CONFORMAL CONNECTIONS AND CONFORMAL TRANSFORMATIONS

BY

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Introduction. In the previous paper [5], we have studied the groups of projective transformations of affinely connected manifolds by the application of the theory of normal projective connections. The main purpose of the present paper is to study the conformal properties of complete Riemannian spaces of some special type by the application of the theory of normal conformal connections.

We shall introduce a family of Riemannian spaces, the elements of which are characterized by conditions on the Ricci tensor fields and will be called of type $\mathfrak{S}$ (Definition 4). Einstein spaces are of type $\mathfrak{S}$ and, in general, a Riemannian space of type $\mathfrak{S}$ is locally isomorphic with the direct product of two Einstein spaces of different signs. Our main result (Theorem 1) states that two conformally equivalent Riemannian metrics on a manifold which are complete and of type $\mathfrak{S}$ necessarily coincide, except the case where either of the two metrics is an Einstein metric with the vanishing or negative Ricci tensor field. Some results are also obtained in the exceptional case. We see from this theorem that the group of all conformal transformations of a complete Riemannian space of type $\mathfrak{S}$ coincides with the group of all isometries, except the case of an Einstein space with the vanishing or negative Ricci tensor field (Theorem 2).

The proof of Theorem 1 is based on the theory of normal conformal connections; it has been suggested by a result of K. Yano and S. Sasaki [8] stating that the conformal holonomy group of a Riemannian space of type $\mathfrak{S}$ fixes a point or a sphere. In §§1–4, we shall show how to construct the normal conformal connection when an arbitrary class of conformally equivalent Riemannian metrics is given. The conditions for the normal conformal connection are exhibited in Proposition 5.

Finally we mention that our formulation of the normal conformal connection is efficacious for many other problems concerning the group of conformal transformations or the conformal holonomy group, for example, for such a problem as is treated in [8], cited above. We want to take up these problems at another occasion.

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Remarks about notations and terminologies. Throughout this paper, we follow, in principle, the notations and terminologies adopted in the author's

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paper [5]. By a manifold (a mapping) we shall always mean that of class C^∞.

Let M be a manifold. We shall denote by M_p the tangent vector space to M at a point p of M and by $\mathfrak{X}(M)$ the set of all vector fields on M. $\mathfrak{X}(M)$ may be considered as a vector space over the field of all real numbers.

Let M and N be two manifolds and let f be a mapping of M into N. For each tangent vector X to M, fX denotes the result of applying the differential of f to X (in Chevalley's notation, $fX = df(X)$). Now, assume that N is a Lie group. For each $p \in M$ and $X \in M_p$, we denote by $f^{-1}df(X)$ the result of applying the differential of the mapping $M \ni q \to f^{-1}(p)f(q) \in N$ to X. Then, $f^{-1}df(X)$ is naturally identified with an element of the Lie algebra $\mathfrak{n}$ of N and the mapping $X \to f^{-1}df(X)$ defines an $\mathfrak{n}$-valued 1-form on M, which we denote by $f^{-1}df$.

Let N be a manifold and let $u(t)$ be a curve in M. We denote by $u'(t)$ the tangent vector to $u(t)$ at t. In the case where N is a Lie group, we denote by $u(t)^{-1}u'(t)$ the result of applying the differential of the left translation $\sigma \to u(t)^{-1}\sigma$ to $u'(t)$. If we set $X(t) = u(t)^{-1}u'(t)$, then $X(t)$ may be considered as a curve in the Lie algebra $\mathfrak{n}$ of N. Conversely, for each curve $X(t)$ in $\mathfrak{n}$ there exists one and only one curve $u(t)$ in N such that $u(t)^{-1}u'(t) = X(t)$ and $u(0) = e$ (the unit element of N) [6, p. 29]. In §§5 and 6, we make use of this fact.

Let $P(M, G)$ be a principal fiber bundle over a base space M with structure group G. For each $\sigma \in G$, we denote by R_σ the right translation of P onto itself which corresponds to σ. For each $z \in P$, G_z denotes the subspace of P_z which is tangent to the fiber through z. Let $\mathfrak{g}$ be the Lie algebra of G. For each $A \in \mathfrak{g}$, A^* denotes the vector field on P induced by the one-parameter group $R_{a(t)}$, where $a(t) = \exp tA$. Then, G_z is equal to the subspace of P_z composed of all the elements A^*_z where A runs over $\mathfrak{g}$.

Let $P(M, G)$ and $P'(M, G')$ be two principal fiber bundles over the same base space M. Let f be a homomorphism of G into G'. A mapping f of P into P' is called a homomorphism corresponding to the homomorphism f of G into G', if, for each $z \in P$ and $\sigma \in G$, $f(z \cdot \sigma) = f(z) \cdot f(\sigma)$ and if it induces the identity transformation of M onto itself. In the case where G is a subgroup of G', a homomorphism f of P into P' corresponding to the injection of G into G' is called an injection. If we identify P with a submanifold of P' by the injection, we shall say that P is a subbundle of P' or P is contained in P'.

1. **Möbius space.** In this section, we define the Möbius space as a homogeneous space and study the homogeneous structure from an infinitesimal point of view.

(1.1) Let F_{n+2} be an $(n+2)$-dimensional vector space over the field R of all real numbers. We consider a fixed decomposition of F_{n+2}:

$$F_{n+2} = F_0 + F_n + F_\infty,$$

where F_0 and F_∞ are 1-dimensional subspaces of F_{n+2} and F_n is an n-dimensional subspace of F_{n+2}. Choose, once for all, bases ξ_0 and ξ_∞ in F_0 and F_∞

respectively. Denote by F_n^* the dual space of F_n and by $\langle \xi, E\rangle$ the product between $\xi \in F_n$ and $E \in F_n^*$.

(1.2) We consider an inner product $\phi(X, Y)$ in F_{n+2} satisfying the following conditions:

(i) $\phi(\xi_0, \xi_0) = \phi(\xi_\infty, \xi_\infty) = \phi(\xi_0, \xi) = \phi(\xi_\infty, \xi) = 0, \phi(\xi_0, \xi_\infty) = -1$ where $\xi \in F_n$;

(ii) The restriction of ϕ to F_n is positive definite. We shall denote by ϕ the linear isomorphism of F_n onto F_n^* defined by $\langle \xi, \phi(\xi')\rangle = \phi(\xi, \xi')$. Let P_{n+1} be the $(n+1)$-dimensional projective space constructed from F_{n+2}. We denote by M_n the quadric in P_{n+1} defined by the quadratic equation in F_{n+2}: $\phi(X, X) = 0$. We denote by C the cone in $F_{n+2}-(0)$ defined by the same equation and by ω the projection of C onto M_n. We set $o = \omega(\xi_0)$ and $\infty = \omega(\xi_\infty)$.

(1.3) Let N be the subgroup of the general linear group of F_{n+2} which leaves the inner product $\phi(X, Y)$ invariant and let ω be the projection of N into the projective group of P_{n+1}. If we set $M(n) = \omega(N)$, we see that $M(n)$ leaves M_n invariant; more precisely, $M(n)$ operates on M_n as follows:

$$\omega(\sigma)\omega(X) = \omega(\sigma X) \quad \text{for all } \sigma \in N \text{ and } X \in C.$$

An important result is that $M(n)$, considered as a transformation group on M_n, is effective and transitive on M_n. Thus M_n may be represented as a homogeneous space

$$M_n = M(n)/M'(n),$$

where $M'(n)$ denotes the isotropy group of $M(n)$ at o. We call this homogeneous space the n-dimensional Möbius space. It is homeomorphic with the n-dimensional sphere. $M(n)$ is called the Möbius group of dimension n.

(1.4) Let $O(n)$ be the orthogonal group of F_n with respect to the inner product ϕ and let $S(n)$ be the similarity group of F_n. Let R_+ be the multiplicative group of all positive real numbers. We know that $S(n)$ is the subgroup of the general linear group of F_n composed of all the elements $\lambda\sigma$ where $\lambda \in R_+$ and $\sigma \in O(n)$. If we identify R_+ with a subgroup of $S(n)$ by the isomorphism $R_+ \ni \lambda \rightarrow \lambda 1 \in S(n)$, where 1 denotes the unit element of $S(n)$, then we have the following expression

$$S(n) = O(n) \cdot R_+ \text{ (direct product)}.$$

We denote by s and ρ the corresponding projections of $S(n)$ onto $O(n)$ and R_+ respectively.

(1.5) Now we construct a Lie group, denoted by $L(S(n), F_n^*)$, from $S(n)$ and F_n^* as follows: The underlying manifold is equal to $S(n) \times F_n^*$ and the operation of multiplication is given by $(\tau, E)\cdot(\tau', E') = (\tau\tau', {}^t\tau' E + E')$, where $\tau, \tau' \in S(n)$ and $E, E' \in F_n^*$. Namely, $L(S(n), F_n^*)$ is identical with the so-called abelian extension of $S(n)$ with respect to the star representation of $S(n)$. We show that $L(S(n), F_n^*)$ may be identified with $M'(n)$. Indeed, we define an isomorphism ψ of $L(S(n), F_n^*)$ into N by $\psi(\sigma)=\tilde{\tau}\cdot\tilde{E}$ for all $\sigma=(\tau, E)$ $\in S(n)\times F_n^*$, where $\tilde{\tau}$ and $\tilde{E}$ denote the elements of N defined respectively as follows:

$$\tilde{\tau}\xi_0 = \rho(\tau)^{-1}\xi_0,\ \tilde{\tau}\eta = s(\tau)\eta,\ \tilde{\tau}\xi_\infty = \rho(\tau)\xi_\infty;$$

$$\tilde{E}\xi_0 = \xi_0,\ \tilde{E}\eta = \langle\eta, E\rangle\xi_0 + \eta,\ \tilde{E}\xi_\infty = 2^{-1}\langle\phi^{-1}E, E\rangle + \phi^{-1}E + \xi_\infty,$$

where $\eta\in F_n$. We see easily that $\omega\circ\psi$ gives an isomorphism of $L(S(n), F_n^*)$ with $M'(n)$. In the following, we identify $S(n)$ with a subgroup of $M'(n)$ by the isomorphism $S(n)\ni\tau\rightarrow(\tau, 0)\in M'(n)$ and we write $\exp E=(1, E)$. With this notation, we see that every element σ of $M'(n)$ is expressed uniquely as

$$\sigma = l(\sigma)\cdot\exp E(\sigma),$$

where $l(\sigma)$ is in $S(n)$ and $E(\sigma)$ in F_n^*. The correspondence $\sigma\rightarrow l(\sigma)$ gives a homomorphism of $M'(n)$ onto $S(n)$. Finally we have the following diagram:

$$\begin{array}{ccccc} M(n) & \xrightarrow{\text{inj.}} & M'(n) & \xrightarrow{l} & S(n) \\ & & & \underset{\text{inj.}}{\nwarrow} & s\downarrow\uparrow\text{inj.} \\ & & & & O(n) \end{array}$$

where inj. means injection and where $s\circ l(\sigma)=\sigma$ for all $\sigma\in O(n)$.

