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THE GAUSS MAP FOR SURFACES: PART 1. THE AFFINE CASE

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ABSTRACT. Let M be a connected oriented surface and let G_2^c be the Grassmannian of oriented 2-planes in Euclidean $(2 + c)$-space, $\mathbf{E}^{2+c}$. Smooth maps $t\colon M \to G_2^c$ are studied to determine whether or not they are Gauss maps. Both local and global results are obtained. If t is a Gauss map of an immersion $X\colon M \to \mathbf{E}^{2+c}$, we study the extent to which t uniquely determines X under certain circumstances.

Let $X\colon M \to \mathbf{E}^{n+c}$ be an immersion of an oriented n-dimensional manifold into Euclidean $(n + c)$-space. Associated to X is the tangent plane map $t\colon M \to G_n^c$, where G_n^c is the Grassmannian of oriented n-planes through the origin of $\mathbf{E}^{n+c}$. The map t assigns to $p \in X$ the oriented n-plane $dX(T_pM)$, where T_pM is the tangent space to M at p. This generalizes the classical Gauss map for surfaces in $\mathbf{E}^3$, so we call t the Gauss map of X. On the other hand, suppose $t\colon M \to G_n^c$ is a smooth map. Is t the Gauss map of an immersion, or even locally a Gauss map, and to what extent does t determine X? In **[9]** we consider this question for $n = 2$ and $c = 2$ under the assumption that t is an immersion. Y. A. Aminov **[2]** studied the same existence question under essentially the same assumptions as imposed in **[9]**. More recently, Hoffman and Osserman **[4, 5]** studied a closely related question for $n = 2$ and c arbitrary under the additional assumption that M is a Riemann surface, i.e., M possesses a conformal structure. In this paper, we again consider the given question, but only under the assumption that M is a surface, i.e., $n = 2$. In Part 2 of this paper, we take a second look at the results of Hoffman and Osserman **[4, 5]** from the point of view established in this paper.

Some of the main results in this paper deal with maps t which are not immersions. Theorem 1 states a necessary and sufficient condition for a rank 1 map $t\colon M \to G_2^c$ —i.e., $\operatorname{rank}(t_{*|p}) = 1$ for all $p \in M$—defined on a simply connected plane domain M to be a Gauss map. A corollary to Theorem 1 is that any rank 1 map $t\colon M \to G_2^1 = S^2(1)$ defined on a simply connected plane domain is a Gauss map. In Theorem 2 we establish a sufficient condition on t in order for t to be a Gauss map on a neighborhood of a point p given that $\operatorname{rank}(t_{*|p}) = 1$.

We also show that the theory developed in **[9]** for immersions $t\colon M \to G_2^2$ holds exactly in the same form for immersions $t\colon M \to G_2^c$, where $c \geqslant 2$, if what we call

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the vector bundle of admissible maps, α, has rank equal to 2. The assumption that t is an immersion is necessary in order for t to be a Gauss map once we assume that α has rank equal to 2.

For any c, G_2^c is in a natural way a submanifold of $S^C(1)$, the unit sphere of dimension $C = \frac{1}{2}(c+2)(c+1) - 1$ and radius 1. We let $\bar{q}$ denote the second fundamental form of G_2^c in $S^C(1)$. Theorem 3 deals with t: $M \to G_2^c$ for which the bundle α has rank equal to 1 (and for which necessarily $c \geqslant 3$). If $t_*(TM)$ contains no asymptotic vectors of $\bar{q}$, then the bundle α induces a unique conformal structure on M with the property that if X: $M \to \mathbf{E}^{2+c}$ is an immersion with Gauss map t, then X induces the same conformal structure on M. This implies the following: If X: $M \to \mathbf{E}^{2+c}$ $(c \geqslant 3)$ is an inflection point free immersion, then the Gauss map of X already determines the conformal structure to be induced on M by X. Finally Theorem 4 states that an immersion X: $M \to \mathbf{E}^{2+c}$ $(c \geqslant 3)$ is determined up to translation and homothety by its Gauss map if the set of inflection points of X on M is nowhere dense. Similar results of this sort involve as well either the induced metric [**1**], the induced conformal structure [**4**], or convexity assumptions [**8**].

The main question and the theorems stated with the exception of Theorem 3 belong to the theory of affine differential geometry since they are independent of the chosen Euclidean structure on $(n+c)$-space. However, for primarily computational reasons it is convenient to use the available Euclidean structure. But note, in fact, that the objects of principal interest in this paper, the kernels of linear transformations, the signs of determinants of linear transformations or quadratic forms, the signatures of quadratic forms, etc., are independent of the chosen Euclidean structure on $\mathbf{E}^{n+c}$ or the induced inner production $\Lambda\mathbf{E}^{n+c}$, the exterior algebra of $\mathbf{E}^{n+c}$.

If ξ is a vector bundle over a manifold N and $p \in N$, we let ξ_p denote the fibre over p. In the case of the tangent bundle of N, TN, we write T_pN for $(TN)_p$. If S is a set and f: $\xi \to S$ is a map, we denote the restriction of f to ξ_p by f_p. Also $\Lambda^k(\xi)$ denotes the bundle of k-forms on N with values in ξ.

We use (,) to denote the inner product on $\mathbf{E}^{2+c}$ and the inner product on the exterior algebra of $\mathbf{E}^{2+c}$ induced from the one on $\mathbf{E}^{2+c}$. Throughout this paper M denotes a connected oriented C^∞ surface. Also all maps are C^∞ unless noted otherwise.

1. Preliminary remarks. Let V be a linear space. We use $G_2(V)$ to denote the Grassmannian of oriented 2-planes in V. Our primary interest is with $V = \mathbf{E}^{2+c}$, $c > 0$, and we abbreviate $G_2(\mathbf{E}^{2+c})$ by G_2^c. The Grassmannian G_2^c is viewed as the set of unit decomposable 2-vectors in $\Lambda^2\mathbf{E}^{2+c}$. On the other hand, $\Lambda^2\mathbf{E}^{2+c}$ is identified with $\mathbf{E}^{C+1}$, where $C = \frac{1}{2}(c+2)(c+1) - 1$; thus $G_2^c \subset S^C(1) \subset \mathbf{E}^{C+1}$, where $S^C(1)$ is the unit sphere centered at the origin of $\mathbf{E}^{C+1}$. Let $\bar{g}_0$ be the metric induced on G_2^c from $\mathbf{E}^{C+1}$. This is the standard metric on G_2^c (see e.g. [**6**] and §2 of this paper for other descriptions of this metric).

Let $\pi \in G_2^c$; by a "π-adapted" frame of $\mathbf{E}^{2+c}$ we mean a positively oriented orthonormal frame $e_1, e_2, \ldots, e_{2+c}$ of $\mathbf{E}^{2+c}$ such that e_1, e_2 is a positively oriented frame of π. Let U be an open subset of G_2^c; a frame field $e_1, e_2, \ldots, e_{2+c}$ of $\mathbf{E}^{2+c}$ defined on U is said to be "adapted" if $e_1(\pi), e_2(\pi), \ldots, e_{2+c}(\pi)$ is π-adapted for each $\pi \in U$. We use the notation $\pi^\perp$ to denote the oriented c-plane in $\mathbf{E}^{2+c}$ with positively oriented frame $e_3, e_4, \ldots, e_{2+c}$ if $e_1, e_2, e_3, \ldots, e_{2+c}$ is a π-adapted frame. Thus for each $\pi \in G_2^c$ we have the orthogonal splitting $\mathbf{E}^{2+c} = \pi \oplus \pi^\perp$ with corresponding orthogonal projections $(\cdots)^\top : \mathbf{E}^{2+c} \to \pi$ and $(\cdots)^\perp : \mathbf{E}^{2+c} \to \pi^\perp$.

