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CALCULUS

Basic Concepts for High Schools

Translated from the Russian
by
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PREFACE

> Many objects are obscure to us not because our perceptions are poor, but simply because these objects are outside of the realm of our conceptions.
>
> *Kosma Prutkov*

CONFESSION OF THE AUTHOR. My first acquaintance with calculus (or mathematical analysis) dates back to nearly a quarter of a century. This happened in the Moscow Engineering Physics Institute during splendid lectures given at that time by Professor D. A. Vasilkov. Even now I remember that feeling of delight and almost happiness. In the discussions with my classmates I rather heatedly insisted on a simile of higher mathematics to literature, which at that time was to me the most admired subject. Sure enough, these comparisons of mine lacked in objectivity. Nevertheless, my arguments were to a certain extent justified. The presence of an inner logic, coherence, dynamics, as well as the use of the most precise words to express a way of thinking, these were the characteristics of the prominent pieces of literature. They were present, in a different form of course, in higher mathematics as well. I remember that all of a sudden elementary mathematics which until that moment had seemed to me very dull and stagnant, turned to be brimming with life and inner motion governed by an impeccable logic.

Years have passed. The elapsed period of time has inevitably erased that highly emotional perception of calculus which has become a working tool for me. However, my memory keeps intact that unusual happy feeling which I experienced at the time of my initiation to this extraordinarily beautiful world of ideas which we call higher mathematics.

CONFESSION OF THE READER. Recently our professor of mathematics told us that we begin to study a new subject which he called calculus. He said that this subject is a foundation of higher mathematics and that it is going to be very difficult. We have already studied real numbers, the real line, infinite numerical sequences, and limits of sequences. The professor was indeed right saying that comprehension of the subject would present difficulties. I listen very carefully to his explanations and during the same day study the relevant pages of my textbook. I seem to understand everything, but at the same time have a feeling of a certain dissatisfaction. It is difficult for me to construct a consistent picture out of the pieces obtained in the classroom. It is equally difficult to remember exact wordings and definitions, for example, the definition of the limit of sequence. In other words, I fail to grasp something very important.

Perhaps, all things will become clearer in the future, but so far calculus has not become an open book for me. Moreover, I do not see any substantial difference between calculus and algebra. It seems that everything has become rather difficult to perceive and even more difficult to keep in my memory.

COMMENTS OF THE AUTHOR. These two confessions provide an opportunity to get acquainted with the two interlocutors in this book. In fact, the whole book is presented as a relatively free-flowing dialogue between the AUTHOR and the READER. From one discussion to another the AUTHOR will lead the inquisitive and receptive READER to different notions, ideas, and theorems of calculus, emphasizing especially complicated or delicate aspects, stressing the inner logic of proofs, and attracting the reader's attention to special points. I hope that this form of presentation will help a reader of the book in learning new definitions such as those of *derivative*, *antiderivative*, *definite integral*, *differential equation*, etc. I also expect that it will lead the reader to better understanding of such concepts as *numerical sequence*, *limit of sequence*, and *function*. Briefly, these discussions are intended to assist pupils entering a novel world of calculus. And if in the long run the reader of the book gets a feeling of the intrinsic beauty and integrity of higher mathematics or even is appealed to it, the author will consider his mission as successfully completed.

Working on this book, the author consulted the existing manuals and textbooks such as *Algebra and Elements of Analysis* edited by A. N. Kolmogorov, as well as the specialized textbook by N. Ya. Vilenkin and S. I. Shvartsburd *Calculus*. Appreciable help was given to the author in the form of comments and recommendations by N. Ya. Vilenkin, B. M. Ivlev, A. M. Kisin, S. N. Krachkovsky, and N. Ch. Krutitskaya, who read the first version of the manuscript. I wish to express gratitude for their advice and interest in my work. I am especially grateful to A. N. Tarasova for her help in preparing the manuscript.

