On the differential invariants of a family of diffusion equations

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Abstract

The equivalence transformation algebra $L_{\mathcal{E}}$ for the class of equations $u_t - u_{xx} = f(u, u_x)$ is obtained. After getting the differential invariants with respect to $L_{\mathcal{E}}$, some results which allow to linearize a subclass of equations are showed. Equations like the standard deterministic KPZ equation fall in this subclass.

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1. Introduction

In some previous papers [1], [2], [3] the differential invariants for the family of equations $u_{tt} - u_{xx} = f(u,\, u_t,\, u_x)$ have been obtained and applied in order to characterize some linearizable subclasses of that equations.

Here, we consider the following diffusion equations:

$$u_t - u_{xx} = f(u,\, u_x), \tag{1.1}$$

which arise in several problems of mathematical physics.

By using the invariance Lie infinitesimal criterion [4], we construct the algebra $L_{\mathcal{E}}$ of the equivalence transformations. These transformations have the property to change any element of a family of PDEs to a PDE which belongs to the same family. An equivalence transformation maps solutions of an equation of the family to solutions of the transformed equation.

Following the method proposed by N.H. Ibragimov in [5], [6] and successively applied in [7], we calculate the differential invariants with respect to the equivalence transformations of the family (1.1).

Starting from these results, we characterize a subclass of equations (1.1) which can be linearized through an equivalence transformation. In this subclass falls the standard deterministic Kardar-Parisi-Zhang (KPZ) equation [8], [9] which models the relaxation of an initially rough surface to a flat surface.

The outline of the paper is the following. In Section 2, we obtain the infinitesimal equivalence generator of equations (1.1). In Section 3, we look for differential invariants and, by following the infinitesimal method [5], [6], we show that the family of equations (1.1) does not admit differential invariants of order zero and one, while second order differential invariants are found. Finally, in Section 4, these last ones are used in order to characterize a subclass of the family (1.1) which can be mapped, by an equivalence transformation, in the Fourier's equation. The conclusions are given in Section 5.

2. Algebra $\mathbf{L}_{\mathcal{E}}$

We recall that a transformation of the type

$$t = t(\tau,\, \sigma,\, v), \quad x = x(\tau,\, \sigma,\, v), \quad u = u(\tau,\, \sigma,\, v), \tag{2.1}$$

which is locally a C^∞-diffeomorphism and changes the original equation into a new equation having the same differential structure but a different form of the function f, is an equivalence transformation [4] (hereafter ET) for the equations (1.1). An invariance transformation can be regarded as particular ET such that the transformed function f has the same form. In the following we consider only continuous groups of equivalence transformations.

The direct search for the equivalence transformations through the finite form of the transformation is connected with considerable computational difficulties. The use of the Lie infinitesimal criterion, suggested by Ovsiannikov [4], gives an algorithm to find the infinitesimal generators of the ETs that overcame these problems.

In order to obtain a continuous group of ETs of equations (1.1), we consider, by following [4], the arbitrary function f as a dependent variable and apply the Lie infinitesimal invariance criterion to the following system:

$$\begin{aligned} u_t - u_{xx} &= f, \\ f_t = f_x = f_{u_t} &= 0, \end{aligned} \tag{2.2}$$

where the last three equations of (2.2) are usually called *auxiliary equations* and give the independence of f on t, x and u_t.

The infinitesimal equivalence generator Y has the following form:

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \mu \frac{\partial}{\partial f}, \tag{2.3}$$

where ξ^1, ξ^2 and η are sought depending on t, x and u, while μ depends on t, x, u, u_t, u_x and f, the components ζ_1 and ζ_2, as known, are given by

$$\zeta_1 = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \quad \zeta_2 = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) \tag{2.4}$$

The operators D_t and D_x denote the total derivatives with respect to t and x:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + ..., \tag{2.5}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + ... \tag{2.6}$$

