\right].$$

It is clear that $J_0(0) = 1$ and $J_n(0) = 0$ for $n \geqslant 1$. A plot of the first few Bessel functions of the first kind of integer order is given in Figure 6.3.

For Bessel functions of the second kind we merely state a few of the properties. The most relevent property is that they are singular at the origin. In fact

$$Y_0(x) \sim C \ln x, \qquad 3Y_1(x) \sim \frac{C}{x} \qquad 1(\text{as } x \to 0^+).$$

A plot of the first few Bessel functions of the second kind of integer order is given in Figure 6.4.

Lemma 46. *Bessel functions of the first kind satisfy the following recurrence relations:*

$$\frac{d}{dx}\left[x^{-\lambda} J_\lambda(x)\right] = -x^{-\lambda} J_{\lambda+1}(x), \qquad \frac{d}{dx}\left[x^{\lambda} J_\lambda(x)\right] = x^{\lambda} J_{\lambda-1}(x).$$

Proof.
Exercise.

Zeros of Bessel Functions.

Consider Bessel's equation written in standard form:

$$u'' + \frac{1}{x}u' + (1 - \frac{\lambda^2}{x^2})u = 0.$$

For large x this equation resembles $u'' + u = 0$, and so one might expect that Bessel functions behave like $\sin x$ and $\cos x$ for large x. This is indeed the case. We will show that Bessel functions are oscillatory and that the spacing between the zeros approaches π. To this end we make a transformation

$$w(x) := \sqrt{x}u(x).$$

Then we have

$$x^2u'' + xu' + (x^2 - \lambda^2)u = x^{-1/2}\{x^2w'' + (x^2 + \frac{1}{4} - \lambda^2)w\}.$$

Thus

$$x^2u'' + xu' + (x^2 - \lambda^2)u = 0 \quad \Longrightarrow \quad \{x^2w'' + (x^2 + \frac{1}{4} - \lambda^2)w\} = 0.$$

We now consider a phase plane analysis: $\{w, w'\} \to \{R, \theta\}$

$$w(x) = R(x)\cos\theta(x), \qquad w'(x) = -R(x)\sin\theta(x). \tag{6.13}$$

Then we have

$$\begin{aligned} w' &= R'\cos\theta - R\theta'\sin\theta = -R\sin\theta, \\ w'' &= -R'\sin\theta - R\theta'\cos\theta = -(1 + \frac{1}{4x^2} - \frac{\lambda^2}{x^2})R\cos\theta, \end{aligned}$$

which can be rewritten as

$$\begin{bmatrix} \cos\theta & -R\sin\theta \\ -\sin\theta & -R\cos\theta \end{bmatrix} \begin{bmatrix} R' \\ \theta' \end{bmatrix} = \begin{bmatrix} -R\sin\theta \\ -(1 + \frac{1}{4x^2} - \frac{\lambda^2}{x^2})R\cos\theta \end{bmatrix},$$

and easily solved for R' and θ'

$$R'(x) = \frac{C_\lambda}{x^2}\sin\theta(x)\cos\theta(x), \tag{6.14}$$

$$\theta'(x) = 1 + \frac{C_\lambda}{x^2}\cos^2\theta(x), \tag{6.15}$$

where $C_\lambda := \frac{1}{4} - \lambda^2$. We consider the above equations subject to the following initial conditions:

$$R(x_0) = R_0 \neq 0, \qquad \theta(x_0) = \theta_0, \quad x_0 > 0.$$

A couple of remarks are in order. The transformation to "polar like" coordinates in the phase plane has resulted in nonlinear ODEs in R and θ. It is usually not a wise move to transform a linear equation to a nonlinear one. In this case however it is not so much an explicit solution that we are after, which we can't get anyway, so much as the qualitative behaviour of the solutions. Further examination of Equations (6.14) and (6.15) reveals that they are partially decoupled so that in principle one would solve Eq. (6.15) for θ, then plug θ into Eq. (6.14) to get a linear ODE for R. The resulting ODE for R is easily solved

$$R(x) = R_0 e^{C_\lambda \int_{x_0}^{x} \frac{\sin\theta(\xi)\cos\theta(\xi)}{\xi^2}\,d\xi}.$$

We now make some observations.

$$R_0 \neq 0 \quad \Longrightarrow \quad R(x) \neq 0 \;\forall x.$$

Therefore it follows from (6.13) that

$$w(x) = 0 \qquad 1\text{only if} \qquad 1\cos\theta(x) = 0, \quad \text{i.e. only if} \qquad 1\theta(x) = (2n{-}1)\frac{\pi}{2} \qquad 1(\text{an odd multiple of } \frac{\pi}{2})$$

Integrating Eq. (6.15) yields

$$\theta(x) - \theta(x_0) = x - x_0 + \int_{x_0}^{x} \frac{\cos^2(\theta(\xi))}{\xi^2}\,d\xi.$$

Further manipulation yields.

$$|\theta(x) - \theta(x_0) - (x - x_0)| = |C_\lambda| \left| \int_{x_0}^{x} \frac{\cos^2(\theta(\xi))}{\xi^2}\,d\xi \right| \leqslant |C_\lambda| \int_{x_0}^{x} \frac{d\xi}{\xi^2} = |C_\lambda| \left(\frac{1}{x_0} - \frac{1}{x} \right) \leqslant \frac{|C_\lambda|}{x_0}.$$

Letting $x \to \infty$ we get

$$\lim_{x\to\infty} |\theta(x) - \theta(x_0) - (x - x_0)| \leqslant \frac{|C_\lambda|}{x_0} < \infty$$

from which it follows that

$$\lim_{x\to\infty} \theta(x) = +\infty.$$

Thus

$$\forall n \;\exists x_n \text{ such that } \theta(x_n) = (2n-1)\frac{\pi}{2}.$$

In other words, w has infinitely many zeros x_n, with $x_n \to \infty$.

To determine the spacing between zeros consider

$$\pi = \theta(x_{n+1}) - \theta(x_n) = x_{n+1} - x_n + C_\lambda \int_{x_n}^{x_{n+1}} \frac{\cos^2\theta(\xi))}{\xi^2}\,d\xi.$$

Taking the limit of the above integral as $n \to \infty$ we get

$$0 \leqslant \lim_{n\to\infty} \int_{x_n}^{x_{n+1}} \frac{\cos^2\theta(\xi))}{\xi^2}\,d\xi \leqslant \lim_{n\to\infty} \int_{x_n}^{x_{n+1}} \frac{d\xi}{\xi^2} = \lim_{n\to\infty} \left(\frac{1}{x_n} - \frac{1}{x_{n+1}} \right) = 0. \qquad (\text{since } x_n \to \infty)$$

Therefore

$$\lim_{n\to\infty} (x_{n+1} - x_n) = \pi.$$

6.3 Heat Equation (2-d)

Recall the general heat equation;

$$\frac{\partial u}{\partial t} = \frac{1}{\rho c}\vec{\nabla}\cdot(K\vec{\nabla}u) + \frac{h}{\rho c}. \qquad 6(\text{heat equation})$$

If the material properties are constant and there is no internal heat generation, then the heat equation simplifies to

$$\frac{\partial u}{\partial t} = k\nabla^2 u.$$

We now consider an example of a 2–dimensional heat equation for a circular disk.

EXAMPLE 6.3. Consider the following problem governing the temperature of a circular plate:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\nabla^2 u, \qquad 0<\rho<a,\ -\pi<\phi<\pi,\ t>0,\\ u(a,\phi,t) &= f(\phi),\\ u(\rho,\phi,0) &= g(\rho,\phi).\end{aligned}$$

As before, we consider the domain Ω in the $\rho\phi$-plane. The following extra conditions for the artificial boundaries are required:

A1: $u(\rho,\pi,t) = u(\rho,-\pi,t)$ and $\frac{\partial u}{\partial \phi}(\rho,\pi,t) = \frac{\partial u}{\partial \phi}(\rho,-\pi,t)$;

A2: $|u(\rho,\phi,t)|$ bounded as $\rho \to 0^+$.

Since the boundary conditions are not homogeneous, we look for a solution of the form

$$u(\rho,\phi,t) = v(\rho,\phi) + w(\rho,\phi,t).$$

This is analogous to the method we employed earlier for nonhomogeneous problems. Plug this into the problem to get

$$\begin{aligned}\frac{\partial w}{\partial t} &= k(\nabla^2 v + \nabla^2 w),\\ v(a,\phi) + w(a,\phi,t) &= f(\phi),\\ v(\rho,\phi) + w(\rho,\phi,0) &= g(\rho,\phi).\end{aligned}$$

We split this into two problems in the obvious way

$$\begin{aligned}\nabla^2 v &= 0, & \frac{\partial w}{\partial t} &= k\nabla^2 w,\\ v(a,\phi) &= f(\phi), & w(a,\phi,t) &= 0,\\ v(\rho,\pi) &= v(\rho,-\pi), & w(\rho,\pi,t) &= w(\rho,-\pi,t),\\ \frac{\partial v}{\partial \phi}(\rho,\pi) &= \frac{\partial v}{\partial \phi}(\rho,-\pi), & \frac{\partial w}{\partial \phi}(\rho,\pi,t) &= \frac{\partial w}{\partial \phi}(\rho,-\pi,t),\\ |v(\rho,\phi)| &\text{ bounded as } \rho\to 0^+, & w|(\rho,\phi,t)| &\text{ bounded as } \rho\to 0^+,\\ & & w(\rho,\phi,0) &= g(\rho,\phi) - v(\rho,\phi) =: h(\rho,\phi).\end{aligned}$$

The problem for v is Laplace's equation in a disk. The solution to this problem was obtained earlier. It is

$$v(\rho,\phi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,\frac{a^2-\rho^2}{a^2-2a\rho\cos(\phi-\xi)+\rho^2}\,d\xi. \qquad \text{(Poisson's integral formula)}$$

For the equation in w we try separation of variables: $w(\rho,\phi,t)=\Psi(\rho,\phi)T(t)$. This leads to

$$\frac{T'}{kT}=\frac{\nabla^2\Psi}{\Psi}=-\lambda. \quad \text{(const.)}$$

We get one equation for T and one for Ψ:

$$T'+\lambda kT=0, \qquad \begin{aligned} \nabla^2\Psi &= -\lambda\Psi,\\ \Psi(a,\phi) &= 0,\\ \Psi(\rho,\pi) &= \Psi(\rho,-\pi),\\ \frac{\partial\Psi}{\partial\phi}(\rho,\pi) &= \frac{\partial\Psi}{\partial\phi}(\rho,-\pi),\\ |\Psi(\rho,\phi)| &\text{ bounded as } \rho\to 0. \end{aligned}$$

The solution to the equation for T is

$$T(t)=e^{-\lambda kt}.$$

¿From the last theorem of the last chapter, we know that the eigenvalues of the problem for Ψ are non-negative. So, letting $\lambda=\mu^2$, the equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\phi^2}=-\mu^2\Psi.$$

We again try separation of variables: $\Psi(\rho,\phi)=R(\rho)S(\phi)$. Plug into the equation to get

$$\frac{\rho(\rho R')'}{R}+\mu^2\rho^2=-\frac{S''}{S}=\nu. \quad \text{(const.)}$$

We get an ODE for R and one for S:

$$\begin{aligned} \rho^2R''+\rho R'+(\mu^2\rho^2-\nu)R &= 0,\\ R(a) &= 0,\\ |R(\rho)| &< \infty, \end{aligned} \qquad \begin{aligned} S''+\nu S &= 0,\\ S(\pi) &= S(-\pi),\\ S'(\pi) &= S'(-\pi). \end{aligned}$$

We have solved this equation for S before. Nontrivial solutions exist only for non-negative ν. Let $\nu=\omega^2$. The solutions are:

$$\begin{aligned} \omega=\omega_n &= n, \qquad 6n=0,1,2,\ldots,\\ S_n(\phi) &= a_n\cos(n\phi)+b_n\sin(n\phi). \end{aligned}$$

The equation for R resembles, but is not quite, Bessel's equation. If we make the change of independent variable $x = \mu\rho$, then $x\dfrac{d}{dx} = \rho\dfrac{d}{d\rho}$ and the equation becomes

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0, \qquad \text{(Bessel's equation of order } n\text{)}$$

the solution of which is

$$R_n = C_nJ_n(x) + D_nY_n(x), \quad \text{hence} \quad R_n(\rho) = C_nJ_n(\mu\rho) + D_nY_n(\mu\rho).$$

The boundary condition $|R_n(\rho)| < \infty$ implies that $D_n = 0$ resulting in

$$R_n(\rho) = J_n(\mu\rho).$$

The other boundary condition yields

$$R_n(a) = 0 \quad \Longrightarrow \quad J_n(\mu a) = 0 \quad \Longrightarrow \quad \mu = \mu_{nk} = \frac{\alpha_{nk}}{a},\ k = 1, 2, \ldots \text{ (where } \alpha_{nk} \text{ is the } k^{th} \text{ zero of } J_n\text{)}$$

Thus we have

$$R_{nk}(\rho) = J_n(\mu_{nk}\rho), \quad T_{nk}(t) = e^{-k\alpha_{nk}t}, \quad \text{and} \quad S_n(\phi) = a_n\cos(n\phi) + b_n\sin(n\phi).$$

Combining these gives

$$\Psi_{nk}(\rho,\phi) = J_n(\mu_{nk}\rho)[A_{nk}\cos(n\phi) + B_{nk}\sin(n\phi)].$$

If we define

$$\widetilde{\Psi}_{nk}(\rho,\phi) := J_n(\mu_{nk}\rho)\cos(n\phi), \quad \widehat{\Psi}_{nk}(\rho,\phi) := J_n(\mu_{nk}\rho)\sin(n\phi), \quad n = 0, 1, 2, \ldots,\ k = 1, 2, \ldots,$$

then $\widetilde{\Psi}_{nk}$ and $\widehat{\Psi}_{nk}$ are 2–dimensional eigenfunctions and

$$\Psi_{nk}(\rho,\phi) = A_{nk}\widetilde{\Psi}_{nk}(\rho,\phi) + B_{nk}\widehat{\Psi}_{nk}(\rho,\phi).$$

A solution to the w equation (applying superposition) can we written as

$$w(\rho,\phi,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\Psi_{nk}(\rho,\phi)T_{nk}(t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[A_{nk}\widetilde{\Psi}_{nk}(\rho,\phi) + B_{nk}\widehat{\Psi}_{nk}(\rho,\phi)]e^{-k\mu_{nk}^2t}.$$

Applying the remaining initial condition: $w(\rho,\phi,0) = h(\rho,\phi)$ we get

$$h(\rho,\phi) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[A_{nk}\widetilde{\Psi}_{nk}(\rho,\phi) + B_{nk}\widehat{\Psi}_{nk}(\rho,\phi)], \quad \text{where} \quad \langle\Box,\triangle\rangle = \iint_{\Omega}\Box\triangle\,dA.$$

Therefore the coefficients are given by

$$A_{nk} = \frac{\left\langle h, \widetilde{\Psi}_{nk} \right\rangle}{\|\widetilde{\Psi}_{nk}\|^2} = \frac{1}{\|\widetilde{\Psi}_{nk}\|^2} \int_{-\pi}^{\pi} \int_0^a h(\rho,\phi) J_n(\mu_{nk}\rho) \cos(n\phi)\, \rho\, d\rho\, d\phi,$$

$$B_{nk} = \frac{\left\langle h, \widehat{\Psi}_{nk} \right\rangle}{\|\widehat{\Psi}_{nk}\|^2} = \frac{1}{\|\widehat{\Psi}_{nk}\|^2} \int_{-\pi}^{\pi} \int_0^a h(\rho,\phi) J_n(\mu_{nk}\rho) \sin(n\phi)\, \rho\, d\rho\, d\phi.$$

The norms in the above expressions can be evaluated explicitly as follows:

$$\|\widetilde{\Psi}_{nk}\|^2 = \int_{-\pi}^{\pi} \int_0^a J_n^2(\mu_{nk}\rho) \cos^2(n\phi)\, \rho\, d\rho\, d\phi = \left(\int_{\pi}^{\pi} \cos^2(n\phi)\, d\phi \right) \left(\int_0^a \rho J_n^2(\mu_{nk}\rho)\, d\rho \right) =: a_n b_{nk},$$

where

$$a_n = \int_{-\pi}^{\pi} \cos^2(n\phi)\, d\phi = \begin{cases} \pi, & \text{if } n \neq 0, \\ 2\pi, & \text{if } n = 0, \end{cases}$$

$$b_{nk} = \int_0^a \rho J_n^2(\mu_{nk}\rho)\, d\rho = \int_0^a \rho J_n^2(\frac{\alpha_{nk}}{a}\rho)\, d\rho = \frac{a^2}{2} J_{n+1}^2(\alpha_{nk}). \qquad \text{(exercise)}$$

Therefore

$$\|\widetilde{\Psi}_{0k}\|^2 = \pi a^2 J_1^2(\alpha_{0k}), \quad \|\widetilde{\Psi}_{nk}\|^2 = \frac{\pi a^2}{2} J_{n+1}^2(\alpha_{nk}),\ n \neq 0$$

with similar expressions for $\widehat{\Psi}_{nk}$. The coefficients may now be written as

$$A_{nk} = \frac{2 - \delta_{0n}}{\pi a^2 J_{n+1}(\alpha_{nk})} \int_{-\pi}^{\pi} \int_0^a h(\rho,\phi) J_n(\mu_{nk}\rho) \cos(n\phi)\, \rho\, d\rho\, d\phi,$$

$$B_{nk} = \frac{2 - \delta_{0n}}{\pi a^2 J_{n+1}(\alpha_{nk})} \int_{-\pi}^{\pi} \int_0^a h(\rho,\phi) J_n(\mu_{nk}\rho) \sin(n\phi)\, \rho\, d\rho\, d\phi.$$

Suppose we consider the special case with no angular dependence, in other words the case with $f(\phi) = 0$ and $g(\rho,\phi) = g(\rho)$. Then $v(\rho,\phi) = 0$, $h(\rho,\phi) = g(\rho)$ and

$$\begin{aligned} A_{nk} &= \frac{1}{\|\widetilde{\Psi}_{nk}\|^2} \int_{-\pi}^{\pi} \int_0^a g(\rho) J_n(\mu_{nk}\rho) \cos(n\phi)\, \rho\, d\rho\, d\phi \\ &= \frac{1}{\|\widetilde{\Psi}_{nk}\|^2} \left(\int_{-\pi}^{\pi} \cos(n\phi)\, d\phi \right) \left(\int_0^a \rho\, g(\rho) J_n(\mu_{nk}\rho)\, d\rho \right) \\ &= \frac{2\pi\delta_{0n}}{\|\widetilde{\Psi}_{nk}\|^2} \int_0^a \rho\, g(\rho) J_n(\mu_{nk}\rho)\, d\rho. \end{aligned}$$

Therefore

$$A_{0k} = \frac{2}{a^2 J_1^2(\alpha_{0k})} \int_0^a \rho\, g(\rho) J_n(\mu_{nk}\rho)\, d\rho, \quad A_{nk} = 0 \qquad 1\text{for } n \neq 0, \quad \text{and} \quad B_{nk} = 0.$$

In this case the final solution to the problem reduces to

$$u(\rho, \phi, t) = \sum_{k=1}^{\infty} A_{0k} \widetilde{\Psi}_{0k}(\rho, \phi) e^{-k\mu_{0k}^2 t} = \sum_{k=1}^{\infty} A_{0k} J_0(\mu_{0k}\rho) e^{-k\mu_{0k}^2 t}.$$

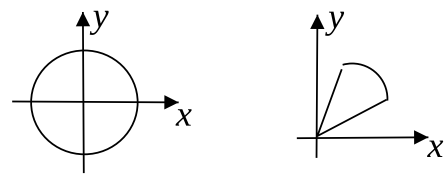


Figure 6.1: Geometries more suited to polar coordinates.

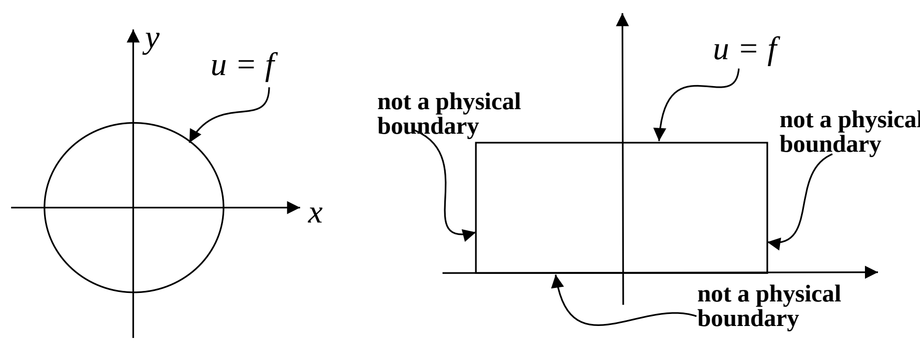


Figure 6.2: Representations of Ω in the xy-plane and the $\rho\phi$-plane.

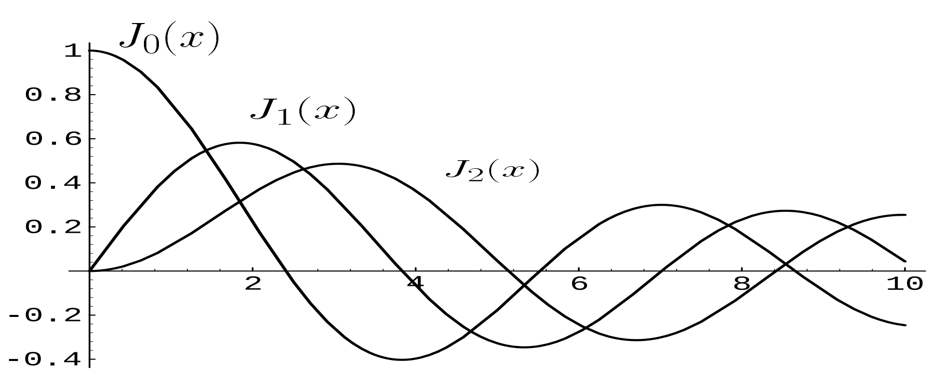


Figure 6.3: Bessel functions of the first kind of integer order.

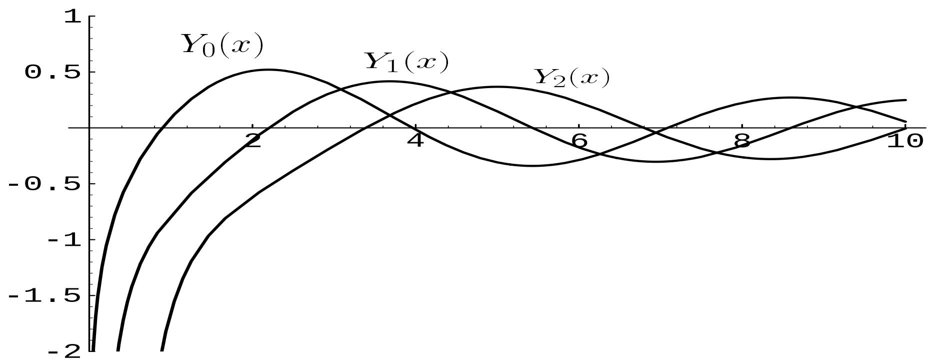


Figure 6.4: Bessel functions of the second kind of integer order.

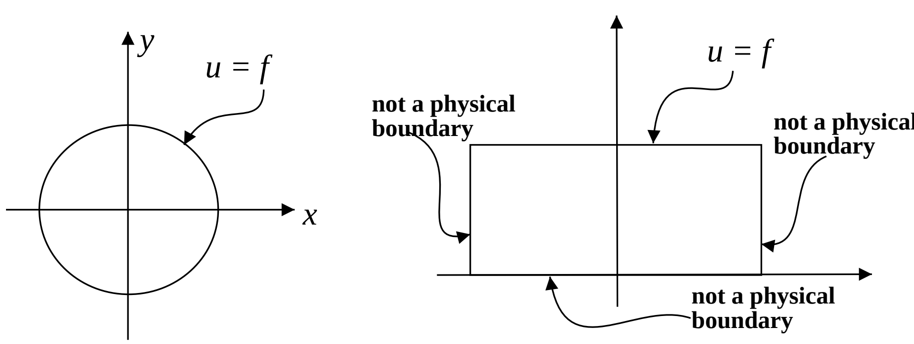


Figure 6.5: Representations of Ω in the xy-plane and the $\rho\phi$-plane.

Chapter 7

Problems in Spherical Coordinates

June 17, 2010

7.1 Spherical Coordinates

Spherical coordinates $\{r, \theta, \phi\}$ are related to Cartesian coordinates $\{x, y, z\}$ as follows

$$\begin{aligned} x &= r\sin\theta\cos\phi, & r &\geqslant 0, \\ y &= r\sin\theta\sin\phi, & -\pi &< \phi \leqslant \pi, \\ z &= r\cos\theta, & 0 &\leqslant \theta \leqslant \pi. \end{aligned}$$

The Jacobian determinant for the transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

which indicates that the transformation is singular at $r = 0$ and $\theta = 0, \pi$ (i.e. along the entire z-axis). The Laplacian in spherical coordinates is

$$\begin{aligned} \nabla^2 u &= \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right\} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \csc^2\theta\frac{\partial^2 u}{\partial\phi^2}\right). \end{aligned}$$

Consider the homogeneous version of either the three-dimensional heat equation or the three-dimensional wave equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\nabla^2 u, & \frac{\partial^2 u}{\partial t^2} &= c^2\nabla^2 u, \\ &+\text{b.c.'s}, & &+\text{b.c.'s}, \\ &+\text{i.c.'s}. & &+\text{i.c.'s}. \end{aligned}$$

If we apply separation of variables $u(r,\theta,\phi,t) = \Psi(r,\theta,\phi)T(t)$, we get

$$\frac{T'}{kT} = \frac{\nabla^2\Psi}{\Psi} = -\widetilde{\lambda}, \qquad \text{or} \qquad \frac{T''}{c^2T} = \frac{\nabla^2\Psi}{\Psi} = -\widetilde{\lambda}.$$

Either way, we end up with a three-dimensional eigenvalue problem for Ψ

$$\begin{aligned} \nabla^2\Psi &= -\widetilde{\lambda}\Psi, \\ &+ \text{b.c.'s.} \end{aligned}$$

As was the case for two-dimensional eigenvalue problems, this three-dimensional eigenvalue problem, for either Dirichlet or Neumann boundary conditions, will only have non-negative eigenvalues. So we set $\widetilde{\lambda} = \mu^2$ and consider the problem

$$\nabla^2\Psi = -\mu^2\Psi.$$

We look for separated solutions of the form $\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$. Plug this into the equation to get

$$\begin{aligned} \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2R'\right)Y + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\} &= -\mu^2 RY \\ \frac{(r^2R')'}{R} + \mu^2r^2 = -\frac{1}{Y\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] &= \lambda \end{aligned}$$

We get an ODE for R and a PDE for Y:

$$(r^2R')' + (\mu^2r^2 - \lambda)R = 0, \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y = 0.$$

Solutions to the equation for Y are called *spherical harmonics*. We apply separation of variables again: $Y(\theta,\phi) = S(\theta)Q(\phi)$. Plug into the equation for Y to get

$$\begin{aligned} \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S')Q + \frac{1}{\sin^2\theta}SQ'' + \lambda SQ &= 0 \\ \frac{\sin\theta(\sin\theta S')'}{S} + \lambda\sin^2\theta = -\frac{Q''}{Q} &= \nu. \end{aligned}$$

Thus we get an ODE in S and an ODE in Q. Collecting all of the equations we have

$$\begin{aligned} &(r^2R')' + (\mu^2r^2 - \lambda)R = 0, && r > 0, \\ &\sin\theta(\sin\theta\, S')' + (\lambda\sin^2\theta - \nu)S = 0, && 0 < \theta < \pi, \\ &Q'' + \nu Q = 0, && -\pi < \phi < \pi. \end{aligned}$$

The first of these equations is

$$\boxed{r^2R'' + 2rR' + (\mu^2r^2 - \lambda)R = 0}. \qquad \text{(spherical Bessel's equation)}$$

In the second of these equations, if we let $x = \cos\theta$ and $v(x) = S(\theta)$, then $0 < \theta < \pi$ implies that $-1 < x < 1$ and we get

$$\boxed{(1 - x^2)v'' - 2xv' + \left(\lambda - \frac{\nu}{1 - x^2}\right)v = 0}. \qquad \text{(associated Legendre equation)}$$

For the special case $\nu = 0$ this reduces to

$$\boxed{(1-x^2)v'' - 2xv' + \lambda v = 0}. \qquad \text{(Legendre's equation)}$$

Before we can go any further in trying to solve the heat equation or wave equation in spherical coordinates, we need to get some understanding of the behaviour of the solutions to the spherical Bessel and Legendre's equations.

7.2 Legendre Functions

7.2.1 Legendre Polynomials

Consider Legendre's equation:

$$(1-x^2)v'' - 2xv' + \lambda v = 0, \qquad -1 < x < 1.$$

Clearly $x = 0$ is an ordinary point of Legendre's equation and $x = \pm 1$ are regular singular points. In applications, we require bounded solutions at $x = \pm 1$ (which correspond to $\theta = 0, \pi$, which correspond to the positive and negative z-axis in physical space).

Since $x = 0$ is an ordinary point, all solutions will be analytic at $x = 0$ so we may expand them in a Taylor series:

$$v(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Plug into the equation to get

$$\sum_{n=0}^{\infty} \{(n+2)(n+1)a_{n+2} + [\lambda - n(n+1)]a_n\} x^n = 0.$$

Therefore the a_n's satisfy the following recurrence relation:

$$a_{n+2} = -\frac{\lambda - n(n+1)}{(n+2)(n+1)} a_n, \qquad n = 0, 1, 2, \ldots.$$

Writing out the first few explicitly we get

$$a_2 = -\frac{\lambda}{2} a_0, \qquad a_3 = -\frac{\lambda - 2}{3 \cdot 2} a_1,$$

$$a_4 = -\frac{\lambda - 2\cdot 3}{4 \cdot 3} a_2 = \frac{\lambda(\lambda - 2\cdot 3)}{4\cdot 3\cdot 2} a_0, \qquad a_5 = -\frac{\lambda - 3\cdot 4}{5\cdot 4} a_3 = \frac{(\lambda-2)(\lambda - 3\cdot 4)}{5!} a_1,$$

$$a_6 = -\frac{\lambda - 4\cdot 5}{6\cdot 5} a_4 = -\frac{\lambda(\lambda - 2\cdot 3)(\lambda - 4\cdot 5)}{6!} a_0, \quad a_7 = -\frac{(\lambda-2)(\lambda-3\cdot 4)(\lambda - 5\cdot 6)}{7!} a_1.$$

Therefore

$$v(x) = a_0 \left[1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda - 2\cdot 3)}{4!}x^4 - \frac{\lambda(\lambda - 2\cdot 3)(\lambda - 4\cdot 5)}{6!}x^6 + \cdots\right]$$
$$+ a_1 \left[x - \frac{\lambda - 2}{3!}x^3 + \frac{(\lambda - 2)(\lambda - 3\cdot 4)}{5!}x^5 - \frac{(\lambda - 2)(\lambda - 3\cdot 4)(\lambda - 5\cdot 6)}{7!}x^7 + \cdots\right].$$

The even and odd parts are linearly independent. The quantities in square brackets are multiples of what are called *Legendre functions*.

Special Case: ($\lambda = N(N+1)$, $N \in \mathbb{Z}^+$) For this case the recurrence relation becomes

$$a_{n+2} = -\frac{N(N+1) - n(n+1)}{(n+2)(n+1)}a_n, \quad \text{with} \quad a_{N+2} = 0.$$

In this case the series terminates giving a polynomial. When suitably scaled, these are called *Legendre polynomials*, denoted P_N. The scaling is chosen so that $P_N(1) = 1$. The first few Legendre polynomials are listed below:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Lemma 47. *Legendre polynomials satisfy the following properties:*

1. $P_n(1) = 1, \qquad 1\ P_n(-1) = (-1)^n.$
2. $\langle P_n, P_m\rangle = 0$ *if* $n \neq m$, *where* $\langle \square, \triangle\rangle = \int_{-1}^{1} \square\triangle\, dx.$
3. $\|P_n\|^2 = \dfrac{2}{2n+1}.$
4. $P_n(x) = \dfrac{1}{2^n\, n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n. \qquad$ *(Rodrigues' formula)*
5. $\dfrac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$, *for* $|x| \leqslant 1\ |t| < 1. \quad$ *(generating function)*
6. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad 1n = 1, 2, \ldots.$
7. $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x).$

For the case $\lambda = N(N+1)$ one of the series terminates, but the other one does not. The functions represented by the other series, when suitably scaled, are denoted Q_N, and are called *Legendre functions of the second kind.* The general solution of Legendre's equation is then given by

$$v(x) = c_1 P_N(x) + c_2 Q_N(x).$$

The first few Legendre functions of the second kind are listed below:

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1, \qquad Q_2(x) = \frac{1}{4}(3x^2-1)\ln\frac{1+x}{1-x} - \frac{3}{2}x.$$

In fact the general formula is

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - W_{n-1}(x), \quad \text{where} \quad W_{n-1}(x) = \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(x)P_{n-k}(x).$$

Notice that Legendre functions of the second kind are singular at ± 1: $\lim_{x\to\pm 1}|Q_n(x)| = \infty$. In fact, the only bounded solutions are Legendre polynomials. We have the following result which we state without proof.

Theorem 48. *Legendre's equation has bounded solutions on the interval* $[-1,1]$ *if and only if*

$$\lambda = n(n+1), \qquad n = 0, 1, 2, \ldots.$$

7.2.2 Associated Legendre Functions

Consider Legendre's equation:

$$(1-x^2)v'' - 2xv' + \lambda v = 0. \tag{7.1}$$

We can derive the associated Legendre equation from Legendre's equation by the following rather convoluted argument. Recall the Leibniz rule for derivatives of a product

$$\frac{d^m}{dx^m}(fg) = \sum_{j=0}^{m}\binom{m}{j} f^{(j)}g^{(m-j)} = fg^{(m)} + mf'g^{(m-1)} + \frac{m(m-1)}{2}f''g^{(m-2)} + \cdots + mf^{(m-1)}g' + f^{(m)}g.$$

We differentiate each of the terms in (7.1) m times.

$$\begin{aligned}
&\frac{d^m}{dx^m}[(1-x^2)v''] = (1-x^2)v^{(m+2)} - 2mxv^{(m+1)} - m(m-1)v^{(m)},\\
&\frac{d^m}{dx^m}(2xv') = 2xv^{(m+1)} + 2mv^{(m)},\\
&\frac{d^m}{dx^m}[\lambda v] = \lambda v^{(m)}.
\end{aligned}$$

We now differentiate Legendre's equation m times to get

$$(1-x^2)v^{(m+2)} - (2m+2)xv^{(m+1)} + [\lambda - m(m+1)]v^{(m)} = 0.$$

Let $w(x) = (1-x^2)^{m/2} v^{(m)}$. Then

$$\begin{aligned}
w' &= (1-x^2)^{\frac{m}{2}} v^{(m+1)} - mx(1-x^2)^{\frac{m}{2}-1} v^{(m)}, \\
(1-x^2)w' &= (1-x^2)^{\frac{m}{2}+1} v^{(m+1)} - mx(1-x^2)^{\frac{m}{2}} v^{(m)}, \\
[(1-x^2)w']' &= (1-x^2)^{\frac{m}{2}+1} v^{(m+2)} - (2m+2)x(1-x^2)^{\frac{m}{2}} v^{(m+1)} + [m^2x^2(1-x^2)^{\frac{m}{2}-1} - m(1-x^2)^{\frac{m}{2}}]v^{(m)} \\
&= (1-x^2)^{\frac{m}{2}} \{(1-x^2)v^{(m+2)} - (2m+2)xv^{(m+1)} + [\frac{m^2x^2}{1-x^2} - m]v^{(m)}\} \\
&= (1-x^2)^{\frac{m}{2}} \{-\lambda + m(m+1) + \frac{m^2x^2}{1-x^2} - m\} v^{(m)} \\
&= [-\lambda + \frac{m^2}{1-x^2}]w.
\end{aligned}$$

Therefore w satisfies the equation

$$[(1-x^2)w']' + [\lambda - \frac{m^2}{1-x^2}]w = 0$$

$$(1-x^2)w'' - 2xw' + [\lambda - \frac{m^2}{1-x^2}]w = 0. \tag{7.2}$$

Eq. (7.2) is the associated Legendre equation with $\nu = m^2$.

$$v \text{ a solution of (7.1)} \quad \Longrightarrow \quad w \text{ is a solution of (7.2).}$$

As with Legendre's equation, we have the following result for the associated Legendre's equation:

Theorem 49. *The associated Legendre's equation has bounded solutions on the interval* $[-1, 1]$ *if and only if*

$$\lambda = n(n+1), \qquad n = 0, 1, 2, \ldots.$$

Associated Legendre functions of the first kind are defined as follows:

$$P_n^m(x) := (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m}(P_n(x)), \qquad 0 \leqslant m \leqslant n.$$

Note that $P_n^0(x) = P_n(x)$ for all n and $P_n^m(x) \equiv 0$ if $m > n$. The first few associated Legendre functions are

$$\begin{aligned}
P_1^1(x) &= (1-x^2)^{\frac{1}{2}}, & P_2^2(x) &= 3(1-x^2), \\
P_2^1(x) &= 3x(1-x^2)^{\frac{1}{2}}, & P_3^2(x) &= 15x(1-x^2), \\
P_3^1(x) &= \frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1), & P_3^3(x) &= 15(1-x^2)^{\frac{3}{2}}.
\end{aligned}$$

We may extend the definition of associated Legendre functions to negative values of m by using Rodrigues' formula in the definition above. Thus, we redefine associated Legendre functions of the first kind

$$P_n^m(x) := \frac{1}{2^n\, n!}(1-x^2)^{\frac{m}{2}}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n, \qquad -n \leqslant m \leqslant n.$$

Associated Legendre functions of the second kind are defined as follows:

$$Q_n^m(x) := (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}(Q_n(x)), \qquad 0 \leqslant m \leqslant n.$$

Lemma 50. *Associated Legendre functions of the first kind satisfy the following properties:*

1. $\|P_n^m\|^2 = \int_{-1}^{1}(P_n^m(x))^2\, dx = \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}.$
2. $\langle P_k^m, P_l^m\rangle = \int_{-1}^{1} P_k^m(x)P_l^m(x)\, dx = 0 \qquad$ *1 if* $k \neq l$.

7.3 Spherical Bessel Functions

Consider the radial equation obtained earlier with $\lambda = n(n+1)$:

$$r^2R'' + 2rR' + (\mu^2r^2 - n(n+1))R = 0.$$

Let $x = \mu r$ and $u(x) = R(r)$. Then we get

$$x^2u'' + 2xu' + (x^2 - n(n+1))u = 0. \qquad \text{(spherical Bessel's equation)}$$

This resembles Bessel's equation, but differs because of the 2 in the second term. We can transform it into Bessel's equation as follows: let $u(x) = x^\gamma w(x)$. Plug into the equation to get

$$x^2w'' + 2(\gamma+1)xw' + [x^2 - n(n+1) + \gamma(\gamma+1)]w = 0.$$

Now set $2(\gamma+1) = 1$. Thus $\gamma = -1/2$ and the equation becomes

$$x^2w'' + xw' + [x^2 - (n+\frac{1}{2})^2]w = 0.$$

This is Bessel's equation of order $n + \frac{1}{2}$. One solution is

$$w(x) = J_{n+\frac{1}{2}}(x), \quad \text{therefore} \quad u(x) = \frac{w(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}}J_{n+\frac{1}{2}}(x).$$

The spherical Bessel functions of order n are defined as

$$j_n(x) := \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x), \qquad \text{(first kind)}$$

$$y_n(x) := \sqrt{\frac{\pi}{2x}}Y_{n+\frac{1}{2}}(x). \qquad \text{(second kind)}$$

The first few spherical Bessel functions are listed below:

$$j_0(x) = \frac{\sin x}{x}, \qquad y_0(x) = -\frac{\cos x}{x},$$
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x},$$
$$j_2(x) = \left(\frac{3}{x^2} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x, \qquad y_2(x) = \left(-\frac{3}{x^2} + \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x.$$

Lemma 51. *Spherical Bessel functions satisfy the following properties:*

1. $j_0(0) = 1, \quad j_n(0) = 0, \; n \geqslant 1.$
2. $y_n(x) \to -\infty$ *as* $x \to 0^+$.
3. $j_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \qquad y_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right).$
4. $\int_0^{\alpha_{nm}} j_n^2(\mu r)\, r^2\, dr = \frac{\pi}{4}\alpha_{nm}^2 J'^2_{n+\frac{1}{2}}(\alpha_{nm}\mu), \quad$ *where* $\quad j_n(\alpha_{nm}) = 0.$
5. $\frac{1}{x}\sin\sqrt{x^2+2xt} = \sum_{n=0}^{\infty} \frac{(-1)^n y_{n-1}(x)}{n!} t^n, \quad 2|t| < |x|.$
6. $\frac{1}{x}\cos\sqrt{x^2-2xt} = \sum_{n=0}^{\infty} \frac{j_{n-1}(x)}{n!} t^n, \quad 2|t| < |x|.$

7.4 Laplace's Equation (3-d)

We now conside an example of Laplace's equation in a sphere.

EXAMPLE 7.1. (Laplace's equation in a sphere)

$$\begin{aligned} \nabla^2 u &= 0, && 0 < r < a, \\ u(a,\theta,\phi) &= f(\theta,\phi) && -\pi < \phi < \pi, \\ & && 0 < \theta < \pi. \end{aligned}$$

Analogous to what was done before, we consider the domain in the $r\theta\phi$-space. Several of the boundaries of this rectanglar region are not "real physical" boundaries. So, similar to the approach we adopted earlier, we make the following additional assumptions:

A1: $u(r,\theta,\pi) = u(r,\theta,-\pi)$ and $\frac{\partial u}{\partial \phi}(r,\theta,\pi) = \frac{\partial u}{\partial \phi}(r,\theta,-\pi)$;

A2: $|u(r,\theta,\phi)|$ bounded everywhere.

We apply separation of variables: $u(r,\theta,\phi) = R(r)S(\theta)Q(\phi)$. The equation becomes

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right\} = 0$$
$$(r^2R')'SQ + \frac{RQ}{\sin\theta}(\sin\theta\, S')' + \frac{RS}{\sin^2\theta}Q'' = 0$$
$$\frac{(r^2R')R'}{R} + \frac{(\sin\theta\, S')'}{\sin\theta\, S} + \frac{Q''}{Q\sin^2\theta} = 0$$
$$\frac{(r^2R')R'}{R} = -\frac{(\sin\theta\, S')'}{\sin\theta\, S} - \frac{Q''}{Q\sin^2\theta} = \lambda$$

This leads to

$$\boxed{(r^2R')' - \lambda R = 0}, \qquad \frac{\sin\theta(\sin\theta\, S')'}{S} + \lambda\sin^2\theta = -\frac{Q''}{Q} = \nu.$$

The second of these leads to

$$\boxed{\sin\theta(\sin\theta\, S')' + (\lambda\sin^2\theta - \nu)S = 0}, \qquad \boxed{Q'' + \nu Q = 0}.$$

We first solve the equation for Q with the appropriate boundary conditions:

$$\left.\begin{matrix} Q'' + \nu Q = 0 \\ Q(\pi) = Q(-\pi) \\ Q'(\pi) = Q'(-\pi) \end{matrix}\right\} \implies \begin{cases} \nu = m^2, \qquad 1m = 0, 1, 2, \ldots, \\ Q_m(\phi) = \alpha_m\cos(m\phi) + \beta_m\sin(m\phi). \end{cases}$$

If we let $x = \cos\theta$ and $v(x) = S(\theta)$, then the equation for S becomes

$$(1-x^2)v'' - 2xv' + [\lambda - \frac{m^2}{1-x^2}]v = 0.$$

¿From (A2) we get

$$|u(r,\theta,\phi)| \text{ bdd.} \implies |S(\theta)| \text{ bdd.} \implies |v(x)| \text{ bdd.} \implies \lambda = n(n+1).$$

Therefore v satisfies

$$(1-x^2)v'' - 2xv' + [n(n+1) - \frac{m^2}{1-x^2}]v = 0.$$

Hence

$$v(x) = c_n^m P_n^m(x) + d_n^m Q_n^m(x).$$

But again $|v(x)|$ bounded implies that $d_n^m = 0$ so that

$$v(x) = c_n^m P_n^m(x), \qquad S_n^m(\theta) = c_n^m P_n^m(\cos\theta).$$

The equation for R becomes

$$r^2R'' + 2rR' - n(n+1)R = 0. \qquad \text{(Cauchy–Euler equation)}$$

Looking for a solution of the form $R(r) = r^\alpha$ implies that

$$\alpha^2 + \alpha - n(n+1) = 0 \quad \Longrightarrow \quad \alpha = n, -1-n \quad \Longrightarrow \quad R(r) = c_1r^n + c_2r^{-1-n}.$$

But $|R(r)|$ bounded implies $c_2 = 0$, hence $R(r) = r^n$. Therefore

$$u_n^m(r,\theta,\phi) = r^nP_n^m(\cos\theta)(\alpha_n^m\cos(m\phi) + \beta_n^m\sin(m\phi)).$$

Applying superposition yields

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^nP_n^m(\cos\theta)(\alpha_n^m\cos(m\phi) + \beta_n^m\sin(m\phi)).$$

Apply the one remaining genuine boundary condition $u(a,\theta,\phi) = f(\theta,\phi)$ to get

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a^nP_n^m(\cos\theta)(\alpha_n^m\cos(m\phi) + \beta_n^m\sin(m\phi)).$$

If we let

$$\widehat{Y}_n^m(\theta,\phi) := P_n^m(\cos\theta)\cos(m\phi), \quad \widetilde{Y}_n^m(\theta,\phi) := P_n^m(\cos\theta)\sin(m\phi), \qquad \text{(spherical harmonics)}$$

then

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a^n(\alpha_n^m\widehat{Y}_n^m(\theta,\phi) + \beta_n^m\widetilde{Y}_n^m(\theta,\phi)).$$

The coefficients are given by

$$a^n\alpha_n^m = \frac{\left\langle f, \widehat{Y}_n^m\right\rangle}{\|\widehat{Y}_n^m\|^2}, \qquad a^n\beta_n^m = \frac{\left\langle f, \widetilde{Y}_n^m\right\rangle}{\|\widetilde{Y}_n^m\|^2},$$

where the inner product is given by

$$\langle f, g\rangle := \int_{-\pi}^{\pi}\int_0^{\pi} f(\theta,\phi)g(\theta,\phi)\,\sin\theta\,d\theta\,d\phi.$$

We have

$$\|\widehat{Y}_n^m\|^2 = \int_{-\pi}^{\pi}\int_0^{\pi}(P_n^m(\cos\theta))^2\cos^2(m\phi)\sin\theta\,d\theta\,d\phi = \left(\int_{-\pi}^{\pi}\cos^2(m\phi)\,d\phi\right)\left(\int_0^{\pi}(P_n^m(\cos\theta))^2\sin\theta\,d\theta\right)$$

$$= \left\{\begin{matrix} \pi & \text{if } m \neq 0 \\ 2\pi & \text{if } m = 0 \end{matrix}\right\} \cdot \int_{-1}^{1}(P_n^m(x))^2\,dx = \begin{cases} \dfrac{2\pi}{2n+1}\cdot\dfrac{(n+m)!}{(n-m)!} & \text{if } m \neq 0 \\ \dfrac{4\pi}{2n+1} & \text{if } m = 0 \end{cases}.$$

Similarly

$$\|\widetilde{Y}_n^m\|^2 = \begin{cases} \dfrac{2\pi}{2n+1} \cdot \dfrac{(n+m)!}{(n-m)!} & \text{if } m \neq 0 \\ \dfrac{4\pi}{2n+1} & \text{if } m = 0 \end{cases}.$$

The final solution is

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} r^n P_n^m(\cos\theta)(\alpha_n^m \cos(m\phi) + \beta_n^m \sin(m\phi)),$$

where

$$\alpha_n^m = (2-\delta_{m0})\frac{2n+1}{4\pi} \cdot \frac{(n-m)!}{(n+m)!} \int_{-\pi}^{\pi} \int_0^{\pi} f(\theta,\phi) P_n^m(\cos\theta) \cos(m\phi) \sin\theta \, d\theta \, d\phi,$$
$$\beta_n^m = (2-\delta_{m0})\frac{2n+1}{4\pi} \cdot \frac{(n-m)!}{(n+m)!} \int_{-\pi}^{\pi} \int_0^{\pi} f(\theta,\phi) P_n^m(\cos\theta) \sin(m\phi) \sin\theta \, d\theta \, d\phi.$$

Special Case: $(f(\theta,\phi) = f(\theta))$
In this case we get

$$\begin{aligned} \alpha_n^m &= \frac{\left\langle f, \widehat{Y}_n^m \right\rangle}{a^n \|\widehat{Y}_n^m\|^2} = \int_{-\pi}^{\pi} \int_0^{\pi} f(\theta) P_n^m(\cos\theta) \cos(m\phi) \sin\theta \, d\theta \, d\phi \\ &= \frac{\left\langle f, \widehat{Y}_n^m \right\rangle}{a^n \|\widehat{Y}_n^m\|^2} \left(\int_{-\pi}^{\pi} \cos(m\phi) \, d\phi \right) \left(\int_0^{\pi} f(\theta) P_n^m(\cos\theta) \sin\theta \, d\theta \right) = 0 \qquad \text{if } m \neq 0, \\ \beta_n^m &= \frac{\left\langle f, \widetilde{Y}_n^m \right\rangle}{a^n \|\widetilde{Y}_n^m\|^2} = \int_{-\pi}^{\pi} \int_0^{\pi} f(\theta) P_n^m(\cos\theta) \sin(m\phi) \sin\theta \, d\theta \, d\phi \\ &= \frac{\left\langle f, \widetilde{Y}_n^m \right\rangle}{a^n \|\widetilde{Y}_n^m\|^2} \left(\int_{-\pi}^{\pi} \sin(m\phi) \, d\phi \right) \left(\int_0^{\pi} f(\theta) P_n^m(\cos\theta) \sin\theta \, d\theta \right) = 0 \qquad \text{for all } m. \end{aligned}$$

Thus we have

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n \left(\frac{r}{a}\right)^n P_n(\cos\theta),$$

where

$$c_n = \frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta.$$

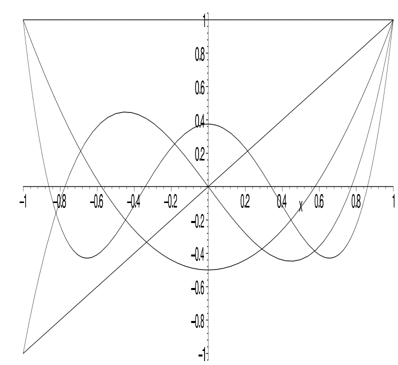


Figure 7.1: Legendre polynomials

Chapter 8

Problems in Infinite Domains

June 17, 2010

8.1 Fourier Integrals

Consider $f \in PC(-\ell, \ell)$ with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell})),$$

where

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos(\frac{n\pi x}{\ell})\, dx, \qquad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin(\frac{n\pi x}{\ell})\, dx.$$

We want to generalize this to the case where $f \in PC(\mathbb{R})$. That is, we wish to consider the limiting case where $\ell \to \infty$.

<u>Heuristic motivation</u>

We begin by introducing some notation. Let

$$\omega_n := \frac{n\pi}{\ell}, \qquad A_\ell(\omega) := \frac{1}{\pi} \int_{-\ell}^{\ell} f(x) \cos \omega x\, dx, \qquad A(\omega) := \lim_{\ell \to \infty} A_\ell(\omega),$$

$$\Delta\omega := \omega_{n+1} - \omega_n = \frac{\pi}{\ell}, \qquad B_\ell(\omega) := \frac{1}{\pi} \int_{-\ell}^{\ell} f(x) \sin \omega x\, dx, \qquad B(\omega) := \lim_{\ell \to \infty} B_\ell(\omega).$$

Then

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos(\omega_n x)\, dx = \frac{\pi}{\ell} A_\ell(\omega_n) = A_\ell(\omega_n)\Delta\omega;$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin(\omega_n x)\, dx = \frac{\pi}{\ell} B_\ell(\omega_n) = B_\ell(\omega_n)\Delta\omega.$$

Now, taking the limit $\ell \to \infty$, we get

$$\lim_{\ell\to\infty}\left\{\sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell})\right\} = \lim_{\ell\to\infty}\left\{\sum_{n=1}^{\infty} A_\ell(\omega_n)\cos(\omega_n x)\Delta\omega\right\} \stackrel{?}{=} \int_0^\infty A(\omega)\cos(\omega x)\,d\omega,$$

$$\lim_{\ell\to\infty}\left\{\sum_{n=1}^{\infty} b_n \cos(\frac{n\pi x}{\ell})\right\} = \lim_{\ell\to\infty}\left\{\sum_{n=1}^{\infty} B_\ell(\omega_n)\cos(\omega_n x)\Delta\omega\right\} \stackrel{?}{=} \int_0^\infty B(\omega)\cos(\omega x)\,d\omega.$$

The last dubious equalities are suggested by the fact that the sums resemble Riemann sums and the limits resemble the original definitions of a Riemann integral. So is it legitimate to write

$$f(x) \stackrel{?}{\sim} \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\,d\omega.$$

The answer is yes. The derivation is given below. We will need a few lemmas.

Lemma 52. $\displaystyle\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}.$

Proof.

Let F be the Laplace transform of $\dfrac{\sin\omega}{\omega}$. That is, let

$$F(x) := \int_0^\infty e^{-\omega x}\frac{\sin\omega}{\omega}\,d\omega.$$

Then

$$F(0) = \int_0^\infty \frac{\sin\omega}{\omega}\,d\omega, \qquad \text{and} \qquad F(\infty) = \lim_{x\to\infty} F(x) = 0.$$

Differentiating F we get

$$F'(x) = \int_0^\infty \frac{\partial}{\partial x}\left\{e^{-\omega x}\frac{\sin\omega}{\omega}\right\}d\omega = -\int_0^\infty e^{-\omega x}\sin\omega\,d\omega = \frac{e^{-\omega x}}{1+x^2}[x\sin\omega + \cos\omega]\Big|_{\omega=0}^{\omega=\infty} = \frac{-1}{1+x^2}.$$

Integrating F' we get

$$F(x) = F(x) - 0 = F(x) - F(\infty) = \int_\infty^x F'(\xi)\,d\xi = -\int_\infty^x \frac{d\xi}{1+\xi^2} - \tan^{-1}\xi\Big|_\infty^x = -\tan^{-1}x + \frac{\pi}{2}$$

Thus

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = F(0) = \frac{\pi}{2}.$$

Lemma 53. $\displaystyle\int_0^\infty \frac{\sin\omega\xi}{\omega}\,d\omega = \begin{cases} \frac{\pi}{2}, & \textit{if } \xi > 0, \\ 0, & \textit{if } \xi = 0, \\ -\frac{\pi}{2}, & \textit{if } \xi < 0. \end{cases}$

<u>*Proof*</u>.
If $\xi = 0$, the result is obvious. If $\xi > 0$ then

$$\int_0^\infty \frac{\sin \omega\xi}{\omega}\, d\omega = \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \qquad \text{(using } x = \omega\xi\text{)}$$

On the other hand, if $\xi < 0$ then

$$\int_0^\infty \frac{\sin \omega\xi}{\omega}\, d\omega = -\int_0^\infty \frac{\sin x}{x}\, dx = -\frac{\pi}{2}. \qquad \text{(using } x = -\omega\xi\text{)}$$

Lemma 54. *If f is piecewise continuous, then*

$$\lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} f(\xi)\, d\xi = \frac{f(x+) + f(x-)}{2}.$$

<u>*Proof*</u>.

$$\lim_{h\to 0} \frac{\int_{x-h}^{x+h} f(\xi)\, d\xi}{2h} = \lim_{h\to 0} \frac{f(x+h) - f(x-h)(-1)}{2} = \frac{f(x+) + f(x-)}{2}.$$

We now define the following function

$$K(x,h) := \frac{2}{\pi}\int_0^\infty \frac{\sin \omega h}{\omega} \cos \omega x\, d\omega.$$

Then we have

$$\begin{aligned}
K(x,h) &= \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}[\sin \omega(x+h) - \sin \omega(x-h)]\, d\omega \\
&= \frac{1}{\pi}\left\{\int_0^\infty \frac{\sin \omega(x+h)}{\omega}\, d\omega - \int_0^\infty \frac{\sin \omega(x-h)}{\omega}\, d\omega\right\} \\
&= \frac{1}{\pi}\begin{cases} -\frac{\pi}{2} - (-\frac{\pi}{2}) & x < -h \\ \frac{\pi}{2} - (-\frac{\pi}{2}) & -h < x < h \\ \frac{\pi}{2} - (\frac{\pi}{2}) & x > h \end{cases} \qquad \text{(using Lemma 53)} \\
&= \frac{1}{\pi}\begin{cases} 0 & x < -h \\ \pi & -h < x < h \\ 0 & x > h \end{cases} \\
&= \begin{cases} 1 & |x| < h \\ 0 & |x| > h \end{cases}.
\end{aligned}$$

Note that K is an even function in x.

We are now ready to give the main result

Theorem 55. *If*

(i) $f \in PC^1(a,b)$ *on any interval* (a,b)*; and*

(ii) $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$,

then

$$\frac{f(x+)+f(x-)}{2} = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx.$$

Proof.
We have

$$\begin{aligned}
\frac{f(x+)+f(x-)}{2} &= \lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h} f(\xi)\,d\xi \qquad \text{(using Lemma 54)}\\
&= \lim_{h\to 0}\frac{1}{2h}\int_{-\infty}^{\infty} f(\xi)K(x-\xi,h)\,d\xi\\
&= \lim_{h\to 0}\frac{1}{2h}\int_{-\infty}^{\infty} f(\xi)\left\{\frac{2}{\pi}\int_0^\infty \frac{\sin\omega h}{\omega}\cos\omega(x-\xi)\,d\omega\right\}d\xi \quad \text{(using the definition of } K)\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\infty f(\xi)\left[\lim_{h\to 0}\frac{\sin\omega h}{\omega h}\right]\cos\omega(x-\xi)\,d\omega\,d\xi\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\infty f(\xi)\cos\omega(x-\xi)\,d\omega\,d\xi\\
&= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)[\cos\omega x\cos\omega\xi + \sin\omega x\sin\omega\xi]\,d\xi\,d\omega\\
&= \frac{1}{\pi}\int_0^\infty\left\{\left(\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\,d\xi\right)\cos\omega x + \left(\int_{-\infty}^{\infty} f(\xi)\sin\omega\xi\,d\xi\right)\sin\omega x\right\}d\omega\\
&= \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega.
\end{aligned}$$

The expression

$$\int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

is called the *Fourier integral* representaion of f, and $A(\omega)$ and $B(\omega)$ are called the *Fourier integral coefficients* of f.

EXAMPLE 8.1. Calculate the Fourier integral of $f(x) = e^{-|x|}$.

Solution:

We have

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|x|}\cos\omega x\,dx = \frac{2}{\pi}\int_{0}^{\infty} e^{-|x|}\cos\omega x\,dx = \frac{2}{\pi}\frac{e^{-x}}{1+\omega^2}(\omega\sin\omega x - \sin\omega x)\bigg|_{x=0}^{\infty} = \frac{2}{\pi(1+\omega^2)},$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|x|}\sin\omega x\,dx = 0. \qquad \text{(since } f \text{ is even)}$$

Hence

$$e^{-|x|} = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\,d\omega.$$

Suppose f is defined on $(0,\infty)$. We define

$$f_e(x) := \begin{cases} f(x) & x>0 \\ f(-x) & x<0 \end{cases} \qquad \text{(even extension of } f\text{)},$$

$$f_o(x) := \begin{cases} f(x) & x>0 \\ -f(-x) & x<0 \end{cases} \qquad \text{(odd extension of } f\text{)}.$$

Then

$$f_e(x) \sim \int_0^\infty A(\omega)\cos\omega x\,d\omega, \qquad \text{(Fourier cosine integral)}$$

$$f_o(x) \sim \int_0^\infty A(\omega)\sin\omega x\,d\omega. \qquad \text{(Fourier sine integral)}$$

where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx, \qquad B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx.$$

8.2 Fourier Transform

There is also a complex form of the Fourier integral. Notice the symmetries of the Fourier integral coefficients:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx, \qquad A(-\omega) = A(\omega),$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx, \qquad B(-\omega) = -B(\omega).$$

Manipulating the Fourier integral we get

$$
\begin{aligned}
f(x) &= \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega \\
&= \frac{1}{2}\int_0^\infty [A(\omega)(e^{i\omega x}+e^{-i\omega x}) - iB(\omega)(e^{i\omega x}-e^{-i\omega x})]\,d\omega \\
&= \frac{1}{2}\int_0^\infty [A(\omega)-iB(\omega)]e^{i\omega x}\,d\omega + \frac{1}{2}\int_0^\infty [A(\omega)+iB(\omega)]e^{-i\omega x}\,d\omega \\
&= \frac{1}{2}\int_0^{-\infty} [A(-\mu)-iB(-\mu)]e^{i(-\mu)x}\,d(-\mu) + \frac{1}{2}\int_0^\infty [A(\omega)+iB(\omega)]e^{-i\omega x}\,d\omega \\
&= \frac{1}{2}\int_{-\infty}^0 [A(\mu)+iB(\mu)]e^{-i\mu x}\,d\mu + \frac{1}{2}\int_0^\infty [A(\mu)+iB(\mu)]e^{-i\mu x}\,d\mu \\
&= \frac{1}{2}\int_{-\infty}^\infty [A(\mu)+iB(\mu)]e^{-i\mu x}\,d\mu = \int_{-\infty}^\infty F(\mu)e^{-i\mu x}\,d\mu,
\end{aligned}
$$

where

$$
F(\mu) := \frac{1}{2}[A(\mu)+iB(\mu)] = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)[\cos\mu x + i\sin\mu x]\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{i\mu x}\,dx.
$$

The expression $F(\omega)$ is called the *Fourier Transform* of f and is sometimes denoted by $\widehat{f}(\omega)$. Thus we have

$$
\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{i\omega x}\,dx, && \text{(Fourier Transform)} \\
f(x) &= \int_{-\infty}^\infty F(\omega)e^{-i\omega x}\,d\omega. && \text{(inverse Fourier Transform)}
\end{aligned}
$$

Remark: Unfortunately, there is no unique way to define the Fourier transform. What makes the Fourier transform so powerful is the relation between transfrom and inverse transform, the detailed definition of the transform, however, allows various choices. Hence in some textbooks you will find Fourier Transform and inverse Fourier Transform with different factors in front:

$$
F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{i\omega x}\,dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\omega)e^{-i\omega x}\,d\omega.
$$

In other textbooks the sign of the exponential might be flipped:

$$
F(\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\omega x}\,dx, \qquad f(x) = \int_{-\infty}^\infty F(\omega)e^{i\omega x}\,d\omega.
$$

EXAMPLE 8.2. If $f(x) = e^{-|x|}$, then its Fourier Transform is

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{i\omega x}\,dx = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{x(1+i\omega)}\,dx + \int_{0}^{\infty} e^{-x(1-i\omega)}\,dx\right\}$$
$$= \frac{1}{2\pi}\left\{\frac{e^{x(1+i\omega)}}{1+i\omega}\Big|_{x=-\infty}^{0} - \frac{e^{-x(1-i\omega)}}{1-i\omega}\Big|_{x=0}^{\infty}\right\} = \frac{1}{2\pi}\left(\frac{1}{1+i\omega}+\frac{1}{1-i\omega}\right) = \frac{1}{\pi(1+\omega^2)}.$$

Therefore

$$e^{-|x|} = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{1+\omega^2}\,d\omega.$$

For f defined on $(0,\infty)$, we can calculate the Fourier Transform for the even and odd extensions f_e and f_o.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_e(x)e^{i\omega x}\,dx = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} f_e(x)e^{i\omega x}\,dx + \int_{0}^{\infty} f_e(x)e^{i\omega x}\,dx\right\}$$
$$= \frac{1}{2\pi}\left\{\int_{0}^{\infty} f(x)e^{-i\omega x}\,dx + \int_{0}^{\infty} f(x)e^{i\omega x}\,dx\right\}$$
$$= \frac{1}{2\pi}\int_{0}^{\infty} f(x)(e^{i\omega x}+e^{-i\omega x})\,dx = \frac{1}{\pi}\int_{0}^{\infty} f(x)\cos\omega x\,dx.$$

Note that F is an even function. To invert this we have

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega = \int_{-\infty}^{0} F(\omega)e^{-i\omega x}\,d\omega + \int_{0}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$$
$$= \int_{0}^{\infty} F(\omega)e^{i\omega x}\,d\omega + \int_{0}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$$
$$= \int_{0}^{\infty} F(\omega)(e^{i\omega x}+e^{-i\omega x})\,d\omega = \int_{0}^{\infty} 2F(\omega)\cos\omega x\,d\omega.$$

A similar calculation holds for the odd extension. So, for a function defined on $(0,\infty)$, we define

$$F_c(\omega) := \frac{2}{\pi}\int_{0}^{\infty} f(x)\cos\omega x\,dx, \qquad \text{(Fourier cosine transform)}$$
$$f(x) = \int_{0}^{\infty} F_c(\omega)\cos\omega x\,d\omega, \qquad \text{(inverse Fourier cosine transform)}$$
$$F_s(\omega) := \frac{2}{\pi}\int_{0}^{\infty} f(x)\sin\omega x\,dx, \qquad \text{(Fourier sine transform)}$$
$$f(x) = \int_{0}^{\infty} F_s(\omega)\sin\omega x\,d\omega. \qquad \text{(inverse Fourier sine transform)}$$

8.3 Applications

Here we will give examples of solving the heat equation and the wave equation on infinite domains. First the heat equation.

EXAMPLE 8.3. Consider the heat equation:

$$\begin{aligned} u_t &= hu_{xx}, \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= e^{-|x|}. \end{aligned}$$

We require the additional assumption that the solution be bounded for all $x \in \mathbb{R}$.

Solution 1.
Apply separation of variables. Look for a solution of the form $u(x,t) = X(x)T(t)$. Plug this into the equation to get:

$$\frac{T'}{kT} = \frac{X''}{X} = -\mu.$$

This leads to ODEs is X and T:

$$\begin{aligned} X'' + \mu X &= 0, \qquad\qquad T' + \mu T = 0, \\ |X(x)| &< \text{bdd}. \end{aligned}$$

The equation for X will have bounded solutions only for $\mu \geqslant 0$. So, set $\mu = \omega^2$ to get

$$X'' + \omega^2 X = 0, \quad \Longrightarrow \quad X(x;\omega) = e^{\pm i\omega x}.$$

Note that, without finite boundary conditions, there is no restriction on ω. Thus $\omega \in \mathbb{R}$ is arbitrary. We now have

$$T(t;\omega) = e^{-k\omega^2 t} \qquad \text{and} \qquad u(x,t;\omega) = F(\omega)e^{-i\omega x}e^{-k\omega^2 t}.$$

The function $u(x,t;\omega)$ is a bounded solution to the heat equation for any F and for any $\omega \in \mathbb{R}$. We now apply a generalized principle of superposition. In earlier problems we took a liear combination of solutions over all eigenvalues. In those problems the spectrum (the set of all eigenvalues) was discrete and so the linear combination consisted of a discrete sum. In this case the spectrum is continuous (all $\omega \in \mathbb{R}$) and so the appropriate linear combination will consist of a continuous sum (i.e. an integral). We get

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}e^{-k\omega^2 t}\, d\omega.$$

We now apply the initial condition $u(x,0) = e^{-|x|}$ to get

$$e^{-|x|} = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\, d\omega.$$

Thus F must be the Fourier transform of $e^{-|x|}$, hence

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{i\omega x}\,dx = \frac{1}{\pi(1+\omega^2)}. \qquad \text{(from Example 8.2)}$$

Therefore, the final solution to the problem, in integral form, is

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega x - k\omega^2 t}}{1+\omega^2}\,d\omega.$$

<u>Solution 2</u>.

We now solve the problem in a slightly different manner. Let $U(\omega,t)$ br the Fourier transform of $u(x,t)$:

$$U(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx.$$

Then

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}\,d\omega, \\ u_t(x,t) &= \int_{-\infty}^{\infty} U_t(\omega,t)e^{-i\omega x}\,d\omega, \\ u_{xx}(x,t) &= \int_{-\infty}^{\infty} (-\omega^2)U(\omega,t)e^{-i\omega x}\,d\omega. \end{aligned}$$

Plug into the equation:

$$u_t - ku_{xx} = 0 \quad \Longrightarrow \quad \int_{-\infty}^{\infty} [U_t(\omega,t) + k\omega^2 U(\omega,t)]e^{-i\omega x}\,d\omega = 0.$$

Set the integrand to zero:

$$\frac{\partial U}{\partial t} = -k\omega^2\,U \quad \Longrightarrow \quad U(\omega,t) = F(\omega)e^{-k\omega^2 t} \quad \Longrightarrow \quad u(x,t) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}e^{-k\omega^2 t}\,d\omega.$$

Apply the initial condition $u(x,0) = e^{-|x|}$ to get

$$e^{-|x|} = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega.$$

Thus F must be the Fourier transform of $e^{-|x|}$, hence

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{i\omega x}\,dx = \frac{1}{\pi(1+\omega^2)}. \qquad \text{(from Example 8.2)}$$

Therefore, the final solution to the problem, in integral form, is

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega x - k\omega^2 t}}{1+\omega^2}\,d\omega.$$

We now consider a problem involving the wave equation.

EXAMPLE 8.4.

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

We use an approach analogous to the second solution given in the previous example. Let $U(\omega, t)$ be the Fourier transform of $u(x,t)$, and let F and G be the Fourier transforms of f and g respectively. Then

$$U(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx \qquad u(x,t) = \int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}\,d\omega,$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx, \qquad f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega, \tag{8.1}$$

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{i\omega x}\,dx. \qquad g(x) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}\,d\omega. \tag{8.2}$$

Differentiating u we get

$$u_{tt}(x,t) = \int_{-\infty}^{\infty} U_{tt}(\omega,t)e^{-i\omega x}\,d\omega, \qquad u_{xx}(x,t) = -\int_{-\infty}^{\infty} \omega^2 U(\omega,t)e^{-i\omega x}\,d\omega.$$

It is clear that $F(\omega) = U(\omega, 0)$ and $G(\omega) = U_t(\omega, 0)$. Plug into the equation:

$$u_{tt} - c^2 u_{xx} = 0 \quad \Longrightarrow \quad \int_{-\infty}^{\infty} [U_{tt}(\omega,t) + c^2\omega^2 U(\omega,t)]e^{-i\omega x}\,d\omega = 0.$$

We get the following initial value problem for U:

$$\begin{aligned} &\frac{\partial^2 U}{\partial t^2} + c^2\omega^2\, U = 0, \\ &U(\omega,0) = F(\omega), \qquad 1U_t(\omega,0) = G(\omega). \end{aligned}$$

The solution to the ODE is

$$U(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t).$$

The initial conditions imply

$$A(\omega) = F(\omega) \qquad \text{and} \qquad B(\omega) = \frac{G(\omega)}{c\omega}.$$

Therefore

$$u(x,t) = \int_{-\infty}^{\infty} [F(\omega)\cos(c\omega t) + G(\omega)\frac{\sin(c\omega t)}{c\omega}]e^{-i\omega x}\,dx$$

However, further manipulation yields

$$\begin{aligned}\int_{-\infty}^{\infty} F(\omega)\cos(c\omega t)e^{-i\omega x}\,d\omega &= \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)[e^{ic\omega t}+e^{-ic\omega t}]e^{-i\omega x}\,d\omega \\ &= \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)[e^{-i\omega(x-ct)}+e^{-i\omega(x+ct)}]\,d\omega \\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]. \qquad \text{(using (8.1))}\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} G(\omega)\frac{\sin(c\omega t)}{c\omega}e^{-i\omega x}\,d\omega &= \frac{1}{2}\int_{-\infty}^{\infty} G(\omega)\frac{e^{ic\omega t}-e^{-ic\omega t}}{ic\omega}e^{-i\omega x}\,d\omega \\ &= \frac{1}{2}\int_{-\infty}^{\infty} G(\omega)\frac{e^{-i\omega(x-ct)}-e^{-i\omega(x+ct)}}{ic\omega}\,d\omega \\ &= \frac{1}{2c}\int_{-\infty}^{\infty} G(\omega)\left(\int_{x-ct}^{x+ct} e^{-i\omega\xi}\,d\xi\right)d\omega \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct}\int_{-\infty}^{\infty}\left(G(\omega)e^{-i\omega\xi}\,d\omega\right)d\xi \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi. \qquad \text{(using (8.2))}\end{aligned}$$

Therefore we recover D'Alembert's solution obtained earlier

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$$

We now redo the previous example involving the heat equation and show how to obtain an explicit representation of the solution, for an arbitrary initial condition, involving only one integration and not involving the Fourier transform. To do this we need the following result:

Lemma 56. *Let $a>0$ be constant. Then*

$$\int_{-\infty}^{\infty} e^{-a\omega^2-i\omega x}\,d\omega = \sqrt{\frac{\pi}{a}}e^{-\frac{x^2}{4a}}.$$

Proof.
Exercise.

