An introduction to quantized Lie groups and algebras

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Abstract

We give a selfcontained introduction to the theory of quantum groups according to Drinfeld highlighting the formal aspects as well as the applications to the Yang-Baxter equation and representation theory. Introductions to Hopf algebras, Poisson structures and deformation quantization are also provided. After having defined Poisson-Lie groups we study their relation to Lie-bi algebras and the classical Yang-Baxter equation. Then we explain in detail the concept of quantization for them. As an example the quantization of sl_2 is explicitly carried out. Next we show how quantum groups are related to the Yang-Baxter equation and how they can be used to solve it. Using the quantum double construction we explicitly construct the universal R-matrix for the quantum sl_2 algebra. In the last section we deduce all finite dimensional irreducible representations for q a root of unity. We also give their tensor product decomposition (fusion rules) which is relevant to conformal field theory.

Introduction

In the beginning of the eighties important progress was being made in the field of quantum integrable field theories. Crucial to this was a quantum mechanical version of the well known inverse scattering method used so successfully in the theory of integrable non-linear evolution equations like the Korteweg de Vries equation (KdV). In [1] P.Kulish and Y.Reshetikhin showed that the quantum linear problem of the quantum sine-Gordon equation was not associated with the Lie algebra sl_2 as in the classical case, but with a deformation of this algebra. Other work showed [2] that deformations of Lie algebraic structures were not special to the quantum sine-Gordon equation and it seemed that they were part of a more general theory.

It was V.I.Drinfeld who showed that a suitable algebraic quantization of so called Poisson Lie groups reproduced exactly the deformed algebraic structures encountered in the theory of quantum inverse scattering [3, 4, 5]. At approximately the same time M.Jimbo derived the same relations coming from a slightly different direction [7, 8]. The new algebraic structures were called quantized universal enveloping algebras (QUEA) and have been the object of intense investigation by mathematicians as well as physicists. The main reason that $QUEA$ are of such great importance is that they are closely related to the so called quantum Yang-Baxter equation which plays a prominent role in many areas of research such as knot theory, solvable lattice models, conformal field theory and quantum integrable systems. There was great excitement when string theorists found out that the decomposition of tensor product representations of the QUEAs resembled very much the fusion rules of certain conformal field theories. This triggered a great deal of research on the relation between quantum groups and chiral conformal algebras [9]. Unfortunately the relation turned out to be not as straightforward as hoped and even today there is no completely satisfying answer to the question how precisely conformal field theory is related to quantum groups. At the moment research focusses on the representation theory of QUEAs.

A different approach to quantum groups and one with a completely different background (C*-algebra theory) was initiated by S.L.Woronowicz who initially called them pseudogroups [10, 11, 12, 13]. The crucial ingredient in this approach is the Gelfand-Naimark theorem which roughly states that any commutative C^*-algebra with unit element is isomorphic to an algebra of all continuous functions on some compact topological manifold. If the topological space is a topological group then the space of functions on it picks up extra structure. Generalizing the basic properties of function spaces on topological groups to non-commutative C^*-algebras the interpretation in terms of an underlying manifold is gone but we can still persue the theory. The motivation for this is that in the commutative case all algebraic information on the topological group is contained in the (extra) structure of its function space which means that we can study the group manifold itself or its function space, it does not make any difference. In the non-commutative case we only have information on the function space (if you insist on calling it that), but this only means that you no longer have the same information in two different disguises. This approach to quantum groups is the one which is most popular among pure mathematicians (algebraists).

Even though the two approaches have different origins they are closely related [20]. The main differences are that in the Drinfeld approach one does not quantize the space of functions on a group (at least not directly) but the universal enveloping algebra of the Lie algebra which is dual to the space of functions. However a more important difference is that in the Woronowicz approach there is no mention of a Poisson structure while they play a prominent role in the Drinfeld approach.

Up to now Drinfeld's approach to quantum groups has received the most attention in the physics literature. The present paper is meant to give a non-specialist introduction to this approach and its applications providing proofs and derivations where they are omitted in literature. In section 1 we review the basic facts of Hopf algebras which is the language in which the theory of quantum groups is written. We also show that the space of functions on a Lie group is a commutative Hopf algebra. As mentioned above it contains all essential information on G, and its dual space is shown to be 'almost equal' to the universal enveloping algebra of the Lie algebra of G, which is also a Hopf algebra.

We thus go from the group G , via the space of smooth functions on G to the universal enveloping algebra of the Lie algebra of G. The reason for taking this path becomes clear in sections 2 and 3 where we introduce and discuss in detail Poisson and co-Poisson structures. If the Lie group happens to be the phase space of some classical dynamical system, then the space of functions on it carries a Poisson bracket. This in turn induces a co-Poisson bracket on the universal enveloping algebra because of the duality between these two spaces. In short, the universal enveloping algebra of a Lie group which is also a classical phase space is a co-Poisson Hopf algebra (section 3). In section 4 there is then defined a suitable form of quantization for such an object which involves, as is well known in physics, replacing a (co)Poisson bracket by a (co)commutator. Since the first four sections are rather abstract we consider in section 5 the case of sl_2 as an example. Using the definition of quantization given in section 4 we quantize sl_2 explicitly giving the so called q-deformed algebra $\mathcal{U}_q(sl_2)$. In section 6 we use this algebra to explicitly solve the Yang-Baxter equation which is of interest to many areas of physics. The solution R we thus obtain is a 'universal object' i.e. it is $\mathcal{U}_q(sl_2)\otimes\mathcal{U}_q(sl_2)$ valued. In order to make it more useful to physical applications we need to consider representations of $\mathcal{U}_q(sl_2)$ which then turn R into a matrix (a so called R-matrix). This is the subject of the last section where we consider the representation theory of $\mathcal{U}_q(sl_2)$ at 'q a root of unity'. We only consider the representation theory in this regime because it seems the most interesting from a physical point of view. We deduce all finite dimensional irreducible representations of $\mathcal{U}_q(sl_2)$ and also give the structure of their tensor product representations. We also hint on a relation between this and the fusion rules of certain conformal field theories.

We have tried to keep the paper as selfcontained as possible. The reader is assumed to know something about Lie algebras and differential geometry.

1 Hopf algebras

Any selfcontained review paper on quantum groups is bound to start with the definition and elementary properties of Hopf algebras [14, 15]. The reason for this is that, as we will see in detail, the algebra of functions on a Lie group (which is the object of any quantization attempt) is a commutative Hopf algebra. As we already mentioned in the introduction, this Hopf algebra contains all the information on the algebraic structure of the Lie group. Therefore quantizing the Lie group as a manifold and as an algebraic structure means deforming the Hopf algebra structure of the function space while maintaining the fact that it is a Hopf algebra.

Instead of just giving the definition of a Hopf algebra, which is rather involved, we will introduce its structure step by step. First let us give a definition of an algebra.

Definition 1 *An algebra is a linear space A together with two maps*

$$m \;:\; A\otimes A \to A \tag{1}$$
$$\eta \;:\; \mathbf{C} \to A \tag{2}$$

such that

1. *m and η are linear*

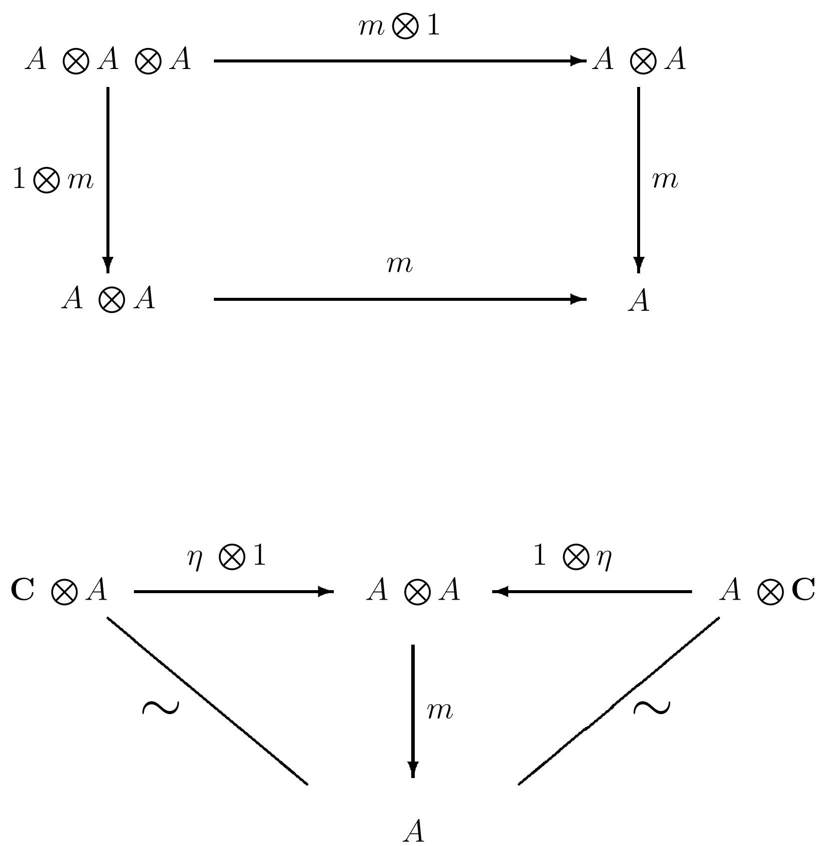

Figure 1: The commutativity of these diagrams expresses the algebra axioms

2. $m(m\otimes 1) = m(1\otimes m)$ *(associativity)*

3. $m(1\otimes\eta) = m(\eta\otimes 1) = id$ *(unit)*

We will usually denote $m(a\otimes b)$ by $a.b$.

Properties 2 and 3 imply that the diagrams in figure 1 commute. (here $\sim$ refers to the fact that $\mathbf{C}\otimes A \equiv \mathbf{C}\otimes_{\mathbf{C}} A \simeq A$ by the obvious isomorphism $\lambda\otimes a = \lambda a$ (for $\lambda\epsilon\mathbf{C}$, and $a\epsilon A$)). Property 3 is an unusual way of saying that A has a unit element, for let $\alpha\otimes a\epsilon A\otimes\mathbf{C} \simeq A$, then $m(\eta\otimes 1)(\alpha\otimes a) = \eta(\alpha)a$ which by property 3 is equal to αa. This means that $\eta(\alpha) = \alpha.1$ where 1 is the unit element of A.

A precise name for the algebras defined above would be unital associative algebras , however we will simply call them algebras. It is easy to think of all sorts of examples (group algebras, rings, fields etc.).

Let (A, m_A, η_A) and (B, m_B, η_B) be algebras, then the tensor product space $A\otimes B$ is naturally endowed with the structure of an algebra. The multiplication $m_{A\otimes B}$ on $A\otimes B$

is defined by

$$m_{A\otimes B} = (m_A \otimes m_B)(1\otimes\tau\otimes 1) \tag{3}$$

where τ is the so called flip map $\tau(a\otimes b) = b\otimes a$. More explicitly this multiplication on $A\otimes B$ reads $(a_1\otimes b_1).(a_2\otimes b_2) = (a_1.a_2)\otimes(b_1.b_2)$. It follows that the set of algebras is closed under taking tensor products.

Even though these abstract algebras are of considerable importance themselves, people (that is physicists) are primarily interested in their representation theory. By a representation we mean the usual, i.e. a homomorphism from the algebra A to an algebra of linear operators on some vectorspace. For completeness we give the definition.

Definition 2 *Let (A, m, η) be an algebra, V a linear space and ρ a map from A to the space of linear operators in V. (V, ρ) is called a representation of A if*

1. *ρ is linear*
2. *$\rho(xy) = \rho(x)\rho(y)$*

In physics it often happens that one has to compose two representations (for example when adding angular momenta). This happens when two physical systems each within a certain representation interact (for example two spin 1/2 particles). Mathematically this means that one must consider tensorproduct representations of the underlying abstract algebra. Let us now try to define tensor product representations for the algebras defined above.