The prolongation of operator (2.3), which we need in order to require the invariance of (2.2), is

$$\widetilde{Y} = Y + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \omega_t\frac{\partial}{\partial f_t} + \omega_x\frac{\partial}{\partial f_x} + \omega_{u_t}\frac{\partial}{\partial f_{u_t}}, \tag{2.7}$$

where (see e.g. [7], [10])

$$\zeta_{22} = D_x(\zeta_2) - u_{tt}D_x(\xi^1) - u_{xx}D_x(\xi^2), \tag{2.8}$$
$$\omega_t = \widetilde{D}_t(\mu) - f_u\widetilde{D}_t(\eta) - f_{u_x}\widetilde{D}_t(\zeta_2), \tag{2.9}$$
$$\omega_x = \widetilde{D}_x(\mu) - f_u\widetilde{D}_x(\eta) - f_{u_x}\widetilde{D}_x(\zeta_2), \tag{2.10}$$
$$\omega_{u_t} = \widetilde{D}_{u_t}(\mu) - f_u\widetilde{D}_{u_t}(\eta) - f_{u_x}\widetilde{D}_{u_t}(\zeta_2), \tag{2.11}$$

while $\widetilde{D}_t$, $\widetilde{D}_x$ and $\widetilde{D}_{u_t}$ are defined by:

$$\widetilde{D}_t = \frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + f_{tt}\frac{\partial}{\partial f_t} + f_{tx}\frac{\partial}{\partial f_x} + ..., \tag{2.12}$$
$$\widetilde{D}_x = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + f_{tx}\frac{\partial}{\partial f_t} + f_{xx}\frac{\partial}{\partial f_x} + ..., \tag{2.13}$$
$$\widetilde{D}_{u_t} = \frac{\partial}{\partial u_t} + f_{u_t}\frac{\partial}{\partial f} + f_{tu_t}\frac{\partial}{\partial f_t} + f_{xu_t}\frac{\partial}{\partial f_x} + ... \tag{2.14}$$

Applying the operator (2.7) to the system (2.2) and following the well known algorithm (see e. g. [10], [11]) we obtain

$$\begin{aligned} Y \;=\;& (c_0 + c_1 t)\frac{\partial}{\partial t} + \left(\frac{1}{2}c_1 x + c_2 t + c_3\right)\frac{\partial}{\partial x} + \varphi(u)\frac{\partial}{\partial u} + \\ &+ (-c_1 u_t - c_2 u_x + \varphi' u_t)\frac{\partial}{\partial u_t} + \left(-\frac{1}{2}c_1 u_x + \varphi' u_x\right)\frac{\partial}{\partial u_x} + \\ &+ \left(-c_1 f - c_2 u_x + \varphi' f - \varphi'' u_x^2\right)\frac{\partial}{\partial f}, \end{aligned} \tag{2.15}$$

where c_0, c_1, c_2 and c_3 are arbitrary constants, φ is an arbitrary function of u and the prime denotes the differentiation with respect to u. So, we have found that the Lie algebra $L_{\mathcal{E}}$ for the class of equations (1.1) is infinite-dimensional and generates an infinite continuous group $G_{\mathcal{E}}$ of equivalence transformations spanned by the following operators:

$$
\begin{aligned}
&Y_0 = \frac{\partial}{\partial t}, \qquad Y_1 = t\frac{\partial}{\partial t} + \frac{1}{2}\,x\frac{\partial}{\partial x} - f\frac{\partial}{\partial f} - u_t\frac{\partial}{\partial u_t} - \frac{1}{2}\,u_x\frac{\partial}{\partial u_x},\\
&Y_2 = t\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial f} - u_x\frac{\partial}{\partial u_t}, \qquad Y_3 = \frac{\partial}{\partial x},\\
&Y_\varphi = \varphi\frac{\partial}{\partial u} + \left(\varphi' f - \varphi'' u_x^2\right)\frac{\partial}{\partial f} + \varphi' u_t\frac{\partial}{\partial u_t} + \varphi' u_x\frac{\partial}{\partial u_x}.
\end{aligned}
$$

3. Search for differential invariants

Following [5]-[7], we recall that, for the family of equations (1.1), a *differential invariant of order s* is a function J, of the independent variables t, x, the dependent variable u and its derivatives u_t, u_x, as well as of the function f an its derivatives of maximal order s, invariant with respect to the equivalence group $G_\mathcal{E}$.

