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2007 Phys. Scr. 76 665

(http://iopscience.iop.org/1402-4896/76/6/013)

Phys. Scr. **76** (2007) 665–668 doi:10.1088/0031-8949/76/6/013

# Lie symmetry analysis of the quantum Zakharov equations

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Received 6 September 2007
Accepted for publication 18 September 2007
Published 7 November 2007
Online at stacks.iop.org/PhysScr/76/665

**Abstract**
The Lie point symmetries of the one-dimensional quantum Zakharov (qZ) system of equations are considered, which is a general model to describe the coupling between the Langmuir and the ion-acoustic waves in a quantum setting. It is demonstrated that the Lie symmetries of the qZ system are exactly similar to those of the classical Zakharov equations. Further, similarity reductions are conducted based on the obtained Lie symmetries. A pure general periodic similarity ion-acoustic wave solution is obtained with the presence of constant linear and time-dependent nonlinear shears and time-dependent background, where the quantum effect increases the period of the waves.

PACS numbers: 04.20.Jb, 02.20.Sv, 52.35.−g

## 1. Introduction

Recently, Garcia *et al* [1] obtained a one-dimensional (1D) quantum Zakharov (qZ) system of equations

$$\mathrm{i}\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}-H^2\frac{\partial^4 E}{\partial x^4}=nE, \tag{1}$$

$$\frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial x^2}+H^2\frac{\partial^4 n}{\partial x^4}=\frac{\partial^2 |E|^2}{\partial x^2}, \tag{2}$$

to model the nonlinear interaction between quantum Langmuir waves and quantum ion-acoustic waves in an electron-ion dense quantum plasma. In the dimensionless equations (1) and (2), $E=E(x,t)$ is the Langmuir envelope electric field, $n=n(x,t)$ is the density fluctuation, $H=\hbar\omega_{\mathrm{i}}/\kappa_{\mathrm{B}}T_{\mathrm{e}}$ ($\hbar$ is the Planck constant divided by $2\pi$, $\kappa_{\mathrm{B}}$ is the Boltzmann constant, $\omega_{\mathrm{i}}$ is the ion plasma frequency, and $T_{\mathrm{e}}$ is the electron temperature) is the quantum parameter representing the ratio between the ion plasmon energy and the electron thermal energy. The effect of this quantum correction is to introduce higher-order dispersion. The classical limit $H\equiv 0$ leads the above quantum system to the original classical Zakharov equations [2], which, in 1D, one dimension, are written as

$$\mathrm{i}\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}=nE, \tag{3}$$

$$\frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial x^2}=\frac{\partial^2 |E|^2}{\partial x^2}. \tag{4}$$

Several aspects related to the qZ system have been studied. In [1] where the qZ system was first derived, the system was applied to the four-wave interactions and the decay instability, respectively. A kinetic description of the qZ equations was introduced in [3], by applying the Wigner transform [4] to the Langmuir propagation equation and then the modulational instability was performed to the resulting equations, which revealed that the modulational instability is enhanced due to the combination of partial coherence and quantum corrections. The existence of quantum solitons of the qZ system was investigated in [5] and several quantum solitary wave solutions were presented including the bright solitons, gray solitons, W-solitons and M-solitons (unfortunately, their results are incorrect, reasons will be given below). Recently, Haas [6] presented approximate solutions of equations (1) and (2) through a variational formulation and a trial function method.

It has been pointed out in [1] that a huge amount of physical and mathematical aspects already assessed in the classical Zakharov equations certainly have quantum counterparts, which ask for equally careful investigations. Therefore, in this paper, we will continue the investigation on the qZ equations by focusing on the Lie symmetries and similarity solutions. The work is organized as follows. In section 2, we obtain the Lie point symmetries of equations (1) and (2). It is found that the Lie symmetry vector fields are exactly the same as those of the classical Zakharov equations presented in [7]. The corresponding similarity solutions and similarity reduction equations are given in section 3. Section 4 contains a brief conclusion.