(1.6) Let $\mathfrak{s}(n)$ be the Lie algebra of $S(n)$ which is identified with a subalgebra of the Lie algebra of all endomorphisms of F_n. Then, corresponding to the decomposition of $S(n)$ given in (1.4), we have the decomposition of $\mathfrak{s}(n)$:

$$\mathfrak{s}(n) = \mathfrak{o}(n) + R.$$

For each $A\in\mathfrak{s}(n)$, we denote by $A_{\mathfrak{o}(n)}$ and A_R the $\mathfrak{o}(n)$- and R-components of A respectively.

(1.7) In what follows, we study the Lie algebra of $M(n)$. Let $\mathfrak{n}$ be the Lie algebra of N which is identified with the Lie algebra composed of all the endomorphisms of F_{n+2} which leave the inner product ϕ in F_{n+2} invariant. First of all, $\mathfrak{n}$ may be identified with the Lie algebra of $M(n)$, because the kernel of ω: $N\rightarrow M(n)$ is a discrete subgroup of N. Now consider the formal direct sum $\mathfrak{m}(n)$ of the three vector spaces F_n, $\mathfrak{s}(n)$ and F_n^*:

$$\mathfrak{m}(n) = F_n + \mathfrak{s}(n) + F_n^*.$$

We now define an isomorphism ψ of $\mathfrak{m}(n)$ onto $\mathfrak{n}$ by the following formulae:

$$\psi(A)\xi_0 = -S_R\xi_0 + \xi, \qquad \psi(A)\xi_\infty = \phi^{-1}E + S_R\xi_\infty,$$

$$\psi(A)\eta = \langle\eta, E\rangle\xi_0 + S_{\mathfrak{o}(n)}\eta + \phi(\xi, \eta)\xi_\infty,$$

where $A=\xi+S+E$, $\xi\in F_n$, $S\in\mathfrak{s}(n)$, $E\in F_n^*$ and $\eta\in F_n$. ψ being an isomorphism of $\mathfrak{m}(n)$ onto $\mathfrak{n}$, we can transfer the structure of the Lie algebra to $\mathfrak{m}(n)$ in such a way that ψ becomes an isomorphism of Lie algebras: The bracket operation of $\mathfrak{m}(n)$ is given as follows:

(i) For $\xi, \xi'\in F_n$, $A, B\in\mathfrak{s}(n)$ and $E, E'\in F_n^*$, $[\xi, \xi']=0$, $[E, E']=0$, $[A, B]=AB-BA$, $[A, \xi]=A\xi$, $[A, E]=-{}^tAE$;

(ii) For $\xi\in F_n$ and $E\in F_n^*$, $[\xi, E]$ belongs to $\mathfrak{s}(n)$: $[\xi, E]_R=\langle\xi, E\rangle$; $[\xi, E]_{\mathfrak{o}(n)}=\langle\eta, E\rangle\xi-\phi(\xi, \eta)\phi^{-1}E$,

where η is in F_n. In the following, we identify $\mathfrak{m}(n)$, provided with this bracket operation, with the Lie algebra of $M(n)$.

(1.8) We have $\exp A=\omega(\exp\psi(A))$, that is,

$$\exp A\cdot\omega(X) = \omega(\exp\psi(A)\cdot X) \qquad \text{for all } A\in\mathfrak{m}(n) \text{ and } X\in C.$$

In particular, we have $\exp\xi\cdot o=\omega(\xi_0+\xi+\phi(\xi, \xi)/2\xi_\infty)$. If we set $E_n=M_n-\{\infty\}$, the mapping $\xi\to\exp\xi\cdot o$ gives a one-to-one correspondence of F_n with E_n. This mapping is known as the stereographic projection of the Euclidean space into the sphere. Finally we remark that the notation exp, introduced in (1.5), is legitimate, because we have $\exp\psi(E)=\psi((1, E))$ for each $E\in F_n^*$.

(1.9) The Lie algebra $\mathfrak{m}'(n)$ of $M'(n)$ is given by

$$\mathfrak{m}'(n) = \mathfrak{o}(n) + R + F_n^*.$$

(1.10) The decomposition of $\mathfrak{m}(n)$

$$\mathfrak{m}(n) = F_n + \mathfrak{m}'(n)$$

is fundamental for our later considerations. For each $A\in\mathfrak{m}(n)$, we denote by A_{F_n} and $A_{\mathfrak{m}'(n)}$ the F_n- and $\mathfrak{m}'(n)$-components of A respectively. We have the following formulae on the adjoint representation of $M'(n)$ in $\mathfrak{m}(n)$:

$$\text{ad }\sigma\xi = \sigma\xi;\ \text{ad }\sigma E = {}^t\sigma^{-1}E;\ \ \text{ad }(\exp E)A = A + [E, A];$$
$$\text{ad }(\exp E)\xi = \xi + [E, \xi] + 2^{-1}[E, [E, \xi]],$$

where $\sigma\in S(n)$, $\xi\in F_n$, $A\in\mathfrak{s}(n)$ and $E\in F_n^*$. It follows that, for each $\sigma\in M'(n)$ and $\xi\in F_n$, $l(\sigma)=(\text{ad }\sigma\xi)_{F_n}=\text{ad }l(\sigma)\xi$, from which we see that l may be considered as the homomorphism of the isotropy group $M'(n)$ of $M(n)$ at o onto the linear isotropy group.

(1.11) Let J be a linear mapping of F_n into F_n^*. We now define a bilinear mapping $\tilde{J}$ of $F_n\times F_n$ into $\mathfrak{gl}(F_n)$ (the Lie algebra of all endomorphisms of F_n) by

$$\tilde{J}(\xi, \xi')\eta = -([\xi, J(\eta)] + [J(\xi), \eta])_{\mathfrak{o}(n)}\xi' \qquad \text{for all } \eta\in F_n.$$

Later on, we shall use

LEMMA 1. *For any bilinear function $Q(\xi, \xi')$ on $F_n\times F_n$, there exists one and only one linear mapping J of F_n into F_n^* such that*

$$\mathrm{Tr}(\tilde{J}(\xi, \xi')) = Q(\xi, \xi') \qquad \textit{for all } \xi, \xi' \in F_n. \tag{1}$$

More precisely, J is given by

$$\langle \xi, J(\xi') \rangle = \frac{1}{n-2}\left(Q(\xi, \xi') - \frac{Q_0 \cdot \phi(\xi, \xi')}{2(n-1)}\right), \tag{2}$$

where we have set $Q_0 = \sum_{i=1}^{n} Q(\xi_i, \xi_i)$, (ξ_i) *being an orthogonal base of* F_n *with respect to the inner product* ϕ.

2. **Orthogonal bundles and conformal $S(n)$-bundles.** In the following, we shall denote by M a connected manifold which satisfies the second countability axiom, and we always assume that the dimension n of M is $\geqq 3$.

We shall say that two Riemannian metrics g and $\bar{g}$ on a manifold M are conformally equivalent, if there exists a positive function λ on M such that

$$\bar{g} = \lambda^2 g.$$

λ will be called the associated function of $\bar{g}$ with respect to g. Now, fix a Riemannian metric g on M. A transformation f of M onto itself is, by definition, a conformal transformation of g, if f^*g and g are conformally equivalent. The group $C(g)$ of all the conformal transformations of g contains the group $I(g)$ of all the isometries of g.

Let M be a manifold and let P_L be the bundle of frames of M. Then, P_L may be considered as a principal fiber bundle over the base space M with the general linear group of F_n as structure group. Each element x of P_L gives a linear isomorphism of F_n with M_p such that $(x\cdot\sigma)\cdot\xi = x\cdot(\sigma\xi)$ for all $\sigma \in GL(n)$ and $\xi \in F_n$, where we have set $\pi_L(x) = p$, π_L being the projection of P_L onto M.

We know that to each Riemannian metric on M there corresponds a principal fiber bundle P_0 over the base space M with structure group $O(n)$ which is a subbundle of P_L. Such a bundle is usually called an orthogonal bundle. Given a Riemannian metric g on M, P_0 is defined as the subset of P_L composed of all the elements x such that $g(x\cdot\xi, x\cdot\xi') = \phi(\xi, \xi')$ for all $\xi, \xi' \in F_n$.

