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On definition of an admitted Lie group for functional differential equations

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Abstract

The manuscript is devoted to applications of group analysis to functional differential equations. It is given a definition of an admitted Lie group for such type of equations and some examples of applications of this definition are studied. The way for constructing an admitted Lie group is similar to the way developed for differential equations: first, one has to construct determining equations, then to split these equations with respect to arbitrary elements, and then to find the general solution of these equations. Particularly, for delay differential equations the process of splitting determining equations and solving them is similar to the case of differential equations. The proposed approach can also be applied for finding an equivalence group, contact and Lie–Bäcklund transformations.

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1. Introduction

In mathematical modelling for studying a real phenomena one uses differential equations. Depending on the problem, these equations take various forms, such as ordinary differential equations, partial differential equations, integro differential equations, functional differential equations and many others. Functional differential equations are classified as of retarded, neutral, or advanced type. The simplest type of retarded differential equations is a type of delay differential equations, where some derivatives of the unknown functions at time t are expressed through their values at earlier instants. Delay differential equations appear in problems with delaying links

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where certain information processing is needed. Concrete examples of delay differential equations are frequent in populations dynamics and bioscience problems, in control problems, electrical networks containing lossless transmission lines. The theory and applications of functional differential equations can be found in many books, for example, [1–5] and many others. Note that beginning with the late 1940's, the theory of delay differential equations is being developed rapidly.

One of the methods for studying properties of differential equations is group analysis. The modern state of group analysis is reviewed in [6]. Group analysis side by side with constructing exact solutions provides a regular procedure for mathematical modelling by classifying differential equations with respect to arbitrary elements. The application of group analysis implies some steps. One of the most important steps is a construction of an admitted Lie group. After finding an admitted Lie group one can apply it for constructing exact solutions.

We should note here that many of partial differential equations have been studied. The algorithmic approach of group analysis, developed for partial differential equations, can not be directly applied to equations with nonlocal terms. The main obstacle for this is their nonlocality: the nonlocality does not allow using the manifold's approach for defining an admitted Lie group. However, in this case one can try to use another definition of an admitted symmetry group, also given by Lie [7]: a Lie group is admitted if it maps any solution of equations into a solution of the same equations. In the sense of applications of group analysis for constructing solutions this definition of an admitted Lie group is more appropriate: it excludes the possibilities [8] where an equation admits a Lie group, but there is no any solution of this equation. This definition was also applied earlier for studying integro differential equations [9–11]. [1] Since some retarded equations are of the type of integro differential equations, it gave us an idea for using similar approach in the case of functional differential equations and particularly for delay differential equations.

The used definition of an admitted Lie group allows constructing determining equations. For solving the determining equations one uses the method of splitting them with respect to arbitrary elements. In the case of differential equations the arbitrary elements are parametrical derivatives. In the case of analytical systems the parametrical derivatives are dictated by the Cauchy–Kovalevskaya theorem for a Cauchy type systems and by the Cartan–Khäler theorem for involutive systems. For other types of equations an answer on the question about arbitrary elements is obtained on the base of an theorem of existence of the Cauchy problem.

Here we also mention an approach developed in [12] where a delay differential equation is replaced by an underdetermined system of differential equations for which the classical group analysis is applied. It should be noted that the reduction to underdetermined system widens a class of equations admitted by a given Lie group. But an extension of equations narrows a set of admitted Lie groups.

This manuscript is devoted to illustrate possibilities of direct applications of group analysis to delay differential equations.

[1] See also [6].

2. A definition of an admitted Lie group and determining equations

As it was mentioned the main difficulty for applying group analysis to functional differential equations arises from the nonlocal terms present in these equations. To overcome this difficulty the following method is suggested. For the sake of simplicity the method is described for functional differential equations with one independent variable

$$S \equiv x'(t) - F(t, x_t) = 0. \quad (1)$$

Here the notations accepted in literature on functional differential equations are used. [2] If $\chi(t)$ is a function defined at least on $[t-r,t]$, then one defines a new function $\chi_t : [-r,0] \to D$ by

$$\chi_t(s) = \chi(t+s), \quad s \in [-r,0],$$

where D is an open subset in $\mathbb{R}^n$, J is some interval in $\mathbb{R}$, F is a functional. For delay differential equations

$$F(t,\chi_t) \equiv f(t, \chi(g_1(t)), \ldots, \chi(g_m(t))),$$

where $f : [t_0, \beta) \times D^m \to \mathbb{R}^n$, and $g_j(t) \leqslant t$ for $t_0 \leqslant t \leqslant \beta$ for each $j = 1, \ldots, m$. A continuous function $\chi(t)$, $t \in [t_0 - r, t_0 + \beta)$ is called a solution of delay differential equations [3] (1) if it is differentiable in the interval (t_0, β) and satisfies Eq. (1) in the interval (t_0, β). The value $\chi'(t_0)$ is understood as the right-hand derivative.