CONTENTS

DIALOGUE ONE

INFINITE NUMERICAL SEQUENCE

AUTHOR. Let us start our discussions of calculus by considering the definition of an *infinite numerical sequence* or simply a *sequence*.

We shall consider the following examples of sequences:

$$1, 2, 4, 8, 16, 32, 64, 128, \ldots \tag{1}$$

$$5, 7, 9, 11, 13, 15, 17, 19, \ldots \tag{2}$$

$$1, 4, 9, 16, 25, 36, 49, 64, \ldots \tag{3}$$

$$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, 2\sqrt{2}, \ldots \tag{4}$$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \ldots \tag{5}$$

$$2, 0, -2, -4, -6, -8, -10, -12, \ldots \tag{6}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots \tag{7}$$

$$1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots \tag{8}$$

$$1, -1, \frac{1}{3}, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{7} \ldots \tag{9}$$

$$1, \frac{2}{3}, \frac{1}{3}, \frac{4}{4}, \frac{1}{5}, \frac{6}{7}, \frac{1}{7}, \frac{8}{9}, \ldots \tag{10}$$

Have a closer look at these examples. What do they have in common?

READER. It is assumed that in each example there must be an infinite number of terms in a sequence. But in general, they are all different.

AUTHOR. In each example we have eight terms of a sequence. Could you write, say, the ninth term?

READER. Sure, in the first example the ninth term must be 256, while in the second example it must be 21.

AUTHOR. Correct. It means that in all the examples there is a *certain law*, which makes it possible to write down the ninth, tenth, and other terms of the sequences. Note, though, that if there is a *finite number* of terms in a sequence, one may fail to discover the law which governs the infinite sequence.

READER. Yes, but in our case these laws are easily recognizable. In example (1) we have the terms of an infinite geometric progression with common ratio 2. In example (2) we notice a sequence of odd numbers starting from 5. In example (3) we recognize a sequence of squares of natural numbers.

AUTHOR. Now let us look at the situation more rigorously. Let us enumerate all the terms of the sequence in sequential order, i.e. 1, 2, 3, . . ., n, ... There is a certain law (a rule) by which each of these natural numbers is *assigned* to a certain number (the corresponding term of the sequence). In example (1) this arrangement is as follows:

$$\begin{array}{cccccccccl} 1 & 2 & 4 & 8 & 16 & 32 & \ldots & 2^{n-1} & \ldots & \text{(terms of the sequence)} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & & \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n & \ldots & \text{(position numbers of the terms)} \end{array}$$

In order to describe a sequence it is sufficient to indicate the term of the sequence corresponding to the number n, i.e. to write down the term of the sequence occupying the nth position. Thus, we can formulate the following definition of a sequence.

Definition:

We say that there is an infinite numerical sequence if every natural number (position number) is unambiguously placed in correspondence with a definite number (term of the sequence) by a specific rule.

This relationship may be presented in the following general form

$$\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & \ldots & y_n & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & \\ 1 & 2 & 3 & 4 & 5 & \ldots & n & \ldots \end{array}$$

The number y_n is the nth term of the sequence, and the whole sequence is sometimes denoted by a symbol (y_n).

READER. We have been given a somewhat different definition of a sequence: a sequence is a function defined on a set of natural numbers (integers).

AUTHOR. Well, actually the two definitions are equivalent. However, I am not inclined to use the term "function" too early. First, because the discussion of a function will come later. Second, you will normally deal with somewhat different functions, namely those defined not on a set of integers but on the real line or within its segment. Anyway, the above definition of a sequence is quite correct.

Getting back to our examples of sequences, let us look in each case for an *analytical expression* (*formula*) for the nth term. Go ahead.

READER. Oh, this is not difficult. In example (1) it is $y_n = 2^n$. In (2) it is $y_n = 2n + 3$. In (3) it is $y_n = n^2$. In (4) it is $y_n = \sqrt{n}$. In (5) it is $y_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$. In (6) it is $y_n = 4 - 2n$. In (7) it is $y_n = \frac{1}{n}$. In the remaining three examples I just do not know.

