Homogeneous spaces and equivariant embeddings

D. A. Timashev

January 10, 2006

Contents

Introduction

Groups entered mathematics as transformation groups. From the works of Caley and Klein it became clear that any geometric theory studies the properties of geometric objects that are invariant under the respective transformation group. This viewpoint culminated in the celebrated Erlangen program [Kl]. An important feature of each one of the classical geometries—affine, projective, euclidean, spherical, and hyperbolic—is that the respective transformation group is transitive on the underlying space. Another feature of these examples is that the transformation groups are linear algebraic and their action is regular. In this way algebraic homogeneous spaces arise in geometry.

Another source for algebraic homogeneous spaces are varieties of geometric figures or tensors of certain type. Examples are provided by Grassmannians, flag varieties, varieties of quadrics, of triangles, of matrices with fixed rank etc. These homogeneous spaces are of great importance in algebraic geometry. They were explored intensively, starting with the works of Chasles, Schubert, Zeuthen et al, which gave rise to the enumerative geometry and intersection theory.

Homogeneous spaces play an important role in representation theory, since representations of algebraic groups are often realized in spaces of sections or cohomologies of line (or vector) bundles over homogeneous spaces. The geometry of a homogeneous space can be used to study representations of the respective group, and conversely. A bright example is the Borel–Weil–Bott theorem [Dem3] and Demazure's proof of the Weyl character formula [Dem1].

In the study of an algebraic homogeneous space G/H, it is often useful by standard reasons of algebraic geometry to pass to a G-equivariant completion, or more generally, to an *embedding*, i.e., a G-variety X containing a dense open orbit isomorphic to G/H.

An example is provided by the following classical problem of enumerative algebraic geometry: compute the number of plane quadrics tangent to 5 given ones. Equivalently, one has to compute the intersection number of certain

5 divisors on the space of quadrics $\mathrm{PSL}_3/\mathrm{PSO}_3$, which is an open orbit in $\mathbb{P}^5 = \mathbb{P}(S^2\mathbb{C}^3)$. To solve our enumerative problem, we pass to a good compactification of $\mathrm{PSL}_3/\mathrm{PSO}_3$. Namely, consider the closure X in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$ of the graph of a rational map sending a quadric to the dual one. Points of X are called *complete quadrics*. It happens that our 5 divisors intersect the complement of the open orbit in X properly. Hence the sought number is just the intersection number of the 5 divisors in X, which is easier to compute, because X is compact.

Embeddings of homogeneous spaces arise naturally as orbit closures, when one studies arbitrary actions of algebraic groups. Such questions as normality of the orbit closure, the nature of singularities, adherence of orbits, the description of orbits in the closure of a given orbit etc are of importance.

Embeddings of homogeneous spaces of reductive algebraic groups are the subject of this survey. The reductivity assumption is natural for two reasons. First, reductive groups have a good structure and representation theory, and a deep theory of embeddings can be developed under this restriction. Secondly, most applications to algebraic geometry and representation theory deal with homogeneous spaces of reductive groups. However, homogeneous spaces of non-reductive groups and their embeddings are also considered. They arise naturally even in the study of reductive group actions as orbits of Borel and maximal unipotent subgroups and their closures. (An example: Schubert varieties.)

The main topics of our survey are:

- The description of all embeddings of a given homogeneous space.
- The study of geometric properties of embeddings: affinity, (quasi)projectivity, divisors and line bundles, intersection theory, singularities etc.
- Application of homogeneous spaces and their embeddings to algebraic geometry, invariant theory, and representation theory.
- Determination of a "good" class of homogeneous spaces, for which the above problems have a good solution. Finding and studying natural invariants that distinguish this class.

Now we describe briefly the content of the survey.

In Chapter 1 we recall basic facts on algebraic homogeneous spaces and consider basic classes of homogeneous spaces: affine, quasiaffine, projective. We give group-theoretical conditions that distinguish these classes. Also bundles and fibrations over a homogeneous space G/H are considered. In particular, we compute $\mathrm{Pic}(G/H)$.

In Chapter 2 we introduce and explore two important numerical invariants of G/H—the complexity and the rank. The *complexity* of G/H is the codimension of a generic B-orbit in G/H, where $B \subseteq G$ is a Borel subgroup. The *rank* of G/H is the rank of the lattice $\Lambda(G/H)$ of weights of rational B-eigenfunctions on G/H. These invariants are of great importance in the theory of embeddings. Homogeneous spaces of complexity ≤ 1 form a "good" class. It was noted by Howe [Ho] and Panyushev [Pan7] that a number of invariant-theoretic problems admitting a nice solution have a certain homogeneous space of complexity ≤ 1 in the background.

Complexity and rank may be defined for any action $G : X$. We prove some semicontinuity results for complexity and rank of G-subvarieties in X. General methods for computing complexity and rank of X were developed by Panyushev, see [Pan7, §§1–2]. We describe them in this chapter, paying special attention to the case $X = G/H$. The formulas for complexity and rank are given in terms of the geometry of the doubled action $G : X \times X^*$ and of cotangent bundle T^*X.

The general theory of embeddings developed by Luna and Vust [LV] is the subject of Chapter 3. The basic idea of Luna and Vust is to patch all embeddings $X \hookleftarrow G/H$ together in a huge prevariety and consider particular embeddings as Noetherian separated open subsets determined by certain conditions. It appears, at least for normal embeddings, that X is determined by the collection of closed G-subvarieties $Y \subseteq X$, and each Y is determined by the collection of B-stable divisors containing Y. This leads to a "combinatorial" description of embeddings, which can be made really combinatorial in the case of complexity ≤ 1. In this case, embeddings are classified by certain collections of convex polyhedral cones, as in the theory of toric varieties [Ful2] (which is in fact a particular case of the Luna–Vust theory). The geometry of embeddings is also reflected in these combinatorial data, as in the toric case. In fact the Luna–Vust theory is developed here in more generality as a theory of G-varieties in a given birational class (not necessarily containing an open orbit).

G-invariant valuations of the function field of G/H correspond to G-stable divisors on embeddings of G/H. They play a fundamental role in the Luna–Vust theory as a key ingredient of the combinatorial data used in the classification of embeddings. In Chapter 4 we explore the structure of the set of invariant valuations, following Knop [Kn3], [Kn5]. This set can be identified with a certain collection of convex polyhedral cones patched together along their common face. This face consists of *central* valuations—those that are zero on B-invariant functions. It is a solid rational polyhedral cone in $\Lambda(G/H) \otimes \mathbb{Q}$ and a fundamental domain of a crystallographic reflection group $W(G/H)$, which is called a *little Weyl group of* G/H. The cone of central valuations and the little Weyl group are linked with the geometry of the cotangent bundle.

Spaces of complexity 0 form the most remarkable subclass of homogeneous spaces. Their embeddings are called *spherical varieties*. They are studied in Chapter 5. Grassmannians, flag varieties, determinantal varieties, varieties of quadrics, of complexes, algebraic symmetric spaces are examples of spherical varieties. We give several characterizations of spherical varieties from the viewpoint of algebraic transformation groups, representation theory, and symplectic geometry. We consider important classes of spherical varieties: symmetric spaces, reductive algebraic monoids, horospherical varieties, toroidal and wonderful varieties. The Luna–Vust theory is much more developed in the spherical case by Luna, Brion, Knop, Pauer et al. We consider the structure of the Picard group of a spherical variety, the intersection theory and its applications to enumerative geometry, the cohomology of coherent sheaves, and a powerful technique of Frobenius splitting, which leads to deep conclusions on geometry and cohomology of spherical varieties by reduction to positive characteristic.

The theory of embeddings of homogeneous spaces is rather new and far from being complete. This survey does not cover all developments and deeper interactions with other areas. Links for further reading may be found in the bibliography. We also recommend the surveys [Kn2], [Bri6], [Bri14] on spherical varieties and the monograph [Pan7] on complexity and rank in invariant theory.

A reader is supposed to be familiar with basic concepts of commutative algebra, algebraic geometry, algebraic groups, and invariant theory. Our basic sources in these areas are [Ma], [Sha] and [Har], [Hum], [PV] and [MFK], respectively. More special topics are covered by Appendices.

Structure of the survey

The paper is divided in chapters, chapters are subdivided in sections, and sections are subdivided in subsections. A link 1.2 refers to Subsection (or Theorem, Lemma, Definition etc) 2 of Section 1. We try to give sketches of the proofs, if they are not very long or technical.

Notation and conventions

We work over an algebraically closed base field $\Bbbk$. A part of our results are valid over an arbitrary characteristic, but we impose the assumption $\operatorname{char}\Bbbk = 0$ whenever it simplifies formulations and proofs. Let p denote the characteristic exponent of $\Bbbk$ ($= \operatorname{char}\Bbbk$, or 1 if $\operatorname{char}\Bbbk = 0$).

Throughout the paper, G denotes a reductive connected linear algebraic group, unless otherwise specified. We always may assume that G is of simply connected type, i.e., a product of a torus and a simply connected semisimple group. When we study the geometry of a given homogeneous space $\mathcal{O}$ and its embeddings, we often fix a base point $o \in \mathcal{O}$ and denote $H = G_o$, thus identifying $\mathcal{O}$ with G/H.

Algebraic groups are denoted by capital Latin letters, and their tangent Lie algebras by the respective lowercase Gothic letters.

Topological terms refer to the Zariski topology, unless otherwise specified.

By a *general point* of an algebraic variety we mean a point in a certain dense open subset (depending on considered situation), in contrast with the *generic point*, which is the dense schematic point of an irreducible algebraic variety.

We use the following general notation.

$A^\times$ is the unit group of an algebra A.

$\operatorname{Quot} A$ is the field of quotients of A.

$\Bbbk[S] \subseteq A$ is the subalgebra generated by a subset $S \subseteq A$.

$\mathfrak{X}(H)$ is the character group of an algebraic group H, i.e., the group of homomorphisms $H \to \Bbbk^\times$ written additively.

$\mathfrak{X}^*(H)$ is the set of (multiplicative) one-parameter subgroups of H, i.e., homomorphisms $\Bbbk^\times \to H$.

$H : M$ denotes an action of a group H on a set M. As a rule, it is a regular action of an algebraic group on an algebraic variety.

M^H is the set of fixed elements under an action $H : M$.

$M^{(H)}$ is the set of all (nonzero) H-eigenvectors in a linear representation $H : M$.

$M_\chi = M_\chi^{(H)} \subseteq M$ is the subspace of H-eigenvectors of the weight $\chi \in \mathfrak{X}(H)$.

$\Bbbk[X]$ is the algebra of regular functions on an algebraic variety X.

$\Bbbk(X)$ is the field of rational functions on X.

$\mathcal{O}(\delta) = \mathcal{O}_X(\delta)$ is the line bundle corresponding to a Cartier divisor δ on X or, more generally, the reflexive sheaf corresponding to a Weil divisor δ.

$X/\!\!/H = \operatorname{Spec} \Bbbk[X]^H$, where $H : X$ is an action of an algebraic group on an affine variety, and $\Bbbk[X]^H$ is finitely generated.

Other notation is gradually introduced in the text.

Chapter 1

Algebraic homogeneous spaces

In this chapter, G denotes an *arbitrary* linear algebraic group (neither supposed to be connected nor reductive), $H \subseteq G$ a closed subgroup. We begin in §1 with the definition of an algebraic homogeneous space G/H as a geometric quotient, and prove its quasiprojectivity. We also prove some elementary facts on tangent vectors and G-equivariant automorphisms of G/H. In §2, we describe the structure of G-fibrations over G/H and compute $\operatorname{Pic}(G/H)$. Some related representation theory is discussed there: induction, multiplicities, the structure of $\Bbbk[G]$. Basic classes of homogeneous spaces are considered in §3. We prove that G/H is projective iff H is parabolic, and consider criteria of affinity of G/H. Quasiaffine G/H correspond to observable H, which may be defined by several equivalent conditions (see Theorem 3.6).

1 Homogeneous spaces

We begin with basic definitions.

Definition 1.1. An algebraic group action $G : \mathcal{O}$ is *transitive* if $\forall x, y \in \mathcal{O}\ \exists g \in G :\ y = gx$. In this situation, $\mathcal{O}$ is said to be a *homogeneous space.*

A *pointed homogeneous space* is a pair $(\mathcal{O}, o)$, where $\mathcal{O}$ is a homogeneous space and $o \in \mathcal{O}$. The natural map $\pi : G \to \mathcal{O}$, $g \mapsto go$, is called the *orbit map.*

A basic property of algebraic group actions is that each orbit is a locally closed subvariety and thence a homogeneous space in the sense of Definition 1.1. Homogeneous spaces are always smooth and quasiprojective, by Sumihiro's Theorem A1.3. The next definition provides a universal construction of algebraic homogeneous spaces.

Definition 1.2. The space G/H equipped with the quotient topology and a structure sheaf $\mathcal{O}_{G/H}$ which is the direct image of the sheaf $\mathcal{O}_G^H$ of H-invariant (w.r.t. the H-action on G by right translations) regular functions on G is called the *(geometric) quotient* of G modulo H.

Theorem 1.1. (1) *$(G/H, \mathcal{O}_{G/H})$ is a quasiprojective homogeneous algebraic variety.*

(2) *For any pointed homogeneous space $(\mathcal{O}, o)$ such that $G_o \supseteq H$, the orbit map $\pi : G \to \mathcal{O}$ factors through $\bar{\pi} : G/H \to \mathcal{O}$.*

(3) *$\bar{\pi}$ is an isomorphism iff $G_o = H$ and π is separable.*

Proof. To prove (1), we use the following theorem of Chevalley [Hum, 11.2]:

> There exists a rational G-module V and a 1-dimensional subspace $L \subseteq V$ such that
>
> $$H = N_G(L) = \{g \in G \mid gL = L\}$$
> $$\mathfrak{h} = \mathfrak{n}_{\mathfrak{g}}(L) = \{\xi \in \mathfrak{g} \mid \xi L \subseteq L\}$$

Let $x \in \mathbb{P}(V)$ correspond to L; then it follows that $H = G_x$ and $\mathfrak{h} = \operatorname{Ker} d_x\pi$, where $\pi : G \to Gx$ is the orbit map. By a dimension argument, $d_x\pi$ is surjective, whence π is separable. Further, Gx is homogeneous, whence smooth, and π is smooth [Har, III.10.4], whence open [Har, Ch.III, Ex.9.1].

Let $U \subseteq Gx$ be an open subset. We claim that each $f \in \Bbbk[\pi^{-1}(U)]^H$ is the pullback of some $h \in \Bbbk[U]$. Indeed, consider the rational map $\phi = (\pi, f) : G \dashrightarrow Gx \times \mathbb{A}^1$ and put $Z = \overline{\phi(G)}$. The projection $Z \to Gx$ is separable and generically bijective, whence birational. Therefore $f \in \phi^*\Bbbk[Z]$ descends to $h \in \Bbbk(U)$, $f = \pi^*h$. If h has the nonzero divisor of poles D, then f has the nonzero divisor of poles π^*D, a contradiction. It follows that $Gx \simeq G/H$ is a geometric quotient.

The universal property (2) is an obvious consequence of the definition. Moreover, any morphism $\phi : G \to Y$ constant on H-orbits factors through $\bar{\phi} : G/H \to Y$.

Finally, (3) follows from the separability of the quotient map $G \to G/H$: π is separable iff $\bar{\pi}$ is so, and $G_o = H$ means that $\bar{\pi}$ is bijective, whence birational and, by equivariance and homogeneity, isomorphic. □

Remark 1.1. In (2), if $G_o = H$ and π is not separable, then $\bar{\pi}$ is bijective purely inseparable and finite [Hum, 4.3, 4.6]. The schematic fiber $\pi^{-1}(o)$ is then a non-reduced group subscheme of G containing H as the reduced part. The homogeneous space $\mathcal{O}$ is uniquely determined by this subscheme [DGr].

Remark 1.2. If $H \lhd G$, then G/H is equipped with the structure of a linear algebraic group with usual properties of the quotient group. Indeed, in the notation of Chevalley's theorem, we may assume that $V = \bigoplus_{\chi \in \mathfrak{X}(G)} V_\chi$ and consider the natural linear action $G : \mathrm{L}(V)$ by conjugation. The subspace $E = \prod \mathrm{L}(V_\chi)$ of operators preserving each V_χ is G-stable, and the image of G in $\mathrm{GL}(V)$ is isomorphic to G/H. See [Hum, 11.5] for details.

Recall that the *isotropy representation* for an action $G : X$ at $x \in X$ is the natural representation $G_x : T_x X$ by differentials of translations. For a quotient, the isotropy representation has a simple description:

Proposition 1.1. *$T_{eH} G/H \simeq \mathfrak{g}/\mathfrak{h}$ as H-modules.*

The isomorphism is given by the differential of the (separable) quotient map $G \to G/H$. The right-hand representation of H is the quotient of the adjoint representation of H in $\mathfrak{g}$.

Now we describe the group $\mathrm{Aut}_G(G/H)$ of G-equivariant automorphisms of G/H.

Proposition 1.2. $\mathrm{Aut}_G(G/H) \simeq N(H)/H$ *is an algebraic group acting on G/H regularly and freely. The action $N(H)/H : G/H$ is induced by the action $N(H) : G$ by right translations: $(nH)(gH) = gn^{-1}H$, $\forall g \in G,\ h \in N(H)$.*

Proof. The regularity of the action $N(H)/H : G/H$ is a consequence of the universal property of quotients. Clearly, this action is free. Conversely, if $\phi \in \mathrm{Aut}_G(G/H)$, then $\phi(eH) = nH$, and $n \in N(H)$, because the ϕ-action preserves stabilizers. Finally, $\phi(gH) = g\phi(eH) = gnH$, $\forall g \in G$. □

2 Fibrations, bundles, and representations

The concept of associated bundle is fundamental in topology. We consider its counterpart in algebraic geometry in a particular case.

Let Z be an H-variety. Then H-acts on $G \times Z$ by $h(g, z) = (gh^{-1}, hz)$.

Definition 2.1. The quotient space $G *_H Z = (G \times Z)/H$ equipped with the quotient topology and a structure sheaf which is the direct image of the sheaf of H-invariant regular functions is called the *homogeneous fiber bundle* over G/H associated with Z.

The G-action on $G \times Z$ by left translations of the first factor commutes with the H-action and factors to a G-action on $G *_H Z$. We denote by $g * z$ the image of (g, z) in $G *_H Z$ and identify $e * z$ with z. The embedding

$Z \hookrightarrow G *_H Z$, $z \mapsto e * z$, solves the universal problem for H-equivariant morphisms of Z into G-spaces.

The homogeneous bundle $G *_H Z$ is G-equivariantly fibered over G/H with fibers gZ, $g \in G$. The fiber map is $g * z \to gH$. This explains the terminology.

Theorem 2.1 ([BB2], [PV, 4.8]). *If Z is covered by H-stable quasiprojective open subsets, then $G *_H Z$ is an algebraic G-variety, and the fiber map $G *_H Z \to G/H$ is locally trivial in étale topology.*

The proof is based on the fact that the fibration $G \to G/H$ is locally trivial in étale topology [Se1]. We shall always suppose that the assumption of the theorem is satisfied when we consider homogeneous bundles. The assumption is satisfied, e.g., if Z is quasiprojective, or normal and H is connected (by Sumihiro's theorem). If H is reductive and Z is affine, then $G *_H Z \simeq (G \times Z)/\!\!/H$ is affine.

The universal property of homogeneous bundles implies that any G-variety mapped onto G/H is a homogeneous bundle over G/H. More precisely, a G-equivariant map $\phi : X \to G/H$ induces a bijective G-map $G *_H Z \to X$, where $Z = \phi^{-1}(eH)$. If ϕ is separable, then $X \simeq G *_H Z$. In particular, any G-subvariety $Y \subseteq G *_H Z$ is G-isomorphic to $G *_H (Y \cap Z)$.

Since homogeneous bundles are locally trivial in étale topology, a number of local properties such as regularity, normality, rationality of singularities etc is transferred from Z to $G *_H Z$ and back. The next lemma indicates when a homogeneous bundle is trivial.

Lemma 2.1. *$G *_H Z \simeq G/H \times Z$ if the H-action on Z extends to a G-action.*

Proof. The isomorphism is given by $g * z \mapsto (gH, gz)$. □

If the fiber is an H-module, then the homogeneous bundle is locally trivial in Zariski topology. By the above, any G-vector bundle over G/H is G-isomorphic to $G *_H M$ for some finite-dimensional rational H-module M. The respective sheaf of sections $\mathcal{L}(M)$ is described in the following way.

Proposition 2.1. *For any open subset $U \subseteq G/H$, we have $\mathrm{H}^0(U, \mathcal{L}(M)) \simeq \mathrm{Mor}_H(\pi^{-1}(U), M)$, where $\pi : G \to G/H$ is the quotient map.*

Proof. It is easy to see that the pullback of $G *_H M \to G/H$ under π is a trivial vector bundle $G \times M \to G$. Hence for $\forall s \in \mathrm{H}^0(U, \mathcal{L}(M))$ we have $\pi^* s \in \mathrm{Mor}(\pi^{-1}(U), M)$, and clearly $\pi^* s$ is H-equivariant. Conversely, any H-morphism $\pi^{-1}(U) \to M$ induces a section $U \to G *_H M$ by the universal property of the quotient. □

If $H : M$ is an infinite-dimensional rational module, we may *define* a quasicoherent sheaf $\mathcal{L}(M) = \mathcal{L}_{G/H}(M)$ on G/H by the formula of Proposition 2.1. The functor $\mathcal{L}_{G/H}(\cdot)$ establishes an equivalence between the category of rational H-modules and that of G-sheaves on G/H.

Any G-line bundle over G/H is G-isomorphic to $G *_H \Bbbk_\chi$, where $\Bbbk_\chi = \Bbbk$ with the H-action via a character $\chi \in \mathfrak{X}(H)$. This yields a homomorphism $\mathfrak{X}(H) \to \operatorname{Pic} G/H$, $\chi \mapsto \mathcal{L}(\chi) = \mathcal{L}(\Bbbk_\chi)$. Its kernel consists of characters that correspond to different G-linearizations of the trivial line bundle $G/H \times \Bbbk$ over G/H.

If G is connected, then these characters are exactly the restrictions to H of characters of G. Indeed, a fiberwise linear G-action on $G/H \times \Bbbk$ is a multiplication by an algebraic cocycle $c : G \times G/H \to \Bbbk^\times$, $c(g_1g_2, x) = c(g_1, g_2x)c(g_2, x)$ for $\forall g_1, g_2 \in G$, $x \in G/H$. For connected G, we have $c(g, x) = \chi(g)\lambda(x)$, because an invertible function on a product of two irreducible varieties is a product of invertible functions on factors [KKV]. Now it is easy to deduce from the cocycle property that $\lambda(x) \equiv 1$ and $\chi \in \mathfrak{X}(G)$. Conversely, if $\chi \in \operatorname{Res}^G_H \mathfrak{X}(G)$, then $G *_H \Bbbk_\chi \simeq G/H \times \Bbbk_\chi$ by Lemma 2.1.

Consider the universal cover $\widetilde{G} \to G$ (see Appendix A1). By $\widetilde{H}$ denote the inverse image of H in $\widetilde{G}$; then $G/H \simeq \widetilde{G}/\widetilde{H}$. Since any line bundle over G/H is $\widetilde{G}$-linearizable (Corollary A1.1), we obtain the following theorem of Popov [Po2], [KKV].

Theorem 2.2. $\operatorname{Pic}_G(G/H) \simeq \mathfrak{X}(H)$. *If G is connected, then* $\operatorname{Pic} G/H \simeq \mathfrak{X}(\widetilde{H})/\operatorname{Res}^{\widetilde{G}}_{\widetilde{H}} \mathfrak{X}(\widetilde{G})$.

(Here Pic_G denotes the group of G-linearized invertible sheaves.)

Example 2.1. Let G be a connected reductive group, $B \subseteq G$ a Borel subgroup. Then $\operatorname{Pic} G/B$ is isomorphic to the weight lattice of the root system of G.

Let X be a G-variety, and $Z \subseteq X$ an H-stable subvariety. By the universal property, we have a G-equivariant map $\mu : G *_H Z \to X$, $\mu(g * z) = gz$.

Proposition 2.2. *If H is parabolic, then μ is proper and GZ is closed in X.*

Proof. The map μ factors as $\mu : G *_H Z \xhookrightarrow{\iota} G *_H X \simeq G/H \times X$ (Lemma 2.1) $\xrightarrow{\pi} X$, where ι is a closed embedding and π is a projection along a complete variety by Theorem 3.1. □

Example 2.2. Let $\mathfrak{N} \subseteq \mathfrak{g}$ be the set of nilpotent elements and $U = \mathrm{R_u}(B)$, a maximal unipotent subgroup of G. Then the map $G *_B \mathfrak{u} \to \mathfrak{N}$ is proper and birational, see, e.g., [PV, 5.6]. This is a well-known Springer's resolution of singularities of $\mathfrak{N}$.

Now we discuss some representation theory related to homogeneous spaces and to vector bundles over them.

We always deal with rational modules over algebraic groups (see Appendix A1) and often drop the word "rational". As usual in representation theories, we may define functors of induction and restriction on categories of rational modules. Let H act on G by right translations, and M be an H-module.

Definition 2.2. A G-module $\operatorname{Ind}_H^G M = \operatorname{Mor}_H(G, M) \simeq (\Bbbk[G] \otimes M)^H$ is said to be *induced* from $H : M$ to G. It is a rational G-$\Bbbk[G/H]$-module. By definition, we have $\operatorname{Ind}_H^G M = \mathrm{H}^0(G/H, \mathcal{L}(M))$.

A G-module N considered as an H-module is denoted by $\operatorname{Res}_H^G N$.

Example 2.3. $\operatorname{Ind}_H^G \Bbbk = \Bbbk[G/H]$, where $\Bbbk$ is the trivial H-module. More generally, $\operatorname{Ind}_H^G \Bbbk_\chi = \Bbbk[G]_{-\chi}$, $\forall \chi \in \mathfrak{X}(H)$.

Clearly, Ind_H^G is a left exact functor from the category of rational H-modules to that of rational G-modules. The functor Res_H^G is exact. We collect basic properties of induction in the following

Theorem 2.3. (1) *If M is a G-module, then* $\operatorname{Ind}_H^G M \simeq \Bbbk[G/H] \otimes M$.

(2) *(Frobenius reciprocity) For rational modules $G : N$, $H : M$, we have*

$$\operatorname{Hom}_G(N, \operatorname{Ind}_H^G M) \simeq \operatorname{Hom}_H(\operatorname{Res}_H^G N, M)$$

(3) *For any H-module M,* $(\operatorname{Ind}_H^G M)^G \simeq M^H$.

(4) *If M, N are rational algebras, then* (1) *and* (3) *are isomorphisms of algebras, and* (2) *holds for equivariant algebra homomorphisms.*

Proof. (1) The isomorphism $\iota : \operatorname{Mor}_H(G, M) \xrightarrow{\sim} \operatorname{Mor}(G/H, M)$ is given by $\iota(m)(gH) = g \cdot m(g)$, $\forall m \in \operatorname{Mor}_H(G, M)$. The inverse mapping is $\mu \mapsto m$, $m(g) = g^{-1}\mu(gH)$, $\forall \mu \in \operatorname{Mor}(G/H, M)$.

(2) The isomorphism is given by the map $\Phi \mapsto \phi$, $\forall \Phi : N \to \operatorname{Mor}_H(G, M)$, where $\phi : N \to M$ is defined by $\phi(n) = \Phi(n)(e)$, $\forall n \in N$. The inverse map $\phi \mapsto \Phi$ is given by $\Phi(n)(g) = \phi(g^{-1}n)$.

(3) Any G-invariant H-equivariant morphism $G \to M$ is constant, and its image lies in M^H. Alternatively, one may apply the Frobenius reciprocity to $N = \Bbbk$.

(4) It is easy. $\square$

Remark 2.1. The union of (1) and (3) yields the following assertion: if M is a G-module, then $(\Bbbk[G/H] \otimes M)^G \simeq M^H$. This is often called the *transfer principle*, because it allows to transfer information from $\Bbbk[G/H]$ to M. For example, if G is reductive, $\Bbbk[G/H]$ is finitely generated, and $M = A$ is a finitely generated G-algebra, then A^H is finitely generated. Other applications are discussed below. A good treatment of induced modules and the transfer principle can be found in [Gro2].

We are interested in the G-module structure of $\Bbbk[G/H]$ and of global sections of line bundles over G/H.

For any two rational G-modules V, M ($\dim V < \infty$), put

$$m_V(M) = \dim \operatorname{Hom}_G(V, M),$$

the *multiplicity* of V in M. If V is simple and M completely reducible (e.g., G is an algebraic torus or a reductive group in characteristic zero), then $m_V(M)$ is the number of occurrences of V in a decomposition of M into simple summands.

For any G-variety X and a G-line bundle $\mathcal{L} \to X$, we abbreviate:

$$m_V(X) = m_V(\Bbbk[X]), \qquad m_V(\mathcal{L}) = m_V(\mathrm{H}^0(X, \mathcal{L}))$$

Here is a particular case of Frobenius reciprocity:

Corollary 2.1. $m_V(G/H) = \dim(V^*)^H$, $m_V(\mathcal{L}(\chi)) = \dim(V^*)^{(H)}_{-\chi}$

Proof. We have $\mathrm{H}^0(G/H, \mathcal{L}(\chi)) = \operatorname{Ind}_H^G \Bbbk_\chi$, whence

$$\operatorname{Hom}_G\bigl(V, \mathrm{H}^0(G/H, \mathcal{L}(\chi))\bigr) = \operatorname{Hom}_H(V, \Bbbk_\chi) = (V^*)^{(H)}_{-\chi}$$

The first equality follows by taking $\chi = 0$. □

A related problem is to describe the module structure of $\Bbbk[G]$. Namely, G itself is acted on by $G \times G$ via $(g_1, g_2)g = g_1 g g_2^{-1}$. Hence $\Bbbk[G]$ is a $(G \times G)$-algebra.

Every finite-dimensional G-module V generates a $(G \times G)$-stable subspace $\mathrm{M}(V) \subset \Bbbk[G]$ spanned by matrix entries $f_{\omega,v}(g) = \langle \omega, gv \rangle$ ($v \in V$, $\omega \in V^*$) of the representation $G \to \mathrm{GL}(V)$. Clearly $\mathrm{M}(V)$ is the image of a $(G \times G)$-module homomorphism $V^* \otimes V \to \Bbbk[G]$, $\omega \otimes v \mapsto f_{\omega,v}$, and $\mathrm{M}(V) \simeq V^* \otimes V$ is a simple $(G \times G)$-module whenever V is simple.

Matrix entries behave well w.r.t. algebraic operations:

$$\mathrm{M}(V) + \mathrm{M}(V') = \mathrm{M}(V \oplus V'), \qquad \mathrm{M}(V) \cdot \mathrm{M}(V') = \mathrm{M}(V \otimes V') \tag{2.1}$$

The inversion of G sends $\mathrm{M}(V)$ to $\mathrm{M}(V^*)$.

Proposition 2.3. $\Bbbk[G] = \bigcup \mathrm{M}(V)$, *where V runs through all finite-dimensional G-modules.*

Proof. Take any finite-dimensional G-submodule $V \subset \Bbbk[G]$ w.r.t. the G-action by right translations. We claim $V \subset \mathrm{M}(V)$. Indeed, let $\omega \in V^*$ be defined by $\langle\omega, v\rangle = v(e)$, $\forall v \in V$; then $\forall v \in V, g \in G :\ v(g) = f_{\omega,v}(g)$. □

Theorem 2.4. *Suppose* $\operatorname{char} \Bbbk = 0$ *and G is reductive. Then there is a $(G \times G)$-module isomorphism*

$$\Bbbk[G] = \bigoplus \mathrm{M}(V) \simeq \bigoplus V^* \otimes V$$

where V runs through all simple G-modules.

Proof. All the $\mathrm{M}(V) \simeq V^* \otimes V$ are pairwise non-isomorphic simple $(G \times G)$-modules. By Proposition 2.3 and (2.1) they span the whole $\Bbbk[G]$. □

Remark 2.2. Corollary 2.1 can be derived from Theorem 2.4 by taking H-(semi)invariants from the right.

The dual object to the coordinate algebra of G provides a version of the group algebra for algebraic groups.

Definition 2.3. The (*algebraic*) *group algebra* of G is $\mathcal{A}(G) = \Bbbk[G]^*$ equipped with the multiplication law coming from the comultiplication in $\Bbbk[G]$.

For finite G we obtain the usual group algebra. Generally, $\mathcal{A}(G)$ can be described by finite-dimensional approximations. The group algebra $\mathcal{A}(V)$ of a finite-dimensional G-module V is defined as the linear span of the image of G in $\mathrm{L}(V)$. Note that $\mathcal{A}(V)$ is the $(G \times G)$-module dual to $\mathrm{M}(V)$. We have $\mathcal{A}(V) = \mathrm{L}(V)$ whenever V is simple. Given a subquotient module V' of V, there is a canonical epimorphism $\mathcal{A}(V) \twoheadrightarrow \mathcal{A}(V')$. Therefore the algebras $\mathcal{A}(V)$ form an inverse system over all V ordered by the relation of being a subquotient. It readily follows from Proposition 2.3 that $\mathcal{A}(G) \simeq \varprojlim \mathcal{A}(V)$. One deduces that $\mathcal{A}(G)$ is a universal ambient algebra containing both G and $\mathrm{U}\mathfrak{g}$, the (restricted) envelopping algebra of $\mathfrak{g}$ [DGa].

Definition 2.4. The algebra $\mathcal{A}(G/H)$ of all G-equivariant linear endomorphisms of $\Bbbk[G/H]$ is called the *Hecke algebra* of G/H, or of (G, H).

Remark 2.3. If $\operatorname{char} \Bbbk = 0$ and G is reductive, then $\mathcal{A}(V) = \prod \mathrm{L}(V_i)$ over all simple G-modules V_i occurring in V with positive multiplicity. Furthermore, $\mathcal{A}(G) = \prod \mathrm{L}(V_i)$ and $\mathcal{A}(G/H) = \prod \mathrm{L}(V_i^H)$ over all simple V_i by Theorem 2.4 and Schur's lemma.

Proposition 2.4 (E. B. Vinberg). *If* $\operatorname{char}\Bbbk = 0$ *and* H *is reductive, then* $\mathcal{A}(G/H) \simeq \mathcal{A}(G)^{H\times H}$. *In particular, the above notation is compatible for* $H = \{e\}$.

Proof. First consider the case $H = \{e\}$. The algebra $\mathcal{A}(V)$ acts on $\mathcal{A}(V)^* = \mathrm{M}(V)$ by right translations: $af(x) = f(xa), \forall a, x \in \mathcal{A}(V),\ f \in \mathrm{M}(V)$. These actions commute with the G-action by left translations and merge together into a G-equivariant linear $\mathcal{A}(G)$-action on $\Bbbk[G]$.

Conversely, every G-equivariant linear map $\phi : \Bbbk[G] \to \Bbbk[G]$ preserves all the spaces $\mathrm{M}(V)$. Indeed, it follows from the proof of Proposition 2.3 by applying the inversion that $W \subseteq \mathrm{M}(W^*)$ for any G-submodule $W \subset \Bbbk[G]$. For $W = \mathrm{M}(V)$ one easily deduces $W = \mathrm{M}(W^*)$ and $\phi W \subseteq \mathrm{M}(\phi W^*) \subseteq \mathrm{M}(W^*)$.

The restriction of ϕ to $\mathrm{M}(V)$ is the right translation by some $a_V \in \mathcal{A}(V)$. These a_V give rise to $a \in \mathcal{A}(G)$ representing ϕ on $\Bbbk[G]$. Hence the group algebra coincides with the Hecke algebra of G.

In the general case, every linear G-endomorphism ϕ of $\Bbbk[G]^H$ extends to a unique $a \in \mathcal{A}(G)^{H\times H}$, which annihilates the right-H-invariant complement of $\Bbbk[G]^H$ in $\Bbbk[G]$. □

If G is a connected reductive group, $B \subseteq G$ a Borel subgroup, then isomorphism classes of simple G-modules are indexed by B-*dominant weights*, which form a subsemigroup $\mathfrak{X}_+ \subseteq \mathfrak{X}(B)$ (the intersection of $\mathfrak{X}(B)$ with the positive Weyl chamber). Any simple G-module V contains a unique, up to proportionality, B-eigenvector (a *highest vector*) of weight $\lambda \in \mathfrak{X}_+$ (the *highest weight*) [Hum, §31]. The highest weight of V^* is $\lambda^* = -w_G\lambda$, where w_G is the longest element of the Weyl group.

By Corollary 2.1,

$$m_V(\mathcal{L}_{G/B}(\mu)) = \dim(V^*)^{(B)}_{-\mu} = \begin{cases} 1, & \mu = -\lambda^* \\ 0, & \text{otherwise} \end{cases}$$

It follows that $V^*(\lambda) = \operatorname{Ind}_B^G \Bbbk_{-\lambda}$ contains a unique simple G-module (of highest weight λ^*) whenever $\lambda \in \mathfrak{X}_+$, otherwise $V^*(\lambda) = 0$. The dual G-module $V(\lambda) = (\operatorname{Ind}_B^G \Bbbk_{-\lambda})^*$ is called a *Weyl module* [Jan, II.2].

Put $m_\lambda(M) = m_{V(\lambda)}(M)$ for brevity.

Proposition 2.5. $m_\lambda(M) = \dim M^{(B)}_\lambda$

Proof. As G/B is a projective variety (Theorem 3.1), $V(\lambda) = \mathrm{H}^0(G/B, \mathcal{L}(-\lambda))^*$ is finite-dimensional. If $\dim M < \infty$, then

$$\operatorname{Hom}_G(V(\lambda), M) \simeq \operatorname{Hom}_G(M^*, V^*(\lambda)) \simeq \operatorname{Hom}_B(M^*, \Bbbk_{-\lambda}) \simeq M^{(B)}_\lambda$$

However, any rational G-module M is a union of finite-dimensional submodules. $\square$

Thus $V(\lambda)$ can be characterized as the universal covering G-module of highest weight λ: the generating highest vector in $V(\lambda)$ is given by evaluation of $\mathrm{H}^0(G/B, \mathcal{L}(-\lambda))$ at eB.

By Corollary 2.1, we have

$$m_\lambda(G/H) = \dim V^*(\lambda)^H, \qquad m_\lambda(\mathcal{L}_{G/H}(\chi)) = \dim V^*(\lambda)^{(H)}_{-\chi}. \tag{2.2}$$

In characteristic zero, complete reducibility yields:

Borel–Weil theorem. *If* $\operatorname{char} \Bbbk = 0$*, then* $V(\lambda)$ *is a simple* G*-module of highest weight* λ *and* $V^*(\lambda) \simeq V(\lambda^*)$*.*

Furthermore, Theorem 2.4 yields

$$\Bbbk[G] \simeq \bigoplus_{\lambda \in \mathfrak{X}_+} V(\lambda^*) \otimes V(\lambda) \tag{2.3}$$

In arbitrary characteristic, Formula (2.3) is no longer true, but $\Bbbk[G]$ possesses a "good" $(G \times G)$-module filtration with factors $V^*(\lambda) \otimes V^*(\lambda^*)$ [Don], [Jan, II.4.20].

Notice that all the dual Weyl modules are combined in a multigraded algebra

$$\Bbbk[G/U] = \bigoplus_{\lambda \in \mathfrak{X}_+} \Bbbk[G]^{(B)}_\lambda \simeq \bigoplus_{\lambda \in \mathfrak{X}_+} V^*(\lambda) \tag{2.4}$$

called the *covariant algebra* of G. Here, as above, $U = \mathrm{R_u}(B)$. The covariant algebra is an example of a multiplicity free G-algebra, in the sense of the following

Definition 2.5. A G-module M is said to be *multiplicity-free* if $m_\lambda(M) \leq 1$, $\forall \lambda \in \mathfrak{X}_+$.

The multiplication in the covariant algebra has a nice property:

Lemma 2.2 ([Jan, II.14.20]). $V^*(\lambda) \cdot V^*(\mu) = V^*(\lambda + \mu)$

The inclusion "$\subseteq$" in the lemma is obvious since the $V^*(\lambda)$ are the homogeneous components of $\Bbbk[G/U]$ w.r.t. an algebra grading. In characteristic zero, the reverse inclusion stems from the fact that the $V^*(\lambda)$ are simple G-modules and $\Bbbk[G/U]$ is an integral domain.

3 Classes of homogeneous spaces

We answer the following question: when is a homogeneous space $\mathcal{O}$ projective or (quasi)affine? First, we reduce the question to a property of the pair (G, H), where $H = G_o$ is the stabilizer of a point $o \in \mathcal{O}$.

Lemma 3.1. *$\mathcal{O}$ is projective, resp. (quasi)affine iff G/H has this property.*

Proof. We may assume $\operatorname{char} \Bbbk = p > 0$. The natural map $\iota : G/H \to \mathcal{O}$ is finite bijective purely inseparable (Remark 1.1). For completeness and affinity, we conclude by [Har, III, ex.4.2].

For quasiaffinity, we argue as follows. First note that $\Bbbk(G/H)^{p^s} \subseteq \iota^*\Bbbk(\mathcal{O})$ for some $s \geq 0$. Furthermore, $\Bbbk[G/H]^{p^s} \subseteq \iota^*\Bbbk[\mathcal{O}]$ (If $f \in \Bbbk(\mathcal{O})$, then $\iota^* f$ has poles on G/H.) Assume $\mathcal{O}$ is an open subset of an affine variety Y. Let B be the integral closure of $\iota^*\Bbbk[Y]$ in $\Bbbk(G/H)$. Then B is finitely generated, and $X = \operatorname{Spec} B$ contains G/H as an open subset. Conversely, if G/H is open in an affine variety X, then $A = \Bbbk[X] \cap \iota^*\Bbbk(\mathcal{O})$ is finite over $\Bbbk[X]^{p^s}$. Hence A is finitely generated, and $X = \operatorname{Spec} A$ contains $\mathcal{O}$ as an open subset. $\square$

In the sequel, we may assume $\mathcal{O} = G/H$.

Lemma 3.2. *If $G \supseteq H \supseteq K$ and G/H, H/K are projective, resp. (quasi)-affine, then G/K is projective, resp. (quasi)affine.*

Proof. The natural map $\phi : G/K \to G/H$ transforms after a faithfully flat base change $G \to G/H$ to the projection $\pi : G/K \times_{G/H} G \simeq H/K \times G \to G$. If H/K is projective (resp. affine), then π is proper (resp. affine), whence ϕ is proper (resp. affine). If in addition G/H is complete (resp. affine), then G/K is complete (resp. affine), too. Another proof for projective and affine cases relies on Theorems 3.1 and 3.5 below. In the quasiaffine case, lemma follows from Theorem 3.6(4). $\square$

Theorem 3.1. *G/H is projective iff H is parabolic.*

Proof. If G/H is quasiprojective, then a Borel subgroup $B \subseteq G$ has a fixed point $gH \in G/H$, by the Borel fixed point Theorem [Hum, 21.2]. Hence $H \supseteq g^{-1}Bg$ is parabolic.

To prove the converse, consider an exact representation $G : V$. The induced action of G on the variety of complete flags in V has a closed orbit. Its stabilizer B is solvable, and we may assume $B \subset H$. By Lemma 3.1, G/B is complete, hence G/H is complete. $\square$

A group-theoretical characterization of affine homogeneous spaces is not known at the moment. We give several sufficient conditions of affinity and a criterion for reductive G.

Lemma 3.3. *The orbits of a unipotent group G on an affine variety X are closed, whence affine.*

Proof. For any $x \in X$, consider closed affine subvarieties $Y = \overline{Gx} \subseteq X$ and $Z = Y \setminus Gx$. Since $\mathbb{I}(Z) \lhd \Bbbk[Y]$ is a G-submodule, the Lie–Kolchin theorem implies $\exists f \in \mathbb{I}(Z)^G$, $f \neq 0$. However f is a nonzero constant on Gx, whence on Y. Thus $Z = \emptyset$. □

Theorem 3.2. *G/H is affine if G is solvable.*

Proof. We may assume that G, H are connected. First suppose G is unipotent. Take a representation $G : V$ such that $\exists v \in V :\ G[v] \simeq G/H$. Then H normalizes $\langle v \rangle$. But $\mathfrak{X}(H) = 0$, whence $G_v = H$ and $Gv \simeq G/H$. We conclude by Lemma 3.3.

In the general case, $G = T \leftthreetimes U$ and $H = S \leftthreetimes V$, where U, V are unipotent radicals and T, S are maximal tori of G, H. We have $U \supset V$ and may assume that $T \supset S$. It is easy to see that $G/H \simeq T *_S U/V = (T \times U/V) /\!\!/ S$ is affine. □

The following notion is often useful in the theory of homogeneous spaces.

Definition 3.1. We say that H is *regularly embedded* in G if $\mathrm{R_u}(H) \subseteq \mathrm{R_u}(G)$.

For example, any subgroup of a solvable group is regularly embedded. The next theorem generalizes Theorem 3.2.

Theorem 3.3. *G/H is affine if H is regularly embedded in G.*

Proof. As $\mathrm{R_u}(G)$ is normal in G, the quotient $G/\mathrm{R_u}(G)$ is affine. By Theorem 3.2, $\mathrm{R_u}(G)/\mathrm{R_u}(H)$ is affine. Thence by Lemma 3.2, $G/\mathrm{R_u}(H)$ is affine. By the Main Theorem of GIT (see Appendix A2), $G/H = (G/\mathrm{R_u}(H)) /\!\!/ (H/\mathrm{R_u}(H))$ is affine, because $H/\mathrm{R_u}(H)$ is reductive. □

Weisfeiler proved [Wei] that any subgroup H of a connected group G is regularly embedded in some parabolic subgroup $P \subseteq G$. (See also [Hum, 30.3].) Thus G/H is a fibration with the projective base G/P and affine fiber P/H.

The following theorem is often called Matsushima's criterion. It was proved for $\Bbbk = \mathbb{C}$ by Matsushima [Mat] and Onishchik [Oni1], and in the general case by Richardson [Ri1].

Theorem 3.4. *G/H is affine if H is reductive. If G is reductive, the converse is also true.*

Proof. If H is reductive, then by the Main Theorem of GIT, $G/H \simeq G/\!\!/H$ is affine. A simple proof of the converse see in [Lu2] ($\operatorname{char} \Bbbk = 0$) or in [Ri1]. $\square$

The lacking of a group-theoretical criterion of affinity is partially compensated by a cohomological criterion.

Theorem 3.5. *G/H is affine iff Ind_H^G is exact.*

Proof. Recall that $\operatorname{Ind}_H^G(M) = \mathrm{H}^0(G/H, \mathcal{L}(M))$, the sheaf $\mathcal{L}(M)$ is quasicoherent, and the functor $\mathcal{L}(\cdot)$ is exact. If G/H is affine, then by Serre's criterion, Ind_H^G is exact. For a proof of the converse, see [Gro2, §6]. $\square$

The class of quasiaffine homogeneous spaces is of interest in invariant theory and representation theory. If G/H is quasiaffine, then the subgroup H is called *observable.* Observable subgroups are exactly the stabilizers of vectors in rational G-modules, since any quasiaffine G-variety can be equivariantly embedded in a G-module [PV, 1.2].

Example 3.1. By Chevalley's theorem, H is observable if $\mathfrak{X}(H) = 0$. In particular a unipotent subgroup is observable.

Example 3.2 ([BHM]). If $\mathrm{R}(H)$ is nilpotent, then H is observable.

It is easy to see that an intersection of observable subgroups is again observable. Therefore for any $H \subseteq G$, there exists a smallest observable subgroup $\widehat{H} \subseteq G$ containing H. It is called the *observable hull* of H. Clearly, for any rational G-module M we have $M^H = M^{\widehat{H}}$. This property illustrates the importance of observable subgroups in invariant theory, see [PV, 3.7].

We give several characterizations of observable subgroups in the next theorem, essentially due to Białynicki-Birula, Hochschild, and Mostow [BHM].

Theorem 3.6. *The following conditions are equivalent:*

(1) *G/H is quasiaffine.*

(2) *G^0/H^0 is quasiaffine.*

(3) $\operatorname{Quot} \Bbbk[G/H] = \Bbbk(G/H)$.

(4) *Any finite-dimensional H-module is embedded as an H-submodule in a finite-dimensional G-module.*

(5) $\forall \chi \in \mathfrak{X}(H) : \quad \Bbbk[G]_\chi \neq 0 \implies \Bbbk[G]_{-\chi} \neq 0$ *(In other words, the semigroup of weights of H-eigenfunctions on G is actually a group.)*

Proof. (1) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) We have $\Bbbk(G/H) = \Bbbk(G/\widehat{H})$, whence $H = \widehat{H}$.

(1) $\Longleftrightarrow$ (2) We may assume that G is connected. The map $G/H^0 \to G/H$ is a Galois covering with the Galois group $\Gamma = H/H^0$. If G/H is open in an affine variety X, then G/H^0 is open in $Y = \operatorname{Spec} A$, where A is the integral closure of $\Bbbk[X]$ in $\Bbbk(G/H^0)$. Conversely, if G/H^0 is open in affine Y, then G/H is open in $X = \operatorname{Spec} \Bbbk[Y]^\Gamma$.

(1) $\Longrightarrow$ (5) For a nonzero $p \in \Bbbk[G]_\chi$, consider its zero set $Z \subset G$. The quotient morphism $\pi : G \to G/H$ maps Z onto a proper closed subset of G/H. Hence $\exists f \in \Bbbk[G/H] : \ f|_{\pi(Z)} = 0$. By Nullstellensatz, $\pi^* f^n = pq$ for some $n \in \mathbb{N},\ q \in \Bbbk[G]_{-\chi}$.

(5) $\Longrightarrow$ (4) First note that a 1-dimensional H-module $W = \Bbbk_\chi$ can be embedded in a G-module V iff $\Bbbk[G]_\chi \neq 0$. (Any function $f(g) = \langle \omega, gv \rangle$, where $w \in W$, $\omega \in V^*$, belongs to $\Bbbk[G]_\chi$.)

Now for any finite-dimensional H-module W, consider the embedding $W \hookrightarrow \operatorname{Mor}(H, W)$ taking each $w \in W$ to the orbit morphism $g \mapsto gw$, $g \in H$. It is H-equivariant w.r.t. the H-action on $\operatorname{Mor}(H, W)$ by right translations of an argument. The restriction of morphisms yields a projection $\operatorname{Mor}(G, W) \to \operatorname{Mor}(H, W)$, and we may choose a finite-dimensional H-submodule $N \subset \operatorname{Mor}(G, W)$ mapped onto W. Embed N into a finite-dimensional G-submodule $M \subset \operatorname{Mor}(G, W)$ and put $U = \operatorname{Ker}(N \to W)$, $m = \dim U$. Then $\bigwedge^m U \hookrightarrow \bigwedge^m M$ and $W \otimes \bigwedge^m U \hookrightarrow \bigwedge^{m+1} M$. By (5) and the above remark, $\bigwedge^m U^*$ is embedded in a G-module. We conclude by $W \simeq (W \otimes \bigwedge^m U) \otimes \bigwedge^m U^*$.

(4) $\Longrightarrow$ (1) By Chevalley's theorem, H is the projective stabilizer of a vector v in some G-module V. As an H-module, $\langle v \rangle \simeq \Bbbk_\chi$ for some $\chi \in \mathfrak{X}(H)$. Then $\Bbbk_{-\chi}$ can be embedded in a G-module, i.e., $\exists G : W,\ w \in W$ such that H acts on w via $-\chi$. It follows that $G_{v \otimes w} = H$. $\square$

Surprisingly, quasiaffine homogeneous spaces admit a group-theoretical characterization. Recall that a *quasiparabolic* subgroup of a connected group is the stabilizer of a highest weight vector in an irreducible representation.

Theorem 3.7 ([Sukh]). *$H \subseteq G$ is observable iff H^0 is regularly embedded in a quasiparabolic subgroup of G^0.*

Chapter 2

Complexity and rank

We retain general conventions of our survey. In particular, G denotes a reductive connected linear algebraic group. We begin with local structure theorems, which claim that a G-variety may be covered by affine open subsets stable under parabolic subgroups of G, and describe the structure of these subsets. In §5, we define two numerical invariants of a G-variety related to the action of a Borel subgroup of G—the complexity and the rank. We reduce their computation to a generic orbit on X (i.e., a homogeneous space) and prove some basic results including the semicontinuity of complexity and rank w.r.t. G-subvarieties. We also introduce the notion of the weight lattice and consider the connection of complexity with the growth of multiplicities in $\Bbbk[X]$ for quasiaffine X. The relation of complexity and modality of an action is considered in §6. In §7, we introduce the class of horospherical varieties defined by the property that all isotropy groups contain a maximal unipotent subgroup of G. The computation of complexity and rank is fairly simple for them. On the other hand, any G-variety can be contracted to a horospherical one of the same complexity and rank.

General formulae for complexity and rank are obtained in §8 as a by-product of the study of the cotangent action $G : T^*X$ and the doubled action $G : X \times X^*$. These formulae involve generic stabilizers of these actions. The particular case of a homogeneous space $X = G/H$ is considered in §9. In §10, we classify homogeneous spaces of complexity and rank ≤ 1. An application to the problem of decomposing tensor products of representations is considered in §11. Decomposition formulae are obtained from the description of the G-module structure of coordinate algebras on double cones of small complexity.

4 Local structure theorems

Those algebraic group actions can be effectively studied which are more or less reduced to linear or projective representations and their restrictions to stable subvarieties in representation spaces. Therefore it is natural to restrict our attention to the following class of actions.

Definition 4.1. A regular algebraic group action $G : X$ (or a G-variety X) is *good* if X can be covered by G-stable quasiprojective open subsets X_i such that $G : X_i$ is the restriction of the projective representation of G in an ambient projective space.

Example 4.1. Consider a rational projective curve X obtained from $\mathbb{P}^1$ by identifying $0, \infty \in \mathbb{P}^1$ in an ordinary double point (a Cartesian leaf). A $\Bbbk^\times$-action on $\mathbb{P}^1$ with the fixed points $0, \infty$ goes down to X. This action is not good. (Otherwise, there is a $\Bbbk^\times$-stable hyperplane section of X in an ambient $\mathbb{P}^n$ that does not contain the double point. But there are no other $\Bbbk^\times$-fixed points on X.)

The reason for the action of Example 4.1 fails to be good is non-normality of X.

Example 4.2. If X is a G-stable subvariety of a normal G-variety (e.g., X is itself normal), then $G : X$ is good by Sumihiro's theorem (A1.3).

The normalization or the equivariant Chow lemma [PV, Th.1.3] reduce the study of arbitrary algebraic group actions to good ones. In the sequel, only good actions are considered unless otherwise specified.

Now let G be a connected reductive group. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Put $U = B'$, a maximal unipotent subgroup of G.

In order to describe the local structure of good G-actions, we begin with a helpful technical construction in characteristic zero due to Brion–Luna–Vust and Grosshans.

Let V be a finite-dimensional G-module with a lowest weight vector v, and $\omega \in V^*$ be the dual highest weight vector such that $\langle v, \omega \rangle \neq 0$. Let $P = G_{\langle\omega\rangle} = L \rightthreetimes P_{\mathrm{u}}$ be the projective stabilizer of ω, where $L \supseteq T$ is a Levi subgroup and $P_{\mathrm{u}} = \mathrm{R_u}(P)$. Then $P^- = G_{\langle v\rangle} = L \rightthreetimes P_{\mathrm{u}}^-$ is the opposite parabolic to P, where $P_{\mathrm{u}}^- = \mathrm{R_u}(P^-)$.

Put $\mathring{V} = V \setminus \langle\omega\rangle^\perp$, $W = \langle\mathfrak{p}_{\mathrm{u}}^-\omega\rangle^\perp$, and $\mathring{W} = W \cap \mathring{V}$. (Here ${}^\perp$ denotes the annihilator in the dual subspace.)

Lemma 4.1 ([BLV], [Gro1]). *In characteristic zero, the action $P : \overset{\circ}{V}$ gives rise to an isomorphism*

$$P_{\mathrm{u}} \times \overset{\circ}{W} = P *_L \overset{\circ}{W} \xrightarrow{\sim} \overset{\circ}{V}$$

Proof. Consider level hyperplanes $V_c = \{x \in V \mid \langle x, \omega\rangle = c\}$, $W_c = W \cap V_c$. We have

$$\overset{\circ}{V} = \bigsqcup_{c \neq 0} V_c = \Bbbk^\times v + V_0,$$

similarly for $\overset{\circ}{W}$. Note that $W_0 = \langle \mathfrak{g}\omega\rangle^\perp$ is P-stable. Affine hyperplanes $V_c \subset V$ and $V_c/W_0 \subset V/W_0$ are P-stable. It suffices to show that the induced action $P_{\mathrm{u}} : V_c/W_0$ is transitive and free whenever $c \neq 0$. But $V_0 = \mathfrak{p}_{\mathrm{u}} v \oplus W_0$, whence $cv + W_0$ has the dense P_{u}-orbit in V_c/W_0 with the trivial stabilizer. It remains to note that all orbits of a unipotent group on an affine variety are closed (Lemma 3.3). □

Theorem 4.1 ([Kn1, 2.3], [Kn3, 1.2], [BLV]). *Let X be a good G-variety and $Y \subseteq X$ a G-stable subvariety. Then there exists a unique parabolic subgroup $P = P(Y) \supseteq B$ with a Levi decomposition $P = L \leftthreetimes P_{\mathrm{u}}$, $L \supseteq T$, $P_{\mathrm{u}} = \mathrm{R_u}(P)$, and a T-stable locally closed affine subvariety $Z \subseteq X$ such that:*

(1) *$\overset{\circ}{X} = PZ$ is an affine open subset of X.*

(2) *The action $P_{\mathrm{u}} : \overset{\circ}{X}$ is proper and has a geometric quotient $\overset{\circ}{X}/P_{\mathrm{u}} = \operatorname{Spec} \Bbbk[\overset{\circ}{X}]^{P_{\mathrm{u}}}$.*

(3) *A natural map $P_{\mathrm{u}} \times Z \to \overset{\circ}{X}$, $(g, z) \mapsto gz$, and the quotient map $Z \to \overset{\circ}{X}/P_{\mathrm{u}}$ are finite and surjective.*

(3)′ *In characteristic zero, Z may be chosen to be L-stable and such that the action $P : \overset{\circ}{X}$ gives rise to an isomorphism*

$$P_{\mathrm{u}} \times Z = P *_L Z \xrightarrow{\sim} \overset{\circ}{X}$$

(4) *$\overset{\circ}{Y} = Y \cap \overset{\circ}{X} \neq 0$, and the kernel L_0 of the natural action $L = P/P_{\mathrm{u}} : \overset{\circ}{Y}/P_{\mathrm{u}}$ contains L'. Moreover, $\overset{\circ}{Y}/P_{\mathrm{u}} \simeq L/L_0 \times C$, where the torus L/L_0 acts on C trivially. In characteristic zero, $Y \cap Z \simeq L/L_0 \times C$.*

Proof. We will assume $\operatorname{char} \Bbbk = 0$. (For the general case, see [Kn3, 1.2, §2].) Replacing X by an open G-subvariety, we may assume that X is quasiprojective, Y is closed in X, and there is a very ample G-line bundle $\mathcal{L}$ on X. Then X is G-equivariantly embedded in $\mathbb{P}(V)$, $V = \mathrm{H}^0(X, \mathcal{L})^*$.

Let $\overline{X}, \overline{Y}$ be the closures of X, Y in $\mathbb{P}(V)$. We can find a homogeneous B-eigenform ω in coordinates on V that vanishes on $\overline{X} \setminus X$ and on any given closed B-subvariety $D \subset Y$, but not on Y. (Take a nonzero B-eigenform in the ideal of $D \cup (\overline{X} \setminus X)$ in the homogeneous coordinate ring of $(\overline{X} \setminus X) \cup \overline{Y}$, and extend it to $\overline{X}$ by complete reducibility of G-modules.) Replacing $\mathcal{L}$ by its power, we may assume $\omega \in \mathrm{H}^0(X, \mathcal{L})^{(B)}$.

Now $\mathring{X} = X_\omega$ is an affine open subset of X. By Lemma 4.1,

$$\mathring{X} \simeq P *_L Z = P_{\mathrm{u}} \times Z,$$

where $P = G_{\langle\omega\rangle}$ and $Z = \mathbb{P}(\mathring{W}) \cap X$.

If we choose for D a (maybe reducible) B-stable divisor in Y whose stabilizer P is the smallest possible one, then any $(B \cap L)$-stable divisor of $Y \cap Z$ is L-stable. It follows that each highest weight function in $\Bbbk[Z]$ is L-semiinvariant, whence L' acts on Z trivially. Taking D sufficiently large, we may replace Z by an open subset L-isomorphic to $L/L_0 \times C$ (with the trivial action on C).

To complete the proof, note that P is uniquely determined by the conditions of the theorem. Namely, $P = P(Y)$ equals the smallest stabilizer of a B-stable divisor in Y. $\square$

Corollary 4.1. *Let $P = P(X)$ be the smallest stabilizer of a B-stable divisor in X. Then there exists a T-stable (L-stable if* $\operatorname{char}\Bbbk = 0$*) locally closed affine subset $Z \subseteq X$ such that $\mathring{X} = PZ$ is an open affine subset of X, the natural maps $P_{\mathrm{u}} \times Z \to \mathring{X}$, $Z \to \mathring{X}/P_{\mathrm{u}}$ are finite and surjective (isomorphic if* $\operatorname{char}\Bbbk = 0$*), and $\mathring{X}/P_{\mathrm{u}} \simeq L/L_0 \times C$, where $L \supseteq L_0 \supseteq L'$ and the L-action on C is trivial.*

Example 4.3. Let $X = \mathbb{P}(S^2\Bbbk^{n*})$ be the space of quadrics in $\mathbb{P}^{n-1}$, $\operatorname{char}\Bbbk \neq 2$. Then $G = \mathrm{GL}_n(\Bbbk)$ acts on X by linear variable changes with the orbits $\mathcal{O}_1, \dots, \mathcal{O}_n$, where $\mathcal{O}_r$ is the set of quadrics of rank r, and $\overline{\mathcal{O}_1} \subset \dots \subset \overline{\mathcal{O}_n} = X$. Choose the standard Borel subgroup $B \subseteq G$ of upper-triangular matrices and the standard maximal torus $T \subseteq B$ of diagonal matrices.

(1) Put $Y = \overline{\mathcal{O}_1}$, the unique closed G-orbit in X, which consists of double hyperplanes. In the notation of Lemma 4.1 and Theorem 4.1, we have $V = S^2\Bbbk^{n*} \ni v = x_1^2$, $V^* = S^2\Bbbk^n \ni \omega = e_1^2$. (Here $e_1, \dots, e_n$ form the standard basis of $\Bbbk^n$ and $x_1, \dots, x_n$ are the standard coordinates.) Then P is a standard parabolic subgroup of matrices of the form

$$\begin{array}{|c|c|}\hline L & P_{\mathrm{u}} \\ \hline \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \underbrace{\qquad L \qquad}_{n-1} \\ \hline \end{array}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\}{\scriptstyle n-1}$$

(We indicate the Levi decomposition of P at the figure.) $\mathring{V}$ is the set of quadratic forms $q = cx_1^2 + \dots$, $c \neq 0$, $\mathfrak{p}_{\mathrm{u}}v = \{a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \mid a_{ij} \in \Bbbk\}$, and W is the space of forms $q = cx_1^2 + q'(x_2, \dots, x_n)$, where $c \in \Bbbk$, q' is a quadratic form in $x_2, \dots, x_n$.

Now $\mathring{X}$ is the set of quadrics given by an equation $x_1^2 + \dots = 0$, Z consists of quadrics with an equation $x_1^2 + q'(x_2, \dots, x_n) = 0$, and $Y \cap Z = \{\langle x_1^2 \rangle\}$. Lemma 4.1 or Theorem 4.1 say that every quadratic form with nonzero coefficient at x_1^2 can be moved by P_{u}, i.e., by a linear change of x_1, to a form containing no products x_1x_j, $j > 1$. This is the first step in the Lagrange method of transforming a quadric to the normal form.

(2) More generally, put $Y = \mathcal{O}_r$. It is easy to see that $P(Y)$ is the group of matrices of the form

$$\begin{array}{c|c} \overbrace{\begin{matrix} * & & * \\ & \ddots & \\ 0 & & * \end{matrix}}^{r} & * \\ \hline 0 & * \end{array}$$

Clearly, $P(Y) = G_{\langle \omega \rangle}$, where ω is the product of the first r upper-left corner minors of the matrix of a quadratic form. Then $\mathring{X}$ is the set of quadrics, where ω does not vanish, i.e., having non-degenerate intersection with all subspaces $\{x_k = \dots = x_n = 0\}$, $k \leq r + 1$. Further, Z consists of quadrics with an equation $c_1x_1^2 + \dots + c_rx_r^2 + q'(x_{r+1}, \dots, x_n) = 0$, $c_i \neq 0$, and $Y \cap Z = \{\langle c_1x_1^2 + \dots + c_rx_r^2 \rangle \mid c_i \neq 0\}$. The Levi subgroup $L = (\Bbbk^\times)^r \times \mathrm{GL}_{n-r}(\Bbbk)$ acts on $Y \cap Z$ via the first factor, and $Y \cap Z = (\Bbbk^\times)^r \times \{\langle x_1^2 + \dots + x_r^2 \rangle\}$. Theorem 4.1 says that each quadric with nonzero first r upper-left corner minors transforms by a unitriangular linear variable change to the form $c_1x_1^2 + \dots + c_rx_r^2 + q'(x_{r+1}, \dots, x_n)$—this is nothing else, but the Gram–Schmidt orthogonalization method.

A refined version of the local structure theorem was proved by Knop in characteristic zero.

Let X be a G-variety. We call any formal $\Bbbk$-linear combination of prime Cartier divisors on X a $\Bbbk$-*divisor*. Let $D = a_1D_1 + \dots + a_sD_s$ be a B-stable $\Bbbk$-divisor, and $P = P[D]$ be the stabilizer of its support $D_1 \cup \dots \cup D_s$. Replacing G by a finite cover, we may assume that the line bundles $\mathcal{O}(D_i)$ are G-linearized. Let $s_i \in \mathrm{H}^0(X, \mathcal{O}(D_i))^{(B)}$ be the sections of B-weights λ_i such that $\operatorname{div} s_i = D_i$, and set $\lambda_D = \sum a_i\lambda_i$. We say that D is *regular* if $\langle\lambda_D,\alpha^\vee\rangle \neq 0$ for any root α such that $\mathfrak{g}_\alpha \subseteq \mathfrak{p}[D]_{\mathrm{u}}$. (For example, any effective B-stable Cartier divisor is regular.)

Define a morphism $\psi_D : X \setminus D \to \mathfrak{g}^*$ by the formula

$$\langle\psi_D(x),\xi\rangle = \sum a_i \frac{\xi s_i}{s_i}(x), \qquad \forall \xi \in \mathfrak{g}$$

Theorem 4.2 ([Kn5, 2.3]). *The map ψ_D is a P-equivariant fibration over the P-orbit of λ_D considered as a linear function on a maximal torus $\mathfrak{t} \subseteq \mathfrak{b}$ and extended to $\mathfrak{g}$ by putting $\langle\lambda_D,\mathfrak{g}_\alpha\rangle = 0$, $\forall\alpha$. The stabilizer $L = P_{\lambda_D}$ is the Levi subgroup of P containing T. In particular, $X \setminus D \simeq P *_L Z$, where $Z = \psi^{-1}(\lambda_D)$.*

Other versions of the local structure theorem can be found in [BL] and in §13, §15.

5 Complexity and rank of G-varieties

As before, G is a connected reductive group with a fixed Borel subgroup B, a maximal unipotent subgroup $U = B'$, and a maximal torus $T \subseteq B$. Let X be an irreducible G-variety.

Definition 5.1. The *complexity* $c_G(X)$ of the action $G : X$ is the codimension of a generic B-orbit in X. By the lower semicontinuity of the function $x \mapsto \dim Bx$, $c_G(X) = \min_{x\in X} \operatorname{codim} Bx$. By the Rosenlicht theorem [PV, 2.3], $c_G(X) = \operatorname{tr.deg} \Bbbk(X)^B$.

The *weight lattice* $\Lambda(X)$ (resp. the *weight semigroup* $\Lambda_+(X)$) is the set of weights of all rational (regular) B-eigenfunctions on X. It is a sublattice in the weight lattice $\mathfrak{X}(B) = \mathfrak{X}(T)$ (a submonoid in the monoid $\mathfrak{X}_+$ of dominant weights, respectively).

The integer $r_G(X) = \operatorname{rk}\Lambda(X)$ is the *rank* of $G : X$.

We usually drop the subscript G in the notation of complexity and rank.

Complexity, rank, and the weight lattice are birational invariants of an action. Replacing X by a G-birationally equivalent variety, we may always assume that X is good, normal, quasiprojective, or smooth, when required. These invariants are very important in studying the geometry of the action $G : X$ and the related representation and compactification theory. Here we examine the most basic properties of complexity and rank.

Example 5.1. Let X be a projective homogeneous G-space. By the Bruhat decomposition, U has a dense orbit in X (a *big cell*). Hence $c(X) = r(X) = 0$.

Example 5.2. Assume $G = T$ and let T_0 be the kernel of the action $T : X$. Then X contains an open T-stable subset $\mathring{X} = T/T_0 \times Z$. Hence $c(X) = \dim Z = \dim \Bbbk(X)^T$, $\Lambda(X) = \mathfrak{X}(T/T_0)$, $r(X) = \dim T/T_0$.

Example 5.3. Let G act on $X = G$ by left translations. Then $c(X) = \dim G/B = \dim U$ is the number of positive roots of G. By Formula (2.4), $\Lambda_+(X) = \mathfrak{X}_+$.

It is easy to prove the following

Proposition 5.1. $c(X) + r(X) = \min_{x \in X} \operatorname{codim} Ux = \operatorname{tr.deg} \Bbbk(X)^U$

(Just apply Example 5.2 to the T-action on the rational quotient X/U.)

Complexity and weight lattice (semigroup) are monotonous by inclusion. More precisely, we have

Theorem 5.1 ([Kn7, 2.3]). *For any closed G-subvariety $Y \subseteq X$, $c(Y) \leq c(X)$, $r(Y) \leq r(X)$, and $Y = X$ iff the equalities hold. Furthermore, $\Lambda(Y) \subseteq \frac{1}{q}\Lambda(X)$ and if X is affine, then $\Lambda_+(Y) \subseteq \frac{1}{q}\Lambda_+(X)$, where q is a sufficiently big power of the characteristic exponent of $\Bbbk$ (= $\operatorname{char}\Bbbk$, or 1 if $\operatorname{char}\Bbbk = 0$).*

The proof relies on a helpful lemma of Knop:

Lemma 5.1. *Let $Y \subseteq X$ be a G-subvariety and p be the characteristic exponent of $\Bbbk$. Then*

$$\forall f \in \Bbbk(Y)^{(B)}\ \exists \widetilde{f} \in \mathcal{O}_{X,Y}^{(B)}\ \exists q = p^N :\ f^q = \widetilde{f}|_Y$$

Proof. Applying normalization, we may assume that X is good and even projective and Y is closed in X. Embed X G-equivariantly in a projective space and let $\widehat{X}, \widehat{Y}$ be the cones over X, Y. These cones are $\widehat{G} = G \times \Bbbk^\times$-stable ($\Bbbk^\times$ acts by homotheties), and f is pulled back to $\Bbbk(\widehat{Y})^{(\widehat{B})}$, where $\widehat{B} = B \times \Bbbk^\times$. Thence $f = F_1/F_2$, where $F_i \in \Bbbk[\widehat{Y}]^{(\widehat{B})}$ are homogeneous B-semiinvariant polynomials. By Corollary A2.1, $F_i^q = \widetilde{F}_i|_{\widehat{Y}}$ for some $\widetilde{F}_i \in \Bbbk[\widehat{X}]^{(\widehat{B})}$, $q = p^N$. Now $\widetilde{f} = \widetilde{F}_1/\widetilde{F}_2$ is pulled down to a rational B-eigenfunction on X such that $\widetilde{f}|_Y = f^q$. ∎

Proof of the theorem. Lemma 5.1 implies that $q\Lambda(Y) \subseteq \Lambda(X)$ and that $\Bbbk(Y)^B$ is a purely inseparable extension of the residue field of $\mathcal{O}_{X,Y}^B$, whence the inequalities and the inclusion of weight lattices. The inclusion of weight semigroups stems from Corollary A2.1.

Now suppose $c(Y) = c(X)$ and $r(Y) = r(X)$. As in Lemma 5.1, we may assume that X, Y are closed G-subvarieties in a projective space and consider the cones $\widehat{X}, \widehat{Y}$ over X, Y. We have $c_{\widehat{G}}(\widehat{X}) = c_G(X)$ and $r_{\widehat{G}}(\widehat{X}) = r_G(X)+1$, in view of an exact sequence

$$0 \longrightarrow \Lambda(X) \longrightarrow \widehat{\Lambda}(\widehat{X}) \longrightarrow \mathfrak{X}(\Bbbk^\times) = \mathbb{Z} \longrightarrow 0,$$

where $\widehat{\Lambda}$ is the weight lattice relative to $\widehat{B}$. Similar equalities hold for $\widehat{Y}$.

By assumption and Proposition 5.1, $\operatorname{tr.deg} \Bbbk(\widehat{Y})^U = \operatorname{tr.deg} \Bbbk(\widehat{X})^U$. But a rational U-invariant function on an affine variety is the quotient of two U-invariant polynomials [PV, Th.3.3], whence $\Bbbk[\widehat{Y}]^U$ and $\Bbbk[\widehat{X}]^U$ have the same transcendence degree. By Lemma A2.1, $\Bbbk[\widehat{Y}]^U$ is a purely inseparable finite extension of $\Bbbk[\widehat{X}]^U|_{\widehat{Y}}$, whence $\Bbbk[\widehat{X}]^U$ restricts to $\widehat{Y}$ injectively. Therefore the ideal of $\widehat{Y}$ contains no nonzero U-invariants, hence is zero. It follows that $\widehat{Y} = \widehat{X}$, whence $Y = X$. □

On the other side, there is a general procedure of "enlarging" a variety which preserves complexity, rank, and the weight lattice.

Definition 5.2. Let G, G_0 be connected reductive groups. We say that a G-variety X is obtained from a G_0-variety X_0 by *parabolic induction* if $X = G *_Q X_0$, where $Q \subseteq G$ is a parabolic subgroup acting on X_0 via an epimorphism $Q \twoheadrightarrow G_0$.

Proposition 5.2. *$c_G(X) = c_{G_0}(X_0)$, $r_G(X) = r_{G_0}(X_0)$, $\Lambda(X) = \Lambda(X_0)$.*

The proof is easy.

The weight lattice is actually an attribute of a generic G- (and even B-) orbit.

Proposition 5.3 ([Kn7, 2.6]). *$\Lambda(X) = \Lambda(Gx)$ for all x in an open subset of X.*

Proof. Replacing X by the rational quotient X/U, we reduce the problem to the case $G = B = T$. Let $f_1, \dots, f_r \in \Bbbk(X)$ be rational T-eigenfunctions whose weights generate $\Lambda(X)$. If all f_i are defined and nonzero at x, then by Lemma 5.1, $\Lambda(X) = \Lambda(Tx)$. (In this case, we may assume $q = 1$ in Lemma 5.1, since T is linearly reductive.) □

Corollary 5.1. *The function $x \to r(Gx)$ is lower semicontinuous on X.*

Using Lemma 5.1, Arzhantsev proved the following

Proposition 5.4 ([Arzh, §2]). *The function $x \to c(Gx)$ is lower semicontinuous on X.*

In the affine case, the weight semigroup is a more subtle invariant of an action than the weight lattice.

Proposition 5.5. *For quasiaffine X, $\Lambda(X) = \mathbb{Z}\Lambda_+(X)$.*

Proof. Any rational B-eigenfunction on X is a quotient of two polynomials: $f = f_1/f_2$. By the Lie-Kolchin theorem, there exists a nonzero B-semiinvariant linear combination $\sum \lambda_i(b_i f_2)$, $\lambda_i \in \Bbbk$, $b_i \in B$. Then $f = \widetilde{f}_1/\widetilde{f}_2$, where $\widetilde{f}_j = \sum \lambda_i(b_i f_j)$ are polynomial B-eigenfunctions on X. □

Proposition 5.6. *For affine X, the semigroup $\Lambda_+(X)$ is finitely generated.*

Proof. The semigroup $\Lambda_+(X)$ is the semigroup of weights for the T-weight decomposition of $\Bbbk[X]^U$, the latter algebra being finitely generated by Theorem A2.1(1). □

In characteristic zero, the complexity controls the growth of multiplicities in the spaces of global sections of G-line bundles on X.

Theorem 5.2. (1) *If X is affine and $\Bbbk[X]^G = \Bbbk$ (e.g., X contains an open G-orbit), then $c(X)$ is the minimal integer c such that $m_{n\lambda}(X) = O(n^c)$ for every dominant weight λ.*

(2) *If X is projective, then $c(X)$ is the minimal integer c such that $m_{n\lambda}(\mathcal{L}^n) = O(n^c)$ for any line bundle $\mathcal{L}$ on X and any dominant weight λ.*

Proof. (1) By Proposition 2.5, $m_\lambda(X) = \dim \Bbbk[X]^U_\lambda$. Replacing X by $X/\!\!/U$, we may assume that $G = B = T$. Put

$$A(\lambda) = \bigoplus_{n\geq 0} \Bbbk[X]_{n\lambda}$$

Then $A(\lambda) \simeq (\Bbbk[X] \otimes \Bbbk[t])^T$, where the indeterminate t has the T-weight $-\lambda$. The function field $K = \operatorname{Quot} A(\lambda)$ is purely transcendental of degree 1 over $K^B \subseteq \Bbbk(X)^B$, whence $\operatorname{tr.deg} A(\lambda) \leq c + 1$.

If λ lies in the interior of the cone $\mathbb{Q}_+\Lambda_+(X)$, then $\operatorname{tr.deg} A(\lambda) = c + 1$. Indeed, we have $\lambda = \sum l_i\lambda_i$, where λ_i are the generators of $\Lambda_+(X)$ and l_i are rational positive numbers. Any $f \in K^B$ is expressible as $f = h_1/h_2$, where $h_j \in \Bbbk[X]_\mu$ for some μ. Now $\mu = \sum m_i\lambda_i$, and for n sufficiently large, $n_i = nl_i - m_i$ are positive integers. Then $f = \widetilde{h}_1/\widetilde{h}_2$, where $\widetilde{h}_j = h_j \prod f_i^{n_i} \in \Bbbk[X]_{n\lambda}$, $f_i \in \Bbbk[X]_{\lambda_i}$. Thus $K^B = \Bbbk(X)^B$, q.e.d.

By the above, $A(\lambda)$ is a finitely generated graded algebra of Krull dimension $d \leq c + 1$, and the equality holds for general λ. We conclude by a standard result of dimension theory, that $\dim A(\lambda)_n$ grows as a polynomial in n of degree d, at least for n sufficiently divisible.

(2) We have $m_{n\lambda}(\mathcal{L}^n) = \dim \mathrm{H}^0(X, \mathcal{L}^n)^U_{n\lambda}$. For a sufficiently ample G-line bundle $\mathcal{M}$, the line bundle $\mathcal{L} \otimes \mathcal{M}$ is very ample, and for nonzero $s \in \mathrm{H}^0(X, \mathcal{M})^U_{\mu}$, we have an inclusion

$$\mathrm{H}^0(X, \mathcal{L}^n)^U_{n\lambda} \hookrightarrow \mathrm{H}^0(X, (\mathcal{L} \otimes \mathcal{M})^n)^U_{n(\lambda+\mu)}$$

provided by $(\cdot \otimes s^n)$. Thus it suffices to consider very ample $\mathcal{L}$. Consider the respective projective embedding of X, and let $\widehat{X}$ be the affine cone over X. Then $\widehat{G} = G \times \Bbbk^\times$ acts on $\widehat{X}$ (here $\Bbbk^\times$ acts by homotheties), and $m_{n\lambda}(\mathcal{L}^n) = m_{(n\lambda,n)}(\widehat{X})$ (at least for $n \gg 0$). The assertion (1) applied to $\widehat{X}$ concludes the proof. $\square$

Example 5.4. Let $\mathcal{O} = G/H$ be a quasiaffine homogeneous space such that $\Bbbk[\mathcal{O}]$ is finitely generated. Then $X = \operatorname{Spec} \Bbbk[\mathcal{O}]$ contains $\mathcal{O}$ as an open orbit. By Theorem 5.2(1), $m_{n\lambda}(X) = \dim V(n\lambda^*)^H$ grows as $n^{c(\mathcal{O})}$ for general λ, and no faster for any λ.

E.g., for $H = \{e\}$, $c(G)$ is the number of positive roots, and $\dim V(n\lambda^*)$ is the polynomial in n of degree $\leq c(G)$ by the Weyl dimension formula.

Remark 5.1. For G-varieties of complexity ≤ 1, more precise results on multiplicities are obtained, see §16, §25.

6 Complexity and modality

The notion of modality was introduced in the works of Arnold on the theory of singularities. The modality of an action is the maximal number of parameters in a continuous family of orbits. More precisely,

Definition 6.1. Let $H : X$ be an algebraic group action. The integer

$$d_H(X) = \min_{x \in X} \operatorname*{codim}_X Hx = \operatorname{tr.deg} \Bbbk(X)^H$$

is called the *generic modality* of the action. The *modality* of $H : X$ is the number $\operatorname{mod}_H X = \max_{Y \subseteq X} d_H(Y)$, where Y runs through H-stable irreducible subvarieties of X.

Note that $c(X) = d_B(X)$.

It may happen that the modality is greater than the generic modality of an action. For example, the natural action $\mathrm{GL}_n(\Bbbk) : \mathrm{L}_n(\Bbbk)$ by left multiplication has an open orbit, whereas its modality equals $[n^2/4]$. Indeed, $\mathrm{L}_n(\Bbbk)$ is covered by finitely many locally closed $\mathrm{GL}_n(\Bbbk)$-stable subsets $Y_{i_1,\dots,i_k}$, where $Y_{i_1,\dots,i_k}$ is the set of matrices of rank k with linearly independent columns

$i_1, \dots, i_k$. Therefore an orbit in $Y_{i_1,\dots,i_k}$ depends on $k(n-k)$ parameters, which are the coefficients of linear expressions of the remaining $n-k$ columns by the columns $i_1, \dots, i_k$. The maximal number of parameters is obtained for $k = [\frac{n+1}{2}]$.

Replacing $\mathrm{GL}_n(\Bbbk)$ by the group $B_n(\Bbbk)$ of non-degenerate upper-triangular matrices and $\mathrm{L}_n(\Bbbk)$ by the space $\overline{B_n(\Bbbk)}$ of all upper-triangular matrices shows that the same thing may happen for a solvable group action. The action $B_n(\Bbbk) : \overline{B_n(\Bbbk)}$ has an open orbit, but infinitely many orbits in its complementary.

Remarkably, for a G-variety X and the restricted action $B : X$, the modality equals the generic modality (=the complexity) of the action. This result was obtained by Vinberg [Vin1] with the aid of Popov's technique of contracting to a horospherical variety (cf. §8). We present a proof due to Knop [Kn7], who developed some earlier ideas of Matsuki. A basic tool is an action of a certain monoid on the set of B-stable subvarieties.

Let $W = N_G(T)/T$ be the Weyl group of G. By the Bruhat decomposition, the only irreducible closed $B \times B$-stable subvarieties in G are the Schubert varieties $D_w = \overline{BwB}$, $w \in W$.

Definition 6.2 ([Kn7, §2], [RS1]). The *Richardson–Springer monoid* (*RS-monoid*) of G is the set of all Schubert subvarieties in G with the multiplication as of subsets in G. Equivalently, RS-monoid is the set W with a new multiplication $*$ defined by $D_{v*w} = D_v D_w$. We denote the set W equipped with this product by W^*.

Clearly, W^* is an associative monoid with the unity e. It is easy to describe W^* by generators and relations. Namely, W is defined by generators $s_1, \dots, s_l$ (simple reflections) and relations $s_i^2 = e$ and

$$\underbrace{s_i s_j s_i \dots}_{n_{ij}\text{ terms}} = \underbrace{s_j s_i s_j \dots}_{n_{ij}\text{ terms}} \qquad \text{(braid relations)},$$

where (n_{ij}) is the Coxeter matrix of W. The monoid W^* has the same generators and relations $s_i^2 = s_i$ and braid relations. If $w = s_{i_1} \dots s_{i_n}$ is a reduced decomposition of $w \in W$, then $w = s_{i_1} * \dots * s_{i_n}$ in W^*. All these assertions follow from standard facts on multiplication of Schubert cells in G [Hum, §29].

Let $\mathfrak{B}(X)$ be the set of all closed irreducible B-stable subvarieties in X. The RS-monoid acts on $\mathfrak{B}(X)$ in a natural way: $w * Z = D_w Z$ is B-stable and closed as the image of $D_w *_B Z$ under the proper morphism $G *_B X \simeq G/B \times X \to X$.

Proposition 6.1. *$c(w * Z) \geq c(Z)$, $r(w * Z) \geq r(Z)$ for any $Z \in \mathfrak{B}(X)$.*

Proof. It suffices to consider the case of a simple reflection $w = s_i$. In this case, $D_w = P_i$ is the respective minimal parabolic subgroup of G If Z is P_i-stable, there is nothing to prove. Otherwise, the map $P_i *_B Z \to P_i Z$ is generically finite, and we may replace $s_i * Z$ by $P_i *_B Z$ and further, by an open subset $Bs_iB *_B Z = B *_{B_i} s_iZ$, where $B_i = B \cap_i Bs_i^{-1}$. Therefore the complexity (rank) of s_iZ equals the complexity (resp. rank) of s_iZ w.r.t. the B_i-action or of Z w.r.t. the action of $s_i^{-1}Bs_i \subseteq B$. The assertion follows. □

Theorem 6.1. *For any B-stable irreducible subvariety $Y \subseteq X$, we have $c(Y) \leq c(X)$, $r(Y) \leq r(X)$. In particular,* $\operatorname{mod}_H(X) = d_H(X)$, *where $H = B$ or U.*

Proof. Follows from Proposition 6.1, Theorem 5.1, and Proposition 5.1. □

Corollary 6.1 ([Vin1], [Bri1]). *Every spherical variety contains finitely many B-orbits.*

In the spherical case, elements of $\mathfrak{B}(X)$ are just B-orbit closures. The set of all B-orbits on a spherical variety, identified with $\mathfrak{B}(X)$, is an interesting combinatorial object. It is finite and partially ordered by the adherence relation $\preceq$ (=inclusion of orbit closures). This partial order is compatible with the action of the RS-monoid and with the dimension function in the following sense:

(1) $\mathcal{O} \preceq s_i * \mathcal{O}$

(2) $\mathcal{O}_1 \preceq \mathcal{O}_2 \implies s_i * \mathcal{O}_1 \preceq s_i * \mathcal{O}_2$

(3) $\mathcal{O}_1 \prec \mathcal{O}_2 \implies \dim \mathcal{O}_1 < \dim \mathcal{O}_2$

(4) $\mathcal{O} \prec s_i * \mathcal{O} \implies \dim(s_i * \mathcal{O}) = \dim \mathcal{O} + 1$

(5) (One step property) $(s_i * \mathcal{O})_{\preceq} = W_i * \mathcal{O}_{\preceq}$, where $W_i = \{e, s_i\}$ is a minimal standard Coxeter subgroup in W, and $\mathcal{O}_{\preceq} = \{\mathcal{O}' \in \mathfrak{B}(X) \mid \mathcal{O}' \preceq \mathcal{O}\}$ is the closure of $\mathcal{O}$.

This compatibility imposes strong restrictions on the adherence of B-orbits on a spherical homogeneous space $X = G/H$. By (5), it suffices to know the closures of the minimal orbits, i.e., such $\mathcal{O} \in \mathfrak{B}(G/H)$ that $\mathcal{O} \neq w * \mathcal{O}'$ for $\forall \mathcal{O}' \neq \mathcal{O}$, $w \in W^*$. If all minimal orbits have the same dimension then they are closed.

Example 6.1. For $H = B$, the B-orbits are the Schubert cells $B[w] \subset G/B$, $w \in W$, and their closures are the Schubert subvarieties $S_w = D_w/B$ in G/B. By standard facts on the multiplication of Bruhat cells, $B[e] = \{[e]\}$

is the unique minimal B-orbit. Whence $S_w = \overline{s_{i_1} * \cdots * s_{i_n} * B[e]}$ ($w = s_{i_1} \dots s_{i_n}$ is a reduced decomposition) $= W_{i_1} * \cdots * W_{i_n} * B[e] = P_{i_1} \dots P_{i_n}[e] = (B \sqcup Bs_{i_1}B) \dots (B \sqcup Bs_{i_n}B)[e] = \bigsqcup Bs_{j_1} \dots s_{j_k}B[e] = \bigsqcup_{v=s_{j_1}\dots s_{j_k}} B[v]$ over all subsequences $(j_1, \dots, j_k)$ of $(i_1, \dots, i_n)$. This is a well-known description of the Bruhat order on W.

Example 6.2. If G/H is a symmetric space, i.e., H is a fixed point set of an involution, up to connected components, then G/H is spherical (Theorem 26.1) and all minimal B-orbits have the same dimension [RS1]. A complete description of B-orbits, of the W^*-action, and of the adherence relation is obtained in [RS1] (cf. Proposition 26.3).

Example 6.3. For $H = TU'$, the space G/H is spherical, but the minimal B-orbits have different dimensions. However, the adherence of B-orbits is completely determined by the W^*-action with the aid of properties (1)–(5). The set $\mathfrak{B}(G/H)$, the W^*-action, and the adherence relation are described in [Tim1].

Conjecture ([Tim1]). *For any spherical homogeneous space G/H, there is a unique partial order on $\mathfrak{B}(G/H)$ satisfying* (1)–(5).

By Theorem 6.1, the complexity of a G-variety equals the maximal number of parameters determining a continuous family of B-orbits on X. Generally, continuous families of G-orbits depend on a lesser number of parameters. However, a result of Akhiezer shows that the complexity of a G-action is the maximal modality in the class of all actions birationally G-isomorphic to the given one.

Theorem 6.2 ([Akh3]). *There exists a G-variety X' birationally G-isomorphic to X such that* $\operatorname{mod}_G X' = c(X)$.

Proof. Let $f_1, \dots, f_c$ be a transcendence base of $\Bbbk(X)^B/\Bbbk$. We may replace X by a birationally G-isomorphic normal projective variety. Consider an ample G-line bundle $\mathcal{L}$ on X. Replacing $\mathcal{L}$ by a power, we may find a section $s_0 \in \mathrm{H}^0(X, \mathcal{L})^{(B)}$ such that $\operatorname{div} s_0 \geq \operatorname{div}_\infty f_i$ for $\forall i$. Put $s_i = f_i s_0 \in \mathrm{H}^0(X, \mathcal{L})^{(B)}$.

Take a G-module M generated by a highest weight vector m_0 and such that there is a homomorphism $\psi_i : M \to \mathrm{H}^0(X, \mathcal{L})$, $\psi_i(m_0) = s_i$. Let $m_0, \dots, m_n$ be its basis of T-eigenvectors with the weights $\lambda_0, \dots, \lambda_n$. Let $E = \langle e_0, \dots, e_c \rangle$ be a trivial G-module of dimension $c + 1$. A homomorphism $\psi : E \otimes M \to \mathrm{H}^0(X, \mathcal{L})$, $e_i \otimes m \mapsto \psi_i(m)$, gives rise to the rational G-equivariant map $\phi : X \dashrightarrow \mathbb{P}((E \otimes M)^*)$. In projective coordinates,

$$\psi(x) = [\cdots : \psi_i(m_j)(x) : \cdots]$$

Take a one-parameter subgroup $\gamma \in \mathfrak{X}^*(T)$ such that $\langle\alpha,\gamma\rangle > 0$ for each positive root α. If all $s_i(x) \neq 0$, then

$$\begin{aligned}\psi(\gamma(t)x) &= [\cdots : t^{-\langle\lambda_j,\gamma\rangle}\psi_i(m_j)(x) : \cdots]\\ &= [\cdots : t^{\langle\lambda_0-\lambda_j,\gamma\rangle}\psi_i(m_j)(x) : \cdots] \longrightarrow [s_0(x) : \cdots : s_c(x) : 0 : \cdots : 0]\end{aligned}$$

as $t \to 0$, because $\lambda_0 - \lambda_j$ is a positive linear combination of positive roots for $\forall j > 0$. Thus

$$\lim_{t\to 0}\gamma(t)\psi(x) = \big([s_0(x) : \cdots : s_c(x)], [m_0^*]\big) \in \mathbb{P}(E^*) \times \mathbb{P}(M^*) \hookrightarrow \mathbb{P}((E \otimes M)^*)$$

(the Segre embedding), where $m_0^*, \dots, m_n^*$ is the dual basis of M^*.

Let $X' \subseteq X \times \mathbb{P}((E \otimes M)^*)$ be the closure of the graph of ϕ. By the above, $Y = X' \cap (X \times \mathbb{P}(E^*) \times \mathbb{P}(M^*))$ contains points of the form

$$x_0 = \lim_{t\to 0}\gamma(t)(x, \psi(x)) = \big(\lim_{t\to 0}\gamma(t)x, [s_0(x) : \cdots : s_c(x)], [m_0^*]\big)$$

The G-equivariant projection $Y \to \mathbb{P}(E^*)$ maps x_0 to $[s_0(x) : \cdots : s_c(x)]$, hence is dominant, because $f_i = s_i/s_0$ are algebraically independent on X. Thence the generic modality of any component of Y dominating $\mathbb{P}(E^*)$ is greater or equal to c. □

Corollary 6.2 ([Akh2]). *A homogeneous space $\mathcal{O}$ is spherical iff any G-variety X with an open orbit isomorphic to $\mathcal{O}$ has finitely many G-orbits.*

7 Horospherical varieties

There is a nice class of G-varieties, which is easily accessible for study from the viewpoint of the local structure, complexity, and rank.

Definition 7.1. A subgroup $S \subseteq G$ is *horospherical* if S contains a maximal unipotent subgroup of G. A G-variety X is called *horospherical* if the stabilizer of any point on X is horospherical. In other words, $X = GX^U$.

Remark 7.1. In the definition, it suffices to require that the stabilizer of a general point is horospherical. Indeed, this implies that GX^U is dense in X. On the other hand, X^U is B-stable, whence GX^U is closed by Proposition 2.2.

Example 7.1. Consider a Lobachevsky space L^n in the hyperbolic realization, i.e., $L^n \subseteq \mathbb{R}^{n+1}$ is an upper pole of a hyperboloid $\{x \in \mathbb{R}^{n+1} \mid (x, x) = 1\}$ in an $(n + 1)$-dimensional pseudo-Euclidean space of signature $(1, n)$. The group $(\operatorname{Isom} L^n)^0 = \mathrm{SO}^+_{1,n}$ acts transitively on the set of horospheres in L^n.

Fix a horosphere $H^{n-1} \subset L^n$ and let $\langle e_1 \rangle \in \partial L^n \subseteq \mathbb{RP}^n$ be its center lying on the absolute. The vector $e_1 \in \mathbb{R}^n$ is isotropic and its projective stabilizer P is a parabolic subgroup of $\mathrm{SO}^+_{1,n}$. Take a line $\ell \subset L^n$ orthogonal to H^{n-1}. It intersects ∂L^n in two points $\langle e_1 \rangle, \langle e_2 \rangle$. The group P contains a one-parameter subgroup A acting in $\langle e_1, e_2 \rangle$ by hyperbolic rotations and trivially on $\langle e_1, e_2 \rangle^\perp$. Then A acts on ℓ by translations and the complementary subgroup $S = P'$ is the stabilizer of H^{n-1}. In the matrix form,

$$S = \left\{ \left. \begin{array}{|c|c|c|} \hline 1 & 0 & u^\top A \\ \hline 0 & 1 & 0 \cdots 0 \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & u & \text{\Large A} \\ \hline \end{array} \;\right|\; A \in \mathrm{SO}_{n-1},\ u \in \mathbb{R}^{n-1} \right\}$$

Recall that H^{n-1} carries a Euclidean geometry, and $S = (\operatorname{Isom} H^{n-1})^0$, where $\mathrm{R_u}(S)$ acts by translations and a Levi subgroup of S by rotations fixing an origin. Clearly, $S(\mathbb{C})$ is a horospherical subgroup of $\mathrm{SO}_{1,n}(\mathbb{C})$, which explains the terminology.

For any parabolic $P \subseteq G$, let $P = L \rightthreetimes P_{\mathrm{u}}$ be its Levi decomposition, and L_0 be any intermediate subgroup between L and L'. Then a subgroup $S = L_0 \rightthreetimes P_{\mathrm{u}}$ is horospherical. Conversely,

Lemma 7.1. *Let $S \subseteq G$ be a horospherical subgroup. Then $P = N_G(S)$ is parabolic, and for a Levi decomposition $P = L \rightthreetimes P_{\mathrm{u}}$, $S = L_0 \rightthreetimes P_{\mathrm{u}}$, where $L' \subseteq L_0 \subseteq L$.*

Proof. Embed S regularly in a parabolic $P \subseteq G$. Since S is horospherical, $S_{\mathrm{u}} = P_{\mathrm{u}}$, and S/P_{u} contains a maximal unipotent subgroup of $P/P_{\mathrm{u}} \simeq L$, whence $S/P_{\mathrm{u}} \simeq L_0 \supseteq L'$. Now it is clear that $S = L_0 \rightthreetimes P_{\mathrm{u}}$ and $P = N_G(S)$, because P normalizes S and $N_G(S)$ normalizes S_{u}. $\square$

In the sequel, assume $\operatorname{char} \Bbbk = 0$ for simplicity.

Horospherical varieties can be characterized in terms of the properties of multiplication in the algebra of regular functions. For any G-module M and any $\lambda \in \mathfrak{X}_+$, let $M_{(\lambda)}$ denote the isotypic component of type λ in M.

Proposition 7.1 ([Po4, §4]). *A quasiaffine G-variety X is horospherical iff $\Bbbk[X]_{(\lambda)} \cdot \Bbbk[X]_{(\mu)} \subseteq \Bbbk[X]_{(\lambda+\mu)}$ for $\forall \lambda, \mu \in \Lambda_+(X)$.*

The local structure of a horospherical action is simple.

Proposition 7.2. *A horospherical G-variety X contains an open G-stable subset $\mathring{X} \simeq G/S \times C$, where $S \subseteq G$ is horospherical and $G : C$ is trivial.*

Proof. By a theorem of Richardson [PV, Th.7.1], Levi subgroups of stabilizers of general points on X are conjugate and unipotent radicals of stabilizers form a continuous family of subgroups in G. Now it is clear from Lemma 7.1, that a horospherical subgroup may not be deformed outside its conjugacy class, whence stabilizers of general points are all conjugate to a certain $S \subseteq G$. Replacing X by an open G-stable subset yields $X \simeq G *_P X^S$, where $P = N_G(S)$ [PV, 2.8]. But P acts on X^S via a torus P/S, hence X^S is locally P-isomorphic to $P/S \times C$, where P acts on C trivially [PV, 2.6]. $\square$

To any G-variety X, one can relate a certain horospherical subgroup of G. Recall that by Corollary 4.1, X contains an open affine subset $\mathring{X} \simeq P_{\mathrm{u}} \times A \times C$, where P_{u} is the unipotent radical of a parabolic subgroup $P = P(X)$ and $A = L/L_0$ is a quotient torus of a Levi subgroup $L \subseteq P$. Then $S(X) = L_0 \leftthreetimes P_{\mathrm{u}}$ is the normalizer of a generic U-orbit in X.

Definition 7.2. The *horospherical type* of X is the opposite horospherical subgroup $S = S(X)^- = L_0 \leftthreetimes P_{\mathrm{u}}^-$, where P_{u}^- is the unipotent radical of the opposite parabolic subgroup P^- intersecting P in L.

Example 7.2. The horospherical type of a horospherical homogeneous space G/S is S, because G contains an open "big cell" $P_{\mathrm{u}} \times L \times P_{\mathrm{u}}^-$, where $P^- = N_G(S) = L \leftthreetimes P_{\mathrm{u}}^-$. For general horospherical varieties, the horospherical type is the (conjugation class of) the stabilizer of general position (Proposition 7.2).

Complexity, rank and weight lattice can be read off the horospherical type. Namely, it follows from Corollary 4.1 that $c(X) = \dim X - \dim G + \dim S$, $\Lambda(X) = \mathfrak{X}(A)$, $r(X) = \dim A$, where $A = P^-/S$.

Every G-action can be deformed to a horospherical one of the same type. This construction, called the *horospherical contraction*, was suggested by Popov [Po4]. We review the horospherical contraction in characteristic zero referring to [Gro2, §15] for arbitrary characteristic.

First consider an affine G-variety X. Choose a one-parameter subgroup $\gamma \in \mathfrak{X}^*(T)$ such that $\langle \gamma, \lambda \rangle \geq 0$ for any dominant weight and any positive root λ. Then $\Bbbk[X]^{(n)} = \bigoplus_{\langle \gamma, \lambda \rangle \leq n} \Bbbk[X]_{(\lambda)}$ is a G-stable filtration of $\Bbbk[X]$. The algebra $\operatorname{gr} \Bbbk[X]$ is finitely generated and has no nilpotents. It is easy to see using Proposition 7.1 that $X_0 = \operatorname{Spec} \operatorname{gr} \Bbbk[X]$ is a horospherical variety of the same type as X. Moreover, $\Bbbk[X_0]^U \simeq \Bbbk[X]^U$ and $\Bbbk[X_0] \simeq \Bbbk[X]$ as G-modules. (Note that $S(X)$ may be described as the common stabilizer of all $f \in \Bbbk[X]^{(B)}$.)

Furthermore, X_0 may be described as the zero-fiber of a flat family over $\mathbb{A}^1$ with a generic fiber X. Namely, let $E = \operatorname{Spec} \bigoplus_{n=0}^{\infty} \Bbbk[X]^{(n)} t^n \subseteq \Bbbk[X][t]$. The natural morphism $\delta : E \to \mathbb{A}^1$ is flat and $G \times \Bbbk^\times$-equivariant, where G acts on $\mathbb{A}^1$ trivially and $\Bbbk^\times$ acts by homotheties. Now $\delta^{-1}(t) \simeq X$ for $\forall t \neq 0$, and $\delta^{-1}(0) \simeq X_0$.

If X is an arbitrary G-variety, then we may replace it by a birationally G-isomorphic projective variety, build an affine cone $\widehat{X}$ over X, and perform the above construction for $\widehat{X}$. Passing again to a projectivization and taking a sufficiently small open G-stable subset, we obtain

Proposition 7.3. *There exists a smooth $G \times \Bbbk^\times$-variety E and a smooth $G \times \Bbbk^\times$-morphism $\delta : E \to \mathbb{A}^1$ such that $X_t = \delta^{-1}(t)$ is G-isomorphic to an open smooth G-stable subset of X for $\forall t \neq 0$, and X_0 is a smooth horospherical variety of the same type as X.*

8 Geometry of cotangent bundle

To a smooth G-variety X, we relate a Poisson G-action on the cotangent bundle T^*X equipped with a natural symplectic structure. Remarkably, the invariants of $G : X$ introduced in §5 are closely related to the symplectic geometry of T^*X and to the respective moment map. In particular, one obtains effective formulae for complexity and rank involving symplectic invariants of $G : T^*X$. This theory was developed by Knop in [Kn1]. To the end of this chapter, we assume $\operatorname{char} \Bbbk = 0$.

Let X be a smooth variety. A standard symplectic structure on T^*X [Arn, §37] is given by a 2-form $\omega = d\mathbf{p} \wedge d\mathbf{q} = \sum dp_i \wedge dq_i$, where $\mathbf{q} = (q_1, \dots, q_n)$ is a tuple of local coordinates on X, and $\mathbf{p} = (p_1, \dots, p_n)$ is an impulse, i.e., tuple of dual coordinates in a cotangent space. In a coordinate-free form, $\omega = d\ell$, where a 1-form ℓ on T^*X is given by $\langle \ell(\alpha), \xi \rangle = \langle \alpha, d\pi(\xi) \rangle$, $\forall \xi \in T_\alpha T^*X$, and $\pi : T^*X \to X$ is the canonical projection.

This symplectic structure defines the Poisson bracket of functions on T^*X. Another way to define this Poisson structure is to consider the sheaf $\mathcal{D}_X$ of differential operators on X. There is an increasing filtration $\mathcal{D}_X = \bigcup \mathcal{D}_X^m$ by the order of a differential operator and the isomorphism $\operatorname{gr} \mathcal{D}_X \simeq \mathrm{S}^\bullet \mathcal{T}_X = \pi_* \mathcal{O}_{T^*X}$ given by the symbol map. Since $\operatorname{gr} \mathcal{D}_X$ is commutative, the commutator in $\mathcal{D}_X$ induces the Poisson bracket on $\mathcal{O}_{T^*X}$ by the rule

$$\{\partial_1 \bmod \mathcal{D}_X^{m-1}, \partial_2 \bmod \mathcal{D}_X^{n-1}\} = [\partial_1, \partial_2] \bmod \mathcal{D}_X^{m+n-2}, \qquad \forall \partial_1 \in \mathcal{D}_X^m,\ \partial_2 \in \mathcal{D}_X^n$$

If X is a G-variety, then the symplectic structure on T^*X is G-invariant, and for $\forall \xi \in \mathfrak{g}$, the velocity field $\xi*$ on T^*X has a Hamiltonian $H_\xi = \xi*$, the respective velocity field on X considered as a linear function on T^*X [Arn, App.5]. Furthermore, the action $G : T^*X$ is Poisson, i.e., the map $\xi \mapsto H_\xi$

is a homomorphism of $\mathfrak{g}$ to the Poisson algebra of functions on T^*X [Arn, App.5]. The comorphism $\Phi : T^*X \to \mathfrak{g}^*$,

$$\langle \Phi(\alpha), \xi \rangle = H_\xi(\alpha) = \langle \alpha, \xi x \rangle, \qquad \forall \alpha \in T^*_x X,\ \xi \in \mathfrak{g}^*,$$

is called the *moment map*. By $M_X \subseteq \mathfrak{g}^*$ we denote the closure of its image. Also set $L_X = M_X /\!\!/ G$.

The moment map is G-equivariant [Arn, App.5]. Clearly $\langle d_\alpha \Phi(\nu), \xi \rangle = \omega(\nu, \xi\alpha)$ for $\forall \nu \in T_\alpha T^*X,\ \xi \in \mathfrak{g}$. It follows that $\operatorname{Ker} d_\alpha \Phi = (\mathfrak{g}\alpha)^\angle$, $\operatorname{Im} d_\alpha \Phi = (\mathfrak{g}_\alpha)^\perp$, where $^\angle$ and $^\perp$ denote the skew-orthocomplement and the annihilator in $\mathfrak{g}^*$, respectively.

Example 8.1. If $X = G/H$, then $T^*X = G *_H \mathfrak{h}^\perp$ and $\Phi(g * \alpha) = g\alpha$, $\forall g \in G,\ \alpha \in \mathfrak{h}^\perp$. Thus $M_{G/H} = \overline{G\mathfrak{h}^\perp}$. In the general case, for $\forall x \in X$, the moment map restricted to $T^*X|_{Gx}$ factors as $\Phi : T^*X|_{Gx} \twoheadrightarrow T^*Gx \to M_{Gx} \subseteq M_X$. We shall see below that for general x, $M_{Gx} = M_X$.

