Inductive Approach to Cartan's Moving Frame Method with Applications to Classical Invariant Theory.

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ABSTRACT

This thesis is devoted to algorithmic aspects of the implementation of Cartan's moving frame method to the problem of the equivalence of submanifolds under a Lie group action. We adopt a general definition of a moving frame as an equivariant map from the space of submanifolds to the group itself and introduce two algorithms, which simplify the construction of such maps. The first algorithm is applicable when the group factors as a product of two subgroups $G = BA$, allowing us to use moving frames and differential invariants for the groups A and B in order to construct a moving frame and differential invariants for G. This approach not only simplifies the computations, but also produces the relations among the invariants of G and its subgroups. We use the groups of the projective, the affine and the Euclidean transformations on the plane to illustrate the algorithm. We also introduce a recursive algorithm, allowing, provided the group action satisfies certain conditions, to construct differential invariants order by order, at each step normalizing more and more of the group parameters, at the end obtaining a moving frame for the entire group. The development of this algorithm has been motivated by the applications of the moving frame method to the problems of the equivalence and symmetry of polynomials under linear changes of variables. In the complex or real case these problems can be reduced and, in theory, completely solved as the problem of the equivalence of submanifolds. Its solution however involves algorithms based on the Gröbner basis computations, which due to their complexity, are not always feasible. Nevertheless, some interesting new results were obtained, such as a classification of ternary cubics and their groups of symmetries, and the necessary and sufficient conditions for a homogeneous polynomial in three variables to be equivalent to $x^n + y^n + z^n$.

Contents

Introduction.

Elie Cartan's method of equivalence is a natural development of the Felix Klein Erlangen program (1872), which describes geometry as the study of invariants of group actions on geometric objects. Cartan formulated the problem of the equivalence of submanifolds under a group of transformations and introduced the method of moving frames, which allows one to construct differential invariants under a group action [6]. The functional relations among the invariants provide a key to the classification of submanifolds under a prescribed group of transformations. Classically, a moving frame is an equivariant map from the space of submanifolds (or more rigorously, from the corresponding jet bundle) to the bundle of frames. Exterior differentiation of this map produces an invariant coframe, which is used to construct a number of differential invariants sufficient to solve the equivalence problem. Considering moving frame constructions on homogeneous spaces, Griffiths [20] and Green [19] observed that a moving frame can be viewed as an equivariant map from the space of submanifolds to the group itself. As pointed out in [20], one of the classical moving frames, the Frénet frame, is in fact a map from the space of curves to the Euclidean group. Adopting this observation as a general definition of a moving frame, Fels and Olver [12], [13] generalized the method for arbitrary, not necessarily transitive, finite-dimensional Lie group actions on a manifold introducing, for the first time, a completely algorithmic way for their construction.

In the first chapter we give an overview of the Cartan's solution to the problem of the equivalence of submanifolds. We also describe a general algorithm for construction of the moving frames and differential invariants developed by Fels and Olver [13]. Following this method, however, one might have to prolong the action to the jet spaces of high order before obtaining any invariants, while the earlier methods [20], [19] allow one to construct invariants order by order. We combine the advantages of

both approaches in the recursive algorithm presented in Chapter 2. Not surprisingly, the construction of moving frames and differential invariants is simpler when the acting group has fewer parameters. Thus, it is desirable to use the results obtained for a subgroup $H \subset G$ to construct a moving frame and differential invariants for the entire group G. The inductive algorithm from Chapter 2 allows us to perform this for the groups that factor as a product. Using this algorithm, one obtains at the same time the relations among the invariants of group G and its subgroup H. An illustrative example is induction from the Euclidean action on plane curves to the special affine action, and then to the action of the entire projective group. As a by-product, one obtains the expression of the affine invariants in terms of the Euclidean ones and the projective invariants in terms of the affine ones. The actions of all three groups play an important role in computer image processing [10], [31].

Equipped with all of these tools we approach the problem of the equivalence and symmetry of polynomials under linear changes of variables in Chapter 3. In fact, applications to classical invariant theory have served as the initial motivation for the development of the algorithms from Chapter 2. Two polynomials are said to be equivalent if there is a linear change of variables that transforms one into the other. The group of symmetries of a polynomial consists of all linear changes of variables that leave the polynomial unchanged. It is desirable to describe the classes of equivalent polynomials and classify the corresponding symmetry groups. We concentrate on polynomials over complex numbers although indicate how to adopt the results to real polynomials. This problem has been traditionally approached by algebraic or algebraic-geometry tools [18], [21], [25], [35].

Inspired by the ideas of Olver's book [28] we approach the problem using a differential geometric method of moving frames. We consider the graph of a polynomial $F(\mathbf{x})$ in m variables as a submanifold in $(m+1)$-dimensional complex (or real) space therefore reducing the question to equivalence problems for submanifolds. In the polynomial case, differential invariants can be chosen to be rational functions, and the polynomial relations among them can be found via elimination algorithms based on Gröbner basis computations. Hence, the problem of the equivalence and symmetry of polynomials can be completely solved, at least in theory. In practice, however, we are confronted with the complexity of Gröbner basis computations, which signif-

icantly limits our ability to solve specific problems. We start with the simplest case of polynomials in two variables, reproducing results from the paper by Peter Olver and myself [2]. We provide a MAPLE code which determines the dimension of the symmetry group of a given polynomial and in the case when the symmetry group is finite, computes it explicitly. Computations become challenging, even in the next case of polynomials in three variables. In order to construct a complete set of differential invariants we apply the recursive algorithm from Chapter 2. In some cases, the relations among the invariants are successfully computed via algorithms based on Gröbner basis computations, while in the other cases, this computation does not seem to be feasible. Nevertheless, some interesting new results were obtained, such as a classification of ternary cubics and their groups of symmetries, and necessary and sufficient conditions for a homogeneous polynomial in three variables to be equivalent to $x^n + y^n + z^n$.

Cartan's method for solving the problem of equivalence for submanifolds was formulated in the category of smooth manifolds. Hence, its direct applicability is restricted to polynomials over complex or real numbers. It would be a worthwhile and interesting project to reformulate the method of moving frames in the algebraic-geometry language, so that it can be applied to the problem of the equivalence and symmetry of algebraic varieties over fields of arbitrary characteristics.

Chapter 1

The Equivalence Problem for Submanifolds.

Two manifolds are said to be equivalent under a transformation group if one can be mapped to the other by an element of the group. Symmetry can be considered as self-equivalence. The group of the symmetries, or the isotropy group, of a submanifold consists of the transformations which map the submanifold to itself. Given a transformation group one would like to find classes of equivalent manifolds and to describe the group of symmetries of a given submanifold.

A local solution to this problem for a group acting on a homogeneous space was presented by E. Cartan in [6] and is known as the method of moving frames. Classically, a moving frame is an equivariant map from the space of submanifolds (or more rigorously from the corresponding jet bundle) to the bundle of frames. Exterior differentiation of this map produces an invariant coframe, which is used to construct a number of differential invariants sufficient to solve the equivalence problem.

The method of moving frames was generalized by Fels and Olver [13] for arbitrary (not necessarily transitive) finite-dimensional Lie group actions on a manifold. It relies on the definition of moving frame as an equivariant map from the space of submanifolds to the group itself, which could be also found in Griffiths [20] and Green [19]. As pointed out by Griffiths in [20], one of the classical moving frames, the Frénet frame, is in fact a map from the space of curves to the Euclidean group.

We remark that the problem of the equivalence for submanifolds is one of many problems which can be reformulated in terms of an exterior differential system and reduced to a question of its integrability [4]. Other important examples include

the equivalence and symmetry problems for differential equations and for variational functionals. Cartan's method consists of rewriting the problem in terms of differential forms in a way that it becomes intrinsic with respect to group action. Then the exterior differentiation of this system produces the sufficient number of differential invariants. The equivalence problem can be solved by examining the functional relations among these invariants. Although not completely algorithmic this approach has led to solving numerous equivalence problems e.g., [5], [7], [15], [23] and [26] including problems which involve infinite dimensional pseudo-groups of transformation.

In what follows we consider the local problem of equivalence and symmetry under finite dimensional Lie group of transformation on a smooth manifold. In the first section we review some basic definitions and results about Lie group actions which we will use later.

1.1 Lie Group Actions on Manifolds.

Definition 1.1.1. A smooth map $w : G \times M \to M$ defines an *action* of a group G on a manifold M if it satisfies the following properties:

$$w(e, z) = z, \qquad w(a, w(b, z)) = w(ab, z), \tag{1.1}$$

for any $z \in M$, $a, b \in G$. When it does not lead to confusion an action will be denoted as multiplication: $w(a, z) = a \cdot z$

We adopt the definition of local group actions from [26].

Definition 1.1.2. A *local group of transformations* on M is given by a (local) Lie group G and an open subset $\mathcal{U} \subset G \times M$, such that $\{e\} \times M \subset \mathcal{U} \subset G \times M$, which is the domain of definition of the group action, and a smooth map $w \colon \mathcal{U} \to M$ such that if $(g, z) \in \mathcal{U}$ the so is $(g^{-1}, w(g, z))$ and two properties (1.1) are satisfied whenever w is defined.

The following definition extends the notion connectness to local transformations:

Definition 1.1.3. A group of transformations G acting on M is called *connected* if the following requirements hold:

(i) G is a connected Lie group and M is a connected manifold;

(ii) the domain $\mathcal{U} \subset G \times M$ of the group action is connected;

(iii) for each $z \in M$ the local group $G_z = \{g \in G | (g, z) \in \mathcal{U}\}$ is connected.