AUTHOR. Let us look at example (8). One can easily see that if n is an even integer, then $y_n = \frac{1}{n}$, but if n is odd, then $y_n = n$. It means that

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n = 2k \\ n & \text{if } n = 2k-1 \end{cases}$$

READER. Can I, in this particular case, find a *single* analytical expression for y_n?

AUTHOR. Yes, you can. Though I think you needn't. Let us present y_n in a different form:

$$y_n = a_n n + b_n \frac{1}{n}$$

and demand that the coefficient a_n be equal to unity if n is odd, and to zero if n is even; the coefficient b_n should behave in quite an opposite manner. In this particular case these coefficients can be determined as follows:

$$a_n = \frac{1}{2}[1 - (-1)^n]; \quad b_n = \frac{1}{2}[1 + (-1)^n]$$

Consequently,

$$y_n = \frac{n}{2}[1-(-1)^n] + \frac{1}{2n}[1+(-1)^n]$$

Do in the same manner in the other two examples.

READER. For sequence (9) I can write

$$y_n = \frac{1}{2n}[1-(-1)^n] - \frac{1}{2(n-1)}[1+(-1)^n]$$

and for sequence (10)

$$y_n = \frac{1}{2n}[1-(-1)^n] + \frac{n}{2(n+1)}[1+(-1)^n]$$

AUTHOR. It is important to note that an analytical expression for the nth term of a given sequence is not necessarily a unique method of defining a sequence. A sequence can be defined, for example, by *recursion* (or the *recurrence method*) (Latin word *recurrere* means to run back). In this case, in order to define a sequence one should describe the first term (or the first several terms) of the sequence and a recurrence (or a recursion) relation, which is an expression for the nth term of the sequence via the preceding one (or several preceding terms).

Using the recurrence method, let us present sequence (1) as follows

$$y_1 = 1; \qquad y_n = 2y_{n-1}$$

READER. It's clear. Sequence (2) can be apparently represented by formulas

$$y_1 = 5; \qquad y_n = y_{n-1} + 2$$

AUTHOR. That's right. Using recursion, let us try to determine one interesting sequence

$$y_1 = 1; \qquad y_2 = 1; \qquad y_n = y_{n-2} + y_{n-1}$$

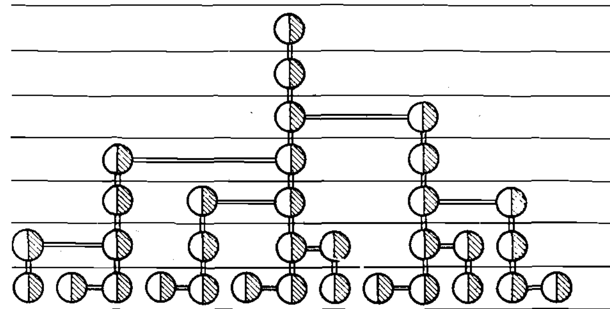
Its first terms are

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots \qquad (11)$$

This sequence is known as the *Fibonacci sequence* (or *numbers*).

READER. I understand, I have heard something about the problem of Fibonacci rabbits.

AUTHOR. Yes, it was this problem, formulated by Fibonacci, the 13th century Italian mathematician, that gave the name to this sequence (11). The problem reads as follows. A man places a pair of newly born rabbits into a warren and wants to know how many rabbits he would have over a certain period of time. A pair of rabbits will start producing offspring two months after they were born and every following month one new pair of rabbits will appear. At the beginning (during the first month) the man will have in his warren only one pair of rabbits ($y_1 = 1$); during the second month he will have the same pair of rabbits ($y_2 = 1$); during the third month the offspring will appear, and therefore the number of the pairs of rabbits in the warren will grow to two ($y_3 = 2$); during the fourth month there will be one more reproduction of the first pair ($y_4 = 3$); during the fifth month there will be offspring both from the first and second couples of rabbits ($y_5 = 5$), etc. An increase of the number of pairs in the warren from month to month is plotted in Fig. 1. One can see that the numbers of pairs of rabbits counted at the end of each month form sequence (11), i.e. the Fibonacci sequence.



n	y_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13

Symbol ◍ denotes one pair of rabbits

Fig. 1.