The cohomomorphism Φ^* exists in two versions—a commutative and a non-commutative one. Let $\mathrm{U}\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{D}(X)$ be the algebra of differential operators on X. The action $G : X$ induces a homomorphism $\Phi^* : \mathrm{U}\mathfrak{g} \to \mathcal{D}(X)$ mapping each $\xi \in \mathfrak{g}$ to a 1-order differential operator $\xi *$ on X. The map Φ^* is a homomorphism of filtered algebras, and the associated graded map

$$\operatorname{gr} \Phi^* : \operatorname{gr} \mathrm{U}\mathfrak{g} \simeq \mathrm{S}^\bullet \mathfrak{g} = \Bbbk[\mathfrak{g}^*] \longrightarrow \operatorname{gr} \mathcal{D}(X) \subseteq \Bbbk[T^*X], \qquad \xi \mapsto H_\xi$$

is the commutative version of the cohomomorphism. The isomorphism $\operatorname{gr} \mathrm{U}\mathfrak{g} \simeq \mathrm{S}^\bullet \mathfrak{g}$ is provided by the Poincaré–Birkhoff–Witt theorem, and the embedding $\operatorname{gr} \mathcal{D}(X) \subseteq \Bbbk[T^*X]$ is the symbol map.

On the sheaf level, we have the homomorphisms $\Phi^* : \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X$, $\mathcal{O}_X \otimes \mathrm{U}\mathfrak{g} \to \mathcal{D}_X$. Let $\mathcal{G}_X = \Phi^*(\mathcal{O}_X \otimes \mathfrak{g})$ denote the *action sheaf* (generated by velocity fields) and $\mathcal{U}_X = \Phi^*(\mathcal{O}_X \otimes \mathrm{U}\mathfrak{g})$. Clearly,

$$T^\mathfrak{g} X := \operatorname{Spec}_{\mathcal{O}_X} \mathrm{S}^\bullet \mathcal{G}_X = \overline{\operatorname{Im}(\pi \times \Phi)} \subseteq X \times \mathfrak{g}^*$$

The moment map factors as

$$\Phi : T^*X \longrightarrow T^\mathfrak{g} X \xrightarrow{\overline{\Phi}} \mathfrak{g}^*$$

The (non-empty) fibers of $\pi \times \Phi$ are affine translates of the conormal spaces to G-orbits. Generic fibers of $T^\mathfrak{g} X \to X$ are the cotangent spaces $\mathfrak{g}_x^\perp = T^*_x Gx$ to generic orbits. The morphism $\overline{\Phi}$ is called the *localized moment map* [Kn6, §2].

Example 8.2. If X is a smooth completion of a homogeneous space $\mathcal{O} = G/H$, then $T^{\mathfrak{g}}X \supset T^*\mathcal{O}$ and $\overline{\Phi}$ is a proper map extending $\Phi : T^*\mathcal{O} \to \mathfrak{g}^*$. Thus one compactifies the moment map of a homogeneous cotangent bundle.

Definition 8.1. A smooth G-variety X is called *pseudo-free* if $\mathcal{G}_X$ is locally free or $T^{\mathfrak{g}}X$ is a vector bundle over X. In other words, the rational map $X \dashrightarrow \mathrm{Gr}_k(\mathfrak{g}) \simeq \mathrm{Gr}_{n-k}(\mathfrak{g}^*)$, $x \mapsto [\mathfrak{g}_x] \mapsto [\mathfrak{g}_x^{\perp}]$ ($n = \dim \mathfrak{g}$, $k = \dim G_x$ for $x \in X$ in general position) extends to X, i.e., generic isotropy subalgebras degenerate at the boundary to specific limits.

Example 8.3. For trivial reasons, X is pseudo-free if the action $G : X$ is generically free. Also, X is pseudo-free if all G-orbits in X have the same dimension: in this case $T^{\mathfrak{g}}X = \bigsqcup_{Gx \subseteq X} T^*Gx$ [Kn6, 2.3].

It is instructive to note that every G-variety X has a pseudo-free resolution of singularities $\check{X} \to X$: just consider the closure of the graph of $X \dashrightarrow \mathrm{Gr}_k(\mathfrak{g})$ and take for $\check{X}$ its equivariant desingularization.

It is easy to see that $\operatorname{gr} \Phi^*$ behaves well on the sheaf level on pseudo-free varieties.

Proposition 8.1 ([Kn6, 2.6]). *If X is pseudo-free, then the filtrations on $\mathcal{U}_X$ induced from $\mathrm{U}\mathfrak{g}$ and $\mathcal{D}_X$ coincide and $\operatorname{gr} \Phi^* : \mathcal{O}_X \otimes \mathrm{S}^{\bullet}\mathfrak{g} \to \mathrm{S}^{\bullet}\mathcal{T}_X$ is surjective onto $\operatorname{gr} \mathcal{U}_X \simeq \mathrm{S}^{\bullet}\mathcal{G}_X$.*

Sometimes it is useful to replace the usual cotangent bundle by its logarithmic version [Dan, §15], [Oda, 3.1]. Let $D \subset X$ be a divisor with normal crossings (e.g., smooth). The sheaf $\Omega^1_X(\log D)$ of differential 1-forms with at most logarithmic poles along D is locally generated by df/f with f invertible outside D. It contains Ω^1_X and is locally free. The respective vector bundle $T^*X(\log D)$ is said to be the *logarithmic cotangent bundle*. The dual vector bundle $TX(-\log D)$, called the *logarithmic tangent bundle*, corresponds to the subsheaf $\mathcal{T}_X(-\log D) \subset \mathcal{T}_X$ of vector fields preserving the ideal sheaf $\mathcal{I}_D \lhd \mathcal{O}_X$.

If D is G-stable, then the velocity fields of G on X are tangent to D, i.e., $\mathcal{G}_X \subseteq \mathcal{T}_X(-\log D)$. By duality, we obtain the logarithmic moment map $\Phi : T^*X(\log D) \to T^{\mathfrak{g}}X \to \mathfrak{g}^*$ extending the usual one on $T^*(X \setminus D)$.

Remark 8.1. We assume that X is smooth in order to use the notions of symplectic geometry. However, the moment map may be defined for any G-variety X. As the definition is local and M_X is a G-birational invariant of X, we may always pass to a smooth open subset of X and conversely, to a (maybe singular) G-embedding of a smooth G-variety.

We are going to describe the structure of M_X. We do it first for horospherical varieties. Then we contract any G-variety to a horospherical one and show that this contraction does not change M_X.

Let X be a horospherical variety of type S. It is clear from Proposition 7.2 that $M_X = M_{G/S}$. The moment map factors as

$$\Phi_{G/S} : G *_S \mathfrak{s}^\perp \xrightarrow{\pi_A} G *_{P^-} \mathfrak{s}^\perp \xrightarrow{\overline{\Phi}} \mathfrak{g}^*,$$

where $P^- = N_G(S)$, and π_A is the quotient map modulo $A = P^-/S$. By Proposition 2.2, the map $\overline{\Phi}$ is proper, whence $M_{G/S} = G\mathfrak{s}^\perp$. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via a non-degenerate G-invariant inner product on $\mathfrak{g}$, then $\mathfrak{s}^\perp$ is identified with $\mathfrak{a} \oplus \mathfrak{p}_\mathrm{u}^-$ and is retracted onto $\mathfrak{a}$ by a certain one-parameter subgroup of $Z(L)$. It follows that $\mathfrak{a} \simeq \mathfrak{a}^*$ intersects all closed G-orbits in $M_{G/S}$, and $L_{G/S} = \pi_G(\mathfrak{a}^*)$, where $\pi_G : \mathfrak{g}^* \to \mathfrak{g}^* /\!\!/ G$ is the quotient map.

Finally, generic fibers of $\overline{\Phi}$ are finite. Indeed, it suffices to find at least one finite fiber. But $\overline{\Phi}^{-1}(G\mathfrak{p}_\mathrm{u}^-) = G *_{P^-} \mathfrak{p}_\mathrm{u}^-$ maps onto $G\mathfrak{p}_\mathrm{u}^-$ with finite generic fibers by a theorem of Richardson [McG, 5.1].

We sum up in

Theorem 8.1. *For any horospherical G-variety X of type S, the natural map $G *_{P^-} \mathfrak{s}^\perp \to M_X = G\mathfrak{s}^\perp$ is generically finite, proper and surjective, and $L_X = \pi_G(\mathfrak{a}^*)$.*

We have already seen that horospherical varieties, their cotangent bundles and moment maps are easily accessible for study. A deep result of Knop says that the closure of the image of the moment map depends only on the horospherical type.

Theorem 8.2. *Assume that X is a G-variety of horospherical type S. Then $M_X = M_{G/S}$.*

In the physical language, the idea of the proof is to apply quantum technique to classical theory. We study the homomorphism $\Phi_X^* : \mathrm{U}\mathfrak{g} \to \mathcal{D}(X)$ and show that its kernel $\mathcal{I}_X \lhd \mathrm{U}\mathfrak{g}$ depends only on the horospherical type. Then we deduce that $I_X = \operatorname{Ker} \operatorname{gr} \Phi_X^* = \mathbb{I}(M_X) \lhd \Bbbk[\mathfrak{g}^*]$ depends only on the type of X, which is the desired assertion. We retain the notation of Proposition 7.3.

Lemma 8.1. $\mathcal{I}_X = \mathcal{I}_{X_0} = \mathcal{I}_{G/S}$.

In the affine case, lemma stems from a G-module isomorphism $\Bbbk[X] \simeq \Bbbk[X_0]$. Indeed, Φ_X^* depends only on the G-module structure of $\Bbbk[X]$. The general case is deduced from the affine one [Kn1, 5.1] with the help of affine cones and of the next lemma.

Put $\mathcal{M}_X = \operatorname{Im}\Phi_X^* \subseteq \mathcal{D}(X)$. By the previous lemma, $\mathcal{M}_X \simeq \mathcal{M}_{X_0} \simeq \mathcal{M}_{G/S}$.

Lemma 8.2. *$\mathcal{M}_{G/S}$ is a finite $\mathrm{U}\mathfrak{g}$-module, and $\operatorname{gr}\mathcal{M}_{G/S}$ is a finite $\Bbbk[\mathfrak{g}^*]$-module.*

Proof. The first assertion follows from the second one. To prove the second assertion, observe that $A = P^-/S$ acts on G/S by G-automorphisms. Therefore $\mathcal{M}_{G/S} \subseteq \mathcal{D}(G/S)^A$ and $\operatorname{gr}\mathcal{M}_{G/S} \subseteq \Bbbk[T^*G/S]^A = \Bbbk[G *_{P^-} \mathfrak{s}^\perp]$, the latter being a finite $\Bbbk[\mathfrak{g}^*]$-module by Theorem 8.1. □

The restriction maps $\mathcal{M}_E \xrightarrow{\sim} \mathcal{M}_{X_t}$ agree with Φ^* and do not rise the order of a differential operator. We identify $\mathcal{M}_E$ and $\mathcal{M}_{X_t}$ via this isomorphism and denote by $\operatorname{ord}_E \partial$ ($\operatorname{ord}_{X_t}\partial$) the order of a differential operator ∂ on E (resp. on X_t).

Theorem 8.2 follows from

Lemma 8.3. *On $\mathcal{M}_X \simeq \mathcal{M}_E \simeq \mathcal{M}_{X_0}$, $\operatorname{ord}_X \partial = \operatorname{ord}_E \partial = \operatorname{ord}_{X_0}\partial$ for $\forall\partial$.*

Proof. The first equality is clear, because an open subset of E is G-isomorphic to $X \times \Bbbk^\times$. It follows from Lemma 8.2 that the orders of a given differential operator on E and on X_0 do not differ very much. Indeed, $\mathcal{M}_{X_0} = \mathcal{M}_{G/S} = \sum(\mathrm{U}\mathfrak{g})\partial_i$ and $\operatorname{gr}\mathcal{M}_{X_0} = \operatorname{gr}\mathcal{M}_{G/S} = \sum \Bbbk[\mathfrak{g}^*]\partial_i$ for some $\partial_1, \dots, \partial_s \in \mathcal{M}_{X_0}$. Put $d_i = \operatorname{ord}_{X_0}\partial_i$ and $d = \max_i \operatorname{ord}_E \partial_i$. If $\operatorname{ord}_{X_0}\partial = n$, then $\partial = \sum u_i\partial_i$ for some $u_i \in U^{n-d_i}\mathfrak{g}$, hence $\operatorname{ord}_E \partial \le n + d$.

However, if $\operatorname{ord}_{X_0}\partial < \operatorname{ord}_E \partial$, then $\operatorname{ord}_{X_0}\partial^{d+1} < \operatorname{ord}_E \partial^{d+1} - d$, a contradiction. □

The explicit description of M_X in terms of the horospherical type allows to examine invariant-theoretic properties of the action $G : M_X$.

In the above notation, put $M = Z_G(\mathfrak{a}) \supseteq L$. Every G-orbit in M_X is of the form Gx, $x \in \mathfrak{s}^\perp = \mathfrak{a} \oplus \mathfrak{p}_{\mathrm{u}}^-$. Consider the Jordan decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$. The (unique) closed orbit in Gx is Gx_{s}. Moving x by P_{u}^-, we may assume $x_{\mathrm{s}} \in \mathfrak{a}$. If x is a general point, then $G_{x_{\mathrm{s}}} = M$, $\mathfrak{z}(x_{\mathrm{s}}) = \mathfrak{m}$, thence $x_{\mathrm{n}} \in \mathfrak{m} \cap \mathfrak{p}_{\mathrm{u}}^-$.

The concept of a general point can be specified as follows: consider the principal open stratum $\mathfrak{a}^{\mathrm{pr}} \subseteq \mathfrak{a}$ obtained by removing all proper intersections with kernels of roots and with W-translates of $\mathfrak{a}$. Then $G_\xi = M$ for $\forall\xi \in \mathfrak{a}^{\mathrm{pr}}$, and G-orbits intersect $\mathfrak{a}^{\mathrm{pr}}$ in orbits of a finite group $W(\mathfrak{a}) = N_G(\mathfrak{a})/M$ acting freely on $\mathfrak{a}^{\mathrm{pr}}$. Furthermore,

$$M_X^{\mathrm{pr}} := \pi_G^{-1}\pi_G(\mathfrak{a}^{\mathrm{pr}}) \simeq G *_{N(\mathfrak{a})} (\mathfrak{a}^{\mathrm{pr}} + \mathfrak{M}), \qquad \text{where } \mathfrak{M} = N(\mathfrak{a})(\mathfrak{m} \cap \mathfrak{p}_{\mathrm{u}}^-)$$

is a nilpotent cone in $\mathfrak{m}$.

Definition 8.2. The Poisson action $G : T^*X$ is said to be *symplectically stable* if the action $G : M_X$ is stable, i.e., generic G-orbits in M_X are closed. By the above discussion, symplectic stability is equivalent to $M = L$.

This class of actions is wide enough.

Proposition 8.2. *If X is quasiaffine, then T^*X is symplectically stable.*

Proof. The horospherical contraction X_0 and a typical orbit G/S therein are quasiaffine, too. If $M \supset L$, then there is a root α w.r.t a maximal torus $T \subseteq L$ such that $\alpha|_{\mathfrak{a}} = 0$ and $\mathfrak{g}_\alpha \subseteq \mathfrak{p}_{\mathrm{u}}^-$. The respective coroot $\alpha^\vee$ lies in $\mathfrak{l}_0$. Let $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \langle\alpha^\vee\rangle \oplus \mathfrak{g}_{-\alpha}$ be the corresponding $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$. Then $\mathfrak{s}_\alpha \cap \mathfrak{s} = \langle\alpha^\vee\rangle \oplus \mathfrak{g}_{-\alpha}$ is a Borel subalgebra in $\mathfrak{s}_\alpha$, and an orbit in G/S of the respective subgroup $S_\alpha \subseteq G$ is isomorphic to $\mathbb{P}^1$, a contradiction with quasiaffinity. □

Remark 8.2. A symplectically stable action is stable, i.e., generic orbits of $G : T^*X$ are closed. Indeed, Φ is smooth along $\Phi^{-1}(x)$ for general $x \in M_X$, whence $(\mathfrak{g}\alpha)^\angle = \operatorname{Ker} d_\alpha\Phi = T_\alpha\Phi^{-1}(x)$ have one and the same dimension $\dim T^*X - \dim M_X$ for $\forall\alpha \in \Phi^{-1}(x)$. It follows that all orbits over Gx are closed in $\Phi^{-1}(Gx)$.

The Poisson G-action on T^*X provides two important invariants:

Definition 8.3. The *defect* $\operatorname{def} T^*X$ is the defect of the symplectic form restricted to a generic G-orbit.

The *corank* $\operatorname{cork} T^*X$ is the rank of the symplectic form on the skew-orthogonal complement to the tangent space of a generic G-orbit.

In other words,

$$\operatorname{def} T^*X = \dim(\mathfrak{g}\alpha)^\angle \cap \mathfrak{g}\alpha$$
$$\operatorname{cork} T^*X = \dim(\mathfrak{g}\alpha)^\angle/(\mathfrak{g}\alpha)^\angle \cap \mathfrak{g}\alpha$$

for general $\alpha \in T^*X$.

The cohomomorphism $\operatorname{gr}\Phi^*$ maps $\Bbbk[\mathfrak{g}^*]^G$ onto a Poisson-commutative subalgebra $\mathcal{A}_X \subseteq \Bbbk[T^*X]^G$ isomorphic to $\Bbbk[L_X]$. Skew gradients of functions in $\mathcal{A}_X$ commute, are G-stable, and both skew-orthogonal and tangent to G-orbits. Indeed, for $\forall f \in \mathcal{A}_X$, $\alpha \in T^*X$, df is zero on $\mathfrak{g}\alpha$ (since f is G-invariant) and on $(\mathfrak{g}\alpha)^\angle = \operatorname{Ker} d_\alpha\Phi$ (because f is pulled back under Φ).

Those skew gradients generate a flow of G-automorphisms preserving G-orbits on T^*X, which is called a *G-invariant collective motion*. The restriction of this flow to $G\alpha$ is a connected Abelian subgroup (in fact, a torus) $A_\alpha \subseteq \operatorname{Aut}_G(G\alpha)$ with the Lie algebra $\mathfrak{a}_\alpha \subseteq \mathfrak{n}(\mathfrak{g}_\alpha)/\mathfrak{g}_\alpha$.

For general α, $\Phi^{-1}\Phi(G\alpha)$ are level varieties for $\mathcal{A}_X$, because G-invariant regular functions separate generic G-orbits in $M_X \subseteq \mathfrak{g}^*$. It follows that

$$\operatorname{Ker} d_\alpha\mathcal{A}_X = T_\alpha\Phi^{-1}\Phi(G\alpha) = (\mathfrak{g}\alpha) + (\mathfrak{g}\alpha)^\angle, \qquad \text{and}$$
$$\mathfrak{a}_\alpha = (\mathfrak{g}\alpha)^\angle \cap (\mathfrak{g}\alpha) = T_\alpha(G\alpha \cap \Phi^{-1}\Phi(\alpha)) \simeq \mathfrak{g}_{\Phi(\alpha)}/\mathfrak{g}_\alpha$$

In particular, $\mathfrak{g}_{\Phi(\alpha)} \supset \mathfrak{g}_\alpha \supset \mathfrak{g}'_{\Phi(\alpha)}$.

The defect of $G : T^*X$ is the dimension of the invariant collective motion: $\operatorname{def} T^*X = \dim \mathfrak{a}_\alpha$ for general $\alpha \in T^*X$.

The next theorem links the geometry of X and the symplectic geometry of T^*X.

Theorem 8.3. *Put* $n = \dim X$, $c = c(X)$, $r = r(X)$. *Then* $\dim M_X = 2n - 2c - r$, $\operatorname{def} T^*X = d_G(M_X) = r$, $d_G(T^*X) = 2c + r$, $\operatorname{cork} T^*X = 2c$.

Proof. In the notation of Definition 7.2, we have a decomposition $\mathfrak{g} = \mathfrak{p}_\mathrm{u} \oplus \mathfrak{l}_0 \oplus \mathfrak{a} \oplus \mathfrak{p}_\mathrm{u}^-$, where $\mathfrak{s} = \mathfrak{l}_0 \oplus \mathfrak{p}_\mathrm{u}^-$ is (the Lie algebra of) the horospherical type of X, and $\mathfrak{s}^\perp$ is identified with $\mathfrak{a} \oplus \mathfrak{p}_\mathrm{u}^-$ via $\mathfrak{g} \simeq \mathfrak{g}^*$. By Corollary 4.1, $\dim \mathfrak{p}_\mathrm{u} = \dim \mathfrak{p}_\mathrm{u}^- = n - c - r$, whence by Theorems 8.1–8.2, $\dim M_X = \dim G/P^- + \dim \mathfrak{s}^\perp = \dim \mathfrak{p}_\mathrm{u} + \dim \mathfrak{p}_\mathrm{u}^- + \dim \mathfrak{a} = 2n - 2c - r$, and $d_G(M_X) = \dim L_X = \dim \mathfrak{a} = r$. For general $\alpha \in T^*X$, we have $d_G(T^*X) = \operatorname{codim} G\alpha = \dim(\mathfrak{g}\alpha)^\angle = \dim \Phi^{-1}(\alpha) = 2n - \dim M_X = 2c + r$, and $\operatorname{def} T^*X = \dim G\alpha \cap \Phi^{-1}\Phi(\alpha) = \dim G_{\Phi(\alpha)}/G_\alpha = \dim G\alpha - \dim G\Phi(\alpha) = (2n - d_G(T^*X)) - (\dim M_X - d_G(M_X)) = d_G(M_X) = r$. Finally, $\operatorname{cork} T^*X = d_G(T^*X) - \operatorname{def} T^*X = 2c$. ∎

Another application of the horospherical contraction and of the moment map is the existence of the stabilizer of general position for the G-action in a cotangent bundle.

Theorem 8.4 ([Kn1, §8]). *Stabilizers in* G *of general points in* T^*X *are conjugate to a stabilizer of the open orbit of* $M \cap S$ *in* $\mathfrak{m} \cap \mathfrak{p}_\mathrm{u}^-$, *in the above notation. In the symplectically stable (e.g., quasiaffine) case, generic stabilizers of* $G : T^*X$ *are conjugate to* L_0.

Proof. We prove the first assertion for horospherical X. The general case is derived form the horospherical one with the aid of the horospherical contraction using Theorem 8.2 and the invariant collective motion, see [Kn1, 8.1], or Remark 23.3 for the symplectically stable case.

We may assume $X = G/S$. A generic stabilizer of $G : T^*G/S$ equals a generic stabilizer of $S : \mathfrak{s}^\perp = \mathfrak{a} \oplus \mathfrak{p}_\mathrm{u}^-$. Take a general point $x \in \mathfrak{s}^\perp$ and let $x = x_\mathrm{s} + x_\mathrm{n}$ be the Jordan decomposition. Moving x by P_u^-, we may assume that x_s is a general point in $\mathfrak{a}$. Then $S_{x_\mathrm{s}} = M \cap S$, $x_\mathrm{n} \in \mathfrak{m} \cap \mathfrak{p}_\mathrm{u}^-$, and

$S_x = (M \cap S)_{x_{\mathrm{n}}}$ is the stabilizer of a general point in $\mathfrak{m} \cap \mathfrak{p}_{\mathrm{u}}^-$. But $M \cap S$ has the same orbits in $\mathfrak{m} \cap \mathfrak{p}_{\mathrm{u}}^-$ as a parabolic subgroup $M \cap P^- \subseteq M$, because these two groups differ by a central torus in M. By [McG, Th.5.1], $M \cap P^-$ has an open orbit in the Lie algebra of its unipotent radical $\mathfrak{m} \cap \mathfrak{p}_{\mathrm{u}}^-$, which proves the first assertion of the theorem.

If X is symplectically stable, then $M = L$, whence the second assertion. $\square$

The last two theorems reduce the computation of complexity and rank to studying generic orbits and stabilizers of a *reductive* group. Namely, it suffices to know generic G-modalities of T^*X and M_X. We have a formula

$$(8.1) \qquad 2c(X) + r(X) = d_G(T^*X) = 2\dim X - \dim G + \dim G_*,$$

where G_* is the stabilizer of general position for $G : T^*X$. For quasiaffine X,

$$(8.2) \qquad r(X) = \operatorname{rk} G - \operatorname{rk} G_*$$

Furthermore, $\Lambda(X)$ is the group of characters of T vanishing on $T \cap G_*$, where T is a maximal torus normalizing G_*. For homogeneous spaces, everything is reduced even to *representations* of reductive groups, see §9.

Now we explain another approach to computing complexity and rank based on the theory of doubled actions [Pan1], [Pan7, Ch.1]. This approach is parallel to Knop's one and coincides with the latter in the case of G-modules.

Let θ be a Weyl involution of G relative to T, i.e., an involution of G acting on T as an inversion. Then $\theta(P) = P^-$ for any parabolic $P \supseteq B$.

Example 8.4. If $G = \mathrm{GL}_n(\Bbbk)$, or $\mathrm{SL}_n(\Bbbk)$, and T is the standard diagonal torus, then we may put $\theta(g) = w_G(g^\top)^{-1}w_G^{-1}$, where w_G permutes the standard basis of $\Bbbk^n$ in the reverse order: $w_G e_i = e_{n+1-i}$.

Definition 8.4. The *dual* G-variety X^* is a copy of X equipped with a twisted G-action: $gx^* = (\theta(g)x)^*$, where $x \mapsto x^*$ is a fixed isomorphism $X \xrightarrow{\sim} X^*$.

The diagonal action $G : X \times X^*$ is called the *doubled action* w.r.t. $G : X$.

Remark 8.3. If V is a G-module, then V^* is the dual G-module with a fixed linear G^θ-isomorphism $V \xrightarrow{\sim} V^*$. Similarly, $\mathbb{P}(V)^* \simeq \mathbb{P}(V^*)$. If $X \subseteq \mathbb{P}(V)$ is a quasiprojective G-variety, then $X^* \subseteq \mathbb{P}(V^*)$.

Remark 8.4. For a G-module V, the doubled G-variety $V \oplus V^*$ is nothing else, but the cotangent bundle T^*V.

The following theorems are parallel to Theorems 8.4 and 8.3.

Theorem 8.5. *Stabilizers in G of general points in $X \times X^*$ are conjugate to L_0.*

Theorem 8.6. *Let G_* be the stabilizer of general position for $G : X \times X^*$. Then*

$$2c(X) + r(X) = d_G(X \times X^*) = 2\dim X - \dim G + \dim G_* \tag{8.3}$$
$$r(X) = \operatorname{rk} G - \operatorname{rk} G_* \tag{8.4}$$

and $\Lambda(X)$ is the group of characters of a maximal torus T normalizing G_* which vanish on $T \cap G_*$.

Proofs. Consider an open embedding $P_{\mathrm{u}} \times Z \hookrightarrow X$ from Corollary 4.1. Then $P_{\mathrm{u}}^- \times Z^* \hookrightarrow X^*$, where Z^* is the dual L-variety to Z. One deduces that $G *_L (Z \times Z^*) \hookrightarrow X \times X^*$ is an open embedding, and the generic stabilizer of $L : Z \times Z^*$ equals L_0, whence the assertion of Theorem 8.5. Theorem 8.6 follows from Theorem 8.5 with the aid of Corollary 4.1. □

Example 8.5. If $X = G/P$ is a projective homogeneous space, then $X^* = G/P^-$ and the stabilizer of general position for $G : X \times X^*$ equals $P \cap P^- = L$.

We have a nice invariant-theoretic property of doubled actions on affine varieties.

Theorem 8.7 ([Pan6, 1.6], [Pan7, 1.3.13]). *If X is affine, then generic G-orbits on $X \times X^*$ are closed.*

For a G-module V, a stabilizer of general position for the doubled G-module (or the cotangent bundle) $V \oplus V^*$ can be found by an effective recursive algorithm relying on the Brion–Luna–Vust construction (see §4). In the notation of Lemma 4.1, we have an isomorphism

$$G *_L (\mathring{W} \times \mathring{W}^*) \xrightarrow{\sim} \mathring{V} \times \mathring{V}^* \subset V \oplus V^*$$

A stabilizer of general position for $G : V \oplus V^*$ equals that for $L : W \oplus W^*$. Replacing $G : V$ by $L : W$, we apply the Brion–Luna–Vust construction again, and so on. We obtain a descending sequence of Levi subgroups $L_i \subseteq G$ and L_i-submodules $W_i \subseteq V$. As the semisimple rank of L_i decreases, the sequence terminates, and on the final s-th step, L_s' acts on W_s trivially. Then G_* is just the kernel of $L_s : W_s$.

Example 8.6. Let $G = \mathrm{Spin}_{10}(\Bbbk)$ and $V = V(\omega_4)$ be one of its half-spinor representations. In the notation of [OV], the positive roots of G are $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, the simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\alpha_5 = \varepsilon_4 + \varepsilon_5$, where $1 \le i < j \le 5$. The weights of V are $\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_5)$, where the number of minuses is odd, and $\omega_4^* = \omega_5 = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_5)$.

Let P_1 be the projective stabilizer of a highest weight vector $v_1^* \in V^*$. It's Levi subgroup L_1 has $\alpha_1, \dots, \alpha_4$ as simple roots. The weights of $(\mathfrak{p}_1^-)_{\mathrm{u}} v_1^*$ are of the form $\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_5)$ with 2 minuses, and the weights of $W_1 = ((\mathfrak{p}_1^-)_{\mathrm{u}} v_1^*)^\perp$ have 1 or 5 minuses. Clearly, L_1 is of type $\mathbf{A}_4$, and W_1 is the direct sum of a 1-dimensional L_1-submodule of the weight $\frac{1}{2}(-\varepsilon_1 - \cdots - \varepsilon_5)$ and of a 5-dimensional L_1-submodule with highest weight $\frac{1}{2}(\varepsilon_1 + \dots \varepsilon_4 - \varepsilon_5)$.

Take a highest weight vector $v_2^* \in W_1^*$ of highest weight $\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_5)$, and let P_2 be its projective stabilizer. The second Levi subgroup $L_2 \subset P_2$ has $\alpha_2, \alpha_3, \alpha_4$ as simple roots, the weights of $(\mathfrak{p}_2^-)_{\mathrm{u}} v_2^*$ are $\frac{1}{2}(-\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_5)$ with exactly 1 plus, and $W_2 = ((\mathfrak{p}_2^-)_{\mathrm{u}} v_2^*)^\perp$ has the weights $\frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_5)$, $\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_5)$.

It is easy to see that L_2' is exactly the kernel of $L_2 : W_2$. Thus our algorithm terminates, and we obtain $G_* \simeq L_2' \simeq \mathrm{SL}_4$, $\Lambda(V) = \mathfrak{X}(T/T \cap L_2') = \langle \omega_1, \omega_5 \rangle$ (here $\omega_1 = \varepsilon_1$), $r(V) = 2$, and $c(V) = 0$. Moreover, $\Bbbk[V]_1 = V^* = V(\omega_5)$ and $\Bbbk[V]_2 = S^2 V(\omega_5) = V(2\omega_5) \oplus V(\omega_1)$, whence $\Lambda_+(V) = \langle \omega_1, \omega_5 \rangle$.

9 Complexity and rank of homogeneous spaces

We apply the methods developed in §8 to computing complexity and rank of homogeneous spaces.

The cotangent bundle of G/H is identified with $G *_H \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp \simeq (\mathfrak{g}/\mathfrak{h})^*$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. The representation $H : \mathfrak{h}^\perp$ is the *coisotropy representation* at $eH \in G/H$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via a non-degenerate G-invariant inner product on $\mathfrak{g}$, then $\mathfrak{h}^\perp$ is just the orthogonal complement of $\mathfrak{h}$.

If H is reductive, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as H-modules. In particular, the isotropy and coisotropy representations are isomorphic.

The following theorem is a reformulation of Theorems 8.3–8.4 for homogeneous spaces.

Theorem 9.1. *Generic stabilizers of the coisotropy representation are all conjugate to a certain subgroup $H_* \subseteq H$. For complexity and rank of G/H,* *we have the equations:*

$$\begin{aligned}
(9.1)\qquad 2c(G/H)+r(G/H) &= d_H(\mathfrak{h}^\perp) = \dim\mathfrak{h}^\perp - \dim H + \dim H_* \\
&= \dim G - 2\dim H + \dim H_* \\
(9.2)\qquad r(G/H) &= \dim G_\alpha - \dim H_\alpha \\
(9.3)\qquad 2c(G/H) &= \dim G\alpha - 2\dim H\alpha
\end{aligned}$$

where $\alpha \in \mathfrak{h}^\perp$ is a general point considered as an element of $\mathfrak{g}^*$. If H is observable (e.g., reductive), then $H_* = L_0$ is the Levi subgroup of the horospherical type of H,

$$(9.4)\qquad r(G/H) = \operatorname{rk} G - \operatorname{rk} H_*,$$

and $\Lambda(G/H)$ is the group of characters of T vanishing on $T \cap H_*$.

Proof. Generic modalities and stabilizers of the actions $G : T^*G/H$ and $H : \mathfrak{h}^\perp$ coincide. This implies all assertions except (9.2), (9.3). We have $\Phi(e*\alpha) = \alpha$ and $G_{e*\alpha} = H_\alpha$, whence the r.h.s. of (9.2) is the dimension of the invariant collective motion, which yields (9.2). Subtracting (9.2) from (9.1) yields (9.3). □

Formulae (9.1), (9.4) are most helpful, especially for reductive H, because stabilizers of general position for reductive group representations are known, e.g., from Elashvili's tables [Ela1], [Ela2].

Example 9.1. Let $G = \mathrm{Sp}_{2n}(\Bbbk)$, $n \geq 2$, and $H = \mathrm{Sp}_{2n-2}(\Bbbk)$ be the stabilizer of a general pair of vectors in a symplectic space, say $e_1, e_2 \in \Bbbk^{2n}$, where a (standard) symplectic form on $\Bbbk^{2n}$ is $\omega = \sum x_{2i-1} \wedge x_{2i}$. Then $\operatorname{Ad} G = \mathrm{S}^2\Bbbk^{2n}$, $\operatorname{Ad} H = \mathrm{S}^2\Bbbk^{2n-2}$ and $\operatorname{Ad} G|_H = \mathrm{S}^2\Bbbk^2 \oplus \Bbbk^2 \otimes \Bbbk^{2n-2} \oplus \mathrm{S}^2\Bbbk^{2n-2}$, where $\Bbbk^2 = \langle e_1, e_2\rangle$ and $\Bbbk^{2n-2} = \langle e_3, \dots, e_{2n}\rangle$. Hence the coisotropy representation of H equals $\Bbbk^{2n-2} \oplus \Bbbk^{2n-2} \oplus \Bbbk^3$, where $\Bbbk^{2n-2}$ is the tautological and $\Bbbk^3$ a trivial representation of H.

Clearly, $H_* = \mathrm{Sp}_{2n-4}(\Bbbk)$ is the stabilizer of $e_3, e_4 \in \Bbbk^{2n-2}$. It follows that $r(G/H) = 2$ and $2c(G/H) + r(G/H) = 2(2n-2) + 3 - (n-1)(2n-1) + (n-2)(2n-3) = 4$, whence $c(G/H) = 1$. Furthermore, the standard diagonal torus $T = \{t = \operatorname{diag}(t_1, t_1^{-1}, \dots, t_n, t_n^{-1})\} \subseteq G$ normalizes H_*, and $\Lambda(G/H) = \mathfrak{X}(T/T \cap H_*) = \langle \omega_1, \omega_2\rangle$, where $\omega_1(t) = t_1$, $\omega_2(t) = t_1t_2$ are the first two fundamental weights of G.

Theorem 9.1 reduces the computation of complexity and rank of affine homogeneous spaces to finding stabilizers of general position for reductive group representations. The next theorem does the same thing for arbitrary homogeneous spaces.

Consider a regular embedding $H \subseteq Q$ in a parabolic subgroup $Q \subseteq G$. Let $K \subseteq M$ be Levi subgroups of H and Q. Clearly, K acts on $Q_{\mathrm{u}}/H_{\mathrm{u}}$ by conjugations, and this action is isomorphic to a linear action $K : \mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}$ via the exponential map.

We may assume that $M \supseteq T$ and $B \subseteq Q^-$. Then $B(M) = B \cap M$ is a Borel subgroup in M. We may assume that $eK \in M/K$ is a general point, i.e., $M_* = K \cap \theta(K)$ is a stabilizer of general position for $M : M/K \times (M/K)^*$, and $B(M_*) = B(M) \cap K$ is a stabilizer of general position for $B(M) : M/K$. By Theorem 8.5, M_* and $B(M_*)$ are normalized by T, and $B(M_*)^0$ is a Borel subgroup in M_*^0. M_* may be non-connected, but it is a direct product of M_*^0 and a finite subgroup of T. The notions of complexity, rank and weight lattice generalize to M_*-actions immediately.

Theorem 9.2 ([Pan4, 1.2], [Pan7, 2.5.20]). *With the above choice of H, Q, and K among conjugates,*

$$\begin{aligned} c_G(G/H) &= c_M(M/K) + c_{M_*}(Q_{\mathrm{u}}/H_{\mathrm{u}}) \\ r_G(G/H) &= r_M(M/K) + r_{M_*}(Q_{\mathrm{u}}/H_{\mathrm{u}}) \end{aligned}$$

and there is a canonical exact sequence of weight lattices

$$0 \longrightarrow \Lambda_M(M/K) \longrightarrow \Lambda_G(G/H) \longrightarrow \Lambda_{M_*}(Q_{\mathrm{u}}/H_{\mathrm{u}}) \longrightarrow 0$$

Proof. As $B \subseteq Q^-$, the B- and even U-orbit of eQ is open in G/Q. Hence codimensions of generic orbits and weight lattices for the actions $B : G/H \simeq G *_Q Q/H$ and $B \cap Q = B(M) : Q/H$ are equal. Further, $Q/H \simeq M *_K Q_{\mathrm{u}}/H_{\mathrm{u}}$. It follows with our choice of K that the codimension of a generic $B(M)$-orbit in M/K is the sum of the codimension of a generic $B(M)$-orbit in M/K and of a generic $B(M_*)$-orbit in $Q_{\mathrm{u}}/H_{\mathrm{u}}$, whence the formula on complexities.

Furthermore, stabilizers of general position of the actions $B : G/H$, $B(M) : Q/H$, $B(M_*) : Q_{\mathrm{u}}/H_{\mathrm{u}}$ are all equal to $B(L_0) = B \cap L_0$, where L_0 is the T-regular Levi subgroup of the horospherical type of G/H. The equalities $\Lambda(G/H) = \mathfrak{X}(B/B(L_0))$, $\Lambda(M/K) = \mathfrak{X}(B(M)/B(M_*))$, $\Lambda(Q_{\mathrm{u}}/H_{\mathrm{u}}) = \mathfrak{X}(B(M_*)/B(L_0))$ imply the assertions on ranks and weight lattices. ☐

Thus the computation of complexity and rank of G/H is performed in two steps:

(1) Compute the group $M_* \subseteq K$, which is by Theorem 8.4 a stabilizer of general position for the coisotropy representations $K : \mathfrak{k}^\perp$ (the orthocomplement in $\mathfrak{m}$). This can be done using, e.g., Elashvili's tables.

(2) Compute the stabilizer of general position for $M_* : \mathfrak{q}_u/\mathfrak{h}_u \oplus (\mathfrak{q}_u/\mathfrak{h}_u)^*$ using, e.g., an algorithm at the end of §8.

Complexity and rank are read off these stabilizers.

Example 9.2. Let $G = \mathrm{Sp}_{2n}(\Bbbk)$, $n \geq 3$, and H be the stabilizer of a general triple of vectors in a symplectic space, say $e_1, e_2, e_3 \in \Bbbk^{2n}$, in the notation of Example 9.1. We choose $K = \mathrm{Sp}_{2n-4}(\Bbbk)$, the stabilizer of $e_1, e_2, e_3, e_4 \in \Bbbk^{2n}$, for a Levi subgroup of H. The unipotent radical of $\mathfrak{h}$ is

$$\mathfrak{h}_u = \left\{ \left(\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & z & v^\top\Omega \\ 0 & 0 & 0 & 0 & 0\cdots 0 \\ \hline 0 & 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & v & 0 \end{array}\right) \;\middle|\; z \in \Bbbk,\ v \in \Bbbk^{2n-4} \right\}$$

where Ω is the matrix of the symplectic form on $\Bbbk^{2n-4} = \langle e_5, \ldots, e_{2n} \rangle$. For Q we take the stabilizer of a flag $\langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \Bbbk^{2n}$. Then

$$M = \left\{ \left(\begin{array}{cccc|c} t_1 & & & & \\ & t_1^{-1} & & 0 & 0 \\ & & t_2 & & \\ 0 & & & t_2^{-1} & \\ \hline & & 0 & & A \end{array}\right) \;\middle|\; t_1, t_2 \in \Bbbk^\times,\ A \in \mathrm{Sp}_{2n-4}(\Bbbk) \right\}$$

$$\mathfrak{q}_u = \left\{ \left(\begin{array}{cc|cc|c} 0 & x & -y_2 & y_1 & u^\top\Omega \\ 0 & 0 & 0 & 0 & 0\cdots 0 \\ \hline 0 & y_1 & 0 & z & v^\top\Omega \\ 0 & y_2 & 0 & 0 & 0\cdots 0 \\ \hline \begin{matrix}0\\ \vdots\\ 0\end{matrix} & u & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & v & 0 \end{array}\right) \;\middle|\; x, y_1, y_2, z \in \Bbbk,\ u, v \in \Bbbk^{2n-4} \right\}$$

Clearly, $M/K \simeq (\Bbbk^\times)^2$, and $M_* = K = \mathrm{Sp}_{2n-4}(\Bbbk)$ acts on $\mathfrak{q}_u$ by left multiplication of $u, v \in \Bbbk^{2n-4}$ by $A \in \mathrm{Sp}_{2n-4}(\Bbbk)$. It follows that $\mathfrak{q}_u/\mathfrak{h}_u = \Bbbk^{2n-4} \oplus \Bbbk^3$, a sum of the tautological and a trivial $\mathrm{Sp}_{2n-4}(\Bbbk)$-module.

We deduce that $c(M/K) = 0$, $r(M/K) = 2$, and $\Lambda(M/K) = \langle \omega_1, \omega_2 \rangle$. A generic stabilizer of $M_* : \mathfrak{q}_u/\mathfrak{h}_u \oplus (\mathfrak{q}_u/\mathfrak{h}_u)^*$ equals $\mathrm{Sp}_{2n-6}(\Bbbk)$ (=the stabilizer of $e_1, \ldots, e_6$), whence $\Lambda(\mathfrak{q}_u/\mathfrak{h}_u) = \langle \overline{\omega}_3 \rangle$, where $\overline{\omega}_3$ is the first fundamental weight of $\mathrm{Sp}_{2n-6}(\Bbbk)$, or equivalently, the restriction to the diagonal torus in $\mathrm{Sp}_{2n-6}(\Bbbk)$ of the third fundamental weight $\omega_3(t) = t_1t_2t_3$ of $\mathrm{Sp}_{2n}(\Bbbk)$. It follows that $r(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}) = 1$, $2c(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}) + r(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}) = 2(2n-4+3) - (n-2)(2n-3) + (n-3)(2n-5) = 7$, hence $c(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}) = 3$. We conclude that $c(G/H) = r(G/H) = 3$, and $\Lambda(G/H) = \langle\omega_1, \omega_2, \omega_3\rangle$.

10 Spaces of small rank and complexity

The term "complexity" is justified by the fact that homogeneous spaces of small complexity are more accessible for study. In particular, a good compactification theory can be developed for homogeneous spaces of complexity ≤ 1, see Chapter 3. On the other hand, rank and complexity are not completely independent invariants of a homogeneous space. In this section, we discuss the interactions between rank and complexity, paying special attention to homogeneous spaces of small rank and complexity. We begin with a simple

Proposition 10.1. $r(G/H) = 0$ *iff* H *is parabolic, i.e.,* G/H *is projective.*

Proof. The "only if" implication follows from the Bruhat decomposition, cf. Example 5.1. Conversely, if $r(G/H) = 0$, then H contains a maximal torus of G. Replacing H by a conjugate, we may assume that $H \supseteq T$ and $B \cap H$ has the minimal possible dimension. We claim that $H \supseteq B^-$. Otherwise, there is a simple root α such that $\mathfrak{h} \not\supseteq \mathfrak{g}_{-\alpha}$. Let $P_\alpha = L_\alpha \leftthreetimes N_\alpha$ be the respective minimal parabolic subgroup with the T-regular Levi decomposition. Then $B = B_\alpha \leftthreetimes N_\alpha$, where $B_\alpha = B \cap L_\alpha$ is a Borel subgroup in L_α, and $H \cap L_\alpha = B_\alpha$ or T. In both cases, we may replace B_α by a conjugate Borel subgroup $\widetilde{B}_\alpha$ in L_α so that $\dim H \cap \widetilde{B}_\alpha < \dim H \cap B_\alpha$. Then for $\widetilde{B} = \widetilde{B}_\alpha N_\alpha$ we have $\dim H \cap \widetilde{B} < \dim H \cap B$, a contradiction. □

In particular, homogeneous spaces of rank zero have complexity zero. This can be generalized to the following general inequality between complexity and rank.

Theorem 10.1 ([Pan1, 2.7], [Pan7, 2.2.10]). $2c(G/H) \leq \operatorname{Cox} G \cdot r(G/H)$, *where* $\operatorname{Cox} G$ *is the maximum of the Coxeter numbers of simple components of* G.

Observe that if G is a simple group and $H = \{e\}$, then the inequality becomes an equality, since $c(G) = \dim U$, $r(G) = \operatorname{rk} G$, and $\operatorname{Cox} G = 2\dim U/\operatorname{rk} G$. This inequality is rather rough, and various examples create an impression that the majority of homogeneous spaces have either small complexity or large rank. In particular, Panyushev proved that $r(G/H) = 1$ implies $c(G/H) \leq 1$.

Proposition 10.2 ([Pan5]). *If $r(G/H) = 1$, then either $c(G/H) = 0$, or G/H is obtained from a homogeneous $\mathrm{SL}_2(\Bbbk)$-space with finite stabilizer by parabolic induction, whence $c(G/H) = 1$.*

Spherical homogeneous spaces of rank 1 where classified by Akhiezer [Akh1] and Brion [Bri5], see Proposition 30.4 and Table 5.10. The above proposition says that, besides the spherical case, there is only one essentially new example of rank 1, namely $\mathrm{SL}_2(\Bbbk)$ acting on itself by left multiplications. (Factorizing by a finite group preserves complexity and rank.) The proof (and the classification) is based on Theorem 9.2. Homogeneous spaces of rank 1 are also characterized in terms of equivariant completions, see Proposition 30.4 and Remark 30.3.

Homogeneous spaces of small complexity are much more numerous. Here classification results concern mainly the case, where H is reductive, i.e., G/H is affine. For simple G, affine homogeneous spaces of complexity 0 were classified by Krämer [Krä] and of complexity 1 by Panyushev [Pan2], [Pan7, Ch.3]. A complete classification of spherical affine homogeneous spaces was obtained by Mikityuk [Mik] and Brion [Bri2], with a final stroke put by Yakimova [Yak1]. We expose their results in Tables 2.1–2.3, see also Table 5.9. Since the computation of complexity and rank of a given homogeneous space represents no difficulties by Theorems 9.1–9.2, the main problem of classification is to "cut off infinity".

Clearly, complexity and rank of G/H do not change if we replace G by a finite cover and/or H by a subgroup of finite index. Thus complexity and rank depend only on the local isomorphism class of G/H, i.e., on the pair $(\mathfrak{g}, \mathfrak{h})$. Therefore we may assume that H is connected.

If G is not semisimple, then it decomposes in an almost direct product $G = G' \cdot Z$, where G' is its (semisimple) commutator subgroup and Z is the connected center of G. It is easy to see that complexities of G/H, G/HZ, and $G'/(HZ \cap G')$ are equal. Therefore it suffices to solve the classification problem for semisimple G.

An initial arithmetical restriction on a subgroup $H \subseteq G$ such that $c(G/H) \leq c$ is that

$$\dim H \leq \dim U - c \tag{10.1}$$

A more subtle restriction is based on the notion of d-decomposition [Pan2]. A triple of reductive groups (L, L_1, L_2) is called a *d-decomposition* if $d_{L_1 \times L_2}(L) = d$, where $L_1 \times L_2$ acts on L by left and right multiplications. Clearly,

(L, L_1, L_2) remains a d-decomposition if one permutes L_1, L_2 or replaces them by conjugates. Besides, $d_{L_1}(L/L_2) = d_{L_2}(L/L_1) = d$. By [Lu1], generic orbits of each one of the actions $L_1 \times L_2 : L$, $L_1 : L/L_2$, $L_2 : L/L_1$ are closed. In particular, 0-decompositions are indeed decompositions: $L = L_1 \cdot L_2$. They were classified by Onishchik [Oni2]. Some special kinds of 1-decompositions of classical groups occurring in the classification of homogeneous spaces of complexity 1 (see below) were described by Panyushev [Pan2].

Let $H \subseteq F$ be reductive subgroups of G, and F_* be the stabilizer of general position for the coisotropy representation $F : \mathfrak{f}^\perp$. We may assume that $o = eF$ is a general point for the B-action on G/F, so that $\dim Bo$ is maximal and B_o^0 is a Borel subgroup in F_*^0. Immediately,

Proposition 10.3. *We have*

$$c_G(G/H) = c_G(G/F) + c_{F_*}(F/H) \geq c_G(G/F) + d_{F_*}(F/H)$$

In particular, if $c(G/H) \leq c$, then (F, F_*, H) is a d-decomposition for some $d \leq c$.

The latter assertion is the keystone in a method of classifying affine spherical homogeneous spaces of simple groups suggested by Mikityuk and extended by Panyushev to the case of complexity one. Let us explain its core.

Let G be a simple algebraic group. Maximal connected reductive subgroups $F \subset G$ are known due to Dynkin [GOV, Ch.6, §3]. We choose among them those with $c(G/F) \leq 1$ and search for reductive $H \subset F$ such that still $c(G/H) \leq 1$.

If G is exceptional, then either $c(G/F) = 0$ or $c(G/F) \geq 2$. For spherical F, sorting out those $H \subset F$ which satisfy (10.1) gives only 4 new subgroups with $c(G/H) \leq 1$ (Nos. 11 of Table 2.1 and 15–17 of Table 2.2).

If G is classical, then inequality (10.1) gives a finite list of subgroups. Again $c(G/F) \neq 1$ with only one exception $G = \mathrm{Sp}_4(\Bbbk)$, $F = \mathrm{SL}_2(\Bbbk)$ embedded in $\mathrm{Sp}_4(\Bbbk)$ by a 4-dimensional irreducible representation (No. 8 of Table 2.2). Here (10.1) becomes an equality, and F cannot be reduced. Sorting out $H \subset F$ with $c(G/H) \leq 1$ is based on (10.1) and on the fact that (F, F_*, H) is a decomposition or 1-decomposition. Here we find 22 new subgroups (Nos. 1–9 of Table 2.1 and 1–7, 9–14 of Table 2.2).

If G/H is a symmetric space, i.e., $H = (G^\sigma)^0$, where σ is an involutive automorphism of G, then $c(G/H) = 0$ (Theorem 26.1). Symmetric spaces are considered in §26 and classified in Table 5.9.

Up to a local isomorphism, all non-symmetric affine homogeneous spaces of simple groups with complexity 0 are listed in Table 2.1 and those of complexity 1 in Table 2.2. In the tables, we use the following notation.

Fundamental weights of simple groups are numbered as in [OV]. By ω_i we denote fundamental weights of G, by $\omega_i', \omega_i'', \dots$ those of simple components of H, and by ε_i basic characters of the central torus of H. We drop an index for a group of rank 1.

The column "$H \hookrightarrow G$" describes the embedding in terms of the restriction to H of the minimal representation of G (the tautological representation for classical groups). We use the multiplicative notation for representations: irreducible representations are indicated by their highest weights expressed in basic weights multiplicatively (i.e., products instead of sums, powers instead of multiples, 1 for the trivial one-dimensional representation, etc.), and "+" stands for the sum of representations.

The rank of G/H is indicated in the column "$r(G/H)$", and the column "$\Lambda_+(G/H)$" contains a minimal system of generators for the weight semigroup.

Table 2.1: Spherical affine homogeneous spaces of simple groups

No.	G	H	$H \hookrightarrow G$	$r(G/H)$	$\Lambda_+(G/H)$
1	SL_n	$\mathrm{SL}_m \times \mathrm{SL}_{n-m}$ $(m < n/2)$	$\omega_1' + \omega_1''$	$m+1$	$\omega_1 + \omega_{n-1}, \dots, \omega_{m-1} + \omega_{n-m+1}$, ω_m, ω_{n-m}
2	SL_{2n+1}	$\mathrm{Sp}_{2n} \times \Bbbk^\times$	$\omega_1'\varepsilon + \varepsilon^{-2n}$	$2n-1$	$\sum k_i\omega_i$ $\sum\limits_{i \text{ odd}} (2n+1-i)k_i = \sum\limits_{i \text{ even}} ik_i$
3	SL_{2n+1}	Sp_{2n}	$\omega_1' + 1$	$2n$	$\omega_1, \dots, \omega_{2n}$
4	Sp_{2n}	$\mathrm{Sp}_{2n-2} \times \Bbbk^\times$	$\omega_1' + \varepsilon + \varepsilon^{-1}$	2	$2\omega_1, \omega_2$
5	SO_{2n+1}	GL_n	$\omega_1'\varepsilon + \omega_{n-1}'\varepsilon^{-1} + 1$	n	$\omega_1, \dots, \omega_{n-1}, 2\omega_n$
6	SO_{4n+2}	SL_{2n+1}	$\omega_1' + \omega_{2n}'$	$n+1$	$\omega_2, \omega_4, \dots, \omega_{2n}, \omega_{2n+1}$
7	SO_{10}	$\mathrm{Spin}_7 \times \mathrm{SO}_2$	$\omega_3' + \varepsilon + \varepsilon^{-1}$	4	$2\omega_1, \omega_2, \omega_4, \omega_5$
8	SO_9	Spin_7	$\omega_3' + 1$	2	ω_1, ω_4
9	SO_8	$\mathbf{G}_2$	$\omega_1' + 1$	3	$\omega_1, \omega_3, \omega_4$
10	SO_7	$\mathbf{G}_2$	ω_1'	1	ω_3
11	$\mathbf{E}_6$	$\mathbf{D}_5$	$\omega_1' + \omega_5' + 1$	3	$\omega_1, \omega_5, \omega_6$
12	$\mathbf{G}_2$	$\mathbf{A}_2$	$\omega_1' + \omega_2' + 1$	1	ω_1

Now we describe spherical affine homogeneous spaces of semisimple groups.

We say that G/H is *decomposable* if, up to a local isomorphism, $G = G_1 \times G_2$, $H = H_1 \times H_2$, and $H_i \subseteq G_i$, $i = 1, 2$. Clearly, $G/H = G_1/H_1 \times G_2/H_2$ is spherical iff G_i/H_i are spherical. Thus it suffices to classify indecomposable spherical spaces.

Table 2.2: Affine homogeneous spaces of simple groups with complexity 1

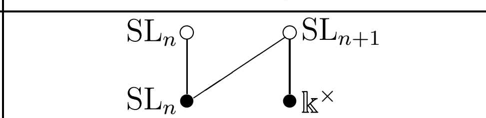
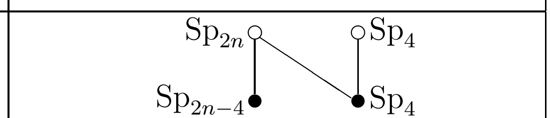
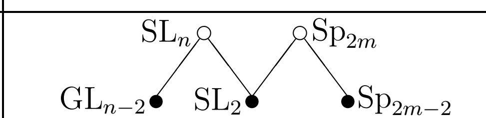
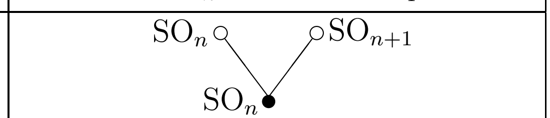
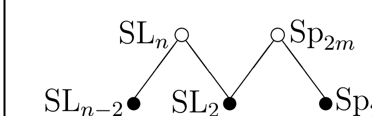
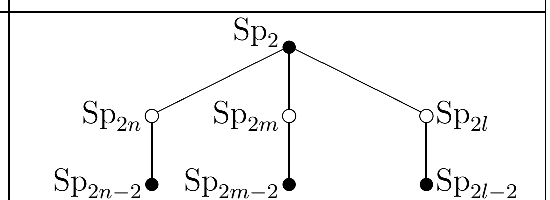
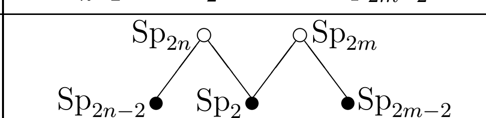
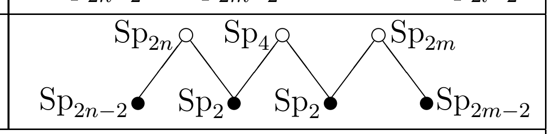
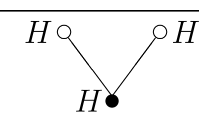
No.	G	H	$H \hookrightarrow G$	$r(G/H)$	$\Lambda_+(G/H)$
1	SL_{2n}	$\mathrm{SL}_n \times \mathrm{SL}_n$	$\omega_1' + \omega_1''$	n	$\omega_1 + \omega_{2n-1}, \dots, \omega_{n-1} + \omega_{n+1}, \omega_n$
2	SL_n $(n \geq 3)$	$\mathrm{SL}_{n-2} \times (\Bbbk^\times)^2$	$\omega_1'\varepsilon_1\varepsilon_2 + \varepsilon_1^{2-n} + \varepsilon_2^{2-n}$	3 $(n > 3)$	$\omega_1 + \omega_{n-1}, \omega_2 + \omega_{n-2}$ $2\omega_1 + \omega_{n-2}, \omega_2 + 2\omega_{n-1}$
				2 $(n = 3)$	$\omega_1 + \omega_2, 3\omega_1, 3\omega_2$
3	SL_n $(n \geq 5)$	$\mathrm{SL}_{n-2} \times \Bbbk^\times$	$\omega_1'\varepsilon + \varepsilon^{d_1} + \varepsilon^{d_2}$ $d_1 \neq d_2$ $d_1 + d_2 = 2 - n$	4	
4	SL_6	$\mathrm{Sp}_4 \times \mathrm{SL}_2 \times \Bbbk^\times$	$\omega_1'\varepsilon + \omega''\varepsilon^{-2}$	5	
5	Sp_{2n}	Sp_{2n-2}	$\omega_1' + 2$	2	ω_1, ω_2
6	Sp_{2n} $(n \geq 3)$	$\mathrm{Sp}_{2n-4} \times \mathrm{SL}_2 \times \mathrm{SL}_2$	$\omega_1' + \omega'' + \omega'''$	3 $(n > 3)$	$\omega_1 + \omega_3, \omega_2, \omega_4$
				2 $(n = 3)$	$\omega_1 + \omega_3, \omega_2$
7	Sp_{2n}	SL_n	$\omega_1' + \omega_{n-1}'$	n	$2\omega_1, \dots, 2\omega_{n-1}, \omega_n$
8	Sp_4	SL_2	${\omega'}^3$	2	$4\omega_1, 3\omega_2$ $4\omega_1 + 2\omega_2, 6\omega_1 + 3\omega_2$
9	SO_n $(n \geq 4)$	SO_{n-2}	$\omega_1' + 2$	2	ω_1, ω_2
10	SO_{2n+1}	SL_n	$\omega_1' + \omega_{n-1}' + 1$	n	$\omega_1, \dots, \omega_n$
11	SO_{4n} $(n \geq 2)$	SL_{2n}	$\omega_1' + \omega_{2n-1}'$	n	$\omega_2, \omega_4, \dots, \omega_{2n}$
12	SO_{11}	$\mathrm{Spin}_7 \times \mathrm{SO}_3$	$\omega_3' + {\omega''}^2$	5	$2\omega_1, 2\omega_2, \omega_3, \omega_4$ $\omega_1 + 2\omega_5, \omega_2 + 2\omega_5$ $\omega_1 + \omega_2 + 2\omega_5$
13	SO_{10}	Spin_7	$\omega_3' + 2$	4	$\omega_1, \omega_2, \omega_4, \omega_5$
14	SO_9	$\mathbf{G}_2 \times \mathrm{SO}_2$	$\omega_1' + \varepsilon + \varepsilon^{-1}$	4	
15	$\mathbf{E}_6$	$\mathbf{B}_4 \times \Bbbk^\times$	$(\omega_1' + 1)\varepsilon^2 + \omega_4'\varepsilon^{-1} + \varepsilon^{-4}$	5	
16	$\mathbf{E}_7$	$\mathbf{E}_6$	$\omega_1' + \omega_5' + 2$	3	$\omega_1, \omega_2, \omega_6$
17	$\mathbf{F}_4$	$\mathbf{D}_4$	$\omega_1' + \omega_3' + \omega_4' + 2$	2	ω_1, ω_2

Let $H \subseteq G$ be a reductive subgroup. We say that G/H is *strictly indecomposable* if G/H' is still indecomposable. All strictly indecomposable spherical affine homogeneous spaces of semisimple (non-simple) groups are listed in Table 2.3. The column "$H \hookrightarrow G$" describes the embedding in the following way. White vertices of a diagram denote simple factors of G and black vertices denote factors of H. (Some factors may vanish for small n.) If a factor H_j of H projects non-trivially to a factor G_i of G, then the respective vertices are joined by an edge. The product of those H_j which project to G_i is a spherical subgroup in G_i, and its embedding in G_i is described in Table 2.1. It follows from Tables 2.1, 2.3 that $\dim Z(H) \leq 1$ for all strictly indecomposable spherical homogeneous spaces G/H.

Now assume that G/H is indecomposable, but not strictly. Then, up to a local isomorphism, $G = G_1 \times \cdots \times G_s$ and $H' = H'_1 \times \cdots \times H'_s$, where H_i are the projections of H to G_i, and G_i/H_i are strictly indecomposable. Furthermore, $H_i = H'_i Z_i$, where Z_i is a one-dimensional central torus, and $H = H'Z$, where $Z \subset Z_1 \times \cdots \times Z_s$ is a subtorus. Since G/H is indecomposable, Z cannot be decomposed as $Z' \times Z''$, where Z', Z'' are the projections of Z to the products of two disjoint sets of factors Z_i.

If G_i/H_i is spherical and $B_i \subset G_i$ is a Borel subgroup such that $\dim B_i \cap H_i$ is minimal, then $\mathfrak{g}_i = \mathfrak{b}_i + \mathfrak{h}'_i$ if G_i/H'_i is spherical, or $\mathfrak{g}_i = (\mathfrak{b}_i + \mathfrak{h}'_i) \oplus \mathfrak{z}_i$ otherwise. It follows that G/H is spherical iff all G_i/H_i are spherical and Z projects onto the product of those Z_i for which G_i/H'_i is not spherical. This completes the classification.

Table 2.3: Spherical affine homogeneous spaces of semisimple groups

<table>
<tr><th>No.</th><th>$H \hookrightarrow G$</th><th>No.</th><th>$H \hookrightarrow G$</th></tr>
<tr><td>1</td><td></td><td>2</td><td></td></tr>
<tr><td>3</td><td></td><td>4</td><td></td></tr>
<tr><td>5</td><td> $(n \geq 5)$</td><td>6</td><td></td></tr>
<tr><td>7</td><td></td><td>8</td><td></td></tr>
<tr><td>9</td><td colspan="3"> (H is any simple group)</td></tr>
</table>

11 Double cones

The theory of complexity and rank can be applied to a fundamental problem of representation theory: decompose a tensor product of two simple G-modules into irreducibles. The idea is to realize this tensor product as a G-submodule in the coordinate algebra of a certain affine G-variety—a double cone—and to compute the algebra of U-invariants on a double cone, which yields the G-module structure of the whole coordinate algebra. In cases of small complexity, the algebra of U-invariants can be effectively computed.

We may and will assume that G is a simply connected semisimple group.

Definition 11.1. Let $\lambda \in \mathfrak{X}_+$ be a dominant weight and $v_{\lambda^*} \in V(\lambda^*)$ be a highest weight vector. A cone $C(\lambda) = \overline{Gv_{\lambda^*}} \subseteq V(\lambda^*)$ is called the *cone of highest weight vectors* (*HV-cone*). Clearly, $C(\lambda) = \overline{Gv_{-\lambda}}$, where $v_{-\lambda} \in V(\lambda^*)$ is a lowest weight vector.

The projectivization of $C(\lambda)$ is a projective homogeneous space $G/P(\lambda^*) \simeq G/P(\lambda)^-$, where $P(\lambda)$ denotes the projective stabilizer of a vector of highest weight λ in $V(\lambda)$.

The following assertions on HV-cones are well known and easily proved [VP, §2], cf. §28.

Proposition 11.1. (1) *$C(\lambda) = Gv_{-\lambda} \cup \{0\}$ is a normal conical variety in $V(\lambda^*)$.*

(2) *$\Bbbk[C(\lambda)]_n \simeq V(n\lambda)$ as a G-module.*

(3) *$C(\lambda)$ is factorial iff λ is a fundamental weight.*

Definition 11.2. A variety $Z(\lambda, \mu) = C(\lambda) \times C(\mu)$ is said to be a *double cone.*

The group $\widehat{G} = G \times (\Bbbk^\times)^2$ acts on $Z(\lambda, \mu)$ in a natural way, where the factors $\Bbbk^\times$ act by homotheties. Thus $\Bbbk[Z(\lambda, \mu)]$ is bigraded and

$$\Bbbk[Z(\lambda, \mu)]_{n,m} = V(n\lambda) \otimes V(m\mu)$$

The algebra $\Bbbk[Z(\lambda, \mu)]^U$ is finitely generated (Theorem A2.1(1)) and $(\mathfrak{X}_+ \times \mathbb{Z}_+^2)$-graded, and it is clear from the above that the knowledge of its (polyhomogeneous) generators and syzygies provides immediately a series of decomposition rules for $V(n\lambda) \otimes V(m\mu)$. Namely, the highest vectors of irreducible summands in $V(n\lambda) \otimes V(m\mu)$ are (linearly independent) products of generators of total bidegree (n, m).

The smaller is the $\widehat{G}$-complexity of $Z(\lambda,\mu)$, the simpler is the structure of $\Bbbk[Z(\lambda,\mu)]^U$. Say, if $Z(\lambda,\mu)$ is $\widehat{G}$-spherical, then $Z(\lambda,\mu)/\!\!/U$ is a toric $\widehat{T}$-variety, where $\widehat{T}=T\times(\Bbbk^\times)^2$. Hence $\Bbbk[Z(\lambda,\mu)]^U$ is the semigroup algebra of the weight semigroup of $Z(\lambda,\mu)$ (cf. Example 15.1). If in addition $Z(\lambda,\mu)$ is factorial, then $Z(\lambda,\mu)/\!\!/U$ is a factorial toric variety, hence $\Bbbk[Z(\lambda,\mu)]^U$ is freely generated by $\widehat{T}$-eigenfunctions of linearly independent weights. This yields a very simple decomposition rule, see below.

Therefore it is important to have a transparent method for computing complexity and rank of double cones. By the theory of doubled actions (§8), the problem reduces to computing the stabilizer of general position for the doubling $Z\times Z^*$ of a double cone $Z=Z(\lambda,\mu)$. This was done by Panyushev in [Pan3], see also [Pan7, Ch.6]. Here are his results.

Let $L(\lambda)$ be the Levi subgroup of $P(\lambda)$ containing T. The character λ extends to $L(\lambda)$. Put $G(\lambda)=\operatorname{Ker}\lambda\subset L(\lambda)$. Denote by $G(\lambda,\mu)$ the stabilizer of general position for $G(\lambda):G/G(\mu)$ and by $L(\lambda,\mu)$ the stabilizer of general position for $L(\lambda):G/L(\mu)$. Recall from [Lu1] that generic orbits of both these actions are closed, hence the codimension of a generic orbit equals the dimension of a categorical quotient:

$$\dim G(\lambda)\backslash\!\!\backslash G/\!\!/G(\mu)=\dim G+\dim G(\lambda,\mu)-\dim G(\lambda)-\dim G(\mu),\tag{11.1}$$
$$\dim L(\lambda)\backslash\!\!\backslash G/\!\!/L(\mu)=\dim G+\dim L(\lambda,\mu)-\dim L(\lambda)-\dim L(\mu),\tag{11.2}$$

where $L_1\backslash\!\!\backslash L/\!\!/L_2$ denotes the categorical quotient of the action $L_1\times L_2:L$ by left and right multiplication. Also put

$$\mathbb{P}(Z)=\mathbb{P}(C(\lambda))\times\mathbb{P}(C(\mu))\simeq G/P(\lambda)^-\times G/P(\mu)^-$$

Theorem 11.1. (1) *The stabilizers of general position for the doubled actions $G:Z\times Z^*$, $G:\mathbb{P}(Z)\times\mathbb{P}(Z^*)$, $\widehat{G}:Z\times Z^*$ are equal to $G(\lambda,\mu)$, $L(\lambda,\mu)$ and $\widehat{G}(\lambda,\mu)=\{\,(g,\lambda(g),\mu(g))\mid g\in L(\lambda,\mu)\,\}$, respectively.*

(2) *Put $V(\lambda,\mu)=\mathfrak{p}(\lambda)_{\mathrm{u}}\cap\mathfrak{p}(\mu)_{\mathrm{u}}$. Then $G(\lambda,\mu)$ and $L(\lambda,\mu)$ are equal to the stabilizers of general position for the doubled actions $G(\lambda)\cap G(\mu):V(\lambda,\mu)\oplus V(\lambda,\mu)^*$, $L(\lambda)\cap L(\mu):V(\lambda,\mu)\oplus V(\lambda,\mu)^*$, respectively.*

The proof of (1) uses the following

Lemma 11.1. *The stabilizers of general position for the doubled actions $G:C(\lambda)\times C(\lambda^*)$, $G:\mathbb{P}(C(\lambda))\times\mathbb{P}(C(\lambda^*))$, $G\times\Bbbk^\times:C(\lambda)\times C(\lambda^*)$ (where $\Bbbk^\times$ acts on $C(\lambda)$ by homotheties) are equal to $G(\lambda)$, $L(\lambda)$ and $\widehat{L}(\lambda)=\{\,(g,\lambda(g))\mid g\in L(\lambda)\,\}$, respectively.*

Proof. Observe that $z = (v_{-\lambda}, v_\lambda) \in C(\lambda) \times C(\lambda^*)$ has stabilizer and projective stabilizer $G(\lambda)$ in G, whence $\operatorname{codim} Gz = 1$, and $\overline{G\langle z\rangle} = C(\lambda) \times C(\lambda^*)$. It follows that $G(\lambda)$ is the stabilizer of general position in G. Other assertions are proved similarly (cf. Example 8.5). □

Proof of Theorem 11.1. (1) We have $Z \times Z^* = (C(\lambda) \times C(\lambda^*)) \times (C(\mu) \times C(\mu^*))$, and similarly for $\mathbb{P}(Z) \times \mathbb{P}(Z^*)$. The stabilizer of general position for any diagonal action $L : X_1 \times X_2$ can be computed in two steps: first find the stabilizers of general position L_i for the actions $L : X_i$, $i = 1, 2$, and then find the stabilizer of general position for $L_1 : L/L_2$. It remains to apply Lemma 11.1 for $L = G$ or $\widehat{G}$ and $X_1 = C(\lambda) \times C(\lambda^*)$ or $\mathbb{P}(C(\lambda)) \times \mathbb{P}(C(\lambda^*))$, $X_2 = C(\mu) \times C(\mu^*)$ or $\mathbb{P}(C(\mu)) \times \mathbb{P}(C(\mu^*))$.

(2) One can prove (2) using Luna's slice theorem, if one observes that the $L(\lambda)$-orbit of $eL(\mu) \in G/L(\mu)$ and the $G(\lambda)$-orbit of $eG(\mu) \in G/G(\mu)$ are closed, and computes the slice module. However, the proof also stems from the theory of doubled actions. It suffices to prove that the actions $B : Z$ (or $B : \mathbb{P}(Z)$) and $B \cap G(\lambda) \cap G(\mu) : V(\lambda, \mu)$ (resp. $B \cap L(\lambda) \cap L(\mu) : V(\lambda, \mu)$) have the same stabilizers of general position. (These are Borel subgroups in the generic stabilizers of double actions.) For computing these stabilizers, we apply the algorithm at the end of §8.

We have $Z \subseteq V = V(\lambda^*) \oplus V(\mu^*)$. Take a B-eigenvector $\omega = (v_\lambda, 0) \in V^*$ and put $\mathring{Z} = Z_\omega$. By Lemma 4.1, $\mathring{Z} \simeq P(\lambda)_{\mathrm{u}} \times Z'$, where $Z' \simeq \langle v_\lambda\rangle \times C(\mu)$ as an $L(\lambda)$-variety. Now take $\omega' = (0, v_\mu)$ and put $\mathring{Z}' = Z'_{\omega'}$. Then $\mathring{Z}' \simeq [L(\lambda) \cap P(\mu)_{\mathrm{u}}] \times Z''$, where $Z'' = \langle v_\lambda\rangle \times \langle v_\mu\rangle \times V(\lambda, \mu)$ as an $L(\lambda) \cap L(\mu)$-variety. This proves our claim on stabilizers of general position. □

We shall denote by c, r, Λ (resp. $\widehat{c}, \widehat{r}, \widehat{\Lambda}$) the complexity, rank and the weight lattice of a G- (resp. $\widehat{G}$-) action. Since maximal unipotent subgroups of G and $\widehat{G}$ coincide, it follows from Proposition 5.1 that

$$c(Z) + r(Z) = \widehat{c}(Z) + \widehat{r}(Z) \tag{11.3}$$

It is also clear that $\widehat{c}(Z) \leq c(Z) \leq \widehat{c}(Z)$. Since an open subset $Gv_{-\lambda} \times Gv_{-\mu} \subset Z$ is a G-equivariant principal $(\Bbbk^\times)^2$-bundle over $\mathbb{P}(Z)$, and $\widehat{\Lambda}(Z) \subseteq \mathfrak{X}(\widehat{G}) = \mathfrak{X}(G) \oplus \mathbb{Z}^2$ projects onto $\mathbb{Z}^2$ with the kernel $\Lambda(Z)$, we have

$$\widehat{c}(Z) = c(\mathbb{P}(Z)) \tag{11.4}$$

$$\widehat{r}(Z) = r(\mathbb{P}(Z)) + 2 \tag{11.5}$$

Theorem 11.1, together with Theorem 8.6, yields

Theorem 11.2. *The following formulae are valid:*

$$2c(Z)+r(Z)=2+\dim G(\lambda)\backslash\!\!\backslash G/\!\!/G(\mu) \qquad r(Z)=\operatorname{rk}G-\operatorname{rk}G(\lambda,\mu)$$
$$2\widehat{c}(Z)+\widehat{r}(Z)=2+\dim L(\lambda)\backslash\!\!\backslash G/\!\!/L(\mu) \qquad \widehat{r}(Z)=\operatorname{rk}\widehat{G}-\operatorname{rk}\widehat{G}(\lambda,\mu)$$

For the proof, just note that $\dim C(\lambda)=\frac{1}{2}(\dim G-\dim G(\lambda)+1)=\frac{1}{2}(\dim\widehat{G}-\dim L(\lambda))$ and recall (11.1)–(11.2).

Corollary 11.1. *The numbers $c, r, \widehat{c}, \widehat{r}$ do not change if one transposes λ and μ or replaces λ (or μ) by the dual weight λ^* (resp. μ^*).*

Indeed, the doubled G-variety $Z\times Z^*$ and the generic stabilizers $G(\lambda,\mu)$, $L(\lambda,\mu)$ do not change.

Corollary 11.2. *For $\mu=\lambda$ or λ^*,*

$$c(Z)=c(G/G(\lambda))+1 \qquad r(Z)=r(G/G(\lambda))$$
$$\widehat{c}(Z)=c(G/L(\lambda)) \qquad \widehat{r}(Z)=r(G/L(\lambda))$$

Proof. It follows from (the proof of) Lemma 11.1 that a generic orbit of $G:Z$ has codimension 1 and is isomorphic to $G/G(\lambda)$, and $G:\mathbb{P}(Z)$ has an open orbit isomorphic to $G/L(\lambda)$. Now apply (11.4)–(11.5). ☐

Now we restrict our attention to factorial double cones. By Proposition 11.1(3), $Z(\lambda,\mu)$ is factorial iff λ,μ are fundamental weights. We shall write $C(i)$, $Z(i,j)$, $P(i)$, … instead of $C(\omega_i)$, $Z(\omega_i,\omega_j)$, $P(\omega_i)$, … . For all simple groups G and all pairs of fundamental weights ω_i,ω_j, complexities and ranks of $Z(i,j)$ w.r.t. the G- and $\widehat{G}$-actions were computed in [Pan3]. All pairs of fundamental weights (ω_i,ω_j) such that $\widehat{c}(Z(i,j))=0,1$ are listed, up to the transposition of i,j and an automorphism of the Dynkin diagram, in Tables 2.4–2.5.

Suppose that $\Bbbk[Z(i,j)]^U$ is minimally generated by bihomogeneous eigenfunctions $f_1,\dots,f_r$ of weights $\lambda_1,\dots,\lambda_r$ and bidegrees $(n_1,m_1),\dots,(n_r,m_r)$. We may assume that f_1,f_2 have the weights ω_i,ω_j and bidegrees $(1,0)$, $(0,1)$. The weights of other generators and their bidegrees are indicated in the columns "Weights" and "Degrees", respectively. Here we assume $\omega_i=0$ whenever $i\neq 1,\dots,\operatorname{rk}G$.

We already noted that if $\widehat{c}(Z(i,j))=0$, then $f_1,\dots,f_r$ are algebraically independent and $(\lambda_1,n_1,m_1),\dots,(\lambda_r,n_r,m_r)$ are linearly independent. If $\widehat{c}(Z(i,j))=1$, then $Z(i,j)/\!\!/U$ is a hypersurface [Pan3, 6.5] and the (unique) syzygy between $f_1,\dots,f_r$ is of the form $P+Q+R=0$, where P,Q,R are all monomials in $f_1,\dots,f_r$ of the same weight λ_0 and bidegree (n_0,m_0) indicated in the column "Syzygy" of Table 2.5.

It follows from the classification that if $i = j$, then $\widehat{c}(Z(i,j)) = 0$ and $m_i = n_i = 1$ for $i = 3, \dots, r$. Hence $\widehat{T}$-eigenspaces in $\Bbbk[Z(i,i)]^U$ are one-dimensional, and the involution transposing the factors of $Z(i,i) = C(i) \times C(i)$ multiplies each $\widehat{T}$-eigenfunction f of bidegree (n,n) by $p(f) = \pm 1$. We call $p(f)$ the *parity* of f. The parities of generators are given in the column "Parity" of Table 2.4. If $f = f_1^{k_1} \dots f_r^{k_r}$ ($k_1 = k_2$), then $p(f) = p(k_3, \dots, k_r) := p(f_3)^{k_3} \dots p(f_r)^{k_r}$.

Table 2.4: Spherical double cones (factorial case)

G	Pair	Weights	Degrees	Parity
$\mathbf{A}_l$	(ω_i, ω_j) $i \le j$	$\omega_{i-k} + \omega_{j+k}$ $k = 1, \dots, \min(i, l+1-j)$	$(1,1)$	$(-1)^k$ for $i = j$
$\mathbf{B}_l$	(ω_1, ω_1)	$0, \omega_2$	$(1,1)$	$1, -1$
	(ω_1, ω_j) $2 \le j \le l-2$	$\omega_{j-1}, \omega_{j+1}$ ω_j	$(1,1)$ $(2,1)$	
	(ω_1, ω_{l-1})	$\omega_{l-2}, 2\omega_l$ ω_{l-1}	$(1,1)$ $(2,1)$	
	(ω_1, ω_l)	ω_l ω_{l-1}	$(1,1)$ $(1,2)$	
	(ω_l, ω_l)	ω_k $k = 0, \dots, l-1$	$(1,1)$	$(-1)^{k(k+1)/2}$
$\mathbf{C}_l$	(ω_1, ω_1)	$0, \omega_2$	$(1,1)$	-1
	(ω_1, ω_j) $2 \le j \le l-1$	$\omega_{j-1}, \omega_{j+1}$ ω_j	$(1,1)$ $(2,1)$	
	(ω_1, ω_l)	ω_{l-1} ω_l	$(1,1)$ $(2,1)$	
	(ω_l, ω_l)	$0, 2\omega_k$ $k = 1, \dots, l-1$	$(1,1)$	$(-1)^l, (-1)^{l-k}$
$\mathbf{D}_l$	(ω_1, ω_1)	$0, \omega_2$	$(1,1)$	$1, -1$
	(ω_1, ω_j) $2 \le j \le l-3$	$\omega_{j-1}, \omega_{j+1}$ ω_j	$(1,1)$ $(2,1)$	
	(ω_1, ω_{l-2})	$\omega_{l-3}, \omega_{l-1} + \omega_l$ ω_{l-2}	$(1,1)$ $(2,1)$	
	(ω_1, ω_{l-1})	ω_l	$(1,1)$	
	(ω_l, ω_l)	ω_{l-2k} $1 \le k \le l/2$	$(1,1)$	$(-1)^l$
	(ω_{l-1}, ω_l)	ω_{l-2k-1} $1 \le k \le (l-1)/2$	$(1,1)$	

Table 2.4: (continued)

G	Pair	Weights	Degrees	Parity
	(ω_2, ω_{l-1})	$\omega_{l-1}, \omega_1 + \omega_l$	$(1,1)$	
		ω_{l-2}	$(1,2)$	
$\mathbf{D}_l$	(ω_3, ω_{l-1})	$\omega_1 + \omega_{l-1}, \omega_2 + \omega_l, \omega_l$	$(1,1)$	
$l \geq 6$		$\omega_{l-3}, \omega_1 + \omega_{l-2}$	$(1,2)$	
		$\omega_2 + \omega_{l-2}$	$(2,2)$	
$\mathbf{D}_5$	(ω_3, ω_4)	$\omega_1 + \omega_4, \omega_2 + \omega_5, \omega_5$	$(1,1)$	
		$\omega_2, \omega_1 + \omega_3$	$(1,2)$	
$\mathbf{E}_6$	(ω_1, ω_1)	ω_2, ω_5	$(1,1)$	$-1, 1$
	(ω_1, ω_2)	$\omega_1 + \omega_5, \omega_3, \omega_6$	$(1,1)$	
		$\omega_2 + \omega_5, \omega_4$	$(2,1)$	
	(ω_1, ω_4)	$\omega_2, \omega_5, \omega_5 + \omega_6$	$(1,1)$	
		ω_3, ω_6	$(2,1)$	
	(ω_1, ω_5)	$0, \omega_6$	$(1,1)$	
	(ω_1, ω_6)	ω_1, ω_4	$(1,1)$	
		ω_2	$(2,1)$	
$\mathbf{E}_7$	(ω_1, ω_1)	$0, \omega_2, \omega_6$	$(1,1)$	$-1, -1, 1$
	(ω_1, ω_6)	ω_1, ω_7	$(1,1)$	
		ω_2	$(2,1)$	
	(ω_1, ω_7)	$\omega_2, \omega_5, \omega_6$	$(1,1)$	
		ω_3, ω_7	$(2,1)$	
		ω_4	$(2,2)$	

For spherical $Z(i,j)$, the algebra $\Bbbk[Z(i,j)]^U$ was computed by Littelmann [Lit]. He observed that a simple structure of $\Bbbk[Z(i,j)]^U$ leads to the following decomposition rules:

Tensor products.

$$V(n\omega_i) \otimes V(m\omega_j) = \bigoplus_{k_1(n_1,m_1)+\cdots+k_r(n_r,m_r)=(n,m)} V(k_1\lambda_1 + \cdots + k_r\lambda_r)$$

Symmetric and exterior squares.

$$\mathrm{S}^2 V(n\omega_i) = \bigoplus_{\substack{k_1+k_3+\cdots+k_r=n \\ p(k_3,\ldots,k_r)=1}} V(2k_1\omega_i + k_3\lambda_3 + \cdots + k_r\lambda_r)$$

$$\bigwedge\nolimits^2 V(n\omega_i) = \bigoplus_{\substack{k_1+k_3+\cdots+k_r=n \\ p(k_3,\ldots,k_r)=-1}} V(2k_1\omega_i + k_3\lambda_3 + \cdots + k_r\lambda_r)$$

Table 2.5: Double cones of complexity one (factorial case)

G	Pair	Weights	Degrees	Syzygy
$\mathbf{B}_l$ $l \geq 4$	(ω_2, ω_l)	$\omega_1 + \omega_l, \omega_l$ $\omega_{l-2}, \omega_{l-1}, \omega_1 + \omega_{l-1}$ $\omega_1 + \omega_{l-1}$	$(1,1)$ $(1,2)$ $(2,2)$	$\omega_1 + \omega_{l-1} + \omega_l$ $(2,3)$
$\mathbf{B}_3$	(ω_2, ω_3)	$\omega_1 + \omega_3, \omega_3$ $\omega_1, \omega_2, \omega_1 + \omega_2$	$(1,1)$ $(1,2)$	$\omega_1 + \omega_2 + \omega_3$ $(2,3)$
$\mathbf{C}_l$ $l \geq 4$	(ω_2, ω_l)	$\omega_{l-2}, \omega_1 + \omega_{l-1}$ $\omega_1 + \omega_{l-1}, 2\omega_1 + \omega_l, \omega_l$ $2\omega_{l-1}$	$(1,1)$ $(2,1)$ $(2,2)$	$2\omega_1 + 2\omega_{l-1} + \omega_l$ $(4,3)$
$\mathbf{C}_3$	(ω_2, ω_3)	$\omega_1, \omega_1 + \omega_2$ $2\omega_1 + \omega_3, \omega_3$ $2\omega_2$	$(1,1)$ $(2,1)$ $(2,2)$	$2\omega_1 + 2\omega_2 + \omega_3$ $(4,3)$
$\mathbf{D}_6$	(ω_4, ω_5)	$\omega_2 + \omega_5, \omega_5, \omega_1 + \omega_6, \omega_3 + \omega_6$ $\omega_4, \omega_2 + \omega_4, \omega_2, \omega_1 + \omega_3$	$(1,1)$ $(1,2)$	$\omega_2 + \omega_4 + \omega_5$ $(2,3)$
$\mathbf{E}_7$	(ω_1, ω_2)	$\omega_1, \omega_1 + \omega_6, \omega_3, \omega_7$ $\omega_2 + \omega_6, \omega_2, \omega_5, \omega_6$	$(1,1)$ $(2,1)$	$\omega_1 + \omega_2 + \omega_6$ $(3,2)$

Restriction.

$$\operatorname{Res}^G_{L(i)} V_G(m\omega_j) = \bigoplus_{k_2 m_2 + \cdots + k_r m_r = m} V_{L(i)}(k_2(\lambda_2 - n_2\omega_i) + \cdots + k_r(\lambda_r - n_r\omega_i))$$

Proofs. The first two rules stem immediately from the above discussion. Indeed, highest weight vectors in $\Bbbk[Z(i,j)]^U_{n,m} = V(n\omega_i) \otimes V(m\omega_j)$ are proportional to monomials $f = f_1^{k_1} \dots f_r^{k_r}$ with $k_1(n_1, m_1) + \cdots + k_r(n_r, m_r) = (n, m)$. The transposition of the factors of $Z(i,i)$ transposes the factors of $\Bbbk[Z(i,i)]^U_{n,n} = V(n\omega_i)^{\otimes 2}$, and f is (skew)symmetric iff $p(f) = 1$ (resp. -1).

To prove the restriction rule, observe that $Z(i,i)_{f_1} = C(i)_{f_1} \times C(j) = P(i) *_{L(i)} (\Bbbk^\times v_{-\omega_i} \times C(j)) = P(i)_{\mathrm{u}} \times \Bbbk^\times v_{-\omega_i} \times C(j)$. Hence $\Bbbk[Z(i,j)]^U_{f_1} \simeq \Bbbk[\Bbbk^\times v_{-\omega_i} \times C(j)]^{U \cap L(i)} \simeq \Bbbk[f_1, f_1^{-1}] \otimes \Bbbk[C(j)]^{U \cap L(i)}$, and $f_2, \dots, f_r$ restrict to a free system of generators $\bar{f}_l(y) = f_l(v_{-\omega_i}, y)$ of $\Bbbk[C(j)]^{U \cap L(i)}$. It remains to remark that $\Bbbk[C(j)]_n \simeq V_G(n\omega_j)$, $U \cap L(i)$ is a maximal unipotent subgroup of $L(i)$, and $\bar{f}_l$ have T-eigenweights $\lambda_l - n_l\omega_i$:

$$t\bar{f}_l(y) = f_l(v_{-\omega_i}, t^{-1}y) = \omega_i(t)^{-n_l} f_l(t^{-1}v_{-\omega_i}, t^{-1}y) = \lambda_l(t)\omega_i(t)^{-n_l}\bar{f}_l(y)$$

$\square$

For the cases $\widehat{c}(Z(i,j)) = 1$, the algebra $\Bbbk[Z(i,j)]^U$ was computed in [Pan7, 6.5]. A decomposition rule for tensor products is of the form:

$$\begin{aligned} V(n\omega_i) \otimes V(m\omega_j) = & \bigoplus_{k_1(n_1,m_1)+\dots+k_r(n_r,m_r)=(n,m)} V(k_1\lambda_1 + \dots + k_r\lambda_r) \\ & - \bigoplus_{l_1(n_1,m_1)+\dots+l_r(n_r,m_r)=(n-n_0,m-m_0)} V(\lambda_0 + l_1\lambda_1 + \dots + l_r\lambda_r) \end{aligned}$$

(Here "$-$" is an operation in the Grothendieck group of G-modules.)

Example 11.1. Suppose $G = \mathrm{SL}_d(\Bbbk)$. Consider a double cone $Z(1,1)$. We have $L(1) = \mathrm{GL}_{d-1}(\Bbbk)$, $V(1,1) = \Bbbk^{d-1}$, and $L(1,1)$ consists of matrices of the form

$$\begin{array}{|c|c|} \hline \begin{matrix} t & 0 \\ 0 & t \end{matrix} & 0 \\ \hline 0 & * \\ \hline \end{array} \qquad (t \in \Bbbk^\times)$$

Its subgroup $G(1,1)$ is defined by $t = 1$. Hence $r = 2$, $\widehat{r} = 3$, $c = 1$, $\widehat{c} = 0$, and $\Lambda(Z(1,1)) = \langle \varepsilon_2 - \varepsilon_1 \rangle = \langle \omega_2 - 2\omega_1 \rangle$, $\widehat{\Lambda}(Z(1,1)) = \Lambda(Z(1,1)) + \langle (\omega_1,1,0),(\omega_1,0,1) \rangle = \langle (\omega_2,1,1),(\omega_1,1,0),(\omega_1,0,1) \rangle$. (Here $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$.)

Since $V(\omega_1)^{\otimes 2} = (\Bbbk^d)^{\otimes 2} = \mathrm{S}^2\Bbbk^d \oplus \bigwedge^2 \Bbbk^d = V(2\omega_1) \oplus V(\omega_2)$, the algebra $\Bbbk[Z(1,1)]^U$ contains eigenfunctions of the weights $(\omega_1,1,0),(\omega_1,0,1),(\omega_2,1,1)$, and a function of the weight $(\omega_2,1,1)$ has parity -1. Clearly, these three functions are algebraically independent (because their weights are linearly independent) and compose a part of a minimal generating system of $\Bbbk[Z(1,1)]^U$. Since $\dim Z(1,1)/\!\!/U = 3$, they generate $\Bbbk[Z(1,1)]^U$.

As a corollary, we obtain decomposition formulae:

$$\begin{aligned} V(n\omega_1) \otimes V(m\omega_1) &= \bigoplus_{0\le k\le \min(n,m)} V((n+m-2k)\omega_1 + k\omega_2) \\ \mathrm{S}^2 V(n\omega_1) &= \bigoplus_{0\le k\le \min(n,m)/2} V((n+m-4k)\omega_1 + 2k\omega_2) \\ \textstyle\bigwedge^2 V(n\omega_1) &= \bigoplus_{0\le k\le \min(n-1,m-1)/2} V((n+m-4k-2)\omega_1 + (2k+1)\omega_2) \end{aligned}$$

For $d = 2$, these are well-known Clebsch–Gordan formulae.

All $\widehat{G}$-spherical double cones $Z = Z(\lambda,\mu)$ were recently classified by Stembridge [Ste]. By (11.4), $\widehat{c}(Z)$ depends only on the parabolics $P(\lambda)$, $P(\mu)$, i.e., on the supports of λ, μ w.r.t. fundamental weights. (The *support* of λ is the set of fundamental weights occurring in the decomposition of λ with nonzero coefficients). When the support of λ is reduced, $P(\lambda)$ increases, whence $\widehat{c}(Z)$ may only decrease. Therefore it suffices to find all pairs of maximal possible supports such that $\widehat{c}(Z) = 0$ for all simple groups.

The case of one-element supports, i.e., where λ, μ are multiples of fundamental weights, is already covered by Table 2.4. All remaining pairs of maximal supports, up to the transposition and an automorphism of the Dynkin diagram, are listed in Table 2.6.

Note that $Z(\lambda, \mu)$ is spherical iff $V(n\lambda) \otimes V(m\mu)$ is multiplicity-free for $\forall n, m$ (see §25).

Table 2.6: Spherical double cones (non-factorial case)

G	$\mathbf{A}_l$				$\mathbf{D}_l$					$\mathbf{E}_6$
λ	ω_1	ω_2	ω_i	ω_i	ω_1	ω_l	ω_l	ω_l	ω_l	ω_1
μ	$\omega_1, \dots, \omega_l$	ω_i, ω_j	ω_1, ω_j	ω_j, ω_{j+1}	ω_i, ω_l	ω_1, ω_{l-1}	ω_1, ω_l	ω_{l-1}, ω_l	ω_1, ω_2	ω_1, ω_5

Chapter 3

General theory of embeddings

Equivariant embeddings of homogeneous spaces are one of the main topics of this survey. The general theory of them was developed by Luna and Vust in a fundamental paper [LV]. However it was noticed in [Tim2] that the whole theory admits a natural exposition in a more general framework, which is discussed in this chapter. The generically transitive case differs from the general one by the existence of a smallest G-variety of a given birational type, namely, a homogeneous space.

In §12 we discuss the general approach of Luna and Vust based on patching all G-varieties of a given birational class together in one huge prevariety and studying particular G-varieties as open subsets in it. An important notion of a B-chart arising in such a local study is considered in §13. A B-chart is a B-stable affine open subset of a G-variety, and any G-variety is covered by (finitely many) G-translates of B-charts. B-charts and their "admissible" collections corresponding to coverings of G-varieties are described in terms of *colored data* arranged of B-stable divisors and G-invariant valuations of a given function field. This leads to a "combinatorial" description of G-varieties in terms of colored data, obtained in §14. In the cases of complexity ≤ 1, considered in §15–§16, this description is indeed combinatorial, namely, in terms of polyhedral cones, their faces, fans and other objects of combinatorial convex geometry.

Divisors on G-varieties are studied in §17. We give criteria for a divisor to be Cartier, finitely generated and ample, describe global sections in terms of colored data. Aspects of the intersection theory on a G-variety are discussed in §18, including the role of B-stable cycles and a formula for the degree of an ample divisor.

12 The Luna–Vust theory

The fundamental problem of classifying algebraic varieties has an equivariant analogue: to describe up to a G-isomorphism all varieties equipped with an action of an algebraic group G. A birational classification of G-varieties (with a given field of G-invariant functions) may be obtained in terms of Galois cohomology [PV, §2]. The second, "biregular", part of the problem may be formulated as follows: to describe all G-actions in a given birational class. More precisely, let K be a fixed function field (i.e., a finitely generated extension of $\Bbbk$), and let G act on K birationally. In other words, K is the function field on some G-variety X. We say that K is a *G-field* and X is a *G-model* of K. The problem is to classify all G-models of K in terms involving certain invariants of K itself (such as valuations etc).

Remark 12.1. If $K^G = \Bbbk$, or equivalently, the G-action on each G-model of K is generically transitive, then there is a minimal G-model $\mathcal{O} = G/H$, which is embedded as a dense orbit in any other G-model of K. The homogeneous space $\mathcal{O}$ determines and is determined by K completely. So the problem may be thought of as classifying all G-equivariant embeddings of $\mathcal{O}$ in terms of invariants of $\mathcal{O}$ itself.

A general approach to this problem was introduced by Luna and Vust [LV]. They considered only embeddings of homogeneous spaces. We will follow [Tim2] in our more general point of view.

All models of K may be glued together into one huge scheme $\mathbb{X} = \mathbb{X}(K)$. By definition, points of $\mathbb{X}$ are local rings that are localizations of finitely generated $\Bbbk$-algebras with quotient field K. Any model X of K (i.e., a variety with $\Bbbk(X) = K$) may be considered as a subset of $\mathbb{X}$, and such subsets define the base of the *Zariski topology* on $\mathbb{X}$. The structure sheaves $\mathcal{O}_X$ are patched together in a structure sheaf $\mathcal{O}_{\mathbb{X}}$. A local ring $\mathcal{O}_{\mathbb{X},Y}$ of $Y \in \mathbb{X}$ in the sense of this sheaf is exactly the ring defining Y as a point of $\mathbb{X}$.

The scheme $\mathbb{X}$ is irreducible, but neither Noetherian nor separated. It can be considered as a prevariety if we consider only closed points $x \in \mathbb{X}$ (i.e., such that the residue field $\Bbbk(x) = \mathcal{O}_x/\mathfrak{m}_x$ of the respective local ring $\mathcal{O}_x = \mathcal{O}_{\mathbb{X},x}$ equals k). Non-closed schematic points are identified with closed irreducible subvarieties $Y \subseteq \mathbb{X}$.

We distinguish in $\mathbb{X}$ open subsets $\mathbb{X}^{\mathrm{reg}}$, $\mathbb{X}^{\mathrm{norm}}$,... of smooth, normal,... points.

From this point of view, a model of K is nothing but a Noetherian separated open subset $X \subseteq \mathbb{X}$.

The birational G-action on K permutes local subrings of K, which yields an action $G : \mathbb{X}$. However this is not an action in the category of schemes or prevarieties. But the action map $\alpha : G \times \mathbb{X} \to \mathbb{X}$ is rational and induces an embedding of function fields $\alpha^* : K = \Bbbk(\mathbb{X}) \hookrightarrow \Bbbk(G \times \mathbb{X}) = \Bbbk(G) \cdot K$. (Here $\Bbbk(G) \cdot K = \mathrm{Quot}(\Bbbk(G) \otimes_k K)$ is a free composite of fields.) It is obvious that G acts on a G-stable open subset $X \subseteq \mathbb{X}$ regularly iff $\alpha^*(\mathcal{O}_{X,x}) \subseteq \mathcal{O}_{G\times X, e\times x}$ for $\forall x \in X$.

Denote by $\mathbb{X}_G$ the set of those $x \in \mathbb{X}$ whose local rings $\mathcal{O}_{\mathbb{X},x}$ are mapped by α^* to $\mathcal{O}_{G\times\mathbb{X}, e\times x}$.

Proposition 12.1. *$\mathbb{X}_G$ is the largest open subset of $\mathbb{X}$ on which G acts regularly.*

Proof. We have only to prove that $\mathbb{X}_G$ is open. In other words, if $\mathcal{O}_{\mathbb{X},x} \hookrightarrow \mathcal{O}_{G\times\mathbb{X}, e\times x}$ for $x = x_0$, then the same thing holds in a neighborhood of x_0. Let $X = \operatorname{Spec} A$ be an affine neighborhood of x_0, where $A = \Bbbk[f_1, \dots, f_s]$ is a finitely generated algebra with quotient field K. Then $\mathcal{O}_{\mathbb{X},x}$ is a localization of A at the maximal ideal of x_0. By assumption, $\alpha^*(f_i)$ are defined in a neighborhood E of $e \times x_0$ (one and the same for $i = 1, \dots, s$), hence α restricts to a regular map $E \to X$. The set of all $x \in X$ such that $e \times x \in E$ is a neighborhood of x_0. In this neighborhood, we have $\alpha(e\times x) = x$, because this holds generically on $\mathbb{X}$. This yields the assertion. ☐

Observe that $\mathfrak{g}$ acts on K by derivations (along velocity fields on $\mathbb{X}_G$ or on any other G-model).

Proposition 12.2 ([LV, 1.4]). *In characteristic zero, $x \in \mathbb{X}_G$ iff $\mathcal{O}_x$ is $\mathfrak{g}$-stable.*

In particular, if $A \subset K$ is a $\mathfrak{g}$-stable finitely generated subalgebra, then any localization of A is $\mathfrak{g}$-stable and consequently $X = \operatorname{Spec} A \subseteq \mathbb{X}_G$.

Example 12.1. Let $G = \Bbbk$ act on $\mathbb{A}^1$ by translations. This yields a birational action $G : K = \Bbbk(t)$, so that $\alpha^*(t) = u + t$ (u is a coordinate on G). A cuspidal curve $X \subset \mathbb{A}^2$ (the Neil parabola) defined by the equation $y^2 = x^3$ becomes a model of K if we put $t = y/x$. The local ring of the singular point $x_0 = (0,0) \in X$ consists of rational functions in $x = t^2$, $y = t^3$ whose denominators have nonzero constant term. But $\alpha^*(t^d) = u^d + du^{d-1}t + \dots$ is not defined at $0 \times x_0 \in G \times \mathbb{X}$ (at least when d does not divide $\operatorname{char} k$), because t is not defined at x_0. Therefore $x_0 \notin \mathbb{X}_G$. All other points of X are in $\mathbb{X}_G$, because they are identified via the normalization map $t \mapsto (t^2, t^3)$ with the respective points of $\mathbb{A}^1$, where G acts regularly.

The standard basic vector $\xi \in \mathfrak{g} = \Bbbk$ acts on K as d/dt, and $\xi(t^d) = dt^{d-1} \in \mathcal{O}_{x_0}$ if $d > 2$. But $\xi x = 2t \notin \mathcal{O}_{x_0}$ if $\operatorname{char} k \neq 2$, in accordance with Proposition 12.2. However in characteristic 2, the algebra $\Bbbk[X] = \Bbbk[x,y]$ is $\mathfrak{g}$-stable and Proposition 12.2 is not applicable.

Example 12.2. Another example of this kind is the birational action of $G = \Bbbk^n$ by translations on a blow-up X of $\mathbb{A}^n$ at 0. All points in the complement to the exceptional divisor are in $\mathbb{X}_G$, since they come from $\mathbb{A}^n$, where G acts regularly. In a neighborhood of a point x_0 on the exceptional divisor, X is defined by local equations $x_i = x_1 y_i$ $(1 < i \le n)$ in $\mathbb{A}^n \times \mathbb{A}^{n-1}$. We have $\alpha^*(x_i) = x_i + u_i$, where u_i are coordinates on G, and $\alpha^*(y_i) = \frac{x_i+u_i}{x_1+u_1} = \frac{x_1y_i+u_i}{x_1+u_1}$ are not defined at $0 \times x_0$. Hence $x_0 \notin \mathbb{X}_G$. On the other hand, the standard basic vectors $\xi_1, \dots, \xi_n$ of $\mathfrak{g} = \Bbbk^n$ act on $K = \Bbbk(x_1, \dots, x_n)$ as $\partial/\partial x_1, \dots, \partial/\partial x_n$, and

$$\xi_i y_j = \begin{cases} 1/x_1, & i = j > 1 \\ -y_j/x_1, & i = 1 < j \\ 0, & \text{otherwise,} \end{cases}$$

so that not all $\xi_i y_j$ are in $\mathcal{O}_{x_0}$.

These two examples are typical in a sense that one obtains "bad" birational actions if one blows up or contracts G-nonstable subvarieties in a variety with a "good" (regular) action.

By Proposition 12.1, a G-model of K is nothing but a G-stable Noetherian separated open subset of $\mathbb{X}_G$. The next theorem gives a way to construct G-models as "G-spans", which we use in §13.

Theorem 12.1 ([LV, 1.5]). *Assume $\mathring{X}$ is an open subset in $\mathbb{X}_G$. Then $X = G\mathring{X}$ is Noetherian (separated) iff $\mathring{X}$ is Noetherian (separated).*

Proof. If $\mathring{X}$ is Noetherian, then $G \times \mathring{X}$ is Noetherian, and $X = \alpha(G \times \mathring{X})$ is Noetherian, too. If X is not separated, then the diagonal Δ_X is not closed in $X \times X$, and the non-empty G-stable subset $\overline{\Delta_X} \setminus \Delta_X$ contains an orbit $\mathcal{O}$. Clearly, $\mathcal{O}$ intersects the two open subsets $\mathring{X} \times X$ and $X \times \mathring{X}$ of $X \times X$. Since $\mathcal{O}$ is irreducible, $\mathcal{O} \cap (\mathring{X} \times X) \cap (X \times \mathring{X}) = \mathcal{O} \cap (\mathring{X} \times \mathring{X}) \subseteq \overline{\Delta_{\mathring{X}}} \setminus \Delta_{\mathring{X}}$ is non-empty, whence $\mathring{X}$ is not separated.

The converse implications are obvious. □

Example 12.3. Let $G = \Bbbk^\times$ act on $\mathbb{A}^1$ by homotheties. Here $K = \Bbbk(t)$ and a generator ξ of $\mathfrak{g} = \Bbbk$ acts on K as $t\frac{d}{dt}$. Put $x = \frac{t}{(1+t)^2}$, $y = \frac{t}{(1+t)^3}$. Then $t \mapsto (x, y)$ is a birational map of $\mathbb{A}^1$ to the Cartesian leaf $\mathring{X} \subset \mathbb{A}^2$ defined by the equation $x^3 = xy - y^2$. This map provides a biregular isomorphism of $\mathbb{A}^1 \setminus \{-1\}$ onto $\mathring{X} \setminus \{x_0\}$, where $x_0 = (0, 0)$ is the singular point of $\mathring{X}$. Therefore $\mathring{X} \setminus \{x_0\} \subseteq \mathbb{X}_G$. One can verify by direct computation that $\alpha^*(x), \alpha^*(y) \in \mathcal{O}_{1 \times x_0}$, whence $x_0 \in \mathbb{X}_G$. In characteristic zero, the situation is simpler, because $\xi x = 2y - x$, $\xi y = y - 3x^2$ imply that the algebra $A = \Bbbk[x, y]$ is

$\mathfrak{g}$-stable, hence $\overset{\circ}{X} = \operatorname{Spec} A \subset \mathbb{X}_G$. Put $X = G\overset{\circ}{X}$. Then G acts on X with 2 orbits $X \setminus \{x_0\}$ and $\{x_0\}$ (an ordinary double point), cf. Example 4.1.

In the study of local geometry of a variety X in a neighborhood of its (irreducible) subvariety Y, we may replace X by any open subset intersecting Y, thus arriving to the notion of a *germ* of a variety in (a neighborhood of) its subvariety. If X is a model of K, then a germ of X in Y is essentially the local ring $\mathcal{O}_{X,Y}$ or the respective schematic point of $\mathbb{X}$.

Definition 12.1. A *G-germ* (of K) is a G-fixed schematic point of $\mathbb{X}_G$ (or a G-stable irreducible subvariety of $\mathbb{X}_G$). The set of all G-germs is denoted by ${}_G\mathbb{X}$; a similar notation ${}_GX$ is used for an arbitrary open subset $X \subseteq \mathbb{X}_G$. A G-model X such that a given G-germ is contained in ${}_GX$ (i.e., intersects X in a G-stable subvariety Y) is called a *geometric realization* or a *model* of the G-germ.

Every G-germ admits a geometric realization: just take its affine neighborhood $\overset{\circ}{X} \subseteq \mathbb{X}_G$ and put $X = G\overset{\circ}{X}$. If $X \subseteq \mathbb{X}_G$ is G-stable, then X and ${}_GX$ determine each other. The Zariski topology is induced on ${}_G\mathbb{X}$, with ${}_GX$ the open subsets. It is straightforward [LV, 6.1] that X is Noetherian iff ${}_GX$ is Noetherian.