3. 1 Differential invariants of order zero.

Here we search for functions

$$J = J(t, x, u, u_t, u_x, f) \tag{3.1}$$

satisfying the invariant condition $Y(J) = 0$.

¿From the invariant tests $Y_0(J) = 0$ and $Y_3(J) = 0$, easily follows that J must depend only on u, u_t, u_x and f.

¿From invariance test $Y_\varphi(J) = 0$, after observing that, being φ an arbitrary function, Y_φ can be splitted in the following three operators

$$\hat{Y}_\varphi = \frac{\partial}{\partial u}, \quad \hat{Y}_{\varphi'} = f\frac{\partial}{\partial f} + u_t\frac{\partial}{\partial u_t} + u_x\frac{\partial}{\partial u_x}, \quad \hat{Y}_{\varphi''} = -u_x^2\frac{\partial}{\partial f},$$

it is a simple matter to get

$$J = J(q), \tag{3.2}$$

with

$$q = \frac{u_t}{u_x}. \tag{3.3}$$

¿From $Y_2(J) = 0$ we get $J_q = 0$, hence the equations (1.1) do not possess differential invariants of zero order.

3. 2 Differential invariants of first order.

The differential invariants of first order involve f_u and f_{u_x} also, so we need the following first prolongation of operator Y:

$$Y^{(1)} \;=\; Y + \omega_u \frac{\partial}{\partial f_u} + \omega_{u_x} \frac{\partial}{\partial f_{u_x}}, \tag{3.4}$$

which, after computing ω_u and ω_{u_x} likewise (2.9)- (2.14), can be written as:

$$\begin{aligned} Y^{(1)} \;&=\; Y + \left(-c_1 f_u + \varphi'' f - \varphi'' u_x f_{u_x} - \varphi''' u_x^2\right) \frac{\partial}{\partial f_u} + \\ &\quad - \left(\frac{1}{2}\, c_1 f_{u_x} + c_2 + 2\varphi'' u_x\right) \frac{\partial}{\partial f_{u_x}}. \end{aligned} \tag{3.5}$$

By observing that:

$$Y_0^{(1)} = Y_0, \quad Y_3^{(1)} = Y_3 \quad \hat{Y}_\varphi^{(1)} = \hat{Y}_\varphi$$

it is a simple matter to ascertain that at this step the search is reduced to look for invariant functions of the form

$$J = J(u_t, u_x, f, f_u, f_{u_x}) \tag{3.6}$$

with respect to the following operators

$$\begin{aligned} Y_1^{(1)} \;&=\; t\frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x} - f\frac{\partial}{\partial f} - u_t \frac{\partial}{\partial u_t} - \frac{1}{2} u_x \frac{\partial}{\partial u_x} - f_u \frac{\partial}{\partial f_u} + \\ &\quad - \frac{1}{2} f_{u_x} \frac{\partial}{\partial f_{u_x}}, \end{aligned} \tag{3.7}$$

$$Y_2^{(1)} \;=\; t\frac{\partial}{\partial x} - u_x \frac{\partial}{\partial f} - u_x \frac{\partial}{\partial u_t} - \frac{\partial}{\partial f_{u_x}}, \tag{3.8}$$

$$\hat{Y}_{\varphi'}^{(1)} \;=\; f\frac{\partial}{\partial f} + u_t \frac{\partial}{\partial u_t} + u_x \frac{\partial}{\partial u_x}, \tag{3.9}$$

$$\hat{Y}_{\varphi''}^{(1)} \;=\; -u_x^2 \frac{\partial}{\partial f} + (f - u_x f_{u_x}) \frac{\partial}{\partial f_u} - 2u_x \frac{\partial}{\partial f_{u_x}}, \tag{3.10}$$

$$\hat{Y}_{\varphi'''}^{(1)} \;=\; -u_x^2 \frac{\partial}{\partial f_u}. \tag{3.11}$$

