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SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS AND THEIR CHARACTERISTIC SURFACES.[1]

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Contents.

§ 1. Introduction.

It is the object of this paper to treat in a general and systematic manner the existence theorems for systems of partial differential equations of first order (Part I), and to study their characteristic surfaces (Part II). Our work includes in particular a detailed treatment of the systems of invariantive type and is so developed as to bring into relationship the methods used in a series of previous papers by one of us[2] and a most interesting paper by Cartan[3] on this same subject.

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[1] Received, June 1, 1932.

[2] T. Y. Thomas, *Determination of affine and metric spaces by their differential invariants.* Math. Ann., 101 (1929), pp. 713–728. *The existence theorems in the problem of the determination of affine and metric spaces by their differential invariants.* Amer. Jour. Math.,

The work in Part II is an extension of the treatment of characteristic surfaces in a series of notes in the *Proceedings of the National Academy of Sciences*[4] and is believed to be the first adequate general treatment of this problem. The method used in proving the existence theorems is patterned after Riquier's[5] theory of orthonomic systems. This has necessitated the development of a theory of sets of monomials which will apply to a system of differential equations, part of which hold only over a subspace; this theory contains as a special case the theory presented by Janet.[6]

Throughout the paper illustrative examples have been given in fine print. In particular we have given for the first time a detailed study of the characteristic surfaces and the associated existence theorems for Einstein's gravitational equations in free space.[7]

PART I.

GENERAL EXISTENCE THEOREMS.

§ 2. Regular Systems of Differential Equations.

Consider a system of L partial differential equations, linear and of the first order in w dependent variables $v_1, \cdots, v_w$ and n independent variables $x^1, \cdots, x^n$, namely

$$\sum_{k=1}^{w} \sum_{\alpha=1}^{n} a_{ik}^{\alpha} \frac{\partial v_k}{\partial x^\alpha} + c_i = 0 \qquad (i = 1, \cdots, L). \tag{2.1}$$

The coefficients a_{ik}^{α} and c_i are functions of x^α and v_k. It is assumed also that the left members of (2.1) are linearly independent in the derivatives of the dependent variables v_k.

Let us suppose that there are $L_1 \leqq L$ equations (2.1) which are independent in the derivatives $\partial v_k / \partial x^1$ and, as the integer L_1 is conceivably

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52 (1930), pp. 225–250. *Invariantive systems of partial differential equations,* Ann. of Math. (2), 31 (1930), pp. 687–713. *Space structure as a boundary value problem.* Ann. of Math. (2), 31 (1930), pp. 714–726. In order to shorten the work references will be made to the above papers frequently. The designation M., Jour., Ann. (1), Ann. (2) respectively will be used when reference is made to the above papers.

[3] Élie Cartan, *Sur la théorie des systèmes en involution et ses applications à la Relativité.* Bull. Sc. Math., 59 (1931), pp. 88–118.

[4] T. Y. Thomas, *On the Unified Field Theory,* Proc. Nat. Acad. Sciences; 16 (1930), Notes I and II, pp. 761–766, 830–835; 17 (1931), Notes III–VI, pp. 48–58, 111–119, 199–210, 325–329. These will be referred to as Proc. Note I, etc.

[5] Riquier, *Les systèmes d'équations aux dérivées partielles,* (1910).

[6] Maurice Janet, *Sur les systèmes d'équations aux dérivées partielles,* Journ. de Math. (8), 3 (1920), pp. 65–144.

[7] The differential equations defining the 3-dimensional characteristic surfaces for Einstein's gravitational equations have been given by T. Levi-Civita, *Caratteristiche e bicaratteristiche delle equazioni gravitazionali di Einstein,* Rend. Accad. Lincei, (6), 11 (1930), pp. 1–11; ibid. pp. 113–121.

dependent on the coördinate system (x), let us suppose that coördinates x^α have been selected for which L_1 will have its maximum value. We can then divide our equations into two sets: a set S_1 consisting of L_1 equations which can be solved for L_1 of the derivatives $\partial v_k/\partial x^1$ and a set S_2 from which these derivatives can be eliminated. Now suppose that the set of equations S_2 is independent in L_2 of the derivatives $\partial v_k/\partial x^2$ and in fact that coördinates x^α are selected so that L_2 has its maximum possible value, under the restriction that the above integer L_1 is unchanged. This makes it possible to divide the set S_2 into two sets: S_2^* and S_3^* such that S_2^*, consisting of L_2 equations, can be solved for L_2 of the derivatives $\partial v_k/\partial x^2$ and such that these latter derivatives can be eliminated entirely from the set S_3^*. Proceeding in this way we arrive at a coördinate system (which is obviously one of an infinity of such coördinate systems) for which our system of equations (2.1) can be put into the form

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} b_{ik\beta}^{\alpha}\frac{\partial v_k}{\partial x^\alpha}+c_{i\beta}=0, \qquad \begin{pmatrix}\beta=1,\cdots,n\\ i=1,\cdots,L_\beta\end{pmatrix}, \tag{2.2}$$

where $b_{ik\beta}^{\alpha}=0$ if $\alpha<\beta$. A system of coördinates x^α with respect to which (2.1) can be put into the form (2.2) in which the integers L_β are characterized by the above mentioned property, is said to be *non-singular;* otherwise the coördinate system is said to be *singular.*

If (2.1) is written in the form

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} b_{ik\beta}^{\alpha}\frac{\partial v_k}{\partial x^\alpha}+c_{i\beta}=0, \qquad \begin{pmatrix}\beta=1,\cdots,n\\ i=1,\cdots,J_\beta\end{pmatrix},$$

where $b_{ik\beta}^{\alpha}=0$ if $\alpha<\beta$, with respect to a singular coördinate system, then $J_1<L_1$ or if $J_i=L_i$ for $i=1,\cdots,r$ then $J_{r+1}<L_{r+1}$. Obviously the inequality $r\leqq n-1$ is here satisfied.

Now assume a non-singular choice of independent variables x^α and make the transformation

$$\bar{x}^\sigma = x^\sigma + m_\tau^\sigma x^\tau, \tag{2.3}$$

where the m_τ^σ are constants. If the dependent variables v_k transform as *scalars* the law of transformation of their derivatives is given by

$$\frac{\partial v_k}{\partial x^\alpha}=\frac{\partial v_k}{\partial \bar{x}^\alpha}+m_\alpha^\sigma\frac{\partial v_k}{\partial \bar{x}^\sigma},$$

and hence equations (2.2) become

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n}(b_{ik\beta}^{\alpha}+m_\sigma^\alpha b_{ik\beta}^{\sigma})\frac{\partial v_k}{\partial \bar{x}^\alpha}+c_{i\beta}=0. \tag{2.4}$$

For a fixed value α belonging to the set $1, \cdots, n-1$ assume that $m^\sigma_\tau = 0$ except when $\sigma = \alpha$. Then if the constants m^α_τ are sufficiently small equations (2.4) for $\beta = \alpha$ must be linearly independent with respect to the derivatives $\partial v_k / \partial \bar{x}^\alpha$. For the above choice of the constants m^σ_τ the coefficients of the derivatives $\partial v_k / \partial \bar{x}^\gamma$ for $\gamma = 1, \cdots, \alpha - 1$, in the equations (2.4) will be equal to the coefficients of the corresponding derivatives $\partial v_k / \partial x^\gamma$ in the system (2.2). Hence the set of forms,

$$\sum_{k=1}^{w} m^\alpha_\sigma b^\sigma_{ik\beta} \frac{\partial v_k}{\partial \bar{x}^\alpha} \qquad (\beta > \alpha, \ \alpha \text{ not summed}),$$

will be linearly dependent on the forms

$$\sum_{k=1}^{w} (b^\alpha_{ik\alpha} + m^\alpha_\sigma b^\sigma_{ik\alpha}) \frac{\partial v_k}{\partial \bar{x}^\alpha} \qquad (\alpha \text{ not summed}),$$

since otherwise the original choice of variables x^α would be singular contrary to hypothesis. Or

$$(2.5) \qquad m^\alpha_\sigma b^\sigma_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{*\alpha}_{ij\beta} (b^\alpha_{jk\alpha} + m^\alpha_\sigma b^\sigma_{jk\alpha}) \qquad (\beta > \alpha \text{ and } \alpha \text{ not summed}).$$

Now take $m^\alpha_\beta = m$, and $m^\sigma_\tau = 0$ otherwise. Then (2.5) gives

$$m\, b^\beta_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{*\alpha}_{ij\beta} (b^\alpha_{jk\alpha} + m\, b^\beta_{jk\alpha}) \quad (\beta > \alpha, \ \alpha \text{ and } \beta \text{ not summed}).$$

Hence, if we let m approach zero, we have

$$(2.6) \qquad b^\beta_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^\alpha_{ij\beta} b^\alpha_{jk\alpha}, \qquad \begin{pmatrix} \beta > \alpha, \ i = 1, \cdots, L_\beta \\ \alpha, \beta \text{ not summed} \end{pmatrix},$$

where

$$\lambda^\alpha_{ij\beta} = \lim_{m \to 0} \left[\frac{\lambda^{*\alpha}_{ij\beta}}{m} \right].$$

More generally, take $m^\alpha_\gamma = m$ for a single index $\gamma \geqq \beta$ and $m^\sigma_\tau = 0$ otherwise. Then (2.5) gives

$$m\, b^\gamma_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{*\alpha}_{ij\beta} (b^\alpha_{jk\alpha} + m\, b^\gamma_{jk\alpha}),$$

and this becomes

$$(2.7) \qquad b^\gamma_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{\alpha\gamma}_{ij\beta} b^\alpha_{jk\alpha} \qquad (\gamma \geqq \beta > \alpha),$$

when m is allowed to approach zero.

It is clear that

$$L_1 \geqq L_2 \geqq L_3 \cdots \geqq L_n,$$

otherwise a transformation of the independent variables x^α, producing merely a permutation of the indices of these variables, would show that the original choice of the variables x^α was singular, contrary to hypothesis.

Now suppose that equations (2.2) for $\beta = 1$ can be solved for the derivatives

$$\frac{\partial v_1}{\partial x^1}, \quad \frac{\partial v_2}{\partial x^1}, \quad \cdots, \quad \frac{\partial v_{L_1}}{\partial x^1},$$

or in other words the matrix of the quantities b^1_{jk1}, where $j, k = 1, \cdots, L_1$, is non-singular. Put $\alpha = 1$, $\beta = 2$ in (2.6) and consider the matrix of the quantities λ^1_{ij2} appearing in these equations. If this matrix is of rank R, then $L_2 \leqq R$; this follows from a theorem in Algebra[8] and the fact that the matrix of the quantities b^2_{ik2} is of rank L_2. Hence $R = L_2$ since R can obviously not be greater than L_2. It then follows from a second theorem in Algebra[9] and (2.6) that the matrix of the quantities b^2_{ik2} for $i = 1, \cdots, L_2$ and $k = 1, \cdots, L_1$ is of rank L_2. Hence equations (2.2) for $\beta = 2$ can be solved for the derivatives

$$\frac{\partial v_1}{\partial x^2}, \quad \cdots, \quad \frac{\partial v_{L_2}}{\partial x^2}$$

after a suitable choice of the indices of the dependent variables v_k has been made. By a continuation of this process it is evident that, if we change the notation for the independent variables v_k in accordance with the following scheme:

$$\begin{array}{ll} v_{i0} & \sim v_{L_1+1}, \cdots, v_w, \\ v_{i1} & \sim v_{L_2+1}, \cdots, v_{L_1}, \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ v_{i\,n-1} & \sim v_{L_n+1}, \cdots, v_{L_{n-1}}, \\ v_{in} & \sim v_1, \quad \cdots, v_{L_n}, \end{array}$$

equations (2.2) can be written

$$\frac{\partial v_{ik}}{\partial x^\alpha} = \sum (x, v) \frac{\partial v_{pq}}{\partial x^\beta} + *, \qquad \begin{pmatrix} k = 1, \cdots, n \\ i = 1, \cdots, w_k \\ \alpha = 1, \cdots, k \end{pmatrix}, \tag{2.8}$$

where

$$\begin{array}{lll} w_0 & = w & - L_1, \\ w_1 & = L_1 & - L_2, \\ w_2 & = L_2 & - L_3, \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ w_{n-1} & = L_{n-1} & - L_n, \\ w_n & = L_n & \end{array} \tag{2.9}$$

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[8] See, for example, Dickson: *Modern Algebraic Theories*, p. 51.

[9] Dickson, *loc. cit.*, p. 51.

and $\beta \geqq \alpha$, $\beta > q$. The coefficients (x, v) in (2.8) depend upon the quantities x and v_{rs}, and the $*$ denotes terms containing no derivatives of the v_{rs}; in the sequel the $*$ will be used to denote terms of lower order than those written down explicitly. A system of the form (2.8) will be said to be *regular*.[10]

Addition of corresponding members of (2.9) gives

$$w = w_0 + w_1 + \cdots + w_n.$$

§ 3. Extension to Tensor Differential Equations.

The results of § 2 can be extended to systems of equations the left members of which are linear in the first derivatives of the components of a tensor T. Thus consider

$$(3.1) \qquad D_{ip\cdots q}^{r\cdots s\alpha} \frac{\partial T_{r\cdots s}^{p\cdots q}}{\partial x^\alpha} + * = 0, \qquad (i = 1, \cdots, L),$$

where the coefficients D are functions of the independent variables x^α and the unknowns T; the same is true of the $*$ terms.

With respect to a non-singular coördinate system (defined as in § 2) equations (3.1) can be written

$$(3.2) \qquad D_{ip\cdots q\beta}^{r\cdots s\alpha} \frac{\partial T_{r\cdots s}^{p\cdots q}}{\partial x^\alpha} + * = 0, \qquad \begin{pmatrix} \beta = 1, \cdots, n \\ i = 1, \cdots, L_\beta \end{pmatrix},$$

where $D = 0$ if $\alpha < \beta$. Under the transformation (2.3) the derivatives of the components of the tensor T transform according to the equations

$$\frac{\partial T_{r\cdots s}^{p\cdots q}}{\partial x^\alpha} = \frac{\partial \overline{T}_{c\cdots d}^{a\cdots b}}{\partial \overline{x}^\sigma} (\delta_\alpha^\sigma + m_\alpha^\sigma) \frac{\partial \overline{x}^c}{\partial x^r} \cdots \frac{\partial \overline{x}^d}{\partial x^s} \frac{\partial x^p}{\partial \overline{x}^a} \cdots \frac{\partial x^q}{\partial \overline{x}^b}.$$

In the coördinate system $(\overline{x})$ equations (3.2) therefore take the form

$$(3.3) \qquad (\overline{D}_{ia\cdots b\beta}^{c\cdots d\alpha} + m_\sigma^\alpha \overline{D}_{ia\cdots b\beta}^{c\cdots d\sigma}) \frac{\partial \overline{T}_{c\cdots d}^{a\cdots b}}{\partial \overline{x}^\alpha} + * = 0$$

with $\overline{D}_{ia\cdots b\beta}^{c\cdots d\alpha} = 0$ if $\alpha < \beta$. Letting α be a particular number of the set $1, \cdots, n-1$ and assuming that $m_\sigma^\tau = 0$ if $\alpha \neq \tau$, we obtain by an argument analogous to that employed in § 2 that

$$(3.4) \quad m_\sigma^\alpha \overline{D}_{ia\cdots b\beta}^{c\cdots d\sigma} = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^{*\alpha} (\overline{D}_{ja\cdots b\alpha}^{c\cdots d\alpha} + m_\sigma^\alpha \overline{D}_{ja\cdots b\alpha}^{c\cdots d\sigma}) \quad (\beta > \alpha, \ \alpha \text{ not summed})$$

[10] Without the restriction $\beta \geqq \alpha$, the system (2.8) is called *regular and immediate* by Méray and Riquier *Sur la convergence des développements des intégrales ordinaires*, Ann. Ec. Norm. Sup., (3), 7, (1890), p. 44.

as a result of the assumption that the original coördinate system (x) is non-singular. Then putting $m^\alpha_\gamma = m$ for a single $\gamma \geqq \beta$ and $m^\sigma_\tau = 0$ otherwise, we obtain, on allowing m to approach zero, that

$$D^{r\cdots s\gamma}_{ip\cdots q\beta} = \sum \lambda^{\alpha\gamma}_{ij\beta} D^{r\cdots s\alpha}_{jp\cdots q\alpha} \qquad (\gamma \geqq \beta > \alpha). \tag{3.5}$$

If all the components $T^{p\cdots q}_{r\cdots s}$ are independent they can be represented by v_k and the system (3.2) together with (3.5) can be written

$$\sum_{k=1}^{w} b^\alpha_{ik\beta} \frac{\partial v_k}{\partial x^\alpha} + c_{i\beta} = 0 \qquad \begin{pmatrix}\beta = 1, \cdots, n \\ i = 1, \cdots, L_\beta\end{pmatrix}, \tag{3.6}$$

$$b^\gamma_{ik\beta} = \sum_j \lambda^{\alpha\gamma}_{ij\beta} b^\alpha_{jk\alpha} \qquad (\gamma \geqq \beta > \alpha), \tag{3.7}$$

where $b^\alpha_{ik\beta} = 0$ if $\alpha < \beta$. Suppose, however, that the components $T^{p\cdots q}_{r\cdots s}$ satisfy linear relations of the form

$$\sum T^{k\cdots l}_{m\cdots n} = 0, \tag{3.8}$$

where the indices $k \cdots l$, $m \cdots n$ are obtainable from $p \cdots q$, $r \cdots s$ by permutations. If, in this case, v_k is used to denote the independent components $T^{p\cdots q}_{r\cdots s}$ when account is taken of (3.8), equations (3.6) and (3.7) will likewise apply. On the basis of the discussion in § 2 the system (3.2) can then be replaced by a system of equations in the regular form (2.8).

§ 4. Application to Invariantive Systems. Affine and Metric Cases.

Consider a system of partial, differential equations of the form

$$T^{\mu\cdots\nu}_{\gamma\cdots\delta}\left(\Gamma^i_{\alpha\beta};\ \frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma};\ \frac{\partial^2 \Gamma^i_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right) = 0, \tag{4.1}$$

where the T's are the components of an absolute or relative tensor invariant of weight W and the $\Gamma^i_{\alpha\beta}$ are components of affine connection. The equations (4.1) will be assumed linear in the second derivatives of the $\Gamma^i_{\alpha\beta}$ and as indicated, to depend on the $\Gamma^i_{\alpha\beta}$ and their first derivatives. Along with this system consider

$$T^{\mu\cdots\nu}_{\gamma\cdots\delta}\left(g_{\alpha\beta};\ \frac{\partial g_{\alpha\beta}}{\partial x^\gamma};\ \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta};\ \frac{\partial^3 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta \partial x^\epsilon}\right) = 0, \tag{4.2}$$

where the T's are now tensor invariants linear in the third derivatives of the components $g_{\alpha\beta}$ of the fundamental metric tensor and depending on the components $g_{\alpha\beta}$ themselves together with their first and second derivatives. In particular (4.1) or (4.2) may consist of a single equation or of several equations in which the left members are scalar invariants.[11] The system (4.1) is completely equivalent to one of the form

$$(4.3)\qquad \sum T'^{\beta\gamma\delta\varepsilon}_{\alpha i}(A)\frac{\partial A^{\alpha}_{\beta\gamma\delta}}{\partial x^{\varepsilon}}+\ast = 0,$$

where the left members are again components of a tensor or scalar invariant. The system is linear in first derivatives of the components $A^{\alpha}_{\beta\gamma\delta}$ of the first normal tensor and the coefficients are functions of $A^{\alpha}_{\beta\gamma\delta}$. Similarly (4.2) can be replaced by a system of the form

$$(4.4)\qquad \sum T^{\alpha\beta\gamma\delta\varepsilon}_{i}(g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta})\frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial x^{\varepsilon}}+\ast = 0,$$

linear in the first derivatives of the components $g_{\alpha\beta,\gamma\delta}$ of the first metric normal tensor with coefficients which are functions of $g_{\alpha\beta,\gamma\delta}$ and the components $g_{\alpha\beta}$ of the fundamental metric tensor. In the system (4.3) the $\ast$ terms depend on the components $A^{\alpha}_{\beta\gamma\delta}$ and $\Gamma^{\alpha}_{\beta\gamma}$; in (4.4) the $\ast$ terms depend on $g_{\alpha\beta}$, $g_{\alpha\beta,\gamma\delta}$ and $\Gamma^{\alpha}_{\beta\gamma}$ where the components $\Gamma^{\alpha}_{\beta\gamma}$ denote the Christoffel symbols for this latter case. In addition to the conditions (4.3) on the derivatives of the components $A^{\alpha}_{\beta\gamma\delta}$ there exists a system of identical relations

$$(4.5)\qquad \frac{\partial A_{ik}}{\partial x^{\alpha}} = \sum \frac{\partial A_{pq}}{\partial x^{r}}+\ast \qquad \begin{pmatrix} k = 1,\cdots, n-2\\ i = 1,\cdots, A_k\\ \alpha = 1,\cdots, k\\ q < r \geqq \alpha \end{pmatrix}$$

in which the summation $\sum$ denotes a linear form in the derivatives $\partial A^{\alpha}_{\beta\gamma\delta}/\partial x^{\varepsilon}$ with constant coefficients. Equations (4.5) express the conditions that the quantities $A^{\alpha}_{\beta\gamma\delta}$ should be components of a normal tensor.[12]

We suppose (4.3) to consist of N independent equations; certain of

[11] T. Y. Thomas and A. D. Michal, *Differential invariants of affinely connected manifolds;* Ann. of Math., (2), 28 (1927), p. 196; also ibid., (2), 28 (1927), p. 631.

[12] In the former treatment, loc. cit. Jour. p. 246, the inequality $q \leqq k$ is established and used in place of the inequality $r \geqq \alpha$ appearing in (4.5). To establish this latter inequality we proceed as follows: The system (6.2) Jour. can be written

$$(\text{a})\qquad \frac{\partial A^{i}_{jk\alpha}}{\partial x^{\beta}} = \frac{\partial A^{i}_{jk\beta}}{\partial x^{\alpha}}+\frac{2}{3}\frac{\partial A^{i}_{j\beta\alpha}}{\partial x^{k}}+\frac{1}{3}\frac{\partial A^{i}_{\alpha\beta j}}{\partial x^{k}}+\frac{1}{3}\frac{\partial A^{i}_{\beta k\alpha}}{\partial x^{j}}-\frac{1}{3}\frac{\partial A^{i}_{\alpha k\beta}}{\partial x^{j}}+\ast,$$

where $\alpha = \mu+1$; $k>\alpha>\beta$; $j\leqq k$. The component $A^{i}_{jk\beta}$ for particular values of the indices j, k, α, β belongs to the group $G_{\beta-1}$; since $\alpha \geqq \beta+1$ the first derivative on the right of this particular equation (a) is not the left member of any equation (a). In the previous paper it is shown that $A^{i}_{j\beta\alpha}$ and $A^{i}_{\alpha\beta j}$ contribute components to the groups these equations may, however, be dependent on the remaining equations of the system (4.3) in consequence of the identities (4.5). These dependent equations which we will assume to be M in number thus possess the property that they are satisfied in consequence of (4.5) and the remaining $N-M$ equations of the system (4.3). If the M dependent equations are excluded, the remainder together with the system of identities (4.5) can be put in the form

$$\frac{\partial A^*_{ik}}{\partial x^\alpha} = \sum (A^*) \frac{\partial A^*_{pq}}{\partial x^r} + * \quad \begin{pmatrix} i = 1, \cdots, A^*_k \\ k = 1, \cdots, n \\ \alpha = 1, \cdots, k \\ q < r \geqq \alpha \end{pmatrix}, \tag{4.6}$$

on the basis of the theory of § 3. The notation A^*_{ik} is used in place of A_{ik} to denote a different grouping of the independent components $A^\alpha_{\beta\gamma\delta}$ from that used in (4.5). Together with the system (4.6) we consider the set of equations[13]

$$\frac{\partial \gamma_{lm}}{\partial x^k} = \frac{\partial \gamma_{pq}}{\partial x^r} + \sum A^*_{pq} + \sum \gamma_{uv}\gamma_{pq} \quad \begin{pmatrix} m = 1, \cdots, n-1 \\ l = 1, \cdots, \gamma_m \\ k = 1, \cdots, m \\ q < r > k \end{pmatrix} \tag{4.7}$$

which define the components A^*_{ik} in terms of the independent components γ_{lm} of the affine connection Γ. Since the inequalities $q < r \geqq \alpha$ and $q < r > k$ are satisfied by the indices of the derivatives in (4.6) and (4.7) respectively, the system composed of (4.6) and (4.7) is regular.

G_μ, $G_{\beta-1}$, G_{j-1} where $j \leqq \mu$. Since k exceeds each of the numbers μ, $\beta-1$, and $j-1$ the derivatives with respect to x^k which appear on the right of any particular equation (a) cannot appear on the left of any equation (a). From the form of the pth and the qth equations, Jour. p. 246, it is seen that the elimination of the derivative $\partial A^i_{\beta k\alpha}/\partial x^j$ where $j \leqq \mu$ from the right of a particular equation (a) cannot bring in any derivatives for which $r < \alpha$ in (4.5). In order to eliminate the last derivative we must use an equation of the form

$$\frac{\partial A^i_{\alpha k\beta}}{\partial x^j} = \cdots, \tag{b}$$

where $j < \beta$. To eliminate the derivative on the left of the particular equation (b) from the right of the corresponding equation (a) we multiply (b) by a factor and add to (a). From the form of (b) it is seen this could not bring any derivatives into the right of (a) for which $r < \alpha$ in (4.5).

A similar discussion can be made for the system Jour. (6.7). The details will be omitted when we come to the analogous discussion in the treatment of the system (4.4).

In case $n = 2$ equations (4.5) and Jour. (6.7), i. e. the following equations (4.8), are satisfied identically, *loc. cit.* Jour., p. 235.

[13] *loc. cit.* Ann. (2), p. 715.

A corresponding discussion can be made for the system (4.4). In place of the set of identities (4.5) we now have a system of the form

$$\frac{\partial B_{lm}}{\partial x^k} = \sum \frac{\partial B_{pq}}{\partial x^r} + * \qquad \begin{pmatrix} m = 1, \cdots, n-2 \\ l = 1, \cdots, B_m \\ k = 1, \cdots, m \\ q < r \geqq k \end{pmatrix} \tag{4.8}$$

which expresses the condition that the quantities B_{lm} constitute the components of the second extension of a fundamental metric tensor. The remarks made in connection with the independence of the equations (4.3) and (4.5) apply equally well to the systems (4.4) and (4.8). The $N-M$ independent equations of the system (4.4) together with the equations (4.8) can be put in the form

$$\frac{\partial B^*_{lm}}{\partial x^\alpha} = \sum (g, B^*) \frac{\partial B^*_{pq}}{\partial x^r} + * \qquad \begin{pmatrix} l = 1, \cdots, B^*_m \\ m = 1, \cdots, n \\ \alpha = 1, \cdots, m \\ q < r \geqq \alpha \end{pmatrix}, \tag{4.9}$$

notation B^*_{lm} being used to denote a regrouping of the components B_{lm}. Equations (4.7) likewise pertain to the present discussion provided that the $\sum A^*_{pq}$ in the right members of these equations be replaced by the corresponding term $\sum B^*_{pq}$ in the components B^*_{pq} of the second extension of the fundamental metric tensor, i. e.

$$\frac{\partial \gamma_{lm}}{\partial x^k} = \frac{\partial \gamma_{pq}}{\partial x^r} + \sum (g)\, B^*_{pq} + \sum \gamma_{uv}\, \gamma_{pq} \qquad \begin{pmatrix} m = 1, \cdots, n-1 \\ l = 1, \cdots, \gamma_m \\ k = 1, \cdots, m \\ q < r > k \end{pmatrix}, \tag{4.10}$$

where the coefficients of the quantities B^*_{pq} are rational functions of the components $g_{\alpha\beta}$. We must now add

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = g_{\sigma\beta}\, \Gamma^\sigma_{\alpha\gamma} + g_{\alpha\sigma}\, \Gamma^\sigma_{\beta\gamma} \tag{4.11}$$

as the expression of the conditions that the $\Gamma^\alpha_{\beta\gamma}$ be Christoffel symbols with respect to the components $g_{\alpha\beta}$ of some fundamental metric tensor. Then, obviously, the equations (4.9), (4.10) and (4.11) comprise a regular system.

§ 5. Application to Invariantive Systems. Vector Case.

There is a certain analogy between the local coördinates which can be introduced into a space of distant parallelism[14] and the normal coördinates of an affinely connected space: Differentiation of the components of a tensor or set of scalar invariants and evaluation at the origin of a system of local coördinates gives rise to a set of absolute scalar invariants. This

[14] *loc. cit.* Proc. Note I.

was treated in a paper by one of us[14] with particular reference to the case of four dimensions but the theory can be extended immediately to the n dimensional case. Now consider a system of equations of the form

$$T^{\mu\nu\cdots\varepsilon}_{\gamma\delta\cdots\eta}\left(h^i_\alpha;\ \frac{\partial h^i_\alpha}{\partial x^\beta};\ \frac{\partial^2 h^i_\alpha}{\partial x^\beta \partial x^\gamma}\right) = 0, \tag{5.1}$$

where the left members are components of a tensor invariant or a set of scalar invariants linear in the second derivatives of the fundamental vectors h^i_α and depending also on first derivatives and the vectors themselves. By a replacement theorem analogous to that for normal coördinates the system (5.1) is seen to be equivalent to one of the form

$$\sum (h^i_\alpha;\ h^i_{j,k}) \frac{\partial h^i_{j,k}}{\partial x^\alpha} + \text{✱} = 0, \tag{5.2}$$

where the summation denotes a linear expression in the derivatives of the invariants $h^i_{j,k}$ with coefficients which are functions of the components $h^i_{j,k}$ and h^i_α; the ✱ terms depend on the $h^i_{j,k}$. In addition there is the set of equations

$$h^i_{j,k,l} + h^i_{k,l,j} + h^i_{l,j,k} = 2\,[h^i_{m,j}\,h^m_{k,l} + h^i_{m,l}\,h^m_{j,k} + h^i_{m,k}\,h^m_{l,j}] \tag{5.3}$$

which expresses the condition that the set of quantities $h^i_{j,k}$ be the invariants arising from a set of fundamental vectors h^i_α. As in § 4 we will suppose that the dependent equations of the system composed of (5.2) and (5.3) have been excluded and the unknowns $h^i_{j,k}$ have been replaced by an independent set K_{lm}.[15] By the theory of § 2 the combined system (5.2) and (5.3) can be put into the form

$$\frac{\partial K^*_{lm}}{\partial x^\alpha} = \sum (h^i_\alpha,\ K^*_{lm}) \frac{\partial K^*_{pq}}{\partial x^r} + \text{✱} \qquad \begin{pmatrix} m = 1, \cdots, n \\ l = 1, \cdots, K^*_m \\ \alpha = 1, \cdots, m \\ q < r \geqq \alpha \end{pmatrix} \tag{5.4}$$

where K^*_{lm} denotes a regrouping of the independent components K_{lm}. The combination of (5.4) and the equations

$$\frac{\partial h^i_\alpha}{\partial x^\beta} = \frac{\partial h^i_\beta}{\partial x^\alpha} + 2\,h^i_{j,k}\,h^j_\alpha\,h^k_\beta \qquad \begin{pmatrix} \alpha = 2, \cdots, n \\ \beta = 1, \cdots, \alpha - 1 \\ i = 1, \cdots, n \end{pmatrix} \tag{5.5}$$

[15] The rule for separating the independent components $h^i_{j,k}$ into groups, which is given in Proc. Note III for the four dimensional case, can be extended immediately to the case of n dimensions. Rule: *The group G_m $(m = 0, 1, \cdots, n-2)$ for the components $h^i_{j,k}$ is composed of all components that can be formed from $h^i_{j,k}$ by taking $k = m+1$ and $i, j = 1, \cdots, n$ subject to the condition $j > m+1$.* There are $K_m = n^2 - nm - n$ components K_{lm} in each group G_m.

which define the set of invariants $h^i_{j,k}$ in terms of the components of the fundamental vectors h^i_α obviously constitute a regular system.