Suppose (ψ_1, V_1) and(ψ_2, V_2) are two representations of an algebra A. How do we define an action of A an $V_1\otimes V_2$ using ψ_1 and ψ_2 ? There are only two reasonable possibilities:

1. The action of $a\epsilon A$ on $v_1\otimes v_2 \epsilon V_1\otimes V_2$ is:

$$a.(v_1\otimes v_2) = (\psi_1(a)v_1)\otimes(\psi_2(a)v_2) \tag{4}$$

2. The action of a on $v_1\otimes v_2$ is

$$a.(v_1\otimes v_2) = \psi_1(a)v_1\otimes v_2 + v_1\otimes\psi_2(a)v_2 \tag{5}$$

Definition 1 certainly does not satisfy the required properties since such a tensor product representation would not be linear. Definition 2 has the problem that the homomorphism property does not hold unless the multiplication on A is antisymmetric (as is the case in for example a Lie algebra). There seems no way out, we have to endow the algebra with some extra structure which makes it possible to define tensor product representations.

Consider a map $\Delta : A \to A\otimes A$ and define the tensor product representation Ψ by

$$\Psi = (\psi_1\otimes\psi_2)\Delta \tag{6}$$

We require of Ψ that it be linear, satisfy the homomorphism property and also that the representations $(V_1\otimes V_2)\otimes V_3$ and $V_1\otimes(V_2\otimes V_3)$ are equal (in this way the set of representations becomes a ring). These requirements lead to the following conditions on Δ :

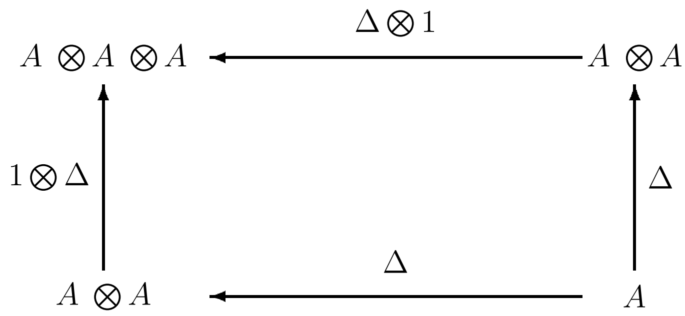

Figure 2: Co-associativity

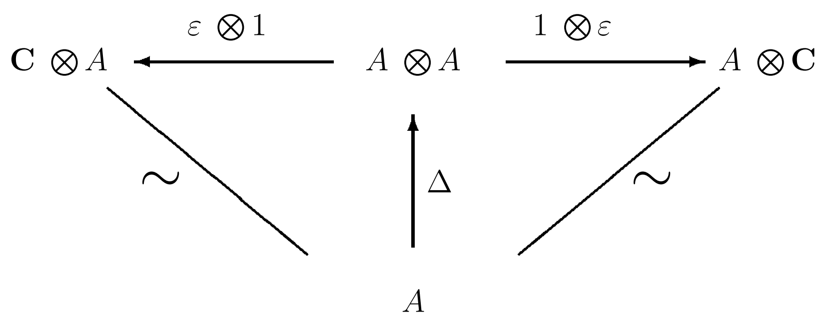

Figure 3: The co-unit

1. Δ is linear.
2. $\Delta(ab) = \Delta(a)\Delta(b)$
3. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

A map $\Delta : A \to A \otimes A$ with these properties is called a co-multiplication. Property 3 means that the diagram in figure 2 commutes. Compare this diagram to the one which expressed the associativity of the multiplication map. The only difference is that all the arrows are reversed. Property 3 is therefore called co-associativity. Property 2 states that Δ is an algebra homomorphism.

Associated with the multiplication on A we had a map η whose properties signaled the existence of a unit in A. In the same way we now define a map, called a co-unit, which has properties that follow by reversing all arrows in the diagram for the map η (see figure 3)

Definition 3 *A co-unit is a map* $\varepsilon : A \to \mathbf{C}$ *such that* $(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = id$

The set $(A, m, \Delta, \eta, \varepsilon)$ is called a bi-algebra if all the above properties are satisfied. It follows from this definition that we know how to define tensor product representations for bi-algebras.

As we will see in the examples certain bi-algebras are related to Lie-groups. It turns out that we can encode all the properties of Lie groups into a particular class of bi-algebras. First of all these bi-algebras are commutative (i.e. $a.b = b.a$). Also they possess an extra structure called an antipode or co-inverse which is (the name already gives it away) related to the fact that every element of a group has an inverse (or put more formally, there exists a map I from the group to itself such that $I(g)g = gI(g) = e$ where e is the unit element of the group). Considering bi-algebras associated to Lie groups as a special case, the properties of the antipode can be generalized. This leads to the following

Definition 4 *A Hopf algebra is a bi-algebra $(A, m, \eta, \Delta, \varepsilon)$ together with a map*

$$S : A \to A \tag{7}$$

with the following property:

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta \circ \varepsilon \tag{8}$$

S is called an antipode.

In order to show the relevance of Hopf algebras to the theory of quantum groups we will now consider two examples which are crucial to the understanding of this theory.

Example 1 *Let G be a compact topological group. Consider the space of continuous functions on G denoted by $C(G)$ together with the following maps:*

- $(f.h)(g) = f(g)h(g)$
- $\Delta(f)(g_1 \otimes g_2) = f(g_1 g_2)$
- $\eta(x) = x1$ *where 1(g)=1 for all $g \epsilon G$*
- $\varepsilon(f) = f(e)$ *where e is the unit element of G*
- $S(f)(g) = f(g^{-1})$

where $g_1, g_2, g \epsilon G$, $x \epsilon \mathbf{C}$ and $f, h \epsilon C(G)$. It is easy to check that the set $C(G)$ together with these maps is a Hopf algebra. Moreover note that it is a commutative Hopf algebra. Therefore we have associated a commutative Hopf algebra to every compact Topological group. You might wonder if is possible conversely to associate a topological group to a given commutative Hopf algebra such that the group which is thus associated to the Hopf algebra $C(G)$ is the group G. As mentioned in the introduction the answer to this question is contained in a theorem by Gelfand and Naimark and is affirmative (actually they proved their theorem within the context of the C^*-algebras so it is necessary that we have given on the Hopf algebra also a $*$-involution and a norm that turn the Hopf algebra into a C^* algebra). The group is constructed as the set of characters of the C^*-Hopf algebra. We will not go into this however.

Doing quantum-groups the non-commutative geometric way means deforming (i.e. making non-commutative) in some way the Hopf algebra of functions on the group. This is studied in detail in [10, 11, 12].

As became clear in the above example the set of commutative Hopf algebras is a very important subset. Another important subset, and in some way a dual one, is the subset of universal enveloping algebras. This is the example we will take a look at next.

Example 2 *Let L be a Lie algebra and $\mathcal{U}(L)$ its universal enveloping algebra, then $\mathcal{U}(L)$ becomes a Hopf algebra if we define*

- *The multiplication is the ordinary multiplication in $\mathcal{U}(L)$.*
- $\Delta(x) = x \otimes 1 + 1 \otimes x$
- $\eta(\alpha) = \alpha 1$
- $\varepsilon(1) = 1$ *and zero on all other elements.*
- $S(x) = -x$

where x is an element of L (considered as a subset of $\mathcal{U}(L)$). Strictly speaking this defines $\Delta, \eta, \varepsilon$ and S only on the subset L of the universal enveloping algebra, however it is easily seen that these maps can be extended uniquely to all of $\mathcal{U}(L)$ such that the Hopf algebra axioms are satisfied everywhere. Note that an arbitrary Lie algebra L itself is not a Hopf algebra because it is not an associative algebra. Also note that the universal enveloping algebras are co-commutative Hopf algebras, that is we have the equality $\Delta = \tau \circ \Delta$ where τ is again the flip operator.

The two examples given above are in a sense dual. In order to explain what we mean by this we will now show that the dual space of a Hopf algebra is again a Hopf algebra. Suppose that $(A, m, \Delta, \eta, \varepsilon, S)$ is a Hopf algebra and that A^* is its dual space, then using the structure maps of A we define the structure maps $(m^*, \Delta^*, \eta^*, \varepsilon^* S^*)$ on A^* as follows:

- $\langle m^*(f \otimes g), x\rangle = \langle f \otimes g, \Delta(x)\rangle$
- $\langle \Delta^*(f), x \otimes y\rangle = \langle f, xy\rangle$
- $\langle \eta^*(\alpha), x\rangle = \alpha.\varepsilon(x)$
- $\varepsilon^*(f) = \langle f, 1\rangle$
- $\langle S^*(f), x\rangle = \langle f, S(x)\rangle$

where f and g are elements of A^* and x, y are elements of A. The brackets $\langle ., .\rangle$ denote the dual contraction between A^* and A, and $\langle f \otimes g, x \otimes y\rangle = \langle f, x\rangle\langle g, y\rangle$. It is easy, using the fact that A is a Hopf algebra, to verify that A^* is also a Hopf algebra. Note that if A is commutative then A^* is co-commutative, and if A is co-commutative then A^* is commutative. This is so because the multiplication on A induces the comultiplication on A^*, and the comultiplication on A induces the multiplication on A^*.

We will now discuss the duality between $\mathcal{U}(L)$ and $C^\infty(G)$. Consider the map

$$\rho : L \to End_{\mathbf{C}}(C^\infty(G)) \tag{9}$$

defined by

$$(\rho(X)\phi)(g) = \frac{d}{dt}(\phi(e^{tX}g)) \mid_{t=0} \tag{10}$$

for $X\epsilon L$, $\phi\epsilon C^\infty(G)$, $g\epsilon G$ (the derivative in g of ϕ in the direction X). This map extends uniquely to a homomorpism

$$\rho : \mathcal{U}(L) \to End_{\mathbf{C}}(C^\infty(G)) \tag{11}$$

(by definition of $\mathcal{U}(L)$). Also consider the right action of G on $C^\infty(G)$

$$R_g : C^\infty(G) \to C^\infty(G) \tag{12}$$

defined by $R_g(\phi)(g') = \phi(g'g)$. We have the following lemma:

Lemma 1 *The map ρ has the following properties.*

1. $R_g \circ \rho(a) = \rho(a) \circ R_g$ *(right invariance)*
2. $\rho(X)(\phi.\psi) = (\rho(X)\phi).\psi + \phi.(\rho(X)\psi)$ *(derivation property)*

for all $a\epsilon\mathcal{U}(L)$, $X\epsilon L$ and $\phi, \psi\epsilon C^\infty(G)$.

Proof: A straightforward calculation gives

$$(\rho(X) \circ R_g)(\phi)(g') = \frac{d}{dt}\phi(e^{tX}g'g) \mid_{t=0}= (R_g \circ \rho(X))(\phi)(g') \tag{13}$$

for $X\epsilon L$. Using the fact that ρ is an algebra homomorphism part 1 of the lemma follows. Part 2 follows immediately from the Leibniz rule. This concludes the proof.

Define the pairing

$$\langle ., . \rangle : C^\infty(G) \times \mathcal{U}(L) \to \mathbf{C} \tag{14}$$

by $\langle \phi, a \rangle = (\rho(a)\phi)(e)$ where e is the unit element of G, $a\epsilon\mathcal{U}(L)$ and $\phi\epsilon C^\infty(G)$.

Theorem 1 *The map*

$$C^\infty(G) \to (\mathcal{U}(L))^* \tag{15}$$

defined by

$$\phi \to \langle \phi, . \rangle \tag{16}$$

is an embedding.

Proof: We have to prove that this map is injective. Suppose $\langle \phi_1, a \rangle = \langle \phi_2, a \rangle$ for all $a\epsilon\mathcal{U}(L)$. Then

$$0 = \langle \phi_1 - \phi_2, a \rangle = (\rho(a)(\phi_1 - \phi_2))(e) \tag{17}$$

Note that $\rho(a)$ is a differential operator of arbitrary order. This is clear from the fact that if $X\epsilon L \subset \mathcal{U}(L)$ then by the above lemma $\rho(X)$ is a derivation which means that it is a vectorfield. A vectorfield can be seen as a differential operator of order 1 on $C^\infty(G)$. An element a of $\mathcal{U}(L)$ can in turn be seen as a linear combination of monomials of elements of L. Since ρ is an algebra homomorphism it follows that $\rho(a)$ is a differential operator of arbitrary order. What we have therefore deduced is that any derivative of the function $\phi_1 - \phi_2$ in the point $e\epsilon G$ is zero. Since a smooth fuction is uniquely determined by its derivatives in e we find $\phi_1 - \phi_2 = 0$. This proves the theorem.