First, as for differential equations, let the symmetry group G of transformations T_a:

$$\bar{t} = T^t(t,x;a), \quad \bar{x} = T^x(t,x;a),$$

depending on a real parameter a, with $t \equiv T^t(t,x;0)$ and $x \equiv T^x(t,x;0)$, map a solution of Eq. (1) to a solution of the same equations. Usually instead of a Lie group one considers the corresponding infinitesimal generator

$$X = \tau(t,x)\partial_t + \eta(t,x)\partial_x,$$

where

$$\tau(t,x) = \frac{\partial T^t}{\partial a}(t,x,0), \quad \eta(t,x) = \frac{\partial T^x}{\partial a}(t,x,0).$$

Let $x = \chi(t)$ be a solution. A parametrical representation of the transformed function $\chi_a(\bar{t})$ is given by the equations $\bar{t} = T^t(t,\chi(t);a)$, $\bar{x} = T^x(t,\chi(t);a)$. In order to find $\chi_a(\bar{t})$, one has to define

$$t = \psi(\bar{t};a), \quad (2)$$

from the equation $\bar{t} = T^t(t,\chi(t);a)$. For differential equations this is guaranteed by a local inverse function theorem. For delay differential equations (1) one has to define the function ψ not only in a neighborhood of the point t, but also in the interval $[t-r,t]$ and in a right-hand neighborhood of t. For obtaining this it is not enough only a local inverse function theorem. Assume that the given Lie group possesses by this property. Hence, the function $\chi_a(\bar{t})$ is defined by $\chi_a(\bar{t}) = T^x(\psi(\bar{t};a), \chi(\psi(\bar{t};a));a)$, and then

[2] See, for example, [4].
[3] See, for example, [5].

$$\frac{d\bar{\chi}(\bar{t})}{d\bar{t}} = (T^x_{,1} + T^x_{,2}\chi'(\psi(\bar{t};a)))\frac{\partial\psi(\bar{t};a)}{\partial\bar{t}},$$

where $f_{,i}$ means the partial derivative of f with respect to ith argument. Thus

$$F(\bar{t},\bar{\chi}_{\bar{t}}) = f(\bar{t},T^x_1,\ldots,T^x_m),$$

where $T^x_i = T^x(\psi(g_i(\bar{t});a),\chi(\psi(g_i(\bar{t});a));a)$, $i = 1,\ldots,m$.

Let $\Xi(t,x_t,x') = x' - F(t,x_t)$. Since $\chi(t)$ and $\chi_a(\bar{t})$ are solutions, then

$$\Xi(t,x_t,x') \equiv 0, \tag{3}$$

for $x = \chi(t)$ and $x = \chi_a(\bar{t})$. Differentiating the functions $\Xi(t,x_t,x')$, where $x = \chi_a(\bar{t})$, with respect to the group parameter a, and taking $a = 0$, one obtains the equations

$$\left.\frac{\partial\Xi(t,x_t,x')}{\partial a}\right|_{a=0} = 0. \tag{4}$$

The left side of these equations is expressed only through the coefficients of the infinitesimal generator X, their derivatives, the function $\chi(t)$ and its derivatives. This we denote by

$$H(\chi,\tau,\eta) \equiv \left.\frac{\partial\Xi(t,x_t,x')}{\partial a}\right|_{a=0}.$$

Thus, Eq. (4) become

$$H(\chi,\tau,\eta) = 0,$$

where $\chi(t)$ is an arbitrary solution of (1).

Note that Eq. (4) coincide with the equations obtained in the result of an action onto (1) by the prolonged canonical Lie–Bäcklund operator [13] equivalent to the generator X:

$$\tilde{X} = (\eta(t,x) - \tau(t,x)x')\partial_x + \cdots$$

The difference in applying a canonical operator for differential equations and for functional differential equations is in the action of the derivative ∂_x: for functional differential equations its action has to be considered in the sense of the Frechet derivative.

Above constructions allow giving the following definition of a Lie group admitted by functional differential equations.

Definition. A Lie group, satisfying the equations

$$H(\chi,\tau,\eta)|_{[S]} = \tilde{X}(S)|_{[S]} = 0, \tag{5}$$

for any solution of (1) is called an admitted Lie group (or Eq. (1) admit this Lie group). Eq. (5) are called determining equations.

