The Variational Bicomplex

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The first main application of the variational bicomplex is to provide a natural geometric context for the calculus of variations. However, applications extend to the general theory of conservation laws for PDE, characteristic classes, Gelfand-Fuks cohomology and more things I don't know anything about.

The Variational Bicomplex!!!

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Conservation Laws,

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Conservation Laws, Lagrangians,

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Conservation Laws, Lagrangians, Euler-Lagrange Eqns, Helmholtz Conds

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$\pi : M \to X$ will be a **fiber bundle** or more generally a fibered manifold.

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For purely local considerations, we may assume $M = X \times U$ where
$X = \mathbb{R}^p$ with coordinates $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^p)$ (*independent variables*)
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Multi-index notation will be used for partial derivatives. If $J = (j_1, \ldots, j_k)$, $1 \leq j_\nu \leq p$, we write $|J| = k$ and

$$\mathbf{u}^\alpha_\mathbf{J} = \mathbf{u}^\alpha_{\mathbf{j_1 \cdots j_k}} = \frac{\partial^\mathbf{k} \mathbf{u}^\alpha}{\partial \mathbf{x}^{\mathbf{j_1}} \cdots \partial \mathbf{x}^{\mathbf{j_k}}}$$

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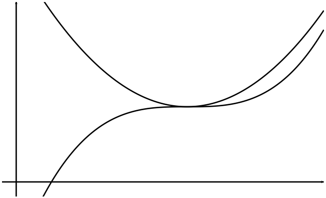
Unless noted otherwise, multi-indices will be symmetric, i.e. $J_1 = J_2$ if one is a reordering of the other. This reflects equality of mixed partials.

Finite Order Jet Bundles

Let $x \in X$,
declare two sections s_1 and s_2 to have **n^{th} order contact at x** if all partial derivatives up to order n of s_1 and s_2 agree at x.
This is an equivalence relation on sections.



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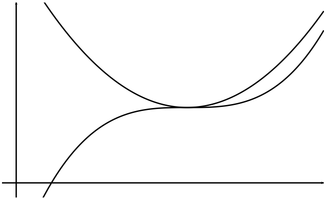
Denote by $\mathbf{j_n s|_x}$ the equivalence class of s: the **n-jet of s at x**. The n-jets constitute a bundle over M, the **jet bundle $\mathbf{J^n(M)}$**.
The jet $\mathbf{j_n s}$ is a section of this bundle.



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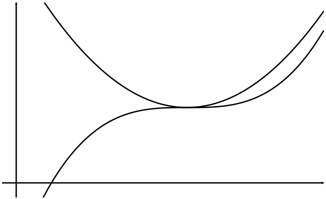

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Adapted local coordinates (x, u) on M induce **local coordinates $(\mathbf{x}, \mathbf{u}, \ldots, \mathbf{u}_\mathbf{J}^\alpha, \ldots) = (\mathbf{x}, \mathbf{u}^{(\mathbf{n})})$ on $\mathbf{J^n(M)}$**, $|J| \leq n$.

Example

Coordinates (x, y, u) on the bundle $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ induce coordinates $(x, y, u^{(2)}) = (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ on $J^2(\mathbb{R}^2 \times \mathbb{R})$.

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Specifying a section s of $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is tantamount to specifying a function $f : \mathbb{R}^2 \to \mathbb{R}$:

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The jet of this section is obtained by differentiating f:

$$j_2 s|_{(x_0,y_0)} = \left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2} \frac{\partial f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2} \right) \Bigg|_{(x_0,y_0)}$$

Variational Calculus

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Fundamental problem of variational calculus

Find the **extrema of the functional**

$$\mathcal{L}[u] = \int_R L(x, u^{(n)}) dx$$

over some class of functions $u = f(x)$.

Example (Minimizing Arclength)

Take $X = \mathbb{R}$ (with coordinate t) and $U = \mathbb{R}^2$ (with coordinates u, v).

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We search for extrema of $\mathcal{L}$ over functions $(u(t), v(t))$ with fixed endpoints.

One expects that a minimum will be a straight line connecting the fixed endpoints. We'll return to this example shortly.

Total Derivatives

Given $F(x, u^{(n)})$ one can "pretend" u is a function of x and take a partial derivative w.r.t. x^j. The result is the **total derivative** $D_{x^j}F$ or D_jF:

$$\mathbf{D_j F} = \frac{\partial \mathbf{F}}{\partial \mathbf{x^j}} + \sum_{\alpha, \mathbf{J}} \mathbf{u}^{\alpha}_{\mathbf{J}\,\mathbf{j}} \frac{\partial \mathbf{F}}{\partial \mathbf{u}^{\alpha}_{\mathbf{J}}} \qquad \text{where } u^{\alpha}_{J\,j} = \frac{\partial u^{\alpha}_J}{\partial x^j}.$$

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$$D_J = D_{j^1} \cdots D_{j^k} \quad \text{and} \quad (-D)_J = (-1)^{|J|} D_J$$

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NOTE: D_j raises the order of F by 1.