EXAMPLE 8.5. Consider the following one–dimensional heat equation:

$$\begin{aligned}u_t &= ku_{xx}, \qquad -\infty<x<\infty,\ t>0,\\ u(x,0) &= f(x).\end{aligned}$$

Let $U(\omega,t)$ and $F(\omega)$ be the Fourier transforms of $u(x,t)$ and $f(x)$ respectively. Then

$$U(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx, \qquad u(x,t) = \int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}\,d\omega$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx, \qquad f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$$

It is clear that $F(\omega) = U(\omega,0)$. Differentiating u we get

$$u_t(x,t) = \int_{-\infty}^{\infty} U_t(\omega,t)e^{-i\omega x}\,d\omega, \qquad u_{xx}(x,t) = \int_{-\infty}^{\infty} -\omega^2 U(\omega,t)e^{-i\omega x}\,d\omega.$$

Plug into the equation:

$$u_t - ku_{xx} = 0 \quad \Longrightarrow \quad \int_{-\infty}^{\infty} [U_t(\omega,t) + k\omega^2 U(\omega,t)]e^{-i\omega x}\,d\omega = 0.$$

We get the following ODE for U:

$$\frac{\partial U}{\partial t} + k\omega^2 U = 0,$$
$$U(\omega,0) = F(\omega).$$

This is easily solved to get

$$U(\omega,t) = F(\omega)e^{-k\omega^2 t}.$$

The solution to the problem is

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega)e^{-k\omega^2 t}e^{-i\omega x}\,d\omega. \tag{8.3}$$

This of course is the solution we obtained in the example before last. However, further manipulation yields

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}e^{-k\omega^2 t}e^{-i\omega x}\,d\xi\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi - i\omega x - k\omega^2 t}\,d\omega\,d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\left\{\int_{-\infty}^{\infty} e^{-i\omega(x-\xi) - k\omega^2 t}\,d\omega\right\}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\left\{\sqrt{\frac{\pi}{kt}}e^{-\frac{(x-\xi)^2}{4kt}}\right\}d\xi \qquad \text{(using Lemma 56)} \\
&= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4kt}}\,d\xi \\
&= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(x-s)e^{-\frac{s^2}{4kt}}\,ds.
\end{aligned}$$

Thus we obtain the *explicit representation*

$$\boxed{u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi}. \tag{8.4}$$

The function

$$\frac{e^{-\frac{(x-\xi)^2}{4kt}}}{2\sqrt{\pi k t}}$$

is called the *Gauss–Weirstrass kernel* or the *fundamental solution* of the heat equation. The explicit representation (8.4) is preferable to the Fourier representation (8.3) for several reasons: (a) it is computationally more direct, requiring only one integration; (b) it makes sense for many functions f for which the Fourier transform is undefined, for example, any bounded continuous function; and (c) it does not require any smoothness in order to satisfy the initial value problem.

Special case: $f(\xi) = u_0$, a constant.

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} u_0 e^{-\frac{s^2}{4kt}}\, ds = u_0.$$

Special case: $f(\xi) = \delta(\xi)$ Dirac δ-function.

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}.$$

8.3.1 Error Function

Another helpful tool to solve the heat equation on the real line $\mathbb{R}$ is the **error function**

$$\operatorname{erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-z^2}\, dz,$$

We illustrate it's use in an example:

Exercise 8.1. ✗
Use convolutions, the error function, and operational properties of the Fourier transform to solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{100} \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 100 & \text{if} \quad -2 < x < 0, \\ 50 & \text{if} \quad 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Transforming the heat equation and the initial conditions, we get the solution to the transformed problem

$$\widehat{u}(\omega,t)=\widehat{f}(\omega)\,e^{-\omega^2 t/100}.$$

Since this is the product of two Fourier transforms, we know that it is $1/2\pi$ times the Fourier transform of a convolution, and using the convolution theorem, the solution can be written as

$$u(x,t)=\frac{1}{2\pi}\mathcal{F}^{-1}\left(e^{-\omega^2 t/100}\right)*f(x).$$

Computing the inverse Fourier transform of $e^{-\omega^2 t/100}$

$$\mathcal{F}^{-1}\left(e^{-\omega^2 t/100}\right)=\int_{-\infty}^{\infty}e^{-\omega^2 t/100}e^{-i\omega x}\,d\omega,$$

we let

$$\frac{\sigma^2}{4}=\frac{\omega^2 t}{100},$$

so that

$$\omega=\frac{\sigma\sqrt{25}}{\sqrt{t}}\qquad\text{and}\qquad d\omega=\frac{\sqrt{25}}{\sqrt{t}}\,d\sigma,$$

and

$$\begin{aligned}\mathcal{F}^{-1}\left(e^{-\omega^2 t/100}\right)&=\frac{\sqrt{25}}{\sqrt{t}}\sqrt{4\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi}}\,e^{-\sigma^2/4}\,e^{-i\sigma\frac{25x}{\sqrt{t}}}\,d\sigma\\&=\frac{\sqrt{25}}{\sqrt{t}}\,\sqrt{4\pi}\,e^{-\frac{25x^2}{t}}\\&=10\sqrt{\frac{\pi}{t}}\,e^{-\frac{25x^2}{t}}.\end{aligned}$$

Therefore, the solution is

$$u(x,t)=\frac{10}{2\pi}\sqrt{\frac{\pi}{t}}\,e^{-\frac{25x^2}{t}}*f(x)=\frac{5}{\sqrt{\pi t}}\int_{-\infty}^{\infty}f(s)\,e^{-\frac{25(x-s)^2}{t}}\,ds,$$

that is,

$$u(x,t)=\frac{5}{\sqrt{\pi t}}\left[\int_{-2}^{0}100\,e^{-\frac{25(x-s)^2}{t}}\,ds+\int_{0}^{1}50\,e^{-\frac{25(x-s)^2}{t}}\,ds\right].$$

We can write the solution

$$u(x,t)=\frac{5}{\sqrt{\pi t}}\left[\int_{-2}^{0}100\,e^{-\frac{25(x-s)^2}{t}}\,ds+\int_{0}^{1}50\,e^{-\frac{25(x-s)^2}{t}}\,ds\right]$$

in terms of the error function

$$\operatorname{erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-z^2}\, dz,$$

by letting $z = \dfrac{5(x-s)}{\sqrt{t}}$, so that $dz = -\dfrac{5}{\sqrt{t}} ds$, then

$$u(x,t) = \frac{100}{\sqrt{\pi}} \int_{\frac{5x}{\sqrt{t}}}^{\frac{5(x+2)}{\sqrt{t}}} e^{-z^2}\, dz + \frac{50}{\sqrt{\pi}} \int_{\frac{5(x-1)}{\sqrt{t}}}^{\frac{5x}{\sqrt{t}}} e^{-z^2}\, dz,$$

that is,

$$u(x,t) = 50\left[\operatorname{erf}\left(\tfrac{5(x+2)}{\sqrt{t}}\right) - \operatorname{erf}\left(\tfrac{5x}{\sqrt{t}}\right)\right] + 25\left[\operatorname{erf}\left(\tfrac{5x}{\sqrt{t}}\right) - \operatorname{erf}\left(\tfrac{5(x-1)}{\sqrt{t}}\right)\right].$$

Chapter 9

Method of Characteristics

June 17, 2010

9.1 Introduction to the Method of Characteristics

The METHOD OF CHARACTERISTICS is an interesting new concept, which we will develop now. It is mathematically quite simple, but it involves an interesting logical twist. To explain this twist, we introduce the idea of an ANCHOR POINT, a concept which is not used in other textbooks.

We begin with a sinple first order partial differential equation on $-\infty < x < +\infty$

$$\frac{\partial z(x,t)}{\partial t} + c\frac{\partial z(x,t)}{\partial x} = 0, \qquad z(x,0) = f(x).$$

Notice that no boundary condition is needed, since we work on the whole real line.

As seen earlier, a solution of a PDE for a function of two variables $z(x,t)$ can be understood as a surface over the (x,t) plane. To parametrize this surface, we use a phantastic idea:

> Find curves $x(t)$ in the (x,t)-plane, such that the PDE can be reduced to an ODE along these curves. These ODEs can be solved and pieced back together to find the solution of the PDE.

Assume there is a curve $x(t)$ and we study z along this curve $z(x(t),t))$. Then by the chain rule we obtain

$$\frac{d}{dt}z(x(t),t) = \frac{\partial z(x(t),t)}{\partial x}\frac{\partial x(t)}{\partial t} + \frac{\partial z(x(t),t)}{\partial t}.$$

Notice, here you can clearly see the difference between a total derivative $\frac{d}{dt}$ and a partial derivative $\frac{\partial}{\partial t}$. Now if we compare the last expression with the original PDE, we see that it would be cool if $\frac{\partial x(t)}{\partial t} = c$. If this would be true, then we have

$$\frac{d}{dt}z(x(t),t) = c\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0$$

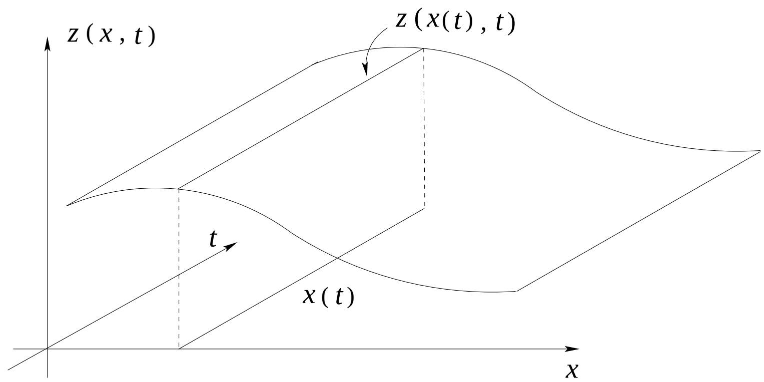


Figure 9.1: Schematic of a typical characteristic curve $x(t)$ and the part $z(x(t), t)$ of the solution surface z which lies directly above

hence the solution is constant along this curve $x(t)$. To make all this happen, we have to solve two ODEs, which are called the CHARACTERISTIC ODEs:

$$\frac{dx(t)}{dt} = c, \qquad \frac{dz(x(t), t)}{dt} = 0.$$

We solve the first one first:

$$x(t) = ct + a, \qquad \text{with an unknown constant } a.$$

We observe that at $t = 0$ we have $x(0) = a$, hence this constant a is the point where the characteristic curve starts. We call it the ANCHOR POINT. If a point (x, t) is given, then we can always find the corresponding anchor point as

$$a = x - ct.$$

Now we solve the second characteristic equation:

$$z(x(t), t) = \text{const}.$$

If z is constant along the characteristic, it must have the same value as at the point where the characteristic starts, i.e. the anchor point. Hence

$$z(x((t), t) = z(x(0), 0) = f(x(0)) = f(a) = f(x - ct)$$

where we used the initial condition $f(x)$. Which means for each given (x, t) we find the solution

$$z(x, t) = f(x - ct).$$

This is a transition to the left with velocity c:

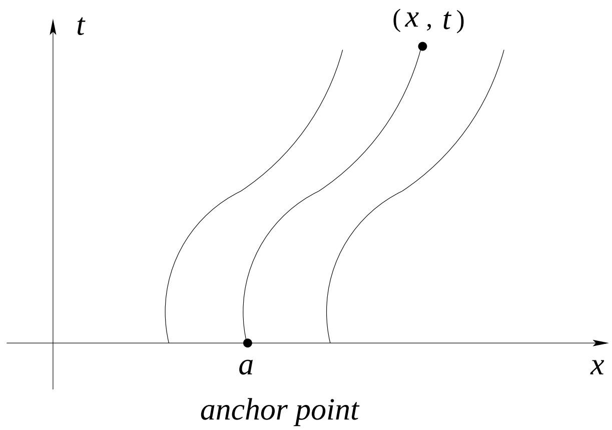


Figure 9.2: Schematic of a typical characteristic curve $x(t)$ and the anchor point. Notice that in general characteristic curves are not necessarily straight lines.

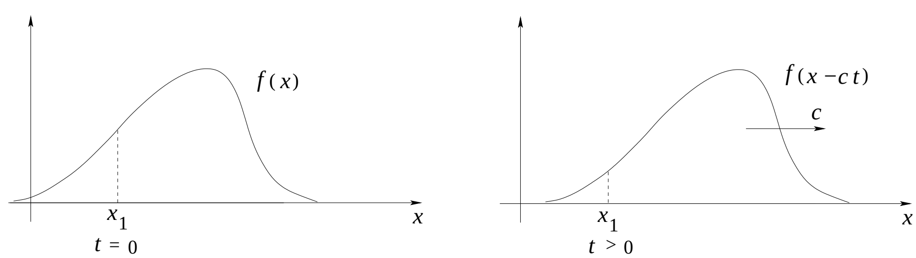


Figure 9.3: Schematic of a travelling wave that moves to the right with constant speed c.

Exercise 9.1.
Solve the following PDE for $z(x,t)$ on $-\infty < x < \infty$

$$z_t + 5z_x = 0, \qquad z(x,0) = e^{-x^2}.$$

Solution: The first characteristic equation is

$$\frac{dx(t)}{dt} = 5$$

which is solved by

$$x(t) = 5t + a, \quad \text{or} \quad a = x - 5t$$

The second characteristic equation is solved by

$$z(x(t),t) = f(a) = e^{-a^2} = e^{-(x-5t)^2}$$

Hence the solution is

$$z(x,t) = e^{-(x-5t)^2}.$$

Now we extend the method to allow for linear source or sink terms.

EXAMPLE 9.1. Solve the following PDE for $u(x,t)$ on $-\infty < x < \infty$:

$$u_t + \alpha u_x + \beta u = 0.$$

Again we look for solutions of the form $u(x(t),t)$. From the chain rule we get

$$\frac{d}{dt}u(x(t),t) = u_x x_t + u_t$$

Hence the characteristic equations become

$$\frac{dx(t)}{dt} = \alpha, \qquad \text{and} \quad \frac{d}{dt}u(x(t),t) = -\beta u(x(t),t).$$

These are solved as

$$x(t) = \alpha t + a, \quad \text{or } a = x - \alpha t$$

and

$$u(x(t),t) = u(x(0),0)e^{-\beta t}$$

Hence for each given (x,t) we find the corresponding anchor point $a = x - \alpha t$ and then the solution is given by

$$u(x,t) = u(a,0)e^{-\beta t} = f(x - \alpha t)e^{-\beta t}.$$

Exercise 9.2.
Solve the following PDE for $u(x,t)$ on $-\infty < x < \infty$:

$$u_t + \sqrt{3}u_x + \beta u = 0, \qquad u(x,0) = \sin^2(x).$$

Solution: The characteristic equations, their solutions and the anchor point are given by

$$\begin{aligned} \frac{dx(t)}{dt} &= \sqrt{3} \qquad & x(t) &= \sqrt{3}t + a, \qquad & a &= x - \sqrt{3}t \\ \frac{du(t)}{dt} &= 16u(t) \qquad & u(t) &= u(0)e^{16t} & & \end{aligned}$$

Then the solution is

$$u(x,t) = (\sin(x - \sqrt{3}t))^2 e^{16t}.$$

9.2 D'Alembert's solution from Method of Characteristics

The method of characteristics can also be used to derive the D'Alembert solution for the one-dimensional wave equation on the whole axis $-\infty < x < \infty$

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

We assume that the solution is twice continuously differentiable, such that $u_{xt} = u_{tx}$.

We will again take a more abstract view and do some algebra with differential operators, whereby we employ the binomial $a^2 - b^2 = (a+b)(a-b) = (a-b)(a+b)$. We can write the wave equation as

$$\begin{aligned} 0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u \\ &= \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u \qquad (9.1) \\ &= \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u \qquad (9.2) \end{aligned}$$

Now we introduce a new variable

$$z(x,t) = \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u(x,t).$$

Then equation (9.1) can be written as

$$0 = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \right) \; z(x,t) = \frac{\partial}{\partial t} z(x,t) - c\frac{\partial}{\partial x} z(x,t).$$

This is, in fact, the simplest first order equation which we used at the beginning of this chapter. Its solution has the form

$$z(x,t) = Q(x+ct)$$

for an appropriate function $Q(x)$.

Now we do the same with (9.2). We introduce

$$v(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \right) \; u(x,t).$$

and obtain

$$0 = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \right) \; v(x,t) = \frac{\partial}{\partial t} v(x,t) - c\frac{\partial}{\partial x} v(x,t).$$

Its solution has the form

$$v(x,t) = P(x-ct)$$

for an appropriate function $P(x)$. Notice that P and Q only depend on one variable:

$$Q(A), \quad A = x+ct, \qquad P(B), \quad B = x-ct.$$

We need anti derivatives of these functions in the following form. Let $G(A)$ and $F(B)$ satisfy

$$cG'(A) = \frac{1}{2}Q(A), \qquad -cF'(B) = \frac{1}{2}P(B).$$

Then

$$\frac{\partial G(x+ct)}{\partial t} = cG'(A) = \frac{1}{2}Q(A) = \frac{1}{2}Q(x+ct)$$

and

$$\frac{\partial F(x-ct)}{\partial t} = -cF'(B) = \frac{1}{2}P(B) = \frac{1}{2}P(x-ct).$$

So far we have

$$z(x,t) = u_t + cu_x = Q(x+ct)$$

and

$$v(x,t) = u_t - cu_x = P(x-ct).$$

Hence

$$u_t = \frac{1}{2}(Q(x+ct) + P(x-ct)) = \frac{\partial G}{\partial t} + \frac{\partial F}{\partial t}$$

which we integrate to

$$u(x,t) = G(x+ct) + F(x-ct) + c_1,$$

where c_1 is a constant of integration. Notice that F and G were defined as antiderivatives of Q and P. Hence they are only unique up to constants and we can absorb the constant c_1 into these functions. For example just define $\tilde{G} = G + \frac{c_1}{2}$ and $\tilde{F} = F + \frac{c_1}{2}$ and then remove the tilde $\tilde{}$. Hence the general solution reads:

$$u(x,t) = G(x+ct) + F(x-ct) \tag{9.3}$$

which is the sum of two traveling waves with speeds $\pm c$, moving in opposite directions.

To find F and G we need the initial conditions. At $t = 0$ we have for the initial displacement

$$f(x) = u(x,0) = G(x) + F(x) \tag{9.4}$$

For the initial velocity we find

$$g(x) = u_t(x,0) = cG'(x) - cF'(x)$$

We differentiate (9.4) once more to get a system for F' and G'

$$G'(x) + F'(x) = f'(x) \tag{9.5}$$
$$cG'(x) - cF'(x) = g(x) \tag{9.6}$$

Adding the equations (9.5) and (9.6) we obtain

$$G'(x) = \frac{1}{2}\left(f'(x) + \frac{g(x)}{c}\right)$$

which gives

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + k_1.$$

Subtracting the equations (9.5) and (9.6) we obtain

$$F'(x) = \frac{1}{2}\left(f'(x) - \frac{g(x)}{c}\right)$$

which gives

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds + k_2.$$

From (9.4) we observe that $k_1 + k_2 = 0$. Then from (9.3) we get the D'Alembert solution

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Part II

Explicitly Solved Problems

Chapter 10

Fourier Series Problems

Exercise 10.1.
Suppose that f is T-periodic and let F be an antiderivative of f, for example,

$$F(x) = \int_0^x f(t)\,dt, \quad -\infty < x < \infty.$$

Show that F is T-periodic if and only if the integral of f over any interval of length T is 0.

Solution: Note that

$$F(x+T) = \int_0^{x+T} f(t)\,dt = \int_0^x f(t)\,dt + \int_x^{x+T} f(t)\,dt = F(x) + \int_x^{x+T} f(t)\,dt$$

for all $x \in \mathbb{R}$. Therefore $F(x+T) = F(x)$ for all $x \in \mathbb{R}$ if and only if

$$\int_x^{x+T} f(t)\,dt = 0$$

for all $x \in \mathbb{R}$. This holds if and only if the integral of f over *any* interval of length T is 0. Since f is T-periodic, then F is T-periodic if and only if

$$\int_0^T f(t)\,dt = 0.$$

Exercise 10.2. ✗✗

Let $f(x) = x - 2\left[\frac{x+1}{2}\right]$, (where [ ] denotes the floor function, i.e. the greatest integer less or equal to the argument), and consider the function

$$h(x) = |f(x)| = \left|x - 2\left[\tfrac{x+1}{2}\right]\right|.$$

(a) Show that h is 2-periodic.

(b) Plot the graph of h.

(c) Generalize (a) by finding a closed formula that describes the $2a$-periodic triangular wave

$$g(x) = |x| \quad \text{if} \quad -a < x < a,$$

and

$$g(x+2a) = g(x) \quad \text{otherwise.}$$

Solution: Note that if we can show that

$$f(x) = x - 2\left[\tfrac{x+1}{2}\right]$$

is 2-periodic, then for any $x \in \mathbb{R}$, we have

$$h(x+2) = |f(x+2)| = |f(x)| = h(x)$$

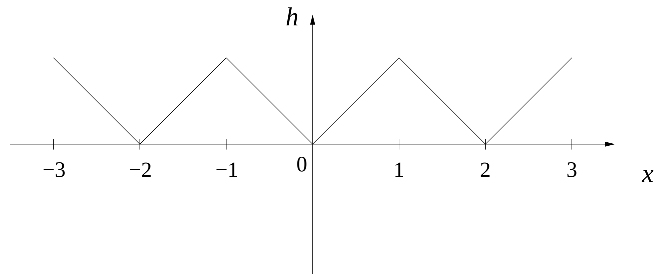
for all $x \in \mathbb{R}$, so that h is also 2-periodic.

(a) Now,

$$\begin{aligned} f(x+2) &= x + 2 - 2\left[\frac{(x+2)+1}{2}\right] \\ &= x + 2 - 2\left[\frac{x+1}{2} + 1\right] \\ &= x + 2 - 2\left(\left[\frac{x+1}{2}\right] + 1\right) \\ &= x - 2\left[\frac{x+1}{2}\right] \\ &= f(x) \end{aligned}$$

and f is 2-periodic, and from the remark above $h = |f|$ is also 2-periodic.

(b) Since $f(x) = x$ for $-1 < x < 1$, then $h(x) = |x|$ for $-1 < x < 1$, and the graph of h is shown below.



(c) In order to find a $2a$-periodic triangular wave, we use the $2a$-periodic function

$$f(x) = x - 2a\left[\frac{x+a}{2a}\right],$$

and note that $f(x) = x$ on the interval $-a < x < a$. We leave it to you to check, exactly as in part (a), that this is $2a$-periodic and that $f(x) = x$ for $-a < x < a$. Therefore,

$$g(x) = \left|x - 2a\left[\frac{x+a}{2a}\right]\right|$$

is a $2a$-periodic triangular wave which is equal to $|x|$ on the interval $-a < x < a$.

Exercise 10.3. ✗

Evaluate

$$\int_0^L \cos\tfrac{n\pi x}{L}\cos\tfrac{m\pi x}{L}\,dx$$

for $n \geqslant 0$, $m \geqslant 0$.

Use the trigonometric identity:

$$\cos a\cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

(consider $a - b = 0$ and $a + b = 0$ separately).

Solution: If $n > m \geqslant 0$, then from the addition formula for the cosine, we have

$$\begin{aligned}
\int_0^L \cos\tfrac{n\pi x}{L}\cos\tfrac{m\pi x}{L}\,dx &= \frac{1}{2}\int_0^L \left[\cos\tfrac{(n-m)\pi x}{L} + \cos\tfrac{(n+m)\pi x}{L}\right]dx \\
&= \frac{L}{2\pi(n-m)}\sin\tfrac{(n-m)\pi x}{L}\Big|_0^L + \frac{L}{2\pi(n+m)}\sin\tfrac{(n+m)\pi x}{L}\Big|_0^L \\
&= \frac{L}{2\pi(n-m)}\sin(n-m)\pi + \frac{L}{2\pi(n+m)}\sin(n+m)\pi \\
&= 0.
\end{aligned}$$

If $n = m > 0$, then

$$\int_0^L \cos\tfrac{n\pi x}{L}\cos\tfrac{m\pi x}{L}\,dx = \int_0^L \cos^2\tfrac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left(1+\cos\tfrac{2m\pi x}{L}\right)\,dx$$

$$= \frac{L}{2} + \frac{L}{2m\pi}\sin\tfrac{2m\pi x}{L}\Big|_0^L = \frac{L}{2}.$$

If $n = m = 0$, then

$$\int_0^L 1\cdot 1\,dx = L.$$

Exercise 10.4. ✗
Evaluate

$$\int_0^L \sin\tfrac{n\pi x}{L}\,\sin\tfrac{m\pi x}{L}\,dx$$

for $n > 0$, $m > 0$ and consider $n = m$ separately.
Use the trignometric identity: $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right].$

Solution: If m and n are positive integers with $m \neq n$, then

$$\int_0^L \sin\tfrac{n\pi x}{L}\,\sin\tfrac{m\pi x}{L}\,dx = \int_0^L \frac{1}{2}\left[\cos\tfrac{(n-m)\pi x}{L} - \cos\tfrac{(n+m)\pi x}{L}\right]\,dx$$

$$= \frac{L}{2(n-m)}\cdot\sin\tfrac{(n-m)\pi x}{L}\Big|_0^L - \frac{L}{2(n+m)}\cdot\sin\tfrac{(n+m)\pi x}{L}\Big|_0^L$$

$$= \frac{L}{2(n-m)}\left[\sin(n-m)\pi - \sin 0\right] - \frac{L}{2(n+m)}\left[\sin(n+m)\pi - \sin 0\right]$$

$$= 0,$$

while if $n = m > 0$, then

$$\int_0^L \sin\tfrac{n\pi x}{L}\,\sin\tfrac{n\pi x}{L}\,dx = \int_0^L \sin^2\tfrac{n\pi x}{L}\,dx$$

$$= \int_0^L \frac{1}{2}\left[1-\cos\tfrac{2n\pi x}{L}\right]\,dx$$

$$= \frac{1}{2}\cdot x\Big|_0^L - \frac{L}{4n\pi}\cdot\sin\tfrac{2n\pi x}{L}\Big|_0^L$$

$$= \frac{L}{2} - \frac{L}{4n\pi}\left[\sin 2n\pi - \sin 0\right]$$

$$= \frac{L}{2}.$$

Therefore

$$\int_0^L \sin\tfrac{n\pi x}{L}\,\sin\tfrac{m\pi x}{L}\,dx = \begin{cases} \dfrac{L}{2} & \text{for} \quad m = n > 0 \\ 0 & \text{for} \quad m > 0,\ n > 0,\ m \neq n. \end{cases}$$

Exercise 10.5. ✗
Compute the Fourier series of the 2π-periodic function f given by

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2, \\ 0 & \text{if} \quad \pi/2 < |x| < \pi, \\ -1 & \text{if} \quad -\pi/2 < x < 0. \end{cases}$$

For which values of x does the Fourier series for f converge? Sketch the graph of the Fourier series.

Solution: Note that f is an odd function on the interval $-\pi < x < \pi$, so that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0,$$

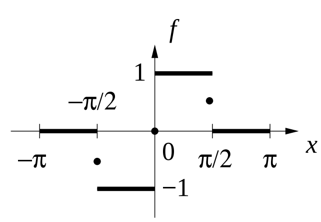
for $n = 1, 2, \ldots$.
We calculate the b_n,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi/2}\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2}\sin nx\,dx \\ &= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]\Bigg|_0^{\pi/2} \\ &= \frac{2}{n\pi}\left[1 - \cos\tfrac{n\pi}{2}\right] \end{aligned}$$

and the Fourier series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin nx.$$

The graph of the Fourier series of f on the interval $-\pi < x < \pi$ is shown below.



Note that the original function f is piecewise smooth and has only a finite jump discontinuity at $x = 0$ and $x = \pm\pi/2$, thus, from the Fourier series representation theorem, the Fourier series of f will converge to 0 at all points $x = 2n\pi,\ n = 0, \pm1, \pm2, \pm3, \ldots$, and for all points $(2n+1)\pi/2$, the Fourier series of f will converge to $(-1)^n\pi/2,\ n = 0, \pm1, \pm2, \pm3, \ldots$. The rest of the graph of the Fourier series can be obtained by translating this graph by an integer multiple of 2π in the x-direction.

Exercise 10.6. ✗✗

Compute the Fourier series of the 2π-periodic function given on $-\pi \leqslant x \leqslant \pi$ by $f(x) = |\cos x|$. For which values of x does the Fourier series for f converge? Sketch the graph of the Fourier series.

Solution: Note that f is even, since

$$f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

for all $x \in \mathbb{R}$, therefore $b_n = 0$ for all $n \geqslant 1$, and we only need to compute a_n for $n \geqslant 0$.
Now,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos x|\,dx = \frac{1}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{1}{\pi}\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx \\ &= \frac{1}{\pi}\sin x\Big|_0^{\pi/2} - \frac{1}{\pi}\sin x\Big|_{\pi/2}^{\pi} = \frac{1}{\pi} - (-1)\frac{1}{\pi} = \frac{2}{\pi}, \end{aligned}$$

and for $n \geqslant 1$, since $\cos nx$ is also an even function, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos x \cos nx\,dx. \end{aligned}$$

If $n = 1$, then

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^{\pi/2} \cos^2 x\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos^2 x\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\,dx \\ &= \frac{2}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = \frac{2}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 0. \end{aligned}$$

Now,

$$2\cos x \cos nx = \cos(n+1)x + \cos(n-1)x$$

so that for $n \neq 1$, we have

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx\,dx &= \frac{1}{\pi}\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x)\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right], \end{aligned}$$

and

$$\begin{aligned}\frac{2}{\pi}\int_{\pi/2}^{\pi}\cos x\cos nx\,dx &= \frac{1}{\pi}\int_{\pi/2}^{\pi}(\cos(n+1)x+\cos(n-1)x)\,dx\\ &= -\frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1}+\frac{\sin(n-1)\pi/2}{n-1}\right].\end{aligned}$$

For $n\neq 1$, we have

$$a_n=\frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1}+\frac{\sin(n-1)\pi/2}{n-1}\right],$$

and if n is odd, then $a_n=0$.
However, if n is even, say $n=2k$, then

$$\begin{aligned}a_{2k} &= \frac{2}{\pi}\left[\frac{\sin(2k+1)\pi/2}{2k+1}+\frac{\sin(2k-1)\pi/2}{2k-1}\right]\\ &= \frac{2}{\pi}\left[\frac{(-1)^k}{2k+1}-\frac{(-1)^k}{2k-1}\right]\\ &= \frac{4}{\pi}\frac{(-1)^k}{4k^2-1}.\end{aligned}$$

and the Fourier series is

$$a_0+\sum_{k=1}^{\infty}a_{2k}\cos 2kx=\frac{2}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos 2kx.$$

Since $f(-\pi)=f(\pi)$, then the piecewise smooth 2π-periodic function with $f(x)=|\cos x|$, $-\pi\leqslant x\leqslant\pi$ is continuous at each $x\in\mathbb{R}$, and therefore the Fourier series converges to $f(x)$ for each $x\in\mathbb{R}$.

Exercise 10.7. ✗
Consider the parabola $f(x)=x^2$ on $-a\leqslant x\leqslant a$ and show that the Fourier series of f is given by

$$\frac{a^2}{3}-\frac{4a^2}{\pi^2}\left[\cos(\pi x/a)-\frac{1}{2^2}\cos(2\pi x/a)+\frac{1}{3^2}\cos(3\pi x/a)-+\cdots\right].$$

Find its values at the points of discontinuity of f.

Solution: Note that since $f(a)=a^2=(-a)^2=f(-a)$, then the piecewise smooth $2a$-periodic function is continuous everywhere, and so has no points of discontinuity.
Also, since f is an even function, then $b_n=0$ for all $n\geqslant 1$, and the Fourier series for f has only cosine terms:

$$a_0+\sum_{n=1}^{\infty}a_n\cos(n\pi x/a)$$

where

$$a_0=\frac{1}{a}\int_0^a f(x)\,dx$$

and

$$a_n = \frac{2}{a}\int_0^a f(x)\cos(n\pi x/a)\;dx$$

for $n \geqslant 1$.
In order to calculate the coefficients a_n, we have

$$a_0 = \frac{1}{a}\int_0^a x^2\,dx = \frac{1}{a}\frac{x^3}{3}\Big|_0^a = \frac{a^2}{3}.$$

For $n \geqslant 1$, we integrate by parts twice to get

$$\begin{aligned} a_n &= \frac{2}{a}\int_0^a x^2\cos(n\pi x/a)\;dx = \frac{2}{a}\left[\frac{a}{n\pi}x^2\sin(n\pi x/a)\;\Big|_0^a - \frac{2a}{n\pi}\int_0^a x\sin(n\pi x/a)\;dx\right] \\ &= \frac{4}{n\pi}\int_0^a x\sin(n\pi x/a)\;dx = \frac{4}{n\pi}\left[-\frac{a}{n\pi}x\cos(n\pi x/a)\;\Big|_0^a + \frac{a}{n\pi}\int_0^a \cos(n\pi x/a)\;dx\right] \\ &= \frac{4a^2}{n^2\pi^2}(-1)^n \end{aligned}$$

for $n = 1, 2, 3, \ldots$. The Fourier series of f is

$$\frac{a^2}{3} - \frac{4a^2}{\pi^2}\left[\cos(\pi x/a) - \frac{1}{2^2}\cos(2\pi x/a) + \frac{1}{3^2}\cos(3\pi x/a) - + \cdots\right],$$

and since f is piecewise smooth and continuous everywhere, the Fourier series given above converges to $f(x)$ for each $x \in \mathbb{R}$.

Exercise 10.8. ✗
Consider the $2a$-periodic function f that is given on the interval $-a < x < a$ by $f(x) = x$. Show that the Fourier series of f is given by

$$\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/a)$$

by differentiating the Fourier series in the previous problem (Exercise 10.7) term by term. Justify your work.

Solution: Since the $2a$-periodic function $F(x)$ in the previous problem is piecewise smooth and continuous everywhere, the Fourier series converges to the function everywhere, and

$$F(x) = \frac{a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x/a)$$

where $F(x) = x^2$ for $-a < x < a$. Since this function also has a piecewise smooth derivative, and

$$F'(x) = 2x = 2\cdot f(x)$$

for $-a < x < a$, then the coefficients in the Fourier series of $F'(x)$ can be obtained by differentiating the above series term-by-term. Therefore, the Fourier series of $F'(x)$ is given by

$$\frac{4a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\pi}{n^2 a}\sin(n\pi x/a) = \frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/a),$$

and the Fourier series of $f(x)$ is

$$\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/a),$$

which converges to $f(x)$ for all $x \neq \pm na$, and to 0 for $x = \pm na$.

Exercise 10.9. ✗✗

Obtain the expansion

$$e^{ax} = \frac{\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

valid for all real numbers $a \neq 0$, and all $-\pi < x < \pi$.

Solution: Note, the solution can be obtained using the standard Fourier series of e^{ax}. However, it turns out that it is easier to use the (equivalent) complex form of the Fourier series, which we do here. If $f(x)$ is a 2π-periodic piecewise smooth function, the complex form of the Fourier series of $f(x)$ is

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where the Fourier coefficients are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt.$$

Here the N^{th} partial sum

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

is the same as the usual partial sum (check this).
Now, if $f(x) = e^{ax}$ for $-\pi < x < \pi$, then

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx = \frac{1}{2\pi}\frac{e^{(a-in)x}}{a-in}\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\frac{e^{(a-in)\pi}-e^{-(a-in)\pi}}{a-in} = \frac{1}{2\pi}\frac{e^{a}(-1)^n - e^{-a}(-1)^n}{a-in} \\ &= \frac{(-1)^n\sinh \pi a}{\pi(a-in)} = \frac{(-1)^n(a+in)\sinh \pi a}{\pi(a^2+n^2)}, \end{aligned}$$

and the Fourier series of f is

$$\frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{(a^2+n^2)} e^{inx} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} (a \cos nx - n \sin nx),$$

where we used the fact that

$$(a+in)e^{inx} = (a+in)(\cos nx + i \sin nx) = (a \cos nx - n \sin nx) + i(a \sin nx + n \cos nx),$$

and the fact that the Fourier series of a real valued function is real valued, so that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} (a \sin nx + n \cos nx) = 0.$$

Since the function f is piecewise smooth and is continuous for $-\pi < x < \pi$, then we have

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} (a \cos nx - n \sin nx)$$

for $-\pi < x < \pi$.

Exercise 10.10. ✗✗
For any complex number $z \in \mathbb{C}$ with $z \neq 1$ show the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

and then use it to derive *Lagrange's trigonometric identity*:

$$1 + \cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\,\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Use the fact that $\cos n\theta = Re\left(e^{in\theta}\right)$.

Solution: If $z \neq 1$, then

$$\begin{aligned}(1-z)(1 + z + z^2 + \cdots z^n) &= 1 + z + z^2 + \cdots + z^n - (z + z^2 + \cdots + z^{n+1}) \\ &= 1 - z^{n+1},\end{aligned}$$

so that

$$1 + z + z^2 + \cdots z^n = \begin{cases} \dfrac{1 - z^{n+1}}{1 - z} & \text{if } z \neq 1 \\ n+1 & \text{if } z = 1. \end{cases}$$

Taking $z = e^{i\theta}$, where $0 < \theta < 2\pi$, then $z \neq 1$, so that

$$\begin{aligned}1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} &= \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} \\ &= \frac{1 - e^{(n+1)i\theta}}{-e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)} \\ &= \frac{-e^{-i\theta/2}\left(1 - e^{(n+1)i\theta}\right)}{2i\sin(\theta/2)} \\ &= \frac{i\left(e^{-i\theta/2} - e^{(n+\frac{1}{2})i\theta}\right)}{2\sin(\theta/2)} \\ &= \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)} + \frac{i}{2\sin(\theta/2)}\left(\cos(\theta/2) - \cos\left(n+\tfrac{1}{2}\right)\theta\right)\end{aligned}$$

Equating real and imaginary parts, we have

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)}$$

for $0 < \theta < 2\pi$, and as an added bonus,

$$\sin\theta + \sin 2\theta + \cdot + \sin n\theta = \frac{1}{2}\cot(\theta/2) - \frac{\cos\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)}$$

for $0 < \theta < 2\pi$.

Exercise 10.11. ✗

Let $f(x) = \cosh x, \quad -\pi \leqslant x \leqslant \pi, \quad f(x+2\pi) = f(x)$.

(a) Find the Fourier series of f.

Note: $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

(b) For which values of $x \in [-\pi, \pi]$ does the Fourier series of f converge to $f(x)$?

(c) Evaluate the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ using part (b).

Solution:

(a) Writing $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, the coefficients in the Fourier series are computed as follows: since $f(x) = \cosh x$ is an even function on the interval $[-\pi, \pi]$, then $b_n = 0$ for all $n \geqslant 1$, and

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh x\, dx = \frac{1}{2\pi}\sinh x\Big|_{-\pi}^{\pi} = \frac{\sinh\pi}{\pi}$$

while for $n \geqslant 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh x \cos nx \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^x + e^{-x}\right)\cos nx \, dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx \, dx \\
&= \frac{1}{2\pi}\left\{\frac{e^x(\cos nx + n\sin nx)}{n^2+1}\right\}\Bigg|_{-\pi}^{\pi} + \frac{1}{2\pi}\left\{\frac{e^{-x}(-\cos nx + n\sin nx)}{n^2+1}\right\}\Bigg|_{-\pi}^{\pi} \\
&= \frac{2(-1)^n}{n^2+1}\,\frac{\sinh \pi}{\pi},
\end{aligned}$$

where we integrated by parts twice to get

$$\int e^x \cos nx \, dx = \frac{e^x(\cos nx + n\sin nx)}{n^2+1} \qquad \text{and} \qquad \int e^{-x}\cos nx \, dx = \frac{e^{-x}(-\cos nx + n\sin nx)}{n^2+1}.$$

The Fourier series for f is therefore

$$\cosh x \sim 2\frac{\sinh \pi}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx\right\}$$

for $-\infty < x < \infty$.

(b) From Dirichlet's theorem, the Fourier series converges to $f(x)$ for all $x \in [-\pi,\pi]$, since f is piecewise smooth on $[-\pi,\pi]$, continuous at each $x \in [-\pi,\pi]$, and $f(-\pi^+) = f(\pi^-)$.

(c) Again, from Dirichlet's theorem, the Fourier series converges to $\cosh \pi$ at $x = \pi$, and so

$$\cosh \pi = 2\frac{\sinh \pi}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos n\pi\right\},$$

therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{\pi\coth\pi - 1}{2}.$$

Exercise 10.12. ✗

Sketch the Fourier series of $f(x)$ and determine the Fourier coefficients for the following functions defined on the interval $-L \leqslant x \leqslant L$,

(a) $f(x) = \begin{cases} 1 & \text{for} \quad |x| < L/2 \\ 0 & \text{for} \quad |x| > L/2 \end{cases}$

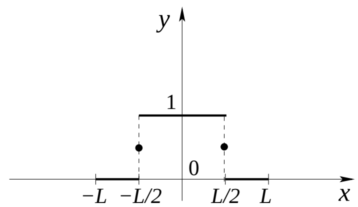
(b) $f(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{if } -L < x < 0 \end{cases}$

Solution:

(a) From Dirichlet's theorem the Fourier series of $f(x)$ converges to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right].$$

The graph of the Fourier series of $f(x)$ on the interval $-L \leqslant x \leqslant L$ is shown below.



Since $f(x)$ is an even piecewise smooth function on the interval $[-L, L]$, it has a Fourier series representation of the form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{\frac{L}{2}} 1\,dx = \frac{1}{2},$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos \tfrac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^{\frac{L}{2}} \cos \tfrac{n\pi x}{L}\,dx = \frac{2}{n\pi}\sin \tfrac{n\pi x}{L}\Big|_0^{\frac{L}{2}} = \frac{2}{n\pi}\sin \tfrac{n\pi}{2}$$

for $n \geqslant 1$, that is,

$$a_n = \begin{cases} \dfrac{2(-1)^k}{\pi(2k+1)}, & \text{if } n = 2k+1 \text{ is odd} \\ 0, & \text{if } n = 2k \text{ is even.} \end{cases}$$

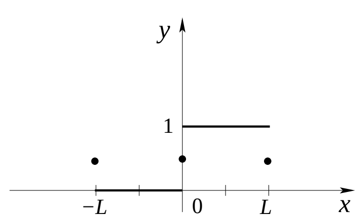
Hence, the Fourier series of $f(x)$ is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \cos \tfrac{(2k+1)\pi x}{L}.$$

(b) Again, from Dirichlet's theorem the Fourier series of $f(x)$ converges to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right].$$

The graph of the Fourier series of $f(x)$ on the interval $-L \leqslant x \leqslant L$ is shown below.



We could compute the Fourier series directly, but instead we observe that $f(x) - \frac{1}{2}$ is an odd piecewise smooth function on the interval $[-L, L]$. Hence it has a Fourier series representation of the form

$$f(x) - \frac{1}{2} \sim \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L},$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \left(f(x) - \frac{1}{2}\right) \sin \tfrac{n\pi x}{L}\, dx = \frac{1}{L}\int_0^L \sin \tfrac{n\pi x}{L}\, dx \\ &= -\frac{1}{n\pi} \cos \tfrac{n\pi x}{L}\bigg|_0^L = -\frac{1}{n\pi}(\cos n\pi - 1) \\ &= \frac{1}{n\pi}[1 - (-1)^n] \end{aligned}$$

for $n \geqslant 1$, that is,

$$b_n = \begin{cases} \dfrac{2}{\pi(2k+1)}, & \text{if } n = 2k+1 \text{ is odd} \\ 0, & \text{if } n = 2k \text{ is even.} \end{cases}$$

Hence the Fourier series of $f(x)$ is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1} \sin \tfrac{(2k+1)\pi x}{L}$$

Exercise 10.13. ✗
Show that finding the Fourier series operation is a linear operation, that is, the Fourier series of

$$c_1 f(x) + c_2 g(x)$$

is the sum of c_1 times the Fourier series of $f(x)$ plus c_2 times the Fourier series of $g(x)$.

Solution: Suppose that the Fourier series of f and g are given by

$$f(x) \sim A_0 + \sum_{n=1}^{\infty}\left(A_n \cos \tfrac{n\pi x}{L} + B_n \sin \tfrac{n\pi x}{L}\right) \qquad \text{and} \qquad g(x) \sim C_0 + \sum_{n=1}^{\infty}\left(C_n \cos \tfrac{n\pi x}{L} + D_n \sin \tfrac{n\pi x}{L}\right)$$

where

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, \quad A_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \tfrac{n\pi x}{L}\, dx, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$, and

$$C_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\, dx, \quad C_n = \frac{1}{L}\int_{-L}^{L} g(x) \cos \tfrac{n\pi x}{L}\, dx, \quad D_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.
If c_1 and c_2 are scalars, and the Fourier series of $c_1 f + c_2 g$ is

$$c_1 f(x) + c_2 g(x) \sim E_0 + \sum_{n=1}^{\infty} \left(E_n \cos \tfrac{n\pi x}{L} + F_n \sin \tfrac{n\pi x}{L} \right),$$

then

$$E_0 = \frac{1}{2L} \int_{-L}^{L} [c_1 f(x) + c_2 g(x)]\, dx = \frac{c_1}{2L} \int_{-L}^{L} f(x)\, dx + \frac{c_2}{2L} \int_{-L}^{L} g(x)\, dx = c_1 A_0 + c_2 C_0.$$

Also,

$$\begin{aligned} E_n &= \frac{1}{L} \int_{-L}^{L} [c_1 f(x) + c_2 g(x)] \cos \tfrac{n\pi x}{L}\, dx \\ &= \frac{c_1}{L} \int_{-L}^{L} f(x) \cos \tfrac{n\pi x}{L}\, dx + \frac{c_2}{L} \int_{-L}^{L} g(x) \cos \tfrac{n\pi x}{L}\, dx \\ &= c_1 A_n + c_2 C_n \end{aligned}$$

for $n \geqslant 1$. Similarly,

$$\begin{aligned} F_n &= \frac{1}{L} \int_{-L}^{L} [c_1 f(x) + c_2 g(x)] \sin \tfrac{n\pi x}{L}\, dx \\ &= \frac{c_1}{L} \int_{-L}^{L} f(x) \sin \tfrac{n\pi x}{L}\, dx + \frac{c_2}{L} \int_{-L}^{L} g(x) \sin \tfrac{n\pi x}{L}\, dx \\ &= c_1 B_n + c_2 D_n \end{aligned}$$

for $n \geqslant 1$. Therefore the Fourier series for $c_1 f + c_2 g$ is

$$\begin{aligned} c_1 f(x) + c_2 g(x) &\sim c_1 A_0 + c_2 C_0 + \sum_{n=1}^{\infty} \left[(c_1 A_n + c_2 C_n) \cos \tfrac{n\pi x}{L} + (c_1 B_n + c_2 D_n) \sin \tfrac{n\pi x}{L} \right] \\ &= c_1 \left[A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \tfrac{n\pi x}{L} + B_n \sin \tfrac{n\pi x}{L} \right) \right] + c_2 \left[C_0 + \sum_{n=1}^{\infty} \left(C_n \cos \tfrac{n\pi x}{L} + D_n \sin \tfrac{n\pi x}{L} \right) \right] \\ &\sim c_1 f(x) + c_2 g(x). \end{aligned}$$

Exercise 10.14.
Show that e^x is the sum of an even function and an odd function.

Solution: We can write

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x,$$

and $\cosh x$ is an even function while $\sinh x$ is an odd function.
In general, if $f(x)$ is an arbitrary function, then we can write

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

where

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

is even, and

$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

is odd.

Exercise 10.15. ✗
Given the function

$$f(x) = \cos \tfrac{\pi x}{a}, \quad 0 \leqslant x < a$$

find the Fourier sine series for f.

Solution: Writing

$$f(x) = \cos \tfrac{\pi x}{a} \sim \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{a},$$

the coefficients b_n in the Fourier sine series are computed as follows:

$$\begin{aligned}
b_n &= \frac{2}{a}\int_0^a \cos \tfrac{\pi x}{a} \sin \tfrac{n\pi x}{a}\, dx = \frac{1}{a}\int_0^a \left(\sin \tfrac{(n+1)\pi x}{a} + \sin \tfrac{(n-1)\pi x}{a} \right) dx \\
&= \frac{1}{\pi}\left(-\frac{1}{n+1} \cos \tfrac{(n+1)\pi x}{a} \Big|_0^a \right) + \frac{1}{\pi}\left(-\frac{1}{n-1} \cos \tfrac{(n-1)\pi x}{a} \Big|_0^a \right) \\
&= \frac{1}{\pi(n+1)}\left((-1)^n + 1\right) + \frac{1}{\pi(n-1)}\left((-1)^n + 1\right) = \frac{1 + (-1)^n}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1} \right) \\
&= \frac{1 + (-1)^n}{\pi} \, \frac{2n}{n^2 - 1}.
\end{aligned}$$

Therefore,

$$b_n = \begin{cases} \dfrac{4n}{\pi(n^2 - 1)} & \text{if} \quad n \quad \text{is even} \\ 0 & \text{if} \quad n \quad \text{is odd}, \quad n \geqslant 3. \end{cases}$$

For $n = 1$,

$$b_1 = \frac{2}{a}\int_0^a \sin \tfrac{\pi x}{a} \cos \tfrac{\pi x}{a}\, dx = \frac{1}{a} \sin^2 \tfrac{\pi x}{a} \Big|_0^a = 0.$$

The Fourier sine series for f is therefore

$$\cos \tfrac{\pi x}{a} \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin \tfrac{2n\pi x}{a}.$$

for $0 \leqslant x < a$.

Exercise 10.16. ✗✗

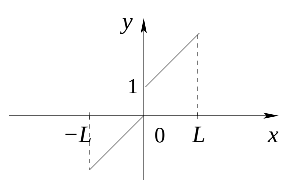
For each of the following functions find the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$, and sketch the appropriate extensions of the functions and all the series involved:

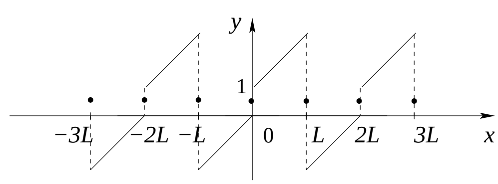
(a) $f(x) = \begin{cases} x & -L < x < 0 \\ 1 + x & 0 < x < L \end{cases}$

(b) $f(x) = \begin{cases} 2, & -L < x < 0 \\ e^{-x} & 0 < x < L \end{cases}$

Solution:

(a) *Fourier Series*: The graphs of $f(x)$ and the Fourier series of $f(x)$ are shown below.





The dfxFourier series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{2L} \int_{-L}^{L} x\, dx + \frac{1}{2L} \int_{0}^{L} 1\, dx = \frac{1}{2L}\, L = \frac{1}{2},$$

since the function x is an odd function on $[-L, L]$.

For $n \geqslant 1$,

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \tfrac{n\pi x}{L}\, dx = \frac{1}{L} \int_{-L}^{L} x \cos \tfrac{n\pi x}{L}\, dx + \frac{1}{L} \int_{0}^{L} \cos \tfrac{n\pi x}{L}\, dx \\ &= \frac{1}{n\pi} \sin \tfrac{n\pi x}{L} \Big|_{0}^{L} = 0, \end{aligned}$$

since $x \cos \frac{n\pi x}{L}$ is an odd function on $[-L, L]$, and

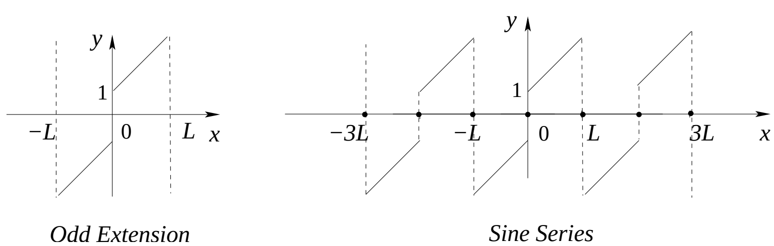
$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{L} x\sin\tfrac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L}\sin\tfrac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\int_{0}^{L} x\sin\tfrac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L}\sin\tfrac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\left[-\frac{L}{n\pi}x\cos\tfrac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\tfrac{n\pi x}{L}\,dx\right] + \frac{1}{L}\left(-\frac{L}{n\pi}\cos\tfrac{n\pi x}{L}\right)\Big|_0^L \\ &= \frac{1}{n\pi}\left[1-(-1)^n\right] - \frac{2L(-1)^n}{n\pi}. \end{aligned}$$

Hence the Fourier series of $f(x)$ is

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n] - 2L(-1)^n}{n}\sin\tfrac{n\pi x}{L}$$

which is a sine series plus a constant term.

Fourier Sine Series: The graphs of the odd extension of $f(x)$ to the interval $[-L, L]$ and the Fourier sine series of $f(x)$ are shown below.



The Fourier sine series representation of $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\tfrac{n\pi x}{L},$$

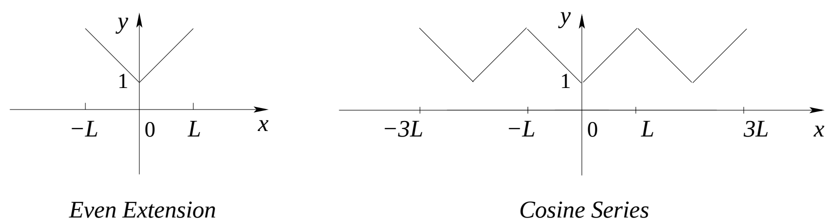
where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\tfrac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L (1+x)\sin\tfrac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\left[-\frac{L}{n\pi}(1+x)\cos\tfrac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L\cos\tfrac{n\pi x}{L}\,dx\right] \\ &= \frac{2}{n\pi}\left[1-(-1)^n\right] - \frac{2L(-1)^n}{n\pi}. \end{aligned}$$

Hence the Fourier sine series of $f(x)$ is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]-L(-1)^n}{n}\sin\tfrac{n\pi x}{L}.$$

Fourier Cosine Series: The graphs of the even extension of $f(x)$ to the interval $[-L, L]$ and the Fourier cosine series of $f(x)$ are shown below.



The Fourier cosine series representation of $f(x)$ is

$$f(x)\sim a_0+\sum_{n=1}^{\infty}a_n\cos\tfrac{n\pi x}{L},$$

where

$$a_0=\frac{1}{L}\int_0^L f(x)\,dx=\frac{1}{L}\int_0^L(1+x)\,dx=\frac{1}{L}\left[x\Big|_0^L+\frac{x^2}{2}\Big|_0^L\right]=1+\frac{L}{2},$$

and for $n\geqslant 1$,

$$\begin{aligned}a_n&=\frac{2}{L}\int_0^L f(x)\cos\tfrac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L(1+x)\cos\tfrac{n\pi x}{L}\,dx\\&=\frac{2}{L}\left[\frac{L}{n\pi}(1+x)\sin\tfrac{n\pi x}{L}\Big|_0^L-\frac{L}{n\pi}\int_0^L\sin\tfrac{n\pi x}{L}\,dx\right]\\&=\frac{2L}{n^2\pi^2}\cos\tfrac{n\pi x}{L}\Big|_0^L=\frac{2L}{n^2\pi^2}\left[(-1)^n-1\right],\end{aligned}$$

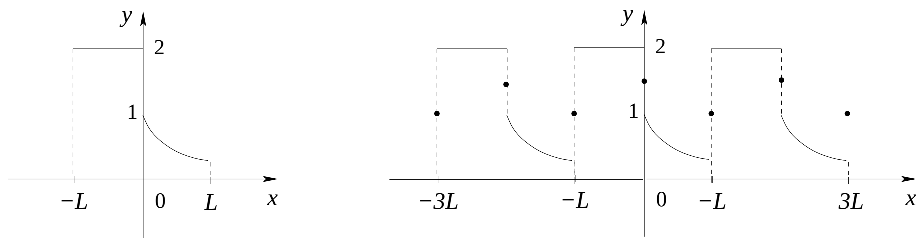
that is,

$$a_n=\begin{cases}-\dfrac{4L}{\pi^2(2k+1)^2}, & \text{if } n=2k+1 \text{ is odd}\\ 0, & \text{if } n=2k \text{ is even.}\end{cases}$$

Hence, the Fourier cosine series of $f(x)$ is

$$1+\frac{L}{2}-\frac{4L}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\tfrac{(2k+1)\pi x}{L}.$$

(b) *Fourier Series*: The graphs of $f(x)$ and the Fourier series of $f(x)$ are shown below.



The Fourier series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0} 2\,dx + \frac{1}{2L}\int_{0}^{L} e^{-x}\,dx = \frac{1}{2L}\left(2L + 1 - e^{-L}\right).$$

Using integration by parts twice, we obtain

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right),$$

so that

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\tfrac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} 2\cos\tfrac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} e^{-x}\cos\tfrac{n\pi x}{L}\,dx \\ &= \frac{2}{n\pi}\sin\tfrac{n\pi x}{L}\Big|_{-L}^{0} + \frac{n\pi}{L^2+n^2\pi^2}e^{-x}\sin\tfrac{n\pi x}{L}\Big|_{0}^{L} - \frac{L}{L^2+n^2\pi^2}e^{-x}\cos\tfrac{n\pi x}{L}\Big|_{0}^{L} \\ &= \frac{L}{L^2+n^2\pi^2}\left[1 - e^{-L}(-1)^n\right] \end{aligned}$$

for $n \geqslant 1$.

Similarly, we have

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right),$$

so that

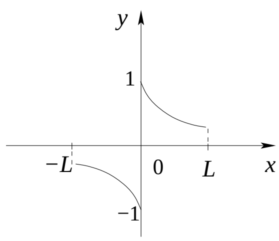
$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} 2\sin\tfrac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} e^{-x}\sin\tfrac{n\pi x}{L}\,dx \\ &= -\frac{2}{n\pi}\cos\tfrac{n\pi x}{L}\Big|_{-L}^{0} - \frac{L}{L^2+n^2\pi^2}e^{-x}\sin\tfrac{n\pi x}{L}\Big|_{0}^{L} - \frac{n\pi}{L^2+n^2\pi^2}e^{-x}\cos\tfrac{n\pi x}{L}\Big|_{0}^{L} \\ &= \frac{n\pi}{L^2+n^2\pi^2}\left[1 - e^{-L}(-1)^n\right] \end{aligned}$$

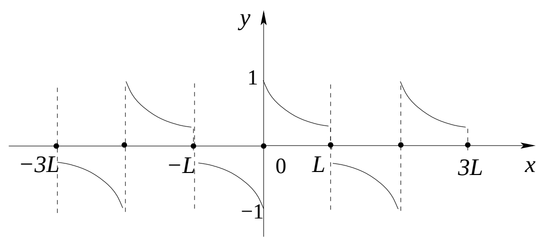
for $n \geqslant 1$.

Hence the Fourier series of $f(x)$ is

$$\frac{2L+1-e^{-L}}{2L}+\sum_{n=1}^{\infty}\frac{1-e^{-L}(-1)^n}{L^2+n^2\pi^2}\left[L\cos\tfrac{n\pi x}{L}+n\pi\sin\tfrac{n\pi x}{L}\right].$$

Fourier Sine Series: The graphs of the odd extension of $f(x)$ to the interval $[-L, L]$ and the Fourier sine series of $f(x)$ are shown below.





The Fourier sine series representation of $f(x)$ is

$$f(x)\sim\sum_{n=1}^{\infty}b_n\sin\tfrac{n\pi x}{L},$$

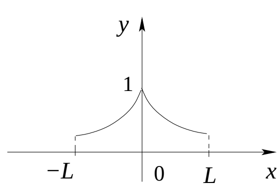
where

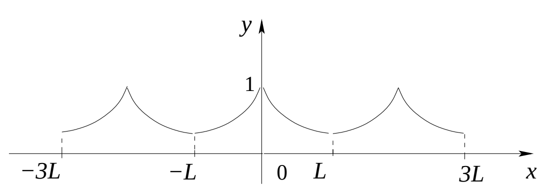
$$\begin{aligned}b_n&=\frac{2}{L}\int_0^L f(x)\sin\tfrac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L e^{-x}\sin\tfrac{n\pi x}{L}\,dx\\&=-\frac{2L}{L^2+n^2\pi^2}e^{-x}\sin\tfrac{n\pi x}{L}\bigg|_0^L-\frac{2n\pi}{L^2+n^2\pi^2}e^{-x}\cos\tfrac{n\pi x}{L}\bigg|_0^L\\&=\frac{2n\pi}{L^2+n^2\pi^2}\left[1-e^{-L}(-1)^n\right]\end{aligned}$$

for $n \geqslant 1$, and the Fourier sine series of $f(x)$ is

$$2\pi\sum_{n=1}^{\infty}\frac{1-e^{-L}(-1)^n}{L^2+n^2\pi^2}\,n\sin\tfrac{n\pi x}{L}.$$

Fourier Cosine Series: The graphs of the even extension of $f(x)$ to the interval $[-L, L]$ and the Fourier cosine series of $f(x)$ are shown below.





The Fourier cosine series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L e^{-x}\,dx = -\frac{1}{L}e^{-x}\bigg|_0^L = \frac{1}{L}\left(1-e^{-L}\right).$$

Since

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right),$$

then

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos \tfrac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L e^{-x}\cos \tfrac{n\pi x}{L}\,dx \\ &= \frac{2n\pi}{L^2+n^2\pi^2}e^{-x}\sin \tfrac{n\pi x}{L}\bigg|_0^L - \frac{2L}{L^2+n^2\pi^2}e^{-x}\cos \tfrac{n\pi x}{L}\bigg|_0^L \\ &= \frac{2L}{L^2+n^2\pi^2}\left[1-e^{-L}(-1)^n\right] \end{aligned}$$

for $n \geqslant 1$, and the Fourier cosine series of $f(x)$ is

$$\frac{1-e^{-L}}{L} + 2L\sum_{n=1}^{\infty}\frac{1-e^{-L}(-1)^n}{L^2+n^2\pi^2}\cos \tfrac{n\pi x}{L}.$$

Exercise 10.17. ✗✗✗

Consider the integral $\displaystyle\int_0^1 \frac{dx}{1+x^2}$.

(a) Evaluate the integral explicitly.

(b) Use the Taylor series of $\dfrac{1}{1+x^2}$ (a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for π.

Solution:

(a) Since

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2},$$

we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\bigg|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

(b) Recall that the geometric series

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots ,$$

that is,

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

converges for all $-1 < t < 1$.

Integrating from 0 to x, where $|x| < 1$, we get

$$\begin{aligned}
\int_0^x \frac{1}{1+t^2}\, dt &= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \bigg|_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},
\end{aligned}$$

and therefore

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

for $-1 < x < 1$, and this is *Gregory's series* for $\tan^{-1} x$, discovered by James Gregory about 1670.

Letting $x \to 1^-$, then a theorem of Abel tells us that

$$\begin{aligned}
\frac{\pi}{4} &= \int_0^1 \frac{1}{1+t^2}\, dt = \lim_{x \to 1^-} \int_0^x \frac{1}{1+t^2}\, dt = \lim_{x \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \lim_{x \to 1^-} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},
\end{aligned}$$

so that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots ,$$

this is *Leibniz's formula* for $\frac{\pi}{4}$, discovered by Leibniz in 1673.

(c) From part (b), we have

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

The convergence is very slow however.

Another proof of Leibniz's formula which doesn't require integrating an infinite series term-by-term is given below.

$$\begin{aligned} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} &= \int_0^1 \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}\right) dx \\ &= \int_0^1 \frac{1 - x^{2n}}{1 + x^2}\, dx \\ &= \int_0^1 \frac{1}{1+x^2}\, dx - \int_0^1 \frac{x^{2n}}{1+x^2}\, dx \end{aligned}$$

and therefore,

$$\left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1}\right)\right| = \int_0^1 \frac{x^{2n}}{1+x^2}\, dx \leqslant \int_0^1 x^{2n}\, dx = \frac{1}{2n+1} \longrightarrow 0$$

as $n \to \infty$.

Exercise 10.18. ✗✗

Assume the function $f(x)$ is continuous on $[-L, L]$.

(a) Under what conditions does $f(x)$ equal its Fourier series for all $x \in [-L, L]$?

(b) Under what conditions does $f(x)$ equal its Fourier sine series for all $x \in [0, L]$?

(c) Under what conditions does $f(x)$ equal its Fourier cosine series for all $x \in [0, L]$?

Hint: What does the Fourier series converge to at the end points of the interval?

(a) From Dirichlet's theorem, we know that for any x_0 with $-L < x_0 < L$, the Fourier series of f converges to $f(x_0)$ since f is continuous at x_0.

We also know that at the endpoints $x = -L$ and $x = L$, the Fourier series converges to

$$\frac{1}{2}\left[f(L^-) + f(-L^+)\right],$$

and if f is continuous at the endpoints, that is, continuous from the left at $x = L$ and continuous from the right at $x = -L$, then the Fourier series converges to

$$\frac{f(L) + f(-L)}{2}$$

at $x = L$ and at $x = -L$, so that the Fourier series converges to $f(x)$ for all $x \in [-L, L]$ if and only if $f(L) = f(-L)$.

(b) Again, from Dirichlet's theorem, if $0 < x_0 < L$, then the Fourier sine series of f converges to $f(x_0)$ since f is continuous at x_0.

If f_{odd} is the odd extension of f to $[-L, L]$, then at $x = 0$, the Fourier sine series of f converges to

$$\frac{1}{2}\left[f_{\text{odd}}(0^-) + f_{\text{odd}}(0^+)\right] = \frac{1}{2}\left[-f(0) + f(0)\right] = 0,$$

and the Fourier sine series converges to f at $x = 0$ if and only if $f(0) = 0$.

If f_{odd} is the odd extension of f to $[-L, L]$, then at $x = L$, the Fourier sine series of f converges to

$$\frac{1}{2}\left[f_{\text{odd}}(L^-) + f_{\text{odd}}(-L^+)\right] = \frac{1}{2}\left[f_{\text{odd}}(L) + f_{\text{odd}}(-L)\right] = \frac{1}{2}\left[f(L) - f(L)\right] = 0,$$

and the Fourier sine series converges to f at $x = L$ if and only if $f(L) = 0$.

(c) From Dirichlet's theorem, if $0, x_0 < L$, then the Fourier cosine series of f converges to $f(x_0)$ since f is continuous at x_0.

If f_{even} is the even extension of f to $[-L, L]$, then at $x = 0$, the Fourier cosine series of f converges to

$$\frac{1}{2}\left[f_{\text{even}}(0^-) + f_{\text{even}}(0^+)\right] = \frac{1}{2}\left[f(0) + f(0)\right] = f(0),$$

and the Fourier cosine series of f converges to f at $x = 0$ if and only if f is continuous from the right at $x = 0$.

If f_{even} is the even extension of f to $[-L, L]$, then at $x = L$, the Fourier cosine series of f converges to

$$\frac{1}{2}\left[f_{\text{even}}(L^-) + f_{\text{even}}(-L^+)\right] = \frac{1}{2}\left[f_{\text{even}}(L) + f_{\text{even}}(-L)\right] \frac{1}{2}\left[f(L) + f(L)\right] = f(L),$$

and the Fourier cosine series of f converges to f at $x = L$ if and only if f is continuous from the left at $x = L$.

Chapter 11

Heat Equation Problems

Exercise 11.1.
For each of the initial value–boundary value problems below, determine whether or not an equilibrium temperature distribution exists and find the values of β for which an equilibrium solution exists.

(a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \qquad \dfrac{\partial u}{\partial x}(0,t) = 1, \qquad \dfrac{\partial u}{\partial x}(L,t) = \beta$

(b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \qquad \dfrac{\partial u}{\partial x}(0,t) = 1, \qquad \dfrac{\partial u}{\partial x}(L,t) = \beta$

(c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x - \beta, \qquad \dfrac{\partial u}{\partial x}(0,t) = 0, \qquad \dfrac{\partial u}{\partial x}(L,t) = 0$

Solution: If the temperature has reached an equilibrium distribution, then u no longer depends on the time t, so that $\dfrac{\partial u}{\partial t} = 0$ and $u = \phi(x)$ is a function of x alone.

(a) In this case the boundary value problem for the equilibrium temperature distribution is

$$\begin{aligned} \phi''(x) + 1 = 0, &\quad 0 \leqslant x \leqslant L \\ \phi'(0) = 1, & \\ \phi'(L) = \beta. & \end{aligned}$$

The general solution is

$$\phi(x) = -\frac{x^2}{2} + Ax + B \qquad \text{with} \qquad \phi'(x) = -x + A$$

for $0 \leqslant x \leqslant L$.

From the first boundary condition, we have $\phi'(0) = A = 1$, while from the second boundary condition we have $\phi'(L) = \beta$, on the other hand, $\phi'(L) = -L + 1$. Therefore there exists an equilibrium temperature distribution if and only if $\beta = 1 - L$. In this case we have

$$u(x) = -\frac{x^2}{2} + x + B, \quad 0 < x < L$$

where B is a constant.

(b) In this case the boundary value problem for the equilibrium temperature distribution is

$$\begin{aligned}\phi''(x) &= 0, \quad 0 \leqslant x \leqslant L\\ \phi'(0) &= 1,\\ \phi'(L) &= \beta.\end{aligned}$$

The general solution is

$$\phi(x) = Ax + B \qquad \text{with} \qquad \phi'(x) = A$$

for $0 \leqslant x \leqslant L$.

From the first boundary condition, we have $\phi'(0) = A = 1$, while from the second boundary condition we have $\phi'(L) = \beta$, and therefore there exists an equilibrium temperature distribution if and only if $\beta = 1$. In this case the equilibrium temperature distribution is

$$u = \phi(x) = x + B, \quad 0 < x < L$$

where B is a constant.

(c) In this case the boundary value problem for the equilibrium temperature distribution is

$$\begin{aligned}\phi''(x) + x - \beta &= 0, \quad 0 \leqslant x \leqslant L\\ \phi'(0) &= 0,\\ \phi'(L) &= 0.\end{aligned}$$

The general solution is

$$\phi(x) = -\frac{x^3}{6} + \frac{\beta x^2}{2} + Ax + B \qquad \text{with} \qquad \phi'(x) = -\frac{x^2}{2} + \beta x + A$$

for $0 \leqslant x \leqslant L$.

