What are Moving Frames?

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A Vague Introduction

Suppose that we have a group of transformations G acting on a space M. What kinds of objects on M do not change when transformed by G?

Example.

$M = \mathbb{R}^2$ $\qquad$ $G = \text{rotations around the origin}$

A function on M depending only on radial distance, $G(x, y) = F(x^2 + y^2)$, will be *invariant*. Evaluating the function at (x, y) before rotating gives the same result as evaluating after rotating: $G(g \cdot (x, y)) = G(x, y)$.

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Are there other such functions? What other kinds of *invariant* objects can we consider? How do we find them? What good are *invariants* anyway?

Moving frames are a clever invention for studying invariants and their applications. The goal of this talk is to describe moving frames and their use in answering the above questions.

Examples of Lie Groups

We will be concerned only with Lie groups. These are groups which are also endowed with a *compatible* manifold structure.

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$GL(n, \mathbb{R}) =$ invertible $n \times n$ matrices over $\mathbb{R}$

$SL(n, \mathbb{R}) = n \times n$ matrices over $\mathbb{R}$ with determinant 1

$SO(n, \mathbb{R}) = n \times n$ orthogonal matrices over $\mathbb{R}$

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More Examples.

Take products, semi-direct products, etc.

$SE(n) = SO(n) \ltimes \mathbb{R}^n$, the special euclidean group

Semi-direct product structure: $(A, v) \cdot (B, w) = (AB, Bv + w)$

The above groups embed in $GL(n, \mathbb{R})$: $\quad (A, v) \mapsto \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$

Group Actions

What Lie groups do best is act on spaces. Our space M will be a manifold, usually just an open subset of $\mathbb{R}^n$.

Example.

$SO(2)$ acts on $\mathbb{R}^2$ by rotation through an angle of φ:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} x\cos\varphi - u\sin\varphi \\ x\sin\varphi + u\cos\varphi \end{pmatrix}$$

Or more opaquely:

$$(\tilde{x}, \tilde{u}) = (x\cos\varphi - u\sin\varphi, x\sin\varphi + u\cos\varphi)$$

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Our space will carry a product structure $M = X \times U$ with variables (x, u). The x we treat as independent variable(s) and the u as dependent.

Group Actions continued

Example. $SE(2)$ acts on $\mathbb{R}^2$ by following the rotation with a translation:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x\cos\varphi - u\sin\varphi + a \\ x\sin\varphi + u\cos\varphi + b \end{pmatrix}$$

The semi-direct product structure is more transparent here:

$$\begin{aligned} (A,v)(B,w)\cdot\begin{pmatrix} x \\ u \end{pmatrix} &= (A,v)\cdot\left(B\begin{pmatrix} x \\ u \end{pmatrix} + w\right) \\ &= AB\begin{pmatrix} x \\ u \end{pmatrix} + Aw + v \\ &= (AB, Aw+v)\cdot\begin{pmatrix} x \\ u \end{pmatrix} \end{aligned}$$

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Notice that this group action is definitely not free. For example,

$$\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Group Actions continued more

Remember that a group action is free if the only group element that fixes anything is the identity:

$$g \cdot x = x \quad \Longrightarrow \quad g = e$$

A group action is free iff its orbits have the same dimension as the group.

Group Actions continued more

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The action of $SO(2)$ on $\mathbb{R}^2$ from the introduction is free if we delete the origin from $\mathbb{R}^2$. $SO(2)$ is a 1-parameter group, and the orbits (excepting the origin), have dimension 1.

The action $SE(2)$ on $\mathbb{R}^2$ cannot be made free so simply. We will return to the topic of freeing up group actions momentarily.

Is the action of $SO(3)$ on $\mathbb{R}^3$ free?

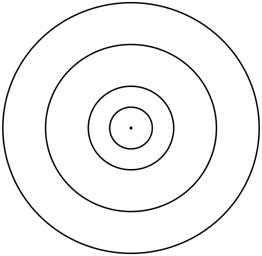

Figure: $SO(2)$ orbits

Group Actions continued more and more!

A possible problem with group actions we'd like to ignore is when orbits get too close to one another.

Example.

Consider the action of $\mathbb{R}$ on the torus $S^1 \times S^1$ given by

$$\lambda \cdot (e^{i\phi}, e^{i\theta}) \mapsto (e^{i(\phi + a\lambda)}, e^{i(\theta + b\lambda)})$$

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The orbit of a point is a line looping around the torus, and if a/b is not a rational multiple of 2π, then this orbit will be dense.