Let M be a connected oriented surface and let t: $M \to G_2^c$ be a smooth map. Let $g_0 = t^*\bar{g}_0$; g_0 is a quadratic form on M and it is positive definite where t_* has maximal rank. When t is an immersion, g_0 is, of course, a riemannian metric. We define a 2-form μ_0 on M as follows: If v_1, v_2 is a positively oriented frame of T_pM, then

$$\mu_0(v_1, v_2) = g_0(v_1, v_1)g_0(v_2, v_2) - g_0(v_1, v_2)^2.$$

On the open subset of M where t_* has maximal rank, μ_0 is an area element, i.e., a positive 2-form.

In this paper, we let μ be a fixed area element on M. In most circumstances the choice of the element of area μ is immaterial to the discussion, but when t is an immersion we may as well and, in fact, will let $\mu = \mu_0$.

A frame field $e_1, e_2, \ldots, e_{2+c}$ of $\mathbf{E}^{2+c}$ defined on a subset of M is said to be "t-adapted" if $e_1(p), e_2(p), \ldots, e_{2+c}(p)$ is $t(p)$-adapted for each p in the subset.

Assume X: $M \to \mathbf{E}^{2+c}$ is an immersion with t: $M \to G_2^c$ its Gauss map, i.e., for all $p \in M$, the differential of X at p, d_pX, maps the oriented 2-dimensional vector space T_pM onto the oriented 2-plane $t(p)$, preserving the orientation. If we view dX as a differential 1-form with values in the trivial bundle $\mathbf{E}_M^{2+c}$ over M with fibre $\mathbf{E}^{2+c}$, i.e., dX is a section in $\Lambda^1(\mathbf{E}_M^{2+c})$, we have as usual $d(dX) = 0$. Therefore, a necessary and (by Poincaré's Lemma) sufficient condition for any section Φ in $\Lambda^1(\mathbf{E}_M^{2+c})$ defined over any simply connected open subset U of M to be the differential dX of an immersion X: $U \to \mathbf{E}^{2+c}$ is given by

$$\text{[C]} \qquad \Phi_p(T_pM) \subset t(p), \quad \det(\Phi_p) > 0, \quad d_p\Phi = 0 \quad \text{for all } p \in U,$$

where $\det(\Phi_p)$ is defined as follows: Let ν_p be the element of area on $t(p)$ induced from $\mathbf{E}^{2+c}$; then $\det(\Phi_p)$ satisfies

$$\Phi_p^*(\nu_p) = \det(\Phi_p)\mu_p.$$

We shall study the conditions of [C] now more closely; without causing confusion we shall consider the map t: $M \to G_2^c$ in the following as an oriented riemannian 2-plane bundle over M, a vector subbundle of $\mathbf{E}_M^{2+c}$ with $t(p) \subset \mathbf{E}^{2+c}$ as its fibre over $p \in M$, the metric being induced from $\mathbf{E}^{2+c}$. Correspondingly we obtain the oriented normal bundle $t^\perp$ of t in $\mathbf{E}_M^{2+c}$, where $t^\perp(p) = (t(p))^\perp$ for all $p \in M$.

In view of the first condition of [C] we introduce the following rank 4 vector subbundle β of the bundle $\Lambda^1(\mathbf{E}_M^{2+c})$, the fibre of which is given by

$$\beta_p = \left\{ \Phi \colon T_pM \to \mathbf{E}^{2+c} \text{ linear map} \middle| \Phi(T_pM) \subset t(p) \right\}$$

for all $p \in M$. Any section Φ in β can be viewed therefore as a differential 1-form with values in $\mathbf{E}^{2+c}$; hence, the vanishing of the differential 2-form $d\Phi$ with values in $\mathbf{E}^{2+c}$ is (because of $\mathbf{E}_M^{2+c} = t \oplus t^{\perp}$) equivalent to

[C′] $$(d\Phi)^{\perp} = 0$$

and

[C″] $$(d\Phi)^{\top} = 0.$$

But $\Phi \to (d\Phi)^{\perp}$ is a tensorial operation (see [**9**, §1]). Therefore we have

LEMMA 1. *There exists a vector bundle homomorphism* C: $\beta \to \Lambda^2(t^{\perp})$ *characterized by* $C(\Phi) = (d\Phi)^{\perp}$ *for every section* Φ *in* β, *and condition* [C′] *reduces to the purely algebraic condition*

[C′] $$C(\Phi) = 0.$$

DEFINITION 1. For each $p \in M$, let $\alpha_p = \ker(C_p\colon \beta_p \to \Lambda^2(t^{\perp}(p)))$. Let $\beta = (B, \pi, M)$, where B is the total space of β and π: $B \to M$ is the projection of B onto M. If $A = \bigcup_{p \in M} \alpha_p$, then $\alpha = (A, (p|A), M)$ is called the *space of admissible maps*. Let a: $M \to \mathbf{Z}$, the integers, be the function defined by

$$a(p) = \dim(\alpha_p).$$

If a is a constant on M, then $\alpha = \ker(C)$ is a vector subbundle of β, the *bundle of admissible maps*. (This terminology is motivated by Lemma 1.)

The second condition in [C], that $\det(\Phi_p) > 0$ for all $p \in U$, motivates the introduction of a function Q: $\alpha \to \mathbf{R}$ defined as follows: For every $p \in M$ the determinant function det: $\beta_p \to \mathbf{R}$ (as defined in [C]) is a nondegenerate quadratic form of signature 0—i.e., it has two positive and two negative eigenvalues—on the 4-dimensional vector space β_p. Therefore the restriction of det to the vector subspace α_p of β_p,

$$Q_p = \det|\alpha_p,$$

is a quadratic form on α_p which, according to the position of α_p in β_p, can be either definite, indefinite, or degenerate. We will use Q to represent its polarization whenever necessary. Finally let $\alpha^+ = \{\Phi \in \alpha | Q(\Phi) > 0\}$. Note that a section Φ in $\Lambda^1(\mathbf{E}_M^{2+c})$ satisfies the first two conditions in [C] as well as [C′] if and only if Φ takes values in α^+. Hence condition [C] on a simply connected open set $U \subset M$ reduces to

[C] $$\Phi_p \in \alpha_p^+ \quad \text{and} \quad (d_p\Phi)^{\perp} = 0 \quad \text{for all } p \in U,$$

where the second condition, [C″], can be viewed by considering independent tangential components of $(d_p\Phi)^{\top}$ as a system of two homogeneous linear first order partial differential equations. Whether the system is undetermined, overdetermined, or like (2) of [**9**], depends on the value of the function a on M.

When there exists a section Φ in $\alpha|U$ satisfying [C] in some neighborhood U of each point $p \in M$ we will say that t *is locally a Gauss map*. When an immersion X: $M \to \mathbf{E}^{2+c}$ exits with t as its Gauss map we say t *is a Gauss map*.

2. The Weingarten map. It is clear from the preceding section that the dimension of α_p, $a(p)$, as well as a knowledge of the behavior of Q_p—i.e., whether Q_p is degenerate, definite, or indefinite—is vitally important. We turn our attention to these items in the present section.