The following example illustrates the above theory by showing how the differential equations for an unrestricted space of distant parallelism can be put into the regular form. In four dimensions the set of invariants $h^i_{3,2}$; $h^i_{4,2}$; $h^i_{4,3}$; $h^i_{2,1}$; $h^i_{3,1}$; $h^i_{4,1}$ are independent, i. e. not connected by a linear relation; when the system of identities (5.3) is referred to these invariants and their derivatives, the matrix of the coefficients of the derivatives is that exhibited in the adjacent table in which

$$\alpha = h^1_1, \quad \beta = h^1_2, \quad \gamma = h^1_3, \quad \delta = h^1_4, \quad a = h^2_1, \quad b = h^2_2$$

j, k, l	$\frac{\partial h^i_{3,2}}{\partial x^1}$	$\frac{\partial h^i_{4,2}}{\partial x^1}$	$\frac{\partial h^i_{4,3}}{\partial x^1}$	$\frac{\partial h^i_{2,1}}{\partial x^1}$	$\frac{\partial h^i_{3,1}}{\partial x^1}$	$\frac{\partial h^i_{4,1}}{\partial x^1}$	$\frac{\partial h^i_{4,3}}{\partial x^2}$
1, 2, 3	$-\alpha$			$-\gamma$	β		
1, 2, 4		$-\alpha$		$-\delta$		β	
1, 3, 4			$-\alpha$		$-\delta$	γ	$-a$
2, 3, 4	$-\delta$	γ	$-\beta$				$-b$

in the contravariant components h^α_i. Rows in this table correspond to equation (5.3) for which the indices j, k, l have the values indicated at the left. Each element in the table is the coefficient of the derivative at the top of the column in which the element appears. The matrix formed by the elements of the first six columns of the table is of rank three. If α and $\alpha b - \beta a$ do not vanish the system (5.3) can be put into the form

$$\text{5 (a)} \qquad \frac{\partial K_{lm}}{\partial x^\alpha} = \sum R(h^i_\alpha)\frac{\partial K_{pq}}{\partial x^r} + \sum R(h^i_\alpha) K^2 \qquad \begin{pmatrix} m = 1, 2 \\ l = 1, \cdots, K_m \\ \alpha = 1, \cdots, m \end{pmatrix}$$

where $R(h^i_\alpha)$ denotes a rational function of the quantities h^i_α and where K_{11} is composed of $h^i_{3,2}$ and $h^i_{4,2}$ and the group K_{12} is composed of $h^i_{4,3}$. [See, Proc. Note III (2.3).] The system composed of (5a) and (5.5) is regular since we have solved (5.3) for the maximum number of derivatives with respect to x^1, namely 12.

It will be proved in § 8 that this latter system is completely integrable and hence by a proper assignment of arbitrary functions in accordance with the general existence theorems of § 6, this system of equations will completely determine the quantities $h^i_{j,k}$ as the scalar invariants arising from a fundamental set of vectors h^i_α.

§ 6. General Existence Theorems.

We shall now impose on the system (2.1) the following two fundamental restrictions.

CONDITION I. *The coefficients* a^α_{ik} *and* c_i *in* (2.1) *are analytic functions in the neighborhood of some set of values* $x^i = q^i$ *and* $v_k = (v_k)_q$ *of their arguments.*

CONDITION II. *The regular system* (2.8) *is completely integrable.*

The first condition carries with it the consequence that the coefficients (x, v) and the ✶ terms in (2.8) are analytic functions in the neighborhood of some set of values p^i and $(v_{ik})_p$, these being values in the neighborhood of q^i and $(v_k)_q$. A system of equations will be said to be completely integrable if the integrability conditions resulting from one differentiation of the system are satisfied identically in the parametric derivatives.[16] It follows from the work of Meray and Riquier[17] that if Conditions I and II are satisfied, the system (2.1) has a unique solution, given by a set of convergent power series expansions, corresponding to the arbitrary assignment of analytic data predicted by the form of the left members of the system (2.8).

EXISTENCE THEOREM. *Suppose that* (2.1) *is a system such that Conditions* I *and* II *are satisfied. Let*

$$\varphi_{ik}(x^{k+1}, \cdots, x^n), \qquad (i = 1, \cdots, w_k),$$

where $k \neq n$ be an arbitrary function of the variables $x^{k+1}, \cdots, x^n$ analytic in the neighborhood of $x^i = p^i$ of their arguments, such that $\varphi_{ik}(p) = (v_{ik})_p$ for all values of the indices for which the φ_{ik} are defined. Then there exists one and only one solution, $v_{ik}(x)$, of (2.1), *each function $v_{ik}(x)$ being analytic in the neighborhood of the set of values $x^i = p^i$,* (1) *such that $v_{in}(p) = (v_{in})_p$ for $i = 1, \cdots, w_n$ and* (2) *such that*

$$v_{i0}(x^1, \cdots, x^n) = \varphi_{i0}(x^1, \cdots, x^n) \quad (i = 1, \cdots, w_0) \qquad \Bigg| \qquad v_{ik}(x^{k+1}, \cdots, x^n) = \varphi_{ik}(x^{k+1}, \cdots, x^n) \begin{bmatrix} i = 1, \cdots, w_k \\ x^1 = p^1, \cdots, x^k = p^k \end{bmatrix}.$$

With regard to the system (4.3) we shall set up the conditions

CONDITION I_A. *The coefficients $T(A)$ and the* ✱ *terms in* (4.3) *are analytic functions in the neighborhood of some set of values $(A_{lm})_q$ of their arguments.*

CONDITION II_A. *The regular system composed of* (4.6) *and* (4.7) *is completely integrable.*

Condition I carries with it the consequence that the coefficients and the ✱ terms in (4.6) will be analytic functions in the neighborhood of some set of values $(\overset{*}{A}_{lm})_p$, this being some set of values in the neighborhood of the set of values $(A_{lm})_q$.

EXISTENCE THEOREM. *Suppose that* (4.3) *is such that the conditions* I_A *and* II_A *are satisfied. Let*

$$\varphi_{ik}(x^{k+1}, \cdots, x^n), \qquad \begin{bmatrix} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, \overset{*}{A}_k \end{bmatrix},$$

where the indices i, k have the same range of values as the indices of the components $\overset{*}{A}_{ik}$ $(k \neq n)$, and

[16] The usual definition of a principle derivative is understood here, namely one which can be obtained by differentiation of a left member of the system, or one of the left members themselves. All other derivatives are parametric.

[17] Meray and Riquier, *loc. cit.*, (1890).

$$\psi_{ik}(x^{k+1}, \cdots, x^n), \qquad \begin{bmatrix} k = 0, \cdots, n-1 \\ i = 1, \cdots, \gamma_k \end{bmatrix}$$

denote functions of the variables $x^{k+1}, \cdots, x^n$ *analytic in the neighborhood of the set of values* $x^i = p^i$ *of their arguments, such that* $\varphi_{ik}(p) = (\overset{*}{A}_{ik})_p$ *for all values of the indices* i *and* k *for which the* φ_{ik} *are defined. Then there exists one and only one affine connection with components* $\Gamma^\alpha_{\beta\gamma} (= \Gamma^\alpha_{\gamma\beta})$ *in a system of coördinates* x^α, *each function* $\Gamma^\alpha_{\beta\gamma}(x)$ *being analytic in the neighborhood of the set of values* $x^i = p^i$, *which constitutes a set of integrals of the system of equations* (4.3) *and which is* (1) *such that* $\overset{*}{A}_{in}(p) = (\overset{*}{A}_{in})_p$ *and* (2) *such that*

$\overset{*}{A}_{i0} = \varphi_{i0}(x^1, \cdots, x^n)$ $[i = 1, \cdots, \overset{*}{A}_0]$	$\gamma_{l0} = \psi_{l0}(x^1, \cdots, x^n)$ $[l = 1, \cdots, \gamma_0]$
$\overset{*}{A}_{ik} = \varphi_{ik}(x^{k+1}, \cdots, x^n)$ $\begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, \overset{*}{A}_k \\ x^1 = p^1, \cdots, x^k = p^k \end{bmatrix}$	$\gamma_{lm} = \psi_{lm}(x^{m+1}, \cdots, x^n)$ $\begin{bmatrix} m = 1, \cdots, n-1 \\ l = 1, \cdots, \gamma_m \\ x^1 = p^1, \cdots, x^m = p^m \end{bmatrix}$.

The corresponding conditions and existence theorems for the system (4.4) can be stated immediately.

CONDITION I_G. *The coefficients* $T(g, B)$ *and the* ✱ *terms in* (4.4) *are analytic functions in the neighborhood of some set of values* $(B_{lm})_q$ *and* $(g_{\alpha\beta})_q = (g_{\beta\alpha})_q$ *of their arguments, the determinant* $|(g_{\alpha\beta})_q|$ *being different from zero.*

CONDITION II_G. *The regular system of equations* (4.9), (4.10) *and* (4.11) *is completely integrable.*

As a consequence of Condition I_G the coefficients and the ✱ terms in (4.9) will be analytic functions in the neighborhood of some set of values $(g_{\alpha\beta})_p$ and $(\overset{*}{B}_{lm})_p$ lying in the neighborhood of the values $(g_{\alpha\beta})_q$ and $(B_{lm})_q$ respectively, such that $(g_{\alpha\beta})_p = (g_{\beta\alpha})_p$ and also such that the determinant $|(g_{\alpha\beta})_p|$ does not vanish.

EXISTENCE THEOREM. *Suppose that* (4.4) *is such that Conditions* I_G *and* II_G *are satisfied. Let*

$$\varphi_{ik}(x^{k+1}, \cdots, x^n), \qquad \begin{bmatrix} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, \overset{*}{B}_k \end{bmatrix},$$

where the indices i, k *have the same ranges of values as the indices of the components* $\overset{*}{B}_{ik}$ $(k \neq n)$ *and*

$$\psi_{ik}(x^{k+1}, \cdots, x^n) \qquad \begin{bmatrix} k = 0, \cdots, n-1 \\ i = 1, \cdots, \gamma_k \end{bmatrix}$$

denote functions of the variables $x^{k+1}, \cdots, x^n$ *analytic in the neighborhood of the set of values* $x^i = p^i$ *of their arguments, such that* $\varphi_{ik}(p) = (\overset{*}{B}_{ik})_p$ *for all values of the indices for which the* φ_{ik} *are defined. Then there exists one, and only one, fundamental metric tensor with components* $g_{\alpha\beta} (= g_{\beta\alpha})$ *in a system of coördinates* x^α *each function* $g_{\alpha\beta}(x)$ *being analytic in the neighborhood of the point* $x^\alpha = p^\alpha$ *which constitutes a set of integrals of the system* (4.4) *and which* (1) *is such that* $g_{\alpha\beta}(p) = (g_{\alpha\beta})_p$ *and* $\overset{*}{B}_{in}(p) = (\overset{*}{B}_{in})_p$ *and* (2) *such that*

$\overset{*}{B}_{i0} = \varphi_{i0}(x^1, \cdots, x^n)$ $[i = 1, \cdots, \overset{*}{B}_0]$	$\gamma_{l0} = \psi_{l0}(x^1, \cdots, x^n)$ $[l = 1, \cdots, \gamma_0]$
$\overset{*}{B}_{ik} = \varphi_{ik}(x^{k+1}, \cdots, x^n)$ $\begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, \overset{*}{B}_k \\ x^1 = p^1, \cdots, x^k = p^k \end{bmatrix}$	$\gamma_{lm} = \psi_{lm}(x^{m+1}, \cdots, x^n)$ $\begin{bmatrix} m = 1, \cdots, n-1 \\ i = 1, \cdots, \gamma_m \\ x^1 = p^1, \cdots, x^m = p^m \end{bmatrix}.$

On the basis of the discussion in § 5, the corresponding existence theorem for spaces of distant parallelism can be stated.

CONDITION I_H. *The coefficients* $(\overset{i}{h}_\alpha, \overset{i}{h}_{j,k})$ *and the* ✶ *terms in the system* (5.2) *are analytic functions in neighborhood of some set values* $(\overset{i}{h}_{j,k})_q [= -(\overset{i}{h}_{k,j})_q]$ *and* $(\overset{i}{h}_\alpha)_q$ *of their arguments, such that the determinant* $|(\overset{i}{h}_\alpha)_q|$ *does not vanish.*

CONDITION II_H. *The regular system composed of* (5.4) *and* (5.5) *is completely integrable.*

Condition I_H implies that the coefficients $(\overset{i}{h}_\alpha, \overset{*}{K}_{lm})$ and the ✶ terms in the system (5.4) are analytic functions in the neighborhood some set of values $(\overset{i}{h}_\alpha)_p$ and $(\overset{*}{K}_{lm})_p$ in the neighborhood of the values $(\overset{i}{h}_\alpha)_q$ and $(\overset{i}{h}_{j,k})_q$ such that the determinant $|(\overset{i}{h}_\alpha)_p|$ does not vanish.

EXISTENCE THEOREM. *Suppose that* (5.2) *is such that conditions* I_H *and* II_H *are satisfied. Let*

$$\varphi_{lm}(x^{m+1}, \cdots, x^n), \qquad \begin{bmatrix} 0 \leqq m \leqq n-1 \\ l = 1, \cdots, \overset{*}{K}_m \end{bmatrix},$$

where l, m *have the same range of values as the indices of the components* $\overset{*}{K}_{lm}$ $(m \neq n)$, *and*

$$\psi_{i\alpha}(x^\alpha, \cdots, x^n) \qquad \begin{bmatrix} \alpha = 1, \cdots, n \\ i = 1, \cdots, n \end{bmatrix}$$

denote functions of the variables $x^{m+1}, \cdots, x^n$ *or* $x^\alpha, \cdots, x^n$ *as the case may be, analytic in the neighborhood of the set of values* $x^\alpha = p^\alpha$ *of their arguments such that* $\varphi_{lm}(p) = (\overset{*}{K}_{lm})_p$ *and* $\psi_{i\alpha}(p) = (\overset{i}{h}_\alpha)_p$ *for all values of the indices for which the functions* φ_{lm} *and* $\psi_{i\alpha}$ *are defined. Then there exists one and only one set of fundamental vectors with components* $\overset{i}{h}_\alpha$ *in* *a system of coördinates x^α each function $h^i_\alpha(x)$ being analytic in the neighborhood of the set of values $x^\alpha = p^\alpha$ which constitutes a set of integrals of the system* (5.2) *and which is* (1) *such that $K^*_{ln}(p) = (K^*_{ln})_p$ and* (2) *such that*

$h^i_1 = \psi_{i1}(x^1, \cdots, x^n)$ $[i = 1, \cdots, n]$	$K^*_{l0} = \varphi_{l0}(x^1, \cdots, x^n)$ $[l = 1, \cdots, K^*_0]$
$h^i_\alpha = \psi_{i\alpha}(x^\alpha, \cdots, x^n)$ $\left[\begin{matrix} i = 1, \cdots, n \\ \alpha = 2, \cdots, n \\ x^1 = p^1, \cdots, x^{\alpha-1} = p^{\alpha-1} \end{matrix}\right]$	$K^*_{lm} = \varphi_{lm}(x^{m+1}, \cdots, x^n)$ $\left[\begin{matrix} l = 1, \cdots, K^*_m \\ m = 1, \cdots, n-1 \\ x^1 = p^1, \cdots, x^m = p^m \end{matrix}\right]$.

§ 7. Functional Systems.

Consider a system of partial differential equations of the form

$$T^{\mu \cdots \nu}_{\varepsilon \cdots \delta}\left(\Gamma^\alpha_{\beta\gamma};\ \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta}\right) = 0, \tag{7.1}$$

where the T's are components of a tensor invariant of the type discussed in § 4, depending on the components $\Gamma^\alpha_{\beta\gamma}$ and their first derivatives. Along with this consider the invariantive system

$$T^{\mu \cdots \nu}_{\varepsilon \cdots \delta}\left(g_{\alpha\beta};\ \frac{\partial g_{\alpha\beta}}{\partial x^\gamma};\ \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma\, \partial x^\delta}\right) = 0, \tag{7.2}$$

where the components T depend on the fundamental metric tensor $g_{\alpha\beta}$ and their first and second derivatives. By the replacement theorem equations (7.1) can be written in terms of the independent components A_{lm} of the affine normal tensor. Hence (7.1) takes the form

$$T(A_{lm}) = 0. \tag{7.3}$$

Assume that the equations (7.3) are independent and N in number; dependent equations are to be discarded from the system.

CONDITION III$^*_A$. *Equations* (7.3) *are such that* (1) *there exists a solution $(A_{lm})_p$ of these equations and* (2) *the components $T(A_{lm})$ of these equations are analytic functions in the neighborhood of the set of values $(A_{lm})_p$.*

By covariant differentiation of (7.3) we obtain a system of the type considered in § 4, namely

$$\frac{\partial T}{\partial A_{lm}} \frac{\partial A_{lm}}{\partial x^\alpha} = \bigstar, \tag{7.4}$$

where the $\bigstar$ indicates a bilinear form in the components T and $\Gamma^\alpha_{\beta\gamma}$. Applying the results of § 4, the system (7.4) together with the identities (4.5) can be replaced by equations of the form

$$(7.5)\qquad \frac{\partial \overset{*}{A}_{ik}}{\partial x^{\alpha}} = \sum (A^*) \frac{\partial \overset{*}{A}_{pq}}{\partial x^{r}} + \bigstar, \qquad \begin{pmatrix} i = 1, \cdots, \overset{*}{A}_k \\ k = 1, \cdots, n \\ \alpha = 1, \cdots, k \\ q < r \geqq \alpha \end{pmatrix},$$

where the $\overset{*}{A}_{lm}$ represent independent components A^i_{jkl}. Corresponding to the previous Condition $\mathrm{I_A}$ we now impose the following

CONDITION $\overset{*}{\mathrm{I}}_{\mathrm{A}}$. *The coefficients* (A^*) *and the* $\bigstar$ *terms in* (7.5) *are analytic functions in the neighborhood of the set of values* $(A_{lm})_p$.

Assuming that (7.5) and (4.7) are completely integrable, these equations can be integrated in accordance with the general existence theorem stated in § 6. In order to see that the functions $\overset{*}{A}_{lm}$ which are formed from the integrals $\Gamma^{\alpha}_{\beta\gamma}(x)$ of the system (7.5) by the process of calculating the components of the first normal tensor, satisfy (7.1) or the equivalent system (7.3) we make use of Condition $\mathrm{III}^{*}_{\mathrm{A}}$. The functions $\overset{*}{A}_{lm}(x)$ are analytic in the neighborhood of $x^i = p^i$ and hence, by Condition $\mathrm{III}^{*}_{\mathrm{A}}$, the components $T(A)$ are analytic, as functions of the x^i, in the neighborhood of the same point. Thus the components T admit power series expansions about the point $x^i = p^i$, the constant terms of which vanish by Condition $\mathrm{III}^{*}_{\mathrm{A}}$. On account of equations (7.4) the remainder of the coefficients in these expansions must vanish likewise, and hence the components T are identically zero as functions of the coördinates x^i.

In an analogous manner the system (7.2) can be replaced by equations of the form

$$(7.6)\qquad T(g_{\alpha\beta};\, B_{lm}) = 0,$$

where the B_{lm} are the independent components of the first metric normal tensor $g_{\alpha\beta,\gamma\delta}$.

CONDITION $\mathrm{III}^{*}_{\mathrm{G}}$. *There exists a solution* $(g_{\alpha\beta})_p = (g_{\beta\alpha})_p$ *and* $(B_{lm})_p$ *of equations* (7.6) *such that* (1) *the determinant* $|(g_{\alpha\beta})_p|$ *does not vanish and* (2) *the components* T *are analytic functions in the neighborhood of the values* $(g_{\alpha\beta})_p$ *and* $(B_{lm})_p$.

By covariant differentiation of (7.6) we obtain

$$(7.7)\qquad \frac{\partial T}{\partial B_{lm}} \frac{\partial B_{lm}}{\partial x^{\alpha}} = \bigstar$$

where the $\bigstar$ denotes a bilinear form in the components T and the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$, and a linear form in the first derivatives of the components $g_{\alpha\beta}$ with coefficients which are functions of the $g_{\alpha\beta}$ and B_{lm}. Following the method of § 4 the system (7.7) is combined with the identities (4.8) to give a system of the form (4.9).

CONDITION I_G^*. *The coefficients* (g, B^*) *and the* ✶ *terms in* (4.9) *are analytic functions in the neighborhood of the set of values* $(g_{\alpha\beta})_P$ *and* $(B_{lm})_P$ *of Condition* III_G^*.

Furthermore it can be shown that the integrals $g_{\alpha\beta}(x)$ of the system (4.9) actually satisfy equations (7.6) in case Conditions I_G^* and III_G^* are satisfied. On the basis of the results of § 5 an analogous treatment can be given for the invariantive system

$$T^{\mu\cdots\nu}_{\varepsilon\cdots\delta}\left(h^i_\alpha;\ \frac{\partial h^i_\alpha}{\partial x^\beta}\right) = 0.$$

As an example illustrating the theory consider Einstein's gravitational equations for free space

$$R_{\alpha\beta} = 0 \tag{7a}$$

which express the fact that the contracted curvature tensor is equal to zero. In a previous paper by T. Y. Thomas, *On the Existence of Integrals of Einstein's Gravitational Equations for free space and their Extension to n-variables*, Proc. Nat. Acad. Sci., 15, (1929), p. 906, the problem of the existence of integrals for the system of equations (7a) has been treated when the components of the fundamental metric tensor $g_{\alpha\beta}$ have the initial values $\pm\delta^\alpha_\beta$. In the present section we shall consider the problem in four dimensions without restricting the initial values of the components $g_{\alpha\beta}$; the coördinates x^i are to be arbitrary. The work of putting the system (7a) in regular form will be carried out in detail.

The system (7a) is completely equivalent to one of the form

$$g^{\gamma\delta}\, g_{\alpha\beta,\,\gamma\delta} = 0 \tag{7b}$$

which corresponds to the system (7.6), and by covariant differentiation, we obtain

$$g^{\gamma\delta}\, g_{\alpha\beta,\,\gamma\delta,\,\varepsilon} = 0. \tag{7c}$$

The system of identities which furnish the conditions that the $g_{\alpha\beta,\,\gamma\delta}$ be the components of the first metric normal tensor formed from some fundamental tensor with components $g_{\alpha\beta}$ are of the form

$$g_{\alpha\beta,\,\gamma\delta,\,\varepsilon} = \frac{1}{2}\,[g_{\varepsilon\beta,\,\gamma\delta,\,\alpha} + g_{\alpha\varepsilon,\,\gamma\delta,\,\beta} + g_{\alpha\beta,\,\varepsilon\delta,\,\gamma} + g_{\alpha\beta,\,\gamma\varepsilon,\,\delta}]. \tag{7d}$$

We shall write out in detail the solved form of these equations since it will be needed in the following work.

Let us adopt the following more convenient notation

$\alpha = g^{11}$, $\beta = g^{22}$, $\gamma = g^{33}$, $\delta = g^{44}$,
$a = g^{12}$, $b = g^{13}$, $c = g^{14}$, $d = g^{23}$, $e = g^{24}$, $f = g^{34}$,

with the understanding that $g^{\alpha\beta}$ is symmetric in α and β; also

$u_1 = g_{11,22}$,	$v_1 = g_{22,33}$,	$w_1 = g_{13,24}$,
$u_2 = g_{11,23}$,	$v_2 = g_{12,33}$,	$w_2 = g_{14,33}$,
$u_3 = g_{11,24}$,	$v_3 = g_{13,22}$,	$w_3 = g_{13,44}$,
$u_4 = g_{11,33}$,	$v_4 = g_{14,22}$,	$w_4 = g_{24,33}$,
$u_5 = g_{11,34}$,	$v_5 = g_{12,44}$,	$w_5 = g_{23,44}$,
$u_6 = g_{11,44}$,	$v_6 = g_{12,34}$,	$w_6 = g_{33,44}$;
	$v_7 = g_{22,34}$,	
	$v_8 = g_{22,44}$,	

where the components u^α constitute the group G_0, v_α constitute G_1 and w_α constitute G_2 according to the rule for grouping independent components given in a former paper *loc. cit.*, Jour. p. 248. Then the system (7d) takes the form

$$(1)\qquad \frac{\partial v_1}{\partial x^1} = \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} + *,$$

$$(2)\qquad \frac{\partial v_2}{\partial x^1} = \frac{\partial u_4}{\partial x^3} - \frac{\partial u_2}{\partial x^3} + *,$$

$$(3)\qquad \frac{\partial v_3}{\partial x^1} = \frac{\partial u_1}{\partial x^3} - \frac{\partial u_2}{\partial x^2} + *,$$

$$(4)\qquad \frac{\partial v_4}{\partial x^1} = \frac{\partial u_1}{\partial x^4} - \frac{\partial u_3}{\partial x^2} + *,$$

$$(5)\qquad \frac{\partial v_5}{\partial x^1} = \frac{\partial u_6}{\partial x^2} - \frac{\partial u_3}{\partial x^4} + *,$$

$$(6)\qquad \frac{\partial v_6}{\partial x^1} = \frac{\partial u_5}{\partial x^2} - \frac{1}{2}\frac{\partial u_3}{\partial x^3} - \frac{1}{2}\frac{\partial u_2}{\partial x^4} + *,$$

$$(7)\qquad \frac{\partial v_7}{\partial x^1} = \frac{\partial v_6}{\partial x^2} + \frac{1}{2}\frac{\partial v_4}{\partial x^3} + \frac{1}{2}\frac{\partial v_3}{\partial x^4} + *,$$

$$(8)\qquad \frac{\partial v_8}{\partial x^1} = \frac{\partial v_5}{\partial x^2} + \frac{\partial v_4}{\partial x^4} + *,$$

$$(9)\qquad \frac{\partial w_1}{\partial x^1} = \frac{\partial u_3}{\partial x^3} - \frac{1}{2}\frac{\partial u_5}{\partial x^2} - \frac{1}{2}\frac{\partial u_2}{\partial x^4} + *,$$

$$(10)\qquad \frac{\partial w_2}{\partial x^1} = \frac{\partial u_4}{\partial x^4} - \frac{\partial u_5}{\partial x^3} + *,$$

$$(11)\qquad \frac{\partial w_3}{\partial x^1} = \frac{\partial u_6}{\partial x^3} - \frac{\partial u_5}{\partial x^4} + *,$$

$$(12)\qquad \frac{\partial w_4}{\partial x^1} = \frac{\partial v_2}{\partial x^4} + \frac{2}{3}\frac{\partial w_1}{\partial x^3} - \frac{2}{3}\frac{\partial v_6}{\partial x^3} + *,$$

$$(13)\qquad \frac{\partial w_5}{\partial x^1} = \frac{\partial v_5}{\partial x^3} - \frac{2}{3}\frac{\partial w_1}{\partial x^4} - \frac{4}{3}\frac{\partial v_6}{\partial x^4} + *,$$

$$(14)\qquad \frac{\partial w_6}{\partial x^1} = \frac{\partial w_3}{\partial x^3} - \frac{\partial w_2}{\partial x^4} + *,$$

$$(15)\qquad \frac{\partial w_1}{\partial x^2} = -\frac{3}{4}\frac{\partial v_4}{\partial x^3} + \frac{3}{4}\frac{\partial v_3}{\partial x^4} - \frac{1}{2}\frac{\partial v_6}{\partial x^2} + *,$$

$$(16)\qquad \frac{\partial w_2}{\partial x^2} = \frac{\partial v_2}{\partial x^4} - \frac{4}{3}\frac{\partial v_6}{\partial x^3} - \frac{2}{3}\frac{\partial w_1}{\partial x^3} + *,$$

$$(17)\qquad \frac{\partial w_3}{\partial x^2} = \frac{\partial v_5}{\partial x^3} + \frac{2}{3}\frac{\partial w_1}{\partial x^4} - \frac{2}{3}\frac{\partial v_6}{\partial x^4} + *,$$

$$(18)\qquad \frac{\partial w_4}{\partial x^2} = \frac{\partial v_1}{\partial x^4} - \frac{\partial v_7}{\partial x^3} + *,$$

$$(19)\qquad \frac{\partial w_5}{\partial x^2} = \frac{\partial v_8}{\partial x^3} - \frac{\partial v_7}{\partial x^4} + *,$$

$$(20)\qquad \frac{\partial w_6}{\partial x^2} = \frac{\partial w_5}{\partial x^3} + \frac{\partial w_4}{\partial x^4} + *.$$

It is to be noticed that the ranges of indices and the inequalities given for (4.8) hold for the above system of equations.

ε	X_{111}	X_{121}	X_{131}	X_{141}	X_{221}	X_{231}	X_{241}	X_{331}	X_{341}	X_{441}
1	α				$-\beta$	$-2d$	$-2e$	$-\gamma$	$-2f$	$-\delta$
2		2α			$2a$	$2b$	$2c$			
3			2α			$2a$		$2b$	$2c$	
4				2α			$2a$		$2b$	$2c$

Table I.

There are 40 equations in the system (7 c), but not all of these equations are independent. In fact, if we multiply through by $g^{\alpha\beta}\, g^{\gamma\delta}$ and sum on the indices $\alpha, \beta, \gamma, \delta$ we find

$$2\, g^{\alpha\beta}\, X_{\varepsilon\alpha\beta} = g^{\alpha\beta}\, X_{\alpha\beta\varepsilon}, \tag{7e}$$

where $X_{\alpha\beta\varepsilon}$ has been used to denote the left member of (7 c). In Table I the matrix of the coefficients of the terms containing the quantities $X_{\alpha\beta 1}$ in (7 e) is exhibited; provided g^{11} is not zero we see that the system (7 e) can be solved for the quantities $X_{1\beta 1}$. Hence the four equations

$$X_{1\beta 1} = 0$$

will be omitted from the system (7 c).

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	α, β	u_1	u_2	u_3	u_4	u_5	u_6	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	w_1	w_2	w_3	w_4	w_5	w_6
1	2, 2	α						γ		$2b$	$2c$			$2f$	δ						
2	2, 3		α					$-d$	$-b$	$-a$			$-2c$	$-e$		$-2c$			$-f$	δ	
3	2, 4			α							$-a$	$-c$		$-d$	$-e$	$2b$			γ	$-f$	
4	3, 3				α			β	$2a$								$2c$		$2e$		δ
5	3, 4					α							$2a$	β			$-b$	$-c$	$-d$	$-e$	$-f$
6	4, 4						α					$2a$			β			$2b$		$2d$	γ
7	1, 1	β	$2d$	$2e$	γ	$2f$	δ														
8	1, 2	$-a$	$-b$	$-c$					γ	$-d$	$-e$	δ	$2f$								
9	1, 3		$-a$		$-b$	$-c$			$-d$	β						$2e$	$-f$	δ			
10	1, 4			$-a$		$-b$	$-c$				β	$-e$	$-2d$			$-2d$	γ	$-f$			

Table II.

If we write equations (7 c) in the form

$$g^{\gamma\delta}\, \frac{\partial\, g_{\alpha\beta,\gamma\delta}}{\partial\, x^{\varepsilon}} = *$$

for a fixed value of ε the matrix of the coefficients of the derivatives $\partial\, g_{\alpha\beta,\gamma\delta}/\partial\, x^{\varepsilon}$ is exhibited in Table II; in this table the indices α, β give the row, and any element is the coefficient of the derivative of the unknown at the top of the column in which it appears When $\varepsilon = 1$ only the first six rows are to be considered, and this set of equations can be solved for the derivatives $\partial\, u_\beta/\partial\, x^1$ provided α is not zero. When $\varepsilon = 2, 3, 4$ it will be possible to solve the corresponding sets of equations taken from (7 c) for the derivatives $\partial\, u_\sigma/\partial\, x^{\varepsilon}$ and $\partial\, v_\tau/\partial\, x^{\varepsilon}$ where $\sigma = 1, \cdots, 6$ and $\tau = 1, 2, 3, 4$ provided the determinant formed from the first ten columns of Table II does not vanish. Expanding this determinant in terms of fourth order minors formed from rows 3, 5, 6, 10 we see that it can be factored into $\alpha^2(\alpha\beta - a^2)$ and a sixth order determinant indicated in Table III, no regard being paid to algebraic sign.

