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Initial Value Problems of the Sine-Gordon Equation and Geometric Solutions

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Abstract. Recent results using inverse scattering techniques interpret every solution $\varphi(x, y)$ of the sine-Gordon equation as a nonlinear superposition of solutions along the axes $x = 0$ and $y = 0$. This has a well-known geometric interpretation, namely that every weakly regular surface of Gauss curvature $K = -1$, in arc length asymptotic line parametrization, is uniquely determined by the values $\varphi(x, 0)$ and $\varphi(0, y)$ of its coordinate angle along the axes. We introduce a generalized Weierstrass representation of pseudospherical surfaces that depends only on these values, and we explicitely construct the associated family of pseudospherical immersions corresponding to it.

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The Sine-Gordon Equation and Initial Value Problems

Let $u\colon D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ represent a differentiable function on some open, simply-connected domain D. In [1] it had already been shown that every solution $u(x, y)$ of the sine-Gordon equation

$$u_{xy} = \sin u \tag{1}$$

represents 'some type of nonlinear superposition of solutions $u_1(x, 0)$ and $u_2(0, y)$', that is, travelling along different characteristics. The purpose of this report is *to obtain* all smooth solutions $u(x, y)$ by algebro-geometric methods which replace the classical ones (such as direct integration, inverse scattering and numerical integration).

A differentiable solution $\varphi(x, y)$ of (1) represents the *Tchebychev angle* (i.e., angle between arc length asymptotic coordinate lines) of a weakly regular pseudospherical surface, measured at the point corresponding to (x, y). By *weakly regular* surface we mean a parametrized surface whose partial velocity vector fields never vanish, but are allowed to coincide at a set of points of measure zero. Obviously, at those singularity points, the parametrization fails to be an immersion.

Thus, every smooth solution $\varphi(x, y)$ of the Equation (1) corresponds to a weakly regular pseudospherical surface. It is known that every such surface is completely determined by a pair of arbitrary smooth functions $\alpha(x)$ and $\beta(y)$, such that $\alpha(x) = \varphi(x, 0)$ and $\beta(y) = \varphi(0, y)$. We view this pair of functions as a *pseudospherical analogue of the Weierstrass representation* from minimal surfaces, and we call it *generalized Weierstrass representation of pseudospherical surfaces*. We deduced this representation by analogy to a method presented in [2]. Our representation simply turned out to depend only on the initial values of the Tchebychev angle, $\alpha(x) = \varphi(x, 0)$ and $\beta(y) = \varphi(0, y)$.

The author of this report found this representation in 1998, while she was a graduate student. At that time, she was not aware of some outstanding works like [1, 3]. No previous paper contained a representation for pseudospherical surfaces of type Weierstrass, and the holomorphic potential of [2] that inspired this approach had only been studied for some harmonic maps (not for the Lorentz-harmonic maps, like in our case).

However, after it was computed in the spirit of [2], this representation turned out to be characterized by the initial conditions of a Goursat problem, so we would now like to recall the following definition.

DEFINITION 1. A *nonlinear hyperbolic system of equations* is a system of partial differential equations for functions $U, V\colon D \to \mathbb{R}$, where $D := [0, x_0] \times [0, y_0]$:

$$V_x = f(U, V), \qquad U_y = g(U, V), \tag{2}$$

with smooth given functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$. We will call *initial value problem for a nonlinear hyperbolic system* the problem consisting of equations in (2), together with the initial conditions

$$U(x, 0) = U_0(x), \qquad V(0, y) = V_0(y), \tag{3}$$

for $(x, y) \in D$. The functions $U_0\colon [0, x_0] \to \mathbb{R}$ and $V_0\colon [0, y_0] \to \mathbb{R}$ are also assumed to be smooth.

PROPOSITION 1 (see [4]). *The initial value problem for a nonlinear hyperbolic system has a unique classical solution.*

For details, see [4], Theorem 1 and its corollary.

Any nonlinear equation of hyperbolic type can be brought to the form (1), by substitutions of type $U = U(u, u_x)$, $V = V(u, u_y)$.

For the particular case of the sine-Gordon equation, one introduces the independent variables $U = u$, $V = u_x$ which satisfy a system of the form (1), namely $U_x = V$, $V_y = \sin U$, with initial conditions (3).

We provide a method of obtaining solutions to such a problem, by solving a simplified ODE system, followed by a loop group factorization.

Since many readers are not familiar with this type of computations, we provided complete arguments for all of our techniques and results, while also striving for brevity.

Geometric Solutions to the Sine-Gordon Equation

We begin our study of surfaces with constant negative Gaussian curvature $K = -1$, called *pseudospherical surfaces*, or *K-surfaces*. We recall that all such surfaces are described by a sine-Gordon equation, with a corresponding Lax system. Let M be the image of $D = [0, x_0] \times [0, y_0]$ through the differentiable map $\psi\colon D \to \mathbb{R}^3$, where ψ represents *a weakly regular asymptotic line parametrization* (i.e., such that the coordinate lines are asymptotic lines, and partial velocities never vanish, so we can assume them to be unitary). An arc length asymptotic line parametrization is also called *Tchebychev parametrization*.

Let φ represent the angle between the asymptotic lines. We will call it a *Tchebychev angle*. Singularities of weakly regular surfaces occur at those values (x, y) where this angle, $\varphi(x, y)$ equals 0 or π. The first fundamental form is [5, 6]:

$$\mathrm{I} = |\,\mathrm{d}\psi|^2 = \mathrm{d}x^2 + 2\cos\varphi\ \mathrm{d}x\ \mathrm{d}y + \mathrm{d}y^2.$$

Let N define the normal vector field to the surface (or Gauss map). Remark that the unit vector field N is orthogonal to $\psi_x, \psi_y, \psi_{xx}, \psi_{yy}$.