If $\pi \in G_2^c$, then it is well known that $T_\pi G_2^c \simeq \mathrm{GL}(\pi, \pi^\perp)$. The isomorphism ι: $T_\pi G_2^c \to \mathrm{GL}(\pi, \pi^\perp)$ can be defined as follows: Let $l \in T_\pi G_2^c$ and $e \in \pi$; then

$$\iota(l)(e) = (l\bar{e})^\perp ,$$

where $\bar{e}$ is an $\mathbf{E}^{2+c}$-valued function defined near π that extends e, i.e., $\bar{e}(\pi) = e$, with $\bar{e}(\lambda) \in \lambda$ for all $\lambda \in G_2^c$ near π, and $(l\bar{e})^\perp = (d\bar{e}(l))^\perp$. In the future, we will suppress the isomorphism ι, sometimes suppress the bar over e, and simply write $l(e) = (le)^\perp$.

When $l \in T_\pi G_2^c$ is viewed as an element of $\mathrm{GL}(\pi, \pi^\perp)$, then $\bar{g}_0(l, l) = \|l\|^2$, the norm squared of l, i.e., if $e_1, e_2, \ldots, e_{2+c}$ is a π-adapted frame of $\mathbf{E}^{2+c}$ and we define reals l_i^α for $i \in \{1, 2\}$, $\alpha \in \{3, 4, \ldots, 2+c\}$ by $l_i^\alpha = (l(e_i), e_\alpha)$, then $g_0(l, l) = \sum_{i,\alpha}(l_i^\alpha)^2$ (cf. [**6**, p. 338] keeping in mind the preceding paragraph).

If t: $M \to G_2^c$ is a smooth map and $u \in T_pM$, then $t_*(u) \in T_{t(p)}G_2^c = \mathrm{GL}(t(p), t^\perp(p))$. For $\Phi \in \beta_p$ and $v \in T_pM$, $\Phi(v) \in t(p)$. It turns out (see [**9**, §2]) that

$$t_*(u)(\Phi(v)) = (u\Phi(v))^\perp , \tag{1}$$

where $u\Phi(v)$ denotes the directional derivative of $\Phi(v)$—actually any extension of $\Phi(v)$ to a section in t defined near p—with respect to u. If Φ is the differential of an immersion X, then the second fundamental form h of X is defined by

$$h(u, v) = (u\Phi(v))^\perp \quad \text{for all } (u, v) \in TM \oplus TM.$$

It follows from (1) that

$$h(u, v) = t_*(u)(\Phi(v)) \quad \text{for all } (u, v) \in TM \oplus TM.$$

Therefore if Φ is an arbitrary element of β_p, the quadratic form h_Φ: $T_pM \times T_pM \to t^\perp(p)$ defined by

$$h_\Phi(u, v) = t_*(u)(\Phi(v)) \quad \text{for all } (u, v) \in T_pM \times T_pM, \tag{2}$$

is called the *second fundamental form of* Φ. One may check (see [**9**, §2]) that α is precisely the set of all Φ for which the second fundamental form h_Φ is symmetric.

DEFINITION 2. For all $v \in T_pM$, $t_*(v) \in \mathrm{GL}(t(p), t^\perp(p))$. By *the span of* $t_{*|p}$, denoted S_p, we mean the span of $\{t_*(v)(e) | v \in T_pM,\ e \in t(p)\}$. Clearly $S_p \subset t^\perp(p)$.

We give another interpretation of S_p. Note that we also view $t_{*|p}$: $T_pM \to T_{t(p)}G_2^c \simeq \mathrm{GL}(t(p), t^\perp(p))$ as the linear map

$$t_{*|p}: T_pM \otimes t(p) \to t^\perp(p),$$

where $t_{*|p}(v \otimes e) = t_{*|p}(v)(e)$ for all $v \in T_pM$ and all $e \in t(p)$. Then $S_p = \mathrm{ran}(t_{*|p})$, the range of $t_{*|p}$. From this point of view the transpose of $t_{*|p}$ is a linear map from $t^\perp(p)^*$ into $[T_pM \otimes t(p)]^*$. Since both $t(p)$ and $t^\perp(p)$ are endowed with an inner product we may identify $t(p)^*$ and $t^{\perp}(p)^*$ with $t(p)$ and $t^{\perp}(p)$, respectively. We denote the transpose of $t_{*|p}$ by A_p and note that with the above identifications

$$A_p\colon t^{\perp}(p) \to T_p^*M \otimes t(p) = \beta_p.$$

Thus we obtain a vector bundle homomorphism $A\colon t^{\perp} \to \beta$. Also we denote the value of A at $z \in t^{\perp}$ by A^z. If $\Phi \in \beta_p$, then (2) implies

$$(h_\Phi(u,v),\, z) = (A^z(u),\, \Phi(v)) \quad \text{for all } u, v \in T_pM.$$

This fact motivates

DEFINITION 3. For any $t\colon M \to G_2^c$ we call $A\colon t^{\perp} \to \beta$ the *Weingarten map* of t. Finally note that $A|S_p$ is an isomorphism and $\operatorname{ran}(A_p) = A(S_p)$ for all $p \in M$; this fact follows immediately from the fact that $t_{*|p}$ and A_p are transposes of one another.

Define a nondegenerate skew-symmetric tensor field $P\colon \beta \oplus \beta \to \mathbf{R}$ as follows: For $\Phi, \Psi \in \beta_p$, let

$$P(\Phi,\Psi) = (\Phi(v_1),\, \Psi(v_2)) - (\Phi(v_2),\, \Psi(v_1)),$$

where $v_1, v_2 \in T_pM$ and $\mu(v_1 \wedge v_2) = 1$.

LEMMA 2. $\alpha_p = \{\Phi \in \beta_p | P(A^z, \Phi) = 0$ *for all* $z \in S_p\}$.

PROOF. The definitions for P given here and in **[9]** are insignificantly different. Noting this, one may check that the proof of the same lemma in **[9]** given under the assumptions that $c = 2$ and t is an immersion holds in this more general setting.

PROPOSITION 1. $\dim(\alpha_p) = 4 - \dim(S_p) \geqslant 4 - c$.

PROOF. Since P is nondegenerate, $\dim(\alpha_p) = 4 - \dim(A(S_p)) = 4 - \dim(S_p)$ since A is an isomorphism on S_p. Also $c \geqslant \dim(S_p)$ since $S_p \subset t^{\perp}(p)$.

COROLLARY. *If* $\dim(S_p) = 4$ *for some* $p \in M$, *then* t *is not locally a Gauss map*.

In the following, if B is a quadratic form, the number of positive eigenvalues of B minus the number of negative eigenvalues of B will be the *signature* of B, denoted $\sigma(B)$.

Let $\alpha_p^{\perp}$ denote the $\det_p$-orthogonal complement of α_p in β_p; then let $Q_p^{\perp} = \det|\alpha_p^{\perp}$. By Lemma 5 of **[9]** we know that $\sigma(Q_p) = -\sigma(Q_p^{\perp})$ since $\sigma(\det_p) = 0$ where, of course, $\det_p$ is viewed as a quadratic form on β_p.

Both P_p and $\det_p$ are nondegenerate and so determine in a standard fashion isomorphisms $\iota_P, \iota_{\det}\colon \beta \to \beta^*$, respectively. Define an isomorphism $I\colon \beta \to \beta$ by setting $I = \iota_{\det}^{-1} \circ \iota_P$. Also set $Q_p' = \det_p|\operatorname{ran}(A_p)$. Then (again see **[9]** for details) the following hold:

$$(1)\quad I(\operatorname{ran}(A_p)) = \alpha_p^{\perp}; \qquad (2)\quad I^*Q_p^{\perp} = Q_p'.$$

Clearly $\sigma(Q_p) = -\sigma(Q_p')$.