Remark 12.2. In characteristic zero, a germ of X in Y is a G-germ iff its local ring $\mathcal{O}_{X,Y}$ is G- and $\mathfrak{g}$-stable (cf. Proposition 12.2).

Germs of normal G-models in G-stable prime divisors play an important role in the Luna–Vust theory. They are identified with the respective G-invariant valuations of K.

Definition 12.2. A discrete $\mathbb{Q}$-valued valuation of $K/\Bbbk$ is called *geometric* if it is a multiple of the valuation corresponding to a prime divisor in a normal model of K. A *G-valuation* is a G-invariant geometric valuation. Denote by $\mathcal{V} = \mathcal{V}(K)$ the set of all G-valuations of $K/\Bbbk$; its structure is considered in Chapter 4, see also §13–§16 below.

The *support* $\mathcal{S}_Y$ of a G-germ Y is the set of $v \in \mathcal{V}$ such that the valuation ring $\mathcal{O}_v$ dominates $\mathcal{O}_{\mathbb{X},Y}$ (i.e., v has center Y in any geometric realization).

The support of a G-germ is non-empty: e.g., if $X \supseteq Y$ is an arbitrary geometric realization of the G-germ, and v is the valuation corresponding to a component of the exceptional divisor in the normalized blow-up of X along Y, then $v \in \mathcal{S}_Y$.

Here is a version of the valuative criterion of separation.

Theorem 12.2. *A G-stable open subset $X \subseteq \mathbb{X}_G$ is separated iff the supports of all its G-germs are disjoint.*

Proof. The closure $\overline{\Delta_X}$ of the diagonal $\Delta_X \subseteq X \times X$ is a G-model of K. The projections of $X \times X$ to the factors induce birational regular G-maps $\pi_i : \overline{\Delta_X} \to X$ $(i = 1, 2)$. If X is not separated and $Y \subseteq \overline{\Delta_X} \setminus \Delta_X$ is a G-orbit, then the orbits $Y_i = \pi_i(Y)$ are distinct for $i = 1, 2$. But $\mathcal{S}_{Y_1} \cap \mathcal{S}_{Y_2} \supseteq \mathcal{S}_Y \neq \emptyset$, a contradiction.

The converse implication follows from the valuative criterion of separation. □

Assume that $K' \subseteq K$ is a subfield containing $\Bbbk$. We have a natural dominant rational map $\phi : \mathbb{X} \dashrightarrow \mathbb{X}' = \mathbb{X}(K')$. If $X \subseteq \mathbb{X}$, $X' \subseteq \mathbb{X}'$ are models of K, K', then $\phi : X \to X'$ is regular iff for any $x \in X$ there exists an $x' \in X'$ such that $\mathcal{O}_x$ dominates $\mathcal{O}_{x'}$. This x' is necessarily unique (because X' is separated), and $x' = \phi(x)$.

Now assume that K' is a G-subfield of K. Suppose that X and X' are G-models of K and K'.

Proposition 12.3. *The natural rational map $\phi : X \to X'$ is regular iff for any G-germ $Y \in {}_G X$ there exists a (necessarily unique) G-germ $Y' \in {}_G X'$ such that $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{X',Y'}$.*

Proof. If $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{X',Y'}$, then there exist finitely generated subalgebras $A \supseteq A'$ such that $\mathcal{O}_{X,Y}$ and $\mathcal{O}_{X',Y'}$ are their respective localizations. Localizing A' and A sufficiently, we may assume that $\mathring{X} = \operatorname{Spec} A$ and $\mathring{X}' = \operatorname{Spec} A'$ are open subsets of X and X' intersecting Y and Y', respectively. The regular map $\mathring{X} \to \mathring{X}'$ extends to the regular map $G\mathring{X} \to G\mathring{X}'$. Since $Y \subseteq X$ is arbitrary, these maps paste together in the regular map $X \to X'$.

The converse implication is obvious: just put $Y' = \overline{\phi(Y)}$. □

The restriction of a G-valuation of K to K' is a G-valuation, and any G-valuation of K' can be extended to a G-valuation of K (Corollary 19.1). Thus the restriction map $\phi_* : \mathcal{V}(K) \to \mathcal{V}(K')$ is well defined and surjective. If $\phi : X \to X'$ is a regular map, then $\phi_*(\mathcal{S}_Y) \subseteq \mathcal{S}_{Y'}$ for any G-germ $Y \subseteq X$ and $Y' = \overline{\phi(Y)}$.

Here is a version of the valuative criterion of properness.

Theorem 12.3 ([LV, 6.4]). *A morphism $\phi : X \to X'$ is proper iff*

$$\bigcup_{Y \subseteq X} \mathcal{S}_Y = \phi_*^{-1}\left(\bigcup_{Y' \subseteq X'} \mathcal{S}_{Y'}\right)$$

Corollary 12.1. *X is complete iff $\bigcup_{Y \subseteq X} \mathcal{S}_Y = \mathcal{V}$ (i.e., each G-valuation has center on X).*

13 B-charts

From now on, G is assumed to be reductive and all G-models to be normal, i.e., to lie in $\mathbb{X}_G^{\mathrm{norm}}$. We have seen in §12 that a G-model X is given by a Noetherian set ${}_G X$ of G-germs whose supports are disjoint. The Noether property means that X is covered by G-spans of finitely many "simple", e.g., affine, open subsets $\mathring{X} \subseteq X$. An important class of such "local charts" is introduced in

Definition 13.1. A *B-chart* of X is a B-stable affine open subset of X. Generally, a *B-chart* is a B-stable affine open subset $\mathring{X} \subset \mathbb{X}_G^{\mathrm{norm}}$.

It follows from the local structure theorem (§4) that any G-germ admits a B-chart $\mathring{X} \subset X$ intersecting Y. Therefore X is covered by finitely many B-charts and their translates. Thus it is important to obtain a compact description for B-charts. We describe their coordinate algebras in terms of their B-stable divisors.

Definition 13.2. Denote by $\mathcal{D} = \mathcal{D}(K)$ the set of prime divisors on X that are not G-stable. The valuation corresponding to a divisor $D \in \mathcal{D}$ is denoted by v_D. Prime divisors that are B-stable but not G-stable, i.e., elements of $\mathcal{D}^B$, are called *B-divisors*.

Let $K_B \subseteq K$ be the subalgebra of rational functions with B-stable divisor of poles on X.

Remark 13.1. The sets $\mathcal{D}$, $\mathcal{D}^B$ and K_B do not depend on the choice of a G-model X. Indeed, a non-G-stable prime divisor on $\mathbb{X}_G^{\mathrm{norm}}$ intersects any G-model $X \subseteq \mathbb{X}_G^{\mathrm{norm}}$, and K_B consists of rational functions defined on $\mathbb{X}_G^{\mathrm{norm}}$ everywhere outside a non-G-stable divisor.

Since $K_B \supseteq \Bbbk[\mathring{X}]$ for any B-chart $\mathring{X}$, it follows that $\operatorname{Quot} K_B = K$.

B-divisors are also called *colors*, and the pair $(\mathcal{V}, \mathcal{D}^B)$ is said to be the *colored equipment* (of K). It is in terms of colored equipment that B-charts, G-germs and G-models are described, as we shall see below.

Remark 13.2. In the generically transitive case, $\mathcal{D}^B$ may be computed as follows. Each $D \in \mathcal{D}$ determines a G-line bundle $\mathcal{L}(\chi) = \mathcal{L}(\Bbbk_\chi)$ over $\mathcal{O} = G/H$ and a section $\eta \in \mathrm{H}^0(\mathcal{O}, \mathcal{L}(\chi)) = \Bbbk[G]^{(H)}_{-\chi}$ defined uniquely up to multiplication by an invertible function on $\mathcal{O}$, i.e., by a scalar multiple of a character of G. The section η may be regarded as an equation for the preimage of D under the orbit map $G \to \mathcal{O}$. Since D is prime, η is indecomposable in the multiplicative semigroup $\Bbbk[G]^{(H)}/\Bbbk^\times \mathfrak{X}(G)$.

Each $f \in \Bbbk(\mathcal{O})$ decomposes as $f = \eta^d \eta_1^{d_1} \dots \eta_s^{d_s}$, where $d, d_1, \dots, d_s \in \mathbb{Z}$ and $\eta, \eta_1, \dots, \eta_s \in \Bbbk[G]^{(H)}$ are pairwise coprime. Then $v_D(f) = d$.

Finally, D is a B-divisor iff η is a $(B \times H)$-eigenfunction. Therefore $\mathcal{D}^B$ is in bijection with the set of generators of $\Bbbk[G]^{(B\times H)}/\Bbbk^\times\mathfrak{X}(G)$.

The "dual" object is the multiplicative group $K^{(B)}$ of rational B-eigenfunctions. There is an exact sequence

$$1 \longrightarrow (K^B)^\times \longrightarrow K^{(B)} \longrightarrow \Lambda \longrightarrow 0 \tag{13.1}$$

where $\Lambda = \Lambda(K)$ is the weight lattice (of any G-model) of K.

In the sequel, we frequenlty use Knop's approximation Lemma 19.2, which is crucial for reducing various questions to B-eigenfunctions. In particular, it implies that G-valuations are determined uniquely by their restriction to $K^{(B)}$ (Corollary 19.3).

Let $\mathring{X}$ be a B-chart. Then $\mathcal{A} = \Bbbk[\mathring{X}]$ is an integrally closed finitely generated algebra, in particular, it is a Krull ring. Therefore

$$\mathcal{A} = \bigcap \mathcal{O}_{\mathring{X},D} \text{ (over all prime divisors } D \subset \mathring{X}) \ = \bigcap_{w\in\mathcal{W}} \mathcal{O}_w \cap \bigcap_{D\in\widetilde{\mathcal{R}}} \mathcal{O}_{v_D},$$

where $\mathcal{O}_v$ is the valuation ring of v, $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{R} \subseteq \mathcal{D}^B$, $\widetilde{\mathcal{R}} = \mathcal{R} \sqcup (\mathcal{D} \setminus \mathcal{D}^B)$. Here the G-valuations $w \in \mathcal{W}$ are determined up to a rational multiple, and we shall ignore this indeterminacy, thus passing to a "projectivization" of $\mathcal{V}$. In particular, we may assume that the group of values of every $w \in \mathcal{W}$ is exactly $\mathbb{Z} \subset \mathbb{Q}$.

The pair $(\mathcal{W}, \mathcal{R})$ is said to be the "colored data" of $\mathring{X}$. A B-chart is uniquely determined by its colored data. Taking another B-chart changes $\mathcal{W}$ and $\mathcal{R}$ by finitely many elements. Hence all possible $\mathcal{W} \sqcup \mathcal{R}$ lie in a certain distinguished class **CD** of equivalent subsets of $\mathcal{V} \sqcup \mathcal{D}^B$ w.r.t. the equivalence relation "differ by finitely many elements".

Conversely, if $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{R} \subseteq \mathcal{D}^B$, and $\mathcal{W} \sqcup \mathcal{R} \in \mathbf{CD}$, then

$$\mathcal{A} = \mathcal{A}(\mathcal{W}, \mathcal{R}) = \bigcap_{w\in\mathcal{W}} \mathcal{O}_w \cap \bigcap_{D\in\widetilde{\mathcal{R}}} \mathcal{O}_{v_D}$$

is a Krull ring. Indeed, for $\forall f \in K$ almost all valuations from $\mathcal{W} \sqcup \mathcal{R}$ (i.e., all but finitely many) vanish on f, since it is true for colored data of B-charts, hence for any subset in the class **CD**.

Example 13.1. $K_B = \mathcal{A}(\emptyset, \emptyset)$

Remark 13.3. Here and below, we identify prime divisors and respective valuations. Thus, for $\mathcal{V}_0 \subseteq \mathcal{W} \sqcup \mathcal{R}$, we write $\langle \mathcal{V}_0, f \rangle \geq 0$ iff $v(f) \geq 0$ for $\forall v \in \mathcal{V}_0$ ($v = v_D$ for $D \in \mathcal{V}_0 \cap \mathcal{R}$), and so on.

Proposition 13.1. (1) *All valuations from* $\widetilde{\mathcal{R}}$ *are essential for* $\mathcal{A}$.

(2) *A valuation* $w \in \mathcal{W}$ *is essential for* $\mathcal{A}$ *iff*

$$\text{(W)} \qquad \exists f \in K^{(B)} : \ \langle \mathcal{W} \sqcup \mathcal{R} \setminus \{w\}, f \rangle \geq 0, \ w(f) < 0$$

Proof. (1) Let X be a smooth G-model of K. Consider the G-line bundle $\mathcal{L} = \mathcal{O}_X(D)$, where $D \in \widetilde{\mathcal{R}}$, and let $\eta \in \mathrm{H}^0(X, \mathcal{L})$ be a section with $\operatorname{div} \eta = D$. Put $f = g\eta/\eta$, where $g \in G$, $gD \neq D$. Then $v_D(f) = -1$, $\langle \widetilde{\mathcal{R}} \setminus \{D\}, f \rangle \geq 0$, and $\langle \mathcal{W}, f \rangle = 0$ by Corollary 19.2. Thus v_D is essential for $\mathcal{A}$.

(2) Assume $w \in \mathcal{W}$ is essential for $\mathcal{A}$; then $\exists f \in K : \ \langle \mathcal{W} \sqcup \widetilde{\mathcal{R}} \setminus \{w\}, f \rangle \geq 0$, $w(f) < 0$. Applying Lemma 19.2, we replace f by a B-eigenfunction and obtain (W). The converse implication is obvious. $\square$

Theorem 13.1. (1) $\operatorname{Quot} \mathcal{A} = K$ *iff*

$$\text{(C)} \quad \forall \mathcal{V}_0 \subseteq \mathcal{W} \sqcup \mathcal{R}, \ \mathcal{V}_0 \text{ finite}, \ \exists f \in K^{(B)} : \ \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0, \ \langle \mathcal{V}_0, f \rangle > 0$$

(2) $\mathcal{A}$ *is finitely generated iff*

$$\text{(F)} \qquad \mathcal{A}^U = \Bbbk\big[\, f \in K^{(B)} \ \big| \ \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0 \,\big] \text{ is finitely generated}$$

(3) *Under the equivalent conditions of* (1)–(2), $\mathring{X} = \operatorname{Spec} \mathcal{A}$ *is a* B*-chart.*

Proof. (1) Assume $\operatorname{Quot} \mathcal{A} = K$. We may assume $\mathcal{V}_0 = \{v\}$; then $\exists f \in \mathcal{A} \subseteq K_B : \ v(f) > 0$. Applying Lemma 19.2, we replace f by an element of $\mathcal{A}^{(B)}$ and obtain (C). Conversely, assume (C) is true and $h \in K_B$. Then we take $\mathcal{V}_0 = \{v \in \mathcal{W} \sqcup \mathcal{R} \mid v(h) \neq 0\}$ and, multiplying h by f^N for $N \gg 0$ (killing the poles), we fall into $\mathcal{A}$. Hence $K_B \subseteq \operatorname{Quot} \mathcal{A}$, and this yields $K = \operatorname{Quot} \mathcal{A}$.

(2) ($\operatorname{char} \Bbbk = 0$) Let X be a smooth G-model of K. Take an effective divisor on X with support $\mathcal{D}^B \setminus \mathcal{R}$ and consider the corresponding section $\eta \in \mathrm{H}^0(X, \mathcal{L})^{(B)}$ of the G-line bundle $\mathcal{L}$. Consider an algebra $R = \bigoplus_{n \geq 0} R_n$, where $R_n = \{\sigma \in \mathrm{H}^0(X, \mathcal{L}^n) \mid \sigma/\eta^n \in \mathcal{A}\}$. Then $\mathcal{A} = \bigcup \eta^{-n} R_n \subseteq \operatorname{Quot} R$. Since every G-valuation of K can be extended to a G-valuation of $\operatorname{Quot} R$ (Corollary 19.1), we see that $R_n = \{\sigma \mid \forall w \in \mathcal{W} : \ w(\sigma) \geq nw(\eta)\}$ is G-stable.

Though $\mathcal{A}$ is not a G-algebra, it is very close to a G-algebra, so that we may apply Lemma 13.1 below and reduce the problem of finite generation to G-algebras.

In the notation of Lemma 13.1, if $\mathcal{A} = \phi(R)$ is finitely generated, then it is easy to construct a finitely generated graded G-subalgebra $S \subseteq R$ such that $\phi(S) = \mathcal{A}$. Hence S^U and also $\mathcal{A}^U = \phi(S^U)$ are finitely generated. Conversely, if $\mathcal{A}^U$ is finitely generated, then we construct a finitely generated graded G-algebra $S = \langle GS^U \rangle$ such that $\phi(S^U) = \mathcal{A}^U$. This yields that $\mathcal{A} = \phi(S)$ is finitely generated.

(3) In characteristic zero, just note that $\mathcal{A}$ is $\mathfrak{g}$-stable, because all $\mathcal{O}_v$ ($v \in \mathcal{W} \sqcup \mathcal{R}$) are. In general, since $\mathcal{A}$ is finitely generated, it follows that $\mathcal{A} = \Bbbk[\eta^{-n}M]$ for a finite-dimensional G-submodule M in some R_n. Let X' be the closure of the image of the natural rational map $X \dashrightarrow \mathbb{P}(M^*)$. Then X' is a G-model and $\mathring{X} = \operatorname{Spec} \mathcal{A} \subseteq X'$. $\square$

Lemma 13.1. *Suppose R is a $\mathbb{Z}_+$-graded G-algebra without zero divisors, $S \subseteq R$ is a G-stable graded subalgebra. Take $\eta \in S_1^{(B)}$ and consider the homomorphism $\phi : R \to \operatorname{Quot} R$, $\phi(\sigma) = \sigma/\eta^n$ for $\forall \sigma \in R_n$. Then $\phi(S)^U = \phi(S^U)$, $\phi(R)^U = \phi(R^U)$, and in characteristic zero $\phi(S) = \phi(R) \iff \phi(S^U) = \phi(R^U)$.*

Proof. Suppose $f = \phi(\sum \sigma_n) \in \phi(S)^U$, $\sigma_n \in S_n$; then $f = \sum \sigma_n/\eta^n = \sum \sigma_n \eta^{n_0-n}/\eta^{n_0}$ (for n_0 sufficiently large) $= \phi(\sum \sigma_n \eta^{n_0-n})$, and $\eta \in S^U$ implies $\sum \sigma_n \eta^{n_0-n} \in S^U$. The same argument shows $\phi(R)^U = \phi(R^U)$.

Now assume $\operatorname{char} \Bbbk = 0$ and $\phi(S) \neq \phi(R)$. The U-module $\phi(R)/\phi(S)$ contains a nonzero U-invariant, which is the image of $f = \sigma/\eta^n$, $\sigma \in R_n$. For $\forall u \in U$ $\exists k \geq 0 :$ $\sigma - u\sigma \in \eta^{-k} S_{n+k}$. Since the U-module generated by σ is finite-dimensional, we may choose k independent of u. Replacing σ by $\sigma\eta^k$, we may assume that σ determines a nonzero element of $(R_n/S_n)^U$. By complete reducibility of G-modules, we may replace σ by an element of R_n^U, without changing σ mod S_n. Then $f \in \phi(R^U) \setminus \phi(S^U)$, and we are done. $\square$

Corollary 13.1. *A pair $(\mathcal{W}, \mathcal{R})$ from* **CD** *is the colored data of a B-chart iff conditions* (C),(F),(W) *are satisfied.*

Note that elements of $\mathcal{D}^B \setminus \mathcal{R}$ are exactly the irreducible components of $X \setminus \mathring{X}$, where $X = G\mathring{X}$. A B-chart is G-stable iff $\mathcal{R} = \mathcal{D}^B$.

Corollary 13.2. *Affine G-models are in bijection with colored data $(\mathcal{W}, \mathcal{D}^B)$ satisfying* (C),(F),(W).

Remark 13.4. In this section, we never use an *apriori* assumption that G-invariant valuations from $\mathcal{W}$ are geometric.

The local structure of B-charts is well understood.

The subgroup $P = N_G(\mathring{X})$ is a parabolic containing B. We have $P = P[\mathcal{D}^B \setminus \mathcal{R}] = \bigcap_{D \in \mathcal{D}^B \setminus \mathcal{R}} P[D]$, where $P[D]$ is the stabilizer of D. In the generically transitive case, if $\eta \in \Bbbk[G]^{(B\times H)}_{\lambda,\chi}$ is an equation of D, then $P[D] = P(\lambda)$ is the parabolic associated with λ.

Let $P = LP_{\mathrm{u}}$ be the Levi decomposition ($L \supseteq T$).

In §17, we prove that the divisor $X \setminus \mathring{X}$ is ample on $X = G\mathring{X}$ (Corollary 17.3). Now Lemma 4.1 implies the following

Proposition 13.2 ([Tim3]). (1) *The action $P_{\mathrm{u}} : \mathring{X}$ is proper and has a geometric quotient $\mathring{X}/P_{\mathrm{u}} = \operatorname{Spec}\Bbbk[\mathring{X}]^{P_{\mathrm{u}}}$.*

(2) *There exists a T-stable (L-stable if $\operatorname{char}\Bbbk = 0$) closed affine subvariety $Z \subseteq \mathring{X}$ such that $\mathring{X} = PZ$ and the natural maps $P_{\mathrm{u}} \times Z \to \mathring{X}$, $Z \to \mathring{X}/P_{\mathrm{u}}$ are finite and surjective.*

(2)′ *In characteristic zero, the P-action on $\mathring{X}$ induces an isomorphism*

$$P_{\mathrm{u}} \times Z = P *_L Z \xrightarrow{\sim} \mathring{X}$$

14 Classification of G-models

We begin with a description of G-germs in terms of colored data.

Consider a G-germ $Y \in {}_G\mathbb{X}^{\mathrm{norm}}$. Let $\mathcal{V}_Y \subseteq \mathcal{V}$, $\mathcal{D}_Y \subseteq \mathcal{D}$ be the subsets corresponding to all B-stable divisors on $\mathbb{X}^{\mathrm{norm}}_G$ containing Y. The pair $(\mathcal{V}_Y, \mathcal{D}^B_Y)$ is said to be the *colored data* of the G-germ.

If the G-germ intersects a B-chart $\mathring{X}$, then $\mathring{Y} = Y \cap \mathring{X}$ is the center of any $v \in \mathcal{S}_Y$, i.e., $v|_{k[\mathring{X}]} \geq 0$, and the ideal $\mathbb{I}(\mathring{Y}) \lhd \Bbbk[\mathring{X}]$ is given by $v > 0$. Conversely, if a G-valuation $v \in \mathcal{V}$ is non-negative on $\Bbbk[\mathring{X}]$, then it determines a G-germ intersecting $\mathring{X}$. If $(\mathcal{W}, \mathcal{R})$ is the colored data of $\mathring{X}$, then $\mathcal{V}_Y \subseteq \mathcal{W}$, $\mathcal{D}^B_Y \subseteq \mathcal{R}$.

Proposition 14.1 ([Kn3, 3.8]). (1) *A G-germ is uniquely determined by its colored data.*

(2) *A G-valuation v is in $\mathcal{S}_Y$ iff*

$$\text{(S)}\qquad \begin{aligned} &\forall f \in K^{(B)} :\ \langle \mathcal{V}_Y \sqcup \mathcal{D}^B_Y, f\rangle \geq 0 \implies v(f) \geq 0 \\ &\textit{and if} > \textit{occurs in the l.h.s., then } v(f) > 0 \end{aligned}$$

Proof. Choose a geometric realization $X \supseteq Y$ and a B-chart $\mathring{X} \subseteq X$ intersecting Y.

(2) Observe that for $f \in K^{(B)}$ we have $f \in \mathcal{O}^{(B)}_{X,Y} \iff \langle \mathcal{V}_Y \sqcup \mathcal{D}^B_Y, f\rangle \geq 0$ and $f \in \mathfrak{m}^{(B)}_{X,Y}$ iff one of these inequalities is strict.

If $\mathcal{O}_v$ dominates $\mathcal{O}_{X,Y}$, then (S) is satisfied. Conversely, if $\exists f \in \mathcal{O}_{X,Y}$ such that $v(f) < 0$, then, applying Lemma 19.2, we replace f by a B-eigenfunction and see that (S) fails. Therefore $\mathcal{O}_v \supseteq \mathcal{O}_{X,Y} \supseteq \Bbbk[\mathring{X}]$ and v has center $Y' \supseteq Y$ on X. If $Y' \neq Y$, then for $\forall v' \in \mathcal{S}_Y$ there is $f \in K$ such that $v'(f) > 0$, $v(f) = 0$. Replacing f by a B-eigenfunction again, we obtain a contradiction with (S). Thus $\mathcal{O}_v$ dominates $\mathcal{O}_{X,Y}$.

(1) Since $\Bbbk[\mathring{X}] \subseteq \mathcal{A} = \mathcal{A}(\mathcal{V}_Y, \mathcal{D}_Y^B) \subseteq \mathcal{O}_{X,Y}$, the local ring $\mathcal{O}_{X,Y}$ is the localization of $\mathcal{A}$ in the ideal $I_Y = \mathcal{A} \cap \mathfrak{m}_{X,Y}$. Take any $v \in \mathcal{S}_Y$; then I_Y is defined in $\mathcal{A}$ by $v > 0$. But $\mathcal{S}_Y$ is determined by $(\mathcal{V}_Y, \mathcal{D}_Y^B)$. □

Now we describe G-germs in a given B-chart $\mathring{X} = \operatorname{Spec} \mathcal{A}$, $\mathcal{A} = \mathcal{A}(\mathcal{W}, \mathcal{R})$.

Theorem 14.1. (1) *$v \in \mathcal{V}$ has a center on $\mathring{X}$ iff*

(V) $$\forall f \in K^{(B)} : \ \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0 \implies v(f) \geq 0$$

(2) *Assume $v \in \mathcal{S}_Y$. A G-valuation $w \in \mathcal{W}$ belongs to $\mathcal{V}_Y$ iff*

(V′) $$\forall f \in K^{(B)} : \ \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0, \ v(f) = 0 \implies w(f) = 0$$

Similarly, $D \in \mathcal{R}$ belongs to $\mathcal{D}_Y^B$ iff

(D′) $$\forall f \in K^{(B)} : \ \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0, v(f) = 0 \implies v_D(f) = 0$$

Proof. (1) v has a center iff $v|_{\mathcal{A}} \geq 0$. This clearly implies (V). On the other hand, if $f \in \mathcal{A}$, $v(f) < 0$, then, applying Lemma 19.2, we replace f by an element of $\mathcal{A}^{(B)}$ see that (V) is false.

(2) Assume $w \in \mathcal{W}$ (or $D \in \mathcal{R}$) belongs to $\mathcal{V}_Y$ (or $\mathcal{D}_Y^B$); then every function $f \in \mathcal{A}$ not vanishing on $\mathring{Y}$ (i.e., $v(f) = 0$) does not vanish on the respective B-stable divisor of $\mathring{X}$ as well (i.e., $w(f) = 0$ or $v_D(f) = 0$). For $f \in \mathcal{A}^{(B)}$, we obtain (V′) (or (D′)).

Conversely, assume $w \notin \mathcal{V}_Y$ (or $D \notin \mathcal{D}_Y^B$); then there exists $f \in \mathcal{A}$ vanishing on the respective B-stable divisor of $\mathring{X}$ (i.e., $w(f) > 0$ or $v_D(f) > 0$) but not on $\mathring{Y}$ (i.e., $v(f) = 0$). Applying Lemma 19.2, we replace f by an element of $\mathcal{A}^{(B)}$ and see that (V′) (or (D′)) is false. □

Summing up, we can construct every G-model in the following way:

(1) Take a finite collection of colored data $(\mathcal{W}_\alpha, \mathcal{R}_\alpha)$ in **CD** satisfying (C),(F). Decrease $\mathcal{W}_\alpha$ if necessary so as to satisfy (W). These colored data determine finitely many B-charts $\mathring{X}_\alpha$.

(2) Compute from $(\mathcal{W}_\alpha, \mathcal{R}_\alpha)$ via conditions (V),(V$'$),(D$'$) the collection of colored data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ of G-germs Y intersecting $\mathring{X}_\alpha$.

(3) Compute the supports $\mathcal{S}_Y$ from $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ using (S).

The G-models $X_\alpha = G\mathring{X}_\alpha$ may be glued together in a G-model X iff the supports $\mathcal{S}_Y$ obtained at Step (3) are disjoint (Theorem 12.2). The collection ${}_GX$ of G-germs is given by Step (2) as the collection of their colored data, which is called the *colored data* of X.

Remark 14.1. We notice in addition that the collection of covering B-charts $\mathring{X}_\alpha$ is of course not uniquely determined. Furthermore, one G-germ may have a lot of different B-charts. For example, in the notation of Theorem 14.1, we may consider a principal open subset $\mathring{X}_f = \{x \mid f(x) \neq 0\}$ in $\mathring{X}$, where $f \in \mathcal{A}^{(B)}$, $v(f) = 0$ (to avoid cutting $\mathring{Y}$ off), i.e., pass from $\mathcal{A}$ to its localization $\mathcal{A}_f$. This corresponds to removing from $(\mathcal{W} \setminus \mathcal{V}_Y) \sqcup (\mathcal{R} \setminus \mathcal{D}_Y^B)$ a finite set $\mathcal{W}_0 \sqcup \mathcal{R}_0$ of those valuations that are positive on f. By (V$'$) and (D$'$), this set may contain any finite number of elements from $(\mathcal{W} \setminus \mathcal{V}_Y) \sqcup (\mathcal{R} \setminus \mathcal{D}_Y^B)$. In particular, if $(\mathcal{W} \setminus \mathcal{V}_Y) \sqcup (\mathcal{R} \setminus \mathcal{D}_Y^B)$ is finite, then there exists a *minimal* B-chart with $\mathcal{W} = \mathcal{V}_Y$, $\mathcal{R} = \mathcal{D}_Y^B$.

Parabolic induction does not change $K^{(B)}$ and $\mathcal{V}$, while $\mathcal{D}^B$ is extended by finitely many colors, whose valuations vanish on K^B, see Proposition 20.4. The G-germs of an induced variety are induced from those of the original variety, and it is easy to prove the following result:

Proposition 14.2. *Parabolic induction does not change the colored data of a G-model.*

15 Case of complexity 0

A practical use of the theory developed in the preceding sections depends on whether the colored equipment of a G-field is accessible for computation and operation or not. It was noted already in [LV] that there is no hope to obtain a transparent classification of G-models from the general description in §14 (maybe except particular examples) if the complexity is > 1. On the other hand, if the complexity is ≤ 1, then an explicit solution to the classification problem is obtained. An appropriate language to operate with the colored equipment is that of convex polyhedral geometry.

We shall write $c(K)$, $r(K)$ for the complexity, resp. rank, of (any G-model of) K. If $c(K) = 0$, then any G-model contains an open B-orbit, hence an open G-orbit $\mathcal{O}$. Homogeneous spaces of complexity zero (=spherical spaces) and their embeddings are studied in details in Chapter 5. Here we classify the embeddings of a given spherical homogeneous space in the framework of the Luna–Vust theory. This classification was first obtained by Luna and Vust [LV, 8.10]. For a modern self-contained exposition, see [Kn2], [Bri14, §3].

Let $\mathcal{O} = G/H$ be a spherical homogeneous space and $K = \Bbbk(\mathcal{O})$. Since $K^B = \Bbbk$, it follows from Corollary 19.3 and the exact sequence (13.1) that G-valuations are identified by restriction to $K^{(B)}$ with $\mathbb{Q}$-linear functionals on the lattice $\Lambda = \Lambda(\mathcal{O})$. The set $\mathcal{V}$ is a convex solid polyhedral cone in $\mathcal{E} = \operatorname{Hom}(\Lambda, \mathbb{Q})$, which is cosimplicial in characteristic zero (see Chapter 4). The set $\mathcal{D}^B$ consists of irreducible components of the complement to the dense B-orbit in $\mathcal{O}$, hence is finite. The restriction to $K^{(B)}$ yields a map $\rho : \mathcal{D}^B \to \mathcal{E}$, which is in general not injective.

Remark 15.1. If $\mathcal{D}^B = \{D_1, \dots, D_s\}$ and $\eta_1, \dots, \eta_s \in \Bbbk[G]^{(B\times H)}$ are the respective indecomposable elements of biweights $(\lambda_1, \chi_1), \dots, (\lambda_s, \chi_s)$, then (λ_i, χ_i) are linearly independent. (Otherwise, there is a linear dependence $\sum d_i(\lambda_i, \chi_i) = 0$, and $f = \eta_1^{d_1} \dots \eta_s^{d_s}$ is a non-constant B-invariant rational function on $\mathcal{O}$.) If $f = \eta_1^{d_1} \dots \eta_s^{d_s} \in K_\lambda^{(B)}$, then $\sum d_i\lambda_i = \lambda$, $\sum d_i\chi_i = 0$, and $\langle \rho(D_i), f\rangle = v_{D_i}(f) = d_i$.

Definition 15.1. The space $\mathcal{E}$ equipped with the cone $\mathcal{V} \subseteq \mathcal{E}$ and with the map $\rho : \mathcal{D}^B \to \mathcal{E}$ is the *colored space* (of $\mathcal{O}$).

Now we consider the structure of colored data and reorganize them in a more convenient way. The proofs are straightforward, as soon as we interpret B-eigenfunctions as linear functionals on $\mathcal{E}$.

The class **CD** consists of finite sets.

Condition (C) means that $\mathcal{W} \sqcup \rho(\mathcal{R})$ generates a strictly convex cone $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ in $\mathcal{E}$ and $\rho(\mathcal{R}) \not\ni 0$.

Condition (W) means that the elements of $\mathcal{W}$ are exactly the generators of those edges of $\mathcal{C}$ that do not intersect $\rho(\mathcal{R})$.

Condition (F) holds automatically: $\mathcal{A}^U$ is the semigroup algebra of $\Lambda \cap \mathcal{C}^\vee$, where $\mathcal{C}^\vee = \{\lambda \in \mathcal{E}^* \mid \langle \mathcal{C}, \lambda\rangle \geq 0\}$ is the dual cone to $\mathcal{C}$. Since $\mathcal{C}^\vee$ is finitely generated, the semigroup $\Lambda \cap \mathcal{C}^\vee$ is finitely generated by Gordan's lemma [Dan, 1.3].

Condition (V) means that $v \in \mathcal{C}$.

Conditions (V′) and (D′) say that $\mathcal{V}_Y$ and $\mathcal{D}_Y^B$ consist of those elements of $\mathcal{W} \sqcup \mathcal{R}$ which lie in the face $\mathcal{C}_Y = \mathcal{C}(\mathcal{V}_Y, \mathcal{D}_Y^B) \subseteq \mathcal{C}$ containing v in its (relative) interior.

Condition (S) means that $v \in \mathcal{V} \cap \operatorname{int} \mathcal{C}_Y$.

Observe that every G-germ Y has a minimal B-chart $\mathring{X}_Y$ with $\mathcal{C} = \mathcal{C}_Y$, $\mathcal{R} = \mathcal{D}_Y^B$ (Remark 14.1). It suffices to consider only such charts.

Definition 15.2. A *colored cone* in $\mathcal{E}$ is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{R} \subseteq \mathcal{D}^B$, $\rho(\mathcal{R}) \not\ni 0$, and $\mathcal{C}$ is a strictly convex cone generated by $\rho(\mathcal{R})$ and finitely many vectors from $\mathcal{V}$.

A colored cone $(\mathcal{C}, \mathcal{R})$ is *supported* if $\operatorname{int}\mathcal{C} \cap \mathcal{V} \neq \emptyset$.

A *face* of $(\mathcal{C}, \mathcal{R})$ is a colored cone $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R} \cap \rho^{-1}(\mathcal{C}')$.

A *colored fan* is a finite set of supported colored cones which is closed under passing to a supported face and such that different cones intersect in faces inside $\mathcal{V}$.

Theorem 15.1. (1) *B-charts are in bijection with colored cones in $\mathcal{E}$.*

(2) *G-germs are in bijection with supported colored cones.*

(3) *G-models are in bijection with colored fans.*

(4) *Every G-model X contains finitely many G-orbits. If $Y_1, Y_2 \subseteq X$ are two G-orbits, then $Y_1 \preceq Y_2$ iff $(\mathcal{C}_{Y_2}, \mathcal{D}^B_{Y_2})$ is a face of $(\mathcal{C}_{Y_1}, \mathcal{D}^B_{Y_1})$.*

Corollary 15.1. *Affine G-models are in bijection with colored cones of the form $(\mathcal{C}, \mathcal{D}^B)$.*

Corollary 15.2. *$\mathcal{O}$ is (quasi)affine iff $\rho(\mathcal{D}^B)$ can be separated from $\mathcal{V}$ by a hyperplane (resp. does not contain 0 and spans a strictly convex cone).*

Corollary 15.3. *A G-model is complete iff its colored fan covers all $\mathcal{V}$.*

Example 15.1 (Toric varieties). Suppose $G = B = T$ is a torus. We may assume $H = \{e\}$. Here $\mathcal{V} = \mathcal{E}$ (see §20) and there are no colors. Hence embeddings of T are in bijection with fans in $\mathcal{E}$, where a *fan* is a finite set of strictly convex polyhedral cones which is closed under passing to a face and such that different cones intersect in faces. Every embedding X of T contains finitely many T-orbits, which correspond to cones in the fan. For any orbit $Y \subseteq X$, the union X_Y of all orbits containing Y in their closure is the minimal T-chart of Y determined by $\mathcal{C}_Y$. We have $\Bbbk[X_Y] = \Bbbk[\mathfrak{X}(T) \cap \mathcal{C}_Y^\vee] \subseteq \Bbbk[T]$. X is complete iff its fan is the subdivision of the whole $\mathcal{E}$.

Equivariant embeddings of a torus are called *toric varieties*. Due to their nice combinatorial description, toric varieties are a good testing site for various concepts and problems of algebraic geometry. Their theory is well developed, see [Dan], [Oda], [Ful2].

Other examples of spherical varieties are considered in Chapter 5.

Now we discuss the functoriality of colored data.

Let $\overline{H} \subseteq G$ be an overgroup of H. Denote by $(\overline{\mathcal{E}}, \overline{\mathcal{V}}, \overline{\mathcal{D}}^B, \overline{\rho})$ the colored space of $\overline{\mathcal{O}} = G/\overline{H}$. The canonical map $\phi : \mathcal{O} \to \overline{\mathcal{O}}$ induces an embedding $\phi^* : \overline{K} \hookrightarrow K$ and a linear map $\phi_* : \mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}$. We have $\phi_*(\mathcal{V}) = \overline{\mathcal{V}}$. If $\mathcal{D}^B_\phi$ is the set of B-divisors in $\mathcal{O}$ mapping dominantly to $\overline{\mathcal{O}}$, then there is a canonical surjection $\phi_* : \mathcal{D}^B \setminus \mathcal{D}^B_\phi \twoheadrightarrow \overline{\mathcal{D}}^B$ such that $\overline{\rho} = \rho\phi_*$.

Definition 15.3. A colored cone $(\mathcal{C}, \mathcal{R})$ in $\mathcal{E}$ *dominates* a colored cone $(\overline{\mathcal{C}}, \overline{\mathcal{R}})$ in $\overline{\mathcal{E}}$ if $\phi_*(\operatorname{int} \mathcal{C}) \subseteq \operatorname{int} \overline{\mathcal{C}}$ and $\phi_*(\mathcal{R} \setminus \mathcal{D}^B_\phi) \subseteq \overline{\mathcal{R}}$. A colored fan $\mathcal{F}$ in $\mathcal{E}$ *dominates* a colored fan $\overline{\mathcal{F}}$ in $\overline{\mathcal{E}}$ if each cone from $\mathcal{F}$ dominates a cone from $\overline{\mathcal{F}}$.

The *support* of $\mathcal{F}$ is $\operatorname{Supp} \mathcal{F} = \bigcup_{(\mathcal{C},\mathcal{R}) \in \mathcal{F}} \mathcal{C} \cap \mathcal{V}$. (Observe that $\{\mathcal{C} \cap \mathcal{V}\}$ is a polyhedral subdivision of $\operatorname{Supp} \mathcal{F}$.)

The next theorem is deduced from the results of §12.

Theorem 15.2 ([Kn2, 4.1–4.2]). *Let $X, \overline{X}$ be the embeddings of $\mathcal{O}, \overline{\mathcal{O}}$ determined by fans $\mathcal{F}, \overline{\mathcal{F}}$. Then ϕ extends to a morphism $X \to \overline{X}$ iff $\mathcal{F}$ dominates $\overline{\mathcal{F}}$. Furthermore, $\phi : X \to \overline{X}$ is proper iff* $\operatorname{Supp} \mathcal{F} = \phi_*^{-1}(\operatorname{Supp} \overline{\mathcal{F}}) \cap \mathcal{V}$.

Proof. If $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{\overline{X},\overline{Y}}$, then clearly $\phi_*(\mathcal{D}^B_Y \setminus \mathcal{D}^B_\phi) \subseteq \overline{\mathcal{D}}^B_{\overline{Y}}$ and $\phi_*(\mathcal{S}_Y) \subseteq \mathcal{S}_{\overline{Y}}$, or equivalently, $(\mathcal{C}_Y, \mathcal{D}^B_Y)$ dominates $(\mathcal{C}_{\overline{Y}}, \overline{\mathcal{D}}^B_{\overline{Y}})$. Conversely, if $(\mathcal{C}_Y, \mathcal{D}^B_Y)$ dominates $(\mathcal{C}_{\overline{Y}}, \overline{\mathcal{D}}^B_{\overline{Y}})$ for some $Y \subseteq X$, $\overline{Y} \subseteq \overline{X}$, then $\mathcal{A} = \mathcal{A}(\mathcal{V}_Y, \mathcal{D}^B_Y) \supseteq \overline{\mathcal{A}} = \mathcal{A}(\overline{\mathcal{V}}_{\overline{Y}}, \overline{\mathcal{D}}^B_{\overline{Y}})$ and $I_{\overline{Y}} = I_Y \cap \overline{\mathcal{A}}$, where $I_Y = \mathcal{A} \cap \mathfrak{m}_{X,Y}$ is defined in $\mathcal{A}$ by $v > 0$, $\forall v \in \mathcal{S}_Y$. Hence $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{\overline{X},\overline{Y}}$.

A criterion of properness is a reformulation of Theorem 12.3. □

Overgroups of H can be classified in terms of the colored space.

Definition 15.4. A *colored subspace* of $\mathcal{E}$ is a pair $(\mathcal{E}_0, \mathcal{R}_0)$, where $\mathcal{R}_0 \subseteq \mathcal{D}^B$ and $\mathcal{E}_0 \subseteq \mathcal{E}$ is a subspace generated as a cone by $\rho(\mathcal{R}_0)$ and some vectors from $\mathcal{V}$.

For example, $(\mathcal{E}_\phi, \mathcal{D}^B_\phi)$ is a colored subspace, where $\mathcal{E}_\phi = \operatorname{Ker} \phi_*$ [Bri14, 3.4].

Theorem 15.3 ([Kn2, 4.4]). *The correspondence $\overline{H} \mapsto (\mathcal{E}_\phi, \mathcal{D}^B_\phi)$ is an order-preserving bijection between overgroups of H with $\overline{H}/H$ connected and colored subspaces of $\mathcal{E}$.*

Example 15.2. If $H = B$, then $\mathcal{E} = \mathcal{V} = 0$ and $\mathcal{D}^B$ is the set of Schubert divisors on G/B, which are in bijection with the simple roots. Hence an overgroup of B is determined by a subset of simple roots—a well-known classification of parabolics.

More generally, parabolic overgroups $P \supseteq H$ are in bijection with subsets $\mathcal{R}_0 \subseteq \mathcal{D}^B$ such that $\rho(\mathcal{R}_0) \cup \mathcal{V}$ generates $\mathcal{E}$ as a cone. Indeed, P is parabolic $\iff r(G/P) = 0 \iff \mathcal{E}(G/P) = 0 \iff \mathcal{E}_\phi = \mathcal{E}$.

One may consider *generalized* colored fans, dropping the assumption that colored cones are strictly convex and their colors do not map to 0. (These are exactly the preimages of usual colored fans in quotients by colored subspaces.) Then there is a bijection between dominant separable G-maps $\mathcal{O} \to X$ to normal G-varieties and generalized colored fans [Kn2, 4.5].

Now we derive some properties of G-orbits (due to Brion) and local geometry of a spherical embedding.

Proposition 15.1. *Suppose X is an embedding of $\mathcal{O}$ and $Y \subseteq X$ an irreducible G-subvariety. Then $c(Y) = 0$, $r(Y) = r(X) - \dim \mathcal{C}_Y = \operatorname{codim} \mathcal{C}_Y$, and $\Lambda(Y) = \mathcal{C}_Y^{\perp} \cap \Lambda(X)$ up to p-torsion.*

Proof. By Theorem 5.1, Y is spherical. By Lemma 5.1, for $\forall f \in \Bbbk(Y)^{(B)}$ there is $\widetilde{f} \in \Bbbk(X)^{(B)}$ such that $\widetilde{f}|_Y = f^q$, where q is a sufficiently big power of p. It remains to note that $\widetilde{f}$ is defined and nonzero on Y iff $\langle \mathcal{V}_Y \cap \mathcal{D}_Y^B, \widetilde{f} \rangle = 0$, i.e., the B-eigenweight of $\widetilde{f}$ lies in $\mathcal{C}_Y^{\perp}$. ☐

The local structure of B-charts is given by Proposition 13.2. For a minimal B-chart $\mathring{X}_Y$, the description can be refined.

Let $P = P[\mathcal{D}^B \backslash \mathcal{D}_Y^B]$ and $P = LP_{\mathrm{u}}$ be its Levi decomposition. Theorem 4.1 yields

Theorem 15.4. *There is a T-stable (L-stable if $\operatorname{char} \Bbbk = 0$) closed subvariety $Z \subseteq \mathring{X}_Y$ such that:*

1. *The natural maps $P_{\mathrm{u}} \times Z \to \mathring{X}_Y$ and $Z \to \mathring{X}/P_{\mathrm{u}}$ are finite and surjective.*
2. *Put $\mathring{Y} = Y \cap \mathring{X}_Y$. Then $\mathring{Y}/P_{\mathrm{u}} \simeq L/L_0$, where $L_0 \supseteq L'$.*
3. *In characteristic zero, $\mathring{X}_Y \simeq P *_L Z = P_{\mathrm{u}} \times Z$, $Y \cap Z \simeq L/L_0$, and there exists an L_0-stable subvariety $Z_0 \subseteq Z$ transversal to $Y \cap Z$ at a fixed point y_0 and such that $Z = L *_{L_0} Z_0$. The varieties Z and Z_0 are affine and spherical, and $r(Z) = r(\mathcal{O})$, $r(Z_0) = \dim \mathcal{C}_Y$.*

The isomorphism $Z \simeq L *_{L_0} Z_0$ stems, e.g., from Luna's slice theorem, or is proved directly: since L/L_0 is a torus, $\Bbbk[Y \cap Z]$ is pulled back to an L-stable subalgebra of $\Bbbk[Z]$, whence an equivariant retraction $Z \to Y \cap Z = L/L_0$ with a fiber Z_0.

Corollary 15.4. $\dim Y = \operatorname{codim} \mathcal{C}_Y + \dim P_{\mathrm{u}}$

Remark 15.2. In characteristic zero, there is a bijection $f \leftrightarrow f|_Z$ between B-eigenfunctions on X and Z, which preserve the order along a divisor. Hence $\mathcal{C}_Y = \mathcal{C}_{Y \cap Z}$ and $\mathcal{D}_Y^B \supseteq \mathcal{D}_{Y \cap Z}^B$. However some colors on X may become L-stable divisors on Z ("a discoloration").

Theorem 15.5. *In characteristic zero, all G-subvarieties $Y \subseteq X$ are normal and have rational singularities (in particular, they are Cohen–Macaulay).*

Proof. By the local structure theorem, we may assume that X is affine. Then $X/\!\!/U$ is an affine toric variety and $Y/\!\!/U$ its T-stable subvariety. It is well known [Ful2, 3.1, 3.5] that $Y/\!\!/U$ is a normal toric variety and has rational (in fact, Abelian quotient) singularities. By Theorem A2.1(3), the same is true for Y. □

A spherical embedding defined by a fan whose colored cones have no colors is called *toroidal*. In particular, toric varieties are toroidal. Conversely, the local structure theorem readily implies that toroidal varieties are "locally toric" (Theorem 29.1). This is the reason for most nice geometric properties which distinguish toroidal varieties among arbitrary spherical varieties. Toroidal varieties are discussed in §29.

16 Case of complexity 1

Here we obtain the classification of G-models in the case $c(K) = 1$. This case splits in two subcases:

(1) *Generically transitive case*: $d_G(K) = 0$. Here any G-model contains a dense G-orbit $\mathcal{O}$ of complexity 1.

(2) *One-parametric case*: $d_G(K) = 1$. Here generic G-orbits in any G-model are spherical and form a one-parameter family. (In fact, all G-orbits are spherical by Proposition 5.4.)

We are interested mainly in the generically transitive case. However the one-parameter case might be of interest, e.g., in studying deformations of spherical homogeneous spaces and their embeddings. There are differences between these two cases (e.g., in the description of B-divisors), but the description of G-models is uniform [Tim2].

First we describe the colored environment.

Since $c(K) = 1$, there is a (unique) non-singular projective curve C such that $K^B = \Bbbk(C)$. Generic B-orbits on a G-model of K are parametrized by an open subset of C. In the generically transitive case, $K \subseteq \Bbbk(G)$ is unirational, because G is a rational variety, which is proved by considering the "big cell" in G. Whence $C = \mathbb{P}^1$ by the Lüroth theorem.

Definition 16.1. For any $x \in C$ consider the half-space $\mathcal{E}_{x,+} = \mathbb{Q}_+ \times \mathcal{E}$. The *hyperspace* (of K) is the union $\breve{\mathcal{E}}$ of all $\mathcal{E}_{x,+}$ glued together along their common boundary hyperplane $\mathcal{E}$, called the *center* of $\breve{\mathcal{E}}$. More formally,

$$\breve{\mathcal{E}} = \bigsqcup_{x\in C} \{x\} \times \mathcal{E}_{x,+} \Big/ \sim$$

where $(x,h,\ell) \sim (x',h',\ell')$ iff $x = x',\ h = h',\ \ell = \ell'$ or $h = h' = 0,\ \ell = \ell'$.

Since Λ is a free Abelian group, the exact sequence (13.1) splits. Fix a splitting $\mathbf{f} : \Lambda \to K^{(B)}$, $\lambda \mapsto \mathbf{f}_\lambda$.

If v is a geometric valuation of K, then $v|_{K^{(B)}}$ is determined by a triple (x,h,ℓ), where $x \in C$, $h \in \mathbb{Q}_+$ satisfy $v|_{K^B} = hv_x$ and $\ell = v|_{\mathbf{f}(\Lambda)} \in \mathcal{E} = \operatorname{Hom}(\Lambda,\mathbb{Q})$. Therefore $v|_{K^B} \in \breve{\mathcal{E}}$. Thus $\mathcal{V}$ is embedded in $\breve{\mathcal{E}}$, and we have a map $\rho : \mathcal{D}^B \to \breve{\mathcal{E}}$ (restriction to $K^{(B)}$). We say that $(\breve{\mathcal{E}},\mathcal{V},\mathcal{D}^B,\rho)$ is the *colored hyperspace*. The valuation v and the respective divisor are called *central* if $v|_{K^{(B)}} \in \mathcal{E}$.

By Theorems 20.1, 21.1, and Corollary 22.2, $\mathcal{V}_x = \mathcal{V} \cap \mathcal{E}_{x,+}$ is a convex solid polyhedral cone in $\mathcal{E}_{x,+}$, simplicial in characteristic 0, and $\mathcal{Z} = \mathcal{V} \cap \mathcal{E}$ is a convex solid cone in $\mathcal{E}$.

By Corollary 20.1, the set $\mathcal{D}^B_x = \mathcal{D}^B \cap \rho^{-1}(\mathcal{E}_{x,+})$ is finite for $\forall x \in C$. In particular, the set of central B-divisors is finite.

For an arbitrary G-model X, consider the rational B-quotient map $\pi : X \dashrightarrow C$ separating generic B-orbits. Thus generic B-orbits determine a one-parameter family of B-stable prime divisors on X parametrized by an open subset $\mathring{C} \subseteq C$. Decreasing $\mathring{C}$ if necessary, we may assume that these divisors do not occur in $\operatorname{div}\mathbf{f}_\lambda$ $(\lambda \in \Lambda)$ and that they are pull-backs of points $x \in \mathring{C}$. Their images in $\breve{\mathcal{E}}$ are the vectors $\varepsilon_x = (1,0) \in \mathcal{E}_{x,+}$. Clearly, $\{\varepsilon_x \mid x \in \mathring{C}\} \in \mathbf{CD}$.

In the generically transitive case, $\pi : \mathcal{O} \dashrightarrow \mathbb{P}^1$ is determined by a one-dimensional linear system of B-divisors. In other words, there is a G-line bundle $\mathcal{L}$ on $\mathcal{O}$ and a two-dimensional subspace M of its B-eigensections which defines this linear system. Elements of M are homogeneous coordinates on $\mathbb{P}^1 = \mathbb{P}(M^*)$. If $\mathcal{O} = G/H$, then $\mathcal{L} = \mathcal{L}(\chi_0)$ and $M = \Bbbk[G]^{(B\times H)}_{(\lambda_0,-\chi_0)}$ for some $\lambda_0 \in \mathfrak{X}_+$, $\chi_0 \in \mathfrak{X}(H)$. Except for finitely many lines, M consists of indecomposable elements corresponding to generic B-divisors.

Indecomposable elements of M and the respective B-divisors are called *regular*. A regular B-divisor $D_x = \pi^*(x)$ is represented in $\breve{\mathcal{E}}$ by a vector $(1,\ell) \in \mathcal{E}_{x,+}$, and $\ell = 0$ for all but finitely many x.

Besides, there is a finite set of one-dimensional subspaces $\Bbbk[G]^{(B\times H)}_{(\lambda_i,-\chi_i)}$, $i = 1,\dots,s$, consisting of indecomposable elements that correspond to other B-divisors. If $\eta_i \in \Bbbk[G]^{(B\times H)}_{(\lambda_i,-\chi_i)}$ divides some $\eta \in \Bbbk[G]^{(B\times H)}_{(\lambda_0,-\chi_0)}$, then $D_i = \operatorname{div}\eta_i$ is represented in $\breve{\mathcal{E}}$ by $(h_i,\ell_i) \in \mathcal{E}_{x,+}$, where $\operatorname{div}\eta = D_x$ and h_i is the multiplicity of η_i in η (or of D_i in D_x). Such η_i and D_i are called *subregular*. Other η_i are called *central* (since D_i are central).

In characteristic zero, the above description of B-divisors allows to compute multiplicities in the spaces of global sections of G-line bundles on $\mathcal{O}$.

Proposition 16.1. *For $\forall\chi \in \mathfrak{X}(H)$ and $\forall\lambda \in \mathfrak{X}_+$, let k_0 be the minimal integer such that $(\lambda, -\chi) = \sum_{i=0}^{s} k_i(\lambda_i, -\chi_i) + (\mu, -\mu)$, where $k_i \geq 0$ and $\mu \in \mathfrak{X}(G)$. Then $m_\lambda(\mathcal{L}(\chi)) = k_0 + 1$.*

Proof. Every $\eta \in \mathrm{H}^0(\mathcal{O}, \mathcal{L}(\chi))^{(B)}_\lambda = \Bbbk[G]^{(B\times H)}_{(\lambda,-\chi)}$ decomposes uniquely as $\eta = \sigma_1 \dots \sigma_{k_0}\eta_1^{k_1} \dots \eta_s^{k_s}$, where $\sigma_j \in \Bbbk[G]^{(B\times H)}_{(\lambda_0,-\chi_0)}$. Therefore $\dim \Bbbk[G]^{(B\times H)}_{(\lambda,-\chi)} = \dim \Bbbk[G]^{(B\times H)}_{(k_0\lambda_0,-k_0\chi_0)}$, and $\Bbbk[G]^{(B\times H)}_{(k_0\lambda_0,-k_0\chi_0)} = \mathrm{S}^{k_0}\Bbbk[G]^{(B\times H)}_{(\lambda_0,-\chi_0)}$ has dimension $k_0 + 1$. □

Corollary 16.1 ([Pan2, 1.2]). *If $\mathfrak{X}(H) = 0$, then $m_\lambda(\mathcal{O}) = k_0 + 1$, where $k_0 = \max\{k \mid \lambda - k\lambda_0 \in \Lambda_+(\mathcal{O})\}$.*

In the one-parameter case, generic B-stable divisors are G-stable, whence $\varepsilon_x \in \mathcal{V}_x$ for $x \in \mathring{C}$.

Lemma 16.1. *In the one-parameter case, all B-divisors are central.*

Proof. If D is a non-central B-divisor, then $v_D(f) > 0$ for some $f \in K^B = K^G$. Hence D is G-stable, a contradiction. □

Every B-divisor intersects a generic G-orbit $\mathcal{O} \subset X$ transversally, and $\mathbf{f}_\lambda$ is defined on $\mathcal{O}$ for $\forall\lambda \in \Lambda$. Hence $\mathcal{E}(K) = \mathcal{E}(\mathcal{O})$ and $\mathcal{D}^B(K)$ is identified with $\mathcal{D}^B(\mathcal{O})$. Furthermore, it follows from §20 that $\mathcal{Z} = \mathcal{V}(\mathcal{O})$ and $\mathcal{V}_x = \mathcal{Z} + \mathbb{Q}_+\varepsilon_x$ for $x \in \mathring{C}$. Thus the colored equipment in the one-parameter case is in the major part determined by the colored equipment of a generic G-orbit: only the structure of $\mathcal{V}_x$ for finitely many $x \in C$ depends on the whole one-parameter family of orbits.

Remark 16.1. Since a splitting $\mathbf{f} : \Lambda \to K^{(B)}$ is not uniquely defined, the maps $\mathcal{V} \hookrightarrow \breve{\mathcal{E}}$, $\rho : \mathcal{D}^B \to \breve{\mathcal{E}}$ are not canonical. But the change of splitting is easily controlled. If $\mathbf{f}'$ is another splitting, then passing from $\mathbf{f}$ to $\mathbf{f}'$ produces a shift of each $\mathcal{E}_{x,+}$: $h' = h$ and $\ell' = \ell + h\ell_x$, where $\langle \ell_x, \lambda\rangle = v_x(\mathbf{f}'_\lambda/\mathbf{f}_\lambda)$. The shifting vectors $\ell_x \in \Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset \mathcal{E}$ have the property that $\sum_{x\in C}\langle \ell_x, \lambda\rangle x$ is a principal divisor on C for $\forall\lambda \in \Lambda$, in particular, $\sum_{x\in C} \ell_x = 0$. Conversely, any collection of integral shifting vectors $\ell_x \in \Lambda^*$ such that $\sum_{x\in C}\langle \ell_x, \lambda\rangle x$ is a principal divisor for $\forall\lambda$ defines a change of splitting. For $C = \mathbb{P}^1$, it suffices to have $\sum \ell_x = 0$.

Now we describe the dual object to the hyperspace.

Definition 16.2. A *linear functional* on the hyperspace is a function ϕ on $\breve{\mathcal{E}}$ such that $\phi_x = \phi|_{\mathcal{E}_{x,+}}$ is a $\mathbb{Q}$-linear functional for $\forall x \in C$ and $\sum_{x\in C}\langle \varepsilon_x, \phi_x\rangle = 0$. A linear functional ϕ is *admissible* if $N\sum_{x\in C}\langle\varepsilon_x,\phi_x\rangle x$ is a principal divisor on C for some $N \in \mathbb{N}$. Denote by $\breve{\mathcal{E}}^*$ the space of linear functionals and by $\breve{\mathcal{E}}^*_{\mathrm{ad}}$ the subspace of admissible functionals on $\breve{\mathcal{E}}$. The set $\operatorname{Ker}\phi = \bigcup_{x\in C}\operatorname{Ker}\phi_x$ is called the *kernel* of $\phi \in \breve{\mathcal{E}}^*$.

If $C = \mathbb{P}^1$, then any linear functional is admissible. Any $f = f_0\mathbf{f}_\lambda \in K^{(B)}$, $f_0 \in K^B$, $\lambda \in \Lambda$, determines an admissible linear functional ϕ by means of $\langle q, \phi_x\rangle = hv_x(f_0) + \langle \ell, \lambda\rangle$, $\forall q = (h,\ell) \in \mathcal{E}_{x,+}$, and f is determined by ϕ uniquely up to a scalar multiple. Conversely, a multiple of any admissible functional is determined by a B-eigenfunction.

Any collection of linear functionals ϕ_x on $\mathcal{E}_{x,+}$ whose restrictions to $\mathcal{E}$ coincide can be deformed to an admissible functional by a “small variation”.

Lemma 16.2. *Let ϕ_x be linear functionals on $\mathcal{E}_{x,+}$ such that $\phi_x|_{\mathcal{E}}$ does not depend on $x \in C$, $\langle\varepsilon_x,\phi_x\rangle = 0$ for all but finitely many x, and $\sum\langle\varepsilon_x,\phi_x\rangle < 0$. Then for any finite subset $C_0 \subset C$ and $\forall \varepsilon > 0$ there exists $\psi \in \breve{\mathcal{E}}^*_{\mathrm{ad}}$ such that $\psi_x \geq \phi_x$ on $\mathcal{E}_{x,+}$ for $\forall x \in C$ with the equality for $x \in C_0$ (in particular, $\psi|_{\mathcal{E}} = \phi_x|_{\mathcal{E}}$) and $|\langle\varepsilon_x,\psi_x\rangle - \langle\varepsilon_x,\phi_x\rangle| < \varepsilon$.*

Proof. The divisor $-N\sum\langle\varepsilon_x,\phi_x\rangle x$ is very ample on C for N sufficiently large. Moving the respective hyperplane section of C, we obtain an equivalent very ample divisor without of the form $\sum n_x x$, $n_x = 0, 1$, $n_x = 0$ whenever $x \in C_0$. Then the divisor $\sum(N\langle\varepsilon_x,\phi_x\rangle + n_x)x$ is principal, and $\psi \in \breve{\mathcal{E}}^*_{\mathrm{ad}}$ defined by $\psi|_{\mathcal{E}} = \phi_x|_{\mathcal{E}}$, $\langle\varepsilon_x,\psi_x\rangle = \langle\varepsilon_x,\phi_x\rangle + n_x/N$, is the desired admissible functional. □

For reorganizing colored data in a way similar to the spherical case, we need some notions from the geometry of the hyperspace.

Definition 16.3. A *cone* in $\breve{\mathcal{E}}$ is a cone in some $\mathcal{E}_{x,+}$.

A *hypercone* in $\breve{\mathcal{E}}$ is a union $\mathcal{C} = \bigcup_{x\in C}\mathcal{C}_x$ of finitely generated convex cones $\mathcal{C}_c = \mathcal{C} \cap \mathcal{E}_{x,+}$ such that

(1) $\mathcal{C}_x = \mathcal{K} + \mathbb{Q}_+\varepsilon_x$ for all but finitely many x, where $\mathcal{K} = \mathcal{C} \cap \mathcal{E}$.

(2) Either (A) $\exists x \in C:\ \mathcal{C}_x = \mathcal{K}$

or (B) $\mathcal{B} = \sum \mathcal{B}_x \subseteq \mathcal{K}$, where $\varepsilon_x + \mathcal{B}_x = \mathcal{C}_x \cap (\varepsilon_x + \mathcal{E})$.

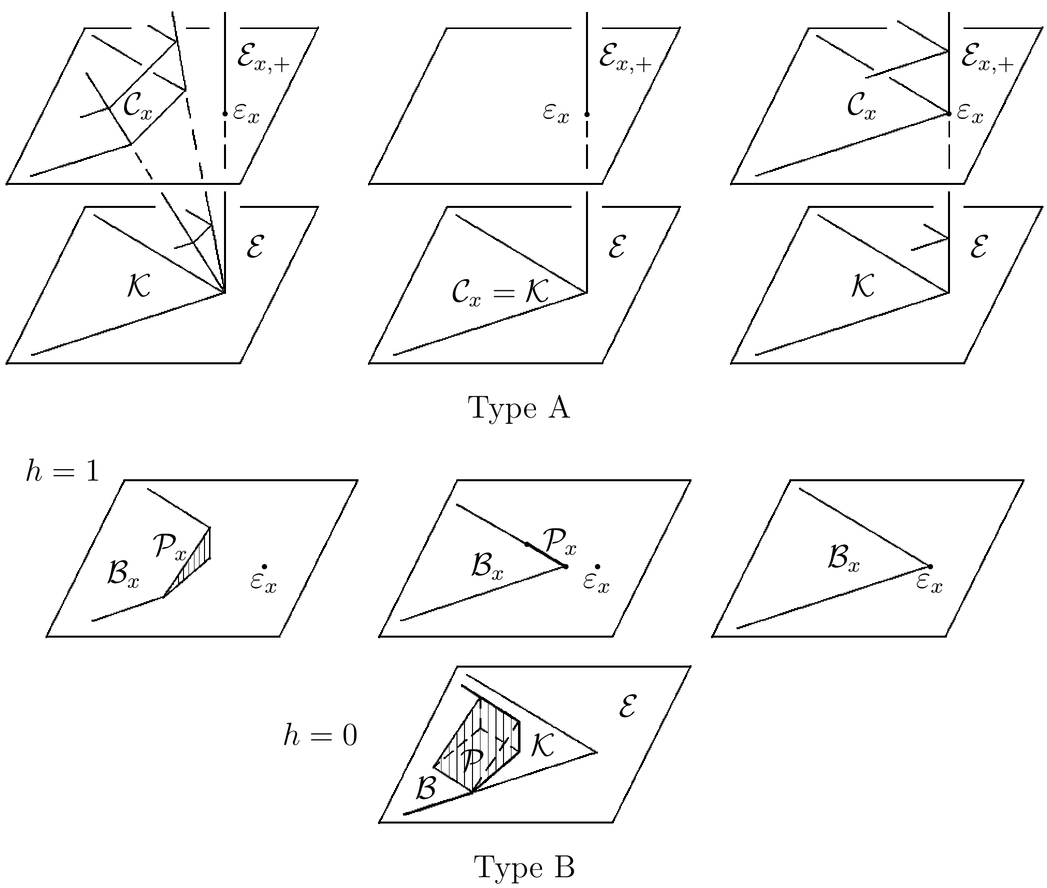
The hypercone is *strictly convex* if all $\mathcal{C}_x$ are and $\mathcal{B} \not\ni 0$.

Remark 16.2. The Minkowski sum $\sum \mathcal{B}_x$ of infinitely many polyhedral domains $\mathcal{B}_x$ is defined as the set of all sums $\sum b_x$, $b_x \in \mathcal{B}_x$, that make sense, i.e., $b_x = 0$ for all but finitely many x. In particular, for a hypercone of type A, $\exists x \in C: \ \mathcal{B}_x = \emptyset \implies \mathcal{B} = \emptyset$.

Definition 16.4. Suppose that $Q \subseteq \check{\mathcal{E}}$ differs from $\{\varepsilon_x \mid x \in \mathring{C}\}$ by finitely many elements. Let $\varepsilon_x + \mathcal{P}_x$ be the convex hull of the intersection points of $\varepsilon_x + \mathcal{E}$ with the rays $\mathbb{Q}_+ q$, $q \in Q$. We say that the hypercone $\mathcal{C} = \mathcal{C}(Q)$, where $\mathcal{C}_x$ are generated by $Q \cap \mathcal{E}_{x,+}$ and $\mathcal{P} = \sum \mathcal{P}_x$, is *generated* by Q.

Remark 16.3. We have $\mathcal{B}_x = \mathcal{P}_x + \mathcal{K}$ and $\mathcal{B} = \mathcal{P} + \mathcal{K}$.

Figure 3.1: Hypercones



Definition 16.5. For a hypercone $\mathcal{C}$ of type B, we define its *interior* $\operatorname{int} \mathcal{C} = \bigcup_{x \in C} \operatorname{int} \mathcal{C}_x \cup \operatorname{int} \mathcal{K}$.

A *face* of a hypercone $\mathcal{C}$ is a face $\mathcal{C}'$ of some $\mathcal{C}_x$ such that $\mathcal{C}' \cap \mathcal{B} = \emptyset$.

A *hyperface* of $\mathcal{C}$ is a hypercone $\mathcal{C}' = \mathcal{C} \cap \operatorname{Ker}\phi$, where $\phi \in \breve{\mathcal{E}}^*$, $\langle \mathcal{C}, \phi\rangle \geq 0$. The hyperface $\mathcal{C}'$ is *admissible* if any such ϕ is admissible.

A hypercone is *admissible* if all its hyperfaces of type B are admissible.

Remark 16.4. A hyperface $\mathcal{C}' \subseteq \mathcal{C}$ is of type B iff $\mathcal{C}' \cap \mathcal{B} \neq \emptyset$. Indeed, $\langle \varepsilon_x + \mathcal{B}_x, \phi_x\rangle \geq 0$ $(\forall x)$ and $\forall x:\ \mathcal{C}'_x \not\subseteq \mathcal{E} \iff \forall x:\ \langle \varepsilon_x + \mathcal{B}_x, \phi_x\rangle \ni 0 \iff \sum\langle \varepsilon_x + \mathcal{B}_x, \phi_x\rangle = \sum\langle \varepsilon_x, \phi_x\rangle + \sum\langle \mathcal{B}_x, \phi_x\rangle = \langle \mathcal{B}, \phi\rangle \ni 0$

Properties of hypercones are similar to properties of convex polyhedral cones. Let $\mathcal{C}$ be a hypercone. There is a separation property:

Lemma 16.3. (1) $q \notin \mathcal{C} \implies \exists \phi \in \breve{\mathcal{E}}^*:\ \langle \mathcal{C}, \phi\rangle \geq 0,\ \langle q, \phi\rangle < 0$

(2) *If $\mathcal{C}$ is strictly convex, then one may assume that ϕ is admissible and $\langle \mathcal{C}_x \setminus \{0\}, \phi_x\rangle > 0$ for any given finite set of x.*

Proof. (1) If $q \in \mathcal{E}_{y,+}$, then we construct a collection of functionals ϕ_x on $\mathcal{E}_{x,+}$ such that $\phi_x|_{\mathcal{E}} = \phi_y|_{\mathcal{E}}$, $\langle q, \phi_y\rangle < 0$, $\langle \mathcal{C}_x, \phi_x\rangle \geq 0$ for $\forall x$ and the equality is reached on $\mathcal{C}_x \setminus \{0\}$. Then $\sum\langle \varepsilon_x + \mathcal{B}_x, \phi_x\rangle = \sum\langle \varepsilon_x, \phi_x\rangle + \langle \mathcal{B}, \phi_y\rangle \geq 0$ and the equality is reached. But $\langle \mathcal{B}, \phi_y\rangle \geq 0 \implies \sum\langle \varepsilon_x, \phi_x\rangle \leq 0$. It remains to modify ϕ_x by a small variation (Lemma 16.2) if necessary.

(2) In the proof of (1), we may assume $\langle \mathcal{K}\setminus\{0\}, \phi_y\rangle > 0 \implies \langle \mathcal{B}, \phi_y\rangle > 0 \implies \sum\langle \varepsilon_x, \phi_x\rangle < 0$, and we may increase finitely many ϕ_x to have $\langle \mathcal{C}_x, \phi_x\rangle \geq 0$. $\square$

This implies a dual characterization of a hypercone:

Lemma 16.4. *For a (strictly convex) hypercone $\mathcal{C} = \mathcal{C}(Q)$, $q \in \mathcal{C}$ iff $\langle Q, \phi\rangle \geq 0 \implies \langle q, \phi\rangle \geq 0$ for all (admissible) ϕ.*

Proof. $\langle Q, \phi\rangle \geq 0 \implies \forall x \in C:\ \langle \varepsilon_x + \mathcal{P}_x, \phi_x\rangle \geq 0 \implies \langle \mathcal{P}, \phi\rangle \geq 0 \implies \langle \mathcal{C}, \phi\rangle \geq 0$. Lemma 16.3 completes the proof. $\square$

For any $v \in \mathcal{C}$, there is a unique face or hyperface of type B containing v in its interior.

Lemma 16.5. *The face or (admissible) hyperface $\mathcal{C}' \subseteq \mathcal{C}$ such that $v \in \operatorname{int}\mathcal{C}'$ is the intersection of (admissible) hyperfaces of $\mathcal{C}$ containing v.*

Proof. If $\langle \mathcal{C}, \phi\rangle \geq 0$, $\langle v, \phi\rangle = 0$, then $\langle \mathcal{C}', \phi\rangle = 0$. (If $\mathcal{C}'$ is a hyperface, then $\langle \mathcal{K}', \phi\rangle = 0 \implies \langle \varepsilon_x + \mathcal{B}'_x, \phi_x\rangle = 0 \implies \langle \mathcal{C}'_x, \phi_x\rangle = 0$ for $\forall x$.) Conversely, if $\mathcal{C}' \subseteq \mathcal{C}_y$ is a face and $q \in \mathcal{C} \setminus \mathcal{C}'$, then we construct an (admissible) functional ϕ such that $\langle \mathcal{C}, \phi\rangle \geq 0$, $\langle \mathcal{C}', \phi\rangle = 0$, $\langle q, \phi\rangle > 0$ as follows. Take ϕ_y on $\mathcal{E}_{y,+}$ such that $\langle \mathcal{C}_y, \phi_y\rangle \geq 0$ and $\mathcal{C}' = \operatorname{Ker}\phi_y \cap \mathcal{C}$. We may include ϕ_y in a collection of functionals ϕ_x on $\mathcal{E}_{x,+}$ such that $\phi_x|_{\mathcal{E}} = \phi_y|_{\mathcal{E}}$, $\langle \mathcal{C}_x, \phi_x\rangle \geq 0$ and the inequality is reached on $\mathcal{C}_x \setminus \{0\}$. But $\mathcal{C}' \cap \mathcal{B} = \emptyset \implies \langle \mathcal{B}, \phi_y\rangle > 0 \implies \sum\langle \varepsilon_x, \phi_x\rangle < 0$, as in Lemma 16.3(2). Then we increase some ϕ_x to obtain $\langle q, \phi\rangle > 0$ and apply Lemma 16.2. $\square$

Now let $(\mathcal{W}, \mathcal{R})$ be colored data from **CD** and consider the hypercone $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ generated by $\mathcal{W} \cup \rho(\mathcal{R})$.

Condition (C) means that $\mathcal{C}$ is strictly convex and $\rho(\mathcal{R}) \not\ni 0$ (Lemma 16.3(2)). We assume it in the sequel.

Condition (W) means that the elements of $\mathcal{W}$ are exactly the generators of those edges of $\mathcal{C}$ that do not intersect $\rho(\mathcal{R})$. (Indeed, (W) $\iff$ $w \notin \mathcal{C}(\mathcal{W} \setminus \{w\}, \mathcal{R})$.)

Condition (F) means that $\mathcal{C}$ is admissible. (This is non-trivial, see [Tim2, Pr.4.1].)

Condition (V) means that $v \in \mathcal{C}$ (Lemma 16.4).

Conditions (V′) and (D′) say that $\mathcal{V}_Y$ and $\mathcal{D}_Y^B$ consist of those elements of $\mathcal{W}$ and $\mathcal{R}$ which lie in the face or hyperface (of type B) $\mathcal{C}_Y = \mathcal{C}(\mathcal{V}_Y, \mathcal{D}_Y^B) \subseteq \mathcal{C}$ such that $v \in \operatorname{int} \mathcal{C}_Y$ (Lemma 16.5).

Condition (S) says that $v \in \mathcal{V} \cap \operatorname{int} \mathcal{C}_Y$ (Lemma 16.5).

Definition 16.6. A *colored hypercone* is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{R} \subseteq \mathcal{D}^B$, $\rho(\mathcal{R}) \not\ni 0$, and $\mathcal{C}$ is a strictly convex hypercone generated by $\rho(\mathcal{R})$ and $\mathcal{W} \subseteq \mathcal{V}$.

A colored hypercone $(\mathcal{C}, \mathcal{R})$ (of type B) is *supported* if $\operatorname{int} \mathcal{C} \cap \mathcal{V} \neq \emptyset$.

A *(hyper)face* of $(\mathcal{C}, \mathcal{R})$ is a colored (hyper)cone $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{C}'$ is a (hyper)face of $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R} \cap \rho^{-1}(\mathcal{C}')$.

A *colored hyperfan* is a finite set of supported colored cones and hypercones of type B whose interiors are disjoint inside $\mathcal{V}$ and which is obtained as the set of all supported (hyper)faces of finitely many colored hypercones.

Theorem 16.1. (1) *B-charts are in bijection with colored hypercones in $\breve{\mathcal{E}}$.*

(2) *G-germs are in bijection with supported colored cones and hypercones of type B. If $Y_1, Y_2 \subseteq X$ are G-subvarieties in a G-model, then $Y_1 \preceq Y_2$ iff $(\mathcal{C}_{Y_2}, \mathcal{D}_{Y_2}^B)$ is a (hyper)face of $(\mathcal{C}_{Y_1}, \mathcal{D}_{Y_1}^B)$.*

(3) *G-models are in bijection with colored hyperfans.*

Corollaries 15.1–15.3 are easily generalized to the case of complexity 1.

Remark 16.5. Let $\mathring{X}$ be a B-chart defined by a colored hypercone $(\mathcal{C}, \mathcal{R})$. Then $\Bbbk[\mathring{X}]^B = \Bbbk$ iff $\mathcal{C}$ is of type B: if $f \in \Bbbk[\mathring{X}]^B$, then the respective $\phi \in \breve{\mathcal{E}}^*_{\mathrm{ad}}$ must be zero on $\mathcal{E}$ and non-negative on $\mathcal{C}$. Thus we have two types of B-charts:

(A) $\Bbbk[\mathring{X}]^B \neq \Bbbk$, or $\mathcal{C}$ is of type A.

(B) $\Bbbk[\mathring{X}]^B = \Bbbk$, or $\mathcal{C}$ is of type B.

There are two types of G-germs:

(A) $\mathcal{C}(\mathcal{V}_Y, \mathcal{D}_Y^B)$ is a colored cone.

(B) $\mathcal{C}(\mathcal{V}_Y, \mathcal{D}_Y^B)$ is a colored hypercone.

A G-germ is of type A iff $\mathcal{V}_Y, \mathcal{D}_Y^B$ are finite, and of type B iff it has a minimal B-chart.

Example 16.1. Suppose that $G = B = T$ is a torus. We may assume (after factoring out by the kernel of the action) that the stabilizer of general position for any T-model is trivial. Since a torus has no non-trivial Galois cohomology (Hilbert's Theorem 90, see [PV, 2.6]), the birational type of the action is trivial, i.e., any T-model is birationally isomorphic to $T \times C$. It follows that $\mathcal{E} = \operatorname{Hom}(\mathfrak{X}(T), \mathbb{Q})$, $\mathcal{V} = \breve{\mathcal{E}}$, $\mathcal{D}^B = \mathcal{D}^B(T) = \emptyset$.

A T-model is given by a set of cones and admissible hypercones of type B with disjoint interiors which consists of all faces and hyperfaces of type B of finitely many admissible hypercones. (The word "colored" is needless here, since there are no colors.) A T-chart $\mathring{X}$ is of type A (type B) iff $\mathring{X}/\!\!/T \subset C$ is an open subset ($\mathring{X}/\!\!/T$ is a point).

If all germs of a T-model X are of type A, then quotient morphisms of its T-charts may be glued together into a regular map $\pi : X \to C$ separating T-orbits of general position. Such T-models were classified by Mumford in [KKMS, Part IV] in the framework of the theory of toroidal embeddings (for this theory see [KKMS, Part II]). The hyperfan of X is a union of fans $\mathcal{F}_x$ in $\mathcal{E}_{x,+}$ having common central part $\mathcal{F} = \{\mathcal{C} \in \mathcal{F}_x \mid \mathcal{C} \subseteq \mathcal{E}\}$ and such that $\mathcal{F}_x$ is a cylinder over $\mathcal{F}$ for $x \neq x_1, \dots, x_s$ (finitely many exceptional points).

It is proved in [KKMS, Part IV] that C is covered by open neighborhoods C_i of x_i such that $\pi^{-1}(C_i) \simeq C_i \times_{\mathbb{A}^1} X_i$, where $\nu_i : C_i \to \mathbb{A}^1$ are etale maps such that $\nu_i^{-1}(0) = \{x_i\}$ and X_i are toric $(T \times \Bbbk^\times)$-varieties with fans $\mathcal{F}_{x_i}$ mapping $\Bbbk^\times$-equivariantly onto $\mathbb{A}^1$.

Remark 16.6. The admissibility of a hypercone is essential for condition (F) as the following example [Kn4] shows.

Let C be a smooth projective curve of genus $g > 0$ and $\delta_i = \sum n_{ix} x$, $i = 1, 2$, be divisors on C having infinite order in $\operatorname{Pic} C$ and such that $\deg \delta_1 = 0$, $\deg \delta_2 > 2g - 2$. Put $\mathcal{L}_i = \mathcal{O}_C(\delta_i)$.

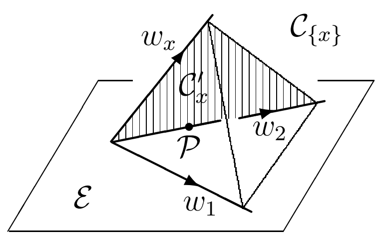
The total space X of $\mathcal{L}_1^* \oplus \mathcal{L}_2^*$ is a $T = (\Bbbk^\times)^2$-model, where the factors $\Bbbk^\times$ act on $\mathcal{L}_i^*$ by homotheties. There are the following T-germs in X: the divisors

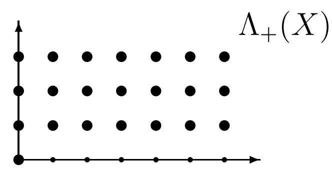
D_i, D_x ($i = 1, 2$, $x \in C$), where D_i is the total space of $\mathcal{L}_j^*$, $\{i, j\} = \{1, 2\}$, and D_x is the fiber of $X \to C$ over $\{x\}$; $Y_{ix} = D_i \cap D_x$; $C = D_1 \cap D_2$; $\{x\} = D_1 \cap D_2 \cap D_x$.

Let $\mathbf{f}_i$ be a rational section of $\mathcal{L}_i$ such that $\operatorname{div} \mathbf{f}_i = \delta_i$. Then $\Lambda(X)$ is generated by the T-weights ω_1, ω_2 of $\mathbf{f}_1, \mathbf{f}_2$. If w_i, w_x are the T-valuations corresponding to D_i, D_x, then $w_1 = (0, 1, 0)$, $w_2 = (0, 0, 1)$, $w_x = (1, n_{1x}, n_{2x})$ in the basis $\varepsilon_x, \omega_1^\vee, \omega_2^\vee$, where $\omega_i^\vee$ are the dual coweights to ω_i.

The algebra $\Bbbk[X]$ is bigraded by the T-action: $\Bbbk[X]_{m,n} = \mathrm{H}^0(C, \mathcal{L}_1^m \otimes \mathcal{L}_2^n) \neq 0$ iff $m \geq 0$, $n > 0$ or $m = n = 0$. Hence $\Lambda_+(X)$ is not finitely generated, and $\Bbbk[X]$ as well. The reason is that $\Bbbk[X]$ is defined by a non-admissible hypercone $\mathcal{C} = \bigcup \mathcal{C}_{\{x\}}$. Indeed, its hyperface $\mathcal{C}' = \bigcup \mathcal{C}_{Y_{2x}}$ is not admissible, because $\mathcal{C}' = \mathcal{C} \cap \operatorname{Ker} \phi$, $\langle \omega_1^\vee, \phi \rangle = 1$, $\langle \omega_2^\vee, \phi \rangle = 0$, $\langle \varepsilon_x, \phi \rangle = -n_{1x}$, and no multiple of $-\delta_1 = \sum \langle \varepsilon_x, \phi \rangle x$ is a principal divisor. See Figure 3.2.

Figure 3.2:





Example 16.2 (SL_2**-embeddings).** Suppose $G = \mathrm{SL}_2(\Bbbk)$, $H = \{e\}$. Then $\mathcal{O} = \mathrm{SL}_2$ has complexity one. Its embeddings were described in [LV, §9].

The elements of G are matrices

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad g_{11}g_{22} - g_{21}g_{12} = 1,$$

and B consists of upper-triangular matrices ($g_{21} = 0$). Let ω be the fundamental weight: $\omega(g) = g_{11}$ for $g \in B$.

All B-divisors in $\mathcal{O}$ are regular. Their equations are the nonzero elements of the two-dimensional subspace $M = \Bbbk[G]_\omega^{(B)}$ generated by $\eta_1(g) = g_{21}$, $\eta_2(g) = g_{22}$. Let $\eta_x = \alpha_1\eta_1 + \alpha_2\eta_2$ be an equation of $x \in \mathbb{P}^1 = \mathbb{P}(M^*)$.

The field $K^B = \Bbbk(\mathbb{P}^1)$ consists of rational functions in η_1, η_2 of degree 0.

The group Λ equals $\mathfrak{X}(B) = \langle \omega \rangle \simeq \mathbb{Z}$. We may take $\mathbf{f}_\omega = \eta_\infty$, where $\infty \in \mathbb{P}^1$ is a certain fixed point.

The set of G-valuations is computed by the method of formal curves (§24). First we determine G-valuations corresponding to divisors with a dense orbit.

Up to a multiple, any such valuation is defined by the formula $v_{x(t)}(f) = \operatorname{ord}_t f(gx(t))$, where $x(t) \in \mathrm{SL}_2(\Bbbk((t)))$ and g is the generic $\Bbbk(\mathrm{SL}_2)$-point of SL_2. By the Iwasawa decomposition (§24), we may even assume $x(t) = \left(\begin{smallmatrix} t^m & u(t) \\ 0 & t^{-m} \end{smallmatrix}\right)$, $u(t) \in \Bbbk((t))$, $\operatorname{ord}_t u(t) = n \le -m$.

The number

$$d = v_{x(t)}(\eta_x) = \operatorname{ord}_t\big((\alpha_1 t^m + \alpha_2 u(t))g_{21} + \alpha_2 t^{-m} g_{22}\big)$$

is constant along $\mathbb{P}^1$ except one x, where it jumps, so that $v_{x(t)} = (h, \ell) \in \mathcal{V}_x$. The following cases are possible:

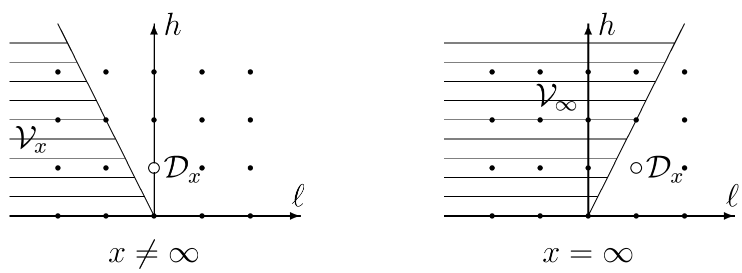
$$m \le n \implies \begin{cases} d = m, & \operatorname{ord}_t(\alpha_1 t^m + \alpha_2 u(t)) = m \\ d \in (m, -m], & \operatorname{ord}_t(\alpha_1 t^m + \alpha_2 u(t)) > m \end{cases} \implies \begin{cases} h \in (0, -2m] \\ \ell = m \text{ (or } m + h) \end{cases}$$

$$m > n \implies \begin{cases} d = n, & \alpha_2 \ne 0 \\ d = m, & \alpha_2 = 0 \end{cases} \implies \begin{cases} h = m - n \\ \ell = n \text{ (or } n + h) \end{cases}$$

(Here $\ell = v_{x(t)}(\mathbf{f}_\omega)$ increases by h if the jump occurs at $x = \infty$.)

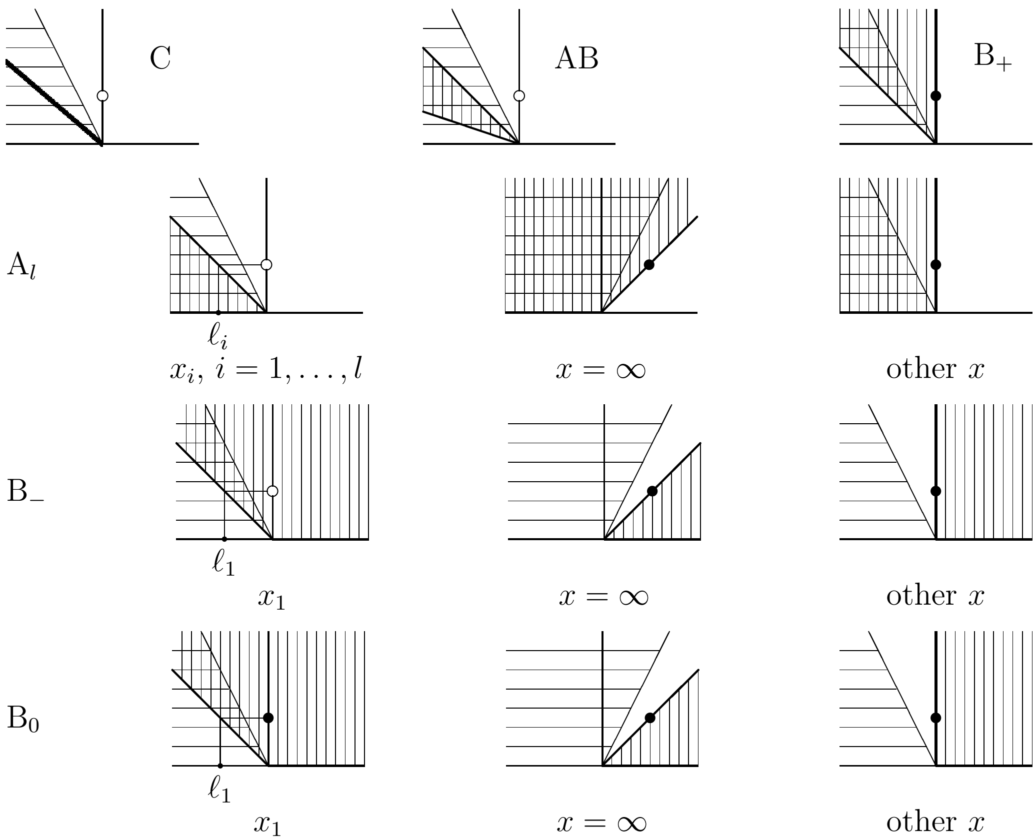
In both cases, we obtain the subset in $\mathcal{E}_{x,+}$ defined by the inequalities $h > 0$, $2\ell + h \le 0$ (or $2\ell - h \le 0$ if $x = \infty$). Thus $\mathcal{V}_x$ is defined by $2\ell + h \le 0$ (or $2\ell - h \le 0$) by §24. The colored equipment is represented in Figure 3.3. (Elements of $\Lambda^* \times \mathbb{Z}_+ \subset \mathcal{E}_{x,+}$ are marked by dots.)

Figure 3.3:



G-germs are given by colored cones or hypercones of type B hatched in the figures below; their colors are marked by bold dots. The notation for G-germs is taken from [LV, §9]. Up to a "change of coordinates" (Remark 16.1), we may assume that for germs of types $\mathrm{C}, \mathrm{AB}, \mathrm{B}_+$ the colored cone lies in $\mathcal{E}_{x,+}$, $x \ne \infty$, and for germs of types $\mathrm{A}_l, \mathrm{B}_-, \mathrm{B}_0$, $x_i \ne \infty$. Moreover, for the hypercone to be strictly convex, we must have $\sum \ell_i < -1$ for A_l, $l \le 1$, and $\ell_1 > -1$ for B_- and B_0.

Figure 3.4: G-germs of SL_2-embeddings



Affine SL_2-embeddings correspond to minimal B-charts of G-germs of type B_0. They were first classified by Popov [Po1]. Embeddings of SL_2/H, where H is finite, were classified in [MJ1]. Embeddings of G/H, where G has semisimple rank 1 and H is finite, were classified in [Tim2, §5].

Example 16.3 (ordered triangles). Suppose $G = \mathrm{SL}_3(\Bbbk)$, $H = T$ is the diagonal torus. Then $\mathcal{O} = G/H$ is the space of ordered triangles on a projective plane. The standard Borel subgroup B consists of upper-triangular matrices $g = (g_{ij})$, $\det g = 1$, $g_{ij} = 0$ for $i > j$. As usual, $\varepsilon_i(g) = g_{ii}$ are the tautological weights of T, $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$ are the fundamental weights, and $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$ are the simple roots. Denote by $\omega_i^\vee$ the fundamental coweights, and let $\rho = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$.

The subregular B-divisors $D_i, \widetilde{D}_i$ are defined by the $(B \times H)$-eigenfunctions $\eta_i(g) = g_{3i}$, $\widetilde{\eta}_i(g) = \left|\begin{smallmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{smallmatrix}\right|$ of biweights $(\omega_2, \varepsilon_i)$, $(\omega_1, -\varepsilon_i)$. $\widetilde{D}_i$ consists of triangles whose i-th side contains the B-fixed point in $\mathbb{P}^2$, and D_i consists of triangles whose i-th vertex lies on the B-fixed line.

The functions $\eta_i\widetilde{\eta}_i$ generate the two-dimensional subspace $M = \Bbbk[G]^{(B\times H)}_{(\rho,0)}$, $\eta_1\widetilde{\eta}_1 + \eta_2\widetilde{\eta}_2 + \eta_3\widetilde{\eta}_3 = 0$. Let $x_i \in \mathbb{P}^1 = \mathbb{P}(M^*)$ be the points corresponding to $\eta_i\widetilde{\eta}_i$. The regular B-divisors D_x, $x \neq x_1, x_2, x_3$, are defined by equations $\eta_x = \alpha_1\eta_1\widetilde{\eta}_1 + \alpha_2\eta_2\widetilde{\eta}_2 + \alpha_3\eta_3\widetilde{\eta}_3$.

The group $\Lambda = \langle\alpha_1, \alpha_2\rangle$ is the root lattice, $\mathbf{f}_{\alpha_1} = \widetilde{\eta}_1\widetilde{\eta}_2\widetilde{\eta}_3/\eta_\infty$, $\mathbf{f}_{\alpha_2} = \eta_1\eta_2\eta_3/\eta_\infty$, where $\infty \in \mathbb{P}^1$ is a certain fixed point.

By §24, any G-valuation corresponding to a divisor with dense G-orbit is proportional to $v_{x(t)}$, where

$$x(t) = \begin{pmatrix} 1 & t^m & u(t) \\ 0 & 1 & t^n \\ 0 & 0 & 1 \end{pmatrix},$$

and we may assume $m, n, r = \operatorname{ord}_t u(t) \leq 0$. Computing the values $v_{x(t)}(\eta_x)$ as in Example 16.2, one finds that the set of G-valuations $v = (h, \ell) \in \mathcal{E}_{x,+}$ corresponding to divisors with dense G-orbit is determined by the inequalities $a_1, a_2 \leq 0 \leq h$ (if $x = x_i$) or $a_1, a_2 \leq -2h \leq 0$ (if $x = \infty$) or $a_1, a_2 \leq -h \leq 0$ (otherwise), and $h = 0 \implies a_1$ or $a_2 = 0$, where $\ell = a_1\omega_1^\vee + a_2\omega_2^\vee$. Hence $\mathcal{V}_x$ is determined by the same inequalities without any restrictions for $h = 0$.

The colored equipment is represented in Figure 3.5. (The intersections of $\mathcal{V}$ with the hyperplane sections $\mathcal{E} = \{h = 0\}$ and $\{h = 1\}$ of $\mathcal{E}_{x,+}$ are hatched.)

The space of ordered triangles has three natural completions: $(\mathbb{P}^2)^3$, $(\mathbb{P}^{2*})^3$, and

$$X = \bigl\{(p_1, p_2, p_3, l_1, l_2, l_3) \bigm| p_j \in \mathbb{P}^2,\ l_i \in \mathbb{P}^{2*},\ p_j \in l_i \text{ whenever } i \neq j\bigr\}.$$

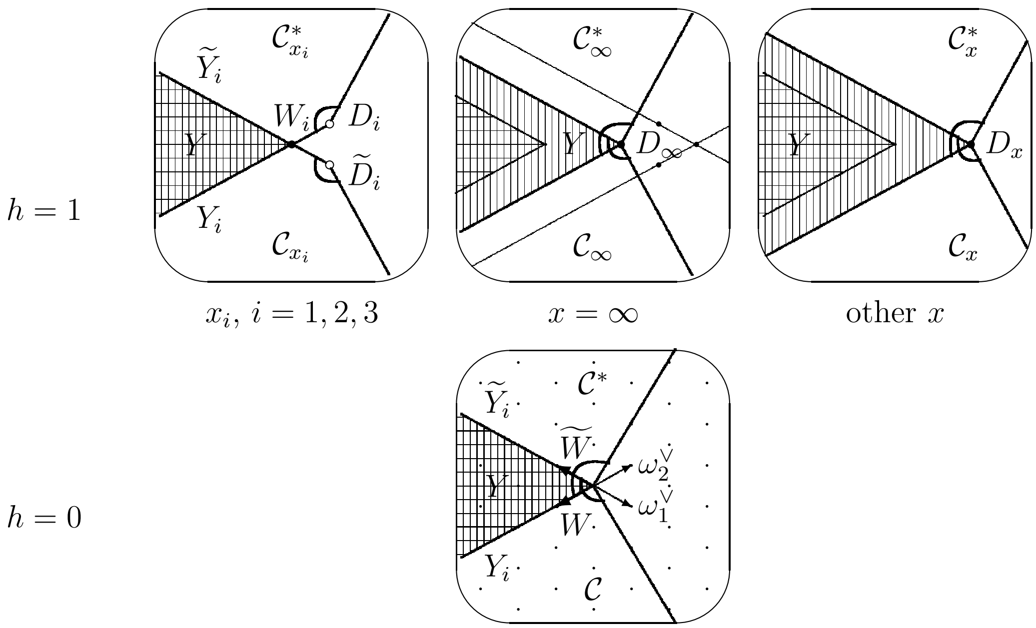
If z_{kj} are the homogeneous coordinates of p_j in $\mathbb{P}^2$, and y_{ik} are the dual coordinates of l_i in $\mathbb{P}^{2*}$, then X is determined by 6 equations $(y \cdot z)_{ij} = 0$, $(i \neq j)$ in $(\mathbb{P}^2)^3 \times (\mathbb{P}^{2*})^3$. One verifies that the Jacobian matrix is non-degenerate everywhere on $X \setminus Y$, where $Y \subset X$ is given by the equations $p_1 = p_2 = p_3$, $l_1 = l_2 = l_3$, and $\operatorname{codim}_X Y = 3$. Thence, by Serre's normality criterion, X is a normal complete intersection, smooth outside Y. It contains the following G-subvarieties of degenerate triangles:

W_i: $p_j = p_k$ and $l_j = l_k$, $\{i, j, k\} = \{1, 2, 3\}$. (A divisor.)

$\widetilde{W}$: p_1, p_2, p_3 are collinear and $l_1 = l_2 = l_3$. (The proper pullback of the divisor $\{\det z = 0\}$ in $(\mathbb{P}^2)^3$.)

W: $p_1 = p_2 = p_3$ and l_1, l_2, l_3 pass through this point. (The proper pullback of the divisor $\{\det y = 0\}$ in $(\mathbb{P}^{2*})^3$.)

Figure 3.5:



$\widetilde{Y}_i$: $p_j = p_k$ and $l_1 = l_2 = l_3$ (codim $= 2$).

Y_i: $l_j = l_k$ and $p_1 = p_2 = p_3$ (codim $= 2$).

Y: $p_1 = p_2 = p_3$ and $l_1 = l_2 = l_3$ (codim $= 3$).

Note that η_i and $\widetilde{\eta}_i$ may be regarded as certain homogeneous coordinates in the i-th copy of $\mathbb{P}^2$, resp. $\mathbb{P}^{2*}$, restricted to $\mathcal{O}$:

$$\begin{cases} \eta_i = z_{3i}, \\ \widetilde{\eta}_i = \begin{vmatrix} z_{2j} & z_{2k} \\ z_{3j} & z_{3k} \end{vmatrix}, \end{cases} \qquad \text{or dually,} \qquad \begin{cases} \eta_i = \begin{vmatrix} y_{j1} & y_{j2} \\ y_{k1} & y_{k2} \end{vmatrix}, \\ \widetilde{\eta}_i = y_{i1}, \end{cases}$$

and η_∞ is a 3-form in the matrix entries of y or z. Then

$$\mathbf{f}_{\alpha_1} = \frac{\widetilde{\eta}_1(y)\widetilde{\eta}_2(y)\widetilde{\eta}_3(y)}{\eta_\infty(y)} = \frac{\widetilde{\eta}_1(z)\widetilde{\eta}_2(z)\widetilde{\eta}_3(z)}{\eta_\infty(z)\det z}$$
$$\mathbf{f}_{\alpha_2} = \frac{\eta_1(y)\eta_2(y)\eta_3(y)}{\eta_\infty(y)\det y} = \frac{\eta_1(z)\eta_2(z)\eta_3(z)}{\eta_\infty(z)}$$

One easily deduces that $\mathbf{f}_{\alpha_1}, \eta_x$ ($\forall x \in C$) are regular and do not vanish along W, $\mathbf{f}_{\alpha_2}, \eta_x$ along $\widetilde{W}$, and $\mathbf{f}_{\alpha_1}$ (resp. $\mathbf{f}_{\alpha_2}$) has the 1-st order pole along $\widetilde{W}$ (resp. W). Hence the G-valuations of $W, \widetilde{W}, W_i$ are $-\omega_2^\vee, -\omega_1^\vee, \varepsilon_{x_i}$.

Since X is complete and contains the minimal G-germ Y (the closed orbit), we have $X = G\mathring{X}$, where $\mathring{X}$ is the minimal B-chart of Y determined by the colored hypercone $(\mathcal{C}_Y, \mathcal{D}_Y^B)$ of type B such that $\mathcal{C}_Y \supseteq \mathcal{V}$ and $\mathcal{V}_Y = \{-\omega_1^\vee, -\omega_2^\vee, \varepsilon_{x_1}, \varepsilon_{x_2}, \varepsilon_{x_3}\}$. It is easy to see from Figure 3.5 that there exists a unique such hypercone and $\mathcal{D}_Y^B = \{D_x \mid x \neq x_1, x_2, x_3\}$. Its (hyper)faces corresponding to various G-germs of X (including Y) are indicated in Figure 3.5 by the same letters.

A similar argument shows that $(\mathbb{P}^2)^3$ is defined by the colored hypercone $(\mathcal{C}, \{\widetilde{D}_i, D_x \mid i = 1,2,3,\ x \neq x_1, x_2, x_3\})$ and $(\mathbb{P}^{2*})^3$ by $(\mathcal{C}^*, \{\widetilde{D}_i, D_x \mid i = 1,2,3,\ x \neq x_1, x_2, x_3\})$.

The space $\mathrm{SL}_3(\Bbbk)/N(T)$ of unordered triangles and its completion is studied in [Tim3, §9]. The resolution of singularities of X was studied already by Schubert with applications to enumerative geometry, see [CF], [Tim3, §9], and §18.

We say that a G-model X *is of type A* if it contains no G-germs of type B, i.e., any G-orbit in X is contained in finitely many B-stable divisors. For any X, there is a canonical proper birational morphism $\nu : \widehat{X} \to X$ such that $\widehat{X}$ is of type A and ν is isomorphic in codimension 1. (Just subdivide each hypercone $\mathcal{C}$ from the hyperfan of X by $\mathcal{K} = \mathcal{C} \cap \mathcal{E}$.)

In characteristic zero, singularities of G-models of type A are good.

Theorem 16.2. *If X is of type A, then all G-subvarieties $Y \subseteq X$ are normal and have rational singularities.*

Proof. By the local structure theorem, we may assume that X is affine and of type A, i.e., $\Bbbk[X]^B \neq k$. Passing to the categorical quotient by U, we may assume that $G = B = T$. In the notation of Example 16.1, we may replace X by X_i and assume that X is an affine toric $(T \times \Bbbk^\times)$-variety such that $X /\!\!/ T \simeq \mathbb{A}^1$ ($\Bbbk^\times$-equivariantly). Then each T-stable closed subvariety of X is either $(T \times \Bbbk^\times)$-stable or lying in the fiber of the quotient map $X \to \Bbbk$ over a nonzero point, which is a toric T-variety. Thus the question is reduced to the case of toric varieties. □

17 Divisors

In the study of divisors on G-models, we may restrict our attention to B-stable ones, by the following result.

Proposition 17.1. *Let a connected solvable algebraic group B act on a variety X. Then any Weil divisor on X is rationally equivalent to a B-stable one.*

Proof. Replacing X by X^{reg}, we may assume that X is smooth. Replacing X by a B-stable open subset, we may assume that X is quasiprojective. Then any Weil divisor δ on X is Cartier. Furthermore, δ is the difference of two globally generated divisors. Therefore we may assume that δ is globally generated. The line bundle $\mathcal{O}(\delta)$ is B-linearizable by Theorem A1.2, and the B-module $\mathrm{H}^0(X, \mathcal{O}(\delta))$ contains a nonzero B-eigensection σ. The divisor $\operatorname{div}\sigma$ is B-stable and equivalent to δ. □

Remark 17.1. The proposition is true for any algebraic cycle, see Theorem 18.1.

Our first aim is to describe Cartier divisors.

For any Cartier divisor δ on X, we shall always equip the respective line bundle with a G-linearization (see Appendix A1).

Lemma 17.1 ([Kn5, 2.2]). *Any prime divisor $D \subset X$ that does not contain a G-orbit of X is globally generated Cartier.*

Proof. Let $\iota : X^{\mathrm{reg}} \hookrightarrow X$ be the inclusion of the subset of smooth points. Then $D \cap X^{\mathrm{reg}}$ is Cartier on X^{reg}, and $D \cap X^{\mathrm{reg}} = \operatorname{div}\eta$ for some $\eta \in \mathrm{H}^0(X^{\mathrm{reg}}, \mathcal{O}(D \cap X^{\mathrm{reg}}))$. As X is normal, $\mathcal{L} = \iota_*\mathcal{O}(D \cap X^{\mathrm{reg}})$ is a trivial line bundle on $X \setminus D$. As G acts on $\mathcal{L}$, the set of points where $\mathcal{L}$ is not invertible is G-stable and contained in D, hence empty. Therefore $\mathcal{L}$ is a line bundle on X.

If we regard η as an element of $\mathrm{H}^0(X, \mathcal{L})$, then $D = \operatorname{div}\eta$, because the equality holds on X^{reg} and $\operatorname{codim}_X(X \setminus X^{\mathrm{reg}}) > 1$. Furthermore, $\mathcal{L}$ is generated by $g\eta$, $g \in G$, because the set of their common zeroes is $\bigcap_{g\in G} gD = \emptyset$. The assertion follows. □

The following criterion says that a B-stable divisor is Cartier iff it is determined by a local equation in a neighborhood of a *general* point of each G-subvariety.

Theorem 17.1. *Suppose δ is a B-stable divisor on X. Then δ is Cartier iff for any G-subvariety $Y \subseteq X$ there exists $f_Y \in K^{(B)}$ such that each prime divisor $D \supseteq Y$ occurs in δ with multiplicity $v_D(f_Y)$.*

Proof. Suppose δ is locally principal in general points of G-subvarieties. Take any G-orbit $Y \subseteq X$ and a B-chart $\mathring{X} \subseteq X$ intersecting Y. Replacing δ by $\delta - \operatorname{div} f_Y$, we may assume that no component of δ contains Y. Let $D_1, \dots, D_n$

be the components of δ intersecting $\mathring{X}$ and $w_i = v_{D_i}$, $i = 1, \dots, n$. We have either $w_i \notin \mathcal{V}_Y$, or $D_i \notin \mathcal{D}_Y^B$ (depending on whether D_i is G-stable or not). By (V$'$) or (D$'$), $\exists f_i \in \Bbbk[\mathring{X}]^{(B)}: \ f_i|_Y \neq 0, \ w_i(f_i) > 0$. Now we may replace $\mathring{X}$ by its localization at $f_1 \dots f_n$ and assume that $\mathring{X}$ intersects no component of δ. By Lemma 17.1, δ is Cartier on $G\mathring{X}$, whence on X.