Euler-Lagrange Equations

To find extrema perform a **variation of u**: let u_ϵ be a family of functions which agree outside a compact $K \subset R$. If $u_0 = u$ is an extremal, then $\left.\frac{d\mathcal{L}[u_\epsilon]}{d\epsilon}\right|_{\epsilon=0} = 0$.

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Each v^α is (essentially) arbitrary, so **an extremum** u **must satisfy** the

Euler-Lagrange equations $\mathbf{E}_\alpha \mathbf{L} = \sum_{\mathbf{J}} (-\mathbf{D})_{\mathbf{J}} \frac{\partial \mathbf{L}}{\partial \mathbf{u}^\alpha_{\mathbf{J}}} = \mathbf{0}$

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Recall that we seek extrema of $\mathcal{L}[u,v] = \int_a^b \sqrt{(u_t)^2 + (v_t)^2} dt$.

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$$\begin{aligned}
(-D)_t \frac{\partial L}{\partial u_t} &= -D_t\left[u_t(u_t^2 + v_t^2)^{-1/2}\right] \\
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Thus the Euler-Lagrange equations are satisfied iff

$$(u_{tt} = 0 \text{ or } v_t = 0) \text{ and } (v_{tt} = 0 \text{ or } u_t = 0)$$

Differential Forms on Finite Order Jet Bundles

A **basic differential *k*-form** on $J^n(M)$ has a local coordinate expression

$$\mathbf{F(x, u^{(n)}) dx^{i_1} \wedge \cdots \wedge dx^{i_a} \wedge du^{\alpha_1}_{J_1} \wedge \cdots \wedge du^{\alpha_b}_{J_b}} \qquad \text{where } a + b = k.$$

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Let $u = f(x, y)$ (i.e. pull back by the section $s(x) = (x, y, f(x, y))$), then

$$(j_2 s)^* \theta_x = d\left(\frac{\partial f}{\partial x}\right) - \frac{\partial^2 f}{\partial x^2}dx - \frac{\partial^2 f}{\partial x \partial y}dy = 0$$

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Smooth functions on $J^\infty(M)$ factor through some finite order $J^n(M)$:

$$F : J^\infty \to \mathbb{R} \implies F = f \circ \pi^\infty_n \text{ for some smooth } f : J^n \to \mathbb{R}$$

Write $\mathbf{F[x, u]}$ for a function on $J^\infty(M)$.

Differential Forms on Infinite Order Jet Bundles

The pullback by the projection $\pi_k^n : J^n(M) \to J^k(M)$ gives a mapping

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ω may be interpreted as a form on some $J^n(M)$ of arbitrary finite order.
$\to$ From now on we write J^n or J^∞ instead of $J^n(M)$ or $J^\infty(M)$.

The Bigrading on $\Omega^*(J^\infty)$

We can define **basic contact forms** in $\Omega^1(J^\infty)$ without restriction on $|J|$:

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An arbitrary form in $\Omega^k(J^\infty)$ is a linear combination of the new basic k-forms

$$F[x,u]dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta_{J_1}^{\alpha_1} \wedge \cdots \wedge \theta_{J_s}^{\alpha_s} \qquad (\star)$$

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Now, fix r, s. Define the space $\mathbf{\Omega^{r,s}(J^\infty)}$ of forms of type (r, s) to be all linear combinations of expressions of the form $(\star)$.

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$\Omega^*(J^\infty)$ is a direct sum (the **bigrading** on $\Omega^*(J^\infty)$):

$$\Omega^*(J^\infty) = \bigoplus_{r,s} \Omega^{r,s}(J^\infty)$$

Example (rewriting a form in terms of the bigrading)

Consider $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with coordinates x, y and u^1, u^2. We choose a form on $J^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, and write it in terms of the bigrading:

$$\begin{aligned} dx \wedge du_x^1 \wedge du^2 &= dx \wedge (\theta_x^1 + u_{xx}^1 dx + u_{xy}^1 dy) \wedge (\theta^2 + u_x^2 dx + u_y^2 dy) \\ &= dx \wedge \theta_x^1 \wedge \theta^2 + u_y^2 dx \wedge \theta_x^1 \wedge dy + u_{xy}^1 dx \wedge dy \wedge \theta^2 \end{aligned}$$