Definition 1.1.4. The *orbit* $\mathcal{O}_z$ through a point $z \in M$ is the image of the map $w_z : G \to M$ given by $w_z(g) = w(g, z)$. Each point of M belongs to a unique orbit. If there is only one orbit on M, then $\mathcal{O}_z = M$ for all z and the action is called *transitive*.

The differential $dw_z : TG|_e \to TM|_z$ maps the Lie algebra of G to the tangent space at the point z. Let $X \in \mathfrak{g} = TG|_e$ then $\hat{X}(z) = dw_z(X)$ is a smooth vector field on M called an *infinitesimal generator* of the G-action:

$$\exp(tX) \cdot z = \exp(t\hat{X}, z),$$

where $\exp(t\hat{X}, z)$ is the flow of the vector field $\hat{X}$.

Definition 1.1.5. The *isotropy* group of a subset $S \subset M$ is

$$G_S = \{g \in G | gS = S\}.$$

The *global isotropy subgroup* is the subgroup

$$G_S^* = \bigcap_{x \in S} G_x = \{g \in G | gS = S\}$$

consisting of those group elements which fix all points in S.

Definition 1.1.6. A group G acts *effectively* if different elements have different actions, or equivalently $G_M^* = \{e\}$. The action of G is locally effective if G_M^* is a discrete subgroup of G. A group G acts *effectively on subsets* if the global isotropy subgroup of each open subset $U \subset M$ is trivial.

If a group does not act effectively then we can replace its action with the action of the quotient group G/G_M. This action is well defined, effective and essentially the same as the action of G. Clearly if G acts *effectively on subsets* then G acts effectively. The converse statement is true in analytic category, however as example 2.3 in [13] shows, it is not valid in general in smooth category.

Definition 1.1.7. A transformation group *acts freely* if the isotropy subgroup of each point is trivial and *locally freely* if the isotropy group of each point is discrete.

If the group acts (locally) freely then the dimension of each orbit $\mathcal{O}_z$ is equal to the dimension of the group. In this case the map w_z defines a (local) diffeomorphism between G and the orbit $\mathcal{O}_z$.

An action of a Lie group G on a manifold M induces the standard linear representation on the space of smooth functions $F \colon M \to \mathbb{R}$:

$$(gF)(gz) = F(z), \tag{1.2}$$

where $g \in G$, $z \in M$.

Definition 1.1.8. A function $F(z)$ is *invariant* if it is a fixed point of the standard representation above, that is

$$F(gz) = F(z). \tag{1.3}$$

We say that F is a *local invariant* if it is defined on an open subset of M, and/or the equality (1.3) holds only for g in a neighborhood of identity in G.

Definition 1.1.9. The *symmetry group* G_F of a function $F(z)$ is the isotropy group of F under representation (1.2):

$$G_F = \{g \in G | F(gz) = F(z)\}$$

Definition 1.1.10. The action of a group G on M is called *semi-regular* if all its orbits have the same dimension. The action is called *regular* if, in addition, each point $z \in M$ has arbitrarily small neighborhood whose intersection with each orbit is a connected subset thereof.

Let G act semi-regularly on an m-dimensional manifold M and let s be the dimension of the orbits, then the infinitesimal generators of the G-action form an integrable distribution of the dimension s. The orbits of G are the integral manifolds for this distribution. By Frobenius' theorem, coordinates $(x^1, \ldots, x^s, y^1, \ldots, y^{m-s})$ on a chart $U \subset M$ can be chosen so that each orbit is a level set of the last $m-s$ coordinates: $y^i = c_i$, $i = 1, \ldots, m-s$. The functions y^i form a complete set of functionally independent local invariants. Thus the number of functionally independent local invariants for a semi-regular action of a Lie group equals to the difference between the dimension of the manifold and the dimension of the orbits. Let $\mathcal{S}$ be a level set of the first s coordinates $x^i = c_i, i = 1, \ldots, s$, then the submanifold $\mathcal{S}$ has codimension s and is

transversal to each orbit in U. Thus $\mathcal{S}$ intersects each orbit in a discrete set of points. If the action is regular and U is sufficiently small, then $\mathcal{S}$ intersects each orbit only once.

Definition 1.1.11. Suppose G acts semi-regularly on an m-dimensional manifold M with s-dimensional orbits. A (local) *cross-section* is an $(m-s)$-dimensional manifold $\mathcal{S} \subset M$ such that $\mathcal{S}$ intersects each orbit transversally. The cross-section is *regular* if $\mathcal{S}$ intersects each orbit at most once.

We conclude this preliminary section with Lie's infinitesimal criterion of invariance:

Theorem 1.1.12. *Let G be a connected group of transformations acting on a manifold M. A function $I\colon M \to \mathbb{R}$ is invariant under G if and only if for every infinitesimal of generator X of the G-action:*

$$X[I] \equiv 0.$$

Remark 1.1.13. *Throughout the thesis we will, without saying it explicitly, consider the group actions, cross-sections and invariants to be local. Nevertheless in order to shorten the statements and formulas we will keep global notation. For instance we will write $w : G \times M \to M$ instead of $w\colon \mathcal{U} \to M$ for a local action, or we will call $\mathcal{S}$ a regular cross-section on M meaning that it is a regular cross-section in some open subset $U \subset M$.*

1.2 Jet Spaces and Differential Invariants.

In the case when the group G acts transitively on M, there are clearly no non-constant invariants. Nevertheless if we consider the transitive action of the group of Euclidean motions on the plane we can find important geometric invariant: the curvature $\kappa = \frac{u_{xx}}{\sqrt{1+u_x^2}^3}$ of an embedded curve $u = u(x)$. We notice that κ depends not only on a point on the curve but also on the derivatives of u with respect to x. This is an example of a *differential invariant*, which is an ordinary invariant function on the prolonged space (or jet space). Differential invariants were used by Lie in his work on the symmetry reduction of differential equations. The formal definition of

jet spaces was first given by Ehresmann. We are going to give a brief description of the geometric structure of jet spaces, for more details see [1], [26], [27].

Definition 1.2.1. Given a smooth manifold M of dimension m and an integer $p < m$, the *k-th order jet bundle* $J^k = J^k(M, p)$ is a fiber bundle over M, such that a fiber over a point $z \in M$ consists of the set of equivalence classes of p-dimensional submanifolds of M with *k-th* order contact at z. In particular $J^0 = M$.

Remark 1.2.2. In the case when M has itself a fiber bundle structure $M \to B$ with a p-dimensional base, then its k-th jet bundle $J^k M$ can be defined by sections $s \colon B \to M$ under the equivalence relations of *k-th* order contact at $z \in M$. Then the jet bundle $J^k = J^k(M, p)$ from Definition 1.2.1 can be called the *extended jet bundle*, since it also includes jets of p-dimensional submanifolds of M that are not transversal to the fibers over B. The fiber bundle $J^k M \to M$ is an open dense subset of the extended bundle $J^k(M, p)$.

There is a natural projection $\pi_l^k : J^k \to J^l$. The inverse sequence of topological spaces (J^k, π_k^l) determine an inverse limit space $J^\infty = J^\infty(M, p)$ together with projection maps $\pi_k^\infty : J^\infty \to J^k$. The space J^∞ is called infinite jet bundle over M. In the same manner the tangent bundle TJ^∞ can be defined as the inverse limit of topological spaces TJ^k under the projections $(\pi_l^k)_* = d\pi_l^k$. For $l < k$ we identify functions and differential forms on J^l with functions and forms on J^k under the pull-back $(\pi_l^k)^*$. The smooth functions and forms on J^∞ are defined as the direct limit of the space of smooth functions and forms on on J^k.

Let U be a chart of M with coordinates $(x^1, \ldots x^p, u^1, \ldots, u^q)$ so that $p + q = \dim M$. We say that a submanifold $S \subset M$ is *transverse* with respect to this coordinates if the restriction of the forms $dx^1, \ldots, dx^p$ to S is a coframe on S. Any transverse submanifold can be locally described as a graph $u^\alpha = f^\alpha(x)$, $\alpha = 1, \ldots, q$. Let $U^0 \subset U$ be a union of transverse submanifolds of U and $U^k \subset J^k$ be a subset such that $\pi_0^k(U^k) = U^0$. It is not difficult to see that U^k is an open subset of J^k, which can be parameterized by the set of independent variables $\{x^1, \ldots, x^p\}$, the set $\{u^1, \ldots, u^q\}$ of dependent variables and coordinates u_J^α which correspond to the derivatives of the dependent variables with respect to the independent ones, where the subscript $J = (j_1, \ldots, j_p)$ is a multi-index, such that $|J| = j_1 + \cdots + j_p \leq k$, $j_i \geq 0$.

Definition 1.2.3. Let S be a p-dimensional submanifold of M, then its k-th prolongation $j^k(S)$ is a p-dimensional submanifold of $J^k(M,p)$, defined by the k-th jets of S. In local coordinates on an open set U^k it is the graph of equations

$$u^\alpha = f^\alpha(x), \qquad u_J^\alpha = \frac{\partial^k f^\alpha}{\partial^{j_1} x^1 \dots \partial^{j_p} x^p},$$

where $\alpha = 1, \dots, q$ and $J = (j_1, \dots, j_p)$ are all possible multi-indices, such that $|J| \le k$.