The following obvious result is due to Lie (around 1870) and is of crucial importance (see also [5]):

THEOREM 1. *Every pseudospherical surface has a one-parameter family of deformations preserving the second fundamental form*

$$\mathrm{II} = \sin\varphi \cdot \mathrm{d}x\ \mathrm{d}y,$$

the Gaussian curvature $K = -1$, *and the angle* φ *between the asymptotic lines. The deformation is generated by the transformation* $x \mapsto x^* = \lambda^{-1}x$ *and* $y \mapsto y^* = \lambda y$, $\lambda > 0$. *(Angle is preserved in the sense that* $\varphi^*(x^*, y^*) = \varphi(x(x^*), y(y^*))$.*)*

We will refer to this simple change of coordinates as the *Lie–Lorentz transformation*. Lie–Lorentz transformations of a certain pseudospherical immersion represent its *associated family*, denoted as $\psi^\lambda\colon D \to \mathbb{R}^3$. In order to define an orthonormal frame on the surface, we consider the so-called curvature line coordinates, defined by $u_1 = x + y$, $u_2 = x - y$. Partial velocities with respect to u_1 and u_2 are orthogonal. This reparametrization diagonalizes both the first and the second fundamental form. The eigenvectors of the shape operator are the orthonormal vectors e_1 and e_2, called principal directions.

DEFINITION 2. For any (weakly regular) pseudospherical immersion $\psi\colon D \to \mathbb{R}^3$, we identify the *orthonormal standard frame* $F = \{\psi, e_1, e_2, N\}$ with the SO(3)-valued function (e_1, e_2, N) defined at every point of the surface.

We will generically call *rotated frame* F_θ the frame obtained by rotating the standard frame F by the angle $\theta(x, y)$ around N, in the tangent plane.

In particular for $\theta = \varphi/2$, where $\varphi(x, y)$ is the Tchebychev angle between the asymptotic directions, the resulting frame is denoted $\mathcal{U} := F_{\varphi/2}$ and is called the *normalized frame* associated with the standard frame F (see [7], p. 18). Expressed in Tchebychev coordinates, the normalized frame $\mathcal{U}$ is oriented just like F, and consists of ψ, ψ_x, a unit vector orthogonal to ψ_x, $\psi_x^\top$, and the unit normal N.

Finally, we will call *extended normalized frame* the normalized frame $\mathcal{U}^\lambda = \mathcal{U}(x, y, \lambda)$ corresponding to the immersion ψ^λ, obtained via Lie–Lorentz transformation of coordinates from the immersion ψ.

It is convenient to use 2×2 matrices instead of 3×3 ones. Therefore, we recall the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

We identify the SO(3)-valued extended normalized frame $\mathcal{U}^\lambda$ with the SU(2)-valued *function* $\mathcal{U}$ defined on the same domain D, with the initial condition $\mathcal{U}(0, 0, \lambda) = I$, via the spinor correspondences between e_k ($k = 1, 2, 3$) and matrices $\mathcal{U} \cdot i\sigma_k \cdot \mathcal{U}^{-1}$. We have the following (see [1, 4, 5, 8]):

THEOREM 2. *The extended normalized frame $\mathcal{U}^\lambda$ is a* SU(2)*-valued function of* $\lambda > 0$*, which satisfies the Lax differential system*

$$\partial_x \mathcal{U}^\lambda = \mathcal{U}^\lambda \cdot \mathcal{A}, \qquad \partial_y \mathcal{U}^\lambda = \mathcal{U}^\lambda \cdot \mathcal{B}, \tag{5}$$

where

$$\mathcal{A} = \frac{i}{2}\begin{pmatrix} \varphi_x & -\lambda \\ -\lambda & -\varphi_x \end{pmatrix}, \qquad \mathcal{B} = \frac{i}{2}\lambda^{-1}\begin{pmatrix} 0 & \mathrm{e}^{-i\varphi} \\ \mathrm{e}^{i\varphi} & 0 \end{pmatrix} \tag{6}$$

The compatibility condition for the system is $\mathcal{A}_y - \mathcal{B}_x - [\mathcal{A}, \mathcal{B}] = 0$, *which can be rewritten as* $\varphi_{xy} = \sin\varphi$.

Conversely, given a smooth solution $\varphi(x, y)$ *of the sine-Gordon equation, there exists a unique solution* $\mathcal{U}(x, y, \lambda)$ *of the Lax system. Moreover, this solution is real analytic in* λ.

Harmonic Maps and the Generalized Weierstrass Representation

For a complete characterization of harmonicity in the context of pseudospherical surfaces, we recommend [9]. Let us remark that the wave equation $u_{xy} = 0$ over the xy-plane can be understood as *harmonicity condition* with respect to the Lorentz metric $\mathrm{d}x \cdot \mathrm{d}y$. A well-known fact is the following: if M is a weakly regular surface with $K < 0$, then M, considered with its second fundamental form II as a metric, represents a *Lorentzian* 2-manifold (M, II). The Gauss map $N\colon (M, \mathrm{II}) \to S^2$ is Lorentz-harmonic (i.e., $N_{xy} = \rho \cdot N$, where ρ is a certain real-valued function) *iff* the curvature $K < 0$ is constant.

