The Fundamental Theorem of Calculus and the Poincaré Lemma

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Introduction

The Fundamental Theorem of Calculus is usually stated in two parts. The "evaluation" part gives

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

It can be viewed as a method of calculating information about the boundary values of a function, when you know the rates of change of the function in the interior of its domain. This part generalizes to the (generalized) Stokes' Theorem in higher dimensions.

The other part of the Fundamental Theorem gives

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x) \tag{$*$}$$

for f continuous. Are there higher dimensional versions of this part? Indeed they have been known for a long time as the Poincaré Lemma, which we describe. In addition to its intrinsic interest, equation (*) allows us to calculate many wonderful things, such as the speed and altitude of an object falling under the acceleration of gravity. The same theorem is used to define useful functions such as the logarithm, ln, and the error function, erf. In fact it provides a solution to every differential equation

$$y'(x) = f(x) \quad \text{with } y(x_0) = y_0$$

in which f is a given continous function. The solution is

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\, dt.$$

The Poincaré Lemma similarly gives solutions to certain partial differential equations. The usual statement of the Poincaré Lemma, briefly, is that a closed form on a starshaped set is exact. We do not assume familiarity with this terminology. Our goal here is to describe a vector field version. We assume no knowledge of differential forms except in the Appendix.

The Poincaré Lemma

For the Poincaré Lemma in the differential form language see Spivak [7] or the nice inductive development in Yap [8]. Samelson [5] has pointed out that it ought to be called Volterra's Theorem. Here is a vector version of the Poincaré Lemma.

Poincaré Lemma. If $\mathbf{v}$ is a smooth vector field and f a smooth scalar field defined in a ball in $\mathbb{R}^3$ centered at the origin, then the following relations hold.

$$\mathbf{v}(\mathbf{r}) = \nabla \left(\int_0^1 \mathbf{v}(t\mathbf{r}) \cdot \mathbf{r}\, dt \right) + \int_0^1 \operatorname{curl} \mathbf{v}(t\mathbf{r}) \times t\mathbf{r}\, dt \tag{1}$$

$$\mathbf{v}(\mathbf{r}) = \operatorname{curl} \left(\int_0^1 (\mathbf{v}(t\mathbf{r}) \times t\mathbf{r})\, dt \right) + \int_0^1 t^2 \mathbf{r} \operatorname{div} \mathbf{v}(t\mathbf{r})\, dt \tag{2}$$

$$f(\mathbf{r}) = \operatorname{div} \left(\int_0^1 t^2 \mathbf{r} f(t\mathbf{r})\, dt \right) \tag{3}$$

Here $\mathbf{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ is the usual notation for points of $\mathbb{R}^3$.

A proof is given in the next section, where we also offer some contexts for the meaning of the various terms.

But first, we focus on equation (3) because it is easy to prove and immediately useful. It says that every smooth scalar field $f(x, y, z)$ defined in a ball is the divergence of some vector field. For example, we can guess that x^2z is the divergence of the vector field $\frac{1}{3}x^3z\mathbf{i}$, and also of the field $x^2yz\mathbf{j}$, and others. But what about something a little harder, say $\sin(xyz)$? A straightforward computation of the integral in equation (3) produces the field

$$\frac{1-\cos(xyz)}{3xyz}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

The fraction extends to a smooth function along the planes where $xyz = 0$. The divergence of this field is indeed $\sin(xyz)$.

Proof of equation (3):

Let

$$C_f(\mathbf{r}) = \int_0^1 t^2\mathbf{r}f(t\mathbf{r})\,dt.$$

Since f is smooth we have by the chain rule

$$\operatorname{div} C_f(\mathbf{r}) = \int_0^1 (t^2 \cdot 3f + t^3\mathbf{r}\cdot\nabla f)\,dt = \int_0^1 \frac{\partial(t^3 f(t\mathbf{r}))}{\partial t}\,dt = f(\mathbf{r}).$$

That concludes the proof of equation (3). The reader might wish to check that an n-dimensional version of (3) also holds provided that one replaces the factor t^2 in the integral by t^{n-1}.