DEFINITION 1. *A conformal $S(n)$-bundle over a manifold M is a principal fiber bundle over the base space M with structure group $S(n)$ which is a subbundle of the bundle of frames of M.*

We first show that to each class $\mathfrak{C}$ of conformally equivalent Riemannian metrics on M there corresponds a conformal $S(n)$-bundle P_S over M. Let $\mathfrak{C}$ be a class of conformally equivalent Riemannian metrics. Fixing an element g of $\mathfrak{C}$, P_S is defined as the subset of P_L composed of all the elements x as follows: There exists a real number ρ such that

$$g(x\cdot\xi, x\cdot\xi') = \rho\cdot\phi(\xi, \xi') \qquad \text{for all } \xi, \xi' \in F_n.$$

Clearly P_S does not depend on the choice of g and the structure of P_S as a principal fiber bundle is naturally induced from that of P_L. From the definition of P_S, we see that the assignment $g \rightarrow P_0$ gives a one-to-one correspondence of $\mathfrak{C}$ with the set of all the orthogonal bundles over M which are contained in P_S. In the same way, the assignment $\mathfrak{C} \rightarrow P_S$ gives a one-to-one. correspondence of the set of all classes of conformally equivalent Riemannian metrics with the set of all the conformal $S(n)$-bundles.

Let P_S be a conformal $S(n)$-bundle over a manifold M and let P_0 be an arbitrary orthogonal bundle contained in P_S. We define a homomorphism s of P_S onto P_0 corresponding to the homomorphism s of $S(n)$ onto $O(n)$ defined in (1.4), by the requirement that $s(x)=x$ for all $x \in P_0$. Now consider a second Riemannian metric $\bar{g}$ which is conformally equivalent to g. Let $\bar{P}_0$ be the corresponding orthogonal bundle and let $\bar{s}$ be the corresponding homomorphism of P_S onto $\bar{P}_0$. Let λ be the associated function of $\bar{g}$ with respect to g. Then, we have

(2.1) $$s(x) = \bar{s}(x) \cdot \lambda \circ \pi_S(x) \qquad \text{for all } x \in P_S,$$

where π_S denotes the projection of P_S onto M and $\lambda \circ \pi_S(x)$ is identified with an element of $R_+ \subset S(n)$.

In the following, we shall give some definitions and formulations about an orthogonal bundle and connections in it which we need in the subsequent sections.

Let P_0 be an orthogonal bundle over a manifold M. An affine connection in P_0 (in the sense of Cartan connection) is usually called a Euclidean connection in P_0. This may be formulated as follows [5, p. 6]: A Euclidean connection in P_0 is a linear mapping B_0 of F_n into $\mathfrak{X}(P_0)$ which satisfies the following conditions:

(E.1) $P_{0x} = B_{0x} + O(n)_x$, where B_{0x} denotes the subspace of P_{0x} composed of all the elements $B_0(\xi)_x$ where $\xi \in F_n$;

(E.2) $R_\sigma B_0(\xi) = B_0(\sigma^{-1}\xi)$;

(E.3) $\theta B_0(\xi)_x = \xi$, where θ denotes the F_n-valued 1-form on P_0 defined by $\theta(X) = x^{-1} \cdot \pi_0 X$ for all $X \in P_{0x}$ and $x \in P_0$.

The assignment $x \rightarrow B_{0x}$ defines a connection in P_0, which we call the linear connection in P_0 associated with the Euclidean connection B_0. By (E.1), there exist, for each $x \in P_0$, bilinear mappings T_x and R_x of $F_n \times F_n$ into F_n and $\mathfrak{o}(n)$ respectively such that

$$-[B_0(\xi), B_0(\xi')]_x = B_0(T_x(\xi, \xi'))_x + R_x(\xi, \xi')^*_x.$$

T_x (resp. R_x) corresponds to the torsion (resp. the curvature) tensor field of the Euclidean connection. There exists one and only one Euclidean connection in P_0, called the Riemannian connection in P_0, such that $T_x = 0$ for all $x \in P_0$.

Let B_0 be a Euclidean connection in P_0. We now define, for each $x \in P_0$, a bilinear function S_x on $F_n \times F_n$ by

$$S_x(\xi, \xi') = \mathrm{Tr}(\tilde{R}_x(\xi, \xi')),$$

where $\tilde{R}_x$ denotes the bilinear mapping of $F_n \times F_n$ into $\mathfrak{gl}(F_n)$ defined by $\tilde{R}_x(\xi, \xi')\eta = R_x(\xi, \eta)\xi'$ for all $\eta, \xi, \xi' \in F_n$. If we denote by $S(X, Y)$ the ordinary Ricci tensor field of the Euclidean connection, then we have

$$S_x(\xi, \xi') = S(x\cdot\xi, x\cdot\xi').$$

Finally, we define, for each $x \in P_0$, a linear mapping J_x of F_n into F_n^* by the formula

$$(2.2) \qquad \langle \xi, J_x(\xi')\rangle = \frac{1}{n-2}\left(S_x(\xi, \xi') - \frac{S_0(x)}{2(n-1)}\phi(\xi, \xi')\right),$$

where $S_0(x) = \sum_{i=1}^n S_x(\xi_i, \xi_i)$, (ξ_i) being an orthogonal base of F_n. S_0 corresponds to what is usually called the scalar curvature.

3. **Conformal $M'(n)$-bundle associated with a conformal $S(n)$-bundle.**

PROPOSITION 1. *For each conformal $S(n)$-bundle P_S over a manifold M there exists a collection (P', l, α) as follows: P' is a principal fiber bundle over the base space M with structure group $M'(n)$; l is a homomorphism of P' onto P_S corresponding to the homomorphism l of $M'(n)$ onto $S(n)$ defined in* (1.5); *α is an R-valued 1-form on P' having the following two properties*:

(i) *$\alpha(A^*) = A_R$ for all $A \in \mathfrak{m}'(n)$, where A_R denotes the R-component of A in the decomposition of $\mathfrak{m}'(n)$ given in* (1.9);

(ii) *$R_\sigma^*\alpha = \alpha + [\mathrm{ad}\, l(\sigma)^{-1}\theta, E(\sigma)]_R$ for all $\sigma \in M'(n)$, where θ denotes the F_n-valued 1-form on P' defined by $\theta(X) = l(z)^{-1}\pi X$ for each $X \in P'_z$ and $z \in P'$, π being the projection of P' onto M, and where $\sigma = l(\sigma)\cdot\exp E(\sigma)$, $E(\sigma) \in F_n^*$.*

Proof. Fix an orthogonal bundle $P_0 \subset P_S$ and let s be the homomorphism of P_S onto P_0 defined in §2. By making use of P_0 and the injection of $O(n)$ into $M'(n)$, we get a principal fiber bundle P' over the base space M with structure group $M'(n)$ together with an injection h of P_0 into P'. Next, we define l to be the homomorphism of P' onto P_S such that $l \circ h(x) = x$ for all $x \in P_0$. α is defined as follows: Since, for each $z \in P'$, z and $h \circ s \circ l(z)$ lie in the same fiber of P', it follows that there is a mapping a of P' into $M'(n)$ such that $z = h \circ s \circ l(z)\cdot a(z)$. By applying $s \circ l$ to this formula and using the fact that $l \circ h(x) = x$ for all $x \in P_0$, we have $s(a(z)) = e$, from which it follows that there exist mappings ρ and E of P' into R_+ and F_n^* respectively such that $a(z) = \rho(z)\cdot\exp E(z)$. Now define α to be $\alpha = \rho^{-1}d\rho + [\theta, E]_R$. That the form α, just defined, has the properties (i) and (ii) follows from the following Lemma 2.

LEMMA 2.
(i) $\rho(z\cdot\sigma) = \rho(z)\cdot\rho \circ l(\sigma)$, $E(z\cdot\sigma) = \mathrm{ad}\, l(\sigma)^{-1}E(z) + E(\sigma)$;
(ii) $R_\sigma^*\theta = \mathrm{ad}\, l(\sigma)^{-1}\theta$, $\theta(A^*) = 0$.

LEMMA 3. *Let V be a manifold and let f and g be two mappings of V into P'. Assume that there exist mappings ρ and E of V into $R_+ \subset M'(n)$ and $F_n^* \subset \mathfrak{m}'(n)$ respectively such that*

$$f(u) = g(u)\cdot\rho(u)\cdot\exp E(u) \qquad \text{for all } u\in V.$$

Then, for each $u\in V$ *and* $X\in V_u$, *we get*

$$fX = R_{a(u)}\circ gX + (\rho^{-1}d\rho(X))^*_{f(u)} + ([\rho^{-1}d\rho(X), E(u)] + b^{-1}db(X))^*_{f(u)},$$

where $a(u)=\rho(u)\cdot\exp E(u)$ *and* $b(u)=\exp E(u)$. $\rho^{-1}d\rho$ *(resp.* $b^{-1}db$*) is an* R-(resp. F_n^*-) *valued* 1*-form on* V.

PROPOSITION 2. *Let* (P', l, α) *be a collection having the properties in Proposition* 1. *Then, for each orthogonal bundle* $P_0\subset P_S$ *there exists one and only one injection* h *of* P_0 *into* P' *such that*

(i) $l\circ h(x)=x$ *for all* $x\in P_0$;

(ii) $h^*\alpha=0$.

Proof. We first show that there exists at least one injection, say h_0, of P_0 into P' such that $l\circ h_0(x)=x$ for all $x\in P_0$. For this purpose, we introduce a fiber bundle, denoted by $P'/S(n)$, over M, which is the quotient space by the following equivalence relation $\sim$ in P': $z'\sim z$ if and only if there is a $\tau\in S(n)$ such that $z'=z\cdot\tau$. The standard fiber of $P'/S(n)$ is given by $M'(n)/S(n)$ (the space of left cosets of $M'(n)$ modulo $S(n)$), which is obviously homeomorphic to a Euclidean space; hence $P'/S(n)$ admits a (differentiable) section g over M. Now denoting by p the projection of P' onto $P'/S(n)$, we see that, for each $x\in P_0$, there is a unique element $h_0(x)$ of P' such that $l(h_0(x))=x$ and $p(h_0(x))=g(\pi x)$; clearly the correspondence $x\to h_0(x)$ defines an injection of P_0 into P' such that $l\circ h_0(x)=x$ for all $x\in P_0$, which proves our assertion.