It is also well known that if $M = (D, \psi)$ is, as usual, a pseudospherical surface given by a Tchebychev immersion $\psi\colon D \to \mathbb{R}^3$, then the frame $\mathcal{U}\colon D \to$ SU(2) represents a lift of the Gauss map of $N\colon D \to S^2$, via the canonical projection relative to the base point e_3, namely $\pi\colon \mathrm{SU}(2) \to \mathrm{S}^2 \cong \mathrm{SU}(2)/\mathrm{S}^1$. From this lifting, it follows (see, for example [5]) that the maps N and $\mathcal{U}$ are related by the identification $N \equiv \mathcal{U} \cdot i\sigma_3 \cdot \mathcal{U}^{-1}$.

A very important result obtained by Sym [10] allows us to obtain the immersion (up to a rigid motion), once we have the expression of the extended frame. This is presented in several papers, (e.g. [12]):

THEOREM 3. *Starting from a given solution $\varphi(x, y)$ of the sine-Gordon equation, let us consider the initial value problem of the Lax system with the initial condition $\mathcal{U}(0, 0, \lambda) = \mathcal{U}_0$. Let $\mathcal{U}(\lambda)$ be the solution to this initial value problem. Then $\mathcal{U}(\lambda)$ represents the extended frame corresponding to the Tchebychev immersion $\psi^\lambda = \mathrm{d}/\mathrm{d}t\mathcal{U}^\lambda \cdot (\mathcal{U}^\lambda)^{-1}$, where $\lambda = \mathrm{e}^t$.*

By this result, once we have the extended frame, we can reconstruct the surface. Since the frame is just a lift $\mathcal{U}$ of the Gauss map N, we infer that we could reconstruct everything starting from the Gauss map. However, there is a freedom in the frame given by a gauge action. Namely, let us act on the extended normalized frame $\mathcal{U}$ via a rotation matrix $\mathcal{R}$. The result is called *gauged frame $\hat{\mathcal{U}}$*.

$$\hat{\mathcal{U}} = \mathcal{R}(0, 0)^{-1} \cdot \mathcal{U} \cdot \mathcal{R}. \tag{7}$$

It will be convenient for our purposes to fix a base point $x_0 \in D$, e.g. $x_0 = (0, 0)$, and impose $\mathcal{U}(x_0, \lambda) = I$. We will use this assumption from now on. Also note that the orthonormal frame F^λ represents a gauged frame of the normalized frame $\mathcal{U}^\lambda$, via a rotation $\mathcal{R}$ of angle $\theta = -\varphi/2$. We have the following consequence of Theorem 3:

COROLLARY 1. *If F^λ represents the orthonormal frame corresponding to the associate family of immersions ψ^λ, then*

$$\psi^\lambda = \mathcal{R}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}F^\lambda (F^\lambda)^{-1}\right)\mathcal{R},$$

where $\lambda = \mathrm{e}^t$ and $\mathcal{R}$ is the rotation of angle $-\varphi(x, y)/2$.

Let us introduce the Cartan connection $\omega^\lambda := -(\mathcal{U}^\lambda)^{-1}\,\mathrm{d}\mathcal{U}^\lambda = \mathcal{A}\,\mathrm{d}x + \mathcal{B}\,\mathrm{d}y$, with $\mathcal{A}$ and $\mathcal{B}$ given by formula (6). That is,

$$\omega^\lambda = \frac{i}{2}\begin{pmatrix} \varphi_x & -\lambda \\ -\lambda & -\varphi_x \end{pmatrix}\mathrm{d}x + \frac{i}{2}\lambda^{-1}\begin{pmatrix} 0 & \mathrm{e}^{-i\varphi} \\ \mathrm{e}^{i\varphi} & 0 \end{pmatrix}\mathrm{d}y \tag{8}$$

Obviously, ω^λ represents a $\Lambda\mathrm{su}(2)$-valued form, and then it decomposes into a diagonal, respectively off-diagonal part as $\omega^\lambda = \omega_0 + \omega_1$, according to the Cartan decomposition of su(2).

The following is a well-known result (see [12, 13]):

PROPOSITION 2. *There is a one-to-one correspondence between the space of Lorentz-harmonic maps from D to S^2 and the equivalence classes of admissible connections, under the action of the gauge action introduced earlier. Moreover, every admissible connection ω corresponds to its associated loop ω^λ satisfying the flatness condition*

$$\mathrm{d}\omega^\lambda + \omega^\lambda \wedge \omega^\lambda = 0. \tag{9}$$

Further, let $\omega_0 = \omega_0' + \omega_0''$ and $\omega_1 = \lambda^{-1}\omega_1' + \lambda\omega_1''$ be the usual splittings into (1, 0) and, respectively, (0, 1) forms, that is

$$\omega_0' = \frac{i}{2}\begin{pmatrix} \varphi_x & 0 \\ 0 & -\varphi_x \end{pmatrix}\mathrm{d}x, \qquad \omega_0'' = \mathrm{O}, \qquad \omega_1' = \frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i\varphi} \\ \mathrm{e}^{i\varphi} & 0 \end{pmatrix}\mathrm{d}y,$$
$$\omega_1'' = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathrm{d}x. \tag{10}$$

In this context, we now introduce the twisted loop algebra of those Laurent polynomials in λ with coefficients in su(2) that are fixed under the $Ad(\sigma_3)$-automorphism, that is,

$$\Lambda\mathrm{su}(2)^{\mathrm{alg}} = \{X : \mathbb{R}_* \to \mathrm{su}(2);\ \mathrm{X}(-\lambda) = \sigma_3 \cdot \mathrm{X}(\lambda) \cdot \sigma_3\}.$$