Application. One of the Maxwell equations

$$\operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon_0}$$

relates electric field $\mathbf{E}$ to charge density ρ (ϵ_0 is a physical constant). Taking a particular family of functions ρ, we can use equation (3) to see that Coulomb's inverse square field

$$\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

may be expressed as a limit of the solutions $C_{\frac{\rho}{\epsilon_0}}$.

Let $\rho(\mathbf{r}) = h^3w(h|\mathbf{r}|)$, where h is a positive parameter and w is a smooth function. Assume also that $\int_0^\infty 4\pi w(s)s^2\,ds = q$ is finite. Then the total charge

$$\iiint_{\mathbb{R}^3} \rho\,dV = 4\pi\int_0^\infty h^3w(hs)\,s^2ds = q$$

is independent of h. The (nonunique) solution to $\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}$ given by equation (3) is

$$\mathbf{E}(\mathbf{r}) = \int_0^1 t^2 \mathbf{r} \frac{\rho(t\mathbf{r})}{\epsilon_0}\, dt = \int_0^1 t^2 \mathbf{r} \frac{h^3 w(ht|\mathbf{r}|)}{\epsilon_0} dt.$$

Substituting $s = h|\mathbf{r}|t$ when $\mathbf{r} \neq \mathbf{0}$, this gives

$$\mathbf{E}(\mathbf{r}) = \int_0^{h|\mathbf{r}|} \left(\frac{s}{h|\mathbf{r}|}\right)^2 \mathbf{r} h^3 \frac{w(s)}{\epsilon_0} \frac{ds}{h|\mathbf{r}|} = \frac{\mathbf{r}}{\epsilon_0 |\mathbf{r}|^3} \int_0^{h|\mathbf{r}|} w(s) s^2\, ds.$$

Now for each $\mathbf{r} \neq \mathbf{0}$, let h approach infinity. This concentrates the charge toward the origin, and $\mathbf{E}(\mathbf{r})$ approaches

$$\frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3}.$$

This application also serves as a reminder that a pointwise limit of smooth fields need not be smooth, and that the nonsmooth limit may have scientific importance.

Proof of the Lemma

We begin a systematic development of the Poincaré Lemma.

The Work Operator

We start with a vector field $\mathbf{v}$ and attempt to solve the differential equation

$$\nabla w = \mathbf{v}$$

As motivation, think of $\mathbf{v}$ as a force field, and remember that force times distance gives energy. So a reasonable guess is that $\mathbf{v}$ might be the gradient of a work integral. Parametrize a line segment from $\mathbf{0}$ to each point $\mathbf{r}$ and interpret the line integral

$$W_{\mathbf{v}}(\mathbf{r}) = \int_{\mathbf{0}}^{\mathbf{1}} \mathbf{v}(\mathbf{tr}) \cdot \mathbf{r}\, \mathbf{dt}$$

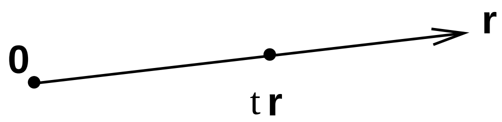

Figure 1: The segment from $\mathbf{0}$ to $\mathbf{r}$ parametrized.

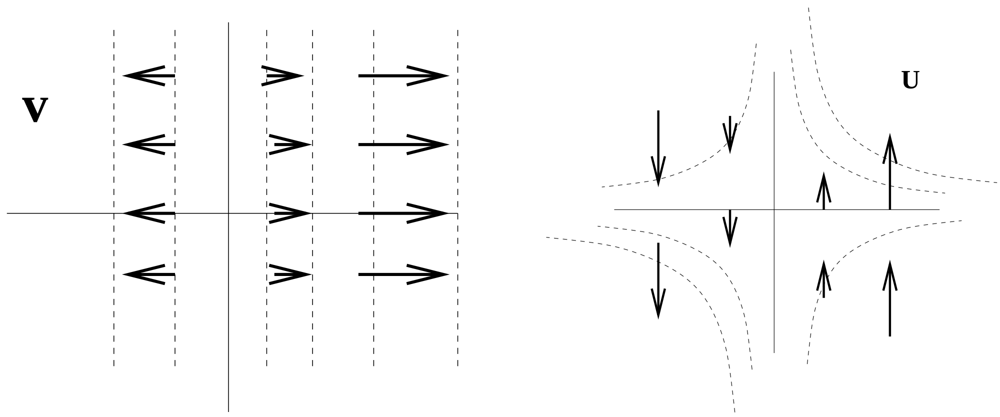

Figure 2: Two vector fields are shown with level sets and gradients of their work integrals.

as the work done by $\mathbf{v}$ along the segment. No assumption is made about $\mathbf{v}$ being "conservative" or independent of path. Rather we choose the specific set of straight segments along which to integrate, and assume only that $\mathbf{v}$ is smooth there.