Now we shall prove the existence of h having the properties (i) and (ii). Let B_0 be the Riemannian connection in P_0. Defining a mapping E of P_0 into F_n^* by $\langle\xi, E(x)\rangle=[\xi, E(x)]_R=-h^*\alpha(B_0(\xi)_x)$ for all $x\in P_0$ and $\xi\in F_n$, we define h by $h(x)=h_0(x)\cdot\exp E(x)$. First of all, we see that $l\circ h(x)=x$ for all $x\in P_0$. By applying Lemma 3 to the case where $V=P_0$, $f=h$, $g=h_0$ and $X=B_0(\xi)_x$, we have

$$hB_0(\xi)_x = R_{b(x)}\circ h_0B_0(\xi)_x + (b^{-1}db(B_0(\xi)_x))^*_{h(x)},$$

where $b(x)=\exp E(x)$. Applying α to this formula and using the properties of α and the fact that $h_0^*\theta(B_0(\xi)_x)=\xi$, we have

$$h^*\alpha(B_0(\xi)_x) = h_0^*\alpha(B_0(\xi)_x) + [\xi, E(x)]_R = 0; \tag{3.1}$$

furthermore, we have $h^*\alpha(A_x^*)=A_R=0$ for all $A\in\mathfrak{o}(n)$. It follows immediately that $h^*\alpha=0$.

Now assume that there exist two injections h and h_0 which both have the properties (i) and (ii). It follows from property (i) that there is a mapping E of P_0 into F_n^* such that $h(x)=h_0(x)\cdot\exp E(x)$ for all $x\in P_0$. Then, by the first equality of (3.1), we see that $\langle\xi, E(x)\rangle=0$ for all $\xi\in F_n$ and $x\in P_0$, that is, $E(x)=0$, whence $h=h_0$. q.e.d.

PROPOSITION 3. *Let (P', l, α) be a collection having the properties in Proposition 1. Let P_0 be an orthogonal bundle contained in P_S and let h be the corresponding injection of P_0 into P' whose existence is assured by Proposition 2. Then, there exist mappings ρ and E of P' into R_+ and F_n^* respectively such that*

$$z = h \circ s \circ l(z) \cdot \rho(z) \cdot \exp E(z)$$

and, using these ρ and E, α is expressed as

$$\alpha = \rho^{-1}d\rho + [\theta, E]_R.$$

Proof. By applying Lemma 3 to the case where $V = P'$, $f =$ the identity transformation of P' onto itself, $g = h \circ s \circ l$ and $u = z$, we obtain

$$X = R_{a(z)} \circ h \circ s \circ lX + (\rho^{-1}d\rho(X))_z^* + ([\rho^{-1}d\rho(X), E(z)] + b^{-1}db(X))_z^*.$$

Applying α to this formula and using the properties of α and the fact that $h^*\alpha = 0$, we have $\alpha(X) = [\mathrm{ad}\, \rho(z)^{-1}\theta(h \circ s \circ lX),\ E(z)]_R + \rho^{-1}d\rho(X)$. By an analogous argument, we have $\theta(X) = \mathrm{ad}\, \rho(z)^{-1}\theta(h \circ s \circ lX)$. q.e.d.

By Propositions 2 and 3 we have easily

PROPOSITION 4. *Let (P', l, α) and $(\bar{P}', \bar{l}, \bar{\alpha})$ be two collections both having the properties in Proposition 1 for the same conformal $S(n)$-bundle P_S. Then, there exists a unique isomorphism f of $\bar{P}'$ onto P' such that $l \circ f = \bar{l}$ and $f^*\alpha = \bar{\alpha}$.*

These observations lead us to the following

DEFINITION 2. *Let P_S be a conformal $S(n)$-bundle over a manifold M. The conformal $M'(n)$-bundle associated with P_S is the principal fiber bundle P' together with a homomorphism l and a form α having the properties in Proposition 1* [1; 7; 3; 2].

The prototype of conformal $M'(n)$-bundles is given by the Möbius space M_n: By the homogeneous structure of $M_n = M(n)/M'(n)$, the Möbius group $M(n)$ may be considered as a principal fiber bundle P' over the base space M_n with structure group $M'(n)$. At the same time, since the linear isotropy group of $M(n)$ at o is identified with $S(n)$, the homogeneous structure yields a conformal $S(n)$-bundle P_S over M_n together with a homomorphism l of P' onto P_S. Moreover, we have a left invariant form α on $P'(=M(n))$ such that $\alpha(X) = X_R$ for all $X \in \mathfrak{m}(n)$, where X_R denotes the R-component of X in the decomposition of $\mathfrak{m}(n)$ given in (1.6) and (1.7), and it can be proved that it has the properties stated in Proposition 1. Therefore, we see that P' together with l and α, obtained in this way, gives the conformal $M'(n)$-bundle associated with P_S, and it is considered as the prototype of conformal $M'(n)$-bundles.

REMARK. Let P_S be a conformal $S(n)$-bundle over a manifold M and let P' together with l and α be the corresponding conformal $M'(n)$-bundle. Given an orthogonal bundle $P_0 \subset P_S$, the existence of the injection h of P_0 into P' assured by Proposition 2 corresponds to that of the so-called "Veblen's repère" which is uniquely determined by the Riemannian metric on M which corresponds to P_0 and by a choice of a coordinate system [7; 3].

Let P' be a conformal $M'(n)$-bundle over a manifold M. For each point p of M, the fiber over p of the associated fiber bundle of P' with standard fiber M_n is called the tangent Möbius space at p, which we shall denote by $M_n(p)$. Every element z of P' gives a one-to-one correspondence of M_n with $M_n(p)$ such that $(z\cdot\sigma)\cdot u = z\cdot(\sigma u)$ for all $\sigma\in M'(n)$ and $u\in M_n$, where $\pi(z)=p$. The origin p^* of the tangent Möbius space at p is the point of $M_n(p)$ defined by $p^* = z\cdot o$, where z is an element of P' such that $\pi(z)=p$. Clearly, the definition is consistent. By making use of P' and the injection of $M'(n)$ into $M(n)$, we define a principal fiber bundle P over the base space M with structure group $M(n)$. In this case, we identify P' with a subbundle of P.

Finally we have the following diagram:

$$\begin{array}{ccccc} P & \overset{\text{inj.}}{} & P' & \overset{l}{} & P_S \\ & & h & s & \text{inj.} \\ & & & & P_0 \end{array}$$

4. Normal conformal connection associated with a conformal $S(n)$-bundle.

PROPOSITION 5. *Let P_S be a conformal $S(n)$-bundle over a manifold M and let P' together with l and α be the corresponding conformal $M'(n)$-bundle. Then, there exists a unique linear mapping B of F_n into $\mathfrak{X}(P')$ which satisfies the following conditions*:

(C.1) $B_z + M'(n)_z = P'_z$ *(direct sum), where B_z denotes the subspace of P'_z composed of all the elements $B(\xi)_z$ where ξ runs over F_n*;

(C.2) $R_\sigma B(\xi) = B((\mathrm{ad}\,\sigma^{-1}\xi)_{F_n}) + (\mathrm{ad}\,\sigma^{-1}\xi)^*_{\mathfrak{m}'(n)}$;

(C.3) $\theta B(\xi)=\xi$, *where θ denotes the F_n-valued 1-form on P' defined in Proposition* 1;

(C.4) $\alpha B(\xi)=0$;

(C.5) *For each orthogonal bundle $P_0\subset P_S$, let h be the corresponding injection of P_0 into P'. Then we have*

$$hB_0(\xi)_x = B(\xi)_{h(x)} + J_x(\xi)^*_{h(x)},$$

where B_0 denotes the Riemannian connection in P_0 associated with P_0 and J_x the linear mapping of F_n into F_n^* given by formula (2.2).

The above proposition corresponds to Proposition 1 of [5] which is concerned with the normal projective connection associated with a class of projectively equivalent affine connections. One will see that the proof of Proposition 5 is parallel to that of Proposition 1 cited above.

Proof of Proposition 5. The proof is divided into three steps.

I. A linear mapping B of F_n into $\mathfrak{X}(P')$ will be called a conformal connection in the conformal $M'(n)$-bundle P' if it satisfies conditions (C.1), (C.2), (C.3) and (C.4) (see Definition 3).

Let B be a conformal connection in P'. We see from condition (C.1) that, for each $z \in P'$, there exist bilinear mappings T_z and A_z of $F_n \times F_n$ into F_n and $\mathfrak{m}'(n)$ respectively such that

$$-[B(\xi), B(\xi')]_z = B(T_z(\xi, \xi'))_z + A_z(\xi, \xi')_z^*.$$

Denoting by $W_z(\xi, \xi')$ the $\mathfrak{o}(n)$-component of $A_z(\xi, \xi')$ in the decomposition of $\mathfrak{m}'(n)$ given in (1.9), we have the following two formulae (cf. [5, p. 8]).

$$\begin{aligned} &\text{(i) } T_{z\cdot\sigma}(\xi, \xi') = \operatorname{ad} l(\sigma)^{-1} T_z(\operatorname{ad} l(\sigma)\xi, \operatorname{ad} l(\sigma)\xi'); \\ (4.1) \qquad &\text{(ii) If } T_z = 0 \quad \text{for all } z \in P', \\ &W_{z\cdot\sigma}(\xi, \xi') = \operatorname{ad} l(\sigma)^{-1} W_z(\operatorname{ad} l(\sigma)\xi, \operatorname{ad} l(\sigma)\xi'). \end{aligned}$$