Now suppose δ is Cartier, and $Y \subseteq X$ is a G-subvariety. By Sumihiro's Theorem A1.3, there is an open G-stable quasiprojective subvariety $X_0 \subseteq X$ intersecting Y. The restriction of δ to X_0 may be represented as the difference of two globally generated divisors, hence we may replace X by X_0 and assume that δ is globally generated.

It follows that the annihilator of Y in $\mathrm{H}^0(X, \mathcal{O}(\delta))$ is a proper G-submodule, whence there is a section $\sigma \in \mathrm{H}^0(X, \mathcal{O}(\delta))^{(B)}$ such that $\sigma|_Y \neq 0$. Therefore δ is principal on the B-stable open subset X_σ intersecting Y, and we may take for f_Y the equation of δ on X_σ. □

By Theorem 17.1, a Cartier divisor on X is determined by the following data:

(1) a collection of rational B-eigenfunctions f_Y given for each G-germ $Y \in {}_G X$ and such that $w(f_{Y_1}) = w(f_{Y_2})$, $v_D(f_{Y_1}) = v_D(f_{Y_2})$, $\forall w \in \mathcal{V}_{Y_1} \cap \mathcal{V}_{Y_2}$, $D \in \mathcal{D}_{Y_1}^B \cap \mathcal{D}_{Y_2}^B$.

(2) a collection of integers m_D, $D \in \mathcal{D}^B \setminus \bigcup_{Y \subseteq X} \mathcal{D}_Y^B$, only finitely many of them being nonzero (m_D is the multiplicity of D in the divisor).

Remark 17.2. It suffices to specify the local equations f_Y only for closed G-orbits $Y \subseteq X$: if a G-subvariety $Y \subseteq X$ contains a closed orbit Y_0, then we may put $f_Y = f_{Y_0}$.

When a Cartier divisor is replaced by a rationally equivalent one, the local equations f_Y are replaced by $f_Y f$ for some $f \in K^{(B)}$, and m_D are replaced by $m_D + v_D(f)$.

In the case of complexity ≤ 1, the data (1)–(2) are retranslated to the language of convex geometry.

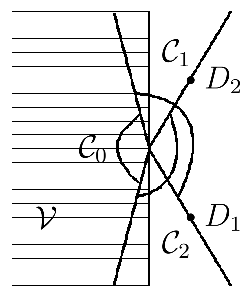
Consider first the spherical case. Each f_Y defines a function ψ_Y on the cone $\mathcal{C}_Y$, which is the restriction of a linear functional $\lambda_Y \in \Lambda$. We may assume that $f_{Y_1} = f_{Y_2}$ if $\mathcal{C}_{Y_1}$ is a face of $\mathcal{C}_{Y_2}$, whence $\psi_{Y_1} = \psi_{Y_2}|_{\mathcal{C}_{Y_1}}$. In particular, ψ_Y paste together in a piecewise linear function on $\bigcup_{Y \subseteq X} (\mathcal{C}_Y \cap \mathcal{V})$. A collection $\psi = (\psi_Y)$ of functions ψ_Y on $\mathcal{C}_Y$ with the above properties is called an *integral piecewise linear function* on the colored fan $\mathcal{F}$ of X.

Note that generally ψ is *not* a well-defined function on $\bigcup_{\mathcal{C} \in \mathcal{F}} \mathcal{C}$ as the following example shows.

Example 17.1 ([Pau3]). Let $G = \mathrm{SL}_3(\Bbbk)$, $H = \mathrm{SL}_2(\Bbbk)$ = the common stabilizer of $e_1 \in \Bbbk^3$, $x_1 \in (\Bbbk^3)^*$. Then $\mathcal{O} = G/H$ is defined in $\Bbbk^3 \oplus (\Bbbk^3)^*$ by an equation $\langle v, v^* \rangle = 1$. The B-divisors $D_1, D_2 \in \mathcal{D}^B$ are defined by the restrictions η_1, η_2 of linear B-eigenfunctions on $\Bbbk^3 \oplus (\Bbbk^3)^*$ of B-weights ω_1, ω_2. Here $\eta_1, \eta_2 \in K^{(B)}$ and $\Lambda = \langle \omega_1, \omega_2 \rangle$, whence $\rho(D_i) = \alpha_i^\vee$.

A one-dimensional torus acting on the summands of $\Bbbk^3 \oplus (\Bbbk^3)^*$ by the weights ± 1 commutes with G and preserves $\mathcal{O}$. The respective grading of $\Bbbk(\mathcal{O})$ determines two G-valuations $v_\pm(f) = \pm \deg f_\mp$, where $f_\pm$ is the highest/lowest degree term of $f \in \Bbbk[\mathcal{O}]$. Since $\deg \eta_i = (-1)^{i-1}$, we have $v_\pm = \pm\alpha_1^\vee \mp \alpha_2^\vee$. It follows easily from Corollary 15.2 that the colored space looks like in Figure 3.6.

Figure 3.6:



Take a fan $\mathcal{F}$ determined by the cones $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ on the figure. Since $\mathcal{C}_1 \cap \mathcal{C}_2$ is a solid cone, a piecewise linear function on $\mathcal{F}$ defines a function on $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{E}$ iff it is linear.

Let $\mathrm{PL}(\mathcal{F})$ be the group of all integral piecewise linear functions on $\mathcal{F}$, and $\mathrm{L}(\mathcal{F})$ be its subgroup of linear functions $\psi = (\lambda|_{\mathcal{C}_Y})$, $\lambda \in \Lambda$.

The above discussion yields the following exact sequences:

$$0 \longrightarrow \mathbb{Z}\left(\mathcal{D}^B \setminus \bigcup_{Y \subseteq X} \mathcal{D}_Y^B\right) \longrightarrow \mathrm{CaDiv}(X)^B \longrightarrow \mathrm{PL}(\mathcal{F}) \longrightarrow 0 \tag{17.1}$$

$$\Lambda \cap \mathcal{F}^\perp \longrightarrow \mathrm{PrDiv}(X)^B \longrightarrow \mathrm{L}(\mathcal{F}) \longrightarrow 0 \tag{17.2}$$

where $\mathrm{CaDiv}(\cdot)$ and $\mathrm{PrDiv}(\cdot)$ denote the groups of Cartier and principal divisors, respectively, and $\mathcal{F}^\perp$ is the annihilator of the union of all cones in $\mathcal{F}$.

Theorem 17.2 ([Bri4, 3.1]). *There is an exact sequence*

$$\Lambda \cap \mathcal{F}^\perp \longrightarrow \mathbb{Z}\left(\mathcal{D}^B \setminus \bigcup_{Y \subseteq X} \mathcal{D}_Y^B\right) \longrightarrow \mathrm{Pic}\, X \longrightarrow \mathrm{PL}(\mathcal{F})/\mathrm{L}(\mathcal{F}) \longrightarrow 0$$

If X contains a complete G-orbit, then $\operatorname{Pic} X$ *is free Abelian of finite rank.*

Proof. The exact sequence is a consequence of (17.1)–(17.2). If $Y \subseteq X$ is a complete G-orbit, then $\mathcal{F}^{\perp} \subseteq \mathcal{C}_Y^{\perp} = 0$ by Propositions 15.1 and 10.1. Then it is easy to see that $\mathrm{PL}(\mathcal{F})/\mathrm{L}(\mathcal{F})$ has finite rank and no torsion, whence the second assertion. $\square$

A spherical G-variety X having only one closed orbit $Y \subseteq X$ is called *simple*. Its fan consists of all supported colored faces of $(\mathcal{C}_Y, \mathcal{D}_Y^B)$.

Corollary 17.1. *If X is simple with the closed orbit $Y \subseteq X$, then there is an exact sequence*

$$\Lambda \cap \mathcal{C}_Y^{\perp} \longrightarrow \mathbb{Z}(\mathcal{D}^B \setminus \mathcal{D}_Y^B) \longrightarrow \operatorname{Pic} X \longrightarrow 0$$

Corollary 17.2. *If X is simple and the closed orbit $Y \subseteq X$ is complete, then* $\operatorname{Pic} X = \mathbb{Z}(\mathcal{D}^B \setminus \mathcal{D}_Y^B)$ *is free Abelian.*

Example 17.2. If $X = G/P$ is a generalized flag variety, then $\operatorname{Pic} X$ is freely generated by the Schubert divisors $D_\alpha = \overline{B[w_G r_\alpha]}$, $\alpha \in \Pi \setminus I$, where $I \subseteq \Pi$ is the set of simple roots defining the parabolic $P \supseteq B$.

Example 17.3 ([Bri4]). Let $G = \mathrm{SL}_3(\Bbbk)$, $H = N_G(\mathrm{SO}_3(\Bbbk)) = \mathrm{SO}_3(\Bbbk) \times \mathbf{Z}_3$, where $\mathbf{Z}_3 = Z(\mathrm{SL}_3(\Bbbk))$. Then $\mathcal{O} = G/H$ is the space of conics in $\mathbb{P}^2$. The coisotropy representation is the natural representation of $\mathrm{SO}_3(\Bbbk)$ in traceless symmetric matrices, whence $H_* = \left\{ \left(\begin{smallmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{smallmatrix} \right) \right\}$ is the Klein 4-group, and $\Lambda(\mathcal{O}) = \langle 2\alpha_1, 2\alpha_2 \rangle$, where α_i are simple roots of $\mathrm{SL}_3(\Bbbk)$. By (9.1), $\mathcal{O}$ is spherical.

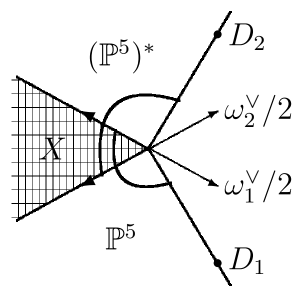
We may consider $\mathcal{O}$ as the projectivization of the open subset of non-degenerate quadratic forms in $\mathrm{S}^2(\Bbbk^3)^*$. The two $(B \times H)$-eigenfunctions $\eta_1(q) = q_{11}$, $\eta_2(q) = \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix}$ $(q \in \mathrm{S}^2(\Bbbk^3)^*)$ of biweights $(2\omega_i, 2i\varepsilon)$, where ω_i are the fundamental weights of G and ε is the weight of $\mathbf{Z}_3$ in $\Bbbk^3$, define the two B-divisors D_1, D_2. They impose the conditions that a conic passes through the B-fixed point, resp. is tangent to the B-fixed line. Since $\mathbf{f}_{2\alpha_1} = \eta_1^2/\eta_2$, $\mathbf{f}_{2\alpha_2} = \eta_2^2/\eta_1 \in K^{(B)}$, and their weights $2\alpha_1, 2\alpha_2$ generate Λ, there are no other B-divisors. (Indeed, if $\eta \in \Bbbk[G]^{(B\times H)}_{(\lambda,\chi)}$, then either $\chi = 0$, $\lambda \in \Lambda$, or $\chi = 2i\varepsilon$, $\lambda - 2\omega_i \in \Lambda$, hence η is proportional to the product of η_1, η_2 and their inverses.) Furthermore, $\Lambda^* = \langle \omega_1^\vee/2, \omega_2^\vee/2 \rangle$ and $\rho(D_i) = \alpha_i^\vee/2$.

The complement to $\mathcal{O}$ in $(\mathbb{P}^5)^* = \mathbb{P}(\mathrm{S}^2(\Bbbk^3)^*)$ is a G-stable prime divisor $\{\det q = 0\}$, and the respective G-valuation is $-\omega_2^\vee/2 \in \mathcal{E}$, because in homogeneous coordinates $\mathbf{f}_{2\alpha_1}(q) = \eta_1(q)^2/\eta_2(q)$, $\mathbf{f}_{2\alpha_2}(q) = \eta_2(q)^2/\eta_1(q) \det q$. The unique closed orbit $Y = \{\operatorname{rk} q = 1\}$ has the colored data $\mathcal{V}_Y = \{-\omega_2^\vee/2\}$, $\mathcal{D}_Y^B = \{D_2\}$.

Similarly, we embed $\mathcal{O}$ in $\mathbb{P}^5 = \mathbb{P}(\mathrm{S}^2\Bbbk^3)$ by the map $q \to q^\vee$ (=the adjoint matrix of q) sending a conic to the dual one. Here the unique closed orbit $Y^\vee = \{\operatorname{rk} q^\vee = 1\}$ has the colored data $\mathcal{V}_{Y^\vee} = \{-\omega_1^\vee/2\}$, $\mathcal{D}_{Y^\vee}^B = \{D_1\}$.

Since $\mathbb{P}^5, (\mathbb{P}^5)^*$ are complete, the cones $\mathcal{C}_Y$ and $\mathcal{C}_{Y^\vee}$ contain $\mathcal{V}$, whence $\mathcal{V}$ is generated by $-\omega_1^\vee/2, -\omega_2^\vee/2$. The colored equipment of $\mathcal{O}$ is represented at Figure 3.7.

Figure 3.7:



The closure X of the diagonal embedding $\mathcal{O} \hookrightarrow \mathbb{P}^5 \times (\mathbb{P}^5)^*$ is called the *space of complete conics*. It is determined in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$ by the equation "$q \cdot q^\vee$ is s scalar matrix", and this implies by direct computations that X is smooth. The unique closed orbit $\widehat{Y} \subseteq X$ has the colored data $\mathcal{V}_{\widehat{Y}} = \{-\omega_1^\vee/2, -\omega_2^\vee/2\}$, $\mathcal{D}_{\widehat{Y}}^B = \emptyset$.

By Corollary 17.2, $\operatorname{Pic} \mathbb{P}^5 \simeq \operatorname{Pic}(\mathbb{P}^5)^* \simeq \mathbb{Z}$ are freely generated by D_1, resp. D_2, and $\operatorname{Pic} X \simeq \mathbb{Z}^2$ is freely generated by D_1, D_2.

Remark 17.3. In fact, the space of smooth conics is a symmetric variety and the space of complete conics is its "wonderful completion", see §26, §30.

In the case of complexity 1, the description of Cartier divisors is similar, but one should speak not only of cones, but also of hypercones $\mathcal{C}_Y$, and of admissible functionals λ_Y which are integral on Λ^* and such that $\sum \langle \varepsilon_x, \lambda_Y \rangle x$ is a principal divisor on C.

In particular, if X is simple, then $\operatorname{Pic} X$ is generated by a finite set $\mathcal{D}^B \setminus \mathcal{D}_Y^B$, where $Y \subseteq X$ is the closed orbit, and Corollary 17.2 is true.

Example 17.4. If X is the space of complete triangles of Example 16.3, then $\operatorname{Pic} X = \langle D_i, \widetilde{D}_i \mid i = 1, 2, 3 \rangle \simeq \mathbb{Z}^6$.

In characteristic zero, the G-module structure of the space of global sections of a Cartier divisor is determined by the set of B-eigensections. If σ is a rational B-eigensection of $\mathcal{O}(\delta)$ such that $\operatorname{div}\sigma = \delta$, then $\mathrm{H}^0(X,\mathcal{O}(\delta))^{(B)} = \{f\sigma \mid f \in K^{(B)},\ \operatorname{div} f + \delta \geq 0\}$. The B-weight of $\eta = f\sigma \in \mathrm{H}^0(X,\mathcal{O}(\delta))^{(B)}$ equals $\lambda + \pi(\delta)$, where λ is the B-weight of f and $\pi(\delta)$ is the B-weight of σ. The multiplicity of $V(\lambda+\pi(\delta))$ in $\mathrm{H}^0(X,\mathcal{O}(\delta))$ equals

$$m_\lambda(\delta) = \dim\{f \in K^B \mid \operatorname{div} f + \operatorname{div}\mathbf{f}_\lambda + \delta \geq 0\} \tag{17.3}$$

The weight $\pi(\delta)$ is defined up to a character of G and may be determined as follows. Consider a generic G-orbit $Y \subseteq X$ and let $\widetilde{\delta}$ be the pull-back on G of $\delta \cap Y$. As G is a factorial variety, $\widetilde{\delta}$ is defined by an equation $F \in k(G)^{(B)}$. Then $\pi(\delta)$ is the weight of F.

In the case $c(X) \leq 1$, the description of $\mathrm{H}^0(X,\mathcal{O}(\delta))$ is given in the language of convex geometry.

If $c(X) = 0$, then the set of highest weights of $\mathrm{H}^0(X,\mathcal{O}(\delta))$ is $\pi(\delta) + \mathcal{P}(\delta) \cap \Lambda$, where

(17.4)
$$\mathcal{P}(\delta) = \left\{ \lambda \in \bigcap_{Y \subseteq X} (-\lambda_Y + \mathcal{C}_Y^\vee) \;\middle|\; \forall D \in \mathcal{D}^B \setminus \bigcup_{Y\subseteq X} \mathcal{D}_Y^B :\ \langle \lambda, \rho(D)\rangle + m_D \geq 0 \right\}$$

and all highest weights occur with multiplicity 1.

Example 17.5. A Schubert divisor $D_{\alpha_i} \subseteq G/P_I$, $\alpha_i \in \Pi \setminus I$, is defined by an equation $\langle v, gv^*\rangle = 0$, $v \in V(\omega_i^*)$, $v^* \in V(\omega_i)$. Hence $\pi(D_{\alpha_i}) = \omega_i^*$. For $\delta = \sum a_i D_{\alpha_i}$, we have $\mathcal{O}(\delta) = \mathcal{L}(-\sum a_i\omega_i)$, $\pi(\delta) = \sum a_i\omega_i^*$, $\mathcal{P}(\delta) = \{0\}$, and $\mathrm{H}^0(G/P_I,\mathcal{O}(\delta)) = V(\sum a_i\omega_i^*)$ (the Borel–Weil theorem, cf. §2).

Example 17.6. Consider $X = \mathbb{P}^{d-1} \times (\mathbb{P}^{d-1})^*$ as a simple projective embedding of a symmetric space $\mathcal{O} = \mathrm{SL}_d/\mathrm{S}(\mathrm{L}_1 \times \mathrm{L}_{d-1})$. Then $X \setminus \mathcal{O}$ is a homogeneous divisor consisting of all pairs (x,y) such that the point x lies in the hyperplane y. It is defined by an equation $\sum x_iy_i = 0$, where $x_1,\dots,x_d$ $(y_1,\dots,y_d)$ are projective coordinates on $\mathbb{P}^{d-1}$ (resp. $(\mathbb{P}^{d-1})^*$). The two B-divisors D, D' are defined by B-eigenfunctions y_1, x_d of biweights $(\omega_1,(d-1)\varepsilon)$, $(\omega_{d-1},(1-d)\varepsilon)$, respectively, where ε generates $\mathfrak{X}(\mathrm{S}(\mathrm{L}_1\times\mathrm{L}_{d-1}))$. One has $\Lambda = \langle \omega_1+\omega_{d-1}\rangle \simeq \mathbb{Z}$, $\mathbf{f}_{\omega_1+\omega_{d-1}}(x,y) = x_dy_1/\sum x_iy_i$. It follows easily that $\mathcal{E} \simeq \mathbb{Q} \supset \mathcal{V} = \mathbb{Q}_-$, $\rho(D) = \rho(D') = 1$.

By Corollary 17.2, $\operatorname{Pic}(X) = \mathbb{Z}D \oplus \mathbb{Z}D'$, and for $\delta = mD + nD'$ we have $\pi(\delta) = m\omega_1 + n\omega_{d-1}$, $\mathcal{P}(\delta) = \{\lambda = -k(\omega_1+\omega_{d-1}) \mid k, m-k, n-k \geq 0\}$. On the other hand, $\mathcal{O}(\delta) = \mathcal{O}(\mathbb{P}^{d-1})n \otimes \mathcal{O}((\mathbb{P}^{d-1})^*)m$, whence $\mathrm{H}^0(X,\mathcal{O}(\delta)) = \mathrm{S}^m\Bbbk^d \otimes \mathrm{S}^n(\Bbbk^d)^* = V(m\omega_1) \otimes V(n\omega_{d-1})$. We obtain a decomposition formula

$$V(m\omega_1) \otimes V(n\omega_{d-1}) = \bigoplus_{0 \leq k \leq \min(m,n)} V((m-k)\omega_1 + (n-k)\omega_{d-1})$$

For other applications to computing tensor product decompositions, including Pieri formulae, see [Bri4, 2.5].

Now assume $c(X) = 1$. Put $\delta = \sum m_D D$ (D runs through all B-stable divisors on X) and $\rho(D) = (h_d, \ell_D) \in \mathcal{E}_{x_D,+}$, $x_d \in C$ (=the smooth projective curve with the function field K^B).

Definition 17.1. A *pseudodivisor* on C is a formal linear combination $\mu = \sum_{x\in C} m_x \cdot x$, where $m_x \in \mathbb{R} \cup \{\pm\infty\}$, and all but finitely many m_x are 0. Put $\mathrm{H}^0(C, \mu) = \{f \in \Bbbk(C) \mid \operatorname{div} f + \mu \geq 0\}$. (Here we assume $\forall c \in \mathbb{R} : c + (\pm\infty) = \pm\infty$.)

If all $m_x \neq -\infty$, then $\mathrm{H}^0(C, \mu)$ is just the space of global sections of the divisor $[\mu] = \sum [m_x] \cdot x$ on $C \setminus \{x \mid m_x = +\infty\}$, otherwise $\mathrm{H}^0(C, \mu) = 0$.

Consider the pseudodivisor

$$\mu = \mu(\delta, \lambda) = \sum_{x\in C} \left(\min_{x_D = x} \frac{\langle\lambda, \ell_D\rangle + m_D}{h_D} \right) x$$

$$\left(\text{Here we assume } \frac{c}{0} = \begin{cases} +\infty, & c \geq 0 \\ -\infty, & c < 0 \end{cases}. \right)$$

Since $\operatorname{div} f = \sum h_D v_{x_D}(f) \cdot D$, $\forall f \in K^B$, and $\operatorname{div} \mathbf{f}_\lambda = \sum \langle\lambda, \ell_D\rangle \cdot D$, it follows from (17.3) that $m_\lambda(\delta) = h^0(\delta, \lambda) := \dim \mathrm{H}^0(C, \mu)$.

We have $h^0(\delta, \lambda) = 0$ outside the polyhedral domain

$$\mathcal{P}(\delta) = \{\lambda \mid \langle\lambda, \ell_D\rangle \geq -m_D \text{ for } \forall D \text{ such that } h_D = 0\} \subseteq \Lambda \otimes \mathbb{R}.$$

If there is $x \in C$ such that $x_D \neq x$ for all D with $h_D > 0$, then $h^0(\delta, \lambda) = \infty$ for $\forall \lambda \in \mathcal{P}(\delta)$, because in this case $h^0(\delta, \lambda)$ is the dimension of the space of sections of a divisor on an affine curve. Otherwise, by the Riemann–Roch theorem,

$$\begin{aligned} h^0(\delta, \lambda) &= \deg[\mu] - g + 1 + h^1(\delta, \lambda) \\ &= A(\delta, \lambda) - \sigma(\delta, \lambda) - g + 1 + h^1(\delta, \lambda), \end{aligned}$$

where g is the genus of C, $h^1(\delta, \lambda) = \dim \mathrm{H}^1(C, [\mu])$,

$$A(\delta, \lambda) = \sum_{x\in C} \left(\min_{x_D = x} \frac{\langle\lambda, \ell_D\rangle + m_D}{h_D} \right)$$

is a piecewise affine concave function of λ, and $\sigma(\delta, \lambda)$ is bounded nonnegative for all δ, λ. Furthermore, as $h^0(\delta, \lambda) \leq \deg[\mu] + 1$ whenever $\deg[\mu] \geq$

0 [Har, ex.IV.1.5], we have $h^1(\delta,\lambda) \le g$ if $A(\delta,\lambda) \ge \sigma(\delta,\lambda)$. Note also that $A(n\delta, n\lambda) = nA(\delta,\lambda)$.

It follows that $h^0(\delta,\lambda) = 0$ if $A(\delta,\lambda) < 0$, and $h^0(\delta,\lambda) = 0$ differs from $A(\delta,\lambda)$ by a globally bounded function whenever $A(\delta,\lambda) \ge 0$, $\lambda \in \mathcal{P}(\delta)$. This gives the asymptotic behaviors of $h^0(\delta,\lambda)$ as $(\delta,\lambda) \to \infty$ in a fixed direction.

Now we give criteria for a Cartier divisor to be globally generated and ample.

Theorem 17.3. *Suppose δ is a Cartier divisor on X determined by the data $\{f_Y\}$, $\{m_D\}$.*

(1) *δ is globally generated iff local equations f_Y can be chosen in such a way that for any G-subvariety $Y \subseteq X$ the following two conditions are satisfied:*

(a) *For any other G-subvariety $Y' \subseteq X$ and each B-stable prime divisor $D \supseteq Y'$, $v_D(f_Y) \le v_D(f_{Y'})$.*

(b) $\forall D \in \mathcal{D}^B \setminus \bigcup_{Y' \subseteq X} \mathcal{D}^B_{Y'} :\ v_D(f_Y) \le m_D$.

(2) *δ is ample iff, after replacing δ by a certain multiple, local equations f_Y can be chosen in such a way that, for any G-subvariety $Y \subseteq X$, there exists a B-chart $\mathring{X}$ of Y such that* (a) *and* (b) *are satisfied and*

(c) *the inequalities therein are strict iff $D \cap \mathring{X} = \emptyset$.*

Proof. (1) δ is globally generated iff for any G-subvariety $Y \subseteq X$, there is $\eta \in \mathrm{H}^0(X, \mathcal{O}(\delta))$ such that $\eta|_Y \ne 0$. We may assume η to be a B-eigensection. This means that $\exists f \in K^{(B)} :\ \operatorname{div} f + \delta \ge 0$, and no $D \supseteq Y$ occurs in $\operatorname{div} f + \delta$ with positive multiplicity. Replacing f_Y by f^{-1} yields the conditions (a)–(b). Conversely, if (a)–(b) hold then $f = f_Y^{-1}$ yields the desired global section.

(2) Suppose δ is ample. Replacing δ by a multiple, we may assume that δ is very ample. Consider the G-equivariant projective embedding $X \hookrightarrow \mathbb{P}(M^*)$ defined by a certain finite-dimensional G-submodule $M \subseteq \mathrm{H}^0(X, \mathcal{O}(\delta))$. Take a G-subvariety $Y \subseteq X$. There exists a homogeneous B-eigenpolynomial in homogeneous coordinates on $\mathbb{P}(M^*)$ (i.e., a section in $\mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes N})^{(B)}$) that vanishes on $\overline{X} \setminus X$ but not on Y. Replacing δ by $N\delta$, we may assume that $\exists \eta \in \mathrm{H}^0(X, \mathcal{O}(\delta))^{(B)} :\ \eta|_{\overline{X}\setminus X} = 0,\ \eta|_Y \ne 0$. Then $\mathring{X} = X_\eta$ is a B-chart of Y, and $\exists f \in K^{(B)} :\ \operatorname{div} f + \delta = \operatorname{div} \eta \ge 0$. It remains to replace f_Y by f^{-1}.

Conversely, assume that the conditions (a)–(c) hold. For any G-subvariety $Y \subseteq X$, there is a section $\eta \in \mathrm{H}^0(X, \mathcal{O}(\delta))^{(B)}$ determined by f_Y^{-1}, and $\mathring{X} = X_\eta$ is a B-chart of Y. We may pick finitely many B-charts $\mathring{X}_\alpha$ of this kind in such a way that $G\mathring{X}_\alpha$ cover X. Let $\eta_\alpha \in \mathrm{H}^0(X, \mathcal{O}(\delta))^{(B)}$ be the respective global sections. Then

$$\Bbbk[\mathring{X}_\alpha] = \bigcup_{n\geq 0} \eta_\alpha^{-n}\mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes n}) = \Bbbk\left[\frac{\sigma_{\alpha,1}}{\eta_\alpha^{n_\alpha}}, \dots, \frac{\sigma_{\alpha,s_\alpha}}{\eta_\alpha^{n_\alpha}}\right]$$

for some $n_\alpha, s_\alpha \in \mathbb{N}$, $\sigma_{\alpha,i} \in \mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes n_\alpha})$. Replacing δ by a multiple, we may assume $n_\alpha = 1$.

Take the finite-dimensional G-submodule $M \subseteq \mathrm{H}^0(X, \mathcal{O}(\delta))$ generated by $\eta_\alpha, \sigma_{\alpha,i}$, $\forall \alpha, i$. The respective rational map $X \dashrightarrow \mathbb{P}(M^*)$ is G-equivariant and defined on $\mathring{X}_\alpha$, hence everywhere. Moreover, $\phi^{-1}(\mathbb{P}(M^*)_{\eta_\alpha}) = \mathring{X}_\alpha$ and $\phi|_{\mathring{X}_\alpha}$ is a closed embedding in $\mathbb{P}(M^*)_{\eta_\alpha}$. Therefore ϕ is a locally closed embedding and δ is very ample. $\square$

Remark 17.4. If X is complete and δ is very ample, then the conditions (a)–(c) hold for δ itself.

Corollary 17.3. *If $\mathring{X}$ is a B-chart and $X = G\mathring{X}$, then a divisor $\sum_{D\subseteq X\setminus\mathring{X}} m_D D$ is globally generated (ample) iff all $m_D \geq 0$ ($m_D > 0$). In particular, X is quasiprojective.*

Corollary 17.4. *If X is simple and $Y \subseteq X$ is the closed G-orbit, then globally generated (ample) divisor classes in* $\operatorname{Pic} X$ *are those $\delta = \sum_{D\in\mathcal{D}^B\setminus\mathcal{D}^B_Y} m_D D$ with $m_D \geq 0$ ($m_D > 0$). In particular, any simple G-variety is quasiprojective. (This also stems from Sumihiro's theorem.)*

In the case of complexity ≤ 1, conditions (a)–(b) mean that $\lambda_Y \leq \psi_{Y'}$ on $\mathcal{C}_{Y'}$ and $\langle\lambda_Y, \rho(D)\rangle \leq m_D$, and (c) means that the inequalities therein are strict outside $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ and $\mathcal{R}$.

The description of globally generated and ample divisors on spherical X in terms of piecewise linear functions is more transparent if X is complete (or all closed G-orbits $Y \subseteq X$ are complete). Then maximal cones $\mathcal{C}_Y \in \mathcal{F}$ are solid, and λ_Y are determined by ψ_Y.

Definition 17.2. A function $\psi \in \mathrm{PL}(\mathcal{F})$ is (*strictly*) *convex* if $\lambda_Y \leq \psi_{Y'}$ on $\mathcal{C}_{Y'}$ (resp. $\lambda_Y < \psi_{Y'}$ on $\mathcal{C}_{Y'} \setminus \mathcal{C}_Y$) for any two maximal cones $\mathcal{C}_Y, \mathcal{C}_{Y'} \in \mathcal{F}$.

Corollary 17.5. *If X is complete (or all closed G-orbits in X are complete) and spherical, then δ is globally generated (ample) iff ψ is (strictly) convex on $\mathcal{F}$ and $\langle\lambda_Y, \rho(D)\rangle \leq m_D$ (resp. $< m_D$) for any closed G-orbit $Y \subseteq X$ and $\forall D \in \mathcal{D}^B \setminus \bigcup_{Y'\subseteq X} \mathcal{D}^B_{Y'}$.*

Proof. It suffices to note that Y has a unique B-chart $\mathring{X}_Y$ given by the colored cone $(\mathcal{C}_Y, \mathcal{D}_Y^B)$, and conditions (a)–(c) are satisfied for δ iff they are satisfied for its multiple. □

Corollary 17.6. *On a complete spherical variety, every ample divisor is globally generated.*

The above results extend to the case of complexity 1, if all closed G-orbits in X are complete and of type B.

Remark 17.5. It follows from the proof of Theorem 17.3(2) that δ is very ample if $\Bbbk[\mathring{X}_\alpha]$ is generated by $\eta_\alpha^{-1}\mathrm{H}^0(X, \mathcal{O}(\delta))$ for $\forall\alpha$. This may be effectively verified in some cases using Lemma 13.1 for $R = \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes n})$, $S = \bigoplus_{n\geq 0}\big[\mathrm{H}^0(X, \mathcal{O}(\delta))\big]^n$, $\eta = \eta_\alpha$. For example, if X is complete and spherical, then it suffices to verify that for each closed G-orbit $Y \subseteq X$ the polyhedral domain $\lambda_Y + \mathcal{P}(\delta)$ contains the generators of the semigroup $\mathcal{C}_Y^\vee \cap \Lambda$.

Example 17.7. On a generalized flag variety $X = G/P$, globally generated (ample) divisors are distinguished in the set of all B-divisors $\delta = \sum a_i D_{\alpha_i}$ by the conditions $a_i \geq 0$ (resp. $a_i > 0$). Every ample divisor is very ample.

Example 17.8. The variety X defined by the colored fan $\mathcal{F}$ from Example 17.1 is complete, but not projective. Indeed, since $\mathcal{C}_1 \cap \mathcal{C}_2$ is a solid cone (Figure 3.6), a convex piecewise linear function on $\mathcal{F}$ is forced to be linear on $\mathcal{C}_1 \cup \mathcal{C}_2$, whence globally on $\mathcal{E}$. Hence there are no non-principal globally generated divisors on X.

Remark 17.6. If a fan $\mathcal{F}$ in a two dimensional colored space has no colors, then the interiors of all cones in $\mathcal{F}$ are disjoint and there exists a strictly convex piecewise linear function on $\mathcal{F}$. Therefore all toroidal spherical varieties (in particular, all toric varieties) of rank 2 are quasiprojective. However, one can construct a complete, but not projective, toric variety of rank 3 [Ful2, p.71].

Example 17.9. The same reasoning as in Example 17.8 shows that an SL_2-embedding X containing at least two G-germs of types B_-, B_0 (Figure 3.4) is not quasiprojective. (Here $r(X) = 1$, $\dim X = 3$.) On the other hand, if X contains at most one G-germ of type B_- or B_0, then it is easy to construct a strictly convex piecewise linear function on the hyperfan of X, whence X is quasiprojective. (For smooth X, this was proved in [MJ2, 6.4].)

18 Intersection theory

Our basic reference in intersection theory is [Ful1]. We begin our study of algebraic cycles on G-models with the following general result reducing everything to B-stable cycles.

Theorem 18.1 ([FMSS]). *Let a connected solvable algebraic group B act on a variety X. Then the Chow group $A_d(X)$ is generated by the B-stable d-cycles with the relations $[\operatorname{div} f] = 0$, where f is a rational B-eigenfunction on a B-stable $(d+1)$-subvariety of X.*

Proof. Using the equivariant completion of X and the equivariant Chow lemma [PV, Th.1.3], one reduces the assertion to the case of projective X by induction on $\dim X$ with the help of the standard technique of exact sequences [FMSS]. The projective case was handled by Vust [MJ2, 6.1] and Brion [Bri10, 1.3]. The idea is to consider the B-action on the Chow variety Z containing a given effective d-cycle z. Applying the Borel fixed point theorem, we find a B-stable cycle $z_0 \in \overline{Bz}$. An easy induction on $\dim B$ shows that z_0 can be connected with z by a sequence of rational curves, whence is rationally equivalent to z. The assertion on relations is proved by a similar technique, see [Bri10, 1.3] for details. □

This theorem clarifies almost nothing in the structure of Chow groups of general G-varieties, because the set of B-stable cycles is almost as vast as the set of all cycles; however it is very useful for G-varieties of complexity ≤ 1.

Assume X is a unirational G-variety of complexity ≤ 1 or a B-stable subvariety in it. (The assumption of unirationality is needless in the spherical case, since X has an open B-orbit. If $c(X) = 1$, then unirationality means that $K^B = \Bbbk(\mathbb{P}^1)$ for $K = \Bbbk(X)$.)

Corollary 18.1. *$A_*(X)$ is finitely generated. If $U : X$ has finitely many orbits, then $A_*(X)$ is freely generated by U-orbit closures.*

Proof. If $c(X) = 0$, then $B : X$ has finitely many orbits, whence $A_*(X)$ is generated by B-orbit closures. If $c(X) = 1$, then by Theorem 5.1, each irreducible B-stable subvariety $Y \subseteq X$ is either a B-orbit closure or the closure of a one-parameter family of B-orbits. In the second case, Y is one of finitely many irreducible components of $X_k = \{x \in X \mid \dim Bx \leq k\}$, $0 \leq k < \dim X$, and it follows from Lemma 5.1 that an open B-stable subset $\mathring{Y} \subseteq Y$ admits a geometric quotient $\mathring{Y}/B$ which is a smooth rational curve. Hence all B-orbits in $\mathring{Y}$ are rationally equivalent and each B-orbit, except finitely many of them, lies in one of $\mathring{Y}$. Therefore $A_*(X)$ is generated by finitely many B-orbit closures and irreducible components of X_k. □

Corollary 18.2. *If X is complete, then:*

(1) *The cone of effective cycles $A_d^+(X)_{\mathbb{Q}} \subseteq A_d(X) \otimes \mathbb{Q}$ is a polyhedral cone generated by the classes of rational subvarieties.*

(2) *Algebraic equivalence coincides with rational equivalence of cycles on X.*

Proof. (1) Similar to the proof of Theorem 18.1 using Corollary 18.1.

(2) The group of cycles algebraically equivalent to 0 modulo rational equivalence is divisible [Ful1, 19.1.2]. ☐

Corollary 18.3. *If X is smooth and complete (projective if $c(X) = 1$), then the cycle map $A_*(X) \to H_*(X)$ is an isomorphism of free Abelian groups of finite rank. (Here $\Bbbk = \mathbb{C}$, but one may also consider étale homology and Chow groups with corresponding coefficients for arbitrary $\Bbbk$.)*

Proof. If $c(X) = 0$, then it is easy to deduce from Theorem 18.1 that the Künneth map $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$ is an isomorphism for $\forall Y$, and the assertion follows from the fact that $z = \sum(u_i \cdot z)v_i$ for $\forall z \in A_*(X)$, where $\sum u_i \otimes v_i$ is the class of the diagonal in $X \times X$ [FMSS, §3]. If X is projective, then one uses the Białynicki-Birula decomposition [BB1]: X is covered by finitely many B-stable locally closed strata X_i, where each X_i is a vector bundle over a connected component X_i^T of X^T, and either $X_i^T = \mathrm{pt}$ or $X_i^T = \mathbb{P}^1$. This yields a cellular decomposition of X, and we conclude by [Ful1, 19.1.11]. ☐

Remark 18.1. The corollaries extend to an arbitrary variety X with an action of a connected solvable group B having finitely many orbits.

Remark 18.2. If X is not unirational, then Corollaries 18.1, 18.2(1) remain valid after replacing $A_*(X)$ by the group $B_*(X)$ of cycles modulo algebraic equivalence.

Example 18.1. If X is a generalized flag variety or its Schubert subvariety, then $A_*(X) \simeq H_*(X)$ is freely generated by Schubert subvarieties in X.

Now we discuss intersection theory on varieties of complexity ≤ 1.

Let X be a projective G-model of complexity ≤ 1, and $\operatorname{char} \Bbbk = 0$. A method to compute intersection numbers of Cartier divisors on X was introduced by Brion [Bri4, §4] in the spherical case and generalized in [Tim3, §8] to the case of complexity 1.

Put $\dim X = d$, $c(X) = c$ $(= 0, 1)$, $r(X) = r$.

The Néron–Severi group $\mathrm{NS}(X)$ of Cartier divisors modulo algebraic equivalence is finitely generated, and the intersection form is a d-linear form on the finite-dimensional vector space $\mathrm{NS}(X)_{\mathbb{Q}} = \mathrm{NS}(X) \otimes \mathbb{Q}$. This form is reconstructed via polarization from the form $\delta \mapsto \deg_X \delta^d$ on $\mathrm{NS}(X)_{\mathbb{Q}}$ of degree d. Moreover, each Cartier divisor on X is a difference of two ample divisors, whence ample divisors form an open solid convex cone in $\mathrm{NS}(X)_{\mathbb{Q}}$, and the intersection form is determined by values of $\deg \delta^d$ for ample δ.

Retain the notation of §17. Also put $A(\delta,\lambda) \equiv 1$, $\forall \lambda \in \mathcal{E}^*$, if $c = 0$, and $\mathcal{P}_+(\delta) = \{\lambda \in \mathcal{P}(\delta) \mid A(\delta,\lambda) \geq 0\}$

Theorem 18.2. *Suppose δ is an ample B-stable divisor on X. Then*

$$d = c + r + |\Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\pi(\delta))^\perp| + 1, \qquad \textit{and} \tag{18.1}$$

$$\deg \delta^d = d! \int\limits_{\pi(\delta)+\mathcal{P}_+(\delta)} A(\delta, \lambda - \pi(\delta)) \prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\pi(\delta))^\perp} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}\, d\lambda, \tag{18.2}$$

where the Lebesgue measure on $\Lambda \otimes \mathbb{R}$ is normalized so that a fundamental parallelepiped of Λ has volume 1.

Proof. We have

$$\begin{aligned}
\dim \mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes n}) &= \sum_{\lambda \in n(\pi(\delta)+\mathcal{P}(\delta)) \cap \Lambda} \dim V(\lambda) \cdot m_{\lambda - n\pi(\delta)}(n\delta) \\
&= \sum_{\lambda \in (\pi(\delta)+\mathcal{P}(\delta)) \cap \frac{1}{n}\Lambda} \dim V(n\lambda) \cdot m_{n(\lambda - \pi(\delta))}(n\delta) \\
&= \sum_{\lambda \in (\pi(\delta)+\mathcal{P}_+(\delta)) \cap \frac{1}{n}\Lambda} \prod_{\alpha^\vee \in \Delta_+^\vee} \left(1 + n\frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}\right) \qquad \text{if } c = 0, \\
&\text{or} \sum_{\lambda \in (\pi(\delta)+\mathcal{P}_+(\delta)) \cap \frac{1}{n}\Lambda} \prod_{\alpha^\vee \in \Delta_+^\vee} \left(1 + n\frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}\right) \big[nA(\delta, \lambda - \pi(\delta)) \\
&\qquad - \sigma(n\delta, n(\lambda - \pi(\delta))) - g + 1 + h^1(n\delta, n(\lambda - \pi(\delta)))\big]
\end{aligned}$$

if $c = 1$, using the Weyl dimension formula. In both cases,

$$\dim \mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes n}) \sim n^{c+r} \int\limits_{\pi(\delta)+\mathcal{P}_+(\delta)} A(\delta, \lambda - \pi(\delta)) \prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\pi(\delta)+\mathcal{P}_+(\delta))^\perp} n\frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}\, d\lambda$$

On the other hand, the Euler characteristic $\chi(\mathcal{O}(\delta)^{\otimes n}) = \deg(\delta^d) n^d / d! + \dots$ equals $\dim \mathrm{H}^0(X, \mathcal{O}(\delta)^{\otimes n})$ for $n \gg 0$. It remains to note that $\mathcal{P}_+(\delta)$ generates $\Lambda \otimes \mathbb{R}$, because each rational B-eigenfunction on X is a quotient of two B-eigensections of some $\mathcal{O}(\delta)^{\otimes n}$. Therefore $(\pi(\delta)+\mathcal{P}_+(\delta))^\perp = (\Lambda + \mathbb{Z}\pi(\delta))^\perp$. $\square$

Remark 18.3. Formula (18.1) may be proved using the local structure theorem (Corollary 4.1).

Remark 18.4. Formula (18.2) is valid for globally generated δ, because globally generated divisor classes lie on the boundary of the cone of ample divisors in $\mathrm{NS}(X)_{\mathbb{Q}}$ and the r.h.s. of (18.2) depends continuously on δ.

Remark 18.5. The integral in the theorem can be easily computed using a simplicial subdivision of the polyhedral domain $\mathcal{P}_+(\delta)$ and Brion's integration formula [Bri4, 4.2, Rem.(ii)]:

> Suppose F is a homogeneous polynomial of degree p on $\mathbb{R}^r$, and $[a_0,\dots,a_r]$ is a simplex with vertices $a_i \in \mathbb{R}^r$. Then
>
> $$\int\limits_{[a_0,\dots,a_r]} F(\lambda)\,d\lambda = \frac{r!\,\mathrm{vol}[a_0,\dots,a_r]}{(p+1)\dots(p+r)}\Pi_r F(a_0,\dots,a_r)$$
>
> where
>
> $$\Pi_r F(a_0,\dots,a_r) = \frac{1}{p!}\sum_{p_0+\dots+p_r=p}\frac{\partial^p F(a_0t_0+\dots+a_rt_r)}{\partial {t_0}^{p_0}\dots\partial {t_r}^{p_r}}$$

Example 18.2. For toric X, $d=r$, $c=0$, and $\deg\delta^r = r!\,\mathrm{vol}\,\mathcal{P}(\delta)$ [Dan, 11.12.2].

Example 18.3. If $X = G/P_I$ is a generalized flag variety, then each ample divisor δ defines an embedding $X \hookrightarrow V(\lambda)$, $\lambda = \pi(\delta)^*$, and the degree of this embedding equals

$$|\Delta_+ \setminus \Delta_{I,+}|!\prod_{\alpha\in\Delta_+\setminus\Delta_{I,+}}\frac{\langle\lambda,\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle}$$

In particular, the degree of the Plücker embedding $\mathrm{Gr}_m(\Bbbk^n) \hookrightarrow \mathbb{P}(\bigwedge^m \Bbbk^n)$ equals

$$[m(n-m)]!\frac{1!\dots(m-1)!}{(n-m)!\dots(n-1)!}$$

(Schubert [Sch2]).

Example 18.4. Consider the space X of complete conics of Example 17.3. Here $d=5$, $c=0$, $r=2$. If $\delta = a_1D_1 + a_2D_2$ is an ample divisor, then $\pi(\delta) = 2a_1\omega_1 + 2a_2\omega_2$. Writing $\lambda = -2x_1\alpha_1 - 2x_2\alpha_2$, we have $d\lambda = dx_1\,dx_2$, and $\mathcal{P}(\delta) = \{\lambda \mid x_1,x_2 \ge 0,\ 2x_1 \le x_2 + a_1,\ 2x_2 \le x_1 + a_2\}$ is a quadrangle with the vertices $\{0, -a_1\alpha_1, -a_2\alpha_2, -\pi(\delta)\}$.

We have $\pi(\delta)+\lambda = (2a_1 - 4x_1 + 2x_2)\omega_1 + (2a_2 - 4x_2 + 2x_1)\omega_2$, and

$$\begin{aligned}\deg\delta^5 &= 5!\Big(\int_{\mathcal{P}(\delta)} \tfrac{(2a_1-4x_1+2x_2)(2a_2-4x_2+2x_1)(2a_1+2a_2-2x_1-2x_2)}{2}\,dx_1\,dx_2 \\ &= a_1^5 + 10a_1^4a_2 + 40a_1^3a_2^2 + 40a_1^2a_2^3 + 10a_1a_2^4 + a_2^5\end{aligned}$$

Polarizing this 5-form in a_1, a_2, we obtain the intersection form on $\mathrm{NS}(X) = \langle D_1, D_2\rangle$: $\deg D_1^5 = \deg D_2^5 = 1$, $\deg D_1^4 D_2 = \deg D_1 D_2^4 = 2$, $\deg D_1^3 D_2^2 = \deg D_1^2 D_2^3 = 4$ (Chasles [Ch]).

This result can be applied to solving various enumerative problems in the space $\mathcal{O}$ of plane conics. For example, let us find the number of conics tangent to 5 given conics in general position. The set of conics tangent to a given one is a prime divisor $D \subset \mathcal{O}$. It is easy to see that (the closure of) D intersects all G-orbits in X properly. By Kleiman's transversality theorem (see [Har, III.10.8] and below) five general translates $g_i D$ ($g_i \in G$, $i = 1, \dots, 5$) are transversal and intersect only inside $\mathcal{O}$. Thus the number we are looking for equals $\deg_X D^5$.

Using local coordinates, one sees that the degree of (the closure of) D in $\mathbb{P}^5$ or $(\mathbb{P}^5)^*$ equals 6. (Take, e.g., a parabola $\{y = x^2\}$. A conic $\{q(x, y) = 0\}$ is tangent to this parabola iff $q(x, x^2) = 0$ and $\begin{vmatrix} 2x & \frac{\partial q}{\partial x}(x, x^2) \\ -1 & \frac{\partial q}{\partial y}(x, x^2) \end{vmatrix} = 0$ for some x. The resultant of these two polynomials has degree 7 in the coefficients of q. Cancelling it by the coefficient at y^2, we obtain the equation of D of degree 6.) Since $\deg_{\mathbb{P}^5} D_1 = \deg_{(\mathbb{P}^5)^*} D_2 = 1$ and $\deg_{\mathbb{P}^5} D_2 = \deg_{(\mathbb{P}^5)^*} D_1 = 2$, one has $D \sim 2D_1 + 2D_2$ (on X) and $\deg_X D^5 = 2^5(1 + 10 + 40 + 40 + 10 + 1) = 3264$.

Spaces of conics and of quadrics in higher dimensions were studied intensively from the origin of enumerative geometry [Ch], [Sch1], [Sch3]. For a modern approach, see [Sem1], [Sem2], [Tyr], [CGMP], [Bri4].

Example 18.5. Let X be a completion of the space $\mathcal{O}$ of ordered triangles from Example 16.3 with $d = 6$, $c = 1$, $r = 2$. Consider an ample divisor $\delta = a_1 \widetilde{D} + a_2 D$, where $D = D_1 + D_2 + D_3$ imposes the condition that one of vertices of a triangle lies on the B-stable line in $\mathbb{P}^2$ and $\widetilde{D} = \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3$ imposes the condition that a triangle passes through the B-fixed point.

Writing $\lambda = -x_1\alpha_1 - x_2\alpha_2$, we have $d\lambda = dx_1\,dx_2$, $\mathcal{P}(\delta) = \{\lambda \mid x_1, x_2 \geq 0\}$, $\pi(\delta) = 3a_1\omega_1 + 3a_2\omega_2$, and

$$A(\delta, \lambda) = \begin{cases} A_0(\lambda) = x_1 + x_2, & x_i \leq a_i \\ A_1(\lambda) = 3a_1 - 2x_1 + x_2, & 0 \leq x_1 - a_1 \geq x_2 - a_2 \\ A_2(\lambda) = 3a_2 - 2x_2 + x_1, & 0 \leq x_2 - a_2 \geq x_1 - a_1 \end{cases}$$

It follows that $\mathcal{P}_+(\delta) = \{\, \lambda \mid x_1, x_2 \geq 0;\ 2x_1 \leq x_2 + 3a_1;\ 2x_2 \leq x_1 + 3a_2 \,\} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_i$ are quadrangles with vertices

$$\{0, -a_1\alpha_1, -a_1\alpha_1 - a_2\alpha_2, -a_2\alpha_2\},$$
$$\{-a_1\alpha_1, -\tfrac{3a_1}{2}\alpha_1, -\pi(\delta), -a_1\alpha_1 - a_2\alpha_2\},$$
$$\{-a_2\alpha_2, -\tfrac{3a_2}{2}\alpha_2, -\pi(\delta), -a_1\alpha_1 - a_2\alpha_2\},$$

and $A(\delta,\lambda) = A_i(\lambda)$ on $\mathcal{P}_i$. We have $\pi(\delta)+\lambda = (3a_1 - 2x_1 + x_2)\omega_1 + (3a_2 - 2x_2 + x_1)\omega_2$, and

$$\begin{aligned}\deg\delta^6 = 6!\Big(&\int_{\mathcal{P}_0}(x_1+x_2)\cdot\tfrac{(3a_1-2x_1+x_2)(3a_2-2x_2+x_1)(3a_1+3a_2-x_1-x_2)}{2}\,dx_1\,dx_2\\ &+\int_{\mathcal{P}_1}(3a_1-2x_1+x_2)\cdot\tfrac{(3a_1-2x_1+x_2)(3a_2-2x_2+x_1)(3a_1+3a_2-x_1-x_2)}{2}\,dx_1\,dx_2\\ &+\int_{\mathcal{P}_2}(3a_2-2x_2+x_1)\cdot\tfrac{(3a_1-2x_1+x_2)(3a_2-2x_2+x_1)(3a_1+3a_2-x_1-x_2)}{2}\,dx_1\,dx_2\Big)\\ = 90a_1^6 + 1080a_1^5a_2 &+ 4320a_1^4a_2^2 + 6840a_1^3a_2^3 + 4320a_1^2a_2^4 + 1080a_1a_2^5 + 90a_2^6\end{aligned}$$

It follows that $\deg D^6 = \deg\widetilde{D}^6 = 90$, $\deg D^5\widetilde{D} = \deg D\widetilde{D}^5 = 180$, $\deg D^4\widetilde{D}^2 = \deg D^2\widetilde{D}^4 = 288$, $\deg D^3\widetilde{D}^3 = 342$.

Since X has finitely many orbits and $D,\widetilde{D}$ intersect all of them properly, it follows from Kleiman's transversality theorem that any 6 general translates δ_i of $D,\widetilde{D}$ are transversal and intersect only inside $\mathcal{O}$. Thus the number of common points of δ_i in $\mathcal{O}$ equals $\deg_X(\delta_1\dots\delta_6)$. Dividing it by 6 (=the number of ordered triangles corresponding to a given unordered triangle), we obtain the number of triangles satisfying 6 conditions imposed by δ_i. For example, there are $\deg(D^3\widetilde{D}^3)/6 = 57$ triangles passing through 3 given points in general position whose vertices lie on 3 given general lines.

Theorem 18.2 was applied in [Tim3, §10] to computing the degree of a closed 3-dimensional orbit in any SL_2-module.

Brion [Bri9, 4.1] proved a formula similar to (18.2) for the multiplicity of a spherical variety along an orbit in it and deduced a criterion of smoothness for spherical varieties [Bri9, 4.2].

For any complete variety X, there is a canonical pairing $\operatorname{Pic}X\times A_1(X)\to\mathbb{Z}$ given by the degree of a line bundle restricted to a curve in X (and pulled back to its normalization). The following theorem is essentially due to Brion [Bri10].

Theorem 18.3. (1) *If X is a complete unirational G-model of complexity ≤ 1, then $\operatorname{Pic}X \hookrightarrow A_1(X)^* = \operatorname{Hom}(A_1(X),\mathbb{Z})$ via the canonical pairing.*

(2) *If X is complete and spherical, then $\operatorname{Pic}X \xrightarrow{\sim} A_1(X)^*$.*

(3) *If in addition X contains a unique closed G-orbit Y, then $A_1(X)$ is torsion-free, and the basis of $A_1(X)$ dual to the basis $\mathcal{D}^B\setminus\mathcal{D}^B_Y$ of $\operatorname{Pic}X$ consists of (classes of) irreducible B-stable curves. Moreover, these basic curves generate the semigroup $A_1^+(X)$ of effective 1-cycles.*

Proof. Using the equivariant Chow lemma and resolution of singularities, we construct a proper birational G-morphism $\phi : \widehat{X} \to X$, where $\widehat{X}$ is a smooth projective G-variety. In the commutative diagram

$$\begin{array}{ccc} \operatorname{Pic} X & \longrightarrow & A_1(X)^* \\ \downarrow & & \downarrow \\ \operatorname{Pic} \widehat{X} & \longrightarrow & A_1(\widehat{X})^* \end{array}$$

the vertical arrows are injections, and the bottom arrow is an isomorphism by Corollary 18.3 and by Poincarè duality, whence (1). Assertions (1), (3) are proved in [Bri12, §3] using the description of B-stable curves and their equivalences on spherical varieties obtained in [Bri10], [Bri12]. □

Remark 18.6 *([Bri10, 1.6, 2.1]).* On a spherical G-model X, any line bundle $\mathcal{L}$ is G-linearized and any B-stable curve C is the closure of a 1-dimensional B-orbit. Let ∞ be a B-fixed point in the normalization $\mathbb{P}^1$ of C and $0 \in \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ be another T-fixed point. Then T acts on $\mathbb{A}^1 \setminus \{0\}$ via a character $\chi \neq 0$. Let $x, y \in C$ be the images of $0, \infty$ under the normalization map $\nu : \mathbb{P}^1 \to C$, and χ_x, χ_y be the weights of the T-action on $\mathcal{L}_x, \mathcal{L}_y$. Then $\chi_x - \chi_y$ is a multiple of χ, and $\langle \mathcal{L}, C\rangle = \deg \nu^*\mathcal{L}|_C = (\chi_x - \chi_y)/\chi$.

Example 18.6. For a generalized flag variety $X = G/P_I$, $\operatorname{Pic} X$ is freely generated by Schubert divisors $D_{\alpha_i} = \overline{B[w_G s_i]}$, and $A_1(X)$ is freely generated by Schubert curves $C_{\alpha_i} = \overline{B[s_i]} \simeq P_{\alpha_i}/B \simeq \mathbb{P}^1$. We have $\mathcal{O}(D_{\alpha_i}) = G *_{P_I} \Bbbk_{-\omega_i}$ and $\mathcal{O}(D_{\alpha_i})|_{C_{\alpha_j}} = P_{\alpha_i} *_B \Bbbk_{-\omega_i} = \mathcal{O}(1)$ if $i = j$ and $\mathcal{O}(0)$ if $i \neq j$. Here the T-fixed points are $[s_i], [e]$, $\chi = \alpha_i$, and $\langle D_{\alpha_i}, C_{\alpha_j}\rangle = (\chi_{[s_i]} - \chi_{[e]})/\chi = (-s_i\omega_j + \omega_j)/\alpha_i = \delta_{ij}$. Hence the above bases of $\operatorname{Pic} X$ and $A_1(X)$ are dual to each other.

Projective unirational normal varieties of complexity ≤ 1 are well-behaved from the point of view of the Mori theory [KMM].

Theorem 18.4 ([Bri10], [BKn]). *Suppose X is a projective unirational G-model of complexity ≤ 1. Then the cone $\operatorname{NE}(X)$ of effective 1-cycles modulo numerical equivalence is finitely generated by rational B-stable curves and all its faces are contractible. If X is $\mathbb{Q}$-factorial, then each contraction of an extremal ray of $\operatorname{NE}(X)$ isomorphic in codimension 1 can be flipped, and every sequence of directed flips terminates.*

Explicit computations of Chow rings for some smooth completions of classical homogeneous spaces were carried on by several authors. Schubert, Pieri, Giambelli, A. Borel, Kostant, Bernstein–Gelfand–Gelfand, Demazure, Lakshmibai–Musili–Seshadri et al contributed to computing Chow (or cohomology) rings of generalized flag varieties.

Here and below we put $A^k(X) = A_{d-k}(X)$, $d = \dim X$.

Without loss of generality, assume that G is semisimple simply connected. Let $\mathfrak{X} = \mathfrak{X}(B)$ be the weight lattice of G. Every $\lambda \in \mathfrak{X}$ defines an induced line bundle $\mathcal{L}(-\lambda) = G *_B \Bbbk_{-\lambda}$ on G/B, and this gives rise to an isomorphism $\mathfrak{X} \xrightarrow{\sim} \operatorname{Pic} G/B \simeq A^1(G/B)$. Put $S = \mathrm{S}^\bullet(\mathfrak{X} \otimes \mathbb{Q})$.

Theorem 18.5 ([Bor], [Dem2]). (1) [Bor], [Dem2] $A^*(G/B)_\mathbb{Q} \simeq S/SS_+^W$ *(the quotient modulo the ideal generated by W-invariants without constant term)*

(2) [BGG, 5.5] *If $I \subseteq \Pi$, then $A^*(G/P_I)_\mathbb{Q}$ embeds in $A^*(G/B)_\mathbb{Q}$ as $S^{W_I}/S^{W_I}S_+^W$.*

Bernstein–Gelfand–Gelfand [BGG] and Demazure [Dem2] used divided difference operators to introduce certain functionals D_w on S which represent Schubert cells S_w ($w \in W$) via the Poincarè duality. They also found the basis of S/SS_+^W dual to D_w.

Chow rings of toric varieties were computed by Jurkiewicz and Danilov, cf. [Dan, §10]. Namely, if X is a smooth complete toric variety, then $A^*(X) = S_\mathbb{Z}^*(\operatorname{Pic} X)/I$, where the ideal I is generated by monomials $[D_1]\dots[D_k]$ such that D_i are T-stable prime divisors and $D_1 \cap \dots \cap D_k = \emptyset$.

The above examples are spherical (see also §27, §29). In the case of complexity 1, Chow rings of complete SL_2-embeddings (cf. Example 16.2) were computed in [MJ2]. The space of complete triangles, which is a desingularization of the space X of Example 16.3, was studied in [CF], and in particular, its Chow ring was determined there.

Many enumerative problems arise on non-complete homogeneous spaces. Given a homogeneous space $\mathcal{O} = G/H$, typically a space of geometric figures or tensors of certain type, one looks for the number of points satisfying a number of conditions in general position. The set of points satisfying a given condition is a closed subvariety $Z \subset \mathcal{O}$, and the configuration of conditions $Z_1, \dots, Z_s \subset \mathcal{O}$ is put in general position by replacing Z_i by their translates g_iZ_i, where $(g_1, \dots, g_s) \in G \times \dots \times G$ is a general s-tuple. By Kleiman's transversality theorem [Kle], [Har, III.10.8], the cycles $g_1Z_1, \dots, g_sZ_s$ intersect transversally in smooth subvarieties of codimension $\sum \operatorname{codim} Z_i$, i.e., $g_1Z_1 \cap \dots \cap g_sZ_s$ is empty if $\sum \operatorname{codim} Z_i > \dim \mathcal{O}$ and finite if $\sum \operatorname{codim} Z_i = \dim \mathcal{O}$, and the cardinality $|g_1Z_1 \cap \dots \cap g_sZ_s|$ is stable for general $(g_1, \dots, g_s)$. Thus the natural intersection ring for the enumerative geometry of $\mathcal{O}$ is provided by the following

Definition 18.1 ([CP2]). The *intersection number* of irreducible subvarieties $Z_1, \dots, Z_s$ whose codimensions sum up to $\dim \mathcal{O}$ is $(Z_1 \cdot \dots \cdot Z_s)_\mathcal{O} =$

$|g_1Z_1 \cap \cdots \cap g_sZ_s|$ for all $(g_1, \ldots, g_s)$ in a dense open subset of $G \times \cdots \times G$.

This defines a pairing between groups of cycles in $\mathcal{O}$ of complementary dimensions. The *group of conditions* $C^*(\mathcal{O})$ is the quotient of the group of all cycles modulo the kernel of this pairing. Write $[[Z]]$ for the image in $C^*(\mathcal{O})$ of a cycle Z.

Theorem 18.6 ([CP2, 6.3]). *If $\mathcal{O}$ is spherical, then $C^*(\mathcal{O})$ is a graded ring w.r.t. the* intersection product $[[Z]]\cdot[[Z']] = [[gZ \cap g'Z']]$, *where $(g, g') \in G \times G$ is a general pair. Furthermore, $C^*(\mathcal{O}) = \varinjlim A^*(X)$ over all smooth complete G-embeddings $X \hookleftarrow \mathcal{O}$.*

Proof. The proof goes in several steps.

(1) For any subvariety $Z \subset \mathcal{O}$ and any smooth complete G-embedding $X \hookleftarrow \mathcal{O}$, there is a smooth complete G-embedding $X' \hookleftarrow \mathcal{O}$ dominating X such that $\overline{Z}$ intersects all G-orbits in X' properly. First, one constructs a smooth toroidal embedding X' dominating X. Then each G-orbit on X' is a normal intersection of G-stable prime divisors (Theorem 29.2), and one applies a general result [CP2, 4.7] that a cycle on a complete smooth variety can be put in regular position w.r.t. a regular configuration of hypersurfaces by blowing up several intersections of pairs of these hypersurfaces.

(2) If $Z, Z' \subset \mathcal{O}$ intersect all G-orbits in $X \setminus \mathcal{O}$ properly, then $[gZ \cap g'Z'] = [Z] \cdot [Z']$ in $A^*(X)$ for general $g, g' \in G$. Indeed, we may apply Kleiman's transversality theorem to intersections of Z, Z' with each of finitely many G-orbits in X and deduce that gZ and $g'Z'$ intersect properly with each other and $gZ \cap g'Z'$ intersects $X \setminus \mathcal{O}$ properly.

(3) For any $z \in A^*(X)$, use the Chow moving lemma to represent it as $z = \sum m_i[Z_i]$, where $Z_i \subset X$ are closed subvarieties intersecting $\mathcal{O}$. For any subvariety $Z' \subset \mathcal{O}$ of complementary dimension, we may assume by (1) that Z_i, Z' intersect all orbits in $X \setminus \mathcal{O}$ properly and deduce from (2) that $([[Z']], \sum m_i[[Z_i]])_{\mathcal{O}} = \deg_X[Z'] \cdot z$ depends only on z. Thus we have a well-defined map $A^*(X) \to C^*(\mathcal{O})$, $z \mapsto \sum m_i[[Z_i]]$.

(4) This map gives rise to a homomorphism $\varinjlim A^*(X) \to C^*(\mathcal{O})$ by (2). Its surjectivity is obvious, and injectivity follows from the Poincarè duality on X. $\square$

The ring of conditions $C^*(\mathcal{O})$ is also called *Halphen ring* in honor of G.-H. Halphen, who used it in the enumerative geometry of conics, see [CX]. If $\mathcal{O}$ is a torus, then $C^*(\mathcal{O})$ is MacMullen's polytope algebra [FS], [Bri11, 3.3]. The Halphen ring of the space of plane conics was computed in [CX].

Theorem 18.6 reflects an idea exploited already by classics that in solving enumerative problems on $\mathcal{O}$, one has to consider an appropriate completion $X \hookleftarrow \mathcal{O}$ with finitely many orbits such that all "conditions" Z_i under consideration intersect all orbits in $X \setminus \mathcal{O}$ properly. If $\sum \operatorname{codim} Z_i = \dim \mathcal{O}$, then for general g_i, all intersection points of $\bigcap g_i Z_i$ lie in $\mathcal{O}$, and the intersection number equals $\deg_X \prod [Z_i]$. Applications of this idea can be found in Examples 18.4, 18.5.

Generalizing these examples, we describe a method to compute the intersection number of d divisors on a spherical homogeneous space $\mathcal{O}$ of dimension d in characteristic 0. Let δ be a divisor on $\mathcal{O}$. Replacing δ by a G-translate, we may assume that δ contains no colors. Let $h \in K$ be an equation of δ on the open B-orbit in $\mathcal{O}$, which is a factorial variety.

Definition 18.2. The *Newton polytope* of δ is the set

$$\mathcal{N}(\delta) = \{ \lambda \in \mathcal{E}^* \mid \forall v \in \mathcal{V} : \ \langle v, \lambda \rangle \geq v(h), \ \forall D \in \mathcal{D}^B : \ \langle \rho(D), \lambda \rangle \geq v_D(h) \}$$

Remark 18.7. We see below that $\mathcal{N}(\delta)$ is indeed a convex polytope in $\mathcal{E}^*$. If $G = \mathcal{O} = T$ is a torus, then $\mathcal{D}^B = \emptyset$, $h = \sum c_i \lambda_i$, $c_i \in \Bbbk^\times$, $\lambda_i \in \mathfrak{X}(T)$, and $v(h) = -\min \langle v, \lambda_i \rangle$. Thus $\mathcal{N}(\delta) = -\operatorname{conv}\{\lambda_1, \dots, \lambda_s\}$ is a usual Newton polytope.

For any embedding $X \hookleftarrow \mathcal{O}$, we have $\operatorname{div} h = \delta - \delta_X$ on X, where $\delta_X = -\sum_i v_i(h) V_i - \sum_{D \in \mathcal{D}^B} v_D(h) D$ is a B-stable divisor and V_i are G-stable prime divisors on X with valuations $v_i \in \mathcal{V}$.

Theorem 18.7. $\mathcal{N}(\delta) = \bigcap_{X \hookleftarrow \mathcal{O}} \mathcal{P}(\delta_X)$. *If X is complete and δ intersects all G-orbits in X properly, then $\mathcal{N}(\delta) = \mathcal{P}(\delta_X)$.*

Proof. The first assertion is obvious from (17.4), since every G-valuation corresponds to a divisor on some embedding of X. Suppose X is complete and δ intersects all orbits properly. Consider a G-linearized line bundle $\mathcal{L} = \mathcal{O}(\delta) = \mathcal{O}(\delta_X)$. We may extend any G-valuation to a G-valuation of $\bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}^n)$ (Corollary 19.1). If σ is a rational section of $\mathcal{L}$ such that $\operatorname{div} \sigma = \delta$, then $v(\sigma) = 0$ for $\forall v \in \mathcal{V}$, because v has a center on X and δ intersects it properly. Now $\forall \lambda \in \mathcal{P}(\delta_X)\ \exists n \in \mathbb{N}\ \exists \eta \in \mathrm{H}^0(X, \mathcal{L}^n)^{(B)}_{n\lambda} \implies \eta = c\mathbf{f}_{n\lambda}\sigma_X^n = c\mathbf{f}_{n\lambda}\sigma^n/h^n$, where $c \in \Bbbk^\times$, $\operatorname{div} \sigma_X = \delta_X$. Then $v(\eta) = v(\mathbf{f}_{n\lambda}/h^n) \geq 0 \implies \langle v, \lambda \rangle \geq v_D(h) \implies \mathcal{P}(\delta_X) \subseteq \mathcal{N}(\delta)$. □

Corollary 18.4 ([Bri11, 4.2]). *For any effective divisor δ on $\mathcal{O}$,*

$$(\delta^d)_{\mathcal{O}} = d! \int\limits_{\pi(\delta)+\mathcal{N}(\delta)} \prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\pi(\delta))^\perp} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \, d\lambda, \tag{18.3}$$

where $\pi(\delta) = -\sum_{D \in \mathcal{D}^B} v_D(h) \pi(D)$

Proof. Follows from Theorems 18.6, 18.7, and 18.2. □

Remark 18.8. In the toric case, the above formula transforms to $(\delta^d)_{\mathcal{O}} = d!\operatorname{vol}\mathcal{N}(\delta)$. Polarizing this formula, we obtain a theorem of Bernstein [Ber] and Kouchnirenko [Kou]: for any effective divisors $\delta_1, \dots, \delta_d$ on $\mathcal{O}$, the intersection number $(\delta_1 \dots \delta_d)_{\mathcal{O}}$ is $d!$ times the mixed volume of $\mathcal{N}(\delta_1), \dots, \mathcal{N}(\delta_d)$. In the general case, we have a "mixed integral" instead.

Corollary 18.4 may be considered as a generalization of the classical Bézout theorem.

Chapter 4

Invariant valuations

This chapter plays a significant, but auxiliary, role in the general context of our survey. We investigate the set of G-invariant valuations of the function field of a G-variety. We have seen in Chapter 3 that G-valuations are of importance in the embedding theory, because they provide a material for constructing combinatorial objects (colored data) that describe equivariant embeddings.

Remarkably, a G-valuation of a given G-field is uniquely determined by its restriction to the multiplicative group of B-eigenfunctions, the latter being a direct product of the weight lattice and of the multiplicative group of B-invariant functions. Thus a G-valuation is essentially a pair composed by a linear functional on the weight lattice and by a valuation of the field of B-invariants. Under these identifications, we prove in §20 that the set of G-valuations is a union of convex polyhedral cones in certain half-spaces.

The common face of these valuation cones is formed by those valuations, called central, that vanish on B-invariant functions. The central valuation cone controls the situation "over the field of B-invariant functions". For instance, its linear part determines the group of G-automorphisms acting identically on B-invariants.

This cone has another remarkable property: it is a fundamental chamber of a crystallographic reflection group called the little Weyl group of a G-variety. This group is defined in §22 as the Galois group of a certain symplectic covering of the cotangent bundle constructed in terms of the moment map. The little Weyl group is linked with the central valuation cone via the invariant collective motion on the cotangent variety, which is studied in §23.

For practical applications, we must be able to compute the set of G-valuations. For central valuations, it suffices to know the little Weyl group. In §24 we describe the "method of formal curves" for computing G-valuations on a homogeneous space. Informally, one computes the order of functions at infinity along a formal curve approaching to a boundary G-divisor.

Most of the results of this chapter are due to D. Luna and Th. Vust, M. Brion, F. Pauer, and F. Knop. We follow [LV], [Kn3], [Kn5] in our exposition.

19 G-valuations

An algebraic counterpart of a prime divisor on an algebraic variety is the respective valuation of the field of rational functions. Valuations obtained in this way are called geometric (see Appendix A3). We consider invariant geometric valuations.

Let G be a connected algebraic group and K a G-field, i.e., the function field of a G-variety.

Definition 19.1. A *G-valuation* is a G-invariant geometric valuation of $K/\Bbbk$. The set of G-valuations is denoted by $\mathcal{V} = \mathcal{V}(K)$.

The following approximation result is due to Sumihiro.

Proposition 19.1 ([Sum, §4]). *For any geometric valuation v of K there exists a G-valuation $\overline{v}$ such that $\forall f \in K : \ \overline{v}(f) = v(gf)$ for general $g \in G$. If $A \subset K$ is a rational G-algebra, then $\forall f \in A : \ \overline{v}(f) = \min_{g\in G} v(gf)$.*

Proof. We may assume $v = v_D$ for a prime divisor D on a model X of K. Then $v' = v_{G\times D}$ is a geometric valuation of $\Bbbk(G \times X)$. It is clear that $\forall f \in \Bbbk(G \times X) : \ v'(f) = v(f(g, \cdot))$ for general $g \in G$. The rational action $G : X$ induces an embedding $\Bbbk(X) \hookrightarrow \Bbbk(G \times X)$. It is easy to see that $v'|_{\Bbbk(X)} = \overline{v}$ is the desired G-valuation.

To prove the 2-nd assertion, observe that $A_{(d)} = \{f \in A \mid v(f) \geq d\}$ is a filtration of A by linear subspaces, Gf is an algebraic variety and $Gf \cap A_{(d)}$ its closed subvariety, $\forall d \in \mathbb{Q}$. $\square$

Remark 19.1. If v has center $Y \subseteq X$, then $\overline{v}$ has center $\overline{GY}$.

Example 19.1. Let $G = \Bbbk$ act rationally on the blow-up X of $\mathbb{A}^2$ at 0 by translations along a fixed axis. In coordinates, $u(x, y) = (x + u, y)$, $\forall u \in G$, $(x, y) \in \mathbb{A}^2$. The valuation v of $\Bbbk(X)$ corresponding to the exceptional divisor is given on $\Bbbk[\mathbb{A}^2] = \Bbbk[x, y]$ by the order of a polynomial in x, y (i.e., the lowest degree of a homogeneous term) and has center 0 on $\mathbb{A}^2$. But $\overline{v}(f) = \min_u v(f(x + u, y))$ is the order of f in y, so that $\overline{v} = v_D$, where $D = \{y = 0\}$ is (the proper pullback of) the x-axis.

Together with Proposition A3.3, Sumihiro's approximation immediately implies

Corollary 19.1. *Let $K' \subseteq K$ be a G-subfield. The restriction of a G-valuation of K to K' is a G-valuation, and any G-valuation of K' can be extended to a G-valuation of K.*

The next corollary is useful in applications.

Corollary 19.2. *Let X be a G-model of K and $\mathcal{L}$ a G-line bundle on X. Then $\forall \sigma, \eta \in \mathrm{H}^0(X, \mathcal{L})$, $\eta \neq 0$, $\forall g \in G: \ v(\sigma/\eta) = v(g\sigma/\eta)$.*

Proof. Consider a rational G-algebra $R = \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathcal{L}^n)$. Then $\operatorname{Quot} R = K'(\eta)$ is a (purely transcendental) extension of a G-subfield $K' \subseteq K$ (consisting of functions representable as ratio of sections of some $\mathcal{L}^n$). Now apply Corollary 19.1 to extend v to R and conclude by $v(\sigma/\eta) = v(\sigma) - v(\eta) = v(g\sigma) - v(\eta) = v(g\sigma/\eta)$. □

A natural geometric characterization of G-valuations is given by

Proposition 19.2. *Any G-valuation is proportional to v_D for a G-stable prime divisor D on a normal G-model X of K.*

Proof. Let $v \in \mathcal{V}$ and choose $f_1, \dots, f_s \in \mathcal{O}_v$ whose residues generate $\Bbbk(v)$. Take a normal projective G-model X of K and a G-line bundle $\mathcal{L}$ on X such that $f_i = \sigma_i/\sigma_0$ for some $\sigma_0, \dots, \sigma_s \in \mathrm{H}^0(X, \mathcal{L})$. Let $M \subseteq \mathrm{H}^0(X, \mathcal{L})$ be the G-submodule generated by $\sigma_0, \dots, \sigma_s$. The respective rational map $\phi : X \dashrightarrow \mathbb{P}(M^*)$ is G-equivariant. Replacing X by the normalized closure of the graph of ϕ, we may assume that ϕ is a G-morphism. Corollary 19.2 implies $v(M/\sigma_0) \geq 0$, whence the center $Y \subseteq X' = \phi(X)$ of $v|_{\Bbbk(X')}$ intersects an affine chart X'_{σ_0}, and $f_1, \dots, f_s \in \mathcal{O}_{X',Y}$. Therefore, if D is the center of v on X, then $f_1, \dots, f_s \in \mathcal{O}_{X,D}$, whence D is a divisor. □

Here is a relative version of this proposition.

Proposition 19.3. *Suppose a G-valuation v has the center Y on a G-model X of K. Then there exists a normal G-model X' and a projective morphism $\phi : X' \to X$ such that the center of v on X' is a divisor D' and $\phi(D') = Y$.*

Proof. Take any projective G-model X' such that the center of v is a divisor $D' \subset X'$. The rational map $\phi : X' \dashrightarrow X$ is defined on an open subset intersecting D', and $\overline{\phi(D')} = Y$. Now we replace X' by the normalized closure $\widetilde{X}$ of the graph of ϕ in $X' \times X$. Since $\widetilde{X}$ projects onto X' isomorphically over the domain of definition of ϕ, we can lift D' to $\widetilde{X}$. □

From now on, G is a connected reductive group.

Lemma 19.1. *If $A \subset K$ is a rational G-algebra, then $\forall v \in \mathcal{V}$, $f \in A$: $v(f) = \min_{\widetilde{f} \in (M^q)^{(B)}} v(\widetilde{f})/q$, where M is a G-submodule generated by f, and q is a sufficiently big power of the characteristic exponent of $\Bbbk$.*

Proof. As M is generated by f, we have $v(M) \geq v(f)$, whence $v\left((M^q)^{(B)}\right) \geq qv(f)$. To prove that the equality is reached, in characteristic zero ($\implies q = 1$) it suffices to note that M is generated by $M^{(B)} \implies v(f) \in v(M) \geq \min v\left(M^{(B)}\right)$. In the general case, this is not true, and one has to consider powers of M. We organize them in a graded G-algebra $R = \bigoplus_{n \geq 0} M^n$ and consider a graded G-stable ideal $I = \bigoplus I_n \lhd R$, $I_n = \{h \in M^n \mid v(h) > nv(f)\}$.

As $M \not\subseteq I$, there exists $r \in M$ such that $0 \neq r \bmod I \in (R/I)^U$. By Lemma A2.1, $(R/I)^U$ is a purely inseparable finite extension of R^U/I^U. Hence $\exists h \in I_q :\ \widetilde{f} = r^q + h \in R^U_q$, and $v(\widetilde{f}) = v(r^q) = qv(f)$. □

Remark 19.2. In characteristic zero or for $G = T$ Lemma 19.1 yields $v(f) = \min_{\lambda \in \mathfrak{X}_+} v(f_{(\lambda)})$, where $f_{(\lambda)}$ is the projection of f to the isotypic component $A_{(\lambda)}$ of A.

Recall from §13 that $\mathcal{D}$ denotes the set of non-G-stable prime divisors on (any) G-model of K and $K_B \subseteq K$ is the subalgebra of rational functions with poles in $\mathcal{D}^B$.

The following approximation lemma of Knop [Kn3, 3.5] allows to simplify the study of G-valuations by restricting to B-eigenfunctions.

Lemma 19.2. *For any G-valuation $v \in \mathcal{V}$ and any rational function $f \in K_B$ there exists a rational B-eigenfunction $\widetilde{f} \in K^{(B)}$ such that:*

$$\begin{cases} v(\widetilde{f}) = v(f^q) \\ w(\widetilde{f}) \geq w(f^q), & \forall w \in \mathcal{V} \\ v_D(\widetilde{f}) \geq v_D(f^q), & \forall D \in \mathcal{D}^B \end{cases}$$

where q is a sufficiently big power of the characteristic exponent.

Proof. Let X be a normal G-model and $\delta = \operatorname{div}_\infty f$, the divisor of poles on X. Take $\eta \in \mathrm{H}^0(X, \mathcal{O}(\delta))$ such that $\operatorname{div} \eta = \delta \implies \sigma = f\eta \in \mathrm{H}^0(X, \mathcal{O}(\delta))$. Extend all G-valuations to $R = \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{O}(n\delta))$ and consider a G-submodule $M \subseteq \mathrm{H}^0(X, \mathcal{O}(\delta))$ generated by σ. For any B-eigensection $\widetilde{\sigma} \in (M^q)^{(B)}$, put $\widetilde{f} = \widetilde{\sigma}/\eta^q$. Then $v_D(\widetilde{f}) \geq qv_D(f)$, $w(\widetilde{f}) \geq qw(f)$, and $v(\widetilde{f}) = qv(f)$ for some $\widetilde{\sigma} \in (M^q)^{(B)}$ by Lemma 19.1. □

Corollary 19.3. *G-valuations are determined uniquely by their restriction to $K^{(B)}$.*

Proof. As $\operatorname{Quot} K_B = K$, two distinct $v, w \in \mathcal{V}$ differ on some $f \in K_B$, say $v(f) < w(f)$. Lemma 19.2 yields $v(\widetilde{f}) < w(\widetilde{f})$ for some $\widetilde{f} \in K^{(B)}$. $\square$

20 Valuation cones

We have seen in §19 that G-valuations of K are determined by their restriction to $K^{(B)}$. In this section, we give a geometric qualitative description of $\mathcal{V}$ in terms of this restriction.

Let $\mathbf{v}$ be a geometric valuation of K^B. Factoring the exact sequence (13.1) by $\mathcal{O}_{\mathbf{v}}^{\times}$ yields an exact sequence of lattices

$$0 \longrightarrow \mathbb{Z}_{\mathbf{v}} \longrightarrow \Lambda_{\mathbf{v}} \longrightarrow \Lambda \longrightarrow 0 \tag{20.1}$$

where $\mathbb{Z}_{\mathbf{v}}$ is the value group of $\mathbf{v}$. Passing to the dual $\mathbb{Q}$-vector spaces, we obtain

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbb{Q}_{\mathbf{v}} & \longleftarrow & \mathcal{E}_{\mathbf{v}} & \longleftarrow & \mathcal{E} & \longleftarrow & 0 \\ & & |\!\bigcup & & |\!\bigcup & & \| & & \\ 0 & \longleftarrow & \mathbb{Q}_{\mathbf{v},+} & \longleftarrow & \mathcal{E}_{\mathbf{v},+} & \longleftarrow & \mathcal{E} & \longleftarrow & 0 \end{array} \tag{20.2}$$

where $\mathbb{Q}_{\mathbf{v}} = \mathbb{Q}$ and $\mathcal{E}_{\mathbf{v},+}$ is the preimage of the positive ray $\mathbb{Q}_{\mathbf{v},+}$ for $\mathbf{v} \neq 0$, and $\mathbb{Q}_0 = \mathbb{Q}_{0,+} = 0$, $\mathcal{E}_{0,+} = \mathcal{E}_0 = \mathcal{E}$.

Definition 20.1. The *hyperspace* (of K) is the union $\breve{\mathcal{E}} = \bigcup_{\mathbf{v}} \mathcal{E}_{\mathbf{v},+}$, where $\mathbf{v}$ runs over all geometric valuations of K^B considered up to proportionality. More precisely, $\breve{\mathcal{E}} = \mathcal{E}$ in the spherical case, and if $c(K) > 0$, then $\breve{\mathcal{E}}$ is the union of half-spaces $\mathcal{E}_{\mathbf{v},+}$ (over all $\mathbf{v} \neq 0$) glued together along their common boundary hyperplane $\mathcal{E}$, called the *center* of $\breve{\mathcal{E}}$.

Since Λ is a free Abelian group, the exact sequence (13.1) splits. Any splitting of (13.1) gives rise to simultaneous splittings of (20.1), (20.2), $\forall \mathbf{v}$. From time to time, we will fix such a splitting $\mathbf{f} : \Lambda \to K^{(B)}$, $\lambda \mapsto \mathbf{f}_\lambda$.

If v is a geometric valuation of K dominating $\mathbf{v}$, then $v|_{K^{(B)}}$ factors to a linear functional on $\Lambda_{\mathbf{v}}$ non-negative on $\mathbb{Z}_{\mathbf{v},+}$, i.e., an element of $\mathcal{E}_{\mathbf{v},+}$. Therefore $\mathcal{V} \hookrightarrow \breve{\mathcal{E}}$, and there is a restriction map $\rho : \mathcal{D}^B \to \breve{\mathcal{E}}$, which is in general not injective. Put $\mathcal{V}_{\mathbf{v}} = \mathcal{V} \cap \mathcal{E}_{\mathbf{v},+}$ and $\mathcal{D}^B_{\mathbf{v}} = \rho^{-1}(\mathcal{E}_{\mathbf{v},+})$. We say that $(\breve{\mathcal{E}}, \mathcal{V}, \mathcal{D}^B, \rho)$ is the *colored hyperspace*.

Our aim is to describe $\mathcal{V}_{\mathbf{v}}$.

Example 20.1. Assume that $G = B = T$ is a torus. Since every T-action has trivial birational type, there exists a T-model $X = T/T_0 \times C$, where $T_0 = \mathrm{Ker}(T : K)$ and $C = X/T$. We have $\Lambda = \mathfrak{X}(T/T_0)$, $K^T = \Bbbk(C)$, $K_T = K^T[\Lambda]$. By Remark 19.2, there is the only way to extend $v \in \breve{\mathcal{E}}$ to a T-valuation of K: put $v(f) = \min_\lambda v(f_\lambda)$, $\forall f = \sum f_\lambda \in K_T$, $f_\lambda \in K_\lambda^{(T)}$, $\lambda \in \Lambda$.

To prove the multiplicative property, for $\forall f = \sum f_\lambda$, $g = \sum g_\lambda \in K_T$, choose $\gamma \in \mathcal{E}$ such that $\min\langle\gamma,\lambda\rangle$ and $\min\langle\gamma,\mu\rangle$ over all λ with $v(f_\lambda) = \min$, resp. μ with $v(g_\mu) = \min$, are reached at only one point λ_0, resp. μ_0. Then $fg = \sum f_\lambda g_\mu = f_{\lambda_0} g_{\mu_0} + \sum_{(\lambda,\mu)\neq(\lambda_0,\mu_0)} f_\lambda g_\mu$, and for any term of the 2-nd sum we have either $v(f_\lambda g_\mu) > v(f_{\lambda_0} g_{\mu_0})$ or $\langle\gamma, \lambda+\mu\rangle > \langle\gamma, \lambda_0+\mu_0\rangle$. It follows that $v(fg) = v(f_{\lambda_0} g_{\mu_0}) = v(f_{\lambda_0}) + v(g_{\mu_0}) = v(f) + v(g)$. Other properties of a valuation are obvious.

Finally, let $\mathbf{v} = v|_{K^T}$ and consider a short exact subsequence of (13.1):

$$1 \longrightarrow K_0^T \longrightarrow K_0^{(T)} \longrightarrow \Lambda_0 \longrightarrow 0 \tag{20.3}$$

where $K_0^{(T)}$ is the kernel of $v : K^{(T)} \to \mathbb{Q}$, and $K_0^T = \mathcal{O}_{\mathbf{v}}^\times$. Note that any element of K can be written as $f = f_1/f_2$, $f_i \in K_T$, $v(f_2) = 0$. It follows that $\Bbbk(v)$ is the fraction field of $K_T \cap \mathcal{O}_v / K_T \cap \mathfrak{m}_v \simeq \Bbbk(\mathbf{v})[\Lambda_0]$ $\Longrightarrow$ $\mathrm{tr.deg}\, K - \mathrm{tr.deg}\, \Bbbk(v) = \mathrm{tr.deg}\, K^T + \mathrm{rk}\,\Lambda - \mathrm{tr.deg}\, \Bbbk(\mathbf{v}) - \mathrm{rk}\,\Lambda_0 = \mathrm{rk}(K^T)^\times/K_0^T + \mathrm{rk}\,\Lambda/\Lambda_0 = \mathrm{rk}\, K^{(T)}/K_0^{(T)} \leq 1$, hence v is geometric by Proposition A3.2.

We conclude that $\mathcal{V} = \breve{\mathcal{E}}$. By the way, we proved that every T-invariant valuation of K is geometric provided that its restriction to K^T is geometric.

The main result of this section is

Theorem 20.1. *For any geometric valuation $\mathbf{v}$ of K^B, $\mathcal{V}_{\mathbf{v}}$ is a finitely generated solid convex cone in $\mathcal{E}_{\mathbf{v},+}$.*

We prove it in several steps.

Lemma 20.1. *For any G-model X, there are only finitely many B-stable prime divisors $D \subset X$ such that v_D maps to $\mathcal{E}_{\mathbf{v},+}$.*

Proof. Take a sufficiently small B-chart $\mathring{X} \subseteq X$ such that a geometric quotient $\pi : \mathring{X} \to \mathring{X}/B$ exists. Now if v_D maps to $\mathcal{E}_{\mathbf{v},+}$, then either D is an irreducible component of $X \setminus \mathring{X}$ or $D = \pi^{-1}(D_0)$, where D_0 is the center of $\mathbf{v}$ on $\mathring{X}/B$. ☐

Corollary 20.1. *$\mathcal{D}_{\mathbf{v}}^B$ is finite.*

In the study of G-valuations, it is helpful to consider their centers on a sufficiently good projective G-model.

Lemma 20.2. *Let P be the common stabilizer of all colors with a Levi decomposition $P = L \leftthreetimes P_{\mathrm{u}}$, $L \supseteq T$. There exists a projective G-model X, a P-stable open (not necessarily affine) subset $\mathring{X} \subseteq X$, and a T-stable closed subvariety $Z \subseteq \mathring{X}$ such that*

(1) *The action $P_{\mathrm{u}} : \mathring{X}$ is proper and has a geometric quotient.*

(2) *$\mathring{X} = PZ$ and the natural maps $P_{\mathrm{u}} \times Z \to \mathring{X}$, $Z \to \mathring{X}/P_{\mathrm{u}}$ are finite and surjective.*

(2)′ *In characteristic zero,*

$$P_{\mathrm{u}} \times Z = P *_L Z \xrightarrow{\sim} \mathring{X}$$

(3) *The L'-action on $\mathring{X}/P_{\mathrm{u}}$ is trivial.*

(4) *Every G-subvariety $Y \subset X$ intersects $\mathring{X}$, hence Z.*

(5) *If Y is the center of $v \in \mathcal{V}_{\mathbf{v}}$, then $\mathcal{D}_Y^B = \emptyset$, $\mathcal{V}_Y \subset \mathcal{E}_{\mathbf{v},+}$.*

Proof. Take any projective G-model X and choose an ample G-line bundle $\mathcal{L}$ and an eigensection $\sigma \in \mathrm{H}^0(X, \mathcal{L})^{(B)}$ vanishing on sufficiently many colors such that $G_{\langle\sigma\rangle} = P$. Put $M = \langle G\sigma \rangle$ and take a lowest vector $u \in M^*$, $\langle \sigma, u \rangle \neq 0$, $G_{\langle u \rangle} = P^-$, so that $G\langle u \rangle \subseteq \mathbb{P}(M^*)$ is the unique closed orbit.

There is a natural rational G-map $\phi : X \dashrightarrow \mathbb{P}(M^*)$. Replacing X by the normalized closure of its graph in $X \times \mathbb{P}(M^*)$ makes ϕ regular. Put $\mathring{X} = X_\sigma = \phi^{-1}(\mathbb{P}(M^*)_\sigma)$. Then (1), (2), (2)′ follow from the local structure of $\mathbb{P}(M^*)_\sigma$ (cf. Lemma 4.1). Every $B \cap L$-stable divisor on $\mathring{X}/P_{\mathrm{u}}$ is L-stable, whence (3). Every closed G-orbit in X maps onto $G\langle u \rangle$, hence intersects $\mathring{X}$, which yields (4).

To prove (5), we modify the construction of X. First, we may choose σ vanishing on $\forall D \in \mathcal{D}_{\mathbf{v}}^B$. Next, consider an affine model C_0 of K^B such that $\mathbf{v}$ has the center $D_0 \subseteq C_0$ which is either a prime divisor or the whole C_0. Let $\Bbbk[C_0] = \Bbbk[f_1, \dots, f_s]$. We may choose $\mathcal{L}$ and σ so that $f_i = \sigma_i/\sigma$, $\sigma_1, \dots, \sigma_s \in \mathrm{H}^0(X, \mathcal{L})^{(B)}$. Let $M' \subseteq \mathrm{H}^0(X, \mathcal{L})$ be the G-submodule generated by $\sigma, \sigma_1, \dots, \sigma_s$.

As above, we may assume that the natural rational map $\phi' : X \dashrightarrow \mathbb{P}(M'^*)$ is regular. Put $X' = \phi'(X)$. Consider the composed map $\pi : \mathring{X} \to \mathring{X}' = X'_\sigma \to C_0$. By Corollary 19.2, $v(M'/\sigma) \geq 0$, whence the center $Y' = \phi'(Y) \subseteq X'$ of $v|_{\Bbbk(X')}$ intersects the B-chart $\mathring{X}'$. Hence $\mathring{Y} = Y \cap \mathring{X}$ is non-empty and $\overline{\pi(\mathring{Y})} \supseteq D_0$. It follows that $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ maps to $\mathcal{E}_{\mathbf{v},+}$. But any $D \in \mathcal{D}_{\mathbf{v}}^B$ is contained in $\mathbb{V}(\sigma) = X \setminus \mathring{X}$, thence $D \not\supseteq Y$, and we are done. $\square$

Proposition 20.1 ([Kn3]). *A G-invariant valuation of K is geometric iff its restriction to K^B is geometric.*

Proof 1 [Kn3, 3.9, 4.4]. Let v be a nonzero valuation of K such that $v|_{K^B}$ is geometric. Take a projective G-model X as in Lemma 20.2. Then v has the center $Y \subset X$ and $\mathring{Y} = Y \cap \mathring{X} \neq \emptyset$. By Lemma 20.2(3), $\Bbbk(\mathring{X}/P_{\mathrm{u}}) = K^U$ and $\mathring{Y}/P_{\mathrm{u}}$ is the center of $v|_{K^U}$.

Since K^U is a T-field and $(K^U)^T = K^B$, it follows from Example 20.1 that $v|_{K^U}$ is geometric. Now by Proposition 19.3 there exists a projective birational L-morphism $Z' \to \mathring{X}/P_{\mathrm{u}}$ such that the center of $v|_{K^U}$ on Z' is a divisor $D' \subset Z'$. Consider a Cartesian square

$$\begin{array}{ccc} \mathring{X}' & \longrightarrow & \mathring{X} \\ \Big\downarrow & & \Big\downarrow \\ Z' & \longrightarrow & \mathring{X}/P_{\mathrm{u}} \end{array}$$

where horizontal arrows are birational projective P-morphisms, and vertical arrows are P_{u}-quotient maps. Therefore v has a center $D \subset \mathring{X}'$, which is P-stable and maps onto D', whence D is the pull-back of D', i.e., a divisor. This means that v is geometric. □

Proof 2. Here we use the embedding theory of Chapter 3. Assume $\mathbf{v} = v|_{K^B}$. It is easy to construct an affine model C_0 of K^B containing a principal prime divisor $D_0 = \operatorname{div}(t)$ such that either D_0 is the center of $\mathbf{v}$, or $\mathbf{v} = 0$, $t = 1$, $D_0 = \emptyset$, and $\mathring{C} = C_0 \setminus D_0 = \mathring{X}/B$ for a (sufficiently small) B-chart $\mathring{X}$.

If $\mathcal{R}$ is the set of all B-stable prime divisors in $\mathring{X}$ (=preimages of prime divisors in $\mathring{C}$), then $\{v\} \sqcup \mathcal{R} \in \mathbf{CD}$ defines colored data, and we may consider the respective Krull algebra $\mathcal{A} = \mathcal{A}(v, \mathcal{R})$ (cf. §13). Recall that we need not to assume apriori that all G-invariant valuations are geometric (Remark 13.4). Clearly, $\mathcal{A}^U = \Bbbk\big[\, f \in K^{(B)} \bigm| \langle v, f\rangle, \langle \mathcal{R}, f\rangle \geq 0 \,\big] \subseteq \Bbbk[\mathring{C}] \otimes \Bbbk[\Lambda]$ is a subalgebra determined by $v(f) \geq 0$, whence $\mathcal{A}^U = \Bbbk[C_0]\big[\, t^d \mathbf{f}_\lambda \bigm| (d, \lambda) \in \Lambda_{\mathbf{v}},\ \langle v, (d, \lambda)\rangle \geq 0 \,\big]$.

The generating set of $\mathcal{A}^U$ over $\Bbbk[C_0]$ forms a finitely generated semigroup in $\Lambda_{\mathbf{v}}$ consisting of lattice points in the half-space $\{v \geq 0\}$, whence condition (F) holds for $\mathcal{A}$.

To prove (C), we take $f = t^d \mathbf{f}_\lambda$ such that $\langle v, (d, \lambda)\rangle > 0$ and multiply f by $f_0 \in \Bbbk[C_0]$ vanishing on sufficiently many divisors in $\mathcal{R}$.

Conversely, taking $f = t^d \mathbf{f}_\lambda$ such that $\langle v, (d, \lambda)\rangle < 0$ proves (W) for v.

By Corollary 13.1, $X_0 = \operatorname{Spec} \mathcal{A}$ is a B-chart and v is the valuation of a G-stable divisor intersecting X_0. □

Remark 20.1. It is often helpful to assume that $K = \operatorname{Quot} R$, where R is a rational G-algebra. For instance, $R = \Bbbk[X]$, where X is a (quasi)affine G-model of K (if any exists). The general case is reduced to this special one by considering a projectively normal G-model X and taking the affine cone $\widehat{X}$ over X. Then $\widehat{K} = \Bbbk(\widehat{X})$ is a $\widehat{G}$-field, where $\widehat{G} = G \times \Bbbk^\times$ with $\Bbbk^\times$ acting by homotheties. Let us denote various objects related to $\widehat{K}$ in the same way as for K, but equipped with a hat. We have short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (K^B)^\times & \longrightarrow & K^{(B)} & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \Vert & & \mid\!\bigcap & & \curvearrowleft & & \\ 1 & \longrightarrow & (\widehat{K}^{\widehat{B}})^\times & \longrightarrow & \widehat{K}^{(\widehat{B})} & \longrightarrow & \widehat{\Lambda} & \longrightarrow & 0 \end{array}$$

and dual sequences

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbb{Q}_{\mathbf{v},+} & \longleftarrow & \mathcal{E}_{\mathbf{v},+} & \longleftarrow & \mathcal{E} & \longleftarrow & 0 \\ & & \Vert & & \uparrow & & \uparrow & & \\ 0 & \longleftarrow & \mathbb{Q}_{\mathbf{v},+} & \longleftarrow & \widehat{\mathcal{E}}_{\mathbf{v},+} & \longleftarrow & \widehat{\mathcal{E}} & \longleftarrow & 0 \end{array}$$

The set of colors $\widehat{\mathcal{D}}^{\widehat{B}}$ is identified with $\mathcal{D}^B$. The grading of $\Bbbk[\widehat{X}]$ determines two G-valuations $\pm v_0 \in \widehat{\mathcal{E}}$, which generate $\operatorname{Ker}(\widehat{\mathcal{E}} \to \mathcal{E})$. By Corollary 19.1, $\widehat{\mathcal{V}}_{\mathbf{v}}$ surjects onto $\mathcal{V}_{\mathbf{v}}$ and even is the preimage of $\mathcal{V}_{\mathbf{v}}$ by Proposition 20.2 below.

Definition 20.2. Let $f_1, \dots, f_s \in R^{(B)}$ and $f \neq f_1 \dots f_s$ be any highest vector in $\langle Gf_1 \rangle \dots \langle Gf_s \rangle$. Then $f/f_1 \dots f_s$ is called a *tail vector* of R and its weight is called a *tail weight* or just a *tail*. Note that tails are negative linear combinations of simple roots. In characteristic zero tails are the nonzero differences $\mu - \lambda_1 - \dots - \lambda_s$ over all highest weights μ occurring in the isotypic decomposition of $R_{(\lambda_1)} \dots R_{(\lambda_s)}$, $\lambda_1, \dots, \lambda_s \in \Lambda_+$.

Now we proceed in proving Theorem 20.1.

Proposition 20.2. *$\mathcal{V}_{\mathbf{v}}$ is a convex cone in $\mathcal{E}_{\mathbf{v},+}$.*

Proof 1. We may assume that $K = \operatorname{Quot} R$, where R is a rational G-algebra. The general case is reduced to this one by considering the affine cone over a projectively normal G-model as above. Then we prove for $\forall v \in \check{\mathcal{E}}$ that $v \in \mathcal{V}$ iff

(T) v is non-negative on all tail vectors of R

Clearly, this condition is necessary, since $v(\langle Gf_1 \rangle \dots \langle Gf_s \rangle) \geq v(f_1 \dots f_s)$, $\forall v \in \mathcal{V}$, $f_1 \dots f_s \in R$.

Conversely, assume that (T) is satisfied. It can be generalized as follows: let f be any highest vector in $\sum_i \langle Gf_{i1}\rangle \dots \langle Gf_{is}\rangle$, $f_{ij} \in R^{(B)}$; then $v(f) \geq \min_i\{v(f_{i1}) + \dots + v(f_{is})\}$. Indeed, if $f = \sum f_i$, where $f_i \in \langle Gf_{i1}\rangle \dots \langle Gf_{is}\rangle$ are highest vectors of the same weight, then $v(f) \geq \min v(f_i) \geq \min\{v(f_{i1}) + \dots + v(f_{is})\}$ by (T). The general case is reduced to this one, because a certain power f^q belongs to the image of $\mathrm{S}^q(\bigoplus \langle Gf_{i1}\rangle \dots \langle Gf_{is}\rangle)^U$ by Corollary A2.1.

Consider a rational B-algebra $A = \Bbbk[\, gf/h \mid g \in G,\ f,h \in R,\ v(f/h) \geq 0\,]$. The ideal $I = (\, gf/h \mid v(f/h) > 0\,) \lhd A$ is proper. Indeed, each $f \in I^{(B)}$ is a linear combination of $(g_{i1}f_{i1}/h_1)\dots(g_{is}f_{is}/h_s)$, where $v(f_{ij}/h_j) \geq 0$ and $>$ occurs for $\forall i$. By the above $v(fh_1\dots h_s) \geq \min\{v(f_{i1}) + \dots + v(f_{is})\} > v(h_1) + \dots + v(h_s) \implies v(f) > 0$, whence $v > 0$ on $I^{(B)}$.

Take any valuation v' non-negative on A and positive on I, extend it to K, and take the approximating G-valuation $\overline{v}$ (Proposition 19.1). For $\forall f \in R^{(B)}$, $g \in G$ we have $v'(gf) \geq v'(f) \implies \overline{v}(f) = v'(f)$. Now $\forall f,h \in R^{(B)} :\ v(f/h) \geq 0\ (>0) \implies f/h \in A\ (\in I) \implies \overline{v}(f/h) \geq 0\ (>0)$. It follows that DVR's of v and $\overline{v}$ on K^B coincide, hence $\overline{v} \in \mathcal{V}_{\mathbf{v}}$ provided $v \in \mathcal{E}_{\mathbf{v}}$, and $\overline{v}, v$ determine proportional linear functionals on $\Lambda_{\mathbf{v}}$. Thus $v \in \mathcal{V}$. □

Proof 2. It relies on the embedding theory. Let $v_1, v_2 \in \mathcal{V}_{\mathbf{v}}$ be two non-proportional vectors. It suffices to prove that $c_1v_1 + c_2v_2 \in \mathcal{V}_{\mathbf{v}}$, $\forall c_1, c_2 \in \mathbb{Z}_+$.

Take an affine model C_0 of K^B as in the 2-th proof of Proposition 20.1 and consider the algebra $\mathcal{A} = \mathcal{A}(v_1, v_2, \mathcal{R})$. Then $\mathcal{A}^U \subseteq \Bbbk[\mathring{C}] \otimes \Bbbk[\Lambda]$ is distinguished by inequalities $v_i(f) \geq 0$, whence $\mathcal{A}^U = \Bbbk[C_0]\big[\, t^d\mathbf{f}_\lambda \bigm| (d,\lambda) \in \Lambda_{\mathbf{v}},\ \langle v_i, (d,\lambda)\rangle \geq 0,\ i = 1,2 \big]$.

The generating set of $\mathcal{A}^U$ over $\Bbbk[C_0]$ forms a finitely generated semigroup of lattice points in the dihedral cone $\{v_1, v_2 \geq 0\} \subseteq \Lambda_{\mathbf{v}} \otimes \mathbb{Q}$, whence condition (F) holds for $\mathcal{A}$. Conditions (C), (W) are verified in the same way as in Proposition 20.1 by taking $f = t^d\mathbf{f}_\lambda$ with $\langle v_i, (d,\lambda)\rangle, \langle v_j, (d,\lambda)\rangle > 0$ or $\langle v_i, (d,\lambda)\rangle < 0 \leq \langle v_j, (d,\lambda)\rangle$, respectively, $\{i,j\} = \{1,2\}$. Thus by Corollary 13.1, $X_0 = \operatorname{Spec}\mathcal{A}$ is a B-chart and v_i correspond to G-stable divisors $D_i \subset X = GX_0$ intersecting X_0.

We blow up the ideal sheaf $\mathcal{O}(-nc_2D_1) + \mathcal{O}(-nc_1D_2)$, $n \gg 0$, and prove that the exceptional divisor corresponds to $c_1v_1 + c_2v_2$.

The local structure of X_0 provided by Proposition 13.2 allows to replace X_0 by X_0/P_{u} and assume that $G = T$ and $X = X_0$ is an affine T-model of K. We may choose $f_i = t^{d_i}\mathbf{f}_{\lambda_i}$ such that $v_i(f_j) = n\delta_{ij}$, $n \gg 0$. The above ideal sheaf is represented by a proper ideal $I = (f_1^{c_2}, f_2^{c_1}) \lhd \mathcal{A}$. (Indeed, it is easy to see that $v_1 + v_2 > 0$ on $I^{(T)}$.)