It will be convenient to use a certain Banach completion of this algebra. For this purpose, consider the Wiener algebra $\mathcal{G}$ that consists of all Laurent series of parameter λ with complex-valued coefficients, $X(\lambda) = \sum_{k\in\mathbb{Z}} X_k \cdot \lambda^k$, with the property that $\sum_{k\in\mathbb{Z}} |X_k| < \infty$. We define $\|X(\lambda)\| = \sum_{k\in\mathbb{Z}} |X_k|$. It is well known that this Wiener algebra $\mathcal{G}$ is a Banach algebra relative to this norm, and it consists of continuous functions. For a matrix $A(\lambda) \in \mathrm{su}(2, \mathcal{G})$, whose entries are elements of $\mathcal{G}$, we consider the norm $\|A\| = \sum_{i,j=1,2} \|A_{ij}\|$, where A_{ij} denotes the (i, j)-entry of A. It can be checked by a direct computation that $\|AB\| \leqslant \|A\| \cdot \|B\|$ and $|I\| = 1$. We denote by

$$\Lambda\mathrm{su}(2) := (\Lambda\mathrm{su}(2)^{\mathrm{alg}}, \|\cdot\|)$$

the completion of $\Lambda\mathrm{su}(2)^{\mathrm{alg}}$ with respect to this norm. Let us also introduce the twisted loop group

$$\Lambda\mathrm{SU}(2) := \{g \in \mathrm{SU}(2);\ \sigma_3 g(\lambda)\sigma_3 = g(-\lambda)\}.$$

It is well known that $\Lambda\mathrm{SU}(2)$ is a Banach Lie group with Lie algebra Lie $\Lambda\mathrm{SU}(2) = \Lambda\mathrm{su}(2)$. The twisting $(\mathrm{Ad}(\sigma_3)$ invariance) condition on loop algebra $\Lambda\mathrm{su}(2)^{\mathrm{alg}}$ can be replaced by the following characteristic property: in spinor representation, the diagonal part is an even function λ, while the off-diagonal part is an odd function of λ. In order to carry out the construction method of pseudospherical surfaces, we introduce the following subalgebras of $\Lambda\mathrm{su}(2)$:

$$\Lambda^+\mathrm{su}(2) = \{X(\lambda);\ X(\lambda) \text{ contains only nonnegative powers of } \lambda\} \tag{11}$$

$$\Lambda^{-}\mathrm{su}(2) = \{X(\lambda);\ X(\lambda) \text{ contains only nonpositive powers of } \lambda\}, \tag{12}$$

$$\Lambda_{*}^{-}\mathrm{su}(2) = \{X(\lambda);\ X(\infty) = 0\}. \tag{13}$$

The connected Banach loop groups whose Lie algebras are described by definitions given earlier are denoted, respectively, $\Lambda^{+}\mathrm{SU}(2)$, $\Lambda^{-}\mathrm{SU}(2)$ and $\Lambda_{*}^{-}\mathrm{SU}(2)$.

In order to obtain the generalized Weierstrass representation of pseudospherical surfaces, we need to use the following adapted factorization (introduced in [14]):

THEOREM 4 (splitting of Birkhoff type, for real parameter λ). *Let $\tilde{\Lambda}\mathrm{SU}(2)$ be the subset of $\Lambda\mathrm{SU}(2)$ whose elements, as maps defined on $\mathbb{R}_{+}$, admit an analytic extension to $\mathbb{C}_{*}$. It is easy to see that $\tilde{\Lambda}\mathrm{SU}(2)$ is a subgroup of $\Lambda\mathrm{SU}(2)$. Then the multiplication map $\tilde{\Lambda}_{*}^{-}\mathrm{SU}(2) \times \tilde{\Lambda}^{+}\mathrm{SU}(2) \to \tilde{\Lambda}\mathrm{SU}(2)$ represents a diffeomorphism onto the open and dense subset $\tilde{\Lambda}_{*}^{-}\mathrm{SU}(2) \cdot \tilde{\Lambda}^{+}\mathrm{SU}(2)$, called the 'big cell'. In particular, if $g \in \tilde{\Lambda}\mathrm{SU}(2)$ is contained in the big cell, then g has a unique decomposition $g = g_{-}g_{+}$ where $g_{-} \in \tilde{\Lambda}_{*}^{-}\mathrm{SU}(2)$ and $g_{+} \in \tilde{\Lambda}^{+}\mathrm{SU}(2)$. The analogous result holds for the multiplication map $\tilde{\Lambda}_{*}^{+}\mathrm{SU}(2) \times \tilde{\Lambda}^{-}\mathrm{SU}(2) \to \tilde{\Lambda}\mathrm{SU}(2)$.*

This represents a 'linearized' version of the classical *Birkhoff loop group factorization* from [15] (where the splitting was introduced and proved for smooth loops on the unit circle S^1). Note that in [14], the aforementioned theorem was formulated for $\mathrm{SO}(3, \mathbb{R})$, instead of $\mathrm{SU}(2)$. There it was shown that the 'Birkhoff' splitting works for λ on *any straight line of the complex plane*.

The first type of Birkhoff factorization, performed away from a singular set $S_1 \subset D$, allows us to split the extended moving frame $\mathcal{U}^{\lambda}$: $D \to \mathrm{SU}(2)$ into two parts. Recall that the first factor of this splitting is of the form $g_{-} = I + \lambda^{-1}g_{-1} + \lambda^{-2}g_{-2} + \cdots$, while the second factor of the splitting is of the form $g_{+} = g_0 + \lambda g_1 + \lambda^2 g_2 + \cdots$, respectively. Since the 'big cell' is open and $\mathcal{U}^{\lambda}$: $D \to \mathrm{SU}(2)$ is continuous, the set

$$\tilde{D}_1 = \{(x, y);\ \mathcal{U}^{\lambda}(x, y) \text{ belongs to the 'big cell'}\}$$

is open. Note that $(0, 0) \in \tilde{D}_1$. Let $S_1 = D - \tilde{D}_1$ denote the 'singular' set. We have just shown that S_1 is closed and $(0, 0)$ is not an element of the set S_1. Similarly, we have S_2 and $\tilde{D}_2$ for the second splitting.