Although the k-th prolongation $j^k(S)$ of S is a p-dimensional submanifold of $J^k(M,p)$, not every p-dimensional submanifold of $J^k(p,M)$ is the prolongation of a submanifold in M. The exterior differential forms which are identically zero when restricted to the prolongation $j^k(S)$ of any submanifold $S \subset M$ for all k form a differential ideal on J^∞, called the *contact ideal*. In local coordinates a basis for the contact ideal can be written as:

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha dx^i, \qquad \alpha = 1, \dots, q, \quad |J| \ge 0.$$

Thus the cotangent bundle on $J^\infty(U,p)$ splits into two sub-bundles: the horizontal sub-bundle which is spanned by the forms $dx^1, \dots, dx^p$, and the vertical sub-bundle spanned by the contact forms. We emphasize that the contact (vertical) sub-bundle has intrinsic definition, independent of the choice of coordinates, whence the choice of basic horizontal one-forms depends on the choice of independent coordinates.

The differential d on J^∞ also splits into horizontal and vertical components,

$$d = d_H + d_V.$$

The horizontal differential is defined by

$$d_H F = \sum_{i=1}^p (D_i F) \, dx^i,$$

where

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_J u_{J,i} \frac{\partial}{\partial u_J^\alpha}.$$

The operators D_i span a subspace of *total vector fields* in the tangent bundle TJ^∞, which can be defined intrinsically as the set of vector fields annihilated by any contact form.

The vertical differential is defined by

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha} d\theta_J^\alpha.$$

The vector fields $\frac{\partial}{\partial u_J^\alpha}$, which are annihilated by any horizontal form, span the subspace of *vertical vector fields* in TJ^∞. Each one-form on J^∞ splits into horizontal and vertical component, inducing a bi-grading of the exterior differential forms on J^∞. The horizontal differential d_H increases the horizontal degree of a form and the vertical differential d_V increases the vertical degree. This splitting gives rise to a bicomplex of differential forms called the variational bicomplex [34], [33], [36], and [1], an important tool in the study of geometry of differential equations and variational problems.

Definition 1.2.4. Let X be a vector field on M, then there is a unique vector field prX on J^∞, called prolongation of X such that

(i) X and prX agree on functions on M,

(ii) prX preserves the contact ideal: the Lie derivative of a contact form with respect to prX is also a contact form.

The vector field $pr^k X = (\pi_k^\infty)_* prX$ is called k-th prolongation of X.

In terms of local coordinates let

$$X = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

be a vector field on M, then its k-th order *prolongation* is:

$$pr^k(X) = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{|J| \leq k} \varphi_J^\alpha(x, u^{|J|}) \frac{\partial}{\partial u_J^\alpha}, \qquad (1.4)$$

where

$$\varphi_J^\alpha(x, u) = D_J Q^\alpha + \sum_{i=1}^p \xi^i(x, u) u_{J,i}^\alpha.$$

and Q^α denotes the characteristics of the vector field X :

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^p \xi^i(x, u) u_i^\alpha.$$

Any prolonged vector field can be decomposed into total and vertical components:

$$pr(X) = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha,J} \left(D_J Q^\alpha + \sum_{i=1}^{p} \xi^i(x,u)u^\alpha_{J,i} \right) \frac{\partial}{\partial u^\alpha_J}$$

$$= \sum_{i=1}^{p} \xi^i(x,u)D_i + \sum_{\alpha,J} D_J Q^\alpha \frac{\partial}{\partial u^\alpha_J}.$$

After we have defined the jet space we would like extend the action of G on it so that the extended action maps a prolongation of a submanifold of M to the prolongation of its image.

Definition 1.2.5. The *k-th order prolongation* of a smooth transformation $g \in G$ on M is defined by the property

$$g^{(k)} \cdot j^k(S) = j^k(g \cdot S)$$

for any submanifold $S \in M$.

It follows from the definition that the following diagram commutes

$$\begin{CD} J^k @>{g^{(k)}}>> J^k \\ @V{\pi^k_l}VV @VV{\pi^k_l}V \\ J^l @>>{g^{(l)}}> J^l \end{CD}$$

When it is clear from the context that the prolonged transformation is considered we will omit the superscript (k) over g.

Definition 1.2.6. A k-th order *differential invariant* on M under the action of G is a function on $J^k(M,p)$ which is invariant under the k-th prolongation of the group action.

Remark 1.2.7. We consider an l-th order differential invariant for $l < k$ as a k-th order differential invariant under the pull-back $(\pi^k_l)^*$. Thus a k-th order differential invariant might only depend on derivatives of order strictly less than k.

If X is an infinitesimal generator of G-action on M, then its k-th prolongation (1.4) generates the corresponding prolonged transformation on $J^k(M,p)$. From Lie's

criterion 1.1.12 it follows that function $I \colon J^k \to \mathbb{R}$ is a differential invariant if and only if for every infinitesimal of generator X of the G-action

$$pr^k(X)[I] \equiv 0.$$

Theoretically this criterion can be used to find all differential invariants. In practice however it is difficult to use since it requires integration of a system of first order partial differential equation. The advantage of the method of moving frames is that it requires only differentiation not integration.

Let $\mathcal{O}^k$ and $\mathcal{O}^l$ be the orbits of prolonged action on J^k and J^l respectively for $k > l$, then $(\pi_l^k)(\mathcal{O}^k) = \mathcal{O}^l$ and hence the dimension of the orbits can only become larger when we prolong the action to the higher jet spaces. On the other hand the dimension of the orbits is bounded by the dimension of the group G. Hence there is an order of prolongation n at which the maximum possible dimension is attained on an open subset of $\mathcal{V}^n \subset J^n$. If $z^{(k)}$, $k > n$ is a point in J^k such that $\pi_n^k(z^{(k)}) \in \mathcal{V}^n$ then the orbit of the prolonged action through $z^{(k)}$ also has the maximal dimension. We call such points *regular jets* and denote their union in J^∞ as $\mathcal{V}$.

Definition 1.2.8. The minimal order at which the orbits reach maximal dimension is called *the order of stabilization*. The subsets $\mathcal{V}^k \subset J^k$, $\quad k = n, \ldots, \infty$ which consists of the points $z^{(k)}$ such that the orbit through $\pi_n^k z^{(k)}$ has maximal dimension are called *regular*.

The following result (Ovsiannikov [30], Olver [29]) is crucial for moving frame construction.

Theorem 1.2.9. *If the action of G on M is locally effective on subsets then the prolonged action is locally free on $\mathcal{V}^k$, for $k \geq n$, where n is the order of stabilization.*

Since the dimension of the space grows with prolongation and the dimension of orbits stabilizes at order n, then nontrivial local differential invariants are guaranteed to appear at least at the order $n + 1$ by Frobenius' theorem. Although there exists only a finite number of functionally independent differential invariants at each order of prolongation, their total number is infinite since one can prolong up to the infinite

order. Fortunately one can obtain all invariants from a finite *generating set of invariants* by applying *invariant differential* operators. The latter can be constructed as dual vector fields to horizontal contact invariant forms, which are defined as follows:

Definition 1.2.10. A differential one-form ω on J^n is called *contact invariant* if for every $g \in G$ we have $\left(g^{(n)}\right)^* \omega = \omega + \theta_g$ for some contact form θ_g. A set of p linearly independent horizontal contact invariant forms $\{\omega_1, ..., \omega_p\}$ is called a *horizontal contact invariant coframe.*

The freeness of the group action on $\mathcal{V}^n \subset J^n$ guarantees the existence of a contact invariant coframe $\omega^1, \ldots, \omega^p$ on $\mathcal{V}^n$ [11], [30]. The horizontal differential of a function F can be written in terms of this coframe as:

$$d_H F = \sum_{i=1}^p \left(\mathcal{D}_i F\right) \omega^i,$$

where the total vector fields $\mathcal{D}_i$ have an important property: they commute with the prolonged action of G and thus map differential invariants to higher order differential invariants. Operators which possess this property are called *invariant differential operators.* The following theorem [30], [13] asserts that one can produce all differential invariants by applying a finite set of invariant differential operators to a finite set of generating invariants.

Theorem 1.2.11. *Suppose that G is a transformation group and let n be its order of stabilization. Then, in a neighborhood of any regular jet $z^{(n)} \in \mathcal{V}^n$, there exists a contact invariant horizontal coframe $\{\omega_1, ..., \omega_p\}$, and corresponding* invariant differential *operators $\mathcal{D}_1, ..., \mathcal{D}_p$. Moreover, there exists a generating system of differential invariants $I_1, ..., I_l$, of order at most $n + 1$, such that, locally, every differential invariant can written as a function of $I_1, ..., I_l$ and their invariant derivatives:*

$$I = H(\ldots, \mathcal{D}_J I_j, \ldots), \tag{1.5}$$

where $\mathcal{D}_J$ is a certain composition of invariant differential operators.