Now we ask whether $\nabla W_{\mathbf{v}}$ is equal to $\mathbf{v}$. Here are two examples that show it is not equal in general.

Examples. Define two vector fields $\mathbf{v}(\mathbf{r}) = (x, 0, 0)$ and $\mathbf{u}(\mathbf{r}) = (0, x, 0)$. We easily find that $\operatorname{curl} \mathbf{v} = \mathbf{0}$, $\operatorname{curl} \mathbf{u} \neq \mathbf{0}$, $W_{\mathbf{v}}(\mathbf{r}) = \frac{1}{2}x^2$, and $W_{\mathbf{u}}(\mathbf{r}) = \frac{1}{2}xy$. FIGURE 2 shows sections of $\mathbf{v}$ and $\mathbf{u}$ in the plane $z = 0$, together with some level sets of $W_{\mathbf{v}}$ and $W_{\mathbf{u}}$, respectively. It is visually apparent that the gradient of the work integral matches the first field, but not the second.

Let's study the difference between $\mathbf{v}$ and $\nabla W_{\mathbf{v}}$. Let $\mathbf{v}$ be a smooth vector field defined in a ball. At this point it is convenient to change notation slightly. Denote points in the ball as $\mathbf{r} = (x_1, x_2, x_3)$ and components of $\mathbf{v}$ as

v_j, so that $\mathbf{v}(\mathbf{r}) = \mathbf{v}(x_1, x_2, x_3) = (v_1(\mathbf{r}), v_2(\mathbf{r}), v_3(\mathbf{r}))$. For each real number $t \in [0,1]$ the notation $\mathbf{v}(t\mathbf{r})$ means the value of $\mathbf{v}$ at the point $t\mathbf{r}$. Then the first coordinate of $\nabla W_{\mathbf{v}}(\mathbf{r})$ is

$$\frac{\partial}{\partial x_1}\left(\int_0^1 (v_1(t\mathbf{r})x_1 + v_2(t\mathbf{r})x_2 + v_3(t\mathbf{r})x_3)\,dt\right)$$

$$= \int_0^1 \left(\frac{\partial v_1}{\partial x_1}(t\mathbf{r})tx_1 + v_1(t\mathbf{r}) + \frac{\partial v_2}{\partial x_1}(t\mathbf{r})tx_2 + \frac{\partial v_3}{\partial x_1}(t\mathbf{r})tx_3\right) dt$$

by the chain rule and product rule. Regrouping terms, we see that the first coordinate of $\nabla W_{\mathbf{v}}(\mathbf{r})$ is

$$= \int_0^1 \left(\frac{\partial(tv_1(t\mathbf{r}))}{\partial t} - t\frac{\partial v_1}{\partial x_2}(t\mathbf{r})x_2 - t\frac{\partial v_1}{\partial x_3}(t\mathbf{r})x_3 + \frac{\partial v_2}{\partial x_1}(t\mathbf{r})tx_2 + \frac{\partial v_3}{\partial x_1}(t\mathbf{r})tx_3\right) dt,$$

which by the fundamental theorem of calculus is

$$= v_1(\mathbf{r}) + \int_0^1 \left(\left(\frac{\partial v_2}{\partial x_1}(t\mathbf{r}) - \frac{\partial v_1}{\partial x_2}(t\mathbf{r})\right)tx_2 + \left(\frac{\partial v_3}{\partial x_1}(t\mathbf{r}) - \frac{\partial v_1}{\partial x_3}(t\mathbf{r})\right)tx_3\right) dt.$$

The other two coordinates are similar. Thus we get

$$\nabla W_{\mathbf{v}}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) + \int_0^1 (\mathbf{r} \times \operatorname{curl} \mathbf{v})(t\mathbf{r})\,dt.$$

This proves equation (1) in our statement of the Poincaré Lemma. We evidently need to study this last integral.