From the first boundary condition, we have $\phi'(0) = A = 0$, while from the second boundary condition

$$\phi'(L) = -\frac{L^2}{2} + \beta L = 0$$

which implies that $\beta = \dfrac{L}{2}$.

In this case the equilibrium temperature distribution is

$$u(x) = -\frac{x^3}{6} + \frac{Lx^2}{2} + B, \quad 0 < x < L$$

where B is a constant.

Exercise 11.2. ✗

Consider the homogeneous Dirichlet problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant L,\ t \geqslant 0$$

subject to the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0$$

for $t \geqslant 0$, with initial conditions

$$u(x,0) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant \dfrac{L}{2} \\ 2 & \text{for} \quad \dfrac{L}{2} < x \leqslant L \end{cases}$$

for $0 \leqslant x \leqslant L$.

Solution: Since both the partial differential equation and the boundary conditions are linear and homogeneous we can use separation of variables, and write

$$u(x,t) = \phi(x) \cdot h(t)$$

where ϕ depends only on x and h depends only on t. Substituting this into the partial differential equation, we have

$$\phi \cdot h' = k\phi'' \cdot h,$$

and separating variables,

$$\frac{\phi''}{\phi} = \frac{h'}{kh} = -\lambda$$

for a yet unspecified constant λ.

We obtain the two ordinary differential equations

$$\phi'' + \lambda\phi = 0 \qquad \text{and} \qquad h' + \lambda k\, h = 0.$$

Since

$$u(0,t) = \phi(0) \cdot h(t) \qquad \text{and} \qquad u(L,t) = \phi(L) \cdot h(t)$$

we can satisfy the boundary conditions by requiring that $\phi(0) = \phi(L) = 0$, so that $\phi(x)$ must satisfy the boundary value problem

$$\begin{aligned} \phi'' + \lambda\phi &= 0, \quad 0 \leqslant x \leqslant L \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}$$

Now we find those values of λ for which this boundary value problem has a nontrivial solution.

Case 1: $\lambda = 0$

In this case, the differential equation is $\phi'' = 0$, with general solution

$$\phi(x) = Ax + B,$$

where A and B are constants. Applying the boundary condition $\phi(0) = 0$, we get $B = 0$, so that $\phi(x) = Ax$. Applying the second boundary condition $\phi(L) = 0$ we get $A = 0$. In this case the equation has only the trivial solution $\phi(x) \equiv 0$ for $0 \leqslant x \leqslant L$.

Case 2: $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu > 0$

In this case, the differential equation becomes $\phi'' - \mu^2\phi = 0$, with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x$$

where A and B are constants.

Applying the first boundary condition, we have

$$\phi(0) = A \qquad \text{so that} \qquad A = 0,$$

and the solution is

$$\phi(x) = B\sinh\mu x.$$

Applying the second boundary condition

$$\phi(L) = B\sinh\mu L = 0 \qquad \text{so that} \qquad B = 0$$

since $\mu > 0$ and $\sinh\mu L \neq 0$. Therefore, in this case the only solution is $\phi(x) = 0$, and again there are no nontrivial solutions.

Case 3: $\lambda > 0$, say $\lambda = \mu^2$ where $\mu > 0$

In this case, the differential equation becomes $\phi'' + \mu^2\phi = 0$, with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x$$

where A and B are constants.

Applying the first boundary condition, we have

$$\phi(0) = A \qquad \text{so that} \qquad A = 0,$$

and the solution is

$$\phi(x) = B\sin\mu x.$$

Now however, when we apply the second boundary condition

$$\phi(L) = B\sin\mu L = 0$$

in order to get a nontrivial solution, we must require that $B \neq 0$, so that $\sin\mu L = 0$, and $\mu L = n\pi$ for some integer n.

Therefore, in the case of **Dirichlet boundary conditions**, we get a nontrivial solution only for the eigenvalues λ_n, where

$$\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2},$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin \frac{n\pi x}{L}$$

for each integer $n \geqslant 1$. The corresponding solution to the time equation

$$h_n'(t) + \lambda_n k\, h_n(t) = 0$$

is

$$h_n(t) = e^{-\lambda_n kt} = e^{-\frac{n^2\pi^2}{L^2}kt}$$

for $n \geqslant 1$.
For each $n \geqslant 1$, the function

$$u_n(x,t) = \phi_n(x) \cdot h_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt} \sin \tfrac{n\pi x}{L}$$

is a solution to the partial differential equation satisfying both boundary conditions.
Using the superposition principle, we write

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L}\, e^{-\frac{n^2\pi^2}{L^2}kt}$$

for $0 \leqslant x \leqslant L$, and $t \geqslant 0$.
We determine the constants b_n from the initial condition

$$u(x,0) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant \dfrac{L}{2} \\ 2 & \text{for} \quad \dfrac{L}{2} < x \leqslant L \end{cases}$$

for $0 \leqslant x \leqslant L$, and setting $t = 0$ in the expression for $u(x,t)$ above, we have

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

Hence we need to find the Fourier sine series of $u(x,0)$ on the interval $[0, L]$, and the coeffcieints are

$$b_n = \frac{2}{L} \int_0^L u(x,0) \sin \frac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.
Now

$$\int_0^L u(x,0) \sin \frac{n\pi x}{L}\, dx = \int_0^{L/2} \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^L 2 \sin \frac{n\pi x}{L}\, dx,$$

so that

$$b_n = \frac{2}{n\pi} + \frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{4}{n\pi}(-1)^n$$

for all $n \geqslant 1$, and the solution to the initial value problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L}\, e^{-\frac{n^2\pi^2}{L^2}kt}$$

for $0 \leqslant x \leqslant L$, and $t \geqslant 0$.

Exercise 11.3. ✗

Solve the following initial value–boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

$$u(x,0) = 3\sin\tfrac{\pi x}{L} - \sin\tfrac{3\pi x}{L}$$

for $0 \leqslant x \leqslant L$.

Solution: Assuming a solution of the form

$$u(x,t) = \phi(x)\cdot G(t)$$

and separating variables, we get

$$\frac{\phi''(x)}{\phi(x)} = \frac{G'(t)}{k\,G(t)} = -\lambda$$

where λ is a constant. The partial differential equation is reduced to two ordinary differential equations:

Spatial Equation:

$$\phi''(x) + \lambda\phi(x) = 0, \quad 0 \leqslant x \leqslant L$$
$$\phi(0) = 0$$
$$\phi(L) = 0.$$

Time Equation:

$$G'(t) + k\lambda G(t) = 0, \quad t \geqslant 0.$$

Since it has a complete set of homogeneous Dirichlet boundary conditions, we solve the spatial equation first.

The eigenvalues and corresponding eigenfunctions for this problem are (see the previous problem)

$$\lambda_n = \tfrac{n^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_n(x) = \sin\tfrac{n\pi x}{L}$$

for $n = 1, 2, 3, \ldots$.

The solutions to the time equation corresponding to these nontrivial solutions are

$$G_n(t) = e^{-\frac{kn^2\pi^2 t}{L^2}}$$

for $n = 1, 2, 3, \ldots$.

For $n \geqslant 1$, the functions

$$u_n(x,t) = \phi_n(x)\cdot G_n(t) = \sin\tfrac{n\pi x}{L}\cdot e^{-\frac{kn^2\pi^2 t}{L^2}}$$

are also solutions to the partial differential equation satisfying the boundary conditions, and since the partial differential equation and the boundary conditions are linear and homogeneous, by the superposition principle, the function

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L} \cdot e^{-\frac{kn^2\pi^2 t}{L^2}}$$

is also a solution. We determine the constants b_n, for $n \geqslant 1$, using the initial condition, namely

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L} = 3 \sin \tfrac{\pi x}{L} - \sin \tfrac{3\pi x}{L},$$

from the orthogonality of the eigenfunctions on the interval $[0, L]$ we see immediately that the coefficients of the Fourier sine series are

$$b_1 = 3, \quad b_3 = -1, \quad \text{and} \qquad b_n = 0 \quad \text{for} \quad n \neq 1,\, 3.$$

The solution is therefore

$$u(x,t) = 3 \sin \tfrac{\pi x}{L} e^{-\frac{k\pi^2 t}{L^2}} - \sin \tfrac{3\pi x}{L} e^{-\frac{9k\pi^2 t}{L^2}}$$

for $0 \leqslant x \leqslant L,\ t \geqslant 0$.

Exercise 11.4. ✗

Solve the boundary value problem for the one dimensional heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(\pi,t) &= 0, \quad t > 0 \\ u(x,0) &= 30 \sin x, \quad 0 < x < \pi, \end{aligned}$$

and give a brief physical explanation of the problem.

Solution: Using separation of variables, since we have homogeneous Dirichlet boundary conditions, we obtain the solution ($k = 1$ and $L = \pi$ here)

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx,$$

where $b_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x) \sin nx \, dx$ for $n \geqslant 1$.

Now,

$$u(x,0) = f(x) = 30 \sin x$$

for $0 < x < \pi$, that is, $f(x)$ is its own Fourier sine series, so that $b_1 = 30$, and $b_n = 0$ for all $n \geqslant 2$. The solution is

$$u(x,t) = 30 e^{-t} \sin x,$$

and this gives the temperature in a bar whose sides are insulated and whose ends $x = 0$ and $x = \pi$ are kept at 0 temperature, with an initial temperature distribution given by $u(x, 0) = 30 \sin x,\ 0 < x < \pi$.

Exercise 11.5. ✗

Solve the following homogeneous Dirichlet problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\ u(0, t) &= 0, \quad t > 0 \\ u(1, t) &= 0, \quad t > 0 \\ u(x, 0) &= e^{-x}, \quad 0 < x < 1, \end{aligned}$$

and give a brief physical explanation of the problem.

Solution: After separating variables, applying the initial conditions, and using the superposition principle, we obtain the solution

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where

$$b_n = 2 \int_0^1 e^{-x} \sin n\pi x \, dx$$

for $n \geqslant 1$.

Integrating by parts, we get

$$\begin{aligned} \int_0^1 e^{-x} \sin n\pi x \, dx &= -\frac{e^{-x}}{1 + n^2\pi^2} (\sin n\pi x + n\pi \cos n\pi x) \bigg|_0^1 \\ &= \frac{n\pi}{1 + n^2\pi^2} \left[1 + (-1)^{n+1} e^{-1}\right], \end{aligned}$$

so that

$$u(x, t) = 2\pi \sum_{n=1}^{\infty} \frac{n}{1 + n^2\pi^2} \left[1 + (-1)^{n+1} e^{-1}\right] e^{-n^2\pi^2 t} \sin n\pi x,$$

and this gives the temperature in a bar whose sides are insulated and whose ends $x = 0$ and $x = 1$ are kept at 0 temperature, with an initial temperature distribution given by $u(x, 0) = e^{-x},\ 0 < x < 1$.

Exercise 11.6. ✗

Solve the problem of heat transfer in a bar of length 1 with initial heat distribution

$$f(x) = \cos \pi x, \quad 0 < x < 1$$

and no heat loss at either end, where the thermal diffusivity is $k = 1$, that is, solve the initial boundary value problem below:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < 1, \quad t > 0 \\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, && t > 0 \\ &u(x,0) = \cos \pi x, && 0 < x < 1. \end{aligned}$$

Solution: Since both the partial differential equation and the boundary conditions are linear and homogeneous we may use separation of variables, and we write

$$u(x,t) = X(x) \cdot T(t)$$

where X depends only on x and T depends only on t. Substituting this into the partial differential equation, we have

$$X \cdot T' = X'' \cdot T,$$

and separating variables,

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda,$$

where λ denotes the unknown separation constant. We obtain two ordinary differential equations

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + \lambda T = 0.$$

Since

$$\frac{\partial u}{\partial x}(0,t) = X'(0) \cdot T(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(1,t) = X'(1) \cdot T(t)$$

we can satisfy the boundary conditions by requiring that $X'(0) = X'(1) = 0$, so that $X(x)$ must satisfy the boundary value problem

$$\begin{aligned} X'' + \lambda X = 0, \quad 0 < x < 1, \quad t > 0 \\ X'(0) = 0 \\ X'(1) = 1, \end{aligned}$$

with homogeneous **Neumann boundary conditions**.

Since it has a complete set of homogeneous Neumann boundary conditions, we solve the spatial equation first.

The eigenvalues and corresponding eigenfunctions for this problem are

$$\lambda_n = n^2\pi^2 \qquad \text{and} \qquad X_n(x) = \cos n\pi x$$

for $n = 0, 1, 2, 3, \ldots$.
The corresponding solutions to the time equation $T' + n^2\pi^2 T = 0$ are

$$T_n(t) = e^{-n^2\pi^2 t}$$

for $n = 0, 1, 2, 3, \ldots$.
For each $n \geqslant 0$, the product

$$u_n(x,t) = X_n(x) \cdot T_n(t) = e^{-n^2\pi^2 t} \cos n\pi x, \quad 0 < x < 1, \quad t > 0$$

satisfies the heat equation and the boundary conditions, and since they are both linear and homogeneous, then any linear combination does also, so we can use the superposition principle to write

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x$$

and all we need to do now is find the coefficients a_n for $n \geqslant 0$, so that the initial condition is also satisfied. Setting $t = 0$ in the series above, we have

$$\cos \pi x = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x,$$

that is, the a_n's are just the coefficients in the Fourier cosine series for $\cos \pi x$ on the interval $[0,1]$. Since $\cos \pi x$ is its own Fourier cosine series on the interval $[0,1]$, then

$$a_n = \begin{cases} 0 & \text{for} \quad n \neq 1, \\ 1 & \text{for} \quad n = 1. \end{cases}$$

and the solution is

$$u(x,y) = e^{-\pi^2 t} \cos \pi x$$

for $0 < x < 1, \quad t > 0$.

Exercise 11.7. ✗✗

Solve the problem of heat transfer in a bar of length $L = \pi$ and thermal diffusivity $k = 1$, with initial heat distribution $u(x,0) = \sin x$ where one end of the bar is kept at a constant temperature $u(0,t) = 0$, while there is no heat loss at the other end of the bar so that $\dfrac{\partial u}{\partial x}(\pi,t) = 0$, that is, solve the boundary value – initial value problem below:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = 0, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0$$

$$u(x,0) = \sin x, \quad 0 < x < \pi.$$

Solution: Assuming $u(x,t) = X(x) \cdot T(t)$ and separating variables, we get the two ordinary differential equations $X'' + \lambda X = 0$ and $T' + \lambda T = 0$, and the boundary conditions lead to the following boundary value problem for X :

$$\begin{aligned} X'' + \lambda X &= 0, \quad 0 < x < \pi \\ X(0) &= 0 \\ X'(\pi) &= 0 \end{aligned}$$

Arguing as in previous problems, the only nontrivial solutions occur when $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$, and the differential equation becomes

$$X'' + \mu^2 X = 0$$

with general solution

$$X(x) = A\cos\mu x + B\sin\mu x$$

and applying the first boundary condition, we have $A = 0$, so that

$$X(x) = B\sin\mu x \qquad \text{and} \qquad X'(x) = \mu B\cos\mu x.$$

Applying the second boundary condition, we have

$$B\cos\mu\pi = 0,$$

and in order to get nontrivial solutions we must have $\mu\pi = \dfrac{(2n-1)\pi}{2}$ for some positive integer n. The eigenvalues are

$$\lambda_n = \mu_n^2 = \frac{(2n-1)^2}{4}$$

and the corresponding eigenfunctions are

$$X_n(x) = \sin\tfrac{(2n-1)x}{2}$$

while the corresponding solutions to the equation $T' + \mu_n^2 T = 0$ are

$$T_n(t) = e^{-\frac{(2n-1)^2 t}{4}}$$

for $n \geqslant 1$.
For each $n \geqslant 1$, the function

$$u_n(x,t) = X_n(x) \cdot T_n(t) = e^{-\frac{(2n-1)^2 t}{4}} \sin\tfrac{(2n-1)x}{2}$$

satisfies the heat equation and the boundary conditions, and using the superposition principle, we write

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{(2n-1)^2 t}{4}} \sin\tfrac{(2n-1)x}{2}$$

for $0 < x < \pi,\ t > 0$.

Setting $t = 0$, in order to satisfy the initial condition we need

$$\sin x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \tfrac{(2n-1)x}{2} \qquad (*)$$

for $0 < x < \pi$.
Note that $\sin x$ is not of the form

$$\sin \tfrac{(2n-1)x}{2},$$

hence we cannot just say that $\sin x$ is its own Fourier series. Here we have to compute the full generalized Fourier series of $\sin x$ in terms of $\sin \frac{(2n-1)x}{2}$.
In order to determine the coefficients b_n, we use the fact that the functions $\left\{\sin \frac{(2n-1)x}{2}\right\}_{n\geqslant 1}$ are orthogonal on the interval $[0,\pi]$. To see this, note that if $n \neq m$, then

$$\begin{aligned}
\int_0^\pi \sin \mu_m x \sin \mu_n x \, dx &= \frac{1}{2}\int_0^\pi \left[\cos(\mu_m - \mu_n)x - \cos(\mu_m + \mu_n)x\right] dx \\
&= \left.\frac{\sin(\mu_m - \mu_n)x}{2(\mu_m - \mu_n)}\right|_0^\pi - \left.\frac{\sin(\mu_m + \mu_n)x}{2(\mu_m + \mu_n)}\right|_0^\pi \\
&= \frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(m+n)\pi}{2(m+n)} \\
&= 0.
\end{aligned}$$

Also, if $m = n$, then

$$\begin{aligned}
\int_0^\pi \sin^2 \mu_m x \, dx &= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2\mu_m x\right) dx \\
&= \frac{\pi}{2} - \left.\frac{\sin 2\mu_m x}{4\mu_m}\right|_0^\pi \\
&= \frac{\pi}{2} - \frac{\sin(2m-1)\pi}{2(2m-1)} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Multiplying the equation $(*)$ by $\sin \mu_m x$ and integrating from 0 to π, and using the orthogonality result just proven, we have

$$\int_0^\pi \sin x \sin \mu_m x \, dx = b_m \int_0^\pi \sin^2 \mu_m \, dx = \frac{\pi}{2} \cdot b_m,$$

that is,

$$b_m = \frac{2}{\pi}\int_0^\pi \sin x \sin \mu_m x\, dx = \frac{1}{\pi}\int_0^\pi \left[\cos(\mu_m - 1)x - \cos(\mu_n + 1)x\right]\, dx$$

$$= \frac{\sin(\mu_m - 1)x}{\pi(\mu_m - 1)}\bigg|_0^\pi - \frac{\sin(\mu_m + 1)x}{\pi(\mu_m + 1)}\bigg|_0^\pi = \frac{\sin(\mu_m - 1)\pi}{\pi(\mu_m - 1)} - \frac{\sin(\mu_m + 1)\pi}{\pi(\mu_m + 1)}$$

$$= \frac{2}{\pi}\left[\frac{\sin\frac{(2m-3)\pi}{2}}{(2m-3)} - \frac{\sin\frac{(2m+1)\pi}{2}}{(2m+1)}\right] = \frac{2}{\pi}\left[\frac{(-1)^m}{(2m-3)} - \frac{(-1)^m}{(2m+1)}\right],$$

that is,

$$b_m = \frac{8}{\pi}\frac{(-1)^m}{(2m-3)(2m+1)},$$

since $\sin\frac{(2m+1)\pi}{2} = (-1)^m$.
Therefore, the solution is

$$u(x,t) = \frac{8}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{(2n-3)(2n+1)} e^{-\frac{(2n-1)^2 t}{4}} \sin\frac{(2n-1)x}{2}$$

for $0 < x < \pi,\ t > 0$.

Exercise 11.8. ✗
Consider the homogeneous Neumann problem problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0; \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0; \quad u(x,0) = f(x), \quad 0 < x < L.$$

Solve this problem by looking for a solution as a Fourier cosine series in terms of $\cos\frac{n\pi x}{L},\ n \geqslant 0$. Assume that u and $\frac{\partial u}{\partial x}$ are continuous and $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial t}$ are piecewise smooth. Justify all differentiations of infinite series.

Solution: We assume a solution of the form

$$u(x,t) = \sum_{n=0}^\infty a_n(t)\cos\frac{n\pi x}{L}$$

and assuming all derivatives are continuous, we have

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=0}^\infty a_n(t)\left(\frac{n\pi}{L}\right)^2\cos\frac{n\pi x}{L}$$

and since $u(x,t)$ satisfies the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

we have

$$\sum_{n=0}^{\infty} a_n'(t)\cos\tfrac{n\pi x}{L} = -k\sum_{n=0}^{\infty} a_n(t)\left(\tfrac{n\pi}{L}\right)^2\cos\tfrac{n\pi x}{L}.$$

Collecting terms that multiply $\cos\frac{n\pi x}{L}$ for $n \geqslant 0$ for $n \geqslant 1$, and using the fact that these trigonometric functions are linearly independent (they are orthogonal on the interval $[0, L]$), then we get

$$a_n'(t) = -ka_n(t)\left(\tfrac{n\pi}{L}\right)^2,$$

and we can solve these first order linear ordinary differential equations for $a_n(t)$ to get

$$a_n(t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt},$$

and the solution $u(x, t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\cos\tfrac{n\pi x}{L}.$$

Note that if we started the solution using separation of variables, we would arrive at the same formula as above.

Differentiating this with respect to x, we get

$$\frac{\partial u}{\partial x}(x,t) = -\sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(\tfrac{n\pi}{L}\right)\sin\tfrac{n\pi x}{L},$$

and setting $x = 0$, we get

$$0 = \frac{\partial u}{\partial x}(0,t),$$

and the first boundary condition is satisfied.
The solution is now

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\cos\tfrac{n\pi x}{L},$$

and we note that the second boundary condition $\dfrac{\partial u}{\partial x}(L,t) = 0$ is also satisfied, so we only need to find the constants A_n to satisfy the initial condition $u(x, 0) = f(x)$.
Setting $t = 0$ in the above expression for $u(x, t)$, we have

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n\cos\tfrac{n\pi x}{L},$$

and the A_n are the Fourier cosine series coefficients of $f(x)$, so that

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\tfrac{n\pi x}{L}\,dx, \quad n \geqslant 1$$

and for $n = 0$,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

Exercise 11.9. ✗✗

Consider the partial differential equation which describes the temperature u in the problem of heat transport with convection:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x},$$

where k denotes the specific heat and V_0 the convective velocity.

(a) Use separation of variables and show that the resulting spatial equation is **not** of Sturm-Liouville form. Find an appropriate multiplier to get it into Sturm-Liouville form.

(b) Solve the initial value–boundary value problem

$$u(0,t) = 0, \quad t > 0$$

$$u(L,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < L$$

Solution:

(a) Assuming a solution of the form $u(x,t) = \phi(x)\cdot h(t)$, the partial differential equation becomes

$$\phi\, h' = (k\phi'' - V_0\phi')h,$$

and separating variables

$$\frac{h'}{k\,h} = \frac{\phi''}{\phi} - \frac{V_0}{k}\,\frac{\phi'}{\phi} = -\lambda$$

where λ is the separation constant.

Thus, we have the following two ordinary differential equations

$$\phi''(x) - \frac{V_0}{k}\,\phi'(x) + \lambda\,\phi(x) = 0, \quad 0 < x < L$$

and

$$h'(t) + \lambda\,k\,h(t) = 0, \quad t > 0.$$

The spatial equation is not of the form

$$\frac{d}{dx}\Big(p(x)\phi'(x)\Big) + [q(x) + \lambda\sigma(x)]\phi(x) = 0, \quad 0 \leqslant x \leqslant L$$

where $p, q,$ and σ satisfy the conditions for a Sturm-Liouville problem, since we would need

$$p(x) = 1 \qquad \text{and} \qquad p'(x) = -\frac{V_0}{k},$$

and that doesn't work. We can, however, multiply the spatial equation by $e^{-\frac{V_0 x}{k}}$, and obtain

$$\frac{d}{dx}\left(e^{-\frac{V_0 x}{k}}\frac{d\phi}{dx}\right)+\lambda e^{-\frac{V_0 x}{k}}\phi=0,$$

which is of Sturm-Liouville form.

(b) The heat equation with convection satisfies the boundary value – initial value problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}-V_0\frac{\partial u}{\partial x}, \quad 0<x<L, \quad t>0\\
&u(0,t)=0, \quad t>0\\
&u(L,t)=0, \quad t>0\\
&u(x,0)=f(x), \ \ 0<x<L
\end{aligned}$$

Assuming a solution of the form $u(x,t)=\phi(x)\cdot h(t)$ and separating variables, we get the two problems:

$$\begin{aligned}
\phi''(x)-\frac{V_0}{k}\phi'(x)+\lambda\phi(x)&=0, \quad 0<x<L \qquad\qquad h'(t)+\lambda\, k\, h(t)=0, \quad t>0,\\
\phi(0)&=0,\\
\phi(L)&=0,
\end{aligned}$$

Making the transformation

$$y=e^{-\frac{V_0 x}{2k}}\phi,$$

then y satisfies the boundary value problem

$$y''+\left(\lambda-\frac{V_0^2}{4k^2}\right)y=0, \quad 0<x<L$$

$$y(0)=0,$$

$$y(L)=0,$$

which has nontrivial solutions if and only if

$$\lambda-\frac{V_0^2}{4k^2}>0 \qquad \text{and} \qquad \lambda-\frac{V_0^2}{4k^2}=\frac{n^2\pi^2}{L^2}$$

for some integer $n \geqslant 1$, and the corresponding solutions are

$$y_n(x)=\sin\tfrac{n\pi x}{L}.$$

Therefore for the Sturm Liouville problem, the eigenvalues are

$$\lambda_n=\frac{V_0^2}{4k^2}+\frac{n^2\pi^2}{L^2}$$

with corresponding eigenfunctions

$$\phi_n(x) = e^{\frac{V_0 x}{2k}}\, y_n(x) = e^{\frac{V_0 x}{2k}} \sin\tfrac{n\pi x}{L}, \quad 0 < x < L$$

for $n \geqslant 1$.

The corresponding solutions to the time equation are

$$h_n(t) = e^{-\lambda_n kt}$$

for $n \geqslant 1$.

For each $n \geqslant 1$, the products

$$u_n(x,t) = e^{\frac{V_0}{2k}x} e^{-\lambda_n kt} \sin\tfrac{n\pi x}{L}$$

satisfy the partial differential equation as well as the boundary conditions.

From the superposition principle, we write

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{V_0}{2k}x} e^{-\lambda_n kt} \sin\tfrac{n\pi x}{L},$$

and we can satisfy the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n e^{\frac{V_0}{2k}x} \sin\tfrac{n\pi x}{L}$$

using the orthogonality of the eigenfunctions on the interval $[0, L]$ with respect to the weight function

$$\sigma(x) = e^{-\frac{V_0}{k}x}.$$

We have

$$b_n = \frac{2}{L}\int_0^L f(x) e^{-\frac{V_0}{2k}x} \sin\tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.

Therefore the solution to the initial value – boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{V_0}{2k}x} e^{-\lambda_n kt} \sin\tfrac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) e^{-\frac{V_0}{2k}x} \sin\tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.

Exercise 11.10. ✗✗

Consider the homogeneous Neumann problem for heat transport in a nonhomogeneous rod of length L

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u, \quad 0 < x < L, \;\; t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \; 0 < x < L,$$

where $c(x)$, $\rho(x)$, $K_0(x)$, $\alpha(x)$ are continuous functions on the interval $[0,L]$, and K_0, c and ρ are nonnegative.

Assume that the appropriate eigenfunctions of the spatial problem are known and denote them by $\phi_n(x)$.

Show that the eigenvalues of the spatial problem are positive if $\alpha < 0$ and solve the initial value–boundary value problem, briefly discussing $\lim_{t\to\infty} u(x,t)$.

Solution: We use separation of variables. Assume a solution of the form $u(x,t) = \phi(x)h(t)$ and substitute this into the partial differential equation to get

$$c\,\rho\,\phi\,h' = \big(K_0\,\phi'\big)'\,h + \alpha\,\phi\,h.$$

Separating variables,

$$\frac{(K_0\,\phi')'}{c\,\rho\,\phi} + \frac{\alpha}{c\,\rho} = \frac{h'}{h} = -\lambda$$

where λ is the separation constant.

This leads to the two ordinary differential equations:

$$\big(K_0(x)\,\phi'(x)\big)' + \alpha(x)\,\phi(x) + \lambda\,c(x)\,\rho(x)\,\phi(x) = 0, \quad 0 \leqslant x \leqslant L; \qquad h'(t) + \lambda\,h(t) = 0, \quad t > 0.$$

$$\phi'(0) = 0,$$

$$\phi'(L) = 0,$$

The spatial equation is a regular Sturm-Liouville problem with

$$p(x) = K_0(x), \; q(x) = \alpha(x), \quad \text{and} \quad \sigma(x) = c(x)\,\rho(x),$$

all of which are assumed continuous on the closed interval $[0,L]$. In addition, on physical grounds we assume that K_0, c, and ρ are nonnegative and not identically zero on $[0,L]$.

If λ is an eigenvalue with corresponding eigenvector $\phi(x)$, $0 < x < L$, from the boundary conditions we have

$$\big[-p(x)\,\phi(x)\,\phi'(x)\big]\Big|_0^L = 0,$$

and the reduces to

$$\lambda = R(\phi) = \frac{\displaystyle\int_0^L \left[K_0(x)\,\phi'(x)^2 - \alpha(x)\,\phi(x)^2\right]\,dx}{\displaystyle\int_0^L \phi(x)^2 \rho(x)\,c(x)\,dx},$$

and if $\alpha(x) < 0$ for $0 \leqslant x \leqslant L$, then $\lambda \geqslant 0$.
Note that in this case, $\lambda = 0$ is impossible, since that would imply that $\phi(x) = 0$ for all $0 < x < L$, which is a contradiction. Therefore all the eigenvalues are strictly positive.
The boundary value problem for ϕ is a regular Sturm-Liouville problem and has an infinite sequence of eigenvalues and corresponding eigenfunctions $\{(\lambda_n, \phi_n)\}_{n\geqslant 1}$ where the ϕ_n's form a complete orthogonal set of functions in the linear space of piecewise continuous functions on $[0, L]$ with respect to the weight function $\sigma(x) = c(x)\,\rho(x)$.
The corresponding solutions to the time equation are

$$h_n(t) = c_n e^{-\lambda_n t}, \quad t > 0$$

for $n \geqslant 1$.
Using the superposition principle, we can write

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \phi_n(x)$$

for $0 < x < L,\ t > 0$, and this satisfies the partial differential equation and the boundary conditions.
In order to satisfy the initial condition, we use the orthogonality of the eigenfunctions to write the generalized Fourier series

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\displaystyle\int_0^L f(x)\,\phi_n(x)\,c(x)\,\rho(x)\,dx}{\displaystyle\int_0^L \phi_n(x)^2\,c(x)\,\rho(x)\,dx}.$$

Since $\lambda_n > 0$ for all $n \geqslant 1$, then for each term in the series,

$$e^{-\lambda_n t} \longrightarrow 0,$$

as $t \to \infty$, and therefore

$$\lim_{t\to\infty} u(x,t) = 0$$

for each $x \in (0, L)$.

Exercise 11.11. ✗✗✗

Consider the heat equation on a two-dimensional plate occupying the rectangular region $0 < x < L,\ 0 < y < H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve the initial value–boundary value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

$$u(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0.$$

Solution: Since the equation and the boundary conditions are linear and homogeneous we can use separation of variables. Assuming a solution of the form $u(x, y, t) = X(x) \cdot Y(y) \cdot T(t)$ and substituting this into the partial differential equation we have

$$T'(t)X(x)Y(y) = k\left[X''(x)Y(y)T(t) + Y''(y)X(x)T(t)\right],$$

so that

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda$$

where λ is the separation constant. This gives

$$\frac{X''(x)}{X(x)} = -\lambda - \frac{Y''(y)}{Y(y)} = -\tau$$

where τ is another separation constant.
We can satisfy the boundary conditions by requiring that

$$X(0) = X'(L) = 0 \qquad \text{and} \qquad Y'(0) = Y'(H) = 0,$$

and therefore X and Y satisfy the boundary value problems

$$\begin{aligned} X''(x) + \tau X(x) &= 0, \quad 0 < x < L, \\ X(0) &= 0 \\ X'(L) &= 0 \end{aligned} \qquad \begin{aligned} Y''(y) + \alpha Y(y) &= 0, \quad 0 < y < H, \\ Y'(0) &= 0 \\ Y'(H) &= 0 \end{aligned}$$

where $\alpha = \lambda - \tau$, while T satisfies the differential equation

$$T' + \lambda kT = 0, \quad t > 0.$$

The eigenvalues and corresponding eigenfunctions for the X-problem are (see Exercise 11.7)

$$\tau_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \qquad \text{and} \qquad X_n(x) = \sin\tfrac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \ldots$$

while the eigenvalues and corresponding eigenfunctions for the Y-problem are

$$\alpha_m = \left(\frac{m\pi}{H}\right)^2 \qquad \text{and} \qquad Y_m(y) = \cos\frac{m\pi y}{H}, \quad m = 0, 1, 2, 3, \ldots$$

The corresponding solutions to the time equation $T'(t) + \lambda k T(t)$ are

$$T_{nm} = e^{-\lambda_{nm} k t}$$

where for

$$\lambda_{nm} = \tau_n + \alpha_m = \left(\frac{(2n-1)\pi}{2L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

we need to use all possible combinations of indices $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, 3, \ldots$.
The products

$$u_{nm}(x, y, t) = X_n(x) Y_m(y) T_{nm}(t) = \sin\frac{(2n-1)\pi x}{2L} \cos\frac{m\pi y}{H} e^{-\lambda_{nm} k t}$$

satisfy the partial differential equation and the boundary conditions, and by the superposition principle, the function

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \sin\frac{(2n-1)\pi x}{2L} \cos\frac{m\pi y}{H} e^{-\lambda_{nm} k t}$$

also satisfies the partial differential equation and all the boundary conditions.
In order to satisfy the initial condition, we could use the fact that the eigenfunctions

$$\left\{\sin\frac{(2n-1)\pi x}{2L} \cos\frac{m\pi y}{H}\right\}_{n \geqslant 1, m \geqslant 0}$$

form an orthogonal set on the rectangle $[0, L] \times [0, H]$ in $\mathbb{R}^2$. However, we use another method which is similar to the methods used for one-dimensional Fourier series expansions.
Setting $t = 0$ in the expression above for $u(x, y, t)$, we want

$$f(x, y) = u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \sin\frac{(2n-1)\pi x}{2L} \cos\frac{m\pi y}{H}$$

for $0 \leqslant x \leqslant L,\ 0 \leqslant y \leqslant H$, and writing this as

$$f(x, y) = \sum_{n=1}^{\infty} \Bigg(\underbrace{\sum_{m=0}^{\infty} C_{nm} \cos\frac{m\pi y}{H}}_{B_n(y)} \Bigg) \sin\frac{(2n-1)\pi x}{2L},$$

we have

$$f(x,y) = \sum_{n=1}^{\infty} B_n(y) \sin \frac{(2n-1)\pi x}{2L}.$$

This is a Fourier sine series expansion of $f(x,y)$ on the interval $[0,L]$ holding y fixed, and therefore

$$B_n(y) = \frac{2}{L}\int_0^L f(x,y) \sin \frac{(2n-1)\pi x}{2L}\,dx, \quad n \geqslant 1.$$

However, for $n \geqslant 1$,

$$B_n(y) = \sum_{m=0}^{\infty} C_{nm} \cos \frac{m\pi y}{H}$$

is the expansion of $B_n(y)$ on the interval $[0,H]$, so that

$$C_{n0} = \frac{1}{H}\int_0^H B_n(y)\,dy = \frac{2}{LH}\int_0^H \int_0^L f(x,y) \sin \frac{(2n-1)\pi x}{2L}\,dxdy, \quad n \geqslant 1$$

$$C_{nm} = \frac{2}{H}\int_0^H B_n(y) \cos \frac{m\pi y}{H}\,dy = \frac{4}{LH}\int_0^H \int_0^L f(x,y) \sin \frac{(2n-1)\pi x}{2L} \cos \frac{m\pi y}{H}\,dxdy, \quad n,m \geqslant 1.$$

Finally, note that in the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm} \sin \frac{(2n-1)\pi x}{2L} \cos \frac{m\pi y}{H} e^{-\lambda_{nm} kt},$$

all terms in the sum for which either $n \geqslant 1$ or $m \geqslant 1$ contain a factor of

$$e^{-\lambda_{nm} kt}$$

where $\lambda_{nm} > 0$, and as $t \to \infty$, all these terms tend to 0, and therefore

$$\lim_{t\to\infty} u(x,y,t) = 0.$$

Exercise 11.12. ✗✗

Find the temperature distribution in a thin two dimensional plate in the shape of a unit square, with thermal diffusivity $k = 1$ and with insulated faces and edges kept at zero temperature with an initial temperature distribution given by

$$f(x,y) = xy(1-x)(1-y)$$

for $0 \leqslant x, y \leqslant 1$, that is, solve the boundary value – initial value problem given below:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, && 0 < x < 1, \quad 0 < y < 1, \quad t > 0\\
&u(0,y,t) = u(1,y,t) = 0, && 0 < y < 1, \quad t > 0\\
&u(x,0,t) = u(x,1,t) = 0, && 0 < x < 1, \quad t > 0\\
&u(x,y,0) = xy(1-x)(1-y), && 0 < x < 1, \quad 0 < y < 1.
\end{aligned}$$

Solution: After separating variables, applying the boundary conditions, and using the superposition principle, we find the solution has the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{n,m} \sin n\pi x \sin m\pi y \, e^{-c\pi\sqrt{n^2+m^2}\,t}.$$

We evaluate $B_{n,m}$ using the initial condition

$$\begin{aligned} B_{n,m} &= 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin n\pi x \sin m\pi y \, dy\, dx \\ &= \left(2\int_0^1 x(1-x)\sin n\pi x\, dx\right)\cdot\left(2\int_0^1 y(1-y)\sin m\pi y\, dy\right) \\ &= \frac{16\,[1-(-1)^n]\cdot[1-(-1)^m]}{n^3m^3\pi^6} \end{aligned}$$

for $n, m \geqslant 1$, that is,

$$B_{n,m} = \begin{cases} \dfrac{64}{n^3m^3\pi^6} & \text{if both } \quad n,m \quad \text{are odd} \\ 0 & \text{otherwise.} \end{cases}$$

The solution is therefore

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{64}{\pi^6(2n-1)^3(2m-1)^3} \sin(2n-1)\pi x \sin(2m-1)\pi y \, e^{-\left[(2n-1)^2+(2m-1)^2\right]\pi^2 t}$$

for $0 < x < 1,\ 0 < y < 1,\ t > 0.$

Exercise 11.13. ✗✗

Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

subject to the boundary and initial conditions:

$$\begin{aligned} u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= f(x). \end{aligned}$$

Obtain the solution by the method of eigenfunction expansions. Show that the solution approaches a steady-state solution.

Solution: Since the problem already has homogeneous boundary conditions, we consider the corresponding homogeneous problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant L,\ t \geqslant 0$$

$$u(0,t) = 0, \quad t \geqslant 0$$

$$u(L,t) = 0, \quad t \geqslant 0.$$

The eigenvalues and eigenfunctions for this problem are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_n(x) = \sin\tfrac{n\pi x}{L}$$

for $n \geqslant 1$.
We write the solution to the nonhomogeneous problem as an expansion in terms of these eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\tfrac{n\pi x}{L},$$

and determine the coefficients $a_n(t)$ which force this to be a solution to the nonhomogeneous problem.
We will need the eigenfunction expansions for $Q(x)$ and $f(x)$:

$$Q(x) = \sum_{n=1}^{\infty} q_n \sin\tfrac{n\pi x}{L}, \qquad \text{with} \qquad q_n = \frac{2}{\pi}\int_0^L Q(x)\sin\tfrac{n\pi x}{L}\,dx$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\tfrac{n\pi x}{L}, \qquad \text{with} \qquad f_n = \frac{2}{\pi}\int_0^L f(x)\sin\tfrac{n\pi x}{L}\,dx.$$

Substituting these expansions into the nonhomogeneous equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x),$$

we obtain

$$\sum_{n=1}^{\infty}\frac{d\,a_n(t)}{dt}\sin\tfrac{n\pi x}{L} = -\sum_{n=1}^{\infty} k\frac{n^2\pi^2}{L^2}a_n(t)\sin\tfrac{n\pi x}{L} + \sum_{n=1}^{\infty} q_n\sin\tfrac{n\pi x}{L},$$

and using the orthogonality of the eigenfunctions on the interval $[0,L]$, the coefficients $a_n(t)$ satisfy the initial value problem

$$\frac{d\,a_n(t)}{dt} + \frac{n^2\pi^2}{L^2}k\,a_n(t) = q_n, \quad t \geqslant 0$$
$$a_n(0) = f_n$$

for $n \geqslant 1$.
The solution to this initial value problem is

$$a_n(t) = f_n e^{-\frac{n^2\pi^2}{L^2}kt} + q_n\int_0^t e^{-\frac{n^2\pi^2}{L^2}k(t-s)}\,ds,$$

that is,

$$a_n(t) = \frac{q_n}{k\frac{n^2\pi^2}{L^2}} + \left(f_n - \frac{q_n}{k\frac{n^2\pi^2}{L^2}}\right)e^{-\frac{n^2\pi^2}{L^2}kt}, \quad t \geqslant 0$$

for $n \geqslant 1$.

Note that since $k > 0$, we have $\lim_{t\to\infty} a_n(t) = \dfrac{q_n}{k\frac{n^2\pi^2}{L^2}}$ for $n \geqslant 1$.

The solution to the heat equation with a steady source is therefore

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{q_n}{k\frac{n^2\pi^2}{L^2}} + \left(f_n - \frac{q_n}{k\frac{n^2\pi^2}{L^2}} \right) e^{-\frac{n^2\pi^2}{L^2}kt} \right] \sin \tfrac{n\pi x}{L}$$

for $0 \leqslant x \leqslant L$ and $t \geqslant 0$.

For large value of t, this solution approaches $r(x)$ where

$$r(x) = \lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty} \frac{q_n}{k\frac{n^2\pi^2}{L^2}} \sin \tfrac{n\pi x}{L}$$

for $0 \leqslant x \leqslant L$, where

$$q_n = \frac{2}{\pi} \int_0^L Q(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.

Differentiating this twice with respect to x, we see that

$$r''(x) = -\sum_{n=1}^{\infty} \frac{q_n}{k} \sin \tfrac{n\pi x}{L} = -\frac{1}{k}\, Q(x),$$

and since $r(0) = r(L) = 0$, then the function $r(x)$ satisfies the boundary value problem

$$k\frac{d^2r}{dx^2} + Q = 0, \quad 0 \leqslant x \leqslant L$$

$$r(0) = 0$$
$$r(L) = 0,$$

which is exactly the boundary value problem for the steady state solution, that is, $r(x)$ is the steady state or equilibrium solution to the original heat flow problem.

Exercise 11.14. ✗✗✗

Solve the two-dimensional heat equation with circularly symmetric time-independent sources, boundary conditions, and initial conditions (inside a circle):

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} \right) + Q(r)$$

with

$$u(r,0) = f(r) \qquad \text{and} \qquad u(a,t) = T.$$

Solution: As usual with problems involving polar coordinates, we seek solutions that are bounded as $r \to 0^+$, so that $|u(r,t)| \leqslant M$ as $r \to 0^+$.

We first convert the problem into one with homogeneous boundary conditions and then use the method of eigenfunction expansions to solve the nonhomogeneous equation that results.

Step 1: In order to get a problem with homogeneous boundary conditions we write

$$u(r,t) = v(r) + w(r,t)$$

where $v(r)$, the steady-state or equilibrium solution, satisfies

$$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0, \quad 0 \leqslant r \leqslant a,$$

$$v(a) = T, \quad t \geqslant 0,$$

$$|v(r)| \leqslant M, \quad \text{as } r \to 0^+.$$

then

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} = \frac{\partial w}{\partial t},$$

and

$$\nabla^2 u = \nabla^2 v + \nabla^2 w = \nabla^2 w.$$

Therefore, $w(r,t)$ satisfies the boundary value – initial value problem

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r)$$

$$w(a,t) = 0$$

$$w(r,0) = f(r) - v(r)$$

We solve the v equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0$$

to get

$$v(r) = c_1 \log r + c_2,$$

with two integration constants c_1 and c_2.

From the boundedness condition, we have $c_1 = 0$, while from the boundary condition $v(a) = c_2 = T$, so that

$$v(r) = T$$

for $0 \leqslant r \leqslant a$.

Therefore, $u(r,t) = w(r,t) + T$, and w satisfies the nonhomogeneous equation with homogeneous boundary conditions:

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r) \qquad (*)$$

$$w(a,t) = 0$$

$$w(r,0) = f(r) - T.$$

Step 2: Next we find the eigenvalues and eigenfunctions for the corresponding homogeneous problem:

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \tag{**}$$

$$w(a,t) = 0$$

$$|w(r,t)| \quad \text{bounded at} \quad r = 0.$$

Using separation of variables, we assume that $w(r,t) = \phi(r)\cdot T(t)$, and separating variables we get

$$\big(r\phi'\big)' + \lambda\, r\,\phi = 0, \quad 0 \leqslant r \leqslant a$$

$$T' + \lambda\, k\, T = 0, \quad t \geqslant 0.$$

The boundary condition $w(a,t) = 0$ for all $t \geqslant 0$ is satisfied if we require

$$\phi(a) = 0.$$

Also, since $r = 0$ is a singular point of the differential equation for ϕ, we add the requirement

$$|\phi(r)| \quad \text{bounded at} \quad r = 0,$$

which is equivalent to requiring that $|w(r,t)|$ be bounded at $r = 0$.
Thus, ϕ satisfies the boundary value problem

$$\big(r\phi'\big)' + \lambda\, r\,\phi = 0, \quad 0 \leqslant r \leqslant a$$

$$\phi(a) = 0, \tag{†}$$

$$|\phi(r)| \quad \text{bounded at} \quad r = 0.$$

We multiply the equation by r and recognize the equation

$$r^2\phi'' + r\phi' + \lambda\, r^2\phi = 0$$

as Bessel's equation of order zero, for which the function

$$\phi(r) = J_0(\sqrt{\lambda}\, r)$$

is the solution bounded at $r = 0$.
In order to satisfy the boundary condition $\phi(a) = 0$, we must have

$$J_0(\sqrt{\lambda}\, a) = 0,$$

or

$$\sqrt{\lambda_n}\, a = z_n, \quad n = 1,\, 2,\, \ldots$$

where z_n are the positive zeros of the Bessel function J_0.

Therefore the eigenvalues and eigenfunctions of the boundary value problem satisfied by $\phi(r)$ are

$$\lambda_n = \frac{z_n^2}{a^2} \qquad \text{and} \qquad \phi_n(r) = J_0(\sqrt{\lambda_n}\, r)$$

for $n \geqslant 1$.

Step 3: Now we use an eigenfunction expansion for $w(r,t)$ as

$$w(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0(\sqrt{\lambda_n}\, r)$$

and determine the coefficients $a_n(t)$ so that $w(r,t)$ is a solution to the nonhomogeneous equation

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r) \tag{$*$}$$

$$w(a,t) = 0$$

$$w(r,0) = f(r) - T,$$

and this means we will need the Fourier-Bessel Series for $Q(r)$ and $f(r) - T$:

$$Q(r) = \sum_{n=1}^{\infty} q_n J_0(\sqrt{\lambda_n}\, r), \qquad \text{with} \qquad q_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r) Q(r) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr}$$

$$f(r) - T = \sum_{n=1}^{\infty} f_n J_0(\sqrt{\lambda_n}\, r), \qquad \text{with} \qquad f_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)(f(r) - T) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr}.$$

Note that the Sturm-Liouville problem (†) has the weight function $\sigma(r) = r$, hence the factor r in the integrals.

Substituting these expansions into $(*)$, we have

$$\sum_{n=1}^{\infty} \frac{da_n(t)}{dt} J_0(\sqrt{\lambda_n}\, r) = \sum_{n=1}^{\infty} a_n(t)(-\lambda_n) J_0(\sqrt{\lambda_n}\, r) + \sum_{n=1}^{\infty} q_n J_0(\sqrt{\lambda_n}\, r),$$

and using the orthogonality of the eigenfunctions, the coefficients $a_n(t)$ satisfy the linear differential equation

$$\frac{da_n(t)}{dt} + \lambda_n a_n(t) = q_n, \quad t \geqslant 0$$

for $n \geqslant 1$.

From the initial condition

$$w(r,0) = \sum_{n=1}^{\infty} a_n(0) J_0(\sqrt{\lambda_n}\, r) = f(r) - T = \sum_{n=1}^{\infty} f_n J_0(\sqrt{\lambda_n}\, r),$$

using the orthogonality again, we have

$$a_n(0) = f_n$$

for $n \geqslant 1$.
Therefore, $a_n(t)$ satisfies the initial value problem

$$\frac{da_n(t)}{dt} + \lambda_n a_n(t) = q_n, \quad t \geqslant 0$$

$$a_n(0) = f_n$$

for $n \geqslant 1$.
Multiplying by the integrating factor $e^{\lambda_n t}$, the differential equation becomes

$$\frac{d}{dt}\left(a_n(t)e^{\lambda_n t}\right) = q_n e^{\lambda_n t},$$

and integrating,

$$a_n(t)e^{\lambda_n t} - a_n(0) = \int_0^t q_n e^{\lambda_n s}\, ds,$$

so that

$$a_n(t) = a_n(0)e^{-\lambda_n t} + \int_0^t q_n e^{-\lambda_n (t-s)}\, ds = a_n(0)e^{-\lambda_n t} + \frac{q_n}{\lambda_n}\left(1 - e^{-\lambda_n t}\right), \quad t \geqslant 0$$

for $n \geqslant 1$.
<u>Step 4:</u> Putting everything together, the solution is

$$u(r,t) = v(r) + w(r,t) = T + \sum_{n=1}^{\infty} a_n(t) J_0(\sqrt{\lambda_n}\, r),$$

that is,

$$u(r,t) = T + \sum_{n=1}^{\infty}\left[\frac{q_n}{\lambda_n} + \left(f_n - \frac{q_n}{\lambda_n}\right)e^{-\lambda_n t}\right] J_0(\sqrt{\lambda_n}\, r)$$

for $0 \leqslant r \leqslant a,\ t \geqslant 0$, where $\lambda_n = \dfrac{z_n^2}{a^2}$, and

$$q_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r) Q(r) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr},$$

$$f_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)(f(r) - T) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr}$$

for $n \geqslant 1$.

Exercise 11.15. ✗✗

A thin homogeneous bar of length π has poorly insulated sides, so that heat radiates freely from the bar along its entire length. Assuming that the heat transfer coefficient A is constant, and that the temperature T of the surrounding medium is also constant, the temperature $u(x,t)$ in the bar satisfies the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - A(u-T), \quad 0 < x < \pi, \quad t > 0.$$

The ends of the bar are kept at temperature T, and the initial temperature is

$$u(x,0) = x + T, \quad 0 < x < \pi.$$

(a) State the initial value–boundary value problem satisfied by $u(x,t)$.

(b) Transform this problem into a familiar one by setting

$$v(x,t) = e^{At}\left[u(x,t) - T\right]$$

and then finding the initial value–boundary value problem satisfied by $v(x,t)$.

(c) Use the method of separation of variables to solve the problem in part (b), and hence obtain the solution $u(x,t)$ to the original problem.

Solution:

(a) The problem satisfied by the temperature function is

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - A(u-T), \quad 0 < x < \pi, \quad t > 0 \\
&u(0,t) = T, \quad t > 0 \\
&u(\pi,t) = T, \quad t > 0 \\
&u(x,t) = x + T, \quad 0 < x < \pi.
\end{aligned}$$

(b) We let $v(x,t) = e^{At}\left[u(x,t) - T\right]$, so that

$$\begin{aligned}
u &= T + e^{-At}v \\
\frac{\partial u}{\partial t} &= e^{-At}\frac{\partial v}{\partial t} - Ae^{-At}v \\
\frac{\partial^2 u}{\partial x^2} &= e^{-At}\frac{\partial^2 v}{\partial x^2}.
\end{aligned}$$

If u is a solution to the partial differential equation in part (a), then

$$e^{-At}\frac{\partial v}{\partial t} - Ae^{-At}v = e^{-At}\frac{\partial^2 v}{\partial x^2} - Ae^{-At}v,$$

and, since e^{-At} is never zero, v satisfies the one dimension heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

As for the boundary conditions, we have

$$v(0,t) = e^{At}\left[u(0,t) - T\right] = e^{At}\left[T - T\right] = 0,$$

and

$$v(\pi,t) = e^{At}\left[u(\pi,t) - T\right] = e^{At}\left[T - T\right] = 0,$$

while for the initial condition, we have

$$v(x,0) = u(x,0) - T = x + T - T = x.$$

Therefore, $v(x,t) = e^{At}\left[u(x,t) - T\right]$ satisfies the initial value–boundary value problem

$$\begin{aligned}
&\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0\\
&v(0,t) = 0, \quad t > 0\\
&v(\pi,t) = 0, \quad t > 0\\
&v(x,t) = x, \quad 0 < x < \pi.
\end{aligned}$$

(c) Using separation of variables, the solution to the Dirichlet problem in part (b) is

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-n^2 t}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx = \frac{2(-1)^n}{n}$$

for $n \geqslant 1$. Therefore, the solution to the original heat transfer problem is

$$u(x,t) = T + e^{-At}\sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\, \sin nx\, e^{-n^2 t}$$

for $0 < x < \pi,\ t > 0$.

Exercise 11.16. ✗✗✗

Consider the homogeneous Robin problem for heat flow in a homogeneous rod of length a where we have convection at the ends into a medium at zero temperature, and where the initial temperature is $f(x)$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \;\; t > 0$$

$$\frac{\partial u}{\partial x}(0,t) - hu(0,t) = 0, \quad t > 0$$

$$\frac{\partial u}{\partial x}(a,t) + hu(a,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < a,$$

where $h > 0$.

Solution: We use separation of variables. Assume a solution of the form $u(x,t) = \phi(x)T(t)$ and substitute this into the partial differential equation to get

$$\phi\, T' = k\phi''\, T.$$

Separating variables,

$$\frac{\phi''}{\phi} = \frac{T'}{kT} = -\lambda$$

where λ is the separation constant.
This leads to the two ordinary differential equations:

$$\begin{aligned} &\phi''(x) + \lambda\,\phi(x) = 0, \quad 0 \leqslant x \leqslant a; \qquad T'(t) + \lambda\, kT(t) = 0, \quad t > 0.\\ &\phi'(0) - h\phi(0) = 0,\\ &\phi'(a) + h\phi(a) = 0, \end{aligned}$$

The spatial equation is a regular Sturm-Liouville problem with

$$p(x) = 1, \quad q(x) = 0, \quad \text{and} \quad \sigma(x) = 1,$$

all of which are continuous on the closed interval $[0,a]$.
If λ is an eigenvalue with corresponding eigenvector $\phi(x),\ 0 < x < a$, from the boundary conditions we have

$$\big[-p(x)\,\phi(x)\,\phi'(x)\big]\Big|_0^a = h\big[\phi(a)^2 + \phi(0)^2\big],$$

and the Rayleigh quotient reduces to

$$\lambda = R(\phi) = \frac{h\big[\phi(a)^2 + \phi(0)^2\big] + \displaystyle\int_0^a \phi'(x)^2\,dx}{\displaystyle\int_0^a \phi(x)^2\,dx},$$

and $\lambda \geqslant 0$. Note that $\lambda = 0$ is impossible, since that would imply that $\phi(x) = 0$ for all $0 < x < a$, which is a contradiction. Therefore all the eigenvalues are strictly positive.

The boundary value problem for ϕ is a regular Sturm-Liouville problem and has an infinite sequence of eigenvalues and corresponding eigenfunctions $\{(\lambda_n, \phi_n)\}_{n\geqslant 1}$ where the ϕ_n's form a complete orthogonal set of functions in the linear space of piecewise continuous functions on $[0, a]$ with respect to the weight function $\sigma(x) = 1$.

In fact, if we write $\lambda = \mu^2$, where $\mu > 0$, then the boundary value problem becomes

$$\phi'' + \mu^2\phi = 0$$
$$\phi'(0) - h\phi(0) = 0$$
$$\phi'(a) + h\phi(a) = 0,$$

with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x \qquad \text{and} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x.$$

Applying the first boundary condition we have

$$\frac{A}{B} = \frac{\mu}{h},$$

and applying the second boundary condition we have

$$\frac{h^2 - \mu^2}{h}\sin\mu a + 2\mu\cos\mu a = 0,$$

and the boundary value problem has a nontrivial solution if and only if

$$\tan\mu a = \frac{2\mu h}{\mu^2 - h^2}.$$

We determine the eigenvalues from the graphs of the functions

$$f(\mu) = \tan\mu a \qquad \text{and} \qquad g(\mu) = \frac{2\mu h}{\mu^2 - h^2}$$

for $\mu > 0$.

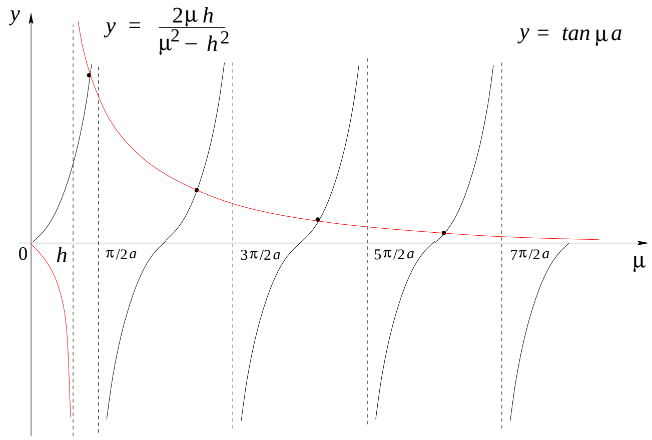
Note that for $\mu > 0$, we have

$$g(\mu) = \frac{2\mu h}{\mu^2 - h^2} = \frac{h}{\mu + h} + \frac{h}{\mu - h},$$

so that

$$g'(\mu) = -\frac{h}{(\mu + h)^2} - \frac{h}{(\mu - h)^2} < 0$$

and g is decreasing on the interval $(0, h)$ and on the interval (h, ∞) and the line $\mu = h$ is a vertical asymptote to the graph. The graphs of g and f are shown below.



From the figure it is clear that there are an infinite number of distinct positive solutions $\mu = \mu_n$ to the equation

$$\tan \mu a = \frac{2\mu h}{\mu^2 - h^2}, \tag{$*$}$$

and the eigenvalues are $\lambda_n = \mu_n^2$, for $n \geqslant 1$.

Since $\lim\limits_{n\to\infty} \mu_n = +\infty$, then

$$\lim_{n\to\infty} \tan \mu_n a = \lim_{n\to\infty} \frac{2\mu_n h}{\mu_n^2 - h^2} = 0,$$

and the roots of the equation $\tan \mu a = \dfrac{2\mu h}{\mu^2 - h^2}$ approach the roots of the equation $\tan \mu a = 0$, that is, for large n,

$$\mu_n a \approx n\pi,$$

and therefore

$$\lambda_n = \mu_n^2 \approx \frac{n^2\pi^2}{a^2}$$

for large n.

The corresponding eigenfunctions are

$$\phi_n(x) = \cos \mu_n x + \tfrac{h}{\mu_n} \sin \mu_n x \tag{$**$}$$

for $n \geqslant 1$.

The corresponding solutions to the time equation are

$$T_n(t) = e^{-k\mu_n^2 t}, \quad t > 0$$

for $n \geqslant 1$.

Using the superposition principle, we can write

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-k\mu_n^2 t}$$

for $0 < x < a,\ t > 0$, and this satisfies the partial differential equation and the boundary conditions.

In order to satisfy the initial condition, we use the orthogonality of the eigenfunctions to write the generalized Fourier series

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x),$$

where

$$a_n = \frac{\displaystyle\int_0^a f(x)\,\phi_n(x)\,dx}{\displaystyle\int_0^a \phi_n(x)^2\,dx}.$$

In order to determine the normalization constant $\displaystyle\int_0^a \phi_n(x)^2\,dx$, we note that

$$\phi_n'' + \mu_n^2 \phi_n = 0,$$

so that

$$\mu_n^2 \int_0^a \phi_n^2\,dx = -\int_0^a \phi_n \phi_n''\,dx = -\phi_n \phi_n'\Big|_0^a + \int_0^a \left(\phi_n'\right)^2 dx$$

However,

$$\mu_n \phi_n(x) = \mu_n \cos \mu_n x + h \sin \mu_n x$$

and

$$\phi_n'(x) = -\mu_n \sin \mu_n x + h \cos \mu_n x$$

so that

$$\mu_n^2 \phi_n(x)^2 + \left(\phi_n'(x)\right)^2 = \mu_n^2 + h^2$$

for $0 \leqslant x \leqslant a$, and integrating we have

$$\mu_n^2 \int_0^a \phi_n(x)^2\,dx + \int_0^a \left(\phi_n'(x)\right)^2 dx = \left(\mu_n^2 + h^2\right) a.$$

Also

$$\mu_n^2 \int_0^a \phi_n(x)^2\,dx - \int_0^a \left(\phi_n'(x)\right)^2 dx = -\phi_n \phi_n'\Big|_0^a,$$

and adding we obtain

$$2\mu_n^2 \int_0^a \phi_n(x)^2\,dx = \left(\mu_n^2 + h^2\right) a - \phi_n \phi_n'\Big|_0^a = \left(\mu_n^2 + h^2\right) a + h\left(\phi_n(a)^2 + \phi_n(0)^2\right). \qquad (***)$$

Since

$$\mu_n^2 \phi_n(x)^2 + \left(\phi_n'(x)\right)^2 = \mu_n^2 + h^2$$

for $0 \leqslant x \leqslant a$, from the first boundary condition we have

$$\mu_n^2 \phi_n(0)^2 + h^2 \phi_n(0)^2 = \left(\mu_n^2 + h^2\right) \phi_n(0)^2 = \mu_n^2 + h^2,$$

so that $\phi_n(0)^2 = 1$. Similarly, from the second boundary condition we have

$$\mu_n^2 \phi_n(a)^2 + h^2 \phi_n(a)^2 = \left(\mu_n^2 + h^2\right) \phi_n(a)^2 = \mu_n^2 + h^2,$$

so that $\phi_n(a)^2 = 1$, and from $(***)$ we have

$$\int_0^a \phi_n(x)^2 \, dx = \frac{\left(\mu_n^2 + h^2\right) a + 2h}{2\mu_n^2}$$

for $n \geqslant 1$.

The solution to the **homogeneous Robin problem** is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\cos \mu_n x + \tfrac{h}{\mu_n} \sin \mu_n x \right) e^{-k\mu_n^2 t}$$

for $0 \leqslant x \leqslant a, \; t \geqslant 0$, where the μ_n's are the positive roots of the transcendental equation

$$\tan \mu a = \frac{2\mu h}{\mu^2 - h^2}$$

and

$$a_n = \frac{2\mu_n^2}{\left(\mu_n^2 + h^2\right) a + 2h} \int_0^a \left(\cos \mu_n x + \tfrac{h}{\mu_n} \sin \mu_n x \right) f(x) \, dx$$

for $n \geqslant 1$.

Exercise 11.17. ✗✗

Solve the following boundary value problem for the steady-state temperature $u(x,y)$ in a thin plate in the shape of a semi-infinite strip when heat transfer to the surroundings at temperature zero takes place at the faces of the plate:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - b\, u = 0, &\qquad 0 < x < \infty, \quad 0 < y < 1 \\
\frac{\partial u}{\partial x}(0,y) = 0, &\qquad 0 < y < 1 \\
u(x,0) = 0, &\qquad 0 < x < \infty \\
u(x,1) = f(x), &\qquad 0 < x < \infty
\end{aligned}$$

where b is a positive constant and $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a. \end{cases}$

Solution: Given the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - b\,u = 0, \qquad 0 < x < \infty, \quad 0 < y < 1$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad 0 < y < 1$$

$$u(x, 0) = 0, \qquad 0 < x < \infty$$

$$u(x, 1) = f(x), \quad 0 < x < \infty$$

where b is a positive constant and $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a. \end{cases}$

We try separation of variables, writing

$$u(x, y) = X(x)\,Y(y),$$

then the partial differential equation becomes

$$X''Y + XY'' - bXY = 0,$$

that is

$$\frac{X''}{X} = -\frac{Y''}{Y} + b = p \qquad \text{(constant)}$$

and we obtain the two ordinary differential equations

$$X'' - p\,X = 0, \quad 0 < x < \infty \qquad\qquad Y'' + (p - b)\,Y = 0, \quad 0 < y < 1$$

$$X'(0) = 0, \qquad\qquad Y(0) = 0,$$

$$|X(x)| \quad \text{bounded as} \quad x \to \infty,$$

case (i) p = 0
The general solution to the equation $X'' = 0$ is

$$X(x) = c_1\,x + c_2$$

and the condition $X'(0) = 0$ implies that $c_1 = 0$, the solution is therefore $X(x) = 1$.
case (ii) $p > 0$, say $p = \mu^2$
The general solution to the equation $X'' - \mu^2\,X = 0$ is

$$X(x) = c_1\,\cosh \mu x + c_2\,\sinh \mu x$$

and the condition $X'(0) = 0$ implies $c_2 = 0$, while the condition $|X(x)|$ bounded as $x \to \infty$ implies that $c_1 = 0$. There are no non-trivial solutions in this case.
case (iii) $p < 0$, say $p = -\lambda^2$
The general solution to the equation $X'' + \lambda^2\,X = 0$ is

$$X(x) = c_1\,\cos \lambda x + c_2\,\sin \lambda x$$

the condition $X'(0) = 0$ implies that $c_2 = 0$, and the solution is $X(x) = c_1 \cos \lambda x$, which is bounded as $x \to \infty$.

Therefore, for **any** $\lambda \geqslant 0$, the function $X_\lambda(x) = \cos \lambda x$ satisfies the differential equation, the boundary condition, and the boundedness condition. In this case we no longer have a discrete spectrum, that is, a discrete set of eigenvalues, and every $\lambda \geqslant 0$ is an eigenvalue.

The corresponding equation for Y is given by

$$\begin{aligned} Y'' - (\lambda^2 + b)\, Y &= 0 \\ Y(0) &= 0 \end{aligned}$$

and has general solution

$$Y(y) = c_1 \sinh\left((1-y)\sqrt{\lambda^2 + b}\right) + c_2 \sinh\left(y\sqrt{\lambda^2 + b}\right).$$

The condition $Y(0) = 0$ implies that $c_1 = 0$, and the solutions are

$$Y_\lambda(y) = \sinh\left(y\sqrt{\lambda^2 + b}\right).$$

Using the superposition principle, we write

$$u(x, y) = \int_0^\infty A(\lambda) \cos \lambda x \sinh\left(y\sqrt{\lambda^2 + b}\right) d\lambda$$

and $u(x, 1) = f(x)$ implies that

$$\begin{aligned} A(\lambda) \sinh \sqrt{\lambda^2 + b} &= \frac{2}{\pi} \int_0^\infty f(x) \cos \lambda x \, dx \\ &= \frac{2}{\pi} \int_0^a \cos \lambda x \, dx \\ &= \frac{2}{\pi \lambda} \sin \lambda a. \end{aligned}$$

Therefore,

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda a \, \cos \lambda x \, \sinh(y\sqrt{\lambda^2 + b})}{\lambda \sinh \sqrt{\lambda^2 + b}} \, d\lambda$$

for $0 < x < \infty, \quad 0 < y < 1$.

Chapter 12

Wave Equation Problems

Exercise 12.1. ✗
Use the method of separation of variables to solve the one dimensional wave equation with homogeneous Dirichlet boundary conditions as given below

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0\\
&u(0,t) = 0, \quad t > 0\\
&u(1,t) = 0, \quad t > 0\\
&u(x,0) = \sin \pi x \cos \pi x, \quad 0 < x < 1,\\
&\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < 1.
\end{aligned}$$

Solution: We assume a solution of the form $u(x,t) = X(x)T(t)$ to obtain two ordinary differential equations:

$$X'' + \lambda\pi^2 X = 0 \qquad \text{and} \qquad T'' + \lambda T = 0,$$

with separation constant λ.
We can satisfy the two boundary conditions by requiring that $X(0) = 0$ and $X(1) = 0$, so that X satisfies the boundary value problem:

$$\begin{aligned}
X'' + \lambda\pi^2 X &= 0, \quad 0 < x < 1\\
X(0) &= 0\\
X(1) &= 0.
\end{aligned}$$

The cases $\lambda = 0$ and $\lambda < 0$ both result in the trivial solution $X(x) = 0$ for all $x \in [0,1]$, and the only nontrivial solution arises when $\lambda > 0$, say $\lambda = \mu^2$, where $\mu \neq 0$. In this case the boundary value problem

$$\begin{aligned}
X'' + \mu^2\pi^2 X &= 0, \quad 0 < x < 1\\
X(0) &= 0\\
X(1) &= 0.
\end{aligned}$$

has general solution

$$X(x) = A\cos \mu\pi x + B \sin \mu\pi x,$$

and applying the first boundary condition, we have $X(0) = A = 0$. Applying the second boundary condition, we have $X(1) = B \sin \mu\pi = 0$, and in order to get a nontrivial solution we must have $\sin \mu\pi = 0$, but this can only happen if $\mu\pi = n\pi$, where n is an integer. The eigenvalues and corresponding eigenfunctions are

$$\lambda_n = n^2\pi^2 \qquad \text{and} \qquad X_n(x) = \sin n\pi x$$

for $n \geqslant 1$.
The corresponding T-equation is

$$T'' + n^2 T = 0$$

with solutions

$$T_n(t) = a_n \cos nt + b_n \sin nt$$

for $n \geqslant 1$.
For each integer $n \geqslant 1$, the function

$$u_n(x,t) = X_n(x) \cdot T_n(t) = \sin n\pi x \,(a_n \cos nt + b_n \sin nt)$$

satisfies the wave equation and the two homogeneous boundary conditions and using the superposition principle, we write

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \,(a_n \cos nt + b_n \sin nt)\,.$$

In order to satisfy the initial conditions, we need

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \tag{1}$$

and

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n b_n \sin n\pi x, \tag{2}$$

that is, the Fourier sine series of $u(x,0)$ and $\dfrac{\partial u}{\partial t}(x,0)$.
Therefore, from (1) we have

$$a_n = 2\int_0^1 u(x,0) \sin n\pi x \, dx$$

and

$$n b_n = 2\int_0^1 \frac{\partial u}{\partial t}(x,0) \sin n\pi x \, dx$$

for $n \geqslant 1$.
Note that $b_n = 0$ for all $n \geqslant 1$, since $\dfrac{\partial u}{\partial t}(x,0) = 0$ for $0 < x < 1$.
Also, we have

$$u(x,0) = \sin \pi x \cos \pi x = \frac{1}{2} \sin 2\pi x,$$

so that $u(x,0)$ is its own Fourier sine series, and

$$a_n = \begin{cases} \dfrac{1}{2} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$$

Therefore, the solution is

$$u(x,t) = \frac{1}{2} \sin 2\pi x \cos 2t$$

for $0 \leqslant x \leqslant 1,\ t \geqslant 0$.