Let P_0 be an orthogonal bundle contained in P_S and let h be the corresponding injection of P_0 into P'. We know from Proposition 3 that there exist mappings ρ and E of P' into R_+ and F_n^* respectively such that

$$z = h \circ s \circ l(z) \cdot \rho(z) \cdot \exp E(z).$$

By applying Lemma 3 to the case where $V = P'$, $f =$ the identity transformation of P' onto itself, $g = h \circ s \circ l$ and $X = B(\xi)_{h(x)}$, we have

$$B(\xi)_{h(x)} = h \circ s \circ lB(\xi)_{h(x)} + A_x(\xi)^*_{h(x)} - J_x(\xi)^*_{h(x)},$$

where we have set $A_x(\xi) = \rho^{-1} d\rho(B(\xi)_{h(x)})$ and $J_x(\xi) = -b^{-1} db(B(\xi)_{h(x)})$. By condition (C.4) and the formula of α in Proposition 3, we have $A_x(\xi) = 0$ for all $\xi \in F_n$. If we define a linear mapping B_0 of F_n into $\mathfrak{X}(P_0)$ by $B_0(\xi)_x = s \circ lB(\xi)_{h(x)}$, then we see from conditions (C.1), (C.2) and (C.3) that it satisfies conditions (E.1), (E.2) and (E.3) for a Euclidean connection in P_0. The Euclidean connection B_0 in P_0 which is obtained in this way will be said to be *induced* by P_0 (with respect to the conformal connection B). For each $x \in P_0$, the correspondence $\xi \rightarrow J_x(\xi)$ defines a linear mapping of F_n into F_n^*, which will be said to be *induced* by P_0. With the above definition, we have

$$(4.2) \qquad hB_0(\xi)_x = B(\xi)_{h(x)} + J_x(\xi)^*_{h(x)}.$$

Let P_0 be an orthogonal bundle contained in P_S and let B_0 (resp. J_x) be the Euclidean connection in P_0 (resp. the linear mapping of F_n into F_n^*) induced by P_0. As we have observed in §2, we can take, for each $x \in P_0$, bilinear mappings T'_x and R_x of $F_n \times F_n$ into F_n and $\mathfrak{o}(n)$ respectively such that

$$-[B_0(\xi), B_0(\xi')]_x = B_0(T'_x(\xi, \xi'))_x + R_x(\xi, \xi')_x^*.$$

T'_x (resp. R_x) corresponds to the torsion (resp. the curvature) tensor field of B_0. Then, we have

$$(4.3)\qquad \begin{aligned} &\text{(i)}\ T_{h(x)}(\xi, \xi') = T'_x(\xi, \xi');\\ &\text{(ii)}\ W_{h(x)}(\xi, \xi') = R_x(\xi, \xi') + ([\xi, J(\xi')] + [J_x(\xi), \xi'])_{\mathfrak{o}(n)} \end{aligned}$$

(cf. [5, p. 10]).

II. Let B be a conformal connection in P'. Now consider the following condition for B:

$$(\mathrm{C.5'})\qquad \begin{aligned} &\text{(i)}\ T_z(\xi, \xi') = 0;\\ &\text{(ii)}\ \mathrm{Tr}(\tilde{W}_z(\xi, \xi')) = 0, \end{aligned}$$

where $\tilde{W}_z(\xi, \xi')$ denotes the bilinear mapping of $F_n \times F_n$ into $\mathfrak{gl}(F_n)$ defined by $\tilde{W}_z(\xi, \xi')\eta = W_z(\xi, \eta)\xi'$ for all $\eta \in F_n$.

We shall show that, if a conformal connection in P' satisfies condition (C.5′), then it satisfies also condition (C.5). Let P_0 be an orthogonal bundle contained in P_S and let B_0 (resp. J_x) be the Euclidean connection in P_0 (resp. the linear mapping of F_n into F_n^*) induced by P_0. By formula (4.2), it is sufficient to show that B_0 is the Riemannian connection in P_0 and J_x is given by formula (2.2). It follows from formula (i) of (4.3) that $T'_x(\xi, \xi') = T_{h(x)}(\xi, \xi') = 0$, which shows that B_0 is the Riemannian connection in P_0. Formula (ii) of (4.3) can be written as

$$(4.4)\qquad \tilde{W}_{h(x)}(\xi, \xi') = \tilde{R}_x(\xi, \xi') - \tilde{J}_x(\xi, \xi');$$

if we observe that $\mathrm{Tr}\,(\tilde{R}_x(\xi, \xi')) = S_x(\xi, \xi')$, formula (ii) of (C.5′) gives $\mathrm{Tr}\,(\tilde{J}_x(\xi, \xi')) = S_x(\xi, \xi')$. Now, by Lemma 1, we see that J_x is given by formula (2.2), which proves our assertion.

III. Now we can prove Proposition 5.

We first prove the uniqueness of B: Fix an orthogonal bundle $P_0 \subset P_S$ and let h be the corresponding injection of P_0 into P'. By Proposition 3, we can take mappings ρ and E such that $z = h \circ s \circ l(z) \cdot \rho(z) \cdot \exp E(z)$. Now, by condition (C.2), we have easily

$$(4.5)\qquad B(\xi)_z = R_{a(z)} B(\mathrm{ad}\ \rho(z)\xi)_{h(x)} - ((\mathrm{ad}\ b(z)^{-1}\xi)_{\mathfrak{m}(n)})^*_z$$

where $x = s \circ l(z)$ and $b(z) = \exp E(z)$. If we take account of the formula $B(\xi)_{h(x)} = hB_0(\xi)_x - J_x(\xi)^*_{h,(x)}$, formula (4.5) implies that B is uniquely determined by the orthogonal bundle P_0 only, which proves the uniqueness of B.

We shall now prove the existence of B: Fix an orthogonal bundle $P_0 \subset P_S$ and let h be the corresponding injection of P_0 into P'. Let B_0 be the Riemannian connection in P_0 and define J_x by formula (2.2). We now define a linear mapping B of F_n into $\mathfrak{X}(P')$ in the following way. First, we define, for each $x \in P_0$ and $\xi \in F_n$, $B(\xi)_{h(x)}$ by the formula $B(\xi)_{h(x)} = hB_0(\xi)_x - J_x(\xi)^*_{h(x)}$. Next, we define, for each $z \in P'$ and $\xi \in F_n$, $B(\xi)_z$ by formula (4.5) by the use of $B(\xi)_{h(x)}$, just defined. Then, the linear mapping B which is obtained in this way satisfies the required conditions. In fact, by condition (E.2) and the fact that $J_{x\cdot\tau}(\xi) = {}^t\tau J_x(\tau\xi)$ for all $x \in P_0$ and $\tau \in O(n)$, we have easily $R_\tau B(\xi)_{h(x)}$

$=B(\mathrm{ad}\,\tau^{-1}\xi)_{h(x\cdot\tau)}$. Therefore, we see that condition (C.2) follows from the following two formulae $\rho(z\cdot\sigma)=\rho(z)\cdot\rho\circ l(\sigma)$ and $b(z)\cdot\sigma=l(\sigma)\cdot b(z\cdot\sigma)$ (see Lemma 2). In order to verify condition (C.1), it is sufficient to deal with the case where z is of the form $z=h(x)$. From the proof of Proposition 3, we know that, for each $X\in P'_{h(x)}$, there exist a $Y\in P_{0x}$ and an $A\in\mathfrak{m}'(n)$ such that $X=hY+A^*_{h(x)}$. We now see that condition (C.1) follows from condition (E.1). Condition (C.3) is obvious from condition (E.3). Condition (C.4) follows from the properties of α. Therefore, we have only to check condition (C.5). For this purpose, it is sufficient, by the argument in II, to show that B satisfies condition (C.5′). By definition of B we see that B_0 and J_x coincide with the ones induced by P_0 with respect to the conformal connection B. From formula (i) of (4.3), we have $T_{h(x)}(\xi,\xi')=0$, because $T'_x(\xi,\xi')=0$ (note that B_0 is the Riemannian connection in P_0). It follows from formula (i) of (4.1) that $T_z(\xi,\xi')=0$. Moreover, by Lemma 1, we have $\mathrm{Tr}\,(\tilde{J}_x(\xi,\xi'))=S_x(\xi,\xi')$ and, using formula (4.4), we have $\mathrm{Tr}(\tilde{W}_{h(x)}(\xi,\xi'))=0$. It follows from formula (ii) of (4.1) that $\mathrm{Tr}(\tilde{W}_z(\xi,\xi'))=0$; thus B satisfies condition (C.5′). Therefore we have completed the proof of Proposition 5.

REMARK. The bilinear mapping T_z (resp. W_z) which appears in the proof of Proposition 5 corresponds to what is usually called the torsion tensor field (resp. the Weyl's conformal curvature tensor field) of the conformal connection B.

REMARK. The proof of Proposition 5 shows that, given a conformal connection B in P', conditions (C.5) and (C.5′) are mutually equivalent.

DEFINITION 3. *Let P_S be a conformal $S(n)$-bundle over a manifold M and let P' together with l and α be the corresponding conformal $M'(n)$-bundle. A conformal connection in P' is a linear mapping B of F_n into $\mathfrak{X}(P')$ which satisfies conditions* (C.1), (C.2), (C.3) *and* (C.4) *in Proposition* 5. *The normal conformal connection associated with the conformal $S(n)$-bundle P_S is the conformal connection B which satisfies condition* (C.5) *in Proposition* 5 *or, equivalently, condition* (C.5′) *in the proof of Proposition* 5 [1; 7; 3; 2].

5. **Conformal development.** This section corresponds to §5 of [5], in which we proved a proposition about projective development. For the definition and fundamental properties of a connection in a principal fiber bundle, we follow the book of K. Nomizu [6].

Let P_S be a conformal $S(n)$-bundle over a manifold M. Let P' be the corresponding conformal $M'(n)$-bundle and let P be the principal fiber bundle over the base space M with structure group $M(n)$ defined in §3. As is well known, a conformal connection B in P' gives rise to a connection Q in P. The horizontal space Q_z at a point z of P' is given by the subspace of P_z composed of all the elements $B(\xi)_z-\xi^*_z$ where ξ runs over $F_n\subset\mathfrak{m}(n)$ (note that P' is identified with a subspace of P).