READER. But in reality the rabbits do not multiply in accordance with such an idealized pattern. Furthermore, as time goes on, the first pairs of rabbits should obviously stop proliferating.

AUTHOR. The Fibonacci sequence is interesting not because it describes a simplified growth pattern of rabbits' population. It so happens that this sequence appears, as if by magic, in quite unexpected situations. For example, the Fibonacci numbers are used to process information by computers and to optimize programming for computers. However, this is a digression from our main topic.

Getting back to the ways of describing sequences, I would like to point out that the *very method chosen to describe a sequence is not of principal importance.* One sequence may be described, for the sake of convenience, by a formula for the nth term, and another (as, for example, the Fibonacci sequence), by the recurrence method. What is important, however, is the method used to describe the *law of correspondence*, i.e. the law by which any natural number is placed in correspondence with a certain term of the sequence. In a

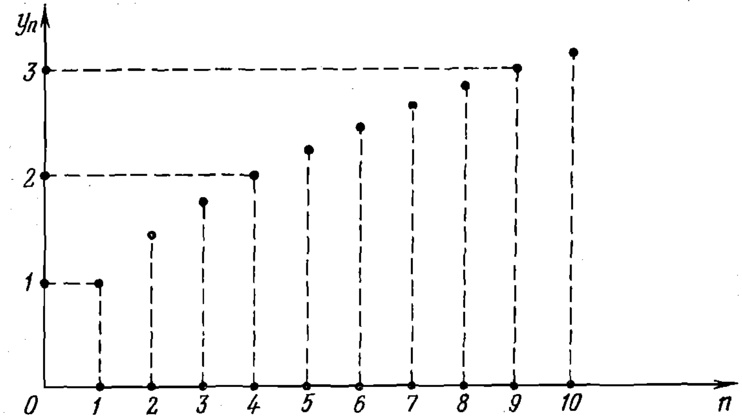


Fig. 2

number of cases such a law can be formulated only by words. The examples of such cases are shown below:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ \ldots \qquad (12)$$

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ \ldots \qquad (13)$$

In both cases we cannot indicate either the formula for the nth term or the recurrence relation. Nevertheless, you can without great difficulties identify specific laws of correspondence and put them in words.

READER. Wait a minute. Sequence (12) is a sequence of *prime numbers* arranged in an increasing order, while (13) is, apparently, a sequence composed of decimal approximations, with deficit, for π.

AUTHOR. You are absolutely right.

READER. It may seem that a numerical sequence differs from a random set of numbers by a presence of an intrinsic *degree of order* that is reflected either by the formula for the nth term or by the recurrence relation. However, the last two examples show that such a degree of order needn't be present.

AUTHOR. Actually, a degree of order determined by a formula (an analytical expression) is not mandatory. It is important, however, to have a law (a rule, a characteristic) of correspondence, which enables one to relate any natural number to a certain term of a sequence. In examples (12) and (13) such laws of correspondence are obvious. Therefore, (12) and (13) are not inferior (and not superior) to sequences

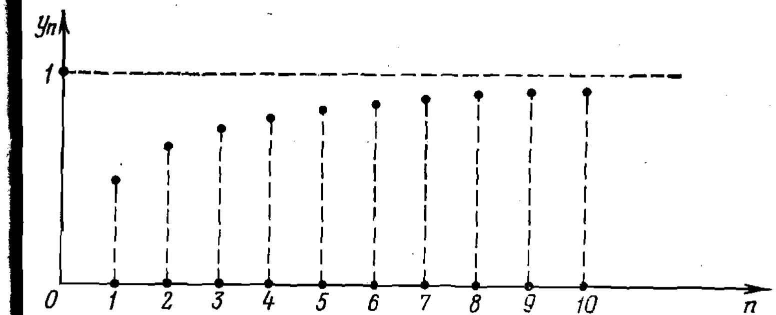


Fig. 3

1)-(11) which permit an analytical description.