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Thus $dx \wedge du^1_x \wedge du^2 \in \Omega^{1,2} \oplus \Omega^{2,1}$ since

$$dx \wedge \theta^1_x \wedge \theta^2 \in \Omega^{1,2}$$

and

$$u^2_y dx \wedge \theta^1_x \wedge dy \text{ and } u^1_{xy} dx \wedge dy \wedge \theta^2 \in \Omega^{2,1}$$

Splitting of the Exterior Derivative

Let $F[x, u]$ be a function on J^∞. The **exterior derivative** is defined as usual:

$$dF = \sum_j \frac{\partial F}{\partial x^j} dx^j + \sum_\alpha \sum_J \frac{\partial F}{\partial u_J^\alpha} du_J^\alpha$$

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We can write dF using contact forms and the total derivative:

$$\begin{aligned} dF &= \sum_j \frac{\partial F}{\partial x^j} dx^j + \sum_\alpha \sum_J \frac{\partial F}{\partial u_J^\alpha} du_J^\alpha \\ &= \sum_j \left(\frac{\partial F}{\partial x^j} + \sum_\alpha \sum_J u_{J\,j}^\alpha \frac{\partial F}{\partial u_J^\alpha} \right) dx^j + \sum_\alpha \sum_J \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha \\ &= \sum_j D_j F \, dx^j + \sum_\alpha \sum_J \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha \end{aligned}$$

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Define

$$\mathbf{d_H F} = \sum_j D_j F \, dx^j \quad \text{and} \quad \mathbf{d_V F} = \sum_\alpha \sum_J \frac{\partial F}{\partial u^\alpha_J} \theta^\alpha_J$$

Splitting of the Exterior Derivative, Continued

Starting with $F \in \Omega^{0,0}$ we found $dF = d_H F + d_V F$, where

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In general, for $\omega \in \Omega^{r,s}$, there is a splitting $\mathbf{d}\omega = \mathbf{d_H}\omega + \mathbf{d_V}\omega$, where

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Example. Consider $\theta^\alpha_J \in \Omega^{0,1}$. Then $d_H\theta^\alpha_J = -\sum_j \theta^\alpha_{Jj} \wedge dx^j$ and $d_V\theta^\alpha_J = 0$. To see this, compute:

$$\begin{aligned} d\theta^\alpha_J &= d\left(du^\alpha_J - \sum_j u^\alpha_{Jj} dx^j\right) \\ &= -\sum_j du^\alpha_{Jj} \wedge dx^j \\ &= -\sum_j \theta^\alpha_{Jj} \wedge dx^j + \sum_{i,j} u^\alpha_{Jji} dx^i \wedge dx^j \\ &= -\sum_j \theta^\alpha_{Jj} \wedge dx^j \end{aligned}$$

Building the Bicomplex

By virtue of $\mathbf{d^2 = 0}$ we have $\mathbf{d_H^2 = 0}$, $\mathbf{d_V^2 = 0}$ and $\mathbf{d_H d_V = -d_V d_H}$.

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$$\Omega^{0,s} \xrightarrow{d_H} \Omega^{1,s} \xrightarrow{d_H} \Omega^{2,s} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{n,s}$$

is a complex, and for fixed r

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is a complex. In summary we have a **bicomplex** of differential forms:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \cdots & & \vdots & & \vdots \\
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Building the Bicomplex, Continued

Let's take a closer look at the bottom right edge of the bicomplex.

$$\begin{array}{ccc} \cdots & \xrightarrow{d_H} & \Omega^{p,1} \\ & & \uparrow{\scriptstyle d_V} \\ \cdots & \xrightarrow{d_H} & \Omega^{p,0} \end{array}$$

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$$d_V(L[x,u]dx^1 \wedge \cdots \wedge dx^p) = \sum_{\alpha,J} \frac{\partial L}{\partial u_J^\alpha} \theta_J^\alpha \wedge dx^1 \wedge \cdots \wedge dx^p$$

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Notice the similarity with the variational derivative:

$$d_V(Ldx) = \sum_{\alpha,J} \frac{\partial L}{\partial u^\alpha_J} \theta^\alpha_J \wedge dx \quad \leftrightarrow \quad \left.\frac{d\mathcal{L}[u_\epsilon]}{d\epsilon}\right|_{\epsilon=0} = \int_R \left\{ \sum_{\alpha,J} \frac{\partial L}{\partial u^\alpha_J} D_J v^\alpha \right\} dx$$

Integration by Parts

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To see this, let $\eta^j = dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^p$ and compute

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&d_H(P[x,u]\theta^\alpha_J \wedge \eta^j) \\
&\qquad = \left(\sum_i D_i P dx^i \right) \wedge \theta^\alpha_J \wedge \eta^j + P \wedge \left(-\sum_i \theta^\alpha_{J\,i} \wedge dx^i \right) \wedge \eta^j \\
&\qquad = D_j P \, dx^j \wedge \theta^\alpha_J \wedge \eta^j - P\theta^\alpha_{J\,j} \wedge dx^j \wedge \eta^j \\
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We'll save the **second way** for another talk.