Now let $F = A^*(\det)$; F is a quadratic form on $t^\perp$. Clearly $\sigma(Q_p) = -\sigma(F_p)$. Since A_p: $S_p \to \operatorname{ran}(A_p)$ is an isomorphism we need only consider $F|S_p$ to determine the signature of F. This proves

LEMMA 3. $\sigma(Q_p) = -\sigma(F|S_p)$ *for all* $p \in M$.

We now define a function k: $M \to \mathbf{R}$ by

$$k(p) = \operatorname{Tr}\left(F|S_p\right) \quad \text{for all } p \in M.$$

The trace, $\operatorname{Tr}(F|S_p)$, is computed with respect to the inner product induced on S_p from $\mathbf{E}^{2+c}$. We set $k(p) = 0$ at any point p where $S_p = \{0\}$, i.e., $t_{*|p} = 0$. The function k is the curvature of the oriented riemannian vector bundle t with respect to μ. It is easy to check that $k(p) = 0$ if $\operatorname{rank}(t_{*|p}) < 2$. If $\operatorname{rank}(t_{*|p}) = 2$ and $\mu_p = \mu_{0|p}$, then $|k(p)| \leqslant 1$; this follows easily from the remarks preceding Lemma 7 of [**9**]. We call k the *pre-Gaussian curvature* of t (with respect to μ). If X is an immersion of M with Gauss map t, then the Gaussian curvature of X, K, is given by $K = \rho(dX)k$, where $\rho(dX) = 1/Q(dX)$ may be interpreted as the Jacobian of the Gauss map, i.e., the ratio of the element of area μ to the element of area induced on M by X.

What is important about k is that $\sigma(F|S_p)$ and $k(p)$ have the same sign under just the right set of circumstances.

We now turn to a case-by-case study of t; each case is determined by $a(p) = \dim(\alpha_p)$. By the corollary to Proposition 1 we need not consider t for which there are points p at which $a(p) = 4$. Also the case $a = 0$ is not interesting in this setting.

3. $a = 3$. In this section we study the consequences of assuming $a(p) = 3$, for some $p \in M$, or assuming $a = 3$ on M.

When $a(p) = 3$ for $p \in M$, Q_p has a positive eigenvalue. This is the case since $\det_p$: $\beta_p \to \mathbf{R}$ has two positive eigenvalues and α_p has codimension 1 in β_p. In particular $\alpha_p^+ \neq \varnothing$. With a little more work, we could show Q_p has eigenvalues of signs $(+, +, -)$, $(+, 0, -)$, or $(+, -, -)$ according as $k(p)$ is negative, zero, or positive.

Now assume $a = 3$ on M. For any $p \in M$, it must be the case that $\operatorname{rank}(t_{*|p}) = 1$ or 2. Of course, if $\operatorname{rank}(t_{*|p}) = 2$, then $\operatorname{rank}(t_*) = 2$ in a neighborhood of p. Therefore, let us first suppose t: $M \to G_2^c$ is an immersion with $a = 3$.

LEMMA 4. *Let* t: $M \to G_2^c$ *be an immersion with* $a = 3$, *i.e.*, S_p *is* 1*-dimensional for all* $p \in M$. *Then there is a subspace* $E^3 \subset \mathbf{E}^{2+c}$ *such that* $t(p) \subset \mathbf{E}^3$ *for all* $p \in M$. *Hence* t *may be viewed as an immersion into* $G_2(\mathbf{E}^3) = S^2(1)$, *which is a submanifold of* G_2^c.

PROOF. The proof is a straightforward exercise in the use of the Cartan structural equations.

If $a = 3$ and $k > 0$ on M, then trivially t is a Gauss map of an immersion X: $M \to \mathbf{E}^3 \subset \mathbf{E}^{2+c}$. In fact, t is the Gauss map of the immersion $X = e_3$ (where e_1, e_2, e_3 is a t-adapted frame of $\mathbf{E}^3$) since $dX = -A^{e_3}$ and $\det(dX) = \det(-A^{e_3}) = k > 0$. In this case t is a Gauss map for any M. If $k < 0$ on M, then necessarily M

must not be closed in order for t to be a Gauss map (since every closed surface in $\mathbf{E}^3$ has a point of positive Gaussian curvature). If M is simply connected and not closed when $k < 0$ on M, then t is a Gauss map. First observe that

$$\det\big(t_*\colon TM \to TG_2(\mathbf{E}^3)\big) < 0$$

if $k < 0$; this follows immediately from the fact that $\det(t_*) = \det(A^{e_3})$, which is easily verified. Then one may give M a conformal structure (with the given orientation on M) for which $t_*\colon TM \to TG_2(\mathbf{E}^3)$ is antiholomorphic. Now use Case 1 of Theoren 2.6 of **[4]** or the results in Part 2 of this paper to prove t is a Gauss map.

Now suppose $t_{*|p}$ has rank 1 for all $p \in M$; we call such a t a *rank* 1 *map*. As we will see in Proposition 2 of §5, it is necessary that $a = 3$ in order for t to be locally a Gauss map. Since each $l \in T_\pi G_2^c$ for $\pi \in G_2^c$ may be viewed as a linear transformation we may speak of l's rank. Clearly, if $t_{*|p}$ has rank 1, then $a(p) = 3$, i.e., $\dim(S_p) = 1$, if and only if $t_*(v)$ has rank 1 for all $v \notin \ker(t_{*|p})$.

THEOREM 1. *Let M be a simply connected plane domain and let $t\colon M \to G_2^c$ be a rank* 1 *map. Then t is a Gauss map if and only if* $\operatorname{rank}(t_*(v)) \leqslant 1$ *for all $v \in TM$.*

PROOF. We have just observed that necessarily $\operatorname{rank}(t_*(v)) \leqslant 1$ for all $v \in TM$.

Now suppose t is a rank 1 map and $\operatorname{rank}(t_*(v)) \leqslant 1$ for all $v \in TM$. By **[10]** there exists an orientation preserving local diffeomorphism $\Psi\colon M \to \Psi(M) \subset \mathbf{R}^2$ which maps the kernel of t_* into vertical vectors in $\mathbf{R}^2$. Let $\Psi = (x, y)$ and note that locally we may use (x, y) as positively oriented coordinates on M so that it makes sense to introduce $\partial/\partial x$, $\partial/\partial y$ and take partial derivatives with respect to x and y. Note that $t_*(\partial/\partial y) = 0$, $t_*(\partial/\partial x) \neq 0$, and hence $\operatorname{rank}(t_*(\partial/\partial x)) = 1$. Since M is simply connected we may find a global t-adapted framing $e_1, e_2, \ldots, e_{2+c}$ with $e_2 \in \ker(t_*(\partial/\partial x))$ and e_3 spanning $\operatorname{ran}(t_*(\partial/\partial x))$. Define 1-forms ω_i^α on M by $\omega_i^\alpha = (de_i, e_\alpha)$ for $i \in \{1,2\}$, $\alpha \in \{3, \ldots, 2 + c\}$. By equation (1), $\omega_i^\alpha(v) = (t_*(v)(e_i), e_\alpha)$ for all vector fields v on M. Set $m = \omega_1^3(\partial/\partial x)$. Since $\omega_1^3(\partial/\partial y) = 0$, it follows that $\omega_1^3 = m\,dx$. Also, $\omega_1^\alpha = 0$ for $\alpha > 3$, and $\omega_2^\alpha = 0$ for all α. Using the Cartan structural equations