The blow-up of I is given in $X \times \mathbb{P}^1$ by the equation $[f_1^{c_2} : f_2^{c_1}] = [t_1 : t_2]$, where t_i are homogeneous coordinates on $\mathbb{P}^1$. The exceptional divisor is given in the open subset $\{t_2 \neq 0\}$ by the equation $f_2 = 0$. Let v_0 be the respective valuation. Up to a power, any $f \in \mathcal{A}^{(T)}$ is represented as $f = f_0 f_1^{k_1} f_2^{k_2}$, $v_i(f_0) = 0$, $k_i \,\vdots\, c_j$, $\{i,j\} = \{1,2\}$. Then $v_0(f) = v_0(f_0 (t_1/t_2)^{k_1/c_2} f_2^{(c_1k_1+c_2k_2)/c_2}) \sim c_1k_1 + c_2k_2$. It follows that $v_0 \sim c_1v_1 + c_2v_2$. ☐

Proposition 20.3. *The cone $\mathcal{V}_{\mathbf{v}}$ is finitely generated.*

Proof. Take a projective G-model X as in Lemma 20.2. Then any $v \in \mathcal{V}_{\mathbf{v}}$ has the center Y on X, and by Lemma 20.2(5), $\mathcal{D}_Y^B = \emptyset$, $\mathcal{V}_Y \subset \mathcal{V}_{\mathbf{v}}$. Condition (S) yields $\forall\lambda \in \Lambda_{\mathbf{v}} : \ \langle \mathcal{V}_Y, \lambda\rangle \geq 0 \implies \langle v, \lambda\rangle \geq 0$. Hence $v \in \mathbb{Q}_+\mathcal{V}_Y$. It remains to note that $\bigcup_Y \mathcal{V}_Y$ is finite by Lemma 20.1. ☐

Proof of Theorem 20.1. Due to Propositions 20.2–20.3 and Theorem 21.1, it remains to prove that any geometric valuation $\mathbf{v} \neq 0$ of K^B extends to a G-valuation of K. For this, we modify the 1-th proof of Proposition 20.2.

Namely, in the definition of A we replace v by $\mathbf{v}$ and assume $f/h \in K^B$. The respective ideal $I \lhd A$ is still proper. For otherwise, 1 is a linear combination of $(g_{i1}f_{i1}/h_1)\dots(g_{is}f_{is}/h_s)$, where $\mathbf{v}(f_{ij}/h_j) \geq 0$ and $>$ occurs for $\forall i$. But the T-weights of all T-eigenvectors in $\langle Gf_{ij}\rangle$ except f_{ij} are obtained from the weight of f_{ij} (=the weight of h_j) by subtracting simple roots. Hence we may assume $g_{ij} = e$, and 1 is a linear combination of $r_i = (f_{i1}/h_1)\dots(f_{is}/h_s)$, $\mathbf{v}(r_i) > 0$, a contradiction.

Now reproducing the arguments of that proof yields a G-valuation $\overline{v}$ such that $\overline{v}|_{K^B} = \mathbf{v}$. ☐

Parabolic induction, which is helpful in various reduction arguments, keeps the colored hyperspace "almost" unchanged. Suppose K is obtained from a G_0-field K_0 by parabolic induction $G \supseteq Q \twoheadrightarrow G_0$, i.e., a G-model X of K is obtained from a G_0-model X_0 of K_0 by this procedure. There is a natural projection $\pi : X = G *_Q X_0 \to G/Q$. We may assume $Q \supseteq B^-$. Then the colors of G/Q are the Schubert divisors $D_\alpha = \overline{B[r_\alpha]}$ ($\alpha \in \Pi$, $r_\alpha \notin Q$). Let us denote various objects related to K_0 in the same way as for K, but with a subscript 0.

Proposition 20.4. *There are natural identifications $\breve{\mathcal{E}} = \breve{\mathcal{E}}_0$, $\mathcal{V} = \mathcal{V}_0$, and $\mathcal{D}^B = \mathcal{D}_0^{B_0} \sqcup \pi^{-1}(\mathcal{D}^B(G/Q))$ such that $\rho = \rho_0$ on $\mathcal{D}_0^{B_0}$ and $\rho(\pi^{-1}(D_\alpha)) \in \mathcal{E}$ is the restriction of $\alpha^\vee$ to Λ.*

Proof. Since $\pi^{-1}(B[e]) \simeq Q_{\mathrm{u}}^- \times X_0$, the restriction to the fiber identifies $K^{(B)}$ with $K_0^{(B_0)}$. Therefore, the hyperspaces are identified and the colors are extended by the pullbacks of the Schubert divisors.

To prove $\mathcal{V} = \mathcal{V}_0$, we construct X_0 as in Lemma 20.2. Then X satisfies the conditions of Lemma 20.2, too. By the condition (5), $\mathcal{V}\cap\mathcal{E}_{\mathbf{v},+} = \mathcal{V}_0\cap\mathcal{E}_{\mathbf{v},+}$.

Finally, $\pi^{-1}(D_\alpha)$ is transversal to the subvariety $X_\alpha = L_\alpha *_{B_\alpha^-} X_0 \subseteq X$ along $r_\alpha * X_0$, where L_α is the Levi subgroup of P_α and $B_\alpha^- = B^-\cap L_\alpha = TU_{-\alpha}$ acts on X_0 via T. The variety X_α is horospherical w.r.t. L_α and intersects generic B-orbits of X. We easily deduce that generic B-orbit closures in X intersect $\pi^{-1}(D_\alpha)$ and the intersections cover a dense subset $Br_\alpha * X_0$. Hence all $f \in K^B$ have order 0 along $\pi^{-1}(D_\alpha)$. To determine $\rho(\pi^{-1}(D_\alpha))$, it suffices to consider the restriction of $K^{(B)}$ to a generic L_α-orbit in X_α, cf. §28. □

21 Central valuations

G-valuations of K that vanish on $(K^B)^\times$ are called *central*. They play a distinguished role among all G-valuations. By (13.1) a central valuation restricted to $K^{(B)}$ factors through a linear functional on Λ. Thus central valuations form a subset $\mathcal{Z} \subset \mathcal{E} = \mathrm{Hom}(\Lambda, \mathbb{Q})$.

If G is linearly reductive (i.e., $\operatorname{char} \Bbbk = 0$ or $G = T$) and $K = \operatorname{Quot} R$ for a rational G-algebra R with the isotypic decomposition $R = \bigoplus_{\lambda\in\Lambda} R_{(\lambda)}$, then any $v \in \mathcal{Z}$ is constant on each isotypic component $R_{(\lambda)}$ (otherwise there would exist two highest vectors $f_1, f_2 \in R^{(B)}$ of the same highest weight λ with $v(f_1) \neq v(f_2)$) and $v|_{R_{(\lambda)}\setminus\{0\}} = \langle v, \lambda\rangle$, $\forall \lambda \in \Lambda$.

Theorem 21.1. *$\mathcal{Z} = \mathcal{V}\cap\mathcal{E}$ is a solid convex polyhedral cone in $\mathcal{E}$ containing the image of the antidominant Weyl chamber.*

Proof. By Remark 20.1, we reduce the question to the case $K = \operatorname{Quot} R$, where R is a rational G-algebra. Condition (T) defining the subset $\mathcal{V} \subset \check{\mathcal{E}}$ transforms under restriction to $\mathcal{E}$ to the following one: $v \in \mathcal{Z}$ iff

$$v \text{ is non-negative on all tails of } R \tag{T$_0$}$$

Since tails are negative linear combinations of simple roots, we see that (T$_0$) determines a convex cone containing the image of the antidominant Weyl chamber, whence a solid cone. We conclude by Proposition 20.3. □

Example 21.1. Let G act on itself by right translations and $K = \Bbbk(G)$. Here $\Lambda = \mathfrak{X}(T)$. Recall from Proposition 2.3 that $\Bbbk[G]$ is the union of subspaces $\mathrm{M}(V)$ of matrix entries over all G-modules V.

In characteristic zero, $\mathrm{M}(V(\lambda)) = \Bbbk[G]_{(\lambda)}$ are the isotypic components of $\Bbbk[G]$ and $\mathrm{M}(V(\lambda))\cdot\mathrm{M}(V(\mu)) = \bigoplus \mathrm{M}(V(\nu))$ over all simple submodules $V(\nu)$ occurring in $V(\lambda)\otimes V(\mu)$ (2.1). Generally, each highest vector $v \in V^{(B)}$ gives rise to highest vectors $f_{\omega,v} \in \mathrm{M}(V)^{(B)}$, $\omega \in V^*$, of the same highest weight.

If $v_\lambda \in V$ is a highest vector of regular highest weight λ, then V contains T-eigenvectors $v_{\lambda-\alpha}$ of weights $\lambda - \alpha$ for all simple roots α, and $v_\lambda \otimes v_{\lambda-\alpha} - v_{\lambda-\alpha} \otimes v_\lambda$ are highest vectors of highest weights $2\lambda - \alpha$ in $V \otimes V$. It follows that all $-\alpha$ occur among tails of $\Bbbk[G]$, whence $\mathcal{Z}$ is the antidominant Weyl chamber.

In characteristic zero, a much more precise information on the structure of $\mathcal{Z}$ can be obtained, see §22.

A special case of Theorem 5.1 distinguishes central and non-central G-valuations in terms of the complexity and the rank of respective G-stable divisors.

Proposition 21.1. *A G-valuation $v \neq 0$ is (non)central iff $c(\Bbbk(v)) = c(K)$, $r(\Bbbk(v)) = r(K) - 1$ ($c(\Bbbk(v)) = c(K) - 1$, $r(\Bbbk(v)) = r(K)$, respectively).*

Proof. Let $Y \subset X$ be a G-stable divisor on a G-model of K corresponding to v. By Lemma 5.1, $\Bbbk(v)^U = \Bbbk(Y)^U$ is a purely inseparable extension of the residue field $\Bbbk(v|_{K^U})$ of $\mathcal{O}_{X,Y}^U$, and similarly for $\Bbbk(v)^B$. Thus by Proposition 5.1, $c(\Bbbk(v)) + r(\Bbbk(v)) = \operatorname{tr.deg} \Bbbk(v)^U = \operatorname{tr.deg} K^U - 1 = c(K) + r(K) - 1$, and $c(\Bbbk(v)) = c(K)$ iff $\Bbbk(v|_{K^B}) = K^B$ iff $v \in \mathcal{Z}$. $\square$

Now we explain the geometric meaning of the linear part of the central valuation cone.

Definition 21.1. A G-automorphism of K acting trivially on K^B is called *central*. Denote by $\operatorname{Cent} K = \operatorname{Ker}(\operatorname{Aut}_G K : K^B)$ the group of central automorphisms.

Theorem 21.2. *There exists the largest connected algebraic subgroup $S \subseteq \operatorname{Cent} K$. It has the following properties:*

(1) *S acts on $\mathcal{V}, \mathcal{D}^B, {}_G\mathbb{X}^{\mathrm{norm}}$ trivially.*

(2) *S acts on every normal G-model of K regularly.*

(3) *There is a canonical embedding $S \hookrightarrow A = \operatorname{Hom}(\Lambda, \Bbbk^\times)$ via the action $S : K^{(B)}$, so that $\operatorname{Hom}(\mathfrak{X}(S), \mathbb{Q}) \subseteq \mathcal{Z} \cap -\mathcal{Z}$.*

(4) *There exists a G-subfield $K' \subseteq K$ with $(K')^U = K^U$ and with the same colored hyperspace as for K such that K is purely inseparable over K' and $\operatorname{Hom}(\mathfrak{X}(S'), \mathbb{Q}) = \mathcal{Z} \cap -\mathcal{Z}$ for the largest connected algebraic subgroup $S' \subseteq \operatorname{Cent} K'$.*

Proof. Let $S \subseteq \operatorname{Cent} K$ be any connected algebraic subgroup. Suppose first that $K = \operatorname{Quot} R$, where R is a rational $(G \times S)$-subalgebra. Without loss of generality, we may assume in the reasoning below that $R = \Bbbk[X]$, where X is a normal (quasi)affine G-model of K acted on by S.

If $f \in R^{(B)}$, then $sf \in R^{(B)}$ has the same weight, $\forall s \in S$, whence $sf/f \in K^B \subseteq K^S$. Hence $s \operatorname{div} f - \operatorname{div} f$ is S-stable. (The divisors are considered on the normalized projective closure of X.) But on the other hand, this divisor has no S-stable components, hence is zero, and $sf \in \Bbbk^\times f$. Therefore $R^{(B)} \subseteq R^{(S)} \implies K^{(B)} \subseteq K^{(S)}$.

This yields a homomorphism $S \to A$. Let S_0 be its connected kernel. Then S_0 acts on R^U trivially. As S_0 commutes with G, it acts on R trivially. (In positive characteristic, this stems from Lemma A2.2.) Hence $S_0 = \{e\}$ and S is a torus. Then every $s \in S$, $s \neq e$, acts non-trivially on $K^{(B)}$: just take a (G-stable) eigenspace of s in R of eigenvalue $\neq 1$ and choose a B-eigenvector there. Thus $S \hookrightarrow A$.

Since S-action multiplies B-eigenfunctions by scalars and G-valuations are determined by their restriction to $K^{(B)}$, the action $S : \mathcal{V}$ is trivial. As any $D \in \mathcal{D}^B$ is a component of $\operatorname{div} f$, $f \in K^{(B)} \subseteq K^{(S)}$, S fixes all colors. Then S fixes all G-germs by Proposition 14.1, whence (1). Assertion (2) stems from (1).

Each one-parameter subgroup $\gamma \in \operatorname{Hom}(\mathfrak{X}(S), \mathbb{Z})$ defines a G-stable grading of R, which gives rise to a central valuation (the order of the lowest homogeneous term) represented by γ: $f_\lambda \mapsto \langle \gamma, \lambda \rangle$, $\forall f_\lambda \in R^{(B)}_\lambda$. This finally yields (3).

Furthermore, any $\tau \in \operatorname{Cent} K$ commutes with γ. Indeed, we have two gradings of R defined by $\gamma, \tau\gamma\tau^{-1}$. They coincide on $R^{(B)}$, whence on the G-subalgebra of R generated by R^U, whence on R. (The last implication is easily deduced from Lemma A2.2.) Therefore any two subtori $S_1, S_2 \subseteq \operatorname{Cent} K$ commute, and the natural homomorphism $S_1 \times S_2 \to \operatorname{Cent} K$ provides a larger subtorus. Since the dimensions of subtori are restricted from above by $\dim(\mathcal{Z} \cap -\mathcal{Z})$, there exists the largest one.

We prove (4) in characteristic zero referring to [Kn3, 8.2] for $\operatorname{char} \Bbbk > 0$. Every lattice vector $\gamma \in \mathcal{Z} \cap -\mathcal{Z}$ defines a G-stable grading of R such that $R_{(\lambda)}$ are homogeneous of degree $\langle \gamma, \lambda \rangle$. Since γ vanishes on tails, this grading respects multiplication and defines a 1-subtorus of central automorphisms. These 1-subtori generate a subtorus $S \subseteq A$ which is the connected common kernel of all tails, and $\operatorname{Hom}(\mathfrak{X}(S), \mathbb{Q}) = \mathcal{Z} \cap -\mathcal{Z}$.

Finally, the general case is reduced to the above by taking a projectively normal G-model X acted on by S and considering the affine cone $\widehat{X}$ over X. (Note that any central subtorus action on X lifts to $\widehat{X}$ if we consider a sufficiently ample projective embedding of X.) By Remark 20.1, $\mathcal{Z} = \widehat{\mathcal{Z}}/\mathbb{Q}v_0$, and $S = \widehat{S}/\Bbbk^\times$, where the central valuation v_0 of $\widehat{K}$ is defined by a 1-subtorus $\Bbbk^\times \subseteq \widehat{S}$ acting on $\widehat{X}$ by homotheties. So all assertions on $\widehat{S}$ transfer to S. □

In the generically transitive case, we can say more.

Proposition 21.2. *If $K = \Bbbk(G/H)$, then* $\operatorname{Cent} K$ *is a quasitorus extended by a finite p-group and* $\dim \operatorname{Cent} K = \dim(\mathcal{Z} \cap -\mathcal{Z})$.

Proof. Since $\operatorname{Aut}_G K = \operatorname{Aut}_G G/H = N(H)/H$ is an algebraic group, $\operatorname{Cent} K$ is an algebraic group as well. Central automorphisms preserve generic B-orbits in G/H, whence there exist finitely many general points $x_1, \dots, x_s \in G/H$ such that $\operatorname{Cent} K \hookrightarrow \operatorname{Aut}_B Bx_1 \times \cdots \times \operatorname{Aut}_B Bx_s$. The latter group is a subquotient of $B \times \cdots \times B$, which explains the structure of $\operatorname{Cent} K$ in view of Theorem 21.2(3). By Theorem 21.2(4), there exists a purely inseparable G-map $G/H \twoheadrightarrow \mathcal{O}$ such that $\dim \operatorname{Cent} \mathcal{O} = \dim(\mathcal{Z} \cap -\mathcal{Z})$. But every G-automorphism of $\mathcal{O}$ lifts to G/H by the universal property of quotients. Hence $\dim \operatorname{Cent} K$ is the same. □

Remark 21.1. If $\operatorname{char} \Bbbk = 0$ and G/H is quasiaffine, then $\operatorname{Cent} G/H$ is canonically embedded in A as the common kernel of all tails of $\Bbbk[G/H]$ and acts on each $\Bbbk[G/H]_{(\lambda)}$ by the character $\lambda \in \Lambda_+(G/H)$.

Example 21.2. In Example 21.1, $\operatorname{Aut}_G G = G$ (acting by left translations) and $\operatorname{Cent} G = \operatorname{Ker}(G : G/B) = Z(G)$.

Example 21.3. If the orbit map $G \to \mathcal{O}$ is not separable, then $\dim \operatorname{Cent} \mathcal{O}$ may be smaller than "the proper value". For instance, let $\operatorname{char} \Bbbk = 2$, $G = \mathrm{SL}_2$, $X = \mathbb{P}(\mathfrak{sl}_2)$. Then X has an open orbit $\mathcal{O}$ with stabilizer U. By Theorem 21.2(2), the central torus S embeds in $\operatorname{Aut}_G X$, but the latter group is trivial. Indeed, each G-automorphism of X lifts to an intertwining operator of $\operatorname{Ad} \mathrm{SL}_2$, but it is easy to see that such an operator has to be scalar. However $\Bbbk(G/U)^B = \Bbbk \implies \operatorname{Cent} G/U = \operatorname{Aut}_G G/U = T \implies \dim(\mathcal{Z} \cap -\mathcal{Z}) = 1$, cf. Theorem 21.3.

We have seen in §8 that horospherical G-varieties play an important role in studying general G-varieties. Apparently they can be characterized in terms of central valuation cones.

Theorem 21.3. *A G-variety X is horospherical iff $\mathcal{Z}(X) = \mathcal{E}(X)$.*

Proof 1 ($\operatorname{char} \Bbbk = 0$) [Po4, §4], [Vin1, §5]. This proof goes back to Popov, who however considered tails of coordinate algebras instead of central valuations, cf. Proposition 7.1. The generically transitive case is due to Pauer [Pau4].

By Remark 20.1, we may assume X to be (quasi)affine. By (T_0), $\mathcal{Z} = \mathcal{E}$ iff $R = \Bbbk[X]$ has no tails. However one proves that a rational G-algebra R has no tails iff $R \simeq (\Bbbk[G/U^-] \otimes R^U)^T = \bigoplus_{\lambda\in\mathfrak{X}_+} \Bbbk[G/U^-]_{(\lambda)} \otimes R^{(B)}_\lambda$. Here $T = \operatorname{Aut}_G G/U^-$ acts on G/U^- by right translations, so that isotypic components $\Bbbk[G/U^-]_{(\lambda)}$ are at the same time T-eigenspaces of weight $-\lambda$ (cf. (2.4)), and the isomorphism is given by $g\mathbf{f}_\lambda \otimes f_\lambda \mapsto gf_\lambda$, $\forall g \in G$, $f_\lambda \in R^{(B)}_\lambda$, where $\mathbf{f}_\lambda \in \Bbbk[G/U^-]^{(B)}_\lambda$, $\mathbf{f}_\lambda(e) = 1$. In our situation, this implies that $X = (G/\!\!/U^- \times X/\!\!/U)/\!\!/T$ is horospherical.

Conversely, if R has tails, then tails do not vanish under restriction of the isotypic decomposition of $R_{(\lambda)} \cdot R_{(\mu)}$ to generic G-orbits. But the coordinate algebra of a horospherical homogeneous space has no tails since isotypic components of $\Bbbk[G/U^-]$ are T-eigenspaces. Hence X is not horospherical. □

Proof 2 [Kn3, 8.5]. If $\mathcal{Z} = \mathcal{E}$, then by Theorem 21.2(4) we may assume that $S = A$ and the geometric quotient X/S exists. Then $r(X/S) = 0$ and by Propositions 5.3, 10.1, orbits of $G : X/S$ are projective homogeneous spaces.

Let $x \in X$ and $x \mapsto \bar{x} \in X/S$. We may assume $G_{\bar{x}} \supseteq U$. Then U preserves Sx, and we have the orbit map $U \to Ux \subseteq Sx$. As U is an affine space (with no non-constant invertible polynomials) and Sx is a torus (whose coordinate algebra is generated by invertibles), this map is constant, whence $Ux = \{x\}$. Thus X is horospherical.

Conversely, for horospherical X put $Z = X^{U^-}$ and consider the natural proper map $X' \to G *_{B^-} Z \twoheadrightarrow X$. There are natural maps $\mathcal{E}(X') \twoheadrightarrow \mathcal{E}(X)$, $\mathcal{Z}(X') \to \mathcal{Z}(X)$. Restriction of functions to Z yields $\Bbbk(X')^U = \Bbbk(Z)$, $\Bbbk(X')^B = \Bbbk(Z)^T$. Then any T-valuation of $\Bbbk(Z)$ extends to a G-valuation of $\Bbbk(X')$. Indeed, we may replace Z by a birationally isomorphic B^--variety and assume that the T-valuation corresponds to a B^--stable divisor $D \subset Z$ (U^- acts trivially); then the desired G-valuation corresponds to $D' = G *_{B^-} D \subset X'$. By Example 20.1, $\mathcal{Z}(X') = \mathcal{E}(X')$, whence $\mathcal{Z}(X) = \mathcal{E}(X)$. □

We conclude this section by the description of G-valuations for the residue field of a central valuation.

Proposition 21.3 ([Kn3, 7.4]). *Let X be a G-model of K, $D \subset X$ a G-stable prime divisor with $v_D \in \mathcal{Z}$, X' the normal bundle of X at D. Then:*

(1) $\breve{\mathcal{E}}(X') = \breve{\mathcal{E}}$ *and* $\mathcal{V}(X') = \mathcal{V} + \mathbb{Q}v_D$*;*

(2) $\breve{\mathcal{E}}(D) = \breve{\mathcal{E}}/\mathbb{Q}v_D$ *and* $\mathcal{V}(D)$ *is the image of* $\mathcal{V}$.

Proof. As usual, we may assume $K = \operatorname{Quot} R$, where R is a rational G-algebra. Then $K' := \Bbbk(X') = \operatorname{Quot} \operatorname{gr} R$, where $\operatorname{gr} R$ is the graded algebra associated with the filtration $R_{(d)} = \{f \in R \mid v_D(f) \geq d\}$ of R. Since v_D is central, it is constant on each B-eigenspace of R^U, whence $R^U \simeq \operatorname{gr}(R^U)$. But $(\operatorname{gr} R)^U$ is a purely inseparable finite extension of $\operatorname{gr}(R^U)$ by Corollary A2.2, hence $(K')^U \supseteq K^U$ is a purely inseparable field extension. This implies $\breve{\mathcal{E}}(X') = \breve{\mathcal{E}}$.

The G-invariant grading of $\operatorname{gr} R$ is defined by a central 1-torus acting on $\mathcal{V}(X')$ trivially by Theorem 21.2(1). Hence it agrees with all G-valuations. Thus $v \in \mathcal{V}(X')$ iff v is non-negative at all tail vectors of the form $\bar{f}_0/\bar{f}_1 \dots \bar{f}_s$, where $\bar{f}_i \in (\operatorname{gr} R)^{(B)}$ are homogeneous elements represented by $f_i \in R$, and $v_D(f_0) = v_D(f_1) + \dots + v_D(f_s)$. Replacing $\bar{f}_i$ by suitable powers, we may assume $f_i \in R^{(B)}$. Thence $\mathcal{V}(X')$ is the set of all $v \in \breve{\mathcal{E}}$ non-negative on tail vectors of R annihilated by v_D, i.e., $\mathcal{V}(X') = \mathcal{V} + \mathbb{Q}v_D$.

By Lemma 5.1, $\Bbbk(D)^U$ is a purely inseparable extension of $\Bbbk(v_D|_{K^U})$, and $\Bbbk(D)^B$ of K^B. Hence $\breve{\mathcal{E}}(D) = \breve{\mathcal{E}}/\mathbb{Q}v_D$. Since X' retracts onto D, $\mathcal{V}(D) = \mathcal{V}(X')/\mathbb{Q}v_D$ by Corollary 19.1. $\square$

22 Little Weyl group

In §8 we found out that important invariants of a G-variety X such as complexity, rank, and weight lattice, which play an essential role in the embedding theory, are closely related to the geometry of the cotangent bundle T^*X. Knop developed these observations further [Kn1], [Kn5] and described the cone of (central) G-valuations as a fundamental chamber for the Galois group of a certain Galois covering of T^*X, called the little Weyl group of X. As this approach requires infinitesimal technique, we assume $\operatorname{char} \Bbbk = 0$ in this and the next section. We retain the notation and conventions of §8.

The Galois covering of T^*X is defined in terms of the moment map. A disadvantage of the moment map Φ is that its image M_X can be non-normal and generic fibers can be reducible. A remedy is to consider the "Stein factorization" of Φ. Let $\widetilde{M}_X$ be the spectrum of the integral closure of $\Bbbk[M_X]$ (embedded via Φ^*) in $\Bbbk(T^*X)$. We may assume that X is smooth, whence T^*X is smooth and normal, and therefore $\Bbbk[\widetilde{M}_X] \subseteq \Bbbk[T^*X]$. It is easy to see that $\Bbbk(\widetilde{M}_X)$ is algebraically closed in $\Bbbk(T^*X)$. Thus Φ decomposes into the product of a finite morphism $\phi : \widetilde{M}_X \to M_X$ and the *normalized moment map* $\widetilde{\Phi} : T^*X \to \widetilde{M}_X$ with irreducible generic fibers. Set $\widetilde{L}_X = \widetilde{M}_X /\!\!/ G$. We have the quotient map $\widetilde{\pi}_G : \widetilde{M}_X \to \widetilde{L}_X$ and the natural finite morphism $\phi /\!\!/ G : \widetilde{L}_X \to L_X$.

The following result illustrates the role of the normalized moment map in equivariant symplectic geometry.

Proposition 22.1. *The fields* $\Bbbk(\widetilde{M}_X)$ *and* $\Bbbk(T^*X)^G$ *are the mutual centralizers of each other in* $\Bbbk(T^*X)$ *w.r.t. the Poisson bracket, and* $\Bbbk(\widetilde{L}_X)$ *is the Poisson center of both* $\Bbbk(\widetilde{M}_X)$ *and* $\Bbbk(T^*X)^G$*.*

Proof. The field $\Bbbk(M_X)$ is generated by Hamiltonians $H_\xi = \Phi^*\xi$, $\xi \in \mathfrak{g}$. Hence $f \in \Bbbk(T^*X)$ Poisson-commutes with $\Bbbk(M_X)$ iff $\{H_\xi, f\} = \xi f = 0$, $\forall \xi \in \mathfrak{g}$, i.e., f is G-invariant. But then f also commutes with $\Bbbk(\widetilde{M}_X)$. Indeed, let μ_h be the minimal polynomial of $h \in \Bbbk(\widetilde{M}_X)$ over $\Bbbk(M_X)$. Then $\{f, \mu_h(h)\} = \mu_h'(h)\{f, h\} = 0 \implies \{f, h\} = 0$. Therefore $\Bbbk(T^*X)^G$ is the centralizer of $\Bbbk(M_X)$ and $\Bbbk(\widetilde{M}_X)$.

Conversely, as generic orbits are separated by invariant functions, $\mathfrak{g}\alpha$ is the common kernel of $d_\alpha f$, $f \in \Bbbk(T^*X)^G$, for general $\alpha \in T^*X$. Hence $\operatorname{Ker} d_\alpha\widetilde{\Phi} = \operatorname{Ker} d_\alpha\Phi = (\mathfrak{g}\alpha)^\angle$ is generated by skew gradients of $f \in \Bbbk(T^*X)^G$. It follows that $h \in \Bbbk(T^*X)$ commutes with $\Bbbk(T^*X)^G$ iff dh vanishes on $\operatorname{Ker} d_\alpha\widetilde{\Phi} = T_\alpha\widetilde{\Phi}^{-1}\widetilde{\Phi}(\alpha)$ iff h is constant on $\widetilde{\Phi}^{-1}\widetilde{\Phi}(\alpha)$, because generic fibers $\widetilde{\Phi}^{-1}\widetilde{\Phi}(\alpha)$ are irreducible. Therefore $\Bbbk(\widetilde{M}_X)$ is the centralizer of $\Bbbk(T^*X)^G$.

Finally, $\Bbbk(\widetilde{L}_X) = \Bbbk(\widetilde{M}_X)^G = \Bbbk(\widetilde{M}_X) \cap \Bbbk(T^*X)^G$, since quotient maps π_G and $\widetilde{\pi}_G$ separate generic orbits. □

Recall the local structure of an open subset of X provided by Corollary 4.1: $\mathring{X} \simeq P_{\mathrm{u}} \times Z$, where $P = P_{\mathrm{u}} \rtimes L$ is a parabolic, and the Levi subgroup L acts on Z with kernel $L_0 \supseteq L'$, so that $Z \simeq A \times C$, $A = L/L_0$.

Generic U-orbits on X coincide with generic P_{u}-orbits and are of the form $P_{\mathrm{u}} \times \{z\}$, $z \in Z$. Generic B-orbits coincide with generic P-orbits and are of the form $P_{\mathrm{u}} \times A \times \{x\}$, $x \in C$. We have $T_xX = \mathfrak{p}_{\mathrm{u}}x \oplus \mathfrak{a}x \oplus T_xC$, $\forall x \in C$.

Generic U- and B-orbits on X form two foliations. Consider the respective conormal bundles $\mathcal{U} \supseteq \mathcal{B}$. They are P-vector bundles defined, e.g., over $\mathring{X}$. It follows from the local structure that $\mathcal{U} \simeq P_{\mathrm{u}} \times T^*Z \simeq P_{\mathrm{u}} \times A \times \mathfrak{a}^* \times T^*C$ and $\mathcal{B} \simeq P_{\mathrm{u}} \times A \times T^*C$ over $\mathring{X}$. We have $\mathcal{U}/\mathcal{B}(x) = (\mathfrak{u}x)^\perp/(\mathfrak{b}x)^\perp = (\mathfrak{b}x/\mathfrak{u}x)^* \simeq (\mathfrak{b}/\mathfrak{u} + \mathfrak{b}_x)^* \subseteq \mathfrak{t}^*$. For $x \in Z$ we have $\mathfrak{b}_x = \mathfrak{b} \cap \mathfrak{l}_0$, whence $\mathcal{U}/\mathcal{B}(x) \simeq \mathfrak{a}^*$. As $\operatorname{Ad} P_{\mathrm{u}}$ acts on $\mathfrak{b}/\mathfrak{u}$ trivially, there is a canonical isomorphism $\mathcal{U}/\mathcal{B}(x) \simeq \mathfrak{a}^*$, $\forall x \in \mathring{X}$. Therefore $\mathcal{U}/\mathcal{B} \simeq \mathfrak{a}^* \times \mathring{X}$ is a trivial bundle over $\mathring{X}$, and we have a canonical projection $\pi : \mathcal{U} \to \mathfrak{a}^*$, which is nothing else, but the moment map for the B-action.

The bundle $\mathcal{U}/\mathcal{B}$ can be lifted (non-canonically) to $\mathcal{U}$ over $\mathring{X}$. Namely consider yet another foliation $\{g(P_{\mathrm{u}}C) \mid g \in P\}$ and let $\mathcal{A}$ be the respective conormal bundle. By the local structure, $\mathcal{A} \simeq P_{\mathrm{u}} \times A \times \mathfrak{a}^* \times C$ over $\mathring{X}$, and $\mathcal{U} = \mathcal{A} \oplus \mathcal{B}$.

The isomorphism $\sigma : \mathfrak{a}^* \to \mathcal{A}(x)$, $x \in C$, defined by the formula

$$\sigma(\lambda) = \begin{cases} \lambda & \text{on } \mathfrak{a}x \simeq \mathfrak{a} \\ 0 & \text{on } \mathfrak{p}_{\mathrm{u}}x \oplus T_xC \end{cases}$$

provides a section for π. It depends on the choice of x and even more—we may replace C by any subvariety in $\mathring{X}^{L_0} = P_{\mathrm{u}}^{L_0} \times A \times C$ intersecting all P-orbits transversally so that x may be any point in $\mathring{X}$ with $P_x = L_0$ and T_xC may be any (L_0-stable) complement to $\mathfrak{p}x$ in T_xX.

Recall that $\mathfrak{a}$ embeds in $\mathfrak{l}$ as the orthocomplement to $\mathfrak{l}_0$. Consider the parabolic subgroup $Q \supseteq P$ having the Levi decomposition $Q_{\mathrm{u}} \rightthreetimes M$, $Q_{\mathrm{u}} \subseteq P_{\mathrm{u}}$, $M = Z_G(\mathfrak{a}) \supseteq L$.

Lemma 22.1. *There is a commutative square of dominant maps*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & \mathfrak{a} \oplus \mathfrak{q}_{\mathrm{u}} \subseteq \mathfrak{g} \simeq \mathfrak{g}^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \textit{projection}} \\ \mathfrak{a}^* & \xrightarrow{\sim} & \mathfrak{a} \end{array}$$

Proof. Take $\alpha \in \mathcal{U}(x)$, $x \in \mathring{X}$. Since all maps are P-equivariant, we may assume $x \in C \implies \mathcal{U}(x) \simeq (\mathfrak{a}x)^* \oplus T_x^*C$ and $\alpha = \sigma(\lambda) + \beta$ for some $\lambda = \pi(\alpha) \in \mathfrak{a}^*$, $\beta \in T_x^*C = \mathcal{B}(x)$. Hence $\langle \alpha, \xi x \rangle = \langle \lambda, \xi \rangle$, $\forall \xi \in \mathfrak{p} \implies \Phi(\alpha) = \lambda \mod \mathfrak{p}^\perp = \mathfrak{p}_{\mathrm{u}}$. Moreover, $\Phi(\alpha) \in (\mathfrak{a} + \mathfrak{p}_{\mathrm{u}})^{L_0} = \mathfrak{a} + \mathfrak{p}_{\mathrm{u}}^{L_0} \subseteq \mathfrak{a} + \mathfrak{q}_{\mathrm{u}}$, because $\mathfrak{p}_{\mathrm{u}}^{L_0} \cap \mathfrak{m} = \mathfrak{p}_{\mathrm{u}}^{L} = 0$. Thus the square is commutative. Finally, for general $\xi \in \mathfrak{a}$ we have $\mathfrak{z}(\xi) = \mathfrak{m} \implies [\mathfrak{q}_{\mathrm{u}}, \xi] = \mathfrak{q}_{\mathrm{u}} \implies \xi + \mathfrak{q}_{\mathrm{u}} = Q_{\mathrm{u}}\xi$ by Lemma 3.3. Therefore $\overline{\Phi(\mathcal{U})} = \mathfrak{a} + \mathfrak{q}_{\mathrm{u}}$. □

Corollary 22.1. *There is a commutative square*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & M_X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_G} \\ \mathfrak{a}^* & \xrightarrow{\pi_G} & L_X = \pi_G(\mathfrak{a}^*) \end{array}$$

Lemma 22.2. *There exists a unique morphism $\psi : \mathfrak{a}^* \to \widetilde{L}_X$ making the following square commutative:*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\widetilde{\Phi}} & \widetilde{M}_X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \widetilde{\pi}_G} \\ \mathfrak{a}^* & \xrightarrow{\psi} & \widetilde{L}_X \end{array}$$

Proof. The uniqueness is evident. Take $\psi = \widetilde{\pi}_G\widetilde{\Phi}\sigma$. The maps $\psi\pi$ and $\widetilde{\pi}_G\widetilde{\Phi}$ coincide on $\sigma(\mathfrak{a}^*)$, and by Corollary 22.1 they map $\forall\alpha \in \mathcal{U}$ to one and the same fiber of $\phi/\!\!/G$. Thus for $\forall\lambda \in \mathfrak{a}^*$ the irreducible subvariety $\pi^{-1}(\lambda) \simeq \mathcal{B}$ is mapped by $\widetilde{\pi}_G\widetilde{\Phi}$ to the (finite) fiber of $\phi/\!\!/G$ through $\psi(\lambda)$, whence to $\psi(\lambda)$. $\square$

The normalization of $L_X = \pi_G(\mathfrak{a}^*)$ equals $\mathfrak{a}^*/W(\mathfrak{a}^*)$, where $W(\mathfrak{a}^*) = N_W(\mathfrak{a}^*)/Z_W(\mathfrak{a}^*)$ is the Weyl group of $\mathfrak{a} \subseteq \mathfrak{g}$. By Lemma 22.2, there is a sequence of dominant finite maps of normal varieties $\mathfrak{a}^* \to \widetilde{L}_X \to \mathfrak{a}^*/W(\mathfrak{a}^*)$. It follows from the Galois theory that $\widetilde{L}_X \simeq \mathfrak{a}^*/W_X$ for a certain subgroup $W_X \subseteq W(\mathfrak{a}^*)$ and the left arrow is the quotient map.

Definition 22.1. The group W_X is called the *little Weyl group* of X. It is a subquotient of W.

By construction, $\widetilde{M}_X$, $\widetilde{L}_X$, and W_X are G-birational invariants of X. They are related to other invariants such as the horospherical type S.

Proposition 22.2 ([Kn1, 6.4], [Kn6, 7.3]). *$\widetilde{M}_{G/S} = \widetilde{M}_X \times_{\widetilde{L}_X} \mathfrak{a}^*$ and $\widetilde{M}_X = \widetilde{M}_{G/S}/W_X$.*

As well as M_X, $\widetilde{M}_X$ and W_X are determined by a generic G-orbit [Kn1, 6.5.4]. For functorial properties of the normalized moment map and the little Weyl group, see [Kn1, 6.5]. For geometric properties of the morphisms $T^*X \to \widetilde{M}_X$, $\widetilde{M}_X \to \widetilde{L}_X$, and $T^*X \to \widetilde{L}_X$, see [Kn1, 7.4, 7.3, 6.6], [Kn6, §§5,7,9]. $\widetilde{M}_X$ has rational singularities [Kn6, 4.3].

Remark 22.1. A non-commutative version of this theory was developed in [Kn6]. Here functions on T^*X are replaced by differential operators on X and $\Bbbk[\mathfrak{g}^*]$ by $\mathrm{U}\mathfrak{g}$. The analogue of $\Bbbk[\widetilde{M}_X]$ consists of *completely regular* differential operators generated by velocity fields locally on X and "at infinity". Invariant completely regular operators form a polynomial ring, which coincides with the center of $\mathcal{D}(X)^G$ whenever X is affine. This ring is isomorphic to $\Bbbk[\rho + \mathfrak{a}^*]^{W_X}$ ("Harish-Chandra isomorphism"), where ρ is half the sum of positive roots and W_X is naturally embedded in $N_W(\rho + \mathfrak{a}^*)$ being thus a subgroup, not only a subquotient, of W.

Example 22.1. Take $X = G$ itself, with G acting by left translations. Here $T^*G \simeq G \times \mathfrak{g}^*$, and the moment map Φ is just the coadjoint action map with irreducible fibers isomorphic to G. We have $A = T$, and $\sigma : \mathfrak{t}^* \simeq \mathfrak{t} \hookrightarrow \mathfrak{b} \simeq \mathfrak{u}^\perp = \mathcal{U}(e)$ is the natural inclusion. Therefore $\widetilde{M}_G = M_G = \mathfrak{g}^*$, $\widetilde{L}_G = L_G = \mathfrak{g}^*/\!\!/G \simeq \mathfrak{t}^*/W$, and $W_G = W$.

Example 22.2. Let $X = G/T$, where G is semisimple. Here $T^*X \simeq G *_T (\mathfrak{u}+\mathfrak{u}^-)$, $A = \mathrm{Ad}_G T$, $M_X = \mathfrak{g}^*$. The subspace $\mathbf{e} + \mathfrak{g}_{\mathbf{f}} \subset \mathfrak{u}+\mathfrak{u}^-$, where $\mathbf{e} \in \mathfrak{u}$, $\mathbf{f} \in \mathfrak{u}^-$, $\mathbf{h} \in \mathfrak{t}$ form a principal $\mathfrak{sl}_2$-triple, is a cross-section for the fibers of $\pi_G\Phi : T^*X \to \mathfrak{g}^*/\!\!/G$. Indeed, $\pi_G : \mathbf{e}+\mathfrak{g}_{\mathbf{f}} \xrightarrow{\sim} \mathfrak{g}^*/\!\!/G$ [McG, 4.2]. Hence $\widetilde{\pi}_G\widetilde{\Phi}(\mathbf{e}+\mathfrak{g}_{\mathbf{f}})$ is a cross-section for the fibers of the finite map $\phi/\!\!/G$. It follows that $\widetilde{L}_{G/T} = L_{G/T}$, thence $\widetilde{M}_{G/T} = M_{G/T}$ and $W_{G/T} = W$.

Example 22.3. Consider a horospherical homogeneous space $X = G/S$. We have seen in Theorem 8.1 that the moment map factors as $\Phi = \overline{\Phi}\pi_A$, where $\overline{\Phi} : G*_{P^-}\mathfrak{s}^\perp \twoheadrightarrow M_{G/S}$ is generically finite proper and $\pi_A : G*_S\mathfrak{s}^\perp \to G*_{P^-}\mathfrak{s}^\perp$ is the A-quotient map. It immediately follows that $\widetilde{M}_{G/S} = \mathrm{Spec}\,\Bbbk[G*_{P^-}\mathfrak{s}^\perp]$ and the natural map $G*_{P^-}\mathfrak{s}^\perp \to \widetilde{M}_{G/S}$ is a resolution of singularities. The natural morphisms $G*_{P^-}\mathfrak{s}^\perp \to G*_{P^-}\mathfrak{a}^* = G/P^-\times\mathfrak{a}^* \to \mathfrak{a}^*$ and $\widetilde{\pi}_G : \widetilde{M}_{G/S} \to \widetilde{L}_{G/S}$ are rational G-quotient maps. Indeed, P^- acts on each fiber $\lambda + \mathfrak{p}_{\mathrm{u}}^-$ of $\mathfrak{s}^\perp \to \mathfrak{a}^*$ generically transitively [McG, 5.5], and fibers of $\widetilde{\pi}_G$ have a dense orbit, because fibers of $\pi_G : M_{G/S} \to L_{G/S}$ do. Passing to rational quotients, we see that $\mathfrak{a}^* \to \widetilde{L}_{G/S}$ is birational, whence isomorphic. Thus $W_{G/S} = \{e\}$.

The last example admits a conversion.

Proposition 22.3. *X is horospherical iff $W_X = \{e\}$.*

Proof. A horospherical variety of type S is birationally G-isomorphic to $G/S \times C$ by Proposition 7.2. Therefore it suffices to consider $X = G/S$, but then $W_X = \{e\}$ by Example 22.3.

Conversely, suppose $W_X = \{e\}$ and consider the morphism $\Psi = \widetilde{\pi}_G\widetilde{\Phi} : T^*X \to \widetilde{L}_X = \mathfrak{a}^*$. Then Ψ^* embeds $\mathfrak{a}$ into the space of fiberwise linear G-invariant functions on T^*X, which restrict to linear functions on $\sigma(\mathfrak{a}^*) \simeq \mathfrak{a}^*$.

Geometrically, $\Psi^*\mathfrak{a}$ is an Abelian subalgebra of G-invariant vector fields on X tangent to G-orbits. Furthermore, $\Psi^*\mathfrak{a} = \pi^*\mathfrak{a} = 0$ on $\mathcal{B}$ by Lemma 22.2, hence $\Psi^*\mathfrak{a}$ is tangent to generic P-orbits. It follows that $\Psi^*\mathfrak{a}$ restricts to Px, $x \in C$, as an Abelian subalgebra in the algebra $(\mathfrak{p}/\mathfrak{p}_x)^{P_x} = \mathfrak{a} \oplus \mathfrak{p}_{\mathrm{u}}^{L_0}$ of P-invariant vector fields on Px, and $\Psi^*\xi(x) = \xi x \mod \mathfrak{p}_{\mathrm{u}}^{L_0}x$, $\forall \xi \in \mathfrak{a}$, whence $\Psi^*\mathfrak{a}$ projects onto $\mathfrak{a}$. As $\mathfrak{z}(\mathfrak{a}) \cap \mathfrak{p}_{\mathrm{u}}^{L_0} = \mathfrak{p}_{\mathrm{u}}^{L} = 0$, $\Psi^*\mathfrak{a}$ is conjugated to $\mathfrak{a}$ by a unique $g_x \in P_{\mathrm{u}}^{L_0}$. Moving each $x \in C$ by g_x, we may assume $\Psi^*\xi(x) = \xi x$, $\forall \xi \in \mathfrak{a}$, $x \in C$ (or $\forall x \in Z = AC$).

Therefore velocity fields of $A : Z$ extend to G-invariant vector fields on X. These vector fields can be integrated to an A-action on X by central automorphisms, which restricts to the natural A-action on $\mathring{X} = P *_L Z$ provided by $A : Z$. (The induced action $A : T^*X$ integrates the invariant collective motion, cf. §23.) We conclude by Theorems 21.2–21.3 that X is horospherical. ∎

Here comes the main result linking the little Weyl group with equivariant embeddings.

Theorem 22.1 ([Kn5, 7.4]). *The little Weyl group W_X acts on $\mathfrak{a}^*$ as a crystallographic reflection group preserving the lattice $\Lambda(X)$, and the central valuation cone $\mathcal{Z}(X)$ is its fundamental chamber in $\mathcal{E} = \mathrm{Hom}(\Lambda(X), \mathbb{Q})$.*

The proof relies on the integration of the invariant collective motion in T^*X and the study of the asymptotic behavior of its projection to X, see §23. The description of $\mathcal{Z} = \mathcal{Z}(X)$ as a fundamental chamber of a crystallographic reflection group was first obtained by Brion [Bri8] in the spherical case and generalized to arbitrary complexity in [Kn3, §9]. In particular, $\mathcal{Z}$ is a cosimplicial cone.

From this theorem, Knop derived the geometric look of all valuation cones.

Corollary 22.2 ([Kn3, §9]). *The cones $\mathcal{V}_{\mathrm{v}}$ are cosimplicial.*

In proving Theorem 22.1, we shall use its formal consequence:

Lemma 22.3. *W_X acts trivially on $\mathcal{Z} \cap -\mathcal{Z}$.*

Proof. By Theorem 21.2, there exists a torus E of central automorphisms such that $\mathfrak{e} = (\mathcal{Z} \cap -\mathcal{Z}) \otimes \Bbbk$ w.r.t. the canonical embedding $\mathfrak{e} \hookrightarrow \mathfrak{a} = \mathcal{E} \otimes \Bbbk$. Consider the action $G^+ = G \times E : X$ and indicate all objects related to this action by the superscript "+". In particular, $A^+ \simeq A$ is the quotient of $A \times E$ modulo the antidiagonal copy of E, and $\mathfrak{X}(A^+) \subset \mathfrak{X}(A) \oplus \mathfrak{X}(E)$ is the graph of the restriction homomorphism $\mathfrak{X}(A) \to \mathfrak{X}(E)$.

Obviously, $\Phi = \tau\Phi^+$, where $\tau : (\mathfrak{g}^+)^* \to \mathfrak{g}^*$ is the canonical projection. It follows that $\widetilde{\Phi} = \widetilde{\tau}\widetilde{\Phi}^+$ for a certain morphism $\widetilde{\tau} : \widetilde{M}_X^+ \to \widetilde{M}_X$. The subalgebra $\Bbbk[\widetilde{M}_X]$ is integrally closed in $\Bbbk[\widetilde{M}_X^+]$, whence $\Bbbk[\widetilde{L}_X]$ is integrally closed in $\Bbbk[\widetilde{L}_X^+]$. On the other hand, we have a commutative square

$$\begin{array}{ccc}
(\mathfrak{a}^+)^* & \xrightarrow[\sim]{\tau} & \mathfrak{a}^* \\
\big\downarrow{\scriptstyle \psi^+} & & \big\downarrow{\scriptstyle \psi} \\
\widetilde{L}_X^+ & \xrightarrow{\widetilde{\tau}/\!\!/G} & \widetilde{L}_X
\end{array}$$

where ψ, ψ^+, and hence $\widetilde{\tau}/\!\!/G$ are finite morphisms. Hence $\widetilde{L}_X^+ \simeq \widetilde{L}_X$ and $W_X^+ = W_X$.

It follows that W_X preserves $(\mathfrak{a}^+)^* \subset \mathfrak{a}^* \oplus \mathfrak{e}^*$ and acts trivially on the 2-nd summand $\mathfrak{e}^* \simeq (\mathfrak{a}^+)^*/(\mathfrak{a}^+)^* \cap \mathfrak{a}^* \simeq \mathfrak{a}^*/(\mathfrak{a}^+)^* \cap \mathfrak{a}^*$. Thus W_X acts trivially on $\mathfrak{e}$ embedded in $\mathfrak{a}$. $\square$

23 Invariant collective motion

The skew gradients of functions in $\Bbbk[L_X]$ (or $\Bbbk[\widetilde{L}_X]$) pulled back to T^*X generate an Abelian flow of G-automorphisms preserving G-orbits, which is called the invariant collective motion, see §8. Restricted to a generic orbit $G\alpha \subset T^*X$, the invariant collective motion gives rise to a connected Abelian subgroup $A_\alpha = (G_{\Phi(\alpha)}/G_\alpha)^0 \subseteq \mathrm{Aut}_G\, G\alpha$. It turns out that $A_\alpha \simeq A$. However, in general, this isomorphism cannot be made canonical in order to produce an A-action on (an open subset of) T^*X integrating the invariant collective motion. This obstruction is overcome by unfolding the cotangent variety by means of a Galois covering with Galois group W_X.

Definition 23.1. The fiber product $\widehat{T}X = T^*X \times_{\widetilde{L}_X} \mathfrak{a}^*$ is called the *polarized cotangent bundle* of X. Since generic fibers of $T^*X \to \widetilde{L}_X$ are irreducible, $\widehat{T}X$ is irreducible. Actually $\widehat{T}X$ is an irreducible component of $T^*X \times_{L_X} \mathfrak{a}^*$, W_X is its stabilizer in $W(\mathfrak{a}^*)$ acting on the set of components, and $\widehat{T}X \to T^*X = \widehat{T}X/W_X$ is a rational Galois cover.

Consider the *principal stratum* $\mathfrak{a}^{\mathrm{pr}} \subseteq \mathfrak{a}^*$ obtained by removing all proper intersections with kernels of coroots and with W-translates of $\mathfrak{a}^*$ in $\mathfrak{t}^*$. The group $W(\mathfrak{a}^*)$ acts on $\mathfrak{a}^{\mathrm{pr}}$ freely. Put $L_X^{\mathrm{pr}} = \pi_G(\mathfrak{a}^{\mathrm{pr}}) = \mathfrak{a}^{\mathrm{pr}}/W(\mathfrak{a}^*)$, the quotient map being an étale finite Galois covering. The preimages of L_X^{pr} in various varieties under consideration will be called *principal strata* and denoted by the superscript “pr”.

In particular, $\widehat{T}^{\mathrm{pr}}X \subset \widehat{T}X$ is a smooth open subvariety (provided that X is smooth) and the projection $\widehat{T}^{\mathrm{pr}}X \twoheadrightarrow T^{\mathrm{pr}}X \subseteq T^*X$ is an étale finite quotient map by W_X. The G-invariant symplectic structure on T^*X is pulled back to $\widehat{T}^{\mathrm{pr}}X$ so that $\widehat{T}^{\mathrm{pr}}X \to T^*X \to M_X$ is the moment map.

The invariant collective motion on $\widehat{T}^{\mathrm{pr}}X$ is generated by the skew gradients of Poisson-commuting functions from $\pi^*\mathfrak{a}$, where $\pi : \widehat{T}^{\mathrm{pr}}X \to \mathfrak{a}^*$ is the other projection. These skew gradients constitute a commutative r-dimensional subalgebra of Hamiltonian vector fields ($r = r(X) = \dim\mathfrak{a}$). Our aim is to show that these vector fields are the velocity fields of a symplectic A-action so that π is the respective moment map [Arn, App.5].

Remark 23.1. In particular, it will follow that the W_X-action on $\mathfrak{a}$ lifts to A, so that $\widehat{T}^{\mathrm{pr}}X$ comes equipped with the Poisson $G \times (W_X \leftthreetimes A)$-action.

Following [Kn5], we shall restrict our considerations to the symplectically stable case (Definition 8.2) for technical reasons.

Proposition 23.1. *$G : T^*X$ is symplectically stable iff $T^*X = \overline{G\mathcal{U}}$.*

Proof. In the notation of Lemma 22.1, $\Phi(G\mathcal{U}^{\mathrm{pr}}) = G(\mathfrak{a}^{\mathrm{pr}} + \mathfrak{q}_{\mathrm{u}}) = G\mathfrak{a}^{\mathrm{pr}}$. Hence density of $G\mathcal{U}$ implies symplectic stability. Conversely, in the symplectically stable case $P_{\mathrm{u}}^{-}\alpha$ is transversal to $\mathcal{U}$ for general $\alpha \in \mathcal{U}$. Indeed, we may assume $\Phi(\alpha) \in \mathfrak{a}^{\mathrm{pr}}$, but then $[\mathfrak{p}_{\mathrm{u}}^{-}, \Phi(\alpha)] = \mathfrak{p}_{\mathrm{u}}^{-}$ is transversal to $\overline{\Phi(\mathcal{U})} = \mathfrak{a} + \mathfrak{p}_{\mathrm{u}}$. Therefore $\dim \overline{P_{\mathrm{u}}^{-}\mathcal{U}} = \dim P_{\mathrm{u}} + \dim \mathcal{U} = \dim T^*X$. $\square$

Suppose that the action $G : T^*X$ is symplectically stable. We have observed in §8 that $M_X^{\mathrm{pr}} \simeq G *_{N(\mathfrak{a})} \mathfrak{a}^{\mathrm{pr}}$. Then $\widetilde{M_X^{\mathrm{pr}}} \simeq M_X^{\mathrm{pr}} \times_{L_X} \widetilde{L}_X \simeq G *_{N_X} \mathfrak{a}^{\mathrm{pr}}$, where $N_X \subseteq N(\mathfrak{a})$ is the extension of W_X by $Z(\mathfrak{a}) = L$. Hence $T^{\mathrm{pr}}X \simeq G *_{N_X} \Sigma$ has a structure of a homogeneous bundle over G/N_X. Fibers of this bundle are called *cross-sections*. They are smooth and irreducible, because generic fibers of $\widetilde{\Phi}$ are irreducible. We may choose a canonical N_X-stable cross-section Σ, namely the unique cross-section in $\Phi^{-1}(\mathfrak{a}^{\mathrm{pr}})$ intersecting $\mathcal{U}$.

Remark 23.2. In fact, $\mathcal{U} \cap \Sigma$ is dense in Σ. Indeed, $\Sigma \cap T^*\mathring{X} \subseteq \mathcal{U}$.

Lemma 23.1. *The kernel of $N_X : \Sigma$ is L_0.*

Proof. By Lemma 22.1, $\Phi(\mathcal{U}^{\mathrm{pr}}) = \mathfrak{a}^{\mathrm{pr}} + \mathfrak{p}_{\mathrm{u}} \simeq P *_L \mathfrak{a}^{\mathrm{pr}} \simeq P_{\mathrm{u}} \times \mathfrak{a}^{\mathrm{pr}}$. Hence $\mathcal{U}^{\mathrm{pr}} = P *_L (\mathcal{U} \cap \Sigma) \simeq P_{\mathrm{u}} \times (\mathcal{U} \cap \Sigma)$. On the other hand, $\mathcal{U}|_{\mathring{X}} = P *_L \mathcal{U}|_Z \simeq P_{\mathrm{u}} \times \mathcal{U}|_Z$, and all the stabilizers of $L : \mathcal{U}|_Z \simeq T^*Z$ are equal to L_0. It follows that generic stabilizers of $L : \Sigma$ are P-conjugate to L_0, hence coincide with L_0. $\square$

Corollary 23.1. *W_X acts on $A = L/L_0$, i.e., preserves the character lattice $\mathfrak{X}(A) = \Lambda(X) \subset \mathfrak{a}^*$.*

Remark 23.3. The lemma implies Theorem 8.4 in the symplectically stable case. This observation was made in [Kn5, §4].

Lemma 23.2. *The A-action integrates the invariant collective motion on Σ.*

Proof. The skew gradient of $\Phi^* f$ ($f \in \Bbbk(M_X)$) at $\alpha \in T^*X$ equals $(d_{\Phi(\alpha)} f) \cdot \alpha$, where $d_{\Phi(\alpha)} f$ is considered as an element of $\mathfrak{g}^{**} = \mathfrak{g}$ up to a shift from $(T_{\Phi(\alpha)} M_X)^{\perp}$. Indeed, the skew gradient of a function is determined by its linear portion at a point, and for linear functions $f \in \mathfrak{g}$ the assertion holds by the definition of Φ.

If $\alpha \in \Sigma$, then $T_{\Phi(\alpha)} M_X = \mathfrak{a} + [\mathfrak{g}, \Phi(\alpha)] = \mathfrak{a} \oplus \mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{p}_{\mathrm{u}}^{-}$, $(T_{\Phi(\alpha)} M_X)^{\perp} = \mathfrak{l}_0$. The differentials $d_{\Phi(\alpha)} f$ of $f \in \Bbbk[L_X^{\mathrm{pr}}]$ generate the conormal space of $G \cdot \Phi(\alpha)$ in M_X at $\Phi(\alpha)$, i.e., $[\mathfrak{g}, \Phi(\alpha)]^{\perp} / (T_{\Phi(\alpha)} M_X)^{\perp} = \mathfrak{l}/\mathfrak{l}_0 = \mathfrak{a}$. Thus the invariant collective motion at α is $\mathfrak{a}_\alpha = \mathfrak{l}\alpha = \mathfrak{a}\alpha$. $\square$

Translation by G permutes cross-sections transitively and extends the A-action to each cross-section. These actions integrate the invariant collective motion, but in general, they do not globalize to a regular A-action on the whole cotangent bundle, due to non-trivial monodromy.

However, unfold $T^{\mathrm{pr}}X$ to $\widehat{T}^{\mathrm{pr}}X = T^{\mathrm{pr}}X \times_{\widetilde{L}_X} \mathfrak{a}^{\mathrm{pr}} \simeq G *_L \widehat{\Sigma}$, where $\widehat{\Sigma} = \{\hat{\alpha} = (\alpha, \Phi(\alpha)) \mid \alpha \in \Sigma\}$. We retain the name "cross-sections" for the fibers of this homogeneous bundle, which are isomorphic to the cross-sections in $T^{\mathrm{pr}}X$. Now there is a natural A-action on $\widehat{T}^{\mathrm{pr}}X$ provided by $A : \widehat{\Sigma} \simeq \Sigma$, which integrates the invariant collective motion on $\widehat{T}^{\mathrm{pr}}X$. The W_X-action on $\widehat{T}^{\mathrm{pr}}X$ is induced from the N_X-action on $G \times \widehat{\Sigma}$ given by $n(g, \hat{\alpha}) = (gn^{-1}, n\hat{\alpha})$, $\forall n \in N_X$, $g \in G$, $\hat{\alpha} \in \widehat{\Sigma}$. We sum up in the following

Theorem 23.1 ([Kn5, 4.1–4.2]). *There is a Poisson $G \times (W_X \ltimes A)$-action on $\widehat{T}^{\mathrm{pr}}X$ with the moment map $\Phi \times \pi : \widehat{T}^{\mathrm{pr}}X \to \mathfrak{g}^* \oplus \mathfrak{a}^*$.*

Proof. It remains only to explain why π is the moment map for the A-action. Take any $\hat{\alpha} \in \widehat{\Sigma}$ over $\alpha \in \Sigma$. By (the proof of) Lemma 23.2, $\forall \xi \in \mathfrak{a}\ \exists f \in \Bbbk[L_X^{\mathrm{pr}}] :\ d_{\Phi(\alpha)} f = \xi \mod \mathfrak{l}_0$. The skew gradient of $\pi^*\xi$ at $\hat{\alpha}$ is pulled back from that of $\Phi^* f$ at α, i.e., from $\xi\alpha$, hence it equals $\xi\hat{\alpha}$. We conclude by G-equivariance. ☐

In particular, the orbit of the invariant collective motion through $\hat{\alpha} \in \widehat{T}^{\mathrm{pr}}X$ over $\alpha \in T^{\mathrm{pr}}X$ is $A\hat{\alpha} = G_{\Phi(\alpha)}\hat{\alpha} \simeq G_{\Phi(\alpha)}/G_\alpha$. For the purposes of the embedding theory it is important to study the projections of these orbits to X and their boundaries.

Definition 23.2. A *flat* in X is $F_\alpha = \pi_X(A\hat{\alpha}) = G_{\Phi(\alpha)}x$, where $\alpha \in T_x^{\mathrm{pr}}X$, $\hat{\alpha} \in \widehat{T}^{\mathrm{pr}}X$ lies over α, and $\pi_X : T^*X \to X$ is the canonical projection. The composed map $A \to A\hat{\alpha} \to F_\alpha$ is called the *polarization* of the flat.

For general α the polarization map is isomorphic: indeed, w.l.o.g. assume $\alpha \in \Sigma \cap T_x^*\mathring{X} \implies G_{\Phi(\alpha)} = L$, $G_{\Phi(\alpha)} \cap G_x = G_\alpha = L_0 \implies F_\alpha \simeq A$. Generic flats are nothing else, but G-translates of L- (or A-) orbits in Z, under appropriate choice of Z. Namely, by Lemma 22.1, there is a commutative diagram

$$\begin{array}{ccc} \mathring{X} \times \mathfrak{a}^* \simeq \mathcal{A} & \xrightarrow{\Phi} & \mathfrak{a} \oplus \mathfrak{p}_{\mathrm{u}} \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow \\ \mathfrak{a}^* & \xrightarrow{\sim} & \mathfrak{a} \end{array} \tag{23.1}$$

For $\forall \lambda \in \mathfrak{a}^{\mathrm{pr}}$ we have $\lambda + \mathfrak{p}_{\mathrm{u}} = P \cdot \lambda \simeq P/L$, hence $\mathring{X} \simeq P *_L Z^\lambda$, where $Z^\lambda = \pi_X(\Phi^{-1}(\lambda) \cap \mathcal{A})$. Clearly, all L-orbits in $Z = Z^\lambda$ are flats. On the other hand, for $\forall \alpha \in \Sigma \cap T_x^*\mathring{X}$ it is easy to construct a subvariety $C \subset \mathring{X}^{L_0}$

through x intersecting all P-orbits transversally such that $\alpha = 0$ on T_xC, whence $x \in Z^{\Phi(\alpha)}$.

The rigidity of torus actions implies that the closures of generic flats are isomorphic.

Proposition 23.2 ([Kn5, §6]). *The closures $\overline{F}_\alpha$ for general $\alpha \in T^{\mathrm{pr}}X$ are A-isomorphic toric varieties, and the W_X-action on $A \simeq F_\alpha$ extends to $\overline{F}_\alpha$.*

Proof. We explain the affine case, the general case being reduced to this one by standard techniques of invariant quasiprojective open coverings and affine cones. Generic flats are G-translates of generic L-orbits in $\overline{\pi_X(\Sigma)}$. We may assume that X is embedded into a G-module. Since $\overline{\pi_X(\Sigma)}$ is N_X-stable, the set of eigenweights of $A = L/L_0$ in $\overline{\pi_X(\Sigma)}$ is W_X-stable. For general $\alpha \in \Sigma$, $\Bbbk[\overline{F}_\alpha]$ is just the semigroup algebra generated by these eigenweights. □

The following result is crucial for interdependence between flats and central valuations. It partially describes the boundary of a generic flat.

Proposition 23.3 ([Kn5, 7.3]). *Let $D \subset X$ be a G-stable divisor with $v = v_D \in \mathcal{Z}$. The closure $\overline{F}_\alpha$ of a generic flat contains A-stable prime divisors $D_{wv} \subseteq D$, $w \in W_X$, that correspond to wv regarded as A-valuations of $\Bbbk(A)$. Furthermore, $\overline{F}_\alpha$ is smooth along D_{wv}.*

Proof. W.l.o.g. we may assume $\alpha \in \Sigma$. The W_X-action on $\overline{F}_\alpha$ is given by $w : \overline{F}_\alpha \to \overline{F}_{n\alpha} \to \overline{F}_\alpha$, where the left arrow is the translation by $n \in N_X$ representing $w \in W_X$ and the right arrow is the unique A-isomorphism mapping $n\alpha$ back to α. Since D is N_X-stable, it suffices to prove the assertion for $w = e$.

Shrinking $\mathring{X}$ if necessary, we find a B-chart X_0 intersecting D such that $\mathring{X} = X_0 \setminus D$.

Lemma 23.3. *The morphism $\mathring{X} \times \mathfrak{a}^* \to \mathfrak{a} + \mathfrak{p}_{\mathrm{u}}$ in* (23.1) *extends to $X_0 \times \mathfrak{a}^*$.*

Proof. Trivializing sections of $\mathcal{A} \simeq \mathring{X} \times \mathfrak{a}^*$ corresponding to $\lambda \in \Lambda$ are $d\mathbf{f}_\lambda/\mathbf{f}_\lambda$, where $\mathbf{f}_\lambda$ are B-eigenfunctions on $\mathring{X}$ that are constant on $P_{\mathrm{u}}C$. These sections extend to sections of $T^*X(\log D)$ over X_0, which trivialize the subbundle $\mathcal{A}(\log D) = \overline{\mathcal{A}} \subseteq T^*X_0(\log D)$. The moment map of $T^*X(\log D)$ restricted to $\mathcal{A}(\log D)$ provides the desired extension. □

Consequently $X_0 \simeq P *_L Z_0$, $Z_0 = \overline{Z^{\Phi(\alpha)}}$, and $F_\alpha = Ax$ is a generic A-orbit in Z_0. The proposition stems from

Lemma 23.4. *After possible shrinking of X_0, $Z_0 \simeq F \times C$, where $F = \overline{Ax}$ is the closure of a generic A-orbit in Z_0.*

Proof. Since v is central and by Lemma 5.1, the restriction of functions identifies $\Bbbk(D)^B \simeq \Bbbk(Z_0 \cap D)^A$ with $\Bbbk(X)^B \simeq \Bbbk(Z_0)^A \simeq \Bbbk(C)$. Hence removing zeroes/poles of a B-invariant function preserves non-empty intersection with D. In particular, we may assume $\Bbbk(Z_0 \cap D)^A = \operatorname{Quot} \Bbbk[Z_0 \cap D]^A$ and $\Bbbk[Z_0 \cap D]^A \simeq \Bbbk[Z_0]^A \simeq \Bbbk[C]$ by shrinking X_0. We have $\Bbbk[Z] = \Bbbk[\, \mathbf{f}_\lambda \mid \lambda \in \Lambda\,] \otimes \Bbbk[C]$ and $\Bbbk[Z_0] \subseteq \Bbbk[\, \mathbf{f}_\lambda \mid \lambda \in \Lambda_0\,] \otimes \Bbbk[C]$, where Λ_0 is the weight semigroup of Z_0. There exist $h_\lambda \in \Bbbk[C]$ such that $h_\lambda \mathbf{f}_\lambda \in \Bbbk[Z_0]$. Shrinking X_0, we may assume $\mathbf{f}_\lambda \in \Bbbk[Z_0]$, $\forall \lambda \in \Lambda_0$ (because Λ_0 is finitely generated). Hence $Z_0 \simeq F \times C$, where $F = \operatorname{Spec} \Bbbk[\Lambda_0]$. □

Now we explain how to deal with non-symplectically stable case.

We may assume X to be quasiprojective. By $\widehat{X}$ denote the cone over X without the origin. In the notation of Remark 20.1, the $\widehat{G}$-action on $T^*\widehat{X}$ is symplectically stable by Proposition 8.2.

The quotient space $T^*\widehat{X}/\Bbbk^\times$ is a vector bundle over X containing T^*X as a subbundle, the quotient bundle being the trivial line bundle. The moment map for $\widehat{G} : T^*\widehat{X}$ factors through $T^*\widehat{X}/\Bbbk^\times$, so that there is a commutative diagram

$$\begin{array}{ccccc} & & T^*\widehat{X} & & \\ & & \downarrow & & \\ T^*X & \subset & T^*\widehat{X}/\Bbbk^\times & \xrightarrow{\Pi} & \Bbbk \\ \downarrow \Phi & & \downarrow \widehat{\Phi} & & \| \\ M_X & \subset & M_{\widehat{X}} & \longrightarrow & \Bbbk \end{array}$$

Here Π is induced by the evaluation at the Euler vector field in $\widehat{X}$, i.e., by the moment map for the $\Bbbk^\times$-action, and the lower right arrow is the projection of $M_{\widehat{X}} \subseteq \hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus \Bbbk$ to $\Bbbk$. T^*X and M_X are the zero-fibers of the respective maps to $\Bbbk$. Also, we have $\mathfrak{a}^* = \hat{\mathfrak{a}}^* \cap \mathfrak{g}^*$. The morphism $\widehat{\Phi}$ factors through $\widetilde{M}_{\widehat{X}}$, hence Φ factors through the zero-fiber M'_X of $\widetilde{M}_{\widehat{X}} \to \Bbbk$. As $M'_X \to M_X$ is finite, there is a commutative diagram

$$\begin{array}{ccccc} T^*X & & \subset & & T^*\widehat{X}/\Bbbk^\times \\ \downarrow \widetilde{\Phi} & & & & \downarrow \\ \widetilde{M}_X & \to & M'_X & \subset & \widetilde{M}_{\widehat{X}} \end{array}$$

Passing to quotients, we obtain

$$\begin{array}{ccc} \mathfrak{a}^* & \subset & \hat{\mathfrak{a}}^* \\ \downarrow & & \downarrow \\ \widetilde{L}_X & \longrightarrow & \widetilde{L}_{\widehat{X}} \end{array}$$

whence $W_X \subseteq W_{\widehat{X}}$. Actually these groups coincide by Theorem 22.1. By Corollary 23.1, $W_{\widehat{X}}$ preserves $\Lambda(\widehat{X})$, whence W_X preserves $\Lambda(X) = \Lambda(\widehat{X}) \cap \mathfrak{a}^*$.

Instead of flats, one considers *twisted flats* defined as projectivizations of usual flats in $\widehat{X}$. The above results on flats and their closures in $\widehat{X}$ descend to twisted flats in X. If T^*X is symplectically stable, then $T^{\mathrm{pr}}X \subset T^{\mathrm{pr}}\widehat{X}/\Bbbk^\times$, and flats are a particular case of twisted flats.

Example 23.1. Let $G = \mathrm{SL}_2$ and $\widehat{X} \subset V(3)$ be the variety of (nonzero) degenerate binary cubic forms (in the variables x, y). Essentially $\widehat{G} = \mathrm{GL}_2$. The form $v = xy^2$ has the open orbit $\widehat{\mathcal{O}} \subset \widehat{X}$ and the stabilizer $\widehat{H} = \left\{\left(\begin{smallmatrix} t^2 & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \mid t \in \Bbbk^\times\right\}$ in $\widehat{G}$. Passing to projectivizations, we obtain a hypersurface $X \subset \mathbb{P}(V(3))$ with the open orbit $\mathcal{O} = G\langle v\rangle$, and $G_{\langle v\rangle} =: H = \left\{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \mid t \in \Bbbk^\times\right\}$. The flats through $\langle v\rangle$ are the orbits of the isotropy groups of $\mathfrak{h}^\perp = \{\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)\}$ in G, i.e., of $L = \{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mid a^2 - b^2 = 1\}$ and its H-conjugates. However the twisted flats are the orbits of the stabilizers in G of non-degenerate matrices from $\hat{\mathfrak{h}}^\perp = \{\left(\begin{smallmatrix} c & * \\ * & 2c \end{smallmatrix}\right) \mid c \in \Bbbk\}$, i.e., the orbits of arbitrary 1-tori in G.

The boundary of the open orbit is a single orbit $X \setminus \mathcal{O} = G\langle v_0\rangle$, $v_0 = y^3$, with the stabilizer $G_{v_0} = B$. Put $Y = \mathbb{P}(\langle v_0, v\rangle)$, a B-stable subspace in X. The natural bijective morphism $\widetilde{X} = G *_B Y \to X$ is a desingularization. The closures of generic twisted flats are isomorphic to $\mathbb{P}^1$ and intersect the boundary divisor $D = \widetilde{X} \setminus \mathcal{O}$ transversally in two points permuted by the (little) Weyl group. Indeed, it suffices to verify it for generic T-orbits in $\mathcal{O}$, which is easy.

Remark 23.4. The fibers $T^cX = \Pi^{-1}(c)$, $c \in \Bbbk \setminus \{0\}$, are called *twisted cotangent bundles* [BoBr, §2]. They carry a structure of affine bundles over X associated with the vector bundle T^*X. Thus each T^cX has a natural symplectic structure. (This is a particular case of symplectic reduction [Arn, App.5] for the $\Bbbk^\times$-action on $T^*\widehat{X}$.) The action $G : T^cX$ is Poisson and symplectically stable: the moment map Φ^c is the composition of $\widehat{\Phi}$ and the projection $\hat{\mathfrak{g}}^* \to \mathfrak{g}^*$, so that $\overline{\operatorname{Im}\Phi^c}$ is identified with the fiber of $M_{\widehat{X}} \to \Bbbk$ over c, and the symplectic stability stems from that of $T^*\widehat{X}$.

The whole theory can be developed for arbitrary G-varieties replacing the usual cotangent bundle by its twisted analogue [Kn5, §9]. If X is quasiaffine, i.e., embedded in a G-module V, then $X \subset \mathbb{P}(V \oplus \Bbbk)$, $\widehat{X} \simeq X \times \Bbbk^\times$, and $T^cX \simeq T^*X$. Therefore in the quasiaffine case the classical theory is included in the twisted one.

Proof of Theorem 22.1. We already know from the above that W_X preserves $\Lambda(X)$ and acts on $\mathcal{E}$. Let $W_X^\# \subset \mathrm{GL}(\mathcal{E})$ be the subgroup generated by reflections at the walls of $\mathcal{Z}$. The first step is to show that $W_X^\# \subseteq W_X$.

Choose a wall of $\mathcal{Z}$ and a vector v in its interior. We may assume $v = v_D$ for a certain G-stable prime divisor $D \subset X$. Consider the normal bundle

X' of X at D. By Proposition 21.3(1), the central valuation cone of X' is a half-space $\mathcal{Z}' = \mathcal{Z} + \mathbb{Q}v$. By Theorem 21.3, X' is not horospherical, whence $W_{X'} \neq \{e\}$ by Proposition 22.3. By Lemma 22.3, $W_{X'}$ acts trivially on $\mathcal{Z}' \cap -\mathcal{Z}'$, whence $W_{X'}$ is generated by the reflection at the chosen face of $\mathcal{Z}$.

On the other hand, X' is deformed to X, i.e., it is the zero-fiber of the $(G \times \Bbbk^\times)$-equivariant flat family $E \to \mathbb{A}^1$ with the other fibers isomorphic to X [Ful1, 5.1]. Since $E \setminus X' \simeq X \times \Bbbk^\times$ and the moment map of $T^*(E \setminus X')$ factors through the projection onto T^*X, we have $\widetilde{M}_X = \widetilde{M}_E$. There is a commutative diagram

$$\begin{array}{ccccc} T^*X' & \leftarrow & T^*E|_{X'} & \subset & T^*E \\ \downarrow & & & & \downarrow \\ M_{X'} & & = & & M_E = M_X \end{array}$$

As $\widetilde{M}_E \to M_E$ is finite and $\Bbbk[\widetilde{M}_{X'}]$ is integrally closed in $\Bbbk[T^*E|_{X'}]$, there is a finite morphism $\widetilde{M}_{X'} \to \widetilde{M}_E$, whence $\mathfrak{a}^* \to \widetilde{L}_{X'} \to \widetilde{L}_E = \widetilde{L}_X$. Thus $W_{X'} \subseteq W_E = W_X$.

At this point we may reduce the problem to the symplectically stable case, because $\widehat{\mathcal{Z}} = \mathcal{Z}(\widehat{X})$ is the preimage of $\mathcal{Z}$ and $W^{\#}_{\widehat{X}} = W^{\#}_X \subseteq W_X \subseteq W_{\widehat{X}}$.

It follows that $W^{\#}_X$ is a finite crystallographic reflection group and $\mathcal{Z}$ is a union of its fundamental chambers. To conclude the proof, it remains to show that different vectors $v_1, v_2 \in \mathcal{Z}$ cannot be W_X-equivalent.

Assume the converse, i.e., $v_2 = wv_1$, $w \in W_X$. W.l.o.g. X contains two G-stable prime divisors D_1, D_2 corresponding to v_1, v_2. (Replace X by the normalized closure of the graph of the birational map $X_1 \dashrightarrow X_2$, where X_i is a complete G-model of K having a divisor with valuation v_i.) Removing $D_1 \cap D_2$, we may assume that D_1, D_2 are disjoint. By Proposition 23.3, the closure of a generic (twisted) flat contains two A-stable prime divisors D_{v_1}, D_{v_2} lying both in D_1 and in D_2, a contradiction. $\square$

Proposition 23.3, together with Theorem 22.1, leads to a description of the whole boundary of a generic flat (to a certain extent).

Definition 23.3. A *source* $Y \subset X$ is the center of a central valuation.

Proposition 23.4 ([Kn5, 7.6]). *Let $F_\alpha \subseteq X$ be a generic (twisted) flat. A vector $v \in \mathcal{E}$, regarded as an A-valuation of $\Bbbk(F_\alpha)$, has a center F_v in $\overline{F}_\alpha$ iff the unique $v' \in W_X v \cap \mathcal{Z}$ has a center $Y \subseteq X$. Furthermore, $F_v \subseteq Y$.*

Proof. Since W_X acts on $\overline{F}_\alpha$, we may assume $v = v'$. Take a G-equivariant completion $\overline{X} \supseteq X$ [Sum] and construct a proper birational G-morphism

$\phi : X' \to \overline{X}$ such that X' contains a divisor D with $v_D = v$ (Proposition 19.3). By Proposition 23.3, the center of v on the closure $\overline{F}'_\alpha$ of F_α in X' is a divisor $D_v \subseteq D$. Hence $\phi(D_v) \subseteq \phi(D)$ is the center of v on the closure $\overline{\overline{F}}_\alpha$ of F_α in $\overline{X}$. It intersects $\overline{F}_\alpha$ (exactly in F_v) iff $\phi(D)$ intersects X (in Y). □

Remark 23.5. A-valuations of $\Bbbk(F_\alpha)$ are determined by one-parameter subgroups of A, see Example 24.2. The open orbit in F_v can be reached from F_α by taking the limits of trajectories of the respective one-parameter subgroup. Thus Proposition 23.4 gives a full picture of the asymptotic behavior of the invariant collective motion.

Corollary 23.2. *There are finitely many sources in a G-variety, and the closure of a generic (twisted) flat intersects all of them.*

Corollary 23.3. *Suppose a quasiaffine G-variety X contains a proper source; then $\Bbbk[X]$ has a G-invariant non-negative grading induced by a certain central one-parameter subgroup.*

Proof. We may assume X to be affine. Corollary 23.2 and the assumptions imply that $\overline{F}_\alpha \neq F_\alpha$ is an affine toric variety acted on by W_X. Its normalization is determined by a strictly convex W_X-stable cone $\mathcal{C} \subset \mathcal{E}$ (Example 15.1). Clearly, $\operatorname{int} \mathcal{C}$ contains a W_X-invariant vector $v \neq 0$. Hence $v \in \mathcal{Z} \cap -\mathcal{Z}$ defines a central one-parameter subgroup γ acting on X by Theorem 21.2, whence a G-invariant grading of $\Bbbk[X]$. Generic γ-orbits in X are contained in generic flats and non-closed therein, because γ contracts F_α to the unique closed orbit in $\overline{F}_\alpha$. Hence the grading is non-negative. □

See [Kn5, §§8–9] and §29 for a deeper analysis of sources, flats, and their closures.

To any G-variety X one can relate a root system $\Delta_X \subset \Lambda(X)$, which is a birational invariant of the G-action. Namely, let $\Pi_X^{\min} \subset \Lambda(X)$ be the set of indivisible vectors generating the rays of the simplicial cone $-\mathcal{Z}(X)^\vee$. It is easy to deduce from Theorem 22.1 that $\Delta_X^{\min} = W_X \Pi_X^{\min}$ is a root system with base $\Pi_X^{\min}$ and the Weyl group W_X, called the *minimal root system* of X. It is a generalization of the (reduced) root system of a symmetric variety (see §26 and Example 30.1). The minimal root system was defined by Brion [Bri8] for a spherical variety and by Knop [Kn8] in the general case.

Remark 23.6. There are several natural root systems related to $G : X$ which generate one and the same Weyl group W_X [Kn8, 6.2, 6.4, 7.5]; $\Delta_X^{\min}$ is the "minimal" one.

Example 23.2. If $X = G$ comes equipped with the G-action by left translations and G' is adjoint, then $\Delta_X^{\min} = \Delta_G$ by Example 21.1. If G' is not adjoint, then $\Delta_X^{\min}$ may differ from Δ_G: this happens iff some roots in Δ_G are divisible in $\mathfrak{X}(T)$. For simple G, $\Delta_G^{\min} = \Delta_G$ unless $G = \mathrm{Sp}_{2n}(\Bbbk)$; in the latter case $\Delta_G, \Delta_G^{\min}$ are of types $\mathbf{C}_n, \mathbf{B}_n$, respectively.

An important result of Knop establishes a relation between the minimal root system and central automorphisms.

Theorem 23.2 ([Kn8, 6.4]). *A quasitorus* $S_X = \bigcap_{\alpha\in\Delta_X^{\min}} \operatorname{Ker}\alpha \subseteq A$ *is canonically embedded in* $\operatorname{Cent} X$.

Note that S_X^0 is the largest connected algebraic subgroup of $\operatorname{Cent} X$ by Theorem 21.2(3), but Theorem 23.2 is much more subtle.

Synopsis of a proof. Standard reductions allow us to assume that X is quasi-affine. It is clear that $S_X \subseteq A^{W_X}$. The action $A : \widehat{T}^{\mathrm{pr}}X$ descends to $A^{W_X} : T^{\mathrm{pr}}X$.

The most delicate part of the proof is to show that the action of S_X extends to T^*X in codimension one. Knop shows that the A-actions on the orbits of the invariant collective motion patch together in an action on T^*X of a smooth group scheme $\mathcal{S}$ over $\widetilde{L}_X$ with connected fibers, and $A^{W_X} \subset \mathcal{S}(\Bbbk(\widetilde{L}_X))$. Furthermore, $s \in A^{W_X}$ induces a rational section of $\mathcal{S} \to \widetilde{L}_X$ which is defined in codimension one whenever $\alpha(s) = 1$, $\forall\alpha \in \Delta_X^{\min}$, whence the claim.

Now S_X acts on an open subset $R \subseteq T^*X$ whose complement has codimension ≥ 2, and this action commutes with G and with homotheties on the fibers. Hence S_X acts by G-automorphisms on $\mathbb{P}(R) \subseteq \mathbb{P}(T^*X)$. Since X is quasiaffine and generic fibers of $\mathbb{P}(R) \to X$ have no non-constant regular functions, we deduce that S_X permutes the fibers. This yields a birational action $S_X : X$ commuting with G and preserving generic flats. The description of generic flats shows that S_X preserves P-orbits in $\mathring{X}$, whence $S_X \hookrightarrow \operatorname{Cent} X$. ∎

24 Formal curves

In the previous sections of this chapter we examined G-valuations on arbitrary G-varieties. However our main interest is in homogeneous varieties. In this section we take a closer look at $\mathcal{V} = \mathcal{V}(\mathcal{O})$, where $\mathcal{O}$ is a homogeneous space.

Namely we describe the subset $\mathcal{V}^1 \subseteq \mathcal{V}$ consisting of G-valuations v such that $\Bbbk(v)^G = \Bbbk$. In geometric terms, if v is proportional to v_D for a G-stable divisor D on a G-embedding $X \hookleftarrow \mathcal{O}$, then $v \in \mathcal{V}^1$ iff D contains a dense G-orbit.

The subset $\mathcal{V}^1$ is big enough. For instance, if $c(\mathcal{O}) = 0$, then $\mathcal{V}^1 = \mathcal{V}$, and if $c(\mathcal{O}) = 1$, then $\mathcal{V}^1 \supseteq \mathcal{V} \setminus \mathcal{Z}$, because in this case $c(\Bbbk(v)) = 0$ by Proposition 21.1. In general, any G-valuation can be approximated by $v \in \mathcal{V}^1$ in a sense [LV, 4.11].

In [LV] Luna and Vust suggested to compute $v(f)$, $v \in \mathcal{V}^1$, $f \in K = \Bbbk(\mathcal{O})$, by restricting f to a (formal) curve in $\mathcal{O}$ approaching to D, in the above notation. More precisely, take a smooth curve $\Theta \subseteq X$ meeting D transversally in x_0, $\overline{Gx_0} = D$. It is clear that $v_D(f)$ equals the order of $f|_{g\Theta}$ at gx_0 for general $g \in G$. More generally, take a germ of a curve $\chi : \Theta \dashrightarrow \mathcal{O}$ that converges to x_0 in X, i.e., χ extends regularly to the base point $\theta_0 \in \Theta$ and $\chi(\theta_0) = x_0$, see Appendix A4. Then

$$v_D(f) \cdot \langle D, \Theta\rangle_{x_0} = v_{\chi,\theta_0}(f) := v_{\theta_0}(\chi^*(gf)) \qquad \text{for general } g \in G, \tag{24.1}$$

where $\langle D, \Theta\rangle_{x_0}$ is the local intersection number [Ful1, Ch.7].

Theorem 24.1. *For any germ of a curve* $(\chi : \Theta \dashrightarrow \mathcal{O}, \theta_0 \in \Theta)$, *Formula* (24.1) *defines a G-valuation $v_{\chi,\theta_0} \in \mathcal{V}^1$, and every $v \in \mathcal{V}^1$ is proportional to some v_{χ,θ_0}. Furthermore, if $X \supseteq \mathcal{O}$ is a G-model of K and $Y \subseteq X$ the center of v, then the germ converges in X to $x_0 \in Y$ such that $\overline{Gx_0} = Y$.*

Proof. The G-action yields a rational dominant map $\alpha : G \times \Theta \dashrightarrow \mathcal{O}$, $(g, \theta) \mapsto g\chi(\theta)$. By construction, v_{χ,θ_0} is the restriction of $v_{G\times\{\theta_0\}} \in \mathcal{V}^1(G\times\Theta)$ to K, whence $v = v_{\chi,\theta_0} \in \mathcal{V}^1$. If v has the center Y on X, then $\alpha : G \times \Theta \dashrightarrow X$ is regular along $G \times \{\theta_0\}$ and $\overline{\alpha(G \times \{\theta_0\})} = Y$, whence χ converges to $x_0 = \alpha(e, \theta_0)$ in the dense G-orbit of Y. □

For computations, it is more practical to adopt a more algebraic point of view, namely to replace germs of curves by germs of formal curves, i.e., by $\Bbbk(\!(t)\!)$-points of $\mathcal{O}$, see Appendix A4.

Any germ of a curve $(\chi : \Theta \dashrightarrow \mathcal{O}, \theta_0 \in \Theta)$ defines a formal germ $x(t) \in \mathcal{O}(\Bbbk(\!(t)\!))$ if we replace Θ by the formal neighborhood of θ_0. We have

$$v_{\chi,\theta_0}(f) = v_{x(t)}(f) := \operatorname{ord}_t f(gx(t)) \qquad \text{for generic } g, \tag{24.2}$$

where "generic" means a sufficiently general point of G (depending on $f \in K$) or the generic $\Bbbk(G)$-point of G (Example A4.2).

The counterpart of Theorem 24.1 is

Theorem 24.2. *For any $x(t) \in \mathcal{O}(\Bbbk(\!(t)\!))$, Formula* (24.2) *defines a G-valuation $v_{x(t)} \in \mathcal{V}^1$, and every $v \in \mathcal{V}^1$ is proportional to some $v_{x(t)}$. Furthermore, if $X \supseteq \mathcal{O}$ is a G-model of K and $Y \subseteq X$ the center of v, then $x(t) \in X(\Bbbk[[t]])$ and $Y = \overline{Gx(0)}$.*

To prove this theorem it suffices to show that $v_{x(t)} = v_{\chi,\theta_0}$ for a certain germ of a curve (χ, θ_0). This stems from the two subsequent lemmas.

Lemma 24.1 ([LV, 4.4]). $\forall g(t) \in G(\Bbbk[[t]]),\ x(t) \in \mathcal{O}(\Bbbk((t))) :\ v_{g(t)x(t)} = v_{x(t)}$

Proof. The G-action on $x(t)$ yields $\Bbbk(\mathcal{O}) \hookrightarrow \Bbbk(G)((t))$, so that $v_{x(t)}$ coincides with ord_t w.r.t. this inclusion. The lemma stems from the fact that $G(\Bbbk[[t]])$ acts on $\Bbbk(G)((t))$ "by right translations" preserving ord_t. □

Lemma 24.2 ([LV, 4.5]). *Every germ of a formal curve in $\mathcal{O}$ is $G(\Bbbk[[t]])$-equivalent to a formal germ induced by a germ of a curve.*

Proof. Since $\mathcal{O}$ is homogeneous, $G(\Bbbk[[t]])$-orbits are open in $\mathcal{O}(\Bbbk((t)))$ in t-adic topology [BT]. Now the lemma stems from Theorem A4.1. □

Germs of formal curves in $\mathcal{O}$ are more accessible if they come from formal germs in G. Luckily, this is "almost" always the case.

Proposition 24.1 ([LV, 4.3]). *For $\forall x(t) \in \mathcal{O}(\Bbbk((t)))$ there exists $n \in \mathbb{N}$ such that $x(t^n) = g(t) \cdot o$ for some $g(t) \in G(\Bbbk((t)))$.*

Proof. Consider the algebraic closure $\overline{\Bbbk((t))}$ of $\Bbbk((t))$. The set $\mathcal{O}(\overline{\Bbbk((t))})$ is equipped with a structure of an algebraic variety over $\overline{\Bbbk((t))}$ with the transitive $G(\overline{\Bbbk((t))})$-action. However $\overline{\Bbbk((t))} = \bigcup_{n=1}^{\infty} \Bbbk((\sqrt[n]{t}))$, whence $x(t) = g(\sqrt[n]{t}) \cdot o$ for some $g(\sqrt[n]{t}) \in G(\Bbbk((\sqrt[n]{t})))$. □

Note that $v_{x(t^n)} = n \cdot v_{x(t)}$. Thus we may describe $\mathcal{V}^1$ in terms of germs of formal curves in G, i.e., points of $G(\Bbbk((t)))$, considered up to left translations by $G(\Bbbk[[t]])$ and right translations by $H(\Bbbk((t)))$. There is a useful structural result shrinking the set of formal germs under consideration:

Iwasawa decomposition [IM]. $G(\Bbbk((t))) = G(\Bbbk[[t]]) \cdot \mathfrak{X}^*(T) \cdot U(\Bbbk((t)))$, *where $\mathfrak{X}^*(T)$ is regarded as a subset of $T(\Bbbk((t)))$.*

Corollary 24.1. *Every $v \in \mathcal{V}^1$ is proportional to (the restriction of) $v_{g(t)}$, $g(t) \in \mathfrak{X}^*(T) \cdot U(\Bbbk((t)))$.*

Let us mention a related useful result on the structure of $G(\Bbbk((t)))$:

Cartan decomposition [IM]. $G(\Bbbk((t))) = G(\Bbbk[[t]]) \cdot \mathfrak{X}^*(T) \cdot G(\Bbbk[[t]])$.

Example 24.1. Suppose that $\mathcal{O} = G/S$ is horospherical. We may assume $S \supseteq U$; then $N(S) \supseteq B$ and $A := \mathrm{Aut}_G \mathcal{O} \simeq N(S)/S = T/T \cap S$ is a torus. Since $\mathcal{O}$ is spherical, $\mathcal{V} = \mathcal{V}^1$. Due to the Iwasawa decomposition, every $v \in \mathcal{V}$ is proportional to some v_γ, $\gamma \in \mathfrak{X}^*(T)$. Let $\overline{\gamma}$ be the image of γ in $\mathfrak{X}^*(A)$. By definition, $v_\gamma(f) = \mathrm{ord}_{t=0} f(g\gamma(t)o) = \mathrm{ord}_{t=0} f(\overline{\gamma}(t) \cdot go)$ is the order of f along generic trajectories of $\overline{\gamma}$ as $t \to 0$. In particular, $\mathcal{V} = \mathfrak{X}^*(A) \otimes \mathbb{Q}$, cf. Theorem 21.3.

Example 24.2. Specifically, let $\mathcal{O} = G = T$ be a torus. Every T-valuation of $\Bbbk(T)$ is proportional to v_γ, $\gamma \in \mathfrak{X}^*(T)$, where v_γ is the order of a function restricted to $s\gamma(t)$ as $t \to 0$ for general $s \in T$. By Theorem 24.1, v_γ has a center Y on a toric variety $X \supseteq T$ iff $\gamma(0) := \lim_{t\to 0} \gamma(t)$ exists and belongs to the dense T-orbit in Y. Thus the lattice points in $\mathcal{V}_Y$ are exactly the one-parameter subgroups of T converging to a point in the dense T-orbit of Y. This is the classical description of the fan of a toric variety [Oda], [Ful2, 2.3].

Other examples can be found in §16.

Chapter 5

Spherical varieties

Although the theory developed in the previous chapters applies to arbitrary homogeneous spaces of reductive groups, and even to more general group actions, it acquires most complete and elegant form for spherical homogeneous spaces and their equivariant embeddings, called spherical varieties. A justification of the fact that spherical homogeneous spaces are a significant mathematical object is that they naturally arise in various fields, such as embedding theory, representation theory, symplectic geometry, etc. In §25 we collect various characterizations of spherical spaces, the most important being: the existence of an open B-orbit, the "multiplicity free" property for spaces of global sections of line bundles, commutativity of invariant differential operators and of invariant functions on the cotangent bundle w.r.t. the Poisson bracket.

Then we examine most interesting classes of spherical homogeneous spaces and spherical varieties in more details. Algebraic symmetric spaces are considered in §26. We develop the structure theory and classification of symmetric spaces, compute the colored data required for description of their equivariant embeddings, study B-orbits and (co)isotropy representation. §27 is devoted to $(G \times G)$-equivariant embeddings of a reductive group G. A particular interest in this class is explained, for example, by an observation that linear algebraic monoids are nothing else but affine equivariant group embeddings. Horospherical varieties of complexity 0 are classified and studied in §28.

Geometric structure of toroidal varieties, considered in §29, is best understood among all spherical varieties, since toroidal varieties are "locally toric". They can be defined by several equivalent properties: their fans are "colorless", they are spherical and pseudo-free, the action sheaf on a toroidal variety is the log-tangent sheaf w.r.t. a G-stable divisor with normal crossings. An important property of toroidal varieties is that they are rigid as G-varieties.

The so-called wonderful varieties are the most remarkable subclass of toroidal varieties. They are canonical completions with nice geometric properties of (certain) spherical homogeneous spaces. The theory of wonderful varieties is developed in §30. Applications include computation of the canonical divisor of a spherical variety and Luna's conceptual approach to the classification of spherical subgroups through the classification of wonderful varieties.

The last §31 is devoted to Frobenius splitting, a technique for proving geometric and algebraic properties (normality, rationality of singularities, cohomology vanishing, etc) in positive characteristic. However, this technique can be applied to zero characteristic using reduction $\bmod p$ provided that reduced varieties are Frobenius split. This works for spherical varieties. As a consequence, one obtains vanishing of higher cohomology of ample or nef line bundles on complete spherical varieties, normality and rationality of singularities for G-stable subvarieties, etc. Some of these results can be proved by other methods, but Frobenius splitting provides a simple uniform approach.

25 Various characterizations of sphericity

Spherical homogeneous spaces can be considered from diverse viewpoints: orbits and equivariant embeddings, representation theory and multiplicities, symplectic geometry, harmonic analysis, etc. The definition and some other implicit characterizations of this remarkable class of homogeneous spaces are already scattered in the text above. In this section, we review these issues and introduce other important properties of homogeneous spaces which are equivalent or closely related to sphericity.

As usual, G is a connected reductive group, $\mathcal{O}$ denotes a homogeneous G-space with the base point o, and $H = G_o$.

Definition–Theorem. *A* spherical homogeneous space $\mathcal{O}$ *(resp. a* spherical subgroup $H \subseteq G$*, a* spherical subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$*, a* spherical pair (G, H) *or* $(\mathfrak{g}, \mathfrak{h})$*) can be defined by any one of the following properties:*

(S1) $\Bbbk(\mathcal{O})^B = \Bbbk$.

(S2) *B has an open orbit in $\mathcal{O}$.*

(S3) *H has an open orbit in G/B.*

(S4) $\exists g \in G: \ \mathfrak{b} + (\operatorname{Ad} g)\mathfrak{h} = \mathfrak{g}$.

(S5) *There exists a Borel subalgebra $\widetilde{\mathfrak{b}} \subseteq \mathfrak{g}$ such that $\mathfrak{h} + \widetilde{\mathfrak{b}} = \mathfrak{g}$.*

(S6) *H acts on G/B with finitely many orbits.*

(S7) *For any G-variety X and $\forall x \in X^H$, $\overline{Gx}$ contains finitely many G-orbits.*

(S8) *For any G-variety X and $\forall x \in X^H$, $\overline{Gx}$ contains finitely many B-orbits.*

The term "spherical homogeneous space" is traced back to Brion, Luna, and Vust [BLV], and "spherical subgroup" to [Krä], though the notions themselves appeared much earlier.

Proof. (S1) $\iff$ (S2) B-invariant functions separate generic B-orbits [PV, 2.3].

(S2) $\iff$ (S3) Both conditions are equivalent to that $B \times H : G$ has an open orbit, where B acts by left and H by right translations, or vice versa.

(S4) and (S5) are just reformulations of (S2) and (S3) in terms of tangent spaces.

(S2) $\implies$ (S8) Gx satisfies (S2), too, and we conclude by Corollary 6.1.

(S8) $\implies$ (S7) Obvious.

(S7) $\implies$ (S2) Stems from Corollary 6.2.

(S8) $\implies$ (S6) B acts on G/H with finitely many orbits, which are in bijection with $(B \times H)$-orbits on G and with H-orbits on G/B.

(S6) $\implies$ (S3) Obvious. □

In particular, spherical spaces are characterized in the framework of embedding theory as those having finitely many orbits in the boundary of any equivariant embedding. The embedding theory of spherical spaces is considered in §15.

Another important characterization of spherical spaces is in terms of representation theory, due to Kimelfeld and Vinberg [VK]. Recall from §2 that the multiplicity of a highest weight λ in a G-module M is

$$m_\lambda(M) = \dim \operatorname{Hom}_G(V(\lambda), M) = \dim M_\lambda^{(B)}$$

In characteristic zero, $m_\lambda(M)$ is the multiplicity of the simple G-module $V(\lambda)$ in the decomposition of M. In positive characteristic, $V(\lambda)$ denotes the respective Weyl module. The module M is said to be multiplicity free if all multiplicities in M are ≤ 1.

Theorem 25.1. *$\mathcal{O}$ is spherical iff the following equivalent conditions hold:*

(MF1) $\mathbb{P}(V(\lambda))^H$ *is finite for* $\forall\lambda \in \mathfrak{X}_+$.

(MF2) $\forall\lambda \in \mathfrak{X}_+,\ \chi \in \mathfrak{X}(H):\ \dim V(\lambda)^{(H)}_\chi \leq 1$

(MF3) *For any G-line bundle $\mathcal{L}$ on $\mathcal{O}$,* $\mathrm{H}^0(\mathcal{O},\mathcal{L})$ *is multiplicity free.*

If $\mathcal{O}$ is quasiaffine, then the last two conditions can be weakened to

(MF4) $\forall\lambda \in \mathfrak{X}_+:\ \dim V(\lambda)^H \leq 1$

(MF5) $\Bbbk[\mathcal{O}]$ *is multiplicity free.*

The spaces satisfying these conditions are called *multiplicity free*.

Proof. (S1) $\iff$ (MF3) If $m_\lambda(\mathcal{L}) \geq 2$, then there exist two non-proportional sections $\sigma_0,\sigma_1 \in \mathrm{H}^0(\mathcal{O},\mathcal{L})^{(B)}_\lambda$. Their ratio $f=\sigma_1/\sigma_0$ is a non-constant B-invariant function. Conversely, any $f \in \Bbbk(\mathcal{O})^B$ can be represented in this way: the G-line bundle $\mathcal{L}$ together with the canonical B-eigensection σ_0 is defined by a sufficiently big multiple of $\operatorname{div}_\infty f$ (cf. Corollary A1.2).

Finally, if $\mathcal{O}$ is quasiaffine, then we may take for $\mathcal{L}$ the trivial bundle: for σ_0 take a sufficiently big power of any B-eigenfunction in $\mathbb{I}(D) \lhd \Bbbk[\mathcal{O}]$, $D = \operatorname{Supp}\operatorname{div}_\infty f \subset \mathcal{O}$. Hence (S1) $\iff$ (MF5).

(MF1) $\iff$ (MF2) Stems from $\mathbb{P}(V(\lambda))^H = \mathbb{P}\left(V(\lambda)^{(H)}\right) = \bigsqcup_\chi \mathbb{P}\left(V(\lambda)^{(H)}_\chi\right)$ (a finite disjoint union).

(MF2) $\iff$ (MF3) If $\mathcal{O}=G/H$ is a quotient space, then this is the Frobenius reciprocity (2.2). Generally, there is a bijective purely inseparable morphism $G/H \to \mathcal{O}$ (Remark 1.1), and $\mathcal{O}$ is spherical iff G/H is so. But we have already seen that the sphericity is equivalent to (MF3).

(MF4) $\iff$ (MF5) is proved in the same way. □

The "multiplicity free" property leads to an interpretation of sphericity in terms of automorphisms and group algebras associated with G. Since the complete reducibility of rational representations is essential here, we assume $\operatorname{char}\Bbbk = 0$ up to the end of this section.

Recall from §2 the algebraic versions of the group algebra $\mathcal{A}(G)$ and the Hecke algebra $\mathcal{A}(\mathcal{O})$.

Theorem 25.2 ([AV], [Vin3]). *An affine homogeneous space $\mathcal{O}$ is spherical iff either of the following equivalent conditions is satisfied:*

(GP1) $\mathcal{A}(\mathcal{O}) = \mathcal{A}(G)^{H\times H}$ *is commutative.*

(GP2) $\mathcal{A}(V)^{H\times H}$ *is commutative for all G-modules V.*

(WS1) *(Selberg condition) The G-action on $\mathcal{O}$ extends to a cyclic extension $\widehat{G} = \langle G, s\rangle$ of G so that (sx, sy) is G-equivalent to (y, x) for general $x, y \in \mathcal{O}$.*

(WS2) *(Gelfand condition) There exists $\sigma \in \operatorname{Aut} G$ such that $\sigma(g) \in Hg^{-1}H$ for general $g \in G$.*

The condition (GP1) is an algebraization of a similar commutativity condition for the group algebra of a Lie group, see [Gel], [Vin3, I.2], and below. The condition (WS2) appeared in [Gel] and (WS1) was first introduced by Selberg in the seminal paper on the trace formula [Sel], and by Akhiezer and Vinberg [AV] in the context of algebraic geometry. The spaces satisfying (WS1)–(WS2) are called *weakly symmetric* and (G, H) is said to be a *Gelfand pair* if (GP1)–(GP2) hold.

Proof. (MF5) $\iff$ (GP1) Stems from Schur's lemma.

(GP1) $\iff$ (GP2) Obvious.

(MF5) $\implies$ (WS1) There exists a Weyl involution $\sigma \in \operatorname{Aut} G$, $\sigma(H) = H$ [AV]. There is a conceptual argument for symmetric spaces and in general a case-by-case verification using the classification from §10. Define $s \in \operatorname{Aut}\mathcal{O}$ by $s(go) = \sigma(g)o$ and $\widehat{G} = G \rightthreetimes \langle s\rangle$ by $sgs^{-1} = \sigma(g)$. The G-action on $\mathcal{O}\times\mathcal{O}$ is extended to $\widehat{G}$ by $s(x, y) = (sy, sx)$.

Consider the $(G\times G)$-isotypic decomposition

$$\Bbbk[\mathcal{O}\times\mathcal{O}] = \bigoplus_{\lambda,\mu\in\Lambda_+(\mathcal{O})} \Bbbk[\mathcal{O}\times\mathcal{O}]_{(\lambda,\mu)}, \qquad \text{where}$$
$$\Bbbk[\mathcal{O}\times\mathcal{O}]_{(\lambda,\mu)} = \Bbbk[\mathcal{O}]_{(\lambda)} \otimes \Bbbk[\mathcal{O}]_{(\mu)} \simeq V(\lambda)\otimes V(\mu)$$

Clearly, s twists the G-action by σ, hence maps $\Bbbk[\mathcal{O}\times\mathcal{O}]_{(\lambda,\mu)}$ to $\Bbbk[\mathcal{O}\times\mathcal{O}]_{(\mu^*,\lambda^*)}$ and preserves each summand of

$$\Bbbk[\mathcal{O}\times\mathcal{O}]^G = \bigoplus_{\lambda\in\Lambda_+(\mathcal{O})} \Bbbk[\mathcal{O}\times\mathcal{O}]^G_{(\lambda,\lambda^*)}$$

However a $\widehat{G}$-invariant inner product $(p, q) \to (pq)^\natural$ on $\Bbbk[\mathcal{O}]$ induces a nonzero pairing between simple G-modules $\Bbbk[\mathcal{O}]_{(\lambda^*)}$ and $\Bbbk[\mathcal{O}]_{(\lambda)}$, whence by duality a $\widehat{G}$-invariant function in $\Bbbk[\mathcal{O}\times\mathcal{O}]_{(\lambda,\lambda^*)}$, which spans $\Bbbk[\mathcal{O}\times\mathcal{O}]^G_{(\lambda,\lambda^*)}$. It follows that s acts trivially on $\Bbbk[\mathcal{O}\times\mathcal{O}]^G$.

But the action $G : \mathcal{O}\times\mathcal{O}$ is stable (Theorem 8.7), whence s preserves generic G-orbits in $\mathcal{O}\times\mathcal{O}$, which is exactly the Selberg condition.

(WS1) $\Longrightarrow$ (WS2) Multiplying s by $g \in G$ preserves the Selberg condition. Also, if $(sx, sy) \sim (y, x)$, then the same is true for any G-equivalent pair. Hence, w.l.o.g., $so = o = x$. Define $\sigma \in \operatorname{Aut} G$ by $\sigma(g) = sgs^{-1}$; then $(so, sgo) = (o, \sigma(g)o) \sim (go, o)$ for general $g \in G$. Hence $g'go = o$, $g'o = \sigma(g)o$, i.e., $g'g = h \in H$, $\sigma(g) = g'h' = hg^{-1}h'$ for some $h' \in H$.

(WS2) $\Longrightarrow$ (WS1) The Gelfand condition implies $\sigma(H) = H$, whence $s \in \operatorname{Aut}\mathcal{O}$, $s(go) = \sigma(g)o$, is a well-defined automorphism. Put $\widehat{G} = G \leftthreetimes \langle s\rangle$, $sgs^{-1} = \sigma(g)$. The Selberg condition is verified by reversing the previous arguments.

(WS2) $\Longrightarrow$ (GP1) The inversion map $g \mapsto g^{-1}$ on G extends to an involutive antiautomorphism of $\mathcal{A}(G)$. Its restriction to $\mathcal{A}(G)^{H\times H}$ coincides with the automorphism induced by σ. Hence $\mathcal{A}(G)^{H\times H}$ is commutative. $\square$

Remark 25.1. Already in the quasiaffine case the classes of weakly symmetric and spherical spaces are not contained in each other [Zor].