So indeed there is a duality between $\mathcal{U}(L)$ and $C^\infty(G)$ because $C^\infty(G)$ can be embeded into $(\mathcal{U}(L))^*$. Using the definition of the dual Hopf algebra given above we can endow $\mathcal{U}(L)$ with a Hopf algebra structure that is induced by the one on $C^\infty(G)$. It is easy to check that this is precisely the Hopf algebra structure of example 2 so the spaces in two examples are not only dual as spaces but also as Hopf algebras.

2 Poisson structures

From a mathematical point of view a classical mechanical system is fixed by giving a phase space, which consists of a smooth manifold together with a closed non-degenerate 2 form (a symplectic form), and a specific function on the manifold which plays the role of a Hamiltonian (determining the dynamics of the system). The symplectic form determines the Poisson bracket. In the spirit of the previous section we translate all the ingredients of classical mechanical systems into an algebraic language by passing on to the space of smooth functions on the phase space. The reason for this is again that this algebraic approach leaves enough room for generalization to non-commutative algebras (in which case there is no longer an interpretation in terms of an underlying manifold). Our first definition will be that of a Poisson algebra.

Definition 5 *A Poisson algebra is a commutative algebra* (A, m, η) *together with a map*

$$\{.,.\} : A \times A \to A \tag{18}$$

such that

1. *A is a Lie algebra with respect to* $\{.,.\}$.
2. $\{ab, c\} = a\{b, c\} + \{a, c\}b$

Obviously the space of smooth functions on a symplectic manifold is a Poisson algebra.

For later use we shall reformulate the defining properties of a Poisson algebra somewhat. By definition of the tensorproduct the map $\{.,.\}$ induces a map $\gamma : A \otimes A \to A$. We can reformulate the properties of $\{.,.\}$ in terms of γ and they read :

1. $\gamma \circ \tau = -\gamma$ (anti-symmetry)
   $\gamma(1\otimes\gamma)(1\otimes1\otimes1 + (1\otimes\tau)(\tau\otimes1) + (\tau\otimes1)(1\otimes\tau)) = 0$ (Jacobi-identity)
2. $\gamma(m\otimes1) = m(1\otimes\gamma)(1\otimes1\otimes1 + \tau\otimes1)$

It is straightforward to derive these identities from the defining properties of a Poisson algebra.

Let us now consider the concept of a Poisson algebra homomorphism.

Definition 6 *Let* $(A, m_A, \{.,.\}_A)$ *and* $(B, m_B, \{.,.\}_B)$ *be Poisson algebras. A Poisson algebra homomorphism is a linear map* f *from* A *to* B *such that*

1. $f(a.b) = f(a)f(b)$
2. $f(\{a,b\}_A) = \{f(a), f(b)\}_B$

where a and b are arbitrary elements of A.

Again we can write the above expressions entirely in terms of m and γ. As one easily verifies the first expression becomes

$$f \circ m_A = m_B \circ (f \otimes f) \tag{19}$$

while the second one reads

$$f \circ \gamma_A = \gamma_B \circ (f \otimes f) \tag{20}$$

The tensor product space $A \otimes B$ of the two Poisson algebras inherits from its constituents a natural Poisson structure. First of all we have to say how to multipy two elements of $A \otimes B$. This is easy

$$(a \otimes b).(c \otimes d) = (a.b) \otimes (c.d) \tag{21}$$

where $a, c \epsilon A$ and $b, d \epsilon B$, or equivalently

$$m_{A \otimes B} = (m_A \otimes m_B)(1 \otimes \tau \otimes 1) \tag{22}$$

where τ is again the flip operator. Second we have to define a Poisson structure such that the axioms of a Poisson algebra are satisfied. The following Poisson bracket does the trick

$$\{a \otimes b, c \otimes d\}_{A \otimes B} = \{a, c\} \otimes bd + ac \otimes \{b, d\} \tag{23}$$

or in other words

$$\gamma_{A \otimes B} = (\gamma_A \otimes m_B + m_A \otimes \gamma_B)(1 \otimes \tau \otimes 1) \tag{24}$$

So the set of Poisson algebras is closed under taking the tensorproduct.

As we argued earlier physicists will primarily be interested in bi-algebras because for them we know how to define tensorproduct representations. For a Poisson algebra we do not want any co-product however because we want a tensor product representation (defined through the co-product) to be a Poisson algebra homomorphism not merely an algebra homomorphism. This is easily seen to give the following condition on Δ

$$\{\Delta(a), \Delta(b)\}_{A \otimes A} = \Delta(\{a, b\}_A) \tag{25}$$

or equivalently

$$\Delta_{A \otimes A} \circ (\Delta \otimes \Delta) = \Delta \circ \gamma_A \tag{26}$$

If this is satisfied then A is called a Poisson bi-algebra. (If the algebra A by accident also carries an antipode, then you give it credit by calling A a Poisson Hopf algebra.)

By now everything has become pretty algebraic and soon we will be able to profit from this. However we still are not where we want to be. First we have to introduce the concept of a co-Poisson structure since we will need this in the theory of quantum groups. As usual the 'co' means that something gets dualized, in this case the Poisson structure. We have to dualize again because of the duality between the universal enveloping algebra of the Lie group and the space of smooth functions on the Lie group. Therefore a Poisson bracket on the Lie group will be a co-Poisson structure on the universal enveloping algebra.

Here is the precise definition of a co-Poisson bi-algebra

Definition 7 *A co-Poisson bi-algebra is a co-commutative bi-algebra $(A, m, \Delta, \eta, \varepsilon)$ together with a map*

$$\delta : A \rightarrow A \otimes A \tag{27}$$

such that

1. $\tau \circ \delta = -\delta$ *(co-antisymmetry)*
2. $(1 \otimes 1 \otimes 1 + (1 \otimes \tau)(\tau \otimes 1) + (\tau \otimes 1)(1 \otimes \tau))(1 \otimes \delta)\delta = 0$ *(co-Jacobi id.)*
3. $(\Delta \otimes 1)\delta = (1 \otimes 1 \otimes 1 + \tau \otimes 1)(1 \otimes \delta)\Delta$ *(co-Leibniz rule)*
4. $(m \otimes m) \circ \delta_{A \otimes A} = \delta \circ m$ *(i.e. m is a co-Poisson homomorphism)*

where $\delta_{A \otimes A} = (1 \otimes \tau \otimes 1)(\delta \otimes \Delta + \Delta \otimes \delta)$ is the co-Poisson structure naturally associated to the tensor product space (compare to eqn.(24)).

Notice that these relations are dual to the ones satisfied by Poisson algebras.

Later in this paper we will define a concept of quantization for both Poisson and co-Poisson algebras, but first we will consider so called Poisson Lie groups which can be interpreted as phase spaces of classical dynamical systems living on group manifolds . As we will see these Poisson Lie groups are closely related to the classical Yang-Baxter equation and it is them that we will ultimately quantize (or more accurately their universal enveloping algebras).

3 Poisson-Lie groups and Lie bi-algebras

In the approach to quantum groups we consider in this paper the basic objects are Poisson-Lie groups [6]. In this section we study some of their properties and show how they are related to Lie bi-algebras and the classical Yang-Baxter equation. As an example we will consider the groups SL_N which will also serve as an illustration of the quantization of Poisson-Lie groups in the next sections.

We start with the definition.

Definition 8 *A Lie group G is called a Poisson Lie group if the space of smooth functions on G is a Poisson Hopf algebra.*

Obviously the Hopf algebra structure of $C^\infty(G)$ is the one given in example 1 and is completely fixed by the structure of the group. It is therefore the Poisson bracket that has to satisfy a certain compatibility relation (see eqn. (25)) which means that not every Poisson structure on a Lie group turns it into a Poisson-Lie group. We will study this compatibility relation in detail below.

The following lemma gives the general form of a Poisson bracket on a Lie group.

Lemma 2 *Let $\{X_\mu\}_{\mu=1}^{dim(G)}$ be a set of right invariant vectorfields on G (i.e. if $R_g(g') = g'g$, then $X_\mu|_g = (R_g)_* X_\mu|_e$ where $(R_g)_*$ is the derivative of R_g, and e is the unit element* *of G). such that $\{X_\mu|_g\}$ is a basis in T_gG for all $g\epsilon G$. Then a Poisson bracket on $C^\infty(G)$ can be written as*

$$\{\phi,\psi\} = \sum_{\mu\nu} \eta^{\mu\nu}(g) X_\mu|_g(\phi)\, X_\nu|_g(\psi) \tag{28}$$

where $g\epsilon G$.

This follows from the fact that $\{\phi, .\} : C^\infty(G) \to C^\infty(G)$ is a derivation which means that it is equal to a vectorfield. Since $\{X_\mu|_g\}$ spans T_gG in every $g\epsilon G$ we can therefore write $\{\phi, .\}(g) = \sum_\mu \gamma^\mu(g) X_\mu|_g$. Applying the same argument to $\{., \psi\}$ the lemma follows.

We can rewrite the form of this Poisson bracket somewhat:

$$\begin{aligned}\{\phi,\psi\}(g) &= \sum_{\mu\nu} \eta^{\mu\nu}(g) X_\mu|_g(\phi)\, X_\nu|_g(\psi) \\ &= \sum_{\mu\nu} \eta^{\mu\nu}(g)(d\phi|_g \otimes d\psi|_g)(X_\mu|_g \otimes X_\nu|_g) \\ &= \eta(g)(d\phi|_g \otimes d\psi|_g)\end{aligned} \tag{29}$$

where $\eta : G \to L\otimes L$ is defined by (L is the Lie algebra of G)

$$g \longmapsto \eta(g) = \sum_{\mu\nu} \eta^{\mu\nu}(g) X_\mu \otimes X_\nu \tag{30}$$

and also $d\phi|_g\ (X_\mu) \equiv d\phi|_g\ (X_\mu|_g)$. Here we have identified the space of right invariant vectorfields with $T_eG = L$. This can be done because $X_\mu|_e$ determines X_μ in any point $g\epsilon G$ by right translation (i.e. the fields X_μ are right invariant which means by definition $X_\mu|_g = (R_g)_* X_\mu|_e$ where $(R_g)_*$ is the derivative of R_g). We can therefore consider $\{X_\mu\}$ to be a basis of L.

Of course the fact that $\{.,.\}$ is a Poisson bracket and also that it is coordinated to the Hopf algebra structure on $C^\infty(G)$ gives the map η certain properties. This what we will investigate next.

Let $C^n(G; L)$ be the space of maps

$$\lambda : G \times \ldots \times G \to L\otimes L \tag{31}$$

We can turn the sequence $\{C^n(G; L)\}_{n=0}^\infty$ into a complex by defining the coboundary operator

$$\delta_G : C^n(G; L) \to C^{n+1}(G; L) \tag{32}$$

as follows

$$\begin{aligned}[\delta_G\lambda](g_1, \ldots, g_{n+1}) &= g_1.\lambda(g_2, \ldots, g_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i \lambda(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ &\quad + (-1)^{n+1}\lambda(g_1, \ldots, g_n)\end{aligned} \tag{33}$$

(where $g_i\epsilon G$ and $\lambda\epsilon C^n(G; L)$). The action of G on $L\otimes L$ (which we used in the definition) is defined by

$$g.(X\otimes Y) = Ad_g X \otimes Ad_g Y \equiv Ad_g^{\otimes 2}(X\otimes Y) \tag{34}$$

where Ad denotes the adjoint action and $X, Y\epsilon L$. It is a straightforeward computation to show that $\delta_G^2 = 0$. The compatibility relation is the subject of the following theorem.