We now define conformal development as follows: Let p be a point of M and let $u(t)$ be a curve in M beginning at p. Let $z(t)$ be a horizontal curve in

P which covers $u(t)$, with respect to the connection Q in P. There is a curve $a(t)$ in $M(n)$ such that $z(t)\cdot a(t)\in P'$ for all t. Then the conformal development of $u(t)$ at p is defined as the curve $u^*(t)=z(0)\cdot a(t)o$ in the tangent Möbius space $M_n(p)$ at p. Clearly the definition is consistent.

We recall here the definition of Euclidean development, cf. [5, p. 14]. Let B_0 be a Euclidean connection in an orthogonal bundle P_0 over M. Let p be a point of M and let $u(t)$ be any curve in M beginning at p. Let $x(t)$ be a horizontal curve in P_0 which covers $u(t)$, with respect to the linear connection associated with the Euclidean connection B_0. There is a curve $\xi(t)$ in F_n such that $B_0(\xi(t))_{x(t)}=x'(t)$. Now the Euclid development of $u(t)$ at p is defined as the curve $v(t)=x(0)\cdot w(t)$ in the tangent Euclidean space M_p at p, where $w(t)=\int_0^t\xi(t)dt$.

The following Proposition 6 may be considered as an alternative description of conformal development in terms of a Euclidean connection.

Let P_S be a conformal $S(n)$-bundle over a manifold M and let P' be the corresponding conformal $M'(n)$-bundle. Let P_0 be any orthogonal bundle and let h be the corresponding injection of P_0 into P'. For each $x\in P_0$, we define J_x by formula (2.2) starting with the Riemannian connection in P_0. Moreover, we identify M_n (resp. F_n) with the tangent Möbius space $M_n(p)$ (resp. the tangent Euclidean space M_p) by $h(x)$ (resp. x). With these preparations, we have

PROPOSITION 6. *Let p be a point of M and let $u(t)$ be a curve in M beginning at p. Fix a point x of P_0 such that $\pi_0(x)=p$. Let $u^*(t)$ (resp. $v(t)$) be the development of $u(t)$ at p into the tangent Möbius space M_n (resp. the tangent Euclidean space F_n) at p with respect to the normal conformal connection in P' (resp. the Riemannian connection in P_0). Then, we have $u^*(t)=a(t)o$, where $a(t)$ denotes the curve in the Möbius group $M(n)$ which is determined by the differential equation*

$$a(t)^{-1}a'(t) = v'(t) + J_{x(t)}(v'(t)) \tag{5.1}$$

with the initial condition $a(0)=e$, where $x(t)$ denotes the lift of $u(t)$ through x with respect to the linear connection associated with the Riemannian connection in P_0.

The proof of this proposition is quite similar to that of Proposition 2 of [5]. We here remark the following point only. *Let $z(t)$ be the lift of $u(t)$ through $h(x)$ with respect to the connection in P which corresponds to the normal conformal connection in P'. Then, we have*

$$z(t)\cdot a(t) = h(x(t)).$$

6. **Some properties of a complete Riemannian space of type $\mathfrak{S}$.** Let g be a Riemannian metric on a manifold M. By a field of projections of rank m we shall mean a tensor field H of type (1, 1) which satisfies the following

conditions: (i) H is parallel with respect to the Riemannian connection, (ii) $H^2=H$, (iii) $g(HX, Y)=g(X, HY)$ for all tangent vectors X, Y and (iv) the rank of H_p is equal to m at each point p of M.

DEFINITION 4. *Let M be an n-dimensional manifold.*

(i) *A Riemannian metric g on M is of type $\mathfrak{S}^{m-1}$ $(0 \leqq m \leqq n)$, if there exists a field H' of projections of rank m and if the Ricci tensor field S of g is expressed as*

$$S = (m-1)g \circ H' - (n-m-1)g \circ H'',$$

where H'' denotes the tensor field of type $(1, 1)$ defined by $H''X=X-H'X$ for all tangent vectors X.

(ii) *A Riemannian metric g on M is of type $\mathfrak{S}^{\infty}$, if the Ricci tensor field is zero. When a Riemannian metric g on M is of type $\mathfrak{S}^{m-1}$ or $\mathfrak{S}^{\infty}$, we shall say that it is of type $\mathfrak{S}$.*

From the definition, we see that a Riemannian space of type $\mathfrak{S}^{-1}$ (resp. $\mathfrak{S}^{n-1}$) is nothing but an Einstein space with the negative (resp. positive) Ricci tensor field, and that, in general, a Riemannian space of type $\mathfrak{S}^{m-1}$ is locally isomorphic to the direct product of two Einstein spaces with the Ricci tensor fields of different signs. Riemannian spaces of type $\mathfrak{S}$ have been studied by S. Sasaki and K. Yano [4; 8] and it was proved that the conformal holonomy group of a space of type $\mathfrak{S}$ fixes an $(m-1)$-dimensional sphere or a point, according as the space is of type $\mathfrak{S}^{m-1}$ or $\mathfrak{S}^{\infty}$ (in the case where $m=0$, the fixed sphere should be considered as an imaginary one). Moreover, they also dealt with the converse problem.

The main purpose of this section is to establish

THEOREM 1. *Let g and $\bar{g}$ be two conformally equivalent Riemannian metrics which are complete and of type $\mathfrak{S}$.*

(i) *If g is of type $\mathfrak{S}^{m-1}$ and $m \geqq 1$, then g and $\bar{g}$ coincide;*

(ii) *If g is of type $\mathfrak{S}^{\infty}$, then the associated function of $\bar{g}$ with respect to g is constant.*

In the above theorem, let f be an arbitrary element of $C(g)$. Then the Riemannian metric f^*g is clearly complete and of type $\mathfrak{S}$. Now, by taking $\bar{g}=f^*g$, we have

THEOREM 2. *Let g be a complete Riemannian metric.*

(i) *If g is of type $\mathfrak{S}^{m-1}$ and $m \geqq 1$, then $C(g)$ and $I(g)$ coincide;*

(ii) *If g is of type $\mathfrak{S}^{\infty}$, then $C(g)$ is homothetic.*

Proof of Theorem 1. The proof is divided into three steps.

I. In the following, by a projection of F_n, we mean an endomorphism H of F_n into itself satisfying (i) $H^2=H$ and (ii) $\phi(H\xi, \xi')=\phi(\xi, H\xi')$ for all ξ, $\xi' \in F_n$. We shall say that a linear mapping J of F_n into F_n^* is of type $\mathfrak{S}^{m-1}$, if there exists a projection H' of F_n which is of rank m and if it is expressed as

$$J = \frac{1}{2}\phi \circ H' - \frac{1}{2}\phi \circ H'',$$

where H'' denotes the endomorphism of F_n into itself defined by $H''\xi = \xi - H'\xi$ for all $\xi \in F_n$.

Fix a linear mapping J of F_n into F_n^* which is of type $\mathfrak{S}^{m-1}$ $(0 \leqq m \leqq n)$ and consider the decomposition of F_n associated with the projection H':

$$F_n = V' + V'',$$

where $V' = H'F_n$ and $V'' = H''F_n$. We denote by $\mathfrak{k}(J)$ the subalgebra of $\mathfrak{m}(n)$ generated by all the elements $\xi + J(\xi)$, where ξ runs over F_n, and denote by $K(J)$ the subgroup of $M(n)$ generated by $\mathfrak{k}(J)$. By the stereographic projection $\xi \to \exp \xi \cdot o$, we define an $(m-1)$-dimensional sphere Φ_J of M_n as follows:

$$\Phi_J = \left\{ \exp \xi' \cdot o \,\middle|\, \phi(\xi', \xi') = 4,\ \xi' \in V' \right\}.$$

Denote by K_J the connected component of $M_n - \Phi_J$ which contains o.

Then we have

Lemma 4. *Let J be a linear mapping of F_n into F_n^* which is of type $\mathfrak{S}^{m-1}$.*

(i) *Every element of $K(J)$ fixes K_J;*

(ii) *Every element p of K_J can be written as*

$$p = \exp (\xi + J(\xi))o \textit{ for some } \xi \in F_n.$$

Proof. Denote by $\mathfrak{l}'$ (resp. $\mathfrak{l}''$) the subspace of $\mathfrak{k}(J)$ composed of all the elements $\xi' + \phi(\xi')/2$ where $\xi' \in V'$ (resp. $\xi'' - \phi(\xi'')/2$ where $\xi'' \in V''$) and consider the subspace $\mathfrak{g}'$ (resp. $\mathfrak{g}''$) of $\mathfrak{m}(n)$ as follows:

$$\mathfrak{g}' = \mathfrak{l}' + \mathfrak{o}(m) \text{ (resp. } \mathfrak{g}'' = \mathfrak{l}'' + \mathfrak{o}(n-m)),$$

where $\mathfrak{o}(m)$ (resp. $\mathfrak{o}(n-m)$) denotes the subalgebra of $\mathfrak{o}(n)$ which operates trivially on V'' (resp. V'). An easy calculation shows that $[\mathfrak{l}', \mathfrak{l}'] = \mathfrak{o}(m)$, $[\mathfrak{l}', \mathfrak{o}(m)] = \mathfrak{l}'$, $[\mathfrak{l}'', \mathfrak{l}''] = \mathfrak{o}(n-m)$, $[\mathfrak{l}'', \mathfrak{o}(n-m)] = \mathfrak{l}''$ and $[\mathfrak{g}', \mathfrak{g}''] = 0$. It follows that $\mathfrak{g}'$ and $\mathfrak{g}''$ are subalgebras of $\mathfrak{m}(n)$ and that $\mathfrak{k}(J)$ is expressed as a direct sum of $\mathfrak{g}'$ and $\mathfrak{g}''$:

$$\mathfrak{k}(J) = \mathfrak{g}' + \mathfrak{g}''.$$

In order to prove (i), it is sufficient to show that the elements of the form $\exp A$ where $A \in \mathfrak{g}'$ or $\mathfrak{g}''$ fix Φ_J. Let p be an arbitrary point of Φ_J. Then we see from (1.8) that it is expressed as $p = \omega(\xi_0 + x' + 2\xi_\infty)$, where $x' \in V'$. If $A \in \mathfrak{g}''$, we have $\psi(A)(\xi_0 + x' + 2\xi_\infty) = 0$, from which it follows that $\exp A \cdot p = p$. Now assume that $A \in \mathfrak{g}'$. We have $\psi(A)(\xi_0 - 2\xi_\infty) = 0$. If we set $\exp \psi(A) \cdot (\xi_0 + x' + 2\xi_\infty) = y_0\xi_0 + y' + y_\infty\xi_\infty$, then we have $\phi(y_0\xi_0 + y' + y_\infty\xi_\infty, \xi_0 - 2\xi_\infty) = 0$, from which it follows that $4y_0^2 = \phi(y', y')$. We have

$$\exp A \cdot p = \omega(y_0\xi_0 + y' + y_\infty\xi_\infty) = \exp ((1/y_0)y') \cdot o \ (\in \Phi_J).$$