$$d\omega_r^s = -\sum_{t=1}^{2+c} \omega_t^s \wedge \omega_r^t \quad \text{for } r, s \in \{1, 2, \ldots, 2 + c\},$$

it follows that there exists a real-valued function f on M such that $\omega_1^2 = f\,dx$ and $\partial f/\partial y = 0$. Note that $e_1 \otimes dx$, $e_1 \otimes dy$, $e_2 \otimes dx$, and $e_2 \otimes dy$ are globally defined linearly independent sections in β. Also $A^{e_3} = -e_1 \otimes \omega_1^3 = -me_1 \otimes dx$. Hence, by Lemma 2, α has a global framing given by $e_1 \otimes dx$, $e_2 \otimes dx$ and $e_2 \otimes dy$. According to condition [C] we look for a global section

$$\Phi = \phi_1^1 e_1 \otimes dx + \phi_1^2 e_2 \otimes dx + \phi_2^2 e_2 \otimes dy$$

in α, where ϕ_1^1, ϕ_1^2, and ϕ_2^2 are unknown real-valued functions on M, satisfying

$$0 = d\Phi(\partial/\partial x, \partial/\partial y) = e_1 \otimes \big(-\partial\phi_1^1/\partial y - f\phi_2^2\big) + e_2 \otimes \big(-\partial\phi_1^2/\partial y + \partial\phi_2^2/\partial x\big)$$

and

$$Q(\Phi) = \phi_1^1\phi_2^2 > 0.$$

If we require, in addition, that $\phi_1^1 = \phi_2^2$, we need only find functions ϕ_1^1 and ϕ_1^2 on M satisfying

$$\partial\phi_1^1/\partial y = -f\phi_1^1 \quad \text{and} \quad \partial\phi_1^2/\partial y = \partial\phi_1^1\partial u,$$

with ϕ_1^1 never 0. Such a solution is

$$\phi_1^1 = \phi_2^2 = e^{-f\cdot y} \quad \text{and} \quad \phi_1^2 = (\partial f/\partial x)(f\cdot ye^{-f\cdot y} + e^{-f\cdot y} - 1)f^{-2}.$$

Hence a globally defined immersion $X\colon M \to \mathbf{E}^{2+c}$ exists with $dX = \Phi$ and thus t is a Gauss map.

COROLLARY. *Let M be a simply connected plane domain and let $t\colon M \to G_2^1$ be a rank 1 map. Then t is a Gauss map.*

PROOF. All nonzero vectors $l \in TG_2^1$ have rank equal to 1.

REMARK. When $c > 1$, it turns out, as you will see in later sections, that a vector $l \in T_\pi G_1^c$ is of rank 1 if and only if it is an asymptotic vector (of the second fundamental form) of G_2^c as a submanifold of $S^C(1)$.

If $a = 3$ and the sign of k is not constant, then it is not immediately clear that t is a Gauss map in a neighborhood of any point p for which $k(p) = 0$. Bleecker and Wilson [3] state without proof that this is so if $c = 1$ (for when $c = 1$, $\operatorname{rank}(t_{*|p}) = 1$ if and only if $a(p) = 3$ and $k(p) = 0$).

THEOREM 2. *Let $t\colon M \to G_2^c$ be a smooth map with $a = 3$. If $\operatorname{rank}(t_{*|p}) = 1$, then there is a neighborhood U of p such that $t|U$ is a Gauss map.*

PROOF. The bundle α is 3-dimensional and we seek three linearly independent sections in α defined near p. Choose $e_1, \ldots, e_{2+c}$ to be a t-adapted frame defined near p such that $e_3(p)$ spans S_p. Define 1-forms ω_i^α as in the proof of the preceding theorem. First

$$A^{e_3} = -\left(e_1 \otimes \omega_1^3 + e_2 \otimes \omega_2^3\right),$$

and since t has rank 1 at p, $\omega_1^3 \wedge \omega_2^3 = 0$ but ω_1^3 and ω_2^3 are not both zero at p. By rotating e_1 and e_2, we may suppose $\omega_1^3 \neq 0$ and $\omega_2^3 = 0$ at p. Let Ψ^1, Ψ^2 be a framing of T^*M near p such that $\Psi^1 = \omega_1^3$ at p. Then define functions m_1 and m_2 by $\omega_2^3 = m_1\Psi^1 + m_2\Psi^2$ and note that $m_1(p) = 0$ and $m_2(p) = 0$.

Now by Lemma 2, $\Phi = e_1 \otimes \phi^1 + e_2 \otimes \phi^2$, for 1-forms ϕ^1, ϕ^2, is a section in α near p if and only if $\phi^1 \wedge \omega_1^3 + \phi^2 \wedge \omega_2^3 = 0$. Clearly $\sigma_1 = -A^{e_3} = e_1 \otimes \omega_1^3 + e_2 \otimes \omega_2^3$, $\sigma_2 = -m_1 e_1 \otimes \Psi^2 + e_2 \otimes \Psi^2$, and $\sigma_3 = e_1 \otimes \omega_2^3 + e_2 \otimes \omega_1^3$ are linearly independent sections in α near p. Any section Φ in α near p can be written $\Phi = \sum_{k=1}^3 u^k\sigma_k$ for suitable real-valued functions u^k defined near p, $k \in \{1,2,3\}$. If we seek Φ which is the differential of an immersion into $\mathbf{E}^{2+c}$ with Gauss map t for which $u^3 = 0$, condition [C] reduces to solving

$$(3) \qquad \begin{cases} (u^1)_2 + m_1(u^2)_1 = \cdots, \\ m_1(u^1)_2 + m_2(u^1)_1 + (u^2)_1 = \cdots \end{cases}$$

with no derivatives of u' on the right side of these equations, subject to

(4) $$(m_2)^2(u^1)^2+\left(1+(m_1)^2\right)u^1u^2>0,$$

where the subscripts 1 and 2 in $(u')_1$, $(u')_2$ are defined by $df=f_1\Psi^1+f_2\Psi^2$. At p, (3) becomes

$$\begin{cases}(u^1)_2=\cdots,\\(u^2)_1=\cdots\end{cases}$$

which shows that (3) is hyperbolic at and hence near p. Also at p (4) becomes $1u^1u^2>0$. Thus we need a solution u^1, u^2 of a hyperbolic system with $u^1u^2>0$ at p. This can be easily accomplished.

COROLLARY. *Let t: $M\to G_2^1$ be a smooth map with t_* never 0. Then t is locally a Gauss map.*

PROOF. Necessarily $a=3$ on M.

To what extent the map t of the last corollary is a Gauss map, even supposing M is simply connected, is unknown to me.

4. The second fundamental form of G_2^c in $S^C(1)$. Let $\bar q$ denote the second fundamental form of G_2^c in $S^C(1)$; $\bar q_\pi$ is a quadratic form on $T_\pi G_2^c$ with values in $\Lambda^2\pi^\perp$, the normal space to G_2^c at π in $S^C(1)$, for all $\pi\in G_2^c$. For $c=1$, $\bar q=0$ since $G_2^1=S^2(1)$. For $c>1$, $\bar q$ has an interesting interpretation when we view $T_\pi G_2^c=\mathrm{GL}(\pi,\pi^\perp)$. If $l\in T_\pi G_2^c$, then $l\wedge l$: $\Lambda^2\pi\to\Lambda^2\pi^\perp$ is the induced transformation on 2-vectors. According to our convention, π is the positive unit 2-vector in $\Lambda^2\pi$; hence $l\wedge l(\pi)\in\Lambda^2\pi^\perp$.