By requiring the invariance of J with respect to the operator $\hat{Y}^{(1)}_{\varphi'''}$ it follows

$$J = J(u_t, u_x, f, f_{u_x}), \tag{3.12}$$

while the invariant test, applied to (3.12)

$$\hat{Y}^{(1)}_{\varphi''}(J) \equiv u_x \frac{\partial J}{\partial f} + 2\,\frac{\partial J}{\partial f_{u_x}} = 0, \tag{3.13}$$

yields that

$$J = J\left(u_t, u_x, p_1\right), \tag{3.14}$$

where

$$p_1 = \frac{f}{u_x} - \frac{f_{u_x}}{2}. \tag{3.15}$$

Acting by operator $\hat{Y}^{(1)}_{\varphi'}$ on the invariant (3.14), one obtains that

$$\hat{Y}^{(1)}_{\varphi'}(J) \equiv u_t \frac{\partial J}{\partial u_t} + u_x \frac{\partial J}{\partial u_x} = 0 \tag{3.16}$$

and hence

$$J = J\left(p_1,\, p_2\right), \tag{3.17}$$

with

$$p_2 = \frac{u_t}{u_x}. \tag{3.18}$$

Finally, from the invariant test $Y^{(1)}_2(J) = 0$ we get

$$J = J(p), \tag{3.19}$$

where

$$p = p_1 - 2p_2 = \frac{f - u_x\, f_{u_x}}{2u_x} - 2\frac{u_t}{u_x}, \tag{3.20}$$

and from $Y^{(1)}_1(J) = 0$ it follows

$$Y^{(1)}_1(J) \equiv \left(-\frac{1}{2}\frac{f}{u_x} + \frac{u_t}{u_x} + \frac{1}{4} f_{u_x}\right) \frac{\partial J}{\partial p} = 0. \tag{3.21}$$

Provided that $-\frac{1}{2}\frac{f}{u_x} + \frac{u_t}{u_x} + \frac{1}{4} f_{u_x} \neq 0$, we get $\frac{\partial J}{\partial p} = 0$. Hence the equations (1.1) do not admit differential invariants of first order.

3. 3 Differential invariants of second order.

In the search for second order differential invariants, because of the function J is sought as depending from f_{uu}, f_{uu_x} and $f_{u_xu_x}$ also, we need the following second prolongation of operator Y:

$$Y^{(2)} = Y^{(1)} + \omega_{uu}\frac{\partial}{\partial f_{uu}} + \omega_{uu_x}\frac{\partial}{\partial f_{uu_x}} + \omega_{u_xu_x}\frac{\partial}{\partial f_{u_xu_x}} \tag{3.22}$$

where [7], [10]:

$$\omega_{uu} = \widetilde{D}_u(\omega_u) - f_{uu}\widetilde{D}_u(\eta) - f_{uu_x}\widetilde{D}_u(\zeta_2), \tag{3.23}$$

$$\omega_{uu_x} = \widetilde{D}_{u_x}(\omega_u) - f_{uu}\widetilde{D}_{u_x}(\eta) - f_{uu_x}\widetilde{D}_{u_x}(\zeta_2), \tag{3.24}$$

$$\omega_{u_xu_x} = \widetilde{D}_{u_x}(\omega_{u_x}) - f_{uu_x}\widetilde{D}_{u_x}(\eta) - f_{u_xu_x}\widetilde{D}_{u_x}(\zeta_2). \tag{3.25}$$

After some calculations we get

$$\begin{aligned} Y^{(2)} \quad &= Y^{(1)} - (\varphi' f_{u_xu_x} + 2\varphi'')\frac{\partial}{\partial f_{u_xu_x}} + \\ &- \left(\frac{1}{2}c_1 f_{uu_x} + \varphi' f_{uu_x} + \varphi'' u_x f_{u_xu_x} - 2\varphi''' u_x\right)\frac{\partial}{\partial f_{uu_x}} + \\ &+ \left[-c_1 f_{uu} - \varphi' f_{uu} + \varphi''(f_u - 2u_x f_{uu_x}) + \varphi'''(f - u_x f_{u_x}) - \varphi^{IV} u_x^2\right]\frac{\partial}{\partial f_{uu}}. \end{aligned} \tag{3.26}$$