We can perform the two splittings on the extended frame $\mathcal{U}^{\lambda}$, independently.

Let $\mathcal{U} = \mathcal{U}^{\lambda}$ be the extended normalized moving frame of a pseudospherical surface and let $(x, y) \in D \setminus (S_1 \cup S_2)$. Then, for some uniquely determined $V_{+} \in \Lambda^{+}\mathrm{SU}(2)$, $V_{-} \in \Lambda^{-}\mathrm{SU}(2)$ and $\mathcal{U}_{-} \in \Lambda_{*}^{-}\mathrm{SU}(2)$, $\mathcal{U}_{+} \in \Lambda_{*}^{+}\mathrm{SU}(2)$, $\mathcal{U}$ can be written as

$$\mathcal{U} = \mathcal{U}_{+} \cdot V_{-} = \mathcal{U}_{-} \cdot V_{+}. \tag{14}$$

Here $\mathcal{U}_{-}$ is an element of the form $\mathcal{U}_{-} = I + \lambda^{-1}\mathcal{U}_{-1} + \lambda^{-2}\mathcal{U}_{-2} + \ldots$, while V_{+} is an element of the form $V_{+} = V_0 + \lambda V_1 + \lambda^2 V_2 + \ldots$, respectively. Analogous expressions can be written for $\mathcal{U}_+$ and V_-, respectively. We will show that, starting from data of type Weierstrass, called normalized potentials η^x and η^y, one can obtain the factors $\mathcal{U}_+$ and $\mathcal{U}_-$ as solutions of a simplified ODE system. These two factors represent the genetic material necessary and sufficient to recreate the frame and then the immersed surface via the Sym formula.

THEOREM 5. *Let $\mathcal{U} = \mathcal{U}^\lambda$, $\mathcal{U}_+$ and $\mathcal{U}_-$ be as above. Then the following systems of differential equations are satisfied:*

$$(\mathcal{U}_+)^{-1} \cdot \partial_x \mathcal{U}_+ = -\lambda \cdot \frac{i}{2} \cdot \begin{pmatrix} 0 & \mathrm{e}^{i(\varphi(0,0)-\varphi(x,0))} \\ \mathrm{e}^{-i(\varphi(0,0)-\varphi(x,0))} & 0 \end{pmatrix} \tag{15}$$

with initial condition $\mathcal{U}_+(x = 0) = I$

and

$$(\mathcal{U}_-)^{-1} \cdot \partial_y \mathcal{U}_- = \lambda^{-1} \cdot \frac{i}{2} \cdot \begin{pmatrix} 0 & \mathrm{e}^{-i\varphi(0,y)} \\ \mathrm{e}^{i\varphi(0,y)} & 0 \end{pmatrix}, \tag{16}$$

with initial condition $\mathcal{U}_-(y = 0) = I$.

Moreover, $\mathcal{U}_+$ does not depend on y and $\mathcal{U}_-$ does not depend on x.

In some other words, $\mathcal{U}_+$ and $\mathcal{U}_-$ are solutions of some first-order systems of differential equations in x and y, respectively.

Proof. We will prove the first statement. Proving the other statement is straightforward.

The first Birkhoff splitting implies $\mathcal{U}_+ = \mathcal{U} \cdot V_-^{-1}$, which after differentiation gives

$$\mathrm{d}\mathcal{U}_+ = \mathrm{d}\mathcal{U} \cdot V_-^{-1} - \mathcal{U} \cdot V_-^{-1} \cdot \mathrm{d}V_- \cdot V_-^{-1}, \tag{17}$$

$$\mathcal{U}_+^{-1}\,\mathrm{d}\mathcal{U}_+ = V_-(\mathcal{U}^{-1}\,\mathrm{d}\mathcal{U})V_-^{-1} - \mathrm{d}V_- \cdot V_-^{-1}. \tag{18}$$

The last equality can also be written as

$$\mathcal{U}_+^{-1}\,\mathrm{d}\mathcal{U}_+ = V_-(\mathcal{A}\,\mathrm{d}x + \mathcal{B}\,\mathrm{d}y)V_-^{-1} - \mathrm{d}V_- \cdot V_-^{-1}. \tag{19}$$

We will use the Lax equations. In the last equality, we compare the coefficient of $\mathrm{d}y$ on the left-hand side with the coefficient of $\mathrm{d}y$ on the right-hand side. The left-hand side clearly contains only positive powers of λ, while the coefficient of $\mathrm{d}y$ on the right-hand side contains nonpositive powers of λ only. Thus, $\mathcal{U}_+$ depends exclusively on x.

Let us now consider the coefficient of $\mathrm{d}x$ in the same equality. The left-hand side contains only positive powers of λ, while the one on the right-hand side, due to the λ-dependence of $\mathcal{A}$, contains one term in λ and no terms in λ^k, with $k > 1$. Next, we can restrict to a sufficiently small interval around $(0, 0)$ on the line $y = 0$. Let now $V_- = \tilde{V}_0 + \lambda^{-1}\tilde{V}_1 + \lambda^{-2}\tilde{V}_2 + \cdots = \tilde{V}_0 \cdot T_-$, with $T_- \in \Lambda_*^- \mathrm{SU}(2)$. But since $\mathcal{U}_+^{-1}(x) \cdot \mathcal{U}_+'(x)$ contains only positive powers of λ, we conclude that