Example 1.2.12. Let us consider the special Euclidean group $SE(2, \mathbb{R}) = SO(2, \mathbb{R}) \ltimes \mathbb{R}^2$ acting on curves in $\mathbb{R}^2$. The group acts freely and transitively on the first jet space J^1 and thus the lowest order invariant, the Euclidean curvature

$$\kappa = \frac{u_{xx}}{\sqrt{1 + u_x^2}^3}$$

appears on the second jet space. The contact invariant coframe consists of a single form, infinitesimal arc-length:

$$ds = \sqrt{1 + u_x^2}\, dx.$$

Higher order differential invariants can be obtained by taking the derivatives of κ with respect to the arc-length, or in other words by applying invariant differential operator $\frac{1}{\sqrt{1+u_x^2}}\, D_x$. From dimensional consideration it is clear that only one new functionally independent invariant appears at each order of prolongation, and thus any differential invariant is a function of $\kappa, \kappa_s = \frac{d\kappa}{ds}, \kappa_{ss} = \frac{d\kappa_s}{ds}$, etc.

The generalized method of moving frame [13] described in Section 1.4 provides a consisting algorithm of constructing differential invariants, a contact invariant coframe, invariant differential operators and recursion formulas (1.5). In the next section we explain the role played by differential invariants in the solution of the equivalence and symmetry problems for submanifolds.

1.3 Equivalence and Symmetry of Submanifolds. Signature Manifolds.

Definition 1.3.1. Let G be a group acting on a manifold M. Two submanifolds S and $\bar{S}$ of dimension p in M are said to be locally *equivalent* if there exist a point $z \in M$ and an element $g \in G$ such that $\bar{S} = g \cdot S$ in a neighborhood of the point $g \cdot z \in \bar{S}$. An element $g \in G$ is a local *symmetry* of S if $S = g \cdot S$ at least in a neighborhood of the point $g \cdot x \in S$.

Both the equivalence and the symmetry problems can be solved by the following construction. Let n be the order of stabilization as in definition 1.2.8, then the group acts locally freely on $\mathcal{V}^n \in J^n(M, p)$ and the differential invariants must appear at least on J^{n+1}. Let $\{I_1, ... I_{N_k}\}$ be a *complete set* of k-th order differential invariants for $k > n$, where by complete set we mean that any other invariant of order k is a function of $\{I_1, ... I_{N_k}\}$. Let $\{\tilde{I}_1, ... \tilde{I}_{N_k}\}$ be the restriction of I's to $j^k(S)$. The functions $\{\tilde{I}_1, ... \tilde{I}_{N_k}\}$ define a map $\phi_k : S \to \mathbb{R}^{N_k}$. Let $t_k = rank(\phi_k)$, then $t_k \leq p$ since $\dim S = p$. It is not difficult to see that the rank t_k does not depend on a choice of a complete system of invariants.

Definition 1.3.2. A submanifold $S \subset M$ is called *regular* if $j^n(S) \subset \mathcal{V}^n$ and the rank t_k, $k \geq l$ does not vary on S.

Definition 1.3.3. Let S be a regular submanifold of M then its *k-th order signature manifold* $\mathcal{C}^k(S)$ is an immersed submanifold $Im(\phi_k) \subset \mathbb{R}^{N_k}$, where ϕ_k is defined as above.

Taking into account Remark 1.2.7 we observe that $t_{n+1} \leq t_{n+2} \leq \cdots \leq t_{n+i}... \leq p$ and hence this sequence stabilizes. In fact once $t_s = t_{s+1}$ for some s, all the subsequent ranks are equal.

Proposition 1.3.4. *If* $t_s = t_{s+1}$ *in the sequence of ranks above then:* $t_s = t_{s+1} = t_{s+2} = \ldots$

Proof. Given that $t_s = t_{s+1}$, we need to prove that $t_{s+1} = t_{s+2}$. Let I be any invariant of order $s+2$, then from theorem 1.2.11 it follows that there exist invariants $I_1, ..., I_N$ of order $s+1$ such that $I = H(\mathcal{D}_{i_1} I_1, \ldots \mathcal{D}_{i_N} I_N)$, where H is some function. From the definition of total vector fields it follows that:

$$\tilde{I} = H\left(\mathcal{D}_{i_1}\tilde{I}_1, \ldots, \mathcal{D}_{i_N}\tilde{I}_N\right), \tag{1.6}$$

where $\tilde{I}, \tilde{I}_1, ..., \tilde{I}_N$ are restrictions of I's to the jet $j^{s+2}(S)$. On the other hand since $t_{s+1} = t_s$ there exist s order invariants $F_1, \ldots F_t$ such that each $\tilde{I}_i$ can be written as a function of the restrictions $\tilde{F}_1, \ldots \tilde{F}_t$ of F's to $j^s(S)$:

$$\tilde{I}_i = H_i(\tilde{F}_1, \ldots \tilde{F}_t). \tag{1.7}$$

By substitution of (1.7) in (1.6) we obtain that

$$\tilde{I} = H\left(\mathcal{D}_{i_1}(H_1(\tilde{F}_1, \ldots, \tilde{F}_t)), \ldots, \mathcal{D}_{i_N}(H_N(\tilde{F}_1, \ldots, \tilde{F}_t))\right). \tag{1.8}$$

We note that $\mathcal{D}_{i_j}(H_j(\tilde{F}_1, \ldots, \tilde{F}_t))$, $j = 1, \ldots, N$ are differential invariants of order $s + 1$ and thus any $s + 2$ order invariant restricted to $j^{s+2}(S)$ can be written as a function of the invariants of order $s + 1$ restricted to $j^{s+1}(S)$. $\qquad \square$

Definition 1.3.5. The minimal order s such that $t_s = t_{s+1}$ is called *differential invariant order* of S and the corresponding rank $t = t_s$ is called *differential invariant rank* of S.

Remark 1.3.6. From Proposition 1.3.4 it follows that $t_n < t_{n+1} < \ldots < t_s = t_{s+1} \leq p$ and thus $s \leq n + p$, where n is the order of stabilization.

From Proposition 1.3.4 and recurrence formulas (1.5) it follows that the signature manifold $\mathcal{C}^{s+1}(S)$ encodes all functional relations among invariants restricted to $j^\infty(S)$. Remark 1.4.6 in the next section gives a geometric description of the signature manifold. As the following two theorems [27], [13] show, the signature manifold of order $s + 1$ plays a crucial role in the solution of the equivalence and symmetry problems for submanifolds .

Theorem 1.3.7. *Let* $S, \bar{S} \in M$ *be two regular p-dimensional submanifolds. Then* S *and* $\bar{S}$ *are (locally) equivalent,* $\bar{S} = gS$ *if and only if they have the same differential order s and their signature manifolds (locally) coincide:* $\mathcal{C}^{s+1}(S) = \mathcal{C}^{s+1}(\bar{S})$

Theorem 1.3.8. *Let* $S \subset M$ *be a regular p-dimensional submanifold of differential invariant rank t with respect to the transformation group G. Then its isotropy group* G_S *is a $(p-t)$-dimensional subgroup of G acting locally freely on S.*

Corollary 1.3.9. *A submanifold* $S \subset M$ *has a discrete symmetry group if and only if its signature manifold has maximal dimension p.*

Remark 1.3.10. The condition $rank(\phi) < p$ is closed and hence a generic submanifold has a discrete group of symmetries.

If the symmetry group is finite then its cardinality can be found from the following theorem [28].

Theorem 1.3.11. *Let* $\dim \mathcal{C}^{s+1}(S) = p$ *and* $c \in \mathcal{C}^{s+1}(S)$ *be a generic point. Then the cardinality of the symmetry group is equal to the cardinality of the preimage of c under ϕ:* $|G_S| = |\phi^{-1}(c)|$.

Example 1.3.12. Let us return to the problem of equivalence and symmetry of curves in $\mathbb{R}^2$ under the action of the special Euclidean group $SE(2, \mathbb{R}) = SO(2, \mathbb{R}) \ltimes \mathbb{R}^2$. In this case the order of stabilization $n = 1$, $\mathcal{V} = J^1$ and so the differential order s of any curve is not greater than two (see Remark 1.3.6). Thus the symmetry and equivalence problem for any submanifold can be solved by considering the signature manifold parameterized by the curvature $\kappa = \frac{u_{xx}}{\sqrt{1+u_x^2}^3}$ and its derivative $\kappa_s = \frac{(1+u_x^2)u_{xxx}-3u_x u_{xx}^2}{(1+u_x^2)^3}$

with respect to the arc-length. If κ, restricted to the jet of a curve, is constant then the signature manifold degenerates to a point $(\kappa, 0)$. The curvature is constant when the curve is either a circle or a line. The symmetry group of a line consists of translations parallel to this line and the symmetry group of a circle consists of rotations around the center of the circle. Both groups are one-dimensional in agreement with Theorem 1.3.8. The curvature of a generic curve is non-constant, and thus the signature manifold is one-dimensional. Let us construct the signature manifolds for two graphs $u = \sin(x)$ and $u = \cos(x)$, which are clearly equivalent under translation by $\frac{\pi}{2}$ in x-direction.

$$u = \cos(x) \quad | \quad u = \sin(x)$$

$$|$$

$$\kappa = -\frac{\cos(x)}{(1 + \sin^2(x))^{(3/2)}} \quad | \quad \kappa = \frac{\sin(x)}{(1 + \cos^2(x))^{(3/2)}}$$

$$|$$

$$\kappa_s = \frac{\sin(x)(1 + \cos^2(x))}{(1 + \sin^2(x))^3} \quad | \quad \kappa_s = -\frac{\cos(x)(1 + \sin^2(x))}{(1 + \cos^2(x))^3}$$
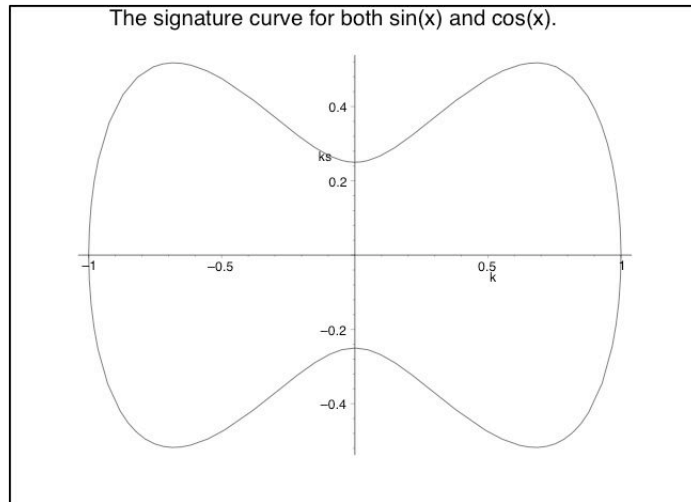
As x varies the signature curve will be traced over and over again with the period 2π. This reflects the fact that the graphs of sine and cosine possess infinite discrete

group of symmetries: translation by $2\pi k$, $k \in \mathbb{Z}$.