By observing that:

$$Y_0^{(2)} = Y_0, \quad Y_3^{(2)} = Y_3 \quad \hat{Y}_\varphi^{(2)} = \hat{Y}_\varphi$$

we ascertain that we must look for invariant functions of the form

$$J = J(u_t, u_x, f, f_u, f_{u_x}, f_{uu}, f_{uu_x}, f_{u_xu_x}) \tag{3.27}$$

which are invariant with respect to the following operators:

$$Y_1^{(2)} = Y_1^{(1)} - f_{uu}\frac{\partial}{\partial f_{uu}} - \frac{1}{2}f_{uu_x}\frac{\partial}{\partial f_{uu_x}}, \tag{3.28}$$

$$Y_2^{(2)} = Y_2^{(1)}, \tag{3.29}$$

$$\hat{Y}_{\varphi'}^{(2)} = Y_{\varphi'}^{(1)} - f_{uu}\frac{\partial}{\partial f_{uu}} - f_{uu_x}\frac{\partial}{\partial f_{uu_x}} - f_{u_xu_x}\frac{\partial}{\partial f_{u_xu_x}}, \tag{3.30}$$

$$\hat{Y}^{(2)}_{\varphi''} = Y^{(1)}_{\varphi''} + (f_u - 2u_x f_{uu_x})\frac{\partial}{\partial f_{uu}} - u_x f_{u_x u_x}\frac{\partial}{\partial f_{uu_x}} - 2\frac{\partial}{\partial f_{u_x u_x}}, \quad (3.31)$$

$$\hat{Y}^{(2)}_{\varphi'''} = Y^{(1)}_{\varphi'''} + (f - u_x f_{u_x})\frac{\partial}{\partial f_{uu}} - 2u_x\frac{\partial}{\partial f_{uu_x}}, \quad (3.32)$$

$$\hat{Y}^{(2)}_{\varphi^{IV}} = -u_x^2\frac{\partial}{\partial f_{uu}}. \quad (3.33)$$

The invariance condition $\hat{Y}^{(2)}_{\varphi^{IV}}(J) = 0$ implies

$$J = J(u_t, u_x, f, f_u, f_{u_x}, f_{uu_x}, f_{u_x u_x}) \quad (3.34)$$

while $\hat{Y}^{(2)}_{\varphi'''}(J) = 0$ yields

$$J = J(u_t, u_x, f, f_{u_x}, f_{u_x u_x}, p_1), \quad (3.35)$$

where

$$p_1 = \frac{f_u}{u_x} - \frac{f_{uu_x}}{2}. \quad (3.36)$$

Likewise, from $Y^{(2)}_2(J) = 0$ we obtain

$$J = J(u_x, f_{u_x u_x}, p_1, p_2, p_3), \quad (3.37)$$

where

$$p_2 = f - u_t, \qquad p_3 = \frac{f}{u_x} - f_{u_x}, \quad (3.38)$$

and from $\hat{Y}^{(2)}_{\varphi'}(J) = 0$,

$$J = J(q_1, q_2, q_3, q_4), \quad (3.39)$$

where

$$q_1 = f_{u_x u_x}(f - u_t), \quad q_2 = \left(\frac{f_u}{u_x} - \frac{f_{uu_x}}{2}\right)(f - u_t), \quad (3.40)$$

$$q_3 = p_3 = \frac{f}{u_x} - f_{u_x}, \quad q_4 = u_x f_{u_x u_x}. \quad (3.41)$$

By applying the operator $\hat{Y}^{(2)}_{\varphi''}$ to the differential invariant given by (3.39), taking into account (3.40-3.41), we get

$$\left(-2\frac{q_1}{q_4} - q_4\right)\frac{\partial J}{\partial q_1} + \left(\frac{1}{2}q_1 - \frac{q_2 q_4}{q_1} + \frac{q_1 q_3}{q_4}\right)\frac{\partial J}{\partial q_2} + \frac{\partial J}{\partial q_3} - 2\frac{\partial J}{\partial q_4} = 0. \quad (3.42)$$