## 2. Lie point symmetries

In order to find the Lie symmetries of the qZ equations, we first write the complex electrical field as $E = u + iv$ with real fields $u$ and $v$. Substituting it into equations (1) and (2) and separating the imaginary and real parts leads to the following equations

$$\frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial x^2} - H^2\frac{\partial^4 v}{\partial x^4} - nv = 0, \tag{5}$$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 u}{\partial x^2} + H^2\frac{\partial^4 u}{\partial x^4} + nu = 0, \tag{6}$$

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} + H^2\frac{\partial^4 n}{\partial x^4} - \frac{\partial^2}{\partial x^2}\left(u^2+v^2\right) = 0. \tag{7}$$

The Lie point symmetries $\sigma_{\mathrm{p}}(\mathrm{p} = u, v, n)$ of the form

$$\sigma_{\mathrm{p}} = X\mathrm{p}_x + T\mathrm{p}_t - Q_{\mathrm{p}}, \tag{8}$$

with $X, T, Q_{\mathrm{p}}$ being functions of the variables ($x, t, u, v$ and $n$), are the solutions of the linearized equations of equations (5)– (7), namely,

$$\frac{\partial \sigma_u}{\partial t} + \frac{\partial^2 \sigma_v}{\partial x^2} - H^2\frac{\partial^4 \sigma_v}{\partial x^4} - \sigma_n v - n\sigma_v = 0, \tag{9}$$

$$\frac{\partial \sigma_v}{\partial t} - \frac{\partial^2 \sigma_u}{\partial x^2} + H^2\frac{\partial^4 \sigma_u}{\partial x^4} + \sigma_n u + n\sigma_u = 0, \tag{10}$$

$$\frac{\partial^2 \sigma_n}{\partial t^2} - \frac{\partial^2 \sigma_n}{\partial x^2} + H^2\frac{\partial^4 \sigma_n}{\partial x^4} - 2\frac{\partial^2}{\partial x^2}\left(u\sigma_u + v\sigma_v\right) = 0. \tag{11}$$

It means that equations (5)–(7) are form invariant under the transformations

$$u \to u + \epsilon\sigma_u, \quad v \to v + \epsilon\sigma_v, \quad n \to n + \epsilon\sigma_n, \tag{12}$$

with a small parameter $\epsilon$.

Substituting (8) into (9)–(11), eliminating the quantities $u_t, v_t, n_{tt}$ and their higher order derivatives by means of (5)–(7), and setting zero for all the coefficients of the independent terms of the polynomials of $u, v, n$ and their partial derivatives, an over-determined set of equations are produced for the unknown functions $X, T$ and $Q_{\mathrm{p}}(\mathrm{p} = u, v$ and $n$). Solving these over-determined equations, we obtain

$$X = c_1, \quad T = c_2, \quad Q_u = \left(\tfrac{1}{2}c_3t^2 + c_4t + c_5\right)v, \tag{13}$$

$$Q_v = -\left(\tfrac{1}{2}c_3t^2 + c_4t + c_5\right)u, \quad Q_n = c_3t + c_4, \tag{14}$$

where $c_i$ ($i = 1, 2, 3, 4$ and 5) are arbitrary constants. The presence of these arbitrary constants leads to a finite-dimensional Lie algebra of symmetries. The corresponding vector field is given by

$$\begin{aligned} V = {} & c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial t} + \left(\tfrac{1}{2}c_3t^2 + c_4t + c_5\right)v\frac{\partial}{\partial u} \\ & - \left(\tfrac{1}{2}c_3t^2 + c_4t + c_5\right)u\frac{\partial}{\partial v} + (c_3t + c_4)\frac{\partial}{\partial n}, \end{aligned} \tag{15}$$

whose Lie algebra is spanned by the following generators

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \tag{16}$$

$$V_4 = vt\frac{\partial}{\partial u} - ut\frac{\partial}{\partial v} + \frac{\partial}{\partial n}, \quad V_5 = \frac{v}{2}t^2\frac{\partial}{\partial u} - \frac{u}{2}t^2\frac{\partial}{\partial v} + t\frac{\partial}{\partial n}, \tag{17}$$

with only two nonvanishing commutators $[V_2, V_4] = V_3$ and $[V_2, V_5] = V_4$. It is seen that the vector field (15) is exactly the same as that of the classical Zakharov equations obtained in [7]. Consequently, the quantum corrections (the entrance of higher-order dispersion—the $H$ terms in equations (1) and (2)) do not have any effect on the symmetries of the underlying system.