$\mathcal{U}_+^{-1}(x) \cdot \mathcal{U}_+'(x)\,\mathrm{d}x = \tilde{V}_0(x, 0) \cdot \omega_1'' \cdot \tilde{V}_0(x, 0)^{-1}$, where ω_1'' is the one from (10). Let us now denote $\tilde{V}_0(x, 0) := V_0$. In order to determine the matrix V_0, one needs to compare the coefficients of the power λ^0 in the same equality. As we pointed out, the left-hand side has positive powers of λ only, while the x-part of right-hand side only contains $-V_0 \cdot \beta_0 \cdot V_0^{-1} - \mathrm{d}V_0 \cdot V_0^{-1}$ as the only term that does not depend on λ, where we denoted

$$\beta_0 = \omega_0'(x, 0) = \frac{i}{2}\begin{pmatrix} \varphi_x(x, 0) & 0 \\ 0 & -\varphi_x(x, 0) \end{pmatrix} \mathrm{d}x.$$

Thus, V_0 is a solution to $\mathrm{d}V_0 = -V_0 \cdot \beta_0$. The solution V_0 of the system must take into account that $\mathcal{U}(0, 0, \lambda) = I$. Thus, $V_0(x) = \mathrm{e}^{\theta(0)-\theta(x)}$, where $\theta(x) := \frac{i}{2}\varphi(x, 0)\sigma_3$. Consequently, we obtain

$$\begin{aligned}(\mathcal{U}_+)^{-1}\mathcal{U}_+'(x) &= -\frac{i}{2}\lambda \cdot V_0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot V_0^{-1} \\ &= -\lambda \cdot \frac{i}{2} \cdot \begin{pmatrix} 0 & \mathrm{e}^{i(\varphi(0,0)-\varphi(x,0))} \\ \mathrm{e}^{-i(\varphi(0,0)-\varphi(x,0))} & 0 \end{pmatrix} \end{aligned} \tag{20}$$

□

DEFINITION 3. We define the *normalized potentials* η^x and η^y via the following

$$(\mathcal{U}_+)^{-1} \cdot \mathcal{U}_+'(x)\,\mathrm{d}x := -\lambda \cdot \eta^x, \tag{21}$$

$$(\mathcal{U}_-)^{-1} \cdot \mathcal{U}_-'(y)\,\mathrm{d}y := -\lambda^{-1} \cdot \eta^y, \tag{22}$$

Clearly, they represent $su(2)$-valued forms in x and y, respectively. Using the theorem we just proved, we obtain the form of the normalized x-potential η^x:

$$\eta^x = \frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{i(\varphi(0,0)-\varphi(x,0))} \\ \mathrm{e}^{-i(\varphi(0,0)-\varphi(x,0))} & 0 \end{pmatrix} \mathrm{d}x \tag{23}$$

By a completely analogous reasoning (the second part of the proof we left to the reader), we obtain the matrix $W_0 = I$ and then the expression of the normalized y-potential:

$$\eta^y = -\frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i\varphi(0,y)} \\ \mathrm{e}^{i\varphi(0,y)} & 0 \end{pmatrix} \mathrm{d}y \tag{24}$$

Note that the normalized potentials η^x and η^y are completely determined by the restrictions of φ to the axes of coordinates. Since $\varphi(x, y)$ is invariant under Lie–Lorentz transformations, these potentials correspond uniquely to each (weakly regular) associate family of surfaces with Gauss curvature -1.

Considering normalized potentials is actually equivalent to giving a Goursat problem for the sine-Gordon hyperbolic system. In the next section, we will use the loop group splitting techniques in order to solve this initial value problem, starting from given normalized potentials.

Gauging the Frame and Its Effect on Potentials

DEFINITION 4. Consider a normalized frame $\mathcal{U}$. For a rotation of smooth angle function $\theta(x, y)$ around e_3, we call *gauged frame* the matrix

$$\hat{\mathcal{U}} = \mathcal{R}_0^{-1} \cdot \mathcal{U} \cdot \mathcal{R},$$

where $\mathcal{R}_0 := \mathcal{R}(0, 0)$.

DEFINITION 5. We define the potentials of the gauged frame $\hat{\mathcal{U}}$, $\hat{\eta}^x$ and $\hat{\eta}^y$, by

$$(\hat{\mathcal{U}}_+)^{-1} \cdot \hat{\mathcal{U}}'_+(x)\,\mathrm{d}x := -\lambda \cdot \hat{\eta}^x, \tag{25}$$

$$(\hat{\mathcal{U}}_-)^{-1} \cdot \hat{\mathcal{U}}'_-(y)\,\mathrm{d}y := -\lambda^{-1} \cdot \hat{\eta}^y, \tag{26}$$

where

$$\hat{\mathcal{U}} = \hat{\mathcal{U}}_+ \hat{V}_- = \hat{\mathcal{U}}_- \hat{V}_+ \tag{27}$$

represent the Birkhoff splittings of the gauged frame $\hat{\mathcal{U}}$.

PROPOSITION 3. *For a normalized frame $\mathcal{U}$ and its gauge-transformed $\hat{\mathcal{U}}$, the corresponding potentials satisfy the relations*

$$\hat{\eta}^x = \mathcal{R}_0^{-1} \cdot \eta^x \cdot \mathcal{R}_0, \qquad \hat{\eta}^y = \mathcal{R}_0^{-1} \cdot \eta^y \cdot \mathcal{R}_0. \tag{28}$$

Proof. A completely straightforward computation, based on easy matrix manipulations and the uniqueness of the splittings yield our formulae.