Theorem 2 *Let η be the map associated to the Poisson structure on $C^\infty(G)$ via the relation (29). Then the compatibility relation (25) between the Poisson bracket and the Hopf algebra structure on $C^\infty(G)$ is equivalent to the cocycle condition on η, i.e.*

$$\delta_G \eta = g_1.\eta(g_2) - \eta(g_1 g_2) + \eta(g_1) = 0 \tag{35}$$

The proof of this theorem is an explicit calculation. If we write $\Delta(\phi) = \sum_i \phi_i^{(1)} \otimes \phi_i^{(2)}$ then by definition of the co-product on $C^\infty(G)$ (see example 1) we have

$$\begin{aligned} \Delta\phi(g_1, g_2) &= \phi(g_1 g_2) \\ &= \sum_i \phi_i^{(1)}(g_1)\, \phi_i^{(2)}(g_2) \end{aligned} \tag{36}$$

Also remembering the definition of the Poisson bracket on the tensor product space $C^\infty(G) \times C^\infty(G)$

$$\{\Delta(\phi), \Delta(\psi)\} = \sum_{ij} \{\phi_i^{(1)}, \psi_j^{(1)}\} \otimes \phi_i^{(2)} \psi_j^{(2)} + \phi_i^{(1)} \psi_j^{(1)} \otimes \{\phi_i^{(2)}, \psi_j^{(2)}\} \tag{37}$$

we find

$$\begin{aligned} \{\Delta(\phi), \Delta(\psi)\}(g_1, g_2) &= \sum_{\mu\nu} \sum_{ij} (\eta^{\mu\nu}(g_1)\, X_\mu|_{g_1}(\phi_i^{(1)})\, X_\nu|_{g_1}(\psi_j^{(1)})\, \phi_i^{(2)}(g_2) \psi_j^{(2)}(g_2) \\ &\quad + \phi_i^{(1)}(g_1) \psi_j^{(1)}(g_1) (\eta^{\mu\nu}(g_2)\, X_\mu|_{g_2}(\phi_i^{(2)})\, X_\nu|_{g_2}(\psi_j^{(2)})) \end{aligned} \tag{38}$$

We also have

$$\begin{aligned} \sum_i X_\mu|_{g_1}(\phi_i^{(1)})\, \phi_i^{(2)}(g_2) &= \frac{d}{dt} \sum_i \phi_i^{(1)}(e^{tX_\mu} g_1)\, \phi_i^{(2)}(g_2)\, |_{t=0} \\ &= \frac{d}{dt} \phi(e^{tX_\mu} g_1 g_2)\, |_{t=0} = d\phi|_{g_1 g_2}\, (X_\mu) \end{aligned} \tag{39}$$

In a similar way we can derive

$$\sum_i \phi_i^{(1)}(g_1)\, X_\mu|_{g_2}(\phi_i^{(2)}) = d\phi|_{g_1 g_2}\, (Ad_{g_1} X_\mu) \tag{40}$$

With these results we arrive at

$$\{\Delta(\phi), \Delta(\psi)\}(g_1, g_2) = d\phi|_{g_1 g_2} \otimes d\psi|_{g_1 g_2}\, (\eta(g_1) + g_1.\eta(g_2)) \tag{41}$$

By definition we have however

$$\Delta\{\phi, \psi\}(g_1, g_2) = (d\phi|_{g_1 g_2} \otimes d\psi|_{g_1 g_2}).\eta(g_1 g_2) \tag{42}$$

Equating these two relations we get the desired result.

So for the Poisson bracket to be coordinated to the Hopf structure the map η must be a 2-cocycle. This gives us one of the properties of η. The other properties, i.e the ones associated to the anti-symmetry and the Jacobi-identity are now easily deduced.

Associated to the map η we define

$$\phi_\eta : L \to L \otimes L \tag{43}$$

by

$$\phi_\eta(X) = \frac{d}{dt}\eta(e^{tX}) \mid_{t=0} \tag{44}$$

One might call ϕ_η the infinitesimal version of η in the unit element of G. The map ϕ_η inherits from η certain properties which are listed in the following theorem.

Theorem 3 *Let η be the map associated to the Poisson structure on the Poisson Lie group G and let ϕ_η be defined by eqn.(44), then*

1. *ϕ_η is co-antisymmetric.*
2. *ϕ_η satisfies the co-Jacobi identity.*
3. *$\phi_\eta([X,Y]) = X.\phi_\eta(Y) - Y.\phi_\eta(X)$*

where L acts on $L\otimes L$ via $X.(Y\otimes Z) = [X,Y]\otimes Z + Y\otimes[X,Z]$ which is the infinitesimal version of the action of G on $L\otimes L$.

Proof: Anti-symmetry of the Poisson bracket gives

$$\begin{aligned} \eta(d\phi\otimes d\psi) &= \sum \eta^{\mu\nu}(g) X_\mu(\phi) X_\nu(\psi) \\ &= -\sum \eta^{\mu\nu}(g) X_\mu(\psi) X_\nu(\phi) \\ &= -(\tau\circ\eta)(d\phi\otimes d\psi) \end{aligned} \tag{45}$$

so indeed we find $\eta = -\tau\circ\eta$. Using the definition of ϕ_η we see that $\tau\circ\phi_\eta = -\phi_\eta$. The proof that ϕ_η satisfies the co-Jacobi identity is similar. The third property is slightly trickier. First note that from the co-cycle condition follows $\eta(e) = 0$ (insert $g_1 = e$ into the cocycle condition). Therefore

$$\begin{aligned} 0 &= \partial_t \eta(e^{tX}e^{-tX}) \mid_{t=0} \\ &= \partial_t \eta(e^{tX}) \mid_{t=0} + \partial_t((Ad^{\otimes 2}_{e^{tX}})\eta(e^{-tX})) \mid_{t=0} \\ &= \phi_\eta(X) + \phi_\eta(-X) \end{aligned} \tag{46}$$

Then we have

$$\begin{aligned} \phi_\eta([X,Y]) &= \frac{d}{ds}\frac{d}{dt}\eta(e^{sX}e^{tY}e^{-sX} \mid_{t=0} \\ &= \frac{d}{ds}\frac{d}{dt}(\eta(e^{tX}) + (Ad^{\otimes 2}_{e^{sX}})\eta(e^{tY}) + Ad^{\otimes 2}_{e^{tX}} Ad^{\otimes 2}_{e^{tY}}\eta(e^{-sX}) \mid_{s,t=0} \\ &= \frac{d}{ds}(Ad^{\otimes 2}_{e^{sX}}) \mid_{s=0} \phi_\eta(Y) + \frac{d}{dt}(Ad^{\otimes 2}_{e^{tY}}) \mid_{t=0} \phi_\eta(-X) \\ &= ad^{\otimes 2}_X \phi(Y) + ad^{\otimes 2}_Y \phi(-X) \\ &= X.\phi_\eta(Y) - Y.\phi_\eta(X) \end{aligned} \tag{47}$$

where we used eqn.(46) in the last step. This proves the theorem.

In general we call a Lie algebra L together with a map $\phi : L \rightarrow L\otimes L$ such that the properties 1, 2 and 3 stated in the theorem are satisfied a Lie bi-algebra. From the foregoing follows that the Lie algebra of a Lie Poisson group is a Lie bi-algebra.

A trivial way to satisfy the co-cycle condition $\delta_G \eta = 0$ is of course to choose η to be a coboundary

$$\eta = \delta_G r \tag{48}$$

for some $r \epsilon L\otimes L$. Let us calculate the Lie co-bracket ϕ_η associated to such an η. From the definition of δ_G we find

$$[\delta_G r](g) = r - g.r \tag{49}$$

so we get

$$\begin{aligned} \phi(X) &= \frac{d}{dt}\eta(e^{tX}) \mid_{t=0} = \frac{d}{dt}(r - e^{tX}.r) \mid_{t=0} \\ &= -X.r \end{aligned} \tag{50}$$

Write for the moment $r = r^{\mu\nu} X_\mu \otimes X_\nu$, then

$$\begin{aligned} \phi(X) &= -r^{\mu\nu}([X, X_\mu]\otimes X_\nu + X_\mu \otimes [X, X_\nu]) \\ &= [r, 1\otimes X + X\otimes 1] \end{aligned} \tag{51}$$

Such a choice for η (and ϕ) trivially satisfies the cocycle condition, which was related to the fact that the Poisson structure and the Hopf structure are related. The co-antisymmetry and co-Jacobi identities for ϕ have not yet been considered. Obviously these will restrict the possible choices for r. We give the conditions on r in a theorem.

Theorem 4 *Let L be a Lie algebra and r an element of $L\otimes L$. Choose an arbitrary basis $\{X_\mu\}$ in L and write $r = r^{\mu\nu} X_\mu \otimes X_\nu$. Also define*

$$r_+ = \frac{1}{2}(r^{\mu\nu} + r^{\nu\mu}) X_\mu \otimes X_\nu \tag{52}$$

$$r_- = \frac{1}{2}(r^{\mu\nu} - r^{\nu\mu}) X_\mu \otimes X_\nu \tag{53}$$

$$r_{12} = r^{\mu\nu} X_\mu \otimes X_\nu \otimes 1 \tag{54}$$

$$r_{13} = r^{\mu\nu} X_\mu \otimes 1 \otimes X_\nu \tag{55}$$

$$r_{23} = r^{\mu\nu} 1 \otimes X_\mu \otimes X_\nu \tag{56}$$

Then the map $\phi : L \rightarrow L\otimes L$ defined by

$$\phi(x) = [r, X\otimes 1 + 1\otimes X] \tag{57}$$

turns L into a Lie bi-algebra if and only if

1. *r_+ is ad-invariant, i.e. $(Ad_g \otimes Ad_g) r_+ = r_+$*
2. *$B = [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}]$ is ad-invariant, i.e. $(Ad_g \otimes Ad_g \otimes Ad_g) B = 0$*

for all $g\epsilon G$. B is called the schouten bracket of r with itself.

The proof of this theorem is a lengthy but straightforward calculation. One has to write out the co-antisymmetry and co-Jacobi identities for the map ϕ given above.

The equation $Ad_g^{\otimes 3}B = 0$ is called the modified (classical) Yang-Baxter equation while the equation $B = 0$ is simply called the classical Yang-Baxter equation [16]. As we will see later the classical Yang-Baxter equation is the classical limit of the even more important quantum Yang-Baxter equation (also known in statistical mechanics as the star-triangle equation).

Let us consider an example.

Example 3 *Let $L = sl_N$ and let r be given by*

$$r = C - \sum_{i<j} e_{ij} \wedge e_{ji} \tag{58}$$

where

$$C = \sum_{i\neq j} e_{ij}\otimes e_{ji} + \sum_{\mu\nu=1}^{N-1} K^{\mu\nu} H_\mu \otimes H_\nu \tag{59}$$

and $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. H_μ are the standard generators of the Cartan subalgebra. $K^{\mu\nu}$ is the inverse of the Cartan matrix.

In the special case of sl_2 this reduces to

$$r = \frac{1}{2} H\otimes H + 2E\otimes F \tag{60}$$

such that ϕ becomes

$$\phi(H) = 0 \tag{61}$$

$$\phi(E) = \frac{1}{2} E \wedge H \tag{62}$$

$$\phi(F) = \frac{1}{2} F \wedge H \tag{63}$$

This map will play a crucial role in the quantization later on.

We can write the Poisson bracket (28) explicitly in terms of r. Since $\eta(g) = r - Ad_g r$ we get

$$\eta^{\mu\nu}(g) X_\mu|_g \otimes X_\nu|_g = r^{\mu\nu}(X_\mu|_g \otimes X_\nu|_g - Ad_g X_\mu|_g \otimes Ad_g X_\nu\,|_g) \tag{64}$$

Now using $X_\mu\,|_g = (R_g)_* X_\mu\,|_e$ and $Ad_g = (L_g)_*(R_g^{-1})_*$ we find $Ad_g X_\mu\,|_g = (L_g)_* X_\mu\,|_e$. Denoting $(R_g)_* X_\mu|_e\,(\phi)$ by $\partial_\mu \phi$ and $(L_g)_* X_\mu|_e$ by $\partial'_\mu \phi$ we get

$$\{f, g\} = r^{\mu\nu}(\partial_\mu f \partial_\nu g - \partial'_\mu f \partial'_\nu g) \tag{65}$$

As we said earlier a Poisson structure on $C^\infty(G)$ induces a co-Poisson structure on $\mathcal{U}(L)$ because of the duality between these two spaces. We conclude that the universal enveloping algebra of a Poisson Lie group G is a co-Poisson Hopf algebra. Denote the co-Poisson structure on $\mathcal{U}(L)$ by δ. We then have the following theorem which gives the relation between δ and ϕ_η.