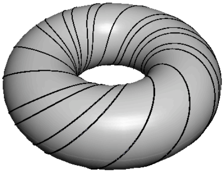

Figure: Getting dense?

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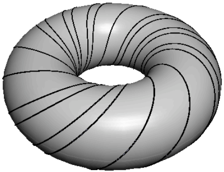
The orbit of a point is a line looping around the torus, and if a/b is not a rational multiple of 2π, then this orbit will be dense.

A group action is regular if each orbit has the same dimension and every point has arbitrarily small neighborhoods which intersect each orbit in a connected set.

A regular group action can't have dense orbits.



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Invariantizing a function: motivation.

Suppose G acts on M. A function $I : M \to \mathbb{R}$ is called an invariant if

$$I(g \cdot z) = I(z) \qquad \text{for all } g \in G \text{ and } z \in M.$$

Geometrically, this means that I is constant on orbits.

Create invariant functions simply by choosing their value on each orbit!
Too good to be true!

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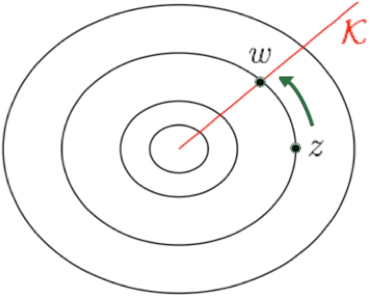
Starting with any function F, choose a cross-section $\mathcal{K}$ to the orbits and define a new, *invariant* function ιF by

$$\iota F(z) = F(w)$$

where w is the point on $\mathcal{K}$ intersecting the orbit of z.



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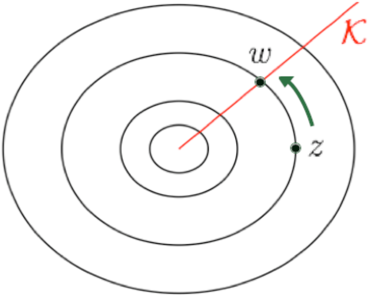
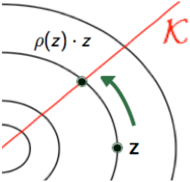


Such a cross-section exists whenever the action is free and regular.

Constructing a Moving Frame Geometrically

Before we define moving frame, we'll build one. Actually, we are almost done building one already.

On the previous slide, we mapped z to the point w on $\mathcal{K}$ intersecting the orbit of z. Since w and z are on the *same orbit*, there is some group element g with $g \cdot z = w$. Thus we can define a map



$$\rho : M \to G \qquad \text{where} \qquad \rho(z) = g.$$

This is a moving frame ! (with cross-section $\mathcal{K}$ for the action of G on M).

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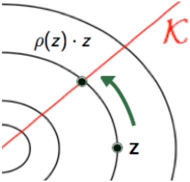

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Notice the following *extremely important* property of the map ρ: if $h \in G$, then

$$\rho(h \cdot z) = \rho(z) h^{-1}.$$

This is called (right) equivariance.

Definition.

A (right) moving frame is a (right) equivariant map $\rho : M \to G$.

Constructing a Moving Frame for $SO(2)$.

- Recall that the formula for the action of $SO(2)$ on $\mathbb{R}^2$ is

$$(\tilde{x}, \tilde{u}) = (x \cos\varphi - u \sin\varphi, x \sin\varphi + u \cos\varphi).$$

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- Given a point (x, u), we look for the φ such that $(\tilde{x}, \tilde{y}) \in \mathcal{K}$, so we solve

$$\tilde{x} = x\cos\varphi - u\sin\varphi = 0$$

for the group parameter φ. This gives the *moving frame* in parameter form

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- Since $\sin\varphi = x/\sqrt{x^2 + u^2}$ and $\cos\varphi = u/\sqrt{x^2 + u^2}$ we can also write the moving frame in *matrix form*:

$$\rho(x, u) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \frac{1}{\sqrt{x^2 + u^2}} \begin{pmatrix} u & -x \\ x & u \end{pmatrix}$$

Constructing a Moving Frame for $SO(2)$ continued.