Exercise 12.2. ✗

Solve the following boundary value – initial value problem for the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0 \\ u(x,0) &= \sin \pi x + \tfrac{1}{2} \sin 3\pi x + 3 \sin 7\pi x, \quad 0 < x < 1, \\ \frac{\partial u}{\partial t}(x,0) &= \sin 2\pi x, \quad 0 < x < 1. \end{aligned}$$

Solution: Similar to the previous problem, we use separation of variables and the superposition principle to get the general solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left(a_n \cos n\pi t + b_n \sin n\pi t \right),$$

where the coefficients are to be determined using the initial conditions. Differentiating, we have

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left(-n\pi a_n \sin n\pi t + n\pi b_n \cos n\pi t \right),$$

and setting $t = 0$, we get

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin n\pi x \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n\pi b_n \sin n\pi x,$$

and again these are just the Fourier sine series of $f(x)$ and $g(x)$, the initial displacement and initial velocity.

From the first initial condition

$$u(x,0) = \sin \pi x + \tfrac{1}{2} \sin 3\pi x + 3 \sin 7\pi x,$$

we see that

$$a_1 = 1, \quad a_3 = \frac{1}{2}, \quad a_7 = 3,$$

and $a_n = 0$ for all other values of n.
From the second initial condition

$$\frac{\partial u}{\partial t}(x,0) = \sin 2\pi x,$$

so that

$$b_n = \begin{cases} \frac{1}{2\pi} & \text{if } n = 2, \\ 0 & \text{if } n \neq 2. \end{cases}$$

Therefore, the solution is

$$u(x,t) = \sin \pi x \cos \pi t + \frac{1}{2\pi} \sin 2\pi x \sin 2\pi t + \frac{1}{2} \sin 3\pi x \cos 3\pi t + 3 \sin 7\pi x \cos 7\pi t$$

for $0 < x < 1,\ t > 0$.

Exercise 12.3. ✗
Show that the solution to the wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\ &u(0,t) = 0, \quad t > 0 \\ &u(L,t) = 0, \quad t > 0 \\ &u(x,0) = f(x), \quad 0 < x < L, \\ &\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < L \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} a_n \left[\sin \tfrac{n\pi(x-ct)}{L} + \sin \tfrac{n\pi(x+ct)}{L} \right]$$

where $a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx,\ n = 1, 2, \ldots.$

Solution: Using separation of variables, the solution to this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin \tfrac{n\pi x}{L} \left(a_n \cos \tfrac{n\pi ct}{L} + b_n \sin \tfrac{n\pi ct}{L} \right)$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L}\, dx \qquad \text{and} \qquad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.
Since $g(x) = 0$ for $0 < x < L$, then $b_n = 0$ for all $n \geqslant 1$, and the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{L} \cos \tfrac{n\pi ct}{L},$$

and since

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right],$$

then

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \left[\sin\tfrac{n\pi(x-ct)}{L} + \sin\tfrac{n\pi(x+ct)}{L}\right]$$

where $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\tfrac{n\pi x}{L}\,dx$.

Exercise 12.4. ✗✗

Consider the homogeneous Dirichlet problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad 0 < x < L$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \quad \text{for} \quad t > 0.$$

Show that the solution can be written as

$$u(x,t) = \frac{1}{2}\Big[F(x-ct) + F(x+ct)\Big],$$

where $F(x)$ is the odd periodic extension of $f(x)$.

Hint: Use separation of variables and

$$\sin a \cos b = \frac{1}{2}\big[\sin(a+b) + \sin(a-b)\big].$$

Solution: We assume a solution of the form

$$u(x,t) = X(t)\cdot T(t)$$

and separate variables to obtain the two ordinary differential equations

$$\begin{aligned} &X''(x) + \lambda X(x) = 0, \quad 0 < x < L && T''(t) + \lambda c^2 T(t) = 0, \quad t > 0 \\ &X(0) = 0 && T'(0) = 0. \\ &X(L) = 0 && \end{aligned}$$

The eigenvalues and eigenfunctions for the X-equation are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad X_n(x) = \sin\tfrac{n\pi x}{L}$$

for $n \geqslant 1$, and the corresponding solutions of the T-equation are

$$T_n(t) = a_n \cos \tfrac{n\pi ct}{L} + b_n \sin \tfrac{n\pi ct}{L}$$

for $n \geqslant 1$.
Using the superposition principle, we write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos \tfrac{n\pi ct}{L} + b_n \sin \tfrac{n\pi ct}{L}) \sin \tfrac{n\pi x}{L},$$

and

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} (-a_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi ct}{L} + b_n \tfrac{n\pi c}{L} \cos \tfrac{n\pi ct}{L}) \sin \tfrac{n\pi x}{L}.$$

We determine the coefficients using the initial conditions and the orthogonality of the eigenfunctions on the interval $[0, L]$.
From the first initial condition we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{L},$$

so that

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$.
From the second initial condition we have

$$0 = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi x}{L},$$

so that $b_n = 0$ for $n \geqslant 1$.
The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi ct}{L} \sin \tfrac{n\pi x}{L} = \sum_{n=1}^{\infty} a_n \left\{ \frac{1}{2} \sin \tfrac{n\pi(x-ct)}{L} + \frac{1}{2} \sin \tfrac{n\pi(x+ct)}{L} \right\}$$

for $0 < x < L$ and $t > 0$.
Note that if $f \in PWS[0, L]$, that is, f is piecewise smooth on the interval $[0, L]$, the series

$$\sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{L}\, dx$$

is the Fourier sine series for f, and converges for <u>all</u> real numbers x, and, except for at most countably many values of x, it converges to the odd periodic extension F of f.
Therefore, assuming the odd periodic extension F is continuous, the solution is

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi(x-ct)}{L} + \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi(x+ct)}{L} = \frac{1}{2} F(x-ct) + \frac{1}{2} F(x+ct)$$

for $0 < x < L$ and $t > 0$.

Exercise 12.5. ✗✗

Consider the homogeneous Dirichlet problem for the wave equation with given initial velocity

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad 0 < x < L$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad t > 0.$$

Show that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(s)\, ds,$$

where $G(x)$ is the odd periodic extension of $g(x)$.
Hint: Use separation of variables and

$$\sin a \sin b = \frac{1}{2}\big[\cos(a-b) - \cos(a+b)\big].$$

Solution: As in the previous exercise, we assume a solution of the form

$$u(x,t) = X(t) \cdot T(t)$$

and separate variables to obtain two ordinary differential equations

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \quad 0 < x < L \qquad & T''(t) + \lambda c^2 T(t) &= 0, \quad t > 0 \\ X(0) &= 0 & T(0) &= 0. \\ X(L) &= 0 & & \end{aligned}$$

As before, the eigenvalues and eigenfunctions for the X-equation are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad X_n(x) = \sin \tfrac{n\pi x}{L}$$

for $n \geqslant 1$, and the corresponding solutions of the T-equation are

$$T_n(t) = a_n \cos \tfrac{n\pi ct}{L} + b_n \sin \tfrac{n\pi ct}{L}$$

for $n \geqslant 1$.
Using the superposition principle, we write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos \tfrac{n\pi ct}{L} + b_n \sin \tfrac{n\pi ct}{L}) \sin \tfrac{n\pi x}{L},$$

and

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} (-a_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi ct}{L} + b_n \tfrac{n\pi c}{L} \cos \tfrac{n\pi ct}{L}) \sin \tfrac{n\pi x}{L}.$$

We determine the coefficients using the initial conditions and the orthogonality of the eigenfunctions on the interval $[0, L]$.
From the first initial condition we have

$$0 = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{L},$$

so that $a_n = 0$ for $n \geqslant 1$.
From the second initial condition we have

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi x}{L},$$

so that

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geqslant 1$, and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi ct}{L} \sin \tfrac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{2} \cos \tfrac{n\pi(x-ct)}{L} - \frac{1}{2} \cos \tfrac{n\pi(x+ct)}{L} \right\}$$

for $0 < x < L$ and $t > 0$.
Now,

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi(x-ct)}{L} + \frac{1}{2} \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi(x+ct)}{L}$$

and if $g \in PWS[0, L]$, that is, g is piecewise smooth on the interval $[0, L]$, then the series

$$\sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L}$$

is the Fourier sine series for g, and converges for <u>all</u> real numbers x, and, except for at most countably many values of x, it converges to the odd periodic extension G of g, that is,

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2} G(x-ct) + \frac{1}{2} G(x+ct).$$

Integrating this from 0 to t, we have

$$\int_0^t \frac{\partial u}{\partial \tau}(x,\tau)\, d\tau = u(x,t) - u(x,0) = u(x,t)$$

since $u(x,0) = 0$. Therefore

$$u(x,t) = \frac{1}{2} \int_0^t G(x-c\tau)\, d\tau + \frac{1}{2} \int_0^t G(x+c\tau)\, d\tau = -\frac{1}{2c} \int_x^{x-ct} G(s)\, ds + \frac{1}{2c} \int_x^{x+ct} G(s)\, ds,$$

where we made the substitution $s = x - c\tau$ in the first integral, and $s = x + c\tau$ in the second integral, and assuming the odd periodic extension of g is continuous, the solution is

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(s)\, ds$$

for $0 < x < L$ and $t > 0$.

Exercise 12.6. ✗✗

Derive d'Alembert's solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0$$

and use it and the superposition principle to solve the wave equation with initial data

$$u(x,0) = e^{-x^2}, \quad \frac{\partial u}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}, \quad -\infty < x < \infty.$$

Solution: Using the change of variables

$$\alpha = x + ct \qquad \text{and} \qquad \beta = x - ct,$$

then from the chain rule we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta},$$

and replacing u by $\dfrac{\partial u}{\partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right),$$

that is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}$$

Again, from the chain rule, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta},$$

and replacing u by $\dfrac{\partial u}{\partial t}$, we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) = c\frac{\partial}{\partial \alpha}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) - c\frac{\partial}{\partial \beta}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right),$$

that is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2},$$

and substituting these expressions into the wave equation, we obtain

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

This equation says that $\dfrac{\partial u}{\partial \beta}$ doesn't depend on α, and therefore

$$\frac{\partial u}{\partial \beta} = g(\beta),$$

where g is an arbitrary differentiable function.
Now, integrating this equation with respect to β, holding α fixed, we get

$$u = \int \frac{\partial u}{\partial \beta}\, d\beta + F(\alpha) = \int g(\beta)\, d\beta + F(\alpha) = F(\alpha) + G(\beta),$$

where F is an arbitrary differentiable function and G is an antiderivative of g.

Finally, using the fact that $\alpha = x + ct$ and $\beta = x - ct$, we get **d'Alembert's solution** to the one-dimensional wave equation:

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary differentiable functions.

Now, in order to solve the original question, we solve the following initial-boundary-value problems, and use the superposition principle to combine them to get a solution to the original problem:

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0, \\ v(x,0) &= e^{-x^2}, \quad -\infty < x < \infty \\ \frac{\partial v}{\partial t}(x,0) &= 0 \quad -\infty < x < \infty, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= c^2 \frac{\partial^2 w}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0, \\ w(x,0) &= 0, \quad -\infty < x < \infty \\ \frac{\partial w}{\partial t}(x,0) &= \frac{x}{(1+x^2)^2} \quad -\infty < x < \infty, \end{aligned} \tag{2}$$

the solution to the original problem is then $u = v + w$. (Check this!!!)

For problem (1), we use the initial conditions to write

$$v(x,0) = e^{-x^2} = F(x) + G(x),$$

so that $F(x) + G(x) = e^{-x^2}$, and

$$\frac{\partial v}{\partial t} = 0 = cF'(x) - cG'(x),$$

so that

$$F(x) - G(x) = C,$$

where C is an arbitrary constant. Therefore,

$$2F(x) = e^{-x^2} + C \qquad \text{and} \qquad 2G(x) = e^{-x^2} - C,$$

and the solution to the first problem is

$$v(x,t) = F(x+ct) + G(x-ct) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right].$$

For problem (2), we use the initial conditions to write

$$w(x,0) = 0 = F(x) + G(x),$$

so that $G(x) = -F(x)$, and

$$\frac{\partial w}{\partial t}(x,0) = \frac{x}{(1+x^2)^2} = cF'(x) - cG'(x),$$

so that $cF'(x) - cG'(x) = 2cF'(x) = \dfrac{x}{(1+x^2)^2}$, and integrating we have

$$2cF(x) = \frac{1}{2}\cdot\frac{-1}{1+x^2} + 2cC,$$

where C is an arbitrary constant. Therefore,

$$F(x) = \frac{-1}{4c(1+x^2)} + C \qquad \text{and} \qquad G(x) = \frac{1}{4c(1+x^2)} - C$$

and the solution to the second problem is

$$w(x,t) = \frac{1}{4c}\left[\frac{-1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right].$$

The solution to the original initial valuer– boundary value problem is then

$$u(x,t) = v(x,t) + w(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] + \frac{1}{4c}\left[\frac{-1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right].$$

Exercise 12.7.
Use d'Alembert's solution of the wave equation to solve the boundary value - initial value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = f(x), \quad -\infty < x < \infty$$
$$\frac{\partial u}{\partial t}(x,0) = g(x), \ -\infty < x < \infty$$

with $f(x) = 0$ and $g(x) = \dfrac{1}{1+x^2}$.

Solution: The boundary value - initial value problem for the displacement of an infinite vibrating string is

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty,\ t > 0 \\
&u(x,0) = f(x), \quad -\infty < x < \infty \\
&\frac{\partial u}{\partial t}(x,0) = g(x), \ \ -\infty < x < \infty
\end{aligned}$$

and the general solution, that is, d'Alembert's solution to the wave equation, is

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

for $-\infty < x < \infty,\ t > 0$, and since $f(x) = 0$ for $-\infty < x < \infty$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+s^2}\,ds.$$

The solution is therefore

$$u(x,t) = \frac{1}{2c}\big[\tan^{-1}(x+ct) - \tan^{-1}(x-ct)\big],$$

for $-\infty < x < \infty,\ t > 0$.

Exercise 12.8.
Use d'Alembert's solution to solve the boundary value problem for the wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \ \ 0 < x < 1,\ t > 0 \\
&u(0,t) = 0, \quad t > 0 \\
&u(1,t) = 0, \quad t > 0 \\
&u(x,0) = 0, \quad 0 < x < 1, \\
&\frac{\partial u}{\partial t}(x,0) = 1, \ \ 0 < x < 1.
\end{aligned}$$

Solution: d'Alembert's solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds$$

where f^* and g^* are the the odd 2-periodic extensions of f and g.
For this problem, we have $c = 1$, and $f(x) = 0$ for $0 < x < 1$, so that $f^*(x) = 0$ for all $x \in \mathbb{R}$.

Also, we have $g(x) = 1$ for $0 < x < 1$, so that

$$g^*(x) = \begin{cases} 1 & \text{for} \quad 0 < x < 1 \\ -1 & \text{for} \quad -1 < x < 0, \end{cases} \quad \text{and} \quad g^*(x+2) = g^*(x) \quad \text{otherwise.}$$

An antiderivative of $g^*(x)$ on the interval $[-1, 1]$ is given by

$$G(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ -x & \text{for} \quad -1 < x < 0, \end{cases}$$

and $G(x+2) = G(x)$ otherwise.
Therefore, the solution is

$$u(x,t) = \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$

where G is as above.

Exercise 12.9.
Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0 \\ &u(0,t) = 0, \qquad t > 0 \\ &u(1,t) = 0, \qquad t > 0 \\ &u(x,0) = 0, \quad\; 0 < x < 1, \\ &\frac{\partial u}{\partial t}(x,0) = \sin \pi x, \quad 0 < x < 1. \end{aligned}$$

Solution: As in the previous problems, d'Alembert's solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\, ds$$

where f^* and g^* are the the odd 2-periodic extensions of f and g.
For this problem, we have $c = 1$, and $f(x) = 0$ for $0 < x < 1$, so that $f^*(x) = 0$ for all $x \in \mathbb{R}$.
Also, we have $g(x) = \sin \pi x$ for $0 < x < 1$, so that

$$g^*(x) = \sin \pi x$$

for $x \in \mathbb{R}$.
An antiderivative of $g^*(x)$ is given by

$$G(x) = -\frac{1}{\pi}\cos \pi x$$

for $x \in \mathbb{R}$.
Therefore, the solution is

$$u(x,t) = \frac{1}{2\pi}\left[\cos \pi(x-t) - \cos \pi(x+t)\right] = \frac{1}{\pi}\sin \pi x \sin \pi t$$

for $0 < x < 1,\ t > 0$.

Exercise 12.10.
Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t \geqslant 0\\
&u(x,0) = x^2, \qquad -\infty < x < \infty\\
&\frac{\partial u}{\partial x}(x,0) = 3 \qquad -\infty < x < \infty.
\end{aligned}$$

Solution: d'Alembert's solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

where $c = 5,\ f(x) = x^2$, and $g(x) = 3$, so that

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left[(x+5t)^2 + (x-5t)^2\right] + \frac{1}{10}\int_{x-5t}^{x+5t} 3\, ds\\
&= \frac{1}{2}\left[x^2 + 10xt + 25t^2 + x^2 - 10xt + 25t^2\right] + \frac{3}{10}\left[x + 5t - x + 5t\right]\\
&= x^2 + 25 + 3t,
\end{aligned}$$

and the solution is

$$u(x,t) = x^2 + 25t^2 + 3t$$

for $-\infty < x < \infty$ and $t > 0$.

Exercise 12.11. ✗✗

Using the one-dimensional wave equation governing the small vertical displacements of a uniform vibrating string,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

derive the conservation of energy for a vibrating string,

$$\frac{dE}{dt} = \rho c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L,$$

where the total enery E is the sum of the kinetic energy and the potential energy,

$$E(t) = \frac{\rho}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{\rho c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Solution: The total energy (potential energy plus kinetic energy) of the string at time t is given by

$$E(t) = \frac{1}{2}\int_0^L \left[T\left(\frac{\partial u}{\partial x}\right)^2 + \rho\left(\frac{\partial u}{\partial t}\right)^2\right] dx = \frac{\rho}{2}\int_0^L \left[c^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx.$$

Using Leibniz's rule, we have

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\left(\frac{\rho}{2}\int_0^L \left[c^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx\right) \\
&= \rho\int_0^L \left[c^2 \frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial t\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial t^2}\right] dx \\
&= \rho\int_0^L \left[c^2 \frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t} + \frac{\partial u}{\partial t}\cdot c^2\frac{\partial^2 u}{\partial x^2}\right] dx \\
&= \rho c^2 \int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\right) dx \\
&= \rho c^2 \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\bigg|_0^L.
\end{aligned}$$

Note that if the string is fixed at both ends, so that

$$\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t) = 0$$

for all $t > 0$, then $E'(t) = 0$ for all $t > 0$, that is, the total energy of the string is conserved.

Exercise 12.12. ✗✗✗

Consider the initial value – boundary value problem (with $h > 0$) given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) - hu(0,t) = 0 \qquad\qquad u(x,0) = f(x)$$

$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad\qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Use separation of variables to do the following:

(a) Show that there are an infinite number of different frequencies of oscillation.

(b) Estimate the leading eigenvalue of the spatial problem and then estimate the large frequencies of oscillation.

(c) Solve the initial value – boundary value problem.

Solution: Since the partial differential equation is linear and homogeneous and the boundary conditions are linear and homogeneous, we can use separation of variables. Assuming a solution of the form

$$u(x,t) = \phi(x) \cdot G(t), \quad 0 \leqslant x \leqslant L, \quad t \geqslant 0$$

and separating variables, we have two ordinary differential equations:

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, \quad 0 \leqslant x \leqslant L, \qquad & G''(t) + \lambda c^2 G(t) = 0, \quad t > 0, \\ \phi'(0) - h\phi(0) = 0 & \\ \phi'(L) = 0 & \end{aligned}$$

(a) We use the Rayleigh quotient to show that $\lambda > 0$ for all eigenvalues λ.

Let λ be an eigenvalue of the Sturm-Liouville problem, and let $\phi(x)$ be the corresponding eigenfunction, then

$$-p(x)\phi(x)\phi'(x)\Big|_0^L = -\phi(L)\phi'(L) + \phi(0)\phi'(0) = h\phi(0)^2 \geqslant 0,$$

and since $q(x) = 0 \leqslant 0$ for all $0 \leqslant x \leqslant L$, then

$$\lambda = \frac{h\phi(0)^2 + \displaystyle\int_0^L \phi'(x)^2\,dx}{\displaystyle\int_0^L \phi(x)^2\,dx} \geqslant 0$$

since $p(x) = \sigma(x) = 1$ for $0 \leqslant x \leqslant L$.

Note that if $\lambda = 0$, then

$$h\phi(0)^2 + \int_0^L \phi'(x)^2\,dx = 0$$

implies that

$$h\phi(0)^2 = 0 \qquad \text{and} \qquad \int_0^L \phi'(x)^2\,dx = 0.$$

Since $h > 0$, this implies that $\phi(0) = 0$, and since ϕ' is continuous on $[0, L]$, that $\phi'(x) = 0$ for $0 \leqslant x \leqslant L$. Therefore $\phi(x)$ is constant on $[0, L]$, so that $\phi(x) = \phi(0) = 0$ for $0 < x < L$, and $\lambda = 0$ is not an eigenvalue. Hence all of the eigenvalues λ of this Sturm-Liouville problem satisfy $\lambda > 0$.

If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the differential equation becomes $\phi'' + \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x, \qquad \text{with} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x$$

for $0 < x < L$.

Applying the first boundary condition,

$$\phi'(0) - h\phi(0) = -hA + B\mu = 0,$$

so that $B = \dfrac{h}{\mu}A$.

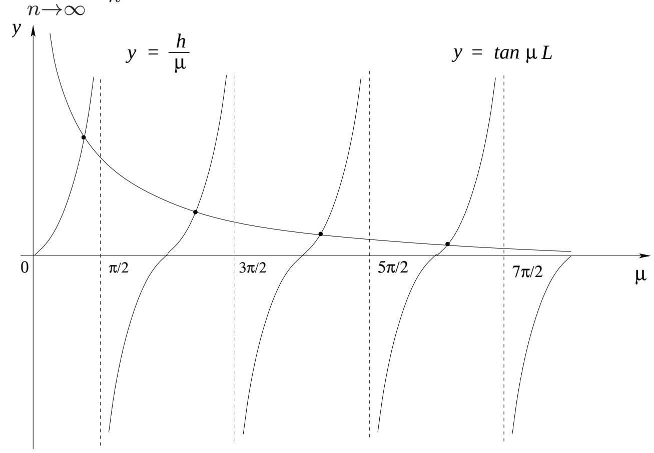
Applying the second boundary condition,

$$\phi'(L) = -\mu A\sin\mu L + \mu B\cos\mu L = A\left(-\mu\sin\mu L + h\cos\mu L\right) = 0,$$

and the boundary value problem has a nontrivial solution if and only if

$$\tan\mu L = \frac{h}{\mu}.$$

From the figure below it is clear that there are an infinite number of distinct eigenvalues $\lambda_n = \mu_n^2$, and that $\lim\limits_{n\to\infty} \lambda_n = +\infty$.



(b) Note that since $\lim_{n\to\infty} \mu_n = +\infty$, then

$$\lim_{n\to\infty} \tan \mu_n L = \lim_{n\to\infty} \frac{h}{\mu_n} = 0,$$

and the roots of the equation

$$\tan \mu_n L = \frac{h}{\mu_n}$$

are approaching the roots of the equation $\tan \mu_n L = 0$, that is, for large n, we have

$$\mu_n \approx \frac{n\pi}{L},$$

and therefore for large n the eigenvalues are

$$\lambda_n = \mu_n^2 \approx \frac{n^2\pi^2}{L^2}.$$

The frequency of oscillation refers to the frequency arising from the solution of the corresponding time equation

$$G_n(t) = a_n \cos \mu_n ct + b_n \sin \mu_n ct,$$

and one period of oscillation corresponds to $\mu_n cT = 2\pi$, and

$$T = \frac{2\pi}{\mu_n c},$$

and the frequency of oscillation is

$$\nu = \frac{1}{T} = \frac{\mu_n c}{2\pi},$$

and for large n,

$$\nu \approx \frac{nc}{2L}.$$

(c) The solutions to the spatial problem are

$$\phi_n(x) = \cos \mu_n x + \frac{h}{\mu_n} \sin \mu_n x, \quad 0 \leqslant x \leqslant L$$

and the corresponding solutions to the time equation are

$$G_n(t) = a_n \cos \mu_n ct + b_n \sin \mu_n ct, \quad t \geqslant 0$$

and from the superposition principle, the function

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \cdot G_n(t) = \sum_{n=1}^{\infty} \left(\cos \mu_n x + \frac{h}{\mu_n} \sin \mu_n x \right) (a_n \cos \mu_n ct + b_n \sin \mu_n ct)$$

satisfies the partial differential equation and the boundary conditions.

Since the spatial problem is a regular Sturm-Liouville problem, then the eigenfunctions are orthogonal on the interval $[0, L]$, and we use this fact to satisfy the initial conditions

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{and} \qquad g(x) = \sum_{n=1}^{\infty} b_n \mu_n c \phi_n(x),$$

where the generalized Fourier coefficients are given by

$$a_n = \frac{\displaystyle\int_0^L f(x)\phi_n(x)\,dx}{\displaystyle\int_0^L \phi_n(x)^2\,dx} \qquad \text{and} \qquad b_n = \frac{\displaystyle\int_0^L g(x)\phi_n(x)\,dx}{\displaystyle\mu_n c \int_0^L \phi_n(x)^2\,dx}$$

for $n \geqslant 1$.

Exercise 12.13. ✗✗

Solve the problem for a vibrating square membrane with side length 1, where the vibrations are governed by the following indexTwo dimension wave equationtwo dimensional wave equation:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), && 0 < x < 1, \quad 0 < y < 1, \quad t > 0\\
&u(0, y, t) = u(1, y, t) = 0, && 0 \leqslant y \leqslant 1, \quad t \geqslant 0\\
&u(x, 0, t) = u(x, 1, t) = 0, && 0 \leqslant x \leqslant 1, \quad t \geqslant 0\\
&u(x, y, 0) = \sin \pi x \sin \pi y, && 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1\\
&\frac{\partial u}{\partial t}(x, y, 0) = \sin \pi x, && 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1.
\end{aligned}$$

Solution: Separating variables, we write $u(x, y, t) = \phi(x, y) \cdot T(t)$, and substitute this into the wave equation

$$\pi^2 \frac{T''}{T} = \frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = -\lambda,$$

to obtain the two equations

$$T'' + \frac{\lambda}{\pi^2} T = 0 \qquad \text{and} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi.$$

Separating variables again in the second equation, we write $\phi(x, y) = X(x) \cdot Y(y)$, and substitute this into the equation, to obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda,$$

that is,

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\kappa$$

where κ is a second separation constant. The boundary conditions give rise to the two boundary value problems

$$\begin{aligned} X'' + \kappa X &= 0 \qquad & Y'' + (\lambda - \kappa)Y &= 0 \\ X(0) &= 0 & Y(0) &= 0 \\ X(1) &= 0 & Y(1) &= 0. \end{aligned}$$

We find nontrivial solutions to the X equation first, since it involves only one separation constant. As in previous problems, there are nontrivial solutions if and only if $\kappa_n = n^2\pi^2$ and the eigenfunctions are

$$X_n(x) = \sin n\pi x$$

for $n \geqslant 1$.

For each $n \geqslant 1$, the function Y satisfies the boundary value problem

$$\begin{aligned} Y'' + (\lambda - n^2\pi^2)Y &= 0 \\ Y(0) &= 0 \\ Y(1) &= 0, \end{aligned}$$

and as in previous problems, this has nontrivial solutions if and only if $\lambda - n^2\pi^2 = m^2\pi^2$, that is $\lambda = (n^2 + m^2)\pi^2$, and the eigenfunctions are

$$Y_m(y) = \sin m\pi y$$

for $m \geqslant 1$.

For each $n, m \geqslant 1$, the function

$$\phi_{n,m}(x, y) = \sin n\pi x \cdot \sin m\pi y$$

satisfies the equation for ϕ, as well as the four boundary conditions.

The solutions of the equation $T'' + \dfrac{\lambda}{\pi^2}T = 0$ corresponding to the separation constant $\lambda = (n^2 + m^2)\pi^2$ are

$$T_{n,m} = A_{n,m} \cos \sqrt{n^2 + m^2}\, t + B_{n,m} \sin \sqrt{n^2 + m^2}\, t$$

and for each $n, m \geqslant 1$, the function

$$u_{n,m}(x, y, t) = \phi_{n,m}(x, y) \cdot T_{n,m}(t) = \sin n\pi x \sin m\pi y \left(A_{n,m} \cos \sqrt{n^2 + m^2}\, t + B_{n,m} \sin \sqrt{n^2 + m^2}\, t \right)$$

satisfies the wave equation and all four boundary conditions. Using the superposition principle, we write the solution as

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin n\pi x \sin m\pi y \left(A_{n,m} \cos \sqrt{n^2 + m^2}\, t + B_{n,m} \sin \sqrt{n^2 + m^2}\, t \right).$$

We evaluate the constants $A_{n,m}$ and $B_{n,m}$ using the initial conditions. Setting $t = 0$ in the above expression for $u(x, y, t)$ we see that

$$\sin \pi x \sin \pi y = u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m} \sin n\pi x \sin m\pi y,$$

so that

$$A_{n,m} = \begin{cases} 1 & \text{for} \quad n = m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Differentiating the expression for $u(x, y, t)$ with respect to t, and setting $t = 0$, we see that

$$\sin \pi x = \frac{\partial u}{\partial t}(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sqrt{n^2 + m^2}\, B_{n,m} \sin n\pi x \sin m\pi y,$$

that is,

$$\sin \pi x = \sum_{n=1}^{\infty} \sin n\pi x \left(\sum_{m=1}^{\infty} \sqrt{n^2 + m^2}\, B_{n,m} \sin m\pi y \right),$$

and we need

$$\sum_{m=1}^{\infty} \sqrt{1 + m^2}\, B_{1,m} \sin m\pi y = 1, \qquad \text{and} \qquad \sum_{m=1}^{\infty} \sqrt{n^2 + m^2}\, B_{n,m} \sin m\pi y = 0 \quad \text{if} \quad n \neq 1.$$

Therefore, we may take $B_{n,m} = 0$ for all $n \neq 1$, while for $n = 1$, we want $\sqrt{1 + m^2}\, B_{1,m}$ to be the coefficients in the Fourier sine series of the function $f(x) = 1,\ 0 \leqslant x \leqslant 1$, that is,

$$B_{1,m} = \frac{2}{\sqrt{1 + m^2}} \int_0^1 \sin m\pi y \, dy = \frac{2}{m\pi\sqrt{1 + m^2}} \left[1 - (-1)^m\right]$$

for $m \geqslant 1$.
Therefore,

$$u(x, y, t) = \sin \pi x \sin \pi y \cos \sqrt{2}\pi t + \sum_{m=1}^{\infty} \frac{2\left[1 - (-1)^m\right]}{m\pi\sqrt{1 + m^2}} \sin \pi x \, \cos m\pi y \, \sin \sqrt{1 + m^2}\, t$$

for $0 < x, y < 1,\ t > 0$.

Exercise 12.14. ✗✗

Solve the wave equation for a vibrating radially symmetric circular membrane

$$\frac{\partial^2 u}{\partial t^2} = \frac{4}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad 0 \leqslant r \leqslant 1,\ t \geqslant 0$$

$$u(1, t) = 0,$$

$$u(r, 0) = 5 J_0(z_3\, r)$$

$$\frac{\partial u}{\partial t}(r, 0) = 0, \quad 0 \leqslant r \leqslant 1,$$

where $J_0(z)$ denotes the Bessel function of the first kind of order zero, and z_n denotes the n^{th} zero of $J_0(z)$.

Solution: We use separation of variables, assuming $u(r,t) = \phi(r) \cdot T(t)$, the wave equation above becomes

$$\frac{4}{r}\left(r\phi'\right)' \cdot T = \phi \cdot T'',$$

and dividing by $4\,\phi \cdot T$, the variables are separated, and we get

$$\frac{(r\phi'(r))'}{r\phi(r)} = \frac{T''(t)}{4\,T(t)}.$$

The two sides of this equation must be a constant, say $-\lambda$, which yields two ordinary differential equations

$$\left(r\phi'\right)' + \lambda\, r\, \phi = 0, \quad 0 \leqslant r \leqslant 1$$

$$T'' + 4\lambda\, T = 0, \quad t \geqslant 0.$$

The boundary condition $u(1,t) = 0$ for all $t \geqslant 0$ is satisfied if we require

$$\phi(1) = 0.$$

Also, since $r = 0$ is a singular point of the differential equation for ϕ, we add the requirement

$$|\phi(r)| \quad \text{bounded at} \quad r = 0,$$

which is equivalent to requiring that $|u(r,t)|$ be bounded at $r = 0$.
Thus, ϕ satisfies the boundary value problem

$$\left(r\phi'\right)' + \lambda\, r\, \phi = 0, \quad 0 \leqslant r \leqslant 1$$

$$\phi(1) = 0,$$

$$|\phi(r)| \quad \text{bounded at} \quad r = 0.$$

We multiply the equation by r and recognize the equation

$$r^2\phi'' + r\phi' + \lambda\, r^2\phi = 0$$

as Bessel's equation of order zero, for which the function

$$\phi(r) = J_0(\sqrt{\lambda}\, r)$$

is the solution bounded at $r = 0$.
In order to satisfy the boundary condition $\phi(1) = 0$, we must have

$$J_0(\sqrt{\lambda}) = 0,$$

or

$$\sqrt{\lambda_n} = z_n, \quad n = 1, 2, \ldots$$

where z_n are the zeros of the function J_0.

Therefore the eigenvalues and eigenfunctions of the boundary value problem satisfied by $\phi(r)$ are

$$\lambda_n = z_n^2 \qquad \text{and} \qquad \phi_n(r) = J_0(\sqrt{\lambda_n}\, r)$$

for $n \geqslant 1$.
For these values of λ which give a nontrivial solution to the boundary value problem for ϕ, the differential equation for T is

$$T''(t) + 4\lambda_n\, T(t) = 0$$

with general solution

$$T_n(t) = a_n \cos 2\sqrt{\lambda_n}\, t + b_n \sin 2\sqrt{\lambda_n}\, t,$$

for $n \geqslant 1$.
For each $n \geqslant 1$, the product solution

$$u_n(r,t) = \phi_n(r) \cdot T_n(t)$$

to the original partial differential equation satisfies the boundary condition $u(1,t) = 0$ and the boundedness condition $|u(r,t)|$ bounded at $r = 0$ for all $t \geqslant 0$.
Using the isuperposition principle we write the solution as

$$u(r,t) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_n}\, r) \left[a_n \cos 2\sqrt{\lambda_n}\, t + b_n \sin 2\sqrt{\lambda_n}\, t \right].$$

The initial conditions are satisfied if

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r) = 5 J_0\left(z_3\, r\right)$$

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} 2 b_n \sqrt{\lambda_n} J_0(\sqrt{\lambda_n}\, r) = 0,$$

for $0 \leqslant r \leqslant 1$.
Using the fact that the eigenfunctions $\{J_0(\sqrt{\lambda_n}\, r)\}_{n \geqslant 1}$ are orthogonal on the interval $[0,1]$ with respect to the weight function $\sigma(r) = r$, we see that $a_n = 0$ for all $n \neq 3$, and $a_3 = 5$, while $b_n = 0$ for all $n \geqslant 1$. Therefore the solution is

$$u(r,t) = 5 J_0\left(z_3\, r\right) \cos 2 z_3 t$$

for $0 \leqslant r \leqslant 1,\ t \geqslant 0$, where z_3 is the third zero of $J_0(z)$.

Exercise 12.15. ✗✗✗

Find the solution for the vibrating circular membrane in polar coordinates

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 100\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), && 0 < r < 1, \quad t > 0 \\
&u(1,t) = 0, && t > 0 \\
&u(r,0) = 1 - r^2, && 0 < r < 1 \\
&\frac{\partial u}{\partial t}(r,0) = 1, && 0 < r < 1.
\end{aligned}$$

You may use the formula

$$\int_0^a x^{p+1} J_p\left(\frac{\alpha}{a}x\right)\, dx = \frac{a^{p+2}}{\alpha} J_{p+1}(\alpha)$$

where α is a positive zero of $J_p(z)$, the Bessel function of the first kind of order p.

Solution: Since $f(r) = 1 - r^2$ and $g(r) = 1$ are radially symmetric, we may assume that the solution does not depend on θ (we can show this by separating variables and applying periodicity conditions in θ). Also, we expect periodic functions in t, and in order to separate variables we write $u(r,t) = R(r) \cdot T(t)$, and obtain the problems

$$\begin{aligned}
rR'' + R' + \lambda^2 rR = 0, \quad 0 < r < 1 \\
R(a) = 0, \\
|R(r)| < M, \text{ for } r \to 0^+,
\end{aligned}$$

where the last condition is added to exclude singular solutions (physically, a singular solution would correspond to a ripping membrane). The time equation is

$$T'' + 100\lambda^2 T = 0, \quad t > 0.$$

The differential equation in the radial problem is known as Bessel's equation of order 0, hence the solutions to the first problem are

$$R(r) = J_0(\lambda r), \quad r > 0,$$

where J_0 is the Bessel function of order 0 of the first kind. The boundary condition $u(1,t) = 0$ for all $t > 0$ can be satisfied by requiring that $R(1) = 0$, that is, $J_0(\lambda) = 0$, so that λ must be a root of the Bessel function J_0. Now, J_0 has infinitely many positive zeros, and we write them as

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \cdots,$$

and therefore we have nontrivial solutions to the boundary value problem if and only if

$$\lambda_n = \alpha_n,$$

$n = 1, 2, 3, \ldots$. These are the eigenvalues of the boundary value problem, and the corresponding eigenfunctions are

$$R_n(r) = J_0\left(\alpha_n r\right),$$

for $n = 1, 2, 3 \ldots$.

The solution to the differential equation for T corresponding to $\lambda_n = \alpha_n$ is given by

$$T_n(t) = A_n \cos 10\lambda_n t + B_n \sin 10\lambda_n t,$$

and the functions

$$u_n(r,t) = \left(A_n \cos 10\lambda_n t + B_n \sin 10\lambda_n t\right) J_0(\lambda_n r)$$

satisfy the wave equation and the boundary condition for each $n = 1, 2, \ldots$.

Using the superposition principle, we write the solution as a Fourier-Bessel series

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos 10\lambda_n t + B_n \sin 10\lambda_n t\right) J_0(\lambda_n r), \qquad (*)$$

and evaluate the coefficients A_n and B_n from the initial conditions. In order to do this, we need the orthogonality conditions

$$\int_0^1 r J_0(\lambda_n r) J_0(\lambda_m r)\, dr = 0$$

for $n \neq m$. In order to see this, we recall that R_n and R_m satisfy the equations

$$\begin{aligned} \left(r R_n'\right)' + \lambda_n^2 r R_n &= 0 \\ \left(r R_m'\right)' + \lambda_m^2 r R_m &= 0 \end{aligned}$$

and multiplying the first equation by R_m and the second equation by R_n and subtracting, we get

$$\left(r R_n'\right)' R_m - \left(r R_m'\right)' R_n = (\lambda_m^2 - \lambda_n^2) r R_n R_m,$$

that is,

$$\left(r(R_m R_n' - R_n R_m')\right)' = (\lambda_n^2 - \lambda_m^2) r R_n R_m,$$

and integrating this last equation from 0 to 1 and using the fact that $R_m(1) = R_n(1) = 0$, we have

$$(\lambda_n^2 - \lambda_m^2) \int_0^1 r R_n(r) R_m(r)\, dr = 0$$

for $n \neq m$, and since $\lambda_n \neq \lambda_m$, we have

$$\int_0^1 r J_0\left(\alpha_n r\right) J_0\left(\alpha_m r\right)\, dr = 0 \qquad (**)$$

for $n \neq m$, and the eigenfunctions are orthogonal with respect to the weight function r on the interval $[0, 1]$.

In order to determine the coefficient A_n from the initial condition, we also need to know the value of

$$\int_0^1 r R_n(r)^2\, dr,$$

and we can determine this by considering the differential equation satisfied by R_n, namely,

$$\left(rR_n'\right)' + \lambda_n^2 r R_n = 0,$$

and multiplying this by $2rR_n'$ to get

$$\frac{d}{dr}\left[(rR_n')^2\right] + 2\lambda_n^2 r^2 R_n R_n' = 0,$$

and integrating both terms we get

$$\left(rR_n'(r)\right)^2\Big|_0^1 + \lambda_n^2\left[r^2 R_n(r)^2\Big|_0^1 - \int_0^1 2rR_n(r)^2\,dr\right] = 0,$$

where we integrated by parts in the second integral. Since $R_n(1) = 0$, we get

$$R_n'(1)^2 - \lambda_n^2\int_0^1 2rR_n(r)^2\,dr = 0,$$

that is,

$$\int_0^1 rR_n(r)^2\,dr = \frac{1}{2\lambda_n^2}R_n'(1)^2 = \frac{1}{2}J_0'(\lambda_n)^2 = \frac{1}{2}J_1(\lambda_n)^2 \qquad (***)$$

for $n = 1, 2, 3, \ldots$. Where we have used the identity $J_0'(r) = -J_1(r)$.
Now we can use the initial conditions to determine the coefficients in the solution $(*)$. Setting $t = 0$, multiplying by $rR_m(r)$, and integrating from 0 to 1, we get

$$\int_0^1 rf(r)R_m(r)\,dr = A_m\int_0^1 rR_m(r)^2\,dr = A_m\frac{J_1(\lambda_m)^2}{2},$$

and since $f(r) = 1 - r^2$, we have

$$A_m = \frac{2}{J_1(\lambda_m)^2}\int_0^1 r(1-r^2)R_m(r)\,dr = \frac{2}{J_1(\lambda_m)^2}\int_0^1 r(1-r^2)J_0(\lambda_m r)\,dr$$

for $m = 1, 2, 3, \ldots$.
If we make the substitution $s = \lambda_m r$ in the last integral, we get

$$\int_0^1 r(1-r^2)J_0(\lambda_m r)\,dr = \frac{1}{\lambda_m^4}\int_0^{\lambda_m} s(\lambda_m^2 - s^2)J_0(s)ds,$$

and integrating by parts with $u = \lambda_m^2 - s^2$ and $dv = J_0(s)s\,ds$ so that

$$v = \int sJ_0(s)\,ds = sJ_1(s),$$

we get

$$\int_0^1 r(1-r^2)J_0(\lambda_m r)\,dr = \frac{2}{\lambda_m^4}\int_0^{\lambda_m} J_1(s)s^2\,ds = \frac{2}{\lambda_m^4}s^2 J_2(s)\Big|_0^{\lambda_m} = \frac{2}{\lambda_m^2}J_2(\lambda_m),$$

for $m = 1, 2, 3, \ldots$, where we used the identity

$$\int x^{p+1} J_p(x)\, dx = x^{p+1} J_{p+1}(x) + C.$$

Therefore,

$$A_m = \frac{2}{J_1(\lambda_m)^2} \int_0^1 r(1-r^2) J_0(\lambda_m r)\, dr = \frac{4 J_2(\lambda_m)}{\lambda_m^2 J_1(\lambda_m)^2},$$

and finally, since λ_m is a zero of J_0, from the identity

$$J_0(x) + J_2(x) = \frac{2}{x} J_1(x),$$

we have

$$A_m = \frac{8}{\lambda_m^3 J_1(\lambda_m)}$$

for $m = 1, 2, 3, \ldots$, and

$$1 - r^2 = f(r) = \sum_{n=1}^{\infty} \frac{8}{\lambda_m^3 J_1(\lambda_m)} J_0(\lambda_n r), \quad 0 < r < 1$$

is the Fourier-Bessel expansion for the initial displacement.
In order to compute the B_n's, we differentiate $(*)$ with respect to t and then set $t = 0$ to get

$$1 = g(r) = \frac{\partial u}{\partial t}(r, 0) = \sum_{n=1}^{\infty} 10 \lambda_n B_n J_0(\lambda_n r),$$

and a similar argument to that above shows that

$$B_m = \frac{1}{5 \lambda_m^2 J_1(\lambda_m)}$$

for $m = 1, 2, 3, \ldots$, therefore the solution is

$$u(r, t) = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{5 \lambda_n^3 J_1(\lambda_n)} \left[40 \cos(10 \lambda_n t) + \lambda_n \sin(10 \lambda_n t) \right]$$

for $0 \leqslant r \leqslant 1$, and $t \geqslant 0$.

Exercise 12.16. ✗✗

Find a solution for the vibrating circular membrane given below:

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right), \quad 0 < r < 1, \quad t > 0$$

$$u(1,t) = 0, \qquad t > 0$$

$$u(r,0) = 0, \qquad 0 < r < 1$$

$$\frac{\partial u}{\partial t}(r,0) = J_0(\alpha_3 r), \qquad 0 < r < 1,$$

where α_3 denotes the third root of the Bessel function J_0. You may use the formula

$$\int_0^a J_p^2\left(\frac{\alpha}{a}x\right) x\,dx = \frac{a^2}{2} J_{p+1}^2(\alpha)$$

where α is a positive zero of $J_p(z)$.

Solution: As in the previous problem (12.15), the solution is

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n t + B_n \sin \lambda_n t)\, J_0(\lambda_n r),$$

where λ_n is the n^{th} positive root of the Bessel function J_0.
In this case, however, $u(r,0) = f(r) = 0$ for $0 < r < 1$, so that $A_n = 0$ for all $n \geqslant 1$. We use the initial condition

$$\frac{\partial u}{\partial t}(r,0) = J_0(\alpha_3 r), \quad 0 < r < 1$$

and the orthogonality to determine the B_n's, as in the previous problem, we have

$$B_n = \frac{2}{\lambda_n J_1(\lambda_n)^2} \int_0^1 r J_0(\lambda_3 r) J_0(\lambda_n r)\, dr = 0$$

for all $n \neq 3$, while for $n = 3$, we have

$$B_3 = \frac{2}{\lambda_3 J_1(\lambda_3)^2} \int_0^1 r J_0(\lambda_3 r)^2\, dr = \frac{2}{\lambda_3 J_1(\lambda_3)^2} \cdot \frac{1}{2} J_1(\lambda_3)^2 = \frac{1}{\lambda_3}$$

and the solution is

$$u(r,t) = \frac{1}{\lambda_3} J_0(\lambda_3 r) \sin \lambda_3 t$$

for $0 \leqslant r \leqslant 1$, and $t \geqslant 0$.

Exercise 12.17. ✗✗

Solve the wave equation on a disk of radius $a > 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad \text{subject to} \qquad \frac{\partial u}{\partial r}(a, \theta, t) = 0$$

with initial conditions

$$u(r, \theta, 0) = 0, \qquad \frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r).$$

Solution: Since neither the boundary conditions nor the initial conditions depend on the variable θ, we look for a solution that is also independent of θ, say $u = u(r, t)$. In this case the problem becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad \text{subject to} \qquad \frac{\partial u}{\partial r}(a, t) = 0$$

with initial conditions

$$u(r, 0) = 0, \qquad \frac{\partial u}{\partial t}(r, 0) = \beta(r).$$

Separating variables, we write $u(r, t) = R(r) \cdot T(t)$, then R and T satisfy the following boundary value and initial value problems, respectively,

$$\begin{aligned} (rR')' + \lambda r R = 0, \quad 0 < r < a, \qquad & T'' + \lambda c^2 T = 0, \quad t > 0, \\ R'(a) = 0 \qquad & T(0) = 0 \\ |R(0)| < \infty \qquad & \end{aligned}$$

We solve the singular Sturm-Liouville problem for R first. The Rayleigh quotient is

$$\lambda = \frac{-rRR'\Big|_0^a + \displaystyle\int_0^a r(R')^2\, dr}{\displaystyle\int_0^a r\, R^2\, dr},$$

and from the boundedness and boundary condition,

$$-rRR'\Big|_0^a = -aR(a)R'(a) = 0,$$

so that $\lambda \geqslant 0$, that is, there are no negative eigenvalues.

case 1. If $\lambda = 0$, then the differential equation is

$$(rR')' = 0,$$

with general solution

$$R(r) = A \log r + B.$$

Applying the boundedness condition, we have $A = 0$, and the eigenfunction is

$$R_0(r) = 1$$

for $0 < r < a$.
<u>case 2</u>. If $\lambda > 0$, then the differential equation is

$$(rR')' + \lambda rR = 0,$$

which is Bessel's equation of order 0, with general solution

$$R(r) = AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r).$$

Applying the boundedness condition, we have $B = 0$, and the solution is

$$R(r) = AJ_0(\sqrt{\lambda}r),$$

and applying the boundary condition, we have

$$R'(a) = A\sqrt{\lambda}J_0'(\sqrt{\lambda}a) = 0,$$

so that $\sqrt{\lambda}a = z_n$, the n^{th} positive root of $J_0'(z)$.
The eigenvalues and corresponding eigenfunctions in this case are

$$\lambda_{n0} = \left(\frac{z_n}{a}\right)^2 \qquad \text{and} \qquad R_{n0} = J_0(\sqrt{\lambda_{n0}})$$

for $n = 1, 2, 3, \ldots$
The corresponding time equation is

$$T''(t) + \lambda c^2 T = 0, \quad t > 0$$
$$T(0) = 0.$$

If $\lambda = 0$, the equation is $T''(t) = 0$, with general solution

$$T(t) = At + B,$$

and from the initial condition we have $T(0) = B = 0$, and we may take $A = 1$, so that

$$T_0(t) = t$$

for $t > 0$.
If $\lambda > 0$, the equation is $T'' + \lambda_{n0}T = 0$ with general solution

$$T(t) = A\cos\sqrt{\lambda}ct + B\sin\sqrt{\lambda}ct,$$

and from the initial condition we have $T(0) = A = 0$, and we may take

$$T_{n0}(t) = \sin\sqrt{\lambda_{n0}}ct$$

for $t > 0$.
Using the superposition principle, we write

$$u(r,t) = c_0 t + \sum_{n=1}^{\infty} c_n J_0(\sqrt{\lambda_{n0}}r) \cdot \sin\sqrt{\lambda_{n0}}ct$$

for $0 < r < a,\ t > 0$.
Finally, applying the nonhomogeneous initial condition, we have

$$\beta(r) = \frac{\partial u}{\partial t}(r,0) = c_0 + \sum_{n=1}^{\infty} c_n J_0(\sqrt{\lambda_{n0}}\, r)\sqrt{\lambda_{n0}}\, c$$

for $0 < r < a$, and using the orthogonality of the eigenfunctions

$$c_0 = \frac{\displaystyle\int_0^a \beta(r)\, r\, dr}{\displaystyle\int_0^a r\, dr} = \frac{2}{a^2}\int_0^a \beta(r)\, r\, dr$$

and

$$c_n = \frac{\displaystyle\int_0^a \beta(r) J_0(\sqrt{\lambda_{n0}}\, r)\, r\, dr}{\displaystyle\sqrt{\lambda_{n0}}\, c \int_0^a J_0(\sqrt{\lambda_{n0}}\, r)^2 r\, dr}$$

for $n = 1, 2, 3, \ldots$.

Exercise 12.18. ✗✗✗
Solve the wave equation for a "pie-shaped" membrane of radius a and angle $\frac{\pi}{3} (= 60^\circ)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Assume that $\lambda > 0$. Determine the natural frequencies of oscillation if the boundary conditions are

$$u(r,0,t) = 0, \qquad u\left(r, \frac{\pi}{3}, t\right) = 0, \qquad \frac{\partial u}{\partial r}(a,\theta,t) = 0.$$

Solution: The wave equation in polar coordinates is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$

and assuming a solution of the form $u(r,\theta,t) = R(r)\cdot\Theta(\theta)\cdot T(t)$, we have

$$\frac{T''}{c^2 T} = \frac{1}{rR}\left(rR'\right)' + \frac{1}{r^2\Theta}\Theta'' = -\lambda$$

where λ is the separation constant. This gives

$$\frac{\Theta''}{\Theta} = -\lambda r^2 - \frac{r}{R}\left(rR'\right)' = -\tau$$

where τ is another separation constant.

We can satisfy the boundary conditions in θ,

$$u(r,0,t)=0 \qquad \text{and} \qquad u\left(r,\frac{\pi}{3},t\right)=0$$

for all $0\leqslant r\leqslant a$ and $t\geqslant 0$, by requiring that

$$\Theta(0)=0 \qquad \text{and} \qquad \Theta\left(\frac{\pi}{3}\right)=0.$$

Also, if the solution is to be bounded we need to require the boundedness condition

$$|u(0,\theta,t)|<\infty,$$

for all $0\leqslant\theta\leqslant\frac{\pi}{3}$, and $t\geqslant 0$. We can satisfy the boundedness condition as well as the boundary condition that

$$\frac{\partial u}{\partial r}(a,\theta,t)=0$$

for $0\leqslant\theta\leqslant\frac{\pi}{3}$ and $t\geqslant 0$, by requiring that

$$R'(a)=0 \qquad \text{and} \qquad |R(0)|<\infty.$$

Therefore R and Θ satisfy the boundary value problems

$$\begin{aligned} r(rR')'+(\lambda r^2-\tau)R=0,\quad 0\leqslant r\leqslant a, &\qquad \Theta''+\tau\Theta=0,\quad 0\leqslant\theta\leqslant\frac{\pi}{3},\\ R'(a)=0 &\qquad \Theta(0)=0\\ |R(0)|<\infty &\qquad \Theta\left(\frac{\pi}{3}\right)=0 \end{aligned}$$

while T satisfies the differential equation

$$T''+\lambda c^2T=0,\quad t\geqslant 0.$$

Since the problem for Θ has a complete set of homogeneous boundary conditions, we solve this first. The eigenvalues and corresponding eigenfunctions are

$$\tau_m^2=(3m)^2 \qquad \text{and} \qquad \Theta_m(\theta)=\sin 3m\theta$$

for $m=1,2,\dots$

The boundary value problem for the corresponding functions $R(r)$ is

$$\begin{aligned} r(rR')'+(\lambda r^2-9m^2)R=0,\quad 0\leqslant r\leqslant a,\\ R'(a)=0\\ |R(0)|<\infty \end{aligned}$$

Making the substitution $z=\sqrt{\lambda}r$, this equation becomes

$$z^2\frac{d^2R}{dz^2}+z\frac{dR}{dz}+(z^2-9m^2)=0$$

which is Bessel's equation of order $3m$ for $m = 0, 1, 2, \ldots$, and the general solution to the R-equation is

$$R(r) = c_1 J_{3m}(\sqrt{\lambda} r) + c_2 Y_{3m}(\sqrt{\lambda} r)$$

where $J_{3m}(\sqrt{\lambda} r)$ is the Bessel function of the first kind of order $3m$, and $Y_{3m}(\sqrt{\lambda} r)$ is the Bessel function of the second kind of order $3m$.

Applying the boundedness condition $|R(0)| < \infty$, then $c_2 = 0$, and the solutions are

$$R(r) = c_1 J_{3m}(\sqrt{\lambda} r)$$

for $0 \leqslant r \leqslant a$.

Applying the boundary condition $R'(a) = 0$, we have

$$R'(a) = c_1 J'_{3m}(\sqrt{\lambda} a) = 0,$$

and we have a nontrivial solution if and only if $J'_{3m}(\sqrt{\lambda} a) = 0$, that is,

$$\sqrt{\lambda} a = \omega_{3m,n}$$

for $n = 1, 2, 3, \ldots$, where $\omega_{3m,n}$ denotes the value of z for which $J_{3m}(z)$ has minima and maxima in increasing order, that is, the positive roots of $J'_{3m}(z)$.

Therefore the eigenvalues are

$$\lambda_{3m,n} = \left(\frac{\omega_{3m,n}}{a}\right)^2$$

with corresponding eigenfunctions

$$R_{3m,n} = J_{3m}\left(\sqrt{\lambda_{3m,n}} r\right)$$

for $m \geqslant 0$, $n \geqslant 1$. We note that $\lambda_{3m,n} > 0$ and for a fixed m, the eigenfunctions

$$R_{3m,n} = J_{3m}\left(\frac{\omega_{3m,n}}{a} r\right), \quad n = 1, 2, 3, \ldots$$

are orthogonal on the interval $0 \leqslant r \leqslant a$ with weight function $\sigma(r) = r$, so that

$$\int_0^a J_{3m}\left(\frac{\omega_{3m,k}}{a} r\right) J_{3m}\left(\frac{\omega_{3m,\ell}}{a} r\right) r\, dr = 0$$

for $k \neq \ell$.

The corresponding solutions to the time equation

$$T'' + \lambda_{3m,n} c^2 T = 0, \quad t \geqslant 0,$$

are given by

$$T_{3m,n}(t) = A_{3m,n} \cos\left(\sqrt{\lambda_{3m,n}} ct\right) + B_{3m,n} \sin\left(\sqrt{\lambda_{3m,n}} ct\right), \quad t \geqslant 0.$$

Thus, the frequencies of oscillation $\nu_{m,n}$ satisfy

$$\frac{\sqrt{\lambda_{3m,n}} c}{\nu_{m,n}} = 2\pi,$$

that is,

$$\nu_{m,n} = \frac{\sqrt{\lambda_{3m,n}}c}{2\pi} = \frac{\omega_{3m,n}c}{2\pi a}$$

for $m \geqslant 1$ and $n \geqslant 1$.

Exercise 12.19. ✗✗✗

(a) Show that small displacements $u(x,t)$ of a hanging chain of length L are governed by the initial value - boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = g\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right), \quad 0 < x < L,\ t > 0,$$

$$u(L,t) = 0, \quad t > 0,$$

$$u(x,t) \text{ bounded, as } x \to 0^+, \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 < x < L,$$

$$\frac{\partial u}{\partial t}(x,0) = v(x), \quad 0 < x < L.$$

where g is the acceleration due to gravity (assumed constant).

(b) Solve the initial value - boundary value problem above.

Solution:

(a) First we give a more detailed description of the **Hanging Chain Problem**. A flexible chain of length L and density (mass per unit length) ρ is fixed at the upper end ($x = L$) and allowed to make small vibrations in a vertical plane. We let $u(x,t)$ be the deflection from the vertical in this plane. In the equilibrium position, the weight of the chain below a point x is equal to the tension in the chain:

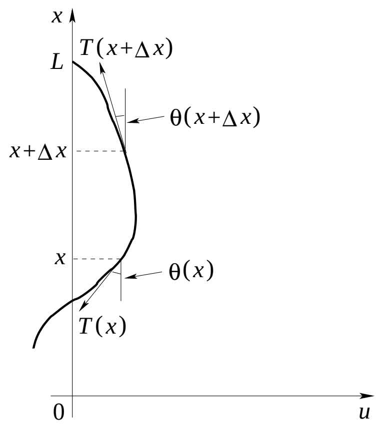
$$T_0(x) = \rho g x,$$

where g is the gravitational acceleration (assumed constant and acting vertically downward).

At time $t > 0$, the horizontal displacement of the chain at the point x is given by $u = u(x,t)$, and applying Newton's second law to the portion of the chain between x and $x + \Delta x$, we obtain

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} \doteq T(x+\Delta x)\sin(\theta(x+\Delta x)) - T(x)\sin(\theta(x)), \tag{$*$}$$

where $\theta(x)$ is the angle between the vertical and the tension $T(x)$, as in the figure.



For small displacements, we have

$$\sin(\theta(x)) \doteq \frac{\partial u}{\partial x}(x,t) \qquad \text{and} \qquad \sin(\theta(x+\Delta x)) \doteq \frac{\partial u}{\partial x}(x+\Delta x,t),$$

and dividing by Δx in $(*)$, we have

$$\rho\frac{\partial^2 u}{\partial t^2} \doteq \frac{1}{\Delta x}\left[T(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t) - T(x)\frac{\partial u}{\partial x}(x,t)\right]. \qquad (**)$$

Letting $\Delta x \to 0$ in $(**)$, in the limit we get the equation

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x)\frac{\partial u}{\partial x}\right].$$

For small displacements, the tension $T(x)$ is approximately the equilibrium tension $T_0(x) = \rho g x$, and the partial differential equation governing *small* displacements of the hanging chain is

$$\frac{\partial^2 u}{\partial t^2} = g\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right), \qquad (***)$$

for $0 < x < L$, and $t > 0$.

Since the chain is fixed at $x = L$, one boundary condition is

$$u(L,t) = 0$$

for $t > 0$.

Also, since the displacement is to remain bounded as $x \to 0^+$, we require the boundedness condition

$$|u(x,t)| \leqslant M, \quad \text{for all} \quad t > 0$$

as $x \to 0^+$.

Finally, to obtain a unique solution to this problem, we need the initial conditions

$$u(x,0) = f(x) \qquad \text{the initial shape of the chain}$$

$$\frac{\partial u}{\partial t}(x,0) = v(x) \qquad \text{the initial velocity of the chain.}$$

The displacement of the hanging chain satisfies the initial value – boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = g\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right), \quad 0 < x < L,\ t > 0$$

$$u(L,t) = 0, \quad t > 0$$

$$\lim_{x\to 0^+} |u(x,t)| \leqslant M, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < L$$

$$\frac{\partial u}{\partial t}(x,0) = v(x), \quad 0 < x < L.$$

(b) Since the partial differential equation and the boundary condition are both linear and homogeneous, we can use separation of variables.

Step 1: Assuming a solution of the form $u(x,t) = X(x)\cdot G(t)$ and substituting this into the partial differential equation, we get

$$X\cdot G'' = g(xX''\cdot G + X'\cdot G),$$

and separating variables,

$$\frac{xX'' + X'}{X} = \frac{G''}{gG} = -\lambda$$

where λ is the separation constant.

We obtain the two problems:

$$\begin{aligned} xX''(x) + X'(x) + \lambda X(x) = 0, &\quad 0 < x < L \\ X(L) = 0, & \\ |X(0)| < \infty. & \end{aligned} \qquad\qquad G''(t) + \lambda\, g\, G(t) = 0, \quad t > 0$$

The spatial problem can be written in Sturm-Liouville form

$$\frac{d}{dx}\left(x\frac{dX}{dx}\right) + \lambda X = 0,$$

where $p(x) = x$, $q(x) = 0$, and $\sigma(x) = 1$, for $0 < x < L$. Note that this is a singular Sturm-Liouville problem since $p(0) = 0$, and the boundary conditions are not of Sturm-Liouville type.

However, if λ is an eigenvalue of this problem with corresponding eigenfunction X, then the relationship between λ and X still holds, that is, λ is given by the Rayleigh quotient

$$\lambda = R(X) = \frac{-xX(x)X'(x)\Big|_0^L + \int_0^L xX'(x)^2\,dx}{\int_0^L X(x)^2\,dx},$$

and from the boundary condition and the boundedness condition we have

$$\lambda = \frac{\int_0^L xX'(x)^2\,dx}{\int_0^L X(x)^2\,dx} \geqslant 0,$$

so that all of the eigenvalues of this singular Sturm-Liouville problem are nonnegative. We leave it as an exercise to show that $\lambda = 0$ is not an eigenvalue, and in fact, all of the eigenvalues are positive.

Therefore we can write $\lambda = \mu^2$ where $\mu \neq 0$, and the boundary value problem for X becomes

$$xX'' + X' + \mu^2 X = 0, \quad 0 < x < L$$
$$X(L) = 0,$$
$$|X(0)| < \infty.$$

We can transform this into Bessel's equation of order zero, by letting

$$s = 2\sqrt{x} \qquad \text{and} \qquad \varphi(s) = X(x(s)),$$

then from the chain rule we have

$$\frac{dX}{dx} = \frac{d\varphi}{ds}\cdot\frac{ds}{dx} = \frac{1}{\sqrt{x}}\cdot\frac{d\varphi}{ds},$$

and

$$\frac{d^2X}{dx^2} = \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\cdot\frac{d\varphi}{ds}\right) = \frac{1}{x}\cdot\frac{d^2\varphi}{ds^2} - \frac{1}{2x^{\frac{3}{2}}}\cdot\frac{d\varphi}{ds}.$$

Therefore

$$\frac{dX}{dx} = \frac{2}{s}\cdot\frac{d\varphi}{ds}$$
$$\frac{d^2X}{dx^2} = \frac{4}{s^2}\cdot\frac{d^2\varphi}{ds^2} - \frac{4}{s^3}\cdot\frac{d\varphi}{ds},$$

and

$$xX'' + X' + \mu^2 X = 0$$

implies that

$$\frac{s^2}{4}\left(\frac{4}{s^2}\cdot\frac{d^2\varphi}{ds^2} - \frac{4}{s^3}\cdot\frac{d\varphi}{ds}\right) + \frac{2}{s}\cdot\frac{d\varphi}{ds} + \mu^2\varphi = 0,$$

that is,

$$\frac{d^2\varphi}{ds^2} - \frac{1}{s}\cdot\frac{d\varphi}{ds} + \frac{2}{s}\cdot\frac{d\varphi}{ds} + \mu^2\varphi = 0,$$

so that

$$s^2\frac{d^2\varphi}{ds^2} + s\frac{d\varphi}{ds} + \mu^2 s^2\varphi = 0,$$

which is Bessel's equation of order zero.

The general solution is

$$\varphi(s) = AJ_0(\mu s) + BY_0(\mu s),$$

and from the boundedness condition at $x = 0$, we must have $B = 0$, so that

$$\varphi(s) = AJ_0(\mu s)$$

for $0 < s < 2\sqrt{L}$.

Applying the boundary condition $\varphi(2\sqrt{L}) = 0$, we have a nontrivial solution if and only if

$$\varphi(2\sqrt{L}) = AJ_0(\mu 2\sqrt{L}) = 0,$$

and $A \neq 0$, that is, if and only if $\mu_n 2\sqrt{L} = z_n$, the n^{th} positive zero of J_0.

The eigenvalues are

$$\lambda_n = \mu_n^2 = \left(\frac{z_n}{2\sqrt{L}}\right)^2,$$

and the corresponding eigenfunctions are

$$\varphi_n(s) = J_0\left(\frac{z_n s}{2\sqrt{L}}\right)$$

for $n \geqslant 1$.

Note that the functions $\left\{J_0\left(\frac{z_n s}{2\sqrt{L}}\right)\right\}_{n\geqslant 1}$ are orthogonal on the interval $0 \leqslant s \leqslant 2\sqrt{L}$ with respect to the weight function $w(s) = s$, so that

$$\int_0^{2\sqrt{L}} J_0\left(\frac{z_n s}{2\sqrt{L}}\right) J_0\left(\frac{z_m s}{2\sqrt{L}}\right) s\, ds = 0$$

if $n \neq m$.

Since $x = \dfrac{s^2}{4}$, then $dx = \dfrac{1}{2} s\, ds$, and we have

$$\int_0^{L} J_0\left(z_n\sqrt{\frac{x}{L}}\right) J_0\left(z_m\sqrt{\frac{x}{L}}\right) dx = 0$$

if $n \neq m$.

Therefore the eigenfunctions

$$X_n(x) = J_0\left(z_n\sqrt{\frac{x}{L}}\right)$$

for $n \geqslant 1$ are orthogonal on the interval $0 \leqslant x \leqslant L$ with respect to the weight function $\sigma(x) = 1$.

Note that after the substitution $x = \dfrac{s^2}{4}$, the normalization

$$\int_0^{2\sqrt{L}} J_0^2\left(\frac{z_n s}{2\sqrt{L}}\right) s\, ds = 2LJ_1^2(z_n)$$

becomes

$$\int_0^L J_0^2\left(z_n\sqrt{\frac{x}{L}}\right)\,dx = LJ_1^2(z_n)$$

for $n \geqslant 1$.

The corresponding time equation is

$$G'' + g\left(\frac{z_n}{2\sqrt{L}}\right)^2 G = 0,$$

with general solution

$$G_n(t) = A_n \cos\left(\sqrt{\frac{g}{L}}\,\frac{z_n}{2}\,t\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\,\frac{z_n}{2}\,t\right)$$

for $n \geqslant 1$.

Using the superposition principle we can write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\cdot G_n(t) = \sum_{n=1}^{\infty} J_0\left(z_n\sqrt{\frac{x}{L}}\right)\left[A_n \cos\left(\sqrt{\frac{g}{L}}\,\frac{z_n}{2}\,t\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\,\frac{z_n}{2}\,t\right)\right]$$

for $0 < x < L,\ t > 0$.

Finally, we use the initial conditions and the orthogonality of the eigenfunctions to determine the constants A_n and B_n.

Step 3: For $t = 0$, we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n J_0\left(z_n\sqrt{\frac{x}{L}}\right)$$

with

$$A_n = \frac{\int_0^L f(x) J_0\left(z_n\sqrt{\frac{x}{L}}\right)\,dx}{\int_0^L J_0^2\left(z_n\sqrt{\frac{x}{L}}\right)\,dx} = \frac{\int_0^L f(x) J_0\left(z_n\sqrt{\frac{x}{L}}\right)\,dx}{LJ_1^2(z_n)}$$

for $n \geqslant 1$, and

$$v(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n\sqrt{\frac{g}{L}}\,\frac{z_n}{2} J_0\left(z_n\sqrt{\frac{x}{L}}\right)$$

with

$$B_n = \frac{2}{z_n J_1^2(z_n)\sqrt{gL}}\int_0^L v(x) J_0\left(z_n\sqrt{\frac{x}{L}}\right)\,dx$$

for $n \geqslant 1$.

Chapter 13

Laplace Equation Problems

Exercise 13.1.
Show that the function

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is harmonic, that is, it is a solution to the three dimensional Laplace equation $\Delta u = 0$.

Solution: By symmetry, we need only calculate the derivatives with respect to one of the variables, say x, and obtain the other derivatives by permuting the variables. For example,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}},$$

so that

$$\frac{\partial u}{\partial y} = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{and} \qquad \frac{\partial u}{\partial z} = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}.$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right) = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}},$$

so that

$$\frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{(2x^2 - y^2 - z^2) + (2y^2 - x^2 - z^2) + (2z^2 - x^2 - y^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0,$$

that is, u satisfies Laplace's equation $\Delta u = 0$.

Exercise 13.2.
Compute the Laplacian of the function

$$u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

in polar coordinates. Decide if the given function satisfies Laplace's equation $\Delta u = 0$.

Solution: Note that in polar coordinates $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, so that

$$u(r, \theta) = \theta,$$

and since

$$\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0,$$

then Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 \theta}{\partial \theta^2} = 0,$$

and $u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ does satisfy Laplace's equation.

Exercise 13.3.
Compute the Laplacian of the function

$$u(x, y) = \ln(x^2 + y^2)$$

in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.

Solution: Note that in polar coordinates, $r^2 = x^2 + y^2$, so that

$$u(r, \theta) = \ln r^2 = 2 \ln r,$$

and

$$\frac{1}{r}\frac{\partial u}{\partial r} = \frac{2}{r^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial r^2} = -\frac{2}{r^2},$$

and since $\frac{\partial^2 u}{\partial \theta^2} = 0$, then

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = -\frac{2}{r^2} + \frac{2}{r^2} = 0$$

and $u(x, y) = \ln(x^2 + y^2)$ does satisfy Laplace's equation.

Exercise 13.4. ✗

Solve Laplace's equation inside a rectangle:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < L,\ 0 < y < H$$

subject to the boundary conditions

$$u(0,y) = g(y), \quad u(x,0) = 0,$$
$$u(L,y) = 0, \quad u(x,H) = 0,$$

for $0 < x < L,\ 0 < y < H$.