When use is made of Laplace's expansion in terms of third order minors it is found that the determinant of Table III can be written

	$-\alpha$	$-\beta$			$2a$
$-\alpha$		$-\gamma$		$2b$	
$-\beta$	$-\gamma$		$2d$		
		d	α	$-a$	$-b$
	b		$-a$	β	$-d$
a			$-b$	$-d$	γ

Table III.

$$-2\begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix}^2 .$$

Hence if the quantity

$$2(g^{11})^2 \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix} \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix}^2 \tag{21}$$

does not vanish the system of equations (7c), excluding the dependent equations $X_{1\beta 1} = 0$ can be written in the form

$$\frac{\partial u_\alpha}{\partial x^1} = \sum (g) \frac{\partial v_\beta}{\partial x^1} + \sum (g) \frac{\partial w_\gamma}{\partial x^1} + \bigstar, \tag{7f}$$

$$\left.\begin{matrix} \dfrac{\partial u_\alpha}{\partial x^2} \\ \dfrac{\partial v_\delta}{\partial x^2} \end{matrix}\right\} = \sum (g) \frac{\partial v_\varepsilon}{\partial x^2} + \sum (g) \frac{\partial w_\gamma}{\partial x^2} + \bigstar, \tag{7g}$$

$$\left.\begin{matrix} \dfrac{\partial u_\alpha}{\partial x^3} \\ \dfrac{\partial v_\delta}{\partial x^3} \end{matrix}\right\} = \sum (g) \frac{\partial v_\varepsilon}{\partial x^3} + \sum (g) \frac{\partial w_\gamma}{\partial x^3} + \bigstar, \tag{7h}$$

$$\left.\begin{matrix} \dfrac{\partial u_\alpha}{\partial x^4} \\ \dfrac{\partial v_\delta}{\partial x^4} \end{matrix}\right\} = \sum (g) \frac{\partial v_\varepsilon}{\partial x^4} + \sum (g) \frac{\partial w_\gamma}{\partial x^4} + \bigstar, \tag{7i}$$

where $\alpha, \gamma = 1, \cdots, 6$; $\beta = 1, \cdots, 8$; $\delta = 1, \cdots, 4$; $\varepsilon = 5, \cdots, 8$. Equations (1) to (14) can be used to eliminate their left members from the right hand members of (7f) and from the resulting equations eliminate the left members of (7g), (7h), (7i). Thus (7f) is replaced by a system (F) having no derivatives of $g_{\alpha\beta,\gamma\delta}$ with respect to x^1 on the right and also no left members of the systems (7g), (7h), (7i). Equations (15) to (20) can be used to eliminate their left members from the right members of (F) and (7g); then use equations (7h) and (7i) to eliminate their left members from the resulting equations. Thus (F) and (7g) are replaced by systems (φ) and (G) respectively such that the system composed of (φ), (G), (7h), (7i) is solved for its left members and contains no left members of equations (1) to (20) on the right. Hence the systems (φ), (G), (7h), (7i) can be used to eliminate their left members from the right hand sides of equations (1) to (20) and we have the combined system (7 c) and (7 d) solved for as many derivatives as there are independent equations in the system. Furthermore adopting the notation

$$B^*_{l4} \sim u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ v_1,\ v_2,\ v_3,\ v_4;$$
$$B^*_{l2} \sim w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6;$$
$$B^*_{l1} \sim v_5,\ v_6,\ v_7,\ v_8$$

the combined system (7 c) and (7 d) can be replaced by one of the form

$$\text{(7 j)} \qquad \frac{\partial B^*_{lm}}{\partial x^\alpha} = \sum (g) \frac{\partial B^*_{pq}}{\partial x^r} + \bigstar, \qquad \begin{pmatrix} m = 1, 2, 4 \\ \alpha = 1, \cdots, m \\ l = 1, \cdots, B^*_m \\ r > q \end{pmatrix},$$

where $B^*_1 = 4$, $B^*_2 = 6$, $B^*_4 = 10$. It is also evident from the method used in eliminating the left members from the right hand sides of the two systems that the inequality $r \geqq \alpha$ is satisfied and hence the system composed of (4.10), (4.11) and (7 j) is regular.

§ 8. Identities. Conditions for Complete Integrability.

A system of partial differential equations of the form (2.1) can be replaced by a system of the form

$$\text{(8.1)} \qquad X_{ikj} \equiv \frac{\partial v_{ik}}{\partial x^j} - F^{pqr}_{ikj} \frac{\partial v_{pq}}{\partial x^r} + \bigstar = 0, \qquad \begin{pmatrix} k = 1, \cdots, n \\ i = 1, \cdots, w_k \\ j = 1, \cdots, k \end{pmatrix}$$

provided the coördinate system is non-singular. The index r is subject to the conditions $q < r \geqq j$. It must therefore be possible to put equations (2.1) in the form

$$\text{(8.2)} \qquad Z_\alpha \equiv b^{ikj}_\alpha \frac{\partial v_{ik}}{\partial x^j} - b^{ikj}_\alpha F^{pqr}_{ikj} \frac{\partial v_{pq}}{\partial x^r} + \bigstar = 0, \quad (\alpha = 1, \cdots, L),$$

where $L = \sum k w_k$ and the matrix of the quantities $b^{ikj}_\alpha (x, v)$ is non-singular, i. e. the Z_α are identically equal to the left members of (2.1). If the system (8.1) is completely integrable the set of equations

$$\text{(8.3)} \qquad \frac{\partial X_{ikj}}{\partial x^l} - \frac{\partial X_{ikl}}{\partial x^j} = 0 \qquad \begin{pmatrix} l, j \leqq k \\ l < j \end{pmatrix}$$

must be satisfied identically when the left members are expressed in terms of the parametric derivatives determined by (8.1), i. e. there must exist

$$A = \sum_{k=2}^{n} w_k\, C(k, 2)$$

equations[18] of the type

$$\text{(8.4)} \qquad \frac{\partial X_{ikj}}{\partial x^l} - \frac{\partial X_{ikl}}{\partial x^j} + M^{IKJB}_{ikjl} \frac{\partial X_{IKJ}}{\partial x^B} = 0,$$

where I, J, K have the same range as i, j, k and $B > K$, these equations being satisfied identically in consequence of (8.1). The quantities M are functions of x^α and v_k.

[18] $C(k, 2)$ is used to denote the number of combinations without repetitions of k things taken 2 at a time.

Conversely suppose that there are A independent equations of the form

$$(8.5)\qquad \mathcal{A}_\sigma^{\alpha d}\frac{\partial Z_\alpha}{\partial x^d}=0 \qquad \begin{pmatrix}\sigma=1,\cdots,A\\ \alpha=1,\cdots,L\\ d=1,\cdots,n\end{pmatrix}$$

which are satisfied identically in consequence of $Z_\alpha=0$; the rank of the matrix of the quantities $\mathcal{A}(x, v)$ therefore has its maximum value A. If we make use of the relations

$$Z_\alpha = b_\alpha^{ikj}X_{ikj},$$

the equations (8.5) can be written

$$(8.6)\qquad \mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,\delta_d^c\frac{\partial X_{ikj}}{\partial x^c}+\sum X_{ikj}=0.$$

The indices c, d run from 1 to n and the system (8.6) possesses the property that it is satisfied in consequence of $Z_\alpha=0$, i. e. $X_{ikj}=0$. Since the matrix of the quantities $b_\alpha^{ikj}\,\delta_d^c$ is non-singular the equations in (8.6) are independent in the derivatives $\partial X_{ikj}/\partial x^l$. If the equations (8.6) are to be satisfied identically in consequence of $X_{ikj}=0$ the coefficients of the second derivatives of the unknowns v_{ik} must vanish identically. The system (8.6) can be written

$$(8.7)\qquad \mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,\delta_d^c\frac{\partial^2 v_{ik}}{\partial x^j\,\partial x^c}-\mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,F_{ikj}^{pqr}\,\delta_d^c\frac{\partial^2 v_{pq}}{\partial x^r\,\partial x^c}+\ast=0.$$

Since $r>q$ and $j\leqq k$, it is evident from the form of (8.7) that

$$(8.8)\qquad \mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,\delta_d^l+\mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikl}\,\delta_d^j\equiv 0,$$

where $l\leqq j\leqq k$. Consequently the equations (8.6) can be written

$$(8.9)\qquad \mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,\delta_d^l\left(\frac{\partial X_{ikj}}{\partial x^l}-\frac{\partial X_{ikl}}{\partial x^j}\right)+\mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,\delta_d^\beta\frac{\partial X_{ikj}}{\partial x^\beta}+\sum X=0,$$

where $l<j\leqq k$ and $\beta>k$. Suppose that the matrix of quantities $\mathcal{A}_\sigma^{\alpha d}\, b_\alpha^{ikj}\,\delta_d^l$ is singular; it follows that (8.9) can be replaced by a system which contains at least one equation of the form

$$(8.10)\qquad N^{ikj\beta}\frac{\partial X_{ikj}}{\partial x^\beta}+\sum X=0 \qquad (\beta>k).$$

By differentiation of the system (8.1) we obtain

$$(8.11)\qquad \frac{\partial X_{ikj}}{\partial x^\beta}\equiv\frac{\partial^2 v_{ik}}{\partial x^j\,\partial x^\beta}-F_{ikj}^{pqr}\frac{\partial^2 v_{pq}}{\partial x^r\,\partial x^\beta}+\ast=0 \qquad \begin{pmatrix}q<r\geqq j\\ \beta>k\geqq j\end{pmatrix}.$$

By the method used in a former paper,[19] equations (8.11) can be replaced by a system of the form

[19] *loc. cit.*, Jour. p. 246.

(8.12) $$Y_{ikj\beta} \equiv \frac{\partial^2 v_{ik}}{\partial x^\beta \partial x^j} - E^{pqrs}_{ikj\beta} \frac{\partial^2 v_{pq}}{\partial x^r \partial x^s} + \star = 0 \qquad \begin{pmatrix} r, s > q \\ \beta > k \geqq j \end{pmatrix},$$

where the indices i, k, j, β have the same range as in (8.11). It is to be noticed that throughout this process the equations obtained by differentiation of (8.1) with respect to x^l where $l \leqq k$ are not needed. Furthermore relations of the form

$$\frac{\partial X_{ikj}}{\partial x^\beta} = D^{IKJB}_{ikj\beta} Y_{IKJB}$$

must be satisfied where the matrix of the quantities D is non-singular. Hence (8.10) can be written

(8.13) $$N^{ikj\beta} D^{IKJB}_{ikj\beta} Y_{IKJB} + \sum X = 0.$$

However from the form of (8.12) it is clear that the equation (8.13) is not identically zero in consequence of $X = 0$ unless the coefficient $N^{ikj\beta} D^{IKJB}_{ikj\beta}$ is identically zero. This implies that the quantities N are all zero and hence that the equations (8.6) are not all independent. Therefore the matrix of the quantities $A^{\alpha d}_\sigma b^{ikj}_\alpha \delta^l_d$ in (8.8) must have a non-vanishing determinant from which it follows that (8.3) are satisfied in consequence of $X_{ikj} = 0$ and $\partial X_{ikj}/\partial x^\beta \; (\beta > k)$, i. e. the system (8.1) is completely integrable.

Hence a necessary and sufficient condition for complete integrability of (8.1) is that there exist independent identities of the type (8.5) equal in number to the number of integrability conditions of (8.1). That there cannot exist more than A independent identities of the type (8.5) follows from the form of (8.9). The existence of A', $(A' > A)$, independent identities of the type (8.5) would lead to A' independent identities (8.6). Since there are only A distinct left members of (8.3), it follows that (8.9), which is equivalent to (8.6), could be replaced by a system containing at least one equation of the form (8.10). The above argument shows that the quantities N must be identically zero, i. e. equations (8.6) or in other words (8.5) would not be independent.

As an application of the theory of this section consider the system of equations 5 (a) together with a system of the type (5.5), i. e.

8 (a) $$\frac{\partial h^i_\alpha}{\partial x^\beta} - \frac{\partial h^i_\beta}{\partial x^\alpha} - 2\, h^i_{j,k}\, h^j_\alpha\, h^k_\beta = 0, \qquad \begin{pmatrix} \alpha = 2, 3, 4 \\ \beta = 1, \cdots, \alpha - 1 \\ i = 1, 2, 3, 4 \end{pmatrix}.$$

The above system 8 (a) can be solved for

$$\frac{\partial h^i_2}{\partial x^1}, \frac{\partial h^i_3}{\partial x^1}, \frac{\partial h^i_3}{\partial x^2}, \frac{\partial h^i_4}{\partial x^1}, \frac{\partial h^i_4}{\partial x^2}, \frac{\partial h^i_4}{\partial x^3}$$

from which it follows that there are 16 integrability conditions arising from 8 (a) alone. By differentiating the equations 8 (a) with respect to x^γ and permuting the indices α, β, γ cyclically and adding the three equations thus obtained, we arrive at the system (5.3), i. e.

$$8\,(b) \qquad h^i_{j,k,l} + h^i_{k,l,j} + h^i_{l,j,k} - 2\,[h^i_{m,j}\, h^m_{k,l} + h^i_{m,l}\, h^m_{j,k} + h^i_{m,k}\, h^m_{l,j}] = 0,$$

where the indices j, k, l have the sets of values 1, 2, 3; 1, 2, 4; 1, 3, 4; 2, 3, 4. Equations 8 (b) form a set of 16 equations of the type (8.5) which are satisfied identically in consequence of 5 (a); in fact 5 (a) is but a different form of 8 (b). From 5 (a) we see that there will be four more integrability conditions or 20 in all for the combined system 5 (a) and 8 (a). Denote the left members of 8 (b) by X^i_{jkl} and consider the four equations

$$8\,(c) \qquad X^i_{123,4} + X^i_{142,3} + X^i_{134,2} + X^i_{324,1} = 0,$$

where the comma denotes absolute differentiation. It can be shown that equations 8 (c) are satisfied identically in consequence of 8 (a) and 5 (a). Hence the system of identities 8 (c) and 8 (b) numbering 20 in all, insures the complete integrability of 8 (a) and 5 (a).

When the system of field equations

$$\Delta_k\, h^i_{j,k} = 0,$$

[see Proc. Note III], is combined with the system 5 (a), a regular system (3.3) Proc. Note III is obtained [that (3.3) is regular follows from the fact that all derivatives with respect to x^1 appear on the left]. The 8 integrability conditions arising from (3.3) Proc. Note III taken together with the 16 conditions arising from 8 (a) make 24 in all. The corresponding set of 24 independent identities involving derivatives of the left members of the system is made up of equations 8 (c), 8 (b), and the equations (3.4) Proc. Note II, the latter stating that the divergence of the left members of the field equations vanishes identically.

§ 9. A Sufficient Condition for Complete Integrability.

To find a set of identities (8.5) which will insure complete integrability requires considerable work in the case of certain systems of differential equations. This difficulty can be met in some measure at least by use of a sufficient condition for complete integrability given by one of us in a former treatment;[20] this condition for complete integrability can be applied with facility to a large class of invariantive systems and in particular to the type of invariantive system which serves as field equations in the theory of relativity. In this section we shall consider this method of establishing complete integrability from the standpoint of the present paper.

In § 7 it was noted that a certain number M of the equations (7.4) may be linearly dependent on the remainder of (7.4) in consequence of the equations of the system (4.5). Let us express this dependence as follows

$$(9.1) \qquad \sum (A) \frac{\partial T}{\partial x^\alpha} + \ast = 0 \qquad (M \text{ equations}).$$

From the theory of § 3 it follows that a coördinate system can be found in which (9.1) can be put in regular form, i. e. can be written

[20] *loc. cit.* Ann. (1), p. 690 et. seq.

$$(9.2)\qquad \frac{\partial T_{ik}}{\partial x^{\alpha}} = \sum(A)\frac{\partial T}{\partial x} + \ast, \qquad \begin{pmatrix} k = 1, \cdots, n \\ \alpha = 1, \cdots, k \\ i = 1, \cdots, M_k \end{pmatrix},$$

where $M = \sum k M_k$. From the fact that (9.2) is regular it follows that all derivatives of order r of the left members of (9.2) vanish in consequence of the vanishing of the components T, all derivatives of components T of orders 1 to r inclusive, and the remainder of the derivatives of the components T of order $r+1$.[21] Now set up the following condition with reference to the numbers A_i^* which arise in the system (7.5).[22]

CONDITION IV_A^*. *The numbers A_i^* are such that*

$$A_0^* = A_0 - N + \sum_1^n M_\alpha,$$
$$A_i^* = A_i - M_i \qquad (i = 1, \cdots, n-1),$$
$$M_{n-1} = 0, \qquad M_n = 0.$$[23]

Since $\sum_0^n A_i^*$ is equal to $\sum_0^{n-2} A_i$, it follows that $A_n^* = N$ when use is made of Condition IV_A^*. A comparison with the set of conditions given previously[24] and the related discussion, shows that if Conditions I_A^*, III_A^*, IV_A^* are satisfied, the combined system (7.5) and (4.7) will be completely integrable.

Conditions similar to IV_A^* can be applied to invariantive systems of the metric and vector types.

As an illustration of the use of this method the reader is referred to a proof that the system of equations (7j) together with (4.10) and (4.11) is completely integrable, *loc. cit.* Ann. (1), p. 713; also Ann. (2) p. 725. This enables us to state the following

EXISTENCE THEOREM. *Let*

$$\varphi_{lm}(x^{m+1}, \cdots, x^4) \qquad \begin{bmatrix} m = 1, 2 \\ l = 1, \cdots, B_m^* \end{bmatrix}$$

and

$$\psi_{lm}(x^{m+1}, \cdots, x^4) \qquad \begin{bmatrix} m = 0, 1, 2, 3 \\ l = 1, \cdots, \gamma_m \end{bmatrix}$$

[21] Cf. *loc. cit.*, Jour. p. 246.

[22] It is immaterial whether or not the coördinate system (x) to which the system (9.2) is referred is identical with the coördinate system (x) to which (4.6) is referred.

[23] It may be noted that $M_{n-1} = 0$ follows from $A_{n-1}^* = A_{n-1} - M_{n-1}$ since $A_{n-1} = 0$ and the A_i^* and M_i are zero or positive.

[24] *loc. cit.*, Ann. (1), p. 690. If Conditions I_A^*, III_A^*, IV_A^* are satisfied, it follows that Conditions I–V in Ann. (1) are likewise satisfied; Condition I, Ann. (1) is equivalent to Condition III_A^*; Condition II, Ann. (1) follows from the fact that (9.2) is regular; Condition III and IV, Ann. (1) follow from Condition IV_A^*; Condition V, Ann. (1) is equivalent to Condition I_A^*.

denote functions of the variables $x^{m+1}, \cdots, x^4$ *analytic in the neighborhood of* $x^i = p^i$ *such that* $\varphi_{lm}(p) = (\overset{*}{B}_{lm})_p$ *for* $m = 1, 2$. *The constants* $(g_{\alpha\beta})_p = (g_{\beta\alpha})_p$ *and* $(\overset{*}{B}_{lm})_p$ *where* $m = 1, 2, 4$ *are to be chosen* (1) *such that they satisfy equations* (7 b), (2) *such that the determinant* $|(g_{\alpha\beta})_p|$ *does not vanish, and* (3) *such that quantity* (21) *does not vanish. Then there exists one and only one fundamental metric tensor with components* $g_{\alpha\beta}$ $(= g_{\beta\alpha})$ *each function* $g_{\alpha\beta}(x)$ *being analytic in the neighborhood of* $x^i = p^i$, *which constitutes a set of integrals of the system* (7 b) *and which is* (1) *such that* $g_{\alpha\beta}(p) = (g_{\alpha\beta})_p$ *and* $\overset{*}{B}_{l4}(p) = (\overset{*}{B}_{l4})_p$, *where* $l = 1, \cdots, 10$ *and* (2) *such that*

$$\gamma_{l0} = \psi_{l0}(x^1, \cdots, x^4) \quad [l = 1, 2, 3, 4] \quad \Bigg| \quad \gamma_{lm} = \psi_{lm}(x^{m+1}, \cdots, x^4) \begin{bmatrix} m = 1, 2, 3 \\ l = 1, \cdots, \gamma_m \\ x^1 = p^1, \cdots, x^m = p^m \end{bmatrix} \quad \Bigg| \quad \overset{*}{B}_{lm} = \varphi_{lm}(x^{m+1}, \cdots, x^4) \begin{bmatrix} m = 1, 2 \\ l = 1, \cdots, \overset{*}{B}_m \\ x^1 = p^1, \cdots, x^m = p^m \end{bmatrix}.$$

§ 10. Existence Theorems in Normal Coördinates.

It is not possible to choose a priori some of the components $\Gamma^\alpha_{\beta\gamma}$ arbitrarily as functions of a part of the variables independently of the arbitrary functions corresponding to the components $\overset{*}{A}_{lm}$ and at the same time to characterize the coördinates as normal coördinates y^i. However the discussion of this problem will utilize many former results. Let it suffice to say that in constructing the power series expansions for the components $\overset{*}{A}_{lm}$ by the use of equations (4.6) that the initial conditions $\Gamma^\alpha_{\beta\gamma} = 0$ at $y^i = 0$ are to be imposed, also that the derivatives $\partial \Gamma^i_{jk}/\partial x^l$ at $y^i = 0$ be equal to $A^i_{jkl}(0)$ and that the higher derivatives are to be determined by equations of the type $M(2.10)$.[25] If the system composed of (4.6) and (4.7) is completely integrable the power series expansions for the $\overset{*}{A}_{lm}$ and for the components $\Gamma^i_{\alpha\beta}$ will be determined uniquely for a suitable assignment of arbitrary data. The Γ series

$$(10.1) \qquad \Gamma^i_{\alpha\beta} = (A^i_{\alpha\beta\gamma})_0\, y^\gamma + \cdots$$

will be shown to converge in § 11.

EXISTENCE THEOREM. *Let* (4.6) *be a regular system which satisfies Conditions* $\mathrm{I_A}$ *and* $\mathrm{II_A}$. *Also let*

$$\varphi_{ik}(y^{k+1}, \cdots, y^n) \qquad \begin{bmatrix} 0 \leq k \leq n-1 \\ i = 1, \cdots, \overset{*}{A}_k \end{bmatrix},$$

where the indices i, k *have the same range of values as the indices of the components* $\overset{*}{A}_{ik}$ $(k \neq n)$, *denote a function of the variables* $y^{k+1}, \cdots, y^n$ *analytic in the neighborhood of* $y^i = 0$ *such that* $\varphi_{ik}(0) = (\overset{*}{A}_{ik})_0$ *for all*

[25] Cf. Jour. p. 246.

values of the indices for which the φ_{ik} *are defined. Then there exists one, and only one, set of components* $\Gamma^i_{\alpha\beta}\,(=\Gamma^i_{\beta\alpha})$ *of affine connection given by the convergent power series* (10.1), *which constitute a set of integrals of the system* (4.6) *and which is* (1) *such that* $A^*_{in}(0) = (A^*_{in})_0$ *where* $i = 1, \cdots, A^*_n$ *and* (2) *such that*

$$A^*_{i0} = \varphi_{i0}(y^1, \cdots, y^n) \quad [i = 1, \cdots, A^*_0] \qquad\qquad A^*_{ik} = \varphi_{ik}(y^{k+1}, \cdots, y^n) \quad \begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, A^*_k \\ y^1 = 0, \cdots, y^k = 0 \end{bmatrix}.$$

Similarly for the metric case equations (4.9) are to be used in calculating the coefficients of the power series expansions of the components B^*_{lm}. Throughout this process the initial values $g_{\alpha\beta}(0) = g_{\beta\alpha}(0) = (g_{\alpha\beta})_0$ and $\partial g_{\alpha\beta}/\partial y^\gamma = 0$ at $y^i = 0$ are to be imposed; also let $\partial^2 g_{\alpha\beta}/\partial y^\gamma\, \partial y^\delta$ at $y^i = 0$ be equal to $(g_{\alpha\beta,\gamma\delta})_0$ and determine the higher derivatives by equations of the type $M(2.16)$. If the system composed of equations (4.9), (4.7), and (4.10) is completely integrable a unique power series

$$g_{\alpha\beta} = g_{\alpha\beta}(0) + \frac{1}{2!}\, g_{\alpha\beta,\gamma\delta}(0)\, y^\gamma\, y^\delta + \cdots \tag{10.2}$$

is determined and this will be shown to converge in § 11.

EXISTENCE THEOREM. *Let* (4.9) *be a regular system which satisfies Conditions* I_G *and* II_G. *Also let*

$$\varphi_{ik}(y^{k+1}, \cdots, y^n) \qquad \begin{bmatrix} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, B^*_k \end{bmatrix},$$

where the indices i, k *have the same range of values as the indices of the components* $B^*_{ik}\,(k \neq n)$, *denote a function of the variables* $y^{k+1}, \cdots, y^n$ *analytic in the neighborhood of* $y^i = 0$ *such that* $\varphi_{ik}(0) = (B^*_{ik})_0$ *for all values of the indices for which the* φ_{ik} *are defined. Then there exists one, and only one set of components* $g_{\alpha\beta}\,(= g_{\beta\alpha})$ *of a fundamental metric tensor, given by the convergent power series* (10.2) *which constitutes a set of integrals of the system of equations* (4.9) *and which is* (1) *such that* $B^*_{in}(0) = (B^*_{in})_0$ *where* $i = 1, \cdots, B^*_n$ *and* (2) *such that*

$$B^*_{i0} = \varphi_{i0}(y^1, \cdots, y^n) \quad [i = 1, \cdots, B^*_0] \qquad\qquad B^*_{ik} = \varphi_{ik}(y^{k+1}, \cdots, y^n) \quad \begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, B^*_k \\ y^1 = 0, \cdots, y^k = 0 \end{bmatrix}.$$

For spaces of distant parallelism a corresponding existence theorem can be stated where the integrals $h^i_\alpha(x)$ will constitute a set of fundamental vectors in a system of local coördinates z^i. In calculating the coefficients of the power series expansions for the invariants $h^i_{j,k}$ with the aid of equations (5.4) the limited conditions $h^i_{|j|} = \delta^i_j$ at $z^i = 0$ and $\partial h^i_{|j|}/\partial z^k = (h^i_{j,k})_0$ at $z^i = 0$ are to be imposed; the higher derivatives of the components $h^i_{|j|}$ are to be determined by equations of the type Proc. Note II (4.2). If the system of equations composed of (5.4) and (5.5) is completely integrable, the power series expansions for the invariants $h^i_{j,k}$ and also the $h^i_{|j|}$ series of the form

$$h^i_{|j|} = \delta^i_j + h^i_{j,k}(0)\, z^k + \cdots \tag{10.3}$$

will be unique; the convergence of this latter series will be proved in § 11.

EXISTENCE THEOREM. *Let* (5.4) *be a regular system which satisfies Conditions* $\mathrm{I_H}$ *and* $\mathrm{II_H}$. *Also let*

$$\varphi_{lm}(z^{m+1}, \cdots, z^n) \qquad \begin{bmatrix} m = 0, \cdots, n-1 \\ l = 1, \cdots, K^*_m \end{bmatrix},$$

where the indices l, m have the same range of values as the indices of the components K^*_{lm} $(m \neq n)$, denote a function of the variables $z^{m+1}, \cdots, z^n$ analytic in the neighborhood of $z^i = 0$, such that $\varphi_{lm}(0) = (K^*_{lm})_0$ for all values of the indices for which the φ_{lm} are defined. Then there exists one, and only one, set of components $h^i_{|j|}$ of the fundamental vectors given by the convergent power series (10.3) *which constitute a set of integrals of the system* (5.4) *and which is* (1) *such that $K^*_{in}(0) = (K^*_{in})_0$ where $i = 1, \cdots, K^*_n$ and* (2) *such that*

$$K^*_{i0} = \varphi_{i0}(z^1, \cdots, z^n) \quad [i = 1, \cdots, K^*_0] \qquad \Bigg| \qquad K^*_{im} = \varphi_{im}(z^{m+1}, \cdots, z^n) \begin{bmatrix} m = 1, \cdots, n-1 \\ i = 1, \cdots, K^*_m \\ z^1 = 0, \cdots, z^m = 0 \end{bmatrix}.$$

The extension to functional systems can be made in accordance with the discussion in § 7.

§ 11. Convergence Proofs.

The equations (4.6) of the present theory have exactly the same form as the system of equations (2.7) Ann. (I) except that in (4.6) there may be a group of unknowns having derivatives with respect to $(n-1)$ of the y's appearing in the left members of the system, which was not the case in the former treatment. It is easily seen however that this greater generality will necessitate no change in the convergence proof of the Γ series. Hence the reader is referred to the former treatment for a proof of the convergence of the series (10.1). Similar remarks apply to convergence of the expansions (10.2) but the convergence of the series (10.3) remains to be demonstrated.

Consider the system of partial differential equations.

$$(11.1a)\qquad \frac{b_m}{a_h}\frac{\partial \mathfrak{K}^*_{lm}}{\partial z^h} = DF(z) + \frac{\mu\,\Theta(\mathfrak{K}^*, \mathfrak{H})}{1-\varepsilon\,\Theta(\mathfrak{K}^*, \mathfrak{H})},$$

$$(11.1b)\qquad \frac{1}{a_k}\frac{\partial \mathfrak{H}^i_j}{\partial z^k} = W \sum b_q\, \mathfrak{K}^*_{pq}\, \mathfrak{H}^c_d\, \mathfrak{H}^g_f$$

in which a_h, b_m and μ are positive constants for which more exact values will be fixed later. The positive constants D and W are to be chosen so that

$$(11.2)\qquad D \geqq \frac{b_m}{a_h}, \quad W \geqq \frac{2}{a\,b},$$

where a and b are the least of the a_h and b_m respectively. The indices p, q, l, m in (11.1) assume all possible values as indices of the independent components K^*_{lm} and all other indices have values from 1 to n. All summations in (11.1) represent the sum of all terms obtainable from the representative term by giving different values to the indices involved. The quantity ε in (11.1a) is a positive constant less than unity and is to be considered as the sum of α positive constants $\nu_1 + \cdots + \nu_\alpha$. The integer α in the sum $\nu_1 + \cdots + \nu_\alpha$ for a particular equation (11.1a) in which the values of the indices in its left member are equal to the corresponding indices in the left member of some equation (5.4), is to be equal to the number of different derivatives $\partial K^*_{pq}/\partial x^r$ in the right member of the said equation of the system (5.4). The integer α for an equation (11.1a) which does not correspond to an equation of (5.4) can be taken to have the value unity. The function $F(z)$ in (11.1a) is defined by the equation

$$F(z) = \frac{\Omega}{1-\sum a_k z^k},$$

where the positive constants Ω and a_k are to be chosen so that the expression

$$(11.3)\qquad \frac{\Omega}{1-a_{h+1} z^{h+1} - \cdots - a_n z^n}$$

is dominant for each derivative $\partial \varphi_{lm}/\partial z^h\,(h > m)$ of the K^*_m functions $\varphi_{lm}\,(l = 1, \cdots, K^*_m)$. Finally

$$\Theta(\mathfrak{K}^*, \mathfrak{H}) = \left\{1 - \sum b_q\,[\mathfrak{K}^*_{pq} - (\mathfrak{K}^*_{pq})_0] - \sum \frac{1}{\varrho}\,[\mathfrak{H}^i_j - (\mathfrak{H}^i_j)_0]\right\}^{-1},$$

where $\sum$ denotes a summation over all possible values of the indices p, q, i, j. The positive constant ϱ is to be chosen later. Equations (11.1) constitute a completely integrable system of total differential equations.

Hence according to the well known theorem for the existence of solutions of systems of total differential equations, there exists a unique solution $\mathfrak{K}^*_{lm}(z)$, $\mathfrak{H}^i_j(z)$ of the system (11.1) such that these integrals assume an arbitrary set of initial values $(\mathfrak{K}^*_{lm})_0$ and $(\mathfrak{H}^i_j)_0$. We shall choose the initial values so that the inequalities

$$(11.4) \qquad (\mathfrak{H}^i_j)_0 \geqq \delta^i_j$$

and

$$(11.5) \qquad (\mathfrak{K}^*_{lm})_0 \geqq |(K^*_{lm})_0|$$

are satisfied, where $(\mathfrak{H}^i_j)_0 = \mathfrak{H}^i_j(0)$ and $(\mathfrak{K}^*_{lm})_0 = \mathfrak{K}^*_{lm}(0)$. All derivatives at $z^i = 0$ of the integrals $\mathfrak{K}^*_{lm}$ and $\mathfrak{H}^i_j$ are then positive since any derivative of one of the sets of terms in the right member of (11.1) evaluated at $z^i = 0$ is a polynomial composed entirely of positive terms.[26] In consequence of the dominant property of the function $F(z)$ and the first inequality of (11.2) it follows from (11.1a) that

$$(11.6) \qquad \mathfrak{K}^*_{lm/h_1\cdots h_s}(0) \geqq \left| \frac{\partial^s \varphi_{lm}}{\partial z^{h_1} \cdots \partial z^{h_s}}(0) \right|,$$

where $h_1, \cdots, h_s > m$ and the notation $\mathfrak{K}^*_{lm/h_1\cdots h_s}$ denotes the ordinary derivative of the function $\mathfrak{K}^*_{lm}$.