Now recall the explicit formulae (23) and (24) of the normalized potentials η^x and η^y, respectively. The asymmetry in the expressions came from ‘normalizing’ the original orthonormal potential F, that is, rotating it by the angle $(\varphi(x, y))/2$. In order to correct that, we have to gauge the frame appropriately, that is rotate it ‘back’ with the angle $-(\varphi(x, y))/2$, while making sure that the initial condition $\mathcal{U}(0, 0, \lambda) = I$ is still satisfied. □

PROPOSITION 4. *By gauging the normalized extended frame $\mathcal{U}$ via the rotation $\mathcal{R}$ of angle $\theta := -\varphi(x, y)/2$, we obtain, modulo a constant rotation, the original orthonormal frame $\hat{\mathcal{U}} = F = (e_1, e_2, N) = F(x, y, 1)$ and its extension $F(x, y, \lambda)$ via coordinate transformation. The potentials that correspond to the frame F are*

$$\tilde{\eta}^x = \mathcal{R}_0^{-1} \cdot \eta^x \cdot \mathcal{R}_0, \qquad \tilde{\eta}^y = \mathcal{R}_0^{-1} \cdot \eta^y \cdot \mathcal{R}_0. \tag{29}$$

Proof. Based on the previous proposition, the proof is straightforward. Let us consider the normalized frame $\mathcal{U}$, whose gauge correspondent is $\hat{\mathcal{U}} = F$. The potentials are linked via the aforementioned formula, where $\mathcal{R}_0$ represent the specific rotation of constant angle $\theta(0, 0) = -(\varphi(0, 0))/2$.

Consequently, we obtain the potentials corresponding to the orthonormal frame F. Denoting $\varphi_0 := \varphi(0,0)$, the potentials corresponding to the frame F are given by

$$\tilde{\eta}^x = \frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i(\varphi(x,0)-\varphi_0)} \\ \mathrm{e}^{i(\varphi(x,0)-\varphi_0)} & 0 \end{pmatrix} \mathrm{d}x,$$
$$\tilde{\eta}^y = -\frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i(\varphi(0,y)-\varphi_0)} \\ \mathrm{e}^{i(\varphi(0,y)-\varphi_0)} & 0 \end{pmatrix} \mathrm{d}y. \tag{30}$$

Remark the symmetry of the two potentials of the frame F. This is an advantage over the potentials corresponding to the normalized frame $\mathcal{U}$.

These symmetric, 'denormalized', potentials are of a simpler, more general form that we can use for the unconstrained pair of type Weierstrass.

Note that at the origin $x = y = 0$, the two potentials equal $i\sigma_1/2$ and $-i\sigma_1/2$, respectively. □

Constructing Pseudospherical Surfaces from Given Potentials

We now introduce symmetric potentials ξ^x and ξ^y of a general form. We will show that there is a one-to-one correspondence between these potentials and associated families of pseudospherical immersions.

DEFINITION 6. Let $\alpha\colon D^x = \{x \mid (x,0) \in D\} \to \mathbb{R}$, $\beta\colon D^y = \{y \mid (0,y) \in D\} \to \mathbb{R}$ be smooth functions, such that $\alpha(0) = \beta(0)$. Let

$$\xi^x = \frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i(\alpha(x)-\alpha(0))} \\ \mathrm{e}^{i(\alpha(x)-\alpha(0))} & 0 \end{pmatrix} \mathrm{d}x,$$
$$\xi^y = -\frac{i}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i(\beta(y)-\beta(0))} \\ \mathrm{e}^{i(\beta(y)-\beta(0))} & 0 \end{pmatrix} \mathrm{d}y. \tag{31}$$

We call ξ^x and ξ^y *symmetric potentials* and we use the same terminology and notations for their 3×3 correspondents.

We are now ready to prove the following theorem.

THEOREM 6. *Let $\hat{\mathcal{U}}_+(y,\lambda) \in \tilde{\Lambda}^*_- \mathrm{SO}(3)_P$ and $\hat{\mathcal{U}}_-(x,\lambda) \in \tilde{\Lambda}^*_+ \mathrm{SO}(3)_P$ be the respective solutions of the following initial value problems:*

$$(\hat{\mathcal{U}}_+)^{-1}\hat{\mathcal{U}}'_+(x)\,\mathrm{d}x = -\lambda\xi^x, \quad \hat{\mathcal{U}}_+(x=0) = I, \tag{32}$$

$$(\hat{\mathcal{U}}_-)^{-1}\hat{\mathcal{U}}'_-(y)\,\mathrm{d}y = -\lambda^{-1}\xi^y, \quad \hat{\mathcal{U}}_-(y=0) = I, \tag{33}$$

where ξ^x and ξ^y are given by (31). Consider the set

$$\tilde{D} := \{(x,y) \in D^x \times D^y;\ \hat{\mathcal{U}}_-(y)\cdot\hat{\mathcal{U}}_+(x) \in \tilde{\Lambda}^*_- \mathrm{SO}(3)_P \cdot \tilde{\Lambda}^*_+ \mathrm{SO}(3)_P\}.$$

In $\tilde{D}$, we perform the Birkhoff splitting

$$\hat{\mathcal{U}}_-^{-1}(y) \cdot \hat{\mathcal{U}}_+(x) = \hat{V}_+(x, y) \cdot \hat{V}_-^{-1}(x, y), \tag{34}$$

where $\hat{V}_+ \in \tilde{\Lambda}_+^* \mathrm{SO}(3)_P$ and $\hat{V}_- \in \tilde{\Lambda}_-^* \mathrm{SO}(3)_P$

Let

$$\hat{\mathcal{U}} := \hat{\mathcal{U}}_- \hat{V}_+ = \hat{\mathcal{U}}_+ \hat{V}_- \tag{35}$$

Then, $\hat{\mathcal{U}}$ represents the orthonormal frame F of an associated family of pseudospherical surfaces in Tchebychev net, whose Tchebychev angle $\varphi(x, y)$ verifies the conditions $\varphi(x, 0) = \alpha(x)$ and $\varphi(0, y) = \beta(y)$.

