Marius Sophus Lie

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Marius Sophus Lie was a Norwegian mathematician who largely created the theory of continuous symmetry, and applied it to the study of geometric structures and differential equations. Lie's principal tool, and one of his greatest achievements, was the discovery that continuous transformation groups (now called Lie groups) could be better understood by "linearizing" them, and studying the corresponding generating vector fields (the so-called infinitesimal generators). The generators obey a linearized version of the group law called the commutator bracket, and have the structure of what we today, in honour of Lie, call a Lie algebra.

The Early Days (1842-1869)

Sophus Lie's father was Johann Herman Lie, a Lutheran minister. His parents had six children and Sophus, was born on 17 December 1842 in Nordfjord, Norway, was the youngest of the six.. He first attended school in the town of Moss, which is a port in south-eastern Norway, on the eastern side of the Oslo Fjord [11]. In 1857, he entered

Nissen's Private Latin School in Christiania, capital and largest city of Norway which became Kristiania, then Oslo in 1925,. While at this school he decided to take up a military career, but his eyesight was not sufficiently good so he gave up the idea and entered University of Christiania and he came to the Royal Norwegian Frederik's University in Kristian (Oslo) in September 1859 at age of 16.

At university Lie studied a broad science course. There was certainly some mathematics in this course, and Lie attended lectures by Sylow, one of the first in the world to lecture on Galois' group theory, in 1862. Although not on the permanent staff, Sylow taught a course, substituting for Broch, in which he explained Abel's and Galois' work on algebraic equations. Lie also attended lectures by Carl Bjerknes on mathematics, so he certainly had teachers of considerable quality, yet he graduated in 1865 without having shown any great ability for the subject, or any great liking for it. The one thing he knew he wanted was an academic career and he thought for a while that astronomy might be the right topic. He learnt some mechanics, wondered whether botany or zoology or physics might be the right subjects and in general became rather confused.

However, there are signs that from 1867 he began to read more and more mathematics. In January, He read Descartes' *Géometrie* and some weeks later Euclid's *Elementa* and the 40th Volume of Grunert's *Archiv der Mathematik und Physik* (Archives of mathematics and physics) with an article of "*Essai d'une exposition rationelle des principes fondamenteux de la géométrie élémantaire*"(An essay of a rational exposure of the fundamental principles of the elementary geometry). The type of mathematics that Lie would study became more clearly defined during 1868 when he avidly read papers on geometry by Plücker and Poncelet. Then, Lie wrote a short mathematical paper in 1869, but the world of mathematics was too cautious to quickly accept Lie's revolutionary notions. At his own and his friend Ernst Motzfeldt's expense, his first scientific paper *Repräsentation der Imaginären der Plangeometrie* (Representation of the imaginary of plan geometry) was accepted to publish by *Crelle's Journal* after the Academy of Science in Christiania was reluctant to publish his work. He sent letters to Theodor Reye and Rudolf Friedrich Alfred Clebsch, two mathematicians of Prussia, an independent state from the 17th century until 1871 which was named Borussia in Latin, still attempting to gain recognition for his ideas.

In Berlin and Paris 1869-1871

Lie went to Prussia and visited Göttingen at the end of the year 1869, and then he went to Berlin which was the centre of the mathematical world at that time. He met Weierstrass, Kronecker and Kummer. Lie was not attracted to the style of Weierstrass's mathematics which dominated Berlin. His interests fitted more closely with Kummer, and Lie could lecture on his own results in his seminar because he had a lot of time for his own work and received high praise from him after he was able to correct some errors that Kummer had made in his work on line congruences of degree 3. Lie received replies from Reye and Clebsch to his earlier letters which greatly encouraged him. Reye wrote to him that he found his works “extremely interesting”.

However, in Berlin most important to Lie, was his meeting with Felix Klein (1849 - 1925). It was easy to see that these two would instantly find common ground in mathematics since Klein had been a student of Plücker, and Lie, although he never met Plücker, always said that he felt like Plücker's student. They were acquainted with each other’s works and there were several points of contact between them: Despite the common link through Plücker's line geometry, Lie and Klein were rather different in character as Freudenthal points. The algebraist Klein was fascinated by the peculiarities of charming problems; the analyst Lie, parting from special cases, sought to understand a problem in its appropriate generalization [1].

In the spring of 1870 Lie and Klein were together again in Paris. There they met Darboux, Chasles and Camille Jordan. Jordan seems to have succeeded in a way that Sylow did not, for Jordan made Lie realise how important group theory was for the study of geometry. Lie started to develop ideas which would later appear in his work on transformation groups. He began to discuss with Klein these new ideas on groups and geometry and he would collaborate later with Klein in publishing several papers. This joint work had as one of its outcomes Klein's characterization of geometry in his *Erlangen Program*, an influential research programme and manifesto was published under the title *Vergleichende Betrachtungen über neuere geometrische Forschungen*

(Comparative views about the newer geometrical research), of 1872 as properties invariant under a group action. In Paris, Lie discovered contact transformations. These transformations allowed a 1-1 correspondence between lines and spheres in such a way that tangent spheres correspond to intersecting lines.