For any smooth field $\mathbf{v}$ define the vector field

$$F_{\mathbf{v}}(\mathbf{r}) = -\int_0^1 (\mathbf{r} \times \mathbf{v})(t\mathbf{r})\,dt.$$

The "F" is for flux, as we will understand soon. We can rewrite equation (1) then as

$$\mathbf{v} = \nabla W_{\mathbf{v}} + F_{\operatorname{curl} \mathbf{v}}. \tag{1'}$$

In particular we see that if the curl of $\mathbf{v}$ is zero, then $\mathbf{v}$ is the gradient of $W_{\mathbf{v}}$.

Next we indicate a context for the F construction.

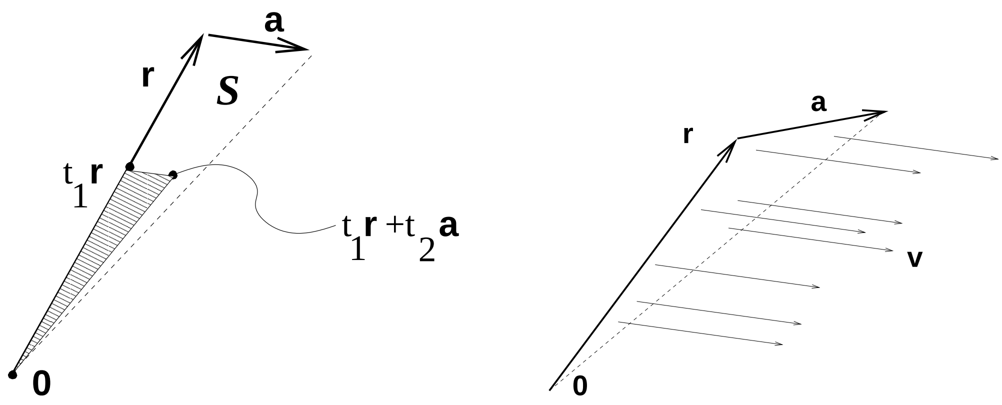

Figure 3: A parametrized triangle and some flux through it.

The Flux Operator

We have seen that equation (1) can be motivated by thinking of vector field $\mathbf{v}$ as a force field. In this section we think of $\mathbf{v}$ instead as the velocity or momentum of a fluid, gas or liquid, and consider the flux of $\mathbf{v}$ across various surface elements. This gives a context to explore the meaning of $F_{\mathbf{v}}$. Let $\mathbf{a}$ be a vector based at point $\mathbf{r}$. Consider the flux of $\mathbf{v}$ through the oriented triangle S which has vertices at $\mathbf{0}$, $\mathbf{r}$, and $\mathbf{r}+\mathbf{a}$. Parametrize the triangle by labeling points as $t_1\mathbf{r}+t_2\mathbf{a}$, $(0 \le t_2 \le t_1 \le 1)$ as in FIGURE 3. Then $\mathbf{r}\times\mathbf{a}$ is in the oriented normal direction.

Next we show that the flux of $\mathbf{v}$ through S is

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,d\sigma = F_{\mathbf{v}}(\mathbf{r})\cdot\mathbf{a} + o(\mathbf{a})$$

where the "little o" term means that $\lim_{|\mathbf{a}|\to 0}\frac{o(\mathbf{a})}{|\mathbf{a}|} = 0$. This equation is the reason that we refer to F as a flux operator.

proof:

The flux of $\mathbf{v}$ through triangle S is

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,d\sigma = \int_0^1\int_0^{t_1} \mathbf{v}(t_1\mathbf{r}+t_2\mathbf{a})\cdot(\mathbf{r}\times\mathbf{a})\,dt_2 dt_1.$$