Now we shall prove (ii). For this purpose, we define four subsets M_m, M_{n-m}, S_m^0 and S_m^∞ of M_n as follows: $M_m=\{\exp \xi'\cdot o \mid \xi'\in V'\}\cup\{\infty\}$, $M_{n-m}=\{\exp \xi''\cdot o \mid \xi''\in V''\}\cup\{\infty\}$, $S_m^0=\{\exp \xi'\cdot o \mid \phi(\xi',\ \xi')<4\}$ and $S_m^\infty=\{\exp \xi'\cdot o \mid \phi(\xi',\xi')>4\}\cup\{\infty\}=\{\exp \phi(\xi')\cdot o \mid \phi(\xi',\xi')<1\}$. Then we have easily

$$K_J = M_n - \Phi_J = M_{n-m}\cup\{M_n - (M_m\cup M_{n-m})\}\cup S_m^0\cup S_m^\infty.$$

Since we have, for each $\xi\in F_n$,

$$\exp\,(\xi + J(\xi)) = \exp\,(\xi' + \phi(\xi')/2)\,\exp\,(\xi'' - \phi(\xi'')/2),$$

where $\xi'=H'\xi$ and $\xi''=H''\xi$, the statement of (ii) follows from the verification of the following four cases:

(a) For each $p\in M_{n-m}$, there is a $\xi''\in V''$ such that

$$p = \exp\,(\xi'' - \phi(\xi'')/2)\cdot o;$$

(b) For each $p\in M_n-(M_m\cup M_{n-m})$, there are a $\xi'\in V'$ and a $q\in M_{n-m}$ such that $p=\exp\,(\xi'+\phi(\xi')/2)\cdot q$;

(c) For each $p\in S_m^0$, there is a $\xi'\in V'$ such that $p=\exp\,(\xi'+\phi(\xi')/2)\cdot o$;

(d) For each $p\in S_m^\infty$, there is a $\xi'\in V'$ such that $p=\exp\,(\xi'+\phi(\xi')/2)\cdot\infty$.

We write down the proof for case (b) only, since the other cases can be treated similarly by using Lemma 5. Let p be an arbitrary point of $M_n-(M_m\cup M_{n-m})$. We see that p is expressed as $p=\exp x\cdot o$, where $x\in F_n$. If we set $x'=H'x$ and $x''=H''x$, then we have $x'\neq 0$ and $x''\neq 0$. Now define an element ξ' of V' and an element y'' of V'' as follows: $\xi'=(\theta/(\phi(x',x'))^{1/2})x'$ and $y''=(\beta/(\phi(x'',x''))^{1/2})x''$, where β and θ are respectively given by

$$\beta = \frac{\phi(x,x) - 4 + ((\phi(x,x) - 4)^2 + 16\phi(x'',x''))^{1/2}}{2(\phi(x'',x''))^{1/2}}$$

and

$$\sin\frac{\theta}{2} = \left(\frac{4(\beta - (\phi(x'',x''))^{1/2})}{(\phi(x'',x''))^{1/2}(4+\beta^2)}\right)^{1/2}.$$

Then, we have $p=\exp x\cdot o=\exp\,(\xi'+2^{-1}\phi(\xi'))\cdot\exp y''\cdot o$. Indeed, if we set

$$\rho = 1 + (\sinh\theta/2)^2 + 4^{-1}(\sinh\theta/2)^2\phi(y'',y''),$$

then we have

$$\rho x = (\sinh\theta/\theta)(1 + 4^{-1}\phi(y'',y''))\xi' + y''.$$

Therefore, by Lemma 5, we have

$$\exp A'\left(\xi_0 + y'' + \frac{\phi(y'',y'')}{2}\xi_\infty\right) = \rho\left(\xi_0 + x + \frac{\phi(x,x)}{2}\xi_\infty\right)$$

where $A'=\psi(\xi'+\phi(\xi')/2)$. But, in view of (1.8), this is nothing but the desired formula. Since $q=\exp y''\cdot o\in M_{n-m}$, we have completed the proof of b). q.e.d.

LEMMA 5.

(i) *For each* $\xi\in F_n$, *set* $A=\psi(\xi+\phi(\xi)/2)$. *Then, we have*

$$\exp A = 1 + (\sinh\theta/\theta)A + (\cosh\theta - 1/\theta^2)A^2,$$

where $\theta=(\phi(\xi,\xi))^{1/2}$;

(ii) *For each* $\xi\in F_n$, *set* $A=\psi(\xi-\phi(\xi)/2)$. *Then, we have* $\exp A=1+(\sin\theta/\theta)A-((\cos\theta-1)/\theta^2)A^2$, *where* $\theta=(\phi(\xi,\xi))^{1/2}$.

We shall use

LEMMA 6. *Let* J and $\bar{J}$ *be two linear mappings of* F_n *into* F^* *which are both of type* $\mathfrak{S}^{m-1}$, *where* $m\geqq 1$. *We assume that there exist an element* ρ *of* R_+ *and an element* E *of* F_n^* *such*

$$\rho\cdot\exp E\Phi_J = \Phi_{\bar{J}}.$$

Then, we have $\rho=1$ *and* $E=0$.

Proof. Let H' and $\bar{H}'$ be the projections of F_n defining J and $\bar{J}$ respectively. If we set $V'=H'F_n$ and $\bar{V}'=\bar{H}'F_n$, then, in view of the definition of Φ_J and $\Phi_{\bar{J}}$, we see that there is a mapping $\xi\to\eta$ of $\{\xi\in V'\,|\,\phi(\xi,\xi)=4\}$ onto $\{\eta\in\bar{V}'\,|\,\phi(\eta,\eta)=4\}$ such that

$$\rho\cdot\exp E\cdot\exp\xi\cdot o = \exp\eta\cdot o.$$

But, by (1.8), we see that this is equivalent to the following two formulae:

(i) $$\rho^2 = 1 + \langle\xi, E\rangle + \langle\phi^{-1}E, E\rangle;$$

(ii) $$\rho\eta = \xi + 2\phi^{-1}E.$$

Observing that $-\xi\in\{\xi\in V'\,|\,\phi(\xi,\xi)=4\}$, it follows from formula (i) that $\rho^2 = 1 + \langle\phi^{-1}E, E\rangle$. Moreover, from formulae (i) and (ii), we have $\rho^2=1+\rho\langle\eta, E\rangle-\langle\phi^{-1}E, E\rangle$; as above, it follows that $\rho^2=1-\langle\phi^{-1}E, E\rangle$. Therefore, we have $\rho=1$ and $E=0$. q.e.d.

II. Let g be a Riemannian metric on a manifold M. Let P_S be the corresponding conformal $S(n)$-bundle and let P' together with l and α be the conformal $M'(n)$-bundle associated with P_S. Let P_0 be the orthogonal bundle corresponding to g and let h be the corresponding injection of P_0 into P'. Under the assumption that g is of type $\mathfrak{S}$, we define, at each point p of M, two subsets $\Phi_g(p)$ and $K_g(p)$ of the tangent Möbius space $M_n(p)$ at p as follows:

The case where g is of type $\mathfrak{S}^{m-1}$. In this case, the Ricci tensor field S of g is expressed as $S=(m-1)g\circ H'-(n-m-1)g\circ H''$. For each $x\in P_0$, we denote by H'_x the endomorphism of F_n into itself which is defined by $H'_x=x^{-1}\cdot H'\cdot x$. Let B_0 be the Riemannian connection in P_0. If we observe that

$S(x\cdot\xi,\ x\cdot\xi')=S_x(\xi,\ \xi')$ and $g(x\cdot\xi,\ x\cdot\xi')=\phi(\xi,\ \xi')$, an easy calculation shows that the linear mapping J_x which is defined by formula (2.2) starting with the Riemannian connection B_0 is given by the formula $J_x=2^{-1}\phi\circ H_x'-2^{-1}\phi\circ H_x''$, where $H_x''\xi=\xi-H_x'\xi$ for all $\xi\in F_n$. This means that J_x is of type $\mathfrak{S}^{m-1}$. Therefore, the argument in II applies to J_x. We now define $\Phi_g(p)$ and $K_g(p)$ as follows:

$$\Phi_g(p) = h(x)\cdot\Phi_{J_x}; \qquad K_g(p) = h(x)\cdot K_{J_x},$$

where x is an element of P_0 such that $\pi_0(x)=p$. The definition does not depend on the choice of $x\in P_0$ such that $\pi_0(x)=p$. Indeed, if we set $V_x'=H_x'F_n$, then we have $V_{x\cdot\sigma}'=\sigma^{-1}V_x'$ for each $x\in P_0$ and $\sigma\in O(n)$. It follows immediately from the definition of Φ_J and K_J that $\Phi_{J_{x\cdot\sigma}}=\sigma^{-1}\Phi_{J_x}$ and $K_{J_{x\cdot\sigma}}=\sigma^{-1}K_{J_x}$.