While Lie and Klein thought deeply about mathematics in Paris, the political situation between France and Prussia was deteriorating. The popularity of Napoleon III, the French emperor, was declining in France and he thought a war with Prussia might change his political fortunes since his advisers having told him that the French Army could defeat Prussia. Bismarck, the Prussian chancellor, saw a war with France as an opportunity to unite the South German states. With both sides feeling that a war was to their advantage, the Franco-Prussian War became inevitable. On 14 July, Bismarck sent a telegram which infuriated the French government and on the 19 July France declared war on Prussia. For Klein, a Prussian citizen who happened to be in Paris when war was declared, there was only one possibility: he had to return quickly to Berlin.

However, Lie was a Norwegian and he was finding mathematical discussions in Paris very stimulating. He decided to remain but became anxious as the German offensive met with only an ineffective French reply. In August, the German army trapped part of the French army in Metz and Lie decided it was time for him to leave and he planned to hike to Italy. He reached Fontainebleau but there he was arrested as a German spy, his mathematics notes being assumed to be top secret coded messages. Only after the intervention of Darboux was Lie released from prison. The French army had surrendered on 1 September, and on 19 September the German army began to blockade Paris. Lie fled again to Italy, and then from there he made his way back to Christiania via Germany so that he could meet and discuss mathematics with Klein.

In 1871 Lie became an assistant at Christiania, having obtained a scholarship, and he also taught at Nissen's Private Latin School in Christiania where he had been a pupil himself. He submitted a dissertation on a class of geometric transformations (written in Norwegian) for his doctorate which was duly awarded in July 1872. The dissertation contained ideas from his first results published in Crelle's Journal and also the work on

contact transformations, a special case of these transformations being a transformation which maps a line into a sphere, which he had discovered while in Paris.

Professor in Christiania 1872-1886

It was clear that Lie was a remarkable mathematician and the University of Christiania reacted in a very positive way, creating a chair for him in 1872.

In Paris, Sophus Lie had become acquainted with Camille Jordan and his work on Galois' theory of group. He had heard Sylow lecture on the theory of groups already in 1862, but it was in Paris that he realized its importance. Safely home again in Christiania he seriously started the study of differential equations. He wrote the theory of differential equations is the most important discipline in modern mathematics.

While Glois had established a theory of algebraic equations, Sophus Lie now developed a theory for differential equations. In the course of 1872 he created his integration method for partial first order differential equations.

While Sophus Lie had arrived at his integration method via geometry, Adolph Mayer (1839-1907), a professor in Leipzig, had arrived at an almost identical ethod via analysis. In autumn 1872, Lie, Mayer and Klein all met in Göttingen, and Mayer invited them to visit him in Leipzig, where they continued to discuss their integration methods. When Sophus Lie published his works on partial differential equations of first order in *Mathematische Annalen,* Mayer helped him with the editing.

In spring 1872, Lie discovered that the theory he had developed for first order partial differential equations was a transformation theory. Therefore, he afterwards developed a complete invariant theory for all finite dimensional groups of contact transformations. He then used this invariant theory in his study of first order differential equations. He discovered that by determining group of the equation it could be integrated and solved by quadrature.

Sophus Lie married in August 1874 with Anna Birch from Risør after they had engaged 2 years ago at Christmas. They had 3 children; Marie, Dagny, Herman.

The famous Norwegian mathematician Abel had died more than 40 years before this (some 14 years before Lie was born) but, despite Abel's short career; his complete works had not been published at that time. It was natural that Norwegian mathematicians would undertake the task. Between 1873 and 1881, Sylow and Lie worked on a new edition of Abel's work in 2 volumes. Lie, however, always claimed that most of the work was done by Ludvig Sylow.

Lie discovered that in the case of straight line, there were only three kinds groups and that those could be converted to groups and that these could be converted to groups of projective transformations. In the case of the plane, on the other hand, he soon encountered serious problems. It took him months of hard calculations to decide all finite groups in the plane. In 1874 he discovered that every differential equation corresponds to a specific group, namely the concept of all contact transformations, which would be called Lie groups later, convert the equation to itself. All leaving behind was his original intention of examining partial differential equations. Later Killing was to examine the Lie algebras associated with Lie groups. He did this quite independently of Lie (and not it would appear in a manner which Lie found satisfactory), and it was Cartan who completed the classification of semi-simple Lie algebras in 1900.