## 3. Similarity reduction solutions

Similarity reduction solutions can be obtained from the characteristic equation

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{dp}}{Q_{\mathrm{p}}}, \quad \mathrm{p} = u, v, n, \tag{18}$$

with $X, T, Q_{\mathrm{p}}$ given by (13) and (14). For the most general generator $V$ (15), we obtain the similarity solutions

$$\begin{aligned} u = {} & A_1\sin\left(\tfrac{1}{6}c_3t^3 + \tfrac{1}{2}c_4t^2 + c_5t\right) + A_2\cos\left(\tfrac{1}{6}c_3t^3\right. \\ & \left. + \tfrac{1}{2}c_4t^2 + c_5t\right), \end{aligned} \tag{19}$$

$$\begin{aligned} v = {} & -A_2\sin\left(\tfrac{1}{6}c_3t^3 + \tfrac{1}{2}c_4t^2 + c_5t\right) \\ & + A_1\cos\left(\tfrac{1}{6}c_3t^3 + \tfrac{1}{2}c_4t^2 + c_5t\right), \end{aligned} \tag{20}$$

$$n = \tfrac{1}{2}c_3t^2 + c_4t + N, \tag{21}$$

where $c_i$ ($i = 2, 3, 4$ and 5) have been redefined, for simplicity without loss of generality, and $A_1$, $A_2$ and $N$ are similarity reduction functions of the similarity variable $\xi = x - c_2t$, satisfying the following similarity reduction equations

$$c_2\frac{\mathrm{d}A_1}{\mathrm{d}\xi} + \frac{\mathrm{d}^2A_2}{\mathrm{d}\xi^2} - H^2\frac{\mathrm{d}^4A_2}{\mathrm{d}\xi^4} + (c_5 - N)A_2 = 0, \tag{22}$$

$$c_2\frac{\mathrm{d}A_2}{\mathrm{d}\xi} - \frac{\mathrm{d}^2A_1}{\mathrm{d}\xi^2} + H^2\frac{\mathrm{d}^4A_1}{\mathrm{d}\xi^4} - (c_5 - N)A_1 = 0, \tag{23}$$

$$(c_2^2 - 1)\frac{\mathrm{d}^2N}{\mathrm{d}\xi^2} + H^2\frac{\mathrm{d}^4N}{\mathrm{d}\xi^4} - \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(A_1^2 + A_2^2\right) + c_3 = 0. \tag{24}$$

Therefore, once the coupled ordinary differential equations (ODEs) are solved, the solutions of the original system (1) and (2) can be obtained via (19)–(21). However, it is still difficult to obtain explicit analytical solutions of the nonlinearly coupled ODEs (22)–(24), hence, in order to investigate quantum effects on the electric field $E$ and the density fluctuation, numerical calculation has to be utilized by imposing appropriate initial and boundary conditions. Instead of doing numerical calculations, let us consider a special case. Evidently, if we take $A_1 = A_2 = 0$, then we have a single decoupled ODE of the similarity reduction function $N$, which has a general solution

$$N = -\frac{B_1 H^2}{c_2^2 - 1}\cos\left(\frac{\sqrt{c_2^2-1}}{H}\xi + \xi_0\right) - \frac{c_3}{2(c_2^2-1)}\xi^2 + B_3\xi + B_2, \qquad (25)$$

for $c_2^2 > 1$, where $\xi_0, B_i (i = 1, 2$ and $3)$ are integration constants. Then, transformed back to the variables of the original system, the exact similarity solution for $n$ reads

$$n = -\frac{B_1 H^2}{c_2^2 - 1}\cos\left(\frac{\sqrt{c_2^2-1}}{H}(x - c_2 t) + \xi_0\right) - \frac{c_3}{2(c_2^2-1)}x^2 + \left(\frac{c_2 c_3}{c_2^2-1}t + B_3\right)x - \frac{c_3}{2(c_2^2-1)}t^2 + (c_4 - c_2 B_3)t + B_2. \qquad (26)$$