Now we characterize sphericity in terms of differential geometry.

Recall from §8 that the action $G : T^*\mathcal{O}$ is Poisson w.r.t. the natural symplectic structure. Thus we have a G-invariant Poisson bracket of functions on $T^*\mathcal{O}$. Homogeneous functions on $T^*\mathcal{O}$ are locally the symbols of differential operators on $\mathcal{O}$, and the Poisson bracket is induced by the commutator of differential operators.

The functions pulled back under the moment map $\Phi : T^*\mathcal{O} \to \mathfrak{g}^*$ are called *collective*. They Poisson-commute with G-invariant functions on $T^*\mathcal{O}$ (Proposition 22.1).

Theorem 25.3. *$\mathcal{O}$ is spherical iff the following equivalent conditions hold:*

(WC1) *Generic orbits of $G : T^*\mathcal{O}$ are coisotropic, i.e., $\mathfrak{g}\alpha \supseteq (\mathfrak{g}\alpha)^{\angle}$ for general $\alpha \in T^*\mathcal{O}$.*

(WC2) *$\Bbbk(T^*\mathcal{O})^G$ is commutative w.r.t. the Poisson bracket.*

(CI) *There exists a complete system of collective functions in involution on $T^*\mathcal{O}$.*

If $\mathcal{O}$ is affine, then these conditions are equivalent to

(Com) *$\mathcal{D}(\mathcal{O})^G$ is commutative.*

The theorem goes back to Guillemin, Sternberg [GS], and Mikityuk [Mik]. The spaces satisfying (WC1)–(WC2) are called *weakly commutative* and those satisfying (Com) are said to be *commutative*.

Proof. (S2) $\iff$ (WC1) By Theorem 8.3, $\operatorname{cork} T^*\mathcal{O} = 2c(\mathcal{O})$ is zero iff $\mathcal{O}$ is spherical, and this means exactly that generic orbits are coisotropic.

(WC1) $\iff$ (WC2) Skew gradients of $f \in \Bbbk(T^*\mathcal{O})^G$ at a point α of general position span $(\mathfrak{g}\alpha)^\angle$. All G-invariant function Poisson-commute iff their skew gradients are skew-orthogonal to each other, i.e., iff $(\mathfrak{g}\alpha)^\angle$ is isotropic.

(GP1) $\iff$ (Com) If $\mathcal{O}$ is quasiaffine, then $\mathcal{D}(\mathcal{O})$ acts faithfully on $\Bbbk[\mathcal{O}]$ by linear endomorphisms. Hence $\mathcal{D}(\mathcal{O})^G$ is a subalgebra in $\mathcal{A}(\mathcal{O})$. It remains to utilize the approximation of linear endomorphisms by differential operators.

Lemma 25.1. *Let X be an smooth affine G-variety.*

(1) *For any linear operator $\phi : \Bbbk[X] \to \Bbbk[X]$ and any finite-dimensional subspace $M \subset \Bbbk[X]$ there exists $\partial \in \mathcal{D}(X)$ such that $\partial|_M = \phi|_M$.*

(2) *If ϕ is G-equivariant, then one may assume $\partial \in \mathcal{D}(X)^G$.*

(3) *Put $\mathcal{I} = \operatorname{Ann} M \lhd \mathcal{D}(X)$; then $\forall f \in \Bbbk[X] : \mathcal{I}f = 0 \implies f \in M$.*

We conclude by Lemma 25.1(2) that $\mathcal{A}(\mathcal{O})$ is commutative iff $\mathcal{D}(\mathcal{O})^G$ is so.

Proof of Lemma 25.1. (1) We deduce it from (3). Choose a basis $f_1, \dots, f_n$ of M. It suffices to construct $\partial \in \mathcal{D}(X)$ such that $\partial f_i = 0$, $\forall i < n$, $\partial f_n = 1$. By (3) there exists $\partial' \in \operatorname{Ann}(f_1, \dots, f_{n-1})$, $\partial' f_n \neq 0$. As $\Bbbk[X]$ is a simple $\mathcal{D}(X)$-module [MR] we may find $\partial'' \in \mathcal{D}(X)$, $\partial''(\partial' f_n) = 1$ and put $\partial = \partial''\partial'$.

(2) W.l.o.g. M is G-stable. Assertion (1) yields an epimorphism of G-$\Bbbk[X]$-modules $\mathcal{D}(X) \twoheadrightarrow \operatorname{Hom}(M, \Bbbk[X])$ given by restriction to M. But taking G-invariants is an exact functor.

(3) The assertion is trivial for $M = 0$ and we proceed by induction on $\dim M$. In the above notation, put $\mathcal{I}' = \operatorname{Ann}(f_1, \dots, f_{n-1})$. For $\forall \partial, \partial' \in \mathcal{I}'$ we have $(\partial f_n)\partial' - (\partial' f_n)\partial \in \mathcal{I}$, whence

$$(\partial f_n)(\partial' f) = (\partial' f_n)(\partial f) \tag{25.1}$$

Taking $\partial' = \xi\partial$, $\xi \in \operatorname{Vect} X$, yields $\xi(\partial f/\partial f_n) = 0 \implies \partial f/\partial f_n = c_\partial =$ const. Substituting this in (25.1) yields $c_{\partial'} = c_\partial = c$ (independent of ∂). Thus $\partial(f - cf_n) = 0 \implies f - cf_n \in \langle f_1, \dots, f_{n-1}\rangle \implies f \in M$. □

(Com) $\Longrightarrow$ (WC2) If $\mathcal{O}$ is affine, then $\operatorname{gr}\mathcal{D}(\mathcal{O}) = \Bbbk[T^*\mathcal{O}]$. By complete reducibility, $\Bbbk[T^*\mathcal{O}]^G = \operatorname{gr}\mathcal{D}(\mathcal{O})^G$ is Poisson-commutative. But the G-action on $T^*\mathcal{O}$ is stable (Remark 8.2), whence $\Bbbk(T^*\mathcal{O})^G = \operatorname{Quot}\Bbbk[T^*\mathcal{O}]^G$ is Poisson-commutative as well.

(CI) $\Longleftrightarrow$ (S2) This equivalence is due to Mikityuk [Mik] (for affine $\mathcal{O}$).

A complete system of Poisson-commuting functions on $M_{\mathcal{O}}$ can be constructed by the method of argument shift [MF1]: choose a regular semisimple element $\xi \in \mathfrak{g}^*$ and consider the derivatives $\partial_\xi^n f$ of all $f \in \Bbbk[\mathfrak{g}^*]^G$. The functions $\partial_\xi^n f$ Poisson-commute and produce a complete involutive system on $Gx \subset \mathfrak{g}^*$ (for general ξ) whenever $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$, where $\operatorname{ind}\mathfrak{g} = d_G(\mathfrak{g}^*)$ [Bol]. In the symplectically stable case, general points $x \in M_{\mathcal{O}}$ are semisimple and $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$. Generally, the equality $\operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$ was conjectured by Elashvili and proved by Charbonnel [Cha] for $\forall x \in \mathfrak{g}^*$.

Since symplectic leaves of the Poisson structure on $M_{\mathcal{O}}$ are G-orbits, there are $(d_G(M_{\mathcal{O}}) + \dim M_{\mathcal{O}})/2 = \dim\mathcal{O} - c(\mathcal{O})$ independent Poisson-commuting collective functions. Thus a complete involutive system of collective functions exists iff $c(\mathcal{O}) = 0$. $\square$

Since $T^*\mathcal{O} = G *_H \mathfrak{h}^\perp$, weak commutativity is readily reformulated in terms of the coadjoint representation [Mik], [Pan1], [Vin3, II.4.1].

Theorem 25.4. *(G,H) is a spherical pair iff general points $\alpha \in \mathfrak{h}^\perp$ satisfy any of the equivalent conditions:*

(Ad1) $\dim G\alpha = 2\dim H\alpha$

(Ad2) *$H\alpha$ is a Lagrangian subvariety in $G\alpha$ w.r.t. the Kirillov form.*

(Ad3) *(Richardson condition)* $\mathfrak{g}\alpha \cap \mathfrak{h}^\perp = \mathfrak{h}\alpha$

The Richardson condition means that $G\alpha \cap \mathfrak{h}^\perp$ is a finite union of open H-orbits [PV, 1.5].

Proof. (WC1) $\Longleftrightarrow$ (Ad1) Recall that the moment map $\Phi : G *_H \mathfrak{h}^\perp \to \mathfrak{g}^*$ is defined via replacing the $*$-action by the coadjoint action (Example 8.1). We have

$$d_G(T^*\mathcal{O}) = \dim\mathcal{O} - \dim H\alpha \qquad \text{and}$$
$$\operatorname{def} T^*\mathcal{O} = \dim G_{\Phi(e*\alpha)}/G_{e*\alpha} = \dim G_\alpha/H_\alpha$$

Hence

$$\operatorname{cork} T^*\mathcal{O} = d_G(T^*\mathcal{O}) - \operatorname{def} T^*\mathcal{O} = \dim G\alpha - 2\dim H\alpha$$

(Ad1) $\iff$ (Ad2) The Kirillov form vanishes on $\mathfrak{h}\alpha$.

(Ad2) $\iff$ (Ad3) Stems from $(\mathfrak{g}\alpha)\cap\mathfrak{h}^\perp = (\mathfrak{h}\alpha)^\angle$, the skew-orthocomplement w.r.t. the Kirillov form. □

Invariant functions on cotangent bundles of spherical homogeneous spaces have a nice structure.

Proposition 25.1 ([Kn1, 7.2]). *If $\mathcal{O} = G/H$ is spherical, then $\Bbbk[T^*\mathcal{O}]^G \simeq \Bbbk[\widetilde{L}_\mathcal{O}] \simeq \Bbbk[\mathfrak{a}^*]^{W_\mathcal{O}}$ is a polynomial algebra; there are similar isomorphisms for fields of rational functions.*

Proof. By Proposition 22.1 and (WC2), $\Bbbk(T^*\mathcal{O})^G \simeq \Bbbk(\widetilde{L}_\mathcal{O}) \simeq \Bbbk(\mathfrak{a}^*)^{W_\mathcal{O}}$. By Lemma 22.2, $\widetilde{\pi}_G\widetilde{\Phi} : T^*\mathcal{O} \to \widetilde{L}_\mathcal{O}$ is a surjective morphism of normal varieties. Therefore any $f \in \Bbbk(T^*\mathcal{O})^G$ having poles on $T^*\mathcal{O}$ must have poles on $\widetilde{L}_\mathcal{O}$, whence $\Bbbk[T^*\mathcal{O}]^G = \Bbbk[\widetilde{L}_\mathcal{O}]$. The latter algebra is polynomial for $W_\mathcal{O}$ is generated by reflections (Theorem 22.1). □

In other words, invariants of the coisotropy representation form a polynomial algebra $\Bbbk[\mathfrak{h}^\perp]^H \simeq \Bbbk[\mathfrak{a}^*]^{W_{G/H}}$ for any spherical pair (G, H).

Remark 25.2. A similar assertion in the non-commutative setup was proved in [Kn6]. Namely, all invariant differential operators on a spherical space $\mathcal{O}$ are completely regular, whence $\mathcal{D}(\mathcal{O})^G$ is a polynomial ring isomorphic to $\Bbbk[\rho + \mathfrak{a}^*]^{W_\mathcal{O}}$ (see Remark 22.1). In particular, every spherical homogeneous space is commutative.

In our considerations G was always assumed to be reductive. However some of the concepts introduced above are reasonable even for non-reductive G assuming H be reductive instead. Some of the above results remain valid:

(1) If $\mathcal{O} = G/H$ is weakly symmetric, then (G, H) is a Gelfand pair.

(2) $\mathcal{O}$ is commutative iff (G, H) is a Gelfand pair.

(3) A commutative space $\mathcal{O}$ is weakly commutative provided that $\Bbbk[\mathfrak{h}^\perp]^H$ separates generic H-orbits in $\mathfrak{h}^\perp$.

The above proofs work in this case: the functor $(\cdot)^G$ is exact on global sections of G-sheaves on $\mathcal{O}$ since $\mathcal{O}$ is affine and $(\cdot)^H$ is exact on rational H-modules, and orbit separation in (3) guarantees $\Bbbk(T^*\mathcal{O})^G = \operatorname{Quot}\Bbbk[T^*\mathcal{O}]^G$. The converse implication in (1) fails, the simplest counterexample being:

Example 25.1 ([Lau]). Put $H = \mathrm{Sp}_{2n}(\Bbbk)$, $G = H \leftthreetimes N$, where $N = \exp \mathfrak{n}$ is a unipotent group associated with the Heisenberg type Lie algebra $\mathfrak{n} = (\Bbbk^{2n} \oplus \Bbbk^{2n}) \oplus \Bbbk^3$, the commutator in $\mathfrak{n}$ being defined by the identification $\bigwedge^2(\Bbbk^{2n} \oplus \Bbbk^{2n})^{\mathrm{Sp}_{2n}(\Bbbk)} \simeq \Bbbk^3 = \mathfrak{z}(\mathfrak{n})$. Then (G, H) is a Gelfand pair, but $\mathcal{O}$ is not weakly symmetric.

Also, the implication (3) fails if the orbit separation is violated. The reason is that there may be too few invariant differential operators. For instance, in the previous example, replace H by $\Bbbk^*$ acting on $\Bbbk^{2n}$ via a character $\chi \neq 0$ and on $\Bbbk^3$ via 2χ. Then $\mathcal{O}$ is not weakly commutative while $\mathcal{D}(\mathcal{O})^G = \Bbbk$.

The classes of weakly symmetric and (weakly) commutative homogeneous spaces were first introduced and examined in Riemannian geometry and harmonic analysis, see the survey [Vin3]. We shall review the analytic viewpoint now.

Quitting a somewhat restrictive framework of algebraic varieties, one may consider the above properties of homogeneous spaces in the category of Lie group actions, making appropriate modifications in formulations. For instance, instead of regular or rational functions one considers arbitrary analytic or differentiable functions. Some of these properties receive new interpretation in terms of differential geometry, e.g., (CI) means that invariant Hamiltonian dynamic systems on $T^*\mathcal{O}$ are completely integrable in the class of Noether integrals [MF2], [Mik].

The situation, where H is a compact subgroup of a real Lie group G, i.e., $\mathcal{O} = G/H$ is a Riemannian homogeneous space, has attracted the main attention of researchers. Most of the above results were originally obtained in this setting.

The properties (MF4), (MF5) are naturally reformulated here in the category of unitary representations of G replacing $\Bbbk[\mathcal{O}]$ by $L^2(\mathcal{O})$. In (GP1) one considers the algebra $\mathcal{A}(G)$ of complex measures with compact support on G. The conditions (WS1), (WS2) are formulated for *all* (not only general) points (which is equivalent for compact H); there is also an infinitesimal characterization of weak symmetry [Vin3, I.1.5].

There are the following implications:

$$\begin{gathered}\text{weakly symmetric space}\\ \Downarrow\\ \text{Gelfand pair}\\ \Updownarrow\\ \text{multiplicity free space}\\ \Updownarrow\\ \text{commutative space}\\ \Updownarrow\\ \text{weakly commutative space}\end{gathered}$$

The implication (WS2) $\Longrightarrow$ (GP1) is due to Gelfand [Gel] and (GP1) $\Longleftrightarrow$ (MF5) was proved in [BGGN]. The equivalence (GP1) $\Longleftrightarrow$ (Com) is due to Helgason [Hel2, Ch.IV, B13] and Thomas [Tho], for a proof see [Vin3, I.2.5]. The implication (Com) $\Longrightarrow$ (WC2) is easy [Vin3, I.4.2] and the converse was recently proved by Rybnikov [Ryb].

A classification of commutative Riemannian homogeneous spaces was obtained by Yakimova [Yak1], [Yak2] using partial results of Vinberg [Vin4] and the classification of affine spherical spaces from §10.

An algebraic homogeneous space $\mathcal{O} = G/H$ over $\Bbbk = \mathbb{C}$ may be considered as a homogeneous manifold in the category of complex or real Lie group actions. At the same time, if (G, H) is defined over $\mathbb{R}$, then $\mathcal{O}$ has a real form $\mathcal{O}(\mathbb{R})$ containing $G(\mathbb{R})/H(\mathbb{R})$ as an open orbit (in classical topology). Thus G/H may be regarded as the complexification of $G(\mathbb{R})/H(\mathbb{R})$, a homogeneous space of a real Lie group $G(\mathbb{R})$.

It is easy to see that G/H is commutative (resp. weakly commutative, multiplicity free, weakly symmetric, satisfies (GP1), (CI)) iff $G(\mathbb{R})/H(\mathbb{R})$ is so. In other words, the above listed properties are stable under complexification and passing to a real form.

This observation leads to the following criterion of sphericity, which is a "real form" of Theorem 25.3.

By Chevalley's theorem, there exists a projective embedding $\mathcal{O} \subseteq \mathbb{P}(V)$ for some G-module V. Assume that G is reductive and $K \subset G$ is a compact real form. Then V can be endowed with a K-invariant Hermitian inner product $(\cdot|\cdot)$, which induces a Kählerian metric on $\mathbb{P}(V)$ and on $\mathcal{O}$ (the *Fubini–Studi metric*). The imaginary part of this metric is a real symplectic form. The action $K : \mathbb{P}(V)$ is Poisson, the moment map $\Phi : \mathbb{P}(V) \to \mathfrak{k}^*$ being defined by the formula

$$\big\langle \Phi(\langle v\rangle), \xi \big\rangle = \frac{1}{2i} \cdot \frac{(v|\xi v)}{(v|v)}, \qquad \forall v \in V,\ \xi \in \mathfrak{k}$$

Theorem 25.5 ([Bri3], [HW], [Akh4, §13]). *$\mathcal{O}$ is spherical iff generic K-orbits in $\mathcal{O}$ are coisotropic w.r.t. the Fubini–Studi form, or equivalently, $C^\infty(\mathcal{O})$ is Poisson-commutative.*

Proof. First note that generic K-orbits in $\mathcal{O}$ are coisotropic iff

$$d_K(\mathcal{O}) = \operatorname{def} \mathcal{O} = \operatorname{rk} K - \operatorname{rk} K_* \tag{25.2}$$

where K_* is the stabilizer of general position for $K : \mathcal{O}$. The condition (25.2) does not depend on the symplectic structure.

If $\mathcal{O}$ is affine, then the assertion can be directly reduced to Theorem 25.3 by complexification. W.l.o.g. $K \cap H$ is a compact real form of H. Using the Cartan decompositions $G = K \cdot \exp i\mathfrak{k}$, $H = (K \cap H) \cdot \exp i(\mathfrak{k} \cap \mathfrak{h})$, we obtain a K-diffeomorphism

$$\mathcal{O} \simeq K *_{K\cap H} i\mathfrak{k}/i(\mathfrak{k} \cap \mathfrak{h}) \simeq T^*(K/K \cap H)$$

Complexifying the r.h.s. we obtain $T^*\mathcal{O}$.

In the general case, it is more convenient to apply the theory of doubled actions (§8).

There exists a Weyl involution θ of G commuting with the Hermitian conjugation $g \mapsto g^*$. The mapping $g \mapsto \overline{g} := \theta(g^*)^{-1}$ is a complex conjugation on G defining a split real form $G(\mathbb{R})$. There exists a $G(\mathbb{R})$-stable real form $V(\mathbb{R}) \subset V$ such that $(\cdot|\cdot)$ takes real values on $V(\mathbb{R})$. The complex conjugation on V, $\mathbb{P}(V)$, or G is defined by conjugating the coordinates or matrix entries w.r.t. an orthonormal basis in $V(\mathbb{R})$.

It follows that the complex conjugate variety $\overline{\mathcal{O}}$ is naturally embedded in $\mathbb{P}(V)$ as a G-orbit. Complexifying the action $K : \mathcal{O}$ we obtain the diagonal action $G : \mathcal{O} \times \overline{\mathcal{O}}$, $g(x, \overline{y}) = (gx, \theta(g)\overline{y})$, $\forall g \in G,\ x, y \in \mathcal{O}$. This action differs slightly from the doubled action, but Theorems 8.5–8.6 remain valid, together with the proofs. Now it follows from (8.3)–(8.4) that $\mathcal{O}$ is spherical iff

$$d_G(\mathcal{O} \times \overline{\mathcal{O}}) = \operatorname{rk} G - \operatorname{rk} G_*$$

where $G_* = K_*(\mathbb{C})$ is the stabilizer of general position for $G : \mathcal{O} \times \overline{\mathcal{O}}$. The latter condition coincides with (25.2). $\square$

26 Symmetric spaces

The concept of a Riemannian symmetric space was introduced by É. Cartan [Ca1], [Ca2]. A (globally) symmetric space is defined as a connected Riemannian manifold $\mathcal{O}$ such that for $\forall x \in \mathcal{O}$ there exists an isometry s_x of $\mathcal{O}$ inverting the geodesics passing through x. Symmetric spaces form a very important class of Riemannian spaces including all classical geometries. The theory of Riemannian symmetric spaces is well developed, see [Hel1].

In particular, it is easy to see that a symmetric space $\mathcal{O}$ is homogeneous w.r.t. the identity component G of the full isometry group, so that $\mathcal{O} = G/H$, where $H = G_o$ is the stabilizer of a fixed base point. The geodesic symmetry $s = s_o$ is an involutive automorphism of $\mathcal{O}$ normalizing G. It defines an involution $\sigma \in \operatorname{Aut} G$ by $\sigma(g) = sgs^{-1}$. From the definition of a geodesic symmetry one deduces that $(G^\sigma)^0 \subseteq H \subseteq G^\sigma$. Furthermore, reducing G to a smaller transitive isometry group if necessary, one may assume that $\mathfrak{g}$ is a reductive Lie algebra. This leads to the following algebraic definition of a symmetric space, which we accept in our treatment.

Definition 26.1. An (algebraic) *symmetric space* is a homogeneous algebraic variety $\mathcal{O} = G/H$, where G is a connected reductive group equipped with a non-identical involution $\sigma \in \operatorname{Aut} G$, and $(G^\sigma)^0 \subseteq H \subseteq G^\sigma$.

Riemannian symmetric spaces are locally isomorphic to real forms (with compact isotropy subgroups) of algebraic symmetric spaces over $\mathbb{C}$.

It is reasonable to impose a restriction $\operatorname{char} \Bbbk \neq 2$ on the ground field.

Remark 26.1. If G is semisimple simply connected, then G^σ is connected [St, 8.2], whence $H = G^\sigma$. On the other side, if G is adjoint, then $G^\sigma = N(H)$ [Vu2, 2.2].

The differential of σ, denoted by the same letter by abuse of notation, induces a $\mathbf{Z}_2$-grading

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{26.1}$$

where $\mathfrak{h}, \mathfrak{m}$ are the (± 1)-eigenspaces of σ.

The subgroup H is reductive [St, §8], thence $\mathcal{O}$ is an affine algebraic variety. More specifically, consider a morphism $\tau : G \to G$, $\tau(g) = \sigma(g)g^{-1}$. Observe that τ is the orbit map at e for the G-action on G by *twisted conjugation*: $g \circ x = \sigma(g)xg^{-1}$. It is not hard to prove the following result.

Proposition 26.1 ([Sp1, 2.2]). $\tau(G) \simeq G/G^\sigma$ *is a connected component of* $\{x \in G \mid \sigma(x) = x^{-1}\}$.

Example 26.1. Let $G = \mathrm{GL}_n(\Bbbk)$ and σ be defined by $\sigma(x) = (x^\top)^{-1}$. Then $G^\sigma = \mathrm{O}_n(\Bbbk)$ and $\tau(G) = \{x \in G \mid \sigma(x) = x^{-1}\}$ is the set of non-degenerate symmetric matrices, which is isomorphic to $\mathrm{GL}_n(\Bbbk)/\mathrm{O}_n(\Bbbk)$.

However, if σ is an inner involution, i.e., the conjugation by a matrix of order 2, then the set of matrices x such that $\sigma(x) = x^{-1}$ is disconnected. The connected components are determined by the collection of eigenvalues of x, which are ± 1.

The local and global structure of symmetric spaces is examined in [KR], [Hel1] (transcendental methods), [Vu1], [Vu2] ($\operatorname{char} \Bbbk = 0$), [Ri2], [Sp1]. We follow these sources in our exposition. The starting point is an analysis of σ-stable tori.

Lemma 26.1. *Every Borel subgroup $B \subseteq G$ contains a σ-stable maximal torus T.*

Proof. The group $B \cap \sigma(B)$ is connected, solvable, and σ-stable. By [St, 7.6] it contains a σ-stable maximal torus T, which is a maximal torus in G, too. $\square$

Corollary 26.1. *Every σ-stable torus $S \subseteq G$ is contained in a σ-stable maximal torus T.*

Proof. Put $T = Z \cdot T'$, where Z is the connected center and T' any σ-stable maximal torus in the commutator subgroup of $Z_G(S)$. $\square$

A σ-stable torus T decomposes into an almost direct product $T = T_0 \cdot T_1$, where $T_0 \subseteq H$ and T_1 is *σ-split*, which means that σ acts on T_1 as the inversion.

Let Δ denote the root system of G w.r.t. T and $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the root subspace corresponding to $\alpha \in \Delta$. One may choose root vectors $e_\alpha \in \mathfrak{g}_\alpha$ in such a way that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ form an $\mathfrak{sl}_2$-triple for $\forall \alpha \in \Delta$. Clearly, σ acts on $\mathfrak{X}(T)$ leaving Δ stable. Choosing e_α in a compatible way allows to subdivide all roots into *complex*, *real*, and *imaginary* (*compact* or *non-compact*) ones, according to Table 5.7.

We fix an inner product on $\mathfrak{X}(T) \otimes \mathbb{Q}$ invariant under the Weyl group $W = N_G(T)/T$ and σ. Then $\mathfrak{X}(T) \otimes \mathbb{Q}$ is identified with $\mathfrak{X}^*(T) \otimes \mathbb{Q}$ and with the orthogonal sum of $\mathfrak{X}(T_0) \otimes \mathbb{Q}$ and $\mathfrak{X}(T_1) \otimes \mathbb{Q}$. The coroots $\alpha^\vee \in \Delta^\vee$ ($\alpha \in \Delta$) are identified with $2\alpha/(\alpha, \alpha)$. Let $\langle \alpha | \beta \rangle = \langle \alpha^\vee, \beta \rangle = 2(\alpha, \beta)/(\alpha, \alpha)$ denote the Cartan pairing on $\mathfrak{X}(T)$ and $r_\alpha(\beta) = \beta - \langle \alpha | \beta \rangle \alpha$ the reflection of β along α.

Two opposite classes of σ-stable maximal tori are of particular importance in the theory of symmetric spaces.

Table 5.7: Root types w.r.t. an involution

α	complex	real	imaginary	
			compact	non-compact
$\sigma(\alpha)$	$\neq \pm\alpha$	$-\alpha$	α	α
$\sigma(e_\alpha)$	$e_{\sigma(\alpha)}$	$e_{-\alpha}$	e_α	$-e_\alpha$

Lemma 26.2. *If* $\dim T_0$ *is maximal possible, then* T_0 *is a maximal torus in* H *and* $Z_G(T_0) = T$. *Moreover,* T *is contained in a* σ*-stable Borel subgroup* $B \subseteq G$ *such that* $(B^\sigma)^0$ *is a Borel subgroup in* H.

Proof. If $Z_G(T_0) \neq T$, then the commutator subgroup $Z_G(T_0)'$ and $(Z_G(T_0)')^\sigma$ have positive dimension. Hence T_0 can be extended by a subtorus in $(Z_G(T_0)')^\sigma$, a contradiction. Now choose a Borel subgroup of H containing T_0 and extend it to a Borel subgroup B of G. Then $B \supseteq T$. If B were not σ-stable, then there would exist a root $\alpha \in \Delta^+$ such that $\sigma(\alpha) \in \Delta^-$. Then $e_{\pm\alpha} + \sigma(e_{\pm\alpha})$ are opposite root vectors in $\mathfrak{h}$ outside the Borel subalgebra $\mathfrak{b}^\sigma$, a contradiction. □

In particular, if T_0 is maximal, then there are no real roots, and σ preserves the set Δ^+ of positive roots (w.r.t. B) and induces a diagram involution $\overline{\sigma}$ of the set $\Pi \subseteq \Delta^+$ of simple roots. If G is of simply connected type, then $\overline{\sigma}$ extends to an automorphism of G so that $\sigma = \overline{\sigma} \cdot \sigma_0$, where σ_0 is an inner automorphism defined by an element of T_0.

Consider the set $\overline{\Delta} = \{\overline{\alpha} = \alpha|_{T_0} \mid \alpha \in \Delta\} \subset \mathfrak{X}(T_0)$. Clearly, $\overline{\Delta}$ consists of the roots of H w.r.t. T_0 and the nonzero weights of $T_0 : \mathfrak{m}$. The restrictions of complex roots belong to both subsets, the eigenvectors being $e_\alpha + \sigma(e_\alpha) \in \mathfrak{h}$, $e_\alpha - \sigma(e_\alpha) \in \mathfrak{m}$, whereas (non-)compact roots restrict to roots of H (resp. weights of $\mathfrak{m}$).

Lemma 26.3. $\overline{\Delta}$ *is a (possibly non-reduced) root system with base* $\overline{\Pi}$. *The simple roots of* H *and the (nonzero) lowest weights of* $H : \mathfrak{m}$ *form an affine simple root system* $\widetilde{\Pi}$, *i.e.,* $\langle\overline{\alpha}|\overline{\beta}\rangle \in \mathbb{Z}_-$ *for distinct* $\overline{\alpha}, \overline{\beta} \in \widetilde{\Pi}$.

Proof. Note that the restriction of $\alpha \in \Delta$ to T_0 is the orthogonal projection to $\mathfrak{X}(T_0)\otimes\mathbb{Q}$, so that $\overline{\alpha} = (\alpha+\sigma(\alpha))/2$. If α is complex, then $\langle\alpha|\sigma(\alpha)\rangle = 0$ or -1 (otherwise $\alpha - \sigma(\alpha)$ would be a real root), In the second case, $2\overline{\alpha} = \alpha + \sigma(\alpha)$ is a non-compact root with a root vector $e_{\alpha+\sigma(\alpha)} = [e_\alpha, \sigma(e_\alpha)]$.

A direct computation shows that $\forall\alpha, \beta \in \Delta : \langle\overline{\alpha}|\overline{\beta}\rangle \in \mathbb{Z}$ and the reflections $r_{\overline{\alpha}}$ preserve $\overline{\Delta}$, see Table 5.8. Hence $\overline{\Delta}$ is a root system. Restricting

Table 5.8: Cartan numbers and reflections for restricted roots

Case	$\langle\overline{\alpha}\vert\overline{\beta}\rangle$	$r_{\overline{\alpha}}(\overline{\beta})$
$\alpha=\sigma(\alpha)$	$\langle\alpha\vert\beta\rangle$	$\overline{r_\alpha(\beta)}$
$\langle\alpha\vert\sigma(\alpha)\rangle=0$	$\langle\alpha\vert\beta\rangle+\langle\sigma(\alpha)\vert\beta\rangle$	$\overline{r_\alpha r_{\sigma(\alpha)}(\beta)}$
$\langle\alpha\vert\sigma(\alpha)\rangle=-1$	$2\langle\alpha\vert\beta\rangle+2\langle\sigma(\alpha)\vert\beta\rangle$	$r_{2\overline{\alpha}}(\overline{\beta})=\overline{r_{\alpha+\sigma(\alpha)}(\beta)}$

to T_0 the expression of $\alpha\in\Delta$ as a linear combination of Π with integer coefficients of the same sign yields a similar expression of $\overline{\alpha}$ in terms of $\overline{\Pi}$. Since $\overline{\Pi}$ is linearly independent, it is a base of $\overline{\Delta}$.

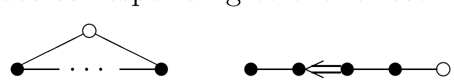
Note that $\overline{\alpha}=\overline{\beta}$ iff $\alpha=\beta$ or $\sigma(\alpha)=\beta$. (Otherwise $\alpha-\beta$ or $\sigma(\alpha)-\beta$ would be a real root, depending on whether $\langle\alpha|\beta\rangle>0$ or $\langle\sigma(\alpha)|\beta\rangle>0$.) Therefore the nonzero weights occur in $\mathfrak{m}$ with multiplicity 1.

To prove the second assertion, it suffices to consider the Cartan numbers $\langle\overline{\alpha}|\overline{\beta}\rangle$ of lowest weights of $\mathfrak{m}$. Assuming $\langle\overline{\alpha}|\overline{\beta}\rangle>0$ yields w.l.o.g. $\langle\alpha|\beta\rangle>0$, whence $\gamma=\alpha-\beta\in\Delta$, $e_\alpha=[e_\beta,e_\gamma]$. If β is non-compact, then $[e_\beta,e_\gamma+\sigma(e_\gamma)]=e_\alpha-\sigma(e_\alpha)$. If β is complex, then γ is complex, too. (Indeed, $\overline{\gamma}$ is not longer than $\overline{\alpha},\overline{\beta}$, i.e., is shorter than any root of Δ.) Then $\beta+\sigma(\gamma),\sigma(\beta)+\gamma\notin\Delta$, whence $[e_\beta-\sigma(e_\beta),e_\gamma+\sigma(e_\gamma)]=e_\alpha-\sigma(e_\alpha)$. In both cases, either $\overline{\beta}$ or $\overline{\alpha}$ is not a lowest weight, a contradiction. $\square$

Remark 26.2. If some of the Cartan numbers of Δ vanish in $\Bbbk$, then the previous arguments concerning lowest weights does not work. The assertion on $\widetilde{\Pi}$ is true only if one interprets lowest weights in the combinatorial sense as those weights of $\mathfrak{m}$ which cannot be obtained from other weights by adding simple roots of H. However this happens only for $G=\mathbf{G}_2$ (if $\operatorname{char}\Bbbk=3$), where the unique (up to conjugation) involution is easy to describe by hand.

In a usual way, the system $\widetilde{\Pi}$ together with the respective Cartan numbers is encoded by an (affine) Dynkin diagram. Marking the nodes corresponding to the simple roots of H by black, and those corresponding to the lowest weights of $\mathfrak{m}$ by white, one obtains the so-called *Kac diagram* of the involution σ, or of the symmetric space $\mathcal{O}$. From the Kac diagram one easily recovers $\mathfrak{h}$ and (at least in characteristic zero) the (co)isotropy representation $H^0:\mathfrak{m}$.

Example 26.2. Let H be diagonally embedded in $G=H\times H$, where σ permutes the factors. Here the Kac diagram is the affine Dynkin diagram of H with the white nodes corresponding to the lowest roots, e.g.:



Now consider an opposite class of σ-stable maximal tori.

Lemma 26.4. *There exist non-trivial σ-split tori.*

Proof. In the converse case σ acts identically on every σ-stable torus. Lemma 26.1 implies that all Borel subgroups are σ-stable. Then all maximal tori are σ-stable and even pointwise fixed, whence σ is identical. $\square$

Lemma 26.5. *If T_1 is a maximal σ-split torus, then $L = Z_G(T_1)$ decomposes into an almost direct product $L = L_0 \cdot T_1$, where $L_0 = L \cap H$.*

Proof. Clearly, L and the commutator subgroup L' are σ-stable. If $L' \not\subseteq H$, then T_1 could be extended by a non-trivial σ-split torus in L' by Lemma 26.4, a contradiction. The assertion follows from $L' \subseteq H$. $\square$

Choose a general one-parameter subgroup $\gamma \in \mathfrak{X}^*(T_1)$ and consider the associated parabolic subgroup $P = P(\gamma)$ with the Lie algebra $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\langle\alpha,\gamma\rangle\geq 0} \mathfrak{g}_\alpha$. Clearly, $L \subseteq P$ is a Levi subgroup and $\mathfrak{p}_\mathrm{u} = \bigoplus_{\langle\alpha,\gamma\rangle>0} \mathfrak{g}_\alpha$. Note that $\sigma(P) = P^-$ (since $\langle\sigma(\alpha),\gamma\rangle = -\langle\alpha,\gamma\rangle$, $\forall\alpha \in \Delta$). In fact, all minimal parabolics having this property are obtained as above [Vu1, 1.2]. It follows that $\mathfrak{h}$ is spanned by $\mathfrak{l}_0$ and $e_\alpha + \sigma(e_\alpha)$ over all $\alpha \in \Delta$ such that $\langle\alpha,\gamma\rangle \geq 0$. This yields:

Iwasawa decomposition. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}_1 \oplus \mathfrak{p}_\mathrm{u}$

As a consequence, we obtain

Theorem 26.1. *Symmetric spaces are spherical.*

Indeed, choosing a Borel subgroup $B \subseteq P$, $B \supseteq T$ yields (S5). There are many other ways to verify this fact. For instance, it is easy to verify the Richardson condition (Ad3): $\forall\xi \in \mathfrak{m} \simeq \mathfrak{h}^\perp$ one has $[\mathfrak{g},\xi] \cap \mathfrak{m} = [\mathfrak{h},\xi]$ for $[\mathfrak{m},\xi] \subseteq \mathfrak{h}$. One can also check the Gelfand condition (WS2) for elements in a dense subset $\tau(G)H \subseteq G$: $g = xh,\ x \in \tau(G),\ h \in \mathfrak{h} \implies \sigma(g) = x^{-1}h = hg^{-1}h$. The multiplicity free property (for compact Riemannian symmetric spaces and unitary representations) was established already by É. Cartan [Ca3, n°17].

The Iwasawa decomposition clarifies the local structure of a symmetric space. Namely, $\mathcal{O}$ contains a dense orbit $P \cdot o \simeq P/L_0 \simeq P_\mathrm{u} \times A$, where $A = T/T \cap H$ is the quotient of T_1 by an elementary Abelian 2-group $T_1 \cap H$. We have $\mathfrak{a} \simeq \mathfrak{t}_1$, $\Lambda(\mathcal{O}) = \mathfrak{X}(A)$, $r(\mathcal{O}) = \dim\mathfrak{a}$. The notation here agrees with Corollary 4.1 and §7.

Lemma 26.6. *All maximal σ-split tori are H^0-conjugate.*

Proof. In the above notation, PH is open in G, whence the H^0-orbit of P is open in G/P. Since P coincides with the normalizer of the open B-orbit in $\mathcal{O}$, all such parabolics are G-conjugate and therefore H^0-conjugate. Hence the Levi subgroups $L = P \cap \sigma(P)$ and finally the maximal σ-split tori $T_1 = (Z(L)^0)_1$ are H^0-conjugate. □

If T_1 is maximal, then every imaginary root is compact and σ maps positive complex or real roots to negative ones. Compact (simple) roots form (the base of) the root system of L.

The endomorphism $\iota = -w_L\sigma$ of $\mathfrak{X}(T)$ preserves Δ^+ and induces a diagram involution of the set Π of simple roots. (Here w_L is the longest element in the Weyl group of L.) Since $w_G w_L \sigma$ preserves Δ^+ and differs from σ by an inner automorphism, it coincides with the diagram automorphism $\overline{\sigma}$, whence $\iota(\lambda) = \overline{\sigma}(\lambda)^*$, $\forall \lambda \in \mathfrak{X}(T)$.

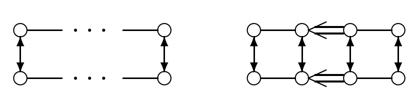
Consider the set $\Delta_{\mathcal{O}} \subset \mathfrak{X}(T_1)$ and the subset $\Pi_{\mathcal{O}} \subset \Delta_{\mathcal{O}}$ consisting of the restrictions $\overline{\alpha} = \alpha|_{T_1}$ of complex and real roots $\alpha \in \Delta$ (resp. $\alpha \in \Pi$) to T_1.

Lemma 26.7. *$\Delta_{\mathcal{O}}$ is a (possibly non-reduced) root system with base $\Pi_{\mathcal{O}}$, called the* (little) root system *of the symmetric space $\mathcal{O}$.*

Proof. The proof is similar to that of Lemma 26.3. The restriction of $\alpha \in \Delta$ to T_1 is identified with the orthogonal projection to $\mathfrak{X}(T_1) \otimes \mathbb{Q}$ given by $\overline{\alpha} = (\alpha - \sigma(\alpha))/2$. We have $\alpha + \sigma(\alpha) \notin \Delta$, $\forall \alpha \in \Delta$. (Otherwise $\alpha + \sigma(\alpha)$ would be a non-compact root.) The involution ι coincides with $-\sigma$ modulo the root lattice of L. One easily deduces that $\overline{\alpha} = \overline{\beta}$ iff $\alpha = \beta$ or $\iota(\alpha) = \beta$ for $\forall \alpha, \beta \in \Pi$ and that $\Pi_{\mathcal{O}}$ is linearly independent. Taking these remarks into account, the proof repeats that of Lemma 26.3 with σ replaced by $-\sigma$. □

The Dynkin diagram of Π with the "compact" nodes marked by black and the remaining nodes by white, where the white nodes transposed by ι are joined by two-headed arrows, is called the *Satake diagram* of the involution σ, or of the symmetric space $\mathcal{O}$. The Satake diagram encodes the embedding of $\mathfrak{h}$ into $\mathfrak{g}$. Besides, it contains information on the weight lattice (semigroup) of the symmetric space (see Propositions 26.4, 26.5).

Example 26.3. The Satake diagram of the symmetric space $\mathcal{O} = H \times H / \operatorname{diag} H$ of Example 26.2 consists of two Dynkin diagrams of H, so that all nodes are white and each node of the 1-st diagram is joined with the respective node of the 2-nd diagram, e.g.:

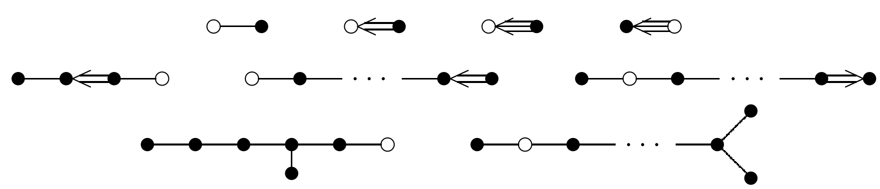


The classification of symmetric spaces goes back to Cartan. To describe it, first note that σ preserves the connected center and either preserves or transposes the simple factors of G. Hence every symmetric space is locally isomorphic to a product of a torus $Z/Z\cap H$, of symmetric spaces $H \times H/\operatorname{diag} H$ with H simple, and of symmetric spaces of simple groups.

Thus the classification reduces to simple G. It can be obtained using either Kac diagrams [Hel1, X.5], [GOV, Ch.3, 3.6–3.11] or Satake diagrams [Sp2], [GOV, Ch.4, 4.1–4.3]. For simple G both Kac and Satake diagrams are connected.

Further analysis shows that the underlying affine Dynkin diagram for the Kac diagram of σ depends only on the diagram involution $\overline{\sigma}$. This diagram is easily recovered form the Dynkin diagram of Π and from $\overline{\sigma}$ using Table 5.8. Since the weight system of $T_0 : \mathfrak{m}$ is symmetric, for each "white" root $\overline{\alpha} \in \widetilde{\Pi}$ there exists a "white" root $\overline{\alpha}_0$ and "black" roots $\overline{\alpha}_1, \dots, \overline{\alpha}_r$ such that $-\overline{\alpha} = \overline{\alpha}_0 + \overline{\alpha}_1 + \cdots + \overline{\alpha}_r$. As $\widetilde{\Pi}$ is bound by a unique linear dependence, the coefficients being positive integers, there exists either unique "white" root, with the coefficient 1 or 2, or exactly two "white" roots, with the coefficients 1. The first possibility occurs exactly for outer involutions, because in this case the weight system contains the zero weight, while the other two possibilities correspond to inner involutions. Using these observations, it is easy to write down all possible Kac diagrams, see Table 5.9.

On the other hand, all apriori possible Satake diagrams can also be classified. One verifies that a Satake diagram cannot be one of the following:



In all cases except the last two, the sum of all simple roots would be a complex root α such that $\alpha + \sigma(\alpha) \in \Delta$, a contradiction. In the remaining two cases, σ would be an inner involution represented by an element $s \in S = Z_G(L_0)^0$. The group S is a simple SL_2-subgroup corresponding to the highest root $\delta \in \Delta$ and T_1 is a maximal torus in S. Replacing T_1 by another maximal torus containing s one obtains $\delta(s) = -1$. However the unique $\alpha \in \Pi$ such that $\alpha(s) = -1$ occurs in the decomposition of δ with an even coefficient, a contradiction.

By a *fragment* of a Satake diagram we mean a ι-stable subdiagram such that no one of its nodes is joined with a black node outside the fragment. A fragment is the Satake diagram of a Levi subgroup in G. It follows that a Satake diagram cannot contain the above listed fragments. Also, if a Satake diagram contains a fragment $\bullet\!\!-\!\cdots\!-\!\!\bullet$ of length > 1, then there are no other black nodes and ι is non-trivial. Having this in mind, it is easy to write down all possible Satake diagrams, see Table 5.9.

Both Kac and Satake diagrams uniquely determine the involution σ. All apriori possible diagrams are realized for simply connected G. It follows that symmetric spaces of simple groups are classified, up to a local isomorphism, by Kac or Satake diagrams.

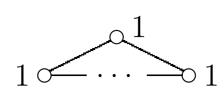
The classification is presented in Table 5.9. The column "σ" describes the involution for classical G in matrix terms. Here

$$I_{n,m} = \begin{pmatrix} -E_m & 0 \\ 0 & E_{n-m} \end{pmatrix}, \quad K_{n,m} = \begin{pmatrix} I_{n,m} & 0 \\ 0 & I_{n,m} \end{pmatrix}, \quad \text{and } \Omega_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

is the matrix of a standard symplectic form fixed by $\mathrm{Sp}_{2n}(\Bbbk)$, where E_k is the unit $(k \times k)$-matrix.

Example 26.4. Let us describe the symmetric spaces of $G = \mathrm{SL}_n(\Bbbk)$. Take the standard Borel subgroup of upper-triangular matrices $B \subset G$ and the standard diagonal torus $T \subset B$. By $\varepsilon_1, \dots, \varepsilon_n$ denote the weights of the tautological representation in $\Bbbk^n$ (i.e., the diagonal entries of T).

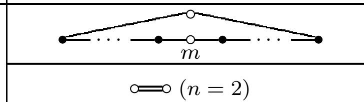
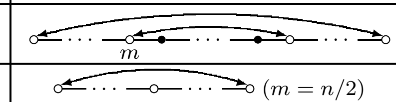
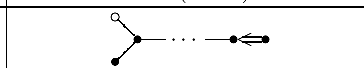
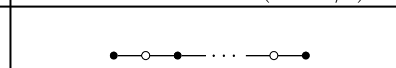
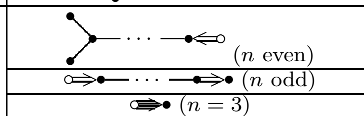
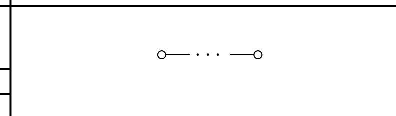
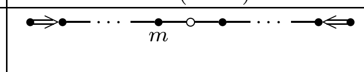
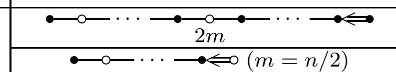
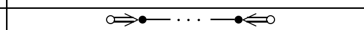
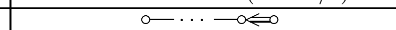
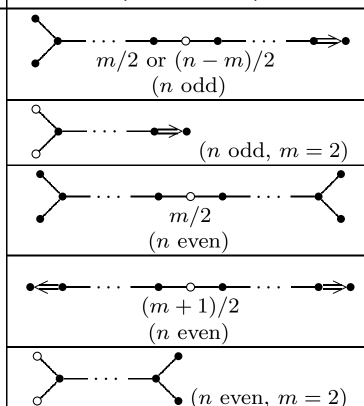
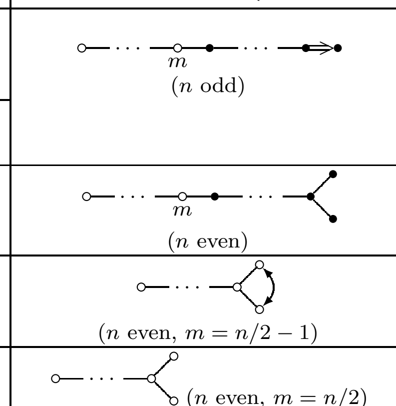
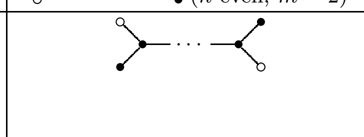
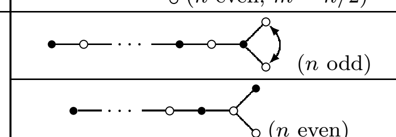
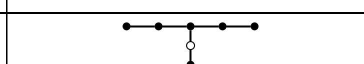
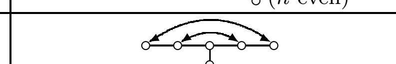
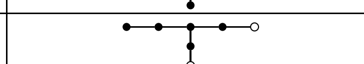
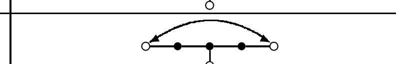
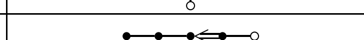
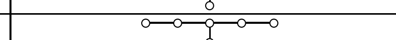
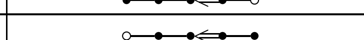
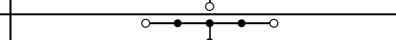
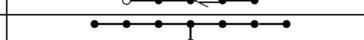
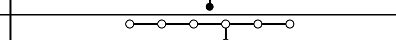
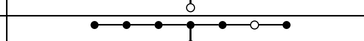
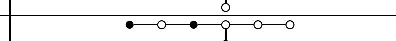
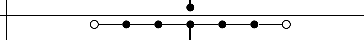
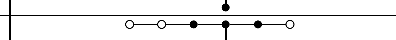
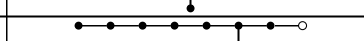
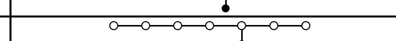
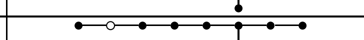
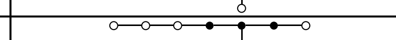
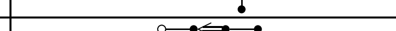
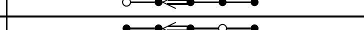
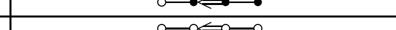
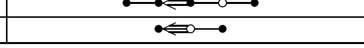
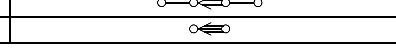
If σ is inner, then $\overline{\Delta} = \Delta$ and the Dynkin diagram of $\widetilde{\Pi}$ is the following one:



The coefficients of the unique linear dependence on $\widetilde{\Pi}$ are indicated at the diagram. It follows that there are exactly two white nodes in the Kac diagram. The involution ι is non-trivial, whence there is at most one black fragment in the Satake diagram, which is located in the middle. Thus we obtain No. 1 of Table 5.9.

The involution σ is the conjugation by an element of order 2 in $\mathrm{GL}_n(\Bbbk)$. In a certain basis, $\sigma(g) = I_{n,m} \cdot g \cdot I_{n,m}$. Then $T_0 = T$, $H = \mathrm{S}(\mathrm{L}_m \times \mathrm{L}_{n-m})$ is embedded in G by the two diagonal blocks, the simple roots being $\varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $i \ne m$, and $\mathfrak{m} = \Bbbk^m \otimes (\Bbbk^{n-m})^* \oplus (\Bbbk^m)^* \otimes \Bbbk^{n-m}$ is embedded in $\mathfrak{g}$ by the two antidiagonal blocks, the lowest weights of the summands being $\varepsilon_m - \varepsilon_{m+1}$, $\varepsilon_n - \varepsilon_1$, in accordance with the Kac diagram.

Table 5.9: Symmetric spaces of simple groups

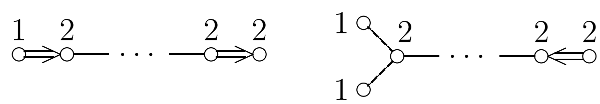
No.	G	H	σ	Kac diagram	Satake diagram	$\Delta_{G/H}$
1	SL_n	$\mathrm{S}(\mathrm{L}_m \times \mathrm{L}_{n-m})$ $(m \le n/2)$	$g \mapsto I_{n,m} g I_{n,m}$	 m $(n=2)$	 m $(m = n/2)$	$\mathbf{BC}_m$ $\mathbf{C}_{n/2}$
2	SL_{2n}	Sp_{2n}	$g \mapsto \Omega_n (g^\top)^{-1} \Omega_n^\top$			$\mathbf{A}_{n-1}$
3	SL_n	SO_n	$g \mapsto (g^\top)^{-1}$	 (n even) (n odd) $(n=3)$		$\mathbf{A}_{n-1}$
4	Sp_{2n}	$\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2(n-m)}$ $(m \le n/2)$	$g \mapsto K_{n,m} g K_{n,m}$	 m	 $2m$ $(m = n/2)$	$\mathbf{BC}_m$ $\mathbf{C}_{n/2}$
5	Sp_{2n}	GL_n	$g \mapsto I_{2n,n} g I_{2n,n}$			$\mathbf{C}_n$
6	SO_n	$\mathrm{SO}_m \times \mathrm{SO}_{n-m}$ $(m \le n/2)$	$g \mapsto I_{n,m} g I_{n,m}$	 $m/2$ or $(n-m)/2$ (n odd) (n odd, $m=2$) $m/2$ (n even) $(m+1)/2$ (n even) (n even, $m=2$)	 m (n odd) m (n even) (n even, $m = n/2-1$) (n even, $m = n/2$)	$\mathbf{B}_m$ $\mathbf{B}_m$ $\mathbf{B}_{n/2-1}$ $\mathbf{D}_{n/2}$
7	SO_{2n}	GL_n	$g \mapsto \Omega_n g \Omega_n^\top$		 (n odd) (n even)	$\mathbf{BC}_{[n/2]}$ $\mathbf{C}_{n/2}$
8	$\mathbf{E}_6$	$\mathbf{A}_5 \times \mathbf{A}_1$				$\mathbf{F}_4$
9	$\mathbf{E}_6$	$\mathbf{D}_5 \times \Bbbk^\times$				$\mathbf{BC}_2$
10	$\mathbf{E}_6$	$\mathbf{C}_4$				$\mathbf{E}_6$
11	$\mathbf{E}_6$	$\mathbf{F}_4$				$\mathbf{A}_2$
12	$\mathbf{E}_7$	$\mathbf{A}_7$				$\mathbf{E}_7$
13	$\mathbf{E}_7$	$\mathbf{D}_6 \times \mathbf{A}_1$				$\mathbf{F}_4$
14	$\mathbf{E}_7$	$\mathbf{E}_6 \times \Bbbk^\times$				$\mathbf{C}_3$
15	$\mathbf{E}_8$	$\mathbf{D}_8$				$\mathbf{E}_8$
16	$\mathbf{E}_8$	$\mathbf{E}_7 \times \mathbf{A}_1$				$\mathbf{F}_4$
17	$\mathbf{F}_4$	$\mathbf{B}_4$				$\mathbf{BC}_1$
18	$\mathbf{F}_4$	$\mathbf{C}_3 \times \mathbf{A}_1$				$\mathbf{F}_4$
19	$\mathbf{G}_2$	$\mathbf{A}_1 \times \mathbf{A}_1$				$\mathbf{G}_2$

In another basis, $\sigma(g) = J_{n,m} \cdot g \cdot J_{n,m}$, where

$$J_{n,m} = \begin{array}{c} \\ m\left\{\vphantom{\begin{matrix}0\\1\end{matrix}}\right. \end{array}\begin{pmatrix} 0 & & \overbrace{\begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{matrix}}^{m} \\ & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \\ \underbrace{\begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{matrix}}_{m} & & 0 \end{pmatrix}\begin{array}{c} \left.\vphantom{\begin{matrix}0\\1\end{matrix}}\right\}m \\ \\ \end{array}$$

Now $T_1 = \{t = \operatorname{diag}(t_1, \dots, t_m, 1, \dots, 1, t_m^{-1}, \dots, t_1^{-1})\}$ is a maximal σ-split torus and the (compact) imaginary roots are $\varepsilon_i - \varepsilon_j$, $m < i \neq j \leq n - m$, in accordance with the Satake diagram. The little root system $\Delta_{\mathcal{O}}$ consists of the nonzero restrictions $\overline{\varepsilon}_i - \overline{\varepsilon}_j$, $1 \leq i, j \leq n$, i.e., of $\pm\overline{\varepsilon}_i \pm \overline{\varepsilon}_j$, $\pm 2\overline{\varepsilon}_i$, and $\pm\overline{\varepsilon}_i$ unless $m = n/2$, $1 \leq i \neq j \leq m$. Thus $\Delta_{\mathcal{O}}$ is of type $\mathbf{BC}_m$ or $\mathbf{C}_{n/2}$.

If σ is outer, then $\overline{\sigma}(\varepsilon_i) = -\varepsilon_{n+1-i}$ and $(T^{\overline{\sigma}})^0 = \{t = \operatorname{diag}(t_1, t_2, \dots, t_2^{-1}, t_1^{-1})\}$. Restricting the roots to this subtorus, we see that $\overline{\Delta}$ consists of $\pm\varepsilon_i' \pm \varepsilon_j'$, $\pm 2\varepsilon_i'$, and $\pm\varepsilon_i'$ for odd n, where ε_i' are the restrictions of ε_i, $1 \leq i \leq n/2$. The Dynkin diagram of $\widetilde{\Pi}$ has one of the following forms:



depending on whether n is odd or even. Therefore the Kac diagram has a unique white node, namely an extreme one.

The involution ι is trivial, whence either all nodes of the Satake diagram are white or the black nodes are isolated from each other and alternate with the white ones, the extreme nodes being black. (Otherwise, there would exist an inadmissible fragment $\circ\!\!-\!\!\!-\!\!\bullet$.) Thus we obtain Nos. 2–3 of Table 5.9.

Any outer involution has the form $\sigma(g) = (g^*)^{-1}$, where $*$ denotes the conjugation w.r.t. a non-degenerate (skew-)symmetric bilinear form on $\Bbbk^n$. In the symmetric case, choosing an orthonormal basis yields $\sigma(g) = (g^\top)^{-1}$, whence $T_1 = T$ is a maximal σ-split torus and $\Delta_{\mathcal{O}} = \Delta$. In a hyperbolic basis, $\sigma(g) = (g^\dagger)^{-1}$, where $\dagger$ denotes the transposition w.r.t. the secondary diagonal. Then $T_0 = (T^{\overline{\sigma}})^0$ is a maximal torus in $H = \mathrm{SO}_n(\Bbbk)$. The space $\mathfrak{m}$ consists of traceless symmetric matrices, and the lowest weight is $-2\varepsilon_1'$.

In the skew-symmetric case, choosing an appropriately ordered symplectic basis yields $\sigma(g) = I_{n,n/2}(g^\dagger)^{-1} I_{n,n/2}$. Here T_0 is a maximal torus in $H = \mathrm{Sp}_n(\Bbbk)$ and $T_1 = \{t = \operatorname{diag}(t_1, t_2, \dots, t_2, t_1) \mid t_1 \dots, t_{n/2} = 1\}$ is a maximal

σ-split torus. The roots of H are $\pm\varepsilon'_i \pm \varepsilon'_j$, $\pm 2\varepsilon'_i$, $1 \le i \ne j \le n/2$, and the lowest weight of $\mathfrak{m}$ is $-\varepsilon'_1 - \varepsilon'_2$. The compact roots are $\varepsilon_i - \varepsilon_{n+1-i}$ $(1 \le i \le n)$, and $\Delta_{\mathcal{O}}$ consists of $\overline{\varepsilon}_i - \overline{\varepsilon}_j$, $1 \le i \ne j \le n/2$, thus having the type $\mathbf{A}_{n/2-1}$.

From now on we assume that T_1 is a maximal σ-split torus.

Consider the Weyl group $W_{\mathcal{O}}$ of the little root system $\Delta_{\mathcal{O}}$.

Proposition 26.2. $W_{\mathcal{O}} \simeq N_{H^0}(T_1)/Z_{H^0}(T_1) \simeq N_G(T_1)/Z_G(T_1)$

Proof. First we prove that each element of $W_{\mathcal{O}}$ is induced by an element of $N_{H^0}(T_1)$. It suffices to consider a root reflection $r_{\overline{\alpha}}$. Let $T_1^{\overline{\alpha}} \subseteq T_1$ be the connected kernel of $\overline{\alpha}$. Replacing G by $Z_G(T_1^{\overline{\alpha}})$ we may assume that $W_{\mathcal{O}} = \{e, r_{\overline{\alpha}}\}$. The same argument as in Lemma 26.6 shows that $P^- = \sigma(P) = hPh^{-1}$ for some $h \in H^0$. It follows that $h \in N_{H^0}(L) = N_{H^0}(T_1)$ acts on $\mathfrak{X}(T_1)$ as $r_{\overline{\alpha}}$.

On the other hand, $N_G(T_1)$ acts on T_1 as a subgroup of the "big" Weyl group $W = N_G(T)/T$. Indeed, any $g \in N_G(T_1)$ normalizes $L = Z_G(T_1)$ and may be replaced by another element in gL normalizing T. Since the Weyl chambers of $W_{\mathcal{O}}$ in $\mathfrak{X}(T_1) \otimes \mathbb{Q}$ are the intersections of Weyl chambers of W with $\mathfrak{X}(T_1) \otimes \mathbb{Q}$, the orbits of $N_G(T_1)/Z_G(T_1)$ intersect them in single points. Thus $N_G(T_1)/Z_G(T_1)$ cannot be bigger than $W_{\mathcal{O}}$. This concludes the proof. ☐

Since $\mathcal{O}$ is spherical, there are finitely many B-orbits in $\mathcal{O}$ (Corollary 6.1). Their structure plays an important role in some geometric problems and, for $\Bbbk = \mathbb{C}$, in the representation theory of the real reductive Lie group $G(\mathbb{R})$ acting on the Riemannian symmetric space $\mathcal{O}(\mathbb{R})$, the non-compact real form of $\mathcal{O}$ [Vog]. The classification and the adherence relation for B-orbits were described in [Sp1], [RS1], [RS2] (cf. Example 6.2). We explain the basic classification result under the assumption $H = G^\sigma$. This is not an essential restriction [RS2, 1.1(b)].

By Proposition 26.1, $\mathcal{O}$ is identified with $\tau(G)$, where G (and B) acts by twisted conjugation.

Proposition 26.3. *The (twisted) B-orbits in $\tau(G) \simeq \mathcal{O}$ intersect $N_G(T)$ in T-orbits. Thus $\mathfrak{B}(\mathcal{O})$ is in bijective correspondence with the set of twisted T-orbits in $N(T) \cap \tau(G)$.*

Proof. Consider a B-orbit $Bgo \subseteq \mathcal{O}$. By Lemma 26.1, replacing g by bg, $b \in B$, one may assume that $g^{-1}Tg$ is a σ-stable maximal torus in $g^{-1}Bg$. This holds iff $\tau(g) \in N(T)$. One the other hand, taking another point $g'o \in Bgo$, $g' = bgh$, $b \in B$, $h \in H$, we have $\tau(g') = \sigma(b)\tau(g)b^{-1} \in N(T)$ iff $\tau(g') = \sigma(t)\tau(g)t^{-1}$, where $b = tu$, $t \in T$, $u \in U$, by standard properties of the Bruhat decomposition [Hum, 28.4]. ☐

There is a natural map $\mathfrak{B}(\mathcal{O}) \to W$, $Bgo \mapsto w$, where $\sigma(B)wB$ is the unique Bruhat cell containing the respective B-orbit $\tau(BgH)$. By Proposition 26.3, $\tau(BgH) \cap N(T) \subseteq wT$. This map plays an important role in the study of B-orbits [RS1], [RS2]. Its image is contained in the set of *twisted involutions* $\{w \in W \mid \sigma(w) = w^{-1}\}$, but in general is neither injective nor surjective onto this set.

Example 26.5. Let $G = \mathrm{GL}_n(\Bbbk)$, $\sigma(g) = (g^\top)^{-1}$, $H = \mathrm{O}_n(\Bbbk)$. Then $\tau(G)$ is the set of non-degenerate symmetric matrices, viewed as quadratic forms on $\Bbbk^n$. The group B of upper-triangular matrices acts on $\tau(G)$ by base changes preserving the standard flag in $\Bbbk^n$. It is an easy exercise in linear algebra that for any inner product on $\Bbbk^n$ one can choose a basis $e_1, \dots, e_n$ compatible with a given flag and having the property that for any i there is a unique j such that $(e_i, e_j) = 1$ and $(e_i, e_k) = 0$, $\forall k \neq j$. The matrix of the quadratic form in this basis is the permutation matrix of the involution transposing i and j. It lies in $N(T)$ (where T is the diagonal torus) and is uniquely determined by the B-orbit of the quadratic form. Thus $\mathfrak{B}(\mathcal{O})$ is in bijection with the set of involutions in $W = \mathrm{S}_n$.

Now we describe the colored equipment of a symmetric space, according to [Vu2].

The weight lattice of a symmetric space is read off the Satake diagram, at least up to an finite extension. Let $Z = Z(G)^0$ and ω_i be the fundamental weights corresponding to the simple roots $\alpha_i \in \Pi$.

Proposition 26.4. *If G is of simply connected type, then*

$$\Lambda(\mathcal{O}) = \mathfrak{X}(Z/Z \cap H) \oplus \left\langle\, \widehat{\omega}_j,\ \omega_k + \omega_{\iota(k)} \,\middle|\, j, k \,\right\rangle \tag{26.2}$$

where j, k run over all ι-fixed, resp. ι-unstable, white nodes of the Satake diagram, and $\widehat{\omega}_j = \omega_j$ or $2\omega_j$, depending on whether the j-th node is adjacent to a black one or not. In the general case, $\Lambda(\mathcal{O})$ is a sublattice of finite index in the r.h.s. of (26.2).

Remark 26.3. The weight lattice $\Lambda(\mathcal{O}) = \mathfrak{X}(T/T \cap H) = \mathfrak{X}(T_1/T_1 \cap H)$ injects into $\mathfrak{X}(T_1)$ via restriction of characters from T to T_1. The space $\mathcal{E} = \mathrm{Hom}(\Lambda(\mathcal{O}), \mathbb{Q})$ is then identified with $\mathfrak{X}^*(T_1) \otimes \mathbb{Q}$. The second direct summand in the r.h.s. of (26.2) is nothing else but the doubled weight lattice $2(\mathbb{Z}\Delta_{\mathcal{O}}^\vee)^*$ of the little root system $\Delta_{\mathcal{O}}$. Indeed, $\widehat{\omega}_j/2$ and $(\omega_k + \omega_{\iota(k)})/2$ restrict to the fundamental weights dual to the simple coroots $\overline{\alpha}_j^\vee = \alpha_j^\vee - \sigma(\alpha_j^\vee)$ or $\alpha_j^\vee$ and $\overline{\alpha}_k^\vee = \alpha_k^\vee - \sigma(\alpha_k^\vee)$.

Proof. W.l.o.g. we may assume that G is semisimple simply connected, whence $H = G^\sigma$. The sublattice $\Lambda(\mathcal{O}) \subseteq \mathfrak{X}(T)$ consisting of the weights vanishing on T^σ, i.e., of $\mu - \sigma(\mu)$, $\mu \in \mathfrak{X}(T)$, is contained in $\mathfrak{X}(T/T_0) = \{\lambda \in \mathfrak{X}(T) \mid \sigma(\lambda) = -\lambda\} = \langle \omega_j,\ \omega_k + \omega_{\iota(k)} \mid j, k\rangle$. The latter lattice injects into $\mathfrak{X}(T_1)$ so that $\Lambda(\mathcal{O})$ is identified with $2\mathfrak{X}(T_1)$. It remains to prove that $\mathfrak{X}(T_1) = (\mathbb{Z}\Delta_{\mathcal{O}}^\vee)^*$ or, equivalently, that $\mathfrak{X}^*(T_1) = \mathbb{Z}\Delta_{\mathcal{O}}^\vee$ is the coroot lattice of the little root system.

We have $\mathfrak{X}^*(T) = \mathbb{Z}\Delta^\vee$ and $\mathfrak{X}^*(T_1) = \mathbb{Z}\Delta^\vee \cap \mathcal{E} \supseteq \mathbb{Z}\Delta_{\mathcal{O}}^\vee$. The *alcoves* (=fundamental polyhedra, see [Bour, IV, §2]) of the affine Weyl group $W_{\mathrm{aff}}(\Delta_{\mathcal{O}}^\vee)$ are the intersections of $\mathcal{E}$ with alcoves of $W_{\mathrm{aff}}(\Delta_{\mathcal{O}}^\vee)$. Hence each alcove of $W_{\mathrm{aff}}(\Delta_{\mathcal{O}}^\vee)$ contains a unique point from $\mathfrak{X}^*(T_1)$. It follows that $\mathfrak{X}^*(T_1)$ coincides with $\mathbb{Z}\Delta_{\mathcal{O}}^\vee$. □

Let $\mathbf{C} = \mathbf{C}(\Delta^+)$ denote the dominant Weyl chamber of a root system Δ (w.r.t. a chosen subset of positive roots Δ^+). The weight semigroup $\Lambda_+(\mathcal{O})$ is contained both in $\Lambda(\mathcal{O})$ and in $\mathbf{C}(\Delta^+)$. Note that $\mathbf{C}(\Delta^+) \cap \mathcal{E} = \mathbf{C}(\Delta_{\mathcal{O}}^+)$.

Proposition 26.5. $\Lambda_+(\mathcal{O}) = \Lambda(\mathcal{O}) \cap \mathbf{C}(\Delta_{\mathcal{O}}^+)$

Proof. Since $\Lambda_+(\mathcal{O})$ is the semigroup of all lattice points in a cone (see §15), it suffices to prove that $\mathbb{Q}_+\Lambda_+(\mathcal{O}) = \mathbf{C}(\Delta_{\mathcal{O}}^+)$. Take any dominant $\lambda \in \Lambda(\mathcal{O})$. We prove that $2\lambda \in \Lambda_+(\mathcal{O})$.

First note that $\lambda = -\sigma(\lambda)$ is orthogonal to compact roots, whence λ is extended to P and $V^*(\lambda) = \operatorname{Ind}_P^G \Bbbk_{-\lambda}$. Consider another dual Weyl module obtained by twisting the G-action by σ: $V^*(\lambda)^\sigma = \operatorname{Ind}_{\sigma(P)}^G \Bbbk_{-\sigma(\lambda)} \simeq V^*(\lambda^*)$. We have the canonical H-equivariant linear isomorphism $\omega : V^*(\lambda)^\sigma \xrightarrow{\sim} V^*(\lambda)$. (If the dual Weyl modules are realized in $\Bbbk[G]$ as in Example 2.3, then ω is just the restriction of σ acting on $\Bbbk[G]$.) In other words, $\omega \in (V^*(\lambda) \otimes V(\lambda^*))^H$. Note that ω maps a T-eigenvector of weight μ to an eigenvector of weight $\sigma(\mu)$. Hence

$$\omega = v_{-\lambda} \otimes v'_{-\lambda} + \sum_{\mu \neq \lambda} v_{\sigma(\mu)} \otimes v'_{-\mu} \tag{26.3}$$

where v_χ, v'_χ denote basic eigenvectors of weight χ in $V^*(\lambda)$ and $V(\lambda^*)$, respectively. Applying the homomorphisms $V(\lambda^*) \to V^*(\lambda)$, $v'_{-\lambda} \mapsto v_{-\lambda}$, and $V^*(\lambda) \otimes V^*(\lambda) \to V^*(2\lambda)$ (induced by multiplication in $\Bbbk[G]$), we obtain a nonzero element $\overline{\omega} \in V^*(2\lambda)^H$, whence $2\lambda \in \Lambda_+(\mathcal{O})$ by (2.2). □

Now we are ready to describe the colors and G-valuations of a symmetric space.

Theorem 26.2. *The B-divisors of a symmetric space $\mathcal{O}$ are represented by the vectors from $\frac{1}{2}\Pi_{\mathcal{O}}^{\vee} \subset \mathcal{E}$ (where $\Pi_{\mathcal{O}}^{\vee}$ is the base of $\Delta_{\mathcal{O}}^{\vee} \subset \mathfrak{X}^*(T_1)$). The valuation cone $\mathcal{V}$ is the antidominant Weyl chamber of $\Delta_{\mathcal{O}}^{\vee}$ in $\mathcal{E}$.*

Corollary 26.2. *$W_{\mathcal{O}}$ is the little Weyl group of $\mathcal{O}$ in the sense of §22.*

Proof. W.l.o.g. G is assumed to be of simply connected type. In the notation of Remarks 13.2, 15.1, each $f \in \Bbbk[\mathcal{O}]_{\lambda}^{(B)}$ is represented as $f = \eta_1^{d_1} \dots \eta_s^{d_s}$, where the η_i are equations of the B-divisors $D_i \in \mathcal{D}^B$, $d_i \in \mathbb{Z}_+$, and $\lambda = \sum d_i\lambda_i$, $\sum d_i\chi_i = 0$, where (λ_i, χ_i) are the biweights of η_i.

In the notation of Proposition 26.4, if $\lambda = \widehat{\omega}_j$ or $\omega_k + \omega_{\iota(k)}$, then $f = \eta_j$, or $\eta_j'\eta_j''$, or η_k, or $\eta_k'\eta_k''$, where the biweights of $\eta_j, \eta_j', \eta_j'', \eta_k, \eta_k', \eta_k''$ are $(\widehat{\omega}_j, 0)$, (ω_j, χ_j), $(\omega_j, -\chi_j)$, $(\omega_k + \omega_{\iota(k)}, 0)$, (ω_k, χ_k), $(\omega_{\iota(k)}, -\chi_k)$, respectively, for some nonzero $\chi_j, \chi_k \in \mathfrak{X}(H)$. In particular, the respective B-divisors $D_j, D_j', D_j'', D_k, D_k', D_k''$ are pairwise distinct, and all B-divisors occur among them since these f's span the multiplicative semigroup $\Bbbk[\mathcal{O}]^{(B)}/\Bbbk[\mathcal{O}]^{\times}$ by Proposition 26.5 and Remark 26.3. The assertion on colors stems now from Remarks 15.1 and 26.3.

Now we treat G-valuations. Take any $v = v_D \in \mathcal{V}$, where D is a G-stable prime divisor on a G-model X of $\Bbbk(\mathcal{O})$. It follows from the local structure theorem that $F = \overline{T_1 o}$ is an $N_H(T_1)$-stable subvariety of X intersecting D in the union of T_1-stable prime divisors D_{wv}, $w \in W_{\mathcal{O}}$, that correspond to wv regarded as T_1-valuations of $\Bbbk(T_1 o)$ (cf. Proposition 23.3). By Theorem 21.1 $\mathcal{V}$ contains the antidominant Weyl chamber. It remains to show as in the proof of Theorem 22.1 that different vectors from $\mathcal{V}$ cannot be $W_{\mathcal{O}}$-equivalent. $\square$

The proof of Theorem 26.2 shows that the map $\rho : \mathcal{D}^B \to \mathcal{E}$ may be non-injective if H is not semisimple. There is a more precise description of B-divisors in the spirit of Proposition 26.3 [Sp1, 5.4], [CS, §4].

It suffices to consider simple G. Assume first that H is connected. For any $\overline{\alpha}^{\vee} \in \Pi_{\mathcal{O}}^{\vee}$ there exist either a unique or exactly two B-divisors mapping to $\overline{\alpha}^{\vee}/2$. They correspond to the twisted T-orbits in $\tau(G) \cap r_{\alpha}T$ (for real α) or in $\tau(G) \cap \left(r_{\sigma(\alpha)}r_{\alpha}T \cup r_{\sigma(\iota(\alpha))}r_{\iota(\alpha)}T\right)$ (for complex α).

If H is semisimple, then such an orbit (and the respective B-divisor D_{α}) is always unique. In particular, $\iota(\alpha) = \alpha$ or $\iota(\alpha) = -\sigma(\alpha) \perp \alpha$.

If H is not semisimple (*Hermitian case*), then inspection of Table 5.9 shows that $\dim Z(H) = 1$ and $\Delta_{\mathcal{O}}$ is of type $\mathbf{BC}_n$ or $\mathbf{C}_n$. The B-divisor mapped to $\overline{\alpha}^{\vee}/2$ is unique except for the case where $\overline{\alpha}^{\vee}$ is the short simple coroot.

In the latter case, if α is complex, then $\tau(G) \cap r_{\sigma(\alpha)}r_{\alpha}T$ and $\tau(G) \cap r_{\sigma(\iota(\alpha))}r_{\iota(\alpha)}T$ are the twisted T-orbits corresponding to the two $\operatorname{Aut}_G \mathcal{O}$-stable

B-divisors $D_\alpha, D_{\iota(\alpha)}$ mapped to $\overline{\alpha}^\vee/2$. Here $\Delta_{\mathcal{O}} = \mathbf{BC}_n$ and $c(G/H') = 0$.

If α is real, then $\tau(G) \cap r_\alpha T$ consists of two twisted T-orbits corresponding to the two B-divisors D'_α, D''_α mapped to $\alpha^\vee/2$ and swapped by $\operatorname{Aut}_G \mathcal{O}$. Here $\Delta_{\mathcal{O}} = \mathbf{C}_n$ and $c(G/H') = 1$.

For disconnected H the divisors $D'_\alpha, D''_\alpha \in \mathcal{D}(G/H^0)^B$ may patch together into a single divisor $D_\alpha \in \mathcal{D}(G/H)^B$.

The (co)isotropy representation $H : \mathfrak{m}$ has nice invariant-theoretic properties in characteristic zero. They were examined by Kostant and Rallis [KR]. From now on assume $\operatorname{char} \Bbbk = 0$.

Semisimple elements in $\mathfrak{m}$ are exactly those having closed H-orbits, and the unique closed H-orbit in $\overline{H\xi}$ ($\xi \in \mathfrak{m}$) is $H\xi_{\mathrm{s}}$. Generic elements of $\mathfrak{m}$ are semisimple. One may deduce it from the fact that $T^*\mathcal{O}$ is symplectically stable (Proposition 8.2) or prove directly: $\mathfrak{g} = \mathfrak{l} \oplus [\mathfrak{g}, \mathfrak{t}_1] \implies \mathfrak{m} = \mathfrak{t}_1 \oplus [\mathfrak{h}, \mathfrak{t}_1] \implies \mathfrak{m} = \overline{H\mathfrak{t}_1}$. This argument also shows that H-invariant functions on $\mathfrak{m}$ are uniquely determined by their restrictions to $\mathfrak{t}_1$. A more precise result was obtained by Kostant and Rallis.

Proposition 26.6 ([KR]). *Every semisimple H-orbit in $\mathfrak{m}$ intersects $\mathfrak{t}_1$ in a $W_{\mathcal{O}}$-orbit. Restriction of functions yields an isomorphism $\Bbbk[\mathfrak{m}]^H \simeq \Bbbk[\mathfrak{t}_1]^{W_{\mathcal{O}}}$.*

Proof. Every semisimple element $\xi \in \mathfrak{m}$ is contained in the Lie algebra of a maximal σ-split torus, hence by Lemma 26.6, $\xi' = (\operatorname{Ad} h)\xi \in \mathfrak{t}_1$ for some $h \in H$. If $\xi \in \mathfrak{t}_1$, then $T_1, h^{-1}T_1h$ are two maximal σ-split tori in $Z_G(\xi)$. Again by Lemma 26.6, $zT_1z^{-1} = h^{-1}T_1h$ for some $z \in Z_G(\xi) \cap H$, whence $h' = hz \in N_H(\mathfrak{t}_1)$, $\xi' = (\operatorname{Ad} h')\xi \in W_{\mathcal{O}}\xi$.

The second assertion is a particular case of Proposition 25.1. It suffices to observe that the surjective birational morphism $\mathfrak{m}/\!\!/H \to \mathfrak{t}_1/W_{\mathcal{O}}$ of two normal affine varieties has to be an isomorphism. □

Global analogues of these results for the H-action on $\mathcal{O}$ (in any characteristic) were obtained by Richardson [Ri2].

It is not by chance that the description of the valuation cone of a symmetric space was obtained by the same reasoning as in §23.

Proposition 26.7 ([Kn5, §6]). *Flats in $\mathcal{O}$ are exactly the G-translates of $T_1 \cdot o$.*

Proof. It suffices to consider flats F_α, $\alpha \in T^{\mathrm{pr}}_o\mathcal{O}$. We have $T^*\mathcal{O} = G *_H \mathfrak{m}$, $\alpha = e * \xi$, $\xi = \Phi(\alpha) \in \mathfrak{m}^{\mathrm{pr}}$. By Proposition 26.6, $\xi \in (\operatorname{Ad} h)\mathfrak{t}_1^{\mathrm{pr}}$, $h \in H^0$. It follows that $G_\xi = hLh^{-1}$, whence $F_\alpha = hLo = hT_1o$. □

The $W_{\mathcal{O}}$-action on the flat $T_1 \cdot o$ comes from $N_H(T_1)$.

In the case $\Bbbk = \mathbb{C}$, flats in $\mathcal{O}$ are (G-translates of) the complexifications of maximal totally geodesic flat submanifolds in a Riemannian symmetric space $\mathcal{O}(\mathbb{R})$ which is a real form of $\mathcal{O}$ [Hel1].

27 Algebraic monoids and group embeddings

Alike algebraic groups, defined by superposing the concepts of an abstract group and an algebraic variety, it is quite natural to consider *algebraic semigroups*, i.e., algebraic varieties equipped with an associative multiplication law which is a regular map.

Example 27.1. All linear operators on a finite-dimensional vector space V form an algebraic semigroup $\mathrm{L}(V) \simeq \mathrm{L}_n(\Bbbk)$ ($n = \dim V$). The operators (matrices) of rank $\leq r$ form a closed subsemigroup $\mathrm{L}^{(r)}(V)$ ($\mathrm{L}_n^{(r)}(\Bbbk)$), a particular example of a determinantal variety.

However the category of *all* algebraic semigroups is immense. (For instance, every algebraic variety X turns into an algebraic semigroup being equipped with the "zero" multiplication $X \times X \to \{0\}$, where $0 \in X$ is a fixed element.) In order to make the theory really substantive, one has to restrict the attention to algebraic semigroups not too far from algebraic groups.

Definition 27.1. An *algebraic monoid* is an algebraic semigroup with unit, i.e., an algebraic variety X equipped with a morphism $\mu : X \times X \to X$, $\mu(x, y) =: x \cdot y$ (the *multiplication law*) and with a distinguished *unity element* $e \in X$ such that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $e \cdot x = x \cdot e = x$, $\forall x, y, z \in X$.

Let $G = G(X)$ denote the group of invertible elements in X. The following elementary result can be found, e.g., in [Rit1, §2].

Proposition 27.1. *G is open in X.*

Proof. Since the left translation $x \mapsto g \cdot x$ by an element $g \in G$ is an automorphism of X, it suffices to prove that G contains an open subset of an irreducible component of X. W.l.o.g. we may assume that X is irreducible. Let p_1, p_2 be the two projections of $\mu^{-1}(e) \subset X \times X$ to X. By the fiber dimension theorem, every component of $\mu^{-1}(e)$ has dimension $\geq \dim X$, and $p_i^{-1}(e) = (e, e)$. Hence p_i are dominant maps and $G = p_1(\mu^{-1}(e)) \cap p_2(\mu^{-1}(e))$ is a dense constructible set containing an open subset of X. □

Corollary 27.1. *G is an algebraic group.*

Those irreducible components of X which do not intersect G do not "feel the presence" of G and their behavior is beyond of control. Therefore it is reasonable to restrict oneself to algebraic monoids X such that $G = G(X)$ is dense in X. In this case, left translations by G permute the components of X transitively and many questions are reduced to the case, where X is irreducible.

Monoids of this kind form an interesting category of algebraic structures closely related to algebraic groups (e.g., they arise as the closures of linear algebraic groups in the spaces of linear operators). The theory of algebraic monoids was created in major part during the last 25 years by M. S. Putcha, L. E. Renner, E. B. Vinberg, A. Rittatore, et al. The interested reader may consult a detailed survey [Ren3] of the theory from the origin up to latest developments. In this section, we discuss algebraic monoids from the viewpoint of equivariant embeddings. A link between these two theories is provided by the following result.

Theorem 27.1 ([Rit1, §2]). (1) *Any algebraic monoid X is a $G \times G$-equivariant embedding of $G = G(X)$, where the factors of $G \times G$ act by left/right multiplication, having a unique closed $G \times G$-orbit.*

(2) *Conversely, any* affine *$G \times G$-equivariant embedding $X \hookleftarrow G$ carries a structure of algebraic monoid with $G(X) = G$.*

Proof. (1) One has only to prove the uniqueness of a closed orbit $Y \subseteq X$. Note that $X \cdot Y \cdot X = \overline{G \cdot Y \cdot G} = Y$, i.e., Y is a (two-sided) *ideal* in X. For any other ideal $Y' \subseteq X$ we have $Y \cdot Y' \subseteq Y \implies Y = Y \cdot Y' \subseteq Y'$. Thus Y is the smallest ideal, called the *kernel* of X.

(2) The actions of the left and right copy of $G \times G$ on X define coactions $\Bbbk[X] \to \Bbbk[G] \otimes \Bbbk[X]$ and $\Bbbk[X] \to \Bbbk[X] \otimes \Bbbk[G]$, which are the restrictions to $\Bbbk[X] \subseteq \Bbbk[G]$ of the comultiplication $\Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$. Hence the image of $\Bbbk[X]$ lies in $(\Bbbk[G] \otimes \Bbbk[X]) \cap (\Bbbk[X] \otimes \Bbbk[G]) = \Bbbk[X] \otimes \Bbbk[X]$, and we have a comultiplication in $\Bbbk[X]$. Now G is open in $X = \overline{G}$ and consists of invertibles. For any invertible $x \in X$, we have $xG \cap G \neq \emptyset$, hence $x \in G$. $\square$

Remark 27.1. Assertion (2) was first proved for reductive G by Vinberg [Vin2] in a different way.

Among general algebraic groups, affine (=linear) ones occupy a privileged position due to their most rich and interesting structure. The same holds for algebraic monoids. We provide two results confirming this observation.

Theorem 27.2 ([Mum, §4]). *Complete irreducible algebraic monoids are just Abelian varieties.*

Theorem 27.3 ([Rit3]). *An algebraic monoid X is affine provided that $G(X)$ is affine.*

This theorem was proved by Renner [Ren1] for quasiaffine X using some structure theory and Rittatore [Rit3] reduced the general case to the quasiaffine one by considering total spaces of certain line bundles over X.

A theorem of Barsotti [Bar] and Rosenlicht [Ros] says that every algebraic group has a unique affine normal subgroup such that the quotient group is an Abelian variety. It is an interesting unsolved problem to obtain an analogous structure result for algebraic monoids.

Theorem 27.4. *Any affine algebraic monoid X admits a closed homomorphic embedding $X \hookrightarrow \mathrm{L}(V)$. Furthermore, $G(X) = X \cap \mathrm{GL}(V)$.*

The proof is essentially the same as that of a similar result for algebraic groups [Hum, 8.6]. Thus the adjectives "affine" and "linear" are synonyms for algebraic monoids, alike for algebraic groups.

In the notation of Theorem 27.4, the space of matrix entries $\mathrm{M}(V)$ generates $\Bbbk[X] \subseteq \Bbbk[G]$. Generally, $\Bbbk[X] \supset \mathrm{M}(V)$ iff the representation $G : V$ is extendible to X. It follows from Theorem 27.4 and (2.1) that

$$\Bbbk[X] = \bigcup \mathrm{M}(V) \tag{27.1}$$

over all G-modules V that are X-modules (cf. Proposition 2.3).

Example 27.2. By Theorem 27.1(2), every affine toric variety X carries a natural structure of algebraic monoid extending the multiplication in the open torus T. By Theorem 27.4, X is the closure of T in $\mathrm{L}(V)$ for some faithful representation $T : V$, i.e., a closed submonoid in the monoid of all diagonal matrices in some $\mathrm{L}_n(\Bbbk)$. The coordinate algebra $\Bbbk[X]$ is the semigroup algebra of the semigroup $\Sigma \subseteq \mathfrak{X}(T)$ consisting of all characters $T \to \Bbbk^\times$ extendible to X. Conversely, every finitely generated semigroup $\Sigma \ni 0$ such that $\mathbb{Z}\Sigma = \mathfrak{X}(T)$ defines a toric monoid $X \supseteq T$.

The classification and structure theory for algebraic monoids is most well developed in the case, where the group of invertibles is reductive.

Definition 27.2. An irreducible algebraic monoid X is called *reductive* if $G = G(X)$ is a connected reductive group.

In the sequel we consider only reductive monoids, thus returning to the general convention of our survey that G is a connected reductive group. By Theorems 27.1, 27.3, reductive monoids are nothing else but $G \times G$-equivariant affine embeddings of G. They were classified by Vinberg [Vin2] in characteristic zero. Rittatore [Rit1] extended this classification to arbitrary characteristic using the embedding theory of spherical homogeneous spaces.

Considered as a homogeneous space under $G \times G$ acting by left/right multiplication, G is a symmetric space (Example 26.2). All σ-stable maximal tori of $G \times G$ are of the form $T \times T$, where T is a maximal torus in G. The maximal σ-split tori are $(T \times T)_1 = \{(t^{-1}, t) \mid t \in T\}$. Choose a Borel subgroup $B \supseteq T$ of G. Then $B^- \times B$ is a Borel subgroup in $G \times G$ containing $T \times T$ and $\sigma(B^- \times B) = B \times B^-$ is the opposite Borel subgroup.

The weight lattice $\Lambda = \mathfrak{X}(T \times T/\operatorname{diag} T) = \{(-\lambda, \lambda) \mid \lambda \in \mathfrak{X}(T)\}$ is identified with $\mathfrak{X}(T)$ and the little root system with $\frac{1}{2}\Delta$. The eigenfunctions $\mathbf{f}_\lambda \in \Bbbk(G)^{(B^- \times B)}$ $(\lambda \in \mathfrak{X}(T))$ are defined on the "big" open cell $U^- \times T \times U \subseteq G$ by the formula $\mathbf{f}_\lambda(u^- t u) = \lambda(t)$. For $\lambda \in \mathfrak{X}_+$ they are matrix entries: $\mathbf{f}_\lambda(g) = \langle v_{-\lambda}, g v_\lambda \rangle$, where $v_\lambda \in V$, $v_{-\lambda} \in V^*$ are highest, resp. lowest, vectors of weights $\pm\lambda$.

By Theorem 26.2, the valuation cone $\mathcal{V}$ is identified with the antidominant Weyl chamber in $\mathfrak{X}^*(T) \otimes \mathbb{Q}$ (this can also be deduced from Example 21.1) and the colors are represented by the simple coroots $\alpha_1^\vee, \dots, \alpha_l^\vee \in \Pi^\vee$. The respective B-divisors are $D_i = \overline{B^- r_{\alpha_i} B}$. Indeed, the equation of D_i in $\Bbbk[\widetilde{G}]$ is $\mathbf{f}_{\omega_i}$, where ω_i denote the fundamental weights.

The results of §24 (in particular, the Cartan decomposition) imply that all $G \times G$-valuations are proportional to $v = v_\gamma$, $\gamma \in \mathfrak{X}^*(T)$. Since $v_{w\gamma} = v_\gamma$, $\forall w \in W = N_G(T)/T$, w.l.o.g. $\gamma \in \mathcal{V}$. Then a direct computation shows $v(\mathbf{f}_\lambda) = \langle \gamma, \lambda \rangle$, $\forall \lambda \in \mathfrak{X}_+$, whence v is identified with γ (as a vector in the valuation cone).

Now Corollary 15.1 yields

Theorem 27.5. *Normal reductive monoids X are in bijection with strictly convex cones $\mathcal{C} = \mathcal{C}(X) \subset \mathfrak{X}^*(T) \otimes \mathbb{Q}$ generated by all simple coroots and finitely many antidominant vectors.*

Remark 27.2. The normality assumption is not so restrictive, because the multiplication on X lifts to its normalization $\widetilde{X}$ turning it into a monoid with the same group of invertibles.

Corollary 27.2. *There are no non-trivial monoids with semisimple group of invertibles.*

Corollary 27.3 ([Put], [Rit1, Pr.9]). *Every normal reductive monoid has the structure $X = (X_0 \times G_1)/Z$, where X_0 is a monoid with zero, and Z is a finite central subgroup in $G(X_0) \times G_1$ not intersecting the factors.*

Proof. Identify $\mathfrak{X}(T) \otimes \mathbb{Q}$ with $\mathcal{E} = \mathfrak{X}^*(T) \otimes \mathbb{Q}$ via a W-invariant inner product. Consider an orthogonal decomposition $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where $\mathcal{E}_0 =$

$\langle\mathcal{C}\cap\mathcal{V}\rangle$, $\mathcal{E}_1=(\mathcal{C}\cap\mathcal{V})^\perp$. It is easy to see that each root is contained in one of the $\mathcal{E}_i$. Then $G=G_0\cdot G_1=(G_0\times G_1)/Z$, where G_i are the connected normal subgroups with $\mathfrak{X}^*(T\cap G_i)=\mathcal{E}_i\cap\mathfrak{X}^*(T)$. Take a reductive monoid $X_0\supseteq G_0$ defined by $\mathcal{C}_0=\mathcal{C}\cap\mathcal{E}_0$. Since $\operatorname{int}\mathcal{C}_0$ intersects $\mathcal{V}(G_0)=\mathcal{V}\cap\mathcal{E}_0$, the kernel of X_0 is a complete variety, hence a single point 0, the zero element w.r.t. the multiplication on X_0. Now X coincides with $(X_0\times G_1)/Z$, because both monoids have the same colored data. □

This classification can be made more transparent via coordinate algebras and representations. Recall from §15 that $\Bbbk[X]^{U^-\times U}=\Bbbk[\mathcal{C}^\vee\cap\mathfrak{X}(T)]$. The algebra $\Bbbk[X]$ itself is given by (27.1). It remains to determine which representations of G extend to X.

Proposition 27.2. *The following conditions are equivalent:*

(1) *The representation $G:V$ is extendible to X.*

(2) *The highest weights of the simple factors of V are in $\mathcal{C}^\vee$.*

(3) *All dominant T-weights of V are in $\mathcal{C}^\vee$.*

Proof. (1) $\Longrightarrow$ (2) Choose a G-stable filtration of V with simple factors and consider the associated graded G-module $\operatorname{gr}V$. If $G:V$ extends to X, then $\operatorname{gr}V$ is an X-module. Hence $\mathbf{f}_\lambda\in\Bbbk[X]$ whenever λ is a highest weight of $\operatorname{gr}V$.

(2) $\Longleftrightarrow$ (3) All T-weights of V are obtained from the highest weights of simple factors by subtracting positive roots. The structure of $\mathcal{C}$ implies that all dominant vectors obtained this way from $\lambda\in\mathcal{C}^\vee$ belong to $\mathcal{C}^\vee$.

(3) $\Longrightarrow$ (1) Assume that $\Bbbk[X]\not\supseteq\mathrm{M}(V)$. Choose $f\in\mathrm{M}(V)$ representing a nonzero $B^-\times B$-eigenvector $\bmod \Bbbk[X]$. Then by Corollary A2.1, $f^q=\mathbf{f}_\lambda$ $\bmod \Bbbk[X]$ for some $\lambda\notin\mathcal{C}^\vee$. It follows that λ/q is a T-weight of V outside $\mathcal{C}^\vee$. □

Corollary 27.4. *If $X\subseteq\mathrm{L}(V)$, then $\mathcal{C}^\vee=\mathcal{K}(V)\cap\mathbf{C}$, where $\mathcal{K}(V)$ denotes the convex cone spanned by the T-weights of V.*

Proof. The proposition implies $\mathcal{C}^\vee\supseteq\mathcal{K}(V)\cap\mathbf{C}$. On the other hand, all $T\times T$-weights of $\Bbbk[X]$ are of the form $(-\lambda,\mu)$, $\lambda,\mu\in\mathcal{K}(V)$, whence $\mathcal{C}^\vee\subseteq\mathcal{K}(V)$. □

In characteristic zero, Proposition 27.2 together with (27.1) yields

$$\Bbbk[X]=\bigoplus_{\lambda\in\mathcal{C}^\vee\cap\mathfrak{X}(T)}\mathrm{M}(V(\lambda)) \tag{27.2}$$

(cf. Theorem 2.4 and (2.3)). In positive characteristic, $\Bbbk[X]$ has a "good" filtration with factors $V^*(\lambda) \otimes V^*(\lambda^*)$ [Do], [Rit2, §4], [Ren3, Cor.9.9].

The embedding theory provides a combinatorial encoding for $G \times G$-orbits in X, which reflects the adherence relation. This description can be made more explicit using the following

Proposition 27.3. *Suppose $X \hookleftarrow G$ is an equivariant embedding. Then $F = \overline{T}$ intersects each $G \times G$-orbit $Y \subset X$ in finitely many T-orbits permuted transitively by W. Exactly one of these orbits $F_Y \subseteq F \cap Y$ satisfies $\operatorname{int} \mathcal{C}_{F_Y} \cap \mathcal{V} \neq \emptyset$; then $\mathcal{C}_{F_Y} = W(\mathcal{C}_Y \cap \mathcal{V}) \cap \mathcal{C}_Y$.*

Remark 27.3. Since T is a flat of G (Proposition 26.7), some of the assertions stem from the results of §23. However, the proposition here is more precise. In particular, it completely determines the fan of F.

Proof. Take any $v \in \mathcal{S}_Y$; then $v = v_\gamma$, $\gamma \in \mathfrak{X}^*(T) \cap \mathcal{V}$, $\exists \lim_{t\to 0} \gamma(t) = \gamma(0) \in Y$. The associated parabolic subgroup $P = P(\gamma)$ contains B^-. Consider the Levi decomposition $P = LP_{\mathrm{u}}$, $L \supseteq T$. One verifies that $(G \times G)_{\gamma(0)} \supseteq (P_{\mathrm{u}}^- \times P_{\mathrm{u}}) \cdot \operatorname{diag} L$. It easily follows that $(B^- \times B)\gamma(0) = \mathring{Y}$ is the open $B^- \times B$-orbit in Y and $F_Y := T\gamma(0) = \mathring{Y}^{\operatorname{diag} T}$ is the unique T-orbit in F intersecting $\mathring{Y}$.

In view of Example 24.2, this implies $\operatorname{int} \mathcal{C}_{F_Y} \supseteq (\operatorname{int} \mathcal{C}_Y) \cap \mathcal{V}$. On the other hand, each T-orbit in $F \cap Y$ is accessed by a one-parameter subgroup $\gamma \in \mathfrak{X}^*(T)$, $\gamma(0) \in Y$. Taking $w \in W$ such that $w\gamma \in \mathcal{V}$ yields $w(T\gamma(0)) = F_Y$. All assertions of the proposition are deduced from these observations. ☐

Now suppose $X \subseteq \mathrm{L}(V)$ and denote $\mathcal{K} = \mathcal{K}(V)$.

Theorem 27.6. *The $G \times G$-orbits in X are in bijection with the faces of $\mathcal{K}$ whose interiors intersect $\mathbf{C}$. The orbit Y corresponding to a face $\mathcal{F}$ is represented by the T-equivariant projector $e_{\mathcal{F}}$ of V onto the sum of T-eigenspaces of weights in $\mathcal{F}$. The cone $\mathcal{C}_Y$ is dual to the corner cone of $\mathcal{K} \cap \mathbf{C}$ at the face $\mathcal{F} \cap \mathbf{C}$, and $\mathcal{D}_Y^B$ consists of simple coroots orthogonal to $\mathcal{F}$.*

Proof. A complete set of T-orbit representatives in $F = \overline{T}$ is formed by the limits of one-parameter subgroups, i.e., by the $e_{\mathcal{F}}$'s over *all* faces $\mathcal{F}$ of $\mathcal{K}$. The respective cones in the fan of F are the dual faces $\mathcal{F}^* = \mathcal{K}^\vee \cap \mathcal{F}^\perp$ of $\mathcal{K}^\vee = W(\mathcal{C} \cap \mathcal{V})$. By Proposition 27.3, the orbits Y are bijectively represented by those $e_{\mathcal{F}}$ which satisfy $\operatorname{int} \mathcal{F}^* \cap \mathcal{V} \neq \emptyset$. This happens iff $\mathcal{F}^*$ lies on a face of $\mathcal{C}$ of the same dimension (namely on $\mathcal{C}_Y$) or, equivalently, $\mathcal{F}$ contains a face of $\mathcal{C}^\vee = \mathcal{K} \cap \mathbf{C}$ of the same dimension (namely $\mathcal{C}_Y^* = \mathcal{F} \cap \mathbf{C}$), i.e., $\operatorname{int} \mathcal{F} \cap \mathbf{C} \neq \emptyset$. The assertion on $(\mathcal{C}_Y, \mathcal{D}_Y)$ stems from the description of a dual face. ☐

Example 27.3. Let $G = \mathrm{GL}_n(\Bbbk)$ and $X = \mathrm{L}_n(\Bbbk)$. For B and T take the standard Borel subgroup of upper-triangular matrices and diagonal torus, respectively. We have $\mathfrak{X}(T) = \langle \varepsilon_1, \dots, \varepsilon_n \rangle$, where the ε_i are the diagonal matrix entries of T. We identify $\mathfrak{X}(T)$ with $\mathfrak{X}^*(T)$ via the inner product such that the ε_i form an orthonormal basis. Let $(k_1, \dots, k_n)$ denote the coordinates on $\mathfrak{X}(T) \otimes \mathbb{Q}$ w.r.t. this basis. The Weyl group $W = S_n$ permutes them.

The weights $\lambda_i = \varepsilon_1 + \dots + \varepsilon_i$ span $\mathfrak{X}(T)$ and $\mathbf{f}_{\lambda_i} \in \Bbbk[X]$ are the upper-left corner i-minors of a matrix. Put $D_i = \{x \in X \mid \mathbf{f}_{\lambda_i}(x) = 0\}$. Then $\mathcal{D}^B = \{D_1, \dots, D_{n-1}\}$, D_i are represented by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\forall i < n$, and D_n is the unique G-stable prime divisor, $v_{D_n} = \varepsilon_n$.

Therefore $\mathcal{C} = \{k_1 + \dots + k_i \geq 0,\ i = 1, \dots, n\}$ is the cone spanned by $\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n$, and $\mathcal{C}^\vee = \{k_1 \geq \dots \geq k_n \geq 0\}$ is spanned by λ_i. The lattice vectors of $\mathcal{C}^\vee$ are exactly the dominant weights of polynomial representations (cf. Proposition 27.2). The lattice vectors of $\mathcal{K} = W\mathcal{C}^\vee = \{k_1, \dots, k_n \geq 0\}$ are all polynomial weights of T.

The $G \times G$-orbits in X are $Y_r = \{x \in X \mid \operatorname{rk} x = r\}$. Clearly, $\mathcal{D}^B_{Y_r} = \{D_i \mid r < i < n\}$ and $\mathcal{C}_{Y_r}$ is a face of $\mathcal{C}$ cut off by the equations $k_1 = \dots = k_r = 0$. The dual face $\mathcal{C}^*_{Y_r}$ of $\mathcal{C}^\vee$ is the dominant part of the face $\mathcal{F}_r = \{k_i \geq 0 = k_j \mid i \leq r < j\} \subseteq \mathcal{K}$, and all faces of $\mathcal{K}$ whose interiors intersect $\mathbf{C} = \{k_1 \geq \dots \geq k_n\}$ are obtained this way. Clearly, the respective projectors $e_{\mathcal{F}_r} = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)$ are the $G \times G$-orbit representatives, and the representatives of all T-orbits in $\overline{T}$ are obtained from $e_{\mathcal{F}_r}$ by the W-action.

In characteristic zero, it is possible to classify (to a certain extent) arbitrary (not necessarily normal) reductive monoids [Vin2] via their coordinate algebras alike (27.2). The question is to describe finitely generated $G \times G$-stable subalgebras of $\Bbbk[G]$ with the quotient field $\Bbbk(G)$. They are of the form

$$\Bbbk[X] = \bigoplus_{\lambda \in \Sigma} \mathrm{M}(V(\lambda)) \tag{27.3}$$

where Σ is a finitely generated subsemigroup of $\mathfrak{X}_+$ such that $\mathbb{Z}\Sigma = \mathfrak{X}(T)$ and the r.h.s. of (27.3) remains closed under multiplication, i.e., all highest weights of $V(\lambda) \otimes V(\mu)$ belong to Σ whenever $\lambda, \mu \in \Sigma$. Such a semigroup Σ is called *perfect*.

Definition 27.3. We say that $\lambda_1, \dots, \lambda_m$ *G-generate* Σ if Σ consists of all highest weights of G-modules $V(\lambda_1)^{\otimes k_1} \otimes \dots \otimes V(\lambda_m)^{\otimes k_m}$, $k_1, \dots, k_m \in \mathbb{Z}_+$. (In particular any generating set G-generates Σ.)

Example 27.4. In Example 27.3, $\Sigma = \mathcal{C}^\vee \cap \mathfrak{X}(T)$ is generated by $\lambda_1, \dots, \lambda_n$ and G-generated by λ_1.

It is easy to see that $X \hookrightarrow \mathrm{L}(V)$ iff the highest weights $\lambda_1, \dots, \lambda_m$ of $G : V$ G-generate Σ. The highest weight theory implies that $\mathcal{K} = \mathcal{K}(V)$ is the W-span of

$$\mathcal{K} \cap \mathbf{C} = (\mathbb{Q}_+\{\lambda_1, \dots, \lambda_m, -\alpha_1, \dots, -\alpha_l\}) \cap \mathbf{C} \tag{27.4}$$

Theorem 27.6 generalizes to this context.

By Theorem A2.1(3), X is normal iff $\Bbbk[X]^{U^- \times U} = \Bbbk[\Sigma]$ is integrally closed iff Σ is the semigroup of all lattice vectors in a polyhedral cone. In general, taking the integral closure yields

$$\mathbb{Q}_+\Sigma = \mathcal{C}^\vee = \mathcal{K} \cap \mathbf{C}$$

where $\mathcal{C} = \mathcal{C}(\widetilde{X})$. Here is a representation-theoretic interpretation: each dominant vector in $\mathcal{K}$ eventually occurs as a highest weight in a tensor power of V, see [Tim4, §2] for a direct proof.

Given $G : V$, the above normality condition for $X \subseteq \mathrm{L}(V)$ is generally not easy to verify, because the reconstruction of Σ from $\{\lambda_1, \dots, \lambda_m\}$ requires decomposing tensor products of arbitrary G-modules. Of course, there is no problem if λ_i already generate $\mathcal{K} \cap \mathfrak{X}_+$—a sufficient condition for normality. Here is an effective necessary condition:

Proposition 27.4 ([Ren2], [Ren3, Th.5.4(b)]). *If X is normal, then $F = \overline{T}$ is normal, i.e., the T-weights of V generate $\mathcal{K} \cap \mathfrak{X}(T)$.*

Proof. We can increase V by adding new highest weights λ_i so that $\lambda_1, \dots, \lambda_m$ will generate $\Sigma = \mathcal{K} \cap \mathfrak{X}_+$. (This operation does not change X and F.) Then $W\{\lambda_1, \dots, \lambda_m\}$ generates $\mathcal{K} \cap \mathfrak{X}(T)$, i.e., $\Bbbk[F] = \Bbbk[\mathcal{K} \cap \mathfrak{X}(T)]$ is integrally closed. □

If $V = V(\lambda)$ is irreducible, then the center of G acts by homotheties, whence $G = \Bbbk^\times \cdot G_0$, where G_0 is semisimple, $\mathfrak{X}(T) \subseteq \mathbb{Z} \oplus \mathfrak{X}(T \cap G_0)$ is a cofinite sublattice, and $\lambda = (1, \lambda_0)$. Recently de Concini showed that $\mathcal{K}(V(\lambda)) \cap \mathfrak{X}_+$ is G-generated by the T-dominant weights of $V(\lambda)$ [Con]. However Σ contains no T-weights of $V(\lambda)$ except λ. It follows that X is normal iff λ_0 is a minuscule weight for G_0 [Con], [Tim4, §12].