The corresponding characteristic equations give

$$J = J(r_1, r_2, r_3), \tag{3.43}$$

where

$$\begin{aligned} r_1 &= \frac{f}{u_x} - f_{u_x} + \frac{1}{2}u_x f_{u_x u_x}, & (3.44)\\ r_2 &= f_u - \frac{1}{2}u_x f_{uu_x} + \frac{1}{2}f f_{u_x u_x} - \frac{1}{2}u_x f_{u_x} f_{u_x u_x} + \frac{1}{4}u_x^2 f_{u_x u_x}^2, & (3.45)\\ r_3 &= \frac{f - u_t}{u_x} - \frac{1}{2}u_x f_{u_x u_x}. & (3.46)\end{aligned}$$

Finally, the invariant test

$$Y_1^{(2)}(J) = 0, \tag{3.47}$$

after some calculations, yields

$$r_1 \frac{\partial J}{\partial r_1} + 2r_2 \frac{\partial J}{\partial r_2} + r_3 \frac{\partial J}{\partial r_3} = 0. \tag{3.48}$$

¿From the corresponding characteristic equations, provided that

$$2f - 2u_t - u_x^2 f_{u_x u_x} \neq 0, \tag{3.49}$$

we get that the general form of second order differential invariants of equation (1.1) is

$$J = J(\lambda_1, \lambda_2), \tag{3.50}$$

with λ_1 and λ_2 given by

$$\begin{aligned} \lambda_1 &= \frac{2f - 2u_x f_{u_x} + u_x^2 f_{u_x u_x}}{2f - 2u_t - u_x^2 f_{u_x u_x}}, & (3.51)\\ \lambda_2 &= \frac{\left(4f_u - 2u_x f_{uu_x} + 2f f_{u_x u_x} - 2u_x f_{u_x} f_{u_x u_x} + u_x^2 f_{u_x u_x}^2\right) u_x^2}{(2f - 2u_t - u_x^2 f_{u_x u_x})^2}. & (3.52)\end{aligned}$$

4. Some Applications

Here we wish use the second order invariants λ_1 and λ_2 in order to bring nonlinear equations of the class (1.1) in linear form by using the equivalence transformations of the admitted group $G_{\mathcal{E}}$.

The search for transformations mapping a non linear differential equation in a linear differential equation has interested several authors. In particular S. Kumei and G. W. Bluman in their pionering paper [12] gave some necessary and sufficient conditions that, by examining the invariance algebra, allow to affirm whether a nonlinear equation is trasformable in linear form.

It is worthwhile noticing that the Kumei-Bluman algorithm (see also [13]) constructing the linearizing map, based on the existence of an admitted infinite parameter Lie group transformations, does not require the knowledge, *a priori*, of a specific linear target equation. The target come out in a natural way during the developments of the algorithm. Here, instead, we search the nonlinear equations of the class (1.1) that can be mapped by an equivalence transformation in a linear equation of the subclass

$$v_\tau - v_{\sigma\sigma} = k_0 v_\sigma, \tag{4.1}$$

with $k_0 = const.$

That is, once fixed *a priori* the target (4.1) we characterize the whole set of equations (1.1) which can be mapped in (4.1).

For the subclass (4.1) the differential invariants λ_1 and λ_2 are zero. So, taking into account (3.51- 3.52), we search the functional forms of $f(u, u_x)$ for which

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0. \end{cases} \tag{4.2}$$

Then, solving

$$2f - 2u_x f_{u_x} + u_x^2 f_{u_x u_x} = 0, \tag{4.3}$$

we get

$$f = u_x^2\, h(u) + h_1(u) u_x \tag{4.4}$$

where h and h_1 are arbitrary functions of u.

By requiring that

$$\lambda_2 \,|_{f = u_x^2\, h(u) + h_1(u) u_x} = 0, \tag{4.5}$$

we get

$$h_1(u) = h_0 \tag{4.6}$$

where h_0 is a constant.

We are able, now, to affirm:

Theorem 1 *An equation belonging to the class (1.1) can be transformed in a linear equation of the form (4.1) by an ET generated by (2.15) if and only if the function f is given by*

$$f = u_x^2\, h(u) + h_0 u_x. \tag{4.7}$$

Proof. From the eqns. (4.3) and (4.5) it follows that the condition (4.7) is necessary.

In order to demonstrate that it is sufficient, we must show that it exists at least a ET transforming the equations

$$u_t - u_{xx} = u_x^2\, h(u) + h_0 u_x \tag{4.8}$$

in (4.1).