The similarity solution (26) represents periodic waves with an arbitrary amplitude for $c_2^2 > 1$, corresponding to supersonic flow. In addition, the presence of arbitrary constants $c_3$, $c_4$, $B_2$ and $B_3$ implies the intrusion of constant linear and time-dependent nonlinear shears and time-dependent background. Two representative periodic similarity ion-acoustic waves are displayed in figure 1, where (a) corresponds to the case of a constant linear shear by setting $c_3 = 0$ and (b) is related to the case of a weak time-dependent nonlinear shear with $c_3 = 0.5$. Quantum effects on the periodic similarity ion-acoustic waves are graphically shown in figure 2 for different quantum parameter values $H = 0.5, 1$ and 3. It is seen that the quantum corrections lead to an increase of the spatial frequency of oscillations. Notice that when $B_3 = c_3 = c_4 = 0$, the solution (26) is identical to the periodic solution reported in [1].

It is noted that we could not obtain nonlinear coherent solutions of the qZ equations (1) and (2) by the Lie group approach. On the other hand, Hass [6] pointed out that quantum effects result in the decaying of the Langmuir solitons, and that the appearance of instabilities of a purely quantum nature might eventually destroy any coherent structures. Therefore, both show that quantum soliton solutions might not exist in qZ equations, which might also be a manifestation that the results presented in [5] are not correct. In fact, it can be easily checked by directly substituting their solutions (equations (13)–(18) in [5]) in the original system. In addition, there are two evident places demonstrating the error of those solutions.

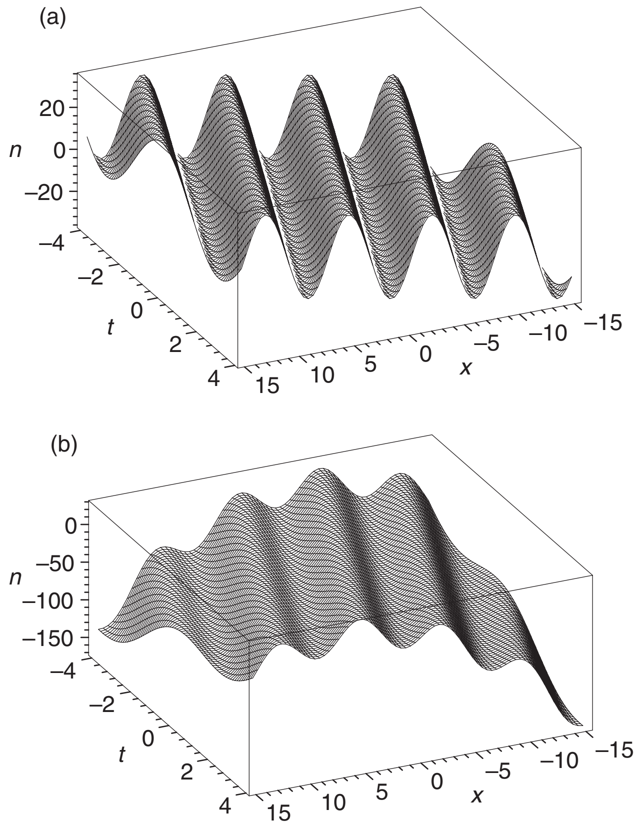


**Figure 1.** Plot of the periodic similarity ion-acoustic waves, given by (26) for the parameter values $\xi_0 = B_2 = 0$, $B_3 = c_4 = H = 1$, $c_2 = 1.3$, $B_1 = 20$ and $c_3 = 0$ for the constant linear shear (a), and $c_3 = 0.5$ for the time-dependent nonlinear shear (b), respectively.

Firstly, equation (8) in [5] is linear, and thus can be solved immediately to give a general solution. However, they just used this equation once and later did not require their solutions to satisfy this simple linear equation. Secondly, the result of their leading order analysis is erroneous because it is contradictory to that of the original system of equations.