By the mean value theorem, $\mathbf{v}(t_1\mathbf{r}+t_2\mathbf{a}) = \mathbf{v}(t_1\mathbf{r}) + O(\mathbf{a})$, where the "big O" term means that $\frac{O(\mathbf{a})}{|\mathbf{a}|}$ is bounded for $\mathbf{a}$ near 0. So the flux is

$$= \int_0^1 \int_0^{t_1} (\mathbf{v}(t_1\mathbf{r})+O(\mathbf{a}))\cdot(\mathbf{r}\times\mathbf{a})\, dt_2 dt_1 = -\left(\int_0^1 \int_0^{t_1} \mathbf{r}\times\mathbf{v}(t_1\mathbf{r})\, dt_2 dt_1\right)\cdot\mathbf{a}+o(\mathbf{a})$$

$$= -\left(\int_0^1 (\mathbf{r}\times\mathbf{v})(t_1\mathbf{r})\, dt_1\right)\cdot\mathbf{a}+o(\mathbf{a}) = F_{\mathbf{v}}(\mathbf{r})\cdot\mathbf{a}+o(\mathbf{a}).$$

Thus $F_{\mathbf{v}}(\mathbf{r})$ is a vector which contains information about the flux of $\mathbf{v}$ through narrow triangles unfurled from the vector $\mathbf{r}$. Further, allow the vector $\mathbf{a}$ to vary in the figure. We see that the flux of $\mathbf{v}$ is maximized when $\mathbf{a}$ is aligned with $F_{\mathbf{v}}$, and the magnitude of $F_{\mathbf{v}}(\mathbf{r})$ is that maximum flux divided by the length $|\mathbf{a}|$.

The Content Operator

Next we prove equation (2) of the Poincaré Lemma. In order to develop that equation, we recall first that the gradient of the work integral gave information leading to equation (1). Motivated by this success we next calculate

$$\operatorname{curl}(F_{\mathbf{v}}) =$$

$$-\operatorname{curl}\int_0^1 (x_2v_3(t\mathbf{r})-x_3v_2(t\mathbf{r}), x_3v_1(t\mathbf{r})-x_1v_3(t\mathbf{r}), x_1v_2(t\mathbf{r})-x_2v_1(t\mathbf{r}))t\, dt$$

$$= -\int_0^1 \Big(x_1v_{2,2}(t\mathbf{r})t - x_2v_{1,2}(t\mathbf{r})t - v_1(t\mathbf{r}) - (x_3v_{1,3}(t\mathbf{r})t - x_1v_{3,3}(t\mathbf{r})t + v_1(t\mathbf{r}))\,, \ldots\Big)t\, dt.$$

We have abbreviated partial derivatives here using comma subscripts. Adding and subtracting $x_1tv_{1,1}$ from the first coordinate, and similarly for the second and third not shown, we see that the $\operatorname{curl}(F_{\mathbf{v}})$ is

$$= -\int_0^1 (x_1t\operatorname{div}\mathbf{v}(t\mathbf{r}) - 2v_1(t\mathbf{r}) - x_1tv_{1,1}(t\mathbf{r}) - x_2tv_{1,2}(t\mathbf{r}) - x_3tv_{1,3}(t\mathbf{r}), \ldots)t\, dt$$

$$= -\int_0^1 \Big(t^2\mathbf{r}\operatorname{div}\mathbf{v}(t\mathbf{r}) - 2\mathbf{v}(t\mathbf{r})t - t^2\frac{d}{dt}(\mathbf{v}(t\mathbf{r}))\Big)\, dt =$$

$$-C_f + \int_0^1 \frac{d}{dt}(\mathbf{v}t^2)\, dt = -C_f + \mathbf{v}.$$

This proves equation (2), and yields a short version of it:

$$\mathbf{v} = \operatorname{curl} F_{\mathbf{v}} + C_{\operatorname{div}\mathbf{v}}. \tag{2'}$$

Application. Consider the partial differential equation

$$\operatorname{curl}\mathbf{u} = \mathbf{v}.$$

Suppose $\mathbf{v}$ is a given smooth vector field in a ball, and suppose there is a smooth solution $\mathbf{u}$. The equality of continuous mixed partial derivatives gives the identity $\operatorname{div}\operatorname{curl} = 0$. Applying this to the PDE we see the necessary condition that $\operatorname{div}\mathbf{v} = 0$. Conversely, assuming that $\operatorname{div}\mathbf{v} = 0$, equation (2') says that one solution is given by $\mathbf{u} = F_{\mathbf{v}}$.