The notation [S] in (5) means that Eq. (5) have to be satisfied for any solution of Eq. (1). This notation is similar to the notation used in [13] for a frame of equations.

We emphasize that one of the main features of the determining equations in the given definition is that they must be satisfied for any solution of Eq. (1). This lets us to split the determining equations with respect to arbitrary elements. Since arbitrary elements of delay differential equations are contained in determining equations by a similar way as for differential equations, the process of solving determining equations for delay differential equations is similar to obtaining solutions of determining equations for differential equations. This will be demonstrated in examples.

Note also that the given definition:

- is free of the requirement for the admitted Lie group to have (2) globally;
- coincides with one of the classical definitions of an admitted Lie group in the case of differential equations (in sense of Lie–Bäcklund representation);
- was also applied for integro-differential equations;
- can be applied for finding an equivalence group, contact and Lie–Bäcklund transformations for functional differential equations.

From one point of view to use solutions for defining an admitted Lie group it seems more difficult for applications than the geometrical definition, [4] but from another point of view this definition allows defining an admitted Lie group for more general objects than differential equations: delay differential equations, functional differential equations, integro-differential equations or even more general. For example, let us consider the functional differential equation studied by Barba [4]

$$y(x)y'(x) = y(y(x)). \tag{6}$$

According to the given definition of an admitted Lie group the determining equation for (6) is

$$\begin{aligned}\tilde{X}(y(x)y'(x) - y(y(x))) = {} & (\eta(x,y(x)) - y'(x)\xi(x,y(x)))y'(x) + y(x)(\eta_x(x,y(x)) + y'(x)\eta_y(x,y(x)) \\ & - y''(x)\xi(x,y(x)) - y'(x)(\xi_x(x,y(x)) + y'(x)\xi_y(x,y(x)))) \\ & - (\eta(x,y(y(x))) - y'(y(x))\xi(x,y(y(x))) + y'(y(x))(\eta(x,y(x)) \\ & - y'(x)\xi(x,y(x)))),\end{aligned}$$

which has to be satisfied for any solution $y = y(x)$ of Eq. (6). Here a generator of the admitted Lie group is $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$. To solve this determining equation one needs to know arbitrariness of the general solution of Eq. (6). Because this analysis is unknown for the authors, we demonstrate solving determining equations for other examples.

3. Examples

3.1. One independent variable

Let us consider the retarded equation

$$x'(t) = \int_{-r}^{0} x_t(s)\,\mathrm{d}s. \tag{7}$$

[4] Although for differential equations there are theorems where studied conditions when these two approaches are equivalent [13].

According to the definition of an admitted Lie group, the determining equation for (7) is

$$\eta_t(t,x(t)) + x'(t)\eta_x(t,x(t)) - x''(t)\tau(t,x(t)) - x'(t)(\tau_t(t,x(t)) + x'(t)\tau_x(t,x(t)))$$
$$= \int_{-r}^{0}(\eta(t+s,x(t+s)) - x'(t+s)\tau(t,x(t+s)))\,\mathrm{d}s,$$

which is satisfied for any solution $x = x(t)$ of Eq. (7). Here a generator of the admitted Lie group is $X = \tau(t,x)\partial_t + \eta(t,x)\partial_x$. To solve this determining equation one needs to deal with an integral equation, which is more difficult for studying than a delay type equations. But one can note that the Cauchy problem

$$x'(t) = \int_{-r}^{0} x_t(s)\,\mathrm{d}s, \quad x_{t_0}(s) = \psi(s), \qquad s \in [t_0 - r, t_0], \tag{8}$$

is equivalent to the Cauchy problem

$$x''(t) = x(t) - x(t-r), \tag{9}$$

$$x'(t_0) = x_1, \quad x_{t_0}(s) = \psi(s), \qquad s \in [t_0 - r, t_0], \tag{10}$$

with an arbitrary continuous function $\psi(s)$ and arbitrary value x_1. Further we study the determining equation corresponding to (9). This determining equation is a delay type. Regarding this equation at the point $(t_0 + 0)$ (limit from the right handed side), on the solution of the Cauchy problem (9) and (10), the determining equation becomes

$$\begin{aligned}\Phi(t_0,x_0,x_1,x_2,x_3) \equiv{}& 2\eta_{tx}(t_0,x_0)x_1 + \eta_{tt}(t_0,x_0) + \eta_{xx}(t_0,x_0)x_1^2 + \eta_x(t_0,x_0)x_0 - \eta_x(t_0,x_0)x_2\\ &- 2\tau_{tx}(t_0,x_0)x_1^2 - \tau_{tt}(t_0,x_0)x_1 - 2\tau_t(t_0,x_0)x_0 + 2\tau_t(t_0,x_0)x_2 - \tau_{xx}(t_0,x_0)x_1^3\\ &- 3\tau_x(t_0,x_0)x_0x_1 + 3\tau_x(t_0,x_0)x_2x_1 - \eta(t_0,x_0) + \eta(t_0-r,x_2) + (\tau(t_0,x_0)\\ &- \tau(t_0-r,x_2))x_3 = 0,\end{aligned}$$