Solution: Since the boundary conditions at $y = 0$ and $y = H$ are homogeneous, we can find a solution using the method of separation of variables.
Writing $u(x,y) = X(x)\,Y(y)$ we obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \qquad \text{(constant)}$$

and hence the two ordinary differential equations

$$X'' - \lambda X = 0 \quad 0 < x < L \qquad \text{and} \qquad Y'' + \lambda Y = 0 \quad 0 < y < H$$
$$X(L) = 0 \qquad\qquad Y(0) = 0$$
$$Y(H) = 0.$$

Solving the boundary value problem for Y, and writing $\lambda_n = \mu_n^2$, the eigenvalues are

$$\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{H^2}$$

and the corresponding eigenfunctions are

$$Y_n(y) = \sin \mu_n y.$$

The corresponding solutions to the equation $X'' - \mu_n^2 X = 0$ are

$$X_n(x) = a_n \cosh \mu_n(L - x) + b_n \sinh \mu_n(L - x),$$

for $n \geqslant 1$, and from the boundary condition $X(L) = 0$, we must have $a_n = 0$ for all $n \geqslant 1$.
Using the superposition principle, we write

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh \mu_n(L - x) \sin \mu_n y.$$

From the boundary condition $u(0,y) = g(y)$, we have

$$g(y) = \sum_{n=1}^{\infty} b_n \sinh \tfrac{n\pi L}{H} \sin \mu_n y$$

so that

$$b_n \sinh \tfrac{n\pi L}{H} = \frac{2}{H} \int_0^H g(y) \sin \mu_n y \, dy,$$

and

$$b_n = \frac{2}{H \sinh \frac{n\pi L}{H}} \int_0^H g(y) \sin \mu_n y \, dy$$

for $n \geqslant 1$.

Exercise 13.5. ✗✗

Solve Laplace's equation inside a rectangle $0 \leqslant x \leqslant L,\ 0 \leqslant y \leqslant H$, with the following boundary conditions:

(a) $\dfrac{\partial u}{\partial x}(0, y) = g(y),\ \dfrac{\partial u}{\partial x}(L, y) = 0,\ u(x, 0) = 0,\ u(x, H) = 0$

(b) $\dfrac{\partial u}{\partial x}(0, y) = 0,\ \dfrac{\partial u}{\partial x}(L, y) = 0,\ u(x, 0) = \begin{cases} 1 & \text{for } 0 < x < L/2, \\ 0 & \text{for } L/2 < x < L, \end{cases}\ \dfrac{\partial u}{\partial y}(x, H) = 0$

Solution:

(a) We assume a solution of the form $u(x, y) = X(x) \cdot Y(y)$, and substitute this into Laplace's equation to obtain

$$X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = 0,$$

so that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda, \quad \text{(constant)}$$

and we have two ordinary differential equations

$$X''(x) - \lambda X(x) = 0 \qquad \text{and} \qquad Y''(y) + \lambda Y(y) = 0.$$

We can satisfy the (homogeneous) boundary conditions by requiring that

$$Y(0) = 0, \quad Y(H) = 0 \qquad \text{and} \qquad X'(L) = 0.$$

Therefore X and Y satisfy the boundary value problems

$$\begin{aligned} X''(x) - \lambda X(x) = 0, \quad 0 \leqslant x \leqslant L \qquad & Y''(y) + \lambda Y(y) = 0, \quad 0 \leqslant y \leqslant H \\ X'(L) = 0 \qquad & Y(0) = 0 \\ & Y(H) = 0. \end{aligned}$$

We solve the complete (Dirichlet) boundary value problem for Y first, the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2$$

with corresponding eigenfunctions

$$Y_n(y) = \sin \tfrac{n\pi y}{H}$$

for $n \geqslant 1$.

The corresponding functions $X(x)$ satisfy the boundary value problem

$$\begin{aligned} X_n'' - \lambda_n X_n &= 0, \quad 0 < x < L \\ X_n'(L) &= 0, \end{aligned}$$

and since the boundary condition at $x = L$ is homogeneous, we choose the following representation for the general solution

$$X_n(x) = A \cosh \tfrac{n\pi(L-x)}{H} + B \sinh \tfrac{n\pi(L-x)}{H}.$$

The condition $X_n'(L) = 0$ implies that $B = 0$, and therefore the solution to the boundary value problem for X is

$$X_n(x) = \cosh \tfrac{n\pi(L-x)}{H}, \quad 0 < x < L$$

for $n \geqslant 1$.

From the superposition principle, the function

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi y}{H} \, \cosh \tfrac{n\pi(L-x)}{H} \tag{$*$}$$

satisfies Laplace's equation in the region $0 < x < L,\ 0 < y < H$, and satisfies all of the boundary conditions except $\dfrac{\partial u}{\partial x}(0,y) = g(y)$.

In order to satisfy this last condition, we use the orthogonality of the eigenfunctions on the interval $0 \leqslant y \leqslant H$. Differentiating ($*$) with respect to x, and setting $x = 0$ we get

$$g(y) = \frac{\partial u}{\partial x}(0,y) = -\sum_{n=1}^{\infty} \frac{n\pi}{H} B_n \sin \tfrac{n\pi y}{H} \, \sinh \tfrac{n\pi L}{H}.$$

Multiplying both sides of this equation by $\sin \frac{m\pi y}{H}$, and integrating over the interval $0 \leqslant y \leqslant H$, we obtain

$$\int_0^H g(y) \sin \tfrac{m\pi y}{H} \, dy = -\sum_{n=1}^{\infty} \frac{n\pi}{H} \sinh \tfrac{n\pi L}{H} B_n \int_0^H \sin \tfrac{m\pi y}{H} \, \sin \tfrac{n\pi y}{H} \, dy$$

and using the orthogonality of the eigenfunctions, we have

$$B_m = \frac{-2}{m\pi \sinh \frac{m\pi L}{H}} \int_0^H g(y) \sin \tfrac{m\pi y}{H} \, dy \tag{$**$}$$

for $m \geqslant 1$.

The solution to Laplace's equation satisfying the given boundary conditions is given by ($*$), where the coefficients $B_m,\ m \geqslant 1$, are given by ($**$).

(b) As above, we assume a solution of the form $u(x,y) = X(x) \cdot Y(y)$ and separate variables we get the boundary value problems

$$X''(x) + \lambda X(x) = 0, \quad 0 \leqslant x \leqslant L \qquad Y''(y) - \lambda Y(y) = 0, \quad 0 \leqslant y \leqslant H$$

$$X'(0) = 0 \qquad\qquad Y'(H) = 0$$

$$X'(L) = 0.$$

We solve the complete (Neumann) boundary value problem for X first, the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

with corresponding eigenfunctions

$$X_n(x) = \cos\tfrac{n\pi x}{L}$$

for $n \geqslant 0$.

The corresponding functions $Y_n(y)$ satisfy the boundary value problem

$$Y_n'' - \lambda_n Y_n = 0, \quad 0 < y < H$$
$$Y_n'(H) = 0,$$

and since the boundary condition at $y = H$ is homogeneous, we choose to represent the general solution as follows

$$Y_n(y) = A\cosh\tfrac{n\pi(H-y)}{L} + B\sinh\tfrac{n\pi(H-y)}{L}.$$

The condition $Y_n'(H) = 0$ implies that $B = 0$, and therefore the solution to the boundary value problem for Y is

$$Y_n(y) = \cosh\tfrac{n\pi(H-y)}{L}, \quad 0 < y < H$$

for $n \geqslant 0$.

From the superposition principle, the function

$$u(x,y) = \sum_{n=0}^{\infty} A_n \cos\tfrac{n\pi x}{L}\,\cosh\tfrac{n\pi(H-y)}{L} \tag{+}$$

satisfies Laplace's equation in the region $0 < x < L$, $0 < y < H$, and satisfies all of the boundary conditions except

$$u(x,0) = f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < L/2, \\ 0 & \text{for} \quad L/2 < x < L. \end{cases}$$

In order to satisfy this condition, we use the orthogonality of the eigenfunctions on the interval $0 \leqslant x \leqslant L$, and setting $y = 0$ we get

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos\tfrac{n\pi x}{L}\,\cosh\tfrac{n\pi H}{L}.$$

Multiplying both sides of this equation by $\cos\frac{m\pi x}{L}$, and integrating over the interval $0 \leqslant x \leqslant L$, we obtain

$$\int_0^L f(x)\cos\tfrac{m\pi x}{L}\,dx = \sum_{n=0}^{\infty}\cosh\tfrac{n\pi H}{L}A_n\int_0^L \cos\tfrac{m\pi x}{L}\,\cos\tfrac{n\pi x}{L}\,dx$$

so that

$$A_0 = \frac{1}{2} \qquad \text{and} \qquad A_m = \frac{2\sin\frac{m\pi}{2}}{m\pi\cosh\frac{m\pi H}{L}} \qquad (++)$$

for $m \geqslant 1$.

From (+) and (++) the solution to Laplace's equation satisfying the given boundary conditions is given by

$$u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2\sin\frac{n\pi}{2}}{n\pi\cosh\frac{n\pi H}{L}}\cos\tfrac{n\pi x}{L}\,\cosh\tfrac{n\pi(H-y)}{L}$$

for $0 < x < L$ and $0 < y < H$.

Exercise 13.6. ✗✗

Solve Laplace's equation for the square $[0,1]\times[0,1]$ with the boundary conditions given below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(x,0) = 0 \qquad 0 \leqslant x \leqslant 1,$$
$$u(x,1) = 100, \qquad 0 \leqslant x \leqslant 1,$$
$$u(0,y) = 0 \qquad 0 \leqslant y \leqslant 1,$$
$$u(1,y) = 100, \qquad 0 \leqslant y \leqslant 1.$$

Solution: We split the original problem into two problems, as below

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$v(x,0) = 0 \qquad 0 \leqslant x \leqslant 1,$$
$$v(x,1) = 100, \qquad 0 \leqslant x \leqslant 1,$$
$$v(0,y) = 0 \qquad 0 \leqslant y \leqslant 1,$$
$$v(1,y) = 0, \qquad 0 \leqslant y \leqslant 1$$

and

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

$$w(x,0) = 0 \qquad 0 \leqslant x \leqslant 1,$$

$$w(x,1) = 0, \qquad 0 \leqslant x \leqslant 1,$$

$$w(0,y) = 0 \qquad 0 \leqslant y \leqslant 1,$$

$$w(1,y) = 100, \qquad 0 \leqslant y \leqslant 1$$

each with one pair of homogeneous boundary conditions (so we can use separation of variables) and the soluton to the original problem is then $u(x,y) = v(x,y) + w(x,y)$.

Now note that we only have to solve one of these problems, say the first, for $v(x,y)$, since we can get the solution to the second problem by interchanging x and y in the solution to the first problem, that is, $w(x,y) = v(y,x)$, so that the solution to the original problem is $u(x,y) = v(x,y) + v(y,x)$.

Writing $v(x,y) = X(x) \cdot Y(y)$, after substituting this into Laplace's equation and separating variables, we have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$$

with separation constant λ. We get the following boundary value problems for X and Y,

$$\begin{aligned} X'' + \lambda X &= 0 \qquad & Y'' - \lambda Y &= 0 \\ X(0) &= 0 & Y(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

We solve the complete x-problem first (it has two boundary conditions). As in previous problems, we have a nontrivial solution for X only if $\lambda = \mu^2 > 0$, and in this case the general solution is

$$X(x) = A\cos\mu x + B\sin\mu x,$$

applying the boundary conditions, we have

$$X(0) = 0 = A, \qquad \text{and} \qquad X(1) = 0 = B\sin\mu.$$

We get a nontrivial solution only when $B \neq 0$, in which case we need $\mu = n\pi$ for some positive integer n, the eigenvalues are $\mu_n^2 = n^2\pi^2$, and the eigenfunctions are

$$X_n(x) = \sin n\pi x$$

for $n \geqslant 1$. For each $n \geqslant 1$, the corresponding equation for Y is $Y'' - \mu_n^2 Y = 0$, with general solution

$$Y(y) = A\cosh n\pi y + B\sinh n\pi y,$$

and applying the boundary condition $Y(0) = 0$, we get $A = 0$, so the corresponding solutions are

$$Y_n(y) = \sinh n\pi y, \quad n \geqslant 1.$$

For each $n \geqslant 1$, the function

$$v_n(x,y) = X_n(x) \cdot Y_n(y) = \sin n\pi x \sinh n\pi y$$

satisfies Laplace's equation and all of the boundary conditions except $v(x,1) = 100$.
Now we use the superposition principle to write

$$v(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y$$

and determine the constants b_n using this last boundary condition, that is,

$$100 = v(x,1) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin n\pi x,$$

and we recognize the constant $b_n \sinh n\pi$ as the Fourier sine series coefficient of the constant function 100 on the interval $[0,1]$, therefore

$$b_n \sinh n\pi = 2\int_0^1 100 \sin n\pi x\, dx = \frac{200}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \dfrac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

The solution to the first problem is therefore

$$v(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh(2n-1)\pi} \sin(2n-1)\pi x \sinh(2n-1)\pi y$$

for $0 \leqslant x, y \leqslant 1$. Interchanging x and y in this solution, we get the solution to the second problem,

$$w(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh(2n-1)\pi} \sin(2n-1)\pi y \sinh(2n-1)\pi x,$$

and the solution to the original problem is therefore

$$u(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh(2n-1)\pi} \left[\sin(2n-1)\pi x \sinh(2n-1)\pi y + \sin(2n-1)\pi y \sinh(2n-1)\pi x\right]$$

for $0 \leqslant x, y \leqslant 1$.

Exercise 13.7. ✗✗

Solve Laplace's equation in the unit square with boundary conditions as given below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

$$u(x,0) = 1 - x \qquad 0 \leqslant x \leqslant 1,$$

$$u(x,1) = x, \qquad 0 \leqslant x \leqslant 1,$$

$$u(0,y) = 0 \qquad 0 \leqslant y \leqslant 1,$$

$$u(1,y) = 0, \qquad 0 \leqslant y \leqslant 1.$$

Solution: As in the previous problem, we divide the problem into two problems:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad 0 < x, y < 1, \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad 0 < x, y < 1,$$

$$v(x,0) = 0, \quad 0 \leqslant x \leqslant 1, \qquad w(x,0) = 1 - x, \quad 0 \leqslant x \leqslant 1,$$

$$v(x,1) = x, \quad 0 \leqslant x \leqslant 1, \qquad w(x,1) = 0, \quad 0 \leqslant x \leqslant 1,$$

$$v(0,y) = 0, \quad 0 \leqslant y \leqslant 1, \qquad w(0,y) = 0, \quad 0 \leqslant y \leqslant 1,$$

$$v(1,y) = 0, \quad 0 \leqslant y \leqslant 1, \qquad w(1,y) = 0, \quad 0 \leqslant y \leqslant 1$$

each with one pair of homogeneous boundary conditions (so we can use separation of variables) and the solution to the original problem is then $u(x,y) = v(x,y) + w(x,y)$.
Note that if we find the solution $v(x,y)$ to the first problem, then the solution to the second problem is

$$w(x,y) = v(1-x, 1-y).$$

We leave it to you to check that $w(x,y)$ satisfies Laplace's equation, and for the boundary conditions, note that

$$\begin{aligned} w(x,0) &= v(1-x,1) = 1-x \\ w(x,1) &= v(1-x,0) = 0 \\ w(0,y) &= v(1,1-y) = 0 \\ w(1,y) &= v(0,1-y) = 0 \end{aligned}$$

so that $w(x,y)$ is a solution to the second problem.
We can use separation of variables as in the previous problem to find the solution $v(x,y)$, and the result is

$$v(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y$$

and the constants b_n are determined from the second boundary condition

$$x = v(x,1) = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin n\pi x$$

so that

$$b_n \sinh n\pi = 2\int_0^1 x \sin n\pi x \, dx = \frac{2(-1)^{n+1}}{n\pi},$$

and

$$b_n = \frac{2(-1)^{n+1}}{n\pi \sinh n\pi}$$

for $n \geqslant 1$.
The solution to the first problem is

$$v(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh n\pi} \sin n\pi x \sinh n\pi y,$$

and the solution to the second problem is

$$w(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sinh n\pi}\sin n\pi(1-x)\sinh n\pi(1-y).$$

The solution to the original problem is given by

$$u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sinh n\pi}\left[\sin n\pi x\sinh n\pi y + \sin n\pi(1-x)\sinh n\pi(1-y)\right]$$

for $0 < x < 1,\ 0 < y < 1$.

Exercise 13.8. ✗
Approximate the temperature at the center of the plate from Exercise (13.7).

Solution: Note that at the center of the plate $x = y = \frac{1}{2}$, and from the previous problem

$$u(\tfrac{1}{2},\tfrac{1}{2}) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sinh n\pi}\sin\tfrac{n\pi}{2}\sinh\tfrac{n\pi}{2}.$$

Now,

$$\sinh n\pi = 2\sinh\tfrac{n\pi}{2}\cosh\tfrac{n\pi}{2},$$

and

$$\sin\tfrac{n\pi}{2} = 0$$

if n is even, while

$$\sin\tfrac{(2k+1)\pi}{2} = (-1)^k$$

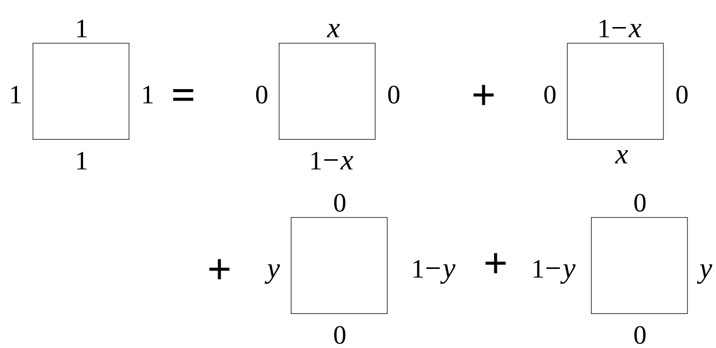
if $n = 2k+1$ is odd.
Therefore,

$$u(\tfrac{1}{2},\tfrac{1}{2}) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\cosh\frac{(2k+1)\pi}{2}}.$$

A geometric symmetry argument as given below shows that this series converges to $\frac{1}{4}$, that is,

$$u(\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{1}{4}.$$

To see this, note that we can decompose the problem when the solution is identically 1 on the boundary of the square into four separate problems as shown in the figure:



By symmetry, each of the four problems has exactly the same value of the solution at the center $(\frac{1}{2}, \frac{1}{2})$, and since the solution to the original problem is identically 1 on the square, then

$$4u(\tfrac{1}{2}, \tfrac{1}{2}) = 1,$$

that is, $u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

Exercise 13.9. ✗✗✗

Solve Laplace's equation inside a circle of radius a,

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \leqslant r \leqslant a, \ -\pi \leqslant \theta \leqslant \pi$$

subject to the boundary condition

$$u(a, \theta) = f(\theta), \quad -\pi \leqslant \theta \leqslant \pi.$$

Hint: You will need to require a boundedness condition on the solution at $r = 0$:

$$|u(0, \theta)| < \infty, \quad -\pi \leqslant \theta \leqslant \pi$$

and periodicity conditions on the solution and its derivative:

$$u(r, -\pi) = u(r, \pi)$$

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

for $r \geqslant 0$.

Solution: This is a classic problem known as the *Dirichlet Problem for Laplace's Equation in a Disk*. We have to solve

$$\Delta u = 0$$

in the disk $D(a) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant a^2\}$.
Here the appropriate coordinate system consists of plane polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. The disk above can then be described as $D(a) = \{(r, \theta) \mid 0 \leqslant r \leqslant a, \ -\pi \leqslant \theta \leqslant \pi\}$.
A formal statement of the problem is given below:

(i) The function $u(r, \theta)$ must satisfy *Laplace's equation in polar coordinates* r, θ, that is,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (*)$$

for $(r, \theta) \in D(a)$.

(ii) In order to ensure that the solution is single-valued, $u(r, \theta)$ must satisfy *periodicity conditions* at $\theta = \pm\pi$, that is,

$$u(r, -\pi) = u(r, \pi)$$

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

for $0 \leqslant r \leqslant a$.

(iii) In order to ensure that the solution is continuous, $u(r,\theta)$ must satisfy *boundedness conditions* at $r=0$, that is,

$$\lim_{r\to 0^+} u(r,\theta) = u(0,\theta) \quad \text{(finite)}$$

for $-\pi \leqslant \theta \leqslant \pi$.

(iv) Finally, the solution must satisfy the *boundary condition* at $r=a$, that is,

$$u(a,\theta) = f(\theta)$$

for $-\pi \leqslant \theta \leqslant \pi$.

The interior Dirichlet problem for Laplace's equation on the disk $D(a)$ models, among other things, the steady-state temperature distribution of a circular plate with top and bottom perfectly insulated, and boundaries held at the temperatures given.
We look for a separable solution, that is, a solution of the form

$$u(r,\theta) = R(r)\Theta(\theta),$$

and substituting this into Laplace's equation $(*)$, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\cdot\Theta + R\cdot\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} = 0,$$

that is,

$$r^2\,R''\cdot\Theta + r\,R'\cdot\Theta + R\cdot\Theta'' = 0.$$

Separating variables, we have

$$\frac{\Theta''}{\Theta} = -\frac{r^2R'' + r\,R'}{R} = -\lambda$$

where λ is the separation constant, and we have two ordinary differential equations:

- The *angle problem*:

$$\Theta'' + \lambda\Theta = 0$$

- The *radius problem*:

$$r^2R'' + r\,R' - \lambda R = 0$$

Note that the periodicity conditions (ii) imply that

$$R(r)\Theta(-\pi) = R(r)\Theta(\pi) \qquad \text{and} \qquad R(r)\Theta'(-\pi) = R(r)\Theta'(\pi)$$

for all $0 \leqslant r \leqslant a$, and in order to obtain a nontrivial solution, we must have

$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi).$$

Therefore, Θ satisfies the regular Sturm-Liouville problem

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ \Theta(-\pi) &= \Theta(\pi) \\ \Theta'(-\pi) &= \Theta'(\pi), \end{aligned}$$

with eigenvalues and corresponding eigenfunctions given by

$$\begin{aligned} &\lambda_0 = 0, \qquad \Theta_0(\theta) = a_0, \quad n = 0 \\ &\lambda_n = n^2 \qquad \Theta_n(\theta) = a_n \cos nx + b_n \sin nx, \quad n \geqslant 1. \end{aligned}$$

The corresponding problem for R_n

$$r^2 R'' + r\, R' - \lambda R = 0$$

is a *Cauchy-Euler equation*, and we assume a solution of the form $R(r) = r^s$, so that

$$R'(r) = sr^{s-1} \qquad \text{and} \qquad R''(r) = s(s-1)r^{s-2},$$

and substituting this into the equation we have

$$s(s-1)r^s + sr^s - \lambda r^s = 0.$$

Assuming $r \neq 0$, we get the characteristic equation

$$s(s-1) + s - \lambda = 0$$

that is, $s^2 = \lambda$, and $s = \pm\sqrt{\lambda}$.
Now we have to consider two cases:

(a) If $n = 0$, then $\lambda_n = 0$, and $s = 0$ is a double root of the characteristic equation, and one solution to the Euler equation is $R(r) = c_1$, that is, a constant solution. In order to find a second linearly independent solution, we consider the original differential equation for $\lambda = 0$,

$$r\frac{d}{dr}\left(r\,\frac{dR}{dr}\right) = 0,$$

integrating,

$$r\,\frac{dR}{dr} = c_2,$$

so that

$$\frac{dR}{dr} = \frac{c_2}{r}$$

and a second independent solution is

$$R(r) = c_2 \log r.$$

The general solution to the radius equation for $\lambda_0 = 0$ is then

$$R_0(r) = c_1 + c_2 \log r$$

for $0 < r \leqslant a$.

(b) If $n > 0$, then $\lambda_n^2 = n$, and $s = \pm n$, and the general solution to the radius equation for $\lambda_n = n^2$ is

$$R_n(r) = c_3 r^n + c_4 r^{-n}$$

for $0 < r \leqslant a$.

From the boundedness condition (iii), we need

$$|u(r,\theta)| < \infty$$

as $r \to 0^+$, so we must have $c_2 = 0$ and $c_4 = 0$, and so

$$R_n(r) = r^n$$

for $n \geqslant 0$.

Using the superposition principle, we write

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right) \qquad (**)$$

and determine the constants from the boundary condition (iv), so that

$$f(\varphi) = u(a,\varphi) = \sum_{n=0}^{\infty} a^n \left(a_n \cos n\varphi + b_n \sin n\varphi\right)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi, \qquad a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi)\cos n\varphi\,d\varphi, \qquad b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi)\sin n\varphi\,d\varphi,$$

for $n \geqslant 1$.

Substituting these values of a_n and b_n into $(**)$, we have

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi + \sum_{n=1}^{\infty}\frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi) r^n \cos n\theta \cos n\varphi\,d\varphi + \sum_{n=1}^{\infty}\frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi) r^n \sin n\theta \sin n\varphi\,d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\left(\cos n\theta\cos n\varphi + \sin n\theta \sin n\varphi\right)\right)d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos n(\theta-\varphi)\right)d\varphi, \end{aligned}$$

and

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos n(\theta-\varphi)\right)d\varphi \qquad (***)$$

for $0 \leqslant r \leqslant a,\ -\pi \leqslant \theta \leqslant \theta$.

We can evaluate the series inside the integral by noting that

$$2\cos n\theta\cos\theta = \cos(n+1)\theta + \cos(n-1)\theta,$$

and if $|b| < 1$, then

$$\begin{aligned}
2\sum_{n=1}^{\infty} b^n \cos n\theta\cos\theta &= \frac{1}{b}\sum_{n=1}^{\infty} b^{n+1}\cos(n+1)\theta + b\sum_{n=1}^{\infty} b^{n-1}\cos(n-1)\theta \\
&= \frac{1}{b}\left[\sum_{n=1}^{\infty} b^n\cos n\theta - b\cos\theta\right] + b\left[\sum_{n=1}^{\infty} b^n \cos n\theta + 1\right] \\
&= \left(\frac{1}{b} + b\right)\sum_{n=1}^{\infty} b^n \cos n\theta + b - \cos\theta,
\end{aligned}$$

so that

$$\left[1 - 2b\cos\theta + b^2\right]\sum_{n=1}^{\infty} b^n\cos n\theta = b\cos\theta - b^2,$$

and

$$\sum_{n=1}^{\infty} b^n \cos n\theta = \frac{b\cos\theta - b^2}{1 - 2b\cos\theta + b^2}.$$

Replacing b by $\dfrac{r}{a}$ and θ by $\theta - \varphi$ under the integral sign in $(***)$, we have

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\frac{ar\cos(\theta-\varphi) - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\right)d\varphi$$

and

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\varphi)}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\,d\varphi \qquad (+)$$

for $0 \leqslant r \leqslant a$, $-\pi \leqslant \theta \leqslant \pi$. This is called *Poisson's integral formula for the disk $D(a)$*, and gives the unique solution to the interior Dirichlet problem for Laplace's equation on the disk.

Exercise 13.10. ✗✗✗

Find the solution of the *exterior Dirichlet problem for a disk*, that is, find a bounded solution to the problem:

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, && a < r < \infty,\ -\pi < \theta < \pi \\
&u(r,\pi) = u(r,-\pi), && a < r < \infty \\
&\frac{\partial u}{\partial \theta}(r,\pi) = \frac{\partial u}{\partial \theta}(r,-\pi), && a < r < \infty \\
&u(a,\theta) = f(\theta), && -\pi < \theta < \pi.
\end{aligned}$$

Solution: A solution to Laplace's equation in polar coordinates which satisfies the periodicity conditions is given by

$$u(r,\theta) = A_0 + B_0\,\log r + \sum_{n=1}^{\infty}\left\{r^n\big(A_n\,\cos n\theta + B_n\,\sin n\theta\big) + \frac{1}{r^n}\big(C_n\,\cos n\theta + D_n\,\sin n\theta\big)\right\},$$

and in order to satisfy the boundedness condition

$$\lim_{r\to\infty} |u(r,\theta)| < \infty,$$

we need $B_0 = A_n = B_n = 0$, for $n = 1, 2, 3, \cdots$.

Therefore,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\frac{1}{r^n}\big(C_n\,\cos n\theta + D_n\,\sin n\theta\big).$$

When $r = a$ we have

$$f(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty}\frac{1}{a^n}\big(C_n\,\cos n\theta + D_n\,\sin n\theta\big),$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi,$$
$$C_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\phi)\,\cos n\phi\,d\phi,$$
$$D_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\phi)\,\sin n\phi\,d\phi$$

for $n = 1, 2, 3\cdots$.

Therefore

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\int_{-\pi}^{\pi} f(\phi)\big\{\cos n\phi\,\cos n\theta + \sin n\phi\,\sin n\theta\big\}\,d\phi,$$

that is,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\Big\{1 + 2\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\,\cos n(\theta-\phi)\Big\}\,d\phi.$$

To evaluate the sum we could use the same method as in the previous problem. However, here we give an alternative method, and set $z = \frac{a}{r}\,e^{i(\theta-\phi)}$, so that

$$z^n = \left(\tfrac{a}{r}\right)^n e^{in(\theta-\phi)} = \left(\tfrac{a}{r}\right)^n\big[\cos n(\theta-\phi) + i\,\sin n(\theta-\phi)\big],$$

and

$$1 + 2\sum_{n=1}^{\infty}\left(\tfrac{a}{r}\right)^n\cos n(\theta-\phi) = Re\left(1 + 2\sum_{n=1}^{\infty} z^n\right).$$

And since $|z| = \frac{a}{r} < 1$, then

$$1 + 2\sum_{n=1}^{\infty} \left(\tfrac{a}{r}\right)^n \cos n(\theta - \phi) = Re\left(1 + \frac{2z}{1-z}\right) = Re\left(\frac{1+z}{1-z}\right) = \frac{r^2 - a^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}.$$

The solution to the exterior Dirichlet problem for the disk is therefore

$$u(r,\theta) = \frac{r^2 - a^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi,$$

for $a < r < \infty, \quad -\pi < \theta < \pi$.
Notice the similarity to the solution for the interior Dirichlet problem for the disk obtained in the previous problem.

Exercise 13.11. ✗✗✗
Solve Laplace's equation inside a circular annulus ($0 < a < r < b$)

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad a < r < b,\ -\pi < \theta < \pi$$

subject to the boundary conditions

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta), \qquad \frac{\partial u}{\partial r}(b,\theta) = g(\theta),$$

for $-\pi < \theta < \pi$.

Solution: Note that we need to include two periodicity conditions to ensure the solution is single valued and continuous (and to get the right number of boundary conditions):

$$u(r,-\pi) = u(r,\pi) \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

for $a \leqslant r \leqslant b$.
We assume a solution of the form $u(r,\theta) = \phi(\theta) \cdot G(r)$, and substitute this into Laplace's equation to get

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

We can satisfy the periodicity conditions by requiring that

$$\phi(-\pi) = \phi(\pi) \qquad \text{and} \qquad \phi'(-\pi) = \phi'(\pi),$$

and we can satisfy the boundary condition $\dfrac{\partial u}{\partial r}(a,\theta) = 0$ by requiring $G'(a) = 0$. and we have two boundary value problems:

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0, \quad a < r < b \qquad \phi''(\theta) + \lambda\phi(\theta) = 0, \quad -\pi < \theta < \pi$$

$$G'(a) = 0 \qquad\qquad \phi(-\pi) = \phi(\pi)$$

$$\phi'(-\pi) = \phi'(\pi).$$

We solve the complete (two periodicity conditions) boundary value problem for ϕ first, again we consider three cases.
case (i): If $\lambda = 0$, the general solution to the differential equation $\phi'' = 0$ is $\phi(\theta) = A\theta + B$, with $\phi'(\theta) = A$. The first periodicity condition implies that

$$-A\pi + B = A\pi + B,$$

so that $A = 0$. The solution is now $\phi(\theta) = B$, and the second periodicity condition is also satisfied, the (nontrivial) solution is $\phi(\theta) = B$. In this case, the eigenvalue is $\lambda_0 = 0$ with corresponding eigenfunction $\phi_0(\theta) = 1$.
case (ii): If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' - \mu^2\phi = 0$ is

$$\phi(\theta) = A\cosh\mu\theta + B\sinh\mu\theta, \qquad \text{with} \qquad \phi'(\theta) = \mu A\sinh\mu\theta + \mu B\cosh\mu\theta.$$

The first periodicity condition implies that

$$A\cosh(-\mu\pi) + B\sinh(-\mu\pi) = A\cosh\mu\pi + B\mu\sinh\mu\pi,$$

and since $\cosh\mu\theta$ is an even function and $\sinh\mu\theta$ is an odd function, then

$$2B\sinh\mu\pi = 0,$$

so that $B = 0$. The solution is now $\phi(\theta) = A\cosh\mu\theta$, and the second periodicity condition implies that

$$\mu A\sinh(-\mu\pi) = \mu A\sinh\mu\pi,$$

so that $2\mu A\sinh\mu\pi = 0$, and so $A = 0$. In this case we have only the trivial solution $\phi(\theta) = 0$, $-\pi < \theta < \pi$.
case (iii): If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' + \mu^2\phi = 0$ is

$$\phi(\theta) = A\cos\mu\theta + B\sin\mu\theta, \qquad \text{with} \qquad \phi'(\theta) = -\mu A\sin\mu\theta + \mu B\cos\mu\theta.$$

The first periodicity condition implies that

$$A\cos(-\mu\pi) + B\sin(-\mu\pi) = A\cos\mu\pi + B\mu\sin\mu\pi,$$

and since $\cos\mu\theta$ is an even function and $\sin\mu\theta$ is an odd function, then

$$2B\sin\mu\pi = 0.$$

The second periodicity condition implies that

$$-\mu A\sin(-\mu\pi) + \mu B\cos(-\mu\pi) = -\mu A\sin\mu\pi + \mu B\cos\mu\pi,$$

so that

$$2\mu A\sin\mu\pi = 0.$$

There is a nontrivial solution if and only if at least one of A and B is nonzero, and the above implies that $\sin\mu\pi = 0$, that is, $\mu\pi = n\pi$ for some integer n. In this case the eigenvalues are $\lambda_n = n^2$, with corresponding eigenfunctions

$$\phi_n(\theta) = \cos n\theta \qquad \text{and} \qquad \phi_n(\theta) = \sin n\theta$$

for $n \geqslant 1$.
If $n \geqslant 1$, and we assume a solution to the corresponding equation

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - n^2 G = 0$$

of the form $G(r) = r^\alpha$, then

$$r\frac{d}{dr}\left(\alpha r^\alpha\right) - n^2 r^\alpha = 0,$$

that is,

$$\alpha^2 r^\alpha - n^2 r^\alpha = 0,$$

so that $\alpha = \pm n$, and we get two linearly independent solutions

$$G_{1n}(r) = r^n \qquad \text{and} \qquad G_{2n}(r) = \frac{1}{r^n}$$

and the general solution is

$$G_n(r) = Ar^n + \frac{B}{r^n},$$

for $n \geqslant 1$.
If $n = 0$, the corresponding differential equation for $G(r)$ is

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0,$$

and we get two linearly independent solutions

$$G_{10}(r) = 1 \qquad \text{and} \qquad G_{20}(r) = \log r,$$

and the general solution is

$$G_0(r) = A + B\log r,$$

From the superposition principle, the function

$$u(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\left[r^n(A_n\cos n\theta + B_n\sin n\theta) + \frac{1}{r^n}(C_n\cos n\theta + D_n\sin n\theta)\right] \qquad (\dagger)$$

with

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{B_0}{r} + \sum_{n=1}^{\infty}\left[nr^{n-1}(A_n\cos n\theta + B_n\sin n\theta) - \frac{n}{r^{n+1}}(C_n\cos n\theta + D_n\sin n\theta)\right]$$

satisfies the periodicity conditions and Laplace's equation in the annular region $a \leqslant r \leqslant b,\ -\pi \leqslant \theta \leqslant \pi$.
We can satisfy the boundary conditions

$$\frac{\partial u}{\partial r}(a, \theta) = f(\theta) \qquad \text{and} \qquad \frac{\partial u}{\partial r}(b, \theta) = g(\theta)$$

for ; $-\pi < \theta < \pi$ by requiring that

$$\begin{aligned} f(\theta) &= \frac{B_0}{a} + \sum_{n=1}^{\infty} \Big[n a^{n-1}(A_n \cos n\theta + B_n \sin n\theta) - \frac{n}{a^{n+1}}(C_n \cos n\theta + D_n \sin n\theta) \Big] \\ g(\theta) &= \frac{B_0}{b} + \sum_{n=1}^{\infty} \Big[n b^{n-1}(A_n \cos n\theta + B_n \sin n\theta) - \frac{n}{b^{n+1}}(C_n \cos n\theta + D_n \sin n\theta) \Big] \end{aligned} \qquad (\dagger\dagger)$$

where the coefficients are determined using the orthogonality of the eigenfunctions

$$\{\, 1,\ \cos\theta,\ \sin\theta,\ \cos 2\theta,\ \sin 2\theta,\ \cos 3\theta,\ \sin 3\theta,\ \cdots \}$$

on the interval $-\pi \leqslant \theta \leqslant \pi$.
Multiplying equations (††) above by the eigenfunction 1 and integrating over the interval $[-\pi, \pi]$, we obtain

$$B_0 = \frac{a}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \qquad \text{and} \qquad B_0 = \frac{b}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta,$$

that is,

$$\int_{-\pi}^{\pi} f(\theta)\, a d\theta = \int_{-\pi}^{\pi} g(\theta)\, b d\theta.$$

Note that this also follows from the divergence theorem, since

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(b, \theta)\, b\, d\theta - \int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(a, \theta)\, a\, d\theta = \int_{\partial D} \text{grad u} \cdot \mathbf{n}\, ds = \iint_D \Delta u\, r dr d\theta = 0,$$

where D is the closed annular region between the circles $r = a$ and $r = b$ and $\mathbf{n}$ is the outward unit normal to the boundary of D.
Multiplying the equations (††) by the appropriate eigenfunctions and integrating over the interval $[-\pi, \pi]$, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta &= n\pi \left(a^{n-1} A_n - \frac{1}{a^{n+1}} C_n \right) \\ \int_{-\pi}^{\pi} g(\theta) \cos n\theta\, d\theta &= n\pi \left(b^{n-1} A_n - \frac{1}{b^{n+1}} C_n \right) \\ \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta &= n\pi \left(a^{n-1} B_n - \frac{1}{a^{n+1}} D_n \right) \\ \int_{-\pi}^{\pi} g(\theta) \sin n\theta\, d\theta &= n\pi \left(b^{n-1} B_n - \frac{1}{b^{n+1}} D_n \right), \end{aligned}$$

and solving for $A_n,\ B_n,\ C_n,$ and D_n, we have

$$A_n = \frac{1}{n\pi(b^{2n} - a^{2n})}\left[b^n \int_{-\pi}^{\pi} g(\theta)\cos n\theta\, b\, d\theta - a^n \int_{-\pi}^{\pi} f(\theta)\cos n\theta\, a\, d\theta\right]$$

$$B_n = \frac{1}{n\pi(b^{2n} - a^{2n})}\left[b^n \int_{-\pi}^{\pi} g(\theta)\sin n\theta\, b\, d\theta - a^n \int_{-\pi}^{\pi} f(\theta)\sin n\theta\, a\, d\theta\right]$$

$$C_n = \frac{a^n b^n}{n\pi(b^{2n} - a^{2n})}\left[a^n \int_{-\pi}^{\pi} g(\theta)\cos n\theta\, b\, d\theta - b^n \int_{-\pi}^{\pi} f(\theta)\cos n\theta\, a\, d\theta\right]$$

$$D_n = \frac{a^n b^n}{n\pi(b^{2n} - a^{2n})}\left[a^n \int_{-\pi}^{\pi} g(\theta)\sin n\theta\, b\, d\theta - b^n \int_{-\pi}^{\pi} f(\theta)\sin n\theta\, a\, d\theta\right]$$

for $n \geqslant 1$.
The solution to the Neumann problem for Laplace's equation in the annulus $a < r < b$ is given by (†), where the coefficients $A_n,\ B_n,\ C_n,$ and D_n for $n \geqslant 1$ are given above, while

$$B_0 = \frac{a}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{b}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta,$$

and A_0 is an arbitrary constant.

Chapter 14

Method of Characteristics Problems

Exercise 14.1. ✗✗

Assume that $u(x,t)$ is the linear density of particulate matter being carried by the wind from a dump truck at the Oil Sands at position $x = 0$ and time t.

The wind is moving in the positive x-direction with a constant speed of k meters/sec, and the particulates are condensing out of the air at a rate $ru(x,t)$, where $r > 0$ is constant.

The density u satisfies the initial value problem

$$\frac{\partial u}{\partial t}(x,t) + k\frac{\partial u}{\partial x}(x,t) = -ru(x,t), \quad 0 < x < \infty, \quad t > 0 \qquad (*)$$

$$u(x,0) = \phi(x), \quad 0 < x < \infty$$

where $\phi(x)$ is the initial distribution of the particle density.

Solve this initial value - boundary value problem using the method of characteristics.

Solution: The method of characteristics reduces the partial differential equation to a pair of ordinary differential equations, one of which is solved for the characteristic curves in the (x,t)-plane along which the solutions to the other equation are easily found.

We write the partial differential equation so that the partial differential operator resembles a directional deriative or a total derivative. For example,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = -ru, \qquad (**)$$

where

$$\frac{dx}{dt} = k.$$

The family of curves with differential equation

$$\frac{dx}{dt} = k$$

are the **characteristic curves** of the partial differential equation $(*)$.

If $x = x(t)$ is a characteristic curve of $(*)$, then along this curve the equation $(**)$ becomes

$$\frac{\partial}{\partial t}\big(u(x(t),t)\big) + \frac{\partial}{\partial x}\big(u(x(t),t)\big) \cdot \frac{dx}{dt} = -ru(x(t),t),$$

that is,

$$\frac{d}{dt}\big(u(x(t),t)\big) = -ru(x(t),t),$$

which is an ordinary differential equation for $u(x(t),t)$.
Letting $v(t) = u(x(t),t)$ for $t \geqslant 0$, then v satisfies the ordinary differential equation

$$\frac{dv}{dt} + rv = 0,$$

that is, a first order, linear, homogeneous, ordinary differential equation, and to solve it we multiply by the integrating factor $M(t) = e^{rt}$, to get

$$\frac{d}{dt}\left(e^{rt}v\right) = e^{rt}\frac{dv}{dt} + re^{rt}v = e^{rt}\left(\frac{dv}{dt} + rv\right) = 0$$

for all $t \geqslant 0$.
Therefore, $e^{rt}v(t)$ is a constant, so that

$$e^{rt}v(t) = e^{r0}v(0) = v(0)$$

that is,

$$e^{rt}u(x(t),t) = u(x(0),0) = \phi(x(0))$$

so that

$$u(x(t),t) = e^{-rt}\phi(x(0))$$

for all $t \geqslant 0$.
Given a point (x,t) in the (x,t)-plane with $t > 0$, there is exactly one characteristic curve that passes through this point, namely,

$$x(t) = kt + x(0),$$

where

$$x = x(t) = kt + x(0),$$

and for this characteristic curve, $x(0) = x - kt$. $x(0)$ is called the **anchor point** of the characteristic through (x,t).
Therefore

$$u(x,t) = e^{-rt}\phi(x(0)) = e^{-rt}\phi(x-kt),$$

and, since (x,t) was arbitrary, then the solution to the initial value problem is given by

$$u(x,t) = e^{-rt}\phi(x-kt)$$

for $0 < x < \infty,\ t > 0$. Finally, we note that $e^{rt}u(x,t)$ is constant along the characteristic curves.

Exercise 14.2. ✗
Use the method of characteristics to solve the initial value problem

$$\frac{\partial w}{\partial t} + 5\frac{\partial w}{\partial x} = e^{3t}, \quad -\infty < x < \infty, \quad t \geqslant 0$$

$$w(x,0) = e^{-x^2}, \quad -\infty < x < \infty.$$

Solution: Let
$$\frac{dx}{dt} = 5,$$
then along the characteristic curve $x(t) = 5t + a$, the partial differential equation becomes
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = e^{3t},$$
so that
$$w(x(t), t) = \frac{1}{3}e^{3t} + K$$
where K is a constant, and $K = w(x(0), 0) - \frac{1}{3}$, so that
$$w(x(t), t) = \frac{1}{3}e^{3t} + w(x(0), 0) - \frac{1}{3} = \frac{1}{3}e^{3t} + w(a, 0) - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3}.$$
Given the point (x, t), let $x = 5t + a$ be the unique characteristic curve passing through this point, then the anchor point is $a = x - 5t$ and the solution is
$$w(x, t) = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-(x-5t)^2} - \frac{1}{3}$$
for $-\infty < x < \infty$ and $t > 0$.

Exercise 14.3. ✗

Use the method of characteristics to solve the initial value problem
$$\frac{\partial w}{\partial t} - x\frac{\partial w}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0$$
$$w(x, 0) = x^3 - 1, \quad -\infty < x < \infty.$$

Solution: Let
$$\frac{dx}{dt} = -x,$$
then along the characteristic curve $x(t) = x_0e^{-t}$, the partial differential equation becomes
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = 0,$$
so that
$$w(x(t), t) = K$$
where K is a constant, and $K = w(x(0), 0)$, so that
$$w(x(t), t) = w(x(0), 0) = w(x_0, 0) = x_0^3 - 1.$$
Given the point (x, t), let $x = x_0e^{-t}$ be the unique characteristic curve passing through this point, then $x_0 = xe^t$ is the anchor point and the solution is
$$w(x, t) = x_0^3 - 1 = x^3e^{3t} - 1$$
for $-\infty < x < \infty$ and $t > 0$.

Exercise 14.4. ✗
Use the method of characteristics to solve the initial value problem

$$\frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} = \sin 2\pi t, \quad -\infty < x < \infty, \quad t > 0$$

$$z(x,0) = \cos x, \quad -\infty < x < \infty.$$

Solution: Let

$$\frac{dx}{dt} = 3,$$

then along the characteristic curve $x(t) = 3t + a$, the partial differential equation becomes

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dx}{dt} = \sin 2\pi t,$$

so that

$$z(x(t),t) = -\frac{1}{2\pi}\cos 2\pi t + K$$

where K is a constant, and

$$z(x(0),0) = -\frac{1}{2\pi} + K = \cos a = \cos(x(t) - 3t),$$

so that

$$K = \cos(x(t) - 3t) + \frac{1}{2\pi}.$$

Given the point (x,t), let $x = 3t + a$ be the unique characteristic curve passing through this point, then the anchor point is $a = x - 3t$ and the solution is

$$z(x,t) = -\frac{1}{2\pi}\cos 2\pi t + \cos(x - 3t) + \frac{1}{2\pi} \qquad (*)$$

for $-\infty < x < \infty$ and $t > 0$.

As a check, we note that for

$$z(x,t) = -\frac{1}{2\pi}\cos 2\pi t + \cos(x - 3t) + \frac{1}{2\pi}$$

we have

$$\frac{\partial z}{\partial t} = \sin 2\pi t + 3\sin(x - 3t)$$

and

$$\frac{\partial z}{\partial x} = -\sin(x - 3t),$$

so that

$$\frac{\partial z}{\partial t} + 3\frac{\partial z}{\partial x} = \sin 2\pi t.$$

Also,

$$z(x,0) = -\frac{1}{2\pi} + \cos x + \frac{1}{2\pi},$$

and $(*)$ is a solution to the given initial value problem.

Exercise 14.5. ✗
Solve the following first-order equation

$$\frac{\partial u}{\partial t} + 3x\frac{\partial u}{\partial x} = 2t, \quad -\infty < x < \infty, \quad t \geqslant 0$$
$$u(x,0) = \ln(1+x^2), \quad -\infty < x < \infty.$$

Solution: Let

$$\frac{dx}{dt} = 3x,$$

then along the characteristic curve $x(t) = ae^{3t}$, the partial differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 2t,$$

so that

$$u(x(t),t) = t^2 + K$$

where K is a constant, and $K = u(x(0),0)$ so that

$$u(x(t),t) = t^2 + u(x(0),0) = t^2 + u(a,0) = t^2 + \ln(1+a^2).$$

Given the point (x,t), let $x = ae^{3t}$ be the unique characteristic curve passing through this point, then the anchor point is $a = xe^{-3t}$ and the solution is

$$u(x,t) = t^2 + \ln\left(1 + x^2e^{-6t}\right)$$

for $-\infty < x < \infty$ and $t > 0$.

Exercise 14.6. ✗
Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t \geqslant 0$$
$$w(x,0) = f(x), \quad -\infty < x < \infty.$$

Solution: Let

$$\frac{dx}{dt} = c,$$

then along the characteristic curve $x(t) = ct + a$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = e^{2x(t)} = e^{2(ct+a)},$$

so that

$$w(x(t),t) = \frac{1}{2c}e^{2ct+2a} + K = \frac{1}{2c}e^{2x(t)} + K$$

where K is a constant, and $K = w(x(0),0) - \frac{1}{2c}e^{2x(0)}$ so that

$$w(x(t),t) = \frac{1}{2c}e^{2x(t)} + f(x(0)) - \frac{1}{2c}e^{2x(0)},$$

that is,

$$w(x(t),t) = \frac{1}{2c}e^{2x(t)} + f(x(t) - ct) - \frac{1}{2c}e^{2(x(t)-ct)}.$$

Given the point (x,t), let $x = ct + a$ be the unique characteristic curve passing through this point, then the anchor point is $a = x - ct$ and the solution is

$$w(x,t) = \frac{1}{2c}e^{2x}\left(1 - e^{-2ct}\right) + f(x - ct)$$

for $-\infty < x < \infty$ and $t > 0$.

Exercise 14.7. ✗
Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1, \quad -\infty < x < \infty, \quad t \geqslant 0$$

$$w(x,0) = f(x), \quad -\infty < x < \infty.$$

Solution: Let

$$\frac{dx}{dt} = t,$$

then along the characteristic curve $x(t) = \frac{t^2}{2} + a$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = 1,$$

so that

$$w(x(t),t) = t + K$$

where K is a constant, and $K = w(x(0),0)$ so that

$$w(x(t),t) = t + w(x(0),0) = t + f(a).$$

Given the point (x,t), let $x = \frac{t^2}{2} + a$ be the unique characteristic curve passing through this point, then the anchor is $a = x - \frac{t^2}{2}$ and the solution is

$$w(x,t) = t + f\left(x - \frac{t^2}{2}\right)$$

for $-\infty < x < \infty$ and $t > 0$.

Exercise 14.8. ✗
Consider

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t \geqslant 0$$

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

Show that the characteristics are straight lines.

Solution: Along the characteristic curve $x = x(t)$ whose differential equation is

$$\frac{dx}{dt} = 2u(x(t),t),$$

the partial differential equation becomes

$$\frac{d}{dt}\left[u(x(t),t)\right] = \frac{\partial u}{\partial t}(x(t),t) + \frac{dx}{dt} \cdot \frac{\partial u}{\partial x}(x(t),t) = 0,$$

so that $u(x(t),t) = \text{ constant } = u(x(0),0)$, and

$$\frac{dx}{dt} = 2u(x(t),t) = 2u(x(0),0),$$

so that

$$x(t) = 2u(x(0),0)t + x(0) = 2f(x(0))t + x(0)$$

for $t \geqslant 0$, and the characteristic curves are the straight lines $x = 2f(x_0)t + x_0$ and intersect the x-axis at the point x_0.

Exercise 14.9. ✗✗
Consider

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t \geqslant 0$$

with

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + x/L & 0 < x < L \\ 2 & x > L. \end{cases}$$

(a) Determine the equations for the characteristics. Sketch the characteristics.

(b) Determine the solution $u(x,t)$. Sketch $u(x,t)$ for t fixed.

Solution:

(a) The equations for the characteristics are

$$x = 2f(x_0)t + x_0,$$

where the parameter x_0 is the intersection of the characteristic with the x-axis for $-\infty < x_0 < \infty$.

(i) For $x_0 < 0$, we have $f(x_0) = 1$, and the characteristics have the equation:

$$x = 2t + x_0.$$

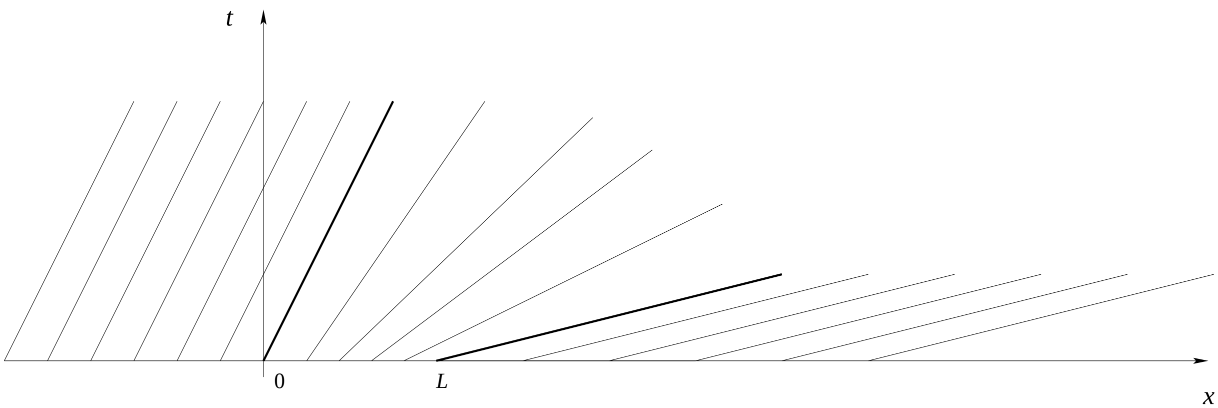
(ii) For $0 < x_0 < L$, we have $f(x_0) = 1 + x_0/L$, and the characteristics have the equation:

$$x = 2(1 + x_0/L)t + x_0.$$

(iii) For $x > L$, we have $f(x_0) = 2$, and the characteristics have the equation:

$$x = 4t + x_0.$$

The characteristics are sketched below.



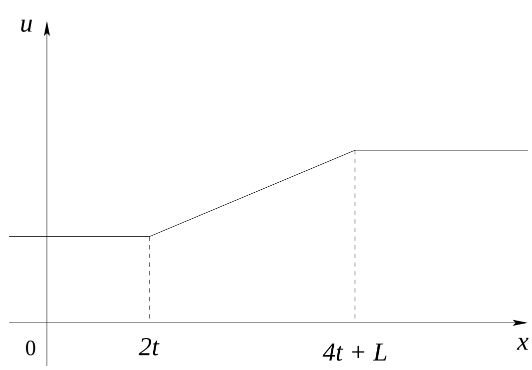
(b) The solution along the characteristic $x = 2f(x_0)t + x_0$ is given by

$$u(x,t) = f(x_0),$$

and considering the cases where $x_0 < 0$, $0 < x_0 < L$, and $L < x_0$, we have

$$u(x,t) = \begin{cases} 1 & \text{for} \quad x < 2t \\ \dfrac{x+L}{L+2t} & \text{for} \quad 2t < x < 4t + L \\ 2 & \text{for} \quad x > 4t + L. \end{cases}$$

For a fixed $t > 0$, the solution is sketched below.



Exercise 14.10. ✗
Derive the general solution of the equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = u, \quad a,\, b \neq 0$$

by using an appropriate change of variables.

Solution: Let

$$\alpha = Ax + Bt \qquad \text{and} \qquad \beta = Cx + Dt,$$

where A, B, C, and D are to be determined so as to reduce the partial differential equation to an ordinary differential equation, which we can then solve.
From the chain rule, we have

$$\frac{\partial u}{\partial x} = A\frac{\partial u}{\partial \alpha} + C\frac{\partial u}{\partial \beta}$$
$$\frac{\partial u}{\partial t} = B\frac{\partial u}{\partial \alpha} + D\frac{\partial u}{\partial \beta}$$

and the original partial differential equation becomes

$$(aB + bA)\frac{\partial u}{\partial \alpha} + (aD + bC)\frac{\partial u}{\partial \beta} = u.$$

Now let $B = -b$, $A = a$, $C = 0$, and $D = 1/a$, then the equation becomes

$$\frac{\partial u}{\partial \beta} - u = 0,$$

and multiplying this equation by $e^{-\beta}$, we have

$$e^{-\beta}\frac{\partial u}{\partial \beta} - e^{-\beta}u = 0,$$

that is,

$$\frac{\partial}{\partial \beta}\left(e^{-\beta}u\right) = 0,$$

and the quantity $e^{-\beta}u$ is independent of β. Therefore, the solution is

$$u = f(\alpha)e^{\beta},$$

where f is an arbitrary function of α. In terms of the original variables, the solution is

$$u(x,t) = f(ax - bt)e^{t/a}.$$

Exercise 14.11. ✗
Solve the partial differential equation on $-\infty < x < \infty$

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 2t + \sin t, \quad -\infty < x < \infty, \quad t > 0$$
$$w(x,0) = x^2 + 5, \quad -\infty < x < \infty.$$

At the end you might check if your solution really satisfies the above problem.

Solution: Let $\dfrac{dx}{dt} = t$, then along the characteristic curve $x(t) = \dfrac{t^2}{2} + a$, the partial differential equation becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} = 2t + \sin t,$$

so that

$$w(x(t), t) = t^2 - \cos t + K$$

where K is a constant, and $K = w(x(0), 0) + 1$, and

$$\begin{aligned} w(x(t), t) &= t^2 - \cos t + w(x(0), 0) + 1 = t^2 - \cos t + w(a, 0) + 1 \\ &= t^2 - \cos t + a^2 + 5 + 1 = t^2 - \cos t + \left(x(t) - \frac{t^2}{2}\right)^2 + 6. \end{aligned}$$

Given a point (x, t) in the x, t-plane, let $x = \dfrac{t^2}{2} + a$ be the unique characteristic curve passing through this point, then the anchor is $a = x - \dfrac{t^2}{2}$ and the solution is

$$w(x, t) = t^2 - \cos t + a^2 + 6 = t^2 - \cos t + \left(x - \frac{t^2}{2}\right)^2 + 6,$$

that is,

$$w(x, t) = t^2 - \cos t + \left(x - \frac{t^2}{2}\right)^2 + 6 \qquad (*)$$

for $-\infty < x < \infty,\ t > 0$.

As a check, note that for this function $w(x, t)$ we have

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 2t + \sin t + 2\left(x - \frac{t^2}{2}\right)(-t) + 2t\left(x - \frac{t^2}{2}\right) = 2t + \sin t.$$

and it satisfies the partial differential equation, while

$$w(x, 0) = x^2 + 5,$$

and $(*)$ is the solution to the problem above.

Exercise 14.12. ✗✗✗

The displacement $u = u(x,t)$ of an infinitely long string is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0.$$

At time $t = 0$ an initial signal is given of the form

$$u(x,0) = f(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ -x+2 & \text{for} \quad 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \qquad -\infty < x < \infty.$$

(a) Solve the above problem.

(b) Sketch the solution for three times t_1, t_2, t_3 with

$$t_1 = 0, \qquad 0 < t_2 < 1, \qquad 1 < t_3.$$

(c) At which time does the signal reach the point $\overline{x} = 11$?

Solution:

(a) D'Alembert's solution to the wave equation is given by

$$u(x,t) = \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} g(s)\,ds,$$

where

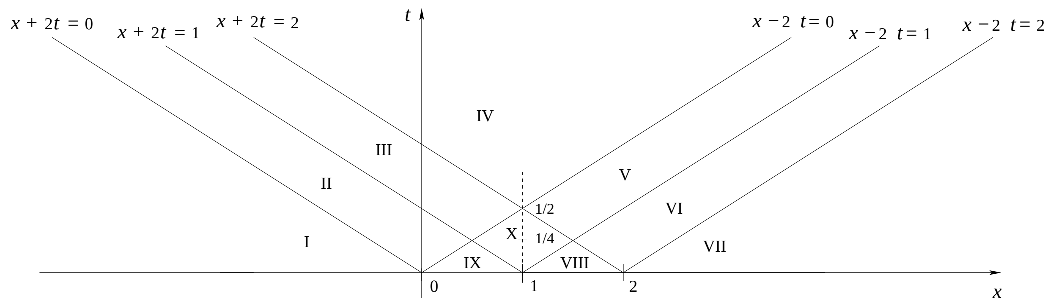
$$f(x+2t) = \begin{cases} x+2t & \text{if} \quad 0 < x+2t < 1 \\ -x-2t+2 & \text{if} \quad 1 < x+2t < 2 \\ 0 & \text{if} \quad x+2t \leqslant 0 \quad \text{or} \quad x+2t \geqslant 2, \end{cases}$$

and

$$f(x-2t) = \begin{cases} x-2t & \text{if} \quad 0 < x-2t < 1 \\ -x+2t+2 & \text{if} \quad 1 < x-2t < 2 \\ 0 & \text{if} \quad x-2t \leqslant 0 \quad \text{or} \quad x-2t \geqslant 2, \end{cases}$$

and $g(x) = 0$ for $-\infty < x < \infty$.

(b) We sketch the solution by considering the following 10 regions in the x,t-plane.



First note that $u = 0$ in regions I, IV, and VII.

Region II: $0 < x + 2t < 1$ and $x - 2t < 0$, that is,

$$-2t < x < 1 - 2t \qquad \text{and} \qquad x < 1 - 2t$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left[x + 2t + 0\right] = \frac{1}{2}\left[x + 2t\right].$$

Region III: $1 < x + 2t < 2$ and $x - 2t < 0$, that is,

$$1 - 2t < x < 2 - 2t \qquad \text{and} \qquad x < 2t$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left[-x - 2t + 2 + 0\right] = \frac{1}{2}\left[2 - x - 2t\right].$$

Region V: $0 < x - 2t < 1$ and $x + 2t > 2$, that is,

$$2t < x < 1 + 2t \qquad \text{and} \qquad x > 2 - 2t$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left[x - 2t + 0\right] = \frac{1}{2}\left[x - 2t\right].$$

Region VI: $1 < x - 2t < 2$ and $x + 2t > 2$, that is,

$$1 + 2t < x < 2 + 2t \qquad \text{and} \qquad x > 2 - 2t$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left[-x + 2t + 2 + 0\right] = \frac{1}{2}\left[2 - x + 2t\right].$$

Region VIII: $1 < x - 2t < 2$ and $1 < x + 2t < 2$, that is,

$$1 + 2t < x < 2 + 2t \qquad \text{and} \qquad 1 - 2t < x < 2 - 2t$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left[-x + 2t + 2 - x - 2t + 2\right] = 2 - x.$$

Region IX: $0 < x - 2t < 1$ and $0 < x + 2t < 1$, that is,

$$2t < x < 1 + 2t \qquad \text{and} \qquad -2t < x < 1 - 2t$$

and the solution in this region is

$$u(x,t) = \frac{1}{2}\left[x + 2t + x - 2t\right] = x.$$

Region X: $0 < x - 2t < 1$ and $1 < x + 2t < 2$, that is,

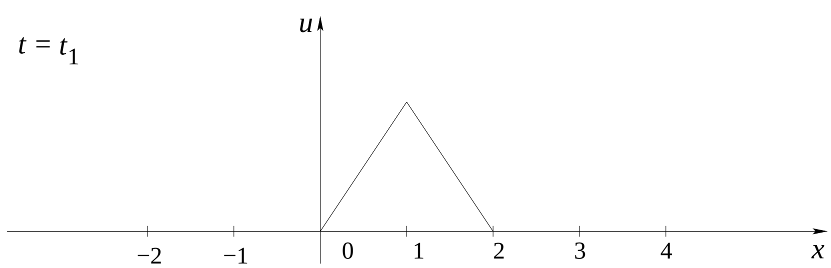
$$2t < x < 1 + 2t \qquad \text{and} \qquad 1 - 2t < x < 2 - 2t$$

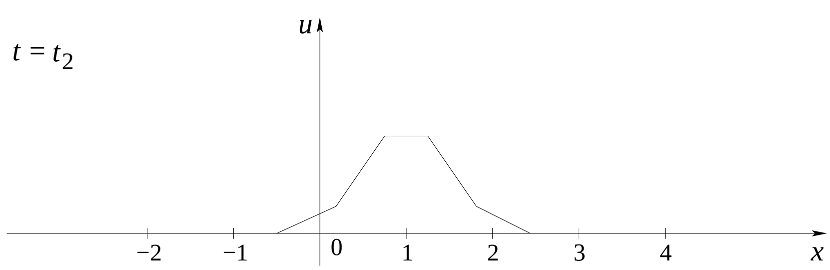
and the solution in this region is

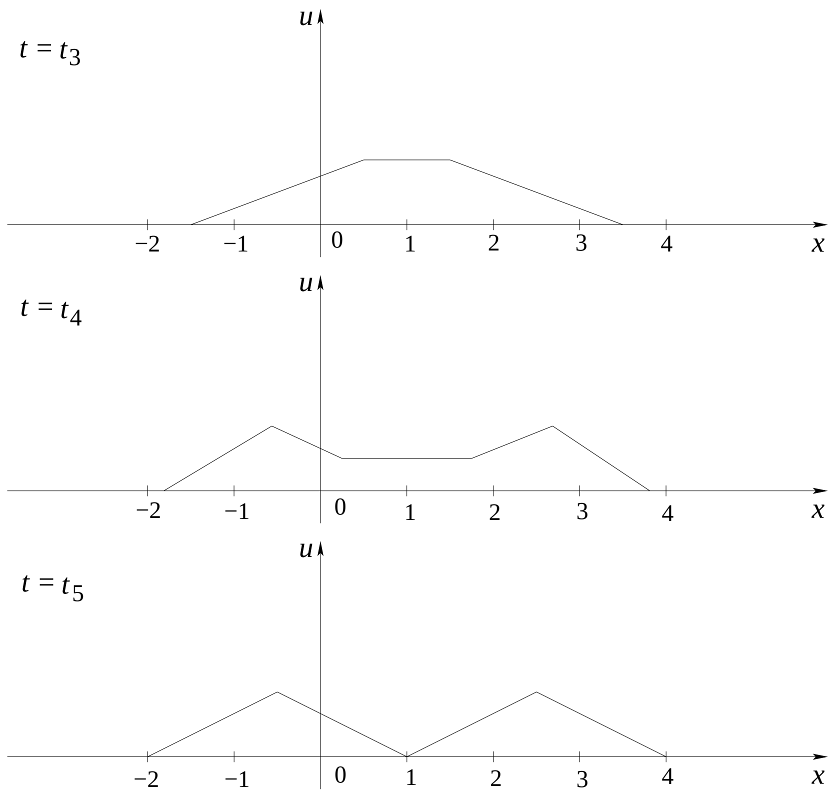
$$u(x,t) = \frac{1}{2}\left[x - 2t - x - 2t + 2\right] = 1 - 2t.$$

We sketch the solution for

$$t_1 = 0, \qquad 0 < t_2 < \frac{1}{4}, \qquad t_3 = \frac{1}{4}, \qquad \frac{1}{4} < t_4 < \frac{1}{2}, \qquad t_5 = \frac{1}{2}.$$







(c) Note that the right-moving signal will reach the point $\overline{x} = 11$ when $x + 2t = 11$, and this characteristic hits the t-axis when $t = \dfrac{11}{2}$.

Exercise 14.13. ✗✗✗

Use the method of characteristics to solve the initial value problem for the one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0$$

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty.$$

Solution: Since the wave operator has constant coefficients, then it can be factored as

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right).$$

If we let

$$v = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$$

and

$$w = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x},$$

where $u = u(x,t)$ is a twice continuously differentiable solution to the one-dimensional wave equation, we obtain the system of partial differential equations

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

$$\frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

since the second mixed partial derivatives are equal.

We solve equation (1) using the method of characteristics. Let

$$\frac{dx}{dt} = c,$$

then equation (1) becomes

$$\frac{dv}{dt} = \frac{\partial v}{\partial x} + \frac{dx}{dt}\frac{\partial v}{\partial x} = 0,$$

and along the characteristic curve $x(t) = ct + k$, v is a constant, and

$$v(x(t),t) = v(x(0),0) = v(k,0) = v(x(t) - ct, 0).$$

Therefore, given a point (x_0, t_0), the solution along the characteristic curve $x(t) - ct = x_0 - ct_0 = k$ is constant and is given by

$$v(x_0,t_0) = v(x(t),t) = v(x(0),0) = v(k,0) = v(x_0 - ct_0, 0) = \phi(x_0 - ct_0)$$

where ϕ is an arbitrary twice continuously differentiable function of a single variable, and since the point (x_0, t_0) was arbitrary, then

$$\frac{\partial u}{\partial t}(x,t) - c\frac{\partial u}{\partial x}(x,t) = v(x,t) = \phi(x - ct) \tag{3}$$

for $-\infty < x < \infty,\ t \geqslant 0$.

Similarly, we solve equation (2) using the method of characteristics, and given a point (x_0, t_0), the solution along the characteristic curve $x(t) + ct = x_0 + ct_0 = k$ is constant and is given by

$$w(x_0,t_0) = w(x(t),t) = w(x(0),0) = w(k,0) = w(x_0 + ct_0, 0) = \psi(x_0 + ct_0)$$

where ψ is an arbitrary twice continuously differentiable function of a single variable, and since the point (x_0, t_0) was arbitrary, then

$$\frac{\partial u}{\partial t}(x,t) + c\frac{\partial u}{\partial x}(x,t) = w(x,t) = \psi(x + ct) \tag{4}$$

for $-\infty < x < \infty,\ t \geqslant 0$. Now we solve equations (3) and (4) using the method of characterstics.

To solve equation (3), we let $\dfrac{dx}{dt} = -c$, and along the characteristic $x(t) = -ct + b$ the differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \phi(x(t) - ct),$$

and letting $F(s)$ be any antiderivative of $\phi(s)$, we have

$$u(x(t), t) = F(x(t) - ct) = F(-2ct + b) \tag{5}$$

along the characteristic curve $x(t) = -ct + b$.

To solve equation (4), we let $\dfrac{dx}{dt} = c$, and along the characteristic $x(t) = ct + a$ the differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \psi(x(t) + ct),$$

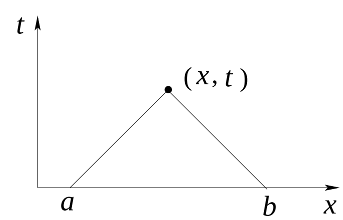
and letting $G(s)$ be any antiderivative of $\psi(s)$, we have

$$u(x(t), t) = G(x(t) + ct) = G(2ct + a) \tag{6}$$

along the characteristic curve $x(t) = ct + a$.

Now let (x, t) be an arbitrary point with $-\infty < x < \infty$, and $t > 0$, as in the figure,

- if the forward facing characteristic $\dfrac{dx}{dt} = c$ passing through (x, t) hits the x-axis at a, then $x - ct = a$ and $2ct + a = x + ct$
- if the backward facing characteristic $\dfrac{dx}{dt} = -c$ passing through (x, t) hits the x-axis at b, then $x + ct = b$ and $-2ct + b = x - ct$



Therefore

$$u(x, t) = F(x - ct) \qquad \text{and} \qquad u(x, t) = G(x + ct)$$

where $F(s)$ is an antiderivative of $\phi(s)$ and $G(s)$ is an antiderivative of $\psi(s)$.

Adding these two equations we get

$$u(x, t) = \frac{1}{2}\left[F(x - ct) + G(x + ct)\right] \tag{7}$$

for $-\infty < x < \infty,\ t \geqslant 0$.

Now, in order to solve the original problem, we solve the following initial value problems, and use the superposition principle to combine them to get the solution:

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial t^2} &= c^2 \frac{\partial^2 u_1}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0, \\
u_1(x,0) &= f(x), \quad -\infty < x < \infty \\
\frac{\partial u_1}{\partial t}(x,0) &= 0 \quad -\infty < x < \infty,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\frac{\partial^2 u_2}{\partial t^2} &= c^2 \frac{\partial^2 u_2}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0, \\
u_2(x,0) &= 0, \quad -\infty < x < \infty \\
\frac{\partial u_2}{\partial t}(x,0) &= g(x) \quad -\infty < x < \infty,
\end{aligned} \tag{9}$$

the solution to the original problem is then $u = u_1 + u_2$.