Since the right members of the equations (11.1a) are all equal, the equations

$$\frac{b_m}{a_h} \frac{\partial \mathfrak{K}^*_{lm}}{\partial z^h} = \frac{b_q}{a_r} \frac{\partial \mathfrak{K}^*_{pq}}{\partial z^r}$$

are satisfied for all values of the indices involved. It is therefore possible to write those equations of (11.1a) which have left members with indices corresponding to the indices of some left member of (5.4) in the form

$$(11.7) \qquad \begin{aligned} \frac{\partial \mathfrak{K}^*_{lm}}{\partial z^h} = \sum_\nu \frac{a_h}{a_r} \frac{b_q}{b_m} \Theta(\mathfrak{K}^*, \mathfrak{H}) \frac{\partial \mathfrak{K}^*_{pq}}{\partial z^r} \\ + DF(z) \frac{a_h}{b_m}[1 - \varepsilon\, \Theta(\mathfrak{K}^*, \mathfrak{H})] + \mu \frac{a_h}{b_m} \Theta(\mathfrak{K}^*, \mathfrak{H}), \end{aligned}$$

where the quantity ν assumes values ν_1 to ν_α so that the first summation in these equations denotes a sum of α terms in which the derivatives are taken to correspond to derivatives in the first summation in the right member of the corresponding equation (5.4). The reader is referred to the former discussion for the proof of the following[27]

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[26] Meray and Riquier, *loc. cit.* p. 48.

[27] *loc. cit.* Ann. (1), p. 698.

LEMMA. *Given any two positive constants P and Q it is possible to assign values, each of which is greater than P to the constants $a_1, \cdots, a_n$ and $b_0, \cdots, b_n$ such that each of the coefficients*

$$\nu \frac{a_h}{a_r} \frac{b_q}{b_m} \tag{11.8}$$

of the derivatives in the right member of (11.7) *will be greater than Q.*

Choose the positive constants P, Q and r so that the expression

$$Q\{1-P(z^{h+1}+\cdots+z^n)\}^{-1}$$

dominates each derivative $\partial\varphi_{lm}/\partial z^h$ $(h>m)$ of the K_m^* functions φ_{lm} $(l=1, \cdots, K_m^*)$ and so that

$$Q\{1-\sum P[\mathfrak{K}_{pq}^*-(\mathfrak{K}_{pq}^*)_0]-\sum \frac{1}{r}[\mathfrak{H}_j^i-(\mathfrak{H}_j^i)_0]\}^{-1}$$

dominates each of the coefficients (h_α^i, K_{lm}^*) in (5.4) and also the terms in (5.4) containing no derivatives. Then choose the positive constants $a_1, \cdots, a_n$ and $b_0, \cdots, b_n$ so that each is greater than P and so that the coefficients (11.8) are each greater than Q; also choose the positive constant $\varrho \leqq r$. Then the coefficients of the derivatives in the right members of the equations (11.7) will dominate the corresponding coefficients in equations (5.4). If the positive constant μ is chosen such that $\mu \geqq BQ/a$ then the quantities $\mu \dfrac{ah}{b_m} \Theta(\mathfrak{K}^*, \mathfrak{H})$ will dominate the terms in the right members of (5.4) containing no derivatives; the quantity a is used to denote the least of the a's and B the greatest of b's. Hence it follows from (11.6) and (11.7) that

$$\mathfrak{K}_{lm/h_1}^*(0) \geqq |K_{lm/h_1}^*(0)|, \tag{11.9}$$

where $h_1 \leqq m$. If the indices i, j, k in (11.1b) determine an independent component $h_{j,k}^i$, then we have from (11.2), (11.5), and the fact that all components $\mathfrak{K}_{pq}^*$ appear in the right member of each equation (11.1b) that

$$\mathfrak{H}_{j/k}^i(0) \geqq |(h_{j,k}^i)_0|. \tag{11.10}$$

The inequality holds also for values of the indices i, j, k which determine a dependent component $h_{j,k}^i$ since, in that case, $h_{j,k}^i$ is merely the negative of an independent component. In order to extend (11.10) to higher derivatives we shall assume the inequalities

$$\mathfrak{H}_{j/k_1\cdots k_s}^i(0) \geqq |h_{j,k_1\cdots k_s}^i(0)| \tag{11.11}$$

for $s < r\ (r \geqq 2)$. In calculating derivatives of second or higher order of the components K^*_{lm} from equations (5.4), we use a process of differentiating equations (5.4) and eliminating derivatives from the right members of the resulting equations, which also appear on the left of equations (5.4) or the equations obtained by differentiating (5.4). Due to the form of the left members of (5.4) this elimination merely involves the substitution for a right hand derivative of its equivalent from some other equation[28] and hence it is easily seen by comparing the two systems of equations, one arising from (5.4) and the other from (11.7), that the inequalities

$$\text{(11.12)} \qquad \mathfrak{K}^*_{lm/h_1\cdots h_s}(0) \geqq |K^*_{lm/h_1\cdots h_s}(0)|$$

hold where $s = 1, \cdots, r$; $k_1 \leqq m$; and h_i, for $i > 1$, is arbitrary. Combining (11.12) and (11.6) it follows that (11.12) is satisfied for all values of the indices involved. Now differentiate equations (5.5) $(r-1)$ times and evaluate at the point $z^i = 0$; we obtain a system of the form

$$h^i_{j,k_1\cdots k_r} - h^i_{k_1,jk_2\cdots k_r} = 2h^i_{j,k_1,k_2\cdots k_r} + *.$$

If we compare these equations with the equations which can be obtained by differentiating (11.1b) it is seen, due to the inequalities (11.2), (11.4), (11.5), (11.11) and (11.12), that

$$\text{(11.13)} \qquad \mathfrak{H}^i_{j,k_1\cdots k_r}(0) \geqq |h^i_{j,k_1\cdots k_r}(0) - h^i_{k_1,jk_2\cdots k_r}(0)|$$

for all values of the indices involved, or interchanging indices

$$\text{(11.14)} \qquad \mathfrak{H}^i_{j,k_1\cdots k_r}(0) \geqq |h^i_{j,k_1\cdots k_r}(0) - h^i_{\mu,\eta_1\cdots\eta_r}(0)|$$

where $\mu\eta_1\cdots\eta_r$ represents any permutation of the indices $jk_1\cdots k_r$. Add together the $(r+1)$ inequalities obtained from (11.14) by allowing the indices $\mu\eta_1\cdots\eta_r$ to assume the cyclical permutations of $jk_1, \cdots, k_r$. The inequalities

$$\text{(11.15)} \qquad (r+1)\,\mathfrak{H}^i_{j/k_1\cdots k_r}(0) \geqq |(r+1)\,h^i_{j,k_1\cdots k_r}(0) - S[h^i_{j,k_1\cdots k_r}(0)]|$$

result and hence (11.11) holds for $s = r$ since the summation S in the right member of (11.15) vanishes.[29] This recurrence process enables one to say that (11.11) and (11.12) are satisfied for $s = 1, 2, \cdots$. Hence the power series expansions for the functions $\mathfrak{H}^i_j$ dominate the corresponding $h^i_{|j}$ series with the result that the latter converge.

[28] *loc. cit.* Jour. (6.5).

[29] *loc. cit.* Proc. Note I (4.9).

PART II.

THEORY OF CHARACTERISTIC SURFACES.

§ 12. Definitions of Characteristic Surfaces.

The point of departure in our study of characteristic surfaces will be the Existence Theorem in § 6 for the system (2.1). The regular form (2.8), which is equivalent to (2.1), shows that if the quantities φ_{i0} and $\partial \varphi_{i0}/\partial x^1$ are assigned over the surface $x^1 = p^1$ and the remaining functions φ_{ik} are assigned as indicated in the statement of the existence theorem, the solution $v_{ik}(x)$ is uniquely determined over $x^1 = p^1$. In a similar manner if the functions

$$\varphi_{ik},\ \frac{\partial \varphi_{ik}}{\partial x^{k+1}},\ \cdots,\ \frac{\partial \varphi_{ik}}{\partial x^{\alpha}} \qquad (k = 0, \cdots, \alpha - 1) \tag{12.1}$$

are assigned over the $n - \alpha$ dimensional surface $x^1 = p^1, \cdots, x^\alpha = p^\alpha$ and the remaining functions φ_{ik} are assigned as indicated in the statement of the existence theorem, the solution $v_{ik}(x)$ is uniquely determined over $x^1 = p^1, \cdots, x^\alpha = p^\alpha$. These remarks apply equally well to a system of the invariantive type.

Let $v_k(x)$ denote a solution of (2.1) and consider the system of equations

$$\sum_{\alpha=1}^{n} \sum_{k=1}^{w} a_{ik}^{*\alpha}(x) \frac{\partial v_k}{\partial x^\alpha} + c_i^*(x) = 0 \qquad (i = 1, \cdots, L), \tag{12.2}$$

where $a_{ik}^{*\alpha}$ and c_i^* are functions of the variables $x^1, \cdots, x^n$ obtained by substituting the integral $v_k(x)$ into the quantities $a_{ik}^{\alpha}(x, v)$ and $c_i(x, v)$ respectively. *A surface C_{n-1} having $x^1 = 0$ as its equation will be called an $n - 1$ dimensional characteristic surface for an integral $v_k(x)$ of the system* (2.1) *if it is impossible to find a coördinate system $(\bar{x})$ defined by the transformation*

$$x^1 = \bar{x}^1, \qquad x^i = f^i(\bar{x}^1, \cdots, \bar{x}^n) \qquad (i = 2, \cdots, n) \tag{12.3}$$

such that (12.2) *can be solved for L_1 derivatives of the set $\partial v^k/\partial \bar{x}^1$ at a point P on the surface C_{n-1}.* If the coefficients a_{ik}^{α} are functions of x^σ alone, the characteristic surfaces C_{n-1} are determined independently of the integrals v_k. An analogous definition and remark applies to the tensor equations (3.1); the same is true of the invariantive systems composed (a) of the equations (4.3), (4.5) and (4.7) for the affine case, (b) of the equations (4.4), (4.8), (4.10) and (4.11) for the metric case, and (c) of the equations (5.2), (5.3) and (5.5) for the vector case. It is to be noted, however, that under case (a) the equations (4.3) and (4.5), under case (b) the equations (4.4) and (4.8), and under case (c) the equations (5.2) and (5.3) are alone of significance in the determination of the characteristic surfaces.[30]

It follows that if the data is assigned over a characteristic surface C_{n-1} of the system (2.1) or (3.1) the general existence theorem will fail to apply.

If the equation of a characteristic surface C_{n-1} of the system (2.1) has the general form $\Phi(x^1, \cdots, x^n) = 0$ we can, by a transformation of coördinates[31]

$$\bar{x}^1 = \Phi(x^1, \cdots, x^n), \qquad \bar{x}^i = x^i \qquad (i = 2, \cdots, n) \tag{12.4}$$

reduce the equation of C_{n-1} to the form $\bar{x}^1 = 0$ with respect to the $(\bar{x})$ coördinate system. In this latter system of coördinates $\bar{x}^\alpha$ the equations (2.1) become

$$\sum_{\beta=1}^{n} \sum_{k=1}^{w} \bar{a}_{ik}^{\beta} \frac{\partial v_k}{\partial \bar{x}^\beta} + \bar{c}_i = 0, \qquad (i = 1, \cdots, L), \tag{12.5}$$

where

$$\bar{a}_{ik}^{\beta} = a_{ik}^{\alpha} \frac{\partial \bar{x}^\beta}{\partial x^\alpha}. \tag{12.6}$$

The fact that $\bar{x}^1 = 0$ is a characteristic surface implies that

$$\bar{W}_1 = 0, \cdots, \bar{W}_r = 0 \tag{12.7}$$

over $\bar{x}^1 = 0$, where the quantities $\bar{W}_i$ are the determinants of order L_1 in the matrix

$$\left\| \begin{matrix} \bar{a}_{11}^{*1} & \cdots & \bar{a}_{1w}^{*1} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \bar{a}_{L1}^{*1} & \cdots & \bar{a}_{Lw}^{*1} \end{matrix} \right\|, \tag{12.8}$$

in which the $\bar{a}_{ik}^{*\alpha}$ are obtained from the quantities $a_{ik}^{*\alpha}$ by the transformation (12.6). The property (12.7) will persist under coördinate transformations. In fact we have

$$\bar{a}_{ik}^{1} = a_{ik}^{\alpha} \Phi_\alpha \tag{12.9}$$

[30] Several definitions of a characteristic surface of a system of equations have appeared in the literature. Cf. Cartan, *loc. cit.*, (3); also N. M. Günther, *On the theory of characteristics of systems of equations with partial derivatives.* St. Petersburg (1913). The above definition has the advantage that it applies to invariantive systems as well as systems in which the unknowns are scalars.

[31] The statement that equations (12.4) define a transformation of coördinates implies that the jacobian determinant of these equations does not vanish identically, i. e. that $\partial \Phi / \partial x^1$ is not identically zero. If such were the case we could, by a relettering of the variables (x), cause the derivative $\partial \Phi / \partial x^1$ to be different from zero; then the system (2.1) or (3.1) could be referred to the new set of variables.

since the a_{ik}^{α} transform as the components of a contravariant vector in the index α and the derivatives Φ_α of the function Φ are the components of a covariant vector. Hence the corresponding terms in (12.8) and the transformed matrix

$$\left\|\begin{matrix} a_{11}^{*\alpha}\,\Phi_\alpha & \cdots & a_{1w}^{*\alpha}\,\Phi_\alpha \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{L1}^{*\alpha}\,\Phi_\alpha & \cdots & a_{Lw}^{*\alpha}\,\Phi_\alpha \end{matrix}\right\| \tag{12.10}$$

are equal at any point P in space; the determinants $W_i\,(i = 1, \cdots, r)$ in the transformed matrix (12.10) therefore vanish over $\Phi = 0$.

Conversely suppose we have a surface $\Phi = 0$ such that $W_1 = 0, \cdots, W_r = 0$ over $\Phi = 0$. Make the transformation (12.4). Then because $\overline{W}_1 = 0, \cdots, \overline{W}_r = 0$ over $\bar{x}^1 = 0$, it is impossible to solve the system (12.5) for L_1 derivatives $\partial v_k / \partial \bar{x}^1$. Under transformations of the type (12.3) namely

$$\bar{x}^1 = \tilde{x}^1, \; \bar{x}^i = f^i(\tilde{x}^1, \cdots, \tilde{x}^n) \qquad (i = 2, \cdots, n) \tag{12.11}$$

we have from (12.6) that $\bar{a}_{ik}^1 = \tilde{a}_{ik}^1$; hence no coördinate system $(\tilde{x})$ can be found in which the system (2.1) can be solved for L_1 derivatives $\partial v_k / \partial \tilde{x}^1$ at a point P on $\tilde{x}^1 = 0$, i. e. $\Phi = 0$ is an $(n-1)$-dimensional characteristic surface C_{n-1}. In other words *a necessary and sufficient condition for the surface* $\Phi = 0$ *to be an* $(n-1)$*-dimensional characteristic surface* C_{n-1} *for an integral* $v_k(x)$ *of the system* (2.1) *is that all the determinants* $W_i\,(i = 1, \cdots, r)$ *of order* L_1 *that can be formed from the matrix* (12.10) *vanish over* $\Phi = 0$.

The above discussion concerning the system (2.1) and its characteristic surface C_{n-1} is likewise applicable to the equations (5.2) and (5.3) of the vector invariantive system composed of (5.2), (5.3) and (5.5) and in fact a corresponding necessary and sufficient condition for the surface $\Phi = 0$ to be a characteristic surface C_{n-1} of the vector invariantive system (5.2), (5.3) and (5.5) can be stated. It is obvious that the quantities W_i whose vanishing constitutes the condition for the surface $\Phi = 0$ to be a characteristic surface C_{n-1} of the system (2.1) are differential parameters since in fact each of the elements of the matrix (12.10) is a differential parameter; an analogous remark applies to the vector invariantive system.[32]

[32] The observation of this fact is facilitated by noting that (5.2) has the particular form

$$\sum (h_{q,r}^{p})\, h_l^{\alpha} \frac{\partial h_{j,k}^{i}}{\partial x^{\alpha}} + \bigstar = 0$$

in which the coefficients $(h_{q,r}^{p})\, h_l^{\alpha}$ are linear and homogeneous in the components h_l^{α} and the index α is to be summed as indicated; hence the coefficients $(h_{q,r}^{p})\, h_l^{\alpha}$ have the contravariant vector transformation in the index α and so correspond exactly to the coefficients a_{ik}^{α} in (2.1).

A surface $C_{n-\alpha}$ having $x^1=0, \cdots, x^\alpha=0$ as its equations will be called an $(n-\alpha)$-dimensional characteristic surface of type β where $\beta=1, \cdots, \alpha$ for an integral v_k of the system (2.1) *if it is impossible to find a coördinate system $(\bar{x})$ defined by the transformation*

$$(12.12)\qquad x^i=\bar{x}^i \ (i=1, \cdots, \alpha); \quad x^j=f^j(\bar{x}^1, \cdots, \bar{x}^n) \ (j=\alpha+1, \cdots, n)$$

such that (12.2) *can be solved for*

$$\begin{array}{lll} L_1 & \text{derivatives:} & \dfrac{\partial v_k}{\partial \bar{x}^1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ L_{\beta-1} & \text{derivatives:} & \dfrac{\partial v_k}{\partial \bar{x}^{\beta-1}} \\ L_\beta & \text{derivatives:} & \dfrac{\partial v_k}{\partial \bar{x}^{\beta}} \end{array}$$

but possible to find a coördinate system $(\bar{x})$ defined by the above transformation such that (12.2) *can be solved for*

$$\begin{array}{lll} L_1 & \text{derivatives:} & \dfrac{\partial v_k}{\partial \bar{x}^1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ L_{\beta-1} & \text{derivatives:} & \dfrac{\partial v_k}{\partial \bar{x}^{\beta-1}} \end{array}$$

at a point P on the surface $C_{n-\alpha}$.

The characteristic surfaces $C_{n-\alpha}$ of the tensor equations (3.1) can be defined in a similar manner by replacing the above derivatives by the derivatives

$$\frac{\partial \bar{T}}{\partial \bar{x}^1}, \cdots, \frac{\partial \bar{T}}{\partial \bar{x}^\beta}$$

and the equations (12.2) by the corresponding system determined by (3.1). When the data is assigned over an $(n-\alpha)$-dimensional characteristic surface $C_{n-\alpha}$ the general existence theorem for the system (2.1) or (3.1) will fail to apply. Analogous definitions and remarks are to be understood to apply to the affine, metric and vector invariantive systems.

Now consider the matrix

$$(12.13)\qquad \left\| \begin{pmatrix} a^{*1}_{11} & \cdots & a^{*1}_{1w} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a^{*1}_{L1} & \cdots & a^{*1}_{Lw} \end{pmatrix} \cdots \begin{pmatrix} a^{*\alpha}_{11} & \cdots & a^{*\alpha}_{1w} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a^{*\alpha}_{L1} & \cdots & a^{*\alpha}_{Lw} \end{pmatrix} \right\|$$

which for brevity will be denoted by $\|a^{*\alpha}_{ik}\|$. We observe that the matrix $\|a^{*\alpha}_{ik}\|$ is composed of the α matrices in the parentheses in (12.13); the first of these matrices in parenthesis will be referred to as the *first sub-matrix* of (12.13), the second as the *second sub-matrix,* etc. Let us denote by Ξ_β any determinant of (12.13) of order

$$\Delta_\beta = \sum_{i=1}^{\beta} L_i,$$

where $\beta = 1, \cdots, \alpha$ that can be formed by selecting L_1 columns of the first sub-matrix, L_2 columns of the second sub-matrix, $\cdots$, L_β columns of the βth sub-matrix of (12.13); particular determinants Ξ_β will be denoted by Ξ_{β_1}, Ξ_{β_2}, etc. The condition that the equations $x^1 = 0, \cdots, x^\alpha = 0$ shall define an $(n-\alpha)$-dimensional characteristic surface $C_{n-\alpha}$ of type β for the system (2.1) is that there exists a determinant $\Xi_{\beta-1}$ which does not vanish but that all determinants Ξ_β shall vanish over this surface. More generally a *necessary and sufficient condition that* $\Phi^i = 0$ *where* $i = 1, \cdots, \alpha$ *should define an* $(n-\alpha)$*-dimensional characteristic surface* $C_{n-\alpha}$ *of type* β *is that there exists a determinant* $\Xi_{\beta-1}$ *in the matrix* $\|\overset{*\sigma}{a_{ik}}\, \Phi^\alpha_\sigma\|$ *which does not vanish where* Φ^μ_ν *denotes the partial derivatives of* Φ^μ $(\mu = 1, \cdots, \alpha)$ *while all determinants* Ξ_β *in this matrix vanish over the surface* $\Phi^i = 0$. This extends the previous condition for $n-1$ dimensional characteristic surfaces; an analogous condition can evidently be stated for the vector invariantive system composed of (5.2), (5.3), and (5.5).

Let us now consider an affine invariantive system (4.3), (4.5) and (4.7) or a metric invariantive system (4.4), (4.8), (4.10) and (4.11); let us in fact represent either the equations (4.3) and (4.5) or the equations (4.4) and (4.8) by writing

$$\sum_{a=1}^{n} \sum_{b=1}^{Q} I^a_{bc} \frac{\partial J_b}{\partial x^a} + * = 0, \tag{12.14}$$

where the coefficients I are functions of the components $A^\alpha_{\beta\gamma\delta}$ in the affine case and functions of the components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$ in the metric case; the quantities J_b where $b = 1, \cdots, Q$ are independent $A^\alpha_{\beta\gamma\delta}$ or $g_{\alpha\beta,\gamma\delta}$ in the affine or metric cases respectively.[33]

If $x^1 = 0, \cdots, x^\alpha = 0$ is an $n-\alpha$ dimensional characteristic surface $C_{n-\alpha}$ of type β where $\beta = 1, \cdots, \alpha$ of an affine or metric invariantive system, this implies that the determinants Ξ_β of order Δ_β in the matrix

$$\left\| \begin{pmatrix} I^1_{11} & \cdots & I^1_{Q1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ I^1_{1W} & \cdots & I^1_{QW} \end{pmatrix} \cdots \begin{pmatrix} I^\alpha_{11} & \cdots & I^\alpha_{Q1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ I^\alpha_{1W} & \cdots & I^\alpha_{QW} \end{pmatrix} \right\| \tag{12.15}$$

[33] For the values of Q in the affine and metric case, see *loc. cit.,* (11) p. 202 and p. 660.

in which W represents the number of equations (12.14), vanish over the surface $x^1 = 0, \cdots, x^\alpha = 0$ for an integral of the system (12.14) but that there exists a determinant $\Xi_{\beta-1}$ which does not vanish over this surface. Now let $U_1, \cdots, U_m$ denote a set of irreducible factors of the determinants Ξ_β in (12.15) such that the vanishing of the $U_1, \cdots, U_m$ implies the vanishing of all determinants Ξ_β. Make an arbitrary transformation of coördinates: $x^i = \Phi^i(\bar{x})$ and replace each component appearing in the U_k by its value in accordance with the tensor law of transformation. Thus the components $A^\alpha_{\beta\gamma\delta}$, $g_{\alpha\beta}$, $g_{\alpha\beta,\gamma\delta}$ will be replaced by expressions which are linear and homogeneous in $\bar{A}^\alpha_{\beta\gamma\delta}$, $\bar{g}_{\alpha\beta}$ and $\bar{g}_{\alpha\beta,\gamma\delta}$ respectively and rational in the derivatives $\partial\Phi^i/\partial\bar{x}^k$ of the coördinate transformation.[34] Hence we shall have

$$\text{Affine case: } U_k(A^\alpha_{\beta\gamma\delta}) = \Psi_k\left(\bar{A}^\alpha_{\beta\gamma\delta};\ \frac{\partial\Phi^1}{\partial\bar{x}^1}, \cdots, \frac{\partial\Phi^n}{\partial\bar{x}^n}\right),$$

$$\text{Metric case: } U_k(g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta}) = \Psi_k\left(\bar{g}_{\alpha\beta};\ \bar{g}_{\alpha\beta,\gamma\delta};\ \frac{\partial\Phi^1}{\partial\bar{x}^1}, \cdots, \frac{\partial\Phi^n}{\partial\bar{x}^n}\right).$$

It is obvious from their method of formations that the functions Ψ_k are differential parameters. If the U_k vanish over the surface $x^1 = 0, \cdots, x^\alpha = 0$, the parameters Ψ_k will vanish over the surface $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ and conversely. Hence, *a necessary and sufficient condition that the equations* $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ *define an* $n-\alpha$ *dimensional characteristic surface of type* β *for an integral* $\Gamma^\alpha_{\beta\gamma}$ *or* $g_{\alpha\beta}$ *of an affine or metric invariantive system respectively, is that* (1) *there exists a differential parameter* Ψ *corresponding to a determinant* $\Xi_{\beta-1}$ *in the matrix* (12.15) *which does not vanish over* $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ *and* (2) *that all the differential parameters* Ψ_k *vanish over the above surface.* It should be observed that in stating the above condition we suppose that the jacobian determinant of the functions $\Phi^1, \cdots, \Phi^n$ occurring in the differential parameters Ψ_k, does not vanish over the surface $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$; also it can be supposed that the functions Φ^i, etc. with which we are concerned in the above necessary and sufficient condition are referred to coördinates (x) instead of the coördinates $(\bar{x})$ as originally considered.

§ 13. Differential Equations of the Characteristic Surfaces.

The conditions obtained by equating to zero the differential parameters Ξ_β formed from the matrix $\| a^{*\sigma}_{ik}\,\Phi^\alpha_\sigma \|$, namely

[34] The denominators of these rational expressions in the derivatives $\partial\Phi^i/\partial\bar{x}^k$ will be the jacobian determinant of the above coördinate transformation and hence will not vanish identically.

$$\Xi_{\beta_1} = 0, \cdots, \Xi_{\beta_m} = 0 \tag{13.1}$$

over $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ constitute a system of first order partial differential equations for the determination of the α independent functions Φ^j. When the functions Φ^j are so determined, the equations $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ define an $n-\alpha$ dimensional characteristic surface $C_{n-\alpha}$ of type β for the integrals $v_k(x)$ of the system (2.1) provided that there exists an expression $\Xi_{\beta-1}$ which does not vanish over the above surface. Analogous remarks apply to the case of the vector invariantive system composed of (5.2), (5.3), and (5.5). Similarly the equations of type

$$\Psi_1 = 0, \cdots, \Psi_m = 0 \tag{13.2}$$

over $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ determine the $n-\alpha$ dimensional characteristic surfaces $C_{n-\alpha}$ of type β of the affine and metric invariantive systems; in case the above equations (13.2) do not possess the required solution, the surfaces $C_{n-\alpha}$ will fail to exist.

The above characteristic surfaces $C_{n-\alpha}$ will have a unique determination in space if and only if the coefficients $\overset{\alpha}{a}_{ik}$ in (2.1) or the coefficients I^a_{bc} in (12.13) are independent of the integrals of the corresponding systems.

If the functions $\Phi^j(x)$ satisfy the partial differential equations (13.1) over the entire x-space, then the equations

$$\Phi^1 = c^1, \cdots, \Phi^\alpha = c^\alpha$$

in which the c's are arbitrary constants within suitable limits, define a family of characteristic surfaces $C_{n-\alpha}$ of type β. A similar remark can be made for the equations (13.2).

§ 14. Sets of Monomials.

The theory of monomials here presented contains as a special case the theory given by Janet[6] in an exposition of Riquier's work. Only properties of monomials will be considered which are essential in the discussion of existence theorems yet to be treated.

By a monomial is meant the product $x_1^{a_1} \cdots x_n^{a_n}$ in which the exponents $a_1, \cdots, a_n$ are positive integers or zero.[35] As the basis of the following discussion we shall consider a finite set (M) of monomials; the monomials of (M) will be supposed to be distinct. Any particular monomial of the set (M) will be denoted by $\bar{M}$, $\bar{\bar{M}}$, etc. and an analogous notation will be used for the monomials of other sets which will be introduced in the following discussion.

[35] Throughout this section and also throughout §§ 15–18 the notation $x_1, \cdots, x_n$ involving *subscripts* will be used for the coördinates.

The above set (M) will be divided into two mutually exclusive sets (L) and (K). The following definitions apply to the sets (L) and (K) respectively.

THE SET (L)

(a) *Multiple.* The monomial $\bar{L} : x_1^{a_1} \cdots x_n^{a_n}$ is a multiple of $\bar{\bar{L}} : x_1^{b_1} \cdots x_n^{b_n}$ if none of the differences

(14.1) $$(a_1 - b_1), \cdots, (a_n - b_n)$$

is negative.

(b) *Multiplier.* The variable x_1 is a multiplier of $\bar{L}$ if the degree of x_1 in $\bar{L}$ is equal to the maximum of the degrees of x_1 in monomials of (M). Similarly $x_i \, (i = 2, \cdots, n)$ is a multiplier of

$$\bar{L}: \quad x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} \; x_i^{\sigma_i} \cdots x_n^{\sigma_n}$$

if σ_i is equal to the maximum of all degrees of the variable x_i in all monomials of (M) having degrees in $x_1, \cdots, x_{i-1}$ equal respectively to the particular values $\sigma_1, \cdots, \sigma_{i-1}$ in $\bar{L}$.

(c) *Non-multiplier.* If x_i is not a multiplier of a monomial $\bar{L}$ it will be called a non-multiplier of $\bar{L}$.

THE SET (K)

Let α have any one of the values $0, 1, \cdots, n-1$; when the value of α has been selected it will be held fixed throughout the discussion of this section.

(a) *Multiple with respect to α.* The monomial $\bar{K}$: $x_1^{a_1} \cdots x_n^{a_n}$ is a multiple with respect to α of $\bar{\bar{K}}$: $x_1^{b_1} \cdots x_n^{b_n}$ if the first α of the differences (14.1) are zero and none of the remaining is negative.

(b) *Multiplier with respect to α.* The variable $x_i \, (i = \alpha + 1, \cdots, n; \; \alpha \geqq 1)$ is a multiplier with respect to α of

$$\bar{K}: \quad x_1^{\tau_1} \cdots x_{i-1}^{\tau_{i-1}} \; x_i^{\tau_i} \cdots x_n^{\tau_n}$$

if τ_i is equal to the maximum of all degrees of the variable x_i in all monomials of (M) having degrees in $x_1, \cdots, x_{i-1}$ equal respectively to the particular values $\tau_1, \cdots, \tau_{i-1}$ in $\bar{K}$. If $\alpha = 0$, the multipliers of $\bar{K}$ are determined as though this monomial were a monomial $\bar{L}$.

(c) *Non-multiplier with respect to α.* If $x_i \, (i = \alpha + 1, \cdots, n)$ is not a multiplier of a monomial $\bar{K}$ it will be called a non-multiplier of $\bar{K}$.

(d) *Variables $x_1, \cdots, x_\alpha$.* The variables $x_1, \cdots, x_\alpha$ will be neither multipliers nor non-multipliers with respect to α of any monomial $\bar{K}$.

For brevity in the following discussion the above phrase "with respect to α" will be omitted since it is to be understood that the value of α, when once selected is to be held fixed throughout this section. It should be noted that if $\alpha = 0$, the above definitions regarding the set (K) become identical with the corresponding definitions for the set (L).

The following example will serve to illustrate the above definitions. Let $\alpha = 2$.

	Monomials	Multipliers	Non-multipliers
	$x^2\ y^3\ z^2$	$y\ z$	$x\ \cdot\ \cdot$
(L)	$x^2\ y^2\ z^2$	$\cdot\ z$	$x\ y\ \cdot$
	$x^2\ y\ z^2$	$\cdot\ z$	$x\ y\ \cdot$
	$x\ y\ z^2$	$y\ z$	$x\ \cdot\ \cdot$
(K)	x^3	$\cdot\ z$	$\cdot\ \cdot\ \cdot$
	$x\ y$	$\cdot\ \cdot$	$\cdot\ \cdot\ z$

We shall now define $n+1$ different sets of monomials $(N_1), (N_2), \cdots, (N_n)$, (N_k) which together will comprise a set (N) of monomials. *The set* (N) *will be said to be complementary to the set* (M). Corresponding to the previous notation we will denote a monomial of (N) by $\overline{N}$, $\overline{\overline{N}}$, etc., a monomial of (N_1) will be denoted by $\overline{N}_1$, $\overline{\overline{N}}_1$, etc. etc.