Check right equivariance:

$$\begin{aligned}\rho(x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta) &= \arctan\left(\frac{x\cos\theta - u\sin\theta}{x\sin\theta + u\cos\theta}\right)\\ &= \arctan\left(\frac{\frac{x}{u} - \tan\theta}{\frac{x}{u}\tan\theta + 1}\right)\\ &= \arctan(x/u) - \theta\end{aligned}$$

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Similarly, for the matrix version of the moving frame

$$\begin{aligned}\rho(\tilde{x}, \tilde{u}) &= \frac{1}{\sqrt{x^2+u^2}}\begin{pmatrix} x\sin\theta + u\cos\theta & -(x\cos\theta - u\sin\theta) \\ x\cos\theta - u\sin\theta & x\sin\theta + u\cos\theta \end{pmatrix}\\ &= \frac{1}{\sqrt{x^2+u^2}}\begin{pmatrix} u & -x \\ x & u \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\end{aligned}$$

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Everything we've done for $SO(2)$ works in general, so long as we are able to choose a suitable cross-section to the orbits of the group action.

Theorem.

A moving frame exists iff the action of G on M is free and regular.

Let's Invariantize!

Remember the original motivation for constructing a moving frame: *if we map to the cross-section, then evaluate our function, the result is invariant.*

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Theorem. Let ρ be a moving frame and let F be a function on M. Then

$$\iota(F)(z) = F(\rho(z) \cdot z) \quad \text{is an invariant.}$$

Proof.

$$\begin{aligned} \iota(F)(g \cdot z) &= F(\rho(g \cdot z)g \cdot z) \\ &= F(\rho(z)g^{-1}g \cdot z) \qquad \text{(by equivariance)} \\ &= F(\rho(z) \cdot z) = \iota(F)(z). \quad \square \end{aligned}$$

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Example. Take $F(x, u) = u$. Then

$$F(\rho(x,u) \cdot (x,u)) = F(0, \sqrt{x^2 + u^2}) = \sqrt{x^2 + u^2}, \text{ radial distance!}$$

This is a good initial answer to the question: *how do we find invariants?*

What About Other Invariants?

We found that the radius function is invariant under rotation, so what? Are there other invariants not so geometrically obvious? Clearly any function of the radius will be an invariant, but are there any more? The answer is deceptively simple.

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Let $I(x, u)$ be an invariant for $SO(2)$. Then, by invariance,

$$\begin{aligned} I(x,u) &= I(\rho(x,u)\cdot(x,u)). \\ &= I(0, \sqrt{x^2+u^2}) \end{aligned}$$

So *any invariant* is just a function of the fundamental invariant

$$\iota(u) = \sqrt{x^2+u^2}.$$

This simple and powerful idea is called the Replacement Theorem.

Setting Group Actions Free

If a group action is not free, the process we've outlined doesn't work.
The action of $SE(2)$ on $\mathbb{R}^2$ is not free, and doesn't have any invariants!

The way to get around this is to have the group act on a larger space. We do this via a process called prolongation.

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Let $SE(2)$ act on $\mathbb{R}^2$ via

$$(\tilde{x}, \tilde{u}) = (x\cos\varphi - u\sin\varphi + a, x\sin\varphi + u\cos\varphi + b)$$

We *prolong* the action of $SE(2)$ to $\mathbb{R}^3$ by adding a coordinate u_x, and letting $SE(2)$ act on u_x as it would on a derivative, i.e. $u_x \mapsto \tilde{u}_{\tilde{x}}$. Now,

$$\tilde{u}_{\tilde{x}} = \tilde{u}_x \cdot x_{\tilde{x}} = \frac{\tilde{u}_x}{\tilde{x}_x} = \frac{\sin\varphi + u_x\cos\varphi}{\cos\varphi - u_x\sin\varphi}$$

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We have found the first prolongation of the action of $SE(2)$:

$$(\tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}) = (x \cos \varphi - u \sin \varphi + a, x \sin \varphi + u \cos \varphi + b, \frac{\sin \varphi + u_x \cos \varphi}{\cos \varphi - u_x \sin \varphi}).$$

Setting Group Actions Free continued

The second prolongation is found similarly:

$$\tilde{u}_{\tilde{x}\tilde{x}} = \left(\frac{\tilde{u}_x}{\tilde{x}_x}\right)_{\tilde{x}} = \frac{1}{\tilde{x}_x}\left(\frac{\tilde{u}_x}{\tilde{x}_x}\right)_x = \frac{u_{xx}}{(\cos\varphi - u_x \sin\varphi)^3}.$$

We have *prolonged* the action of $SE(2)$ to a space with coordinates (x, u, u_x, u_{xx}) called second order jet space $J^2(\mathbb{R}^2)$. The action is now free!

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In general, we may prolong any group action on M to $J^n(M)$ and we can be pretty sure that eventually we will have a free action.

Theorem. With a mild hypothesis (effectiveness of the group action), there is some n such that the prolonged action of G on J^n is free and regular on a dense subset of J^n.