LEMMA 5. *For all $l\in T_\pi G_2^c$*

$$\tfrac12\bar q(l,l)=l\wedge l(\pi).$$

PROOF. Let λ be a unit length decomposable 2-vector in $\Lambda^2\pi^\perp$. It is enough to show that

(5) $$\left(\tfrac12\bar q(l,l),\,\lambda\right)=\left(l\wedge l(\pi),\,\lambda\right).$$

If we let r^λ: $\pi^\perp\to\lambda$ denote orthogonal projection onto λ, then (5) is equivalent to

$$\left(\tfrac12\bar q(l,l),\,\lambda\right)=\det(r^\lambda\circ l).$$

Let $e_1,e_2,\ldots,e_{2+c}$ be an adapted frame field of $\mathbf{E}^{2+c}$ defined near π such that $\lambda=\operatorname{span}\{e_3(\pi),e_4(\pi)\}$. Then, at π, using $l(e_i)=(de_i(l))^\perp$ we obtain

$$\begin{aligned}\det(r^\lambda\circ l)&=\det\left[(l(e_i),\,e_\alpha)\right]_{\substack{i=1,2\\ \alpha=3,4}}=\det\left[(de_i(l),\,e_\alpha)\right]_{\substack{i=1,2\\ \alpha=3,4}}\\&=\det\left[\omega_i^\alpha(l)\right]_{\substack{i=1,2\\ \alpha=3,4}}=\left(\omega_1^3\omega_2^4-\omega_2^3\omega_1^4\right)(l,l).\end{aligned}$$

The 1-forms ω_i^α are defined by $\omega_i^\alpha=(de_i,e_\alpha)$ for $i\in\{1,2\}$, $\alpha\in\{3,\ldots,2+c\}$. On the other hand, one may check that at π

$$(\bar q,\lambda)=-\left(d(e_1\wedge e_2),\,d(e_3\wedge e_4)\right)=2\left(\omega_1^3\omega_2^4-\omega_2^3\omega_1^4\right).$$

Hence $(\bar q(l,l),\,\lambda)=2\det(r^\lambda\circ l)$.

If $t\colon M \to G_2^c$ is a smooth map, we let $q = t^*\bar{q}$. Note that q_p is a quadratic form on T_pM with values in $\Lambda^2 S_p$ since $t_*(v)$ has values just in S_p for all $v \in T_pM$. So if $a(p) = 3$, i.e., $\dim(S_p) = 1$, then $q_p = 0$. When $a(p) = 2$, give S_p an orientation if it does not have one (i.e., if $S_p \neq \pi^\perp$). Let $\tilde{S}_p$ denote this oriented S_p, so by convention $\tilde{S}_p \in \Lambda^2\pi^\perp$; then q_p takes values that are multiples of $\tilde{S}_p$. When this is the case it is natural to redefine q_p to be real-valued, i.e., we set

$$\tfrac{1}{2}q_p(v,v) = \det\big(t_*(v)\colon t(p) \to \tilde{S}_p\big)$$

for all $v \in T_pM$. This determines q_p up to sign, except when $c = 2$; when $c = 2$, $\tilde{S}_p = \pi^\perp$.

Let K_G denote the sectional curvature of G_2^c. When t is an immersion we define $K_t\colon M \to \mathbf{R}$ by $K_t(p) = K_G(t_*(T_pM))$. Also, when t is an immersion, we may view q as the restriction of $\bar{q}$ to M; hence we may use q together with the Gauss curvature equation to compute K_t. In particular when $a(p) = 2$ so that $\bar{q}_{t(p)}|t_*(T_pM)$ takes values only in the S_p direction, we have (remembering that $\bar{q}$ is the second fundamental form of G_2^c in $S^C(1)$)

$$K_t(p) = 1 + \operatorname{Det}(q_p),$$

where $\operatorname{Det}(q_p)$ stands for the determinant of the (real-valued) quadratic form q_p with respect to $g_0 = t^*\bar{g}_0$ at p. Note that $\operatorname{Det}(q_p)$ is well defined even though q_p, in general, is defined up to sign. This yields

LEMMA 6. *If $a(p) = 2$ and t is regular at p (i.e.,* $\operatorname{rank}(t_{*|p}) = 2$), *then* $\operatorname{Det}(q_p) = K_t(p) - 1$.

5. $a = 2$. First we turn our attention to the algebraic consequences of assuming $a(p) = 2$ for $p \in M$. Then we study existence and uniqueness questions for an immersion X with t as its Gauss map assuming $a = 2$ on M.

When $a(p) = 2$, there is no restriction on the possible signs of the eigenvalues of Q_p. We need another invariant beside $\sigma(Q_p)$ to determine its behavior. So choose and fix an inner product on β. By $\operatorname{Det}(Q_p)$ we mean the determinant of Q_p computed using a frame in α_p that is orthonormal with respect to the imposed inner product on β_p. Since S_p lies in a Euclidean space, it carries an inner product induced from that Euclidean space. By $\operatorname{Det}(F|S_p)$ we mean the determinant of $F|S_p$ with respect to that inner product. The next lemma is proved in [**9**, §3]. One may check that the proof does not use the facts that $c = 2$ or t is an immersion as is assumed throughout [**9**].

LEMMA 7. *If $a(p) = 2$, then*

$$\operatorname{sign}\operatorname{Det}\big(Q_p\big) = \operatorname{sign}\operatorname{Det}\big(\,F|S_p\big).$$

Thus Lemmas 3 and 7 tell us how the behavior of $F|S_p$ completely determines the behavior of Q_p. For example, suppose $a(p) = 2$ and $\operatorname{rank}(t_{*|p}) = 1$. Then there exists a nonzero vector $v \in \ker(t_{*|p})$. Clearly $v \in \ker(A^z)$ for all $z \in t^\perp(p)$. Thus $\det(A^z) = 0$ for all $z \in S_p$, i.e., $F|S_p = 0$. Then Lemmas 3 and 7 imply $Q_p = 0$. This proves

LEMMA 8. *If $a(p) = 2$ and* $\operatorname{rank}(t_{*|p}) = 1$, *then* $Q_p = 0$.

When $\operatorname{rank}(t_{*|p}) = 1$, it follows easily from Proposition 1 that $a(p) = 2$ or 3 since necessarily $\dim(S_p) = 1$ or 2. This observation, along with the previous lemma, proves

PROPOSITION 2. *If* $\operatorname{rank}(t_{*|p}) = 1$, *then necessarily* $a(p) = 3$ *in order for* t *to be locally a Gauss map.*

Now we turn out attention to the relationship between the behaviors of Q_p and q_p when $a(p) = 2$ and t is regular at p, i.e., $\operatorname{rank}(t_{*|p}) = 2$.

If B is a quadratic form on a vector space V we call a 1-dimensional subspace of V an *isotropic direction* if it is spanned by an isotropic vector of B. Let $\mathscr{I}B$ be the set of isotropic directions of B. In our situation $\dim(V) = 2$ so that $\mathscr{I}B$ consists of 0, 1, 2, or every 1-dimensional subspace of V. Let $\#\mathscr{I}B$ denote the number of elements in $\mathscr{I}B$; thus $\#\mathscr{I}B = 0, 1, 2$, or ∞.

LEMMA 9. *If* $a(p) = 2$ *and* t *is regular at* p, *then* $\#\mathscr{I}Q_p = \#\mathscr{I}q_p$. *This is equivalent to saying*

$$\operatorname{sign}\operatorname{Det}(Q_p) = \operatorname{sign}\operatorname{Det}(q_p)$$

and, moreover, $Q_p = 0$ *if and only if* $q_p = 0$.

