Application of Group Analysis to Delay Differential Equation

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Abstract. In this paper, Lie group analysis is applied to delay differential equations. It is shown method for constructing and solving determining equations and it is given example.

INTRODUCTION

Delay differential equations (DDEs) play a central role in many mathematical modelings. Unfortunately, most methods for solving particular types of DDEs come from the approach of trial and error. In this paper, Lie group analysis method, the powerful method [1], is applied to DDEs for finding symmetries. The sketch process is not different from the classical one.

The equations to be studied are[1]

$$x' = F(t, x_t), \tag{1}$$

where, for given $t \in [t_0, \beta)$ and given any function $\chi : [\gamma, t] \to D$,

$$F(t, \chi_t) \equiv f(t, \chi(g_1(t)), ..., \chi(g_m(t))),$$

D is an open subset in $\mathbb{R}^n$, x and f are n-vector-valued enough time differentiable functions, $f : [t_0, \beta) \times D^m \to \mathbb{R}^n$, and for each $j = 1, ..., m,\ \gamma \le g_j(t) \le t$ for $t_0 \le t \le \beta$.

Definition 1. A *solution* of equations (1), with initial condition $\theta(t)$ defined on $[\gamma, t_0]$, is a continuous function $x : [\gamma, \beta_1] \to D$, for some $\beta_1 \in (t_0, \beta]$ such that

1. $x(t) = \theta(t)$ for $\gamma \le t \le t_0$, and
2. $x'(t) = F(t, x_t)$ for $t_0 \le t \le \beta_1$.

Remark. We understand $x'(t_0)$ to mean the right-hand derivative.

[1]See, for example, in [2]

Nonlinear Acoustics at the Beginning of the 21st Century
edited by O.V. Rudenko and O.A. Sapozhnikov (Faculty of Physics, MSU, Moscow, 2002)
vol. 1, pp. 607-610

Definition 2. We define a *symmetry group* admitted by DDEs (1) according to Lie's definition of a symmetry group admitted by an equation[2], i.e., it is a group of transformations which transform a solution of DDEs to a solution of the same system.

The plan of the paper is as follows. The next section provides description of constructing and solving determining equations. Then it is given an example.

DETERMINING EQUATIONS OF DDEs

Constructing of Determining Equations

Like the classical method, we will let the symmetry group G of transformations T_a:

$$\bar{t} = T^t(t,x;a),\ \bar{x} = T^x(t,x;a),$$

depending on a real parameter a, $t \equiv T^t(t,x;0)$ and $x \equiv T^x(t,x;0)$, be admitted by DDEs (1). Suppose $x = \chi(t)$ be a solution, we relate this solution with $\bar{\chi}(\bar{t})$ by $\bar{t} = T^t(t,\chi(t);a)$ and $\bar{x} = T^x(t,\chi(t);a)$. In order to write $\bar{x}$ as a function of $\bar{t}$, we first have to derive

$$t = \psi(\bar{t};a) \tag{2}$$

from $\bar{t} = T^t(t,\chi(t);a)$. Compared with ODEs, in order to consider DDEs' solutions, one has to consider functions not only of a neighborhood of an initial point, but of initial interval $[\gamma,t_0]$ and a neighborhood of t_0. Local inverse function theorem is not enough to guarantee obtaining (2), we have to require *"global inverse function theorem"*.

We can write down $\bar{x}$ as a function of $\bar{t}$ by $\bar{x} = \bar{\chi}(\bar{t}) = T^x(\psi(\bar{t};a),\chi(\psi(\bar{t};a));a)$. So

$$\bar{x}' = \frac{d\bar{\chi}(\bar{t})}{d\bar{t}} = (T^x_{,1} + T^x_{,2}\chi'(\psi(\bar{t};a)))\frac{\partial\psi(\bar{t};a)}{\partial\bar{t}}$$

here $f_{,i}$ means the partial derivative of f with respect to i-th argument, and

$$F(\bar{t},\bar{x}_{\bar{t}}) = f(\bar{t},T^x(\psi(g_1(\bar{t});a),\chi(\psi(g_1(\bar{t});a));a),\ldots,T^x(\psi(g_m(\bar{t});a),\chi(\psi(g_m(\bar{t});a));a).$$

Let $\Xi(t,x_t,x') = x' - F(t,x_t)$, then

$$\Xi(t,x_t,x') \equiv 0, \tag{3}$$

for $x = \chi(t)$ and t in some considered interval, and also $\Xi(\bar{t},\bar{x}_{\bar{t}},\bar{x}') \equiv 0$:

$$\left.\frac{\partial\Xi(\bar{t},\bar{x}_{\bar{t}},\bar{x}')}{\partial a}\right|_{a=0} = 0. \tag{4}$$

Definition 3. The equations

$$\left.\frac{\partial\Xi(\bar{t},\bar{x}_{\bar{t}},\bar{x}')}{\partial a}\right|_{a=0,(3)} = 0 \tag{5}$$

are called determining equations.