## 4. Conclusion

The Lie symmetry analysis has been applied on the qZ system of equations, which model the nonlinear interactions between the quantum Langmuir and quantum ion-acoustic waves in an electron–ion dense quantum plasma. It is found that the classical Lie symmetries are exactly the same as those of the classical Zakharov equations, which thus shows that the quantum correction to the classical Zakharov system has no effect on the symmetries of the underlying system. Based on the Lie symmetries, similarity reduction solutions are obtained where the similarity reduction functions are determined by the coupled reduced ODEs.

It is noted that we could not obtain nonlinear coherent soliton solutions of the qZ system of equations. In a special case with $E = 0$, a pure general periodic similarity nonlinear ion-acoustic wave solution is obtained with the presence of constant linear and time-dependent nonlinear shears and time-dependent background. Figure 1 shows the ion-acoustic waves with different shears and background, and figure 2 shows the

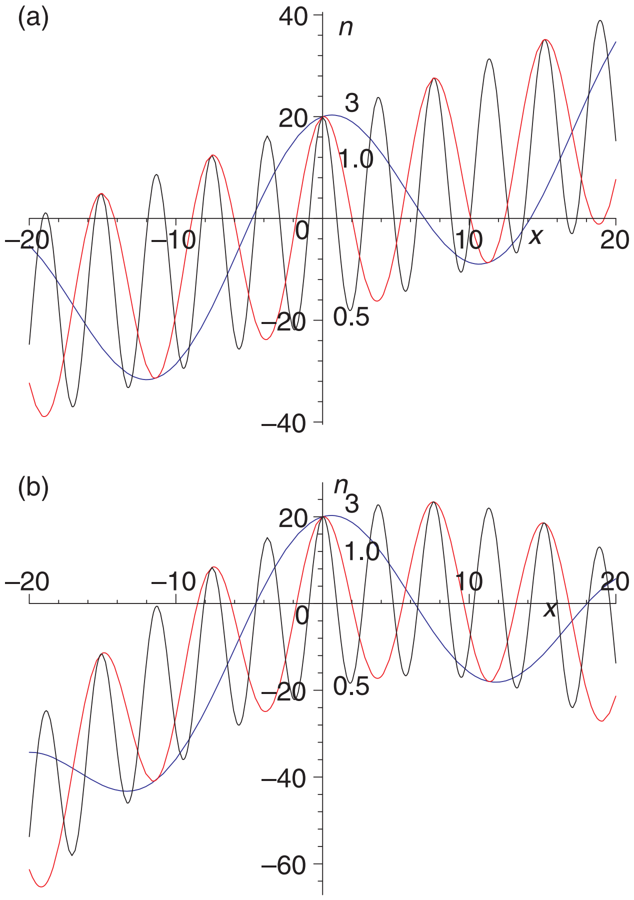


**Figure 2.** Plot of the periodic similarity ion-acoustic waves, given by (26) for the parameter values same as in figure 1, except that $c_3 = 0$ for (a) and $c_3 = 0.1$ for (b) at time $t = 0$, for different quantum parameters $H = 0.5$ (black), 1 (red), and 3 (blue), respectively.

profiles of the waves for different values of $H$, which depend on the ion number density and the Fermi electron temperature. It is found that the quantum effect increases the period of the waves, which is consistent with the results presented in [1].

It is known that the classical qZ equations are nonintegrable though they have envelope soliton solutions [8]. For instance, it has been proved to be a non-Painlevé system [9] because it cannot pass the Painlevé test at the third step for the compatibility condition at the resonance 2 is not satisfied. The qZ system of equations (1) and (2) might be nonintegrable as well since we are still not able to obtain soliton solutions of these equations. Moreover, following [9] we carry out the Painlevé analysis of the qZ equations and find that they cannot pass the Painlevé test at the second step for they have noninteger resonances $r$ determined by $(r^2 - 13r + 60)(r^2 - 9r + 38)(r^2 - 5r + 24) = 0$, besides $r = -1, 0, 6, 7, 9$ and 10.

## Acknowledgments

XYT acknowledges the financial support from the Alexander von Humboldt Foundation. The work was partially supported by Youth Foundation of Shanghai Jiao Tong University and National Natural Science Foundation of China (no. 10475055).

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