In 1876, Sophus Lie returned to his geometrical studies. He turned his thoughts towards minimal surfaces. During the following years, he published a number of papers in the Norwegian *Archiv for Mathematik og Naturvidenskab.* Here he presented the results of his studies of minimal surfaces, geodesic curves, surfaces of constant curvature and translation surfaces, as well as a number on transformation groups.

In 1877, Lie started thinking about writing a book about differential equations of first order which include his recent three papers about Pfaff's Problem, but his plans for a book were never realized since he was spending lots of time for publication of Abel's work.

1822 was an important year for Lie because he returned to his work on differential invariants and his theory of groups. He spent 2 months in Paris, and met with three mathematicians Halphen, Poincaré and Picard to show a sincere interest for his work on differential equations and transformation groups. He was overjoyed when he saw Picard,

in a note in an article in *Comptes rendus* had referred to his work on transformation groups and had predicted at the same time that this work would become very important. Given this encouragement, Sophus Lie noe devoted himself entirely to the work on transformation groups.

This visit was too important for him because he worked without any contact with other mathematicians. Although Lie was producing highly innovative mathematics, he became increasingly sad at the lack of recognition he was receiving in the mathematical world. One reason was undoubtedly his isolation in Christiania, but a second reason was that his papers were not easily understood, partly through his style of writing and partly because his geometrical intuition greatly exceeded that of other mathematicians. Klein, realizing the problems, had the excellent idea of sending Friedrich Engel to Christiania to help Lie. The other reason of this idea was the risk of Lie's plans for writing a larger work on transformation groups never to be finished

Engel had received his doctorate from Leipzig in 1883 having studied under Adolph Mayer writing a thesis on contact transformations. Klein recognized that he was the right man to assist Lie and, at Klein's suggestion, Engel went to work with Lie in Christiania starting in 1884. He worked with Lie for nine months leaving in 1885. Engel then was appointed to Leipzig and, when Klein left the chair at Leipzig in 1886, Lie was appointed to succeed him. The collaboration between Engel and Lie continued for nine years culminating with their joint major publication *Theorie der Transformationsgruppen* (Theory of transformation groups) in three volumes between 1888 and 1893. This was Lie's major work on continuous groups of transformations.

After his understanding of importance of reaching others' researches, he started believing that printing his ideas in the *Norvegian Archiv* had not been enough. From now on the most important thing for Sophus Lie was to publish internationally, and he prepared two articles in 1884 for *Mathematishe Annalen*: *Über Differentialinvarianten* (About Differential variants) and *Allgemeine Untersuchungen über Differantialgleichungen die eine continuirliche endliche*(General investigations over Differential equations some continuirliche finite).

Professor in Leipzig 1886-1898

Felix Klein from 1885 professor in Göttingen, was making a deliberate effort to make Göttingen once more the centre of mathematics in Germany. He regarded Sophus Lie as an important ally, and did all he could to get Lie to succeed him in Leipzig. Sophus Lie was to build up a school of mathematics based on theory of transformation groups.

In 1885, in spite of expressed opposition among the mathematicians in Berlin, the Royal Ministry of Culture and Education in Saxony invited Sophus Lie to fill the vacant chair in geometry in Leipzig. He did not take a long time to accept the invitation.

In April 1886, Lie and his family moved to Leipzig. Sophus Lie became the head of The Mathematical Seminar where his old friend Mayer also worked. The three *privatdozenten* (lecturers, at the seminar), Engel, Friedrich Schur and Eduard Study, were all working on themes connected to Sophus Lie's theories.

Fredrich Engel was Lie's right hand in the work of creating a new school of mathematics based on the theory of transformation groups. The theoretical part of the work consisting of editing the work *Theorie der Transformationsgruppen* which during the period 1888-1893 was published in three volumes through cooperation between Sophus Lie and Friedrich Engel. The practical part was the teaching. While Engel lectured on analysis and differential equations, Lie lectured on geometry.

In the seminars and in the exercises, Sophus Lie and Friedrich Engel presented their own research and tried to motivate the students to become interested in the theory of transformation groups and the students poured in, from Belgium, Poland, Serbia, America, Scandinavia, Germany and especially France. Darbaux, Picard and Poincaré encouraged their students to travel to Leipzig to study geometry, and especially Sophus Lie's theory of continuous groups. At the time that Sophus Lie served as professor in Leipzig, about half the decorate students sat for him for their degree, although he was only one of five professors. This was a heavy burden.

However all was not well, he still felt unrecognised and, as Svare writes in [3] In Leipzig Lie was troubled by constant homesickness. A keen outdoor man, he missed the forests and mountains of Norway.

In 1889, Lie’s strength failed him. Overworked, sleepless and deeply depressed he had to enter a nerve clinic and he stay there until June 1890.