The case where g is of type $\mathfrak{S}^\infty$. In this case, we define $\Phi_g(p)$ and $K_g(p)$ as follows:

$$\Phi_g(p) = h(x)\cdot\infty; \qquad K_g(p) = h(x)\cdot E_n,$$

where x is an element of P_0 such that $\pi_0(x)=p$ and where $E_n=M_n-\{\infty\}$, as was defined in (1.8). The definition is independent of the choice of $x\in P_0$ such that $\pi_0(x)=p$, because $O(n)$ fixes ∞.

In each case, $K_g(p)$ is an open submanifold of $M_n(p)$ which contains the origin p^*, and $\Phi_g(p)$ is the boundary of $K_g(p)$.

LEMMA 7. *Assume that g is a complete Riemannian metric of type* $\mathfrak{S}$.

(i) *At each point p of M, every curve through p in M is developed into the part* $K_g(p)$ *of the tangent Möbius space* $M_n(p)$ *at p with respect to the normal conformal connection associated with* P_S.

(ii) *For each point p of M and each point* q^* *of* $K_g(p)$, *there exists a curve in* $K_g(p)$ *which joins* p^* *(the origin of* $M_n(p)$*) and* q^* *and which admits the development into the base space.*

Proof. Fix a point x of P_0 such that $\pi_0(x)=p$ and identify F_n (resp. M_n) with the tangent Euclidean space M_p (resp. the tangent Möbius space $M_n(p)$) at p by x (resp. $h(x)$). In the following, we use the notation in Proposition 6.

The case where g is of type $\mathfrak{S}^{m-1}$. Let $u(t)$ be an arbitrary curve through p in M. We know from Proposition 6 that the conformal development $u^*(t)$ of $u(t)$ at p is given by $u^*(t)=a(t)o$, where $a(t)$ is the curve in $M(n)$ which is uniquely determined by the differential equation $a(t)^{-1}a'(t)=v'(t)+J_{x(t)}(v'(t))$ with the initial condition $a(0)=e$. Since the Ricci tensor field S of g is parallel, we see that $S_{x(t)}$ is constant for any horizontal curve in P_0 (with respect to the Riemannian connection in P_0). It follows from formula (2.2) that $J_{x(t)}$ is constant, hence, $v'(t)+J_{x(t)}(v'(t))=v'(t)+J_x(v'(t))$ is contained in $\mathfrak{k}(J_x)$. It follows that $a(t)$ is a curve in $K(J_x)$, because $K(J_x)$ is the subgroup of $M(n)$ generated by $\mathfrak{k}(J_x)$. Therefore, by (i) of Lemma 4, we see that

$a(t)o$ is a curve in K_{J_x}, which proves (i) (note the K_{J_x} is identified with $K_g(p)$ by $h(x)$).

Now we shall prove (ii). Let q^* be an arbitrary element of K_{J_x}. We know from (ii) of Lemma 4 that there is an element ξ of F_n such that $q^* = \exp(\xi + J(\xi))$. Now consider the geodesic $u(t)$ of g such that $u'(0) = \xi$ (we identify ξ with a vector in M_p by x). Since the metric g is complete, the geodesic $u(t)$ is defined for any real number t. We show that the conformal development $u^*(t)$ of $u(t)$ at p joins p^* and q^*. $u(t)$ being a geodesic such that $u'(0) = \xi$, the Euclidean development $v(t)$ of $u(t)$ at p is given by $v(t) = t\xi$. Hence, by Proposition 6, the conformal development $u^*(t)$ of $u(t)$ at p is given by $u^*(t) = a(t)o$, where $a(t)^{-1}a'(t) = \xi + J_{x(t)}(\xi) = \xi + J_x(\xi)$, that is, $a(t) = \exp t(\xi + J_x(\xi))$. We have $u^*(1) = a(1)o = q^*$, which proves our assertion, we have thereby proved (ii).

The case where g is of type $\mathfrak{S}^\infty$. The proof is similar to the case where g is of type $\mathfrak{S}^{m-1}$. We remark the following points. The linear mapping J_x of F_n into F_n^* which is defined by formula (2.2) reduces to zero. The stereographic projection $\xi \rightarrow \exp \xi \cdot o$ gives a one-to-one correspondence of F_n with E_n, and E_n is identified with $K_g(p)$ by $h(x)$. We have thus completed the proof of Lemma 7.

REMARK. A theorem of K. Yano and S. Sasaki [4; 8] states that the assignment $p \rightarrow \Phi_g(p)$ is parallel with respect to the normal conformal connection, which is almost evident by Lemma 7, if we remark that $\Phi_g(p)$ is the boundary of $K_g(p)$.

III. Now consider a second Riemannian metric $\bar{g}$ on M which is complete and of type $\mathfrak{S}$ and which is conformally equivalent to g. Let $\overline{P}_0$ and $\bar{h}$ be the corresponding orthogonal bundle and injection of $\overline{P}_0$ into P'. At each point p of M, we define $\Phi_{\bar{g}}(p)$ and $K_{\bar{g}}(p)$ as in II starting with $\bar{g}$.

LEMMA 8. *$\Phi_g(p) = \Phi_{\bar{g}}(p)$ at each point p of M.*

Proof. $\Phi_g(p)$ (resp. $\Phi_{\bar{g}}(p)$) is the boundary of $K_g(p)$ (resp. $K_{\bar{g}}(p)$). Therefore, it is sufficient to prove that $K_g(p) = K_{\bar{g}}(p)$. Let q^* be an arbitrary point of $K_g(p)$. We know from (ii) of Lemma 7 that there is a curve $u^*(t)$ in $K_g(p)$ joining p^* and q^* which admits the development $u(t)$ into the base space. Now, by applying (i) of Lemma 7 to $\bar{g}$, we see that $u^*(t)$ is a curve in $K_{\bar{g}}(p)$ and, in particular, q^* is contained in $K_g(p)$. Thereby we have proved that $K_g(p) \subset K_{\bar{g}}(p)$. In the same way, we have $K_{\bar{g}}(p) \subset K_g(p)$, whence $K_g(p) = K_g(p)$. q.e.d.

By using Lemma 8, we now prove Theorem 1. To do this, we need several formulae concerning the relation between P_0 and $\overline{P}_0$. Let s (resp. $\bar{s}$) be the homomorphism of P_S onto P_0 (resp. $\overline{P}_0$) defined in §2 and let λ be the associated function of $\bar{g}$ with respect to g. By Proposition 3, applied to $\overline{P}_0$, there exist mappings $\bar{\rho}$ and $\overline{E}$ of P' into R_+ and F_n^* respectively such that

$z=\bar{h}\circ\bar{s}\circ l(z)\cdot\bar{\rho}(z)\cdot\exp\overline{E}(z)$ and $\alpha=\bar{\rho}^{-1}d\bar{\rho}+[\theta,\overline{E}]_R$. If we set $\rho=\bar{\rho}\circ h$ and $E=\overline{E}\circ h$, then it follows that

$$h(x)=\bar{h}\circ\bar{s}(x)\cdot\rho(x)\cdot\exp E(x)\qquad\text{for all } x\in P_0; \tag{6.1}$$

$$\rho^{-1}d\rho+[\theta',E]_R=0,\qquad\text{where }\theta'=h^*\theta; \tag{6.2}$$

$$\rho(x)=\lambda\circ\pi_S(x). \tag{6.3}$$

(6.2) follows from the fact that $h^*\alpha=0$ and (6.3) is obtained by applying l to formula (6.1) and taking account of formula (2.1).

(i) *The case where g is of type* $\mathfrak{S}^{m-1}$ ($m\geqq 1$). By Lemma 8, we see that $\bar{g}$ is also of type $\mathfrak{S}^{m-1}$. Fix a point x of P_0 and set $y=\bar{s}(x)$ and $p=\pi_0(x)$. We have

$$\Phi_g(p)=h(x)\cdot\Phi_{J_x}\quad\text{and}\quad\Phi_{\bar{g}}(p)=\bar{h}(y)\cdot\Phi_{\bar{J}_y}$$

where $\bar{J}_y$ denotes the linear mapping of F_n into F_n^* defined by formula (2.2) starting with $\bar{g}$. It follows from Lemma 8 and formula (6.1) that

$$\rho(x)\cdot\exp E(x)\Phi_{J_x}=\Phi_{\bar{J}_y}.$$

Therefore, by Lemma 6, we have $\rho(x)=1$ and hence, by formula (6.3), $\lambda(p)=1$. p being arbitrary, we have $\lambda=1$, whence $g=\bar{g}$.

(ii) *The case where g is of type* $\mathfrak{S}^\infty$. It follows from Lemma 8 that $\bar{g}$ is also of type $\mathfrak{S}^\infty$. Fix a point x of P_0 and set $y=\bar{s}(x)$ and $p=\pi_0(x)$. We have

$$\Phi_g(p)=h(x)\cdot\infty\quad\text{and}\quad\Phi_{\bar{g}}(p)=\bar{h}(y)\cdot\infty.$$

It follows from Lemma 8 and formula (6.1) that

$$\rho(x)\cdot\exp E(x)\cdot\infty=\infty.$$

But, by (1.8), we have

$$\rho(x)\cdot\exp E(x)\cdot\infty=\omega(1/2\rho(x)\langle\phi^{-1}E(x),E(x)\rangle\xi_0+\phi^{-1}E(x)+\rho(x)\xi_\infty);$$

furthermore, we have $\infty=\omega(\xi_\infty)$. It follows immediately that $E(x)=0$. Now, by formula (6.2), we see that ρ is constant and hence, by formula (6.3) that λ is constant, which proves our assertion. Thus we have completed the proof of Theorem 1.

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