1.4 The Method of Moving Frames.

In this section we describe the generalization of the Cartan's method of moving frame by Fels and Olver [13] which provides an algorithm to construct a complete set of differential invariants of any order as well as a contact invariant coframe ω^i, $i = 1, \ldots, p$, corresponding invariant differential operators $\mathcal{D}_i$ and recursion formulas (1.5).

Definition 1.4.1. Let a group G act on a smooth manifold N. A *(right) moving frame* is defined as an equivariant map $\rho : N \to G$, where G acts on itself by right multiplication. In other words the following diagram commutes:

$$
\begin{array}{ccc}
G & \xrightarrow{\;R_{g^{-1}}\;} & G \\
\rho \uparrow & & \uparrow \rho \\
N & \xrightarrow{\;g\;} & N
\end{array}
$$

Theorem 1.4.2. *A local moving frame exists if and only if G acts regularly and (locally) freely on N.*

Proof. We sketch the proof of the sufficiency of these conditions and refer the reader to Theorem 4.4 in [13] for the proof of their necessity. Construction of a moving frame is equivalent to choosing a regular cross-section (see Definition 1.1.11). If an r-dimensional Lie group G acts regularly and locally freely on M then in a neighborhood U of every point $z \in N$ there exists a cross-section $\mathcal{S}$ that intersects each orbit at a unique point. We define a map $\rho : U \to G$ by the condition

$$\rho(z) \cdot z \in \mathcal{S}.$$

If G acts freely on U then the map $\rho(z) \in G$ is well defined, if G acts locally freely then the map on a neighborhood of the identity $V_e \subset G$ is well defined. To show that ρ is equivariant we observe that $\rho(gz) \cdot gz = \rho(z) \cdot z$ for all $g \in G$ and so since the action of $V_e \subset G$ is free then $\rho(gz) = \rho(z)g^{-1}$. In the future we will adopt global notation for local objects and maps (see Remark 1.1.13). $\qquad\square$

Definition 1.4.3. A moving frame on a submanifold $S \subset N$ is the restriction of ρ to S.

We notice that non-constant coordinates of $\rho(z) \cdot z$ provide a complete set of functionally independent local invariants on N. In fact if $f(z)$ is any function on N then its invariantization is defined as $\grave{\imath} f(z) = f(\rho(z) \cdot z)$. In other words the function $\grave{\imath} f(z)$ is obtained by spreading the values of f on $\mathcal{S}$ along the orbits of G and thus it is invariant.

This process can be described as normalization of invariants on $\mathcal{B} = G \times N$, which are called *lifted invariants*. Let the map $w : \mathcal{B} \to N$ be defined by the group action, that is $w(g, z) = g \cdot z$, the map $\sigma : N \to \mathcal{B}$ be defined by $\sigma(z) = (\rho(z), z)$ and the map $\iota : N \to N$ be defined by $\iota(z) = \rho(z) \cdot z$, then the following diagram commutes:

$$
\begin{array}{ccc}
 & \mathcal{B} = G \times N & \\
{\scriptstyle \sigma}\nearrow & & \searrow{\scriptstyle w} \\
N & \xrightarrow{\quad \iota \quad} & N
\end{array}
$$

We notice that the maps w and ι are G-invariant and the map σ is G-equivariant. Let z^i be the i-th coordinate function on $U \subset N$, then $(gz)^i = w^* z^i$, are invariant functions on $\mathcal{B}$, under the action $g(h, z) = (hg^{-1}, g \cdot z)$. Such functions are called *lifted invariants*. Their normalization $I^i = \sigma^* w^* z^i$ include a complete set of functionally independent invariants on N. The invariantization of an arbitrary function on N is defined by its pull-back under the map $\iota = \omega \circ \sigma : N \to N$. Indeed if f is any function on N then $\grave{\imath} f = \iota^*(f) = \sigma^* w^* f$ is an invariant function.

It is clear that if η is any form on N then $\bar{\eta} = w^* \eta$ is an invariant form on $\mathcal{B}$ and $\iota^* \eta = \sigma^*(\bar{\eta})$ is an invariant form on N. The pull back of any coframe on N under ι produces a set of invariant one-forms of codimension r. We can complete this set to an invariant coframe on N by pulling back a left invariant coframe on G under ρ. However, we define invariantization of differential forms in a different manner, for the reasons which become apparent later. We notice that cotangent bundle on $\mathcal{B}$ splits into two subspaces: the cotangent bundle to G spanned by the Maurer-Cartan forms and the cotangent bundle to the manifold N. The differential d on $\mathcal{B}$ splits accordingly $d = d_G + d_N$, leading to a well defined bicomplex of differential forms

on $\mathcal{B}$. Since the action of G on $\mathcal{B}$ is a product of the actions on G itself and the action on N then the action of G on $\mathcal{B}$ preserves this splitting. In particular, if $\bar{\eta}$ is a G-invariant form on $\mathcal{B}$ then so are $\pi_N(\bar{\eta})$ and $\pi_G(\bar{\eta})$, where π_N and π_G are the projection of the forms on $\mathcal{B}$ on purely manifold component and on purely group component respectively. If η is any form on N then $w^*(\eta)$ is an invariant form on $\mathcal{B}$, and so is $\pi_N w^*(\eta)$. Since σ is a G-equivariant map from N to $\mathcal{B}$, then $\sigma^* \pi_N w^*(\eta)$ is an invariant form on N.

Definition 1.4.4. Let η be a differential form on N, then its *invariantization* is defined by the formula:

$$\grave{\imath}(\eta) = \sigma^* \pi_N w^*(\eta). \tag{1.9}$$

In particular, if the forms dz^i form a coframe on N then the forms $\grave{\imath}(dz^i) = \sigma^* \pi_N w^* dz^i = \sigma^* d_N(w^* z^i)$ form a G-invariant coframe on N. Applied to zero-forms (functions) this definition coincide with the one which was given above. It is easy to check that $\grave{\imath}$ is a projection operator, that is it maps any invariant form to itself.

One can apply the process of invariantization to construct differential invariants, contact invariant forms and invariant differential operators on $J^\infty(M, p)$. Due to theorem 1.2.9 there exists high enough order of prolongation n such that G acts locally freely on $\mathcal{V}^n \subset J^n(M, p)$. Thus there exists a local cross-section $\mathcal{S}^n$ and a corresponding local moving frame $\rho : \mathcal{V}^n \to G$. We can extend these cross-section and moving frame to any higher order regular set $\mathcal{V}^k$ by defining $\mathcal{S}^k = \{z^{(k)} | \pi_n^k z^{(k)} \in \mathcal{S}^n\}$ and $\rho(z^{(k)}) = \rho\left(\pi_n^k(z^{(k)})\right)$ for $k = n, \ldots, \infty$. Let the map $w^k : \mathcal{B}^k \to J^k$ be defined by the prolonged group action, that is $w(g, z^{(k)}) = g \cdot z^{(k)}$, the map $\sigma^k : J^k \to \mathcal{B}^k$ be defined by $\sigma(z^{(k)}) = (\rho(z^{(k)}), z^{(k)})$ and the map $\iota^k : J^k \to J^k$ be defined by $\iota(z^{(k)}) = \rho(z^{(k)}) \cdot z^{(k)}$, then the following diagram commutes for all $k = n, \ldots, \infty$:

$$
\begin{array}{ccc}
 & \mathcal{B}^k = G \times J^k & \\
 & \nearrow \quad \searrow & \\
\sigma^k & & w^k \\
 & & \\
J^k & \xrightarrow[\iota^k]{} & J^k
\end{array}
$$

Later we will omit the superscript k for the maps between jet spaces.

The process of invariantization can be defined as in Definition 1.4.4, where the role of d_N is played by the jet differential d_J. In particular if

$$x^i, \ i = 1, \ldots, p, \qquad u_J^\alpha, \ \alpha = 1, \ldots, q$$

are coordinates on J^∞, then the functions

$$y^i = w^*(x^i), \ i = 1, \ldots, p, \qquad v_J^\alpha = w^*(u_J^\alpha), \ \alpha = 1, \ldots, q$$

are lifted invariants on $\mathcal{B}^\infty$ and the functions

$$\mathcal{X}^i = \imath(x^i), \ i = 1, \ldots, p, \qquad I_J^\alpha = \imath(u_J^\alpha), \ \alpha = 1, \ldots, q$$

form a complete set of differential invariants on J^∞.