Later we shall talk about the *geometric image* (or *map*) of a numerical sequence. Let us take two coordinate axes, and y. We shall mark on the first axis integers 1, 2, 3, ..., n, . . ., and on the second axis, the corresponding terms of a sequence, i.e. the numbers y_1, y_2, y_3, . . ., y_n, ... Then the sequence can be represented by a set of points $M(n, y_n)$ on the coordinate plane. For example, Fig. 2 images sequence (4), Fig. 3 images sequence (5), Fig. 4 images sequence (9), and Fig. 5 images sequence (10).

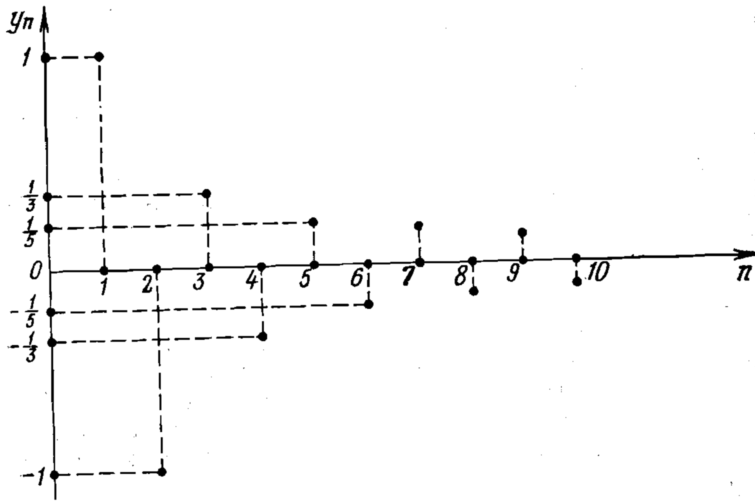


Fig. 4

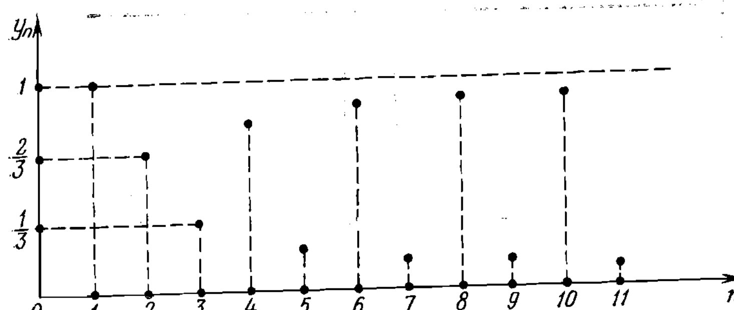


Fig. 5

As a matter of fact, there are other types of geometr images of a numerical sequence. Let us retain, for exampl only one coordinate y-axis and plot on it points y_1, y

y_3, . . ., y_n, . . . which map the terms of a sequence. In Fig. 6 this method of mapping is illustrated for the sequences that have been shown in Figs. 2-5. One has to admit that the latter method is less descriptive in comparison with the former method.