It turns out that Example 27.3 is essentially the unique non-trivial example of a smooth reductive monoid.

Theorem 27.7 (cf. [Ren2], [Tim4, §11]). *Smooth reductive monoids are of the form $X = (G_0 \times \mathrm{L}_{n_1}(\Bbbk) \times \dots \times \mathrm{L}_{n_s}(\Bbbk))/Z$, where $Z \subset G_0 \times \mathrm{GL}_{n_1}(\Bbbk) \times \dots \times \mathrm{GL}_{n_s}$ is a finite central subgroup not intersecting $\mathrm{GL}_{n_1}(\Bbbk) \times \dots \times \mathrm{GL}_{n_s}$.*

Proof. By Corollary 27.3, $X = G_0 *_Z X_0$, where X_0 has the zero element. Thus it suffices to consider monoids with zero. We explain how to handle this case in characteristic zero.

Assume $X \subseteq \mathrm{L}(V)$. There exists a coweight $\gamma \in \operatorname{int}\mathcal{C} \cap \mathcal{V}$, $\gamma \perp \Delta$. It defines a one-parameter subgroup $\gamma(t) \in Z(G)$ contracting V to 0 (as $t \to 0$). The algebra $\mathcal{A} = \mathcal{A}(V)$ spanned by X in $\mathrm{L}(V)$ is semisimple, i.e., a product of matrix algebras, and T_0X is an ideal in $\mathcal{A}$. As X is smooth and the multiplication by $\gamma(t)$ contracts X to 0, the equivariant projection $X \to T_0X$ is an isomorphism. $\square$

We conclude this section by a discussion of arbitrary (not necessarily affine) equivariant embeddings of G. For simplicity, we assume $\operatorname{char}\Bbbk = 0$.

In the same way as a faithful linear representation $G : V$ defines a reductive monoid $\overline{G} \subseteq \mathrm{L}(V)$, a faithful projective representation $G : \mathbb{P}(V)$ (arising from a linear representation of a finite cover of G in V) defines a projective completion $X = \overline{G} \subseteq \mathbb{P}(\mathrm{L}(V))$. These group completions are studied in [Tim4]. There are two main tool to reduce their study to reductive monoids.

First, the cone $\widehat{X} \subseteq \mathrm{L}(V)$ over X is a reductive monoid whose group of invertibles $\widehat{G}$ is the extension of G by homotheties. Conversely, any such monoid gives rise to a projective completion. This allows to transfer some of the above results to projective group completions. For instance, Theorem 27.6 transfers verbatim if we only replace the weight cone $\mathcal{K}(V)$ by the weight polytope $\mathcal{P} = \mathcal{P}(V)$ (=the convex hull of the T-weights of V), see [Tim4, §9] for details.

Another approach, suitable for local study, is to use the local structure theorem. By the above, closed $(G \times G)$-orbits $Y \subset X$ correspond to the dominant vertices $\lambda \in \mathcal{P}$, and the representatives are $y = \langle v_\lambda \otimes v_{-\lambda} \rangle$, where $v_\lambda \in V$, $v_{-\lambda} \in V^*$ are highest, resp. lowest, vectors of weights $\pm\lambda$, $\langle v_\lambda, v_{-\lambda} \rangle \neq 0$. Consider the parabolic $P = P(\lambda)$ and its Levi decomposition $P = LP_{\mathrm{u}}$, $L \supseteq T$. Then $V_0 = \langle v_{-\lambda} \rangle^\perp$ is an L-stable complement to $\langle v_\lambda \rangle$ in V. Put $\mathring{X} = X \setminus \mathbb{V}(\mathbf{f}_\lambda)$.

Lemma 27.1. *$\mathring{X} \simeq P_{\mathrm{u}}^- \times Z \times P_{\mathrm{u}}$, where $Z \simeq \overline{L} \subseteq \mathrm{L}(V_0 \otimes \Bbbk_{-\lambda})$ is a reductive monoid with the zero element y.*

Proof. Applying Lemma 4.1 to $G \times G : \mathrm{L}(V) = V \otimes V^*$, passing to projectivization and intersecting with X, we obtain a neighborhood of the desired structure with $Z = X \cap \mathbb{P}\big(\Bbbk^\times(v_\lambda \otimes v_{-\lambda}) + E_0\big)$, where

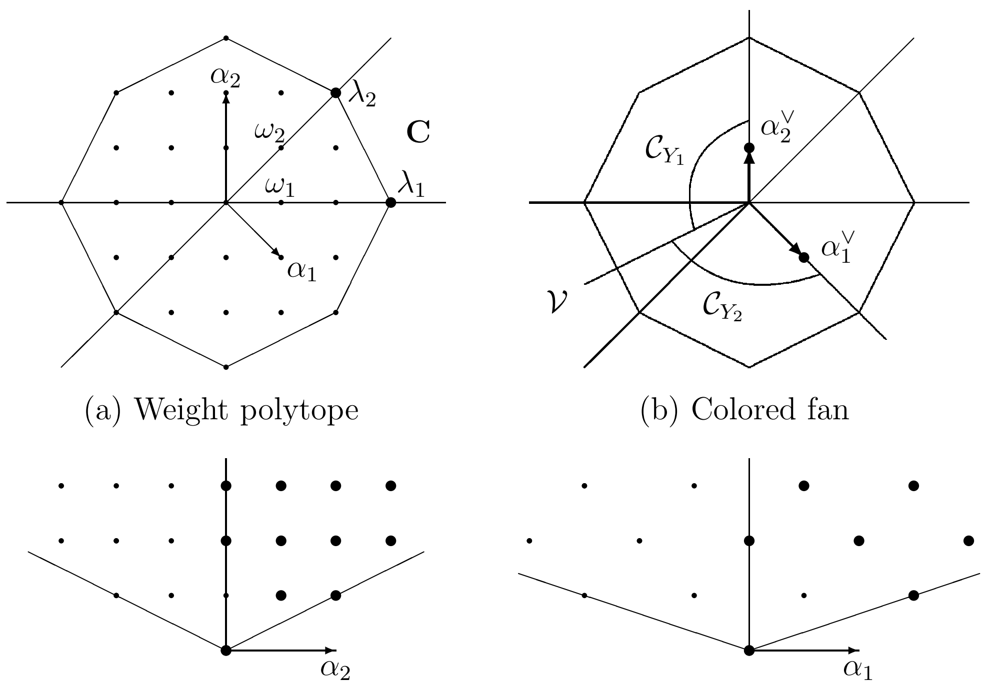
$$E_0 = (\mathfrak{g} \times \mathfrak{g})(v_{-\lambda} \otimes v_\lambda)^\perp = (\mathfrak{g}v_{-\lambda} \otimes v_\lambda + v_{-\lambda} \otimes \mathfrak{g}v_\lambda)^\perp \supseteq V_0 \otimes V_0^* = \mathrm{L}(V_0)$$

Hence $Z = \overline{L} \subseteq \mathbb{P}\big(\Bbbk^\times(v_\lambda \otimes v_{-\lambda}) \oplus \mathrm{L}(V_0)\big) \simeq \mathrm{L}(V_0 \otimes \Bbbk_{-\lambda})$. $\square$

The monoids Z are transversal slices to the closed orbits in X. They can be used to study the local geometry of X. For instance, one can derive criteria for normality and smoothness [Tim4, §§10,11].

Example 27.5. Take $G = \mathrm{Sp}_4(\Bbbk)$, with the simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = 2\varepsilon_2$, and the fundamental weights $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, $\pm\varepsilon_i$ being the weights of the tautological representation $\mathrm{Sp}_4(\Bbbk) : \Bbbk^4$. Let $\lambda_1 = 3\omega_1$, $\lambda_2 = 2\omega_2$ be the highest weights of V. The weight polytope $\mathcal{P}$ is depicted in Figure 5.8(a), the highest weights are indicated by bold dots. There are two

Figure 5.8: A projective completion of $\mathrm{Sp}_4(\Bbbk)$



(c) Slice semigroups

closed orbits $Y_1, Y_2 \subset X$. The respective Levi subgroups are $L_1 = \mathrm{SL}_2(\Bbbk)\times\Bbbk^\times$ and $L_2 = \mathrm{GL}_2(\Bbbk)$, with the simple roots α_2 and α_1, respectively.

Consider the slice monoids Z_i for Y_i. The weight semigroups of $F_i = \overline{T}$ (the closure in Z_i) are plotted by dots in Figure 5.8(c), the bold dots corresponding to the weight semigroups Σ_i of Z_i. (They are easily computed using the Clebsch–Gordan formula.) We can now see that F_i are normal, but Z_i are not, i.e., X is non-normal along Y_1, Y_2. However, if we increase V by adding two highest weights $\lambda_2 = 2\omega_1$, $\lambda_3 = \omega_1 + \omega_2$, then X becomes normal. Its colored fan is depicted in Figure 5.8(b).

The projective completions of adjoint simple groups in projective linear operators on fundamental and adjoint representation spaces were studied in detail in [Tim4, §12]. In particular, the orbital decomposition was described, and normal and smooth completions were identified.

Example 27.6. Suppose $G = \mathrm{SO}_{2l+1}(\Bbbk)$, and $V = V(\omega_i)$ is a fundamental representation. We have a unique closed orbit $Y \subset X$. If $i < l$, then $L \not\simeq \mathrm{GL}_{n_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{n_s}(\Bbbk)$, hence Z and X are singular. But for $i = l$ (the spinor representation), $L \simeq \mathrm{GL}_l(\Bbbk)$ and $V(\omega_l) \otimes \Bbbk_{-\omega_l}$ is L-isomorphic to $\bigwedge^\bullet \Bbbk^l$. It follows that $Z \simeq \mathrm{L}_l(\Bbbk)$, whence X is smooth.

Example 27.7. Suppose that all vertices of $\mathcal{P}$ are regular weights. Then the slice monoids Z are toric and their weight semigroups Σ are generated by the weights $\mu - \lambda$, where μ runs over all T-weights of V. The variety X is toroidal, and normal (smooth) iff each Σ consists of all lattice vectors in the corner cone of $\mathcal{P}$ at λ (resp. Σ is generated by linearly independent weights).

In particular, if $V = V(\lambda)$ is a simple module of regular highest weight, then $\Sigma = \mathbb{Z}(-\Pi)$, whence X is smooth. This is a particular case of a wonderful completion, see §30.

A interesting model for the wonderful completion of G in terms of Hilbert schemes was proposed by Brion [Bri17]. Namely, given a generalized flag variety $M = G/Q$, he proves that the closure $X = \overline{(G \times G)[\operatorname{diag} M]}$ in the Hilbert scheme (or the Chow variety) of $M \times M$ is isomorphic to the wonderful completion. If $G = (\operatorname{Aut} M)^0$ (e.g., if $Q = B$), then X is an irreducible component of the Hilbert scheme (the Chow variety). All fibers of the universal family over X are reduced and Cohen–Macaulay (even Gorenstein if $Q = B$).

Toroidal and wonderful group completions were studied intensively in the framework of the general theory of toroidal and wonderful varieties (see §29–§30) and by their own. De Concini and Procesi [CP3] and Strickland [Str2] computed ordinary and equivariant rational cohomology of smooth toroidal completions over $\Bbbk = \mathbb{C}$ (see also [BCP], [LP]). Brion [Bri15] carried out a purely algebraic treatment of these results replacing cohomology by (equivariant) Chow rings.

The basis of the Chow ring $A(X)$ of a smooth toroidal completion $X = \overline{G}$ is given by the closures of the Białynicki-Birula cells [BB1], which are isomorphic to affine spaces and intersect $G \times G$-orbits in $B^- \times B$-orbits [BL, 2.3]. The latter were described in [Bri15, 2.1]. The $B^- \times B$-orbit closures in X are smooth in codimension 1, but singular in codimension 2 (apart from trivial exceptions arising from $G = \mathrm{PSL}_2(\Bbbk)$) [Bri15, §2]. For wonderful X, the Białynicki-Birula cells are described in [Bri15, 3.3], and those intersecting G (=the closures in X of $B^- \times B$-orbits in G) are normal and Cohen–Macaulay [BPo]. Geometry of $B^- \times B$-orbit closures in X was studied in [Sp3], [Ka].

The class of reductive group embeddings is not closed under degenerations. Alexeev and Brion [AB1], [AB2] introduced a more general class of (*stable*) *reductive varieties* closed under flat degenerations with irreducible (resp. reduced) fibers. Affine (stable) reductive varieties may be defined as affine spherical $G \times G$-varieties X such that $\Lambda(X) = \Lambda(G) \cap \mathcal{S}$ for some subspace $\mathcal{S} \subseteq \Lambda(G) \otimes \mathbb{Q}$ (resp. as seminormal connected unions of reductive varieties); projective (stable) reductive varieties are the projectivizations of affine ones. Affine reductive varieties provide examples of algebraic semigroups without unit.

Alexeev and Brion gave a combinatorial classification and described the orbital decomposition for stable reductive varieties in the spirit of Theorems 27.5, 27.6. They constructed moduli spaces for affine stable reductive varieties embedded in a $G \times G$-module and for stable reductive pairs, i.e., projective stable reductive varieties with a distinguished effective ample divisor containing no $G \times G$-orbit.

An interesting family of reductive varieties was introduced by Vinberg [Vin2]. Consider the group $\widehat{G} = (G \times T)/Z$, where $Z = \{(t^{-1}, t) \mid t \in Z(G)\}$. The cone $\mathcal{C} \subset \mathcal{E}(\widehat{G})$ spanned by (the projections to $\mathcal{E}$ of) $(\alpha_i^\vee, 0)$ and $(-\gamma, \gamma)$, $\gamma \in \mathcal{V}(G)$, defines a normal reductive monoid $\operatorname{Env} G$, called the *enveloping semigroup* of G, with group of invertibles $\widehat{G}$. The projection $\mathcal{E}(\widehat{G}) \to \mathcal{E}(T/Z(G))$ maps $\mathcal{C}$ onto $\mathbf{C}$. Hence by Theorem 15.2 we have an equivariant map $\pi_G : \operatorname{Env} G \to \mathbb{A}^l$, where $G \times G$ acts on $\mathbb{A}^l$ trivially and T acts with the weights $\alpha_1, \dots, \alpha_l$.

The algebra $\Bbbk[\operatorname{Env} G] = \bigoplus_{\chi \in \lambda + \mathbb{Z}_+\Pi} \mathrm{M}(V(\lambda)) \otimes \Bbbk\chi$ is a free module over $\Bbbk[\mathbb{A}^l] = \Bbbk[\mathbb{Z}_+\Pi]$ and $\Bbbk[\operatorname{Env} G]^{U^- \times U} = \Bbbk[\mathfrak{X}_+] \otimes \Bbbk[\mathbb{Z}_+\Pi]$, i.e., all schematic fibers of π_G have the same algebra of $U^- \times U$-invariants $\Bbbk[\mathfrak{X}_+]$. Hence π_G is flat and all its fibers are reduced and irreducible by Theorem A2.1(1), i.e., $\operatorname{Env} G$ is the total space of a family of reductive varieties. (In fact, $\mathbb{A}^l = (\operatorname{Env} G) /\!\!/ (G \times G)$ and π_G is the categorical quotient map.)

It is easy to see that the fibers of π_G over points with nonzero coordinates are isomorphic to G. Degenerate fibers are obtained from G by a deformation of the multiplication law in $\Bbbk[G]$. In particular, the "most degenerate" fiber $\operatorname{As} G := \pi_G^{-1}(0)$, called the *asymptotic semigroup* of G, is just the horospherical contraction of G (see §7). In a sense, the asymptotic semigroup reflects the behavior of G at infinity.

The enveloping semigroup is used in [AB1, 7.5] to construct families of affine reductive varieties with given generic fiber X: $\operatorname{Env} X = (\operatorname{Env} G \times X)/\!\!/G$, where G acts as $\{e\} \times \operatorname{diag} G \times \{e\} \subset G \times G \times G \times G$, so that $\Bbbk[\operatorname{Env} X] = \bigoplus_{\chi \in \lambda + \mathbb{Z}_+\Pi} \Bbbk[X]_{(\lambda)} \otimes \Bbbk\chi \subseteq \Bbbk[X \times T]$. The map π_G induces a flat morphism $\pi_X : \operatorname{Env} X \to \mathbb{A}^l$ with reduced and irreducible fibers.

It was proved in [AB1, 7.6] that π_X is a locally universal family of reductive varieties with generic fiber X, i.e., every flat family of affine reductive varieties with reduced fibers over irreducible base is locally a pullback of π_X. The universal property for enveloping semigroups was already noticed in [Vin2].

Example 27.8. Let us describe the enveloping semigroup of $G = \mathrm{SL}_n(\Bbbk)$, using the notation of Example 27.3. Here $\Lambda_+(\operatorname{Env} G)$ is generated by (ω_i, ω_i), $(0, \alpha_i)$, $i = 1, \dots, n-1$. Recall that $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$ is the highest weight of $\bigwedge^i \Bbbk^n$. Thus $\operatorname{Env} \mathrm{SL}_n(\Bbbk)$ is the closure in $\mathrm{L}(V)$ of the image of $\mathrm{SL}_n(\Bbbk) \times T$ acting on $V = \bigwedge^\bullet \Bbbk^n \oplus \Bbbk^{n-1}$, where $\mathrm{SL}_n(\Bbbk)$ acts on $\bigwedge^\bullet \Bbbk^n$ in a natural way, and T acts on $\bigwedge^k \Bbbk^n$ by the weight $\varepsilon_1 + \dots + \varepsilon_k$ and on $\Bbbk^{n-1}$ by the weights $\varepsilon_i - \varepsilon_{i+1}$. In other words, the image of $\mathrm{SL}_n(\Bbbk) \times T$ consists of tuples of the form

$$(t_1 g, \dots, t_1 \cdots t_k \textstyle\bigwedge^k g, \dots, t_1/t_2, \dots, t_{n-1}/t_n)$$

where $g \in \mathrm{SL}_n(\Bbbk)$, $t = \operatorname{diag}(t_1, \dots, t_n) \in T$ $(t_1 \cdots t_n = 1)$. It follows that

$$\operatorname{Env} \mathrm{SL}_n(\Bbbk) = \Big\{ (a_1, \dots, a_k, \dots, z_1, \dots, z_{n-1}) \;\Big|\; a_k \in \mathrm{L}(\textstyle\bigwedge^k \Bbbk^n),\ z_i \in \Bbbk, \\ a_k \wedge a_l = a_{k+l} \displaystyle\prod_{\substack{i=1,\dots,k \\ j=1,\dots,l}} z_{i+j-1},\ a_n = 1 \Big\}$$

In particular, $\operatorname{Env} \mathrm{SL}_2(\Bbbk) = \mathrm{L}_2(\Bbbk)$ and $\operatorname{As} \mathrm{SL}_2(\Bbbk)$ is the subsemigroup of degenerate matrices.

28 S-varieties

Horospherical varieties of complexity 0 form another class of spherical varieties whose structure and embedding theory is understood better than in the general case.

Definition 28.1. An *S-variety* is an equivariant embedding of a horospherical homogeneous space $\mathcal{O} = G/S$.

This terminology is due to Popov and Vinberg [VP], though they considered only the affine case. General S-varieties were studied by Pauer [Pau1], [Pau2] in the case, where S is a maximal unipotent subgroup of G.

S-varieties are spherical. We shall examine them from the viewpoint of the Luna–Vust theory. In order to apply it, we have to describe the colored space $\mathcal{E} = \mathcal{E}(\mathcal{O})$.

It is convenient to assume $S \supseteq U^-$; then $S = L_0 \leftthreetimes P_{\mathrm{u}}^-$ for a certain parabolic $P \supseteq B$ with the Levi subgroup $L \supseteq L_0 \supseteq L'$ and the unipotent radical P_{u} (Lemma 7.1). We may assume that $L \supseteq T$. Put $T_0 = T \cap L_0$.

We have $\Lambda(\mathcal{O}) = \mathfrak{X}(A)$, where $A = P^-/S \simeq L/L_0 \simeq T/T_0$. By Theorem 21.3, $\mathcal{V}(\mathcal{O}) = \mathcal{E}$. The space $\mathcal{E} = \mathfrak{X}^*(A) \otimes \mathbb{Q}$ may be identified with the orthocomplement of $\mathfrak{X}^*(T_0) \otimes \mathbb{Q}$ in $\mathfrak{X}^*(T) \otimes \mathbb{Q}$. It follows from the Bruhat decomposition that the B-divisors on $\mathcal{E}$ are of the form $D_\alpha = \overline{Br_\alpha o}$, $\alpha \in \Pi \setminus \Pi_0$, where $\Pi_0 \subseteq \Pi$ is the simple root system of L. An argument similar to that in §27 shows that D_α maps to $\overline{\alpha^\vee}$, the image of $\alpha^\vee$ under the projection $\mathfrak{X}^*(T) \to \mathfrak{X}^*(A)$.

Theorem 15.1(3) says that normal S-varieties are classified by colored fans in $\mathcal{E}$, each fan consisting of finitely many colored cones $(\mathcal{C}_i, \mathcal{R}_i)$, so that the cones $\mathcal{C}_i$ form a polyhedral fan in $\mathcal{E}$, $\mathcal{R}_i \subseteq \mathcal{D}^B$, and each $\mathcal{C}_i \setminus \{0\}$ contains all $\overline{\alpha^\vee}$ such that $D_\alpha \in \mathcal{R}_i$. The colored cones in a fan correspond to the G-orbits Y_i in the respective S-variety X, and X is covered by simple open S-subvarieties $X_i = \{x \in X \mid \overline{Gx} \supseteq Y_i\}$.

The following result "globalizing" Theorem 15.4 is a nice example of how the combinatorial embedding theory of §15 helps to clarify the geometric structure of S-varieties. For any G-orbit $Y \subseteq X$ let $P(Y) = P[\mathcal{D}^B \setminus \mathcal{D}^B_Y]$ be the normalizer of generic B-orbits in Y and $S(Y) \subseteq P(Y)$ the normalizer of generic U-orbits, so that $S(Y)^-$ is the stabilizer of $G : Y$ (see §7). The Levi subgroup $L(Y) \subseteq P(Y)$ containing T has the simple root system $\Pi_0 \cup \{\alpha \in \Pi \mid D_\alpha \in \mathcal{D}^B_Y\}$, and $S(Y) = L(Y)_0 \leftthreetimes P(Y)_{\mathrm{u}}$, where the Levi subgroup $L(Y)_0$ is intermediate between $L(Y)$ and $L(Y)'$ and is in fact the common kernel of all characters in $\Lambda(Y)$ or in $\mathfrak{X}(A) \cap \mathcal{C}_Y^\perp$.

Theorem 28.1 (cf. [Pau1, 5.4]). *Let X be a simple normal S-variety with the unique closed G-orbit $Y \subseteq X$.*

(1) *There exists a $P(Y)$-stable affine closed subvariety $Z \subseteq X$ such that $P(Y)_{\mathrm{u}}^-$ acts on Z trivially and $X \simeq G *_{P(Y)^-} Z$.*

(2) *There exists an $S(Y)$-stable closed subvariety $Z_0 \subseteq Z$ with a fixed point such that $Z \simeq P(Y)^- *_{S(Y)^-} Z_0 \simeq L(Y) *_{L(Y)_0} Z_0$ and $X \simeq G *_{S(Y)^-} Z_0$.*

(3) *The varieties Z and Z_0 are equivariant affine embeddings of $L(Y)/L(Y)\cap S$ and $L(Y)_0/L(Y)_0\cap S$ whose weight lattices are $\mathfrak{X}(A)$ and $\mathfrak{X}(A)/\mathfrak{X}(A)\cap\mathcal{C}_Y^{\perp}$, colored spaces are $\mathcal{E}$ and $\mathcal{E}_0 := \langle\mathcal{C}_Y\rangle$, and colored cones coincide with $(\mathcal{C}_Y,\mathcal{D}_Y^B)$.*

Proof. The idea of the proof is to construct normal affine S-varieties Z and Z_0 with the colored data as in (3) and then to verify that the colored data of $L(Y)*_{L(Y)_0}Z_0$ coincide with those of Z and the colored data of $G*_{P(Y)^-}Z$ with those of X. In each case both varieties under consideration are simple normal embeddings of one and the same homogeneous space. The restriction of B-eigenfunctions to the fiber of each homogeneous bundle above preserves the orders along B-stable divisors. It follows that the colored cones of both varieties coincide with the colored cone of the fiber, whence the varieties are isomorphic. Note that Z_0 contains a fixed point since it is determined by a colored cone of full dimension. □

The theorem shows that the local geometry of (normal) S-varieties is completely reduced to the affine case (even to affine S-varieties with a fixed point). Affine S-varieties were studied in [VP] in characteristic 0 and in [Gro2, §17] in arbitrary characteristic.

First note that $\mathcal{O}$ is quasiaffine iff all $\overline{\alpha^\vee}$ are nonzero (whenever $\alpha\in\Pi\setminus\Pi_0$) and generate a strictly convex cone in $\mathcal{E}$ (Corollary 15.2). This holds iff there exists a dominant weight λ such that $\langle\lambda,\Pi^\vee\setminus\Pi_0^\vee\rangle>0$ and $\lambda|_{T_0}=1$, i.e, iff S is regularly embedded in the stabilizer of a highest weight vector of weight λ (cf. Theorem 3.7).

Theorem 28.2. *Let X be the normal affine S-variety determined by a colored cone $(\mathcal{C},\mathcal{D}^B)$. Then*

$$\Bbbk[X]\simeq\bigoplus_{\lambda\in\mathfrak{X}(A)\cap\mathcal{C}^\vee}V^*(\lambda^*)\subseteq\Bbbk[G/S]=\bigoplus_{\lambda\in\mathfrak{X}(A)\cap\mathbf{C}}V^*(\lambda^*)$$

If the semigroup $\mathfrak{X}(A)\cap\mathcal{C}^\vee$ is generated by dominant weights $\lambda_1,\dots,\lambda_m$, then $X\simeq\overline{Gv}\subseteq V(\lambda_1^*)\oplus\dots\oplus V(\lambda_m^*)$, where $v=v_{-\lambda_1}+\dots+v_{-\lambda_m}$ is the sum of respective lowest weight vectors.

Proof. Observe that $R=\bigoplus_{\lambda\in\mathfrak{X}(A)\cap\mathcal{C}^\vee}V^*(\lambda^*)$ is the largest subalgebra of $\Bbbk[G/S]$ with the given algebra of U-invariants $R^U=\Bbbk[X]^U\simeq\Bbbk[\mathfrak{X}(A)\cap\mathcal{C}^\vee]$. Hence $R\supseteq\Bbbk[X]\supseteq\langle G\cdot R^U\rangle$, and the extension is integral by Lemma A2.2. Now $R=\Bbbk[X]$ since $\Bbbk[X]$ is integrally closed.

It is easy to see that $\overline{Gv}$ is an affine embedding of $\mathcal{O}$ such that $\Bbbk[\overline{Gv}]$ is generated by $V^*(\lambda_1^*)\oplus\dots\oplus V^*(\lambda_m^*)\subset\Bbbk[\mathcal{O}]$. By Lemma 2.2, $\Bbbk[\overline{Gv}]=R$. □

Every (even non-normal) affine S-variety with the open orbit $\mathcal{O}$ is realized in a G-module V as $X = \overline{Gv}$, $v \in V^S$. We may assume $V = \langle Gv \rangle$ and decompose $v = v_{-\lambda_1} + \dots + v_{-\lambda_m}$, where $v_{-\lambda_i}$ are lowest vectors of certain antidominant weights $-\lambda_i$.

In characteristic zero, $V \simeq V(\lambda_1^*) \oplus \dots \oplus V(\lambda_m^*)$ and the same arguments as in the proof of Theorem 28.2 show that $\Bbbk[X] = \bigoplus_{\lambda \in \Sigma} V^*(\lambda^*)$, where Σ is the semigroup generated by $\lambda_1, \dots, \lambda_m$, and the dual Weyl modules $V^*(\lambda^*) \simeq \Bbbk[G]^{(B)}_{\lambda^*} \simeq V(\lambda)$ are the (simple) G-isotypic components of $\Bbbk[X]$. It is easy to see that $Gv \simeq \mathcal{O}$ iff $\lambda_1, \dots, \lambda_m$ span $\mathfrak{X}(A)$. Thus we obtain the following

Proposition 28.1 ([VP, 3.1, 3.4]). *In the case* $\operatorname{char} \Bbbk = 0$*, affine S-varieties X with the open orbit $\mathcal{O}$ bijectively correspond to finitely generated semigroups Σ of dominant weights spanning $\mathfrak{X}(A)$, via $\Sigma = \Lambda_+(X)$. The variety X is normal iff the semigroup Σ is saturated, i.e., $\Sigma = \mathbb{Q}_+\Sigma \cap \mathfrak{X}(A)$. Moreover, the saturation $\widetilde{\Sigma} = \mathbb{Q}_+\Sigma \cap \mathfrak{X}(A)$ of Σ corresponds to the normalization $\widetilde{X}$ of X.*

G-orbits in an affine S-variety $X = \overline{Gv} \subseteq V$ have a transparent description "dual" to that in Theorem 15.1(4).

Proposition 28.2 ([VP, Th.8]). *The orbits in X are in bijection with the faces of $\mathcal{C}^\vee = \mathbb{Q}_+\lambda_1 + \dots + \mathbb{Q}_+\lambda_m$. The orbit corresponding to a face $\mathcal{F}$ is represented by $v_{\mathcal{F}} = \sum_{\lambda_i \in \mathcal{F}} v_{\lambda_i}$. The adherence of orbits agrees with the inclusion of faces.*

Proof. We have $X = G\overline{Tv}$ since $\overline{Tv}$ is B-stable. The T-orbits in $\overline{Tv}$ are represented by $v_{\mathcal{F}}$ over all faces $\mathcal{F} \subseteq \mathcal{C}^\vee$, and the adherence of orbits agrees with the inclusion of faces. On the other hand, it is easy to see that the U^--fixed point set in each G-orbit of X is a T-orbit, hence distinct $v_{\mathcal{F}}$ represent distinct G-orbits. ∎

In characteristic zero, one can describe the defining equations of X in V. Let $c = \sum \xi_i \xi_i^* \in \mathrm{U}\mathfrak{g}$ be the Casimir element w.r.t. to a G-invariant inner product on $\mathfrak{g}$, ξ_i, ξ_i^* being mutually dual bases. It is well known that c acts on $V(\lambda^*)$ by a scalar $c(\lambda) = (\lambda + 2\rho, \lambda)$. Note that $c(\lambda)$ depends on λ monotonously w.r.t. the partial order induced by positive roots: if $\lambda = \mu + \sum k_i\alpha_i$, $k_i \geq 0$, then $c(\lambda) = c(\mu) + \sum k_i\big((\lambda + 2\rho, \alpha_i) + (\alpha_i, \mu)\big) \geq c(\mu)$, and the inequality is strict, except for $\lambda = \mu$. The following result is due to Kostant:

Proposition 28.3 ([LT]). *If* $\operatorname{char}\Bbbk = 0$ *and* $\lambda_1, \dots, \lambda_m$ *are linearly independent, then* $\mathbb{I}(X) \lhd \Bbbk[V]$ *is generated by the relations*

$$c(x_i \otimes x_j) = (\lambda_i + \lambda_j + 2\rho, \lambda_i + \lambda_j)(x_i \otimes x_j), \qquad i, j = 1, \dots, m,$$

where x_k *denotes the projection of* $x \in V$ *to* $V(\lambda_k^*)$.

Proof. The algebra $\Bbbk[V] = \bigoplus_{k_1,\dots,k_m} \mathrm{S}^{k_1}V(\lambda_1) \otimes \cdots \otimes \mathrm{S}^{k_m}V(\lambda_m)$ is multigraded and $\mathbb{I}(X)$ is a multihomogeneous ideal. The structure of $\Bbbk[X]$ implies that each homogeneous component $\mathbb{I}(X)_{k_1,\dots,k_m}$ is the kernel of the natural map $\mathrm{S}^{k_1}V(\lambda_1) \otimes \cdots \otimes \mathrm{S}^{k_m}V(\lambda_m) \to V(k_1\lambda_1 + \cdots + k_m\lambda_m)$.

Consider a series of linear endomorphisms $\pi = c - c(\sum k_i\lambda_i)\mathbf{1}$ of the subspaces $\mathrm{S}^{k_1,\dots,k_m}V = \mathrm{S}^{k_1}V(\lambda_1^*) \otimes \cdots \otimes \mathrm{S}^{k_m}V(\lambda_m^*) \subset \mathrm{S}^\bullet V$. Note that $\operatorname{Ker}\pi \simeq V(\sum k_i\lambda_i^*)$ is the highest irreducible component of $\mathrm{S}^{k_1,\dots,k_m}V$, annihilated by $\mathbb{I}(X)_{k_1,\dots,k_m}$, and $\operatorname{Im}\pi \simeq \mathbb{I}(X)^*_{k_1,\dots,k_m}$ is the complementary G-module.

It follows that $\mathbb{I}(X)$ is spanned by the coordinate functions of all $\pi(x_1^{k_1} \cdots x_m^{k_m})$. An easy calculation shows that

$$\pi(x_1^{k_1} \cdots x_m^{k_m}) = \sum_i \frac{k_i(k_i-1)}{2}\pi(x_i^2)x_1^{k_1} \cdots x_i^{k_i-2} \cdots x_m^{k_m} + \sum_{i<j} k_ik_j\pi(x_ix_j)x_1^{k_1} \cdots x_i^{k_i-1} \cdots x_j^{k_j-1} \cdots x_m^{k_m}$$

Thus $\mathbb{I}(X)$ is generated by the relations $\pi(x_ix_j) = 0$, $i, j = 1, \dots, m$. □

If the generators of $\Lambda_+(X)$ are not linearly independent, one has to extend the defining equations of X by those arising from the linear dependencies between the λ_i's, see [Sm-E].

The results of §17 allow to compute the divisor class group of a normal affine S-variety X. Every Weil divisor is rationally equivalent to a B-stable one $\delta = \sum m_\alpha D_\alpha + \sum m_iY_i$, where Y_i are the G-stable prime divisors corresponding to the generators v_i of the rays of $\mathcal{C}$ containing no colors. The divisor δ is principal iff $m_\alpha = \langle\lambda, \alpha^\vee\rangle$ and $m_i = \langle\lambda, v_i\rangle$ for a certain $\lambda \in \mathfrak{X}(A)$. This yields a finite presentation for $\operatorname{Cl} X$. In particular, we have

Proposition 28.4. *An affine S-variety* X *is factorial iff* $\Lambda_+(X)$ *is generated by weights* $\lambda_1 \dots, \lambda_s, \pm\lambda_{s+1}, \dots, \pm\lambda_r$ $(s \leq r)$*, where the* λ_i*'s are linearly independent and the projection* $\mathfrak{X}(T) \to \mathfrak{X}(T \cap G')$ *maps them to distinct fundamental weights or to* 0.

For semisimple G, we conclude that factorial S-varieties are those corresponding to weight semigroups Σ generated by some of the fundamental weights [VP, Th.11].

The simplest class of affine S-varieties is formed by *HV-varieties*, i.e., cones of highest (or lowest) vectors $X = \overline{Gv_{-\lambda}}$, $v_{-\lambda} \in V^{(B^-)}$, see §11. Particular examples are quadratic cones or Grassmann cones of decomposable polyvectors. The above results on affine S-varieties imply Proposition 11.1, which describes basic properties of HV-varieties. It follows from Proposition 28.3 that an HV-cone is defined by quadratic equations in the ambient simple G-module. For a Grassmann cone we recover the Plücker relations between the coordinates of a polyvector.

Now we describe smooth S-varieties in characteristic zero. By Theorem 28.1, the problem is reduced to affine S-varieties with a fixed point, which are nothing else but G-modules with a dense orbit of a U-fixed vector.

Lemma 28.1. *If a G-module V is an S-variety, then $V = V_0 \oplus V_1 \oplus \dots \oplus V_s$ so that $Z = Z(G)^0$ acts on V_0 with linearly independent weights and each V_i $(i > 0)$ is a simple submodule acted on non-trivially by a unique simple factor $G_i \subseteq G$, $G_i \simeq \mathrm{SL}(V_i)$ or $\mathrm{Sp}(V_i)$.*

Proof. Since Z has a dense orbit in $V_0 = V^{G'}$, it acts with linearly independent weights. If G_i acts non-trivially on two simple submodules V_i, V_j, and $v_i \in V_i^{(B)}$, $v_j \in V_j^{(B^-)}$, then the stabilizer of $v_i + v_j$ is not horospherical, i.e., V is not an S-variety. Therefore we may assume that V is irreducible and each simple factor of G acts non-trivially.

Then G acts transitively on $\mathbb{P}(V)$, which implies $G' \simeq \mathrm{SL}(V)$ or $\mathrm{Sp}(V)$ [Oni2]. Indeed, we have $V = \mathfrak{b}v_{-\lambda}$, where $v_{-\lambda} \in V$ is a lowest vector. Hence there exists a unique root δ such that $e_\delta v_{-\lambda} = v_{\lambda^*}$ is a highest vector. One easily deduces that the root system of G is indecomposable and δ is the highest root, so that $\delta = \lambda + \lambda^*$ is the sum of two dominant weights, whence the assertion. □

The colored data of such a G-module V are easy to write down. Namely $\Pi \setminus \Pi_0 = \{\alpha_1, \dots, \alpha_s\}$, where α_i are the first simple roots in some components of Π having the type $\mathbf{A}_l$ or $\mathbf{C}_l$. The weight lattice $\mathfrak{X}(A)$ is spanned by linearly independent weights $\lambda_1, \dots, \lambda_r$, where $\lambda_1, \dots, \lambda_s$ are the highest weights of V_i^*, which project to the fundamental weights ω_i corresponding to α_i, and $\lambda_{s+1}, \dots, \lambda_r$ are the weights of V_0^*, which are orthogonal to Π. The cone $\mathcal{C}$ is spanned by the basis $\overline{\alpha_1^\vee}, \dots, \overline{\alpha_s^\vee}, v_{s+1}, \dots, v_r$ of $\mathfrak{X}^*(A)$ dual to $\lambda_1, \dots, \lambda_r$. Using Theorem 28.1 we derive the description of colored data of arbitrary smooth S-varieties:

Theorem 28.3 (cf. [Pau2, 3.5]). *An S-variety X is smooth iff all colored cones $(\mathcal{C}_Y, \mathcal{D}_Y^B)$ in the colored fan of X satisfy the following properties:*

(1) *$\mathcal{C}_Y$ is generated by a part of a basis of $\mathfrak{X}^*(A)$, and all $\overline{\alpha^\vee}$ such that $D_\alpha \in \mathcal{D}_Y^B$ are among the generators.*

(2) *The simple roots α such that $D_\alpha \in \mathcal{D}_Y^B$ are isolated from each other at the Dynkin diagram of G, and each α is connected with at most one component Π_α of Π_0; moreover, $\{\alpha\} \cup \Pi_\alpha$ has the type $\mathbf{A}_l$ or $\mathbf{C}_l$, α being the first simple root therein.*

The condition (1) is equivalent to the local factoriality of X.

29 Toroidal embeddings

In this section we assume $\operatorname{char} \Bbbk = 0$. Recall that a G-equivariant normal embedding X of a spherical homogeneous space $\mathcal{O} = G/H$ is said to be *toroidal* if $\mathcal{D}_Y^B = \emptyset$ for each G-orbit $Y \subseteq X$. Toroidal embeddings are defined by fans in $\mathcal{V}$ and G-morphisms between them correspond to subdivisions of these fans in the same way as in toric geometry [Ful2]. There is a more direct relation between toroidal and toric varieties. Put $P = P(\mathcal{O})$, with the Levi decomposition $P = LP_{\mathrm{u}}$ and other notation from §7.

Theorem 29.1 ([BPa, 3.4], [Bri14, 2.4]). *A toroidal embedding $X \hookleftarrow \mathcal{O}$ is covered by G-translates of an open P-stable subset*

$$\mathring{X} = X \setminus \bigcup_{D \in \mathcal{D}^B} D \simeq P *_L Z \simeq P_{\mathrm{u}} \times Z$$

where Z is a locally closed L-stable subvariety pointwise fixed by L_0. The variety Z is a toric embedding of $A = L/L_0$ defined by the same fan as X, and the G-orbits in X intersect Z in A-orbits.

Proof. The problem is easily reduced to the case, where X contains a unique closed orbit Y with $\mathcal{C}_Y = \mathcal{V}$, $\mathcal{D}_Y^B = \emptyset$. Such toroidal embeddings, called wonderful, are discussed in §30. Indeed, consider another spherical homogeneous space $\overline{\mathcal{O}} = G/N(H)$. Then $\overline{\mathcal{V}} = \mathcal{V}(\overline{\mathcal{O}}) = \mathcal{V}/(\mathcal{V} \cap -\mathcal{V})$ is strictly convex, whence there exists a wonderful embedding $\overline{X} \hookleftarrow \overline{\mathcal{O}}$. The canonical map $\phi : \mathcal{O} \to \overline{\mathcal{O}}$ extends to $X \to \overline{X}$ by Theorem 15.2. We have $P(\overline{\mathcal{O}}) = P$, and $\mathring{X}, Z$ are the preimages of the respective subvarieties defined for $\overline{X}$. The assertion on fans and orbits is easy, cf. Remark 15.2.

For wonderful X one applies the local structure theorem in a neighborhood of Y: by Theorem 15.4, $\mathring{X} = \mathring{X}_Y \simeq P *_L Z$, where Z is toric since $Z \cap \mathcal{O}$ is a single A-orbit. $\square$

It follows that toroidal varieties are locally toric and have at worst Abelian quotient singularities. They inherit many nice geometric properties from toric varieties. On the other hand, each spherical variety is the image of a toroidal one by a proper birational equivariant map: to obtain this toroidal covering variety, just remove all colors from the fan. This universality of toroidal varieties can be used to derive some properties of spherical varieties from the toroidal case.

A toroidal variety is smooth iff all cones of its fan are simplicial and generated by a part of a basis of $\Lambda(\mathcal{O})^*$: for toric varieties this is deduced from the description of the coordinate algebra [Ful2, 2.1] (cf. Example 15.1) and the general case follows by Theorem 29.1. For a singular toroidal variety one may construct an equivariant desingularization by subdividing its fan, cf. [Ful2, 2.6].

Every (smooth) toroidal variety admits an equivariant (smooth) completion, which is defined by adding new cones to the fan in order to cover all of $\mathcal{V}(\mathcal{O})$. Smooth complete toroidal varieties have other interesting characterizations.

Theorem 29.2 ([BiB]). *For a smooth G-variety X consider the following conditions:*

(1) *X is toroidal.*

(2) *There is a dense open orbit $\mathcal{O} \subseteq X$ such that $\partial X = X \setminus \mathcal{O}$ is a divisor with normal crossings, each orbit $Gx \subset X$ is locally the intersection of several components of ∂X, and G_x has a dense orbit in $T_xX/\mathfrak{g}x$.*

(3) *There is a G-stable divisor $D \subset X$ with normal crossings such that $\mathcal{G}_X = \mathcal{T}_X(-\log D)$.*

(4) *X is spherical and pseudo-free.*

Then (4) $\implies$ (1) $\implies$ (2) $\iff$ (3). *If X is complete or spherical, then all conditions are equivalent.*

G-varieties satisfying the condition (2), resp. (3), are known as *regular* in the sense of Bifet–de Concini–Procesi [BCP], resp. of Ginzburg [Gin].

Proof. (1) $\implies$ (2)&(3) Theorem 29.1 reduces the problem to smooth toric varieties. The latter are covered by invariant affine open charts of the form $X = \mathbb{A}^s \times (\mathbb{A}^1 \setminus 0)^{r-s}$, where $(\Bbbk^\times)^r$ acts in the natural way, so that $D = \partial X$ is the union of coordinate hyperplanes $\{x_i = 0\}$, X is isomorphic to the normal bundle of the closed orbit, and $\mathcal{T}_X(-\log D)$ is a free sheaf spanned by velocity fields $x_1\partial_1, \dots, x_n\partial_n$ ($\partial_i := \partial/\partial x_i$).

(2) $\iff$ (3) First observe that $\mathcal{O} = X \setminus D$ is a single G-orbit iff $\mathcal{G}_{X\setminus D} = \mathcal{T}_{X\setminus D}$. Now consider a neighborhood of any $x \in D$. Due to local nature of the conditions (2), (3), we may assume that all components $D_1, \dots, D_k$ of D contain x. Choose local parameters $x_1, \dots, x_n$ at x such that D_i are locally defined by the equations $x_i = 0$. Let $\partial_1, \dots, \partial_n$ denote the vector fields dual to $dx_1, \dots, dx_n$. Then $\mathcal{T}_X(-\log D)$ is locally generated by $x_1\partial_1, \dots, x_k\partial_k, \partial_{k+1}, \dots, \partial_n$.

Let $Y = D_1 \cap \dots \cap D_k$ and $\pi : N = \operatorname{Spec} \mathrm{S}^\bullet(\mathcal{I}_Y/\mathcal{I}_Y^2) \to Y$ be the normal bundle. There is a natural embedding $\pi^*\mathcal{T}_X(-\log D)|_Y \hookrightarrow \mathcal{T}_N$: each vector field in $\mathcal{T}_X(-\log D)$ preserves $\mathcal{I}_Y$ whence induces a derivation of $\mathrm{S}^\bullet(\mathcal{I}_Y/\mathcal{I}_Y^2)$. The image of $\pi^*\mathcal{T}_X(-\log D)|_X$ is $\mathcal{T}_N(-\log \bigcup N_i)$, where N_i are the normal bundles to Y in D_i. Indeed, $\bar{x}_i = x_i \bmod \mathcal{I}_Y^2$ $(i \le k)$, $\bar{x}_j = \pi^* x_j|_Y$ $(j > k)$ are local parameters on N and $x_i\partial_i, \partial_j$ induce the derivations $\bar{x}_i\bar{\partial}_i, \bar{\partial}_j$. Note that $N = \bigoplus L_i$, where $L_i = \bigcap_{j\ne i} N_j$ are G-stable line subbundles. Hence the G_x-action on $T_xX/T_xY = N(x) = \bigoplus L_i(x)$ is diagonalizable.

Condition (2) implies that Gx is open in Y and the weights of $G_x : L_i(x)$ are linearly independent. This yields velocity fields $\bar{x}_i\bar{\partial}_i$ on $N(x)$ and in transversal directions, which locally generate $\mathcal{T}_N(-\log \bigcup N_i)$. Therefore $\mathcal{T}_X(-\log D)|_Y$ is generated by velocity fields. By Nakayama's lemma, $\mathcal{T}_X(-\log D) = \mathcal{G}_X$ in a neighborhood of x.

Conversely, (3) implies $\mathcal{G}_Y = \mathcal{T}_Y$ and $\mathcal{G}_N = \mathcal{T}_N(-\log \bigcup N_i)$. Hence Gx is open in Y and $N|_{Gx} = G *_{G_x} T_xX/\mathfrak{g}x$ has an open G-orbit. Thus $T_xX/\mathfrak{g}x$ contains an open G_x-orbit.

(2)&(3) $\implies$ (4) Since $\mathcal{T}_X(-\log \partial X)$ is a vector bundle, the implication is trivial provided that X is spherical. It remains to prove that X is spherical if it is complete.

A closed orbit $Y \subseteq X$ intersects a B-chart $\mathring{X} \simeq P *_L Z$, where $L \subseteq P = P(Y)$ is the Levi subgroup and Z is an L-stable affine subvariety intersecting Y in a single point z. Since the maximal torus $T \subseteq L \subseteq G_z = P^-$ acts on $T_zZ \simeq T_zX/\mathfrak{g}z$ with linearly independent weights, $Z \simeq T_zZ$ contains an open T-orbit, whence $\mathring{X}$ has an open B-orbit.

(4) $\implies$ (1) There is a morphism $X \to \mathrm{Gr}(\mathfrak{g})$, $x \mapsto [\mathfrak{h}_x]$, extending the map $x \mapsto [\mathfrak{g}_x]$ on $\mathcal{O}$. If X is not toroidal, then there exists a G-orbit $Y \subset X$ contained in a B-divisor $D \subset X$. Then we have $\mathfrak{b} + \mathfrak{h}_{gy} = \mathfrak{b} + (\operatorname{Ad} g)\mathfrak{h}_y \ne \mathfrak{g}$, $\forall y \in Y$, $g \in G$, i.e., $\mathfrak{h}_y$ is not spherical.

To obtain a contradiction, it suffices to prove that all $\mathfrak{h}_x$ are spherical subalgebras. Passing to a toroidal variety mapping onto X, one may assume that X itself is toroidal. Consider the normal bundle N to $Y = Gx$. Since

$\mathcal{G}_N = \pi^*\mathcal{G}_X|_Y$, $\mathfrak{h}_x$ is the stabilizer subalgebra of general position for $G : N$. But N is spherical, because the minimal B-chart $\mathring{X}$ of Y is P-isomorphic to $N|_{Y\cap\mathring{X}}$. □

Toric varieties and generalized flag varieties form two "extreme" classes of toroidal varieties. A number of geometric and cohomological results generalize from these particular cases to general toroidal varieties. A powerful vanishing theorem was proved by Bien and Brion (1991) and refined by Knop (1992).

Theorem 29.3 ([BiB]). *If X is a smooth complete toroidal variety, then* $\mathrm{H}^i(X, \mathrm{S}^\bullet\mathcal{T}_X(-\log\partial X)) = 0$, $\forall i > 0$.

For flag varieties, this result is due to Elkik (vanishing of higher cohomology of the tangent sheaf was proved already by Bott in 1957). In fact, Bien and Brion proved a twisted version of Theorem 29.3 [BiB, 3.2]: $\mathrm{H}^i(X, \mathcal{L}\otimes\mathrm{S}^\bullet\mathcal{T}_X(-\log\partial X)) = 0$ for all $i > 0$ and any globally generated line bundle $\mathcal{L}$ on X, under a technical condition that the stabilizer H of $\mathcal{O}$ is parabolic in a reductive subgroup of G. (Generally, higher cohomology of globally generated line bundles vanishes on every complete spherical variety, see Corollary 31.1.)

In view of Theorem 29.2, Theorem 29.3 stems from a more general vanishing result of Knop:

Theorem 29.4 ([Kn6, 4.1]). *If X is a pseudo-free equivariant completion of a homogeneous space $\mathcal{O}$, then* $\mathrm{H}^i(X, \mathcal{U}_X^m) = \mathrm{H}^i(X, \mathrm{S}^m\mathcal{G}_X) = 0$, $\forall i > 0$, $m \geq 0$.

Synopsis of a proof. The assertions on $\mathcal{U}_X$ are reduced to those on $\mathrm{S}^\bullet\mathcal{G}_X = \operatorname{gr}\mathcal{U}_X$. Since $\pi_X : T^{\mathfrak{g}}X \to X$ is an affine morphism, the Leray spectral sequence reduces the question to proving $\mathrm{H}^i(T^{\mathfrak{g}}X, \mathcal{O}_{T^{\mathfrak{g}}X}) = 0$. The localized moment map $\overline{\Phi} : T^{\mathfrak{g}}X \to M_X$ factors through $\widetilde{\Phi} : T^{\mathfrak{g}}X \to \widetilde{M}_X$. As $\widetilde{M}_X$ is affine, $\mathrm{H}^i(T^{\mathfrak{g}}X, \mathcal{O}_{T^{\mathfrak{g}}X}) = \mathrm{H}^0\big(\widetilde{M}_X, \mathrm{R}^i\widetilde{\Phi}_*\mathcal{O}_{T^{\mathfrak{g}}X}\big)$, and it remains to prove $\mathrm{R}^i\widetilde{\Phi}_*\mathcal{O}_{T^{\mathfrak{g}}X} = 0$. Here one applies to $\widetilde{\Phi}$ a version of Kollár's vanishing theorem [Kn6, 4.2]:

> If Y is smooth, Z has rational singularities, and $\phi : Y \to Z$ is a proper morphism with connected generic fibers F, which satisfy $\mathrm{H}^i(F, \mathcal{O}_F) = 0$, $\forall i > 0$, then $\mathrm{R}^i\phi_*\mathcal{O}_Y = 0$ for all $i > 0$.

It remains to verify the conditions. The morphism $\widetilde{\Phi}$ is proper by Example 8.2. The variety $\widetilde{M}_X$ has rational singularities by [Kn6, 4.3]. To show vanishing of the higher cohomology of $\mathcal{O}_F$, it suffices to prove that F is unirational [Se2]. Here one may assume $X = \mathcal{O}$, $\widetilde{\Phi} : T^*\mathcal{O} \to \widetilde{M}_{\mathcal{O}}$. Unirationality of the fibers of the moment map is the heart of the proof [Kn6, §5]. □

There is a relative version of Theorem 29.4 asserting $\mathrm{R}^i\psi_*\mathcal{U}_X^m = \mathrm{R}^i\psi_*\mathrm{S}^m\mathcal{G}_X = 0$, $\forall i > 0$, for a proper morphism $\psi : X \to Y$ separating generic orbits, where X is pseudo-free and Y has rational singularities.

The vanishing theorems of Bien–Brion and Knop have a number of important consequences. For instance, on a pseudo-free completion X of $\mathcal{O}$ the symbol map $\operatorname{gr}\mathrm{H}^0(X,\mathcal{U}_X) \to \Bbbk[T^{\mathfrak{g}}X]$ is surjective. In the toroidal case, $\mathrm{H}^1(X, \mathcal{T}_X(-\log\partial X)) = 0$ implies that the pair $(X, \partial X)$ is locally rigid, by deformation theory of Kodaira–Spencer [Bin]. Using this observation, Alexeev and Brion proved Luna's conjecture on rigidity of spherical subgroups.

Theorem 29.5 ([AB3, §3]). *For any (irreducible) G-variety with spherical (generic) orbits, the stabilizers of points in general position are conjugate.*

Proof. Let $\mathcal{X}$ be a G-variety with spherical orbits. Passing to an open subset, we may assume that $\mathcal{X}$ is smooth quasiprojective and there exists a smooth G-invariant morphism $\pi : \mathcal{X} \to Z$ whose fibers contain dense orbits. Regarding $\mathcal{X}$ as a family of spherical G-orbit closures, we may replace it by a birationally isomorphic family of smooth projective toroidal varieties.

Indeed, there is a locally closed G-embedding of $\mathcal{X}$ into $\mathbb{P}(V)$, and therefore into $\mathbb{P}(V) \times Z$, for some G-module V. Replacing $\mathcal{X}$ by its closure and taking a pseudo-free desingularization, we may assume that $\mathcal{X}$ is pseudo-free and π is a projective morphism. By Theorem 29.2, the fibers of π are smooth projective toroidal varieties. Shrinking Z if necessary, we obtain that the G-orbits of non-maximal dimension in $\mathcal{X}$ form a divisor with normal crossings $\partial\mathcal{X} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k$ whose components $\mathcal{D}_i$ are smooth over Z.

Morally, an equivariant version of Kodaira–Spencer theory should imply that all fibers of π are G-isomorphic, which should complete the proof. An alternative argument uses nested Hilbert schemes [Che].

Let X be any fiber of π, with $\partial X = D_1 \cup \cdots \cup D_k$, $D_i = \mathcal{D}_i \cap X$. Applying a suitable Veronese map, we satisfy a technical condition that the restriction map $V^* \to \mathrm{H}^0(X, \mathcal{O}(1))$ is surjective.

The nested Hilbert scheme Hilb parametrizes tuples $(Y, Y_1, \dots, Y_k)$ of projective subvarieties in $\mathbb{P}(V)$ having the same Hilbert polynomials as $X, D_1, \dots, D_k$. The varieties $\mathcal{X}, \mathcal{D}_1, \dots, \mathcal{D}_k$ are obtained as the pullbacks under $Z \to \mathrm{Hilb}$ of the universal families $\mathcal{Y}, \mathcal{Y}_1, \dots, \mathcal{Y}_k \to \mathrm{Hilb}$. The groups $\mathrm{GL}(V)$ and G act on Hilb in a natural way, so that Hilb^G parametrizes tuples of G-subvarieties. Since the centralizer $\mathrm{GL}(V)^G$ of G maps G-subvarieties to G-isomorphic ones, it suffices to prove that the $\mathrm{GL}(V)^G$-orbit of $(X, D_1, \dots, D_k)$ is open in Hilb^G.

This is done by considering tangent spaces. Let $\mathcal{N}_Z, \mathcal{N}_{Z/Z_i}$ denote the normal bundles to Z in $\mathbb{P}(V)$, resp. to Z_i in Z. Then $T_{(X,D_1,\dots,D_k)}\mathrm{Hilb} = \mathrm{H}^0(X, \mathcal{N})$, where $\mathcal{N} \subset \mathcal{N}_X \oplus \mathcal{N}_{D_1} \oplus \cdots \oplus \mathcal{N}_{D_k}$ is formed by tuples $(\xi, \xi_1, \dots, \xi_k)$ of normal vector fields such that $\xi|_{D_i} = \xi_i \bmod \mathcal{N}_{X/D_i}$, $i = 1, \dots, k$. (These vector fields define infinitesimal deformations of $X, D_1, \dots, D_k$, so that the deformation of D_i is determined by the deformation of X modulo a deformation inside X.) There are exact sequences

$$0 \longrightarrow \mathcal{T}_X(-\log \partial X) \longrightarrow \mathcal{T}_{\mathbb{P}(V)}|_X \longrightarrow \mathcal{N} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_X \longrightarrow V \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{T}_{\mathbb{P}(V)}|_X \longrightarrow 0$$

Taking cohomology yields

$$\mathrm{H}^0\left(X, \mathcal{T}_{\mathbb{P}(V)}\right) \longrightarrow T_{(X,D_1,\dots,D_k)}\mathrm{Hilb} \longrightarrow \mathrm{H}^1(X, \mathcal{T}_X(-\log \partial X)) = 0$$
$$V \otimes \mathrm{H}^0(X, \mathcal{O}(1)) \longrightarrow \mathrm{H}^0\left(X, \mathcal{T}_{\mathbb{P}(V)}\right) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) = 0$$

(The first cohomologies vanish by Theorem 29.3 and [Se2], since X is a smooth projective rational variety.) Hence the differential of the orbit map

$$\mathfrak{gl}(V) \simeq V \otimes V^* \longrightarrow V \otimes \mathrm{H}^0(X, \mathcal{O}(1)) \longrightarrow \mathrm{H}^0\left(X, \mathcal{T}_{\mathbb{P}(V)}\right) \longrightarrow T_{(X,D_1,\dots,D_k)}\mathrm{Hilb}$$

is surjective. By linear reductivity of G, the composite map

$$\mathfrak{gl}(V)^G \longrightarrow T_{(X,D_1,\dots,D_k)}(\mathrm{Hilb}^G) \subseteq \left(T_{(X,D_1,\dots,D_k)}\mathrm{Hilb}\right)^G$$

is surjective as well. Hence $(X, D_1 \dots, D_k)$ is a smooth point of Hilb^G and $\mathrm{GL}(V)^G(X, D_1 \dots, D_k)$ is open. ☐

Cohomology rings of smooth complete toroidal varieties (over $\Bbbk = \mathbb{C}$) were computed by Bifet–de Concini–Procesi [BCP], see also [LP] for toroidal completions of symmetric spaces. By Corollary 18.3, cohomology coincides with the Chow ring in this situation. The most powerful approach is through equivariant cohomology or equivariant intersection theory of Edidin–Graham, see [Bri13]. In particular, Chow (or cohomology) rings of smooth (complete) toric varieties and flag varieties are easily computed in this way [BCP, I.4], [Bri11, 2, 3], [Bri13], cf. §18.

The local structure of toroidal varieties can be refined in order to obtain a full description for the closures of generic flats.

Proposition 29.1 ([Kn5, 8.3]). *The closure of a generic twisted flat in a toroidal variety X is a normal toric variety whose fan is the W_X-span of the fan of X.*

Proof. It suffices to choose the toric slice Z in Theorem 29.1 in such a way that the open A-orbit in Z is a generic (twisted) flat F_α. Then $\overline{Z} = \overline{F}_\alpha$ (the closure in X), so that Theorem 29.1 and Proposition 23.2 imply the claim.

If X is smooth and T^*X is symplectically stable, then the conormal bundle to generic U-orbits extends to a trivial subbundle $\mathring{X} \times \mathfrak{a}^* \hookrightarrow T\mathring{X}(\log \partial X)$, the trivializing sections being $d\mathbf{f}_\lambda/\mathbf{f}_\lambda$, $\lambda \in \Lambda$. The logarithmic moment map restricts to $\Phi : \mathring{X} \times \mathfrak{a}^* \to \mathfrak{a} \oplus \mathfrak{p}_\mathrm{u}$, cf. Lemma 23.3. It follows that $\mathring{X} \simeq P *_L Z$, where $Z = \pi_X \Phi^{-1}(\lambda)$, $\lambda \in \mathfrak{a}^{\mathrm{pr}}$, and F_α is the open L-orbit in Z for $\forall \alpha \in \Phi^{-1}(\lambda) \cap T^*\mathcal{O}$.

If X is singular, then it admits a toroidal resolution of singularities $\nu : X' \to X$. Then $\mathring{X}' := \nu^{-1}(\mathring{X}) = X' \setminus \bigcup_{D \in \mathcal{D}^B} D \simeq P *_L Z'$ and $Z' \supseteq F_\alpha$. The map $\Phi : \mathring{X}' \times \mathfrak{a}^* \to \mathfrak{a} \oplus \mathfrak{p}_\mathrm{u}$ descends to $\mathring{X}$, because $\Bbbk[\mathring{X}'] = \Bbbk[\mathring{X}]$. Thus one may put $Z = \nu(Z')$.

If T^*X is not symplectically stable, then passing to affine cones and back to projectivizations yields Z such that the open L-orbit in Z is a twisted flat. □

Example 29.1. If X is a toroidal $G \times G$-embedding of G, then T is a flat and $F = \overline{T}$ is a toric variety whose fan is the W-span of the fan of X (in the antidominant Weyl chamber), cf. Proposition 27.3. For instance, if $X = \overline{G} \subseteq \mathbb{P}(\mathrm{L}(V))$ for a faithful projective representation $G : \mathbb{P}(V)$ with regular highest weights, then the fan of F is formed by the duals to the corner cones of the weight polytope $\mathcal{P}(V)$, and the fan of X is its antidominant part (see §27).

Example 29.2. Consider the variety of complete conics $X \subset \mathbb{P}(\mathrm{S}^2(\Bbbk^3)^*) \times \mathbb{P}(\mathrm{S}^2\Bbbk^3)$ from Example 17.3. The set $F = \{([q],[q^\vee]) \mid q \text{ diagonal}, \ \det q \neq 0\}$ is a flat. Using the Segre embedding $\mathbb{P}(\mathrm{S}^2(\Bbbk^3)^*) \times \mathbb{P}(\mathrm{S}^2\Bbbk^3) \hookrightarrow \mathbb{P}(\mathrm{S}^2(\Bbbk^3)^* \otimes \mathrm{S}^2\Bbbk^3)$ and observing that the T-weights occurring in the weight decomposition of $q \otimes q^\vee$ are $2(\varepsilon_i - \varepsilon_j)$, we conclude that the fan of $\overline{F}$ is the set of all Weyl chambers of $G = \mathrm{SL}_3(\Bbbk)$ together with their faces, while the fan of X consists of the antidominant Weyl chamber and its faces.

30 Wonderful varieties

In the study of a homogeneous space $\mathcal{O}$ it is useful to consider its equivariant completions. The reason is that properties of $\mathcal{O}$ and of related objects (subvarieties and their intersection, functions, line bundles and their sections, etc) often become apparent "at infinity", and equivariant completions of $\mathcal{O}$ take into account the points at infinity. Also, complete varieties behave better than non-complete ones from various points of view (e.g., in intersection theory).

Among all equivariant completions of a spherical homogeneous space $\mathcal{O}$ one distinguishes two opposite classes. Toroidal completions have nice geometry (see §29) and a universal property: each equivariant completion of $\mathcal{O}$ is dominated by a toroidal one. On the other hand, simple completions of $\mathcal{O}$ (i.e., those having a unique closed orbit) are the most "economical" ones: their boundaries are "small". Simple completions exist iff the valuation cone $\mathcal{V}$ is strictly convex.

These two classes intersect in a unique element, called the wonderful completion.

Definition 30.1. A spherical subgroup $H \subseteq G$ is called *sober* if $N(H)/H$ is finite or, equivalently, if $\mathcal{V}(G/H)$ is strictly convex.

The *wonderful embedding* of $\mathcal{O} = G/H$ is the unique toroidal simple complete G-embedding $X \hookleftarrow \mathcal{O}$, defined by the colored cone $(\mathcal{V}, \emptyset)$, provided that H is sober.

The wonderful embedding has a universal property: for any toroidal completion $X' \hookleftarrow \mathcal{O}$ and any simple completion $X'' \hookleftarrow \mathcal{O}$, there exist unique proper birational G-morphisms $X' \to X \to X''$ extending the identity map on $\mathcal{O}$.

Wonderful embeddings were first introduced by De Concini and Procesi [CP1] for symmetric spaces. Their remarkable properties were studied by many researchers (see below) mainly in characteristic zero, though some results in special cases, e.g., for symmetric spaces [CS], are obtained in arbitrary characteristic. For simplicity, we assume $\operatorname{char} \Bbbk = 0$ from now on.

Every spherical subgroup $H \subseteq G$ is contained in the smallest sober overgroup $H \cdot N(H)^0$. This stems, e.g., from the following useful lemma.

Lemma 30.1. *If $H \subseteq G$ is a spherical subgroup, then $N(H) = N(\overline{H})$ for any intermediate subgroup $\overline{H}$ between H and $N(H)$.*

Proof. As $N(H)/H$ is Abelian, we have $N(H) \subseteq N(\overline{H})$. In particular, $N(H) = N(H^0)$. To prove the converse inclusion, we may assume w.l.o.g. that H is connected and $\mathfrak{b} + \mathfrak{h} = \mathfrak{g}$. Then the right multiplication by $N(\overline{H})$ preserves $BH = B\overline{H}$, the unique open $(B \times H)$-orbit in G. Hence the $N(\overline{H})$-action on $\Bbbk(G)$ by right translations of an argument preserves $\Bbbk[G]^{(B\times H)}$ (=the set of regular functions on G invertible on BH). Since this action commutes with the G-action by left translations, it preserves $\Bbbk[G]^{(H)}$, whence $\Bbbk(G/H)$, too. Hence $N(\overline{H})$ acts on G/H by G-automorphisms, i.e., is contained in $N(H)$. □

Now let $H \subseteq G$ be a sober subgroup and X the wonderful embedding of $\mathcal{O} = G/H$. The local structure theorem reveals the orbit structure and local geometry of X: by Theorem 29.1 there are an affine open chart $\mathring{X} = X \setminus \bigcup_{D\in\mathcal{D}^B} D$ and a closed subvariety $Z \subset \mathring{X}$ such that $\mathring{X}$ is stable under

$P = P(\mathcal{O})$, the Levi subgroup $L \subset P$ leaves Z stable and acts on it via the quotient torus $A = L/L_0$, $\mathring{X} \simeq P *_L Z \simeq P_{\mathrm{u}} \times Z$, and each G-orbit of X intersects Z in an A-orbit. Actually $\mathring{X}$ is the unique B-chart of X intersecting all G-orbits.

The affine toric variety Z is defined by the cone $\mathcal{V}$, so that $\Bbbk[Z] = \Bbbk[\mathcal{V}^\vee \cap \Lambda]$, where $\Lambda = \Lambda(\mathcal{O}) = \mathfrak{X}(A)$. The orbits (of $A : Z$ or of $G : X$) are in an order-reversing bijection with the faces of $\mathcal{V}$, and each orbit closure is the intersection of invariant divisors containing the orbit. If $\mathcal{V}$ is generated by a basis of Λ^*, then $Z \simeq \mathbb{A}^r$ with the natural action of $A \simeq (\Bbbk^\times)^r$; the eigenweight set for $A : Z$ is $\Pi_{\mathcal{O}}^{\min}$. Generally, since $\mathcal{V}$ is simplicial (Theorem 22.1), one deduces that $Z \simeq \mathbb{A}^r/\Gamma$ with the natural action of $A \simeq (\Bbbk^\times)^r/\Gamma$, where $\Gamma \simeq \Lambda^*/N$ is the common kernel of all $\lambda \in \Lambda$ in $(\Bbbk^\times)^r = N \otimes \Bbbk^\times$, the sublattice $N \subseteq \Lambda^*$ being spanned by the indivisible generators of the rays of $\mathcal{V}$.

In particular, X is smooth iff $\mathcal{V}$ is generated by a basis of Λ^* iff $\Lambda = \mathbb{Z}\Delta_{\mathcal{O}}^{\min}$. It is a delicate problem to characterize the (sober) spherical subgroups $H \subseteq G$ such that the wonderful embedding $X \hookleftarrow \mathcal{O} = G/H$ is smooth.

Note that $N(H)/H = \operatorname{Aut}_G \mathcal{O}$ acts on a finite set $\mathcal{D}^B$.

Definition 30.2. A spherical subgroup $H \subseteq G$ is called *very sober* if $N(H)/H$ acts on $\mathcal{D}^B$ effectively. (In particular, H is sober, because $(N(H)/H)^0$ leaves $\mathcal{D}^B$ pointwise fixed.) The *very sober hull* of H is the kernel $\overline{H}$ of $N(H) : \mathcal{D}^B$.

Remark 30.1. It is easy to deduce from Lemma 30.1 that $\overline{H}$ is the smallest very sober subgroup of G containing H. The colored space $\overline{\mathcal{E}} = \mathcal{E}(G/\overline{H})$ is identified with $\mathcal{E}/(\mathcal{V} \cap -\mathcal{V})$, the valuation cone is $\overline{\mathcal{V}} = \mathcal{V}/(\mathcal{V} \cap -\mathcal{V})$, and the set of colors $\overline{\mathcal{D}}^B$ is identified with $\mathcal{D}^B$ via pullback.

Observe that $\overline{H}$ is the kernel of $N(H) : \mathfrak{X}(H)$ [Kn8, 7.4]. Indeed, (a multiple of) each B-stable divisor δ on $\mathcal{O}$ is defined by an equation $\eta \in \Bbbk(G)^{(B\times H)}_{(\lambda,\chi)}$, and each $\chi \in \mathfrak{X}(H)$ arises in this way (because every G-line bundle $\mathcal{L}_{G/H}(\chi)$ has a rational B-eigensection). The right multiplication by $n \in N(H)$ maps η to $\eta' \in \Bbbk(G)^{(B\times H)}_{(\lambda,\chi')}$, the equation of $\delta' = n(\delta)$, where $\chi'(h) = \chi(n^{-1}hn)$. Since $\Bbbk(\mathcal{O})^B = \Bbbk$, we have $\chi' = \chi \iff \eta'/\eta = \mathrm{const} \iff \delta' = \delta$.

In particular, $\overline{H} \supseteq Z_G(H)$.

Theorem 30.1 ([Kn8, 7.6, 7.2]). *If H is very sober, then the wonderful embedding $X \hookleftarrow G/H$ is smooth. In particular, X is smooth if $N(H) = H$.*

Remark 30.2. If all simple factors of G are isomorphic to PSL_{n_i}, then very soberness is also a necessary condition for X be smooth [Lu6, 7.1]. This is not true in general: $S^{n-1} = \mathrm{SO}_n/\mathrm{SO}_{n-1}$ and $\mathrm{SL}_4/\mathrm{Sp}_4$ are symmetric spaces of rank 1, hence their wonderful embeddings are smooth (Proposition 30.3), while $\overline{\mathrm{SO}_{n-1}} = \mathrm{O}_n$, $\overline{\mathrm{Sp}_4} = \mathrm{Sp}_4 \cdot Z(\mathrm{SL}_4)$.

Proof. By Theorem 23.2, $S_{\mathcal{O}} = \bigcap_{\alpha\in\Delta_{\mathcal{O}}^{\min}} \operatorname{Ker}\alpha \hookrightarrow \operatorname{Aut}_G \mathcal{O} = N(H)/H$. It suffices to show that $S_{\mathcal{O}}$ fixes all colors; then $S_{\mathcal{O}} = \{e\}$, i.e., $\Delta_{\mathcal{O}}^{\min}$ spans Λ.

Take any $D \in \mathcal{D}^B$. Replacing D by a multiple, we may assume that $\mathcal{O}(D)$ is G-linearized. Consider the total space $\widehat{\mathcal{O}} = \widehat{G}/\widehat{H}$ of $\mathcal{O}(D)^\times$, where $\widehat{G} = G \times \Bbbk^\times$, cf. Remark 20.1. Using the notation of Remark 20.1, we have

$$0 \longrightarrow \Lambda \longrightarrow \widehat{\Lambda} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

$\mathcal{V} = \widehat{\mathcal{V}}/(\widehat{\mathcal{V}} \cap -\widehat{\mathcal{V}})$, and $\Delta_{\widehat{\mathcal{O}}}^{\min} = \Delta_{\mathcal{O}}^{\min}$. Therefore $S_{\mathcal{O}} = S_{\widehat{\mathcal{O}}}/\Bbbk^\times$.

However the pullback $\widehat{D} \subset \widehat{\mathcal{O}}$ of D is principal. Since $S_{\widehat{\mathcal{O}}}$ multiplies the equation of $\widehat{D}$ by scalars, it leaves $\widehat{D}$ stable, whence S_X leaves D stable. ☐

If $N(H) = H$, then $\mathcal{O} \simeq G[\mathfrak{h}]$, the orbit of $\mathfrak{h}$ in $\mathrm{Gr}_k(\mathfrak{g})$, $k = \dim\mathfrak{h}$. The closure $X(\mathfrak{h}) = \overline{G[\mathfrak{h}]} \subseteq \mathrm{Gr}_k(\mathfrak{g})$ is called the *Demazure embedding* of $\mathcal{O}$.

Proposition 30.1 ([Bri8, 1.4]). *The wonderful embedding X is the normalization of $X(\mathfrak{h})$.*

Proof. The decomposition $\mathfrak{g} = \mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{a} \oplus \mathfrak{h}$ yields $\mathfrak{h} = \mathfrak{l}_0 \oplus \langle e_{-\alpha} + \xi_\alpha \mid \alpha \in \Delta^+ \setminus \Delta_L^+\rangle$, where $\xi_\alpha \in \mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{a}$ is the projection of $-e_{-\alpha}$ along $\mathfrak{h}$. Hence

$$\widehat{\mathfrak{h}} = \widehat{\mathfrak{l}}_0 \wedge \bigwedge_{\alpha\in\Delta^+\setminus\Delta_L^+} (e_{-\alpha} + \xi_\alpha) = \widehat{\mathfrak{s}} + \text{terms of higher } T\text{-weights}$$

where $\widehat{\mathfrak{q}} \in \bigwedge^\bullet \mathfrak{g}$ denotes a generator of $[\mathfrak{q}] \in \mathrm{Gr}(\mathfrak{g})$, $\mathfrak{s} = \mathfrak{l}_0 \oplus \mathfrak{p}_{\mathrm{u}}^-$, and the weights of other terms differ from that of $\widehat{\mathfrak{s}}$ by $\sum(\alpha_i + \beta_i)$, $\alpha_i, \beta_i \in \Delta^+ \setminus \Delta_L^+$ or $\beta_i = 0$.

Let $Z(\mathfrak{h})$ be the closure of $T[\mathfrak{h}]$ in the affine chart defined by non-vanishing of the highest weight covector dual to $\widehat{\mathfrak{s}}$. It is an affine toric variety with the fixed point $[\mathfrak{s}]$. Thus $Y = G[\mathfrak{s}] \subset X(\mathfrak{h})$ is a closed orbit. The local structure theorem in a neighborhood of $[\mathfrak{s}]$ provides a B-chart $\mathring{X}(\mathfrak{h}) \subset X(\mathfrak{h})$, $\mathring{X}(\mathfrak{h}) \simeq P_{\mathrm{u}} \times Z(\mathfrak{h})$. Note that for any $[\mathfrak{q}] \in Z(\mathfrak{h}) \setminus T[\mathfrak{h}]$ the subalgebra $\mathfrak{q}$ is transversal to $\mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{a}$ while $\mathfrak{n}(\mathfrak{q}) \cap \mathfrak{a} \neq 0$, whence $\dim G_{[\mathfrak{q}]} > \dim H$. It follows that $\mathring{X}(\mathfrak{h})$ intersects no colors, i.e., $X(\mathfrak{h})$ is toroidal in a neighborhood of Y.

On the other hand, every smooth toroidal embedding of $\mathcal{O}$ maps to $X(\mathfrak{h})$ by Theorem 29.2. It follows that $\widetilde{X(\mathfrak{h})}$ is simple, whence wonderful. ☐

It is an open question whether the Demazure embedding is always smooth.

If $N(H) \neq H$, then the normalization of $X(\mathfrak{h})$ is the wonderful embedding of $G/N(H)$.

Example 30.1. Let G be an adjoint semisimple group and $H = G^\sigma$ a symmetric subgroup. Here $N(H) = H$. We have $\Lambda(\mathcal{O}) = \mathfrak{X}(T/T^\sigma) = \{\mu - \sigma(\mu) \mid \mu \in \mathfrak{X}(T)\}$, where T is a σ-stable maximal torus such that T_1 is a maximal σ-split torus. Hence $\Lambda(\mathcal{O})$ is the root lattice of $2\Delta_{\mathcal{O}}$. Since $\mathcal{V}(\mathcal{O})$ is the antidominant Weyl chamber of $\Delta_{\mathcal{O}}^\vee$ in $\Lambda(\mathcal{O})^* \otimes \mathbb{Q}$ (by Theorem 26.2), $\Delta_{\mathcal{O}}^{\min}$ is the reduced root system associated with $2\Delta_{\mathcal{O}}$. It follows that the wonderful completion X is smooth in this case.

Wonderful completions of symmetric spaces were studied in [CP1], [CS]. In particular, a geometric realization for a wonderful completion as an embedded projective variety was constructed. Let λ be a dominant weight of $\widetilde{G}$ such that $\sigma(\lambda) = -\lambda$ and $\lambda \in \operatorname{int} \mathbf{C}(\Delta_{\mathcal{O}}^+)$. There exists a unique (up to proportionality) $\widetilde{G}^\sigma$-fixed vector $v' \in V^*(\lambda)$. Then $X' = \overline{G[v']} \subseteq \mathbb{P}(V^*(\lambda))$ is the wonderful embedding of $G[v'] \simeq \mathcal{O}$.

Indeed, a natural closed embedding $\mathbb{P}(V^*(\lambda)) \hookrightarrow \mathbb{P}(V^*(2\lambda))$ (given by the multiplication $V^*(\lambda) \otimes V^*(\lambda) \to V^*(2\lambda)$ in $\Bbbk[\widetilde{G}]$) identifies X' with $X'' = \overline{G[v'']}$, where $v'' \in V^*(2\lambda)$ is a unique $\widetilde{G}^\sigma$-fixed vector. As X'' is a simple projective embedding of $G[v'']$, the natural map $\mathcal{O} \to G[v'']$ extends to $X \to X''$. On the other hand, the homomorphism $V^*(\lambda) \otimes V^*(\lambda) \to V^*(2\lambda)$ maps ω to v'', where ω is defined by (26.3). Let Z'' be the closure of $T[v'']$ in the affine chart of $\mathbb{P}(V^*(2\lambda))$ defined by non-vanishing of the highest weight covector of weight 2λ. From (26.3) it is easy to deduce that $Z'' \simeq \mathbb{A}^r$ is acted on by T via the eigenweight set $\Pi_{\mathcal{O}}^{\min}$ and the closed orbit $G[v_{-2\lambda}]$ is transversal to Z'' at $[v_{-2\lambda}]$. Hence $Z \xrightarrow{\sim} Z''$, $\mathring{X} \xrightarrow{\sim} PZ'' \simeq P_{\mathrm{u}} \times Z''$, and finally $X \xrightarrow{\sim} X'' \simeq X'$. (A similar reasoning shows $X \simeq \overline{G[\omega]}$. A slight refinement carries over the construction to positive characteristic [CS].)

Another model for the wonderful completion is the Demazure embedding. First note that $\mathfrak{h} = \mathfrak{l}_0 \oplus \langle e_\alpha + e_{\sigma(\alpha)} \mid \alpha \in \Delta^+ \setminus \Delta_L^+ \rangle$. Arguing as in the proof of Proposition 30.1, we see that $Z(\mathfrak{h}) = \overline{T[\mathfrak{h}]} \simeq \mathbb{A}^r$ is acted on by T with the eigenweights $\alpha - \sigma(\alpha)$, $\alpha \in \Pi$, and $Y = G[\mathfrak{s}]$ is transversal to $Z(\mathfrak{h})$ at $[\mathfrak{s}]$. This yields $\mathcal{C}_Y = \mathcal{V}$. Now the Luna–Vust theory together with the description of the colored data for symmetric spaces implies that $X(\mathfrak{h})$ is wonderful. The varieties $X(\mathfrak{h})$ were first considered by Demazure in the case, where $G = \mathrm{PSL}_n(\Bbbk)$ and H is the projective orthogonal or symplectic group [Dem4].

Using the Demazure embedding, Brion computed the canonical class of any spherical variety.

Proposition 30.2 ([Bri8, 1.6]). *Suppose X is a spherical variety with the open orbit $\mathcal{O} \simeq G/H$. Consider the G-morphism $\phi : \mathcal{O} \to \mathrm{Gr}_k(\mathfrak{g})$, $\phi(o) = [\mathfrak{h}]$,*

$k = \dim H$. *Then a canonical divisor of X is*

$$K_X = -\sum_i D_i - \overline{\phi^*\mathcal{H}} = -\sum_i D_i - \sum_{D \in \mathcal{D}^B} m_D D$$

where D_i runs over all G-stable prime divisors in X, $\mathcal{H}$ is a hyperplane section of $X(\mathfrak{h})$ in $\mathbb{P}(\bigwedge^k \mathfrak{g})$, and $m_D \in \mathbb{N}$.

Explicit formulæ for m_D are given in [Bri12, 4.2].

Proof. Removing all G-orbits of codimension > 1, we may assume that X is smooth and toroidal. Then by Theorem 29.2, ϕ extends to X, and we have an exact sequence

$$0 \longrightarrow \phi^*\mathcal{E} \longrightarrow \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log \partial X) \longrightarrow 0$$

where $\mathcal{E}$ is the tautological vector bundle on $\mathrm{Gr}_k(\mathfrak{g})$. Taking the top exterior powers yields $\omega_X \otimes \mathcal{O}_X(\partial X) \simeq \bigwedge^k \phi^*\mathcal{E} = \mathcal{O}_X(-\phi^*\mathcal{H})$, whence the first expression for K_X. If $\mathcal{H}$ is defined by a covector in $(\bigwedge^k \mathfrak{g}^*)^{(B)}$ dual to $\widehat{\mathfrak{s}}$, then $\mathring{X}(\mathfrak{h}) = X(\mathfrak{h}) \setminus \mathcal{H}$ intersects all G-orbits in open B-orbits, whence $\phi^*\mathcal{H} = \sum m_D D$ with $m_D > 0$ for $\forall D \in \mathcal{D}^B$. □

Using the characterization of ample divisors on complete spherical varieties (Corollary 17.5), one deduces that certain smooth wonderful embeddings (e.g., flag varieties, wonderful completions of symmetric spaces, of affine spherical spaces of rank 1) are Fano varieties (i.e., anticanonical divisor is ample).

Smooth wonderful embeddings can be characterized intrinsically by the configuration of G-orbits.

Theorem 30.2 ([Lu4]). *A smooth complete G-variety X is a wonderful embedding of a spherical homogeneous space iff it satisfies the following conditions:*

(1) *X contains a dense open orbit $\mathcal{O}$.*

(2) *$X \setminus \mathcal{O}$ is a divisor with normal crossings, i.e., its components $D_1, \dots, D_r$ are smooth and intersect transversally.*

(3) *For each tuple $1 \le i_1 < \dots < i_k \le r$, the set $D_{i_1} \cap \dots \cap D_{i_k} \setminus \bigcup_{i \neq i_1, \dots, i_k} D_i$ is a single G-orbit. (In particular, it is non-empty.)*

G-varieties satisfying the conditions of the theorem are called *wonderful varieties.*

Sketch of a proof. Smooth wonderful embeddings obviously satisfy the conditions (1)–(3), as a particular case of Theorem 29.2: the toric slice $Z \simeq \mathbb{A}^r$ is transversal to all orbits and the G-stable prime divisors intersect it in the coordinate hyperplanes.

To prove the converse, consider the local structure of X in a neighborhood of the closed orbit Y which is provided by an embedding of X into a projective space. Let $P = LP_{\mathrm{u}}$ be a Levi decomposition of $P = P(Y)$. There is a B-chart $\mathring{X} \simeq P_{\mathrm{u}} \times Z$ such that Z is a smooth L-stable locally closed subvariety intersecting Y transversally at the unique P^--fixed point z. It is easy to see that a general dominant one-parameter subgroup $\gamma \in \mathfrak{X}^*(Z(L))$ contracts $\mathring{X}$ to z. Hence Z is L-isomorphic to $T_z Z$.

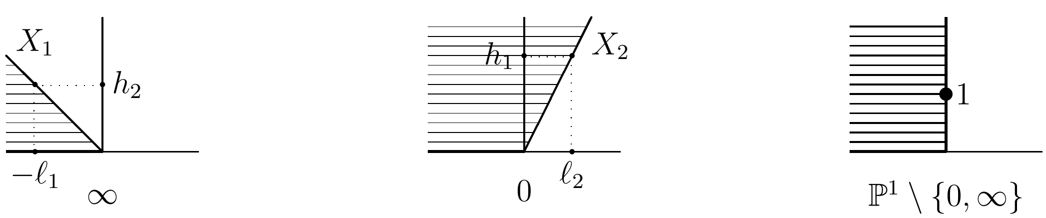
Consider the wonderful subvarieties $X_i = \bigcap_{j\neq i} D_j$ of rank 1 and let $\{\lambda_i\}$ be the T-weights of $T_z(Z \cap X_i)$, $i = 1, \dots, r$. Since $T_z Z = \bigoplus T_z(Z \cap X_i)$, it suffices to prove that $\lambda_1, \dots, \lambda_r$ are linearly independent.

The latter is reduced to the cases $r = 1$ or 2. Indeed, if we already know that X_i and $X_{ij} = \bigcap_{k\neq i,j} D_k$ are wonderful embeddings of spherical spaces, then $\Pi^{\min}_{X_i} = \{\lambda_i\}$ and $\Pi^{\min}_{X_{ij}} = \{\lambda_i, \lambda_j\}$. Thus the λ_i's are positive linear combinations of positive roots located at obtuse angles to each other. This implies the linear independence.

The case $r = 1$ stems from Proposition 30.4.

The case $r = 2$ can be reduced to $G = \mathrm{SL}_2$. Indeed, assuming that λ_1, λ_2 are proportional, we see that $c(X) = r(X) = 1$. By Proposition 10.2, $\mathcal{O}$ is obtained from a 3-dimensional homogeneous SL_2-space by parabolic induction. Let us describe the colored hypercone $(\mathcal{C}_Y, \mathcal{D}^B_Y)$.

Since $T_z Z$ is contracted to 0 by γ, we have $\Lambda(X) = \mathbb{Z}\lambda$, $\lambda_i = h_i\lambda$, where λ is dominant and h_1, h_2 are coprime positive integers. W.l.o.g. $\ell_1 h_1 - \ell_2 h_2 = 1$ for some $\ell_1, \ell_2 \in \mathbb{N}$. Consider $T_z(Z \cap X_i)$ as coordinate axes in $T_z Z \simeq Z$ and extend the respective coordinates to $f_1, f_2 \in \Bbbk(X)^{(B)}$. Then we may put $\mathbf{f}_\lambda = f_2^{\ell_2}/f_1^{\ell_1}$, and $\Bbbk(X)^B = \Bbbk\left(f_2^{h_1}/f_1^{h_2}\right)$. We have the following picture for $(\mathcal{C}_Y, \mathcal{D}^B_Y)$ ($f_2^{h_1}/f_1^{h_2}$ is regarded as affine coordinate on $\mathbb{P}^1$, colors in $\mathcal{D}^B_Y$ are marked by bold dots):



Since $\mathcal{D}^B_Y$ does not contain central colors, Propositions 20.4 and 14.2 imply that X is induced from a wonderful SL_2-variety. However it is easy to see (e.g., from the classification in [Tim2, §5]) that there exist no SL_2-germs with the colored data as above. (Luna uses different arguments in [Lu4].) □

Wonderful varieties play a distinguished role in the study of spherical homogeneous spaces, because they are the canonical completions of these spaces having nice geometric properties. To a certain extent this role is analogous to that of (generalized) flag varieties in the theory of reductive groups. For symmetric spaces this was already observed by de Concini and Procesi [CP1]. For general spherical spaces this principle was developed by Brion, Knop, Luna, et al [BPa], [Bri8], [Kn8], [Lu3], [Lu5], [Lu6].

In particular, wonderful varieties are applied to classification of spherical subgroups. The strategy, proposed by Luna, is to reduce the classification to very sober subgroups, which are stabilizers of general position for wonderful varieties, and then to classify the wonderful varieties.

By Theorem 29.5, there are no continuous families of non-conjugate spherical subgroups, and even more:

Proposition 30.3. *There are finitely many conjugacy classes of sober spherical subgroups $H \subseteq G$.*

Proof. Sober spherical subalgebras of dimension k form a locally closed G-subvariety in $\mathrm{Gr}_k(\mathfrak{g})$. Indeed, the set of spherical subalgebras is open in the variety of k-dimensional Lie subalgebras, and sober subalgebras are those having k-dimensional orbits. Theorem 29.5 implies that this variety is a finite union of locally closed strata such that all orbits in each stratum have the same stabilizer. But the isotropy subalgebras are nothing else but the points of the strata. Hence each stratum is a single orbit, i.e., there are finitely many sober subalgebras, up to conjugation. As for subgroups, there are finitely many ways to extend H^0 by a (finite) subgroup in $N(H^0)/H^0$. □

Note that finiteness fails for non-sober spherical subgroups: H^0 can be extended by countably many quasitori in $N(H^0)/H^0$.

These results create an evidence that spherical subgroups should be classified by some discrete invariants. Such invariants were suggested by Luna, under the names of *spherical systems* and *spherical homogeneous data* (Definition 30.3). They are defined in terms of roots and weights of G and wonderful G-varieties of rank 1.

For spherical homogeneous spaces of rank 1, wonderful embeddings are always smooth. Indeed, they are normal G-varieties consisting of two G-orbits—a dense one and another of codimension 1. Furthermore, spherical homogeneous spaces of rank 1 are characterized by existence of a completion by homogeneous divisors.

Proposition 30.4 ([Akh1], [Bri5]). *The following conditions are equivalent:*

(1) $\mathcal{O} = G/H$ *is a spherical homogeneous space of rank* 1.

(2) *There exists a smooth complete embedding* $X \hookleftarrow \mathcal{O}$ *such that* $X \setminus \mathcal{O}$ *is a union of* G*-orbits of codimension* 1.

Moreover, if $\mathcal{O}$ *is horospherical, then* $X \setminus \mathcal{O}$ *consists of two orbits and* $X \simeq G *_Q \mathbb{P}^1$*, where* $Q \subseteq G$ *is a parabolic acting on* $\mathbb{P}^1$ *via a character. Otherwise* $X \setminus \mathcal{O}$ *is a single orbit and* X *is a wonderful embedding of* $\mathcal{O}$.

Proof. The implication (1) $\Longrightarrow$ (2) and the properties of X easily stem from the Luna–Vust theory: the colored space $\mathcal{E}$ is a line, whence there exists a unique smooth complete toroidal embedding X, which is obtained by adding two homogeneous divisors (corresponding to the two rays of $\mathcal{E}$) if $\mathcal{V} = \mathcal{E}$ and is wonderful if $\mathcal{V}$ is a ray.

To prove (2) $\Longrightarrow$ (1), we consider the local structure of X in a neighborhood of a closed orbit Y. Let $P = LP_{\mathrm{u}}$ be a Levi decomposition of $P = P(Y)$. There is a B-chart $\mathring{X} \simeq P_{\mathrm{u}} \times Z$ such that Z is an L-stable affine curve intersecting Y transversally at the unique P^--fixed point. Note that $T \subseteq L$ cannot fix Z pointwise for otherwise $\mathcal{O}^T$ would be infinite, which is impossible. Hence $T : Z$ has an open orbit, whence (1). □

Remark 30.3. A similar reasoning proves an embedding characterization of arbitrary rank 1 spaces, due to Panyushev [Pan5]: $r(\mathcal{O}) = 1$ iff there exists a complete embedding $X \hookleftarrow \mathcal{O}$ such that $X \setminus \mathcal{O}$ is a divisor consisting of closed G-orbits. Here Z is an affine L-stable subvariety with a pointwise L-fixed divisor $Z \setminus \mathcal{O}$ (provided that Y is a generic closed orbit), which readily implies that generic orbits of $L : Z$ are one-dimensional, whence $r(X) = r(Z) = 1$. On the other hand, it is easy to construct a desired embedding X for a homogeneous space $\mathcal{O}$ parabolically induced from SL_2 modulo a finite subgroup, cf. Proposition 10.2.

Spherical homogeneous spaces G/H of rank 1 were classified by Akhiezer [Akh1] and Brion [Bri5]. It is easy to derive the classification from a regular embedding of H into a parabolic $Q \subseteq G$. In the notation of Theorem 9.2 we have an alternative: either $r(M/K) = 1$, $r_{M_*}(Q_{\mathrm{u}}/H_{\mathrm{u}}) = 0$, or vice versa.

In the first case $H_{\mathrm{u}} = Q_{\mathrm{u}}$, i.e., G/H is parabolically induced from an affine spherical homogeneous rank 1 space M/K. Except the trivial case $M/K \simeq \Bbbk^\times$ (where H is horospherical), K is sober in M and H in G.

In the second case $M = K = M_*$, and $Q_{\mathrm{u}}/H_{\mathrm{u}} \simeq \mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}$ is an M-module such that $(\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}) \setminus \{0\}$ is a single M-orbit. Indeed, $\Bbbk[\mathfrak{q}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}}]^{U(M)}$ is generated by one $B(M)$-eigenfunction, namely a highest weight covector in $(\mathfrak{q}_\mathrm{u}/\mathfrak{h}_\mathrm{u})^*$, whence $\mathfrak{q}_\mathrm{u}/\mathfrak{h}_\mathrm{u}$ is an HV-cone. Therefore M acts on $\mathfrak{q}_\mathrm{u}/\mathfrak{h}_\mathrm{u} \simeq \Bbbk^n$ as $\mathrm{GL}_n(\Bbbk)$ or $\Bbbk^\times \cdot \mathrm{Sp}_n(\Bbbk)$ and the highest weight of $\mathfrak{q}_\mathrm{u}/\mathfrak{h}_\mathrm{u}$ is a negative simple root.

We deduce that every spherical homogeneous space of rank 1 is either horospherical or parabolically induced from a *primitive* rank 1 space $\mathcal{O} = G/H$ with G semisimple and H sober. Primitive spaces are of the two types:

(1) H is reductive.

(2) H is regularly embedded in a maximal parabolic $Q \subseteq G$ which shares a Levi subgroup M with H and $\mathfrak{q}_\mathrm{u}/\mathfrak{h}_\mathrm{u}$ is a simple M-module of type $\mathrm{GL}_n(\Bbbk) : \Bbbk^n$ or $\Bbbk^\times \cdot \mathrm{Sp}_n(\Bbbk) : \Bbbk^n$ generated by a simple root vector.

Primitive spherical homogeneous spaces of rank 1 are listed in Table 5.10. Those of the 1-st type are easy to classify, e.g., by inspection of Tables 2.1, 2.3, and 5.9. We indicate the embedding $H \hookrightarrow G$ by referring to Table 5.9 (2.1) in the (non-)symmetric case. Primitive spaces of the 2-nd type are classified by choosing a Dynkin diagram and its node corresponding to a short simple root α which is adjacent to an extreme node of the remaining diagram, the latter being of type $\mathbf{A}_l$ or $\mathbf{C}_l$. The diagrams are presented in the column "$H \hookrightarrow G$", with the white node corresponding to α.

The wonderful embeddings of spherical homogeneous space of rank 1 are parabolically induced from those of primitive spaces. The latter are easy to describe. For type 1 the construction of Example 30.1 works whenever $N(H) = H$: the wonderful embedding of G/H is realized as $X = \overline{G[v]} \subseteq \mathbb{P}(V(\lambda))$, where $v \in V(\lambda)^{(\widetilde{H})}$, $\lambda \in \Lambda_+(\widetilde{G}/\widetilde{H}^0)$. If $N(H) \neq H$, then X is the projective closure of Gv in $\mathbb{P}(V(\lambda) \oplus \Bbbk)$. The simple minimal root of X is the generator of $\Lambda_+(G/H)$.

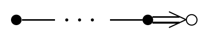
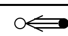
For type 2 the wonderful embedding is $X = G *_Q \mathbb{P}(Q_\mathrm{u}/H_\mathrm{u} \oplus \Bbbk)$. Indeed, Q_u acts on the M-module $Q_\mathrm{u}/H_\mathrm{u}$ by affine translations, whence the projective closure of $Q_\mathrm{u}/H_\mathrm{u}$ consists of two Q-orbits—the affine part and the hyperplane at infinity. Here $\Pi_X^{\min} = \{w_M\alpha\}$.

Also the Demazure embedding is wonderful in all cases where $N(H) = H$.

Simple minimal roots of arbitrary wonderful G-varieties of rank 1 are called *spherical roots* of G. They are non-negative linear combinations of Π_G. Let Σ_G denote the set of all spherical roots. It is a finite set, which is easy to find from the classification of wonderful varieties of rank 1.

Spherical roots of reductive groups of simply connected type are listed in Table 5.11. Namely, $\lambda \in \Sigma_G$ iff it is a spherical root of a simple factor, or a product of two simple factors, indicated in the 1-st column of the table. For each spherical root λ, we indicate the Dynkin diagram of the simple roots occurring in the decomposition of λ with positive coefficients. The

Table 5.10: Wonderful varieties of rank 1

No.	G	H	$H \hookrightarrow G$	$\Pi^{\min}_{G/H}$	Wonderful embedding
1	$\mathrm{SL}_2 \times \mathrm{SL}_2$	SL_2	diagonal	$\omega + \omega'$	$X = \{(x : t) \mid \det x = t^2\} \subset \mathbb{P}(\mathrm{L}_2 \oplus \Bbbk)$
2	$\mathrm{PSL}_2 \times \mathrm{PSL}_2$	PSL_2		$2\omega + 2\omega'$	$\mathbb{P}(\mathrm{L}_2)$
3	SL_n	GL_{n-1}	symmetric No. 1	$\omega_1 + \omega_{n-1}$	$\mathbb{P}^n \times (\mathbb{P}^n)^*$
4	PSL_2	PO_2	symmetric No. 3	$4\omega_1$	$\mathbb{P}(\mathfrak{sl}_2)$
5	Sp_{2n}	$\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$	symmetric No. 4	ω_2	$\mathrm{Gr}_2(\Bbbk^{2n})$
6	Sp_{2n}	$B(\mathrm{Sp}_2) \times \mathrm{Sp}_{2n-2}$		ω_2	$\mathrm{Fl}_{1,2}(\Bbbk^{2n})$
7	SO_n	SO_{n-1}	symmetric No. 6	ω_1	$X = \{(x : t) \mid (x, x) = t^2\} \subset \mathbb{P}^n$
8	SO_n	$\mathrm{S}(\mathrm{O}_1 \times \mathrm{O}_{n-1})$		$2\omega_1$	$\mathbb{P}^{n-1}$
9	SO_{2n+1}	$\mathrm{GL}_n \leftthreetimes \bigwedge^2 \Bbbk^n$		ω_1	$X = \{(V_1, V_2) \mid V_1 \subset V_1^{\perp}\} \subset \mathrm{Fl}_{n,2n}(\Bbbk^{2n+1})$
10	Spin_7	$\mathbf{G}_2$	non-symmetric No. 10	ω_3	$X = \{(x : t) \mid (x, x) = t^2\} \subset \mathbb{P}(V(\omega_3) \oplus \Bbbk)$
11	SO_7	$\mathbf{G}_2$		$2\omega_3$	$\mathbb{P}(V(\omega_3))$
12	$\mathbf{F}_4$	$\mathbf{B}_4$	symmetric No. 17	ω_1	
13	$\mathbf{G}_2$	SL_3	non-symmetric No. 12	ω_1	$X = \{(x : t) \mid (x, x) = t^2\} \subset \mathbb{P}(V(\omega_1) \oplus \Bbbk)$
14	$\mathbf{G}_2$	$N(\mathrm{SL}_3)$		$2\omega_1$	$\mathbb{P}(V(\omega_1))$
15	$\mathbf{G}_2$	$\mathrm{GL}_2 \leftthreetimes (\Bbbk \oplus \Bbbk^2) \otimes \bigwedge^2 \Bbbk^2$		$\omega_2 - \omega_1$	

numbering of the simple roots α_i is according to [OV], and α_i, α'_j denote simple roots of different simple factors. For arbitrary G, Σ_G is obtained from $\Sigma_{\widetilde{G}}$ by removing the spherical roots that are not in the weight lattice of G. Note that if $\lambda, \mu \in \Sigma_G$ are proportional, then $\lambda = 2\mu$ or $\mu = 2\lambda$, and also if $\lambda \in \Sigma_G \setminus \mathbb{Z}\Delta_G$, then $2\lambda \in \Sigma_G \cap \mathbb{Z}\Delta_G$.

More generally, two-orbit complete (normal) G-varieties were classified by Cupit-Foutou [C-F] and Smirnov [Sm-A]. All of them are spherical.

Wonderful varieties of rank 2 were classified by Wasserman [Wa].

For arbitrary wonderful varieties, many questions can be reduced to the case of rank ≤ 2 via the procedure of localization [Lu5], [Lu6, 3.2].

Given a wonderful variety X with the open G-orbit $\mathcal{O}$, there is a bijection $D_i \leftrightarrow \lambda_i$ $(i = 1, \ldots, r)$ between the component set of ∂X and $\Pi^{\min}_X$. Namely λ_i is orthogonal to the facet of $\mathcal{V}$ complementary to the ray which corresponds to D_i. Also, λ_i is the T-weight of $T_z X / T_z D_i$ at the unique B^--fixed point z.

For any subset $\Sigma \subset \Pi^{\min}_X$, put $X^{\Sigma} = \bigcap_{\lambda_i \notin \Sigma} D_i$, the *localization* of X at Σ.

Table 5.11: Spherical roots

G	Σ_G
$\mathbf{A}_l$	$\alpha_i+\dots+\alpha_j$ ($\mathbf{A}_{j-i+1}$, $i\le j$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $(\alpha_1+\alpha_3)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=3$), $\alpha_{i-1}+2\alpha_i+\alpha_{i+1}$ ($\mathbf{D}_3$, $1<i<l$), $(\alpha_1+2\alpha_2+\alpha_3)/2$ ($\mathbf{D}_3$, $l=3$)
$\mathbf{B}_l$	$\alpha_i+\dots+\alpha_j$ ($\mathbf{A}_{j-i+1}$, $i\le j<l$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $(\alpha_1+\alpha_3)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=3,4$), $\alpha_{i-1}+2\alpha_i+\alpha_{i+1}$ ($\mathbf{D}_3$, $1<i<l-1$), $(\alpha_1+2\alpha_2+\alpha_3)/2$ ($\mathbf{D}_3$, $l=4$), $\alpha_i+\dots+\alpha_l$ ($\mathbf{B}_{l-i+1}$, $i<l$), $2\alpha_i+\dots+2\alpha_l$ ($\mathbf{B}_{l-i+1}$, $i<l$), $\alpha_{l-2}+2\alpha_{l-1}+3\alpha_l$ ($\mathbf{B}_3$), $(\alpha_1+2\alpha_2+3\alpha_3)/2$ ($\mathbf{B}_3$, $l=3$)
$\mathbf{C}_l$	$\alpha_i+\dots+\alpha_j$ ($\mathbf{A}_{j-i+1}$, $i\le j<l$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$, $i\le j-2$), $\alpha_{i-1}+2\alpha_i+\alpha_{i+1}$ ($\mathbf{D}_3$, $1<i<l-1$), $\alpha_i+2\alpha_{i+1}+\dots+2\alpha_{l-1}+\alpha_l$ ($\mathbf{C}_{l-i+1}$, $i<l$), $2\alpha_{l-1}+2\alpha_l$ ($\mathbf{C}_2$)
$\mathbf{D}_l$	$\alpha_{i_1}+\dots+\alpha_{i_k}$ ($\mathbf{A}_k$, $k\ge1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $2\alpha_{i_1}+\alpha_{i_2}+\alpha_{i_3}$ ($\mathbf{D}_3$), $(2\alpha_{i_1}+\alpha_{i_2}+\alpha_{i_3})/2$ ($\mathbf{D}_3$, $l=4$), $2\alpha_i+\dots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$ ($\mathbf{D}_{l-i+1}$, $i<l-1$), $\alpha_i+\dots+\alpha_{l-2}+(\alpha_{l-1}+\alpha_l)/2$ ($\mathbf{D}_{l-i+1}$, $i<l-1$), $(\alpha_{l-1}+\alpha_l)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$), $(\alpha_1+\alpha_3)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=4$), $(\alpha_1+\alpha_4)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $l=4$)
$\mathbf{E}_l$	$\alpha_{i_1}+\dots+\alpha_{i_k}$ ($\mathbf{A}_k$, $k\ge1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $2\alpha_{i_1}+\dots+2\alpha_{i_{k-2}}+\alpha_{i_{k-1}}+\alpha_{i_k}$ ($\mathbf{D}_k$, $k\ge3$)
$\mathbf{F}_4$	α_i ($\mathbf{A}_1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_i+\alpha_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $\alpha_i+\alpha_{i+1}$ ($\mathbf{A}_2$, $i\ne2$), $\alpha_2+\alpha_3$ ($\mathbf{C}_2$), $2\alpha_2+2\alpha_3$ ($\mathbf{C}_2$), $\alpha_1+2\alpha_2+\alpha_3$ ($\mathbf{C}_3$), $\alpha_2+\alpha_3+\alpha_4$ ($\mathbf{B}_3$), $2\alpha_2+2\alpha_3+2\alpha_4$ ($\mathbf{B}_3$), $3\alpha_2+2\alpha_3+\alpha_4$ ($\mathbf{B}_3$), $2\alpha_1+3\alpha_2+2\alpha_3+\alpha_4$ ($\mathbf{F}_4$)
$\mathbf{G}_2$	α_i ($\mathbf{A}_1$), $2\alpha_i$ ($\mathbf{A}_1$), $\alpha_1+\alpha_2$ ($\mathbf{G}_2$), $2\alpha_1+\alpha_2$ ($\mathbf{G}_2$), $4\alpha_1+2\alpha_2$ ($\mathbf{G}_2$)
$\mathbf{X}_l\times\mathbf{Y}_m$	$\alpha_i+\alpha'_j$ ($\mathbf{A}_1\times\mathbf{A}_1$), $(\alpha_l+\alpha'_m)/2$ ($\mathbf{A}_1\times\mathbf{A}_1$, $\mathbf{X}=\mathbf{Y}=\mathbf{C}$, $l,m\ge1$)

It is a wonderful variety with $\Pi^{\min}_{X^\Sigma}=\Sigma$, and all colors in $\mathcal{D}(X^\Sigma)^B$ are obtained as irreducible components of $\overline{D}\cap X^\Sigma$, $D\in\mathcal{D}^B$. (To see the latter, observe that every B-divisor on X^Σ is contained in the zeroes of a B-eigenform in projective coordinates, which extends to X by complete reducibility of G-modules.) In particular, the above considered wonderful subvarieties X_i, X_{ij} of ranks $1,2$ are the localizations of X at $\{\lambda_i\}$, $\{\lambda_i,\lambda_j\}$, respectively.