Theorem 5 *The restriction of δ to $L \subset \mathcal{U}(L)$ is equal to ϕ_η.*

Proof: Denote the Poisson structure (28) by γ, i.e. $\{\phi, \psi\} = \gamma(\phi \otimes \psi)$. The map δ is defined by the relation

$$\langle \gamma(\phi \otimes \psi), a \rangle = \langle \phi \otimes \psi, \delta(a) \rangle \tag{66}$$

where $\phi, \psi \epsilon C^\infty(G)$, $a \epsilon \mathcal{U}(L)$ and $\langle ., . \rangle$ denotes the duality between $C^\infty(G)$ and $\mathcal{U}(L)$. Denoting $X_\mu \phi$ by $\partial_\mu \phi$ the left hand side of this equation is equal to

$$\begin{aligned}
\sum_{\mu\nu} \langle \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \psi, X \rangle &= \rho(X)(\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \psi)(e) \\
&= \frac{d}{dt} \eta^{\mu\nu}(e^{tX})\, \partial_\mu \phi(e^{tX})\, \partial_\nu \psi(e^{tX}) \mid_{t=0} \\
&= \frac{d}{dt} \eta^{\mu\nu}(e^{tX}) \mid_{t=0} \partial_\mu \phi(e)\, \partial_\nu \psi(e) \\
&= \phi_\eta^{\mu\nu}(X) X_\mu|_e(\phi)\, X_\nu|_e(\psi)
\end{aligned} \tag{67}$$

for $X \epsilon L \subset \mathcal{U}(L)$. Since

$$\langle \phi, X \rangle = \frac{d}{dt} \phi(e^{tX}) \mid_{t=0} = X|_e(\phi) = d\phi|_e(X) \tag{68}$$

we find $\langle \phi \otimes \psi, \delta(X) \rangle = (d\phi|_e \otimes d\psi|_e)\delta(X)\rangle$ for the right hand side. The theorem follows.

Quantizing Poisson Lie groups can be performed at different levels. One could quantize the Poisson Hopf algebra $C^\infty(G)$, or equivalently the co-Poisson Hopf algebra $\mathcal{U}(L)$. In the next section we will define quantization for both, however we will only persue the quantization of $\mathcal{U}(L)$.

4 Deformation quantization of (co)-Poisson structures

In section 2 we defined (co)-Poisson algebras and in section 3 we saw how they are related to Poisson Lie groups. It is the purpose of this section to define a suitable form of quantization, called deformation quatization [17], for these objects. Using this definition we will quantize the universal enveloping algebra of sl_2, which is a (co)-Poisson Hopf algebra as we saw in the previous section.

Definition 9 *Let $(A_0, m_0, \eta_0, \{., .\})$ be a Poisson algebra over the complex numbers* $\mathbf{C}$. *A quantization of A_0 is a non- commutative algebra (A, m, η) over the ring $\mathbf{C}[[\hbar]]$, where $\hbar$ is a formal parameter (interpreted as Planck's constant), such that*

1. $A/\hbar A \cong A_0$
2. $m_0 \circ (\pi \otimes \pi) = \pi \circ m$
3. $\pi \circ \eta = \eta_0$
4. $\{\pi(a), \pi(b)\} = \pi(\frac{[a,b]}{\hbar})$ *for all* $a, b \epsilon A$.

where π denotes the canonical quotient map

$$\pi : A \to A/\hbar A \cong A_0 \tag{69}$$

and $[.,.]$ denotes the ordinary commutator (with respect to the multiplication m).

Let us consider this definition more closely. First of all , because of property 1, the space A can be seen as the set of polynomials in $\hbar$ with coefficients in A_0. Factoring out $\hbar A$ is then equivalent to putting $\hbar = 0$ which corresponds to the classical limit. However the multiplication on A is not simply the multiplication which we would have obtained had we extended the multiplication on A_0 to A (i.e. if $a = \sum_{i=0}^{\infty} a_i \hbar^i$ and $b = \sum_{j=0}^{\infty} b_j \hbar^j$ then naively extending the multiplication of A_0 to A would have given $a.b = \sum_{ij} a_i b_j \hbar^{i+j}$. This multiplication however is commutative since the multiplication on A_0 in commutative by definition of a Poisson algebra). On A there is a new, non-commutative, multiplication m which we denote by $\star$ (i.e. $m(a \otimes b) \equiv a \star b$). Quite generally one can say that this multiplication is of the form

$$a \star b = \sum_{n=0}^{\infty} f_n(a,b)\hbar^n \tag{70}$$

where $f_n : A \times A \to A$. Property 2 of a quantization is equivalent to saying that in the classical limit ($\hbar \to 0$) m must reduce to m_0 (i.e. $\star$ must reduce to .). This fixes $f_0(a,b) = a.b$. Also m must be associative

$$(a \star b) \star c = a \star (b \star c) \tag{71}$$

which leads to the identity

$$\sum_{n=0}^{\infty} f_n(f_{l-n}(a,b),c) = \sum_{n=0}^{\infty} f_n(a, f_{l-n}(b,c)) \tag{72}$$

(for all l). Furthermore since $1 \star a = a \star 1 = a$ we have

$$f_n(a,1) = f_n(1,a) = a\delta_{n,0} \tag{73}$$

Another point that needs to be cleared up is the $1/\hbar$ factor in the right hand side of property 2, since one might say that this could cause terms with negative powers of $\hbar$ which are not in the algebra A. That everything is alright is contained in the following lemma.

Lemma 3 *The commutator $[a,b]$ is at least of order $\hbar$ for all elements a, b of A.*

The proof is easy and goes as follows: We know that since A_0 is commutative, $0 = [\overline{a}, \overline{b}]$. By property 1 the equivalence class $\overline{a}$ is given by $a + \hbar A$. Therefore $0 = [a + \hbar A, b + \hbar A] = [a,b] + \hbar A$. Again by property 1 this is equal to $\overline{[a,b]}$ which means that $[a,b]$ is in the kernel of π which is equal to $\hbar A$. This proves the lemma.

As far as we know there is no general theorem stating that every Poisson algebra can be quantized in the way descibed above or that a quantization is unique if it exists.

Having defined deformation quantization for Poisson algebras we can dualize this definition in order to get a definition of quantization for co-Poisson bi-algebras. Obviously the dual analogue of a commutator $[.,.]$ ($= m - m \circ \tau$) is the map $\Delta - \tau \circ \Delta$. Motivated by this we come to the following definition.

Definition 10 *Let* $(A_0, m_0, \Delta_0, \eta_0, \varepsilon_0; \delta)$ *be a co-Poisson bi-algebra where* δ *denotes the co-Poisson structure. A quantization of* A_0 *is a non co-commutative bi-algebra* $(A, m, \Delta, \eta, \varepsilon)$ *over the ring* $\mathbf{C}[[\hbar]]$ *such that*

1. $A/\hbar A \cong A_0$
2. $(\pi \otimes \pi) \circ \Delta = \Delta_0 \circ \pi$
3. $m_0 \circ (\pi \otimes \pi) = \pi \circ m$
4. $\pi \circ \eta = \eta_0$
5. $\varepsilon \circ \pi = \varepsilon_0$
6. $\delta(\pi(a)) = \pi(\frac{1}{\hbar}(\Delta(a) - \tau \circ \Delta(a))$ *for all* a *in* A

where π *again denotes the canonical quotient map* $\pi : A \to A_0$.

W.r.t. this definition similar remarks can be made as before (we will not repeat them). We do want to bring another point to the attention of the reader. Even though the quantization of a co-Poisson structure involves primarily the comultiplication (see point 6 in the definition) the other structures in the bi-algebra may also be deformed (i.e. the maps ε, m, η will not simply be the maps $\varepsilon_0, m_0, \eta_0$ extended to A) because in a bi-algebra the different maps are coordinated. The axiom relating the multiplication to the co-multiplication reads

$$\Delta_0(a_0.b_0) = \Delta_0(a_0).\Delta(b_0) \tag{74}$$

and

$$\Delta(a \star b) = \Delta(a) \star \Delta(b) \tag{75}$$

in A_0 and A respectively. From eqn.(75) one immediately sees that given Δ the multiplication m_0 may have to be altered in order to satisfy this relation. The same can be said about the other structure maps because of their relations with the (co)-multiplication.

In the next section we will consider in detail the quantization of the universal enveloping algebra of sl_2 which is a co-Poisson Hopf algebra. Indeed we will find that not only the co-multiplication is deformed but all the other Hopf algebra structures as well.

5 The quantization of $\mathcal{U}(sl_2)$

In this section we will undertake the quantization of the universal enveloping algebra of sl_2 equipped with the co-Poisson structure discribed in section 3. What we will obtain is a non-commutative and non-cocommutative Hopf algebra called the quantized universal enveloping algebra denoted by $\mathcal{U}_q(sl_2)$. This algebra was first introduced by Drinfeld and in a slightly different form (and starting from a different principle) by Jimbo.

As we saw in section 3 the co-Poisson structure of $\mathcal{U}(sl_2)$ is given by the extension to the entire universal enveloping algebra of the map

$$\begin{aligned} \delta(H) &= 0 && (76)\\ \delta(E) &= \frac{1}{2}E\wedge H && (77)\\ \delta(F) &= \frac{1}{2}F\wedge H && (78)\end{aligned}$$

(the Lie algebra sl_2 itself together with this this map was called a Lie bi-algebra). As a space the quantization of the universal enveloping algebra is known, as we saw in the previous paragraph. It is simply the set of (formal) polynomials in $\hbar$ with coefficients in $\mathcal{U}(sl_2)$. What we have to do first is find the coproduct Δ on this new space. This co-product is determined by the following requirements:

1. Δ must be co-associative (or else the quantized algebra will not be a Hopf algebra).
2. $\delta(\pi(a)) = \pi(\frac{1}{\hbar}(\Delta(a) - \tau\circ\Delta(a)))$
3. In the classical limit ($\hbar \to 0$) the coproduct Δ must reduce to the ordinary coproduct on $\mathcal{U}(sl_2)$.

The comultiplication Δ has the general form

$$\Delta = \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\Delta_{(n)} \qquad (79)$$

The third requirement fixes $\Delta_{(0)}$:

$$\begin{aligned} \Delta_{(0)}(H) &= H\otimes 1 + 1\otimes H && (80)\\ \Delta_{(0)}(E) &= E\otimes 1 + 1\otimes E && (81)\\ \Delta_{(0)}(F) &= F\otimes 1 + 1\otimes F && (82)\end{aligned}$$

The second requirement reads as follows:

$$\begin{aligned} \delta(H) &= 0 = \Delta_{(1)}(H) - \tau\circ\Delta_{(1)}(H) && (83)\\ \delta(E) &= \frac{1}{2}E\wedge H = \Delta_{(1)}(E) - \tau\circ\Delta_{(1)}(E) && (84)\\ \delta(F) &= \frac{1}{2}F\wedge H = \Delta_{(1)}(F) - \tau\circ\Delta_{(1)}(F) && (85)\end{aligned}$$

An obvious solution of this system of equations is given by:

$$\begin{aligned} \Delta_{(1)}(H) &= 0 && (86)\\ \Delta_{(1)}(E) &= \frac{1}{4}E\wedge H && (87)\\ \Delta_{(1)}(F) &= \frac{1}{4}F\wedge H && (88)\end{aligned}$$

So using the classical limit and the quantization condition for the Poisson structure we have been able to find the two lowest order elements of Δ. In order to find the higher order terms we can use requirement 1 which states that Δ must be co-associative. Inserting the expansion of Δ into the equation which expresses the coassociativity and collecting terms of order $\hbar^n$ we come to the following recursive relation:

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (\Delta_{(k)} \otimes 1 - 1 \otimes \Delta_{(k)}) \Delta_{(n-k)} = 0 \tag{89}$$

If we know all the $\Delta_{(k)}$ for $k < n$ we get an equation for $\Delta_{(n)}$. In this way we can solve the above equation recursively (remember we already know $\Delta_{(0)}$ and $\Delta_{(1)}$). It is now easy to show by induction that for arbitrary n

$$\Delta_{(n)}(H) = 0 \tag{90}$$

$$\Delta_{(n)}(E) = \frac{1}{4^n}(E \otimes H^n + (-1)^n H^n \otimes E) \tag{91}$$

$$\Delta_{(n)}(F) = \frac{1}{4^n}(F \otimes H^n + (-1)^n H^n \otimes F) \tag{92}$$

solve the recursion relation for all n. Here by H^n is meant $H \star H \star ... \star H$ (n-times) where $\star$ is the deformed multiplication on A. First one has to show that $\Delta_{(n)}(H) = 0$ solves eqn.(89) (this is easy). In the induction step for E and F we need to know what $\Delta_{(N)}(H \star H)$ is. This can be derived from the fact that the quantized algebra must still be a Hopf algebra which implies the identity $\Delta(H \star H) = \Delta(H) \star \Delta(H)$. The l.h.s. of this equation is equal to $\sum_n \frac{\hbar^n}{n!} \Delta_{(n)}(H \star H)$ while the r.h.s. is equal to $\sum_{nm} \frac{h^{n+m}}{n!m!} \Delta_{(n)}(H) \star \Delta_{(m)}(H) = \Delta_{(0)}(H) \star \Delta_{(0)}(H)$. Therefore $\Delta_{(n)}(H \star H) = 0$ for $n > 0$ and $H^2 \otimes 1 + 2H \otimes H + 1 \otimes H^2$ for $n = 0$.