For problem (8), we use the initial conditions to write

$$u_1(x,0) = f(x) = \frac{1}{2}\left[F(x) + G(x)\right],$$

so that

$$F(x) + G(x) = 2f(x),$$

and

$$\frac{\partial u_1}{\partial t} = 0 = \frac{c}{2}\left[F'(x) - G'(x)\right],$$

so that

$$F(x) - G(x) = 2K,$$

where K is an arbitrary constant. Therefore,

$$2F(x) = 2f(x) + 2K \qquad \text{and} \qquad 2G(x) = 2f(x) - 2K,$$

and the solution to the first problem is

$$u_1(x,t) = \frac{1}{2}\left[F(x+ct) + G(x-ct)\right] = \frac{1}{2}\left[f(x+ct) + f((x-ct)\right].$$

For problem (9), we use the initial conditions to write (a different F and G)

$$u_2(x,0) = 0 = \frac{1}{2}\left[F(x) + G(x)\right],$$

so that $G(x) = -F(x)$, and

$$\frac{\partial u_2}{\partial t}(x,0) = g(x) = \frac{c}{2}\left[F'(x) - G'(x)\right],$$

so that $cF'(x) - cG'(x) = 2cF'(x) = 2g(x)$, and integrating we have

$$cF(x) = \int_0^x g(s)\,ds + 2c\widehat{K},$$

where $\widehat{K}$ is an arbitrary constant. Therefore,

$$\frac{1}{2}F(x) = \frac{1}{2c}\int_0^x g(s)\,ds + \widehat{K} \qquad \text{and} \qquad \frac{1}{2}G(x) = -\frac{1}{2c}\int_0^x g(s)\,ds - \widehat{K}$$

and the solution to the second problem is

$$u_2(x,t) = \frac{1}{2c}\left[\int_0^{x+ct} g(s)\,ds - \int_0^{x-ct} g(s)\,ds\right] = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

The solution to the original initial value problem is therefore

$$u(x,t) = u_1(x,t) + u_2(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

for $-\infty < x < \infty,\ t \geqslant 0$.

Exercise 14.14. ✗✗✗

The following hyperbolic model arises as the equation for the moment generating function $A(s,t)$ of a stochastic birth - death process

$$\frac{\partial A}{\partial t}(s,t) - (s-1)\left(bs - \delta(t)\right)\frac{\partial A}{\partial s}(s,t) = 0 \qquad (*)$$

$$A(s,0) = s^{n_0}$$

where b denotes a constant birth rate and $\delta(t)$ a time-dependent death rate. Find $A(s,t)$.

Solution: We use the method of characteristics to solve for $A(s,t)$.
Note that if $s = s(t)$ is a curve in the t,s-plane such that

$$\frac{ds}{dt} = -(s-1)\left(bs - \delta(t)\right), \qquad (**)$$

then from the chain rule, along this curve the partial differential equation $(*)$ becomes

$$\frac{d}{dt}\left[A(s(t),t)\right] = \frac{\partial A}{\partial t}(s(t),t)\cdot\frac{dt}{dt} + \frac{\partial A}{\partial s}(s(t),t)\cdot\frac{ds}{dt} = 0,$$

that is,

$$A(s(t),t) = \text{ constant } = A(s(0),0) = s(0)^{n_0}$$

and we need only solve the characteristic equation

$$\frac{ds(t)}{dt} = -(s-1)\left(bs - \delta(t)\right) \qquad (*\!*\!*)$$

for the equation of the curve passing through the point $(s(0), 0)$.
The differential equation $(***)$ can be written as

$$\frac{ds}{dt} = (1-s)(bs - \delta(t)) = (1-s)(-b + bs + b - \delta(t)) = (1-s)(b - \delta(t)) - b(1-s)^2,$$

that is,

$$\frac{1}{(1-s)^2}\frac{ds}{dt} = \frac{b - \delta(t)}{1-s} - b,$$

and letting $y(t) = \dfrac{1}{1-s(t)}$, we have

$$\frac{dy}{dt} - (b - \delta(t))y = -b. \qquad (\dagger)$$

This is a first-order linear differential equation for $y = y(t)$ which can be solved using an integrating factor or the method of variation of parameters. Multiplying by the integrating factor

$$\Lambda(t) = \exp\left(-\int_0^t (b - \delta(\tau))\, d\tau\right),$$

we have

$$\frac{d}{dt}\left[\Lambda(t)y(t)\right] = -b\Lambda(t),$$

integrating and using the fact that $\Lambda(0) = 1$, we have

$$\Lambda(t)y(t) = y(0) - b\int_0^t \Lambda(\tau)\, d\tau.$$

Therefore,

$$\frac{\Lambda(t)}{1-s(t)} = \frac{1}{1-s(0)} - b\int_0^t \Lambda(\tau)\, d\tau,$$

and we can solve this equation for $s(t)$ and obtain an explicit form for the equation of the characteristic curve $s = s(t)$.
However, since

$$A(s(t), t) = \text{ constant } = A(s(0), 0) = s(0)^{n_0}, \qquad (\dagger\dagger)$$

we need to find the anchor point $(s(0), 0)$. We solve the above equation for $s(0)$ to get

$$s(0) = 1 - \cfrac{1}{\cfrac{\Lambda(t)}{1-s(t)} + b\displaystyle\int_0^t \Lambda(\tau)\, d\tau},$$

and the solution is constant along the characteristic $s = s(t)$, so that

$$A(s(t), t) = A(s(0), 0) = s(0)^{n_0} = \left[1 - \cfrac{1}{\cfrac{\Lambda(t)}{1-s(t)} + b\displaystyle\int_0^t \Lambda(\tau)\, d\tau}\right]^{n_0} \qquad (\dagger\dagger\dagger)$$

where

$$\Lambda(t) = \exp\left(-\int_0^t (b - \delta(\tau))\ d\tau\right).$$

Since the initial value problem

$$\frac{dy}{dt} - (b - \delta(t))y = -b$$

$$y(0) = \frac{1}{1 - s(0)}$$

has a unique solution, given a point (s, t) in the (s, t)-plane, there exists a unique characteristic passing through this point with $s(0) = s_0$, and

$$A(s,t) = A(s_0, 0) = {s_0}^{n_0} = \left[1 - \frac{1}{\dfrac{\Lambda(t)}{1-s} + b\displaystyle\int_0^t \Lambda(\tau)\, d\tau}\right]^{n_0} \tag{+}$$

where

$$\Lambda(t) = \exp\left(-\int_0^t (b - \delta(\tau))\ d\tau\right).$$

This can be simplified somewhat by substituting the expression for $\Lambda(t)$, and we find

$$A(s,t) = \left[1 - \frac{\exp\left(\displaystyle\int_0^t (b - \delta(z))\ dz\right)}{\dfrac{1}{1-s} + b\displaystyle\int_0^t \exp\left[\int_\tau^t (b - \delta(z))\, dz\right]\ d\tau}\right]^{n_0}. \tag{++}$$

Chapter 15

Sturm-Liouville Theory Problems

Exercise 15.1. ✗✗

Given the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

determine the eigenvalues λ and corresponding eigenfunctions if ϕ satisfies the following boundary conditions.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$.

(b) $\phi(0) = 0$ and $\phi(1) = 0$.

(c) $\dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$.

(d) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$.

(e) $\dfrac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$.

(f) $\phi(a) = 0$ and $\phi(b) = 0$ (you may assume that $\lambda > 0$).

(g) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) + \phi(L) = 0$.

Analyze three cases: $\lambda > 0,\ \lambda = 0,\ \lambda < 0$. You may assume that the eigenvalues are real.

Solution:

(a) If ϕ satisfies the boundary value problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0, \quad 0 \leqslant x \leqslant \pi \\ \phi(0) &= 0, \\ \phi(\pi) &= 0 \end{aligned}$$

we have to consider three cases.

case 1: If $\lambda = 0$, the equation is
$$\phi''(x) = 0$$
with general solution
$$\phi(x) = Ax + B, \quad 0 \leqslant x \leqslant \pi.$$
The first boundary condition gives $\phi(0) = B = 0$, and the solution is now
$$\phi(x) = Ax, \quad 0 \leqslant x \leqslant \pi.$$
The second boundary condition gives $\phi(\pi) = A\pi = 0$, so that $A = 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant \pi$.

case 2: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is
$$\phi''(x) - \mu^2\phi(x) = 0$$
with general solution
$$\phi(x) = A\cosh\mu x + B\sinh\mu x.$$
The first boundary condition gives $\phi(0) = A = 0$, and the solution is now
$$\phi(x) = B\sinh\mu x, \quad 0 \leqslant x \leqslant \pi.$$
The second boundary condition gives $\phi(\pi) = B\sinh\mu\pi = 0$, so that $B = 0$, since $\sinh\mu\pi \neq 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant \pi$.

case 3: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is
$$\phi''(x) + \mu^2\phi(x) = 0$$
with general solution
$$\phi(x) = A\cos\mu x + B\sin\mu x.$$
The first boundary condition gives $\phi(0) = A = 0$, and the solution is now
$$\phi(x) = B\sin\mu x, \quad 0 \leqslant x \leqslant \pi.$$
The second boundary condition gives $\phi(\pi) = B\sin\mu\pi = 0$, and if $B = 0$, then again we get the trivial solution. In this case we have a nontrivial solution if and only if $\mu\pi = n\pi$ for some integer n, that is, $\mu = n$ for some integer n.

The eigenvalues are $\lambda_n = n^2$, with corresponding eigenfunctions
$$\phi_n(x) = \sin nx, \quad 0 \leqslant x \leqslant \pi$$
for $n \geqslant 1$.

(b) If ϕ satisfies the boundary value problem
$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, &\quad 0 \leqslant x \leqslant 1 \\ \phi(0) = 0&, \\ \phi(1) = 0& \end{aligned}$$

we have to consider three cases.

<u>case 1</u>: If $\lambda = 0$, the equation is

$$\phi''(x) = 0$$

with general solution

$$\phi(x) = Ax + B, \quad 0 \leqslant x \leqslant 1.$$

The first boundary condition gives $\phi(0) = B = 0$, and the solution is now

$$\phi(x) = Ax, \quad 0 \leqslant x \leqslant 1.$$

The second boundary condition gives $\phi(1) = A = 0$, so that $A = 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant 1$.

<u>case 2</u>: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) - \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x.$$

The first boundary condition gives $\phi(0) = A = 0$, and the solution is now

$$\phi(x) = B\sinh\mu x, \quad 0 \leqslant x \leqslant 1.$$

The second boundary condition gives $\phi(1) = B\sinh\mu = 0$, so that $B = 0$, since $\sinh\mu \neq 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant 1$.

<u>case 3</u>: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) + \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x.$$

The first boundary condition gives $\phi(0) = A = 0$, and the solution is now

$$\phi(x) = B\sin\mu x, \quad 0 \leqslant x \leqslant 1.$$

The second boundary condition gives $\phi(1) = B\sin\mu = 0$, and if $B = 0$, then again we get the trivial solution. In this case we have a nontrivial solution if and only if $\mu = n\pi$ for some integer n.

The eigenvalues are $\lambda_n = (n\pi)^2$, with corresponding eigenfunctions

$$\phi_n(x) = \sin n\pi x, \quad 0 \leqslant x \leqslant 1$$

for $n \geqslant 1$.

(c) If ϕ satisfies the boundary value problem

$$\begin{aligned}\phi''(x) + \lambda\phi(x) = 0, &\quad 0 \leqslant x \leqslant L\\ \phi'(0) = 0,&\\ \phi'(L) = 0&\end{aligned}$$

we have to consider three cases.

<u>case 1</u>: If $\lambda = 0$, the equation is

$$\phi''(x) = 0$$

with general solution

$$\phi(x) = Ax + B, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi'(0) = A = 0$, and the solution is now

$$\phi(x) = B, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = A = 0$. In this case we have a nontrivial solution $\phi(x) = B$ for $0 \leqslant x \leqslant L$.

<u>case 2</u>: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) - \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x, 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi'(0) = \mu B = 0$, so that $B = 0$, and the solution is now

$$\phi(x) = A\cosh\mu x, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = \mu A\sinh\mu L = 0$, so that $A = 0$, since $\sinh\mu L \neq 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant L$.

<u>case 3</u>: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) + \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x.$$

The first boundary condition gives $\phi'(0) = \mu B = 0$, so that $B = 0$, and the solution is now

$$\phi(x) = A\cos\mu x, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = -\mu A\sin\mu L = 0$, and if $A = 0$, then again we get the trivial solution. In this case we have a nontrivial solution if and only if $\mu L = n\pi$ for some integer n, that is, $\mu = \dfrac{n\pi}{L}$ for some integer n.

The eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, with corresponding eigenfunctions

$$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad 0 \leqslant x \leqslant L$$

for $n \geqslant 0$.

Note that this includes the constant solution obtained for $\lambda_0 = 0$.

(d) If ϕ satisfies the boundary value problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0, \quad 0 \leqslant x \leqslant L \\ \phi(0) &= 0, \\ \phi'(L) &= 0 \end{aligned}$$

we have to consider three cases.

case 1: If $\lambda = 0$, the equation is

$$\phi''(x) = 0$$

with general solution

$$\phi(x) = Ax + B, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi(0) = B = 0$, and the solution is now

$$\phi(x) = Ax, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = A = 0$. In this case we have only the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant L$.

case 2: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) - \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x, 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi(0) = A = 0$, and the solution is now

$$\phi(x) = B\sinh\mu x, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = \mu B\cosh\mu L = 0$, so that $B = 0$, since $\mu\cosh\mu L \neq 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant L$.

case 3: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) + \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi(0) = A = 0$, and the solution is now

$$\phi(x) = B \sin \mu x, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = \mu B \cos \mu L = 0$, and if $B = 0$, then again we get the trivial solution. In this case we have a nontrivial solution if and only if

$$\mu L = \frac{(2n-1)\pi}{2}$$

for some integer $n \geqslant 1$, that is,

$$\mu = \frac{(2n-1)\pi}{2L}$$

for some integer $n \geqslant 1$.

The eigenvalues are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L} \right)^2,$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin \frac{(2n-1)\pi x}{2L}, \quad 0 \leqslant x \leqslant L$$

for $n \geqslant 1$.

(e) If ϕ satisfies the boundary value problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, &\quad 0 \leqslant x \leqslant L \\ \phi'(0) = 0, & \\ \phi(L) = 0 & \end{aligned}$$

we have to consider three cases.

case 1: If $\lambda = 0$, the equation is

$$\phi''(x) = 0$$

with general solution

$$\phi(x) = Ax + B, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi'(0) = A = 0$, and the solution is now

$$\phi(x) = B, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) = B = 0$. In this case we have only the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant L$.

case 2: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) - \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A \cosh \mu x + B \sinh \mu x, 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi'(0) = \mu B = 0$, so that $B = 0$, and the solution is now

$$\phi(x) = A\cosh \mu x, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi(L) = A\cosh \mu L = 0$, so that $A = 0$, since $\cosh \mu L \neq 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant L$.

case 3: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) + \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cos \mu x + B\sin \mu x, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi'(0) = \mu B = 0$, so that $B = 0$, and the solution is now

$$\phi(x) = A\cos \mu x, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi(L) = A\cos \mu L = 0$, and if $A = 0$, then again we get the trivial solution. In this case we have a nontrivial solution if and only if

$$\mu L = \frac{(2n-1)\pi}{2}$$

for some integer $n \geqslant 1$, that is,

$$\mu = \frac{(2n-1)\pi}{2L}$$

for some integer $n \geqslant 1$.

The eigenvalues are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2,$$

with corresponding eigenfunctions

$$\phi_n(x) = \cos\frac{(2n-1)\pi x}{2L}, \quad 0 \leqslant x \leqslant L$$

for $n \geqslant 1$.

(f) If ϕ satisfies the boundary value problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, &\quad 0 \leqslant x \leqslant \pi \\ \phi(a) = 0, & \\ \phi(b) = 0 & \end{aligned}$$

we have to consider three cases.

case 1: If $\lambda = 0$, the equation is

$$\phi''(x) = 0$$

with general solution

$$\phi(x) = Ax + B, \quad a \leqslant x \leqslant b.$$

The first boundary condition gives $\phi(a) = Aa + B = 0$, so that $B = -aA$, and the solution is now

$$\phi(x) = A(x - a), \quad a \leqslant x \leqslant b.$$

The second boundary condition gives $\phi(b) = A(b - a) = 0$, so that $A = 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $a \leqslant x \leqslant b$.

case 2: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) - \mu^2 \phi(x) = 0$$

with general solution

$$\phi(x) = Ae^{\mu x} + Be^{-\mu x}, \quad a \leqslant x \leqslant b.$$

The first and second boundary conditions give the following homogeneous system of linear equations for A and B

$$\begin{aligned} Ae^{\mu a} + Be^{-\mu a} &= 0 \\ Ae^{\mu b} + Be^{-\mu b} &= 0 \end{aligned}$$

and the determinant of the coefficient matrix is

$$\begin{vmatrix} e^{\mu a} & e^{-\mu a} \\ e^{\mu b} & e^{-\mu b} \end{vmatrix} = e^{\mu(a-b)} - e^{-\mu(a-b)} = 2 \sinh \mu(a - b) \neq 0,$$

since $a \neq b$. Therefore this system of equations has a unique solution $A = 0, \ B = 0$, and in this case the only solution to the boundary value problem is the trivial solution $\phi(x) = 0$ for $a \leqslant x \leqslant b$.

case 3: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) + \mu^2 \phi(x) = 0$$

with general solution

$$\phi(x) = A \cos \mu x + B \sin \mu x.$$

The first boundary condition gives

$$\phi(a) = A \cos \mu a + B \sin \mu a = 0,$$

while the second boundary condition gives

$$\phi(b) = A \cos \mu b + B \sin \mu b = 0,$$

and we have the following homogeneous system of linear equations for A and B

$$\begin{aligned} A \cos \mu a + B \sin \mu a &= 0 \\ A \cos \mu b + B \sin \mu b &= 0 \end{aligned}$$

and the determinant of the coefficient matrix is

$$\begin{vmatrix} \cos \mu a & \sin \mu a \\ \cos \mu b & \sin \mu b \end{vmatrix} = \cos \mu a \sin \mu b - \sin \mu a \cos \mu b = \sin \mu(b - a).$$

The boundary value problem has the trivial solution if and only if this determinant is nonzero, so we get a nontrivial solution if and only if $\sin\mu(b-a) = 0$, that is, if and only if $\mu(b-a) = n\pi$ for some integer n.

The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2,$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin n\pi\left(\frac{x-a}{b-a}\right), \quad a \leqslant x \leqslant b$$

for $n \geqslant 1$.

(g) If ϕ satisfies the boundary value problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0, \quad 0 \leqslant x \leqslant \pi \\ \phi(0) &= 0, \\ \phi'(L) + \phi(L) &= 0 \end{aligned}$$

we have to consider three cases.

<u>case 1</u>: If $\lambda = 0$, the equation is

$$\phi''(x) = 0$$

with general solution

$$\phi(x) = Ax + B, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi(0) = B = 0$, and the solution is now

$$\phi(x) = Ax, \quad 0 \leqslant x \leqslant L.$$

The second boundary condition gives $\phi'(L) + \phi(L) = A(1+L) = 0$, so that $A = 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant L$.

<u>case 2</u>: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) - \mu^2\phi(x) = 0$$

with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x, \quad 0 \leqslant x \leqslant L.$$

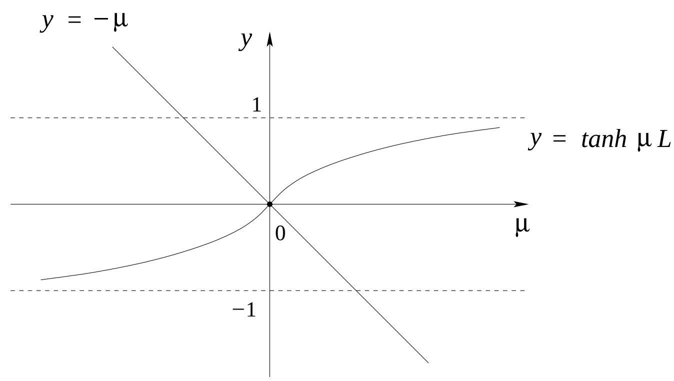
The first boundary condition gives $\phi(0) = A = 0$, and the solution is now

$$\phi(x) = B\sinh\mu x.$$

The second boundary condition gives

$$\phi'(L) + \phi(L) = \mu B\cosh\mu L + B\sinh\mu L = B\cosh\mu L(\mu + \tanh\mu L) = 0.$$

If we look at the graphs of $y = \tanh \mu L$ and $y = -\mu$ below



we see that $\mu + \tanh \mu L = 0$ if and only if $\mu = 0$. Therefore, since $\mu \neq 0$, and $\cosh \mu L \neq 0$, then the second boundary condition implies that $B = 0$. In this case the only solution is the trivial solution $\phi(x) = 0$ for $0 < x < L$.

<u>case 3</u>: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the equation is

$$\phi''(x) + \mu^2 \phi(x) = 0$$

with general solution

$$\phi(x) = A \cos \mu x + B \sin \mu x, \quad 0 \leqslant x \leqslant L.$$

The first boundary condition gives $\phi(0) = A = 0$, and the solution is now

$$\phi(x) = B \sin \mu x.$$

The second boundary condition gives

$$\phi'(L) + \phi(L) = \mu B \cos \mu L + B \sin \mu L = B(\mu \cos \mu L + \sin \mu L) = 0,$$

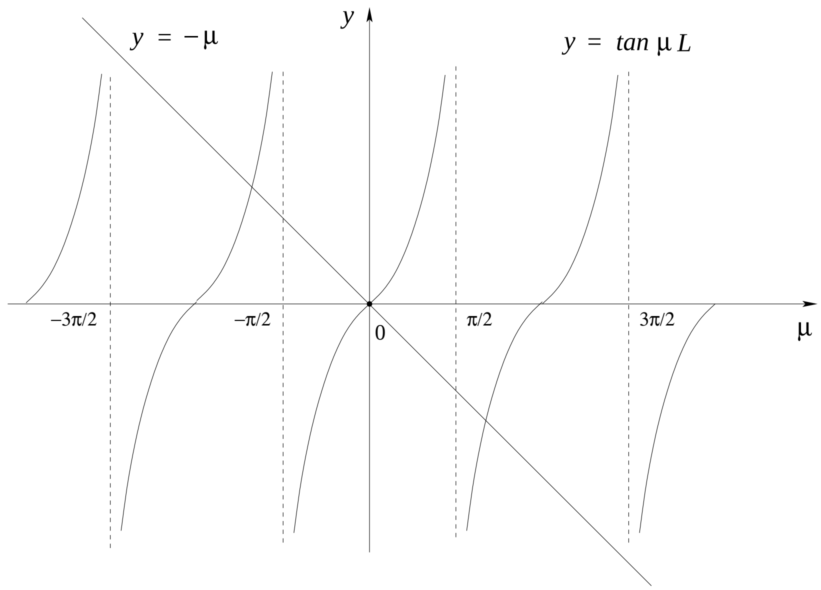
and we have a nontrivial solution if and only if

$$\mu \cos \mu L + \sin \mu L = 0,$$

that is, if and only if

$$\tan \mu L = -\mu.$$

From the graphs of $y = \tan \mu L$ and $y = -\mu$ below



we see that there are an infinite number of solutions μ_n. The eigenvalues are

$$\lambda_n = \mu_n^2$$

where $\tan \mu_n L = -\mu_n$, and the corresponding eigenfunctions are

$$\phi_n(x) = \sin \mu_n x, \quad 0 < x < L$$

for $n \geqslant 1$.

Note: Since $\cot \mu_n L = -\dfrac{1}{\mu_n}$, and $\mu_n \to \infty$ as $n \to \infty$, then the zeros of the equation $\tan \mu_n L = -\mu_n$ approach the zeros of $\cos \mu_n L$, and for *large* n, μ_n is given approximately by

$$\mu_n \approx \frac{(2n-1)\pi}{2L}.$$

Exercise 15.2. ✗
Use the **energy method** to show that there are no negative eigenvalues for the Neumann problem

$$\frac{d^2\phi}{d\phi^2} + \lambda\phi = 0, \quad 0 < x < L$$

$$\frac{d\phi}{dx}(0) = 0$$

$$\frac{d\phi}{dx}(L) = 0$$

This means, multiply the equation by ϕ, integrate and solve for λ. Does the expression for λ look familiar?

Solution: Suppose that λ is an eigenvalue of the boundary value problem with corresponding eigenvector ϕ, then ϕ satisfies the differential equation and the boundary conditions, and multiplying the differential equation by ϕ, we have

$$\phi\,\frac{d^2\phi}{d\phi^2} + \lambda\phi^2 = 0$$

for $0 < x < L$.

Integrating over the interval $[0, L]$, we have

$$\int_0^L \phi\,\frac{d^2\phi}{d\phi^2}\,dx + \lambda\int_0^L \phi^2\,dx = 0,$$

and integrating the first integral by parts,

$$\phi\,\frac{d\phi}{dx}\bigg|_0^L - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx + \lambda\int_0^L \phi^2\,dx = 0,$$

that is,

$$\phi(L)\,\frac{d\phi}{dx}(L) - \phi(0)\,\frac{d\phi}{dx}(0) - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx + \lambda\int_0^L \phi^2\,dx = 0,$$

and since $\dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(L) = 0$, then

$$\lambda\int_0^L \phi^2\,dx - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx = 0,$$

that is,

$$\lambda = \frac{\displaystyle\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\displaystyle\int_0^L \phi^2\,dx} \geqslant 0,$$

and there are no negative eigenvalues for this boundary value problem. Note that the expression for λ is in fact the Rayleigh quotient for the above Sturm-Liouville eigenvalue problem.

Alternative Solution: We can show this explicitly by solving the boundary value problem. Assuming a solution of the form $\phi(x) = e^{rx}$, we get $(r^2 + \lambda)e^{rx} = 0$, and since e^{rx} is never zero, the auxiliary equation is $r^2 + \lambda = 0$.

Now if $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the auxiliary equation $r^2 - \mu^2 = 0$ has two real roots, namely $r_1 = \mu$ and $r_2 = -\mu$.

We have two linearly independent eigenfunctions: $\phi_1(x) = \cosh\mu x$ and $\phi_2(x) = \sinh\mu x$, and the general solution to the differential equation is

$$\phi(x) = A\cosh\mu x + B\sinh\mu x.$$

Differentiating, we have

$$\frac{d\phi}{dx}(x) = \mu A\sinh\mu x + \mu B\cosh\mu x.$$

From the first boundary condition

$$\frac{d\phi}{dx}(0) = \mu B = 0,$$

and since $\mu \neq 0$, then $B = 0$.
From the second boundary condition

$$\frac{d\phi}{dx}(L) = \mu A \sinh \mu L = 0,$$

and since $\mu \neq 0$ and $\sinh \mu L \neq 0$, then $A = 0$ also.
Therefore we have only the trivial solution $\phi(x) \equiv 0$ on $[0, L]$, and $\lambda \geqslant 0$, that is, the boundary value problem has no negative eigenvalues.

Exercise 15.3. ✗✗
Solve the initial value problem

$$\begin{aligned} y'' + 9y &= F(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

where $F(t)$ is the 2π-periodic input function given by its Fourier series

$$F(t) = \sum_{n=1}^{\infty} \left[\frac{\cos nt}{n^2} + (-1)^n \frac{\sin nt}{n} \right].$$

Solution: Since the differential equation is a linear equation with constant coefficients, then the general solution to the nonhomogeneous equation is given by

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the general solution to the corresponding homogeneous equation and $y_p(t)$ is *any* particular solution to the nonhomogeneous equation.
The solution to the homogeneous equation is

$$y_h(t) = c_1 \cos 3t + c_2 \sin 3t$$

where c_1 and c_2 are arbitrary constants.
In order to find a particular solution to the nonhomogeneous equation, we solve the equation

$$y''(t) + 9y(t) = A_n \cos nt + B_n \sin nt$$

for $n \geqslant 0$, where A_n and B_n are the Fourier coefficients of the driving force $F(t)$, that is, $A_0 = B_0 = 0$,

$$A_n = \frac{1}{n^2} \qquad \text{and} \qquad B_n = \frac{(-1)^n}{n}$$

for $n \geqslant 1$.

Note that for $n \neq 3$, from the method of undetermined coefficients, the n^{th} normal mode of vibration is

$$y_n(t) = a_n \cos nt + b_n \sin nt$$

where the constants a_n and b_n are determined from the Fourier coefficients of $F(t)$ to be

$$a_0 = 0, \quad a_n = \frac{1}{n^2(9-n^2)}, \quad b_n = \frac{(-1)^n}{n(9-n^2)}$$

for $n \geqslant 1,\ n \neq 3$.

While for $n = 3$, the term in the driving force has the same frequency as the natural frequency of the system, and we have to solve the nonhomogeneous equation

$$y_3''(t) + 9y_3(t) = A_3 \cos 3t + B_3 \sin 3t.$$

In this case the method of undetermined coefficients suggests a solution of the form

$$y_3(t) = t(a_3 \cos 3t + b_3 \sin 3t).$$

In order to determine the constants a_3 and b_3, we substitute this expression into the differential equation

$$y_3'' + 9y_3 = A_3 \cos 3t + B_3 \sin 3t$$

to obtain

$$a_3 = -\frac{B_3}{6} \qquad \text{and} \qquad b_3 = \frac{A_3}{6}.$$

The particular solution to the nonhomogeneous equation can then be written as

$$y_p(t) = \sum_{\substack{n=1\\ n\neq 3}}^{\infty} \left(\frac{1}{n^2(9-n^2)} \cos nt + \frac{(-1)^n}{n(9-n^2)} \sin nt \right) + \frac{t}{6}\left(-\frac{1}{3}\cos 3t + \frac{1}{3^2}\sin 3t \right),$$

and the general solution to the nonhomogeneous equation is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + \sum_{\substack{n=1\\ n\neq 3}}^{\infty} \left(\frac{1}{n^2(9-n^2)} \cos nt + \frac{(-1)^n}{n(9-n^2)} \sin nt \right) + \frac{t}{6}\left(-\frac{1}{3}\cos 3t + \frac{1}{9}\sin 3t \right)$$

and the constants c_1 and c_2 can now be evaluated using the initial conditions $y(0) = y'(0) = 0$. Applying the initial conditions, we find

$$c_1 = -\sum_{\substack{n=1\\ n\neq 3}}^{\infty} \frac{1}{n^2(9-n^2)} \qquad \text{and} \qquad c_2 = \frac{1}{3^2 \cdot 6} - \frac{1}{3}\sum_{\substack{n=1\\ n\neq 3}}^{\infty} \frac{(-1)^n}{9-n^2}.$$

Exercise 15.4. ✗✗✗

Find the eigenvalues and eigenfunctions of the regular Sturm-Liouville problem

$$\left(x^3 X'\right)' + \lambda\, x\, X = 0 \quad 1 < x < e$$
$$X(1) = 0$$
$$X(e) = 0$$

Solution: Let $X = \dfrac{Y}{x}$, then $X' = \dfrac{Y'}{x} - \dfrac{Y}{x^2}$, so that $x^3 X' = x^2 Y' - x\,Y$, and

$$\big(x^3 X'\big)' = \big(x\,x\,Y'\big)' - Y - x\,Y' = x\,Y' + x\big(x\,Y'\big)' - Y - x\,Y' = x\big(x\,Y'\big)' - Y.$$

Therefore the original boundary value problem is equivalent to the following problem:

$$\begin{gathered}\big(x\,Y'\big)' + \frac{\mu}{x}\,Y = 0 \quad 1 < x < e \\ Y(1) = 0 \\ Y(e) = 0\end{gathered}$$

where $\mu = \lambda - 1$.
Now let $x = e^t$ and $\hat{Y}(t) = Y(e^t)$, then

$$\frac{dY}{dx} = \frac{d\hat{Y}}{dt}\,\frac{dt}{dx} = \frac{1}{x}\,\frac{d\hat{Y}}{dt},$$

that is,

$$x\frac{dY}{dx} = \frac{d\hat{Y}}{dt}.$$

Also,

$$\frac{d}{dx}\left(x\,\frac{dY}{dx}\right) = \frac{d}{dt}\left(\frac{d\hat{Y}}{dt}\right)\frac{dt}{dx} = \frac{1}{x}\,\frac{d^2\hat{Y}}{dt^2} = e^{-t}\,\frac{d^2\hat{Y}}{dt^2},$$

and therefore

$$\frac{d}{dx}\left(x\,\frac{dY}{dx}\right) + \frac{\mu}{x}\,Y = 0 \quad \text{for} \quad 1 < x < e$$

if and only if

$$e^{-t}\,\frac{d^2\hat{Y}}{dt^2} + e^{-t}\mu\,\hat{Y} = 0 \quad \text{for} \quad 0 < t < 1$$

So we have the equivalent regular Sturm-Liouville problem

$$\begin{gathered}\frac{d^2\hat{Y}}{dt^2} + \mu\,\hat{Y} = 0, \quad 0 < t < 1 \\ \hat{Y}(0) = 0 \\ \hat{Y}(1) = 0\end{gathered}$$

with eigenvalues $\mu_n = n^2\pi^2$, and eigenfunctions

$$\hat{Y}_n(t) = \sin n\pi t, \quad 0 < t < 1$$

for $n = 1, 2, 3, \cdots$.
The eigenvalues for the original problem are therefore

$$\lambda_n = 1 + n^2\pi^2,$$

and the corresponding eigenfunctions are

$$X_n(x) = \frac{1}{x}\,\sin(n\pi\log x), \quad 1 < x < e$$

for $n = 1, 2, 3, \cdots$.

Exercise 15.5. ✗

Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

subject to the periodicity conditions

$$\phi(0) = \phi(2\pi) \qquad \text{and} \qquad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

Solution: Again, we consider three cases.

case 1: If $\lambda = 0$, then the equation is $\phi'' = 0$ with general solution $\phi(x) = Ax + B$. From the first periodicity condition $\phi(0) = \phi(2\pi)$ we have

$$\phi(0) = A \cdot 0 + B = A \cdot 2\pi + B,$$

so that $2\pi A = 0$, and $A = 0$. The solution is now

$$\phi(x) = B, \quad 0 \leqslant x \leqslant 2\pi.$$

The second periodicity condition $\phi'(0) = \phi(2\pi)$ holds automatically, since

$$\phi'(0) = 0 = \phi'(2\pi).$$

Therefore $\lambda_0 = 0$ is an eigenvalue with corresponding eigenfunction

$$\phi_0(x) = 1, \quad 0 \leqslant x \leqslant 2\pi.$$

case 2: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$. The differential equation is $\phi'' - \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x, \quad 0 \leqslant x \leqslant 2\pi.$$

From the first periodicity condition

$$\phi(0) = A = A\cosh 2\pi\mu + B\sinh 2\pi\mu = \phi(2\pi),$$

while from the second periodicity condition

$$\phi'(0) = \mu B = \mu A\sinh 2\pi\mu + \mu B\cosh 2\pi\mu = \phi'(2\pi).$$

We have the homogeneous system of linear equations for A and B

$$\begin{aligned}(\cosh 2\pi\mu - 1)\,A + \sinh 2\pi\mu\,B &= 0\\ \sinh 2\pi\mu\,A + (\cosh 2\pi\mu - 1)\,B &= 0,\end{aligned}$$

and the determinant of the coefficient matrix is

$$\begin{vmatrix}\cosh 2\pi\mu - 1 & \sinh 2\pi\mu\\ \sinh 2\pi\mu & \cosh 2\pi\mu - 1\end{vmatrix} = 2(1 - \cosh 2\pi\mu) = -4\sinh^2\pi\mu \neq 0$$

since $\pi\mu \neq 0$, and this system has only the trivial solution $A = B = 0$. In this case the boundary value problem has only the trivial solution $\phi(x) = 0$ for $0 \leqslant x \leqslant 2\pi$.
case 3: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the differential equation is $\phi'' + \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x, \quad 0 \leqslant x \leqslant 2\pi.$$

From the first periodicity condition

$$\phi(0) = A = A\cos 2\pi\mu + B\sin 2\pi\mu = \phi(2\pi),$$

while from the second periodicity condition

$$\phi'(0) = \mu B = -\mu A\sin 2\pi\mu + \mu B\cos 2\pi\mu = \phi'(2\pi).$$

We have the homogeneous system of linear equations for A and B

$$\begin{aligned} (1-\cos 2\pi\mu)\,A + \sin 2\pi\mu\,B &= 0 \\ -\sin 2\pi\mu\,A + (1-\cosh 2\pi\mu)\,B &= 0, \end{aligned} \tag{$*$}$$

and the determinant of the coefficient matrix is

$$\begin{vmatrix} 1-\cos 2\pi\mu & \sin 2\pi\mu \\ -\sin 2\pi\mu & 1-\cos 2\pi\mu \end{vmatrix} = 2(1-\cos 2\pi\mu) = 4\sin^2\pi\mu$$

and this system has a nontrivial solution if and only if this determinant is zero, that is, if and only if $\sin^2\pi\mu = 0$, that is if and only if $\pi\mu = n\pi$ for some integer n.
In this case the boundary value problem has a nontrivial solution if and only if $\mu = n$ for some integer n. The eigenvalues are

$$\lambda_n = \mu_2^2 = n^2,$$

for $n \geqslant 1$.
For these eigenvalues the coefficient matrix in $(*)$ becomes the zero matrix, and both coefficients are undetermined. Hence for each $n \geqslant 1$, we have two linearly independent eigenfunctions:

$$\phi_{1,n}(x) = \cos nx \qquad \text{and} \qquad \phi_{2,n}(x) = \sin nx$$

for $n \geqslant 1$.
Thus, the solution to this eigenvalue problem has eigenvalues

$$\lambda_n = n^2$$

with corresponding eigenfunctions

$$\{\,\cos nx,\, \sin nx\,\}$$

for $n \geqslant 0$.

Exercise 15.6.
Assume that $f(x)$ is an even function and $g(x)$ is an odd function. Show that the set of functions $\{f(x), g(x)\}$ is orthogonal with respect to the weight function

$$w(x) = 1$$

on any symmetric interval $[-a, a]$ containing 0.

Solution: We have

$$\begin{aligned}\int_{-a}^{a} f(x)g(x)\,dx &= \underbrace{\int_{-a}^{0} f(x)g(x)\,dx}_{t=-x} + \int_{0}^{a} f(x)g(x)\,dx \\ &= \int_{0}^{a} f(-t)g(-t)\,dt + \int_{0}^{a} f(x)g(x)\,dx \\ &= -\int_{0}^{a} f(t)g(t)\,dt + \int_{0}^{a} f(x)g(x)\,dx \\ &= 0,\end{aligned}$$

and therefore f and g are orthogonal on the symmetric interval $[-a, a]$ with respect to the weight function $w(x) = 1$.

Exercise 15.7.
Show that the set of Laguerre polynomials $\left\{1,\, 1-x,\, \frac{1}{2}(2-4x+x^2)\right\}$ is orthogonal with respect to the weight function

$$w(x) = e^{-x}$$

on the interval $[0, \infty)$.

Solution: This problem can be solved by pairwise integration of the functions with the weight function e^{-x}. Since this is quite tedious, we give a more elegant method using the gamma function. Recall that for $n \geqslant 0$ we have

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = n!,$$

and therefore

$$< 1, 1-x >= \int_0^\infty (1-x)e^{-x}\,dx = 0! - 1! = 0,$$

$$< 1, \frac{1}{2}(2-4x+x^2) >= 0! - 2\cdot 1! + \frac{1}{2}\cdot 2! = 1 - 2 + 1 = 0$$

and finally,

$$\begin{aligned}< 1-x, \frac{1}{2}(2-4x+x^2) > &=< 1, \frac{1}{2}(2-4x+x^2) > - < x, \frac{1}{2}(2-4x+x^2) > \\ &= 0 - \left(1! - 2\cdot 2! + \frac{3!}{2}\right) \\ &= -1 + 4 - 3 \\ &= 0,\end{aligned}$$

and the set of Laguerre polynomials $\left\{1,\ 1-x,\ \frac{1}{2}(2-4x+x^2)\right\}$ forms an orthogonal set on the interval $[0,\infty)$ with respect to the weight function $w(x)=e^{-x}$.

Exercise 15.8. ✗

Is the set of functions $\left\{\frac{1}{2}(2-4x+x^2),\ -12x+8x^3\right\}$ orthogonal with respect to the weight function

$$w(x)=e^{-x}$$

on the interval $[0,\infty)$?

Solution: These functions are **not** orthogonal with respect to the weight function $w(x)=e^{-x}$ on the interval $[0,\infty)$, in fact,

$$\begin{aligned}
< 8x^3-12x, \frac{1}{2}(x^2-4x+2) > &= 2 < 2x^3-3x, x^2-4x+2 > \\
&= 2\int_0^\infty (2x^3-3x)(x^2-4x+2)e^{-x}\,dx \\
&= 2\int_0^\infty (2x^5-8x^4+x^3+12x^2-6x)e^{-x}\,dx \\
&= 2\,[2\cdot 5!-8\cdot 4!+3!+12\cdot 2!-6\cdot 1!] \\
&= 2\cdot 72 = 144.
\end{aligned}$$

Exercise 15.9.

Given the boundary value problem

$$\begin{aligned}
(1-x^2)y''-2xy'+(1+\lambda x)y &= 0\\
y(-1) &= 0\\
y(1) &= 0,
\end{aligned}$$

on the interval $[-1,1]$. Put the equation in Sturm-Liouville form and decide whether the problem is regular or singular.

Solution: We can rewrite the boundary value problem in the form

$$\begin{aligned}
\left((1-x^2)y'\right)'+(1+\lambda x)y &= 0\\
y(-1) &= 0\\
y(1) &= 0
\end{aligned}$$

and here $p(x)=1-x^2,\ p'(x)=-2x,\ q(x)=1,\ \sigma(x)=x$ are all continuous on the interval $[-1,1]$. Also, $c_1=d_1=1$ and $c_2=d_2=0$.

However, $p(x)=0$ at the endpoints of the interval $[-1,1]$, and $\sigma(0)=0$, so this is a **singular** Sturm-Liouville problem.

Exercise 15.10. ✗

Find the eigenvalues and eigenfunctions of the periodic eigenvalue problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(-\pi) &= y(\pi) \\ y'(-\pi) &= y'(\pi). \end{aligned}$$

Solution:

Case 1: If $\lambda = 0$, then the equation $y'' = 0$ has general solution $y(x) = Ax + B$ with $y' = A$. The first periodicity condition gives

$$-A\pi + B = A\pi + B$$

so that $A = 0$. The second periodicity condition is then automatically satisfied, so there is one nontrivial solution in this case. The eigenvalue is $\lambda = 0$ with corresponding eigenfunction $y_0 = 1$.

Case 2: If $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$, then the differential equation becomes $y'' - \mu^2 y = 0$, and has general solution $y(x) = A\cosh\mu x + B\sinh\mu x$ with $y' = \mu A\sinh\mu x + \mu B\cosh\mu x$. The first periodicity condition gives

$$A\cosh\mu\pi - B\sinh\mu\pi = A\cosh\mu\pi + B\sinh\mu\pi,$$

since $\cosh\mu x$ is an even function and $\sinh\mu x$ is an odd function. We have $2B\sinh\mu\pi = 0$, and since $\sinh\mu\pi \neq 0$, then $B = 0$. The solution is then $y = A\cosh\mu x$, and the second periodicity condition gives

$$-\mu A\sinh\mu\pi = \mu A\sinh\mu\pi,$$

so that $2\mu A\sinh\mu\pi = 0$, and since $\mu \neq 0$, then $\sinh\mu\pi \neq 0$, so we must have $A = 0$. Therefore, there are no nontrivial solutions in this case.

Case 3: If $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$, the differential equation becomes $y'' + \mu^2 y = 0$, and has general solution $y(x) = A\cos\mu x + B\sin\mu x$, with $y'(x) = -A\mu\sin\mu x + B\mu\cos\mu x$.

Applying the first periodicity condition, we have

$$y(-\pi) = A\cos\mu\pi - B\sin\mu\pi = A\cos\mu\pi + B\sin\mu\pi = y(\pi)$$

so that $2B\sin\mu\pi = 0$.

Applying the second periodicity condition, we have

$$y'(-\pi) = A\mu\sin\mu\pi + B\mu\cos\mu\pi = -A\mu\sin\mu\pi + B\mu\cos\mu\pi = y'(\pi)$$

so that $2A\sin\mu\pi = 0$. Therefore, the following equations must hold simultaneously:

$$\begin{aligned} A\sin\mu\pi &= 0 \\ B\sin\mu\pi &= 0 \end{aligned}$$

In order to get a nontrivial solution, we must have either $A \neq 0$, or $B \neq 0$, and if the equations hold, we must have $\sin \mu\pi = 0$. Therefore, μ must be an integer, so that the eigenvalues are

$$\lambda_n = \mu_n^2 = n^2$$

for $n = 1, 2, 3 \ldots$, and the eigenfunctions corresponding to these eigenvalues are $\sin nx$ and $\cos nx$ for $n = 1, 2, 3 \ldots$.
The full set of orthogonal eigenfunctions for the above periodic eigenvalue problem is

$$\{\, 1,\, \cos nx,\, \sin nx;\, n = 1, 2, 3, \ldots \,\}$$

for $0 \leqslant x \leqslant \pi$.

Exercise 15.11. ✗✗
Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0$$
$$y(0) + y'(0) = 0$$
$$y(2\pi) = 0.$$

Solution:
Case 1: If $\lambda = 0$, then the equation $y'' = 0$ has general solution $y(x) = Ax + B$ with $y' = A$. The first boundary condition gives

$$B + A = 0$$

so that $A = -B$. The second boundary condition gives

$$2\pi A + B = 0$$

so that $(2\pi - 1)A = 0$, and $A = -B = 0$, so there are no nontrivial solutions in this case.

Case 2: If $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$, then the differential equation becomes $y'' - \mu^2 y = 0$, and has general solution $y(x) = A\cosh \mu x + B\sinh \mu x$ with $y' = \mu A \sinh \mu x + \mu B \cosh \mu x$. The first boundary condition gives

$$A + \mu B = 0$$

so that $A = -\mu B$. The second boundary condition gives

$$A\cosh 2\pi\mu + B\sinh 2\pi\mu = 0$$

and since $\cosh 2\pi\mu \neq 0$, then

$$B(\tanh 2\pi\mu - \mu) = 0,$$

and in order to get nontrivial solutions we need

$$\tanh 2\pi\mu = \mu.$$

The graphs of $f(\mu) = \tanh 2\pi\mu$ and $g(\mu) = \mu$ intersect at the origin, $\mu = 0$, and since

$$\lim_{\mu\to\infty} \tanh 2\pi\mu = 1 \qquad \text{and} \qquad \lim_{\mu\to-\infty} \tanh 2\pi\mu = -1,$$

and

$$f'(0) = 2\pi > 1 = g'(0),$$

they intersect again in exactly two more points $\mu = \pm\mu_0$, where μ_0 is the positive root of the equation $\tanh 2\pi\mu = \mu$. There is one nontrivial solution in this case, with eigenvalue $\lambda = -(\mu_0)^2$ and the corresponding eigenfunction is

$$\sinh \mu_0 x - \mu_0 \cosh \mu_0 x.$$

Case 3: If $\lambda > 0$, say $\lambda = \mu^2$ where $\mu \neq 0$, then the differential equation becomes $y'' + \mu^2 y = 0$, and has general solution $y(x) = A\cos\mu x + B\sin\mu x$ with $y' = -\mu A\sin\mu x + \mu B\cos\mu x$. The first boundary condition gives

$$y(0) + y'(0) = A + \mu B = 0$$

so that $A = -\mu B$. The second boundary condition gives

$$y(2\pi) = A\cos 2\pi\mu + B\sin 2\pi\mu = 0,$$

and so

$$B\left[\sin 2\pi\mu - \mu\cos 2\pi\mu\right] = 0,$$

and the eigenvalues are $\lambda_n = \mu_n^2$, where μ_n is the n^{th} positive root of the equation $\tan 2\pi\mu = \mu$ (which has an infinite number of solutions $\mu_n,\ n = 1, 2, 3, \dots$).
The corresponding eigenfunctions are

$$y_n = \sin\mu_n x - \mu_n \cos\mu_n x$$

for $n = 1, 2, 3, \dots$.

Exercise 15.12. ✗✗✗
Show that the boundary value problem

$$y'' - \lambda y = 0$$
$$y(0) + y'(0) = 0$$
$$y(1) + y'(1) = 0$$

has one positive eigenvalue. Does this contradict the Theorem below?

Theorem. The eigenvalues of a regular Sturm-Liouville problem are all real and form an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

where $\lambda_n \to \infty$ as $n \to \infty$.

Solution:
Case 1: If $\lambda = 0$, the differential equation $y'' = 0$ has general solution $y = Ax + B$, with $y' = A$. Applying the first boundary condition, we have

$$B + A = 0,$$

so that $B = -A$. Applying the second boundary condition, we have

$$A + B + A = 0,$$

so that $B = -2A$, and therefore $B = 2B$, and $B = A = 0$. Therefore, there are no nontrivial solutions in this case.

Case 2: If $\lambda < 0$, say $\lambda = -\mu^2$ where $\mu \neq 0$, the differential equation becomes $y'' + \mu^2 y = 0$ and has general solution

$$y = A\cos\mu x + B\sin\mu x, \qquad \text{with} \qquad y' = -\mu A\sin\mu x + \mu B\cos\mu x.$$

The first boundary condition gives

$$y(0) + y'(0) = A + \mu B = 0$$

so that $A = -\mu B$. The second boundary condition gives

$$y(1) + y'(1) = A\cos\mu + B\sin\mu - \mu A\sin\mu + \mu B\cos\mu = 0,$$

that is,

$$(\cos\mu - \mu\sin\mu)A + (\sin\mu + \mu\cos\mu)B = 0.$$

The system of linear equations for A and B

$$\begin{aligned} A + \mu B &= 0 \\ (\cos\mu - \mu\sin\mu)A + (\sin\mu + \mu\cos\mu)B &= 0 \end{aligned}$$

has nontrivial solutions if and only if the determinant of the corresponding coefficient matrix is zero, that is,

$$(1 + \mu^2)\sin\mu = 0,$$

that is, if and only if $\sin\mu = 0$. The eigenvalues are $\lambda_n = -(\mu_n)^2 = -n^2$, with corresponding eigenfunctions

$$y_n = \sin nx - n\cos nx$$

for $n = 1, 2, 3, \ldots$.
Case 3: If $\lambda > 0$, say $\lambda = \mu^2$, the differential equation becomes $y'' - \mu^2 y = 0$ and has general solution

$$y = A\cosh\mu x + B\sinh\mu x, \qquad \text{with} \qquad y' = \mu A\sinh\mu x + \mu B\cosh\mu x.$$

The first boundary condition gives

$$y(0) + y'(0) = A + \mu B = 0$$

The second boundary condition gives

$$y(1) + y'(1) = A\cosh\mu + B\sinh\mu + \mu A\sinh\mu + \mu B\cosh\mu = 0,$$

that is,

$$(\cosh\mu + \mu\sinh\mu)A + (\sinh\mu + \mu\cosh\mu)B = 0.$$

The system of linear equations for A and B

$$\begin{aligned} A + \mu B &= 0 \\ (\cosh\mu + \mu\sinh\mu)A + (\sinh\mu + \mu\cosh\mu)B &= 0 \end{aligned}$$

has nontrivial solutions if and only if the determinant of the corresponding coefficient matrix is zero, that is,

$$(1 - \mu^2)\sinh\mu = 0,$$

and since $\sinh\mu \neq 0$, the system has nontrivial solutions if and only if $1 - \mu^2 = 0$, that is, if and only if $\mu = \pm 1$.

Therefore, there is only one positive eigenvalue, namely

$$\lambda = (\pm 1)^2 = 1,$$

with corresponding eigenfunction

$$y = \sinh x - \cosh x.$$

Note: Here the weight function is $\sigma(x) = -1 < 0$, and the problem is not a regular Sturm-Liouville problem, and so this does not contradict the Theorem, since the Theorem does not apply. We can, however, redefine the eigenvalue as $\tilde{\lambda} = -\lambda$, then the problem becomes a regular Sturm-Liouville problem and the Theorem does apply. According to the above computations we get for $\tilde{\lambda}$ the eigenvalues -1, and n^2 for $n = 1, 2, \ldots$.

Exercise 15.13. ✗

Show explicitly that there are no negative eigenvalues for the boundary value problem

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi = 0, &\quad 0 < x < L \\ \frac{d\phi}{dx}(0) = 0& \\ \frac{d\phi}{dx}(L) = 0& \end{aligned}$$

Hint: Multiply the equation by ϕ and integrate.

Solution: Following the hint, we can show this using the differential equation and the boundary conditions. Suppose that λ is an eigenvalue of the boundary value problem with corresponding eigenvector ϕ, then ϕ satisfies the differential equation and the boundary conditions, and multiplying the differential equation by ϕ, we have

$$\phi\,\frac{d^2\phi}{dx^2} + \lambda\phi^2 = 0$$

for $0 < x < L$.
Integrating over the interval $[0, L]$, we have

$$\int_0^L \phi\,\frac{d^2\phi}{dx^2}\,dx + \lambda\int_0^L \phi^2\,dx = 0,$$

and integrating the first integral by parts,

$$\phi\,\frac{d\phi}{dx}\bigg|_0^L - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx + \lambda\int_0^L \phi^2\,dx = 0,$$

that is,

$$\phi(L)\,\frac{d\phi}{dx}(L) - \phi(0)\,\frac{d\phi}{dx}(0) - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx + \lambda\int_0^L \phi^2\,dx = 0,$$

and since $\dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(L) = 0$, then

$$\lambda\int_0^L \phi^2\,dx - \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx = 0,$$

that is,

$$\lambda = \frac{\displaystyle\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\displaystyle\int_0^L \phi^2\,dx} \geqslant 0,$$

and there are no negative eigenvalues for this boundary value problem.

Alternatively, we can show this by solving the boundary value problem. Assuming a solution of the form $\phi(x) = e^{rx}$, we get $(r^2 + \lambda)e^{rx} = 0$, and since e^{rx} is never zero, the auxiliary equation is $r^2 + \lambda = 0$.
Now if $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the auxiliary equation $r^2 - \mu^2 = 0$ has two real roots, namely $r_1 = \mu$ and $r_2 = -\mu$.
We have two linearly independent eigenfunctions: $\phi_1(x) = \cosh\mu x$ and $\phi_2(x) = \sinh\mu x$, and the general solution to the differential equation is

$$\phi(x) = A\cosh\mu x + B\sinh\mu x.$$

Differentiating, we have

$$\frac{d\phi}{dx}(x) = \mu A\sinh\mu x + \mu B\cosh\mu x.$$

From the first boundary condition

$$\frac{d\phi}{dx}(0) = \mu B = 0,$$

and since $\mu \neq 0$, then $B = 0$.
From the second boundary condition

$$\frac{d\phi}{dx}(L) = \mu A \sinh \mu L = 0,$$

and since $\mu \neq 0$ and $\sinh \mu L \neq 0$, then $A = 0$ also, so if $\lambda < 0$, we have only the trivial solution $\phi(x) \equiv 0$ on $[0, L]$. Therefore we must have $\lambda \geqslant 0$, that is, the boundary value problem has no negative eigenvalues.

Exercise 15.14. ✗
Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + \big[\lambda\beta(x) + \gamma(x)\big]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \big[\lambda\sigma(x) + q(x)\big]\phi = 0.$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$?

Solution: Multiplying the differential equation by $H(x)$ we have

$$H\,\frac{d^2\phi}{dx^2} + \alpha\,H\,\frac{d\phi}{dx} + \lambda\,\beta\,H\,\phi + \gamma\,H\,\phi = 0,$$

and we want to determine H so that the first two terms are an exact derivative, that is,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] = H\,\frac{d^2\phi}{dx^2} + \alpha\,H\,\frac{d\phi}{dx},$$

that is,

$$p(x)\,\frac{d^2\phi}{dx^2} + \frac{dp(x)}{dx}\,\frac{d\phi}{dx} = H\,\frac{d^2\phi}{dx^2} + \alpha\,H\,\frac{d\phi}{dx}.$$

Thus, we want

$$p(x) = H(x) \qquad \text{and} \qquad p'(x) = \alpha(x)\,H$$

so that $H(x)$ satisfies the differential equation

$$H'(x) = \alpha(x)\,H(x).$$

If we take

$$p(x) = H(x) = \exp\left(\int \alpha(x)\,dx\right),$$

then the differential equation is in Sturm-Liouville form

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \big[\lambda\sigma(x) + q(x)\big]\phi = 0.$$

where

$$p(x) = \exp\left(\int \alpha(x)\,dx\right), \quad q(x) = \gamma(x)\,\exp\left(\int \alpha(x)\,dx\right), \quad \sigma(x) = \beta(x)\,\exp\left(\int \alpha(x)\,dx\right).$$

Note that $p(x) > 0$ and $\sigma(x) > 0$ provided that $\beta(x) > 0$.

Exercise 15.15. ✗

For the Sturm-Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad 0 < x < L$$
$$\frac{d\phi}{dx}(0) = 0,$$
$$\phi(L) = 0,$$

verify the following general properties:

(a) There are an infinite number of eigenvalues with a smallest but no largest.

(b) The n^{th} eigenfunction has $n-1$ zeros.

(c) The eigenfunctions are complete and orthogonal.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues ?

Solution:

(a) Assuming that the eigenvalues are real, we have to consider the three cases when $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

<u>case 1</u>: If $\lambda = 0$, the general solution to the differential equation $\phi''(x) = 0$ is $\phi(x) = Ax + B$, with $\phi'(x) = A$, and applying the first boundary condition $\phi'(0) = 0$, we have $A = 0$, and the solution is $\phi(x) = B$ for $0 < x < L$. Applying the second boundary condition $\phi(L) = 0$, we have $B = 0$, and the only solution in this case is the trivial solution $\phi(x) = 0$ for $0 < x < L$. Therefore $\lambda = 0$ is not an eigenvalue.

<u>case 2</u>: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' - \mu^2\phi = 0$ is

$$\phi(x) = A\cosh\mu x + B\sinh\mu x \qquad \text{with} \qquad \phi'(x) = \mu A\sinh\mu x + \mu B\cosh\mu x.$$

Applying the first boundary condition $\phi'(0) = \mu B = 0$ implies that $B = 0$, and the solution is now

$$\phi(x) = A\cosh\mu x$$

Applying the second boundary condition $\phi(L) = 0$ implies that $A\cosh\mu L = 0$, so that $A = 0$, and in this case we have only the trivial solution $\phi(x) = 0$ for $0 < x < L$.

case 3: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' + \mu^2\phi = 0$ is

$$\phi(x) = A\cos\mu x + B\sin\mu x \qquad \text{with} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x.$$

Applying the first boundary condition $\phi'(0) = 0$ implies that $\mu B = 0$, so that $B = 0$, and the solution is now

$$\phi(x) = A\cos\mu x$$

Applying the second boundary condition $\phi(L) = 0$ implies that $A\cos\mu L = 0$, and if $A = 0$ we get only the trivial solution. The boundary value problem has a nontrivial solution if and only if $\cos\mu L = 0$, that is, if and only if $\mu L = (n - \frac{1}{2})\pi$ for some integer $n \geqslant 1$, and therefore the eigenvalues are

$$\lambda_n = \left(\tfrac{(2n-1)\pi}{2L}\right)^2$$

with corresponding eigenfunctions

$$\phi_n(x) = \cos\tfrac{(2n-1)\pi x}{2L}$$

for $n = 1, 2, \ldots$.

The eigenvalues are therefore ordered as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

and there are an infinite number of eigenvalues with the smallest one being $\lambda_1 = \frac{\pi^2}{4L^2}$, but there is no largest eigenvalue.

(b) For $n \geqslant 1$, the eigenfunction ϕ_n is given by

$$\phi_n(x) = \cos\tfrac{(2n-1)\pi x}{2L}$$

for $0 < x < L$. Note that

$$\phi_n(0) = 1 \qquad \text{while} \qquad \phi_n(L) = \cos\tfrac{(2n-1)\pi}{2} = 0,$$

and all the zeros of ϕ_n occur in the interval $(0, L]$.

Also, $\phi_n(x) = 0$ exactly when

$$\tfrac{(2n-1)\pi x}{2L} = \tfrac{(2k-1)\pi}{2}$$

for $1 \leqslant k \leqslant n$, that is,

$$x = \left(\tfrac{2k-1}{2n-1}\right)L$$

for $1 \leqslant k \leqslant n$, and the eigenfunction $\phi_n(x) = \cos\frac{(2n-1)\pi x}{2L}$ has exactly n zeros in the interval $(0, L]$, that is, $\phi_m(x)$ has exactly $n - 1$ zeros in the open interval $(0, L)$.

(c) From Dirichlet's theorem we know that every f in the linear space of all piecewise smooth functions on $[0, L]$ has a Fourier series expansion in terms of the eigenfunctions, that is, the eigenfunctions form a complete set in the linear space $\mathcal{PWS}[0, L]$. The eigenfunctions form what is usually called a **Schauder Basis** for the linear space $\mathcal{PWS}[0, L]$.

Finally, we note that

$$\int_0^L \phi_m(x)\phi_n(x)\,dx = \int_0^L \cos\tfrac{(2m-1)\pi x}{2L}\cos\tfrac{(2n-1)\pi x}{2L}\,dx = 0$$

for $m, n \geqslant 1$ with $m \neq n$, and the set of eigenfunctions forms an orthogonal set.

(d) Using the boundary conditions

$$\phi'(0) = 0 \qquad \text{and} \qquad \phi(L) = 0$$

for the regular Sturm-Liouville problem above, we can write the eigenvalues in terms of the corresponding eigenfunctions as follows

$$\lambda_n = R(\phi_n) = \frac{\displaystyle\int_0^L \phi_n'(x)^2\,dx}{\displaystyle\int_0^L \phi_n(x)^2\,dx},$$

and clearly $\lambda_n \geqslant 0$.

If $\lambda_0 = 0$ is an eigenvalue then

$$\lambda_0 = R(\phi_0) = \frac{\displaystyle\int_0^L \phi_0'(x)^2\,dx}{\displaystyle\int_0^L \phi_0(x)^2\,dx} = 0,$$

and then $\phi_0'(x) = 0$ for $0 \leqslant x \leqslant L$, and $\phi_0(x)$ is a constant, and then $\phi_0(L) = 0$ implies that $\phi_0(x) = 0$ for $0 < x < L$, which is a contradiction, and therefore $\lambda_0 = 0$ is **not** an eigenvalue.

Exercise 15.16. ✗

Show that $\lambda > 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi = 0, \quad 0 < x < 1$$

with

$$\frac{d\phi}{dx}(0) = 0 \qquad \text{and} \qquad \frac{d\phi}{dx}(1) = 0.$$

Solution: This is a regular Sturm-Liouville problem with

$$p(x) = 1, \qquad q(x) = -x^2 \leqslant 0, \qquad \text{and} \qquad \sigma(x) = 1$$

for $0 \leqslant x \leqslant 1$, and from the boundary conditions

$$\Big[-p(x)\phi(x)\phi'(x)\Big]\Big|_0^1 = 0,$$

and the Rayleigh quotient reduces to

$$\lambda = R(\phi) = \frac{\displaystyle\int_0^1 \left[\phi'(x)^2 + x^2\phi(x)^2\right]\,dx}{\displaystyle\int_0^1 \phi(x)^2\,dx} \geqslant 0,$$

and all of the eigenvalues are nonnegative.
If $\lambda = 0$ is an eigenvalue and ϕ_0 is the corresponding eigenfunction (and is thus not identically zero on the interval $[0,1]$), then

$$0 = R(\phi_0) = \frac{\displaystyle\int_0^1 \left[\phi_0'(x)^2 + x^2\phi_0(x)^2\right]\,dx}{\displaystyle\int_0^1 \phi_0(x)^2\,dx},$$

assuming that ϕ_0 and ϕ_0' are continuous on the interval $[0,1]$, this implies that

$$\phi_0'(x)^2 = 0 \qquad \text{and} \qquad x^2\phi_0(x)^2 = 0$$

for all $x \in [0,1]$, and this implies that $\phi_0(x) = 0$ for all $x \in [0,1]$, which is a contradiction. Therefore $\lambda_0 = 0$ is **not** an eigenvalue.

Exercise 15.17. ✗
Give an example of an eigenvalue problem where there is more than one eigenfunction corresponding to an eigenvalue.

Solution: Consider the boundary value problem with periodicity conditions as given below.

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad -\pi < x < \pi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{dx}(-\pi) = \frac{d\phi}{dx}(\pi).$$

The eigenvalues are $\lambda_n = n^2$ with corresponding eigenfunctions

$$\phi_n(x) = \cos nx \qquad \text{and} \qquad \psi_n(x) = \sin nx$$

for $n \geqslant 0$.

Therefore there are two linearly independent eigenfunctions for each eigenvalue λ_n for $n \geqslant 1$. For $\lambda_0 = 0$, there is only one eigenfunction, namely, $\phi_0(x) = 1$ for $-\pi < x < \pi$.

Note that the periodicity conditions given here are not boundary conditions of Sturm-Liouville type, they are *mixed boundary conditions*, in the sense that each contains the function or its derivative evaluated at *both* endpoints of the interval.

Exercise 15.18. ✗✗

Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $u\,L(v) - v\,L(u)$ is an exact differential.

(b) Evaluate

$$\int_0^1 [u\,L(v) - v\,L(u)]\;dx$$

in terms of the boundary data for any functions u and v.

(c) Show that

$$\int_0^1 [u\,L(v) - v\,L(u)]\;dx = 0$$

if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = 0 \qquad \phi(1) = 0$$

$$\frac{d\phi}{dx}(0) = 0 \qquad \frac{d^2\phi}{dx^2}(1) = 0.$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [u\,L(v) - v\,L(u)]\;dx = 0.$$

(e) For the eigenvalue problem below using the boundary conditions from (c)

$$\frac{d^4\phi}{dx^4} + \lambda e^x\phi = 0, \quad 0 < x < 1,$$

$$\phi(0) = 0 \qquad \phi(1) = 0$$

$$\frac{d\phi}{dx}(0) = 0 \qquad \frac{d^2\phi}{dx^2}(1) = 0,$$

show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal. What is the weight function?

Solution:

(a) We consider

$$uv^{(4)} = (uv''')' - u'v''' = (uv''')' - (u'v'')' + u''v'', \tag{$*$}$$

and by symmetry,

$$vu^{(4)} = (vu''')' - (v'u'')' + v''u'', \tag{$**$}$$

and subtracting (∗∗) from (∗) we have

$$uL(v) - vL(u) = \left(uv''' - vu''' - u'v'' + v'u''\right)',$$

and $uL(v) - vL(u)$ is an exact differential.

(b) We have

$$\int_0^1 [uL(v) - vL(u)]\, dx = \left[uv''' - vu''' - u'v'' + v'u''\right]\Big|_0^1$$
$$= u(1)v'''(1) - v(1)u'''(1) - u'(1)v''(1) + v'(1)u''(1)$$
$$- u(0)v'''(0) + v(0)u'''(0) + u'(0)v''(0) - v'(0)u''(0).$$

(c) If u and v are any two functions satisfying the boundary conditions

$$\phi(0) = 0, \qquad \phi(1) = 0,$$
$$\phi'(0) = 0, \qquad \phi''(1) = 0.$$

From part (b) each of the first four terms contains either $u(1),\ v(1),\ u''(1),$ or $v''(1)$, each of which is 0, while each of the last four terms contains either $u(0),\ v(0),\ u'(0),$ or $v'(0)$, each of which is also 0.

(d) Another set of boundary conditions for which

$$\int_0^L [uL(v) - vL(u)]\, dx = 0$$

is given by

$$\phi'(0) = 0, \qquad \phi'(1) = 0,$$
$$\phi'''(0) = 0, \qquad \phi'''(1) = 0.$$

(e) Let (λ_n, ϕ_n) and (λ_m, ϕ_m) be distinct eigenvalue - eigenfunction pairs satisfying the boundary value problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0, \quad 0 < x < 1,$$
$$\phi(0) = 0, \qquad \phi(1) = 0,$$
$$\phi'(0) = 0, \qquad \phi''(1) = 0,$$

then we have

$$0 = \int_0^1 \phi_n L(\phi_m) - \phi_m L(\phi_n)\, dx$$
$$= \int_0^1 \left[\phi_n\left(-\lambda_m e^x \phi_m\right) - \phi_m\left(-\lambda_n e^x \phi_n\right)\right]\, dx$$
$$= (\lambda_n - \lambda_m) \int_0^1 \phi_n \phi_m e^x\, dx$$

and if $\lambda_n \neq \lambda_m$, then

$$\int_0^1 \phi_n \phi_m e^x \, dx = 0$$

and ϕ_n and ϕ_m are orthogonal on the interval $[0, 1]$ with respect to the weight function

$$\sigma(x) = e^x$$

for $x \in [0, 1]$.

Exercise 15.19. ✗

Let $u(x) = J_0(\alpha x)$ and $v(x) = J_0(\beta x)$.

(a) Show that $xu'' + u' + \alpha^2 xu = 0$ and $xv'' + v' + \beta^2 xv = 0$.

(b) Show that $[x(u'v - v'u)]' = (\beta^2 - \alpha^2)xuv$.

(c) Show that

$$(\beta^2 - \alpha^2) \int xJ_0(\alpha x)J_0(\beta x)\, dx = x\left[\alpha J_0'(\alpha x)J_0(\beta x) - \beta J_0'(\beta x)J_0(\alpha x)\right].$$

This is one of a set of formulas called *Lommel's integrals.*

(d) Show that if α and β are distinct zeros of $J_0(z)$, then

$$\int_0^1 xJ_0(\alpha x)J_0(\beta x)\, dx = 0$$

so that $J_0(\alpha x)$ and $J_0(\beta x)$ are orthogonal on the interval $[0, 1]$ with respect to the weight function $\sigma(x) = x$.

Solution:

(a) Since $u(x) = J_0(\alpha x)$ and $v(x) = J_0(\beta x)$ are solutions to Bessel's equation of order zero, then

$$\big(xu'(x)\big)' + \alpha^2 xu(x) = 0 \qquad \text{and} \qquad \big(xv'(x)\big)' + \beta^2 xv(x) = 0,$$

that is,

$$xu''(x) + u'(x) + \alpha^2 xu(x) = 0 \qquad \text{and} \qquad xv''(x) + v'(x) + \beta^2 xv(x) = 0$$

for $0 \leqslant x \leqslant 1$.

(b) From part (a), we have

$$xu''(x)v(x) + u'(x)v(x) + \alpha^2 xu(x)v(x) = 0$$

and

$$xv''(x)u(x) + v'(x)u(x) + \beta^2 xv(x)u(x) = 0,$$

and subtracting the second equation from the first, we have

$$xu''(x)v(x) + u'(x)v(x) - (xv''(x)u(x) + v'(x)u(x)) - (\beta^2 - \alpha^2)xu(x)v(x) = 0,$$

that is,

$$(\beta^2 - \alpha^2)xu(x)v(x) = \big(xu'(x)v(x) - xv'(x)u(x)\big)'$$

for $0 \leqslant x \leqslant 1$.