The Breakdowns

Towards the end of the 1880s Lie's relationship with Engel broke down. In 1892 the lifelong friendship between Lie and Klein broke down. When Klein, wanted to print the *Erlangen program* again and write about how it came about, he sent the manuscript to Sophus Lie was dismayed when he saw what Klein had written, and got the impression that his friend now wanted to have his share of what Sophus Lie had regarded as his life’s work. To make thinks clear, he asked Klein to let him borrow the letters he has sent him before the *Erlangen Program* was written, but the letters no longer existed.

It is difficult for any biographer to represent these events, and the events which followed, fairly since there is a great deal of contradictory material in the literature. The reason for this is not hard to understand, for information about Lie was for many years based on [4] which Engel wrote on Lie's death. The position is complicated by the mental difficulties which Lie suffered in 1889. Klein's [5]: “(...) "defence" of Lie's behaviour by referring to the close relationship between genius and madness really created a generally accepted explanation which has survived up to the present. By this act of "defence" Klein did his old friend an incredible injustice.”

The truth is that Lie's behavior was not totally irrational as it has been portrayed, but was indeed motivated by the way that both Engel and Klein had behaved. Purkert in [6] discusses the breakdown of relations between Lie and Engel. He has studied material from the University of Leipzig and believes that Lie changed his attitude toward Engel because Lie still felt a lack of recognition yet he knew that he was in a different class as a creative mathematician to Engel.

In [7] Fritzsche comments on Lie's illness he writes, “Through information about Sophus Lie's illness it is possible to trace consequences that shed light on certain

biographical aspects of his life; for example, his break with Friedrich Engel and Felix Klein. Furthermore, this evidence contradicts the oft-stated opinion that Lie's sickness was brought about by overwork."

Straume in [5] points out why Lie's behaviour towards Klein, with the final breakdown in 1892, was not irrational: "Klein's Erlangen Program from 1872 had not attracted much attention; in fact, it was Lie rather than Klein himself who had influenced the mathematical development envisioned in this Program. ... Klein decided to republish the Program and also write about its origins (in which Lie was much involved), but Lie disagreed strongly with Klein's views on what had happened in the past. It also turned out that Klein burned all the letters he had received from Lie up to 1877 (and thus breaking a previous mutual agreement between them)."

Lie reacted by publicly attacking Klein as writing "I am no pupil of Klein, nor is the opposite the case, although this might be closer to the truth." in the Preface to the third volume of his *Theorie der Transformationsgruppen* in 1893. Certainly Lie was an angry man but he was attacking someone holding such a leading role on the world scene of mathematics that the attack was always more likely to rebound on Lie rather than hurt Klein. Already current research is showing Lie in a much better light over this affair (and therefore Klein in a less good one) than previously reported and all the indications are that further research will prove even more favorable to Lie.

The person who was hardest hit by the conflict between Lie and Klein was Friedrich Engel, due to his lack of permanent position. In 1893 someone was to be appointed to an extraordinary professorship in Königsberg, and Engel was a relevant candidate, but so came the publication of the 3rd volume of *Theorie der Transformationsgruppen* and a letter of Hilbert to Klein, he wrote that Engel had excluded Engel completely, but he had not made any comment in the preface, and Hilbert hold Engel to some extent co-responsible for the incomprehensible and totally useless personal animosity which the third volume of Lie's work on transformation groups was full of. Engel had no choice. Although he was employed at The Mathematical Seminar headed by Sophus Lie he was forced to withdraw from the cooperation with Lie. After

Lie’s death, Engel published his collected treatises about Lie’s works and for the next 40 years he devoted his life to this work.

During his stay at the nerve clinic in 1990, he published *Leibziger Berichte* entitled *Über the Grundlagen der Geometrie.* In 1897 he was awarded the Lobatschewski Prize for this work, being the first person ever to receive it.

Perhaps an indication of Lie's love for his homeland is the fact that he continued to hold his chair in Christiania from his first appointment in 1872, being officially on leave while holding the chair in Leipzig and he returned to Christiania in 1898 to take up a post specially created for him. He produced a report about who should fill his chair, and this is given in full in [6]. Despite Engel being one of the leading workers in Lie's own research field, Purkert believes that Lie's assessment that he lacked creativity was entirely fair. However his health was already deteriorating when he returned to a chair in Christiania in 1898 and he died of pernicious anemia in February 1899 soon after taking up the post.

O'Connor J J , Robertson E F [9] finished their article by quoting from Robert Hermann's preface to [8]: “In reading Lie's work in preparation for my commentary on these translations, I was overwhelmed by the richness and beauty of the geometric ideas flowing from Lie's work. Only a small part of this has been absorbed into mainstream mathematics. He thought and wrote in grandiose terms, in a style that has now gone out of fashion, and that would be censored by our scientific journals! The papers translated here and in the succeeding volumes of our translations present Lie in his wildest and greatest form.”

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