Remark 1.4.5. The set of invariants $\{\mathcal{X}^i, \ldots, \mathcal{X}^p, I_J^\alpha, \ \alpha = 1, \ldots, q \ \}$ is complete in a sense that every other differential invariant can be expressed as a function of these invariants. However exactly r of these invariants are functionally dependent on the others. If the cross-section $\mathcal{S}^n$ is chosen as a level set of r coordinates, then the invariantization of these r coordinates produce constant functions, called *phantom* invariants.

Remark 1.4.6. The k-th order signature manifold of a submanifold S (see Definition 1.3.3) can be described as the image of its k-th prolongation $j^k(S)$ under the projection ι^k. Indeed, for $k = n, \ldots, \infty$ the map ι^k projects J^k onto the subset $\mathcal{S}^k$ of codimension $r = \dim G$, parameterized by a complete set of functionally independent invariants. Thus if $S \subset M$ is any submanifold then its k-th order signature manifold is the projection of the k-th jet of S under ι^k: $\mathcal{C}^k(S) = \iota^k(j^k(S))$ onto the cross-section chosen to define the corresponding moving frame. The symmetry and equivalence theorems 1.3.7, 1.3.8, 1.3.10 have a nice geometrical interpretation in terms of this projection. For instance the dimension of the signature manifold decreases when the jet of submanifold $j^{s+1}(S)$ is not transversal to the prolonged orbits, and hence there are infinitesimal generators of the group action which are tangent to $j^{s+1}(S)$. These infinitesimal generators give rise to the symmetry group of S.

Let us return now to the process of invariantization. The invariantization of the basis form $dx^1, \ldots, dx^p, \theta_J^\alpha$:

$$\begin{aligned}
\varpi^i &= \imath(dx) = \sigma^* d_J w^*(x^i), \ i = 1, \ldots, p, \\
\vartheta_J^\alpha &= \imath(\theta_J^\alpha) = \sigma^* \pi_J w^*(\theta_J^\alpha), \ \alpha = 1, \ldots, q
\end{aligned}$$

produces an invariant coframe on J^∞. We recall that jet differential d_J splits into vertical and horizontal components, thus the differential on $\mathcal{B}^\infty$ splits into three components $d = d_G + d_H + d_V$. Definition 1.4.4 was motivated by the fact that such invariantization preserves the contact ideal: invariantization of a contact form is a contact form. The contact ideal is defined intrinsically, while the choice of horizontal forms depends on the choice of coordinates and is not preserved by invariantization, that is an invariantized horizontal form might gain a vertical component. By projecting the invariantization of a horizontal form to its purely horizontal part we obtain a contact invariant form. Let η be any form on J^∞, then $\bar{\eta} = w^*\eta$ is an invariant form on $\mathcal{B}^\infty$. Let π_H denote the projection of a form on its purely horizontal component. Since the action of G preserves the contact ideal then $\pi_H(w^*\eta)$ is a horizontal contact invariant forms on $\mathcal{B}^\infty$, and $\sigma^*\pi_H w^*(\eta)$ is a horizontal contact invariant form on J^∞. In particular forms

$$dy^i = \pi_H w^*(dx^i) = d_H w^*(x^i), \ \ i = 1, \ldots, p$$

form a horizontal contact invariant coframe on $\mathcal{B}^\infty$. The dual vector fields $\bar{\mathcal{D}}_i$ produce a complete set of *lifted invariant differential operators*, such that $v^\alpha_{J,i} = \bar{\mathcal{D}}_i v^\alpha_J$. The forms

$$\omega^i = \sigma^* d_H w^* (x^i) = \sigma^*(d_H y^i), \ \ i = 1, \ldots, p \tag{1.10}$$

form a horizontal contact invariant coframe on J^∞. In general we can call $\sigma^*\pi_H w^*\eta$ the contact invariantization of a form η on J^∞. The vector fields $\mathcal{D}_i$ dual to the forms ω^i provide a complete set of invariant differential operators. We notice that in contrast with invariant differential operators $\mathcal{D}_i$ on J^∞, the lifted operators $\bar{\mathcal{D}}_i$ commute. The formulas which relate normalized invariants with invariants obtained by invariant differentiation are given in ([13], section 13).

Example 1.4.7. Let us return to the special Euclidean action on the plane as described in 1.3.12:

$$x \ \mapsto \ y = \cos(\alpha)x - \sin(\alpha)u + a,$$
$$u \ \mapsto \ v = \sin(\alpha)x + \cos(\alpha)u + b.$$

The lifted contact invariant form $d_H y = \tau dx$, where $\tau = \cos(\alpha) - \sin(\alpha)u_x$. The lifted invariant differential operator $\bar{\mathcal{D}}_x = \frac{1}{\tau}D_x$ and thus the lifted differential invariants are

given by formulas:

$$
\begin{aligned}
v_1 &= \frac{\sin(\alpha) + \cos(\alpha)u_x}{\tau}, \\
v_2 &= \frac{u_{xx}}{\tau^3}, \\
v_3 &= \frac{\tau u_{xxx} + 3\sin(\alpha)u_{xx}^2}{\tau^5}, \\
v_4 &= \frac{\tau^2 u_{xxxx} + 10\sin(\alpha)\tau u_{xx}u_{xxx} + 15\sin^2(\alpha)u_{xx}^3}{\tau^7}.
\end{aligned}
$$

We note that the lifted invariants define the prolongation of the group action: $u_x \mapsto v_1$, $u_{xx} \mapsto v_2$, etc. The moving frame can be defined on $J^1(\mathbb{R}^2, 1)$ by choosing a cross-section $\{x = 0, \, u = 0, \, u_x = 0\}$, and so an equivariant map $J^1(\mathbb{R}^2, 1) \to SE(2)$ can be found by solving the equations:

$$
\begin{aligned}
y &= \cos(\alpha)x - \sin(\alpha)u + a = 0, \\
v &= \sin(\alpha)x + \cos(\alpha)u + b = 0, \\
v_1 &= \frac{\sin(\alpha) + \cos(\alpha)u_x}{\cos(\alpha) - \sin(\alpha)u_x} = 0.
\end{aligned}
$$

Thus we obtain the moving frame:

$$
\alpha = -\arctan(u_x), \qquad a = -\frac{u_x u + x}{\sqrt{1 + u_x^2}}, \qquad b = \frac{u_x x - u}{\sqrt{1 + u_x^2}}. \tag{1.11}
$$

The corresponding element of the special Euclidean group can be written in a matrix form:

$$
\rho_r = \begin{pmatrix} \dfrac{1}{\sqrt{1+u_x^2}} & \dfrac{u_x}{\sqrt{1+u_x^2}} & -\dfrac{uu_x+x}{\sqrt{1+u_x^2}} \\[2ex] -\dfrac{u_x}{\sqrt{1+u_x^2}} & \dfrac{1}{\sqrt{1+u_x^2}} & \dfrac{xu_x-u}{\sqrt{1+u_x^2}} \\[2ex] 0 & 0 & 1 \end{pmatrix}
$$

The differential invariants are obtained by normalization, that is substitution of the moving frame into the lifted invariants v_k:

$$
\begin{aligned}
I_2 &= \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \\
I_3 &= \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3} \\
I_4 &= \frac{(1 + u_x^2)^2 u_{xxxx} - 10u_x u_{xx}u_{xxx}(1 + u_x^2) + 15u_x^2 u_{xx}^3}{(1 + u_x^2)^{9/2}}
\end{aligned}
$$

The contact invariant differential form is

$$\omega = \sigma^*(d_H y) = \sqrt{1 + u_x^2}\, dx = ds$$

We notice that $I_2 = \kappa$, the Euclidean curvature, $I_3 = \kappa_s = \frac{d\kappa}{ds}$ but $I_4 = \kappa_{ss} + 3\kappa^3$, according to recurrence formulas in [13]. As we have seen in Example 1.3.12 the first two invariants κ and κ_s are sufficient to solve the equivalence problem for curves in Euclidean space.

Remark 1.4.8. The action of the special Euclidean group on J^1 is locally free, however the isotropy group of each point (x, u, u_x) contains one nontrivial transformation:

$$\begin{pmatrix} -1 & 0 & 2x \\ 0 & -1 & 2u \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.12}$$

This is reflected in the ambiguity of normalization for α, which is defined up to the addition of πn. Thus the invariants $I_2, I_3, \ldots$ are local. In particular $I_2 = \kappa$ changes its sign under the transformation (1.12).