[2] See p.101, section 1, chapter 6 in [3]

Solving Determining Equations

Let $\xi(t,x) = \left.\frac{\partial T^t(t,x;a)}{\partial a}\right|_{a=0}$ and $\eta(t,x) = \left.\frac{\partial T^x(t,x;a)}{\partial a}\right|_{a=0}$. From the previous section, we obtain

$$\left.\frac{\partial \bar{\chi}'(\bar{t})}{\partial a}\right|_{a=0} = \eta_{,1}(t,\chi(t)) + (\eta_{,2}(t,\chi(t)) - \tag{6}$$

$$\xi_{,1}(t,\chi(t)))\chi'(t) - \xi_{,2}(t,\chi(t))[\chi'(t)]^2 - \xi(t,\chi(t))\chi''(t),$$

$$\left.\frac{\partial \bar{\chi}(g_\lambda(\bar{t}))}{\partial a}\right|_{a=0} = -\xi(g_\lambda(t),\chi(g_\lambda(t)))\chi'(g_\lambda(t)) + \eta(g_\lambda(t),\chi(g_\lambda(t))).$$

Let Δ and Θ_λ denote the right side terms of equations (6) respectively. Equations (4) become

$$\sum_{\lambda=1}^{m} \Xi_{,\lambda+1}(t,x_t,x')\Theta_\lambda + \Xi_{,m+2}\Delta = 0. \tag{7}$$

To solve determining equations, we have to consider (7) on the manifold (3). Then the involved variable and functions are:

- $t, \chi(t), \chi(g_\lambda(t)), \chi'(g_\lambda(t)), \chi''(g_\lambda(t)), \ldots$
- functions η and ξ of $(t,\chi(t))$ or of $(g_\lambda(t),\chi(g_\lambda(t)))$ and their derivatives $\eta_{,1}, \xi_{,1}, \eta_{,2}, \xi_{,2}, \ldots$

where $\lambda = 1,\ldots,m$.

Determining equations (5) have to be satisfied identically for any t in a neighborhood of t_0, $t \geq t_0$, and for any solution $\chi(t)$ of equations (1). For any initial function $\theta(t), t \in [\gamma, t_0]$, *existence theory* [2] guarantees existence of solution $\chi(t)$, therefore, by arbitrariness of $t \in [\gamma, \beta_1]$ and initial function $\theta(t)$, the determining equations can be split into a system of several equations and can be solved analytically, as it is done for ODEs [4].

Example

Let us consider

$$x''(t) - x(t) + x(t-r) = 0, \tag{8}$$

$$x'(t_0) = x_1, \quad x(s) = \theta(s), \;\; s \in [t_0 - r, t_0], \;\; r > 0.$$

The determining equation for equation (7) is

$$\begin{gathered}
\eta_{,11}(t,\chi(t)) + \chi'(t)(2\eta_{,12}(t,\chi(t)) - \xi_{,11}(t,\chi(t))) + \\
[\chi'(t)]^2(\eta_{,22}(t,\chi(t)) - 2\xi_{,2}(t,\chi(t))) - [\chi'(t)]^3\xi_{,22}(t,\chi(t)) + \\
[\chi(t) - \chi(t-r)](\eta_{,2}(t,\chi(t)) - 2\xi_{,1}(t,\chi(t))) - 3[\chi'(t)(x(t) - x(t-r))]\xi_{,2}(t,\chi(t)) - \\
\xi(t,\chi(t))(\chi'(t) - \chi'(t-r)) + \xi(t,\chi(t))\chi'(t) - \eta(t,\chi(t)) - \\
\xi(t-r,\chi(t-r))\chi'(t-r) + \eta(t-r,\chi(t-r)) = 0
\end{gathered}$$

The generator obtained from solving the determining equation is

$$X = h(t)\partial_t + c_1 x \partial x,$$

where c_1 is an arbitrary constant and the function $h(t)$ is an arbitrary solution of equation (8).

CONCLUSION

In the strength of the theory of existence of solution of the problem for DDEs, the process of solving determining equations for DDEs is similar to obtaining solutions of determining equations for differential equations.

In this paper, it is shown that Lie group analysis can be applied to DDEs. And it can also be applied to other types of DEs, like integro-differential equations, functional differential equations, etc.

ACKNOWLEDGEMENTS

This research was supported by the Royal Golden Jubilee Ph.D. Program of Thailand. The authors are thankful to N.H. Ibragimov for giving reference on [3] and participants of the seminar of L.V. Ovsiannikov for fruitful discussions concerning the topics of the paper.

REFERENCES

1. Ovsiannikov, L.V., *Group Analysis of Differential Equations*, (Nauka, Moscow, 1978), English translation: ed. by W.F. Ames (Academic Press, New York, 1982).
2. Driver, R.D., *Ordinary and Delay Differential Equation applied Mathematical Sciences* (Springer-Verlag, New York, 1977).
3. Lie, S., *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen (Bearbeitet und herausgegeben von Dr. G.Scheffers)*, B.G. Teubner, Leipzig, 1891.
4. Ibragimov, N.H. (ed.), *CRC Handbook of Lie Group Analysis of Differential Equations* (CRC Press, Boca Raton, FL, **1**, 1994, **2**, 1995, **3**, 1996).