The equation $\operatorname{curl}\mathbf{u} = \mathbf{v}$ occurs in fluid mechanics and electromagnetism. In the case of fluids, $\mathbf{u}$ is the velocity field, and the vorticity field ω is given by $\operatorname{curl}\mathbf{u} = \omega$. In the case of electromagnetism there is the magnetic field $\mathbf{B}$, and we find a vector potential $\mathbf{A}$ such that $\operatorname{curl}\mathbf{A} = \mathbf{B}$.

Remark. The identity $\operatorname{div}\operatorname{curl} = 0$ is called the "Poincaré Lemma" in some references, and what we have stated, they call the converse of the lemma. See for example [6] and [1]. The solution $\mathbf{u} = F_{\mathbf{v}}$ which we have given to $\operatorname{curl}\mathbf{u} = \mathbf{v}$ does not seem to have been prominently displayed in many texts. A formula equivalent to it occurs as an exercise in Griffiths [2]. It may also be found in the first chapter in Madsen and Tornehave [4].

Example. We also give an elementary example of equation (2'). Consider a linear vector field $\mathbf{v}(\mathbf{r}) = A\mathbf{r}$, where A is a three by three matrix and we write $\mathbf{r}$ as a column for this purpose. Then $\operatorname{div}(\mathbf{v}) = \operatorname{tr}(A)$ and equation (2) becomes

$$A\mathbf{r} = \operatorname{curl}\,(-\frac{1}{3}\mathbf{r} \times A\mathbf{r}) + \frac{1}{3}\operatorname{tr}(A)\mathbf{r}.$$

Thus $A\mathbf{r}$ is, or is not, a curl depending on the trace of A.

Finally we give a context for understanding the C_f construction. Recall that $C_f(\mathbf{r}) = \int_0^1 t^2 f(t\mathbf{r})\mathbf{r}\, dt$. Imagine that f is the energy density due to the

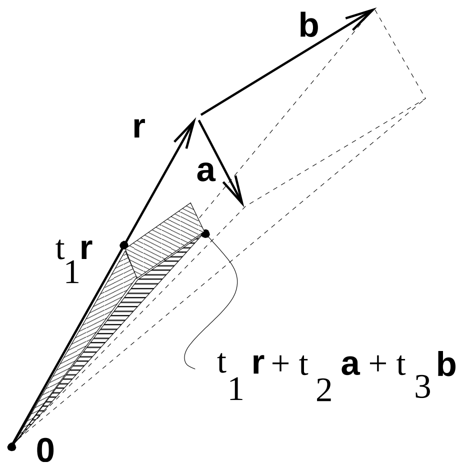

Figure 4: A three-dimensional cone.

temperature within some material. Keeping track of units, then f ought to be the (specific heat)×(density)×(temperature). We show that the energy content of the three-dimensional cone (FIGURE 4) is

$$\iiint f\,dV = C_f(\mathbf{r})\cdot(\mathbf{a}\times\mathbf{b}) + o(\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}).$$

Proof:

Label points of the cone as $(t_1\mathbf{r}+t_2\mathbf{a}+t_3\mathbf{b})$ with $0\le t_3, t_2\le t_1\le 1$, and assume that $(\mathbf{r},\mathbf{a},\mathbf{b})$ is a righthand frame as suggested by the figure. Then the energy

$$\iiint f\,dV = \int_0^1\int_0^{t_1}\int_0^{t_1} f(t_1\mathbf{r}+t_2\mathbf{a}+t_3\mathbf{b})|\det[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]|dt_3dt_2dt_1$$

$$= \int_0^1\int_0^{t_1}\int_0^{t_1}(f(t_1\mathbf{r})+O(\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}))\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})dt_3dt_2dt_1$$

$$= \left(\int_0^1 t_1^2 f(t_1\mathbf{r})\mathbf{r}\,dt_1\right)\cdot(\mathbf{a}\times\mathbf{b}) + o(\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2})$$

$$= C_f(\mathbf{r})\cdot(\mathbf{a}\times\mathbf{b}) + o(\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}).$$

This concludes the proof.