Using these results and the expansion of Δ we find the co-product of the quantized universal enveloping algebra to be

$$\Delta(H) = H \otimes 1 + 1 \otimes H \tag{93}$$

$$\Delta(E) = E \otimes q^H + q^{-H} \otimes E \tag{94}$$

$$\Delta(F) = F \otimes q^H + q^{-H} \otimes F \tag{95}$$

where $q = e^{\hbar/4}$.

Let us pause for a moment to reflect the result. First of all note that in the limit $\hbar \rightarrow 0$ we indeed recover the old co-multiplication of the universal enveloping algebra. Also note that the co-multiplication of the Cartan element is not deformed. On the whole we have found a non co-commutative co-product. As we will see however the non-commutativity of this co-product is under control, i.e. there does exist a relation between Δ and $\tau \circ \Delta$. Therefore the tensorproduct representations $V \otimes W$ and $W \otimes V$ are no longer equal but still equivalent. We will come to this later.

Using the definition of quantization we have found the co-multiplication of the quantized universal enveloping algebra of sl_2. What about the other structures that make the universal enveloping algebra into a Hopf algebra? As we discussed in the previous section the structure maps of a Hopf algebra are coordinated, and changing one of these maps, even by a small amount, may violate the Hopf algebra axioms. Therefore we have to check if the co-multiplication we found for the quantized universal enveloping algebra (QUEA) is still compatible with the other structure maps on the UEA. If not we have to deform the other structure maps in such a way that the totally deformed Hopf algebra still is a Hopf algebra and reduces to the ordinary UEA in the classical limit.

First consider the multiplication map. One of the axioms of a Hopf algebra is that for all a, b we must have

$$\Delta(a \star b) = \Delta(a) \star \Delta(b) \tag{96}$$

In particular this means that the following equalities must hold

$$\begin{aligned} \Delta([H,E]) &= [\Delta(H), \Delta(E)] && (97) \\ \Delta([H,F]) &= [\Delta(H), \Delta(F)] && (98) \\ \Delta([E,F]) &= [\Delta(E), \Delta(F)] && (99) \end{aligned}$$

It is easily verified that if we take $[H, E] = 2E$ and $[H, F] = -2F$ then the first two relations are satisfied. Therefore these commutation relations, which are the ordinary sl_2 relations, are still consistent with the Hopf algebra axioms even after quantization. The situation is different with the third relation however. Working out the right hand side of this relation we get

$$\Delta([E,F]) = [E,F] \otimes q^{2H} + q^{-2H} \otimes [E,F] \tag{100}$$

It is obvious that $[E, F] = H$ is not a solution of this equation because $\Delta(H) = H \otimes 1 + 1 \otimes H$. In finding a solution we also have to remember that in the classical limit we do have to find the relation $[E, F] = H$ back. A commutation relation that satisfies all the requirements is

$$[E,F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}} \tag{101}$$

It is an easy exercise to check relation (100) and the classical limit for this commutation relation.

In the same manner we can proceed with the other structure maps by confronting them with the coordinating axioms. The resulting Hopf algebra looks like this

$$\begin{aligned} [H,E] &= 2E && (102) \\ [H,F] &= -2F && (103) \\ [E,F] &= [H]_q && (104) \\ \Delta(H) &= H \otimes 1 + 1 \otimes H && (105) \\ \Delta(E) &= E \otimes q^H + q^{-H} \otimes E && (106) \\ \Delta(F) &= F \otimes q^H + q^{-H} \otimes F && (107) \\ \varepsilon(E) &= \varepsilon(F) = \varepsilon(H) = 0 && (108) \\ \varepsilon(1) &= 1 && (109) \\ S(E) &= -qE && (110) \\ S(F) &= -q^{-1}F && (111) \\ S(H) &= -H && (112) \end{aligned}$$

where we defined

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{113}$$

This Hopf algebra is called the quantum universal enveloping algebra of sl_2 and is denoted by $\mathcal{U}_q(sl_2)$.

Before moving on a remark is in order. In the above derivation we have carefully evaded all issues of uniqueness. To our knowledge there is no rigorous proof of the uniqueness of the quantization of a given co-Poisson algebra. What the above construction does show however is that in the case of the UEA of sl_2 there does exist a quantization.

The same procedure can be repeated for the other simple algebras. The resulting QUEAs have the same deformation structure as in the sl_2 case.

In the next section we will consider an application of these QUEAs. This will lead us to the so called quasitriangular Hopf algebras which play an important role in relation with the quantum Yang-Baxter equation.

6 Quantum groups and the Yang-Baxter equation

In this section we we will consider the construction of solutions of the quantum Yang-Baxter equation using the quantum group $\mathcal{U}_q(sl_2)$. The construction we use is called the quantum double construction which has been applied successfully to derive the quantum R-matrices for many quantum algebras [3, 18, 21]. Before we come to the quantum double construction we consider its classical analogue.

6.1 The classical double

In this section we consider the classical double construction which allows one to construct a very simple solution of the classical Yang-Baxter equation on a Lie algebra that is constructed out of a bi-algebra an its dual . We do this because the classical double construction is very easy and will give the reader a good idea of what the quantum double is all about.

Again consider a Lie bi-algebra $(L, [.,.], \phi)$. The axioms of the map ϕ are such that the bracket $[.,.]^* : L^* \times L^* \to L^*$ defined by

$$[f, g]^* = (f \otimes g) \circ \phi \tag{114}$$

(where $f, g \epsilon L^*$) turns the dual L^* of L into a Lie algebra. Consider now the vectorspace $D = L \oplus L^*$ on which we define the scalar product

$$\langle (X, f), (Y, g) \rangle = f(Y) + g(X) \tag{115}$$

(where $f, g \epsilon L^*$ and $X, Y \epsilon L$). This space has some very nice properties one of which is contained in the following theorem.

Theorem 6 *There exists a unique Lie algebra structure on D such that*

- *L and L^* are Lie subalgebras of D.*

- $\langle [A,B],C\rangle = \langle A,[B,C]\rangle$ *for all* $A,B,C\epsilon D$.

The only non-trivial part of the proof is defining the bracket $[X,f]$ for $X\epsilon L$ and $f\epsilon L^*$ such that the second property holds. Imposing this property we get the following equations:

$$\langle [X,f],Y\rangle = -\langle f,[X,Y]\rangle \equiv -(ad^*_X f)(Y) \tag{116}$$

and

$$\langle [X,f],g\rangle = -\langle X,[f,g]\rangle = (f\otimes g)\phi(X) \tag{117}$$

which hold for all $Y\epsilon L$ and $g\epsilon L^*$. From this it follows immediately that

$$[X,f] = -ad^*_X f + (f\otimes 1)\circ(X) \tag{118}$$

The only thing left is to check the axioms of a Lie algebra for this bracket. This however is an easy exercise.

If we choose a particular basis $\{X_j\}$ in L and equip L^* with the dual basis $\{f^i\}$ (i.e. $f^i(X_j) = \delta^i_j$) then we can easily verify that the bracket on D discribed in the theorem above can be written as follows:

$$[X_i,X_j] = C^k_{ij}X_k \tag{119}$$
$$[f^i,f^j] = \Gamma^{ij}_k f^k \tag{120}$$
$$[f^i,X_j] = C^i_{jk}f^k - \Gamma^{ik}_j X_j \tag{121}$$

The space D together with this Lie algebra structure and the scalar product is called the Double Lie algebra associated to the Lie bi-algebra $(L,[.,.],\phi)$.

The nice thing of a double Lie algebra is that it is very easy to construct solutions of the classical Yang-Baxter equation on it. The precise statement of this fact is contained in the following theorem.

Theorem 7 *The element* $r = X_i\otimes f^i\epsilon D\otimes D$, *called the canonical element, is a solution of the classical Yang-Baxter equation, i.e.*

$$B = [r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] = 0 \tag{122}$$

The proof is very easy using the commutation relations in the double algebra:

$$B = C^k_{ij}X_k\otimes f^i\otimes f^j + C^i_{jk}X_i\otimes f^k\otimes f^j - \Gamma^{ik}_j X_i\otimes X_k\otimes f^j + \Gamma^{ij}_k X_i\otimes X_j\otimes f^k = 0 \tag{123}$$

which proves the theorem.

Given a Lie bi-algebra it is therefore straightforward to construct a solution of the classical Yang-Baxter equation on its double algebra. The construction we will use in the quantum case is a direct quantum analogue of the classical double construction outlined above.

6.2 Quasi-triangular Hopf algebras

In this section we will be taking a look at the so called (quasi)-triangular Hopf algebras (QTHA) [3, 22]. As we will see QTHAs are closely related to the quantum Yang-Baxter equation. The method we consider in the next section for the construction of solutions of the QYBE makes explicit use of them. The definition of a QTHA is

Definition 11 *Let* $(A, m, \Delta, \eta, \varepsilon, S)$ *be a Hopf algebra and* R *an invertible element of* $A \otimes A$*, then the pair* (A, R) *is called a QTHA if*

1. $\Delta'(a) = R\Delta(a)R^{-1}$
2. $(\Delta \otimes 1)R = R_{13}R_{23}$
3. $(1 \otimes \Delta)R = R_{13}R_{12}$

where $\Delta' = \tau \circ \Delta$ *is the so called opposite comultiplication.*

If we consider some matrix representation $T = (t_{ij})$ of A then relation 1 becomes

$$\mathcal{R}(\rho \otimes \rho)\Delta'(a) = (\rho \otimes \rho)\Delta(a)\mathcal{R} \tag{124}$$

where $\mathcal{R} = (\rho \otimes \rho)R$. $(\rho \otimes \rho)\Delta = (1 \otimes T)(T \otimes 1)$ while $(\rho \otimes \rho)\Delta'$ can be written as $(T \otimes 1)(1 \otimes T)$. Since Δ' is non-cocommutative the elements t_{ij} of T do not commute in general which means that $(T \otimes 1)(1 \otimes T) \neq (1 \otimes T)(T \otimes 1)$. If we write $\tilde{T} = T \otimes 1$ and $\tilde{\tilde{T}} = 1 \otimes T$ then property 1 implies

$$\mathcal{R}\tilde{T}\tilde{\tilde{T}} = \tilde{\tilde{T}}\tilde{T}\mathcal{R} \tag{125}$$

which gives certain permutation relations for the matrix elements of T. This equation plays an important role in the theory of solvable lattice models as well as in the quantum inverse scattering method for integrable quantum field theories.

In relation to the approach we took with respect to Hopf algebras in the first section we can say the following about this definition. In contrast to ordinary Hopf algebras these QTHA have the important property that if V_1 and V_2 are representations then the tensor product representations $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are isomorphic, the isomorphism being provided by the element R. This follows from the fact that the tensor product representation $V_1 \otimes V_2$ is related to the coproduct Δ while the tensorproduct representation $V_2 \otimes V_1$ is related to the opposite comultiplication Δ'. In an arbitrary Hopf algebra Δ and Δ' are not related, however in a QTHA they are related by the invertible element R. This then implies the equivalence stated above.

We come now to the relevance of QTHAs to the quantum Yang-Baxter equation.