The finite form of the ETs generated by (2.15) is:

$$t = \tau\, e^{-\varepsilon_1} - \varepsilon_0, \quad x = (\sigma - \tau\varepsilon_2 - \varepsilon_3)e^{-\frac{1}{2}\varepsilon_1}, \quad u = \psi(v), \tag{4.9}$$

where ψ is an arbitrary function, with $\psi'(v) \neq 0$, and ε_i are arbitrary parameters.

By applying the trasformation (4.9) to the equations (4.8), we get

$$v_\tau - v_{\sigma\sigma} = v_\sigma^2\, \frac{\psi'^2\, h(\psi(v)) + \psi''}{\psi'} + \left(h_0 e^{-\frac{1}{2}\varepsilon_1} - \varepsilon_2\right) v_\sigma. \tag{4.10}$$

By choosing as $\psi(v)$ a solution of ODE

$$\frac{\psi'^2\, h(\psi(v)) + \psi''}{\psi'} = 0, \tag{4.11}$$

the transformed equation (4.10) takes the linear form (4.1) where $k_0 = h_0 e^{-\frac{1}{2}\varepsilon_1} - \varepsilon_2$. □

It is a simple matter to show that it is possible to choose the arbitrary parameters ε_i in order to make $k_0 = 0$. So we can affirm, by assuming, for sake of simplicity, $\varepsilon_1 = 0$ and $\varepsilon_2 = h_0$:

Corallary 1 *The group of ETs*

$$t = \tau - \varepsilon_0, \quad x = \sigma - \tau h_0 - \varepsilon_3, \quad u = H^{-1}(c_0 v + c_1), \tag{4.12}$$

with H^{-1} *denoting the inverse function of* $H(\psi) = \int^{\psi} e^{\int^{w} h(z)dz} dw$ *and with* $c_0 \neq 0$, c_1 *arbitrary constants, maps the equations of the form (4.8) in the equation*

$$v_\tau - v_{\sigma\sigma} = 0.$$

Example 1 We consider the equation

$$u_t - u_{xx} = -u_x^2 \, tg\, u + u_x. \tag{4.13}$$

In this case is $h(u) = -tg\, u$ and $h_0 = 1$, so $H(\psi) = sin\, \psi$ and the transformations (4.12) become

$$t = \tau - \varepsilon_0, \quad x = \sigma - \tau - \varepsilon_3, \quad u = arcsin(c_0 v + c_1). \tag{4.14}$$

It is simple matter to verify that the transformation (4.14) maps equation (4.13) in

$$v_\tau - v_{\sigma\sigma} = 0.$$

Remark The standard deterministic KPZ equation has the form

$$h_t - D h_{zz} = \lambda h_z^2 \tag{4.15}$$

where $h(t, z)$ is the height of the surface at time t above the point z in the reference plane. Dh_{zz} describes diffusional relaxation within the surface. D is the diffusion coefficient. The strength of the nonlinearity λ is proportional to the growth speed.

The above equation, after a trivial change of independent variables $t = t$ and $z = \sqrt{D}x$, reads

$$h_t - h_{xx} = \frac{\lambda}{D} h_x^2. \tag{4.16}$$

A special case of this equation is the Burger's equation in potential form

$$u_t - u_{xx} = u_x^2. \tag{4.17}$$

One can ascertain that the transformation (4.12.III) for (4.16) and (4.17) becomes the well known transformation wich maps the considered equations in the well studied linear Fourier's equation

$$w_t - w_{xx} = 0. \tag{4.18}$$

5. Conclusions

In this paper we considered a family of semilinear diffusion equations and following [5], [6] we have obtained the differential invariants of second order under the equivalence transformations for this family by the infinitesimal method.

As an application, we have proved that a family of generalized diffusion equations can be reduced to the heat equation

$$v_\tau - v_{\sigma\sigma} = 0 \tag{5.1}$$

via appropriate equivalence transformations.

Finally, for special equations as standard deterministic KPZ (Burger's equation in potential form), from the transformation (4.12) we recovery the well known transformation which brings them to the heat equation.

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