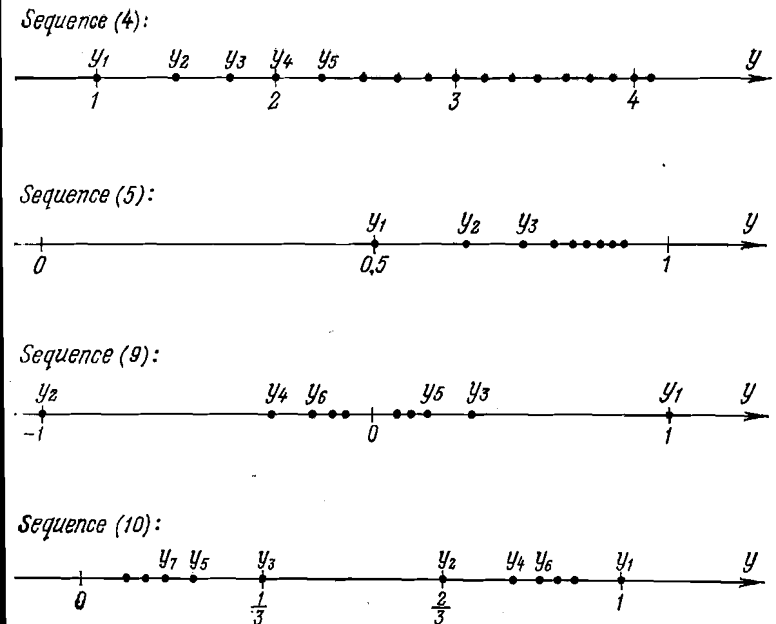


Fig. 6

READER. But in the case of sequences (4) and (5) the second method looks rather obvious.

AUTHOR. It can be explained by specific features of these sequences. Look at them closer.

READER. The terms of sequences (4) and (5) possess the following property: each term is greater than the preceding term

$$y_1 < y_2 < y_3 < \ldots < y_n < \ldots$$

It means that all the terms are arranged on the y-axis according to their serial numbers. As far as I know, such sequences are called *increasing*.

AUTHOR. A more general case is that of *nondecreasing* sequences provided we add the equality sign to the above series of inequalities.

Definition:

A sequence (y_n) *is called nondecreasing if*

$$y_1 \leqslant y_2 \leqslant y_3 \leqslant \ldots \leqslant y_n \leqslant \ldots$$

A sequence (y_n) *is called nonincreasing if*

$$y_1 \geqslant y_2 \geqslant y_3 \geqslant \ldots \geqslant y_n \geqslant \ldots$$

Nondecreasing and nonincreasing sequences come under the name of monotonic sequences.

Please, identify monotonic sequences among examples (1)-(13).

READER. Sequences (1), (2), (3), (4), (5), (11), (12), and (13) are nondecreasing, while (6) and (7) are nonincreasing. Sequences (8), (9), and (10) are not monotonic.

AUTHOR. Let us formulate one more

Definition:

A sequence (y_n) *is bounded if there are two numbers A and B, labelling the range which encloses all the terms of a sequence*

$$A \leqslant y_n \leqslant B \quad (n = 1, 2, 3, \ldots)$$

If it is impossible to identify such two numbers (or, in particular, one can find only one of the two such numbers, either the least or the greatest), such a sequence is *unbounded*.

Do you find bounded sequences among our examples?

READER. Apparently, (5) is bounded.

AUTHOR. Find the numbers A and B for it.

READER. $A = \frac{1}{2}$, $B = 1$.

AUTHOR. Of course, but if there exists even one pair of A and B, one may find any number of such pairs. You could say, for example, that $A = 0$, $B = 2$, or $A = -100$, $B = 100$, etc., and be equally right.

READER. Yes, but my numbers are more accurate.

AUTHOR. From the viewpoint of the bounded sequence definition, my numbers A and B are not better and not worse than yours. However, your last sentence is peculiar. What do you mean by saying "more accurate"?

READER. My A is apparently the greatest of all possible lower bounds, while my B is the least of all possible upper bounds.

AUTHOR. The first part of your statement is doubtlessly correct, while the second part of it, concerning B, is not so self-explanatory. It needs proof.

READER. But it seemed rather obvious. Because all the terms of (5) increase gradually, and evidently tend to unity, always remaining less than unity.

AUTHOR. Well, it is right. But it is not yet evident that $B = 1$ is the least number for which $y_n \leqslant B$ is valid for all n. I stress the point again: your statement is not self-evident, it needs proof.