PROOF. Clearly $\#\mathscr{I}Q_p = \#\mathscr{I}(F|S_p)$ by Lemmas 3 and 7. We will show $\#\mathscr{I}(F|S_p) = \#\mathscr{I}q_p$ to prove the lemma. Let $L \in \mathscr{I}(F|S_p)$ and choose $0 \neq z \in L$. Then $\det(A^z) = F(z) = 0$ by $A^z \neq 0$ since $z \in S_p - \{0\}$. The kernel of A^z is 1-dimensional and independent of the choice of $z \in L - \{0\}$. Let $0 \neq v \in \ker(A^z)$; then $\operatorname{ran}(t_*(v)) \subset S_p$ but is orthogonal to z. Thus $\det(t_*(v)\colon t(p) \to S_p) = 0$. Hence $q(v, v) = 2\det(t_*(v)) = 0$. Therefore $\ker(A^z) \in \mathscr{I}q_p$. Thus we define a function f: $\mathscr{I}(F|S_p) \to \mathscr{I}q_p$ as follows: If $L \in \mathscr{I}(F|S_p)$, then

$$f(L) = \ker(A^z) \quad \text{for } 0 \neq z \in L.$$

We now show f is a bijection. Suppose $\ker(A^z) = \ker(A^w)$, where $S_p = \operatorname{span}\{z, w\}$. If $0 \neq v \in \ker(A^z) = \ker(A^w)$, then $\operatorname{ran}(t_*(v))$ is orthogonal to $\operatorname{span}\{z, w\} = S_p$. Thus $t_*(v) = 0$, contradicting the fact that $\operatorname{rank}(t_{*|p}) = 2$. Hence f is an injection. Now suppose that $\operatorname{span}\{v\} \in \mathscr{I}q_p$. Then there exists $0 \neq z \in S_p$ such that $\operatorname{ran}(t_*(v)) \perp z$; thus $v \in \ker(A^z)$. Thus f is surjective.

An immediate consequence of Lemmas 6 and 9 is the next lemma. This is also proved in **[9]** for $c = 2$ by different means.

LEMMA 10. *If* $a(p) = 2$ *and* t *is regular at* p, *then*

$$\operatorname{sign}\operatorname{Det}(Q_p) = \operatorname{sign}(K_t(p) - 1).$$

Suppose again $a(p) = 2$. Since $\operatorname{sign} k(p) = \operatorname{sign}\operatorname{Tr}(F|S_p) = \operatorname{sign}\sigma(F|S_p)$ when $\operatorname{Det}(F|S_p) \geqslant 0$, we conclude from Lemma 3 that $\operatorname{sign}\sigma(Q_p) = -\operatorname{sign} k(p)$ when $\operatorname{Det}(Q_p) \geqslant 0$. We summarize the results we have obtained.

PROPOSITION 3. *Suppose* $a(p) = 2$ *and* t *is regular at* p. *Then*

$$\operatorname{sign}\operatorname{Det}(Q_p) = \operatorname{sign}(K_t(p) - 1),$$

and if $\mathrm{Det}(Q_p) \geqslant 0$, *then*

$$\operatorname{sign}\sigma(Q_p) = -\operatorname{sign} k(p).$$

COROLLARY. *If* $a(p) = 2$ *and* t *is regular at* p, *then* $\alpha_p^+ \neq \varnothing$ *if and only if* (1) $K_t(p) < 1$, *or* (2) $K_t(p) \geqslant 1$ *and* $k(p) < 0$.

REMARK. When $a(p) = 2$, let $\mathbf{E}^4 = \operatorname{span}(t(p) \cup S_p)$; then $G_2(\mathbf{E}^4)$ is a totally geodesic submanifold of G_2^c passing through $t(p)$. It is straightforward to show that $t_*(T_pM) \subset T_{t(p)}G_2(\mathbf{E}^4)$. Hence, since the sectional curvatures of G_2^2 lie between 0 and 2 inclusive, it is clear that $0 \leqslant K_t(p) \leqslant 2$. Also, since t is an immersion so that $\mu = \mu_0$, we know that $|k(p)| \leqslant 1$. However, arguing as in **[9]** we can show that it is not the case that simultaneously $K_t(p) = 1$ and $k(p) = \pm 1$ since this implies $a(p) = 3$. Otherwise there are no other restrictions on the possible values of $(K_t(p), k(p))$ when $a(p) = 2$.

If $K_t(p) \geqslant 1$ and $k(p) > 0$, then changing the orientation of M changes the sign of $k(p)$ resulting in $\alpha_p^+ \neq \varnothing$. However, if $K_t(p) = 1$ and $k(p) = 0$, then $\alpha_p^+ = \varnothing$ under either orientation; in fact, at such p, Proposition 3 implies $Q_p = 0$. These points at which $K_t = 1$ and $k = 0$ are the essence of the example of Aminov **[2**, pp. 166–168] of a surface in G_2^2 which is not locally the image of a Gauss map.

Let us suppose $a = 2$ on M and t is an immersion (for if t fails to be an immersion then Proposition 3 implies that t is not locally a Gauss map). Also suppose $\alpha_p^+ \neq \varnothing$ for all $p \in M$. Exactly as in **[9]** one may show that the type of the linear system of two partial differential equations one gets by considering two independent components of $(d\Phi)^{\mathsf{T}} = 0$, i.e., condition [C″], depends on sign $\mathrm{Det}(Q_p)$ which equals sign $\mathrm{Det}(q_p)$ by Lemma 9. The next proposition then follows from Lemma 6.

PROPOSITION 4. *Let* $t\colon M \to G_2^c$ *be an immersion with* $a = 2$ *for which* $\alpha_p^+ \neq \varnothing$ *for all* $p \in M$. *Then the type of* $(d\Phi)^{\mathsf{T}} = 0$ *is elliptic, parabolic, or hyperbolic according as* $K_t(p) - 1$ *is positive, zero, or negative.*

REMARK. The characteristic directions of the partial differential equation $(d\Phi)^{\mathsf{T}} = 0$ at p are the isotropic (or, viewing q_p as a second fundamental form of t, the asymptotic) directions of q_p. To see this combine the proof of Lemma 9 with the following fact: A vector v is a characteristic vector of $(d\Phi)^{\mathsf{T}} = 0$ at p (i.e., points in a characteristic direction) if and only if there is a nonzero vector $z \in S_p$ such that $v \in \ker(A^z)$. For the proof of this use Lemma 2 and **[9**, §4].

Now that Propositions 3 and 4 have been established, we may proceed as in **[9]** to prove existence and uniqueness theorems identical to those in **[9]** under the assumptions $a = 2$ and t is an immersion when $K_t < 1$ or when $K_t > 1$ and $k < 0$ regardless of the codimension. The invariant Δ that appears in the uniqueness theorems, Theorem 3, its corollary, and Theorem 5 of **[9]** can still be defined precisely because $a = 2$. Aminov **[2]** has proved similar local existence theorems under the assumptions $c = 2$ and $K_t \neq 1$.

If X: $M \to \mathbf{E}^{2+c}$ is an immersion with $c \geqslant 3$ and with Gauss map t for which $a = 2$, then necessarily at each point of M the span of the values of the second fundamental form is 2-dimensional. Examples of such immersions for $c \geqslant 3$ are minimal immersions whose normal bundle is nowhere flat.