Another kind of localization is defined by choosing a subset $I\subset\Pi$. Let P_I be the respective standard parabolic in G, with the standard Levi subgroup L_I, and $T_I=Z(L_I)^0$. Denote by $Z^I,\mathring{X}^I,X^I$ the sets of T_I-fixed points in Z, $\mathring{X}$, and $X_I:=P_I\mathring{X}$, respectively.

Lemma 30.2. (1) *The contraction by a general dominant one-parameter subgroup $\gamma \in \mathfrak{X}^*(T_I)$ gives a P_I-equivariant retraction $\pi_I : X_I \to X^I$, $\pi_I(x) = \lim_{t\to 0}\gamma(t)x$.*

(2) *X^I is a wonderful L_I-variety with $P(X^I) = P \cap L_I$, $\Pi_{X^I}^{\min} = \Pi_X^{\min} \cap \langle I\rangle$, and the colors of X^I are in bijection, given by the pull-back along π_I, with the P_I-unstable colors of X.*

(3) *$\mathring{X}^I \simeq (P_{\mathrm{u}} \cap L_I) \times Z^I$ is the $(B \cap L_I)$-chart of X^I intersecting all orbits.*

(4) *$\mathrm{R_u}(P_I^-)$ fixes X^I pointwise.*

Proof. It is obvious that π_I contracts $\mathring{X} \simeq P_{\mathrm{u}} \times Z$ onto $\mathring{X}^I \simeq (P_{\mathrm{u}} \cap L_I) \times Z^I$, while the conjugation by $\gamma(t)$ contracts P_I to L_I. Hence π_I extends to a retraction of X_I onto $X^I = L_I\mathring{X}^I$, and $\pi_I^{-1}(\mathring{X}^I) = \mathring{X}$ since $X_I \setminus \mathring{X}$ is closed and γ-stable. Thus the P_I-unstable B-divisors on X intersect X_I and are the pull-backs of the $(B \cap L_I)$-divisors on X^I. The structure of $\mathring{X}^I$ readily implies the remaining assertions on X^I in (2): both G- and B-orbits intersect $\mathring{X}^I$ in the orbits of $(P_{\mathrm{u}} \cap L_I)T$, and $\Pi_X^{\min} \cap \langle I\rangle$ is the set of T-weights of Z^I.

Since $\mathrm{R_u}(P_I^-)$-orbits are connected, it suffices to prove in (4) that $\mathrm{R_u}(P_I^-)x \cap \mathring{X} = \{x\}$, $\forall x \in \mathring{X}^I$. If $gx \in \mathring{X}$ for some $g \in \mathrm{R_u}(P_I^-)$, then $\gamma(t)gx = \gamma(t)g\gamma(t)^{-1}x \to x$ as $t \to \infty$, whence $gx = x$, because $\gamma(t)$ contracts $\mathring{X}$ to $\mathring{X}^I$ as $t \to 0$. □

The wonderful variety X^I is called the *localization* of X at I. It is easy to see that $X^I \subseteq X^\Sigma$, where $\Sigma = \Pi_X^{\min} \cap \langle I\rangle$. Conversely, for any $\Sigma \subseteq \Pi_X^{\min}$ one has $X^\Sigma \simeq G *_Q X^I$, where I is the set of simple roots occurring in the decompositions of $\lambda_i \in \Pi_X^{\min}$ and Q is the parabolic generated by P_I^- and P^-.

It is helpful to extend localization to an arbitrary spherical homogeneous space $\mathcal{O} = G/H$ using an arbitrary smooth complete toroidal embedding $X \hookleftarrow \mathcal{O}$ instead of the wonderful one. For any subset $\Sigma \subseteq \Pi_{\mathcal{O}}^{\min}$ one can find an orbit $\mathcal{O}' \subset X$ such that $\mathcal{C}_{\mathcal{O}'}$ is a solid subcone in the face $\mathcal{V} \cap \Sigma^\perp$ of $\mathcal{V} = \mathcal{V}(\mathcal{O})$. Then $X^\Sigma := \overline{\mathcal{O}'}$ is a wonderful subvariety of X and $\Pi_{X^\Sigma}^{\min} = \Sigma$. In particular, the localization at a single $\lambda \in \Pi_{\mathcal{O}}^{\min}$ yields $\Pi_{\mathcal{O}}^{\min} \subseteq \Sigma_G$.

Also for any $I \subset \Pi$ such that $\Lambda(\mathcal{O}) \not\subset \langle I\rangle$ one can find $X \hookleftarrow \mathcal{O}$ and a general dominant one-parameter subgroup $\gamma \in \mathfrak{X}^*(T_I)$ which is contained in a unique solid cone $\mathcal{C}_Y$ in the fan of X. (It suffices to take care lest the image of $\langle I\rangle^\perp$ in $\mathcal{E}(\mathcal{O})$ should lie in a hyperplane which separates two neighboring solid cones in the fan.) Starting with $\mathring{X} = \mathring{X}_Y$, one defines X^I as above and generalizes Lemma 30.2. If $\Lambda(\mathcal{O}) \subset \langle I\rangle$, then X^I can be defined for any toroidal embedding $X \hookleftarrow \mathcal{O}$ using $\mathring{X} = X \setminus \bigcup_{D\in\mathcal{D}^B} D$. Lemma 30.2 extends to this setup except that X^I may be no longer wonderful.

In particular, the localization at a single root $\alpha \in \Pi$ yields a smooth complete subvariety X^α of rank 1 acted on by $S_\alpha = L'_\alpha \simeq \mathrm{SL}_2(\Bbbk)$ or $\mathrm{PSL}_2(\Bbbk)$. The classification of complete varieties of rank 1, together with Lemma 30.2, allows to subdivide all simple roots in 4 types:

(p) $\alpha \in \Pi_L$. Here X^α is a point and P_α leaves all colors stable.

(b) $\alpha \notin \mathbb{Q}_+\Pi^{\min}_{\mathcal{O}} \cup \Pi_L$. If X^α is wonderful, then $r(X^\alpha) = 0$ whence $X^\alpha = S_\alpha/B \cap S_\alpha \simeq \mathbb{P}^1$; otherwise $r(X^\alpha) = 1$ and $X^\alpha \simeq S_\alpha *_{B\cap S_\alpha} \mathbb{P}^1$, where $B \cap S_\alpha$ acts on $\mathbb{P}^1$ via a character. There is a unique P_α-unstable color $D_\alpha = \overline{\pi_\alpha^{-1}([e])}$ or $\overline{\pi_\alpha^{-1}(e * \mathbb{P}^1)}$.

(a) $\alpha \in \Pi^{\min}_{\mathcal{O}}$. Here $r(X^\alpha) = 1$ and $X^\alpha \simeq \mathbb{P}^1 \times \mathbb{P}^1$. There are two P_α-unstable colors $D^+_\alpha = \overline{\pi_\alpha^{-1}(\mathbb{P}^1 \times [e])}$ and $D^-_\alpha = \overline{\pi_\alpha^{-1}([e] \times \mathbb{P}^1)}$.

(a') $2\alpha \in \Pi^{\min}_{\mathcal{O}}$. Here $r(X^\alpha) = 1$ and $X^\alpha \simeq \mathbb{P}^2 = \mathbb{P}(\mathfrak{s}_\alpha)$. There is a unique P_α-unstable color $D'_\alpha = \overline{\pi_\alpha^{-1}(\mathbb{P}(\mathfrak{b} \cap \mathfrak{s}_\alpha))}$.

The type of a color $D \in \mathcal{D}^B$ is defined as the type of $\alpha \in \Pi$ such that P_α moves D. Using the localization at $\{\alpha, \beta\} \subseteq \Pi$ and the classification of wonderful varieties of rank ≤ 2, one verifies that, as a rule, each $D \in \mathcal{D}^B$ is moved by a unique P_α, with the following exceptions: $D_\alpha = D_\beta$ iff α, β are pairwise orthogonal simple roots of type b such that $\alpha+\beta \in \Pi^{\min}_{\mathcal{O}} \sqcup 2\Pi^{\min}_{\mathcal{O}}$; two sets $\{D^\pm_\alpha\}$ and $\{D^\pm_\beta\}$ may intersect in one color for distinct α, β of type a. In particular, each color belongs to exactly one type. We obtain disjoint partitions $\Pi = \Pi^a \sqcup \Pi^{a'} \sqcup \Pi^b \sqcup \Pi^p$, $\mathcal{D}^B = \mathcal{D}^a \sqcup \mathcal{D}^{a'} \sqcup \mathcal{D}^b$ according to the types of simple roots and colors.

Lemma 30.3. *For $\forall \lambda \in \Lambda(\mathcal{O})$ we have*

$$\begin{aligned} \langle D^+_\alpha, \lambda\rangle + \langle D^-_\alpha, \lambda\rangle &= \langle \alpha^\vee, \lambda\rangle, &\quad& \forall \alpha \in \Pi^a \\ \langle D'_\alpha, \lambda\rangle &= \langle \tfrac{\alpha^\vee}{2}, \lambda\rangle, && \forall \alpha \in \Pi^{a'} \\ \langle D_\alpha, \lambda\rangle &= \langle \alpha^\vee, \lambda\rangle, && \forall \alpha \in \Pi^b \end{aligned}$$

Proof. Let $Y^\alpha \simeq S_\alpha/(B^- \cap S_\alpha) \simeq \mathbb{P}^1$ be a closed S_α-orbit in X^α. Namely Y^α is the diagonal of $X^\alpha \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in type a, a conic in $X^\alpha \simeq \mathbb{P}^2$ in type a', and a section of $X^\alpha \to \mathbb{P}^1$ in type b. Put $\delta_\lambda = \sum_{D \in \mathcal{D}^B} \langle D, \lambda\rangle D$. From the description of P_α-stable and unstable colors, we readily derive $\langle Y^\alpha, \delta_\lambda\rangle = \langle D^+_\alpha, \lambda\rangle + \langle D^-_\alpha, \lambda\rangle$, $2\langle D'_\alpha, \lambda\rangle$, or $\langle D_\alpha, \lambda\rangle$, depending on the type of α.

On the other hand, $\delta_\lambda \sim -\sum \langle v_i, \lambda\rangle D_i$, where D_i runs over all G-stable prime divisors in X and $v_i \in \mathcal{V}$ is the corresponding G-valuation. Since

$\mathcal{O}(D_i)|_{D_i}$ is the normal bundle of D_i, the fiber of $\mathcal{O}(D_i)$ at the B^--fixed point $z \in Y^\alpha$ is T_zX/T_zD_i for $\forall D_i \supseteq Y^\alpha$. Note that the T-weights λ_i of these fibers form the basis of $-\mathcal{C}_Y^\vee$ dual to $-v_i$'s, where $Y = Gz$ is the closed G-orbit in X containing Y^α. Hence $\mathcal{O}(\delta_\lambda)|_{Y^\alpha} = \mathcal{L}(-\sum_{D_i \supseteq Y}\langle v_i, \lambda\rangle\lambda_i) = \mathcal{L}(\lambda)$ and $\langle Y^\alpha, \delta_\lambda\rangle = \deg \mathcal{L}(\lambda) = \langle \alpha^\vee, \lambda\rangle$. The lemma follows. $\square$

These results show that $\mathcal{D}^{a'}, \mathcal{D}^b$ as abstract sets and their representation in $\mathcal{E}(\mathcal{O})$ are determined by Π^p and $\Pi_{\mathcal{O}}^{\min}$. The colors of type a, together with the weight lattice, the parabolic P, and the simple minimal roots, form a collection of combinatorial invariants supposed to identify $\mathcal{O}$ up to isomorphism. Namely $(\Lambda(\mathcal{O}), \Pi^p, \Pi_{\mathcal{O}}^{\min}, \mathcal{D}^a)$ is a homogeneous spherical datum in the sense of the following

Definition 30.3 ([Lu6, §2]). A *homogeneous spherical datum* is a collection $(\Lambda, \Pi^p, \Sigma, \mathcal{D}^a)$, where Λ is a sublattice in $\mathfrak{X}(T)$, $\Pi^p \subseteq \Pi_G$, $\Sigma \subseteq \Sigma_G \cap \Lambda$ is a linearly independent set consisting of indivisible vectors in Λ, and $\mathcal{D}^a$ is a finite set equipped with a map $\rho : \mathcal{D}^a \to \Lambda^*$, which satisfies the following axioms:

(A1) $\langle \rho(D), \lambda\rangle \leq 1$, $\forall D \in \mathcal{D}^a$, $\lambda \in \Sigma$, and the equality is reached iff $\lambda = \alpha \in \Sigma \cap \Pi$ and $D = D_\alpha^\pm$, where $D_\alpha^+, D_\alpha^- \in \mathcal{D}^a$ are two distinct elements depending on α.

(A2) $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee$ on Λ for $\forall \alpha \in \Sigma \cap \Pi$.

(A3) $\mathcal{D}^a = \{D_\alpha^\pm \mid \alpha \in \Sigma \cap \Pi\}$

(Σ1) If $\alpha \in \Pi \cap \frac{1}{2}\Sigma$, then $\langle \alpha^\vee, \Lambda\rangle \subseteq 2\mathbb{Z}$ and $\langle \alpha^\vee, \Sigma \setminus \{2\alpha\}\rangle \leq 0$.

(Σ2) If $\alpha, \beta \in \Pi$, $\alpha \perp \beta$, and $\alpha + \beta \in \Sigma \sqcup 2\Sigma$, then $\alpha^\vee = \beta^\vee$ on Λ.

(S) $\langle \alpha^\vee, \Lambda\rangle = 0$, $\forall \alpha \in \Pi^p$, and the pair (λ, Π^p) comes from a wonderful variety of rank 1 for any $\lambda \in \Sigma$.

A *spherical system* is a triple $(\Pi^p, \Sigma, \mathcal{D}^a)$ satisfying the above axioms with $\Lambda = \mathbb{Z}\Sigma$.

The homogeneous spherical datum of the open orbit in a wonderful variety amounts to its spherical system. It is easy to see that there are finitely many spherical systems for given G.

For the homogeneous spherical datum of $\mathcal{O}$, most of the axioms (A1)–(A3), (Σ1)–(Σ2), (S) are verified using the above results together with some additional general arguments. For instance, the inequality in (Σ1) stems from the fact that $\Sigma = \Pi_{\mathcal{O}}^{\min}$ is a base of a root system $\Delta_{\mathcal{O}}^{\min}$. On the other hand, each axiom involves at most two simple or spherical roots, like the axioms of classical root systems. Thus the localizations at one or two simple or spherical roots reduce the verification to wonderful varieties of rank ≤ 2.

Actually the list of axioms was obtain by inspecting the classification of wonderful varieties of rank ≤ 2, which leads to the following conclusion: spherical systems (homogeneous data) with $|\Sigma| \leq 2$ bijectively correspond to wonderful varieties of rank ≤ 2 (resp. to spherical homogeneous spaces $\mathcal{O} = G/H$ with $r(G/N(H)) \leq 2$). It is tempting to extend this combinatorial classification to arbitrary wonderful varieties and spherical spaces. Luna succeeded to fulfil this program in the case, where all simple factors of G are locally isomorphic to SL_{n_i}.

Theorem 30.3 ([Lu6]). *Suppose G is a reductive group with all simple factors of type* **A***; then there are natural bijections:*

$$\left\{ \begin{array}{c} \textit{spherical homogeneous} \\ G\textit{-spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homogeneous spherical} \\ \textit{data for } G \end{array} \right\} \tag{30.1}$$

$$\{\textit{wonderful } G\textit{-varieties}\} \longleftrightarrow \{\textit{spherical systems for } G\} \tag{30.2}$$

Recently this result was generalized to the groups with the simple factors of types **A** and **D** [Bra] or **A** and **C** (with some technical restrictions) [Pez].

Actually (30.1) was proved by Luna for any G provided that $G/Z(G)$ satisfies (30.2). The basic idea is to replace $\mathcal{O} = G/H$ by $\overline{\mathcal{O}} = G/\overline{H}$. This passage preserves the types of simple roots and colors, and $\Pi^{\min}_{\overline{\mathcal{O}}}$ is obtained from $\Pi^{\min}_{\mathcal{O}}$ by a dilation: some $\lambda \in \Pi^{\min}_{\mathcal{O}} \setminus \mathbb{Q}_+\Pi$ are replaced by 2λ. It is not hard to prove that spherical subgroups H with a fixed very sober hull $\overline{H}$ bijectively correspond to homogeneous spherical data $(\Lambda, \Pi^p, \Sigma, \mathcal{D}^a)$ such that $(\Pi^p, \Pi^{\min}_{\overline{\mathcal{O}}}, \mathcal{D}^a)$ is the spherical system of $\overline{\mathcal{O}}$, $\Lambda \supset \Pi^{\min}_{\overline{\mathcal{O}}}$, and Σ is obtained from $\Pi^{\min}_{\overline{\mathcal{O}}}$ by replacing $\lambda \in \Pi^{\min}_{\overline{\mathcal{O}}} \setminus \mathbb{Q}_+\Pi$ by $\lambda/2$ whenever $\lambda/2 \in \Lambda$ [Lu6, §6].

If (30.2) holds for the adjoint group of G, then $\Pi^{\min}_{\overline{\mathcal{O}}}$ is obtained from $\Pi^{\min}_{\mathcal{O}}$ by the "maximal possible" dilation: every $\lambda \in \Pi^{\min}_{\mathcal{O}} \setminus \mathbb{Q}_+\Pi$ such that $2\lambda \in \Sigma_G$ is replaced by 2λ [Lu6, 7.1]. It follows that the spherical homogeneous datum of $\mathcal{O}$ determines the spherical system of $\overline{\mathcal{O}}$ in a pure combinatorial way. Conversely, this spherical system together with Λ determines $\overline{\mathcal{O}}$ and $\mathcal{O}$ by the above, which proves (30.1).

The proof of (30.2) for adjoint G is much more involved for lacking of a uniform conceptual argument. Let us explain the general scheme. The first stage is to prove that certain geometric operations on wonderful varieties (localization, parabolic induction, direct product, etc) are expressed in a pure combinatorial language of spherical systems. Every spherical system is obtained by these combinatorial operations from a list of *primitive* systems, which are explicitly classified. For primitive spherical systems, the existence and uniqueness of a geometric realization is proved case by case.

Another situation, where the assertions of Theorem 30.3 remain valid, is the classification of solvable spherical subgroups (of arbitrary G) [Lu3]. More precisely, the bijections (30.1)–(30.2) hold for spherical homogeneous spaces with stabilizers contained in a Borel subgroup of G. Spherical data arising here satisfy $\Sigma = \Pi^a$, $\Pi^{a'} = \Pi^p = \emptyset$, $D_\alpha^- \neq D_\beta^\pm$ ($\forall \alpha, \beta \in \Pi^a$, $\alpha \neq \beta$), and $-(\mathbb{Q}_+\Sigma)^\vee + \sum \mathbb{Q}_+\rho(D_\alpha^+) = \Lambda \otimes \mathbb{Q}$.

31 Frobenius splitting

Frobenius splitting is a powerful tool of modern algebraic geometry which allows to prove various geometric and cohomological results by reduction to positive characteristic. This notion was introduced by Mehta and Ramanathan [MR] in their study of Schubert varieties.

Let X be an algebraic variety over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The Frobenius endomorphism $f \mapsto f^p$ of $\mathcal{O}_X$ gives rise to the Frobenius morphism $F : X^{1/p} \to X$, where $X^{1/p} = X$ as ringed spaces but the $\Bbbk$-algebra structure on $\mathcal{O}_{X^{1/p}}$ is defined as $c * f = c^p f$, $\forall c \in \Bbbk$.

If X is a subvariety in $\mathbb{A}^n$ or $\mathbb{P}^n$, then $X^{1/p}$ is, too. The defining equations of $X^{1/p}$ are obtained from those of X by replacing all coefficients by their p-th powers. The Frobenius morphism F is given by raising all coordinates to the power p.

The Frobenius endomorphism may be regarded as an injection of $\mathcal{O}_X$-modules $\mathcal{O}_X \hookrightarrow F_*\mathcal{O}_{X^{1/p}}$, where $F_*\mathcal{O}_{X^{1/p}} = \mathcal{O}_X$ is endowed with another $\mathcal{O}_X$-module structure: $f * h = f^p h$ for any local sections f of $\mathcal{O}_X$ and h of $F_*\mathcal{O}_{X^{1/p}}$.

Definition 31.1. The variety X is said to be *Frobenius split* if the Frobenius homomorphism has an $\mathcal{O}_X$-linear left inverse $\sigma : F_*\mathcal{O}_{X^{1/p}} \to \mathcal{O}_X$, called a *Frobenius splitting*. In other words, σ is a $\mathbf{Z}_p$-linear endomorphism of $\mathcal{O}_X$ such that $\sigma(1) = 1$ and $\sigma(f^p h) = f\sigma(h)$.

For any subvariety $Y \subset X$ one has $\sigma(\mathcal{I}_Y) \supseteq \mathcal{I}_Y$, because $\mathcal{I}_Y \supseteq \mathcal{I}_Y^p$. The splitting σ is *compatible* with Y if $\sigma(\mathcal{I}_Y) = \mathcal{I}_Y$. Clearly, a compatible splitting induces a splitting of Y.

More generally, let δ be an effective Cartier divisor on X, with the canonical section $\eta_\delta \in \mathrm{H}^0(X, \mathcal{O}(\delta))$, $\operatorname{div} \eta_\delta = \delta$. We say that X is *Frobenius split relative to* δ if there exists an $\mathcal{O}_X$-module homomorphism, called a δ-*splitting*, $\sigma_\delta : F_*\mathcal{O}_{X^{1/p}}(\delta) \to \mathcal{O}_X$ such that $\sigma(h) = \sigma_\delta(h\eta_\delta)$ is a Frobenius splitting, or equivalently, $\sigma_\delta(\eta_\delta) = 1$ and $\sigma_\delta(f^p\eta) = f\sigma_\delta(\eta)$ for any local section η of $\mathcal{O}(\delta)$.

The δ-splitting σ_δ is *compatible* with Y if $\operatorname{Supp}\delta$ contains no component of Y (i.e., δ restricts to a divisor on Y) and σ is compatible with Y. Then σ_δ induces a $(\delta \cap Y)$-splitting of Y.

For a systematic treatment of Frobenius splitting and its applications, we refer to a monograph of Brion and Kumar [BKu]. Here we recall some of its most important properties.

Clearly, a Frobenius splitting of X (compatible with Y, relative to δ) restricts to a splitting of every open subvariety $U \subset X$ (compatible with $Y \cap U$, relative to $\delta \cap U$). Conversely, if X is normal and $\operatorname{codim}(X \setminus U) > 1$, then any splitting of U extends to X. In applications it is often helpful to consider $U = X^{\mathrm{reg}}$.

If $\phi : X \to Y$ is a morphism such that $\phi_*\mathcal{O}_X = \mathcal{O}_Y$, then a Frobenius splitting of X descends to a splitting of Y. If the splitting of X is compatible with $Z \subset X$, then the splitting of Y is compatible with $\overline{\phi(Z)}$. For instance, one obtains a splitting of a normal variety X from that of its desingularization.

It is not hard to prove that Frobenius split varieties are *weakly normal*, i.e., every bijective finite birational map onto a Frobenius split variety has to be isomorphic [BKu, 1.2.5].

Proposition 31.1. (1) *Suppose X is a Frobenius split projective variety; then $\mathrm{H}^i(X, \mathcal{L}) = 0$ for any ample line bundle $\mathcal{L}$ on X and $\forall i > 0$.*

(2) *If $Y \subset X$ is a compatibly split subvariety, then the restriction map $\mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(Y, \mathcal{L})$ is surjective.*

(3) *If the splittings above are relative to an ample divisor δ, then the assertions of* (1)–(2) *hold for any numerically effective line bundle.*

(4) *There are relative versions of assertions* (1)–(3) *for a proper morphism $\phi : X \to Z$ stating that $\mathrm{R}^i\phi_*\mathcal{L} = 0$ and $\phi_*\mathcal{L} \to \phi_*(\iota_*\iota^*\mathcal{L})$ is surjective under the same assumptions, with $\iota : Y \hookrightarrow X$.*

Proof. The idea is to embed the cohomology of $\mathcal{L}$ as a direct summand in the cohomology of a sufficiently big power of $\mathcal{L}$. Namely the canonical homomorphism $\mathcal{L} \to F_*F^*\mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_{X^{1/p}}$ has a left inverse $\mathbf{1} \otimes \sigma$, whence $\mathcal{L}$ is a direct summand in $F_*F^*\mathcal{L}$. Taking the cohomology yields a split injection

$$\mathrm{H}^i(X, \mathcal{L}) \hookrightarrow \mathrm{H}^i(X, F_*F^*\mathcal{L}) \simeq \mathrm{H}^i(X^{1/p}, F^*\mathcal{L}) \simeq \mathrm{H}^i(X, \mathcal{L}^{\otimes p}), \qquad \forall i \geq 0$$

(The right isomorphism is only $\mathbf{Z}_p$-linear.) Iterating this procedure yields a split $\mathbf{Z}_p$-linear injection $\mathrm{H}^i(X,\mathcal{L}) \hookrightarrow \mathrm{H}^i(X,\mathcal{L}^{\otimes p^k})$ compatible with the restriction to Y. Thus the assertions (1) and (2) are reduced to the case of the line bundle $\mathcal{L}^{\otimes p^k}$, $k \gg 0$, where the Serre theorem applies [Har, III.5.3].

Similar reasoning applies to (3) making use of a split injection $\mathrm{H}^i(X,\mathcal{L}) \hookrightarrow \mathrm{H}^i(X,\mathcal{L}^{\otimes p} \otimes \mathcal{O}(\delta))$ together with ampleness of $\mathcal{L}^{\otimes p} \otimes \mathcal{O}(\delta)$. The relative assertions are proved by the same arguments. $\square$

Among other cohomology vanishing results for Frobenius split varieties we mention the extension of the Kodaira vanishing theorem [BKu, 1.2.10(i)]: if X is smooth projective and Frobenius split, then $\mathrm{H}^i(X,\mathcal{L} \otimes \omega_X) = 0$ for ample $\mathcal{L}$ and $i > 0$.

Now we reformulate the notion of Frobenius splitting for smooth varieties in terms of differential forms.

The de Rham derivation of $\Omega^\bullet_X$ may be considered as an $\mathcal{O}_X$-linear derivation of $F_*\Omega^\bullet_{X^{1/p}}$. Let $\mathcal{H}^k_X$ denote the respective cohomology sheaves. It is easy to check that $f \mapsto [f^{p-1}df]$ is a $\Bbbk$-derivation of $\mathcal{O}_X$ taking values in $\mathcal{H}^1_X$ (where $[\,\cdot\,]$ denotes the de Rham cohomology class). By the universal property of Kähler differentials, it induces a homomorphism of graded $\mathcal{O}_X$-algebras

$$c : \Omega^\bullet_X \to \mathcal{H}^\bullet_X, \qquad c(f_0 df_1 \wedge \cdots \wedge df_k) = [f_0^p (f_1 \cdots f_k)^{p-1} df_1 \wedge \cdots \wedge df_k],$$

called the *Cartier operator*. Cartier proved that c is an isomorphism for smooth X. (Using local coordinates, the proof is reduced to the case $X = \mathbb{A}^n$, where the verification is straightforward [BKu, 1.3.4].)

Now suppose that X is smooth. Then we have the *trace map*

$$\tau : F_*\omega_{X^{1/p}} \to \omega_X, \qquad \tau(\omega) = c^{-1}[\omega]$$

In local coordinates $x_1,\dots,x_n$, the trace map can be characterized as the unique $\mathcal{O}_X$-linear map taking $(x_1\cdots x_n)^{p-1}dx_1 \wedge \cdots \wedge dx_n \mapsto dx_1 \wedge \cdots \wedge dx_n$ and $x_1^{k_1}\cdots x_n^{k_n} dx_1 \wedge \cdots \wedge dx_n \mapsto 0$ unless $k_1 \equiv \cdots \equiv k_n \equiv p-1 \pmod p$.

Using the trace map, it is easy to establish an isomorphism

$$\mathcal{H}om(F_*\mathcal{O}_{X^{1/p}}, \mathcal{O}_X) \simeq F_*\omega^{1-p}_{X^{1/p}}, \qquad \sigma \leftrightarrow \widehat{\sigma}$$

such that $\sigma(h)\omega = \tau(h\omega^{\otimes p} \otimes \widehat{\sigma})$ for any local sections h of $F_*\mathcal{O}_{X^{1/p}}$ and ω of ω_X. Similarly, for any divisor δ on X we have

$$\mathcal{H}om(F_*\mathcal{O}_{X^{1/p}}(\delta), \mathcal{O}_X) \simeq F_*\omega^{1-p}_{X^{1/p}}(-\delta)$$

This leads to the following conclusion.

Proposition 31.2 ([BKu, 1.3.8, 1.4.10]). *Suppose that X is smooth and irreducible. Then $\sigma \in \mathrm{Hom}(F_*\mathcal{O}_{X^{1/p}}, \mathcal{O}_X)$ is a splitting of X iff the Taylor expansion of $\widehat{\sigma}$ at some (whence any) $x \in X$ has the form*

$$\Bigl((x_1 \cdots x_n)^{p-1} + \sum c_{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n}\Bigr)\,(\partial_1 \wedge \cdots \wedge \partial_n)^{\otimes(p-1)}$$

where the sum is taken over all multiindices $(k_1, \dots, k_n)$ such that $\exists k_i \not\equiv p-1 \pmod p$. (Here x_i denote local coordinates and ∂_i the vector fields dual to dx_i.) If X is complete, then it suffices to have

$$\widehat{\sigma} = ((x_1 \cdots x_n)^{p-1} + \cdots)(\partial_1 \wedge \cdots \wedge \partial_n)^{\otimes(p-1)}$$

The splitting σ is relative to any effective divisor $\delta \le \operatorname{div}\widehat{\sigma}$.

By abuse of language, we shall say that $\widehat{\sigma}$ splits X if σ does. Also, X is said to be *split by a $(p-1)$-th power* if $\alpha^{\otimes(p-1)}$ splits X for some $\alpha \in \mathrm{H}^0(X, \omega_X^{-1})$. This splitting is compatible with $\mathbb{V}(\alpha)$. For instance, a smooth complete variety X is split by the $(p-1)$-th power of α if the divisor of α in a neighborhood of some $x \in X$ is a union of $n = \dim X$ smooth prime divisors intersecting transversally at x.

Example 31.1. Every smooth toric variety X is Frobenius split by a $(p-1)$-th power compatibly with ∂X. For complete X, this stems from the structure of its canonical divisor, given by Proposition 30.2 (which extends to positive characteristic in the toric case). The general case follows by passing to a smooth toric completion. Now toric resolution of singularities readily implies that all normal toric varieties are Frobenius split compatibly with their invariant subvarieties.

Example 31.2 ([Ram], [BKu, Ch.2–3]). Generalized flag varieties are Frobenius split by a $(p-1)$-th power. For $X = G/B$, $\omega_X^{-1} = \mathcal{L}(-2\rho)$ and the splitting is provided by $\alpha = f_\rho \cdot f_{-\rho} \in V^*(2\rho)$, where $f_{\pm\rho} \in V^*(\rho)$ are T-weight vectors of weights $\pm\rho$.

Moreover, this splitting is compatible with all Schubert subvarieties $S_w = \overline{B[w]} \subset X$, $w \in W$. Using the weak normality of S_w and the Bott–Samelson resolution of singularities

$$\begin{gathered}\phi : \check{S} = \check{S}_{\alpha_1,\dots,\alpha_l} := P_{\alpha_1} *_B \cdots *_B P_{\alpha_l}/B \to S_w \\ w = r_{\alpha_1} \cdots r_{\alpha_l}, \quad \alpha_i \in \Pi, \quad l = \dim S_w\end{gathered}$$

with connected fibers and $\mathrm{R}^i\phi_*\mathcal{O}_{\check{S}} = 0$, $\forall i > 0$, one deduces that S_w are normal (Demazure, Seshadri) and have rational singularities (Andersen, Ramanathan). These properties descend to Schubert subvarieties in G/P, $\forall P \supset B$.

Splitting by a $(p-1)$-th power has further important consequences. For instance, the Grauert–Riemenschneider theorem extends to this situation, due to Mehta–van der Kallen [MK]:

> If $\phi : X \to Y$ is a proper birational morphism, X is smooth and split by $\alpha^{\otimes(p-1)}$ such that ϕ is isomorphic on $X \setminus \mathbb{V}(\alpha)$, then $\mathrm{R}^i\phi_*\omega_X = 0$, $\forall i > 0$.

Although the concept of Frobenius splitting is defined in characteristic $p > 0$, it successfully applies to algebraic varieties in characteristic zero via reduction $\bmod p$.

Namely let X be an algebraic variety over an algebraically closed field $\Bbbk$ of characteristic 0. One can find a finitely generated subring $R \subset \Bbbk$ such that X is defined over R, i.e., is obtained from an R-scheme $\mathcal{X}$ by extension of scalars. One may assume that $\mathcal{X}$ is flat over R, after replacing R by a localization. For any maximal ideal $\mathfrak{p} \lhd R$ we have $R/\mathfrak{p} \simeq \mathbb{F}_{p^k}$. The variety $X_\mathfrak{p}$ obtained from the fiber $\mathcal{X}_\mathfrak{p}$ of $\mathcal{X} \to \operatorname{Spec} R$ over $\mathfrak{p}$ by an extension of scalars $\mathbb{F}_{p^k} \to \mathbb{F}_{p^\infty}$ is called a *reduction* $\bmod p$ of X and sometimes denoted simply by X_p (by abuse of notation).

Reductions $\bmod p$ exist and share geometric properties of X (affinity, projectivity, completeness, smoothness, normality, rationality of singularities, etc) for all sufficiently large p. Conversely, a local geometric property of open type (e.g., smoothness, normality, rationality of singularities) holds for X if it holds for X_p whenever $p \gg 0$. Replacing R by an appropriate localization, one may always assume that a given finite collection of algebraic and geometric objects on X (subvarieties, line bundles, coherent sheaves, morphisms, etc) is defined over R, whence specializes to X_p for $p \gg 0$; coherent sheaves may be supposed to be flat over R.

Cohomological applications of reduction $\bmod p$ are based on the semicontinuity theorem [Har, III.12.8], which may be reformulated in our setup as follows:

> If X is complete and $\mathcal{F}$ is a coherent sheaf on X, then $\dim \mathrm{H}^i(X, \mathcal{F}) = \dim \mathrm{H}^i(X_p, \mathcal{F}_p)$ for all $p \gg 0$.

This implies, for instance, that the assertions of Proposition 31.1 hold in characteristic zero provided that X_p are Frobenius split for $p \gg 0$. This is the case, e.g., for Fano varieties. Another case, which is important in the framework of this chapter, are spherical varieties.

Theorem 31.1 ([BI]). *If X is a spherical G-variety in characteristic 0, then X_p is Frobenius split by a $(p-1)$-th power compatibly with all G-subvarieties and relative to any given B-stable effective divisor, for $p \gg 0$.*

Proof. Using an equivariant completion of X and its toroidal desingularization, we may assume that X is smooth, complete, and toroidal. Consider the natural morphism $\phi : X \to X(\mathfrak{h})$, where $\mathfrak{h}$ is a generic isotropy subalgebra for $G : X$. By Proposition 30.2, $\omega_X^{-1} = \mathcal{O}(\partial X + \phi^*\mathcal{H})$, where $\mathcal{H}$ is a hyperplane section of $X(\mathfrak{h})$.

The restriction of $\mathcal{O}(\partial X)$ to a closed G-orbit $Y \subset X$ is the top exterior power of the normal bundle to Y, whence $\omega_Y^{-1} = \omega_X^{-1}|_Y \otimes \mathcal{O}(-\partial X)|_Y = \mathcal{O}(\phi^*\mathcal{H})|_Y$. Since Y is a generalized flag variety, Y_p is split by the $(p-1)$-th power of (the reduction $\bmod p$ of) some $\alpha_Y \in \mathrm{H}^0(Y, \omega_Y^{-1})$. The G-module $\mathrm{H}^0(Y, \omega_Y^{-1})$ being irreducible and $\mathcal{O}(\phi^*\mathcal{H})$ globally generated, the restriction map $\mathrm{H}^0(X, \mathcal{O}(\phi^*\mathcal{H})) \to \mathrm{H}^0(Y, \omega_Y^{-1})$ is surjective and α_Y extends to $\alpha_0 \in \mathrm{H}^0(X, \mathcal{O}(\phi^*\mathcal{H}))$.

We have $\partial X = D_1 \cup \cdots \cup D_k$, where D_i runs over all G-stable prime divisors of Y. It is easy to see from Proposition 31.2 that $\alpha = \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_k$ provides a splitting for X_p, where $\alpha_i \in \mathrm{H}^0(X, \mathcal{O}(D_i))$, $\operatorname{div} \alpha_i = D_i$. Moreover, this splitting is compatible with all $(D_i)_p$ and therefore with all G-subvarieties in X_p, because the latter are unions of transversal intersections of some $(D_i)_p$.

Finally, for any B-stable effective divisor δ we have $\delta \leq (1-p)K_X$ for $p \gg 0$, by Proposition 30.2. Hence the splitting is relative to δ_p by Proposition 31.2. □

It is worth noting that not all spherical varieties in positive characteristic are Frobenius split. Counterexamples are provided by some complete homogeneous spaces with non-reduced isotropy group subschemes [La].

Frobenius splitting of spherical varieties provides short and conceptual proofs for a number of important geometric and cohomological properties. In particular, Theorem 15.5 can be deduced in the following way.

Consider a resolution of singularities $\psi : X' \to X$, where X' is toroidal and quasiprojective. Choose an ample B-stable effective divisor δ on X'; then X'_p is split relative to δ_p for $p \gg 0$. By semicontinuity and Proposition 31.1(4), applied to the trivial line bundle over X'_p, $\mathrm{R}^i\psi_*\mathcal{O}_{X'} = 0$, whence X has rational singularities. By the same reason, $\mathcal{O}_X = \psi_*\mathcal{O}_{X'}$ surjects onto $\psi_*\mathcal{O}_{Y'}$ for any irreducible closed G-subvariety $Y' \subset X'$, whence $\psi_*\mathcal{O}_{Y'} = \mathcal{O}_Y$ for $Y = \psi(Y')$. Since Y' is smooth, Y is normal and has rational singularities by the above.

For any line bundle $\mathcal{L}$ on X denote $\mathcal{L}' = \psi^*\mathcal{L}$. The Leray spectral sequence

$$\mathrm{H}^{i+j}(X', \mathcal{L}') \Longleftarrow \mathrm{H}^i(X, \mathrm{R}^j\psi_*\mathcal{L}') = \mathrm{H}^i(X, \mathcal{L} \otimes \mathrm{R}^j\psi_*\mathcal{O}_{X'})$$

degenerates to $\mathrm{H}^i(X', \mathcal{L}') = \mathrm{H}^i(X, \mathcal{L})$, $\forall i \geq 0$. The same holds for direct images instead of cohomology. Together with Proposition 31.1, applied to

X'_p and $\mathcal{L}'_p$, this proves the following

Corollary 31.1. *Suppose* $\operatorname{char} \Bbbk = 0$*. If* X *is a complete spherical* G*-variety,* $Y \subset X$ *a* G*-subvariety, and* $\mathcal{L}$ *a numerically effective line bundle on* X*, then* $\mathrm{H}^i(X, \mathcal{L}) = 0$*,* $\forall i > 0$*, and the restriction map* $\mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(Y, \mathcal{L})$ *is surjective. More generally, if* X *is spherical and* $\phi : X \to Z$ *is a proper morphism, then* $\mathrm{R}^i\phi_*\mathcal{L} = 0$*,* $\forall i > 0$*, and* $\phi_*\mathcal{L} \to \phi_*(\iota_*\iota^*\mathcal{L})$ *is surjective, where* $\iota : Y \hookrightarrow X$*.*

See [Bri7], [Bri12] for other proofs.

More precise results on Frobenius splitting of spherical varieties and their subvarieties (usually G- or B-orbit closures) are obtained in special cases.

As noted above, generalized flag varieties are Frobenius split compatibly with their Schubert subvarieties, and the latter have rational singularities in positive, hence any (by semicontinuity), characteristic.

Equivariant normal embeddings of G (§27) are Frobenius split compatibly with their $(G \times G)$-subvarieties, in all positive characteristics. For wonderful completions of adjoint semisimple groups, this was established by Strickland [Str1]. The general case is due to Rittatore [Rit2], see also [BKu, Ch.6]. This implies that normal reductive group embeddings have rational singularities (in particular, they are Cohen–Macaulay) and that the coordinate algebras of normal reductive monoids have "good" filtration [Rit2, §4], [BKu, 6.2.13].

Brion and Polo proved that the closures of the Schubert cells in wonderful completions of adjoint semisimple groups (called *large Schubert varieties*) are compatibly split and deduced that they are normal and Cohen–Macaulay [BPo].

De Concini and Springer proved that wonderful embeddings of symmetric spaces for adjoint G are Frobenius split compatibly with their G-subvarieties [CS, 5.9] in odd characteristics. However this splitting is not always compatible with B-orbit closures; in fact, the latter may be neither normal nor Cohen–Macaulay [Bri16].

See [Bri16] for a detailed study of B-orbits in spherical varieties and their closures. This is an area of active current research, with many open questions.

Appendix

A1 Rational modules and linearization

Rational modules are representations of algebraic groups in the category of algebraic varieties.

Definition A1.1. Let G be a linear algebraic group. A finite-dimensional G-module M is called *rational* if the representation map $R : G \to \mathrm{GL}(M)$ is a homomorphism of algebraic groups. The terminology is explained by observing that for $G \subseteq \mathrm{GL}_n(\Bbbk)$ the matrix entries of $R(g)$ are rational functions in the matrix entries of $g \in G$ (the denominator being a power of $\det g$). Generally, a *rational G-module* is a union of finite-dimensional rational submodules.

A G-algebra A is said to be *rational* if it is a rational G-module and G acts on A by algebra automorphisms.

If a rational G-module M is at the same time an A-module and $g(am) = (ga)(gm)$ for $\forall g \in G, a \in A, m \in M$, then M is called a *rational G-A-module*.

By $\mathrm{Mor}(X, M)$ denote the set of all morphisms of an algebraic variety X to a vector space M. (If $\dim M = \infty$, then a morphism $X \to M$ is by definition a morphism to a finite-dimensional subspace of M.) It is a free $\Bbbk[X]$-module: $\mathrm{Mor}(X, M) \simeq \Bbbk[X] \otimes M$. If X is a G-variety and M is a rational G-module, then $\mathrm{Mor}(X, M)$ is a rational G-$\Bbbk[X]$-module.

The $\Bbbk[X]^G$-submodule $\mathrm{Mor}_G(X, M) \simeq (\Bbbk[X] \otimes M)^G$ of equivariant morphisms is called the *module of covariants* on X with values in M. If G is reductive and $\dim M < \infty$, then $\mathrm{Mor}_G(X, M)$ is finite over $\Bbbk[X]^G$ [PV, 3.12].

Another source for infinite-dimensional rational G-modules are functions or global sections of sheaves on G-varieties.

Let X be a G-variety, $\alpha, \pi_X : G \times X \to X$ the action morphism and the projection, and $\mathcal{F}$ a quasicoherent sheaf on X.

Definition A1.2. A *G-linearization* of $\mathcal{F}$ is an isomorphism $\widehat{\alpha} : \pi_X^*\mathcal{F} \xrightarrow{\sim} \alpha^*\mathcal{F}$ inducing a G-action on the set of local sections of $\mathcal{F}$ via isomorphisms $\widehat{\alpha}|_{g \times X} : \mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(gU)$ over all $g \in G$, U open in X.

A *G-sheaf* is a quasicoherent sheaf equipped with a G-linearization.

Theorem A1.1 ([Kem2]). *Given a G-variety X and a G-sheaf $\mathcal{F}$ on X, $\Bbbk[X]$ is a rational G-algebra and $\mathrm{H}^i(X,\mathcal{F})$ are rational G-$\Bbbk[X]$-modules.*

If $\mathcal{F}$ is the sheaf of sections of a vector bundle $F \to X$, then its G-linearization is given by a fiberwise linear action $G : F$ compatible with the projection onto X.

By abuse of language we often make no terminological difference between vector bundles and the respective locally free sheaves of sections since they determine each other.

An important problem is to construct G-linearizations for line bundles on G-varieties. Its treatment goes back to Mumford. Here we follow [KKLV].

Assume G is connected.

Theorem A1.2 ([KKLV, 2.4]). *If G is factorial, i.e., $\operatorname{Pic} G = 0$, then any line bundle $\mathcal{L}$ on a normal G-variety X is G-linearizable.*

We say that an algebraic group $\widetilde{G}$ is a *universal cover* of G if $\widetilde{G}/\mathrm{R_u}(\widetilde{G})$ is a product of a torus and a simply connected semisimple group, and there is an epimorphism $\widetilde{G} \to G$ with finite kernel. Every connected group has a universal cover: it is well known for reductive groups [Hum, §§32,33], [DGr, XXIII], and generally we may put $\widetilde{G} = G \times_{G_{\mathrm{red}}} \widetilde{G_{\mathrm{red}}}$, where $G_{\mathrm{red}} = G/\mathrm{R_u}(G)$.

Corollary A1.1. *Any line bundle $\mathcal{L}$ on X is $\widetilde{G}$-linearizable.*

Indeed, $\widetilde{G}$ is factorial.

Corollary A1.2. *A certain power $\mathcal{L}^{\otimes d}$ of $\mathcal{L}$ is G-linearizable.*

For d one may take the degree of the universal covering or the order of $\operatorname{Pic} G$ [KKLV, 2.4].

The existence of a G-linearization has fundamental consequences in the local description of G-varieties, due to Sumihiro:

Theorem A1.3 ([Sum], [KKLV, §1]). *Let G be a connected group acting on a normal variety X. Then any point $x \in X$ has an open G-stable neighborhood U which admits a locally closed G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$ for some G-module V.*

Proof. Take an affine neighborhood $U_0 \ni x$. The complement $D = X \setminus U_0$ may support no effective Cartier divisor. However if we remove $\bigcap_{g\in G} gD$ from X, then any effective Weil divisor with support D becomes base point free, hence Cartier (cf. Lemma 17.1).

Take $\sigma_0 \in \mathrm{H}^0(X, \mathcal{L})$ such that $U_0 = X_{\sigma_0}$. Then

$$\Bbbk[U_0] = \bigcup_{d \geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes d})/\sigma_0^d = \Bbbk\left[\frac{\sigma_1}{\sigma_0^{d_1}}, \dots, \frac{\sigma_m}{\sigma_0^{d_m}}\right]$$

for some $\sigma_i \in \mathrm{H}^0(X, \mathcal{L}^{\otimes d_i})$, $d_i \in \mathbb{N}$. Replacing $\mathcal{L}$ by a power, we may assume it to be a G-bundle and all $d_i = 1$. Include $\sigma_0, \dots, \sigma_m$ in a finite-dimensional G-submodule $M \subseteq \mathrm{H}^0(X, \mathcal{L})$. The induced rational map $X \dashrightarrow \mathbb{P}(V)$, $V = M^*$, is a locally closed embedding on $U = GU_0$. $\square$

Remark A1.1. If X is itself quasiprojective, then one may take $U = X$. Indeed, a certain power of an ample line bundle on X is G-linearizable, and we can find a finite-dimensional G-stable space of sections inducing a projective embedding of X.

A2 Invariant theory

Let G be a linear algebraic group and A a rational G-algebra. The subject of algebraic invariant theory is the structure of the subalgebra A^G of G-invariant elements.

A geometric view on the subject is to consider an affine G-variety $X = \operatorname{Spec} A$, provided that A is finitely generated. (Note that each rational G-algebra is a union of finitely generated G-stable subalgebras.) If A^G is finitely generated, too, then one may consider $X/\!\!/G := \operatorname{Spec} A^G$ and the natural dominant morphism $\pi = \pi_G : X \to X/\!\!/G$. The variety $X/\!\!/G$, considered together with π, is called the *categorical quotient* of $G : X$, because it is the universal object in the category of G-invariant morphisms from X to affine varieties. This means that every morphism $\phi : X \to Y$ (with affine Y) which is constant on G-orbits fits into a unique commutative triangle:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ X/\!\!/G \arrow[r, "\overline{\phi}"'] & Y \end{tikzcd}$$

Geometric properties of $X/\!\!/G$ and π_G translate into algebraic properties of A^G and of its embedding into A, and vice versa. The case of reductive G is considered by Geometric Invariant Theory (GIT). We collect basic results on invariants and quotients of affine varieties by reductive groups in the following theorem.

Main Theorem of GIT. *Let G be a reductive group and A a rational G-algebra.*

(1) *If A is finitely generated, then A^G is so.*

Under this assumption, put $X = \operatorname{Spec} A$. Then $\pi : X \to X/\!\!/G$ is well defined and has the following properties:

(2) *π is surjective and maps closed G-stable subsets of X to closed subsets of $X/\!\!/G$.*

(3) *$X/\!\!/G$ carries the quotient topology w.r.t. π, and $\mathcal{O}_{X/\!\!/G} = \pi_* \mathcal{O}_X^G$.*

(4) *If $Z_1, Z_2 \subset X$ are disjoint closed G-stable subsets, then $\pi(Z_1) \cap \pi(Z_2) = \emptyset$. In particular, each fiber of π contains a unique closed orbit.*

Thus $X/\!\!/G$ may be regarded as the "variety of closed orbits" for $G : X$. It is not hard to show that $\pi_G : X \to X/\!\!/G$ is the categorical quotient in the category of *all* algebraic varieties.

Finite generation of G-invariants goes back to Hilbert and Weyl (in characteristic zero), the general case is due to Nagata and Haboush. Other assertions are due to Mumford.

If G is *linearly reductive*, i.e., all rational G-modules are completely reducible (e.g., $\operatorname{char} \Bbbk = 0$ or G is a torus), then the proof is considerably simplified [PV, 3.4, 4.4] by using the G-A^G-module decomposition $A = A^G \oplus A_G$, where A_G is the sum of all nontrivial irreducible G-submodules. The respective projection

$$A \mapsto A^G, \qquad f \mapsto f^\natural$$

is known as the *Reynolds operator*. For finite G and $\operatorname{char} \Bbbk = 0$, it is just the group averaging:

$$f^\natural = \frac{1}{|G|} \sum_{g \in G} gf, \qquad \forall f \in A$$

For a complex reductive group G, the Reynolds operator may be defined by averaging over a compact real form of G.

The proof in positive characteristic may be found in [MFK, App. 1A, 1C].

For non-reductive groups the situation is not so nice—even finite generation of invariants fails due to famous Nagata's counterexample [Nag] and results of Popov [Po3]. However for subgroups of reductive groups acting on algebras or affine varieties, there are positive results on finite generation and the structure of invariant algebras and categorical quotients.

Lemma A2.1. *Let G be a reductive group, $H \subseteq G$ be an algebraic subgroup, A be a rational G-algebra, and $I \lhd A$ be a G-stable ideal. Then $(A/I)^H$ is a purely inseparable finite extension of A^H/I^H.*

The lemma is obvious for $\operatorname{char} \Bbbk = 0$, since A/I lifts to a G-submodule of A, whence $(A/I)^H = A^H/I^H$. The proof for $H = G$ may be found in [MFK, Lemma A.1.2] and the general case follows by the transfer principle (Remark 2.1).

Corollary A2.1. *Let M be a G-module and $N \subset M$ a G-submodule. For any $\overline{m} \in (M/N)^{(H)}$ there exist $q = p^n$ and $m \in (\mathrm{S}^q M)^{(H)}$ such that $m \mapsto \overline{m}^q \in \mathrm{S}^q(M/N)$.*

Proof. Just apply Lemma A2.1 to $A = \mathrm{S}^\bullet M$, $I = AN$, replacing H by the common kernel H_0 of all $\chi \in \mathfrak{X}(H)$, and use the fact that H/H_0 is diagonalizable. $\square$

Corollary A2.2. *If A carries a G-stable filtration, then $(\operatorname{gr} A)^H$ is a purely inseparable extension of $\operatorname{gr}(A^H)$.*

Proof. Each homogeneous component of $\operatorname{gr} A$ has the form M/N, where M, N are two successive members of the filtration. It remains to apply Corollary A2.1 to M and N. $\square$

The most important case is $H = U$, a maximal unipotent subgroup in G. U-invariants of rational G-algebras were studied by Hadzhiev, Vust, Popov (in characteristic zero), Donkin, Grosshans (in arbitrary characteristic), et al. We refer to [Gro2] for systematic exposition of the theory.

Lemma A2.2 ([Gro2, 14.3]). *A rational G-algebra is integral over its subalgebra $\langle G \cdot A^U \rangle$.*

The proof relies on Lemma A2.1. In characteristic zero, Lemma A2.2 is trivial, since $A = \langle G \cdot A^U \rangle$ by complete reducibility of G-modules and highest weight theory.

The fundamental importance of U-invariants is explained by the fact that A and A^U share a number of properties.

Theorem A2.1. *Let G be a connected reductive group and A a rational G-algebra.*

(1) *A rational G-algebra A is finitely generated (resp. has no nilpotents, is an integral domain) iff A^U is so.*

(2) *In particular, for any affine G-variety X, the categorical quotient $\pi_U : X \to X/\!\!/U$ is well defined.*

(3) *In characteristic zero, X is normal (has rational singularities) iff $X/\!\!/U$ is so.*

In characteristic zero, finite generation of U-invariants is due to Hadzhiev; other assertions were partially proved by Brion (nilpotents and zero divisors), Vust (normality), Kraft (rationality of singularities), and by Popov in full generality [Po4]. The theorem was extended to arbitrary characteristic by Grosshans. We shall give an outline of the proof scattered in [Gro2].

Finite generation of $A^U \simeq (\Bbbk[G/U] \otimes A)^G$ (see Remark 2.1) stems from that of A and of $\Bbbk[G/U]$. The latter is proved by a representation-theoretic argument (Lemma 2.2) or by providing an explicit embedding of G/U into a G-module, with the boundary of codimension 2 [Gro2, 5.6] (cf. Theorem 28.2).

The other assertions are proved using horospherical contraction (cf. §7).

The algebra A is endowed with a G-stable increasing filtration $A^{(n)}$ such that $\operatorname{gr} A$ has an integral extension $S = (\Bbbk[G/U^-] \otimes A^U)^T$ and $A^U \simeq \operatorname{gr}(A^U) = (\operatorname{gr} A)^U \simeq S^U$, where T acts on G/U^- by right translations. In characteristic zero this filtration is described in §7 (for $A = \Bbbk[X]$) and the general case is considered in [Gro2, §15].

Now finite generation of A^U implies that of S, hence of $\operatorname{gr} A$ (both are finite modules over $\langle G \cdot S^U \rangle$), and finally of A by a standard argument. Moreover, the algebra $R = \bigoplus_{n=0}^{\infty} A^{(n)} t^n \subseteq A[t]$ is finitely generated, too (because $R^U \simeq A^U[t]$) [Gro2, 16.5].

The remaining assertions may be proved for finitely generated A and $X = \operatorname{Spec} A$. As in §7, put $E = \operatorname{Spec} R$ and consider the natural $G \times \Bbbk^\times$-morphism $\delta : E \to \mathbb{A}^1$ with the zero fiber $X_0 = \operatorname{Spec} \operatorname{gr} A$ and other fibers isomorphic to X. Note that $\Bbbk^\times$ contracts E to X_0 (i.e., $\forall x \in E\ \exists \lim_{t \to 0} t \cdot x \in X_0$), because the grading on R is non-negative.

Since δ is flat, the set of $x \in E$ such that the schematic fiber $\delta^{-1}(\delta(x))$ has a given local property of open type (e.g., is reduced, irreducible, normal, or has rational singularity) at x is open in E. The complementary closed subset of E is $\Bbbk^*$-stable, whence it intersects X_0 whenever it is non-empty. It follows that X has the property of open type whenever X_0 has this property.

If an affine $\Bbbk$-scheme Z of finite type is reduced (resp. irreducible or normal), then $Z/\!\!/H$ is so for any algebraic group H acting on Z. (Only normality requires some explanation.) In particular, these properties are inherited by $X/\!\!/U$ from X.

Conversely, if $X/\!\!/U$ has one of these properties, then $\operatorname{Spec} S = (G/\!\!/U^- \times X/\!\!/U)/\!\!/T$ and X_0 have it, too. (For normality, we use the isomorphism

$S \simeq \operatorname{gr} A$ in characteristic zero.) By the above, X has this property. More elementary (and lengthy) arguments are given in [Gro2, §18].

The same reasoning works for rational singularities in characteristic zero, using the facts that $G/\!\!/U$ has rational singularities [Kem1] and that rational singularities are preserved by products [Elk] and categorical quotients modulo reductive groups [Bout] (this guarantees that $X/\!\!/U = (G/\!\!/U \times X)/\!\!/G$ has rational singularities provided that X is so).

A3 Geometric valuations

Let K be a function field, i.e., a finitely generated field extension of $\Bbbk$. By a valuation v of K we always mean a discrete $\mathbb{Q}$-valued valuation of $K/\Bbbk$, i.e., assume the following properties:

(1) $v : K^\times \to \mathbb{Q}$, $v(0) = \infty$

(2) $v(K^\times) \simeq \mathbb{Z}$

(3) $v(\Bbbk^\times) = 0$

(4) $v(fg) = v(f) + v(g)$

(5) $v(f + g) \geq \min(v(f), v(g))$

Remark A3.1. If v is defined only on a $\Bbbk$-algebra A with $\operatorname{Quot} A = K$, then it is extended to K in a unique way by putting $v(f/g) = v(f) - v(g)$, $f, g \in A$.

Our main source in the valuation theory is [ZS, Ch.4, App.2].

Definition A3.1. A valuation v is called *geometric* if there exists a normal variety X with $\Bbbk(X) = K$ (a *model* of K) and a prime divisor $D \subset X$ such that $v(f) = c \cdot v_D(f)$, $\forall f \in K$, for some $c \in \mathbb{Q}_+$. Here $v_D(f)$ is the order of f along D.

To any valuation corresponds a (*discrete*) *valuation ring* (*DVR*) $\mathcal{O}_v = \{f \in K \mid v(f) \geq 0\}$, which is a local ring with the maximal ideal $\mathfrak{m}_v = \{f \in K \mid v(f) > 0\}$ and quotient field K. The *residue field* of v is $\Bbbk(v) = \mathcal{O}_v/\mathfrak{m}_v$.

Example A3.1. If v is geometric, then $\mathcal{O}_v = \mathcal{O}_{X,D}$, $\Bbbk(v) = \Bbbk(D)$.

Properties. (1) *$\mathcal{O}_v$ is a maximal subring of K.*

(2) *$\mathcal{O}_v$ determines v up to proportionality.*

Definition A3.2. Let X be a model of K. A closed irreducible subvariety $Y \subseteq X$ is the *center* of v on X if $\mathcal{O}_v$ dominates $\mathcal{O}_{X,Y}$ (i.e., $\mathcal{O}_v \supseteq \mathcal{O}_{X,Y}$, $\mathfrak{m}_v \supseteq \mathfrak{m}_{X,Y}$, which implies $\Bbbk(v) \supseteq \Bbbk(Y)$).

Example A3.2. A prime divisor $D \subset X$ is the center of the respective geometric valuation.

If $\phi : X \to X'$ is a dominant morphism and v has the center $Y \subseteq X$, then the restriction v' of v to $K' = \Bbbk(X')$ has the center $Y' = \overline{\phi(Y)} \subseteq X'$.

Valuative criterion of separation: X is separated iff any (geometric) valuation has at most one center on X

Valuative criterion of properness: The map $\phi : X \to X'$ is proper iff any (geometric) valuation of K has the center on X provided that its restriction to K' has the center on X'.

Valuative criterion of completeness: X is complete iff any (geometric) valuation has the center on X.

Proposition A3.1. *If X is affine, then v has the center $Y \subseteq X$ iff $v|_{\Bbbk[X]} \geq 0$, and then $\mathbb{I}(Y) = \Bbbk[X] \cap \mathfrak{m}_v$.*

Proposition A3.2. *A valuation $v \neq 0$ is geometric iff* $\operatorname{tr.deg} \Bbbk(v) = \operatorname{tr.deg} K - 1$.

Proof. Assume that $\operatorname{tr.deg} K = n$ and that the residues of $f_1, \dots, f_{n-1} \in \mathcal{O}_v$ form a transcendence base of $\Bbbk(v)/\Bbbk$. Take a nonzero $f_n \in \mathfrak{m}_v$; then $f_1, \dots, f_n$ are easily seen to be a transcendence base of $K/\Bbbk$. Consider an affine variety X such that $\Bbbk[X]$ is the integral closure of $\Bbbk[f_1, \dots, f_n]$ in K. It is easy to show that $v|_{\Bbbk[X]} \geq 0$, whence v has the center $D \subset X$ and $f_1, \dots, f_{n-1} \in \Bbbk[D]$ are algebraically independent. Hence D is a prime divisor, and $\mathcal{O}_v = \mathcal{O}_{X,D}$ implies $v = v_D$ up to a multiple. The converse implication is obvious. $\square$

Proposition A3.3. *Let $\Bbbk \subseteq K' \subseteq K$ be a subfield.*

(1) *If v is a geometric valuation of K, then $v' = v|_{K'}$ is geometric.*

(2) *Any geometric valuation v' of K' extends to a geometric valuation v of K.*

Proof. (1) Take $f_1, \dots, f_k \in \mathcal{O}_v$ whose residues form a transcendence base of $\Bbbk(v)/\Bbbk(v')$. They are algebraically independent over K' (otherwise one can take an algebraic dependence of $f_1, \dots, f_k$ over $\mathcal{O}_{v'}$ with at least one coefficient not in $\mathfrak{m}_{v'}$, and pass to residues obtaining a contradiction). Hence $\operatorname{tr.deg} \Bbbk(v') = \operatorname{tr.deg} \Bbbk(v) - \operatorname{tr.deg}_{\Bbbk(v')} \Bbbk(v) \geq \operatorname{tr.deg} K - 1 - \operatorname{tr.deg}_{K'} K = \operatorname{tr.deg} K' - 1$, and we conclude by Proposition A3.2.

(2) Take a complete normal variety X' with a prime divisor $D' \subset X'$ such that v' is proportional to $v_{D'}$. We may construct a complete normal variety X with $\Bbbk(X) = K$ mapping onto X': take any complete model X of K and replace it by the normalization of the closure of the graph of the rational map $X \dashrightarrow X'$. Let $D \subset X$ be a component of the preimage of D' mapping onto D'. Then we may take $v = v_D$ up to a multiple. $\square$

A4 Schematic points

Given a $\Bbbk$-scheme X, it is often instructive to consider the respective representable functor associating with any $\Bbbk$-scheme S the set $X(S)$ of $\Bbbk$-morphisms $S \to X$, called *S-points* of X. If $S = \operatorname{Spec} A$ is affine, then S-points are called *A-points* and the notation $X(A) := X(S)$ is used.

Example A4.1. If $X \subseteq \mathbb{A}^n$ is an (embedded) affine scheme of finite type, then its A-point is given by an algebra homomorphism $\Bbbk[X] = \Bbbk[t_1, \dots, t_n]/\mathbb{I}(X) \to A$, i.e., by an n-tuple $x = (x_1, \dots, x_n) \in A^n$ satisfying the defining equations of X. A similar description works for quasiaffine schemes.

We require a closer look at this notion in case, where X is an algebraic variety over $\Bbbk$ and A is a local $\Bbbk$-algebra with the maximal ideal $\mathfrak{m}$. Given $\chi \in X(A)$, the closed point of $\operatorname{Spec} A$ is mapped by χ to the generic point of an irreducible subvariety $Y \subset X$ called the *center* of χ. If $\mathring{X} \subseteq X$ is an affine chart meeting Y, then $\chi \in \mathring{X}(A)$. Thus $X(A) = \bigcup \mathring{X}(A)$ over all affine open subsets $\mathring{X} \subseteq X$. From the algebraic viewpoint, an A-point of X is given by an irreducible subvariety $Y \subseteq X$ and a local algebra homomorphism $\mathcal{O}_{X,Y} \to A$, $\mathfrak{m}_Y \to \mathfrak{m}$, or by a homomorphism $\Bbbk[\mathring{X}] \to A$, where $\mathring{X} \subseteq X$ is an affine chart (intersecting Y).

Example A4.2. The *generic point* of an irreducible variety X over $\Bbbk(X)$ has the center X, and $\mathcal{O}_{X,X} \to \Bbbk(X)$ is the identity map. Informally, the coordinates of the generic point are indeterminates bound only by relations that hold identically on X.

Example A4.3. If v is a valuation of $\Bbbk(X)$ with center $Y \subseteq X$, then the inclusion $\mathcal{O}_{X,Y} \subseteq \mathcal{O}_v$ yields an $\mathcal{O}_v$-point of X with center Y.

Example A4.4. Any A-point of a quasiprojective scheme $X \subseteq \mathbb{P}^n$ is at the same time an A-point of $X \cap \mathbb{A}^n$ for a certain affine chart $\mathbb{A}^n \subseteq \mathbb{P}^n$. In view of Example A4.1, A-points of X are identified with tuples $x = (x_0 : \dots : x_n)$, $x_i \in A$, considered up to proportionality, satisfying the defining equations of X, and such that at least one x_i is invertible.

It is quite common in algebraic geometry to consider the case, where A is a field. For applications in §24, we consider points over the function field of an algebraic curve or its formal analogue.

Definition A4.1. A *germ of a curve* in X is a pair (χ, θ_0), where $\chi \in X(\Bbbk(\Theta))$, Θ is a smooth projective curve, and $\theta_0 \in \Theta$. In other words, a germ of a curve is given by a rational map from a curve to X and a fixed base point on the curve.

The germ is said to be *convergent* if $\chi \in X(\mathcal{O}_{\Theta,\theta_0})$, i.e., the rational map $\chi : \Theta \dashrightarrow X$ is regular at θ_0. The point $x_0 = \chi(\theta_0)$ is the *limit* of the germ.

There is a formal analytic analogue of this notion.

Definition A4.2. A *germ of a formal curve* in X is a $\Bbbk((t))$-point of X. A $\Bbbk[[t]]$-point is called a *convergent formal germ*, and its center $x_0 \in X$ is the *limit* of the formal germ.

It is natural to think of a formal germ as of a "parametrized formal analytic curve" $x(t)$ in X. In local coordinates, $x(t)$ is a tuple of Laurent series satisfying the defining equations. If $x(t)$ converges, then its coordinates are power series, and their constant terms are the coordinates of the limit $x_0 =: x(0) = \lim_{t\to 0} x(t)$.

With any germ of a curve $(\theta_0 \in \Theta \dashrightarrow X)$ one can associate a formal germ via the inclusions $\mathcal{O}_{\Theta,\theta_0} \subset \widehat{\mathcal{O}}_{\Theta,\theta_0} \simeq \Bbbk[[t]]$, $\Bbbk(\Theta) \subset \Bbbk((t))$, depending on the choice of a formal uniformizing parameter $t \in \widehat{\mathcal{O}}_{\Theta,\theta_0}$.

Proposition A4.1. *A formal germ is induced by a germ of a curve iff its center has dimension* ≤ 1*.*

Proof. The "only if" direction and the case, where the center is a point, are clear. Suppose the center of a formal germ is a curve $C \subseteq X$. Then $\Bbbk(C) \hookrightarrow \Bbbk((t))$. Choose any $f \in \Bbbk(C)$, $\operatorname{ord}_t f = k > 0$, and consider $s \in \Bbbk[[t]]$, $s^k = f$. Then $\Bbbk(C)(s) = \Bbbk(\Theta)$ is a function field of a smooth projective curve Θ, and $\Bbbk(\Theta) \cap \Bbbk[[t]] = \mathcal{O}_{\Theta,\theta_0}$ for a certain $\theta_0 \in \Theta$, so that $\widehat{\mathcal{O}}_{\Theta,\theta_0} = \Bbbk[[s]] = \Bbbk[[t]]$. $\square$

There is a *t-adic topology* on $X(\Bbbk((t)))$ thinner than the Zariski topology [BT]. For $X = \mathbb{A}^n$, a basic t-adic neighborhood of $x(t) = (x_1(t), \dots, x_n(t))$ consists of all $y(t) = (y_1(t), \dots, y_n(t))$ such that $\operatorname{ord}_t(y_i(t) - x_i(t)) \geq N$, $\forall i = 1, \dots, n$, where $N \in \mathbb{N}$. The t-adic topology on arbitrary varieties is induced from that on affine spaces using affine charts.

An important approximation result is due to Artin:

Theorem A4.1 ([Art]). *The set of formal germs induced by germs of curves is dense in* $X(\Bbbk((t)))$ *w.r.t. the t-adic topology.*

Bibliography

[Akh1] D. N. Akhiezer, *Equivariant completion of homogeneous algebraic varieties by homogeneous divisors*, Ann. Global Anal. Geom. **1** (1983), 49–78.

[Akh2] D. N. Akhiezer, *Actions with a finite number of orbits*, Funct. Anal. Appl. **19** (1985), no. 1, 1–4. **Russian original:** Д. Н. Ахиезер, *О действиях с конечным числом орбит*, Функц. анализ и его прил. **19** (1985), № 1, 1–5.

[Akh3] D. N. Akhiezer, *On modality and complexity of actions of reductive groups*, Russian Math. Surveys **43** (1988), no. 2, 157–158. **Russian original:** Д. Н. Ахиезер, *О модальности и сложности действий редуктивных групп*, Успехи мат. наук **43** (1988), № 2, 129–130.

[Akh4] D. N. Akhiezer, *Spherical varieties*, Schriftenreihe, Heft Nr. 199, Bochum, 1993.

[AV] D. N. Akhiezer, E. B. Vinberg, *Weakly symmetric spaces and spherical varieties*, Transform. Groups **4** (1999), no. 1, 3–24.

[AB1] V. Alexeev, M. Brion, *Stable reductive varieties, I: Affine varieties*, Invent. Math. **157** (2004), no. 2, 227–274.

[AB2] V. Alexeev, M. Brion, *Stable reductive varieties, II: Projective case*, Adv. Math. **184** (2004), no. 2, 380–408.

[AB3] V. Alexeev, M. Brion, *Moduli of affine schemes with reductive group action*, J. Algebra **14** (2005), no. 1, 83–117.

[Arn] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989. **Russian original:** В. И. Арнольд, *Математические методы классической механики*, Наука, Москва, 3-е изд., 1989.

[Art] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math., no. 36, pp. 23–58, 1969.

[Arzh] I. V. Arzhantsev, *On actions of complexity one of the group* SL_2, Math. USSR-Izv. **61** (1997), no. 4, 685–698. **Russian original:** И. В. Аржанцев, *О действиях сложности один группы* SL_2, Изв. РАН, сер. мат. **61** (1997), № 4, 3–18.

[Bar] I. Barsotti, *Structure theorems for group varieties*, Ann. Mat. Pura Appl. (4) **38** (1955), 77–119.

[BGGN] F. A. Berezin, I. M. Gelfand, M. I. Graev, M. A. Naimark, *Representations of groups*, Amer. Math. Soc. Transl. **16** (1960), 325–353. **Russian original:** Ф. А. Березин, И. М. Гельфанд, М. И. Граев, М. А. Наймарк, *Представления групп*, Успехи мат. наук **11** (1956), № 6, 13–40.

[Ber] D. I. Bernstein, *The number of roots of a system of equations*, Funct. Anal. Appl. **9** (1975), 183–185. **Russian original:** Д. И. Бернштейн, *О числе решений системы уравнений*, Функц. анализ и его прил. **9** (1975), № 3, 1–4.

[BGG] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), no. 3, 1–26. **Russian original:** И. Н. Бернштейн, И. М. Гельфанд, С. И. Гельфанд, *Клетки Шуберта и когомологии пространств G/P*, Успехи мат. наук **28** (1973), № 3, 3–26.

[BB1] A. Białynicki-Birula, *Some properties of the decompositions of algebraic varieties determined by actions of a torus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 667–674.

[BB2] A. Białynicki-Birula, *On induced actions of algebraic groups*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 2, 365–368.

[BHM] A. Białynicki-Birula, G. Hochschild, G. D. Mostow, *Extensions of representations of algebraic linear groups*, Amer. J. Math. **85** (1963), 131–144.

[BiB] F. Bien, M. Brion, *Automorphisms and local rigidity of regular varieties*, Compositio Math. **104** (1996), no. 1, 1–26.

[BCP] E. Bifet, C. de Concini, C. Procesi, *Cohomology of regular embeddings*, Adv. Math. **82** (1990), no. 1, 1–34.

[Bin] J. Bingener, *Locale Modulräume in der analytischen Geometrie*, vol. 2, Vieweg, Braunschweig, 2nd edn., 1987.

[Bol] A. V. Bolsinov, *Completeness of families of functions in involution related to compatible Poisson brackets*, Tensor and vector analysis (O. V. Manturov A. T. Fomenko, V. V. Trofimov, eds.), pp. 3–24, Gordon and Breach, Amsterdam, 1998. **Russian original:** А. В. Болсинов, *О полноте семейств функций в инволюции, связанных с согласованными скобками Пуассона*, Тр. семинара по вект. и тенз. анализу **23** (1988), 18–38.

[Bor] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207.

[BoBr] W. Borho, J.-L. Brylinski, *Differential operators on homogeneous spaces, II, Relative enveloping algebras*, Bull. Soc. Math. France **117** (1989), 167–210.

[Bour] N. Bourbaki, *Groupes et algèbres de Lie*, chap. IV–VI, Éléments de Math., Hermann, Paris, 1968.

[Bout] J.-F. Boutot, *Singularités rationnelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987), 65–68.

[Bra] P. Bravi, *Wonderful varieties of type* **D**, Ph.D. thesis, Università di Roma "La Sapienza", Dec. 2003.

[Bri1] M. Brion, *Quelques propriétés des espaces homogènes sphériques*, Manuscripta Math. **55** (1986), 191–198.

[Bri2] M. Brion, *Classification des espaces homogènes sphériques*, Compositio Math. **63** (1987), 189–208.

[Bri3] M. Brion, *Sur l'image de l'application moment*, Séminaire d'algebre Paul Dubreil et Marie-Paule Malliavin, Lect. Notes in Math., vol. 1296, pp. 177–192, 1987.

[Bri4] M. Brion, *Groupe de Picard et nombres charactéristiques des variétés sphériques*, Duke Math. J. **58** (1989), no. 2, 397–424.

[Bri5] M. Brion, *On spherical homogeneous spaces of rank one*, Group actions and invariant theory, CMS Conf. Proc., vol. 10, pp. 31–41, 1989.

[Bri6] M. Brion, *Spherical varieties: an introduction*, Topological methods in algebraic transformation groups (H. Kraft, T. Petrie, G. Schwarz, eds.), Progress in Math., vol. 80, pp. 11–26, Birkhäuser, Basel–Boston–Berlin, 1989.

[Bri7] M. Brion, *Une extension du théorème de Borel–Weil*, Math. Ann. **286** (1990), 655–660.

[Bri8] M. Brion, *Vers une généralisation des espaces symétriques*, J. Algebra **134** (1990), 115–143.

[Bri9] M. Brion, *Sur la géométrie des variétés sphériques*, Comment. Math. Helv. **66** (1991), 237–262.

[Bri10] M. Brion, *Variétés sphériques et théorie de Mori*, Duke Math. J. **72** (1993), no. 2, 369–404.

[Bri11] M. Brion, *Piecewise polynomial functions, convex polytopes and enumerative geometry*, Parameter spaces, Banach Center Publ., vol. 36, pp. 25–44, Inst. of Math., Polish Acad. Sci., Warszawa, 1996.

[Bri12] M. Brion, *Curves and divisors on spherical varieties*, Algebraic groups and Lie groups; a volume of papers in honour of the late R. W. Richardson (G. Lehrer et al, ed.), Austral. Math. Soc. Lect. Ser., vol. 9, pp. 21–34, Cambridge University Press, Cambridge, 1997.

[Bri13] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Representation Theories and Algebraic Geometry (A. Broer, ed.), Nato ASI Series C, vol. 514, pp. 1–37, Montreal, 1997, Kluwer.

[Bri14] M. Brion, *Variétés sphériques*, Notes de la session de la S. M. F. “Opérations hamiltoniennes et opérations de groupes algébriques”, p. 59, Grenoble, 1997, `http://www-fourier.ujf-grenoble.fr/~mbrion/spheriques.ps`.

[Bri15] M. Brion, *The behaviour at infinity of the Bruhat decomposition*, Comment. Math. Helv. **73** (1998), 137–174.

[Bri16] M. Brion, *On orbit closures of spherical subgroups in flag varieties*, Comment. Math. Helv. **76** (2001), 263–299.

[Bri17] M. Brion, *Group completions via Hilbert schemes*, J. Algebraic Geom. **12** (2003), no. 4, 605–626.

[BI] M. Brion, S. P. Inamdar, *Frobenius splitting of spherical varieties*, Algebraic groups and their generalizations: Classical methods (W. J. Haboush, B. J. Parshall, eds.), Proc. Symp. Pure Math., vol. 56, part 1, pp. 207–218, AMS, Providence, 1994.

[BKn] M. Brion, F. Knop, *Contractions and flips for varieties with group actions of small complexity*, J. Math. Sci. Univ. Tokyo **1** (1994), 641–655.

[BKu] M. Brion, S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Math., vol. 231, Birkhäuser, Boston, 2005.

[BL] M. Brion, D. Luna, *Sur la structure locale des variétés sphériques*, Bull. Soc. Math. France **115** (1987), 211–226.

[BLV] M. Brion, D. Luna, Th. Vust, *Espaces homogènes sphériques*, Invent. Math. **84** (1986), 617–632.

[BPa] M. Brion, F. Pauer, *Valuations des espaces homogènes sphériques*, Comment. Math. Helv. **62** (1987), 265–285.

[BPo] M. Brion, P. Polo, *Large Schubert varieties*, Represent. Theory **4** (2000), 97–126.

[BT] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math., no. 41, pp. 5–252, 1972.

[Ca1] É. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264.

[Ca2] É. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **55** (1927), 114–134.

[Ca3] É. Cartan, *Sur la détermination d'un système orthogonal complet dans un espace de Riemann symétrique clos*, Rend. Circ. Mat. Palermo **53** (1929), 217–252.

[CX] E. Casas-Alvero, S. Xambó-Descamps, *The enumerative theory of conics after Halphen*, Lect. Notes in Math., vol. 1196, 1986.

[Cha] J.-Y. Charbonnel, *Propriétés (Q) et (C). Variété commutante*, Bull. Soc. Math. France **132** (2004), 477–508.

[Ch] M. Chasles, *Determination du nombre des sections coniques qui doivent toucher cinq courbes d'ordre quelconque, ou satisfaire à diverses autres conditions*, C. R. Acad. Sci. Paris Sér. I Math. **58** (1864), 222–226.

[Che] J. Cheah, *Cellular decompositions for nested Hilbert schemes of points*, Pacific J. Math. **183** (1998), no. 1, 39–90.

[CF] A. Collino, W. Fulton, *Intersection rings of spaces of triangles*, Mém. Soc. Math. France (N.S.) **38** (1989), 75–117.

[Con] C. de Concini, *Normality and non-normality of certain semigroups and orbit closures*, Invariant theory and Algebraic transformation groups III (R. V. Gamkrelidze, V. L. Popov, eds.), Encyclopaedia Math. Sci., vol. 132, pp. 15–35, Springer-Verlag, Berlin, 2004.

[CGMP] C. de Concini, M. Goresky, R. MacPherson, C. Procesi, *On the geometry of quadrics*, Comment. Math. Helv. **63** (1988), 337–413.

[CP1] C. de Concini, C. Procesi, *Complete symmetric varieties, I*, Invariant theory, Proceedings (F. Gherardelli, ed.), Lect. Notes in Math., vol. 996, pp. 1–44, Montecatini, 1983, Springer-Verlag.

[CP2] C. de Concini, C. Procesi, *Complete symmetric varieties, II*, Algebraic groups and related topics (R. Hotta, ed.), Adv. Studies in Pure Math., vol. 6, pp. 481–513, Kinokuniya, Tokio, 1985.

[CP3] C. de Concini, C. Procesi, *Cohomology of compactifications of algebraic groups*, Duke Math. J. **53** (1986), 585–594.

[CS] C. de Concini, T. A. Springer, *Compactification of symmetric spaces*, Transform. Groups **4** (1999), no. 2–3, 273–300.

[C-F] S. Cupit-Foutou, *Classification of two-orbit varieties*, Comment. Math. Helv. **78** (2003), 245–265.

[Dan] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), no. 2, 97–154. **Russian original:** В. И. Данилов, *Геометрия торических многообразий*, Успехи мат. наук **33** (1978), № 2, 85–134.

[Dem1] M. Demazure, *Sur la formule des caractères de H. Weyl*, Invent. Math. **9** (1969/1970), no. 3, 249–252.

[Dem2] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.

[Dem3] M. Demazure, *A very simple proof of Bott's theorem*, Invent. Math. **33** (1976), no. 3, 271–272.

[Dem4] M. Demazure, *Limites de groupes orthogonaux ou symplectiques*, Preprint, Paris, 1980.

[DGa] M. Demazure, P. Gabriel, *Groupes algébriques I*, Masson/North-Holland, Paris/Amsterdam, 1970.

[DGr] M. Demazure, A. Grothendieck, *Schémas en groupes*, Lect. Notes in Math., vol. 151–153, 1970.

[Don] S. Donkin, *Good filtrations of rational modules for reductive groups*, Proc. Symp. Pure Math., vol. 47, pp. 69–80, 1987.

[Do] S. Doty, *Representation theory of reductive normal monoids*, Trans. Amer. Math. Soc. **351** (1999), 2539–2551.

[Ela1] A. G. Elashvili, *Canonical forms and stationary subalgebras of points in general position for simple linear Lie groups*, Funct. Anal. Appl. **6** (1972), no. 1, 44–53. **Russian original:** А. Г. Элашвили, *Канонический вид и стационарные подалгебры точек общего положения для простых линейных групп Ли*, Функц. анализ и его прил. **6** (1972), № 1, 51–62.

[Ela2] A. G. Elashvili, *Stationary subalgebras of points in general position for irreducible linear Lie groups*, Funct. Anal. Appl. **6** (1972), no. 2, 139–148. **Russian original:** А. Г. Элашвили, *Стационарные подалгебры точек общего положения для неприводимых линейных групп Ли*, Функц. анализ и его прил. **6** (1972), № 2, 65–78.

[Elk] R. Elkik, *Singularités rationnelles et déformations*, Invent. Math. **47** (1978), 139–147.

[Ful1] W. Fulton, *Intersection theory*, Ergeb. Math. Grenzgeb. (3), vol. 2, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1984.

[Ful2] W. Fulton, *Introduction to toric varieties*, Princeton University Press, Princeton, 1993.

[FMSS] W. Fulton, R. MacPherson, F. Sottile, B. Sturmfels, *Intersection theory on spherical varieties*, J. Algebraic Geom. **4** (1995), 181–193.

[FS] W. Fulton, B. Sturmfels, *Intersection theory on toric varieties*, Preprint, 1994.

[Gel] I. M. Gelfand, *Spherical functions on symmetric Riemannian spaces*, Amer. Math. Soc. Transl. **37** (1964), 39–43. **Russian original:** И. М. Гельфанд, *Сферические функции на симметрических римановых пространствах*, Докл. АН СССР **70** (1950), 5–8.

[Gin] V. Ginzburg, *Admissible modules on a symmetric space*, Astérisque **173–174** (1989), 199–255.

[GOV] V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg, *Structure of Lie groups and Lie algebras*, Lie groups and Lie algebras III, Encyclopaedia Math. Sci., vol. 41, Springer, Berlin–Heidelberg–New York, 1994. **Russian original:** Э. Б. Винберг, В. В. Горбацевич, А. Л. Онищик, *Строение групп и алгебр Ли*, Соврем. пробл. мат. Фундам. направл., т. 41, ВИНИТИ, Москва, 1990.

[Gro1] F. D. Grosshans, *Constructing invariant polynomials via Tschirnhaus transformations*, Invariant theory (S. S. Koh, ed.), Lect. Notes in Math., vol. 1278, pp. 95–102, Springer, Berlin–Heidelberg–New York, 1987.

[Gro2] F. D. Grosshans, *Algebraic homogeneous spaces and Invariant theory*, Lect. Notes in Math., vol. 1673, 1997.

[GS] V. Guillemin, S. Sternberg, *Multiplicity-free spaces*, J. Differential Geom. **19** (1984), 31–56.

[Har] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

[Hel1] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.

[Hel2] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, Academic Press, Orlando, 1984.

[Ho] R. Howe, *The First Fundamental Theorem of Invariant Theory and Spherical Subgroups*, Algebraic groups and their generalizations: Classical methods (W. J. Haboush, B. J. Parshall, eds.), Proc. Symp. Pure Math., vol. 56, part 1, pp. 333–346, AMS, Providence, 1994.

[HW] A. T. Huckleberry, T. Wurzbacher, *Multiplicity-free complex manifolds*, Math. Ann. **286** (1990), 261–280.

[Hum] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975.

[IM] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math., no. 25, pp. 5–48, 1965.

[Jan] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, New York, 1987.

[Ka] S. Kato, *A Borel–Weil–Bott type theorem for group completions*, J. Algebra **259** (2003), 572–580.

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model program*, Algebraic geometry (T. Oda, ed.), Adv. Studies in Pure Math., vol. 10, pp. 283–360, Kinokuniya, Tokio, 1987.

[Kem1] G. Kempf, *On the collapsing of homogeneous bundles*, Invent. Math. **37** (1976), 229–239.

[Kem2] G. Kempf, *The Grothendieck–Cousin complex of an induced representation*, Adv. in Math. **29** (1978), 310–396.

[KKMS] G. Kempf, F. Knudson, D. Mumford, B. Saint-Donat, *Toroidal embeddings, I*, Lect. Notes in Math., vol. 339, 1973.

[Kle] S. L. Kleiman, *The transversality of a general translate*, Comment. Math. Helv. **38** (1974), 287–297.

[Kl] F. Klein, *Vergleichende Betrachtungen über neue geometrische Forschungen*, Erlangen, 1972.

[Kn1] F. Knop, *Weylgruppe und Momentabbildung*, Invent. Math. **99** (1990), 1–23.

[Kn2] F. Knop, *The Luna–Vust theory of spherical embeddings*, Proc. Hyderabad Conf. on Algebraic Groups (S. Ramanan, ed.), pp. 225–249, Manoj Prakashan, Madras, 1991.

[Kn3] F. Knop, *Über Bewertungen, welche unter einer reductiven Gruppe invariant sind*, Math. Ann. **295** (1993), 333–363.

[Kn4] F. Knop, *Über Hilbert vierzehntes Problem für Varietäten mit Kompliziertheit eins*, Math. Z. **213** (1993), 33–36.

[Kn5] F. Knop, *The asymptotic behavior of invariant collective motion*, Invent. Math. **116** (1994), 309–328.

[Kn6] F. Knop, *A Harish-Chandra homomorphism for reductive group actions*, Ann. of Math. (2) **140** (1994), 253–288.

[Kn7] F. Knop, *On the set of orbits for a Borel subgroup*, Comment. Math. Helv. **70** (1995), 285–309.

[Kn8] F. Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. **9** (1996), no. 1, 153–174.

[KKLV] F. Knop, H. Kraft, D. Luna, Th. Vust, *Local properties of algebraic group actions*, Algebraische Transformationsgruppen und Invariantentheorie (H. Kraft, P. Slodowy, T. A. Springer, eds.), DMV Seminar, vol. 13, pp. 63–76, Birkhäuser, Basel–Boston–Berlin, 1989.

[KKV] F. Knop, H. Kraft, Th. Vust, *The Picard group of a G-variety*, Algebraische Transformationsgruppen und Invariantentheorie (H. Kraft, P. Slodowy, T. A. Springer, eds.), DMV Seminar, vol. 13, pp. 77–88, Birkhäuser, Basel–Boston–Berlin, 1989.

[KR] B. Kostant, S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[Kou] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.

[Krä] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math. **38** (1979), 129–153.

[LT] G. Lancaster, J. Towber, *Representation-functors and flag-algebras for the classical groups I*, J. Algebra **59** (1979), 16–38.

[Lau] J. Lauret, *Gelfand pairs attached to representations of compact Lie groups*, Transform. Groups **5** (2000), 307–324.

[La] N. Lauritzen, *Splitting properties of complete homogeneous spaces*, J. Algebra **162** (1993), 178–193.

[Lit] P. Littelmann, *On spherical double cones*, J. Algebra **166** (1994), 142–157.

[LP] P. Littelmann, C. Procesi, *Equivariant cohomology of wonderful compactifications*, Operator algebras, unitary representations, enveloping algebras, and Invariant Theory, Progress in Math., vol. 92, pp. 219–262, Birkhäuser, Boston, 1990.

[Lu1] D. Luna, *Sur les orbites fermées des groupes algèbriques réductifs*, Invent. Math. **16** (1972), 1–5.

[Lu2] D. Luna, *Slices étales*, Mém. Soc. Math. France (N.S.) **33** (1973), 81–105.

[Lu3] D. Luna, *Sous-groupes sphériques résolubles*, Prépublication de l'Institut Fourier no. 241, Grenoble, 1993.

[Lu4] D. Luna, *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), no. 3, 249–258.

[Lu5] D. Luna, *Grosses cellules pour les variétés sphériques*, Austral. Math. Soc. Lect. Ser., vol. 9, pp. 267–280, Cambridge University Press, Cambridge, 1997.

[Lu6] D. Luna, *Variétés sphériques de type A*, Inst. Hautes Études Sci. Publ. Math., no. 94, pp. 161–226, 2001.

[LV] D. Luna, Th. Vust, *Plongements d'espaces homogènes*, Comment. Math. Helv. **58** (1983), 186–245.

[Ma] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[Mat] Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J. **16** (1960), 205–216.

[MR] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian rings*, Graduate Studies in Math., vol. 30, AMS, Providence, 2001.

[McG] W. M. McGovern, *The adjoint representation and the adjoint action*, Invariant theory and Algebraic transformation groups II (R. V. Gamkrelidze, V. L. Popov, eds.), Encyclopaedia Math. Sci., vol. 131, pp. 159–238, Springer, Berlin–Heidelberg–New York, 2002.

[MK] V. B. Mehta, W. van der Kallen, *On a Grauert–Riemenschneider vanishing theorem for Frobenius split varieties in characteristic p*, Invent. Math. **108** (1992), 11–13.

[MR] V. B. Mehta, A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. (2) **122** (1985), 22–40.

[Mik] I. V. Mikityuk, *On the integrability of invariant Hamiltonian systems with homogeneous configuration spaces*, Math. USSR-Sb. **57** (1987), no. 3, 527–546. **Russian original:** И. В. Микитюк, *Об интегрируемости инвариантных гамильтоновых систем с однородными конфигурационными пространствами*, Мат. сб. **129** (1986), № 3, 514–534.

[MF1] A. S. Mishchenko, A. T. Fomenko, *Euler equations on finite-dimensional Lie groups*, Math. USSR-Izv. **12** (1978), no. 2, 371–389. **Russian original:** А. С. Мищенко, А. Т. Фоменко, *Уравнения Эйлера на конечномерных группах Ли*, Изв. АН СССР, сер. мат. **42** (1978), № 2, 396–415.

[MF2] A. S. Mishchenko, A. T. Fomenko, *Generalized Liouville method of integration of Hamiltonian systems*, Funct. Anal. Appl. **12** (1978), 113–121. **Russian original:** А. С. Мищенко, А. Т. Фоменко, *Обобщённый метод Лиувилля интегрирования гамильтоновых систем*, Функц. анализ и его прил. **12** (1978), № 2, 46–56.

[MJ1] L. Moser-Jauslin, *Normal embeddings of* $\mathrm{SL}(2)/\Gamma$, Thesis, University of Geneva, 1987.

[MJ2] L. Moser-Jauslin, *The Chow ring of smooth complete* $SL(2)$*-embeddings*, Compositio Math. **82** (1992), no. 1, 67–106.

[Mum] D. Mumford, *Abelian varieties*, Tata Inst. Fund. Research Studies in Math., no. 5, Oxford University Press, London, 1974.

[MFK] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, Springer-Verlag, Berlin–Heidelberg–New York, 3rd edn., 1994.

[Nag] M. Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

[Oda] T. Oda, *Convex bodies and algebraic geometry: an introduction to the theory of toric varieties*, Springer-Verlag, Berlin–Heidelberg–New York, 1988.

[Oni1] A. L. Onishchik, *Complex hulls of compact homogeneous spaces*, Soviet Math. Dokl. **1** (1960), 88–91. **Russian original:** А. Л. Онищик, *Комплексные оболочки компактных однородных пространств*, Докл. АН СССР **130** (1960), № 2, 726–729.

[Oni2] A. L. Onishchik, *Inclusion relations among transitive compact transformation groups*, Amer. Math. Soc. Transl. **50** (1966), 5–58. **Russian original:** А. Л. Онищик, *Отношения включения между транзитивными компактными группами преобразований*, Тр. ММО **11** (1962), 199–242.

[OV] A. L. Onishchik, E. B. Vinberg, *Lie groups and algebraic groups*, Springer, Berlin–Heidelberg–New York, 1990. **Russian original:** Э. Б. Винберг, А. Л. Онищик, *Семинар по группам Ли и алгебраическим группам*, Наука, Москва, 1988.

[Pan1] D. I. Panyushev, *Complexity and rank of homogeneous spaces*, Geom. Dedicata **34** (1990), 249–269.

[Pan2] D. I. Panyushev, *Complexity of quasiaffine homogeneous varieties, t-decompositions, and affine homogeneous spaces of complexity* 1, Lie groups, their discrete subgroups and Invariant Theory (E. B. Vinberg, ed.), Adv. Sov. Math., vol. 8, pp. 151–166, AMS, Providence, 1992.

[Pan3] D. I. Panyushev, *Complexity and rank of double cones and tensor product decompositions*, Comment. Math. Helv. **68** (1993), 455–468.

[Pan4] D. I. Panyushev, *Complexity and nilpotent orbits*, Manuscripta Math. **83** (1994), 223–237.

[Pan5] D. I. Panyushev, *On homogeneous spaces of rank one*, Indag. Math. **6** (1995), 315–323.

[Pan6] D. I. Panyushev, *A restriction theorem and the Poincaré series for U-invariants*, Math. Ann. **301** (1995), 655–675.

[Pan7] D. I. Panyushev, *Complexity and rank of actions in Invariant theory*, J. Math. Sci. (New York) **95** (1999), no. 1, 1925–1985.

[Pau1] F. Pauer, *Normale Einbettungen von G/U*, Math. Ann. **257** (1981), 371–396.

[Pau2] F. Pauer, *Glatte Einbettungen von G/U*, Math. Ann. **262** (1983), 421–429.

[Pau3] F. Pauer, *Plongements normaux de l'espace homogène $SL(3)/SL(2)$*, C. R. du 108$^{\text{ième}}$ Congrès Nat. Soc. Sav., Grenoble, 1983.

[Pau4] F. Pauer, *"Caracterisation valuative" d'une classe de sous-groupes d'un groupe algébrique*, C. R. du 109$^{\text{ième}}$ Congrès Nat. Soc. Sav., vol. 3, Grenoble, 1984.

[Pez] G. Pezzini, *Wonderful varieties of type* **C**, Ph.D. thesis, Università di Roma "La Sapienza", Dec. 2003.

[Po1] V. L. Popov, *Quasihomogeneous affine algebraic varieties of the group* SL(2), Math. USSR-Izv. **7** (1973), no. 4, 793–831. **Russian original:** В. Л. Попов, *Квазиоднородные аффинные алгебраические многообразия группы* SL(2), Изв. АН СССР, сер. мат. **37** (1973), № 4, 792–832.

[Po2] V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles*, Math. USSR-Izv. **8** (1974), no. 2, 301–327. **Russian original:** В. Л. Попов, *Группы Пикара однородных пространств линейных алгебраических групп и одномерные однородные векторные расслоения*, Изв. АН СССР, сер. мат. **38** (1974), № 2, 294–322.

[Po3] V. L. Popov, *Hilbert's theorem on invariants*, Soviet Math. Dokl. **20** (1979), 1318–1322. **Russian original:** В. Л. Попов, *О теореме Гильберта об инвариантах*, Докл. АН СССР **249** (1979), № 3, 551–555.

[Po4] V. L. Popov, *Contractions of actions of reductive algebraic groups*, Math. USSR-Sb. **58** (1987), no. 2, 311–335. **Russian original:** В. Л. Попов, *Стягивания действий редуктивных алгебраических групп*, Мат. сб. **130** (1986), № 3, 310–334.

[PV] V. L. Popov, E. B. Vinberg, *Invariant theory*, Algebraic geometry IV, Encyclopaedia Math. Sci., vol. 55, pp. 123–284, Springer, Berlin–Heidelberg–New York, 1994. **Russian original:** Э. Б. Винберг, В. Л. Попов, *Теория инвариантов*, Соврем. пробл. мат. Фундам. направл., т. 55, с. 137–309, ВИНИТИ, Москва, 1989.

[Put] M. S. Putcha, *Linear algebraic monoids*, London Math. Soc. Lect. Note Ser., no. 133, Cambridge University Press, London, 1988.

[Ram] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Inst. Hautes Études Sci. Publ. Math., no. 65, pp. 61–90, 1987.

[Ren1] L. E. Renner, *Quasi-affine algebraic monoids are affine*, Semigroup Forum **30** (1984), no. 2, 167–176.

[Ren2] L. E. Renner, *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc. **292** (1985), 193–223.

[Ren3] L. E. Renner, *Linear algebraic monoids*, Invariant theory and Algebraic transformation groups V (R. V. Gamkrelidze, V. L. Popov, eds.), Encyclopaedia Math. Sci., vol. 134, Springer-Verlag, Berlin, 2005.

[Ri1] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), 38–41.

[Ri2] R. W. Richardson, *Orbits, invariants and representations associated to involutions of reductive groups*, Invent. Math. **66** (1982), 287–312.

[RS1] R. W. Richardson, T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), 389–436.

[RS2] R. W. Richardson, T. A. Springer, *Combinatorics and geometry of K-orbits on the flag manifold*, Contemp. Math. **153** (1993), 109–142.

[Rit1] A. Rittatore, *Algebraic monoids and group embeddings*, Transform. Groups **3** (1998), no. 4, 375–396.

[Rit2] A. Rittatore, *Reductive embeddings are Cohen–Macaulay*, Proc. Amer. Math. Soc. **131** (2003), 675–684.

[Rit3] A. Rittatore, *On affine algebraic monoids*, Preprint, Jun. 2004.

[Ros] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), no. 2, 401–443.

[Ryb] L. G. Rybnikov, *On the commutativity of weakly commutative homogeneous spaces*, Funct. Anal. Appl. **37** (2003), 114–122. **Russian original:** Л. Г. Рыбников, *О коммутативности слабо коммутативных однородных пространств*, Функц. анализ и его прил. **37** (2003), № 2, 41–51.

[Sch1] H. C. H. Schubert, *Kalkül der abzählenden Geometrie*, 1879, reprinted by Springer-Verlag, 1979.

[Sch2] H. C. H. Schubert, *Anzahlbestimmungen für lineare Räume beliebiger Dimension*, Acta Math. **8** (1886), 97–118.

[Sch3] H. C. H. Schubert, *Allgemeine Anzahlfunctionen für Kegelschnitte, Flächen und Räume zweiten Grades in n Dimensionen*, Math. Ann. **45** (1894), 153–206.

[Sel] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with application to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87.

[Sem1] J. G. Semple, *On complete quadrics, I*, J. London Math. Soc. (2) **23** (1948), 258–267.

[Sem2] J. G. Semple, *On complete quadrics, II*, J. London Math. Soc. (2) **27** (1952), 280–287.

[Se1] J.-P. Serre, *Espaces fibrés algébriques*, Anneaux de Chow et Applications, Séminaire C. Chevalley, E. N. S., Paris, 1958.

[Se2] J.-P. Serre, *On the fundamental group af a unirational variety*, J. London Math. Soc. **34** (1959), 481–484.

[Sha] I. R. Shafarevich, *Basic algebraic geometry*, vol. 1, 2, Springer-Verlag, Berlin, 1994. **Russian original:** И. Р. Шафаревич, *Основы алгебраической геометрии*, т. 1, 2, Наука, Москва, 2-е изд., 1988.

[Sm-A] A. V. Smirnov, *Quasi-closed orbits in projective representations of semisimple complex Lie groups*, Trans. Moscow Math. Soc. (2003), 45–78. **Russian original:** А. В. Смирнов, *Квазизамкнутые орбиты в проективных представлениях полупростых комплексных групп Ли*, Тр. ММО **64** (2003), 193–247.

[Sm-E] E. Yu. Smirnov, *On generating systems for the ideals of S-varieties*, Moscow Univ. Math. Bull. **60** (2005), no. 3, ??–?? **Russian original:** Е. Ю. Смирнов, *О системах порождаюших для идеалов S-многообразий*, Вестник МГУ, сер. 1, мат. мех. (2005), № 3, 3–6.

[Sp1] T. A. Springer, *Some results on algebraic groups with involutions*, Algebraic groups and related topics (R. Hotta, ed.), Adv. Studies in Pure Math., vol. 6, pp. 525–543, Kinokuniya, Tokio, 1985.

[Sp2] T. A. Springer, *The classification of involutions of simple algebraic groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 655–670.

[Sp3] T. A. Springer, *Intersection cohomology of $B \times B$-orbit closures in group compactifications*, J. Algebra **258** (2002), 71–111.

[St] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. **80** (1968).

[Ste] J. R. Stembridge, *Multiplicity-free products and restrictions of Weyl characters*, Represent. Theory **7** (2003), 404–439.

[Str1] E. S. Strickland, *A vanishing theorem for group compactifications*, Math. Ann. **277** (1987), 165–171.

[Str2] E. S. Strickland, *Computing the equivariant cohomology of group compactifications*, Math. Ann. **291** (1991), no. 2, 275–280.

[Sukh] A. A. Sukhanov, *Description of the observable subgroups of linear algebraic groups*, Math. USSR-Sb. **65** (1990), no. 1, 97–108. **Russian original:** А. А. Суханов, *Описание наблюдаемых подгрупп линейных алгебраических групп*, Мат. сб. **137** (1988), № 1, 90–102.

[Sum] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), no. 1, 1–28.

[Tho] E. G. F. Thomas, *An infinitesimal characterization of Gelfand pairs*, Contemp. Math. **26** (1984), 379–385.

[Tim1] D. A. Timashev, *A generalization of the Bruhat decomposition*, Russian Acad. Sci. Izv. Math. **45** (1995), no. 2, 339–352 (English). **Russian original:** Д. А. Тимашёв, *Обобщение разложения Брюа*, Изв. РАН, сер. мат. **58** (1994), № 5, 110–123.

[Tim2] D. A. Timashev, *Classification of G-varieties of complexity* 1, Math. USSR-Izv. **61** (1997), no. 2, 363–397. **Russian original:** Д. А. Тимашёв, *Классификация G-многообразий сложности* 1, Изв. РАН, сер. мат. **61** (1997), № 2, 127–162.

[Tim3] D. A. Timashev, *Cartier divisors and geometry of normal G-varieties*, Transform. Groups **5** (2000), no. 2, 181–204.

[Tim4] D. A. Timashev, *Equivariant compactifications of reductive groups*, Russian Acad. Sci. Sb. Math. **194** (2003), no. 4, 589–616. **Russian original:** Д. А. Тимашёв, *Эквивариантные компактификации редуктивных групп*, Мат. сб. **194** (2003), № 4, 119–146.

[Tyr] J. A. Tyrell, *Complete quadrics and collineations in S_n*, Mathematika **3** (1956), 69–79.

[Vin1] E. B. Vinberg, *Complexity of actions of reductive groups*, Funct. Anal. Appl. **20** (1986), no. 1, 1–11. **Russian original:** Э. Б. Винберг, *Сложность действий редуктивных групп*, Функц. анализ и его прил. **20** (1986), № 1, 1–13.

[Vin2] E. B. Vinberg, *On reductive algebraic semigroups*, Lie Groups and Lie Algebras: E. B. Dynkin Seminar (S. G. Gindikin, E. B. Vinberg, eds.), Amer. Math. Soc. Transl., vol. 169, pp. 145–182, 1995.

[Vin3] E. B. Vinberg, *Commutative homogeneous spaces and co-isotropic symplectic actions*, Russian Math. Surveys **56** (2001), no. 1, 1–60. **Russian original:** Э. Б. Винберг, *Коммутативные однородные пространства и коизотропные симплектические действия*, Успехи мат. наук **56** (2001), № 1, 3–62.

[Vin4] E. B. Vinberg, *Commutative homogeneous spaces of Heisenberg type*, Trans. Moscow Math. Soc. (2003), 45–78. **Russian original:** Э. Б. Винберг, *Коммутативные однородные пространства гейзенбергова типа*, Тр. ММО **64** (2003), 54–89.

[VK] E. B. Vinberg, B. N. Kimelfeld, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funct. Anal. Appl. **12** (1979), 168–174. **Russian original:** Э. Б. Винберг, Б. Н. Кимельфельд, *Однородные области на флаговых многообразиях и сферические подгруппы полупростых групп Ли*, Функц. анализ и его прил. **12** (1978), № 3, 12–19.

[VP] E. B. Vinberg, V. L. Popov, *On a class of quasihomogeneous affine varieties*, Math. USSR-Izv. **6** (1972), no. 4, 743–758. **Russian original:** Э. Б. Винберг, В. Л. Попов, *Об одном классе квазиоднородных аффиных многообразий*, Изв. АН СССР, сер. мат. **36** (1972), № 4, 749–763.

[Vog] D. A. Vogan, *Irreducible characters of semisimple Lie groups III. Proof of the Kazhdan–Lusztig conjecture in the integral case*, Invent. Math. **71** (1983), 381–417.

[Vu1] Th. Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes*, Bull. Soc. Math. France **102** (1974), 317–334.

[Vu2] Th. Vust, *Plongements d'espaces symétriques algébriques: une classification*, Ann. Soc. Math. Polon. Ser. III Mat. Stos. **17** (1990), 165–194.

[Wa] B. Wasserman, *Wonderful varieties of rank two*, Transform. Groups **1** (1996), no. 4, 375–403.

[Wei] B. Yu. Weisfeiler, *On a class of unipotent subgroups in semisimple algebraic groups*, Uspekhi Mat. Nauk **21** (1966), no. 2, 222–223. **Russian original:** Б. Ю. Вейсфейлер, *Об одном классе унипотентных подгрупп полупростых алгебраических групп*, Успехи мат. наук **21** (1966), № 2, 222–223.

[Yak1] O. S. Yakimova, *On weakly symmetric spaces of semisimple Lie groups*, Moscow Univ. Math. Bull. **57** (2002), no. 2, 37–40. **Russian original:** О. С. Якимова, *О слабо симметрических пространствах полупростых групп Ли*, Вестник МГУ, сер. 1, мат. мех. (2002), № 2, 57–60.

[Yak2] O. S. Yakimova, *Gelfand pairs*, Bonner Mathematische Schriften **374** (2005), `http://hss.ulb.uni-bonn.de/diss_online/math_nat_fak/2004/yakimova_oksana`.

[ZS] O. Zariski, P. Samuel, *Commutative algebra*, vol. 1, 2, Van Nostrand, Princeton, 1960.

[Zor] A. A. Zorin, *On relation between spherical and weakly symmetric homogeneous spaces of reductive groups*, Moscow Univ. Math. Bull. **60** (2005), no. 4, ??–?? **Russian original:** А. А. Зорин, *О связи сферичности и слабой симметричности однородных пространств редуктивных групп*, Вестник МГУ, сер. 1, мат. мех. (2005), № 4, 14–18.