I shall note also that "self-evidence" of your statement about $B = 1$ is nothing but your subjective impression; it is not a mathematically substantiated corollary.

READER. But how to prove that $B = 1$ is, in this particular case, the least of all possible upper bounds?

AUTHOR. Yes, it can be proved. But let us not move too fast and by all means beware of excessive reliance on so-called self-evident impressions. The warning becomes even more important in the light of the fact that the boundedness of a sequence does not imply at all that the greatest A or the least B must be known explicitly.

Now, let us get back to our sequences and find other examples of bounded sequences.

READER. Sequence (7) is also bounded (one can easily find $A = 0$, $B = 1$). Finally, bounded sequences are (9) (e.g. $A = -1$, $B = 1$), (10) (e.g. $A = 0$, $B = 1$), and (13) (e.g. $A = 3$, $B = 4$). The remaining sequences are unbounded.

AUTHOR. You are quite right. Sequences (5), (7), (9), (10), and (13) are bounded. Note that (5), (7), and (13) are bounded and at the same time monotonic. Don't you feel that this fact is somewhat puzzling?

READER. What's puzzling about it?

AUTHOR. Consider, for example, sequence (5). Note that each subsequent term is greater than the preceding one. I repeat, each term! But the sequence contains an *infinite number* of terms. Hence, if we follow the sequence far enough, we shall see as many terms with increased magnitude (compared to the preceding term) as we wish. Nevertheless, these values will never go beyond a certain "boundary", which in this case is unity. Doesn't it puzzle you?

READER. Well, generally speaking, it does. But I notice that we add to each preceding term an increment which gradually becomes less and less.

AUTHOR. Yes, it is true. But this condition is obviously insufficient to make such a sequence bounded. Take, for example, sequence (4). Here again the "increments" added to each term of the sequence gradually decrease; nevertheless, the sequence is not bounded.

READER. We must conclude, therefore, that in (5) these "increments" diminish faster than in (4).

AUTHOR. All the same, you have to agree that it is not immediately clear that these "increments" may decrease at a rate resulting in the boundedness of a sequence.

READER. Of course, I agree with that.

AUTHOR. The possibility of infinite but bounded sets was not known, for example, to ancient Greeks. Suffice it to recall the famous paradox about Achilles chasing a turtle.

Let us assume that Achilles and the turtle are initially separated by a distance of 1 km. Achilles moves 10 times faster than the turtle. Ancient Greeks reasoned like this: during the time Achilles covers 1 km the turtle covers 100 m. By the time Achilles has covered these 100 m, the turtle will have made another 10 m, and before Achilles has covered these 10 m, the turtle will have made 1 m more, and so on. Out of these considerations a paradoxical conclusion was derived that Achilles could never catch up with the turtle.

This "paradox" shows that ancient Greeks failed to grasp the fact that a monotonic sequence may be bounded.

READER. One has to agree that the presence of both the monotonicity and boundedness is something not so simple to understand.

AUTHOR. Indeed, this is not so simple. It brings us close to a discussion on the limit of sequence. The point is that if a sequence is both monotonic and bounded, it should necessarily have a limit.

Actually, this point can be considered as the "beginning" of calculus.

DIALOGUE TWO

LIMIT OF SEQUENCE

AUTHOR. What mathematical operations do you know?

READER. Addition, subtraction, multiplication, division, involution (raising to a power), evolution (extracting a root), and taking a logarithm or a modulus.

AUTHOR. In order to pass from elementary mathematics to higher mathematics, this "list" should be supplemented with one more mathematical operation, namely, that of finding the limit of sequence; this operation is called sometimes the limit transition (or passage to the limit). By the way, we shall clarify below the meaning of the last phrase of the previous dialogue, stating that calculus "begins" where the limit of sequence is introduced.

READER. I heard that higher mathematics uses the operations of *differentiation* and *integration*.