(c) Integrating this last expression, we have an indefinite integral

$$\begin{aligned}(\beta^2 - \alpha^2)\int xu(x)v(x)\,dx &= \int \big(xu'(x)v(x) - xv'(x)u(x)\big)'\,dx \\ &= \big(xu'(x)v(x) - xv'(x)u(x)\big),\end{aligned}$$

that is,

$$(\beta^2 - \alpha^2)\int xJ_0(\alpha x)J_0(\beta x)\,dx = x\left[\alpha J_0'(\alpha x)J_0(\beta x) - \beta J_0'(\beta x)J_0(\alpha x)\right].$$

(d) Now, if α and β are distinct zeros of $J_0(z)$, then

$$(\beta^2 - \alpha^2)\int_0^1 xJ_0(\alpha x)J_0(\beta x)\,dx = \alpha J_0'(\alpha)J_0(\beta) - \beta J_0'(\beta)J_0(\alpha) = 0,$$

and since $\alpha \neq \beta$, then

$$\int_0^1 xJ_0(\alpha x)J_0(\beta x)\,dx = 0,$$

so that $J_0(\alpha x)$ and $J_0(\beta x)$ are orthogonal on the interval $[0, 1]$ with respect to the weight function $\sigma(x) = x$.

Exercise 15.20. ✗✗✗

Consider the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad 0 \leqslant x \leqslant 1,$$

$$\phi(0) - \frac{d\phi}{dx}(0) = 0,$$

$$\phi(1) + \frac{d\phi}{dx}(1) = 0.$$

(a) Using the Rayleigh quotient, show that $\lambda \geqslant 0$. Why is $\lambda > 0$?

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

(c) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

(d) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with

$$u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0$$

$$u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0$$

$$u(x,0) = f(x).$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

Solution:

(a) We use the Rayleigh quotient to show that $\lambda > 0$ for all eigenvalues λ.

Let λ be an eigenvalue of the Sturm-Liouville problem above, and let $\phi(x)$ be the corresponding eigenfunction, then

$$-p(x)\phi(x)\phi'(x)\Big|_0^1 = -\phi(1)\phi'(1) + \phi(0)\phi'(0) = \phi(1)^2 + \phi(0)^2 \geqslant 0,$$

and since $q(x) = 0 \leqslant 0$ for all $0 \leqslant x \leqslant 1$, then

$$\lambda = \frac{\phi(0)^2 + \phi(1)^2 + \displaystyle\int_0^1 \phi'(x)^2\,dx}{\displaystyle\int_0^1 \phi(x)^2\,dx} \geqslant 0$$

since $p(x) = \sigma(x) = 1$ for $0 \leqslant x \leqslant 1$.

If $\lambda = 0$, then

$$\phi(0)^2 + \phi(1)^2 + \int_0^1 \phi'(x)^2\,dx = 0,$$

so that $\phi(0) = \phi(1) = 0$ and $\displaystyle\int_0^1 \phi'(x)^2\,dx = 0$, and since ϕ' is continuous on $[0,1]$, then $\phi(x)$ is constant on $[0,1]$, so that $\phi(x) = \phi(0) = 0$ for all $0 \leqslant x \leqslant 1$. Therefore $\lambda = 0$ is **not** an eigenvalue of this boundary value problem, and all the eigenvalues satisfy $\lambda > 0$.

(b) If λ_1 and λ_2 are distinct eigenvalues of this boundary value problem, with corresponding eigenfunctions ϕ_1 and ϕ_2, respectively, then

$$\phi_1'' + \lambda_1\phi_1 = 0 \qquad \text{and} \qquad \phi_2'' + \lambda_2\phi_2 = 0,$$

so that

$$\phi_2\phi_1'' - \phi_1\phi_2'' + (\lambda_1 - \lambda_2)\phi_1\phi_2 = 0,$$

that is,

$$(\phi_2\phi_1' - \phi_1\phi_2')' + (\lambda_1 - \lambda_2)\phi_1\phi_2 = 0,$$

and integrating, we have

$$(\phi_2\phi_1' - \phi_1\phi_2')\Big|_0^1 + (\lambda_1 - \lambda_2)\int_0^1 \phi_1\phi_2\,dx = 0.$$

However,

$$\begin{aligned}(\phi_2\phi_1' - \phi_1\phi_2')\Big|_0^1 &= \phi_2(1)\phi_1'(1) - \phi_1(1)\phi_2'(1) - \phi_2(0)\phi_1'(0) + \phi_1(0)\phi_2'(0)\\ &= -\phi_2(1)\phi_1(1) + \phi_1(1)\phi_2(1) + \phi_2(0)\phi_1(0) - \phi_1(0)\phi_2(0) = 0,\end{aligned}$$

so that

$$(\lambda_1 - \lambda_2)\int_0^1 \phi_1\phi_2\,dx = 0.$$

Since $\lambda_1 \neq \lambda_2$, then

$$\int_0^1 \phi_1(x)\phi_2(x)\,dx = 0,$$

that is, ϕ_1 and ϕ_2 are orthogonal on the interval $[0,1]$.

(c) If $\lambda > 0$, then $\lambda = \mu^2$, where $\mu \neq 0$, and the differential equation is $\phi'' + \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x \qquad \text{and} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x$$

for $0 \leqslant x \leqslant 1$.

From the first boundary condition

$$\phi(0) + \phi'(0) = A - \mu B = 0,$$

and $A = \mu B$.

From the second boundary condition

$$\phi(1) + \phi'(1) = A\cos\mu + B\sin\mu - \mu A\sin\mu + \mu B\cos\mu = 0,$$

that is,

$$B\left[2\mu\cos\mu - (\mu^2 - 1)\sin\mu\right] = 0,$$

and the boundary value problem has a nontrivial solution if and only if

$$\tan\mu = \frac{2\mu}{\mu^2 - 1},$$

that is, if and only if

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

In order to determine the eigenvalues we sketch the graphs of the functions

$$f(\mu) = \tan\mu \qquad \text{and} \qquad g(\mu) = \frac{2\mu}{\mu^2 - 1}$$

for $\mu > 0$.

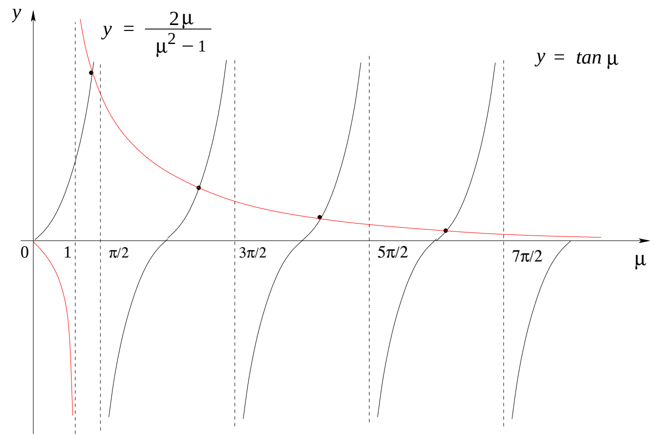
Note that for $\mu > 0$, we have

$$g(\mu) = \frac{2\mu}{\mu^2 - 1} = \frac{1}{\mu + 1} + \frac{1}{\mu - 1},$$

so that

$$g'(\mu) = -\frac{1}{(\mu + 1)^2} - \frac{1}{(\mu - 1)^2} < 0$$

and g is decreasing on the interval $(0, 1)$ and on the interval $(1, \infty)$ and the line $\mu = 1$ is a vertical asymptote to the graph. The graphs of g and f are shown below.



From the figure it is clear that there are an infinite number of distinct solutions μ_n to the equation

$$\tan\mu = \frac{2\mu}{\mu^2 - 1},$$

and the eigenvalues are $\lambda_n = \mu_n^2$, for $n \geqslant 1$.

Since $\lim\limits_{n\to\infty} \mu_n = +\infty$, then

$$\lim_{n\to\infty} \tan\mu_n = \lim_{n\to\infty} \frac{2\mu_n}{\mu_n^2 - 1} = 0,$$

and the roots of the equation $\tan\mu = \dfrac{2\mu}{\mu^2-1}$ approach the roots of the equation $\tan\mu = 0$, that is, for large n,

$$\mu_n \approx n\pi,$$

and therefore

$$\lambda_n = \mu_n^2 \approx n^2\pi^2$$

for large n.

(d) We want to solve the boundary value – initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \\ u(0,t) - \frac{\partial u}{\partial x}(0,t) &= 0 \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) &= 0 \\ u(x,0) &= f(x), \end{aligned}$$

and since both the equation and the boundary conditions are linear and homogeneous, we can use separation of variables.

Assuming a solution of the form $u(x,t) = \phi(x) \cdot h(t)$, and separating variables we get the two ordinary differential equations

$$\begin{aligned} &\phi''(x) + \lambda\phi(x) = 0, \quad 0 \leqslant x \leqslant 1, \qquad h'(t) + \lambda k h(t) = 0, \quad t \geqslant 0,\\ &\phi(0) - \phi'(0) = 0\\ &\phi(1) + \phi'(1) = 0. \end{aligned}$$

From part (a) we know that we have a nontrivial solution if and only if $\lambda > 0$, in which case $\lambda = \mu^2$ where $\mu \neq 0$, and the differential equation becomes $\phi'' + \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x \qquad \text{with} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x.$$

Applying the boundary conditions as in part (c), the only values of μ for which we have a nontrivial solution are those for which

$$\tan\mu = \frac{2\mu}{\mu^2 - 1},$$

and we have an infinite sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

where $\lambda_n = \mu_n^2$. The corresponding eigenfunctions are

$$\phi_n(x) = \mu_n\cos\mu_n x + \sin\mu_n x$$

for $n \geqslant 1$.

The corresponding solutions to the time equation are

$$h_n(t) = e^{-\lambda_n k t},$$

and from the superposition principle the sum

$$u(x,t) = \sum_{n=1}^{\infty} a_n\left(\mu_n\cos\mu_n x + \sin\mu_n x\right) e^{-\mu_n^2 k t}$$

satisfies the partial differential equation and the boundary conditions.

In order to satisfy the initial condition we use the orthogonality of the eigenfunctions from part (c), and write

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n\phi_n(x),$$

where

$$a_n = \frac{\displaystyle\int_0^1 f(x)\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n(x)^2\,dx}$$

for $n \geqslant 1$.

As in Exercise 11.16, you can verify that the normalization constant is given by

$$\int_0^1 \phi_n(x)^2 \, dx = \frac{\mu_n^2 + 3}{2}$$

so that

$$a_n = \frac{2}{\mu_n^2 + 3} \int_0^1 f(x)\phi_n(x) \, dx$$

for $n \geqslant 1$.

Chapter 16

Fourier Transform Problems

NOTE: The Fourier transform is defined as

$$\mathcal{F}(f)(\omega) = \widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt,$$

and the inverse transform is defined as

$$\mathcal{F}^{-1}(\widehat{f})(t) = \int_{-\infty}^{\infty} \widehat{f}(\omega)e^{-i\omega t}\,d\omega.$$

The convolution of two functions $f(x)$ and $g(x)$ is defined to be

$$f * g\,(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

and the convolution theorem says that if $F(\omega)$ and $G(\omega)$ are the Fourier transforms of f and g, respectively, then

$$\mathcal{F}^{-1}\left(F(\omega)G(\omega)\right)(x) = \frac{1}{2\pi} f * g\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-t)g(t)\,dt.$$

Exercise 16.1. ✗
Evaluate the Fourier integral formula for the function

$$f(x) = \begin{cases} 1-\cos x & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: The Fourier integral representation of $f(x)$ is given by

$$f(x) \sim \int_{0}^{\infty} \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)\,d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt \qquad \text{and} \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt.$$

Since $f(x)$ is an even function, then $B(\omega) = 0$ for all ω.

Also, since $f(x)$ is even and $f(x) = 0$ for $|x| \geqslant \frac{\pi}{2}$, then for all $\omega \neq 0$ and $\omega \neq \pm 1$, we have

$$\begin{aligned}
A(\omega) &= \frac{2}{\pi}\int_0^{\pi/2}(1-\cos t)\cos\omega t\,dt \\
&= \frac{2}{\pi}\int_0^{\pi/2}\cos\omega t\,dt - \frac{2}{\pi}\int_0^{\pi/2}\cos t\cos\omega t\,dt \\
&= \frac{2}{\pi}\frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi}\int_0^{\pi/2}\left[\cos(1-\omega)t + \cos(1+\omega)t\right]\,dt \\
&= \frac{2}{\pi}\frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi}\frac{\sin(1-\omega)t}{1-\omega}\bigg|_0^{\pi/2} - \frac{1}{\pi}\frac{\sin(1+\omega)t}{1+\omega}\bigg|_0^{\pi/2} \\
&= \frac{2}{\pi}\frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi}\frac{\sin((1-\omega)\pi/2)}{1-\omega} - \frac{1}{\pi}\frac{\sin((1+\omega)\pi/2)}{1+\omega} \\
&= \frac{2}{\pi}\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{\pi}\left[\frac{1}{1-\omega} + \frac{1}{1+\omega}\right] \\
&= \frac{2}{\pi}\left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right],
\end{aligned}$$

so that

$$A(\omega) = \frac{2}{\pi}\left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right]$$

for $\omega \neq 0, \pm 1$.
If $\omega = 0$, then

$$A(0) = \frac{2}{\pi}\int_0^{\pi/2}(1-\cos t)\,dt = \frac{2}{\pi}\left[\frac{\pi}{2} - \sin(\pi/2)\right] = 1 - \frac{2}{\pi}.$$

If $\omega = \pm 1$, then

$$A(\pm 1) = \frac{2}{\pi}\frac{\sin(\pm\pi/2)}{\pm 1} - \frac{2}{\pi}\int_0^{\pi/2}\cos^2 t\,dt = \frac{2}{\pi} - \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1+\cos 2t}{2}\right)dt = \frac{2}{\pi} - \frac{1}{2}.$$

Note that $A(w)$ is continuous for all ω.
From Dirichlet's theorem, the integral

$$\frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right]\cos\omega x\,d\omega$$

converges to $1-\cos x$ for all $|x| < \frac{\pi}{2}$, converges to 0 for all $|x| > \frac{\pi}{2}$, and converges to $\frac{1}{2}$ for $x = \pm\frac{\pi}{2}$. Thus, if we redefine $f(\pm\pi/2) = \frac{1}{2}$, then the Fourier integral representation of $f(x)$ is given by

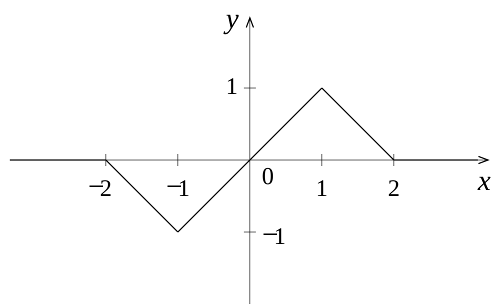
$$\frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right]\cos\omega x\,d\omega = f(x) = \begin{cases} 1-\cos x & \text{for} \quad |x| < \frac{\pi}{2} \\ 0 & \text{for} \quad |x| > \frac{\pi}{2} \\ \frac{1}{2} & \text{for} \quad x = \pm\frac{\pi}{2}. \end{cases}$$

Exercise 16.2. ✗

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} x & \text{if} \quad -1 < x < 1, \\ 2-x & \text{if} \quad 1 < x < 2, \\ -2-x & \text{if} \quad -2 < x < -1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: The graph of $f(x)$ is shown below and it is easy to see that the function $f(x)$ is an odd function.



Therefore, $A(\omega) = 0$ for all ω, and

$$B(\omega) = \frac{2}{\pi}\int_0^2 f(t)\sin\omega t\,dt = \frac{2}{\pi}\int_0^1 t\sin\omega t\,dt + \frac{2}{\pi}\int_1^2 (2-t)\sin\omega t\,dt.$$

Integrating by parts, we have

$$\begin{aligned} B(\omega) &= \frac{2}{\pi}\left[\frac{-t}{\omega}\cos\omega t\bigg|_0^1 + \int_0^1 \frac{\cos\omega t}{\omega}\right] + \frac{2}{\pi}\left[\frac{-2+t}{\omega}\cos\omega t\bigg|_1^2 - \int_1^2 \frac{\cos\omega t}{\omega}\,dt\right] \\ &= \frac{2}{\pi}\left[-\frac{\cos\omega}{\omega} + \frac{\sin\omega t}{\omega^2}\bigg|_0^1\right] + \frac{2}{\pi}\left[\frac{\cos\omega}{\omega} - \frac{\sin\omega t}{\omega^2}\bigg|_1^2\right] \\ &= \frac{2}{\pi}\left[\frac{2\sin\omega}{\omega^2} - \frac{\sin 2\omega}{\omega^2}\right] \\ &= \frac{2}{\pi}\left(\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right), \end{aligned}$$

that is,

$$B(\omega) = \frac{2}{\pi}\left(\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right)$$

for all $\omega \neq 0$.

If $\omega = 0$, then

$$B(0) = \frac{2}{\pi}\int_0^2 f(t)\sin(0\cdot t)\,dt = 0.$$

Since $f(x)$ is continuous everywhere, from Dirichlet's theorem, the Fourier sine integral converges to $f(x)$ for all x, and therefore

$$\frac{2}{\pi}\int_0^\infty \left(\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right)\sin\omega x\,d\omega = f(x)$$

for all $x \in \mathbb{R}$.

Exercise 16.3. ✗
Let

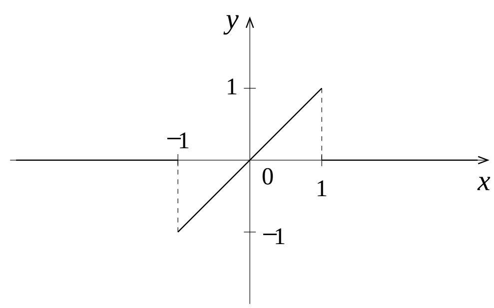
$$f(x) = \begin{cases} x & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Plot the function $f(x)$ and find its Fourier transform.

(b) If $\widehat{f}$ is real valued, plot it; otherwise plot $\left|\widehat{f}\right|$.

Solution:

(a) The graph of the function $f(x)$ is plotted below.



The Fourier transform of $f(x)$ is computed as

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt = \frac{1}{2\pi}\int_{-1}^{1} te^{i\omega t}\,dt \\
&= \frac{1}{2\pi}\left[\frac{t}{i\omega}e^{i\omega t}\bigg|_{-1}^{1} - \frac{1}{i\omega}\int_{-1}^{1} e^{i\omega t}\,dt\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{i\omega}\left(e^{i\omega} + e^{-i\omega}\right) - \frac{1}{(i\omega)^2}e^{i\omega t}\bigg|_{-1}^{1}\right] \\
&= \frac{2}{2\pi i}\left[\left(\frac{e^{i\omega} + e^{-i\omega}}{2\omega}\right) - \left(\frac{e^{i\omega} - e^{-i\omega}}{2i\omega^2}\right)\right] \\
&= \frac{1}{\pi i}\left(\frac{\omega\cos\omega - \sin\omega}{w^2}\right),
\end{aligned}$$

so that

$$\widehat{f}(\omega) = \frac{1}{\pi i}\left(\frac{\omega\cos\omega - \sin\omega}{w^2}\right)$$

for all $\omega \neq 0$.

If $\omega = 0$, then

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-1}^{1} t\,dt = \frac{1}{2\pi}\frac{t^2}{2}\bigg|_{-1}^{1} = 0,$$

and from L'Hospital's rule, we see that $\lim_{\omega\to 0}\widehat{f}(\omega) = 0$ also, so that $\widehat{f}(\omega)$ is continuous at each ω.

(b) Since

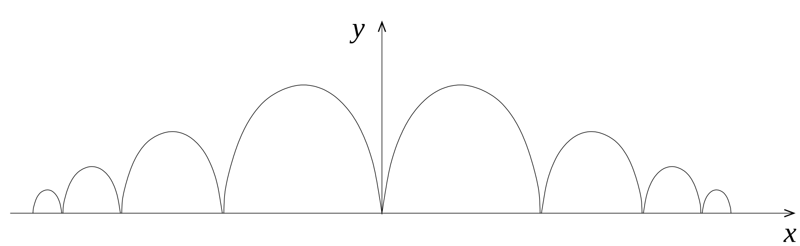
$$\widehat{f}(\omega) = \frac{1}{\pi i}\left(\frac{\omega\cos\omega - \sin\omega}{w^2}\right),$$

then

$$\left|\widehat{f}(\omega)\right| = \frac{1}{\pi}\left|\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right|$$

for all ω.

Note that the zeros of the function $g(\omega) = \sin\omega - \omega\cos\omega$ are precisely the roots of the equation $\tan\omega = \omega$, so the graph of $\left|\widehat{f}(\omega)\right|$ looks something like the figure below.



Exercise 16.4. ✗
Reciprocity relation for the Fourier transform.

(a) From the definition of transforms, explain why

$$\mathcal{F}(f)(x) = \frac{1}{2\pi}\mathcal{F}^{-1}(f)(-x).$$

(b) Use (a) to derive the **reciprocity relation**

$$\mathcal{F}^2(f)(x) = \frac{1}{2\pi}f(-x),$$

where $\mathcal{F}^2(f) = \mathcal{F}\left(\mathcal{F}(f)\right)$.

(c) Conclude the following: f is even if and only if $\mathcal{F}^2(f)(x) = \dfrac{1}{2\pi}f(x)$; and f is odd if and only if $\mathcal{F}^2(f)(x) = -\dfrac{1}{2\pi}f(x)$.

(d) Show that for any f, $\mathcal{F}^4(f) = \dfrac{1}{4\pi^2}f$.

Solution:

(a) Note that the Fourier transform of f is

$$\mathcal{F}(f)(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt,$$

and evaluating this transorm at $\omega = x$, and making a change of variables, we get

$$\mathcal{F}(f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{ixt}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega(-x)}\,d\omega = \frac{1}{2\pi}\mathcal{F}^{-1}(f)(-x),$$

that is,

$$\mathcal{F}(f)(x) = \frac{1}{2\pi}\mathcal{F}^{-1}(f)(-x)$$

for all $x \in \mathbb{R}$.

(b) Let $\widehat{f}$ be the Fourier transform of f, from part (a) we have

$$\mathcal{F}\big(\hat{f}\big)(x) = \frac{1}{2\pi}\mathcal{F}^{-1}\big(\hat{f}\big)(-x) = \frac{1}{2\pi}f(-x),$$

and therefore

$$\mathcal{F}^2(f)(x) = \frac{1}{2\pi}f(-x)$$

for all $x \in \mathbb{R}$.

(c) The function f is even if and only if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, but from part (b), we have f is even if and only if

$$\mathcal{F}^2(f)(x) = \mathcal{F}\big(\hat{f}\big)(x) = \frac{1}{2\pi} f(-x) = \frac{1}{2\pi} f(x)$$

for all $x \in \mathbb{R}$. Similarly, f is odd if and only if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, but again from part (b), we have f is odd if and only if

$$\mathcal{F}^2(f)(x) = \mathcal{F}\big(\hat{f}\big)(x) = \frac{1}{2\pi} f(-x) = -\frac{1}{2\pi} f(x)$$

for all $x \in \mathbb{R}$.

(d) For any integrable f, we have

$$\mathcal{F}^4(f)(x) = \mathcal{F}^2\left(\mathcal{F}^2(f)\right)(x) = \frac{1}{2\pi}\mathcal{F}^2(f)(-x) = \frac{1}{4\pi^2} f(-(-x)) = \frac{1}{4\pi^2} f(x)$$

for all $x \in \mathbb{R}$.

Exercise 16.5. ✗
Basic Properties of Convolutions.
Establish the following properties of convolutions. (These properties can be derived directly from the definitions or by using the operational properties of the Fourier transform.)

(a) $f * g = g * f$ (commutativity).

(b) $f * (g * h) = (f * g) * h$ (associativity).

(c) Let a be a real number and let f_a denote the translate of f by a, that is,

$$f_a(x) = f(x - a).$$

Show that

$$(f_a) * g = f * (g_a) = (f * g)_a.$$

This important property says that convolutions commute with translations.

Solution: The most convenient way to prove these properties are true is to use the uniqueness of the Fourier transform, that is, if f and g are integrable and if $\widehat{f} = \widehat{g}$, then $f = g$. However, we will prove them directly from the definition of the convolution.

(a) Given absolutely integrable functions f and g, we make a simple substitution in the definition of the convolution to get

$$f * g\,(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt = \int_{-\infty}^{\infty} f(s)g(x-s)ds = g * f\,(x),$$

for all $x \in \mathbb{R}$, and therefore $f * g = g * f$.

(b) Let f, g, and h be absolutely integrable, then for each $x \in \mathbb{R}$ we have

$$\begin{aligned}
(f*(g*h))(x) &= \int_{-\infty}^{\infty} f(x-t)(g*h)(t)\,dt \\
&= \int_{-\infty}^{\infty} f(x-t)\left(\int_{-\infty}^{\infty} g(t-s)h(s)\,ds\right)\,dt \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-t)g(t-s)h(s)\,ds\right)\,dt \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(s)f(x-t)g(t-s)\,dt\right)\,ds \quad (v=x-s) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(x-v)f(x-t)g(t-(x-v))\,dt\right)\,dv \quad (u=x-t) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(x-v)f(u)g(v-u)\,du\right)\,dv \\
&= \int_{-\infty}^{\infty} h(x-v)\left(\int_{-\infty}^{\infty} f(u)g(v-u)\,du\right)\,dv \\
&= \int_{-\infty}^{\infty} h(x-v)(g*f)(v)\,dv \\
&= \int_{-\infty}^{\infty} h(x-v)(f*g)(v)\,dv \\
&= (h*(f*g))(x) = ((f*g)*h)(x)
\end{aligned}$$

(c) We use the shift theorem

$$\begin{aligned}
\mathcal{F}(f_a)(\omega) &= \mathcal{F}(f(x-a))(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t-a)e^{i\omega t}\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)e^{i\omega(s+a)}\,ds \\
&= \frac{e^{i\omega a}}{2\pi}\int_{-\infty}^{\infty} f(s)e^{i\omega s}\,ds \\
&= e^{i\omega a}\mathcal{F}(f)(\omega),
\end{aligned}$$

for all ω, so that

$$\mathcal{F}(f_a) = e^{i\omega a}\mathcal{F}(f).$$

We have

$$\begin{aligned}\mathcal{F}\left((f*g)_a(x)\right) &= \mathcal{F}\left((f*g)(x-a)\right)\\ &= e^{i\omega a}\mathcal{F}\left((f*g)(x)\right)\\ &= 2\pi\, e^{i\omega a}\mathcal{F}(f(x))\mathcal{F}(g(x))\\ &= 2\pi\,\mathcal{F}(f_a(x))\mathcal{F}(g(x))\\ &= \mathcal{F}\left(((f_a)*g)(x)\right),\end{aligned}$$

and $\mathcal{F}\left((f*g)_a\right) = \mathcal{F}\left((f_a)*g\right)$. Since the Fourier transform is unique, then $(f*g)_a = (f_a)*g$.

We can also prove this directly, as follows.

$$\begin{aligned}(f*g)_a(x) &= (f*g)(x-a)\\ &= \int_{-\infty}^{\infty} f(x-a-t)g(t)\,dt\\ &= \int_{-\infty}^{\infty} f_a(x-t)g(t)\,dt\\ &= ((f_a)*g)(x)\end{aligned}$$

for all $x \in \mathbb{R}$, so that $(f*g)_a = (f_a)*g$.

Also, since $f*g = g*f$, we have

$$(f*g)_a = (g*f)_a = (g_a)*f = f*(g_a).$$

Exercise 16.6. ✗

Determine the solution of the following initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = e^{-x^2},\ -\infty < x < \infty.$$

Give your answer in the form of an inverse Fourier transform.

Solution: We hold t fixed and take the Fourier transform of the partial differential equation and the initial condition with respect to the space variable to get the initial value problem for $\widehat{u}(\omega,t) = \mathcal{F}(u(x,t))(\omega)$:

$$\frac{d\widehat{u}}{dt}(\omega,t) = -\frac{\omega^2}{4}\widehat{u}(\omega,t),$$

$$\widehat{u}(\omega,0) = \mathcal{F}\big(e^{-x^2}\big)(\omega) = \frac{1}{\sqrt{4\pi}}\,e^{-\frac{\omega^2}{4}}.$$

The general solution to this first-order linear equation is

$$\widehat{u}(\omega, t) = A(\omega)\, e^{-\frac{\omega^2}{4}t},$$

and we can determine the "constant" of integration $A(\omega)$ from the initial condition. Setting $t = 0$, we get

$$\widehat{u}(\omega, 0) = A(\omega) = \frac{1}{\sqrt{4\pi}}\, e^{-\frac{\omega^2}{4}},$$

so that

$$\widehat{u}(\omega, t) = \frac{1}{\sqrt{4\pi}}\, e^{-\frac{\omega^2}{4}}\, e^{-\frac{\omega^2}{4}t} = \frac{1}{\sqrt{4\pi}}\, e^{-\frac{\omega^2}{4}(1+t)}.$$

Taking the inverse transform, the solution is

$$u(x, t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}(1+t)}\, e^{-i\omega x}\, d\omega$$

for $-\infty < x < \infty,\ t \geqslant 0$.

Exercise 16.7.
Use the Fourier transform to solve the following initial value – boundary value problem

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty.$$

Assume that the function f has a Fourier transform.

Solution: Taking the Fourier transform of the partial differential equation and the initial condition with respect to x, we have

$$\frac{d\widehat{u}}{dt}(\omega, t) + i\omega\widehat{u}(\omega, t) = 0,$$

$$\widehat{u}(\omega, 0) = \widehat{f}(\omega).$$

The general solution to this first-order linear equation is

$$\widehat{u}(\omega, t) = A(\omega)\, e^{-i\omega t},$$

and we can determine the "constant" of integration $A(\omega)$ from the transformed initial condition

$$\widehat{u}(\omega, 0) = A(\omega) = \widehat{f}(\omega).$$

Therefore,

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)\, e^{-i\omega t},$$

and taking the inverse Fourier transform, we have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-i\omega t}\, e^{-i\omega x}\, d\omega \\ &= \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-i\omega(x+t)}\, d\omega \\ &= f(x+t), \end{aligned}$$

and the solution is

$$u(x,t) = f(x+t)$$

for $-\infty < x < \infty,\ t \geqslant 0$.

Exercise 16.8. ✗

Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1-x & \text{if} \quad 0 < x < 1, \\ 0 & \text{if} \quad x \geqslant 1. \end{cases}$$

and write $f(x)$ as an inverse cosine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geqslant 0$, is the restriction of an *even* function f_e, then

$$\mathcal{F}_c(f)(\omega) = 2\mathcal{F}(f_e)(\omega)$$

for all $\omega \geqslant 0$.

Solution: The Fourier cosine transform of the function f is given by

$$\widehat{f_c}(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t)\cos\omega t\, dt = \frac{2}{\pi}\int_0^1 (1-t)\cos\omega t\, dt,$$

and this is the same as the Fourier transform of the *even* extension f_e of f to the whole real line $\mathbb{R}$. In this case however, we can evaluate the last integral directly by integration by parts:

$$\begin{aligned} \int_0^1 (1-t)\cos\omega t\, dt &= \int_0^1 \cos\omega t\, dt - \int_0^1 t\cos\omega t\, dt \\ &= \frac{\sin\omega t}{\omega}\bigg|_0^1 - \left[t \cdot \frac{\sin\omega t}{\omega}\bigg|_0^1 - \frac{1}{\omega}\int_0^1 \sin\omega t\, dt \right] \\ &= \frac{\sin\omega}{\omega} - \frac{\sin\omega}{\omega} + \frac{1}{\omega}\left[-\frac{1}{\omega}\cos\omega t\bigg|_0^1 \right] \\ &= \frac{1-\cos\omega}{\omega^2}, \end{aligned}$$

and therefore

$$\widehat{f_c}(\omega) = \frac{2}{\pi}\cdot\frac{1-\cos\omega}{\omega^2}$$

for $\omega > 0$.
Knowing that f_c is absolutely integrable implies that $\widehat{f_c}$ is continuous at $\omega = 0$, and we have

$$\widehat{f_c}(0) = \lim_{\omega\to 0^+} \frac{2}{\pi} \cdot \frac{1-\cos\omega}{\omega^2} = \frac{2}{\pi} \cdot \lim_{\omega\to 0^+} \frac{\sin\omega}{2\omega} = \frac{1}{\pi}$$

by L'Hospital's rule.
Therefore, we have

$$\widehat{f_c}(\omega) = \begin{cases} \dfrac{2}{\pi} \cdot \dfrac{1-\cos\omega}{\omega^2} & \text{for} \quad \omega > 0 \\ \dfrac{1}{\pi} & \text{for} \quad \omega = 0. \end{cases}$$

Since f_e is continuous for all $x \in \mathbb{R}$, from Dirichlet's theorem the inverse Fourier cosine transform of $\widehat{f_c}$ is given by

$$\frac{2}{\pi} \int_0^\infty \frac{1-\cos\omega}{\omega^2} \cdot \cos\omega x \, d\omega = \begin{cases} 1-x & \text{for} \quad 0 \leqslant x < 1 \\ 0 & \text{for} \quad x \geqslant 1. \end{cases}$$

Exercise 16.9. ✗
Find the Fourier sine transform of

$$f(x) = \frac{x}{1+x^2}, \quad x > 0,$$

and write $f(x)$ as an inverse sine transform. Use a known Fourier transform and the fact that if $f(x)$, $x \geqslant 0$, is the restriction of an *odd* function f_o, then

$$\mathcal{F}_s(f)(\omega) = -2i\mathcal{F}(f_o)(\omega)$$

for all $\omega \geqslant 0$.
Hint: Consider the Fourier sine transform of $g(x) = e^{-x}$.

Solution: We can find the Fourier sine transform of the given function using the suggested method, or we can find it directly. To do this, we consider the function

$$g(x) = e^{-x}, \quad x > 0$$

with Fourier sine transform given by

$$\widehat{g}_s(\omega) = \frac{2}{\pi} \int_0^\infty e^{-t} \sin\omega t \, dt$$

and we can evaluate this integral by integrating by parts:

$$\begin{aligned}\int_0^\infty e^{-t}\sin\omega t\,dt &= -\frac{e^{-t}\cos\omega t}{\omega}\Big|_0^\infty - \frac{1}{\omega}\int_0^\infty e^{-t}\cos\omega t\,dt\\ &= \frac{1}{\omega} - \frac{1}{\omega}\left[e^{-t}\cdot\frac{\sin\omega t}{\omega}\Big|_0^\infty + \frac{1}{\omega}\int_0^\infty e^{-t}\sin\omega t\,dt\right]\\ &= \frac{1}{\omega} - \frac{1}{\omega^2}\int_0^\infty e^{-t}\sin\omega t\,dt\end{aligned}$$

so that

$$\left(1+\frac{1}{\omega^2}\right)\int_0^\infty e^{-t}\sin\omega t\,dt = \frac{1}{\omega}.$$

Therefore,

$$\int_0^\infty e^{-t}\sin\omega t\,dt = \frac{\omega}{1+\omega^2}$$

for $\omega \geqslant 0$, so that

$$\widehat{g}_s(\omega) = \frac{2}{\pi}\cdot\frac{\omega}{1+\omega^2}$$

for $\omega \geqslant 0$.

Taking the inverse Fourier sine transform of this, we have

$$g(x) = \int_0^\infty \widehat{g}_s(\omega)\sin\omega x\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin\omega x\,d\omega.$$

Now interchanging x and ω, we have

$$e^{-\omega} = g(\omega) = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}\sin\omega x\,dx,$$

and

$$\widehat{f}_s(\omega) = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}\sin\omega x\,dx = g(\omega) = e^{-\omega}$$

for $\omega \geqslant 0$.

From the above, we can write $f(x)$ as an inverse Fourier sine transform:

$$f(x) = \frac{x}{1+x^2} = \int_0^\infty e^{-\omega}\sin\omega x\,d\omega$$

for $x > 0$.

Exercise 16.10. ✗

Show that

$$\int_0^\infty \frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda = \frac{\pi}{2a}e^{-ax}$$

for $a > 0, \quad x > 0$.

Solution: Let $f(x) = e^{-ax}$ for $x > 0$ and let f_e be the even extension of f to all of $(-\infty, \infty)$. Since f_e is piecewise smooth and agrees with f for $x > 0$, then we can write (again from Dirichlet's theorem)

$$e^{-ax} = \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda, \quad x > 0$$

where

$$A(\lambda) = \frac{2}{\pi} \int_0^\infty e^{-ax} \cos \lambda x \, dx.$$

In order to evaluate $A(\lambda)$, integrating by parts twice, since $a > 0$ we have

$$\begin{aligned}
\int_0^\infty e^{-ax} \cos \lambda x \, dx &= \frac{e^{-ax} \sin \lambda x}{\lambda}\bigg|_0^\infty - \int_0^\infty \frac{(-a)e^{-ax} \sin \lambda x}{\lambda} \, dx \\
&= \frac{a}{\lambda}\left[-\frac{e^{-ax} \cos \lambda x}{\lambda}\bigg|_0^\infty + \int_0^\infty \frac{(-a)e^{-ax} \cos \lambda x}{\lambda} \, dx\right] \\
&= \frac{a}{\lambda}\left[\frac{1}{\lambda} - \frac{a}{\lambda}\int_0^\infty e^{-ax} \cos \lambda x \, dx\right],
\end{aligned}$$

so that

$$\left(1 + \frac{a^2}{\lambda^2}\right) \int_0^\infty e^{-ax} \cos \lambda x \, dx = \frac{a}{\lambda^2}.$$

Therefore

$$\int_0^\infty e^{-ax} \cos \lambda x \, dx = \frac{a}{\lambda^2 + a^2}$$

for $\lambda > 0$ (you should check that this holds for $\lambda = 0$ also). Therefore

$$A(\lambda) = \frac{2a}{\pi(a^2 + \lambda^2)},$$

for $\lambda \geqslant 0$, and

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a \cos \lambda x}{a^2 + \lambda^2} \, d\lambda.$$

for $0 < x < \infty$. Thus,

$$\int_0^\infty \frac{\cos \lambda x}{a^2 + \lambda^2} \, d\lambda = \frac{\pi}{2a} e^{-ax}$$

for $x > 0$.

Exercise 16.11. ✗✗

Give the function $f(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad 0 < x < 1 \\ 2 & \text{for} \quad 1 < x < 2 \\ 0 & \text{for} \quad x > 2. \end{cases}$

(a) Find the Fourier integral formula for $f(x)$.

(b) Find the Fourier sine integral formula for $f(x)$.

(c) Find the Fourier cosine integral formula for $f(x)$.

(d) Find the Fourier transform of $f(x)$.

(e) Find the Fourier sine transform of $f(x)$.

(f) Find the Fourier cosine transform of $f(x)$.

Solution:

(a) The Fourier integral representation of f is given by

$$\frac{f(x^+) + f(x^-)}{2} = \int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)\ d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx \qquad \text{and} \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

for $0 < \omega < \infty$.

We have

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_0^2 f(x)\cos\omega x\,dx \\ &= \frac{1}{\pi}\int_0^1 \cos\omega x\,dx + \frac{1}{\pi}\int_1^2 2\cos\omega x\,dx \\ &= \left.\frac{\sin\omega x}{\pi\omega}\right|_0^1 + \left.\frac{2\sin\omega x}{\pi\omega}\right|_1^2 \\ &= \frac{1}{\pi}\left(-\frac{\sin\omega}{\omega} + \frac{2\sin 2\omega}{\omega}\right) \end{aligned}$$

and

$$
\begin{aligned}
B(\omega) &= \frac{1}{\pi}\int_0^2 f(x)\sin\omega x\,dx \\
&= \frac{1}{\pi}\int_0^1 \sin\omega x\,dx + \frac{1}{\pi}\int_1^2 2\sin\omega x\,dx \\
&= \frac{1}{\pi\omega}\left(1-\cos\omega+2\cos\omega-2\cos 2\omega\right) \\
&= \frac{1}{\pi\omega}\left(1+\cos\omega-2\cos 2\omega\right)
\end{aligned}
$$

so that

$$
\frac{f(x^+)+f(x^-)}{2} = \int_0^\infty \left(\frac{\sin\omega}{\pi\omega}\left[4\cos\omega-1\right]\cos\omega x + \frac{1}{\pi\omega}\left(3+\cos\omega-4\cos^2\omega\right)\sin\omega x\right)d\omega
$$

for $-\infty < x < \infty$.

(a') For those of you that prefer the complex Fourier integral representation of f, we have

$$
\frac{f(x+)+f(x^-)}{2} = \int_{-\infty}^\infty F(\omega)e^{-i\omega x}\,d\omega
$$

where

$$
F(\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)e^{i\omega\xi}\,d\xi,
$$

and

$$
\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_0^1 e^{i\omega\xi}\,d\xi + \frac{1}{2\pi}\int_1^2 2e^{i\omega\xi}\,d\xi \\
&= \frac{1}{2\pi i\omega}\left[2e^{2i\omega}-e^{i\omega}-1\right],
\end{aligned}
$$

so that

$$
\frac{f(x^+)+f(x^-)}{2} = \frac{1}{2\pi i}\int_{-\infty}^\infty \left(\left[\frac{2e^{2i\omega}-e^{i\omega}-1}{\omega}\right]e^{-i\omega x}\right)d\omega
$$

for $-\infty < x < \infty$.

(b) The Fourier sine integral formula for $f(x)$ is

$$
f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega
$$

where

$$
B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx.
$$

For the function f above, we have

$$\begin{aligned}
B(\omega) &= \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx\\
&= \frac{2}{\pi}\int_0^2 f(x)\sin\omega x\,dx\\
&= \frac{2}{\pi}\int_0^1 \sin\omega x\,dx + \frac{2}{\pi}\int_1^2 2\sin\omega x\,dx\\
&= -\frac{2}{\pi\omega}\cos\omega x\Big|_0^1 - \frac{4}{\pi\omega}\cos\omega x\Big|_1^2\\
&= -\frac{2}{\pi\omega}\cos\omega + \frac{2}{\pi\omega} - \frac{4}{\pi\omega}(\cos 2\omega - \cos\omega),
\end{aligned}$$

so that

$$B(\omega) = \frac{2}{\pi\omega}(1+\cos\omega - 2\cos 2\omega)$$

for $0 < \omega < \infty$.

The Fourier sine integral formula for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{(1+2\cos\omega - \cos 2\omega)}{\omega}\sin\omega x\,d\omega$$

for $0 < x < \infty$.

(c) The Fourier cosine integral formula for $f(x)$ is

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$$

where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx.$$

For the function f above, we have

$$\begin{aligned}
A(\omega) &= \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx\\
&= \frac{2}{\pi}\int_0^2 f(x)\cos\omega x\,dx\\
&= \frac{2}{\pi}\int_0^1 \cos\omega x\,dx + \frac{2}{\pi}\int_1^2 2\cos\omega x\,dx\\
&= \frac{2}{\pi\omega}\sin\omega x\Big|_0^1 + \frac{4}{\pi\omega}\sin\omega x\Big|_1^2\\
&= \frac{2}{\pi\omega}\sin\omega + \frac{4}{\pi\omega}(\sin 2\omega - \sin\omega),
\end{aligned}$$

so that

$$A(\omega) = \frac{2}{\pi\omega}(\sin 2\omega - 2\sin\omega)$$

for $0 < \omega < \infty$.

The Fourier cosine integral formula for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{(\sin 2\omega - 2\sin\omega)}{\omega}\cos\omega x\, d\omega$$

for $0 < x < \infty$.

(d) The Fourier transform of $f(x)$ is

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)e^{i\omega\xi}\, d\xi$$

for $-\infty < x < \infty$, and for the function given we have

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_0^1 e^{i\omega\xi}\, d\xi + \frac{1}{2\pi}\int_1^2 2e^{i\omega\xi}\, d\xi \\ &= \frac{1}{2\pi i\omega}\left[2e^{2i\omega} - e^{i\omega} - 1\right], \end{aligned}$$

that is,

$$F(\omega) = \frac{1}{2\pi i\omega}\left[2e^{2i\omega} - e^{i\omega} - 1\right]$$

for $-\infty < x < \infty$.

(e) The Fourier sine transform of $f(x)$ is

$$F_s(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx = \frac{2}{\pi}\left(\frac{1 + 2\cos\omega - \cos 2\omega}{\omega}\right)$$

for $0 < \omega < \infty$.

(f) The Fourier cosine transform of $f(x)$ is

$$F_c(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx = \frac{2}{\pi}\left(\frac{\sin 2\omega - 2\sin\omega}{\omega}\right)$$

for $0 < \omega < \infty$.

Exercise 16.12. ✗

Use Fourier transforms to find the solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 100 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

in terms of the error function.

Solution: Let $\widehat{u}(\omega, t) = \mathcal{F}(u(x,t))$, taking the transform of both sides of the partial differential equation we have

$$\frac{\partial \widehat{u}(\omega, t)}{\partial t} = (i\omega)^2 \widehat{u}(\omega, t) = -\omega^2 \widehat{u}(\omega, t),$$

and the initial condition gives

$$\widehat{u}(\omega, 0) = \widehat{f}(\omega) = \frac{1}{2\pi} \int_{-1}^{1} 100\, e^{i\omega x}\, dx.$$

The solution to this ordinary differential equation is

$$\widehat{u}(\omega, t) = \widehat{u}(\omega, 0)\, e^{-\omega^2 t}$$

and from the convolution theorem we have

$$u(x,t) = \frac{1}{2\pi} \mathcal{F}^{-1}\left(e^{-\omega^2 t}\right) * f(x) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} 100\, e^{-\frac{(x-s)^2}{4t}}\, ds.$$

The solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} 100\, e^{-\frac{(x-s)^2}{4t}}\, ds$$

for $-\infty < x < \infty,\ t > 0$, can be written in terms of the error function by making the substitution

$$z = \frac{x-s}{\sqrt{4t}} \qquad \text{and} \qquad dz = -\frac{ds}{\sqrt{4t}} ds,$$

when $s = -1$, then $z = \dfrac{x+1}{\sqrt{4t}}$, and when $s = 1$, then $z = \dfrac{x-1}{\sqrt{4t}}$, so that

$$\begin{aligned} u(x,t) &= \frac{100}{\sqrt{4\pi t}} (-\sqrt{4t}) \int_{\frac{x+1}{\sqrt{4t}}}^{\frac{x-1}{\sqrt{4t}}} e^{-z^2}\, dz \\ &= \frac{100}{\sqrt{\pi}} \left(\int_0^{\frac{x+1}{\sqrt{4t}}} e^{-t^2}\, dt - \int_0^{\frac{x-1}{\sqrt{4t}}} e^{-t^2}\, dt \right) \\ &= \frac{100}{2} \left[\operatorname{erf}\left(\frac{x+1}{\sqrt{4t}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4t}}\right) \right], \end{aligned}$$

therefore

$$u(x,t) = 50 \left[\operatorname{erf}\left(\frac{x+1}{\sqrt{4t}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4t}}\right) \right]$$

for $-\infty < x < \infty,\ t > 0$, where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$$

is the error function.

Exercise 16.13. ✗

(a) Show that the Fourier transform is a linear operator; that is, show that

$$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = c_1 F(\omega) + c_2 G(\omega)$$

(b) Show that $\mathcal{F}\left[f(x)g(x)\right] \neq F(\omega)G(\omega)$.

Solution:

(a) If the Fourier transforms of f and g both exist, and c_1 and c_2 are constants, then

$$\begin{aligned}
\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(c_1 f(x) + c_2 g(x)\right)e^{i\omega x}\,dx \\
&= \frac{c_1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx + \frac{c_2}{2\pi}\int_{-\infty}^{\infty} g(x)e^{i\omega x}\,dx \\
&= c_1\mathcal{F}\left(f(x)\right) + c_2\mathcal{F}\left(g(x)\right),
\end{aligned}$$

that is,

$$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = c_1\mathcal{F}\left(f(x)\right) + c_2\mathcal{F}\left(g(x)\right)$$

and the Fourier transform is a linear operator.

(b) Let f and g are functions such that $\mathcal{F}(f(x)) = F(\omega)$ and $\mathcal{F}(g(x)) = G(\omega)$ both exist, for example,

$$f(x) = g(x) = \begin{cases} 1, & \text{for} \quad |x| < a \\ 0, & \text{for} \quad |x| > a \end{cases}$$

then

$$F(\omega) = G(\omega) = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega x}\,dx = \frac{\sin\omega a}{\pi\omega}.$$

Now let $h(x) = f(x)\cdot g(x)$ for $-\infty < x < \infty$, clearly $h(x) = f(x) = g(x)$ for all x, and

$$H(\omega) = \frac{\sin\omega a}{\pi\omega} \neq \frac{\sin^2\omega a}{\pi^2\omega^2} = F(\omega)\cdot\widehat{G}(\omega).$$

Exercise 16.14.
If $F(\omega)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. This result is known as the **Shift Theorem** for Fourier transforms.

Solution: We have

$$\begin{aligned}\mathcal{F}^{-1}\left(e^{i\omega\beta}F(\omega)\right) &= \int_{-\infty}^{\infty} F(\omega)e^{i\omega\beta}e^{-i\omega x}\,d\omega \\ &= \int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x-\beta)}\,d\omega \\ &= f(x-\beta).\end{aligned}$$

Exercise 16.15. ✗

(a) Solve

$$\begin{aligned}&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u, \quad -\infty < x < \infty, \quad t > 0 \\ &u(x,0) = f(x), \qquad -\infty < x < \infty.\end{aligned}$$

(b) Does your solution suggest a simplifying transformation?

Solution:

(a) If $u(x,t)$ is the solution to

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} - \gamma u, \quad -\infty < x < \infty,\ t > 0 \\ u(x,0) &= f(x), \quad -\infty < x < \infty,\end{aligned}$$

let

$$\widehat{u}(\omega,t) = \mathcal{F}(u(x,t)) \qquad \text{and} \qquad \widehat{u}(\omega,0) = \widehat{f}(\omega),$$

then $\widehat{u}(\omega,t)$ satisfies the initial value problem

$$\begin{aligned}\frac{d\widehat{u}}{dt} &= -(k\omega^2+\gamma)\widehat{u}, \quad t \geqslant 0 \\ \widehat{u}(\omega,0) &= \widehat{f}(\omega),\end{aligned}$$

with solution

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-(k\omega^2+\gamma)t} = \widehat{f}(\omega)e^{-k\omega^2 t}e^{-\gamma t}.$$

The solution to the partial differential equation is

$$\begin{aligned}u(x,t) &= \mathcal{F}^{-1}\left(\widehat{u}(\omega,t)\right) \\ &= \mathcal{F}^{-1}\left(\widehat{f}(\omega)e^{-k\omega^2 t}e^{-\gamma t}\right) \\ &= e^{-\gamma t}\mathcal{F}^{-1}\left(\widehat{f}(\omega)e^{-k\omega^2 t}\right) \qquad \text{(since } \mathcal{F}^{-1} \text{ is linear)} \\ &= e^{-\gamma t}\,\frac{1}{2\pi}\,f * g\,(x,t)\end{aligned}$$

where

$$g(x,t) = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}.$$

Therefore

$$u(x,t) = e^{-\gamma t} \int_{-\infty}^{\infty} f(s) \frac{e^{-(x-s)^2/4kt}}{\sqrt{4\pi kt}}\, ds$$

for $-\infty < x < \infty$ and $t > 0$.

(b) If we multiply the solution above by $e^{\gamma t}$, we find

$$e^{\gamma t} u(x,t) = \frac{1}{2\pi}\, f * g\,(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(s)\, e^{-(x-s)^2/4kt}\, ds,$$

which looks like the solution to a homogeneous heat equation.

Indeed, if we define

$$w(x,t) = e^{\gamma t} u(x,t),$$

then

$$\begin{aligned} \frac{\partial w}{\partial t} &= \gamma e^{\gamma t} u + e^{\gamma t} \frac{\partial u}{\partial t} \\ &= \gamma w + e^{\gamma t} \left(k \frac{\partial^2 u}{\partial x^2} - \gamma u \right) \\ &= \gamma w + k \frac{\partial^2 w}{\partial x^2} - \gamma w, \end{aligned}$$

so that

$$\frac{\partial w}{\partial t} = k \frac{\partial^2 w}{\partial x^2}$$
$$w(x,0) = f(x)$$

for $-\infty < x < \infty, \quad t > 0$.

Exercise 16.16. ✗✗

Solve

$$\begin{aligned} &\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, && 0 < x < \infty, \; t > 0 \\ &\frac{\partial u}{\partial x}(0,t) = 0, && t > 0 \\ &u(x,0) = f(x), && 0 < x < \infty. \end{aligned}$$

Solution: Since the boundary condition is a Neumann condition, we use the Fourier cosine transform. Let

$$\widetilde{u}(\omega, t) = F_c(u(x,t)) = \frac{2}{\pi}\int_0^\infty u(x,t)\cos\omega x\,dx,$$

and

$$\widetilde{f}(\omega) = F_c(f(x)) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx,$$

then

$$F_c\left(\frac{\partial u}{\partial t}\right) = \frac{\partial \widetilde{u}}{\partial t}(\omega, t),$$

and

$$F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{2}{\pi}\frac{\partial u}{\partial x}(0,t) - \omega^2\widetilde{u}(\omega, t),$$

and from the boundary condition, $\dfrac{\partial u}{\partial x}(0,t) = 0$, so that

$$F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\omega^2\widetilde{u}(\omega, t).$$

After taking the Fourier cosine transform of both sides of the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

the transform $\widetilde{u}(\omega, t)$ satisfies the initial value problem

$$\frac{d\widetilde{u}}{dt}(\omega, t) + k\omega^2\widetilde{u}(\omega, t) = 0$$

$$\widetilde{u}(\omega, 0) = \widetilde{f}(\omega),$$

with solution

$$\widetilde{u}(\omega, t) = \widetilde{u}(\omega, 0)e^{-\omega^2 kt} = \widetilde{f}(\omega)e^{-\omega^2 kt}$$

for $-\infty < \omega < \infty$ and $t > 0$.
Therefore

$$u(x,t) = \int_0^\infty \widetilde{f}(\omega)e^{-\omega^2 kt}\cos\omega x\,d\omega$$

for $0 < x < \infty$ and $t > 0$.
Note that each of the functions $\widetilde{f}(\omega)$, $e^{-\omega^2 kt}$, and $\cos\omega x$ in the integrand is an odd function of ω, so that

$$\int_0^\infty \widetilde{f}(\omega)e^{-\omega^2 kt}\cos\omega x\,d\omega = \frac{1}{2}\int_{-\infty}^\infty \widetilde{f}(\omega)e^{-\omega^2 kt}\cos\omega x\,d\omega.$$

Since $\sin\omega x$ is an odd function of ω, then

$$\int_{-\infty}^\infty \widetilde{f}(\omega)e^{-\omega^2 kt}\sin\omega x\,d\omega = 0,$$

and we can write the solution $u(x,t)$ as

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{-\infty}^{\infty} \widetilde{f}(\omega) e^{-\omega^2 kt}(\cos\omega x - i\sin\omega x)\,d\omega \\ &= \int_{-\infty}^{\infty} \frac{\widetilde{f}(\omega)}{2}\, e^{-\omega^2 kt} e^{-i\omega x}\,d\omega, \end{aligned}$$

that is,

$$u(x,t) = \mathcal{F}^{-1}\left(\frac{\widetilde{f}(\omega)}{2}\, e^{-\omega^2 kt}\right). \tag{$*$}$$

Let f_{even} be the even extension of $f(x)$ to $(-\infty,\infty)$, then

$$\begin{aligned} \frac{\widetilde{f}(\omega)}{2} &= \frac{1}{2}\,\frac{2}{\pi}\int_0^{\infty} f(x)\cos\omega x\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{\text{even}}(x)\cos\omega x\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{\text{even}}(x)(\cos\omega x + i\sin\omega x)\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{\text{even}}(x) e^{i\omega x}\,dx \\ &= \mathcal{F}\left(f_{\text{even}}(x)\right), \end{aligned}$$

so that

$$\frac{\widetilde{f}(\omega)}{2} = \mathcal{F}\left(f_{\text{even}}(x)\right). \tag{$**$}$$

From $(*)$ and $(**)$ it follows that $u(x,t)$ is the solution to the initial value – boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ &\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0 \\ &u(x,0) = f_{\text{even}}(x), \quad -\infty < x < \infty, \end{aligned}$$

and therefore

$$u(x,t) = f_{\text{even}} * G\,(x,t)$$

where $G(x,t)$ is the heat kernel or Gaussian kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-x^2/4kt}.$$

The solution is then

$$\begin{aligned} u(x,t) &= f_{\text{even}} * G\,(x,t) \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f_{\text{even}}(s) e^{-(x-s)^2/4kt}\, ds \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} f(s) \left(e^{-(x+s)^2/4kt} + e^{-(x-s)^2/4kt} \right) ds, \end{aligned}$$

so that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} f(s) \left(e^{-(x+s)^2/4kt} + e^{-(x-s)^2/4kt} \right) ds$$

for $0 < x < \infty,\ t > 0$.

Exercise 16.17. ✗✗✗

Solve the circularly symmetric diffusion equation on an infinite two- dimensional domain:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad r > 0,\ t > 0$$

$$u \quad \text{bounded as } r \to 0^+, \quad t > 0$$

$$r\, u\, \frac{\partial u}{\partial r} \to 0 \quad \text{as } r \to \infty, \quad t > 0$$

$$u(r,0) = f(r), \quad r > 0.$$

Solution: We solve this problem using separation of variables, we write

$$u(r,t) = \varphi(r) \cdot T(t),$$

so that

$$\varphi(r) \cdot T'(t) = \frac{k}{r} \left(r\varphi'(r)\right)' \cdot T(t) = \left(k\varphi''(r) + \frac{k}{r}\,\varphi'(r)\right) \cdot T(t),$$

and separating variables, we have

$$\frac{T'(t)}{kT(t)} = \frac{\varphi''(r) + \frac{1}{r}\,\varphi'(r)}{\varphi(r)} = -\lambda \quad \text{(constant)}.$$

Thus, we obtain the following two ordinary differential equations

$$\varphi'' + \frac{1}{r}\,\varphi' + \lambda\varphi = 0,$$

$$T' + \lambda k T = 0.$$

Note that the boundedness conditions on φ are satisfied if φ satisfies the following singular Sturm-Liouville problem

$$\varphi'' + \frac{1}{r}\varphi' + \lambda\varphi = 0, \quad 0 < r < \infty$$
$$\varphi(r) \quad \text{bounded as } r \to 0^+,$$
$$r\varphi(r)\varphi'(r) \to 0 \quad \text{as } r \to \infty.$$

Multiplying by r, the spatial problem can be written in the form

$$r\varphi'' + \varphi' + \lambda r\varphi = 0, \quad 0 < r < \infty$$
$$\varphi(r) \quad \text{bounded as } r \to 0^+,$$
$$r\varphi(r)\varphi'(r) \to 0 \quad \text{as } r \to \infty.$$

We solve the singular Sturm-Liouville problem for φ first. The Rayleigh quotient is

$$\lambda = \frac{-r\varphi\varphi'\Big|_0^\infty + \displaystyle\int_0^\infty r(\varphi')^2\,dr}{\displaystyle\int_0^\infty r\,\varphi^2\,dr},$$

and from the boundedness conditions,

$$-r\varphi\varphi'\Big|_0^\infty = -\lim_{r\to\infty} r\varphi(r)\varphi'(r) = 0,$$

so that $\lambda \geqslant 0$, that is, there are no negative eigenvalues.
case 1. If $\lambda = 0$, then the differential equation is

$$(r\varphi')' = 0,$$

with general solution

$$\varphi(r) = A\log r + B.$$

Applying the boundedness condition, we have $A = 0$, and the eigenfunction is

$$\varphi_0(r) = 1$$

for $0 < r < \infty$.
case 2. If $\lambda > 0$, then the differential equation is

$$(r\varphi')' + \lambda r\varphi = 0,$$

which is Bessel's (parametric) equation of order 0, with general solution

$$\varphi(r) = AJ_0\left(\sqrt{\lambda}r\right) + BY_0\left(\sqrt{\lambda}r\right).$$

Applying the boundedness condition, we have $B = 0$, and the solution may be written

$$\varphi(r) = J_0(\mu r)$$

for $0 < r < \infty$, where $\lambda = \mu^2$.
The corresponding T equation

$$T' + \mu^2 k T = 0$$

has solution

$$T(t) = e^{-\mu^2 k t}$$

for $t \geqslant 0$.
Therefore, for each $\mu \geqslant 0$, the function

$$u(r,t,\mu) = J_0(\mu r)\, e^{-\mu^2 k t}, \quad 0 < r < \infty,\ t > 0$$

satisfies the partial differential equation and the boundedness condition, and from the superposition principle, we write

$$u(r,t) = \int_0^\infty A(\mu) J_0(\mu r)\, e^{-\mu^2 k t} \mu\, d\mu,$$

and this satisfies (formally) the diffusion equation as well as the boundedness conditions. The only thing not satisfied is the initial condition

$$u(r,0) = f(r), \quad 0 < r < \infty$$

so we want

$$f(r) = u(r,0) = \int_0^\infty A(\mu) J_0(\mu r)\, \mu\, d\mu$$

for $0 < r < \infty$.
In order to determine the coefficients $A(\mu)$, we have a theorem analogous to Dirichlet's theorem, called *Hankel's integral theorem*

Theorem. Given a function f defined on the interval $(0,\infty)$, which is piecewise continuous and of bounded variation on every finite subinterval $[a,b]$, where $0 < a < b < \infty$, and such that

$$\int_0^\infty \sqrt{r}\, |f(r)|\, dr < \infty,$$

then for each $r > 0$ we have

$$\frac{1}{2}\left[f(r^+) + f(r^-)\right] = \int_0^\infty A(\lambda) J_0(\lambda r)\, \lambda\, d\lambda$$

where

$$A(\lambda) = \int_0^\infty f(r) J_0(\lambda r)\, r\, dr.$$

Note: The coefficient

$$A(\lambda) = \int_0^\infty f(r) J_0(\lambda r)\, r\, dr$$

is called the **Fourier-Bessel transform** of $f(r)$, or the **Hankel transform** of $f(r)$.

The solution to the circularly symmetric diffusion equation on an infinite 2-dimensional domain is therefore given by

$$u(r,t) = \int_0^\infty A(\mu) J_0\,(\mu r)\, e^{-\mu^2 kt} \mu\, d\mu, \quad 0 < r < \infty,\ t > 0$$

where

$$A(\mu) = \int_0^\infty f(r) J_0\,(\mu r)\, r\, dr$$

for $\mu > 0$.

Chapter 17

Four Sample Midterm Examinations

17.1 Midterm Exam 1

Exercise 17.1. ✗

Find the values of λ^2 for which the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda^2 u = 0, \quad 0 < x < \frac{\pi}{2}$$

$$u(0) = 0$$

$$\int_0^{\frac{\pi}{2}} u(t)\, dt = 0$$

has nontrivial solutions.

Exercise 17.2. ✗

Let $f(x) = \cos^2 x, \quad 0 \leqslant x \leqslant \pi$, and $f(x + 2\pi) = f(x)$ otherwise.

(a) Find the Fourier sine series for f on the interval $[0, \pi]$.

Hint: For $n \geqslant 1$

$$\int \cos^2 x \sin nx\, dx = -\frac{1}{2n}\cos nx + \frac{1}{4}\int [\sin(n+2)x + \sin(n-2)x]\; dx.$$

(b) Find the Fourier cosine series for f on the interval $[0, \pi]$.

(c) For which values of x in $[0, \pi]$ do the series in (a) and (b) converge to $f(x)$?

Exercise 17.3. ✗

Let $v(x)$ be the steady-state solution to the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + r = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0$$

$$u(0,t) = T_0, \quad t > 0$$

$$\frac{\partial u}{\partial x}(a,t) = 0, \quad t > 0$$

where r is a constant. Find and solve the boundary value problem for the steady-state solution $v(x)$.

17.2 Midterm Exam 2

Exercise 17.4. ✗

The neutron flux u in a sphere of uranium obeys the differential equation

$$\frac{\lambda}{3}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + (k-1)A\,u = 0$$

for $0 < r < a$, where λ is the effective distance traveled by a neutron between collisions, A is called the absorption cross section, and k is the number of neutrons produced by a collision during fission. In addition, the neutron flux at the boundary of the sphere is 0.

(a) Make the substitution

$$u = \frac{v}{r} \qquad \text{and} \qquad \mu^2 = \frac{3(k-1)A}{\lambda}$$

and show that $v(r)$ satisfies $\dfrac{d^2v}{dr^2} + \mu^2\,v = 0,\ 0 < r < a.$

(b) Find the general solution to the differential equation in part (a) and then find $u(r)$ that satisfies the boundary condition and boundedness condition:

$$u(a) = 0 \qquad \text{and} \qquad \lim_{r\to 0^+} |u(r)| \quad \text{bounded.}$$

(c) Find the critical radius, that is, the smallest radius a for which the solution is not identically 0.

Exercise 17.5. ✗✗

Show that

$$|\sin x| = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos nx$$

for $-\infty < x < \infty$.

Hint: Using the identity $\sin A\,\cos B = \dfrac{1}{2}\big[\sin(A+B)+\sin(A-B)\big]$, find the Fourier cosine series of the function $f(x) = \sin x$, for $0 \leqslant x \leqslant \pi$, and then use Dirichlet's theorem.

Exercise 17.6. ✗✗

Solve Laplace's equation in the square $0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi$ with the boundary conditions given below

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi$$

$$u(0, y) = 0, \quad 0 \leqslant y \leqslant \pi$$

$$u(\pi, y) = 0, \quad 0 \leqslant y \leqslant \pi$$

$$u(x, 0) = 0, \quad 0 \leqslant x \leqslant \pi$$

$$u(x, \pi) = 1, \quad 0 \leqslant x \leqslant \pi.$$

17.3 Midterm Exam 3

Exercise 17.7. ✗✗

Consider the following eigenvalue problem on the interval $[0,1]$:

$$u''(x) + 2u'(x) - u(x) + \lambda\,(x+1)^2 e^{-2x}\,u(x) = 0$$
$$u'(0) = 0$$
$$u'(1) = 0$$

(a) Explain the meaning of *eigenvalue problem*.

(b) Show that this eigenvalue problem is not of Sturm-Liouville type.

(c) Multiply the above equation by e^{2x} to obtain a Sturm-Liouville problem. Identify $p(x)$, $q(x)$, and $\sigma(x)$.

(d) Use the Rayleigh quotient to show that the leading eigenvalue is positive, that is, $\lambda_1 > 0$.

(e) Find an upper bound for the leading eigenvalue.

Exercise 17.8. ✗✗

Consider Laplace's equation for the steady state temperature distribution in a square plate of side length 1.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$$

$$u(0,y) = 1, \quad u(1,y) = 1 \quad \text{for} \quad 0 \leqslant y \leqslant 1$$

$$\frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,1) = 0 \quad \text{for} \quad 0 \leqslant x \leqslant 1.$$

Obviously, the solution is $u(x,y) = 1$ for $0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$. Show that this is the case using separation of variables.

17.4 Midterm Exam 4

Exercise 17.9. ✗

Solve the normalized wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant \pi, \quad t \geqslant 0$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \geqslant 0$$

$$u(x,0) = \sin x, \quad \frac{\partial u}{\partial t}(x,0) = \sin x, \quad 0 \leqslant x \leqslant \pi.$$

Exercise 17.10. ✗✗

Consider the regular Sturm-Liouville problem

$$\varphi''(x) + \lambda\,\varphi(x) = 0, \quad 0 \leqslant x \leqslant 1$$

$$\varphi(0) = 0$$

$$\varphi(1) - h\,\varphi'(1) = 0$$

where $h > 0$.
Show that there is a single negative eigenvalue λ_0 if and only if $h < 1$. Find λ_0 and the corresponding eigenfunction $\varphi_0(x)$.
Hint: Assume $\lambda = -\mu^2$ for some real number $\mu \neq 0$.

Exercise 17.11. ✗

Consider Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in a semi-circular disk of radius a centered at the origin with boundary conditions

$$u(r,0) = 0, \quad 0 < r \leqslant a,$$

$$u(r,\pi) = 0, \quad 0 < r \leqslant a,$$

$$u(a,\theta) = \sin\theta, \quad 0 \leqslant \theta \leqslant \pi,$$

$$|u(r,\theta)| < \infty \quad \text{as} \quad r \to 0^+.$$

Solve this problem using separation of variables.

Chapter 18

Four Sample Final Examinations

18.1 Final Exam 1

Exercise 18.1.
Assume that $f(x)$ is absolutely integrable and a is a given real constant. Show that

$$\mathcal{F}\left(e^{iax} f(x)\right)(\omega) = \widehat{f}(\omega - a).$$

✗✗

Exercise 18.2.
Hermite's differential equation reads

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty$$

(a) Multiply by e^{-x^2} and bring the differential equation into Sturm-Liouville form. Decide if the resulting Sturm-Liouville problem is regular or singular.

(b) Show that the Hermite polynomials

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x$$

are eigenfunctions of the Sturm-Liouville problem and find the corresponding eigenvalues.

(c) Use an appropriate weight function and show that H_1 and H_2 are orthogonal on the interval $(-\infty, \infty)$ with respect to this weight function.

✗

Exercise 18.3.
Find all functions ϕ for which $u(x,t) = \phi(x - ct)$ is a solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad -\infty < x < \infty$$

where k and c are constants.

Exercise 18.4. ✗✗

Consider the regular Sturm-Liouville problem

$$\begin{aligned} \phi'' + \lambda^2 \phi = 0 \qquad & 0 \leqslant x \leqslant \pi \\ \phi'(0) = 0 & \\ \phi(\pi) = 0 & \end{aligned}$$

(a) Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

(b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Given the function $f(x) = \dfrac{\pi^2 - x^2}{2}$, $0 < x < \pi$, find the eigenfunction expansion for f.

(d) Show that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots$$

Exercise 18.5. ✗✗

Given the following initial boundary value problem for the heat equation on $[0, 1]$.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{9}\frac{\partial^2 u}{\partial x^2} - 2u \\ u(0,t) &= 0, \\ u(1,t) &= 0 \\ u(x,0) &= 7 \sin 3\pi x \end{aligned}$$

(a) If $u(x,t)$ is the solution to the problem above, find an initial boundary value problem satisfied by

$$w(x,t) = e^{2t} u(x,t).$$

(b) Solve the problem found in part (a) for $w(x,t)$.

(c) Find the solution $u(x,t)$ to the original problem.

(d) Find the time T_1 such that $u(x,t) < 1$ for every $x \in [0,1]$ and every $t > T_1$.

18.2 Final Exam 2

Exercise 18.6. ✗

Let $0 < a < \pi$, given the function

$$f(x) = \begin{cases} \dfrac{1}{2a} & \text{if} \quad |x| < a \\ 0 & \text{if} \quad x \in [-\pi, \pi], \quad \text{and} \quad |x| > a \end{cases}$$

find the Fourier series for f and use Dirichlet's convergence theorem to show that

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$$

for $0 < a < \pi$.

Exercise 18.7. ✗

Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 7 \sin 3x$$

subject to the initial and boundary conditions:

$$u(0,t) = 0, \qquad u(\pi, t) = 0, \qquad \text{and} \qquad u(x,0) = 5 \sin 3x.$$

Solve this problem using the method of eigenfunction expansions. Show that the solution approaches a steady-state solution as $t \to \infty$.

Exercise 18.8. ✗

Consider torsional oscillations of a homogeneous cylindrical shaft. If $\omega(x,t)$ is the angular displacement at time t of the cross section at x, then

$$\frac{\partial^2\omega}{\partial t^2} = a^2\frac{\partial^2\omega}{\partial x^2}, \qquad 0 \leqslant x \leqslant L, \quad t > 0.$$

where the initial conditions are

$$\omega(x,0) = f(x), \qquad \text{and} \qquad \frac{\partial\omega}{\partial t}(x,0) = 0, \quad 0 \leqslant x \leqslant L,$$

and the ends of the shaft are fixed elastically:

$$\frac{\partial\omega}{\partial x}(0,t) - \alpha\,\omega(0,t) = 0, \qquad \text{and} \qquad \frac{\partial\omega}{\partial x}(L,t) + \alpha\,\omega(L,t) = 0, \quad t > 0$$

with α a positive constant.