This suggests one meaning of C_f: a radial vector field whose flux through any small rectangle based at $\mathbf{r}$ is the "f content" of the subtended cone. But why, heuristically, should we believe that this has divergence equal to f? Consider a solid region D having rectangle $R = \mathbf{a} \times \mathbf{b}$ based at point $\mathbf{r}$ as one face, and a rescaled copy $R' = \mathbf{a}' \times \mathbf{b}'$ based at $\mathbf{r}'$, where $\mathbf{a}' = (1+\epsilon)\mathbf{a}$, $\mathbf{b}' = (1+\epsilon)\mathbf{b}$, and $\mathbf{r}' = (1+\epsilon)\mathbf{r}$, $\epsilon > 0$. That is, D is a segment of a cone like that in FIGURE 4. The field C_f is radial, so the flux of C_f out through the boundary surface of D is the same as the flux through R' minus the flux through R. By the divergence theorem this is equal to the integral of $\operatorname{div} C_f$ over D, and by our calculation it is also well approximated by the integral of f over D, for $\mathbf{a}$ and $\mathbf{b}$ small. So the divergence of C_f is f.

Conclusion. The Poincaré Lemma deserves to be better known as a generalization of the Fundamental Theorem of Calculus to higher dimensions. Like the Fundamental Theorem, it has broad implications that help us solve various equations and derive important relationships. It is not too advanced for consideration in the vector calculus course.

Appendix on the Relation to Forms

Some readers may want to know how the preceeding discussion relates to differential forms. We assume here a slight familiarity with differential forms.

Functions f are by definition 0-forms, but when we think of such applications as heat energy content then it is better to let f give rise to the 3-form $f\,dx_1 \wedge dx_2 \wedge dx_3$ which assigns to any small volume element its energy content. Similarly a vector field $\mathbf{v}$ can give us a 2-form $v_1 dx_2 \wedge dx_3 + v_2 dx_3 \wedge dx_1 + v_3 dx_1 \wedge dx_2$ when we think of flux of momentum field $\mathbf{v}$, or a 1-form $v_1 dx_1 + v_2 dx_2 + v_3 dx_3$ if we think of work done by force field $\mathbf{v}$.

The exterior derivatives and cone operations fit a nice pattern:

$$
\begin{array}{rccccccc}
\text{field:} & f & \leftarrow & \mathbf{v} & \leftarrow & \mathbf{v} & \leftarrow & f \\
\text{cone operator:} & & W & & F & & C & \\
& & & & & & & \\
\text{form dimension:} & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 \\
\text{derivative:} & & \nabla & & \text{curl} & & \text{div} &
\end{array}
$$

The modern Poincaré Lemma includes a formula which looks like

$$\omega = dI\omega + Id\omega \tag{4}$$

where the exterior derivative d covers all the cases grad, curl, div, and the cone integral I represents the work, flux, and content constructions W, F, and C. Equation (4) includes our three equations (1), (2), and (3). For example, given a vector field $\mathbf{v}$ we can associate the 2-form

$$\omega = \sum_{i_1<i_2} \omega_{i_1 i_2} dx^{i_1} \wedge dx^{i_2} = v_1(\mathbf{r})dy \wedge dz + v_2(\mathbf{r})dz \wedge dx + v_3(\mathbf{r})dx \wedge dy$$

Then the I operation, see Spivak [7], gives the 1-form

$$
\begin{aligned}
I\omega &= \sum_{i_1<i_2} \sum_{\alpha=1}^{2} (-1)^{\alpha-1} \Big(\int_0^1 t^{2-1} \omega_{i_1 i_2}(tx)\, dt \Big) x^{i_\alpha} dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\alpha}} \wedge \ldots \wedge dx^{i_2} \\
&= \int_0^1 tv_1(t\mathbf{r})\, dt(y\, dz - z\, dy) + \int_0^1 tv_2(t\mathbf{r})\, dt(z\, dx - x\, dz) \\
&\quad + \int_0^1 tv_3(t\mathbf{r})\, dt(x\, dy - y\, dx)
\end{aligned}
$$

This exactly matches our F operation, in the sense that it is the 1-form we associate with $F_{\mathbf{v}}$ in the diagram above.

Roughly speaking, in every case a $(k+1)$-form is modified to assign values to a k-dimensional object by integrating: the cone of a k-dimensional object has dimension $k+1$, so that the integral makes sense. The "cone" terminology came from Hubbard [3].

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