AUTHOR. These operations, as we shall see, are in essence nothing but the variations of the limit transition.

Now, let us get down to the concept of the *limit of sequence*. Do you know what it is?

READER. I learned the definition of the limit of sequence However, I doubt that I can reproduce it from memory.

AUTHOR. But you seem to "feel" this notion somehow? Probably, you can indicate which of the sequences discussed above have limits and what the value of the limit is in each case.

READER. I think I can do this. The limit is 1 for sequence (5), zero for (7) and (9), and π for (13).

AUTHOR. That's right. The remaining sequences have no limits.

READER. By the way, sequence (9) is not monotonic ... .

AUTHOR. Apparently, you have just remembered the end of our previous dialogue where it was stated that if a sequence is both monotonic and bounded, it has a limit.

READER. That's correct. But isn't this a contradiction?

AUTHOR. Where do you find the contradiction? Do you think that from the statement "If a sequence is both monotonic and bounded, it has a limit" one should necessarily draw a reverse statement like "If a sequence has a limit, it must be monotonic and bounded"? Later we shall see that a necessary condition for a limit is only the boundedness of a sequence. The monotonicity is not mandatory at all; consider, for example, sequence (9).

Let us get back to the concept of the limit of sequence. Since you have correctly indicated the sequences that have limits, you obviously have some understanding of this concept. Could you formulate it?

READER. A limit is a number to which a given sequence tends (converges).

AUTHOR. What do you mean by saying "converges to a number"?

READER. I mean that with an increase of the serial number, the terms of a sequence converge very closely to a certain value.

AUTHOR. What do you mean by saying "very closely"?

READER. Well, the difference between the values of the terms and the given number will become infinitely small. Do you think any additional explanation is needed?

AUTHOR. The definition of the limit of sequence which you have suggested can at best be classified as a subjective impression. We have already discussed a similar situation in the previous dialogue.

Let us see what is hidden behind the statement made above. For this purpose, let us look at a rigorous definition of the limit of sequence which we are going to examine in detail.

Definition:

The number a is said to be the limit of sequence (y_n) if for any positive number ε there is a real number N such that for all $n > N$ the following inequality holds:

$$|y_n - a| < \varepsilon \qquad (1)$$

READER. I am afraid, it is beyond me to remember such a definition.

AUTHOR. Don't hasten to remember. Try to comprehend this definition, to realize its *structure* and its *inner logic*. You will see that every word in this phrase carries a definite and necessary content, and that no other definition of the limit of sequence could be more succinct (more delicate, even).

First of all, let us note the logic of the sentence. A certain number is the limit provided that for any $\varepsilon > 0$ *there is* a number N such that for all $n > N$ inequality (1) *holds*. In short, it is necessary that for any ε a certain number N *should exist*.

Further, note two "delicate" aspects in this sentence. First, the number N should exist for *any* positive number ε. Obviously, there is an infinite set of such ε. Second, inequality (1) should hold always (i.e. for each ε) for *all* $n > N$. But there is an equally infinite set of numbers n!

READER. Now, the definition of the limit has become more obscure.

AUTHOR. Well, it is natural. So far we have been examining the definition "piece by piece". It is very important that the "delicate" features, the "cream", so to say, are spotted from the very outset. Once you understand them, everything will fall into place.

In Fig. 7a there is a graphic image of a sequence. Strictly speaking, the first 40 terms have been plotted on the graph. Let us assume that if any regularity is noted in these 40 terms, we shall conclude that the regularity does exist for $n > 40$.

Can we say that this sequence converges to the number a (in other words, the number a is the limit of the sequence)?

READER. It seems plausible.

AUTHOR. Let us, however, act not on the basis of our impressions but on the basis of the definition of the limit of sequence. So, we want to verify whether the number a is the limit of the given sequence. What does our definition of the limit prescribe us to do?

READER. We should take a positive number ε.

AUTHOR. Which number?

READER. Probably, it must be small enough.