The Set (N)

(a) *The set* (N_1) is composed of all monomials x_1^β where β is a positive integer or zero which is less than the maximum of the exponents of x_1 in (M) and which does not appear among the exponents of x_1 in (M).

(b) *The set* (N_i) where $i = 2, \cdots, n$ is composed of all monomials that can be formed from

$$x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}}\, x_i^\beta$$

by taking $\sigma_1, \cdots, \sigma_{i-1}$ as any system of exponents of $x_1, \cdots, x_{i-1}$ respectively of a monomial of (M) and β as a positive integer or zero such that (1) β is less than the maximum exponent of x_i of monomials of (M) of the form

$$x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} \cdots$$

and (2) β does not appear among the exponents of x_i in these monomials of (M).

(c) *The set* (N_k). Consider a monomial $\overline{K}$: $x_1^{a_1} \cdots x_n^{a_n}$ which would have at least one of the variables $x_1, \cdots, x_\alpha$ as a multiplier if all of the monomials of (M) were assumed to belong to the set (L). Let $x_{\nu_1}, \cdots, x_{\nu_r}$ where $\nu_i \leqq \alpha$ and $\alpha = 1, \cdots, n-1$ be the hypothetical multipliers of this monomial $\overline{K}$ and form the r $(\leqq \alpha)$ monomials

$$x_1^{a_1} \cdots x_{\nu_i}^{a_{\nu_i}+1} \cdots x_n^{a_n} \qquad (i = 1, \cdots, r).$$

The totality of these latter monomials constitutes the set (N_k). For $\alpha = 0$, the set (N_k) is not defined.

(d) *Multipliers of* $\bar{N}_1$. The variables $x_2, \cdots, x_n$ are multipliers for any monomial $\bar{N}_1$.

(e) *Multipliers of* $\bar{N}_i$ *where* $i = 2, \cdots, n$. The multipliers of a monomial $\bar{N}_i$ having the quantities $\sigma_1 \cdots \sigma_{i-1}$ as exponents of the variables $x_1, \cdots, x_{i-1}$ respectively are (1) the variables $x_{i+1}, \cdots, x_n$ and (2) those variables of the set $x_1, \cdots, x_{i-1}$ which would be multipliers of a monomial of (M) of the form

$$x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} \cdots$$

if all the monomials of (M) were considered to belong to the set (L).

(f) *Multipliers of* $\bar{N}_k$. The multipliers of any monomial

$$\bar{N}_k: \quad x_1^{a_1} \cdots x_{\nu_i}^{a_{\nu_i}+1} \cdots x_n^{a_n}$$

are the multipliers of the corresponding monomial $\bar{K}$: $x_1^{a_1} \cdots x_n^{a_n}$ when all monomials of (M) are considered to belong to the set (L) with the exception of $x_{\nu_1}, \cdots, x_{\nu_{i-1}}$.

Let us find the complementary set to the set (M) treated in the above example.

	Complementary sets			Multipliers		
(N_1)	1			·	y	z
(N_2)	x			·	·	z
	x^2			·	·	z
(N_3)	x	y	z	·	y	·
	x^2	y^3	z	·	y	·
	x^2	y^3		·	y	·
	x^2	y^2	z	·	·	·
	x^2	y^2		·	·	·
	x^2	y	z	·	·	·
	x^2	y		·	·	·
(N_K)	x^4			x	y	z
	x^3	y		·	y	z
	x	y^2		·	y	·

The class $(\bar{\mathfrak{M}})$ *of a monomial* $\bar{M}$ *is the set of all monomials which can be obtained by forming the product of* $\bar{M}$ *and an arbitrary monomial in its multipliers.* It is to be observed that the class $(\bar{\mathfrak{M}})$ as so defined, contains the monomial $\bar{M}$ in case $\bar{M}$ possesses a multiplier and that in this case the class $(\bar{\mathfrak{M}})$ is composed of infinitely many monomials. *If* $\bar{M}$ *possesses no multipliers it will be assumed that* $(\bar{\mathfrak{M}})$ *consists of the single monomial* $\bar{M}$. Analogous definitions and remarks apply to the classes $(\bar{\mathfrak{N}}_i)$ where $i = 1, \cdots, n$ and $(\bar{\mathfrak{N}}_k)$ corresponding to a monomial $\bar{N}_i$ and $\bar{N}_k$ respectively.

We will say that a monomial of the class $(\bar{\mathfrak{M}})$, $(\bar{\mathfrak{N}}_i)$ or $(\bar{\mathfrak{N}}_k)$ arises from the monomial $\bar{M}$, $\bar{N}_i$ or $\bar{N}_k$ as basis. To designate a particular monomial of the class $(\bar{\mathfrak{M}})$ we will use the symbol $\bar{\mathfrak{M}}$, etc. The set of all classes $(\bar{\mathfrak{M}})$ will be denoted by $(\mathfrak{M})$; similarly the set of all classes $(\bar{\mathfrak{N}}_i)$ and $(\bar{\mathfrak{N}}_k)$ will be denoted by $(\mathfrak{N})$.

THEOREM I. *An arbitrary monomial P in $x_1, \cdots, x_n$ belongs to one and only one class $(\bar{\mathfrak{M}})$ or $(\bar{\mathfrak{N}})$.*

In proving this theorem we shall assume $\alpha \neq 0$; this is always permissable since by a change of notation a set (M) for which $\alpha = 0$ can be replaced by a set $(\tilde{M})$ for which the set $(\tilde{K})$ is vacuous. In case $n = 1$ the theorem follows immediately: all monomials in (M) then belong to the set (L) and any monomial P, i. e. x_1^a either belongs to (M) or (N) or is a multiple of x_1^b where $b (\geqq 0)$ is the highest power of x_1 in (M). Hence we shall assume the theorem for $n-1$ variables and prove it for n variables $(n \geqq 2)$. Let $e_1 < e_2 < \cdots < e_m$ be the exponents of x_1 in (M) and denote by $\lambda (\geqq 1)$ the degree of x_1 in P. The induction proof will be divided into the following cases.

Case 1. *λ is less than e_m and different from e_i $(i < m)$.* For this case P cannot belong to a class $(\bar{\mathfrak{L}})$ since x_1 is a multiplier only for those monomials of (L) for which the degree in x_1 is e_m; P cannot belong to a class $(\bar{\mathfrak{K}})$ since x_1 is not a multiplier of any monomial $\bar{K}$; P cannot belong to a class $(\bar{\mathfrak{N}}_k)$ since x_1 is a multiplier only for those monomials of the set (N_K) for which the degree in x_1 is e_m; likewise P cannot belong to a class $(\bar{\mathfrak{N}}_i)$ where $i = 2, \cdots, n$. Finally it is evident that P belongs to one and only one class $(\bar{\mathfrak{N}}_1)$.

Case 2. *λ is equal to e_h for $h < m$.* Here P can obviously arise only from some monomial $\bar{M}^\lambda$ or $\bar{N}^\lambda$ in (M) or (N) respectively for which x_1 has the exponent λ. Dividing a monomial $\bar{M}^\lambda$ or $\bar{N}^\lambda$ by x_1^λ let us denote the resulting monomial in the variables $x_2, \cdots, x_n$ by $\bar{M}^{*\lambda}$ or $\bar{N}^{*\lambda}$ respectively. The monomials $\bar{L}^{*\lambda}$ and $\bar{K}^{*\lambda}$ belong to the sets $(L^{*\lambda})$ and $(K^{*\lambda})$ of $(M^{*\lambda})$ for which $\alpha^* = \alpha - 1$ will be assumed. If $\alpha^* = 0$, $(M^{*\lambda})$ will be considered as a set $(\tilde{M})$ for which $(\tilde{K})$ is vacuous in accordance with the previous procedure; however the designations $\bar{L}^{*\lambda}$ and $\bar{K}^{*\lambda}$ will continue to be employed as indicative of the origin of these monomials from monomials $\bar{L}^\lambda$ and $\bar{K}^\lambda$ respectively. Then observe that the monomials $\bar{N}^{*\lambda}$ are complementary to the monomials of the set $(M^{*\lambda})$; also that the multipliers of a monomial $\bar{M}^{*\lambda}$ in the set $(M^{*\lambda})$ are the same as the multipliers of the corresponding monomial $\bar{M}^\lambda$ in (M); likewise that the multipliers of a monomial $\bar{N}^{*\lambda}$ in the set $(N^{*\lambda})$ are the same as the multipliers of the corresponding monomial $\bar{N}^\lambda$ in the set (N).

Now the monomial P/x_1^λ in the variables $x_2, \cdots, x_n$ arises from one and only one monomial $\bar{M}^{*\lambda}$ or $\bar{N}^{*\lambda}$ as basis since the above theorem is assumed for $n-1$ variables; hence P arises from the corresponding monomial $\bar{M}^\lambda$ or $\bar{N}^\lambda$ as basis, i. e. P belongs to the class $(\bar{\mathfrak{M}}^\lambda)$ or $(\bar{\mathfrak{N}}^\lambda)$. If P belongs to a class $(\bar{\mathfrak{M}}^\lambda)$ or $(\bar{\mathfrak{N}}^\lambda)$ the monomial P/x_1^λ belongs to the corresponding class $(\bar{\mathfrak{M}}^{*\lambda})$ or $(\bar{\mathfrak{N}}^{*\lambda})$ respectively; hence if P belonged to more than one class $(\bar{\mathfrak{M}}^\lambda)$ or $(\bar{\mathfrak{N}}^\lambda)$ the monomial P/x_1^λ would belong to more than one class $(\bar{\mathfrak{M}}^{*\lambda})$ or $(\bar{\mathfrak{N}}^{*\lambda})$ contrary to hypothesis.

Case 3. *λ is equal to or greater than e_m.* In this case the monomial P can arise only from a monomial $\bar{M}$ or $\bar{N}$ in which the exponent of x_1 is e_m or e_m+1. Just as in Case 2 let us consider the sets (M^{*e_m}) and (N^{*e_m}) obtained by dividing the monomials $\bar{M}^{e_m}$ and $\bar{N}^{e_m}$ by $x_1^{e_m}$ respectively. Then observe that the multipliers of the monomial $\bar{M}^{*e_m}$ in the set (M^{*e_m}) are the same as those for the corresponding monomial $\bar{M}^{e_m}$ in (M) with the exception of x_1 if $\bar{M}^{e_m}$ is a monomial $\bar{L}^{e_m}$; for the monomial $\bar{K}^{e_m}$ the variable x_1 does not occur as a multiplier. Also the monomials $\bar{N}^{*e_m}$ are complementary to the monomials of the set (M^{*e_m}); the multipliers of a monomial $\bar{N}^{*e_m}$ in the set (N^{*e_m}) are the same as those for the corresponding $\bar{N}^{e_m}$ in the set (N) with the exception of x_1. Now the monomial P/x_1^λ arises from one and only one monomial $\bar{M}^{*e_m}$ or $\bar{N}^{*e_m}$ as basis since the theorem is assumed for $n-1$ variables. If P/x_1^λ arises from a monomial $\bar{L}^{*e_m}$ or $\bar{N}_i^{*e_m}$, the monomial P can arise from the corresponding monomial $\bar{L}^{e_m}$ or $\bar{N}_i^{e_m}$. If P/x_1^λ arises from a monomial $\bar{K}^{*e_m}$ the monomial P arises from the corresponding $\bar{K}^{e_m}$ if $\lambda = e_m$ and from the monomial $\bar{N}_K^{e_m+1}$ as basis if $\lambda > e_m$; if P/x_1^λ arises from a monomial $\bar{N}_K^{*e_m}$, the monomial P arises from the corresponding monomial $\bar{N}_K^{e_m}$ if $\lambda = e_m$ and from the monomial $\bar{N}_K^{e_m+1}$ as basis if $\lambda > e_m$. Conversely suppose P belongs to any two of the classes $(\bar{\mathfrak{L}}^{e_m})$, $(\bar{\mathfrak{N}}_i^{e_m})$, $(\bar{\mathfrak{K}}^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m+1})$ with the exception of two classes $(\bar{\mathfrak{K}}^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m+1})$ of the monomials $\bar{K}^{e_m}$, $\bar{N}_K^{e_m}$, and $\bar{N}_K^{e_m+1}$ respectively where $\bar{N}_K^{e_m}$ and $\bar{N}_K^{e_m+1}$ correspond to the monomial $\bar{K}^{e_m}$, then P/x_1^λ belongs to two classes $(\bar{\mathfrak{M}}^{*e_m})$ and $(\bar{\mathfrak{N}}^{*e_m})$, contrary to hypothesis. It is easily seen that P cannot belong to two of the excluded classes $(\bar{\mathfrak{K}}^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m})$ and $(\bar{\mathfrak{N}}_K^{e_m+1})$.

The set (M) is said to be complete if any monomial P which is a multiple of at least one monomial of the set (M) belongs to the class $(\bar{\mathfrak{M}})$. It follows from Theorem I that the multiples of a complete set (M) are separated into a finite number of mutually exclusive classes, namely the classes $(\bar{\mathfrak{M}})$; monomials which are not multiples of any $\bar{M}$ are likewise separated into a finite number of mutually exclusive classes, i. e. the classes $(\bar{\mathfrak{N}})$.

The set (M) *is said to be strongly complete if any monomial* P *which is a multiple of at least one monomial of the set* (L) *belongs to the class* $(\mathfrak{L})$ *and any monomial* Q *which is a multiple of at least one monomial of the set* (K) *belongs to the class* $(\mathfrak{M})$.[36] A set which is strongly complete is also complete; the definition of strong completeness is introduced since Theorem II below gives a convenient test for strong completeness while we do not have such a test for completeness.

THEOREM II. *A necessary and sufficient condition that a set* (M) *be strongly complete is that the product of any monomial* $\bar{L}$ *by one of its non-multipliers belongs to the class* $(\mathfrak{L})$ *and the product of any* $\bar{K}$ *by one of its non-multipliers belongs to the class* $(\mathfrak{M})$.

The necessity of the condition in the theorem follows immediately. The product of any monomial $\bar{L}$ by one of its non-multipliers is a multiple of $\bar{L}$ and hence belongs to the class $(\mathfrak{L})$. Likewise the product of any monomial $\bar{K}$ by one of its non-multipliers is a multiple of $\bar{K}$ and hence belongs to the class $(\mathfrak{M})$.

Let us prove the sufficiency condition for the case $n=1$; this implies $\alpha=0$. Suppose that $e_1<e_2<\cdots<e_m$ are the exponents of x_1 in the set (L) and $d_1<d_2<\cdots<d_q$ are the exponents of x_1 in the set (K). Then the class $(\mathfrak{M})$ consists of the monomials

$$x_1^{d_1}, \cdots, x_1^{d_q};\ \ x_1^{e_1}, \cdots, x_1^{e_m};\ \ x_1^{a+1}, x_1^{a+2}, \cdots \tag{14.2}$$

where a is the maximum exponent in (M). But the monomials

$$x_1^{e_1+1}, \cdots, x_1^{e_{m-1}+1}$$

belong to the class $(\mathfrak{L})$ by hypothesis. Hence $e_1+1=e_2, \cdots, e_{m-1}+1=e_m$ and the exponents $e_1, \cdots, e_m$ are consecutive integers. Suppose $e_m \neq a$, then x_1 is a non-multiplier for $x_1^{e_m}$ and so $x_1^{e_m+1}$ belongs to the class $(\mathfrak{L})$ by hypothesis. However upon examining (14.2) we find that if $e_m \neq a$ the monomial $x_1^{e_m+1}$ could only belong to the class $(\mathfrak{K})$ or $(\mathfrak{N}_1)$ and we have a contradiction. Hence $e_m=a$. By hypothesis $x_1^{d_q+1}$ then belongs to a class $(\overline{\mathfrak{M}})$ so that $d_q+1=e_1$ and $d_1, \cdots, d_q$ are also consecutive integers. A set of monomials of this type is easily seen to be strongly complete.

In order to prove the sufficiency condition for $n>1$ we assume that every product of an $\bar{L}$ by any one of its non-multipliers belongs to the class $(\mathfrak{L})$ and that every product of a $\bar{K}$ by any one of its non-multipliers belongs to the class $(\mathfrak{M})$; then we show that (1) the product of every monomial $\bar{L}$ by an arbitrary monomial in $x_1, \cdots, x_n$ belongs to the

[36] When $\alpha=0$, this definition introduces a distinction between the sets (L) and (K).

class $(\mathfrak{L})$ and (2) the product of every monomial $\bar{K}$ by an arbitrary monomial in $x_{\alpha+1}, \cdots, x_n$ belongs to the class $(\mathfrak{M})$. Let us assume the theorem for $n-1$ variables and prove it for n variables.

Again denote by $\bar{M}^{*\lambda}$ the quotient of $\bar{M}^{\lambda}$ by x_1^{λ}. The monomials $\bar{L}^{*\lambda}$ and $\bar{K}^{*\lambda}$ will belong to the sets $(L^{*\lambda})$ and $(K^{*\lambda})$ with $\alpha^* = \alpha - 1$ for $\alpha \geqq 1$ and $\alpha^* = 0$ for $\alpha = 0$. Then those variables $x_2, \cdots, x_n$ which are multipliers or non-multipliers for $\bar{M}^{\lambda}$ in the set (M) are precisely the multipliers or non-multipliers respectively for $\bar{M}^{*\lambda}$ in $(M^{*\lambda})$. Hence the product of any monomial $\bar{\mathfrak{L}}^{*\lambda}$ of the class arising from $\bar{L}^{*\lambda}$ in the set $(M^{*\lambda})$ and x_1^{λ} belongs to the class $(\bar{\mathfrak{L}}^{\lambda})$. Similarly the product of any monomial $\bar{\mathfrak{K}}^{*\lambda}$ of the class arising from $\bar{K}^{*\lambda}$ in the set $(M^{*\lambda})$ and x_1^{λ} belongs to the class $(\bar{\mathfrak{K}}^{\lambda})$. By assumption the product of every $\bar{L}^{\lambda}$ by one of its non-multipliers selected from $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{L})$; this product $x_\beta \bar{L}^{\lambda}$ is of degree λ in x_1 and hence must belong to a class $(\bar{\bar{\mathfrak{L}}}^{\lambda})$, i. e. the class arising from a monomial $\bar{\bar{L}}^{\lambda}$ in (M). Hence

$$x_1^{\lambda} x_\beta \bar{L}^{*\lambda} = x_1^{\lambda} p \bar{\bar{L}}^{*\lambda},$$

where p is a monomial in multipliers of $\bar{\bar{L}}^{*\lambda}$ in the set $(M^{*\lambda})$. That is (1) the product of a monomial $\bar{L}^{*\lambda}$ by a non-multiplier of $\bar{L}^{*\lambda}$ in the set $(M^{*\lambda})$ belongs to the class $(\bar{\bar{\mathfrak{L}}}^{*\lambda})$ of a monomial $\bar{\bar{L}}^{*\lambda}$ in $(M^{*\lambda})$. Similarly the product of $\bar{K}^{\lambda}$ by one of its non-multipliers x_β belongs by assumption to the class $(\bar{\bar{\mathfrak{M}}}^{\lambda})$. Hence

$$x_1^{\lambda} x_\beta \bar{K}^{*\lambda} = x_1^{\lambda} p \bar{\bar{M}}^{*\lambda},$$

where p is a monomial in multipliers of $\bar{\bar{M}}^{*\lambda}$ in the set $(M^{*\lambda})$. That is (2) the product of a monomial $\bar{K}^{*\lambda}$ by a non-multiplier of $\bar{K}^{*\lambda}$ in the set $(M^{*\lambda})$ belongs to the class $(\bar{\bar{\mathfrak{M}}}^{*\lambda})$ of a monomial $\bar{\bar{M}}^{*\lambda}$ in $(M^{*\lambda})$. By assumption for $n-1$ variables it follows therefore that the set $(M^{*\lambda})$ is complete on account of statements (1) and (2). Hence we have the strongly product of a monomial $\bar{L}^{*\lambda}$ by an arbitrary monomial in $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{L}^{*\lambda})$; that the product of a monomial $\bar{K}^{*\lambda}$, with $\alpha = 0$, by an arbitrary monomial $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{M}^{*\lambda})$; that the product of a monomial $\bar{K}^{*\lambda}$ with $\alpha \geqq 1$ by an arbitrary monomial in $x_{\alpha+1}, \cdots, x_n$ belongs also to the class $(\mathfrak{M}^{*\lambda})$. Thus we have proved that for $\alpha \geqq 1$ a multiple of a monomial $\bar{K}$ belongs to the class $(\mathfrak{M})$, that the product of a monomial $\bar{K}$, with $\alpha = 0$, by an arbitrary monomial in $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{M})$, and that the product of a monomial $\bar{L}$ by an arbitrary monomial in $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{L})$. It remains to prove that the products of a monomial $\bar{L}$ and a monomial $\bar{K}$, with

$\alpha = 0$, by an arbitrary monomial of degree $\varrho\,(\geqq 1)$ in x_1 belong to the class ($\mathfrak{L}$) and the class ($\mathfrak{M}$) respectively.

Now assume that the product of a monomial $\bar{L}$ by an arbitrary monomial of degree $\varrho\,(\geqq 0)$ in x_1 belongs to the class ($\mathfrak{L}$) and let us seek to prove the property for $\varrho + 1$. By this hypothesis the product of a monomial $\bar{L}$ by an arbitrary monomial of degree $\varrho + 1$ in x_1 can be considered as the product of x_1 and a particular monomial $\bar{\bar{\mathfrak{L}}}$. If the degree of $\bar{\bar{\mathfrak{L}}}$ is less than e_m the monomial $\bar{\bar{\mathfrak{L}}}$ belongs to a class arising from a monomial $\bar{\bar{L}}$ for which x_1 is not a multiplier; $x_1\,\bar{\bar{L}}$ is therefore at most of degree e_m in x_1. Hence $x_1\,\bar{\bar{\mathfrak{L}}}$ is the product of $x_1\,\bar{\bar{L}}$ and a monomial in $x_2, \cdots, x_n$. But $x_1\,\bar{\bar{L}}$ belongs to the class ($\mathfrak{L}$) by hypothesis since x_1 is a non-multiplier of $\bar{\bar{L}}$. Hence $x_1\,\bar{\bar{L}}$ is the product of a monomial $\bar{\bar{\bar{L}}}$ and a monomial in $x_2, \cdots, x_n$. By the above paragraph the monomial $x_1\,\bar{\bar{\mathfrak{L}}}$ therefore belongs to the class ($\mathfrak{L}$). If the degree of $\bar{\bar{\mathfrak{L}}}$ is equal to or greater than e_m the monomial $\bar{\bar{\mathfrak{L}}}$ belongs to the class ($\bar{\bar{\mathfrak{L}}}$) arising from a monomial $\bar{\bar{L}}$ for which x_1 is a multiplier and hence $x_1\,\bar{\bar{\mathfrak{L}}}$ belongs to the same class ($\bar{\bar{\mathfrak{L}}}$). In other words we have proved that the product of a monomial $\bar{L}$ by an arbitrary monomial belongs to a class ($\bar{\bar{\mathfrak{L}}}$). The argument used here can be applied without modification to show that the product of a monomial $\bar{K}$, with $\alpha = 0$, by an arbitrary monomial belongs to the class ($\mathfrak{M}$) and the theorem is proved.

Upon examination of the set (M) in the above example, one finds that the monomial $x^3\,y^3\,z^2$ does not belong to any class ($\mathfrak{M}$) and hence the set (M) is not complete.

The following example exhibits a set (M) with $\alpha = 1$ which is complete but not strongly complete.

	Monomials			Multipliers			Non-multipliers	
(L)	x^2	y^2	z	x	y	z	$\cdot$	$\cdot$
	x^2	y	z	x	$\cdot$	z	$\cdot$	y
	x	y	z	$\cdot$	$\cdot$	z	x	y
(K)	x	y^2		$\cdot$	y	z	$\cdot$	$\cdot$

It is evident by inspection that the above set (M) is complete. The requirement that the product of $\bar{L}$ by a non-multiplier belong to ($\mathfrak{L}$) would mean that the monomial $x\,y^2\,z$ would belong to the class ($\mathfrak{L}$); however the monomial $x\,y^2\,z$ belongs to the class ($\mathfrak{K}$).

The set (M) *will be said to be normal if no monomial* $\bar{K}$ *when considered as an* $\bar{L}$, *is a multiple of any monomial* $\bar{\bar{L}}$ *in the set* (L). In the applications of the theory of monomials we shall have occasion to deal with normal sets (M).

We are now going to give a procedure for obtaining a strongly complete set (M_ν) from an arbitrary normal set (M_1) such that the multiples of the monomials of (M_ν) and (M_1) are identical. Those products of a monomial $\bar{L}_1$ by one of its non-multipliers which do not belong to the class $(\mathfrak{L}_1)$ will be added to the set (L_1) to form the set (M_2). On account of the fact that (M_1) is normal the set (M_2) will be composed of distinct monomials, etc. We must show that after a finite number of such operations A we will arrive at a set (M_μ) after which no new monomials will be added by this process. Then form the product of a monomial $\overline{K}_\mu$ with one of its non-multipliers and if this product does not belong to the class $(\mathfrak{M}_\mu)$ add the monomial to (K_μ) to form $(M_{\mu+1})$. Let this operation be denoted by B.

Consider the numbers $\sigma_1, \cdots, \sigma_n$ equal respectively to the maximum exponents of the corresponding variables $x_1, \cdots, x_n$ in the set (M_1). The operation A cannot add a monomial P: $x_1^{\tau_1} \cdots x_n^{\tau_n}$ with $\tau_\beta > \sigma_\beta$ for any value of $\beta\ (= 1, \cdots, n)$. For suppose such were the case; let x_γ be the variable $x_1, \cdots, x_n$ for which $\tau_\gamma = \sigma_\gamma + 1$; then we have

$$P \equiv x_\gamma \overline{L}_1 \equiv x_1^{\tau_1} \cdots x_{\gamma-1}^{\tau_{\gamma-1}} x_\gamma^{\sigma_\gamma+1} \cdots,$$

where x_γ is a non-multiplier of $\overline{L}_1$. But x_γ is not a non-multiplier of $\overline{L}_1$ since the exponent of x_γ in $\overline{L}_1$ is equal to the maximum of the exponents of x_γ in (M_1). Since there are only a finite number of monomials P with $\tau_i \leqq \sigma_i\ (i = 1, \cdots, n)$ we must after a finite number of operations A arrive at a set (M_μ) after which the operations A will cease to add new monomials. Similar remarks apply to the operation B.

Now consider the set (M_μ) formed from (M_1) by operations A, the set (M_μ) being such that no new monomials can be added by further operations A. It is evident that the set (M_μ) is normal. Let us show that the property of normality is likewise preserved when we make an operation B upon the set (M_μ). Suppose that the monomial

$$\overline{K}_\mu:\quad x_1^{a_1} \cdots x_\sigma^{a_\sigma - 1} \cdots x_n^{a_n}$$

possesses the non-multiplier x_σ and that the monomial

$$\overline{K}_{\mu+1}:\quad x_1^{a_1} \cdots x_\sigma^{a_\sigma} \cdots x_n^{a_n}$$

is added to the set (M_μ) by the operation B. Let us assume that $\overline{K}_{\mu+1}$ is a multiple of a monomial $\overline{L}_\mu$ when $\overline{K}_{\mu+1}$ is considered as belonging to the set (L_μ), i. e. that the set $(M_{\mu+1})$ is not normal. Hence we have

$$\overline{L}_\mu \equiv x_1^{b_1} \cdots x_n^{b_n} \qquad (b_j \leqq a_j).$$

If $b_1 < a_1$ it follows that x_1 is a non-multiplier for $\overline{L}_\mu$. Since operation A can add no new monomials to the set (L_μ) the monomial $x_1 \overline{L}_\mu$ must belong to the class $(\overline{\overline{\mathfrak{L}}}_\mu)$ with the basis

$$\overline{\overline{L}}_\mu:\quad x_1^{b_1+1} x_2^{c_2} \cdots x_n^{c_n} \qquad \begin{pmatrix} b_1 + 1 \leqq a_1 \\ c_j \leqq b_j \leqq a_j \\ j = 2, \cdots, n \end{pmatrix}.$$

If $b_1+1<a_1$, we repeat the process until we have finally that the monomial $\overline{K}_{\mu+1}$ is a multiple of

$$\tilde{L}_\mu:\quad x_1^{a_1}x_2^{d_2}\cdots x_n^{d_n}\qquad (d_j\leqq a_j;\ j=2,\cdots,n).$$

Then repeat the process for the variables $x_2,\cdots,x_\sigma$ each time comparing the monomial in question of the set (L_μ) with monomial $\overline{K}_\mu$ to determine whether or not the variable $x_2,\cdots,x_\sigma$ in question is a multiplier. By repetition we have finally that the monomial $\overline{K}_{\mu+1}$ is multiple of

$$\tilde{\tilde{L}}_\mu:\quad x_1^{a_1}\cdots x_\sigma^{a_\sigma}x_{\sigma+1}^{e_{\sigma+1}}\cdots x_n^{e_n}\quad \begin{pmatrix} e_j\leqq a_j\\ j=\sigma+1,\cdots,n\end{pmatrix}.$$

Let x_τ be the first of the variables $x_{\sigma+1},\cdots,x_n$ such that (1) $e_\tau<a_\tau$ and (2) x_τ is a non-multiplier for $\tilde{\tilde{L}}_\mu$. Then $x_\tau\,\tilde{\tilde{L}}_\mu$ belongs to the class $(\hat{\mathfrak{L}}_\mu)$ with basis

$$\hat{L}_\mu:\quad x_1^{a_1}\cdots x_\sigma^{a_\sigma}x_{\sigma+1}^{e_{\sigma+1}}\cdots x_\tau^{e_\tau+1}x_{\tau+1}^{s_{\tau+1}}\cdots x_n^{s_n}\quad (s_i\leqq a_i;\ i=\tau+1,\cdots,n).$$

A continuation of this process will show that $\overline{K}_{\mu+1}$ belongs to the class of the monomial L_μ^*: $x_1^{a_1}\cdots x_\sigma^{a_\sigma}\cdots$, where each of the variables x_ν of the set $x_{\sigma+1},\cdots,x_n$ is either a multiplier for L_μ^* or else appears in L_μ^* with the exponent a_ν. Thus we have a contradiction and $\overline{K}_{\mu+1}$ is not added by operation B.

If a monomial $\overline{\overline{K}}_{\mu+1}$ is added by operation B then repeat operation A on the set $(M_{\mu+1})$ until no new monomials are added by further operations; then repeat operation B once, etc., until finally we arrive at a strongly complete set (M_ν). Since operations A and B add only multiples of monomials $\overline{L}$ and $\overline{K}$ to the sets (L) and (K) respectively in the process of forming (M_ν), it is evident that the multiples of the monomials of (M_1) and (M_ν) are identical.

The set (M) treated in the above example is normal. If one adds the monomials $x^3y^3z^2$, $x^3y^2z^2$, x^3yz^2 to the set (L_1) to form the set (L_2) and then adds the monomial xyz to the set (K_2) to form the set (K_3) it is seen that the resulting set satisfies the condition of Theorem II and hence (M_3) is complete.

	Monomials	Multipliers	Non-Multipliers
(L_3)	$x^3\ y^3\ z^2$	$x\ y\ z$	$\cdot\ \cdot\ \cdot$
	$x^3\ y^2\ z^2$	$x\ \cdot\ z$	$\cdot\ y\ \cdot$
	$x^3\ y\ z^2$	$x\ \cdot\ z$	$\cdot\ y\ \cdot$
	$x^2\ y^3\ z^2$	$\cdot\ y\ z$	$x\ \cdot\ \cdot$
	$x^2\ y^2\ z^2$	$\cdot\ \cdot\ z$	$x\ y\ \cdot$
	$x^2\ y\ z^2$	$\cdot\ \cdot\ z$	$x\ y\ \cdot$
	$x\ y\ z^2$	$\cdot\ y\ z$	$x\ \cdot\ \cdot$
(K_3)	x^3	$\cdot\ \cdot\ z$	$\cdot\ \cdot\ \cdot$
	$x\ y\ z$	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ z$
	$x\ y$	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ z$

A monomial $x_1^{a_1} \cdots x_n^{a_n}$ is said to be of *higher or lower rank* than a different monomial $x_1^{b_1} \cdots x_n^{b_n}$ according as the first of the differences (14.1) which is not zero is positive or negative.