where $x_0 = \psi(t_0)$, $x_{-1} = x_2 = \psi(t_0 - r)$, $x_3 = \psi'(t_0 - r)$. Here also the derivatives

$$x''(t_0) = x(t_0) - x(t_0 - r), \quad x'''(t_o) = x_1 - x'(t_o - r),$$

found from Eq. (9) and its first derivative, are substituted. Since the function $\psi(s)$ and the value x_1 are arbitrary and because of the theorem of existence [2] of a solution of the Cauchy problem (9) and (10), one can account that the values t_0, x_0, x_1, x_2 and x_3 are arbitrary in the determining equation: one considers the determining equation with substituted in it the solution of this Cauchy problem. The arbitrariness of the values t_0, x_0, x_1, x_2 and x_3 allows splitting the determining equation. Because further analysis of the determining equation is similar to the classical analysis of solving determining equations for differential equations we present only result of calculations. The general solution of the determining equations corresponds to the generator

$$X = (c_1x + h(t))\partial_x + c_2\partial_t,$$

where c_1, c_2 are arbitrary constants and the function $h(t)$ is an arbitrary solution of Eq. (9).

3.2. Two independent variables

Let us study the equation

$$u_t + uu_x = g(u, \bar{u}), \tag{11}$$

where $u = u(x,t)$, $\bar{u} = u(x,t-r)$, $u_t = u_t(x,t)$, $u_x = u_x(x,t)$. It is assumed that $g_{\bar{u}} \neq 0$ (otherwise it is not a delay differential equation). The determining equation of the admitted Lie group is

$$\bar{u}_t g_{\bar{u}}(\bar{\tau} - \tau) + u_x(-\eta_t - \eta_u g - \eta_x u + \tau_t u + \tau_u g u + \tau_x u^2 + \zeta) + \bar{u}_x g_{\bar{u}}(\bar{\eta} - \eta) - g_u \zeta - g_{\bar{u}} \bar{\zeta} \\ - \tau_t g - \tau_u g^2 - \tau_x g u + \zeta_t + \zeta_u g + \zeta_x u = 0, \tag{12}$$

where $\bar{u}_t = u_t(x,t-r)$, $\bar{u}_x = u_x(x,t-r)$, $\bar{\tau} = \tau(x,t-r,\bar{u})$, $\bar{\eta} = \eta(x,t-r,\bar{u})$, $\bar{\zeta} = \zeta(x,t-r,\bar{u})$, $\tau = \tau(x,t,u)$, $\eta = \eta(x,t,u)$, $\zeta = \zeta(x,t,u)$. The Cauchy problem for equation (11) with the initial values

$$u(x,s) = \psi(x,s), \quad s \in [t_0 - r, t_0],$$

has a solution [14] for any arbitrary value t_0 and any arbitrary given function $\psi(x,s)$, $s \in [t_0 - r, t_0]$. In the strength of the arbitrariness of the value t_0 and the function $\psi(x,s)$ one can consider the variables $\bar{u}_t$, $\bar{u}_x$, u_x, u, $\bar{u}$, x, and t as independent and arbitrary in the determining Eq. (12). Splitting the determining equation with respect to $\bar{u}_t$, $\bar{u}_x$ and u_x one obtains

$$g_{\bar{u}}(\bar{\tau} - \tau) = 0, \quad g_{\bar{u}}(\bar{\eta} - \eta) = 0,$$
$$\zeta - \eta_t - \eta_u g - \eta_x u + \tau_t u + \tau_u g u + \tau_x u^2 = 0, \tag{13}$$
$$- g_u \zeta - g_{\bar{u}} \bar{\zeta} - \tau_t g - \tau_u g^2 - \tau_x g u + \zeta_t + \zeta_u g + \zeta_x u = 0.$$