Theorem 8 *If* (A, R) *is a quasi-triangular Hopf algebra then* R *satisfies the QYBE*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{126}$$

The way to prove this theorem is to calculate $(1\otimes\Delta')R$ in two ways. Write $R = R_i^{(1)}\otimes R_i^{(2)}$, where summation over i is understood, then we find

$$\begin{aligned}(1\otimes\Delta')(R) &= R_i^{(1)}\otimes\Delta'(R_i^{(2)}) \\ &= R_i^{(1)}\otimes R\Delta(R_i^{(2)})R^{-1} \\ &= (1\otimes R)(R_i^{(1)}\otimes\Delta(R_i^{(2)}))(1\otimes R^{-1}) \\ &= R_{23}(1\otimes\Delta)RR_{23}^{-1} \\ &= R_{23}R_{13}R_{12}R_{23}^{-1}\end{aligned} \tag{127}$$

On the other hand we have

$$\begin{aligned}(1\otimes\Delta')R &= (1\otimes\tau)(1\otimes\Delta)R \\ &= (1\otimes\tau)R_{13}R_{12} \\ &= R_{12}R_{13}\end{aligned} \tag{128}$$

Equating these two results we get the required result.

In the applications of the Yang-Baxter equation to knot theory and conformal field theory the R-matrix satisfies another relation associated to the fact that the braid of figure 4 is equivalent to the trivial braid. This relation reads $R_{12}R_{21} = 1$.

Definition 12 *A quasi-triangular Hopf algebra is called triangular if the element R satisfies the extra relation*

$$R_{12}R_{21} = 1 \tag{129}$$

It is easy to think of an example of a QTHA albeit a rather trivial one. Take for example the universal enveloping algebra of any Lie algebra together with the element $R = 1\otimes 1$. In fact this is a triangular Hopf algebra. The reason that this example is trivial is obviously the fact that universal enveloping algebras are co-commutative. For $\mathcal{U}_q(sl_2)$ however $R = 1\otimes 1$ is not a quasitriangular structure since $\Delta' \neq \Delta$ so the first axiom is violated. For $\hbar = 0$ however it must again be a solution which leads us to make the following ansatz for R

$$R = 1\otimes 1 + \sum_{n=1}^{\infty} R^{(n)}\hbar^n \tag{130}$$

Inserting this into the axioms 1 to 3 of a quasi-triangular structure we get a set of recursive relations for the elements R^n. Solving these relations leads to a solution R that satisfies all requirements. This method of constructing R is however extremely cumbersome for other cases than sl_2. In the next section we therefore take a look at a systematic method for constructing these R-matrices.

6.3 The quantum double construction

In this section we will show that the QUEA $\mathcal{U}_q(sl_2)$ is a quasi-triangular Hopf algebra. We will prove this by explicitly constructing an element R satisfying the axioms of a quasi-triangular Hopf algebra. The construction we use is the quantum version of the classical double construction we considered earlier.

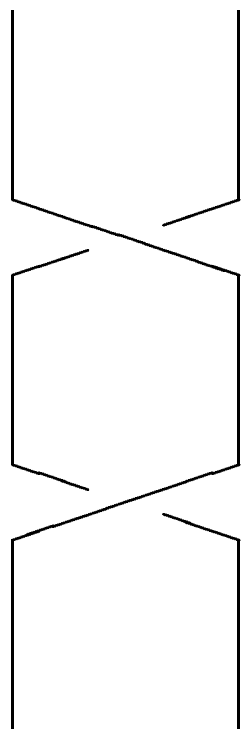

Figure 4: This braid is equivalent to the trivial braid

We start with some notational matters. Let A be a Hopf algebra and A^* its dual Hopf algebra. Replacing the comultiplication on A^* by the opposite comultiplication we obtain a new Hopf algebra denoted by A^o. Choose in A a basis $\{e_i\}$ and let $\{e^i\}$ be the dual basis. Then we denote

$$e_i.e_j = m(e_i \otimes e_j) = m_{ij}^k e_k \tag{131}$$
$$\Delta(e_i) = \Delta_i^{kl} e_k \otimes e_l \tag{132}$$
$$S(e_i) = S_i^k e_k \tag{133}$$

where we used the summation convention for repeated indices. The relations in the algebra A^o then become

$$e^i.e^j = \Delta_k^{ij} e^k \tag{134}$$
$$\Delta(e^i) = m_{lk}^i e^k \otimes e^l \tag{135}$$
$$S(e^i) = S_k^i e^k \tag{136}$$

where we used the definitions of the structure maps of the dual Hopf algebra in terms of the structure maps of the Hopf algebra itself.

Consider now the space $D(A)$ which, as a vectorspace, is isomorphic to $A \otimes A^o$ and which contains A and A^o as Hopf subalgebras. The structure of this space will become clearer in a moment, however we can say the following. In the classical double construction we considered the space $L \oplus L^*$ which obviously contained L and L^* as Lie subalgebras. In the quantum case however we are always working at the level of universal enveloping algebras. At this level the classical double construction gives a similar structure as the one we encounter here because of the isomorphism

$$\mathcal{U}(L_1 \oplus L_2) \simeq \mathcal{U}(L_1) \otimes \mathcal{U}(L_2) \tag{137}$$

so the universal enveloping algebra of the double $D = L \oplus L^*$ is isomorphic to $\mathcal{U}(L) \otimes \mathcal{U}(L^*)$ as a vectorspace and contains $\mathcal{U}(L)$ and $\mathcal{U}(L^*)$ as subalgebras.

$D(A)$ is not yet an algebra itself because we do not know how to multiply elements of A with A^o (we will come to this later). Consider the element $R = e_i \otimes e^i$ of $D(A) \otimes D(A)$. We have

$$\begin{aligned}(\Delta \otimes 1)(R) &= (\Delta \otimes 1)(e_i \otimes e^i) \\ &= \Delta_i^{kl} e_k \otimes e_l \otimes e^i \\ &= e_k \otimes e_l \otimes e^k.e^l \\ &= R_{13} R_{23}\end{aligned} \tag{138}$$

In the same way we find for this R

$$(1 \otimes \Delta)R = R_{13} R_{12} \tag{139}$$

so two of the three axioms for a quasi-triangular Hopf algebra are satisfied. The remaining axiom $\Delta' = R \Delta R^{-1}$ is easily seen to lead to the equation

$$\Delta_i^{lk} m_{kj}^p (e_l.e^j) = \Delta_i^{kl} m_{jk}^p (e^j.e_l) \tag{140}$$

If we define the multiplication between elements e_i and e^j such that these permutation relations are satisfied, then the element $R = e_i \otimes e^i$ (called the canonical element) defines a quasi-triangular structure on $D(A)$. The pair $(D(A), R)$ is then called the double of A. Using the Hopf algebra axioms we can rewrite the permutation relations into

$$e^i . e_j = m^i_{kl} m^k_{nm} \Delta^{pl}_j \Delta^{sr}_p S^n_s (e_r . e^m) \tag{141}$$

which is the form in which we shall use them later on.

Summarizing we have the following theorem:

Theorem 9 *To every Hopf algebra A there is associated a quasi-triangular Hopf algebra that contains A and A^o as Hopf subalgebras and is isomorphic to $A \otimes A^o$ as a vectorspace.*

We will now apply the above theory to find an R matrix making $\mathcal{U}_q(sl_2)$ into a quasi-triangular Hopf algebra. Apart from the generators (E, F, H) of $\mathcal{U}_q(sl_2)$ (the Chevalley generators) we will also need the algebra in terms of generators (e, f, H) where

$$e = q^{H/2} E \tag{142}$$
$$f = q^{-H/2} F \tag{143}$$
$$H = H \tag{144}$$

The relations of $\mathcal{U}_q(sl_2)$ in terms of these generators are

$$[H, e] = 2e \tag{145}$$
$$[H, f] = -2f \tag{146}$$
$$[e, f] = \frac{2 sinh(\frac{\hbar}{2} H)}{1 - q^{-2}} \tag{147}$$
$$\Delta(e) = 1 \otimes e + e \otimes q^H \tag{148}$$
$$\Delta(f) = f \otimes 1 + q^{-H} \otimes f \tag{149}$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H \tag{150}$$
$$S(e) = -q^{-H} e \tag{151}$$
$$S(f) = -q^{-2} q^H f \tag{152}$$
$$S(H) = -H \tag{153}$$

Considering the relations of $\mathcal{U}_q(sl_2)$ it is not difficult to see that the sets

$$\{H^n E^m F^l\}^{\infty}_{n,m,l=0} \tag{154}$$

and

$$\{H^n e^m f^l\}^{\infty}_{n,m,l=0} \tag{155}$$

are bases of this algebra. Analogous to the case of the classical UEA we define the positive and negative Borel subalgebras

$$\mathcal{U}_q(b_+) = span\{H^n E^m\}^{\infty}_{n,m=0} \tag{156}$$
$$\mathcal{U}_q(b_-) = span\{H^n F^m\}^{\infty}_{n,m=0} \tag{157}$$

What we will do now is construct the double of the positive Borel subalgebra. Let W and Y be the dual elements of H and E respectively, i.e.

$$\begin{aligned} W(H) &= 1 \quad (158) \\ W(E) &= 0 \quad (159) \\ Y(H) &= 0 \quad (160) \\ Y(E) &= 1 \quad (161) \end{aligned}$$

Obviously $\{W^n Y^m\}_{n,m=0}^{\infty}$ is then a basis for the dual of the positive Borel subalgebra, $(\mathcal{U}_q(b_+))^*$, dual to the basis $\{H^n E^m\}$. Using the rule

$$\langle f.g, x\rangle = \langle f\otimes g, \Delta(x)\rangle \quad (162)$$

we can calculate the commutation relation between W and Y. We find

$$[W, Y] = -\frac{\hbar}{2} Y \quad (163)$$

In the same way, using the rule

$$\langle \Delta(f), x\otimes y\rangle = \langle f, xy\rangle \quad (164)$$

we can calculate the co-multiplication on $(\mathcal{U}_q(b_+))^*$. The result of this easy exercise is

$$\begin{aligned} \Delta(W) &= W\otimes 1 + 1\otimes W \quad (165) \\ \Delta(Y) &= 1\otimes Y + Y\otimes e^{-2W} \quad (166) \end{aligned}$$

We see that if we define $\tilde{W} = \frac{4}{\hbar} W$ and $\tilde{Y} = \frac{1-q^2}{\hbar} Y$ and if we transpose the comultiplication in order get the relations of the opposite dual $(\mathcal{U}_q(b_+))^o$, we get

$$\begin{aligned} [\tilde{W}, \tilde{Y}] &= -2\tilde{Y} \quad (167) \\ \Delta(\tilde{W}) &= \tilde{W}\otimes 1 + 1\otimes \tilde{W} \quad (168) \\ \Delta(\tilde{Y}) &= \tilde{Y}\otimes 1 + q^{-\tilde{W}}\otimes \tilde{Y} \quad (169) \end{aligned}$$

Compare these relations to the relations of the negative Borel subalgebra. They are exactly equal if we take $\tilde{W} = H$ and $\tilde{Y} = f$ which leads us to conclude that

$$(\mathcal{U}_q(b_+))^o \cong \mathcal{U}_q(b_-) \quad (170)$$

The quantum double of the positive Borel subalgebra is therefore the Hopf algebra $\mathcal{U}_q(b_+)\otimes\mathcal{U}_q(b_-)$ which is generated by the elements $\{H, e, \hat{H}, f\}$ (where we denote the Cartan element of the negative Borel subalgebra by $\hat{H}$). However, we have not yet calculated the commutation relations between elements of the positive Borel subalgebra and its opposite dual. Using eqn.(141) we can easily calculate these. The resulting set of commutation relations for the double algebra $D(\mathcal{U}_q(b_+))$ is

$$\begin{aligned} [H, e] &= 2e \quad (171) \\ [H, f] &= -2f \quad (172) \\ [\hat{H}, e] &= 2e \quad (173) \\ [\hat{H}, f] &= -2f \quad (174) \\ [e, f] &= [H]_q \quad (175) \\ [H, \hat{H}] &= 0 \quad (176) \end{aligned}$$

From these commutation relations it is immediate that

$$\frac{D(\mathcal{U}_q(b_+))}{\langle H-\hat{H}\rangle} \cong \mathcal{U}_q(sl_2) \tag{177}$$

where $\langle H-\hat{H}\rangle$ is the ideal generated by the element $H-\hat{H}$. The canonical homomorphism is explicitly given by

$$H, \hat{H} \rightarrow H \tag{178}$$
$$e \rightarrow e \tag{179}$$
$$f \rightarrow f \tag{180}$$

We can now easily construct the canonical element of the double. We know that $\{H^n e^m\}$ is a basis of $\mathcal{U}_q(b_+)$ and $\{W^k Y^l\}$ is a basis of $(\mathcal{U}_q(b_+))^o$, however, these two bases are not dual. Consider for a moment the general case again where $\{e_i\}$ and $\{f^i\}$ are (not necessarily dual) bases of A and A^o respectively. Also suppose that $f^i = B^i_j e^j$ where $\{e^i\}$ is the dual basis of $\{e_i\}$. Then the canonical element $R = e_i \otimes e^i$ w.r.t. the bases e_i and f^j is equal to

$$R = (B^{-1})^i_j e_i \otimes f^j \tag{181}$$

and we also have $\langle f^i, e_j\rangle = B^i_j$.