THEOREM III. *The product of any monomial of a complete set by one of its non-multipliers is equal to the product of a monomial of the complete set of higher rank, by multipliers of the latter.*

By definition the product of a monomial $\bar{M}$ of a complete set (M) by one of its non-multipliers is the product of another monomial $\bar{\bar{M}}$ of the set (M) by certain multipliers of the latter. We must show that $\bar{\bar{M}}$ is of higher rank than $\bar{M}$. Consider a monomial $\bar{M}$: $x_1^{a_1} \cdots x_n^{a_n}$ and let x_1 be the non-multiplier in question. In order that $x_1 \bar{M}$ be a multiple of $\bar{\bar{M}}$: $x_1^{b_1} \cdots x_n^{b_n}$ we must have $a_1 + 1 \geqq b_1$. If $\bar{\bar{M}} = \bar{\bar{K}}$, with $\alpha \geqq 1$, x_1 would not be a multiplier for $\bar{\bar{K}}$; the degree of $\bar{\bar{K}}$ in x_1 in that case could not be raised to $a_1 + 1$ by multiplication by multipliers of $\bar{\bar{K}}$. Hence $a_1 + 1 = b_1$. Similarly if $\bar{\bar{M}} = \bar{\bar{L}}$ or if $\bar{\bar{M}} = \bar{\bar{K}}$, with $\alpha = 0$, and if b_1 were less than $a_1 + 1$ then x_1 would not be a multiplier of $\bar{\bar{L}}$ or $\bar{\bar{K}}$ since x_1 is a multiplier only for monomials of the set (M) which contain the highest power of x_1. If x_σ is the non-multiplier of $\bar{M}$ it follows in exactly the same manner that

$$a_1 \geqq b_1, \; a_2 \geqq b_2, \; \cdots, \; a_\sigma + 1 \geqq b_\sigma.$$

By repetition of the above argument

$$a_1 = b_1, \; \cdots, \; a_\sigma + 1 = b_\sigma;$$

the monomial $\bar{\bar{M}}$ is therefore of higher rank than $\bar{M}$.

§ 15. General Existence Theorem.

For the purpose of ordering a system of partial differential equations of the form

$$\text{(15.1)} \qquad \frac{\partial^{a_1 + \cdots + a_n} u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}} = F\left(x;\; u;\; \frac{\partial u}{\partial x};\; \cdots\right)$$

we shall assign to each independent variable and to each unknown s successive cotes which as is well known are represented by zero, positive and negative integers.[37] The qth cote of an arbitrary derivative of u_i is then obtained by adding the qth cote of the unknown u_i and the qth cotes of all variables of differentiation, distinct or not. If the cotes of a derivative z, namely $C_1, \cdots, C_s$ are not all equal to the cotes of a derivative z^*, namely $C_1^*, \cdots, C_s^*$ then z will be said to precede or follow z^*, or to be of lower or higher rank than z^*, according as the first of the differences

$$C_1 - C_1^*, \; \cdots, \; C_s - C_s^*$$

[37] Cotes were introduced and used systematically by Riquier, *loc. cit.*, (5), p. 201.

which is not zero, is negative or positive. The order which is thereby established among the derivatives appearing in (15.1) by the first s cotes is therefore not effected by assigning additional cotes. If the order of all derivatives of the unknowns $u_1, \cdots, u_r$ with respect to variables $x_1, \cdots, x_n$ is not uniquely established by the assignment of the above s cotes ordering the derivatives appearing in (15.1), we can accomplish this by the assignment of additional cotes. For example, if we assign the $n+1$ additional cotes

	x_1	x_2	$\cdots$	x_n	u_1	u_2	$\cdots$	u_r
$s+1$	0	0	$\cdots$	0	1	2	$\cdots$	r
$s+2$	1	0	$\cdots$	0	0	0	$\cdots$	0
$s+3$	0	1	$\cdots$	0	0	0	$\cdots$	0
.	.	.	$\cdots$	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	$\cdots$	.
$s+n+1$	0	0	$\cdots$	1	0	0	$\cdots$	0

all derivatives are uniquely ordered. Thus, if two derivatives $D_i u_\sigma$ and $D_j u_\tau$ have exactly the same set of $s+n+1$ cotes, the equality of the $(s+1)$st cotes implies that the functions u_σ and u_τ are identical and from the equality of the remaining cotes the derivatives are identical.

Let us say that the totality of derivatives of the unknowns $u_1, \cdots, u_r$ of first cote C constitutes the class C as a matter of convenient terminology.

It will be assumed that the above system (15.1) satisfies the following conditions:

(A) The equations (15.1), the unknown functions $u_1, \cdots, u_r$ and the independent variables $x_1, \cdots, x_n$ are finite in number.

(B) The equations (15.1) are composed of two systems of equations, (1) the system R which holds throughout the region D: $|x_i| < c_i$ where the c's are constants and (2) the system S which holds throughout the subspace $\mathfrak{D}$: $x^1 = 0, \cdots, x^\alpha = 0$ of the region D.

(C) The equations R are solved for certain derivatives of the unknowns u and the functions F which constitute the right members of these equations depend on the independent variables $x_1, \cdots, x_n$, the unknowns $u_1, \cdots, u_r$ and the derivatives of the unknowns.[38]

(D) The equations S are solved for certain derivatives which are not left members of R and the functions F which constitute the right members of these equations depend on the independent variables $x_{\alpha+1}, \cdots, x_n$ and the derivatives of the unknowns which are not left members of R or S.

(E) The functions F which constitute the right members of the system (15.1) are analytic functions in the neighborhood of the values

[38] The term *derivative* is to be interpreted as including the unknowns u themselves.

$$(15.2)\qquad x = 0, \quad u = (u)_0; \qquad \frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_0; \cdots$$

of their arguments.

(*F*) Cotes, s in number, are assigned to the unknowns $u_1, \cdots, u_r$ and the independent variables $x_1, \cdots, x_n$ such that (1) the first cote of each variable (x) is unity and (2) the derivatives of the functions (u) are uniquely ordered.

(*G*) The order established by the above assignment of cotes is such that any derivative z in the right member of an equation of the system R except those for which the derivative $\partial F/\partial z$ vanishes throughout the subspace $\mathfrak{D}$, precedes the derivative in the left member of the equation; also any derivative in the right member of an equation of the system S, precedes the corresponding left member.

(*H*) In any equation of the system R the class of the derivatives in the right member does not exceed the class of the derivative in the left member of the equation.

It is to be noted that if a derivative z precedes a derivative z^*, that $\partial z/\partial x^\sigma$ will precede $\partial z^*/\partial x^\sigma$; likewise z will precede z^* if z is of lower class than z^*. All derivatives of a particular unknown u_i in the class C will moreover be of the same order.

Let us now associate with the derivatives of the unknown u_i which appear in the left members of the system (15.1), the set (M_i) of monomials $x_1^{a_1} \cdots x_n^{a_n}$ corresponding to these derivatives; the set (L_i) of (M_i) will be composed of those monomials which correspond to derivatives of u_i in the left members of the system R and the set (K_i) of (M_i) will be composed of monomials corresponding to derivatives of u_i in the left members of the equations of the system S.

(I) The set (M_i) of monomials is strongly complete.

A system of equations (15.1) *will be said to be normal if the above conditions $A, \cdots, I$ are satisfied.*

By differentiation of an equation of a normal system (15.1) we obtain an equation which satisfies conditions G and H. Let z^* denote the derivative in the left member of an equation R or S and let z be a derivative in the right member of the same equation such that z precedes the derivative z^*. By differentiation of this equation we obtain

$$(15.3)\qquad \frac{\partial z^*}{\partial x^i} = \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x^i} + \cdots \equiv F'$$

where the derivatives z and $\partial z/\partial x^i$ in the right member precede the derivative in the left member. Suppose now that z follows z^* for an equation of the system R; this implies that $\partial F/\partial z = 0$ throughout the subspace $\mathfrak{D}$. Hence

$$\frac{\partial F'}{\partial z'} = \frac{\partial F}{\partial z} = 0 \quad \text{throughout } \mathfrak{D},$$

where the derivative $\partial z/\partial x^i$ has been denoted by z'. In other words the equation (15.3) satisfies properties G and H. Hence G and H will be satisfied by the system of equations which result from a normal system (15.1) by the addition of a finite number of equations obtained from (15.1) by partial differentiation.

Differentiating the equations of the system (15.1) with respect to multipliers (x) of the corresponding monomials, each principle derivative of an unknown $u_1, \cdots, u_r$ will be obtained as the left member of one and only one of the resulting equations; this follows from the characteristic property of a complete system of monomials. Equations so formed by differentiation of an equation of the system R are to be added to R and those formed by differentiation of an equation of S are to be added to S. Thus we arrive at a system of equations consisting (1) of the system R and all equations which were added to R and (2) of the system S and all equations which were added to the system S; we shall refer to these as the systems $\tilde{R}$ and $\tilde{S}$ respectively. It is evident that a derivative z cannot appear both in the left and right member of an equation of $\tilde{R}$ or $\tilde{S}$. Now consider the system E composed of all equations of $\tilde{R}$ and $\tilde{S}$ whose left members belong to classes C of serial number not exceeding N; we shall refer to a particular equation of the system E by E_1, E_2, etc. Let z^* be the derivative of lowest rank which appears in the right member of an equation E_1 of the system $\tilde{R}$ and in the left member of an equation E_2 of $\tilde{R}$; using the equation E_2 we eliminate the derivative z^* from the right member of E_1. Similarly if the derivative z^* appears in the right member of an equation E_3 of the system $\tilde{S}$ and also in the left member of an equation E_4 of $\tilde{R}$ or $\tilde{S}$, the equation E_4 is to be used to eliminate z^* from the right member of E_3. By repetition of this process we finally reach a system $\mathfrak{E}$ equivalent to the system E and such that $\mathfrak{E}$ involves only parametric derivatives in its right members; more precisely if we denote by $\mathfrak{E}_R$ and $\mathfrak{E}_S$ the equations of $\mathfrak{E}$ corresponding to $\tilde{R}$ and $\tilde{S}$ respectively, then the right members of $\mathfrak{E}_R$ depend only on parametric derivatives of the system R and the right members of $\mathfrak{E}_S$ depend only on parametric derivatives of the system (15.1). The fact that we can limit our considerations to the system E in the process of forming the system $\mathfrak{E}$ is a consequence of the above result that properties G and H are preserved under differentiation. It is easily seen that conditions analogous to $A, \cdots, H$ are satisfied by the system $\mathfrak{E}$.

We are looking for power series of the type

$$(15.4)\qquad u_i = \sum_{a=0}^{\infty} \frac{B_{a_1\cdots a_n}}{a_1!\cdots a_n!} x_1^{a_1}\cdots x_n^{a_n}$$

which will satisfy (15.1). Let us first, however, consider the problem of determining a solution $u_1, \cdots, u_r$ of the system (15.1) such that each equation of (15.1) is satisfied when we equate to zero the non-multipliers (x) of the corresponding monomial. For brevity the system (15.1) so modified will be referred to as the system U^*. It is evident that the equations of the above system $\mathfrak{E}$, evaluated at $x_\beta = 0$, can be arranged in the increasing order of the principle derivatives in their left members and such that any equation will involve in its right member only parametric derivatives of the system (15.1). Allowing the number N to increase without limit we therefore see that for a given assignment of the initial values of the parametric derivatives, the power series (15.4), i. e. the formal solution $u_1, \cdots, u_r$ of the system U^* will be uniquely determined. But all parametric derivatives of (15.1) at $x_\beta = 0$ are uniquely determined if the derivatives of each unknown u_i corresponding to monomials of the complementary set (N_i), are assigned as arbitrary functions of the multipliers (x) of the associated monomials, these functions being analytic in the neighborhood of the values $x_\beta = 0$ of their arguments. The functions thus associated with an unknown u_i will be called (following Riquier) the initial determination of the unknown u_i; the sum of the initial determinations of an unknown u_i will be denoted by φ_i. Hence the formal power series solution (15.4) of the system U^* is uniquely determined by the assignment of the initial determinations; the convergence of these series will be proved in § 17.[39]

Let us now adopt the notation

$$(15.5)\qquad \bar{Z}_i \equiv D_i u - F = 0$$

for an equation of the system (15.1); here $D_i u$ denotes the derivative in the left member of the equation (15.1) in question. According as the expression $\bar{Z}_i$ corresponds to an equation of the system R or the system S, it will be denoted by $\bar{R}_i$ or $\bar{S}_i$ respectively. We shall speak of the derivative $D_i u$ as the *first term* of the corresponding expression $\bar{Z}_i$. For convenience in terminology we will speak of the cotes, the rank and the class of the first term of an expression $\bar{Z}_i$ as the cotes, the rank, and the class respectively of the expression $\bar{Z}_i$ itself; an analogous terminology

[39] While the condition of strong completeness is imposed by condition I it may be observed that the less restrictive condition of completeness of the sets (M_i) is sufficient for the determination of these power series.

will be applied to an expression obtained from $\overline{Z}_i$ by differentiation. Likewise the multipliers and non-multipliers (x) of the monomial corresponding to the first term of an expression $\overline{Z}_i$ will be referred to as the multipliers and non-multipliers (x) of the expression $\overline{Z}_i$ itself. We shall also denote by $\overline{\mathfrak{Z}}_i$ either the expression $\overline{Z}_i$ or an expression resulting by differentiation of $\overline{Z}_i$ with respect to its multipliers; more particularly $\overline{\mathfrak{Z}}_i$ will be denoted by $\overline{\mathfrak{R}}_i$ or $\overline{\mathfrak{S}}_i$ according as $\overline{\mathfrak{Z}}_i$ was obtained from an expression $\overline{R}_i$ or $\overline{S}_i$ respectively.

The derivative of the first term of an expression $\overline{R}_1$ with respect to a non-multiplier x_b of $\overline{R}_1$ is identical with the first term of some expression $\overline{\mathfrak{R}}_2$ by condition I. Furthermore the difference

$$\frac{d\overline{R}_1}{dx_b} - \overline{\mathfrak{R}}_2 \equiv \Phi\,(Du,\ x), \tag{15.6}$$

where the derivative of $\overline{R}_1$ denotes the result of differentiating the expression $\overline{R}_1$ considered as a function of the independent variables (x) alone, is such that (1) the derivative $\partial\Phi/\partial(Du)$ vanishes throughout the subspace $\mathfrak{D}$ for any derivative Du in Φ which is of higher rank than the first term of $d\overline{R}_1/dx_b$ and (2) the class of any derivative Du in Φ is at most equal to that of the first term of $d\overline{R}_1/dx_b$. Similarly, if x_c is a non-multiplier of $\overline{S}_1$, the expression

$$\frac{d\overline{S}_1}{dx_c} - \overline{\mathfrak{Z}}_2 \equiv \Psi(Du,\ x), \tag{15.7}$$

where the first terms of $d\overline{S}_1/dx_c$ and $\overline{\mathfrak{Z}}_2$ are identical, is such that any derivative Du in Ψ precedes the first term of the expression $d\overline{S}_1/dx_c$.

Now use the equations $\mathfrak{E}_R$ to eliminate all principle derivatives of the system R from the function Φ in (15.6); also use the combined system $\mathfrak{E}$ to eliminate all principle derivatives of (15.1) from the function Ψ in (15.7). Suppose that

$$\Phi(Du,\ x) \rightarrow \Phi^*(Du,\ x), \qquad \Psi(Du,\ x) \rightarrow \Psi^*(Du,\ x)$$

as the result of this elimination. *The normal system* (15.1) *will be said to be completely integrable if all functions* Φ^* *and* Ψ^* *which can be constructed from the equations of the system* (15.1), *vanish identically.*

In the following section it will be shown that a completely integrable normal system (15.1) is satisfied by the power series solution (15.4) of the system U^*.

EXISTENCE THEOREM. *A completely integrable normal system* (15.1) *is satisfied by a unique set of analytic functions* $u_1, \cdots, u_r$ *determined by the assignment of the initial determinations predicted by the form of the left members of the system.*

§ 16. Equivalence of the Solutions of a Completely Integrable Normal System and its Associated System U^*.

We now consider the problem of showing that the power series solution (15.4) of the system U^* satisfies the normal system (15.1) provided that this latter system is completely integrable.

Corresponding to an equation $\bar{\mathfrak{R}}_i = 0$ or $\bar{\mathfrak{S}}_j = 0$ we have identities of the form

$$(16.1) \quad \text{(a)} \quad D_i u \equiv \bar{\mathfrak{R}}_i + \mathfrak{F}(Du, x), \qquad \text{(b)} \quad D_j u \equiv \bar{\mathfrak{S}}_j + \mathfrak{F}(Du, x)$$

derivable by differentiation of (15.5), where the functions $\mathfrak{F}$ in (16.1a) and (16.1b) have properties regarding the derivatives Du analogous to those stated in connection with equations (15.6) and (15.7) respectively. Hence equations of the type (16.1a) can be used to eliminate principle derivatives Du of the system R which occur in the functions $\mathfrak{F}$ in these equations; the infinite system so obtained as a result of this elimination furnishes equations which can be used in turn to eliminate principle derivatives Du of the system R from the functions Φ in equations of the type (15.6). The equation (15.6) thereby becomes

$$(16.2) \qquad \frac{d\bar{R}_1}{dx_b} - \bar{\mathfrak{R}}_2 \equiv \tilde{\Phi}(\bar{\mathfrak{R}}, Du, x),$$

where (a) the derivatives Du in $\tilde{\Phi}$ are parametric for the system R, (b) the class of an expression $\bar{\mathfrak{R}}$ in $\tilde{\Phi}$ is at most equal to the class of $d\bar{R}_1/dx_b$ and (c) any expression $\bar{\mathfrak{R}}$ in $\tilde{\Phi}$ which does not precede $d\bar{R}_1/dx_b$ is such that the derivative $\partial\tilde{\Phi}/\partial\bar{\mathfrak{R}}$ vanishes throughout the subspace $\mathfrak{D}$. Similarly from (15.7) we have

$$(16.3) \qquad \frac{d\bar{S}_1}{dx_c} - \bar{\mathfrak{Z}}_2 \equiv \tilde{\Psi}(\bar{\mathfrak{R}}, \bar{\mathfrak{S}}, Du, x),$$

where (d) the derivatives Du in $\tilde{\Psi}$ are parametric for the system (15.1) and (e) any expression $\bar{\mathfrak{R}}$ or $\bar{\mathfrak{S}}$ in $\tilde{\Psi}$ precedes $d\bar{S}_1/dx_c$.

If (15.1) is completely integrable the functions $\tilde{\Phi}$ and $\tilde{\Psi}$ must be such that they vanish in consequence of the vanishing of the expressions $\bar{\mathfrak{R}}$ and $\bar{\mathfrak{S}}$ appearing in these functions; this follows as a result of the way in which the above functions Φ^* and Ψ^* were obtained.

Now consider the system of equations composed of all equations of the type (16.2) and (16.3) which can be formed from (15.1) and into these equations substitute the solution $u_1, \cdots, u_r$ of the system U^*. The resulting system

$$(16.4) \qquad \frac{\partial \bar{Z}_i^*}{\partial x_j} = D_k \bar{Z}_l^* + \Pi(D\bar{Z}^*, x)$$

will be regarded as a system of equations for the determination of the unknowns $\overline{Z}_i^*$ where $\overline{Z}_i^*$ denotes the result of substituting the solution $u_1, \cdots, u_r$ of the system U^* in the expression $\overline{Z}_i$; it will furthermore be understood that the variables $x_1 = \cdots = x_\alpha = 0$ in an expression $\overline{S}_i^*$. In any equation (16.4) the first, second and third terms indicated result respectively from the first, second and third terms of the corresponding equation (16.2) or (16.3). We shall show in the following that (16.4) is a normal system, i.e. that these equations satisfy the above conditions $A, \cdots, I$ provided that we consider those equations (16.4) which contain a derivative $\partial \overline{R}_i^* / \partial x_j$ in their left member as the system R and the remaining equations (16.4) as the system S of condition B.

The monomials of the set (M_i) corresponding to the derivatives of an unknown $\overline{R}_i^*$ which appear in the left members of (16.4) all belong to the set (L_i). It can be shown that any set (M) of monomials of this type in which each monomial consists of a single variable x_β, is strongly complete.[40] Since each monomial of the set (M_i) for an unknown $\overline{S}_i^*$ belongs to the set (K_i) and furthermore since each of these monomials consists of one of the variables $x_{\alpha+1}, \cdots, x_n$ it follows that the set (M_i) can be considered as a set (M) in the $n-\alpha$ variables $x_{\alpha+1}, \cdots, x_n$ where all monomials of (M) belong to the set (L); hence (M_i) is strongly complete by the above argument. The condition I is therefore satisfied by the system (16.4).

Let us assign the cotes for the system (16.4) in the following manner:

(α) The variables x_β will be given the same set of s cotes as in the normal system (15.1).

(β) The unknowns $\overline{Z}_i^*$ will be given the same set of s cotes as $\overline{Z}_i$ in the system (15.1).

(γ) The variables x_β will each be given an $(s+1)$st cote of zero.

(δ) Suppose that the expressions $\overline{Z}_i$ which have derivatives of the same unknown u_j as their first terms, are denoted by $\overline{Z}_1^{(j)}, \overline{Z}_2^{(j)}, \cdots$ when arranged in the order of decreasing rank of the monomials corresponding to their first terms. The unknown $\overline{Z}_i^{*(j)}$ will be given an $(s+1)$st cote equal to the integer i.

The first s cotes of the derivative $D_k \overline{Z}_l^*$ in the right member of an equation (16.4) will be identical with the first s cotes of the derivative in the left member of this equation; however the $(s+1)$st cote of $D_k \overline{Z}_l^*$ will be less than the $(s+1)$st cote of the derivative in the left member in consequence of the above conditions (γ) and (δ) and Theorem III of § 14. Hence $D_k \overline{Z}_l^*$ precedes the derivative in the left member of (16.4). Also

[40] cf. Janet, *loc. cit.*, (6), p. 85.

the derivatives $D\bar{Z}^*$ in the function Π of (16.4) either precede the derivative in the left member of this equation or are such that the derivative $\partial \Pi/\partial(D\bar{Z}^*)$ vanishes over the subspace $\mathfrak{D}$; these results follow in consequence of the above statements (c) and (e) made in connection with equations (16.2) and (16.3) respectively. Hence the system (16.4) satisfies condition G.

Consider those equations (16.4) which involve a derivative of an $\bar{R}_i^*$ in their left members; any derivative of an unknown $\bar{Z}_i^*$ appearing in the right member of one of these equations will have class at most equal to the class of the corresponding left member in consequence of statement (b) under equation (16.2). Condition H is therefore satisfied by (16.4).

To see that equations (16.4) are actually solved for the derivatives $\partial \bar{Z}_i^*/\partial x_j$ appearing in the left members of these equations we have merely to note that x_j is a non-multiplier of the monomial corresponding to the first term of $\bar{Z}_i$ while derivatives in the right members of these equations involve only differentiations with respect to multipliers (x) of the corresponding monomials; hence no derivative occurring in the right member of an equation (16.4) can appear on the left of any equation (16.4) and conversely. Hence conditions C and D are satisfied by (16.4).

If the $s+1$ sets of cotes given by conditions α, β, γ, δ do not uniquely order all derivatives of $\bar{Z}_i^*$, the ordering can be made unique by the assignment of additional cotes in accordance with the discussion in § 15. Condition F can therefore be considered to be satisfied by (16.4). The remaining conditions A, B, E are obviously satisfied by (16.4) on account of the method of formation of these equations.

Consider the system V^* obtained from (16.4) by equating to zero in each equation the non-multipliers (x) of the monomial corresponding to the left member of the equation. Hence by the theory of § 15 there exist a power series solution $\bar{Z}_i^*$ of the system V^* which is uniquely determined by the initial determinations in the sense of § 15. In the functions $\bar{S}_i^*$ of the solution $\bar{Z}_i^*$ so obtained let us put $x_1 = \cdots = x_\alpha = 0$. It then follows from the form of (16.4) that the solution $\bar{Z}_i^*$ as so modified also satisfies the system (16.4) and hence is to be regarded as the solution of (16.4) as above considered; hereafter by the solution $\bar{Z}_i^*$ of the system (16.4) this latter solution will be understood.

Now consider the set (M_i) of monomials corresponding to an unknown $\bar{R}_i^*$; all monomials $x_p, \cdots, x_q$ of this set (M_i) then belong to the set (L_i). The set complementary to (M_i) consists of one monomial $\bar{N}_q:1$ with multipliers consisting (1) of the variables $x_{q+1}, \cdots, x_n$ and (2) of those variables $x_1, \cdots, x_{q-1}$ which do not belong to the set $x_p, \cdots, x_{q-1}$. Hence the initial determination of each $\bar{R}_i^*$ will be a function θ of the multipliers (x) of the corresponding expression $\bar{R}_i$ in the normal system

(15.1). Similarly the initial determination of each function $\bar{S}_i^*$ will be a function ψ of the multipliers (x) of the corresponding expression $\bar{S}_i$ in (15.1).

We wish to prove that the expressions $\bar{Z}_i$ reduce identically to zero when the unknowns $u_1, \cdots, u_r$ in $\bar{Z}_i$ are replaced by the solution $u_1, \cdots, u_r$ of the system U^*. To prove this we observe (1) that when the non-multipliers (x) of $\bar{R}_i$ are equated to zero, the expression $\bar{R}_i^* \equiv 0$ and when the variables $x_1, \cdots, x_\alpha$ together with the non-multipliers (x) of $\bar{S}_i$ are equated to zero, the expression $\bar{S}_i^* \equiv 0$, (2) when the non-multipliers (x) of $\bar{Z}_i$ are equated to zero, the function $\bar{Z}_i^*$ reduces to its initial determination and (3) that the analytic functions $\bar{Z}_i^* \equiv 0$ satisfy (16.4) and the initial determinations of this solution of (16.4) vanish identically. This proves the above statement and shows that the solution $u_1, \cdots, u_r$ of the system U^* satisfies (15.1).

§ 17. Convergence Proof.

Denote by b and B the minimum and maximum values of the classes of the derivatives appearing in the left members of the equations of the system U^*. Call the totality of equations of the system U^* whose left members are in the class b, the system T_b. The system T_{b+1} will be composed of all equations with principle derivatives of the class $b+1$ in their left members; if such an equation is lacking in the system U^* it can be obtained by differentiating a suitable equation of U^*. Continuing in this way we obtain the sequence of systems

$$T_b, \; T_{b+1}, \; \cdots, \; T_B, \; T_{B+1}, \; \cdots.$$

The equations of this sequence fall into two component systems, namely (1) the system composed of R^* and those equations obtainable by differentiation of the system R^* and (2) the system S^* and those equations resulting by differentiation of S^*; the first of these component systems will be referred to as *the extended system* R^* and second as *the extended system* S^*. It is evident that any equation of the system T_{B+1} is linear in derivatives of class $B+1$. By the method of elimination used in forming the system $\mathfrak{E}$ in § 15, the systems $T_b, \cdots, T_{B+1}$ can be replaced by equivalent systems $T_b^*, \cdots, T_{B+1}^*$ which evidently determine uniquely the principle derivatives of (15.1) at $x_\beta = 0$ in classes b to $B+1$ inclusive when the initial determinations are assigned.

Consider the transformation

$$u_i = \bar{u}_i + \varphi_i + \sum \frac{C_{a_1 \cdots a_n}}{a_1! \cdots a_n!} x_1^{a_1} \cdots x_n^{a_n} \tag{17.1}$$

of the unknowns $u_1, \cdots, u_r$ in which the summation $\sum$ is limited to those $C_{a_1 \cdots a_n}$ which give at $x_\beta = 0$ the values of those principle derivatives of the unknown u_i in classes b to B inclusive. Substituting (17.1) into the system (15.1), this latter system goes over into an analogous system involving the corresponding derivatives of $\bar{u}_1, \cdots, \bar{u}_r$ in its left members. On account of the form of (17.1) it is easily seen (1) that the principle derivatives of the unknowns $\bar{u}_1, \cdots, \bar{u}_r$ in classes b to B inclusive must be equal to zero at $x_\beta = 0$ and (2) that the initial determinations of the unknowns $\bar{u}_1, \cdots, \bar{u}_r$ vanish identically. It can therefore be assumed without loss of generality that the system (15.1) satisfies the following condition.

(J) The principle derivatives of $u_1, \cdots, u_r$ in classes b to B inclusive vanish at $x_\beta = 0$ and the initial determinations vanish identically.

Let us now write the system T^*_{B+1} in the form

$$D_i u_\sigma = \sum p_{i\sigma j\mu} D_j u_\mu + q_{i\sigma}; \tag{17.2}$$

we observe that this system has the following properties

(a) The derivatives $D_i u_\sigma$ in the left members of (17.2) are all in the class $B+1$.

(b) All principle derivatives of the system U^* in classes C where $C > B+1$ are determined once and only once by differentiation of the left members of (17.2).

(c) The right member of an equation (17.2) is linear in derivatives $D_j u_\mu$ of class $B+1$; the coefficients p and the q terms may contain derivatives of class not exceeding B.

(d) Any derivative $D_j u_\mu$ in the right member of an equation of R^* of (17.2) which follows the derivative $D_i u_\sigma$ in the left member of the equation, has a coefficient p which vanishes throughout the domain $\mathfrak{D}$.

(e) The coefficients p and the q terms in the right member of any equation (17.2) are analytic functions in the neighborhood of the values

$$x = 0, \quad Du = 0, \cdots \tag{17.3}$$

of their arguments.

We shall show that when the principle derivatives of $u_1, \cdots, u_r$ and the initial determinations are assigned for the normal system (15.1) in accordance with condition J, that the unique power series determinations of the $u_1, \cdots, u_r$ by the system U^*, converge. It will then follow that the series for the $u_1, \cdots, u_r$ as determined by the system U^* without the condition J, will converge, since these latter series for $u_1, \cdots, u_r$ are related to the above convergent series by an equation of the form (17.1).

The above convergence proof will depend on two lemmas which we will now proceed to state. Let us associate with the unknowns $u_1, \cdots, u_r$ the positive constants $\zeta_1, \cdots, \zeta_r$ and with the variables $x_1, \cdots, x_n$ the positive constants $\xi_1, \cdots, \xi_n$ such that $\xi_i > 1$. Then corresponding to a derivative

$$\frac{\partial^{a_1+\cdots+a_n} u_i}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}$$

we can construct the quantity

$$\zeta_i \xi_1^{a_1} \cdots \xi_n^{a_n}. \tag{17.4}$$

Let us likewise choose r^2 positive constants λ_σ^μ where $\sigma, \mu = 1, \cdots, r$, such that

$$\sum_{\mu=1}^{r} \lambda_\sigma^\mu < 1.$$

We then have the following[41]

LEMMA I. *A set of positive constants ζ, ξ where ($\xi > 1$) can be selected so that the ratio of any quantity* (17.4) *corresponding to a derivative of class C, to the quantity* (17.4) *corresponding to a derivative of lower rank of the class C, is greater than a given positive number Q.*

LEMMA II. *The determinant*

$$|\delta_\sigma^\mu - \lambda_\sigma^\mu|$$

is positive and the cofactors of order $r-1$ are all positive or zero.