We conclude this example with the discussion on how the procedure described above corresponds to the classical definition of the Euclidean curvature. The Frénet frame consists of the unit tangent T and the unit normal N attached to each point on a curve $(x, u(x))$, with consistent orientation of the frame. This produces a map $\rho_l : J^1(\mathbb{R}^2, 1) \to SE(2)$. Indeed, the pair (T, N) defines a rotation matrix at each point, and a point $(x, u(x))$ defines a translation vector at each point. This map is equivariant with respect to the action of the group on itself by *left* multiplication, and so it is called a *left* moving frame. The element of the Euclidean group assigned to each point can be written in matrix form:

$$\rho_l = \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & -\frac{u_x}{\sqrt{1+u_x^2}} & x \\ \frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & u \\ 0 & 0 & 1 \end{pmatrix}$$

In this example as well as in general the *left* moving frame is the group inverse of the corresponding *right* moving frame: $\rho_l = (\rho_r)^{-1}$. In matrix form the Frénet equations can be written as:

$$\left(\frac{dT}{ds}, \frac{dN}{ds}, \frac{dX}{ds}\right) = (T, N, X) \begin{pmatrix} 0 & -\kappa & 1 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $X = (x, u, 1)^t$. We recall that $\rho_l = (T, N, X)$ and thus

$$\rho_l^{-1} \frac{d}{ds}(\rho_l)\, ds = \begin{pmatrix} 0 & -\kappa ds & ds \\ \kappa ds & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a pull-back of the Maurer-Cartan forms on $SE(2)$ to the jet of a curve under the G-equivariant map ρ_l. The elements of this matrix are contact invariant forms and their ratio produce the second order differential invariant κ.

Chapter 2

Inductive Construction of Moving Frames.

In this chapter we present two modifications of the moving frame method which were motivated by its practical implementation. We call the first modification a recursive construction because in contrast to the algorithm from chapter one it allows to construct differential invariants order by order. At each step we normalize more and more of the group parameters at the end obtaining a moving frame for the group G. In the next chapter we use this algorithm to construct a complete set of the differential invariants for ternary cubics transformed by linear changes of variables.

The second modification can be used when the group G factors as a product of two subgroups: $G = AB$, such that $A \cap B$ is discrete. In this case invariants and moving frames for A and B can be used to construct invariants and a moving frame for G. Such approach not only simplifies the computations but also produces as a by-product the relations among the invariants of G and its subgroups. We notice that the group of the Euclidean motions on the plane is a factor of the group of the special affine motions. In its turn the special affine motions is a factor of the group of projective transformations on the plane. We use these groups to illustrate our algorithm. As a consequence we obtain the affine curvature in terms of the Euclidean and the projective curvature in terms of the special affine and also the relations among corresponding differential operators.

2.1 Recursive Construction of Moving Frames.

We first sketch the main idea of the algorithm. Assume that G acts on M regularly but not freely. We notice that if $\mathcal{S} \subset M$ is a cross-section to the orbits of G and a smooth map $\rho \colon M \to G$ is defined by the condition $\rho(z) \cdot z \in \mathcal{S}$ for $z \in M$, then the non-constant coordinates of $\rho(z) \cdot z$ provide a complete set of zero order invariants. However in order to build a moving frame recursively we require that $\mathcal{S}$ satisfies a certain condition, namely each point of $\mathcal{S}$ has the same isotropy group. We postpone the discussion of the existence of such a cross-section until later and assume that H_1 is the isotropy group of each point in $\mathcal{S}$. In this case the map ρ_0 defined by the condition $\rho_0(z) \cdot z \in \mathcal{S}$ is a G-equivariant map from M to the right cosets $H_1 \backslash G$. We prolong the action of G to the first order and define the set $\mathcal{S}^1 = \{z^{(1)} | \pi_0^1(z^{(1)}) \in \mathcal{S}\}$. The set $\mathcal{S}^1$ is invariant under the action of H_1 and we assume that there is a cross-section $\mathcal{S}_1^1 \subset \mathcal{S}^1$ with a constant isotropy group H_2. We use this cross-section to define a map $\rho_1 \colon \mathcal{S}^1 \to H_2 \backslash H_1$, by the condition $\rho_1(z^{(1)}) \cdot z^{(1)} \in \mathcal{S}_1^1$. The map $\rho_1\left(\rho_0(z^{(1)}) \cdot z^{(1)}\right)\rho_0(z^{(1)})$ is a G-equivariant map from J^1 to $H_2 \backslash G$. The non-constant coordinates of $\rho_1\left(\rho_0(z^{(1)}) \cdot z^{(1)}\right)\rho_0(z^{(1)}) \cdot z^{(1)}$ provide a complete set of the first order invariants. We continue this process by prolonging the action of H_2 to the next order, or we may prolong by several orders at once if we wish. The algorithm terminates at the order where the isotropy group becomes trivial. For regular jets this happens at the order of stabilization of the group. This procedure resembles in many ways the algorithm presented by M. Green [19] for constructing moving frames for curves in homogeneous spaces, however taking advantage of the generalized approach by Fels and Olver [13], we can apply our algorithm to construct a moving frame for submanifolds of any dimension under more general (not necessarily transitive) group actions.

Following [16] we call a cross-section with a constant isotropy group a *slice.* The obvious necessary condition for the existence of a slice in a neighborhood U of $z_0 \in M$ is that the isotropy groups of any two points in U are conjugate by an element of G. There is a simple counterexample of this phenomenon ([35], Example 1,§ 7).

Example 2.1.1. Let $\mathbb{R}^2$ act on $\mathbb{R}^2$ by

$$x \to x + au + b, \qquad u \to u$$

The isotropy group of a point (x_0, u_0) is defined by the condition $au_0 + b = 0$. The orbits are lines parallel to the x-axis. All the points that lie on the same orbit have equal isotropy groups. On the other hand the isotropy groups of two points from different orbits are not equal and they are not conjugate because the group is commutative.

However, one can find slices for many common group actions. In particular slices exist if the group action is proper [16].

Definition 2.1.2. The action of G is called *proper* if the map $\theta : G \times M \to M \times M$ defined by $\theta(g, z) = (g \cdot z, z)$ is proper. In other words if $K \subset M \times M$ is compact then so is $\theta^{-1}(K) \subset G \times M$.

Since $\theta^{-1}(z, z) = (H, z)$ where H is an isotropy group of z, if the action is proper, then the isotropy group of each point is compact.

Proposition 2.1.3. *Let G act regularly on a manifold M and let $\mathcal{S}$ be a slice. Then the condition $\rho(z) \cdot z \in \mathcal{S}$ for $z \in M$ defines a map $[\rho] : M \to H \backslash G$ which is G-equivariant.*

Proof. Let group elements g_1 and g_2 be such that $g_1 \cdot z \in \mathcal{S}$ and $g_2 \cdot z \in \mathcal{S}$. Since each orbit intersects the slice $\mathcal{S}$ at one point, then: $g_1 \cdot z = g_2 \cdot z$ and so $g_2^{-1} g_1$ belongs to the isotropy group G_z of z. On the other hand since H is the isotropy group of $g_2 \cdot z$, then $G_z = g_2^{-1} H g_2$. Thus $g_2^{-1} g_1 \in g_2^{-1} H g_2$ and hence $g_1 \in H g_2$. We have proved that the map $[\rho]$ is well defined.

To show the equivariance of $[\rho]$ we need to prove that $[\rho](g \cdot z) = [\rho](z) g^{-1}$. Let us choose $q \in [\rho](z)$ and $\tilde{q} \in [\rho](g \cdot z)$ By the construction of $[\rho]$ one has

$$\tilde{q} g \cdot z = q \cdot z \in \mathcal{S}.$$

It follows that $q^{-1} \tilde{q} g \in G_z = q^{-1} H q$, or equivalently

$$\tilde{q} \in H q g^{-1}.$$

Since $[\rho](z) = Hq$ and $[\rho](gz) = H\tilde{q}$ we have proved that $[\rho](g \cdot z) = [\rho](z) g^{-1}$. $\qquad \square$

We can extend $[\rho]$ to a G-equivariant map on J^k by $[\rho](z^{(k)}) = [\rho]\left(\pi_0^k(z^{(k)})\right)$. Locally we can choose a section $s \colon H \backslash G \to G$, such that $s[H] = e$ and define the map

$$\rho_s = s \circ [\rho] : J^k(M, p) \to G. \tag{2.1}$$

The map ρ_s is not G-equivariant, but it is *G-equivariant up to the left action of H*, and so the functions $f\left(\rho_s(z^{(k)}) \cdot z^{(k)})\right)$, where f is any function on J^k are invariant up to the left action of H.

Proposition 2.1.4. *There exists element $h \in H$ such that*

$$\rho_s(g \cdot z^{(k)}) = h\rho_s(z^{(k)})g^{-1}.$$

Proof. Since $[\rho]$ is a G-equivariant map from J^k to $H\backslash G$ then

$$\rho_s(g \cdot z^{(k)}) = s[\rho(g \cdot z^{(k)})] = s[\rho(z^{(k)})g^{-1}] = s[H\rho_s(z^{(k)})g^{-1}] \in H\rho_s(z^{(k)})g^{-1}.$$

Thus there exists $h \in H$ such that $\rho_s(g \cdot z^{(k)}) = h\rho_s(z^{(k)})g^{-1}$. $\qquad\square$

Let $\mathcal{S}^k = \{z^{(k)} | \pi_0^k(z^{(k)}) \in \mathcal{S}\}$ be the subset of J^k which projects to $\mathcal{S}$. In other words the set $\mathcal{S}^k$ is a pull-back of the fiber bundle $J^k \to M$ under the inclusion $\mathcal{S} \to M$. By construction $\mathcal{S}^k$ is invariant under the prolongation of the H-action.