The Euler Operator

Using $\equiv$ for equivalence modulo $\operatorname{Im} d_H$, we find

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\begin{aligned}
d_V(L dx) &= \sum_{\alpha,J} \frac{\partial L}{\partial u_J^\alpha} \theta_J^\alpha \wedge dx \\
&\equiv \sum_{\alpha,J} (-D)_J \frac{\partial L}{\partial u_J^\alpha} \theta^\alpha \wedge dx \\
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Define $\mathcal{F}^{\mathbf{1}} = \Omega^{n,1} / \operatorname{Im} d_H$ and π the projection $\pi : \Omega^{n,1} \to \mathcal{F}^1$. $\mathcal{F}^1$ is called the space of source forms.

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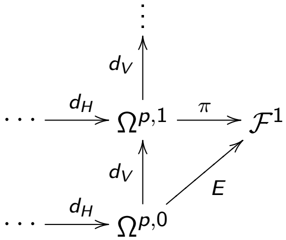
$$\pi(d_V(Ldx)) \equiv \sum_\alpha E_\alpha(L)\, \theta^\alpha \wedge dx$$

We write $\pi \circ \mathbf{d_V} = \mathbf{E}$ and call E the **Euler operator**.

Building the Bicomplex, Continued

We build $\mathcal{F}^1$ and the Euler operator E into our bicomplex:



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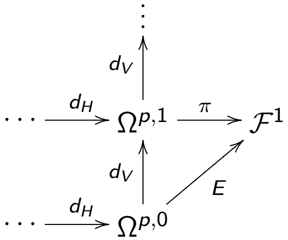

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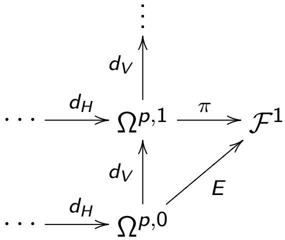

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Use d_V to define $\delta_V : \mathcal{F}^s \to \mathcal{F}^{s+1}$ by mapping equivalence representatives:

$$\delta_V(\omega + \operatorname{Im} d_H) = d_V\omega + \operatorname{Im} d_H$$

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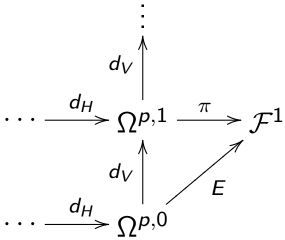

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Since $d_V d_H = -d_H d_V$, δ_V is well-defined, and since $d_V^2 = 0$, $\delta_V^2 = 0$.

Building the Bicomplex, almost done

We can now incorporate $\mathcal{F}^s$ and δ_V into our bicomplex:

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
\uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
\Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \longrightarrow & \mathcal{F}^2 \\
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\uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & \nearrow E & \\
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Apart from some minor ornamentation, this is the **variational bicomplex**.

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\end{array}
$$

Apart from some minor ornamentation, this is the **variational bicomplex**.

The edge of this bicomplex is of particular interest, and is called the **Euler-Lagrange complex**.

The Euler-Lagrange Complex

Theorem. The cohomology of the Euler-Lagrange complex is equal to the de Rham cohomology of M. In particular, if $M = \mathbb{R}^p \times \mathbb{R}^q$, the **Euler-Lagrange complex is exact**.

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Let's examine an easy consequence: suppose that L is such that the Euler-Lagrange equations are automatically satisfied. Thus $E(L\,dx) = 0$. By exactness, this means that $L\,dx = d_H\omega$ for some $\omega \in \Omega^{n-1,0}$.

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A general $\omega \in \Omega^{p-1,0}$ has the form $\sum_j P_j[x,u]\eta^j$, so

$$d_V(L\,dx) = 0 \iff d_H\omega = Ldx \iff L = \sum_j (-1)^{j-1} D_j P_j dx$$

Thus L is a null Lagrangian if and only if $L = \sum_j D_j\Big((-1)^{j-1} P_j\Big)$

Concluding Remarks