Let $H_{i\sigma\mu}$ be the number of the derivatives of the unknown u_μ of class $B+1$ which appear in the right of an equation with left member $D_i u_\sigma$. Also let $P_{i\sigma j\mu}$ be the absolute values of the constant terms of the power series expansions of $p_{i\sigma j\mu}$, about the values (17.3). Denote the largest of the numbers

$$P_{i\sigma j\mu} H_{i\sigma\mu} / \lambda_\sigma^\mu$$

by Q. Taking the class C in Lemma I to be the class $B+1$ we can then choose positive constants ζ, ξ in accordance with this lemma so that

$$\frac{\lambda_\sigma^\mu \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{H_{i\sigma\mu} \zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}} > P_{i\sigma j\mu}, \tag{17.5}$$

where $D_i u_\sigma$ is of higher rank than $D_j u_\mu$. Evidently the ratio (17.5) is valid also for a derivative $D_j u_\mu$ in the right member of an equation (17.2) which follows the corresponding left member on account of the above property (d). Now let G be an upper bound to the absolute values of the terms of the expansions of p, q about the values (17.3). Then choose r^2

[41] Janet, *loc. cit.*, (6), p. 144 and p. 141. In the proof of Lemma I use is made of the fact that without changing the ordering of the derivatives effected by an assignment of cotes, it is possible to replace the given set of cotes by a set composed entirely of positive integers. Cf. Riquier, *loc. cit.* (5) p. 253.

numbers ε_σ^μ, positive or zero, where $\sigma, \mu = 1, \cdots, r$ so as to satisfy the conditions

$$\frac{\lambda_\sigma^\mu + \varepsilon_\sigma^\mu}{H_{i\sigma\mu}} \cdot \frac{\zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{\zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}} \geqq G \tag{17.6}$$

for all combinations of the indices such that $D_j u_\mu$ is a derivative of class $B+1$ in the right member of an equation (17.2) of which $D_i u_\sigma$ is the left member. Finally choose the r positive constants ν_σ so that

$$\nu_\sigma \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n} > G, \tag{17.7}$$

where $D_i u_\sigma$ is any principle derivative of class $B+1$.

Consider the system

$$D_i Y_\sigma = \sum \frac{1}{H_{i\sigma\mu}} \left(\frac{\lambda_\sigma^\mu + \varepsilon_\sigma^\mu}{1-\tau} - \varepsilon_\sigma^\mu \right) \frac{\zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{\zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}} D_j Y_\mu + \frac{\nu_\sigma \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{1-\tau} \tag{17.8}$$

which has the same form as the system (17.2). Let

$$\tau = \frac{\xi_1 x_1 + \cdots + \xi_n x_n + \sum D Y}{\varrho},$$

where the summation $\sum D Y$ denotes the sum of all derivatives corresponding to derivatives Du of class not exceeding B, and ϱ is a small positive number such that the function

$$\frac{G}{1 - \dfrac{\sum x + \sum D Y}{\varrho}} \tag{17.9}$$

dominates the coefficients p and the terms q in an equation of (17.2). Since $\xi_i > 1$ the function $G/1-\tau$ has the same dominating properties. Hence the terms $q_{i\sigma}$ in equations (17.2) are dominated by the corresponding terms in (17.8) in consequence of the inequalities (17.7). If $G_1 + G_2 \geqq G$ and G_1 exceeds the absolute value of the constant $P_{i\sigma j\mu}$ then

$$\frac{G_1 + G_2}{1-\tau} - G_2$$

dominates the function $p_{i\sigma j\mu}$. Corresponding to a coefficient $p_{i\sigma j\mu}$ in the system (17.2) put

$$G_1 = \frac{\lambda_\sigma^\mu \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{H_{i\sigma\mu} \zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}}, \qquad G_2 = \frac{\varepsilon_\sigma^\mu \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{H_{i\sigma\mu} \zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}};$$

then it follows that the p's in (17.2) are dominated by the corresponding coefficients in (17.8) when use is made of (17.5) and (17.6).

Let us put

$$Y_\sigma = \zeta_\sigma \overline{Y}_\sigma$$

and regard the quantities $\overline{Y}_\sigma$ as functions of the single variable y where

$$y = \xi_1 x_1 + \cdots + \xi_n x_n.$$

Denoting the ith derivative of $\overline{Y}_\sigma$ where $i = i_1 + \cdots + i_n$ by $\overline{Y}_\sigma^{(i)}$ we have

$$D_i Y_\sigma = \overline{Y}_\sigma^{(i)} \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}.$$

The quantity τ becomes a function $\overline{\tau}$ depending on the variable y and derivatives of $\overline{Y}_\sigma$ of order not exceeding h_σ, where $h_\sigma = B - c_\sigma$, c_σ being the first cote of u_σ. Equations (17.8) then become

$$\overline{Y}_\sigma^{(h_\sigma+1)} = \sum_{\mu=1}^{r} \left(\frac{\lambda_\sigma^\mu + \varepsilon_\sigma^\mu}{1-\overline{\tau}} - \varepsilon_\sigma^\mu \right) \overline{Y}_\mu^{(h_\mu+1)} + \frac{\nu_\sigma}{1-\overline{\tau}}. \tag{17.10}$$

Let us seek to find a solution $\overline{Y}_\sigma(y)$ of this system, satisfying the initial conditions

$$y = 0, \quad \overline{Y}_\sigma = 0, \quad \overline{Y}_\sigma^{(1)} = 0, \cdots, \overline{Y}_\sigma^{(h_\sigma)} = 0 \quad (\sigma = 1, \cdots, r). \tag{17.11}$$

Since $\overline{\tau}$ is equal to zero for the initial values of its arguments, the equations (17.10) can be solved for the derivatives $\overline{Y}_\sigma^{(h_\sigma+1)}$ on account of Lemma II. We thus obtain a system of ordinary equations of the form

$$\overline{Y}_\sigma^{(h_\sigma+1)} = \Omega\,[y, \overline{Y}, \overline{Y}^{(1)}, \cdots, \overline{Y}^{(h_\sigma)}] \tag{17.12}$$

valid in the neighborhood of the above values (17.11); this system has a unique analytic solution $\overline{Y}_\sigma(y)$ satisfying the initial conditions. Furthermore the coefficients of the power series expansions of the solution $\overline{Y}_\sigma$ of (17.12) about the value $y = 0$, are either positive or zero. The first $h_\sigma + 1$ terms of this series are zero in consequence of (17.11); the coefficients of all other terms in these series are positive. Let us write the equations (17.10) in the form

$$\overline{Y}_\sigma^{(h_\sigma+1)} = \overline{\tau}\, \overline{Y}_\sigma^{(h_\sigma+1)} + \sum_{\mu=1}^{r} (\lambda_\sigma^\mu + \overline{\tau}\, \varepsilon_\sigma^\mu)\, \overline{Y}_\mu^{(h_\mu+1)} + \nu_\sigma. \tag{17.13}$$

The initial values of the derivatives $\overline{Y}_\sigma^{(h_\sigma+1)}$ satisfy the equations

$$[\overline{Y}_\sigma^{(h_\sigma+1)}]_0 - \sum_{\mu=1}^{r} \lambda_\sigma^\mu [\overline{Y}_\mu^{(h_\mu+1)}]_0 = \nu_\sigma; \tag{17.14}$$

from Lemma II and the fact that the ν_σ are all positive the quantities $[\overline{Y}_\sigma^{(h_\sigma+1)}]_0$ determined by (17.14) are all positive. Differentiating the equations (17.13) and evaluating at $y = 0$, we see likewise that the quantities $[\overline{Y}_\sigma^{(h_\sigma+2)}]_0$ are all positive, etc. It follows that the coefficients of the power series expansions of the corresponding solution $Y_\sigma(x)$ of the system (17.8) are all positive or zero and hence these expansions dominate the corresponding expansions determined by the equations (17.2) or the system U^*.

§ 18. Reduction to Normal Form.

In consequence of condition I of § 15, the derivatives for which the system S is solved in accordance with condition D, are parametric for the system R. Hence the condition D can be replaced by the somewhat stronger condition.

(D^*). The system S is solved for certain derivatives which are parametric for the system R and the functions F which constitute the right members of these equations depend on the independent variables $x_{\alpha+1}, \cdots, x_n$ and derivatives which are not left members of the system R.

Conversely suppose that the conditions A, B, C, D^*, E, F, G, H are satisfied by a system of equations (15.1). By the condition D^* the set (M_i) of monomials associated with unknown u_i as in § 15 will be normal. Making use of the process developed in § 14 we can therefore form a strongly complete set (M_i^*) of monomials having the same multiples as the set (M_i). By differentiation of the equations of (15.1) which involve in their left members derivatives of u_1, we can deduce a system of equations whose left members will be in one to one reciprocal correspondence with the monomials of the strongly complete set (M_1^*). This process is now to be repeated for the derivatives of the functions $u_2, \cdots, u_r$ in turn which appear in the left members of (15.1). Equations so formed by differentiation of an equation of the system R are to be added to R and those formed by differentiation of an equation of S are to be added to S. In this manner we arrive at a system of equations which can be solved for the derivatives in their left members by the application of the process of elimination used in § 15 in the derivation of the system $\mathfrak{E}$; we shall refer to the solved form of these equations as the *extended system* (15.1). From the discussion in § 15 it is obvious that the extended system (15.1) satisfies conditions $A, \cdots, H$ and the condition I is satisfied on account of the method of formation of these equations. Hence the extended system (15.1) is normal.

§ 19. Applications of the General Existence Theorem.

We saw in § 12 that the general existence theorems of § 6 failed to apply when we attempted to take a characteristic surface $C_{n-\alpha}$ as an $(n-\alpha)$-dimensional surface bearing a portion of the arbitrary data of the problem. The existence theoretic problem connected with the characteristic surface $C_{n-\alpha}$ defined by the equations $x^1 = 0, \cdots, x^\alpha = 0$ for the case of the system (2.1) is the problem of determining a solution $v_k(x)$ of this system such that

$$\Xi_{\beta_1}(x, v) = 0, \cdots, \Xi_{\beta_m}(x, v) = 0 \tag{19.1}$$

over the surface $C_{n-\alpha}$. The theory of normal systems of § 15 has immediate application to this problem. Let us suppose that the combined system (2.1) and (19.1) can be put into a completely integrable normal form (15.1) where equations (2.1) give rise to the system R and equations (19.1) give rise to the system S. The solution $v_k(x)$ so obtained will then satisfy the system (19.1) throughout the subspace $\mathfrak{D}$, i. e. the surface $x^1 = 0, \cdots, x^\alpha = 0$ with the result that this latter surface is an $(n-\alpha)$-dimensional characteristic surface $C_{n-\alpha}$ of type β for the solution $v_k(x)$, provided that there exists a determinant $\Xi_{\beta-1}(x, v)$ which does not vanish over this surface. In accordance with the remarks of § 12 the above theory applies equally well to a vector invariantive system. The invariantive systems composed of (4.3), (4.5), (4.7) for the affine case and (4.4), (4.8), (4.10), (4.11) for the metric case permit an analogous treatment when the equations (19.1) are replaced by the systems

$$\text{(19.2)} \qquad \begin{aligned} &\text{(a) Affine case: } U_k(A^\alpha_{\beta\gamma\delta}) = 0, \\ &\text{(b) Metric case: } U_k(g_{\alpha\beta};\, g_{\alpha\beta,\gamma\delta}) = 0 \end{aligned}$$

respectively.

In the practical application of this theory to systems of equations the work is usually facilitated by use of a particular assignment of cotes which is said to put the derivatives into *canonical order* and which orders all derivatives uniquely. We have illustrated this assignment of cotes by the accompanying scheme. The ordering of

	x^1	x^2	$\cdots$	x^{n-1}	x^n	v_1	v_2	$\cdots$	v_w
1	1	1	$\cdots$	1	1	0	0	$\cdots$	0
2	1	0	$\cdots$	0	0	0	0	$\cdots$	0
3	0	1	$\cdots$	0	0	0	0	$\cdots$	0
.	.	.	$\cdots$	.	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	.	$\cdots$	.
n	0	0	$\cdots$	1	0	0	0	$\cdots$	0
$n+1$	0	0	$\cdots$	0	1	0	0	$\cdots$	0
$n+2$	0	0	$\cdots$	0	0	1	2	$\cdots$	w

derivatives by the canonical assignment of cotes amounts to arranging the derivatives first according to increasing order; those derivatives of the same order are arranged according to the number of differentiations with respect to x^1; those derivatives of the same order and having the same number of differentiations with regard to x^1 are arranged according to the number of differentiations with regard to x^2; etc. Finally those derivatives corresponding to the same monomial are ordered according to the index k of the unknown v_k by the $(n+2)$nd cote.

As an illustration of the application of the theory of normal systems we can consider the treatment of the above mentioned problem for the case of the system of field equations in Proc. Note V. It is easily seen that the set of monomials corresponding to the derivatives of any one of the unknowns appearing in the left members of the system composed of equations (4.1), (4.2), (4.3), (5.1), (5.3) in Proc. Note V is strongly complete and in fact that this system is normal, and that the derivatives can be put in canonical order; the complete integrability of this system can be shown by the method of § 20. The convergence proof given in § 17 supplies the proof omitted in the former treatment.

In the discussion of the example in § 7 we found that if $g^{11} = 0$ at the initial point $x^i = 0$, the existence theorem did not apply. In view of the general theory presented in § 12, § 13 let us now see whether $g^{11} = 0$ over $x^1 = 0$ will define $x^1 = 0$ as a characteristic surface C_8. In other words in the notation of § 7, if $\alpha = 0$ is it possible to solve the combined system (1)–(20) and (7 c) for L_1 (= 20) derivatives $\partial g_{\alpha\beta,\gamma\delta}/\partial x^1$? Consider the matrix of the coefficients of the derivatives of $u_1, \cdots, u_6$ with respect to x^1 in the combined system (1)–(20) and (7 c); since all sixth order determinants in this matrix contain α as a factor, it is impossible to solve the system (1)–(20) and (7 c) for 20 (= L_1) derivatives of independent components $g_{\alpha\beta,\gamma\delta}$ with respect to x^1. Hence a surface $\Phi = 0$ such that

$$g^{\alpha\beta}\frac{\partial\Phi}{\partial x^\alpha}\frac{\partial\Phi}{\partial x^\beta} = 0 \tag{19.3}$$

over $\Phi = 0$ for some integral $g_{\alpha\beta}(x)$ must be a characteristic surface for the system (7 c).

Let us assume $\alpha = 0$ over $x^1 = 0$ and

$$\begin{vmatrix} \alpha & a \\ a & \beta \end{vmatrix} \neq 0, \qquad \begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix} \neq 0 \tag{19.4}$$

at the point $x^i = 0$ on this surface. As a consequence of these assumptions we have

$$a \neq 0, \qquad a^2\gamma - 2abd + b^2\beta \neq 0$$

at $x^i = 0$. The determinant of the columns 5, 6, 7, 8 in Table I is

$$8a(a^2\gamma - 2abd + b^2\beta).$$

Hence we shall drop the equations

$$X_{221} = 0, \qquad X_{231} = 0, \qquad X_{241} = 0, \qquad X_{331} = 0$$

from the system (7 c), thus making rows 5–10 in Table II the coefficients of the derivatives in the equations $X_{\alpha\beta 1} = 0$ which remain in (7 c). In Table II the determinant

$$\begin{vmatrix} 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 11 & 12 \end{vmatrix}$$

formed from rows 5, 6, 7, 8, 9, 10 and columns 1, 2, 3, 4, 11, 12 is equal to $4a^2(a^2\gamma - 2abd + b^2\beta)$ disregarding algebraic sign; hereafter a similar notation for determinants will be used without further explanation. Hence let us solve the remaining equations $X_{\alpha\beta 1} = 0$ for the derivatives of the unknown $u_1, u_2, u_3, u_4, v_5, v_6$ with respect to x^1. The determinant in Table II,

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 12 \end{vmatrix} = 8a^2 \begin{vmatrix} \alpha & a \\ a & \beta \end{vmatrix} \begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix}^2,$$

disregarding sign; for expanding in terms of second order minors in rows 5, 6 the above determinant reduces to the product of $4a^2$ and the eighth order determinant which was encountered in § 7. Hence we can solve equations $X_{\alpha\beta\varepsilon} = 0$ where $\varepsilon = 2, 3, 4$ for derivatives of the unknowns $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_5, v_6$ with respect to x^ε. Hence we have the following systems of equations

$$\text{(19a)} \qquad \left.\begin{matrix} \dfrac{\partial u_i}{\partial x^1} \\ \dfrac{\partial v_j}{\partial x^1} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^1} + \sum \frac{\partial v}{\partial x^1} + \sum \frac{\partial w}{\partial x^1} + \ast,$$

$$\text{(19b)} \qquad \left.\begin{matrix} \dfrac{\partial u_i}{\partial x^2} \\ \dfrac{\partial v_k}{\partial x^2} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^2} + \sum \frac{\partial v}{\partial x^2} + \sum \frac{\partial w}{\partial x^2} + \ast,$$

$$\text{(19c)} \qquad \left.\begin{matrix} \dfrac{\partial u_i}{\partial x^3} \\ \dfrac{\partial v_k}{\partial x^3} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^3} + \sum \frac{\partial v}{\partial x^3} + \sum \frac{\partial w}{\partial x^3} + \ast,$$

$$\text{(19d)} \qquad \left.\begin{matrix} \dfrac{\partial u_i}{\partial x^4} \\ \dfrac{\partial v_k}{\partial x^4} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^4} + \sum \frac{\partial v}{\partial x^4} + \sum \frac{\partial w}{\partial x^4} + \ast,$$

where $i = 1, 2, 3, 4$; $j = 5, 6$; $k = 1, \cdots, 6$. The derivatives in the right members of each of the above systems (19a)–(19d) involve unknowns not appearing in the left members of that system.

In accordance with the above theory let us now solve the combined systems (19a)–(19d) and (1)–(20) for derivatives in such a manner that the resulting system will satisfy the conditions for a normal system. Eliminate the left members of equations (1)–(4) and (7)–(14) from the right members of (19a) and use (19b), (19c) and (19d) to eliminate their left members from the right of the resulting system; denote the system which results by $\mathfrak{A}$. Then use equations (15)–(20) to eliminate their left members from the right of (19b) and $\mathfrak{A}$. The equation (15) contains the left member $\partial v_6/\partial x^2$ of an equation (19b) in its right member. However the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 15 \end{vmatrix}$$

in Table II vanishes identically and hence the derivative $\partial w_1/\partial x^2$ does not appear on the right of the equation (19b) having $\partial v_6/\partial x^2$ for its left member; hence the equation (19b) in question can be used to eliminate its left member from the remaining equations (19b) and $\mathfrak{A}$. Thus we can eliminate the left members of (19c) and (19d) from the right of (19b) and $\mathfrak{A}$ giving the systems B and A respectively. Then use A, B, (19c), (19d) to eliminate their left members from the right of equations (1)–(4) and (7)–(20) and we have the combined system consisting of (1)–(4), (7)–(20) and (19a)–(19d), in solved form.

Let us now consider the problem of introducing equations (5) and (6) into the above system so that equations (1)–(20) and (19a)–(19d) will be in solved form. From Table II it can be seen that the system (7c) contains the two equations

$$(19\text{e})\quad \begin{aligned} 2a\frac{\partial v_6}{\partial x^1} &= -\alpha\frac{\partial u_5}{\partial x^1} - \beta\frac{\partial v_7}{\partial x^1} + b\frac{\partial w_2}{\partial x^1} + c\frac{\partial w_3}{\partial x^1} + d\frac{\partial w_4}{\partial x^1} + e\frac{\partial w_5}{\partial x^1} + f\frac{\partial w_6}{\partial x^1} + \bigstar, \\ 2a\frac{\partial v_5}{\partial x^1} &= -\alpha\frac{\partial u_6}{\partial x^1} - \beta\frac{\partial v_8}{\partial x^1} - 2b\frac{\partial w_3}{\partial x^1} - 2d\frac{\partial w_5}{\partial x^1} - \gamma\frac{\partial w_6}{\partial x^1} + \bigstar. \end{aligned}$$

Consider the form of the corresponding equations in the system A. From the form of equations (10)–(14) it is seen that only the first two terms on the right of each equation (19e) can give rise to derivatives with respect to x^1 and x^2. Hence making use of (7) and (8) the equations (19e) become

$$(19\text{f})\quad \begin{aligned} 2a\frac{\partial v_6}{\partial x^1} &= -\alpha\frac{\partial u_5}{\partial x^1} - \beta\frac{\partial v_6}{\partial x^2} + \cdots, \\ 2a\frac{\partial v_5}{\partial x^1} &= -\alpha\frac{\partial u_6}{\partial x^1} - \beta\frac{\partial v_5}{\partial x^2} + \cdots, \end{aligned}$$

where the dots in these and following equations denote derivatives with respect to x^3 and x^4 and terms of lower order; when however, one or more derivatives with respect to x^3 are written down explicitly the dots will denote only derivatives with respect to x^4 and terms of lower order. Since the determinants

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 5 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 6 & 12 \end{vmatrix}$$

in Table II contain α as a factor and since moreover the determinants

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 6 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 5 & 12 \end{vmatrix}$$

in Table II vanish identically, there are two equations of the system (19b) of the form

$$(19\text{g})\quad \begin{aligned} \frac{\partial v_5}{\partial x^2} &= \alpha V\frac{\partial u_6}{\partial x^2} + \text{———} + \frac{\partial(v_7, v_8, w_1, \cdots, w_6)}{\partial x^2} + \bigstar, \\ \frac{\partial v_6}{\partial x^2} &= \text{———} + \alpha V\frac{\partial u_5}{\partial x^2} + \frac{\partial(v_7, v_8, w_1, \cdots, w_6)}{\partial x^2} + \bigstar, \end{aligned}$$

where the blank indicates that the corresponding derivative does not enter; and the notation employed in the second term of each of these equations as well as in certain of the following equations, denotes a linear sum of the derivatives of the indicated unknowns; here likewise V represents a rational function of the $g_{\alpha\beta}$ although not necessarily the same function in the two equations (19g). Hence if we make use of equations (15)–(20) and the equations (19g), the equations (19f) can be put in the form

$$(19\text{h})\quad \begin{aligned} 2a\frac{\partial v_6}{\partial x^1} &= -\alpha\frac{\partial u_5}{\partial x^1} + \alpha V\frac{\partial u_5}{\partial x^2} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots, \\ 2a\frac{\partial v_5}{\partial x^1} &= -\alpha\frac{\partial u_6}{\partial x^1} + \alpha V\frac{\partial(u_5, u_6)}{\partial x^2} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots. \end{aligned}$$

Then equations (19h) can be used to put equations (5) and (6) in form

$$(5')\quad \frac{\partial u_6}{\partial x^2} = \alpha V\frac{\partial u_6}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$(6')\quad \frac{\partial u_5}{\partial x^2} = \alpha V\frac{\partial u_5}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots$$

respectively. Then (5′) and (6′) can be used to eliminate their left members from the solved form of (1)-(4), (7)-(20) and (19a)-(19d). The resulting equations will be denoted by (1′)-(4′), (7′)-(20′) and (19a′)-(19d′) respectively. Thus we have for these equations

$$(19a') \qquad \frac{\partial}{\partial x^1}\begin{Bmatrix} u_1, \cdots, u_4 \\ v_5, v_6 \end{Bmatrix} = \frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$(19b') \qquad \frac{\partial}{\partial x^2}\begin{Bmatrix} u_1, \cdots, u_4 \\ v_1, \cdots, v_6 \end{Bmatrix} = \alpha V \frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$(19c') \qquad \frac{\partial}{\partial x^3}\begin{Bmatrix} u_1, \cdots, u_4 \\ v_1, \cdots, v_6 \end{Bmatrix} = \cdots,$$

$$(19d') \qquad \frac{\partial}{\partial x^4}\begin{Bmatrix} u_1, \cdots, u_4 \\ v_1, \cdots, v_6 \end{Bmatrix} = \cdots;$$

$$\left.\begin{matrix}(1')\text{-}(4') \\ (7')\text{-}(14')\end{matrix}\right\} \qquad \frac{\partial}{\partial x^1}\begin{Bmatrix} v_1, \cdots, v_4 \\ v_7, v_8 \\ w_1, \cdots, w_6 \end{Bmatrix} = \alpha V \frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$(15')\text{-}(20') \qquad \frac{\partial}{\partial x^2}\{w_1, \cdots, w_6\} = \alpha V \frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots.$$

Together with above system we must consider the equations

$$(19i) \qquad \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} + g_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma} \qquad \left(\begin{matrix}\gamma = 1, \cdots, 4 \\ \alpha, \beta = 1, \cdots, 4 \\ \alpha \geqq \beta\end{matrix}\right),$$

$$(19j) \qquad \frac{\partial \gamma_{lm}}{\partial x^\alpha} = \frac{\partial \gamma_{pq}}{\partial x^r} + \ast \qquad \left(\begin{matrix}m = 0, \cdots, 3 \\ l = 1, \cdots, \gamma_m \\ \alpha = 1, \cdots, m \\ r > \alpha\end{matrix}\right),$$

where the notation

$$\gamma_{i0} \sim \Gamma^i_{11},$$
$$\gamma_{i1} \sim \Gamma^i_{12}, \Gamma^i_{22},$$
$$\gamma_{i2} \sim \Gamma^i_{13}, \Gamma^i_{23}, \Gamma^i_{33},$$
$$\gamma_{i3} \sim \Gamma^i_{14}, \Gamma^i_{24}, \Gamma^i_{34}, \Gamma^i_{44}$$

is used (see § 4). In addition, differentiating the condition $g^{11} = 0$ with respect to x^2, x^3, x^4, we have

$$(19k) \qquad \frac{\partial g^{11}}{\partial x^\gamma} = -g^{1\sigma}\Gamma^1_{\sigma\gamma} - g^{1\sigma}\Gamma^1_{\sigma\gamma} = 0 \qquad (\gamma = 2, 3, 4);$$

also differentiating a second time, we have

$$(19l) \qquad \sum_{\sigma=2}^{4} g^{1\sigma}\frac{\partial \Gamma^1_{\sigma\gamma}}{\partial x^\delta} + \ast = 0 \qquad (\gamma \leqq \delta;\ \gamma, \delta = 2, 3, 4).$$

If we expand the equations (19l) and then eliminate the left members of (19j), we find that these six equations can be solved to obtain

$$(19m) \qquad \begin{aligned} \frac{\partial \Gamma^1_{22}}{\partial x^\varepsilon} &= \sum \frac{\partial \gamma_{pq}}{\partial x^r} + \ast && (\varepsilon = 2, 3, 4), \\ \frac{\partial \Gamma^1_{23}}{\partial x^\varepsilon} &= \sum \frac{\partial \gamma_{pq}}{\partial x^r} + \ast && (\varepsilon = 3, 4), \\ \frac{\partial \Gamma^1_{24}}{\partial x^4} &= \sum \frac{\partial \gamma_{pq}}{\partial x^r} + \ast \end{aligned}$$

where the $\sum$ denotes a linear form in parametric derivatives of (19j), and $r \geqq \varepsilon$ for any equation of this system.

Let us now assign canonical cotes in the following manner:

cote	x^1	x^2	x^3	x^4	$u_1, \cdots, u_4$ $v_1, \cdots, v_6$	u_5, u_6 $w_1, \cdots, w_6$	v_7, v_8
1	1	1	1	1	0	0	0
2	1	0	0	0	0	0	0
3	0	1	0	0	0	0	0
4	0	0	1	0	0	0	0
5	0	0	0	1	0	0	0
6	0	0	0	0	3	2	1

In addition we shall take all six cotes of the $g_{\alpha\beta}$ to be zero and the first five cotes of the $\Gamma^{\alpha}_{\beta\gamma}$ as zero; the sixth cote of the Γ's occurring in the left members of (19m) will be taken to be 1 and the sixth cote of all other Γ's will be given the value zero.

With the above assignment of cotes it is seen that the equations (1′)–(20′), (19a′)–(19d′), (19i) and (19j) constitute the system R and (19m) the system S of a normal system. The arbitrary data predicted by the form of the left members of this normal system can be represented by the following scheme

$$
(19\text{n}) \qquad
\begin{array}{lll}
u_5,\ u_6 & \sim J(x^1, x^3, x^4) & \text{for } x^2 = 0, \\
v_7,\ v_8 & \sim K(x^2, x^3, x^4) & \text{for } x^1 = 0, \\
w_1, \cdots, w_6 & \sim L(x^3, x^4) & \text{for } x^1 = x^2 = 0, \\
\Gamma^i_{11} & \sim P(x^1, x^2, x^3, x^4), & \\
\Gamma^i_{12},\ \Gamma^j_{22} & \sim Q(x^2, x^3, x^4) & \text{for } x^1 = 0, \\
\Gamma^i_{13},\ \Gamma^j_{23},\ \Gamma^i_{33} & \sim R(x^3, x^4) & \text{for } x^1 = x^2 = 0, \\
\Gamma^i_{14},\ \Gamma^j_{24},\ \Gamma^i_{34},\ \Gamma^i_{44} & \sim S(x^4) & \text{for } x^1 = x^2 = x^3 = 0,
\end{array}
$$

where $i = 1, \cdots, 4$; $j = 2, 3, 4$. The remaining unknowns, namely

$$
(19\text{o}) \qquad u_1, \cdots, u_4,\ v_1, \cdots, v_6,\ \Gamma^1_{22},\ \Gamma^1_{23},\ \Gamma^1_{24},\ g_{11}, \cdots, g_{44}
$$

can take on arbitrary values at the point $x^i = 0$. That the system in question is completely integrable will be proved in § 20 and hence we can state the following

Existence Theorem: *Let the arbitrary functions represented by* (19n) *and the arbitrary constants* (19o) *be assigned subject to the condition that the initial values at* $x^i = 0$ *are such that* (1) *they satisfy equations* (7b), (2) *the determinant* $|(g_{\alpha\beta})_0|$ *does not vanish,* (3) *the inequalities* (19.4) *are satisfied,* (4) *the quantity* $(g^{11})_0$ *is equal to zero, and* (5) *the equations* (19k) *are satisfied. Then there exists one, and only one, set of functions* $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ *analytic in the neighborhood of* $x^i = 0$ *which gives rise to the given functions indicated by* (19n) *and the given initial values of the constants* (19o), *and which constitutes a set of integrals of equations* (7b) *for which the surface* $x^1 = 0$ *is a characteristic surface.*

It can be observed that the inequalities (19.4) impose no restriction on the integrals $g_{\alpha\beta}(x)$ since a coördinate system can always be selected in which the equation of the characteristic surface C_3 is $x^1 = 0$ and the inequalities (19.4) are satisfied. The above existence theorem will give all the characteristic surfaces C_3, since if the equation (19.3) is not satisfied over $\Phi = 0$ we will be able to find a coördinate system in which the system (7c), (7d), (19i) and (19j) can be put in regular form at some point P on $\Phi = 0$.

If a surface $x^1 = x^2 = 0$ is to be a two dimensional characteristic surface C_2, either one or both of the quantities g^{11} and $g^{11}\, g^{22} - g^{12}\, g^{12}$ must vanish over $x^1 = x^2 = 0$. If this were not the case we could find a coördinate system by a transformation of the type (19.12) for $\alpha = 2$ in which the quantity (21) in § 7 would not be zero at some point P on $x^1 = x^2 = 0$, and hence the system could be put in regular form in the neighborhood of the point P. Let us divide the discussion into four different cases.

Case I. $\alpha = 0$ *for* $x^1 = x^2 = 0$; $\alpha\beta - a^2 \neq 0$ *at* $x^i = 0$.

Case II. $\alpha = \beta = 0$ *for* $x^1 = x^2 = 0$; $a \neq 0$ *at* $x^i = 0$.

In these cases it is evident that an existence theorem could be stated which would be of the same form as the existence theorem for the three dimensional characteristic surface C_3 except that there would be additional arbitrariness in the arbitrary data corresponding to the functions $\Gamma^{\alpha}_{\beta\gamma}$. The surface $x^1 = x^2 = 0$ is a two dimensional characteristic surface C_2 of type 1.

Case III. $\alpha\beta - a^2 = 0$ *for* $x^1 = x^2 = 0$; $\alpha \neq 0$ *at* $x^i = 0$. In this case it is easily seen that a coördinate transformation of the type (12.12) can be made so that the determinant

$$\begin{vmatrix} \bar{\alpha} & \bar{a} & \bar{b} \\ \bar{a} & \bar{\beta} & \bar{d} \\ \bar{b} & \bar{d} & \bar{\gamma} \end{vmatrix} = -\frac{(\bar{\alpha}\,\bar{d} - \bar{a}\,\bar{b})^2}{\bar{\alpha}}$$

does not vanish at some point P on $x^1 = x^2 = 0$. Since at the point P the determinant formed from Table II,

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 15 \end{vmatrix} = 4\,(\alpha\, d - a\, b)^5$$

disregarding sign, it is easily seen that the system can be put into regular form at the point P on $\bar{x}^1 = \bar{x}^2 = 0$ and hence the surface $x^1 = x^2 = 0$ is not a characteristic surface C_2.