A Moving Frame for $SE(2)$

- Recall the action of $SE(2)$ on J^2 sends (x, u, u_x, u_{xx}) to

$$(x\cos\varphi - u\sin\varphi + a, x\sin\varphi + u\cos\varphi + b, \frac{\sin\varphi + u_x\cos\varphi}{\cos\varphi - u_x\sin\varphi}, \frac{u_{xx}}{(\cos\varphi - u_x\sin\varphi)^3})$$

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- Choose a nice cross-section, like $\{x = 0, u = 0, \text{ and } u_x = 0\}$.
- Solve for the group parameters a, b, φ in the equations

$$x\cos\varphi - u\sin\varphi + a = 0 \qquad x\sin\varphi + u\cos\varphi + b = 0$$

$$\frac{\sin\varphi + u_x\cos\varphi}{\cos\varphi - u_x\sin\varphi} = 0$$

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$$(x\cos\varphi - u\sin\varphi + a, x\sin\varphi + u\cos\varphi + b, \frac{\sin\varphi + u_x\cos\varphi}{\cos\varphi - u_x\sin\varphi}, \frac{u_{xx}}{(\cos\varphi - u_x\sin\varphi)^3})$$

- Choose a nice cross-section, like $\{x = 0, u = 0, \text{ and } u_x = 0\}$.
- Solve for the group parameters a, b, φ in the equations

$$x\cos\varphi - u\sin\varphi + a = 0 \qquad x\sin\varphi + u\cos\varphi + b = 0$$

$$\frac{\sin\varphi + u_x\cos\varphi}{\cos\varphi - u_x\sin\varphi} = 0$$

- Arrive at the moving frame (in parameter form)

$$\varphi = -\arctan u_x \quad a = -\frac{x + uu_x}{\sqrt{1 + u_x^2}} \quad b = \frac{xu_x - u}{\sqrt{1 + u_x^2}}$$

Differential Invariants for $SE(2)$

With moving frame in hand, we can invariantize a function on J^2. The result will be functions of (x, u, u_x, u_{xx}) which are invariant under the prolonged transformations, called (second order) differential invariants.

Differential Invariants for $SE(2)$

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$$\begin{aligned}\iota(u_{xx}) &= \rho(x, u, u_x, u_{xx}) \cdot u_{xx} \\ &= \frac{u_{xx}}{(\cos\varphi - u_x \sin\varphi)^3} \qquad (\varphi = -\arctan u_x) \\ &= \frac{u_{xx}}{(1+u_x^2)^{3/2}}.\end{aligned}$$

You may recognize the *curvature* $\kappa = \dfrac{u_{xx}}{(1+u_x^2)^{3/2}}$.

(Insert geometric interpretation here.)

Generating More Differential Invariants

By a simple application of the *Replacement Theorem*, we conclude that any second order differential invariant must be a function of $\iota(u_{xx})$.

The Replacement Theorem will guarantee that any n-th order differential invariant will be a function of the fundamental differential invariants

$$I_2 = \iota(u_{xx}) \quad I_3 = \iota(u_{xxx}) \quad \cdots \quad I_n = \iota(u_{x^n})$$

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We obtain higher order invariants by invariantizing higher order derivatives. What happens if we differentiate fundamental invariants? Nothing good!

The derivative of an invariant might not be invariant. In order to generate more differential invariants, we need to first invariantize our derivative operator. This new operator may be used to generate new higher-order differential invariants.

We illustrate these ideas with a simple example.

Invariantizing the Derivative Operator

Consider the following scaling action of $(\mathbb{R}^+, \cdot)$ on $\mathbb{R}^2$: $(\tilde{x}, \tilde{u}) = (\lambda x, u/\lambda)$.

The prolonged action is $\tilde{u}_{\tilde{x}} = \dfrac{u_x}{\lambda^2}$, $\tilde{u}_{\tilde{x}\tilde{x}} = \dfrac{u_{xx}}{\lambda^3}$, *etc.*

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Choosing the cross-section $\{x = 1\}$ gives the moving frame $\lambda = 1/x$.