We learn from this that we have to calculate the matrix

$$B^{nm}_{kl} = \langle W^k Y^l, H^n e^m\rangle \tag{182}$$

and invert it. Explicit calculation shows that

$$B^{nm}_{kl} = \delta^n_k \delta^m_l \frac{k![l;q^{-2}]!}{\hbar^l} \tag{183}$$

where we used the notation

$$[u;q]! = \prod_{i=1}^{u} \frac{1-q^i}{1-q} \tag{184}$$

Fortunately the matrix B is diagonal which makes inverting it very easy. The result is

$$(B^{-1})^{nm}_{kl} = \frac{\hbar^l}{k![l;q^{-2}]!}\delta^n_k \delta^m_l \tag{185}$$

Finally we can write down the explicit expression for the canonical element R:

$$\begin{aligned} R &= \sum_{kl} \frac{\hbar^l}{k![l;q^{-2}]!} H^k e^l \otimes W^k Y^l \\ &= e^{H\otimes W} \sum_l \frac{\hbar^l}{[l;q^{-2}]!} e^l \otimes Y^l \end{aligned} \tag{186}$$

The image of this element under the canonical homomorphism $D(\mathcal{U}_q(b_+)) \rightarrow \mathcal{U}_q(sl_2)$ is therefore

$$R = q^{\frac{1}{2}H\otimes H} E_{q^{-2}}^{\lambda e\otimes f} \tag{187}$$

where $\lambda = 1 - q^{-2}$ and

$$E_q^x \equiv \sum_{l=0}^{\infty} \frac{x^l}{[l;q]!} \tag{188}$$

is the so called q-deformed exponential.

In the standard (fundamental) representation of sl_2, which is also a representation of $\mathcal{U}_q(sl_2)$, this matrix takes on the form

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-q^{-2} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{189}$$

This concludes the derivation of the quasi-triangular structure of $\mathcal{U}_q(sl_2)$.

6.4 The classical limit

In this paper we have considered two Yang-Baxter equations, the classical YBE and the quantum YBE. One might wonder what the relation between them is. This is contained in the following theorem.

Theorem 10 *Let $R(t)$ be a one parameter family of solutions of the quantum YBE which can be written as a power series of the form*

$$R(t) = 1 \otimes 1 + rt + At^2 + \mathcal{O}(t^3) \tag{190}$$

then r satisfies the classical YBE.

Proof: Inserting the power series (190) into the QYBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{191}$$

we get

$$\begin{aligned} &(r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23})t^2 + \mathcal{O}(t^3) = \\ &\qquad (r_{13}r_{12} + r_{23}r_{12} + r_{23}r_{13})t^2 + \mathcal{O}(t^3) \end{aligned}$$

from which the theorem follows immediately.

The solution of the QYBE constructed above using the quantum double construction can be expanded in a power series of the deformation parameter $\hbar$. The associated solution of the classical YBE is easily found to be (up to overall rescaling which is irrelevant because the YBEs are homogeneous)

$$r = \frac{1}{2} H \otimes H + 2E \otimes F \tag{192}$$

which is the r-matrix considered in example 3.

7 Elements of representation theory

In this last section we shall be taking a look at the representation theory of $\mathcal{U}_q(sl_2)$. We will not go into great detail however and will not prove all the theorems (see for example [23, 24, 25, 26, 27, 28]). What we will do is derive the irreducible (physical) represenions of the algebra $\mathcal{U}_q(sl_2)$ at q a root of unity (i.e. $q^m = 1$ for some m), discuss the indecomposable (non-physical) reps and also the tensorproduct decompositions (fusion rules) of tensorproduct representations. The reason why we consider the representation theory only at q a root of unity is that in the applications of quantum groups to physics it is these representations that appear to be the most important. The representation theory at q not a root of unity was given in [19] and is not too different from the representation theory of ordinary simple Lie algebras. At a root of unity this changes drastically. The presentation we give here will follow closely the paper [26].

Let $q = e^{2\pi i n/m}$ where n, m are natural numbers, $1 \leq n \leq m-1$ and n, m are relatively prime. We set $M = m$ for m =odd and $M = m/2$ for M =even. Consider $\mathcal{U}_q(sl_2)$ at these values of q

$$\begin{aligned}
[H, E] &= E \qquad [H, F] = -F \qquad [E, F] = [2H]_q && (193)\\
\Delta(E) &= E \otimes q^H + q^{-H} \otimes E && (194)\\
\Delta(F) &= F \otimes q^H + q^{-H} \otimes F && (195)\\
\Delta(H) &= H \otimes 1 + 1 \otimes H && (196)
\end{aligned}$$

(where we rescaled $H \to \frac{1}{2}H$) supplemented by the relations

$$E^M = F^M = 0 \qquad (197)$$

which have to be imposed if the quantum R-matrix is to still be defined for these values of q (i.e. we insist on having a quasi-triangular structure because we want the reps $V \otimes W$ and $W \otimes V$ to be equivalent).

Now, let $\rho : \mathcal{U}_q(sl_2) \to End(W)$ be an irreducible representation on a finite dimensional linear space W.

Lemma 4 *W is spanned by eigenvectors of $\rho(H)$.*

Any matrix can be put in upper triangular form by a suitable basis change in W. We put $\rho(H)$ in uppertriangular form. Any upper triangular matrix has at least one eigenvector v, namely $(1, 0, ..., 0)$. Using the commutation rules we easily find

$$\rho(H)(\rho(e^r f^s)v) = (\lambda + r - s)(\rho(e^r f^s)v) \qquad (198)$$

for $\rho(H)v = \lambda v$, so $\rho(e^r f^s)v$ are all eigenvectors with different eigenvalues. Since W is irreducible a basis of W must be contained in this set of vectors (for $0 \leq r, s \leq M-1$). The Lemma follows.

Since $\dim(W) < \infty$ there exists a highest weight vector ψ such that $\rho(e)\psi = 0$ and $\rho(H)\psi = j\psi$, where j is the highest weight. It is easy to see that the set

$$span\{\rho(f^r)\psi \mid 0 \leq r \leq M-1\} \qquad (199)$$

is invariant under $\rho(\mathcal{U}_q(sl_2)$ and is therefore equal to W (or else W would not be irreducible). Also note that $\dim(W) \leq M$. From now on we denote $p = dim(W)$.

Lemma 5
$$[E, F^k] = F^k[k]_q[2H - k + 1]_q \tag{200}$$

The proof of this lemma is a straightforeward calculation using the commutation relations of the algebra.

We now come to the main theorem.

Theorem 11 *The finite dimensional irreducible representations W of $\mathcal{U}_q(sl_2)$ fall into two classes.*

1. *$dim(W) < M$: The inequivalent irreps are labeled by their dimension p and an integer z. They have highest weight*
$$j = \frac{1}{2}(p-1) + \frac{m}{4n}z \tag{201}$$

2. *$dim(W) = M$: The irreps are labeled by a complex number z which ranges over the complex numbers minus the set $\{\mathcal{Z} + \frac{2n}{m}r \mid 1 \leq r \leq M-1\}$ and have highest weight $j = \frac{1}{2}(M-1) + \frac{m}{4n}z$.*

The irreducible representations will be denoted by $\langle p, z\rangle$.

The proof of the theorem is not very difficult. Since the dimension of W is p we have the following two identities.

$$\rho(EF^p)\psi = [p]_q[2j - p + 1]_q\rho(F^{p-1})\psi = 0 \tag{202}$$
$$\rho(EF^r)\psi = [r]_q[2j - p + 1]_q\rho(F^{r-1})\psi \neq 0 \tag{203}$$

$(r < p)$ where we used the previous lemma. First consider the case $p < M$. In that case $[p]_q \neq 0$ so the first equation gives $[2j - p + 1]_q = 0$ which is easily seen to be equivalent to
$$j = j(p, z) = \frac{1}{2}(p-1) + \frac{m}{4n} \tag{204}$$
where z is an arbitrary integer. One can easily show that within the parameters in which p and z are defined we have an equality $j(p, z) = j(q, w)$ if and only if $p = q$ and $z = w$. From this it follows that $\rho(EF^r)\psi \neq 0$ for $r < p$ and also that p and r completely determine the irrep up to isomorphism. For $p = M$ eqn (202) is automatically satisfied which means that the only demand on j is $[2j - r + 1]_q \neq 0$ for $r \leq M - 1$. From this it follows that j can take on any complex value except the values $\{j(p, z) \mid 1 \leq p \leq M - 1; z = integer\}$. Part 2 of the theorem follows if we parametrize j by $j = 1/2(M - 1) + mz/4n$ where z can be any complex number except for the numbers specified in the theorem.

The fact that all irreps have dimension smaller or equal to M does not mean that there are no representations with a higher dimension. In fact if we consider the decomposition of a tensor product representation of two irreducible reps we find not only representations of the type discussed above but also some indecomposable representations. By an indecomposable representation we mean a representation which has invariant subspaces but cannot be written as a direct sum of them. Without proof we now give the decomposition of a tensor product rep of two irreducible representations.

Theorem 12

$$\langle i,z\rangle\otimes\langle j,w\rangle = \bigoplus_{k=|i-j|+1}^{K}\langle k,z+w\rangle \oplus \bigoplus_{l=r,r+2,..}^{i+j-M} I^l_{z+w} \tag{205}$$

where $K = min\{i+j-1, 2M-i-j-1\}$.

The representations I^l_z are the indecomposable ones. A few features of their structure are collected in the next theorem.

Theorem 13 *The indecomposable representations have the following properties:*

- $dim(I^l_z) = 2M$
- I^l_z *is a direct sum of weight spaces* W_μ *with highest weight* $\mu_{HW}(l,z) = \frac{1}{2}(M+p-2) + \frac{m}{4n}z$. *The other weights are* $\{\mu_{HW}, \mu_{HW}-1, ..., \mu_{HW}-(M+p-1)\}$
- $dim(W_\mu) = 2$ *for the weights* μ *such that*

$$\frac{1}{2}(M-p) + (\frac{m}{4n}z) \geq \mu \geq -\frac{1}{2}(M-p) + \frac{m}{4n}z \tag{206}$$

  and $dim(W_\mu) = 1$ *in all other cases.*

If one ignores these indecomposable representations the tensor product decompositions discribed above resemble very much the fusion rules of certain conformal field theories. Before you forget about them however you must check if they form an ideal in the representation ring. That this is indeed the case follows from the following equations

$$\langle j,z\rangle\otimes I^p_z = \bigoplus_{(l,n)} I^l_n \tag{207}$$

$$I^j_z\otimes I^p_w = \bigoplus_{(l,w)} I^l_n \tag{208}$$

The indecomposable representations can nicely be characterized by the fact that their so called quantum dimensions are zero. The quantum dimension χ^1_ρ of a representation ρ is a special case of a character χ^r_ρ (where r is a complex number) defined as

$$\chi^r_\rho = Tr(\rho(q^{2rH})) \tag{209}$$

We find for the representations discussed above

$$\chi^1_{\langle p,z\rangle} = [p]_q e^{i\pi z} \neq 0 \tag{210}$$

$$\chi^1_{\langle p,0\rangle} = > 0 \tag{211}$$

$$\chi^1_{I^j_w} = 0 \tag{212}$$

so that the physical representations always have a positive quantum dimension while the non-physical reps have zero quantum dimension.

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