Case IV. $\alpha = \alpha\beta - a^2 = 0$ *for* $x^1 = x^2 = 0$; $\beta \neq 0$ *at* $x^i = 0$. Evidently the surface $x^1 = x^2 = 0$ is a characteristic surface C_2 for Case IV but the detailed treatment of the corresponding existence theorem will be omitted.

Case V. $\alpha = \beta = a = 0$ *for* $x^1 = x^2 = 0$. From the form of Table II and the equations (1)–(20) it is seen that for this case it is impossible to solve for either of the derivatives $\partial u_1/\partial x^1$ and $\partial u_1/\partial x^2$ and hence the system cannot be replaced by a normal system having arbitrary data of the form that occurs in Cases I and II.

We shall consider in detail the existence theorem for Case V. Let us suppose that

$$\text{(19p)} \qquad b \neq 0, \quad c \neq 0, \quad d \neq 0, \quad e \neq 0, \quad cd - be \neq 0,$$

at $x^i = 0$; these conditions can always be obtained by a transformation of the allowed type (12.12). Then from

$$\text{(19q)} \qquad \text{Table I:} \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \end{vmatrix} = -16\, b^2 (c\, d - b\, e), \text{ over } x^1 = x^2 = 0,$$

$$\text{(19r)} \qquad \text{Table II:} \quad \begin{vmatrix} 1 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 9 & 17 \end{vmatrix} = 8\, b^4 (c\, d - b\, e), \text{ over } x^1 = x^2 = 0,$$

$$\text{(19s)} \qquad \text{Table II:} \quad \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 5 & 8 & 9 & 10 & 15 & 16 & 17 & 18 \end{vmatrix} = 32\, b^5 e\, (c\, d - b\, e)^2,$$
$$\text{over } x^1 = x^2 = 0$$

no regard being paid to algebraic sign in (19r) and (19s). From (19p) it follows that the above three determinants do not vanish at $x^i = 0$. Since the determinant (19q) does not vanish, we can drop equations

$$X_{231} = 0, \quad X_{241} = 0, \quad X_{331} = 0, \quad X_{341} = 0$$

from the system (7c). We can solve the remaining equations $X_{\alpha\beta 1} = 0$ for the derivatives of the unknowns $u_2, u_3, u_4, u_5, v_3, w_3$ with respect to x^1, since the determinant (19r) does not vanish at $x^i = 0$. Since the determinant (19s) does not vanish at $x^i = 0$, we can solve the remaining equations (7c) for derivatives of the unknowns $u_3, u_4, u_5, v_2, v_3, v_4, w_1, \cdots, w_4$ with respect to x^2, x^3, x^4. Arranging the equations so obtained in a form corresponding to (19a)–(19d), we shall denote them for the moment by (19α), (19β), (19γ), (19δ) respectively. By a process similar to that used above the equations (19α)–(19δ) can be combined with the equations (1), (2), (4)–(10), (12)–(15), (19), (20) so as to obtain a system completely solved for the derivatives in the left members of the above equations (19α)–(20), where equation (15) is written

$$(15) \qquad \frac{\partial v_6}{\partial x^2} = -2\frac{\partial w_1}{\partial x^2} - \frac{3}{2}\frac{\partial v_4}{\partial x^3} + \frac{3}{2}\frac{\partial v_3}{\partial x^4} + \bigstar.$$

It is to be noticed that the equation (19β) having $\partial w_1/\partial x^2$ for left member does not contain $\partial v_6/\partial x^2$ on the right and hence the above elimination is possible.

Now consider the equations

$$(3) \qquad \frac{\partial u_2}{\partial x^2} = -\frac{\partial v_3}{\partial x^1} + \frac{\partial u_1}{\partial x^3} + \bigstar,$$

$$(11) \qquad \frac{\partial u_6}{\partial x^3} = \frac{\partial w_3}{\partial x^1} + \frac{\partial u_5}{\partial x^4} + \bigstar,$$

$$(16) \qquad \frac{\partial v_6}{\partial x^3} = -\frac{3}{4}\frac{\partial w_2}{\partial x^2} - \frac{1}{2}\frac{\partial w_1}{\partial x^3} + \frac{3}{4}\frac{\partial v_2}{\partial x^4} + \bigstar,$$

$$(17) \qquad \frac{\partial v_5}{\partial x^3} = \frac{\partial w_3}{\partial x^2} - \frac{2}{3}\frac{\partial w_1}{\partial x^4} + \frac{2}{3}\frac{\partial v_6}{\partial x^4} + \bigstar,$$

$$(18) \qquad \frac{\partial v_7}{\partial x^3} = -\frac{\partial w_4}{\partial x^2} + \frac{\partial v_1}{\partial x^4} + \bigstar.$$

Consider also the following equations selected from (19α)–(19γ)

$$\frac{\partial v_3}{\partial x^1} = -\frac{\gamma}{2b}\frac{\partial v_1}{\partial x^1} - \frac{c}{b}\frac{\partial v_4}{\partial x^1} - \frac{f}{b}\frac{\partial v_7}{\partial x^1} - \frac{\delta}{2b}\frac{\partial v_8}{\partial x^1} + \bigstar,$$

$$\frac{\partial w_3}{\partial x^1} = -\frac{d}{b}\frac{\partial w_5}{\partial x^1} - \frac{\gamma}{2b}\frac{\partial w_6}{\partial x^1} + \bigstar,$$

$$\frac{\partial w_2}{\partial x^2} = -\frac{e}{b}\frac{\partial w_5}{\partial x^2} + V\frac{\partial w_6}{\partial x^2} + \bigstar,$$

$$\frac{\partial w_3}{\partial x^2} = -\frac{d}{b}\frac{\partial w_5}{\partial x^2} + V\frac{\partial w_6}{\partial x^2} + \bigstar,$$

$$\frac{\partial w_4}{\partial x^2} = \frac{c}{b}\frac{\partial w_5}{\partial x^2} + V\frac{\partial w_6}{\partial x^2} + \bigstar,$$

$$\frac{\partial u_3}{\partial x^2} = -\frac{d}{e}\frac{\partial u_2}{\partial x^2} - \frac{(c^2\gamma - 2bcf + b^2\delta)}{2b^2 e}\frac{\partial u_6}{\partial x^3} + \frac{\partial(v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^2} + \bigstar,$$

$$\frac{\partial v_2}{\partial x^2} = \frac{\partial(v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^2} + \bigstar,$$

$$\frac{\partial w_1}{\partial x^\varepsilon} = \frac{c}{2b}\frac{\partial v_5}{\partial x^\varepsilon} + \frac{d}{2b}\frac{\partial v_7}{\partial x^\varepsilon} + \frac{e}{2b}\frac{\partial v_8}{\partial x^\varepsilon} + \frac{f}{2b}\frac{\partial w_5}{\partial x^\varepsilon} + \bigstar. \qquad (\varepsilon = 2, 3).$$

The terms with coefficients which vanish over $x^1 = x^2 = 0$ have not been written down in these equations. Hence equation (3) can be written

$$\text{(19t)} \quad \frac{\partial u_2}{\partial x^2} = \frac{cd}{be}\frac{\partial u_2}{\partial x^2} + \frac{c(c^2\gamma - 2bcf + b^2\delta)}{2b^2 e}\frac{\partial u_6}{\partial x^2} + \frac{\partial(v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^2} + \frac{\partial u_1}{\partial x^3} + \frac{\partial(u_2, u_6, v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^3} + \cdots.$$

Then making use of the solved form of equations (19 α)–(19 δ') and (1), (2), (4)–(10), (12)–(15), (19), (20) we can put the above equations (18), (17), (16), (11), (19t) in the following form

$$\text{(18')} \quad \frac{\partial v_7}{\partial x^3} = -\frac{c}{b}\frac{\partial v_8}{\partial x^3} + V\frac{\partial w_5}{\partial x^3} + \cdots,$$

$$\text{(17')} \quad \frac{\partial v_5}{\partial x^3} = -\frac{d}{b}\frac{\partial v_8}{\partial x^3} + V\frac{\partial w_5}{\partial x^3} + \cdots,$$

$$\text{(16')} \quad \frac{\partial v_6}{\partial x^3} = \left(\frac{be+cd}{2b^2}\right)\frac{\partial v_8}{\partial x^3} + V\frac{\partial w_5}{\partial x^3} + \cdots,$$

$$\text{(11')} \quad \frac{\partial u_6}{\partial x^3} = \frac{\partial(v_8, w_5, w_6)}{\partial x^3} + \cdots,$$

$$\text{(3')} \quad \frac{\partial u_2}{\partial x^2} = \frac{c(c^2\gamma - 2bcf + b^2\delta)}{2b^2(be-cd)}\frac{\partial u_6}{\partial x^2} + \frac{\partial(v_1, v_5, v_7, v_8)}{\partial x^2} + \left(\frac{be}{be-cd}\right)\frac{\partial u_1}{\partial x^3} + \frac{\partial(u_2, v_1, v_8, w_5, w_6)}{\partial x^3} + \cdots.$$

These equations can be used to eliminate their left members from the right of the above solved form of the remaining equations (19 α)–(20) and we can write the equations so obtained in the form:

$$\text{(19}\alpha'\text{)} \quad \frac{\partial}{\partial x^1}\begin{Bmatrix} u_2, u_3, u_4, u_5 \\ v_3, w_3 \end{Bmatrix} = V\frac{\partial u_6}{\partial x^1} + \frac{\partial(u_1, u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots,$$

$$\text{(19}\beta'\text{)} \quad \frac{\partial}{\partial x^2}\begin{Bmatrix} u_3, & u_4, & u_5 \\ v_2, & v_3, & v_4 \\ w_1, & \cdots, & w_4 \end{Bmatrix} = \frac{\partial(u_1, u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots,$$

$$\text{(19}\gamma'\text{)} \quad \frac{\partial}{\partial x^3}\begin{Bmatrix} u_3, & u_4, & u_5 \\ v_2, & v_3, & v_4 \\ w_1, & \cdots, & w_4 \end{Bmatrix} = \cdots,$$

$$\text{(19}\delta''\text{)} \quad \frac{\partial}{\partial x^4}\begin{Bmatrix} u_3, & u_4, & u_5 \\ v_2, & v_3, & v_4 \\ w_1, & \cdots, & w_4 \end{Bmatrix} = \cdots,$$

$$\left.\begin{matrix}\text{(1'), (2')} \\ \text{(4')–(10')} \\ \text{(12')–(14')}\end{matrix}\right\} \quad \frac{\partial}{\partial x^1}\begin{Bmatrix} v_1, v_2 \\ v_4, \cdots, v_8, w_1 \\ w_2, w_4, w_5, w_6 \end{Bmatrix} = \frac{\partial(u_1, u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots,$$

$$\text{(15')} \quad \frac{\partial v_6}{\partial x^2} = -\frac{c}{b}\frac{\partial v_5}{\partial x^2} - \frac{d}{b}\frac{\partial v_7}{\partial x^2} - \frac{e}{b}\frac{\partial v_8}{\partial x^2} + \cdots,$$

$$\text{(19'), (20')} \quad \frac{\partial}{\partial x^3}\{w_5, w_6\} = \cdots,$$

where again the terms with coefficients which vanish over $x^1 = x^2 = 0$ have not been written down.

Over the surface $x^1 = x^2 = 0$, we have

$$\text{(19u)} \quad g^{11} = g^{12} = g^{22} = 0$$

by hypothesis; hence

$$(19\text{v}) \qquad \frac{\partial g^{\mu\nu}}{\partial x^{\varepsilon}} = -g^{\mu\sigma}\Gamma^{\nu}_{\sigma\varepsilon} - g^{\sigma\nu}\Gamma^{\mu}_{\sigma\varepsilon} = 0, \quad (\mu,\ \nu = 1, 2;\ \varepsilon = 3, 4;\ \mu \leqq \nu)$$

over $x^1 = x^2 = 0$. Also over $x^1 = x^2 = 0$, we have

$$g^{13}\frac{\partial \Gamma^1_{3\varepsilon}}{\partial x^{\delta}} + g^{14}\frac{\partial \Gamma^1_{4\varepsilon}}{\partial x^{\delta}} = \bigstar,$$

$$(19\text{w}) \qquad g^{23}\frac{\partial \Gamma^2_{3\varepsilon}}{\partial x^{\delta}} + g^{24}\frac{\partial \Gamma^2_{4\varepsilon}}{\partial x^{\delta}} = \bigstar,$$

$$g^{13}\frac{\partial \Gamma^2_{3\varepsilon}}{\partial x^{\delta}} + g^{14}\frac{\partial \Gamma^2_{4\varepsilon}}{\partial x^{\delta}} + g^{23}\frac{\partial \Gamma^1_{3\varepsilon}}{\partial x^{\delta}} + g^{24}\frac{\partial \Gamma^1_{4\varepsilon}}{\partial x^{\delta}} = \bigstar$$

$$(\varepsilon \leqq \delta;\ \varepsilon,\ \delta = 3, 4).$$

Now eliminate the left members of (19j) from the above system (19w); the system resulting from (19w) with the exception of that equation which results from the last set of equations (19w) where $\varepsilon = \delta = 3$, can be solved so as to obtain

$$\frac{\partial \Gamma^i_{33}}{\partial x^{\varepsilon}} = (g)\,w_6 + \sum (g)\,\Gamma\Gamma \qquad (i = 1,\ 2;\ \varepsilon = 3, 4),$$

$$(19\text{x}) \qquad \frac{\partial \Gamma^i_{34}}{\partial x^4} = (g)\,w_6 + \sum (g)\,\Gamma\Gamma \qquad (i = 1, 2),$$

$$\frac{\partial \Gamma^i_{44}}{\partial x^4} = (g)\,w_6 + \sum (g)\,\Gamma\Gamma \qquad (i = 1, 2),$$

when account is taken of (19p); in these equations the (g) denotes a rational expression in the components $g_{\alpha\beta}$ and the last terms are quadratic in the $\Gamma^{\alpha}_{\beta\gamma}$. Elimination of the left members of (19x) from the equation corresponding to $\varepsilon = \delta = 3$ which was above excluded, gives

$$(19\text{y}) \qquad w_6 = \sum (g)\,\Gamma\Gamma,$$

in consequence of which $(19x)$ becomes

$$(19\text{z}) \qquad \frac{\partial \Gamma^i_{33}}{\partial x^{\varepsilon}} = \sum (g)\,\Gamma\Gamma, \quad \frac{\partial \Gamma^i_{34}}{\partial x^4} = \sum (g)\,\Gamma\Gamma, \quad \frac{\partial \Gamma^i_{44}}{\partial x^4} = \sum (g)\,\Gamma\Gamma$$

$$(i = 1,\ 2;\quad \varepsilon = 3,\ 4).$$

Differentiating (19y) with respect to x^3, x^4 and eliminating the left members of (19i), (19j), and (19z) from the resulting equations, we obtain

$$(19\text{A}) \qquad \frac{\partial w_6}{\partial x^{\varepsilon}} = \left[g_{\mu\nu};\ \gamma_{lm};\ \frac{\partial \gamma_{pq}}{\partial x^r};\ g_{\mu\nu,\sigma\tau}\right] \qquad (\varepsilon = 3, 4;\ r \geqq \varepsilon),$$

where the right member is a rational expression in the quantities indicated.

Cotes will be assigned to the independent variables x, the unknowns $u_1, \cdots, w_6$, the $g_{\alpha\beta}$ and the γ_{lm} as in the above treatment of the 3-dimensional characteristic surface C_3 with the exception of the sixth cotes of the unknowns for which we shall make the following assignments.

unknowns	$g_{\mu\nu}$	γ_{lm}	$v_1,\ w_5,\ w_6$	v_8	$v_5,\ v_7$	u_6	v_6	u_2	u_1	$u_3,\ u_4,\ u_5$ $v_2,\ v_3,\ v_4$ $w_1, \cdots, w_4$
cotes	0	0	1	2	3	4	5	6	7	8

The equation ($19\alpha'$) which contains the derivative $\partial u_2/\partial x^1$ in its left member when written more explicitly, is of the form:

$$\frac{\partial u_2}{\partial x^1} = \frac{c(c^2\gamma - 2bcf + b^2\delta)}{2b^2(be - cd)} \frac{\partial u_0}{\partial x^1} + \frac{\partial(u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots.$$

Hence using this equation and equations (3'), (16'), (15') we can construct two integrability conditions of the form

$$(19\,\mathrm{B}) \quad \begin{aligned} \frac{\partial^2 u_1}{\partial x^1 \partial x^3} &= V \frac{\partial^2 u_6}{\partial x^1 \partial x^3} + \frac{\partial^2(u_1, \cdots, w_6)}{\partial x^\mu \partial x^\nu} + \bigstar, & \left(\begin{matrix} \mu \leqq \nu \\ \nu = 4, \text{ if } \mu = 1 \end{matrix}\right), \\ \frac{\partial^2 v_8}{\partial x^2 \partial x^3} &= \frac{\partial^2(u_1, \cdots, w_6)}{\partial x^\varepsilon \partial x^\delta} + \bigstar, & \left(\begin{matrix} \varepsilon, \delta \neq 1 \\ \varepsilon \leqq \delta \\ \delta = 4, \text{ if } \varepsilon = 2 \end{matrix}\right). \end{aligned}$$

The condition that we be able to solve for the derivative $\partial^2 v_8/\partial x^2 \partial x^3$ is that $3(be - cd)/2b^2$ be different from zero.

The system R composed of ($19\alpha'$)–($19\delta'$), (1')–(20'), ($19i$), ($19j$), (19B) and the system S composed of (19z), (19A) constitute a normal system which will be shown to be completely integrable in § 20. The strongly complete set of monomials for the unknown u_1 is x^1x^3 and for v_8 is x^1 and x^2x^3; all other sets of monomials consist of a single variable. The calculation of the complementary sets and their multipliers shows that the data involving arbitrary functions can be represented by the following scheme:

$$(19\,\mathrm{C}) \quad \begin{array}{lll} u_1, v_1 & \sim P(x^2, x^3, x^4) & \text{for } x^1 = 0, \\ \partial u_1/\partial x^1, u_6 & \sim Q(x^1, x^2, x^4) & \text{for } x^3 = 0, \\ u_2, v_8, w_5 & \sim R(x^3, x^4) & \text{for } x^1 = x^2 = 0, \\ \partial v_8/\partial x^2, v_5, v_7 & \sim S(x^2, x^4) & \text{for } x^1 = x^3 = 0, \\ v_6 & \sim T(x^4) & \text{for } x^1 = x^2 = x^3 = 0, \\ \Gamma^i_{11} & \sim K(x^1, \cdots, x^4) & \\ \Gamma^i_{12}, \Gamma^i_{22} & \sim L(x^2, x^3, x^4) & \text{for } x^1 = 0, \\ \Gamma^i_{13}, \Gamma^i_{23}, \Gamma^j_{33} & \sim M(x^3, x^4) & \text{for } x^1 = x^2 = 0, \\ \Gamma^i_{14}, \Gamma^i_{24}, \Gamma^j_{34}, \Gamma^j_{44} & \sim N(x^4) & \text{for } x^1 = x^2 = x^3 = 0, \\ & (i = 1, 2, 3, 4;\ j = 3, 4). & \end{array}$$

The unknowns

$$(19\,\mathrm{D}) \quad u_3, u_4, u_5, v_2, v_3, v_4, w_1, w_2, w_3, w_4, w_6, \Gamma^k_{33}, \Gamma^k_{34}, \Gamma^k_{44}, g_{11}, \cdots, g_{44} \qquad (k = 1, 2)$$

can take on arbitrary values at the point $x^i = 0$.

EXISTENCE THEOREM: *Let the arbitrary functions represented by* (19C) *and the arbitrary constants* (19D) *be assigned subject to the condition that the initial values at* $x^i = 0$ *are such that* (1) *they satisfy equations* (7b), (2) *the determinant* $|(g_{\alpha\beta})_0|$ *does not vanish,* (3) *the inequalities* (19p) *are satisfied,* (4) *the equations* (19u), (19v) *and* (19y) *are satisfied. Then there exists one, and only one, set of functions* $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ *analytic in the neighborhood of* $x^i = 0$ *which gives rise to the given functions indicated by* (19C) *and the given initial values* (19D), *and which constitutes a set of integrals of equations* (7b) *for which the surface* $x^1 = x^2 = 0$ *is a characteristic surface* C_2.

It should be observed that for an arbitrary integral $g_{\alpha\beta}(x)$ of the field equations (7b), a characteristic surface of the type specified by the above existence theorem will not exist; characteristic surfaces of this type will only exist for special integrals of the equations (7b).

An analogous treatment could evidently be made for the characteristic curves C_1. We will however not consider the further treatment of these curves as we believe that the above discussion sufficiently illustrates the procedure to be adopted.

§ 20. Criterium of Complete Integrability.

The method of establishing complete integrability which was discussed in § 9 can be extended to the normal systems which were applied to the treatment of the characteristic surface problem for an invariantive system of partial differential equations. In describing this extension let us consider in particular the affine invariantive case for which we have the equations (4.3), (4.5), (4.7) and in addition a system of the type (19.2a). The process of forming the normal system may necessitate the addition of equations obtained by differentiation of the above system; the normal system being thus formed, the construction of its conditions of integrability will necessitate further differentiations. Let us suppose that the construction of the above normal system and its conditions of integrability involves D_1, D_2, D_3, D_4 differentiations of the systems (4.3), (4.5), (4.7), (19.2a) respectively. Now assume that an upper bound L to the number of conditions on the derivatives of the unknowns in the normal system which are imposed by (4.3), (19.2a), and those conditions obtained by D_1 differentiations of (4.3) and by D_4 differentiations of (19.2a), is established in some manner. Let us also assume that the integrability conditions taken at the point $x^1 = 0$ involve M derivatives, parametric for the normal system. Then if it is known that for an unrestricted affine space there are $L+M$ derivatives of the sort appearing in the integrability conditions which can have arbitrary values at a point P of space, the normal system must be completely integrable. If this were not the case there would be less than M arbitrary derivatives at the point $x^i = 0$ taking account of the original equations (4.3), (4.5), (4.7), (19.2a) and those conditions obtained from them by the above differentiations D, and hence there would be less than $L+M$ arbitrary derivatives for an unrestricted space, contrary to hypothesis. Analogous remarks apply to the cases of the metric and vector invariantive systems.

Consider for example the normal system which arose in § 19 in the treatment of the characteristic surface C_2 of the invariantive system (7b). It can be seen from the form of the arbitrary data (19n) and (19o) that the integrability conditions of the normal system (1′)-(20′), (19a′)-(19d′), (19i), (19j), and (19m) involve

	$K(4,2)$	or	10	derivatives	$g_{\mu\nu}$,
	$4K(4,2)$	„	40	„	γ_{lm},
	$4K(4,1)+7K(3,1)+11K(2,1)+15K(1,1)$...	„	74	„	$\partial \gamma_{lm}/\partial x^\mu$,
(20.1)	$4K(4,2)+7K(3,2)+11K(2,2)+15K(1,2)$...	„	130	„	$\partial^2 \gamma_{lm}/\partial x^\mu \partial x^\nu$,
	$K(4,2)\ K(4,2)-4K(4,3)$	„	20	„	B_{lm},
	$4K(3,1)+6K(2,1)$	„	24	„	$\partial B_{lm}/\partial x^\mu$,
	$4K(3,2)+6K(2,2)$	„	42	„	$\partial^2 B_{lm}/\partial x^\mu \partial x^\nu$

or a total of 340 parametric derivatives at the point $x^i = 0$; this number 340 is therefore the number M in the above discussion. At a point P of an unrestricted space there are

	$K(4,2)$	or	10	components	$g_{\mu\nu}$,
	$4\,K(4,2)$	„	40	„	$\Gamma^i_{\alpha\beta}$,
	$4\,K(4,3)$	„	80	„	$\Gamma^i_{\alpha\beta\gamma}$,
(20.2)	$4\,K(4,4)$	„	140	„	$\Gamma^i_{\alpha\beta\gamma\delta}$,
	$K(4,2)\;K(4,2)-4\,K(4,3)$	„	20	„	$g_{\alpha\beta,\,\gamma\delta}$,
	$K(4,2)\;K(4,3)-4\,K(4,4)$	„	60	„	$g_{\alpha\beta,\,\gamma\delta\varepsilon}$,
	$K(4,2)\;K(4,4)-4\,K(4,5)$	„	126	„	$g_{\alpha\beta,\,\gamma\delta\varepsilon\mu}$

or a total of 476 components which can take on arbitrary values. When the quantities of the type (20.1) are known at the point $x^i=0$,[42] the components (20.2) are determined at $x^i=0$; conversely when the quantities of the type (20.2) are known at $x^i=0$, the quantities of the type (20.1) are determined at this point. Hence it follows that there are 476 quantities of the type (20.1) which can take on arbitrary values at a point in the unrestricted space.[43] The maximum number of conditions that can be obtained from the system (7b) by a single differentiation is 36 and by two differentiations is 84, account being taken of the identities (7c). By two differentiations of $g^{11}=0$ over $x^1=0$, we obtain 6 conditions and by three differentiations, we obtain 10 conditions. Hence the upper bound L is $36+84+6+10$ or 136 and the number $L+M$ is equal to $136+340$ or 476 which was shown above to be equal to the number of arbitrary derivatives of the type (20.1) at a point of the unrestricted space. It follows therefore that the normal system composed of (1′)–(20′), (19a′)–(19d′), (19i), (19j) and (19m) is completely integrable.

A similar argument applies in the case of the existence theorems mentioned in connection with Cases I and II in § 19. The condition $g^{11}=0$ over $x^1=0$ is simply replaced by the condition $g^{11}=0$ over $x^1=x^2=0$ in Case 1 and by $g^{11}=g^{22}=0$ over $x^1=x^2=0$ in Case II. Obvious changes in the above numbers would then result.

In the normal system treated under Case V the above argument must be modified to a larger extent since second derivatives appear in the system. The integrability conditions at $x^i=0$ when expressed in terms of parametric derivatives will involve the following

	$K(4,2)$	or	10	derivatives	$g_{\mu\nu}$,
	$4K(4,2)$	„	40	„	γ_{lm},
	$4K(4,1)+8K(3,1)+10K(2,1)+12K(1,1)$	„	72	„	$\partial\gamma_{lm}/\partial x^\mu$,
	$4K(4,2)+8K(3,2)+10K(2,2)+12K(1,2)$	„	130	„	$\partial^2\gamma_{lm}/\partial x^\mu\partial x^\nu$,
(20.3)	$4K(4,3)+8K(3,3)+10K(2,3)+12K(1,3)$	„	212	„	$\partial^3\gamma_{lm}/\partial x^\mu\partial x^\nu\partial x^\delta$,
	$4K(4,4)+8K(3,4)+10K(2,4)+12K(1,4)$	„	322	„	$\partial^4\gamma_{lm}/\partial x^\mu\partial x^\nu\partial x^\delta\partial x^\varepsilon$,
	$K(4,2)\;K(4,2)-4K(4,3)$	„	20	„	B_{lm},
	$3K(3,1)+5K(2,1)+2+K(1,1)$	„	22	„	$\partial B_{lm}/\partial x^\mu$,
	$3K(3,2)+5K(2,2)+K(3,1)+K(2,1)+K(1,2)$	„	39	„	$\partial^2 B_{lm}/\partial x^\mu\partial x^\nu$,
	$3K(3,3)+5K(2,3)+K(3,2)+K(2,2)+K(1,3)$	„	60	„	$\partial^3 B_{lm}/\partial x^\mu\partial x^\nu\partial x^\delta$

or a total of $927=M$ parametric derivatives. To the 476 components in (20.2) which can have arbitrary values at a point P of space, we must add the

[42] *loc. cit.* Ann. (1), p. 720.

[43] *loc. cit.* Ann. (1), p. 690.

$4K(4,5)$	or 244 components		$\Gamma^{i}_{\alpha\beta\gamma\delta\varepsilon}$,
$4K(4,6)$	„ 336	„	$\Gamma^{i}_{\alpha\beta\gamma\delta\varepsilon\mu}$.
$K(4,2)\,K(4,5)-4K(4,6)$	„ 224	„	$g_{\alpha\beta,\gamma\delta\varepsilon\mu\nu}$,

which makes a total of 1260 components; hence there are 1260 derivatives of the type (20.3) which can take on arbitrary values at a point P of the unrestricted space. The maximum number of conditions imposed by the equations (7b) is $36+84+160$ or 280 since now three differentiations are allowed. From the equations $g^{11}=g^{12}=g^{22}=0$ over $x^1=x^2=0$ we obtain a maximum of $8+12+15+18$ or 53 conditions. Hence $L=280+53$ or 333 and $L+M$ is equal to the above number 1260. The system is therefore completely integrable.

§ 21. A Geometrical Interpretation.

Consider the problem treated in § 19, namely that of determining a solution $v_k(x)$ of the system (2.1) such that the surface $x^1=\cdots=x^\alpha=0$ is a characteristic surface $C_{n-\alpha}$ for the integral $v_k(x)$. Let us suppose that the combined system (2.1) and (19.1) has been put into a completely integrable normal form with the canonical assignment of cotes.

Let us first consider the case of an $(n-1)$-dimensional characteristic surface C_{n-1}. Suppose that some one of the functions belonging to the arbitrary data, let us say $\psi(x)$, is defined over a surface $x^{a_1}=\cdots=x^{a_\nu}=0$ where none of the a_i are equal to 1. Since the assignment of cotes is canonical we can confine ourselves to differentiations with respect to $x^2, x^3, \cdots, x^n$ in determining the power series expansions of the integrals $v_k(x)$ over the surface $x^1=0$. Moreover if we differentiate with respect to x^1 all equations of the system for which this differentiation is possible, we can thereafter confine ourselves to differentiations with respect to $x^2, \cdots, x^n$ so as to determine over the surface $x^1=0$, all first derivatives of the integrals $v_k(x)$. Now take the function $\psi(x)$ of the form

$$(x^1)^\mu\,\psi_\mu(x)+(x^1)^{\mu+1}\,\psi_{\mu+1}(x)+\cdots,$$

where the functions $\psi_i(x)$ are independent of x^1 and $\mu-2$ is equal to the maximum order of any derivative appearing in the normal system. From what we have just said regarding the determination of the integrals $v_k(x)$ and their first derivatives over the surface $x^1=0$ it is clear that the values of these quantities over $x^1=0$ will be independent of the choice of the above functions $\psi_\mu, \psi_{\mu+1}, \cdots$. Thus there exists an infinite number of sets of different integrals $v_k(x)$ such that each set of integrals and their first derivatives assume the same values over the characteristic surface $x^1=0$.

In particular the above discussion applies when the function $\psi(x)$ is defined over the entire x-space; if, however, $\psi(x)$ is defined only over a subspace then the completely arbitrary functions, i. e. those functions defined over the entire x-space, which are part of the arbitrary data, will be the same for each of the above sets of integrals $v_k(x)$. Under this latter condition, i. e. the definite assignment of data defined over the entire x-space, the values of the $v_k(x)$ and their first derivatives over the surface $x^1 = 0$ will uniquely determine a solution $v_k(x)$ if this surface is not a characteristic surface.

If the arbitrary data of a normal system obtained in connection with an $n - \alpha$ dimensional characteristic surface problem contains a function which is defined over a surface $x^{b_1} = \cdots = x^{b_\gamma} = 0$, where $\gamma \geqq \alpha$ and at least one of the indices $b_i > \alpha$, then we can have an infinite number of sets of integrals $v_k(x)$ such that each set of integrals and their first derivatives assume the same values over the characteristic surface $x^1 = \cdots = x^\alpha = 0$. In particular the arbitrary data defined over the entire x-space and all surfaces of dimensionality greater than $n - \alpha$, may be the same for each of the above sets of integrals $v_k(x)$. Under these conditions the integrals $v_k(x)$ would be determined uniquely if the surface $x^1 = \cdots = x^\alpha = 0$ were not a characteristic surface. Analogous remarks apply to the invariantive systems.

The examples of § 19 will serve to illustrate the above remarks. In the case of the $(n-1)$-dimensional characteristic surface defined by equations (19.3) the arbitrary data (19 n) contains a function $J(x^1 x^3 x^4)$ defined over $x^2 = 0$ so that the above interpretation is possible. The existence theorems mentioned in Case I and Case II will however not permit the corresponding interpretation for the $(n-2)$-dimensional characteristic surface. For Case V we have an arbitrary function $S(x^2, x^4)$ defined over $x^1 = x^3 = 0$ so that the interpretation applies.

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