Some open problems in mathematics

Christoph Thiele

December 16, 2005

These are some of my favorite open problems in mathematics.

1 Tri-linear Hilbert transform

Let α be an irrational number. For compactly supported smooth functions f_1, f_2, f_3, f_4 on $\mathbf{R}$ define

$$\Lambda(f_1, f_2, f_3, f_4) = \int_{\mathbf{R}} p.v. \int_{\mathbf{R}} f_1(x-t) f_2(x+t) f_3(x-\alpha t) \frac{dt}{t} f_4(x) dx$$

Here the principal value is defined as

$$p.v. \int_{\mathbf{R}} \dots \frac{dt}{t} = \lim_{\epsilon \to 0} \int_{\mathbf{R} \setminus [-\epsilon, \epsilon]} \dots \frac{dt}{t}$$

Prove or disprove (here $\|f\|_4$ denotes the L^4 norm)

Conjecture 1 *There is a constant C independent of $f_1, \dots, f_4$ such that*

$$|\Lambda(f_1, f_2, f_3, f_4)| \leq C \|f_1\|_4 \|f_2\|_4 \|f_3\|_4 \|f_4\|_4$$

To trace some background information start with [2].

2 Non-linear Carleson theorem

Let V be a function in $L^2(\mathbf{R})$. Then for $k \geq 0$, by elementary methods, the ordinary differential equation

$$f'' + Vf = -kf$$

has a two dimensional space of classical solutions (with absolutely continuous derivatives) satisfying the o.d.e. almost everywhere on $\mathbf{R}$. Prove or disprove:

Conjecture 2 *For $V \in L^2(\mathbf{R})$ there is a set of measure zero in $\mathbf{R}^+$ such that for $k \in \mathbf{R}^+$ not in this set all solutions to the above o.d.e are bounded (L^∞) functions.*

To trace some background information start with [2].

3 Sharp Beurling constant

The Beurling operator is the pricipal value convolution with $1/z^2$ in the complex plane. Problem: What is the exact norm of this operator in $L^p(\mathbf{R}^2)$.

There are several reformulations of this problem. Define $L(x)$ to be $|x| - |\overline{x}|^2$ for $|x| + |\overline{x}| < 1$ and $2|x| - 1$ otherwise. The prove $\int L(\nabla u) \geq 0$ for all $u \in W_0^{1/2}$. (See Baernstein, Montgomery-Smith: Some conjectures about integral means).

References

[1] Thiele, C. *Singular integrals meet modulation invariance* Proceedings ICM 2002 Beijing, Volume II

[2] Muscalu C., Tao T., Thiele C. *A Carleson type theorem for a Cantor group model of the scattering transform.* Preprint