Since $g_{\bar{u}} \neq 0$, then $\tau(x,t-r,\bar{u}) = \tau(x,t,u)$, $\eta(x,t-r,\bar{u}) = \eta(x,t,u)$. Because u and $\bar{u}$ are independent in the last equalities, then the functions $\tau(x,t,u)$ and $\eta(x,t,u)$ do not depend on the variable u and $\tau(x,t-r) = \tau(x,t)$, $\eta(x,t-r) = \eta(x,t)$. From the third equation of (13) one finds the function $\zeta(x,t,u)$:

$$\zeta = \eta_t + u(\eta_x - \tau_t) - \tau_x u^2. \tag{14}$$

Substituting it and found ζ into the last equation of (13) one has

$$-(g_u + g_{\bar{u}})\eta_t - (g_u u + g_{\bar{u}} \bar{u} - g)(\eta_x - \tau_t) + (g_u u^2 + g_{\bar{u}} \bar{u}^2 - 3gu)\tau_x - g\tau_t + \eta_{tt} - \tau_{xx} u^3 \\ + (\eta_{xx} - 2\tau_{tx})u^2 + (2\eta_{tx} - \tau_{tt})u = 0. \tag{15}$$

This gives that the kernel of admitted Lie groups consists of the transformations corresponding to the generators

$$X_1 = \partial_x, \quad X_2 = \partial_t.$$

Because the main goal of this manuscript is not to do full group classification of Eq. (11) we consider only some particular cases of the function $g(u,\bar{u})$.

The first case is the case where the function $g(u,\bar{u})$ is a linear function

$$g = k_1 u + k_2 \bar{u} + k_3 \qquad (k_2 \neq 0).$$

In this case the only nontrivial extension of the kernel is for $k_3 = 0$:

$$X = x\partial_x + u\partial_u.$$

Another case of the function $g(u,\bar{u})$ that we study here is the case where the functions $\eta(x,t)$, $\tau(x,t)$ are linear with respect to x and t. Because of $\tau(x,t-r) = \tau(x,t)$, $\eta(x,t-r) = \eta(x,t)$, then

$$\eta = c_1x + c_2, \quad \tau = c_3x + c_4, \quad \zeta = u(c_1 - c_3u).$$

Here the function ζ is defined by (13). Note that an extension of the kernel of admitted Lie groups exists only for $c_1^2 + c_3^2 \neq 0$. Eq. (14) becomes

$$g_u u(c_1 - c_3u) + g_{\bar{u}}\bar{u}(c_1 - c_3\bar{u}) = g(c_1 - 3c_3u). \tag{16}$$

Let us analyze different choices of the constants c_1 and c_2.

If $c_1 = 0$, then

$$g = u^3\Phi\left(\frac{1}{u} - \frac{1}{\bar{u}}\right),$$

and a nontrivial admitted generator is

$$X = x\partial_t - u^2\partial_u.$$

Here and later the function Φ is an arbitrary function of one argument.

If $c_1 \neq 0$ and $c_3 = 0$, then one has

$$g = u\Phi\left(\frac{\bar{u}}{u}\right),$$

and an extension is given by the generator

$$X = x\partial_x + u\partial_u.$$

If $c_1 \neq 0$ and $c_3 \neq 0$, then

$$g = u(ku-1)^2\Phi\left(\frac{\bar{u}(ku-1)}{u(k\bar{u}-1)}\right)$$

and an extension is given by the generator

$$X = x\partial_x + kx\partial_t - u(ku-1)\partial_u,$$

where $k = c_3/c_1 \neq 0$.

Remark. Slight differences are in studying the delay differential equation

$$u_t + \bar{u}u_x = g(u, \bar{u}).$$

For example, for a linear function $g = k_1u + k_2\bar{u} + k_3$ one obtains that an extension of the kernel of admitted Lie groups is possible only for $k_3 = 0$. If $k_1 \neq 0$, then there is only one extension

$$X = t\partial_t + x\partial_x,$$

if $k_1 = 0$, there is one more admitted generator

$$X = x\partial_t + k_2x^2\partial_x - u(u - 2k_2x)\partial_u.$$

4. Conclusion

The definition of an admitted Lie group for functional differential equations, proposed in the manuscript, does not require using a global inverse function theorem as it was in [15]. The approach is similar to the approach, developed for differential equations: first, one has to construct determining equations, then, to split these equations with respect to arbitrary elements, which are dictated by the theorem of existence of the Cauchy problem, and then to find the general solution of the determining equations.

Practical construction of determining equations is performed by using the canonical Lie–Bäcklund's representation of an infinitesimal generator and acting by it on the original equation. The derivatives with respect to the dependent variables should be understood in terms of the Frechet derivatives. One should remember that determining equations have to be satisfied for any solution of the original system of equations.

In the strength of the theory of existence of solution of the Cauchy problem for delay differential equations, the process of solving determining equations for them is similar to obtaining the general solution of determining equations for differential equations.

The proposed approach can also be applied for finding an equivalence group, discrete, contact and Lie–Bäcklund transformations.

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