We have differential invariants

$$I = \iota(u) = xu, \quad I_1 = \iota(u_x) = x^2 u_x, \quad I_2 = \iota(u_{xx}) = x^3 u_{xx}, \quad etc.$$

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As noted earlier, the derivatives

$$\frac{d}{dx}(xu) = u + xu_x, \quad \frac{d}{dx}(x^2 u_x) = 2xu_x + x^2 u_{xx}, \quad etc.$$

might not be invariant. For example,

$$\tilde{u} + \tilde{x}\tilde{u}_{\tilde{x}} = \frac{1}{\lambda}(u + xu_x).$$

Invariantizing the Derivative Operator

But, we can invariantize the operator $\dfrac{d}{dx}$ just like a function

$$\mathcal{D} = \frac{d}{d\tilde{x}}\bigg|_{\lambda=1/x} = \frac{dx}{d\tilde{x}}\bigg|_{\lambda=1/x}\frac{d}{dx} = x\frac{d}{dx}$$

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Curiously, a quantity of great interest is the difference between the fundamental invariant I_k and the invariant derivative $\mathcal{D}(I_{k-1})$. From the above we see that

$$\mathcal{D}(I) - I_1 = I, \qquad \mathcal{D}(I_1) - I_2 = 2I_1, etc.$$

These recurrence relations illuminate the structure of the algebra of differential invariants, and can be useful in applications (as we will see).

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$$xu_x^2 - 3uu_x + 9x^2 = 0.$$

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Step 1. Let x, u be functions of a new parameter s. Our ODE becomes:

$$x\left(\frac{u_s}{x_s}\right)^2 - 3u\left(\frac{u_s}{x_s}\right) + 9x^2 = 0.$$

We are free to choose the parametrization arbitrarily!

Using Moving Frames to Solve an ODE

Step 2. Notice that $\mathbb{R}$ is a symmetry group of this equation with action

$$(\tilde{s}, \tilde{x}, \tilde{u}) = (s, e^{2\alpha}x, e^{3\alpha}u).$$

Choosing the cross-section $\{x = 1\}$ we get the moving frame

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Step 3. Write

$$J = \iota(x) \quad J_1 = \iota(x_s) \quad I = \iota(u) \quad I_1 = \iota(u_s).$$

Invariantizing the entire differential equation gives

$$\iota\left(x\left(\frac{u_s}{x_s}\right)^2 - 3u\left(\frac{u_s}{x_s}\right) + 9x^2\right) = J\left(\frac{I_1}{J_1}\right)^2 - 3I\left(\frac{I_1}{J_1}\right) + 9J^2 = 0.$$

Using Moving Frames to Solve an ODE

By our choice of cross-section, $J = 1$. Since we are free to choose our parametrization, we may also let $J_1 = 1$.

To clarify: $J_1 = \iota(x_s) = \dfrac{x_s}{x}$, so choose $x(s)$ satisfying $x_s = x$.

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Our equation now becomes much simpler!

$$J\left(\frac{I_1}{J_1}\right)^2 - 3I\left(\frac{I_1}{J_1}\right) + 9J^2 = 0 \implies (I_1)^2 - 3I \cdot I_1 + 9 = 0$$

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Before solving our equation, we must relate I_1 and $\dfrac{dI}{ds}$. There are several methods to find the *recurrence relation*

$$\frac{dI}{ds} = I_1 - \frac{3}{2}J_1 \cdot I = I_1 - \frac{3}{2} \cdot I \qquad (\text{using } J_1 = 1)$$

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Rewriting our equation again:

$$\left(\frac{dI}{ds} + \frac{3}{2}I\right)^2 - 3I \cdot \left(\frac{dI}{ds} + \frac{3}{2}I\right) + 9 = 0$$

Using Moving Frames to Solve an ODE

A little algebra and we have the separable equation

$$\left(\frac{dI}{ds}\right)^2 - \frac{9}{4} \cdot I^2 + 9 = 0.$$

The general solution is

$$I = c_0 e^{3s/2} + \frac{1}{c_0} e^{-3s/2} \qquad c_0 \text{ a constant of integration.}$$

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Step 4. Recall that

$$I = \iota(u) = \frac{u}{x^{3/2}} \qquad \text{and} \qquad x = c_1 e^s.$$

Thus

$$\frac{u}{x^{3/2}} = c_0 \left(\frac{x}{c_1}\right)^{3/2} + \frac{1}{c_0}\left(\frac{x}{c_1}\right)^{-3/2},$$

so

$$u = cx^3 + \frac{1}{c} \qquad c = \frac{c_0}{c_1^{3/2}}.$$

Concluding Remarks

You've been lied to. No one would ever solve an ODE this way because explicit computation of a moving frame is difficult. Instead, there are clever ways to compute things using only the chosen cross-section and an infinitesimal characterization of the group action. This story is even more interesting, but necessarily more complicated.

To find the truth you should check my references.

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