Proof. Proposition 1 shows the existence and uniqueness of a solution φ to the initial value problem $\varphi_{xy} = \sin\varphi$, $\varphi(x, 0) = \alpha(x)$, $\varphi(0, y) = \beta(y)$. Let $\hat{\mathcal{U}} = F$ be the orthonormal frame corresponding to the Tchebychev parametrization of angle φ. Formulae (30) give the symmetric potentials $\tilde{\eta^x}$ and $\tilde{\eta^y}$ corresponding to this frame F, as being identical with the symmetric potentials ξ^x and ξ^y assigned by (31).

In order to obtain φ explicitly as a solution, we first integrate (uniquely) (25) and (26), and obtain $\hat{\mathcal{U}}_+$ and $\hat{\mathcal{U}}_+$. Since $\varphi(0, 0) = \alpha(0) = \beta(0)$ is provided, so is $\mathcal{R}_0$. We use $\hat{\mathcal{U}}_- = \mathcal{R}_0^{-1} \mathcal{U}_- \mathcal{R}_0$ and $\hat{\mathcal{U}}_+ = \mathcal{R}_0^{-1} \mathcal{U}_+ \mathcal{R}_0$ to obtain $\mathcal{U}_+$ and $\mathcal{U}_-$. Next, the Birkhoff splitting

$$\mathcal{U}_-^{-1}(y) \cdot \mathcal{U}_+(x) = V_+(x, y) \cdot V_-^{-1}(x, y), \tag{36}$$

provides V_+, V_- uniquely. Hence, the normalized frame $\mathcal{U} = \mathcal{U}_- \cdot V_+$ via formula (27), is obtained in a unique way. We apply the Sym formula, and obtain the associated family of immersions

$$\psi^\lambda = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{U}^\lambda (\mathcal{U}^\lambda)^{-1}, \tag{37}$$

where $\lambda = \mathrm{e}^t$. Finally, the map $\varphi(x, y)$ represents the angle of this parametrization, and can be written explicitly. □

Remark 1. The K-Lab contains a numerical implementation of this algorithm. Starting from two arbitrary potentials of the form (31) (i.e., pair of initial functions $\alpha(x)$ and $\alpha(y)$), it computes and models the corresponding family of associated surfaces.

Note that factorizations are possible only in the ‘big cell’, which is an open and dense subset of the domain. The K-lab algorithm contains an in-built numerical method that ‘jumps’ the singularities once they are detected, and thus allows construction and visualization of all regular patches.

COROLLARY 2. *The correspondence between the pair of symmetric potentials, and the family of associated pseudospherical surfaces of angle φ is a bijection.*

Proof. Let Σ be the map from the set of associated families of pseudospherical surfaces in Tchebychev net into the set of all pairs of potentials of general form (31). In essence, Σ maps the angle φ to the pair of potentials from (30), which in particular are of the form (31).

On the other hand, we have a reverse procedure. Theorem 6 constructs a map from any pair of potentials (31) to a certain family of immersions of angle φ, via the frame $\hat{\mathcal{U}}$. We will denote this map by Ω. The proof of Theorem 6 shows that the map Ω is well defined.

The construction in Theorem 6 shows that $\Sigma \circ \Omega = id$, which is the same with showing that every pair of potentials (31) is of the form (30), for a uniquely determined angle φ that defines a family of pseudospherical immersions ψ^{λ}.

The uniqueness of the construction method from Theorem 6 also shows that $\Omega \circ \Sigma = \mathrm{id}$.

This completes the proof of the Corollary. □

EXAMPLE (Amsler's Surface). In Tchebychev net parametrization, this surface corresponds to an angle $\varphi(x, y)$ that is constant on both x- and y-axes. For some well-known surfaces, like the pseudosphere, the Tchebychev angle $\varphi(x, y)$ is easily written as a trigonometric function of x and y. This is not the case for the Amsler surface. On the other hand, we can rewrite the sine-Gordon equation in a very simple form [13]: Let $t := xy$ with $(x, y) \in D = \mathbb{R}^2$. If we express $\varphi(x, y) = h(xy)$, with $h\colon \mathbb{R} \to (0, \pi)$ a differentiable function, then For Amsler surfaces, the sine-Gordon equation is written as the second-order differential equation

$$th''(t) + h'(t) = \sin(h(t)).$$

A change of function $w = \mathrm{e}^{i\psi}$ transforms the aforementioned equation into the so-called third Painleve equation. Since $\varphi(x, y)$ is smooth, a straightforward calculation yields

$$\varphi(0, 0) = \varphi(x, 0) = \varphi(0, y) := \varphi_0$$

for every pair $(x, y) \in D$. Amsler [11] showed that the solution $\varphi(x, y) = h(xy)$ oscillates near π when $t > 0$ and near 0 when $t < 0$. He also proved that the surface has two cuspidal edges corresponding to $\varphi = 0$ and $\varphi = \pi$, respectively.

We note the two straight lines contained in the Amsler surface, corresponding to $x = 0$ and $y = 0$. As an obvious consequence of the angle being constant along the axes, the symmetric potentials (50) of the Amsler surface can be written as

$$\tilde{\eta}^x = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{d}x, \qquad \tilde{\eta}^y = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{d}y. \tag{38}$$

For an interactive visualization of Amsler surfaces obtained using the generalized Weierstrass representation (60, 61) and computational loop-group splittings, see http://www.gang.umass.edu/gallery/k/kgallery0201.html.

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