(a) Why is it possible to use separation of variables to solve this problem?

(b) Use separation of variables and show that one of the resulting problems is a regular Sturm-Liouville problem.

(c) Show that all of the eigenvalues of this regular Sturm-Liouville problem are positive.

Note: You do not need to solve the initial value problem, just answer the questions (a), (b), and (c).

Exercise 18.9. ✗

(a) Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t \geqslant 0$$

$$w(x,0) = \frac{1}{2}\,e^{2x}, \quad -\infty < x < \infty.$$

(b) For which values of c does this initial value problem have a time-independent solution?

18.3 Final Exam 3

Exercise 18.10.
Assume that $f''(t)$ is absolutely integrable and

$$\lim_{t\to\infty} f(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} f'(t) = 0.$$

Show that

$$\mathcal{F}_s\left(f''\right)(\omega) = -\omega^2 \mathcal{F}_s\left(f\right)(\omega) + \frac{2}{\pi}\,\omega f(0).$$

Exercise 18.11. ✗✗
Legendre's differential equation reads

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \qquad -1 < x < 1$$

(a) Write the differential equation in Sturm-Liouville form. Decide if the resulting Sturm-Liouville problem is regular or singular.

(b) Show that the first four Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

are eigenfunctions of the Sturm-Liouville problem and find the corresponding eigenvalues.

(c) Use an appropriate weight function and show that P_1 and P_2 are orthogonal on the interval $(-1, 1)$ with respect to this weight function.

Exercise 18.12. ✗
Find all functions ϕ for which $u(x,t) = \phi(x + ct)$ is a solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

where k and c are constants.

Exercise 18.13. ✗✗

Let

$$f(x) = \begin{cases} \cos x & |x| < \pi, \\ 0 & |x| > \pi. \end{cases}$$

(a) Find the Fourier integral of f.

(b) For which values of x does the integral converge to $f(x)$?

(c) Evaluate the integral

$$\int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1 - \lambda^2}\, d\lambda$$

for $-\infty < x < \infty$.

18.4 Final Exam 4

Exercise 18.14. ✗

A fluid occupies the half plane $y > 0$ and flows past (left to right, approximately) a plate located near the x-axis. If the x and y components of the velocity are $U_0 + u(x,y)$ and $v(x,y)$, respectively where U_0 is the constant free-stream velocity, then under certain assumptions, the equations of motion, continuity, and state can be reduced to

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \left(1 - M^2\right)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (*)$$

valid for all $-\infty < x < \infty,\ 0 < y < \infty$.
Suppose there exists a function ϕ (called the *velocity potential*), such that

$$u = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v = \frac{\partial \phi}{\partial y}.$$

(a) State a condition under which the first equation in $(*)$ above becomes an identity.

(b) Show that the second equation in $(*)$ above becomes (assuming the free-stream Mach number M is a constant) a partial differential equation for ϕ which is elliptic if $M < 1$ or hyperbolic if $M > 1$.

Exercise 18.15. ✗

Besides linear equations, some nonlinear equations can also result in *traveling wave solutions* of the form

$$u(x,t) = \phi(x - ct).$$

Fisher's equation, which models the spread of an advantageous gene in a population, where $u(x,t)$ is the density of the gene in the population at time t and location x, is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Show that Fisher's equation has a solution of this form if ϕ satisfies the nonlinear ordinary differential equation

$$\phi'' + c\phi' + \phi(1-\phi) = 0.$$

Exercise 18.16. ✗✗

Given the regular Sturm-Liouville problem

$$\begin{aligned}\phi''(x) + \lambda^2\phi(x) = 0, \quad 0 \leqslant x \leqslant \pi \\ \phi(0) = 0, \\ \phi(\pi) = 0.\end{aligned}$$

(a) Find the eigenvalues λ_n^2 and corresponding eigenfunctions $\phi_n(x)$ for this problem.

(b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal on the interval $[0, \pi]$.

(c) Use the method of eigenfunction expansions to find the solution to the boundary value problem

$$\begin{aligned}u''(x) = -x, \quad 0 \leqslant x \leqslant \pi \\ u(0) = 0, \\ u(\pi) = 0.\end{aligned}$$

(d) Solve the problem in (c) by direct integration and use this result to show that

$$\frac{x(\pi^2 - x^2)}{6} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}$$

for $-\pi \leqslant x \leqslant \pi$.

Exercise 18.17. ✗✗

Find the solution to Laplace's equation on the rectangle:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \\ u(0, y) = 1, \quad 0 < y < b \\ u(a, y) = 1, \quad 0 < y < b \\ \frac{\partial u}{\partial y}(x, 0) = 0, \quad 0 < x < a \\ \frac{\partial u}{\partial y}(x, b) = 0, \quad 0 < x < a\end{aligned}$$

using the method of separation of variables. Is your solution what you expected?

Exercise 18.18. ✗

Solve the following initial value problem for the damped wave equation

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0\\ u(x,0) &= \frac{1}{1+x^2}, \quad -\infty < x < \infty\\ \frac{\partial u}{\partial t}(x,0) &= 1, \quad -\infty < x < \infty.\end{aligned}$$

Hint: Do not use separation of variables, instead solve the initial value – boundary value problem satisfied by $w(x,t) = e^t \cdot u(x,t)$.

Appendix A

Higher Dimensional Fourier Transforms

Notice: In this appendix, the Fourier transform has been defined using $e^{-i\lambda x}$ whereas in the previous section we used $e^{i\lambda x}$. These definitions are equivalent in the sense that the results will be the same. The computations, however, need to be adjusted to reflect the sign change. We will make these changes in future reviews.

One can define n-dimensional Fourier transforms and inverse Fourier transforms as follows:

$$F(\vec{\mu}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\vec{x})e^{-i\vec{\mu}\cdot\vec{x}}\,dx_1\ldots dx_n$$
$$f(\vec{x}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(\vec{\mu})e^{i\vec{\mu}\cdot\vec{x}}\,d\mu_1\ldots d\mu_n,$$

where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$.

EXAMPLE A.1. Consider the following two–dimensional heat equation:

$$u_t = k(u_{xx} + u_{uu}), \qquad -\infty < x, y < \infty,\ t > 0,$$
$$u(x, y, 0) = f(x, y).$$

Let $U(\mu, \lambda, t)$ and $F(\mu, \lambda)$ be the Fourier transforms of $u(x, y, t)$ and $f(x, y)$ respectively. Then

$$U(\mu,\lambda,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)e^{-i(\mu x+\lambda y)}\,dx\,dy, \quad u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\mu,\lambda,t)e^{i(\mu x+\lambda y)}\,d\mu\,d\lambda,$$
$$F(\mu,\lambda) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(\mu x+\lambda y)}\,dx\,dy, \qquad f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\lambda)e^{i(\mu x+\lambda y)}\,d\mu\,d\lambda.$$

It is clear that $F(\mu, \lambda) = U(\mu, \lambda, 0)$. Differentiating u we get

$$u_t(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_t(\mu,\lambda,t)e^{i(\mu x+\lambda y)}\,d\mu\,d\lambda,$$
$$\nabla^2 u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -(\mu^2+\lambda^2)U(\mu,\lambda,t)e^{i(\mu x+\lambda y)}\,d\mu\,d\lambda.$$

Plug into the equation to get

$$u_t - ku_{xx} = 0 \quad\Longrightarrow\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{U_t + k(\mu^2+\lambda^2)U\}e^{i(\mu x+\lambda y)}\,d\mu\,d\lambda = 0.$$

This gives the following ODE for U:

$$\frac{\partial U}{\partial t} + k(\mu^2 + \lambda^2)U = 0, \qquad U(\mu, \lambda, 0) = F(\mu, \lambda).$$

Therefore

$$U(\mu, \lambda, t) = F(\mu, \lambda)e^{-k(\mu^2+\lambda^2)t}.$$

Thus, the solution to the heat equation is

$$\begin{aligned}
u(x, y, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu, \lambda)e^{-k(\mu^2+\lambda^2)t}e^{i(\mu x+\lambda y)}\, d\mu\, d\lambda \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)e^{-i(\mu\xi+\lambda\eta)}e^{-k(\mu^2+\lambda^2)t}e^{i(\mu x+\lambda y)}\, d\xi\, d\eta\, d\mu\, d\lambda \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\left\{\int_{-\infty}^{\infty} e^{-k\mu^2 t+i\mu(x-\xi)}\, d\mu\right\}\left\{\int_{-\infty}^{\infty} e^{-k\lambda^2 t+i\lambda(y-\eta)}\, d\lambda\right\} d\xi\, d\eta \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\left(\sqrt{\frac{\pi}{kt}}e^{-\frac{(x-\xi)^2}{4kt}}\right)\left(\sqrt{\frac{\pi}{kt}}e^{-\frac{(y-\eta)^2}{4kt}}\right) d\xi\, d\eta \\
&= \frac{1}{4\pi kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)e^{-\frac{1}{4kt}[(x-\xi)^2+(y-\eta)^2]}\, d\xi\, d\eta.
\end{aligned}$$

Special case: $f(\xi, \eta) = \delta(\xi)\delta(\eta)$.

$$u(x, y, t) = \frac{1}{4\pi kt}e^{-\frac{x^2+y^2}{4kt}}.$$

Before going on to solve the 3–dimensional wave equation, the following example will prove useful.

EXAMPLE A.2. Find the 3–dimensional Fourier transform of

$$f(x, y, z) = \begin{cases} 1, & (x, y, z) \in \Omega_R \\ 0, & (x, y, z) \notin \Omega_R \end{cases}$$

where $\Omega_R := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leqslant R^2\}$.

Solution:
We have

$$F(\mu, \lambda, \nu) = \frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} f(x, y, z)e^{-i(\mu x+\lambda y+\nu z)}\, dV_x = \frac{1}{(2\pi)^3}\iiint_{\Omega_R} e^{-i(\mu x+\lambda y+\nu z)}\, dV_x.$$

To evaluate this integral we change to spherical coordinates with the z-axis oriented in the direction of the fixed vector $\vec{\mu} = (\mu, \lambda, \nu)$. Then we get

$$\mu x + \lambda y + \nu z = \vec{\mu} \cdot \vec{x} = |\vec{\mu}|\,|\vec{x}| \cos\theta$$

and the Fourier transform becomes

$$F(\vec{\mu}) = F(\mu,\lambda,\nu) = \frac{1}{(2\pi)^3}\iiint\limits_{\Omega_R} e^{-i\vec{\mu}\cdot\vec{x}}\,dV_x$$
$$= \frac{1}{(2\pi)^3}\int_0^R\int_{-\pi}^{\pi}\int_0^{\pi} e^{-i|\vec{\mu}|r\cos\theta} r^2\sin\theta\,d\theta\,d\phi\,dr$$
$$= \frac{1}{(2\pi)^2}\int_0^R\int_0^{\pi} e^{-i|\vec{\mu}|r\cos\theta} r^2\sin\theta\,d\theta\,dr = \frac{1}{(2\pi)^2}\int_0^R \left.\frac{e^{-i|\vec{\mu}|r\cos\theta}}{i|\vec{\mu}|r}\right|_0^{\pi} r^2\,dr$$
$$= \frac{2}{(2\pi)^2|\vec{\mu}|}\int_0^R r\sin(|\vec{\mu}|r)\,dr = \frac{2}{(2\pi)^2|\vec{\mu}|^3}[\sin(|\vec{\mu}|R) - |\vec{\mu}|R\cos(|\vec{\mu}|R)].$$

For any R, let $S_R := \partial\Omega_R$. That is $S_R = \{(x,y,z)\in\mathbb{R}^3 | x^2+y^2+z^2 = R^2\}$. The following lemma will also prove useful.

Lemma 57.

$$\iint\limits_{S_R}(\cdot)d\sigma = \frac{d}{dR}\iiint\limits_{\Omega_R}(\cdot)dV.$$

Proof.
The surface integral in spherical coordinates is given by

$$\iint\limits_{S_R}(\cdot)d\sigma = \int_{-\pi}^{\pi}\int_0^{\pi}(\cdot)R^2\sin\theta\,d\theta\,d\phi.$$

The volume integral in spherical coordinates is given by

$$\iiint\limits_{\Omega_R}(\cdot)dV = \int_0^R\int_{-\pi}^{\pi}\int_0^{\pi}(\cdot)r^2\sin\theta\,d\theta\,d\phi\,dr = \int_0^R\left(\iint\limits_{S_r}(\cdot)\,d\sigma\right)dr.$$

The result follows by differentiating with respect to R.

We now apply this lemma to a result obtained in the previous example:

$$\iint\limits_{S_R} e^{i\vec{\mu}\cdot\vec{x}}\,d\sigma_x = \frac{d}{dR}\iiint\limits_{\Omega_R} e^{i\vec{\mu}\cdot\vec{x}}\,dV_x = \frac{d}{dR}\left\{\frac{4\pi}{|\vec{\mu}|^3}[\sin(|\vec{\mu}|R) - |\vec{\mu}|R\cos(|\vec{\mu}|R)]\right\} = \frac{4\pi}{|\vec{\mu}|}R\sin(|\vec{\mu}|R).$$

This result, more conveniently written

$$\iint\limits_{S_R} e^{i\vec{\mu}\cdot\vec{x}}\,d\sigma_x = 4\pi R^2\frac{\sin(|\vec{\mu}|R)}{|\vec{\mu}|R} \tag{A.1}$$

will be used in the next example.

EXAMPLE A.3. Consider the following 3–dimensional wave equation:

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u, \qquad (x,y,z)\in\mathbb{R}^3,\ t>0,\\ u(x,y,z,0) &= f(x,y,z),\\ u_t(x,y,z,0) &= g(x,y,z). \end{aligned}$$

For convenience we denote $\vec{x} = (x, y, z)$ and $\vec{\mu} = (\mu, \lambda, \nu)$. Let U, F, and G represent the 3–dimensional Fourier transforms of u, f, and g respectively. Then we have

$$\begin{aligned} U(\vec{\mu},t) &= \frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3} u(\vec{x},t)e^{-i\vec{\mu}\cdot\vec{x}}\,dV_x, & u(\vec{x},t) &= \iiint\limits_{\mathbb{R}^3} U(\vec{\mu},t)e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu,\\ F(\vec{\mu}) &= \frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3} f(\vec{x})e^{-i\vec{\mu}\cdot\vec{x}}\,dV_x, & f(\vec{x}) &= \iiint\limits_{\mathbb{R}^3} F(\vec{\mu})e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu,\\ G(\vec{\mu}) &= \frac{1}{(2\pi)^3}\iiint\limits_{\mathbb{R}^3} g(\vec{x})e^{-i\vec{\mu}\cdot\vec{x}}\,dV_x, & g(\vec{x}) &= \iiint\limits_{\mathbb{R}^3} G(\vec{\mu})e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu. \end{aligned} \tag{A.2}$$

Differentiating u we get

$$\begin{aligned} u_{tt}(\vec{x},t) &= \iiint\limits_{\mathbb{R}^3} U_{tt}(\vec{\mu},t)e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu,\\ u_{xx}(\vec{x},t) &= -\iiint\limits_{\mathbb{R}^3} \mu^2 U(\vec{\mu},t)e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu,\\ u_{yy}(\vec{x},t) &= -\iiint\limits_{\mathbb{R}^3} \lambda^2 U(\vec{\mu},t)e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu,\\ u_{zz}(\vec{x},t) &= -\iiint\limits_{\mathbb{R}^3} \nu^2 U(\vec{\mu},t)e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu. \end{aligned}$$

Plug into the wave equation to get

$$\iiint\limits_{\mathbb{R}^3} [U_{tt}(\vec{\mu},t) + c^2(\mu^2+\lambda^2+\nu^2)U(\vec{\mu},t)]e^{i\vec{\mu}\cdot\vec{x}}\,dV_\mu = 0.$$

This leads to the following initial value problem for U:

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} + c^2|\vec{\mu}|^2 U &= 0,\\ U(\vec{\mu},0) &= F(\vec{\mu}),\\ \frac{\partial U}{\partial t}(\vec{\mu},0) &= G(\vec{\mu}). \end{aligned}$$

The solution is easily obtained:

$$U(\vec{\mu},t) = F(\vec{\mu})\cos(c|\vec{\mu}|t) + G(\vec{\mu})\frac{\sin(c|\vec{\mu}|t)}{c|\vec{\mu}|}.$$

Therefore the solution to the wave equation is

$$u(\vec{x},t) = \iiint_{\mathbb{R}^3} [F(\vec{\mu})\cos(c|\vec{\mu}|t) + G(\vec{\mu})\frac{\sin(c|\vec{\mu}|t)}{c|\vec{\mu}|}]\, e^{i\vec{\mu}\cdot\vec{x}}\, dV_\mu. \tag{A.3}$$

This is the Fourier transform representation of the solution to the wave equation. We now derive the 3–dimensional analogue of d'Alembert's solution. To this end, recall (A.1). Replace R with ct and re-arrange (A.1) to get:

$$\frac{\sin(c|\vec{\mu}|t)}{c|\vec{\mu}|} = \frac{1}{4\pi c^2 t}\iint_{S_{ct}} e^{i\vec{\mu}\cdot\vec{\xi}}\, d\sigma_\xi.$$

Multiply by $G(\vec{\mu})e^{i\vec{\mu}\cdot\vec{x}}$ and integrate over $\mathbb{R}^3$ to get

$$\begin{aligned}\iiint_{\mathbb{R}^3} G(\vec{\mu})\frac{\sin(c|\vec{\mu}|t)}{c|\vec{\mu}|}e^{i\vec{\mu}\cdot\vec{x}}\, dV_\mu &= \frac{1}{4\pi c^2 t}\iiint_{\mathbb{R}^3}\iint_{S_{ct}} G(\vec{\mu})\, e^{i\vec{\mu}\cdot\vec{x}}\, e^{i\vec{\mu}\cdot\vec{\xi}}\, d\sigma_\xi\, dV_\mu \\ &= \frac{1}{4\pi c^2 t}\iint_{S_{ct}}\left(\iiint_{\mathbb{R}^3} G(\vec{\mu})\, e^{i\vec{\mu}\cdot(\vec{x}+\vec{\xi})}\, dV_\mu\right) d\sigma_\xi \\ &= \frac{1}{4\pi c^2 t}\iint_{S_{ct}} g(\vec{x}+\vec{\xi})\, d\sigma_\xi. \qquad \text{(using (A.2))}\end{aligned}$$

For the other part of the solution we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} F(\vec{\mu})\cos(c|\vec{\mu}|t)e^{i\vec{\mu}\cdot\vec{x}}\, dV_\mu &= \frac{\partial}{\partial t}\iiint_{\mathbb{R}^3} F(\vec{\mu})\frac{\sin(c|\vec{\mu}|t)}{c|\vec{\mu}|}e^{i\vec{\mu}\cdot\vec{x}}\, dV_\mu \\ &= \frac{\partial}{\partial t}\left\{\frac{1}{4\pi c^2 t}\iint_{S_{ct}} f(\vec{x}+\vec{\xi})\, d\sigma_\xi\right\}.\end{aligned}$$

Therefore the solution to the wave equation is

$$u(\vec{x},t) = \frac{\partial}{\partial t}\left\{\frac{1}{4\pi c^2 t}\iint_{S_{ct}} f(\vec{x}+\vec{\xi})\, d\sigma_\xi\right\} + \frac{1}{4\pi c^2 t}\iint_{S_{ct}} g(\vec{x}+\vec{\xi})\, d\sigma_\xi. \tag{A.4}$$

This formula is due to Poisson, but is known as *Kirchhoff's formula*, and is the 3–dimensional analogue of d'Alembert's solution.

It is worthwhile to compare the two forms of the solution to the wave equation. To evaluate Eq. (A.3) a six fold integration is required: a triple integral to evaluate F and G, and then another triple integral to get the solution u. However, to evaluate Eq. (A.4), only a double (surface) integral is required. For this reason Kirchhoff's formula is by far the more desirable way to represent the solution.

We can write Kirchhoff's formula in a more compact form if we define the following *mean value operator*:

$$M_R[f] := \frac{1}{4\pi R^2} \iint\limits_{S_R} f(\vec{x} + \vec{\xi}\,)\, d\sigma_\xi.$$

This integral operator gives the mean value of f on the surface of the sphere S_R. Kirchhoff's formula may now be written as

$$u(\vec{x}, t) = \frac{\partial}{\partial t}(tM_{ct}[f]) + tM_{ct}[g]. \tag{A.5}$$

EXAMPLE A.4. Consider the following 2–dimensional wave equation:

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u, \qquad (x, y) \in \mathbb{R}^2,\ t > 0, \\ u(x, y, 0) &= f(x, y), \\ u_t(x, y, 0) &= g(x, y). \end{aligned}$$

In order to solve this 2–dimensional problem we use the solution of the 3–dimensional problem and make use of the fact that the relevant functions are independent of z. Consider the expression

$$M_R[g] = \frac{1}{4\pi R^2} \iint\limits_{S_R} g(x + \xi, y + \eta)\, d\sigma_\xi.$$

Since the integrand is independent of z (and ζ), the integral is just double the integral over the upper hemisphere of S_R. For the upper hemisphere of $S_R :\ \xi^2 + \eta^2 + \zeta^2 = R^2$ we have

$$\zeta = \sqrt{R^2 - \xi^2 - \eta^2}, \quad \Longrightarrow \quad d\sigma_\xi = \sqrt{1 + \left(\frac{\partial \zeta}{\partial \xi}\right)^2 + \left(\frac{\partial \zeta}{\partial \eta}\right)^2}\, d\xi\, d\eta = \frac{R}{\sqrt{R^2 - \xi^2 - \eta^2}}\, d\xi\, d\eta.$$

Hence

$$M_R[g] = \frac{1}{4\pi R^2} \iint\limits_{S_R} g(x + \xi, y + \eta)\, d\sigma_\xi = \frac{1}{2\pi R} \iint\limits_{D_R} \frac{g(x + \xi, y + \eta)}{\sqrt{R^2 - \xi^2 - \eta^2}}\, d\xi\, d\eta,$$

where

$$D_R := \{(\xi, \eta) \in \mathbb{R}^2 | \xi^2 + \eta^2 \leqslant R^2\}.$$

Apply this result to both terms in (A.5) to get

$$u(x, y, t) = \frac{1}{2\pi c} \left\{ \frac{\partial}{\partial t} \iint\limits_{D_{ct}} \frac{f(x + \xi, y + \eta)}{\sqrt{R^2 - \xi^2 - \eta^2}}\, d\xi\, d\eta + \iint\limits_{D_{ct}} \frac{g(x + \xi, y + \eta)}{\sqrt{R^2 - \xi^2 - \eta^2}}\, d\xi\, d\eta \right\}.$$

It is worth while comparing the solutions for the 2 and 3 dimensional wave equations:

$$u(\vec{x}, t) = \frac{1}{4\pi c^2} \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t} \iint\limits_{S_{ct}} f(\vec{x} + \vec{\xi}\,)\, d\sigma_\xi \right) + \frac{1}{t} \iint\limits_{S_{ct}} g(\vec{x} + \vec{\xi}\,)\, d\sigma_\xi \right\}, \tag{3-d}$$

$$u(\vec{x}, t) = \frac{1}{2\pi c} \left\{ \frac{\partial}{\partial t} \iint\limits_{D_{ct}} \frac{f(\vec{x} + \vec{\xi}\,)}{\sqrt{R^2 - |\vec{\xi}\,|^2}}\, d\xi\, d\eta + \iint\limits_{D_{ct}} \frac{g(\vec{x} + \vec{\xi}\,)}{\sqrt{R^2 - |\vec{\xi}\,|^2}}\, d\xi\, d\eta \right\}. \tag{2-d}$$

This representation displays an important property of the three dimensional wave equation. This property is known as *Huygens' principle* and it may be stated as follows: If the initial data f and g have compact support, i.e. they are identically zero outside of a sufficently large set, then the solution $u(x, y, z, t) = 0$ for sufficently large time t. This is clear since the expression for the solution contains only surface integrals over a sphere of radius ct which, for sufficently large t, is so large so as to not intersect the set where $f \neq 0$ and $g \neq 0$. Thus, a limited initial disturbance is experienced by an observer for a finite duration.

For the two dimensional wave equation the situation is quite different. In this case the expression for the solution contains integrals over the *interior* of a circle of radius ct. What this means is that the integration will always cover the region where $f \neq 0$ and $g \neq 0$, even though the boundary of the circle extends beyond the region where $f \neq 0$ and $g \neq 0$. As a consequence, Huygens' principle is *not valid* for the two dimensional wave equation. For example, a pebble dropped in a pond of water will create a wave motion on the surface of the water. An observer positioned r units away from the initial disturbance will sense the disturbance at time $t = r/c$ later. However, after this time the disturbance experienced by the observer will continue to be non zero for all subsequent time. This is the phenonenon of a *wake* behind the initial disturbance. This wake phenonenon is a property of the two dimensional wave motion. Huygens' principle can be restated to say that, in three dimensional wave motion, no wake is present.

As one final example with the wave equation, we show that d'Alembert's solution for the one dimensional wave motion actually is the one dimensional version of Kirchhoff's formula.

EXAMPLE A.5. Consider the following 1–dimensional wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

In order to solve this 1–dimensional problem we use the solution of the 3–dimensional problem and make use of the fact that the relevant functions are independent of y and z. Consider the expression

$$M_R[g] = \frac{1}{4\pi R^2} \iint\limits_{S_R} g(x+\xi)\, d\sigma_\xi.$$

If we parameterize S_R with spherical coordinates

$$\xi = R\cos\theta, \quad \eta = R\sin\theta\cos\phi, \quad \zeta = R\sin\theta\sin\phi$$

then the expression for $M_R[g]$ becomes

$$\begin{aligned} M_R[g] &= \frac{1}{4\pi R^2} \iint\limits_{S_R} g(x+\xi)\, d\sigma_\xi = \frac{1}{4\pi R^2} \int_{-\pi}^{\pi} \int_0^{\pi} g(x + R\cos\theta)\, R^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{1}{2} \int_0^{\pi} g(x + R\cos\theta)\, \sin\theta\, d\theta = \frac{1}{2R} \int_{-R}^{R} g(x+\xi)\, d\xi. \end{aligned}$$

Apply this result to both terms in (A.5) to get

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left\{\frac{1}{2c}\int_{-ct}^{ct} f(x+\xi)\,d\xi\right\} + \frac{1}{2c}\int_{-ct}^{ct} g(x+\xi)\,d\xi \\ &= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, \end{aligned}$$

which is just d'Alembert's solution.

We now use Fourier transforms to find a solution to Laplace's equation in a half-plane.

EXAMPLE A.6. Consider the following 2–dimensional Laplace equation:

$$\begin{aligned} \nabla^2 u &= 0, \qquad x\in\mathbb{R},\ y>0,\ t>0, \\ u(x,0) &= f(x). \end{aligned}$$

We require the additional assumption that $|u(x,y)|<\infty$. Let $U(\mu,y)$ and $F(\mu)$ be the Fourier transforms of $u(x,y)$ and $f(x)$ respectively. Then

$$\begin{aligned} U(\mu,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)e^{-i\mu x}\,dx, & u(x,y) &= \int_{-\infty}^{\infty} U(\mu,y)\,e^{i\mu x}\,d\mu, \\ F(\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\mu x}\,dx, & f(x) &= \int_{-\infty}^{\infty} F(\mu)\,e^{i\mu x}\,d\mu. \end{aligned}$$

Plug into the equation to get

$$u_{xx}+u_{yy}=0 \quad\Longrightarrow\quad \int_{-\infty}^{\infty}[-\mu^2 U(\mu,y)+U_{yy}(\mu,y)]e^{i\mu x}\,d\mu = 0.$$

This leads to the following problem for U:

$$\frac{\partial^2 U}{\partial y^2} - \mu^2 U = 0, \qquad 3U(\mu,0) = F(\mu).$$

Notice that this is a second order ODE with only one auxiliary condition. The second condition we use is that we require that U be bounded. The soluton to the ODE itself is

$$U(\mu,y) = a(\mu)e^{\mu y} + b(\mu)e^{-\mu y}.$$

¿From the boundedness condition we get

$$|U(\mu,y)|<\infty \quad\Longrightarrow\quad \begin{cases} a(\mu)=0, & \text{for } \mu>0 \\ b(\mu)=0, & \text{for } \mu<0 \end{cases} \quad\Longrightarrow\quad U(\mu,y) = A(\mu)e^{-|\mu|y}.$$

The other condition then gives

$$U(\mu,0) = F(\mu) \quad\Longrightarrow\quad U(\mu,y) = F(\mu)e^{-|\mu|y}.$$

The soluton to the problem is

$$
\begin{aligned}
u(x, y) &= \int_{-\infty}^{\infty} F(\mu) e^{-|\mu| y} e^{i\mu x} \, d\mu \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-|\mu| y} e^{i\mu x} e^{-i\mu\xi} \, d\xi \, d\mu \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-\infty}^{\infty} e^{-|\mu| y} e^{i\mu(x-\xi)} \, d\mu \right) d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-\infty}^{\infty} e^{-|\mu| y} [\cos(\mu(x-\xi)) + i \sin(\mu(x-\xi))] \, d\mu \right) d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(2 \int_{0}^{\infty} e^{-\mu y} \cos(\mu(x-\xi)) \, d\mu \right) d\xi \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \left(\frac{e^{-\mu y}}{(x-\xi)^2 + y^2} [-y \cos(\mu(x-\xi)) + (x-\xi) \sin(\mu(x-\xi))] \right) \Bigg|_{\mu=0}^{\infty} d\xi \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y \, f(\xi)}{(x-\xi)^2 + y^2} \, d\xi
\end{aligned}
$$

This is Poisson's integral formula for the half plane.

Appendix B

Generalizations to Higher Dimensions

B.1 Classification of PDEs in $\mathbb{R}^n$

For the classification of second order linear PDEs in two variables in Chapter 1, we wrote the PDE in matrix form, and introduced the symbol A of a differential operator Chapter 1Introduce the following notation:

$$\left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\right)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} u = au_{xx} + 2bu_{xy} + cu_{yy}.$$

Then we classified the equations according to the sign pattern of the eigenvalues of the matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

We use the same approach for higher dimensional second order equations.
Now consider a 2$^{\text{nd}}$ order PDE in n independent variables $x_1, x_2, \ldots, x_n$:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\,\frac{\partial^2 u}{\partial x_i \partial x_j} = F(x, u, u_{x_1}, \ldots, u_{x_n}), \tag{B.1}$$

where $x = (x_1, x_2, \ldots, x_n)$. Since the classification only depends on the highest order terms (2nd order terms), we summarized all lower order terms in a big function F. The above PDE may be written in matrix form as

$$\partial_{\mathbf{x}}{}^T \mathbf{A}\,\partial_{\mathbf{x}} u = \widetilde{F},$$

where the symbol

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is a real symmetric matrix, and the vector of derivatives $\partial_{\mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix}^T$.

We may now generalize the classification scheme to second order PDEs of any dimension.

Definition 58. *Eq. (B.1) is said to be*
(i) **elliptic** *if all eigenvalues of* $\mathbf{A}$ *have the same sign;*

(ii) **parabolic** *if at least one eigenvalue of* $\mathbf{A}$ *is zero;*
(iii) **hyperbolic** *if one eigenvalue of* $\mathbf{A}$ *has a sign different from the others; and*
(iv) **ultrahyperbolic** *if two or more eigenvalues of* $\mathbf{A}$ *have a sign different from the others.*

Note that ultrahyperbolic equations occur only for dimension $n \geqslant 4$.

EXAMPLE B.1. For the 3-d Laplace equation $\nabla^2 u = 0$ (i.e. $u_{xx} + u_{yy} + u_{zz} = 0$) we have

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} u = 0.$$

To get the eigenvalues:

$$|\mathbf{A} - \mu \mathbf{I}| = 0 \quad \Longrightarrow \quad (\mu - 1)^3 = 0 \quad \Longrightarrow \quad \mu_1 = 1,\ \mu_2 = 1,\ \mu_3 = 1.$$

Therefore the PDE is elliptic.

EXAMPLE B.2. For the 3-d heat equation $u_t = \nabla^2 u$ we have

$$\begin{bmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} u = 0.$$

To get the eigenvalues:

$$|\mathbf{A} - \mu \mathbf{I}| = 0 \quad \Longrightarrow \quad \mu(\mu + 1)^3 = 0 \quad \Longrightarrow \quad \mu_1 = 0,\ \mu_2 = -1,\ \mu_3 = -1,\ \mu_4 = -1.$$

Therefore the PDE is parabolic.

EXAMPLE B.3. For the 3-d wave equation $u_{tt} = \nabla^2 u$ we have

$$\begin{bmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} u = 0.$$

To get the eigenvalues:

$$|\mathbf{A} - \mu \mathbf{I}| = 0 \quad \Longrightarrow \quad (\mu - 1)(\mu + 1)^3 = 0 \quad \Longrightarrow \quad \mu_1 = 1,\ \mu_2 = -1,\ \mu_3 = -1,\ \mu_4 = -1.$$

Therefore the PDE is hyperbolic.

B.2 Adjoint Operators

In this section we assume that $\Omega \subset \mathbb{R}^N$ is a bounded set, that $x = (x_1, x_2, \ldots, x_N)$, and that we have some inner product defined:

$$\langle f, g \rangle := \int_\Omega f(x)\, g(x)\, w(x)\, dV, \qquad (dV = dx_1\, dx_2 \ldots dx_N)$$

where w is some non-negative weight function. Suppose we have a function $f\colon \Omega \to \mathbb{R}$. We define the support of f as follows:

Definition 59. *The* **support** *of the function* $f\colon \Omega \to \mathbb{R}$*, denoted* $\operatorname{supp}(f)$*, is*

$$\operatorname{supp}(f) := \overline{\{x \in \Omega;\ f(x) \neq 0\}}.$$

EXAMPLE B.4. Suppose $\Omega = [-4, 4]$, and $f(x) = x^2$, $g(x) = 1 - x^2$ and $h(x) = \begin{cases} \sin(x) & 0 < x < 4 \\ 0 & -4 < x \leqslant 0 \end{cases}$.

Then

$$\begin{aligned}
\operatorname{supp}(f) &= \overline{\{x \in \Omega;\ f(x) \neq 0\}} = \overline{[-4,0) \cup (0,4]} = [-4,4] = \Omega; \\
\operatorname{supp}(g) &= \overline{\{x \in \Omega;\ g(x) \neq 0\}} = \overline{[-4,-1) \cup (-1,1) \cup (1,4]} = [-4,4] = \Omega; \\
\operatorname{supp}(h) &= \overline{\{x \in \Omega;\ h(x) \neq 0\}} = \overline{(0,\pi) \cup (\pi,4)} = [0,4].
\end{aligned}$$

We introduce some more notation. Let $\Omega \subset \mathbb{R}^N$ be open.
$C(\Omega) = \{f : \Omega \subset \mathbb{R}^N \to \mathbb{R};\ f \text{ is continuous}\}$;
$C^n(\Omega) = \{f : \Omega \subset \mathbb{R}^N \to \mathbb{R};\ \partial f/\partial x_i \in C^{n-1}(\Omega),\ i = 1, \ldots, N\}, \quad n \in \mathbb{N}$;
$C^\infty(\Omega) = \cap_{n=1}^\infty C^n(\Omega)$;
$C_0(\Omega) = \{f \in C(\Omega);\ \operatorname{supp}(f) \text{ is compact.}\}$;
$C_0^n(\Omega) = C_0(\Omega) \cap C^n(\Omega)$;
$C_0^\infty(\Omega) = C_0(\Omega) \cap C^\infty(\Omega)$.

Note that $\phi \in C_0^\infty(\Omega)$ implies that ϕ and all of its derivatives are zero near $\partial\Omega$. Functions in $C_0^\infty(\Omega)$ are sometimes called "test functions".

EXAMPLE B.5. Let $\Omega = (-2, 2)$ and let

$$\phi(x) = \begin{cases} e^{1/(x^2-1)} & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases}.$$

Then $\operatorname{supp}(\phi) = [-1, 1]$ is compact. It can be verified that all derivatives of ϕ exist so that $\phi \in C_0^\infty(-2, 2)$. The first couple of derivatives of ϕ are:

$$\phi'(x) = \begin{cases} \frac{2x}{(x^2-1)^2} e^{1/(x^2-1)} & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases}, \qquad \phi''(x) = \begin{cases} 2\frac{3x^4-1}{(x^2-1)^4} e^{1/(x^2-1)} & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases}.$$

Suppose we have a linear operator $L\colon C^n(\Omega) \to C^m(\Omega)$. We define the adjoint as follows:

Definition 60. *The* **formal adjoint** *of L, denoted L^*, is the linear operator that satisfies the following:*

$$\langle Lu, \phi \rangle = \langle u, L^*\phi \rangle \quad \forall u \in C^n(\Omega),\ \forall \phi \in C_0^\infty(\Omega).$$

That such an adjoint always exists will not be proven here. We merely illustrate with several examples. First a couple of one dimensional examples.

EXAMPLE B.6. Suppose $L\colon C^1(a,b) \to C(a,b)$ is defined by $L := d/dx$. Then we have

$$\begin{aligned}\langle Lu, \phi \rangle &= \int_a^b (Lu)\phi \, dx = \int_a^b u'(x)\phi(x)\, dx = u(x)\phi(x)\Big|_a^b - \int_a^b u(x)\phi'(x)\, dx \\ &= u(b)\phi(b) - u(a)\phi(a) - \int_a^b u(L\phi)\, dx = 0 - 0 - \langle u, L\phi \rangle = \langle u, -L\phi \rangle .\end{aligned}$$

Therefore $L^* = -L = -d/dx$.

EXAMPLE B.7. Suppose $L\colon C^2(a,b) \to C(a,b)$ is defined by $L := d^2/dx^2$. Then we have

$$\begin{aligned}\langle Lu, \phi \rangle &= \int_a^b (Lu)\phi \, dx = \int_a^b u''(x)\phi(x)\, dx = u'(x)\phi(x)\Big|_a^b - \int_a^b u'(x)\phi'(x)\, dx \\ &= u'(b)\phi(b) - u'(a)\phi(a) - \left\{ u(x)\phi'(x)\Big|_a^b - \int_a^b u(x)\phi''(x)\, dx \right\} \\ &= -[u(b)\phi'(b) - u(a)\phi'(a)] + \int_a^b u(x)\phi''(x)\, dx \\ &= \langle u, L\phi \rangle .\end{aligned}$$

Therefore $L^* = L = d^2/dx^2$.

This prompts the following definition:

Definition 61. *A linear operator L is said to be* **formally self-adjoint** *if $L = L^*$.*

Now for a few higher dimensional examples.

EXAMPLE B.8. Suppose $L : C^2(\Omega) \to C(\Omega)$, where $\Omega \subset \mathbb{R}^3$, is defined by $L := \nabla^2$. Before proceeding with the main calculation, we need to use a couple of vector identities. We have

$$\begin{aligned}\vec{\nabla} \cdot (\phi \vec{\nabla} u) &= \vec{\nabla}\phi \cdot \vec{\nabla} u + \phi \nabla^2 u, \\ \vec{\nabla} \cdot (u \vec{\nabla} \phi) &= \vec{\nabla} u \cdot \vec{\nabla} \phi + u \nabla^2 \phi.\end{aligned}$$

Subtracting one from the other we get

$$\phi \nabla^2 u = u \nabla^2 \phi + \vec{\nabla} \cdot (\phi \vec{\nabla} u - u \vec{\nabla} \phi). \tag{B.2}$$

Now, proceeding to calculate the adjoint, we have

$$\begin{aligned}
\langle Lu, \phi \rangle &= \iiint\limits_{\Omega} (Lu)\phi \, dV = \iiint\limits_{\Omega} (\nabla^2 u)\phi \, dV \\
&= \iiint\limits_{\Omega} [u\nabla^2\phi + \vec{\nabla} \cdot (\phi\vec{\nabla} u - u\vec{\nabla}\phi)] \, dV \qquad 7(\text{using (B.2)}) \\
&= \iiint\limits_{\Omega} u\nabla^2\phi \, dV + \iint\limits_{\partial\Omega} (\phi\vec{\nabla} u - u\vec{\nabla}\phi) \cdot \vec{n} \, d\sigma \qquad (\text{using the divergence theorem}) \\
&= \left\langle u, \nabla^2\phi \right\rangle + 0 = \langle u, L\phi \rangle . \qquad 12(\text{since } \phi \in C_0^\infty(\Omega))
\end{aligned}$$

Therefore $L^* = L = \nabla^2$. The Laplacian is formally self-adjoint.

EXAMPLE B.9. Suppose $L\colon C^1(\Omega) \to C(\Omega)$, where $\Omega \subset \mathbb{R}^n$, is defined by $L := \sum\limits_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$ with $b_i \in C^1(\Omega)$. We have

$$\begin{aligned}
\langle Lu, \phi \rangle &= \int_\Omega \left(\sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) \right) \phi(x) \, dV = \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i}(x)\phi(x) \, dV \\
&= \int_\Omega \sum_{i=1}^{n} \left[\frac{\partial}{\partial x_i}(b_i u\phi) - u\frac{\partial}{\partial x_i}(b_i\phi) \right] dV \\
&= \int_{\partial\Omega} \sum_{i=1}^{n} b_i u\phi n_i \, d\sigma - \int_\Omega \sum_{i=1}^{n} u\frac{\partial}{\partial x_i}(b_i\phi) \, dv \qquad (\text{using the divergence theorem}) \\
&= 0 - \int_\Omega u \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i\phi) \right) dv = \left\langle u, -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i\phi) \right\rangle = \langle u, L^*\phi \rangle .
\end{aligned}$$

Therefore $L^*\phi = -\sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i\phi)$.

EXAMPLE B.10. Suppose $L\colon C^2(\Omega) \to C(\Omega)$, where $\Omega \subset \mathbb{R}^n$, is defined by $L := \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ with $a_{ij} \in C^2(\Omega)$. It can be shown, in a manner similar to the calulation of the previous example (exercise), that the adjoint is given by $L^*\phi = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\phi)$.

One can easily show that the adjoint operator satisfies the following property:

$$(L_1 + L_2)^* = L_1^* + L_2^*.$$

Using this property, together with the results of the last two examples, we see that the adjoint for the general 2nd order linear differential operator

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

is given by

$$L^*u = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)u) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i(x)u) + c(x)u.$$

Under what conditions is L self-adjoint? It can be show that L is self-adjoint if

$$b_i(x) = \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}.$$

In this case we have

$$Lu = L^*u = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + c(x)u.$$

EXAMPLE B.11. If we let $n = 3$ and $a_{ij}(x) = \delta_{ij}p(x)$ in the previous expression, then

$$Lu = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x_i}(\delta_{ij}p(x)\frac{\partial u}{\partial x_j}) + c(x)u = \sum_{i=1}^3 \frac{\partial}{\partial x_i}(p(x)\frac{\partial u}{\partial x_j}) + c(x)u = \vec{\nabla} \cdot (p(x)\vec{\nabla} u) + c(x)u.$$

Thus, the differential operator that appears in the general heat equation is self-adjoint.

We now define a certain bilinear functional:

Definition 62. *Let* $J\colon C^n(\Omega) \times C^n(\Omega) \to \mathbb{R}$ *be given by*

$$J(u, v) := \langle Lu, v\rangle - \langle u, L^*v\rangle .$$

Notice that $J(u, \phi) = \langle Lu, \phi\rangle - \langle u, L^*\phi\rangle = 0$ for all $\phi \in C_0^\infty(\Omega)$ but that, in general, $J(u, v) \neq 0$.

EXAMPLE B.12. Suppose $L\colon C^2(a, b) \to C(a, b)$ is defined by $L := d^2/dx^2$. Then we saw before that $L = L^*$ and we have

$$\begin{aligned} J(u, v) = \langle Lu, v\rangle - \langle u, Lv\rangle &= \int_a^b (u''v - uv'')\, dx \\ &= \int_a^b [(u'v)' - u'v' - (uv')' + u'v']\, dx = \int_a^b [(u'v)' - (uv')']\, dx \\ &= (u'v - uv')\Big|_a^b = u'(b)v(b) - u(b)v'(b) - u'(a)v(a) + u(a)v'(a). \end{aligned}$$

Boundary value problems consist of a PDE together with appropriate boundary conditions. Let $\Omega \subset \mathbb{R}^n$. Define the following "linear boundary operator":

$$B\colon C^2(\partial\Omega) \to C(\partial\Omega) \quad \text{by} \quad Bu := \alpha u\Big|_{\partial\Omega} + \beta\frac{\partial u}{\partial n}\Big|_{\partial\Omega}.$$

If $\Omega = (a, b)$, then $\partial\Omega = \{a\} \cup \{b\}$ (a disconnected set) so that B splits into two parts:

$$\begin{aligned} B_1u &= \alpha_{11}(a)u(a) - \beta_{11}(a)u'(a) + \alpha_{12}(b)u(b) - \beta_{12}(b)u'(b), \\ B_2u &= \alpha_{21}(a)u(a) + \beta_{21}(a)u'(a) + \alpha_{22}(b)u(b) + \beta_{22}(b)u'(b). \end{aligned}$$

Typically B_1 applies at $x = a$ and B_2 applies at $x = b$, in which case this reduces to

$$\begin{aligned} B_1 u &= \alpha_{11}(a)u(a) - \beta_{11}(a)u'(a) = \alpha_1(a)u(a) - \beta_1(a)u'(a), \\ B_2 u &= \alpha_{22}(b)u(b) + \beta_{22}(b)u'(b) = \alpha_2(b)u(b) + \beta_2(b)u'(b). \end{aligned}$$

Consider the following linear homogeneous boundary value problem with linear homogeneous boundary conditions:

$$\begin{aligned} Lu &= 0, \qquad x \in \Omega \subset \mathbb{R}^n, \\ Bu &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{B.3}$$

Solutions to this problem will lie in the set $\mathcal{M} := \{u \in C^2(\Omega) |\ Bu = 0\}$. It is easily shown that the set $\mathcal{M}$ is a vector subspace of $C^2(\Omega)$.
We now define the following set associated with $\mathcal{M}$:

$$\mathcal{M}^* := \{v \in C^2(\Omega) |\ J(u, v) = 0\ \forall u \in \mathcal{M}\}.$$

There exists an operator B^* such that $\mathcal{M}^*$ can be rewritten as

$$\mathcal{M}^* = \{v \in C^2(\Omega) |\ B^* v = 0\}.$$

We call B^* the **adjoint boundary** operator.

EXAMPLE B.13. Suppose $L\colon C^2(a, b) \to C(a, b)$ is defined by $L := d^2/dx^2$ and the boundary operator is given by

$$B_1 u = u'(a) - u(b), \quad B_2 u = u'(b).$$

Then $\mathcal{M}$ and $\mathcal{M}^*$ are given by

$$\begin{aligned} \mathcal{M} &= \{u \in C^2(a, b) |\ u'(a) = u(b),\ u'(1) = 0\}, \\ \mathcal{M}^* &= \{v \in C^2(a, b) |\ J(u, v) = 0\ \forall u \in \mathcal{M}\}. \end{aligned}$$

$$\begin{aligned} J(u, v) = \langle Lu, v\rangle - \langle u, Lv\rangle &= \int_a^b (u''v - uv'')\, dx \\ &= \int_a^b [(u'v)' - u'v' - (uv')' + u'v']\, dx = \int_a^b [(u'v)' - (uv')']\, dx \\ &= (u'v - uv')\Big|_a^b = u'(b)v(b) - u(b)v'(b) - u'(a)v(a) + u(a)v'(a). \end{aligned}$$

Thus we have

$$\begin{aligned} u \in \mathcal{M} \quad \Longrightarrow \quad J(u, v) &= u'(b)v(b) - u(b)v'(b) - u'(a)v(a) + u(a)v'(a) \\ &= 0 - u(b)v'(b) - u(b)v(a) + u(a)v'(a) \\ &= -u(b)[v'(b) + v(a)] + u(a)v'(a). \end{aligned}$$

Therefore

$$v \in \mathcal{M}^* \quad \Longrightarrow \quad J(u, v) = 0\ \forall u \in \mathcal{M} \quad \Longrightarrow \quad \begin{cases} v'(b) + v(0) = 0 \\ v'(a) = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} B_1^* v = v(a) + v'(b), \\ B_2^* v = v'(a), \end{cases}$$

and we get

$$\mathcal{M}^* = \{v \in C^2(\Omega) |\ v(a) + v'(b) = 0,\ v'(a) = 0\}.$$

The **adjoint problem** to (B.3) is defined as

$$L^*u = 0, \qquad x \in \Omega \subset \mathbb{R}^n,$$
$$B^*u = 0, \qquad x \in \partial\Omega.$$

A boundary value problem is said to be **self-adjoint** iff $L^* = L$ and $B^* = B$.

B.3 Finite Fourier Transforms

Suppose we have two linear differential operators K and L, where
K involves only time derivatives (either first or second order) and
L involves only spatial derivatives.

Consider the following nonhomogeneous PDE with nonhomogeneous initial and boundary conditions.

$$\begin{aligned} Ku + Lu &= F(x,t), && 3x = (x_1, x_2, \dots, x_n) \in \Omega,\ t > 0, \\ Bu &= g(x,t), && 3x \in \partial\Omega,\ t > 0, \\ u(x,0) &= f(x), && x \in \Omega, \\ u_t(x,0) &= \widehat{f}(x), && x \in \Omega. \quad \text{(if } K \text{ is of second order)} \end{aligned} \tag{B.4}$$

In addition, consider the following associated eigenvalue problem:

$$L\phi = \mu\phi,$$
$$B\phi = 0,$$

with an appropriate inner product $\langle \cdot, \cdot \rangle$. Denote the eigenvalues and normalized eigenfunctions by μ_k and ϕ_k, $(k = 1, 2, \dots)$ respectively. Then we have

$$L\phi_k = \mu_k\phi_k,$$
$$B\phi_k = 0,$$

where $\langle \phi_k, \phi_j \rangle = \delta_{kj}$.
We now look for a solution to (B.4) of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\phi_k(x).$$

Then u_k must satisfy $u_k = \langle u, \phi_k \rangle$. We call the collection $\{u_k\}_{k=1}^{\infty}$ the **finite Fourier transform** of u.
Expand F, f, and $\widehat{f}$ in eigenfunction expansions:

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t)\phi_k(x), \qquad F_k = \langle F, \phi_k \rangle\,,$$

$$f(x) = \sum_{k=1}^{\infty} f_k\phi_k(x), \qquad f_k = \langle f, \phi_k \rangle\,,$$

$$\widehat{f}(x) = \sum_{k=1}^{\infty} \widehat{f}_k\phi_k(x), \qquad \widehat{f}_k = \left\langle \widehat{f}, \phi_k \right\rangle.$$

We have

$$\begin{aligned}\langle Lu, \phi_k \rangle &= \langle u, L^* \phi_k \rangle + J(u, \phi_k) \\ &= \langle u, L\phi_k \rangle + J(u, \phi_k) \qquad \text{(assuming } L^* = L) \\ &= \langle u, \mu_k \phi_k \rangle + J(u, \phi_k) \\ &= \mu_k \langle u, \phi_k \rangle + J(u, \phi_k) \\ &= \mu_k u_k + H_k(t),\end{aligned}$$

where $H_k(t) := J(u, \phi_k)$. Going back to the original PDE $Ku + Lu = F$, we get

$$\begin{gathered}\langle Ku, \phi_k \rangle + \langle Lu, \phi_k \rangle = \langle F, \phi_k \rangle \\ K \langle u, \phi_k \rangle + \mu_k \langle u, \phi_k \rangle + J(u, \phi_k) = \langle F, \phi_k \rangle \\ Ku_k(t) + \mu_k u_k(t) = F_k(t) - H_k(t).\end{gathered}$$

The initial conditions become

$$\begin{aligned}u_k(0) &= \langle u(x,0), \phi_k(x) \rangle = \langle f(x), \phi_k(x) \rangle = f_k, \\ u_k'(0) &= \langle u_t(x,0), \phi_k(x) \rangle = \left\langle \widehat{f}(x), \phi_k(x) \right\rangle = \widehat{f}_k.\end{aligned}$$

Thus, we get the following sequence of ODEs for u_k:

$$\begin{aligned}Ku_k(t) + \mu_k u_k(t) &= F_k(t) - H_k(t), \\ u_k(0) &= f_k, \\ u_k'(0) &= \widehat{f}_k.\end{aligned} \tag{B.5}$$

EXAMPLE B.14. Consider the following nonhomogeneous heat equation:

$$\begin{aligned}u_t &= u_x x - \gamma^2 (u - c), \qquad 0 < x < \ell, \\ u(x,0) &= f(x), \qquad 9c, c_L, c_R, \gamma \text{ const.} \\ u(0,t) &= c_L, \qquad 1u(\ell,t) = c_R.\end{aligned}$$

This problem is of the form of problem (B.4) with

$$K = \frac{\partial}{\partial t}, \quad L = -\frac{\partial^2}{\partial x^2} + \gamma^2, \quad F(x,t) = \gamma^2 c, \quad Bu = \begin{cases} B_1 u = u(0,t) = c_L \\ B_2 = u(\ell,t) = c_R \end{cases}.$$

The associated eigenvalue problem is

$$\left.\begin{aligned}L\phi &= \mu\phi \\ B\phi &= 0\end{aligned}\right\} \quad \Longrightarrow \quad \begin{cases} -(\phi'' - \gamma^2 \phi) = \mu\phi \\ \phi(0) = \phi(\ell) = 0 \end{cases}.$$

This is easily solved to get

$$\begin{aligned}\mu_k &= \gamma^2 + \frac{k^2 \pi^2}{\ell^2}, \qquad 1k = 1, 2, \ldots, \qquad &&\text{(eigenvalues)} \\ \phi_k(x) &= \sqrt{\frac{2}{\ell}} \sin\left(\frac{k\pi x}{\ell}\right). &&\text{(eigenfunctions)}\end{aligned}$$

We have

$$J(u,v) = \langle Lu, v\rangle - \langle u, L^*v\rangle = \langle Lu, v\rangle - \langle u, Lv\rangle = \int_0^\ell (vLu - uLv)\,dx$$
$$= -\int_0^\ell (u_{xx}v - uv_{xx})\,dx = -\int_0^\ell [(u_xv)_x - (uv_x)_x]\,dx = -(u_xv - uv_x)\Big|_0^\ell$$
$$= -[u_x(\ell,t)v(\ell,t) - u(\ell,t)v_x(\ell,t) - u_x(0,t)v(0,t) + u(0,t)v_x(0,t)],$$

therefore

$$H_k(t) = J(u,\phi_k) = -[u_x(\ell,t)\phi_k(\ell) - u(\ell,t)\phi_k'(\ell) - u_x(0,t)\phi_k(0) + u(0,t)\phi_k'(0)]$$
$$= -\sqrt{\frac{2}{\ell}}\,\frac{k\pi}{\ell}[c_L - (-1)^k c_R],$$

$$F_k(t) = \langle F, \phi_k\rangle = \gamma^2 c\sqrt{\frac{2}{\ell}}\int_0^\ell \sin(\frac{k\pi x}{\ell})\,dx = \sqrt{\frac{2}{\ell}}\gamma^2 c\frac{\ell}{k\pi}[1 - (-1)^k],$$

$$f_k = \langle f, \phi_k\rangle = \sqrt{\frac{2}{\ell}}\int_0^\ell f(x)\sin(\frac{k\pi x}{\ell})\,dx =: \sqrt{\frac{2}{\ell}}\,a_k.$$

Eq. (B.5) for u_k becomes

$$u_k' + (\gamma^2 + \frac{k^2\pi^2}{\ell^2})u_k = \sqrt{\frac{2}{\ell}}\,\frac{\ell}{k\pi}\lambda_k, \quad k = 1, 2, \ldots,$$
$$u_k(0) = f_k,$$

where $\lambda_k = \gamma^2[1 - (-1)^k]c + \frac{k^2\pi^2}{\ell^2}[c_L - (-1)^k c_R]$. This ODE is easily solved to get

$$u_k(t) = \sqrt{\frac{2}{\ell}}\left\{\frac{\lambda_k}{\omega_k(\gamma^2 + \omega_k^2)} + \left(a_k - \frac{\lambda_k}{\omega_k(\gamma^2 + \omega_k^2)}\right)e^{-(\gamma^2 + \omega_k^2)t}\right\},$$

where $\omega_k := \frac{k\pi}{\ell}$. The final solution is

$$u(x,t) = \sum_{k=1}^\infty u_k(t)\phi_k(x) = \frac{2}{\ell}\sum_{k=1}^\infty \left\{\frac{\lambda_k}{\omega_k(\gamma^2 + \omega_k^2)} + \left(a_k - \frac{\lambda_k}{\omega_k(\gamma^2 + \omega_k^2)}\right)e^{-(\gamma^2 + \omega_k^2)t}\right\}\sin(\omega_k x).$$

EXAMPLE B.15. Consider the following problem governing the temperature of a circular plate:

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \qquad 0 < \rho < a,\ -\pi < \phi < \pi,\ t > 0,$$
$$u(a,\phi,t) = f(\phi),$$
$$u(\rho,\phi,0) = g(\rho,\phi).$$

As before, we consider the domain Ω in the $\rho\phi$-plane. The following extra conditions for the artificial boundaries are required:

A1: $u(\rho,\pi,t) = u(\rho,-\pi,t)$ and $\frac{\partial u}{\partial \phi}(\rho,\pi,t) = \frac{\partial u}{\partial \phi}(\rho,-\pi,t)$;

A2: $|u(\rho,\phi,t)|$ bounded as $\rho \to 0^+$.

This problem is of the form of problem (B.4) with

$$K = \frac{\partial}{\partial t}, \quad L = -k\nabla^2, \quad F(x,t) = \gamma^2 c, \quad Bu = u(a,\theta,t).$$

The associated eigenvalue problem is

$$\left.\begin{matrix} L\phi = \mu\phi \\ B\phi = 0 \end{matrix}\right\} \quad \Longrightarrow \quad \begin{cases} -k\nabla^2\phi = \mu\phi \\ u(a,\theta,t) = 0 \end{cases}.$$

This we solved earlier. The eigenfunctions are

$$\widehat{\phi}_{nm}(r,\theta) = \frac{\widehat{\psi}_{nm}(r,\theta)}{\|\widehat{\psi}_{nm}\|^2}, \qquad \widetilde{\phi}_{nm}(r,\theta) = \frac{\widetilde{\psi}_{nm}(r,\theta)}{\|\widetilde{\psi}_{nm}\|^2},$$

where

$$\widehat{\psi}_{nm}(r,\theta) = J_n(\lambda_{nm} r)\cos(n\theta), \quad \widetilde{\psi}_{nm}(r,\theta) = J_n(\lambda_{nm} r)\sin(n\theta), \quad \lambda_{nm} = \frac{\alpha_{nm}}{a}, \quad J_n(\alpha_{nm}) = 0.$$

We have

$$J(u,v) = \langle Lu, v\rangle - \langle u, Lv\rangle = -k \iint_\Omega [(\nabla^2 u)v - u(\nabla^2 v)]dA$$
$$= -k \iint_\Omega [\vec{\nabla} \cdot [v\vec{\nabla} u - u\vec{\nabla} v]dA = -k \int_{\partial\Omega} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds.$$

Using the fact that

$$\frac{\partial \widehat{\psi}_{nm}}{\partial r} = \lambda_{nm} J_n'(\lambda_{nm} r)\cos(n\theta), \qquad \frac{\partial \widetilde{\psi}_{nm}}{\partial r} = \lambda_{nm} J_n'(\lambda_{nm} r)\sin(n\theta),$$

we get

$$\widehat{H}_{nm} = J(u,\widehat{\phi}_{nm}) = -ka\int_{-\pi}^{\pi}[\widehat{\phi}_{nm}(a,\theta)\frac{\partial u}{\partial r}(a,\theta,t) - u(a,\theta,t)\frac{\partial \widehat{\phi}_{nm}}{\partial r}(a,\theta)]\,d\theta$$
$$= -ka\lambda_{nm}\frac{J_{n+1}(\alpha_{nm})}{\|\widehat{\psi}_{nm}\|^2}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta,$$

$$\widetilde{H}_{nm} = J(u,\widetilde{\phi}_{nm}) = -ka\int_{-\pi}^{\pi}[\widetilde{\phi}_{nm}(a,\theta)\frac{\partial u}{\partial r}(a,\theta,t) - u(a,\theta,t)\frac{\partial \widetilde{\phi}_{nm}}{\partial r}(a,\theta)]\,d\theta$$
$$= -ka\lambda_{nm}\frac{J_{n+1}(\alpha_{nm})}{\|\widetilde{\psi}_{nm}\|^2}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta,$$

$$\widehat{g}_{nm} = \left\langle g,\widehat{\phi}_{nm}\right\rangle = \iint_{\Omega} g(r,\theta)\widehat{\phi}_{nm}(r,\theta)\,dA = \frac{1}{\|\widehat{\psi}_{nm}\|^2}\int_{-\pi}^{\pi}\int_0^a g(r,\theta)J_n(\lambda_{nm}r)\cos(n\theta)\,r\,dr\,d\theta,$$
$$\widetilde{g}_{nm} = \left\langle g,\widetilde{\phi}_{nm}\right\rangle = \iint_{\Omega} g(r,\theta)\widetilde{\phi}_{nm}(r,\theta)\,dA = \frac{1}{\|\widetilde{\psi}_{nm}\|^2}\int_{-\pi}^{\pi}\int_0^a g(r,\theta)J_n(\lambda_{nm}r)\sin(n\theta)\,r\,dr\,d\theta.$$

We look for a solution of the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[\widehat{u}_{nm}(t)\widehat{\phi}_{nm}(r,\theta) + \widetilde{u}_{nm}(t)\widetilde{\phi}_{nm}(r,\theta)],$$

where

$$\widehat{u}_{nm} = \left\langle u,\widehat{\phi}_{nm}\right\rangle, \qquad \widetilde{u}_{nm} = \left\langle u,\widetilde{\phi}_{nm}\right\rangle.$$

Manipulating the equation we get

$$\begin{aligned} u_t + Lu = 0 \quad &\Longrightarrow \quad \left\langle u_t,\widehat{\phi}_{nm}\right\rangle + \left\langle Lu,\widehat{\phi}_{nm}\right\rangle \\ &\Longrightarrow \quad \frac{\partial}{\partial t}\left\langle u,\widehat{\phi}_{nm}\right\rangle + \left\langle u,L\widehat{\phi}_{nm}\right\rangle + J(u,\widehat{\phi}_{nm}) = 0 \\ &\Longrightarrow \quad \widehat{u}_{nm}' + k\lambda_{nm}^2\widehat{u}_{nm} + \widehat{H}_{nm} = 0. \end{aligned}$$

This is a first order ODE for $\widehat{u}_{nm}$, with initial condition $\widehat{u}_{nm}(0) = \widehat{g}_{nm}$. This is easily solved to giving

$$\widehat{u}_{nm}(t) = \left(\widehat{g}_{nm} + \frac{\widehat{H}_{nm}}{k\lambda_{nm}^2}\right)e^{-k\lambda_{nm}^2 t} - \frac{\widehat{H}_{nm}}{k\lambda_{nm}^2}.$$

A similar calculation for $\widetilde{\phi}_{nm}$ yields:

$$\widetilde{u}_{nm}(t) = \left(\widetilde{g}_{nm} + \frac{\widetilde{H}_{nm}}{k\lambda_{nm}^2}\right)e^{-k\lambda_{nm}^2 t} - \frac{\widetilde{H}_{nm}}{k\lambda_{nm}^2}.$$

The solution is now complete.

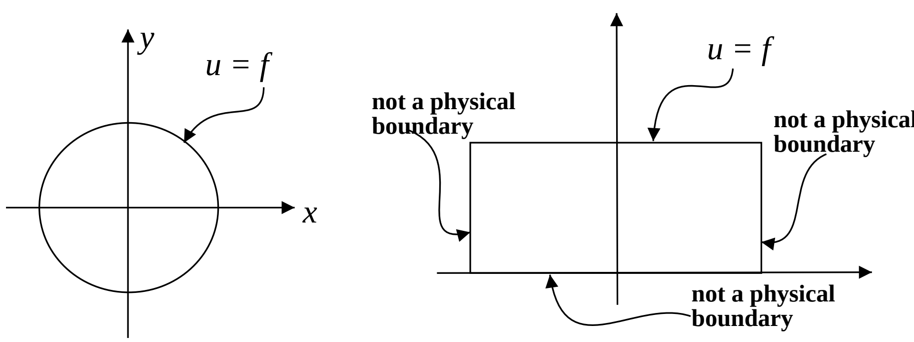


Figure B.1: Representations of Ω in the xy-plane and the $\rho\phi$-plane.

Appendix C

Gamma Function

We summarize a few properties of the gamma function. For $z \in \mathbb{C}$, with $\Re\{z\} > 0$, define

$$\Gamma_+(z) := \int_0^\infty e^{-t} t^{z-1}\, dt.$$

Then Γ_+ is analytic in $\{z \in \mathbb{C};\, \Re\{z\} > 0\}$. Integration by parts yields a recurrence relation

$$\Gamma_+(z+1) = \int_0^\infty e^{-t} t^{z}\, dt = -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t} t^{z-1}\, dt = z\Gamma_+(z). \quad \text{(using } u = t^z\text{)}$$

Repeated use of this recurrence relation yields

$$\Gamma_+(z+n) = (z+n-1)\cdots(z+1)z\Gamma_+(z) \quad \Longrightarrow \quad \Gamma_+(z) = \frac{\Gamma_+(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}.$$

This allows us to extend the definition to the right half plane by analytic continuation.

Definition.

$$\Gamma(z) := \begin{cases} \Gamma_+(z) & \Re\{z\} > 0, \\ \dfrac{\Gamma_+(z+n)}{z(z+1)(z+2)\cdots(z+n-1)} & -n < \Re\{z\} \leqslant -n+1, \qquad 1z \neq -n+1, \qquad 1n = 1,2,3,\ldots. \end{cases}$$

The gamma function Γ is analytic in the complex plane except at $z = 0, -1, -2, \ldots$ where it has simple poles. A graph of the gamma function for real values of its argument is given below.
We derive a couple more useful results. For $\alpha \in \mathbb{R}$, with $0 < \alpha < 1$, we have

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt = \int_0^\infty e^{-y^2} (y^2)^{\alpha-1} 2y\, dy = 2\int_0^\infty e^{-y^2} y^{2\alpha-1}\, dy. \quad \text{(using } t = y^2\text{)}$$

Therefore

$$\begin{aligned}
\Gamma(\alpha)\Gamma(1-\alpha) &= \left(2\int_0^\infty e^{-y^2} y^{2\alpha-1}\, dy\right)\left(2\int_0^\infty e^{-x^2} x^{1-2\alpha}\, dx\right)\\
&= 4\int_0^\infty \int_0^\infty \left(\frac{y}{x}\right)^{2\alpha-1} e^{-(x^2+y^2)}\, dx\, dy\\
&= 4\int_0^{\frac{\pi}{2}} \int_0^\infty (\tan\theta)^{2\alpha-1} e^{-r^2}\, r\, dr\, d\theta\\
&= s\int_0^{\frac{\pi}{2}} (\tan\theta)^{2\alpha-1}\, d\theta\\
&= \int_0^\infty \frac{\xi^{\alpha-1}}{1+\xi}\, d\xi = \frac{\pi}{\sin\pi\alpha}, \qquad \text{(using } \xi = \tan^2\theta\text{)}
\end{aligned}$$

where the last equality follows from residue theory. Evaluating at $\alpha = 1/2$ gives:

$$\Gamma(\frac{\pi}{2})\Gamma(\frac{\pi}{2}) = \frac{\pi}{\sin(\pi/2)} = \pi \quad \Longrightarrow \quad \Gamma(\frac{\pi}{2}) = \sqrt{\pi}.$$

Extending by analytic continuation yields

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

One last formula is the following:

$$\int_0^\infty e^{-t^\mu}\, dt = \frac{1}{\mu}\int_0^\infty e^{-\xi}\xi^{\frac{1}{\mu}-1}\, d\xi = \frac{1}{\mu}\Gamma(\frac{1}{\mu}) = \Gamma(\frac{1+\mu}{\mu}).$$

The main properties of the gamma function are summarized in

Theorem 63. *The gamma function satisfies the following:*

1. $\Gamma(z+1) = z\Gamma(z)$.
2. $\Gamma(1) = 1$.
3. $\Gamma(n+1) = n\Gamma(n) = n!, \quad$ *for* $n = 1, 2, 3, \ldots$.
4. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin\pi z}$.

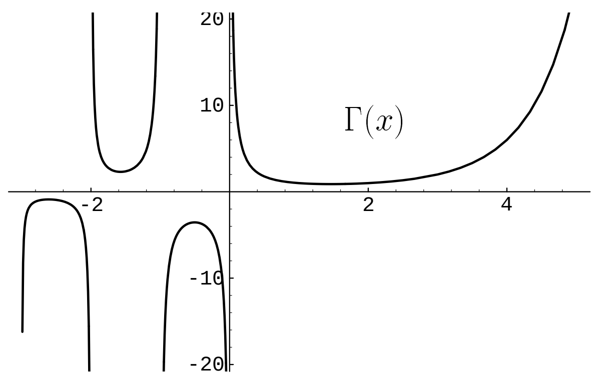


Figure C.1: A plot of $y = \Gamma(x)$.

Appendix D

Useful Formulas

1. Method of Characteristics for first order equations:

$$u(x,t) = u(x(t),t)$$

   First find $x(t)$, then find $u(x(t),t)$.

2. D'Alemberts formula for the wave equation on $(-\infty,\infty)$:

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

3. Fourier series on $[-L,L]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$$
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx$$

4. Fourier cosine series on $[0,L]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)$$
$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$
$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos(n\pi x/L)\,dx$$

5. Fourier sine series on $[0, L]$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

6. Separation of variables:

   (a) Write $u(x,t) = X(x)\cdot T(t)$.

   (b) Solve the Sturm-Liouville problem for $X(x)$.

   (c) Solve the corresponding time problem for $T(t)$.

   (d) Use superposition.

   (e) Use the initial conditions.

7. Laplacian in polar coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

8. Generalized eigenfunction expansion with a weight function $w(x)$:

$$f(x) = \sum_{i=1}^{\infty} \frac{\int_a^b f(t)\phi_i(t)w(t)\,dt}{\int_a^b \phi_i(t)^2 w(t)\,dt}\,\phi_i(x)$$

9. Fourier-integral formula on $(-\infty, \infty)$:

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)\,d\omega$$
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

10. Fourier-cosine-integral formula on $[0, \infty)$:

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$$
$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$

11. Fourier-sine-integral formula on $[0, \infty)$:

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega$$
$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

12. Fourier transform on $(-\infty, \infty)$:

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$
$$\mathcal{F}^{-1}(\hat{f})(x) = f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}\,d\omega$$

13. Fourier cosine transform on $[0, \infty)$:

$$\mathcal{F}_c(f)(\omega) = \hat{f}_c(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(x)\cos\omega x\,dx$$
$$f(x) = \int_{0}^{\infty} \hat{f}_c(\omega)\cos\omega x\,d\omega$$

14. Fourier sine transform on $[0, \infty)$:

$$\mathcal{F}_s(f)(\omega) = \hat{f}_s(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(x)\sin\omega x\,dx$$
$$f(x) = \int_{0}^{\infty} \hat{f}_s(\omega)\sin\omega x\,d\omega$$

15. Gauss kernel

$$g(x,t) = \frac{1}{\sqrt{2Dt}}\,e^{-\frac{x^2}{4Dt}}.$$

16. Error function

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_{0}^{z} e^{-x^2}\,dx.$$