6. $a = 1$. When $a(p) = 1$, $\sigma(Q_p)$ completely determines the behavior of Q_p, and $\sigma(Q_p)$ is positive, zero, or negative according as the signs of the eigenvalues of $F|S_p$ are

$$(-1, -1, +1), \quad (-1, 0, +1), \quad \text{or} \quad (-1, +1, +1).$$

Note that t is necessarily regular at p when $a(p) = 1$ since S_p is 3-dimensional. If we view $t_*(T_pM) \subset \mathrm{GL}(t(p), t^\perp(p))$, then, in fact, $t_*(T_pM)$ can contain at most one linearly independent rank 1 linear transformation, because if $t_*(T_pM)$ had a basis of rank 1 linear transformations, then $\dim(S_p) \leqslant 2$. When $t_*(T_pM)$ contains the rank 1 linear transformation $t_*(v)$ for $v \in T_pM$, then $q(v, v) = 0$, i.e., q_p has an isotropic direction. On the other hand if $t_*(T_pM)$ has no rank 1 linear transformations, clearly q_p has no isotropic directions.

LEMMA 11. *Suppose $a(p) = 1$; q_p has an isotropic direction if and only if $Q_p = 0$.*

PROOF. Suppose there exists $v \in T_pM$ such that $t_*(v)$ has rank 1. Then there is $\lambda \in G_2(S_p)$ such that $\mathrm{ran}(t_*(v)) \perp \lambda$. Equivalently, $A^z(v) = 0$ for all $z \in \lambda$. Thus $\det(A^z) = 0$ for all $z \in \lambda$; hence $F|\lambda = 0$. But $F|S_p$ can have a 2-dimensional isotropic subspace only if it has a zero eigenvalue, so $Q_p = 0$.

Now suppose $Q_p = 0$. Since $F|S_p$ has a zero eigenvalue there exists $0 \neq z \in S_p$ such that $F(z, w) = 0$ for all $w \in S_p$; hence $\det(A^z, A^w) = 0$ for all $w \in S_p$. In particular, $\det(A^z) = 0$. Let $0 \neq v \in \ker(A^z)$ and suppose u, v is a basis for T_pM. Then $\det(A^z, A^w) = 0$ implies $A^z \wedge A^w(u \wedge v) = A^z(u) \wedge A^w(v) = 0$. If $0 \neq e \in t(p)$ is orthogonal to $A^z(u) \neq 0$, then $A^w(v) \perp e$ for all $w \in \lambda$. Hence $t_*(v)(e) \perp \lambda$ and thus $t_*(v)$ has rank 1.

REMARK. Hence if $a(p) = 1$ and q_p has an isotropic direction at $p \in M$, then t is not locally a Gauss map.

Equivalently, Lemma 11 states that q_p has no isotropic directions precisely when Q_p is definite. Also note that changing the orientation of T_pM (when Q_p is definite) will change Q_p from being positive definite to being negative definite or vice versa. Hence there is just one orientation on T_pM such that Q_p is positive definite.

PROPOSITION 5. *Let t be the Gauss map of an immersion X: $M \to E^{2+c}$ with $c \geqslant 3$. If there is a point p of M which is not an inflection point of X, then t is not locally a Guass map of M with the opposite orientation.*

PROOF. If p is a point that is not an inflection point, i.e., the span of the values of the second fundamental form at p is 3-dimensional, then $a(p) = 1$. Since t is a Gauss map, Q_p must be positive definite when M has its given orientation.

Note that when Q_p is definite, α_p induces a conformal structure on T_pM since each nonzero $\Phi \in \alpha_p$ pulls back the same conformal structure to T_pM (since all nonzero $\Phi \in \alpha_p$ are multiples of one another); in fact, the conformal structure is the one with the orientation making Q_p positive definite. Finally notice that if X is an immersion with Gauss map t, then X induces on T_pM the conformal structure just described since necessarily $d_pX \in \alpha_p$.

THEOREM 3. *Let* t: $M \to G_2^c$ *be a smooth map for which* $a = 1$ (*and hence* $c \geqslant 3$). *If* q *has no isotropic* (*or asymptotic*) *directions on* M, *then the bundle* α *induces a unique conformal structure on* M *with the property that if* X: $M \to \mathbf{E}^{2+c}$ *is an immersion with Gauss map* t, *then* X *induces the same conformal structure on* M.

REMARK. This is clearly not an affine theorem since the induced conformal structure depends on the Euclidean structure $\mathbf{E}^{2+c}$.

By considering independent tangential components of $(d\Phi)^{\mathsf{T}}$ condition [C″], i.e., $(d\Phi)^{\mathsf{T}} = 0$, becomes a system of two linear partial differential equations in one unknown function (since $a = 1$). It is overdetermined and thus we ought to look for an integrability condition. But it is most natural to look for the integrability condition using the conformal structure induced on M by α (since we necessarily assume Q is positive definite on M). We therefore postpone any existence theorems to Part 2 of this paper where we consider the conformal aspects of our general problem.

Suppose X: $M \to \mathbf{E}^{2+c}$ is an immersion with Gauss map t. Since t already determines the conformal structure to be induced on M at p by X when $a(p) = 1$, we ought to expect that two immersions with the same Gauss map differ at most by a translation and homothety if $a = 1$ at most points of M. This observation is suggested by the uniqueness results of Hoffman and Osserman [**4**] or Part 2 of this paper.

THEOREM 4. *Let* X_i: $M \to \mathbf{E}^{2+c}$ *for* $i \in \{1, 2\}$ *be immersions with* $c \geqslant 3$ *which have the same Gauss map. If the set of inflection points of* X_1 (*and hence* X_2) *is nowhere dense, then* $X_2 = sX_1 + f$ *for some nonzero scalar* $s \in \mathbf{R}$ *and some* $f \in \mathbf{E}^{2+c}$.

PROOF. Both X_1 and X_2 have the same set of inflection points (since they have the same Gauss map); let M^* be the complement of the set of inflection points. M^* is a dense open set. Since $a = 1$ on M^*, $dX_1^{-1} \circ dX_2$: $T_pM \to T_pM$ is a multiple of the identity for all $p \in M^*$. By continuity, $dX_1^{-1} \circ dX_2$ is a multiple of the identity for all $p \in M$. Define a smooth function s: $M \to \mathbf{R}$ by $dX_2 = s\,dX_1$. Hence X_1 and X_2 induce the same conformal structure on M as well as have the same Gauss map. Since the mean curvature does not vanish on M^* (because points where the mean curvature vanishes are inflection points when $c \geqslant 3$), s is constant on each component of M^* by Theorem 2.5 of [**4**]. Thus $ds = 0$ on M^*; by continuity $ds = 0$ on M. Thus $s =$ constant and the result follows.

REMARK. Generically, when $c \geqslant 3$, the set of inflection points of an immersion X: $M \to \mathbf{E}^{2+c}$ has codimension $\geqslant 1$ and so is nowhere dense. Thus generically immersions of surfaces into Euclidean spaces of dimension greater than 4 are essentially uniquely determined by their Gauss maps alone.

REMARK. Some of the results in this paper also appear in two papers by Yu. A. Aminov, *Determining a surface in* $\mathbf{E}^4$ *from its degenerate Grassmann transform*, Ukrain. Geom. Sb. **26** (1983), 6–13 (Russian), and *Restoration of a two-dimensional surface in n-dimensional Euclidean space from its Grassmann transform*, Mat. Zametki **36** (1984), 223–228 (Russian). Aminov's methods are markedly different from those employed in this paper.

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