Let the map $w : \mathcal{B}^k \to J^k$ be defined by the prolonged group action, that is $w(g, z^{(k)}) = g \cdot z^{(k)}$, the map $\sigma_0 \colon J^k \to \mathcal{B}^k$ be defined by $\sigma_0(z^{(k)}) = (\rho_s(z^{(k)}), z^{(k)})$ and the map $\iota_0 : J^k \to J^k$ be defined by $\iota_0(z^{(k)}) = \rho_s(z^{(k)}) \cdot z^{(k)}$ for $k = n, \ldots, \infty$, then we obtain a commutative diagram similar to the one in Section 1.4 of Chapter 1:

$$
\begin{array}{ccc}
 & \mathcal{B}^k = G \times J^k & \\
 \nearrow^{\sigma_0} & & \searrow^{w} \\
 J^k \xrightarrow{\quad\iota_0\quad} & & J^k
\end{array}
$$

If f is any function on J^k then due to the proposition above the function $\iota_0^*(f)$ is invariant under G up to an element of H. We note that ι_0 projects J^k to $\mathcal{S}^k$ and thus non-constant coordinates of $\iota_0(z^{(k)})$, restricted to $\mathcal{S}^k$ are transformed exactly in the same way as coordinate functions on $\mathcal{S}^k$. More generally if f is any function on J^k then $\iota_0^*(f)$ restricted to $\mathcal{S}^k$ equals to f restricted to $\mathcal{S}^k$ and so as functions on $\mathcal{S}^k$ they are transformed exactly in the same way. This trivial observation is used in both algorithms and so we state it as a proposition.

Proposition 2.1.5. *Assume that $S \subset N$ be a submanifold of N invariant under the action of the group H and there is a smooth projection $\iota : N \to S$, that is $\iota(z) \in S$ for any point $z \in N$ and $\iota(\tilde{z}) = \tilde{z}$ for any point $\tilde{z} \in S$. Let $f(z)$ be function on N and let $\tilde{f}$ be its restriction to S. We define a function $F(z)$ by the formula $F(z) = f(\iota(z))$ and denote its restriction to S as $\tilde{F}$. Then $\tilde{F} = \tilde{f}$ and hence $h \cdot \tilde{F} = h \cdot \tilde{f}$ for all $h \in H$.*

Remark 2.1.6. Note that the proposition above *does not* assert that $h \cdot F = h \cdot f$ on N, or equivalently, in general $f(\iota(h \cdot z)) \neq f(h \cdot z)$.

The following proposition asserts that $S^n \cap V^n \neq \emptyset$ and hence one can construct a moving frame:

$$\rho_H : S^n \to H. \tag{2.2}$$

Proposition 2.1.7. *Assume that S is a cross-section for an action of a group G on a manifold M. Let n be the order of stabilization, V^n be the regular set and $S^n = \{z^{(n)} | \pi_0^n(z^{(n)}) \in S\} \subset J^n(M, p)$, then $S^n \cap V^n \neq \emptyset$.*

Proof. The statement $S^n \cap V^n = \emptyset$ implies that $\pi_0^n(V^n) \cap S = \emptyset$, that is there are no p-dimensional submanifolds of M passing through any point $z \in S$ such that its n-th prolongation $j^n(S)$ at z belongs to V^n. But this means that for any $g \in G$, there are no submanifolds through the point $g \cdot z$ that give rise to a regular n-th jet. Thus $\{G \cdot S\} \cap \pi_0^n(V^n) = \emptyset$. The set $G \cdot S$ is an open subset of M and hence we arrive to a contradiction with the assertion that the set V^n is dense in J^n if the action of G [30]. $\qquad\square$

Remark 2.1.8. Although the map $\rho_H : S^n \to H$ is H-equivariant, its extension on J^n defined by $\rho_H(\iota_0(z^{(n)}))$ is *not* an H-equivariant map from J^n to H.

Proposition 2.1.9. *The map $\rho_G : J^n(M, p) \to G$ defined by*

$$\rho_G(z^{(n)}) = \rho_H(\iota_0(z^{(n)})) \, \rho_s(z^{(n)}). \tag{2.3}$$

is G-equivariant.

Proof. We recall that $\iota_0(z^{(n)}) = \rho_s(z^{(n)}) \cdot z^{(n)}$. From Proposition 2.1.4 we know that for an element $g \in G$ there is an element $h \in H$ such that

$$\rho_s(g \cdot z^{(n)}) = h \rho_s(z^{(n)}) g^{-1}.$$

and so

$$\iota_0(g \cdot z^{(n)}) = h\rho_s(z^{(n)}) \cdot z^{(n)} = h \cdot \iota_0(z^{(n)}).$$

Thus

$$
\begin{aligned}
\rho_G(g \cdot z^{(n)}) &= \rho_H\left(\iota_0(g \cdot z^{(n)})\right)\rho_s(g \cdot z^{(n)}) \\
&= \rho_H\left(h \cdot \iota_0(z^{(n)})\right)h\rho_s(z^{(n)})g^{-1} = \rho_H\left(\iota_0(z^{(n)})\right)\rho_s(z^{(n)})g^{-1}.
\end{aligned}
$$

In the last equality we have used H-equivariance of the map ρ_H.   □

Since ρ_G is a moving frame, then non-constant coordinates of $\iota_G = \rho_G(z^{(n)}) \cdot z^{(n)}$ provide a complete set of n-th order differential invariants.

Let $[g]$ denote the equivalence class of g in $H\backslash G$, let $U_{H\backslash G}$ be a neighborhood of H in $H\backslash G$ where the local section $s\colon H\backslash G \to G$ is defined, and let $e \in U_G \subset G$ be the preimage of $U_{H\backslash G}$ under the canonical projection $G \to H\backslash G$. Then the following two maps are local diffeomorphisms:

$$
\begin{aligned}
\phi\colon U_G \times J^\infty \to H \times U_{H\backslash G} \times J^\infty &\quad:\quad \phi(g, z^{(\infty)}) = \left(g(s[g])^{-1}, [g], z^{(\infty)}\right), \\
\psi\colon H \times U_{H\backslash G} \times J^\infty \to &\quad:\quad \psi(h, [g], z^{(\infty)}) = \left(h\, s[g],\, z^{(\infty)}\right).
\end{aligned}
$$

We can summarize the recursive construction in the following commutative diagram:

where the maps w are defined by the group actions:

$$
\begin{aligned}
w_0(h, [g], z^{(\infty)}) &= (h, s[g] \cdot z^{(\infty)}), \\
w_H(h, z^{(\infty)}) &= h \cdot z^{(\infty)}, \\
w_G(g, z^{(\infty)}) &= g \cdot z^{(\infty)}.
\end{aligned}
$$

The maps σ are defined from the maps ρ (see formulas (2.1, 2.2, 2.3):

$$\sigma_H(z^{(\infty)}) = \left(\rho_H(\rho_s(z^{(\infty)}) \cdot z^{(\infty)}), z^{(\infty)}\right),$$
$$\sigma_0(h, z^{(\infty)}) = \left(h, [\rho_s(z^{(\infty)})], z^{(\infty)}\right),$$
$$\sigma_G(g, z^{(\infty)}) = \left(\rho_G(z^{(\infty)}), z^{(\infty)}\right),$$

and the maps ι are projections:

$$\iota_0(h, z^{(\infty)}) = (h, \rho_s(z^{(\infty)}) \cdot z^{(\infty)}),$$
$$\iota_G(z) = \rho_H(\rho_s(z^{(\infty)}) \cdot z^{(\infty)}) \cdot z^{(\infty)}.$$

Example 2.1.10. Let the special rotation group $G = SO(3, \mathbb{R})$ act on $M = \mathbb{R}^3 \backslash \{0\} \times \mathbb{R}$ by rotations on the independent variables x, y, z and the trivial action on the dependent variable u.

The action of $SO(3, \mathbb{R})$ on $\mathbb{R}^3$ is not free. The isotropy group of every point on the positive half of the z-axis consists of the rotations around the z-axis. On the other hand each orbit of $SO(3, \mathbb{R})$ intersects the positive half of the z-axis at the unique point, and hence it can serve as a slice $\mathcal{S}$ with the isotropy group $H \simeq SO(2, \mathbb{R})$

Our first step is to construct the map $\rho_s : M \to G$ such that $\rho_s(p) \cdot p \in \mathcal{S}$ for each $p \in M$ and $\rho_s(p_0) = \mathrm{I}$ for any $p_0 \in \mathcal{S}$.

Each coset $G \backslash H$ can be represented by the product of two rotations: $R_x(\theta)$ with respect to the x-axis and $R_y(\tau)$ with respect to the y-axis. In matrix form this can be written as

$$\begin{pmatrix} \cos\tau & 0 & -\sin\tau \\ 0 & 1 & 0 \\ \sin\tau & 0 & \cos\tau \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\tau & -\sin\tau\sin\theta & -\sin\tau\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ \sin\tau & \cos\tau\sin\theta & \cos\tau\cos\theta \end{pmatrix}$$

We choose the first rotation $R_x(\theta)$ so that it brings an arbitrary point $p = (x, y, z)$ to the upper xz-plane. It can be achieved by choosing $\theta = \arctan(\frac{y}{z})$, then $\tilde{p} = R_x(\theta) \cdot p = (x, 0, \sqrt{z^2 + y^2})$. We choose the rotation $R_y(\tau)$ so that it brings $\tilde{p}$ to the z-axis. We take $\tau = \arctan(\frac{x}{\sqrt{z^2+y^2}})$ and then $R_y(\tau) \cdot \tilde{p} = (0, 0, \sqrt{z^2 + y^2 + x^2})$ lies

on the positive z-axis. In the matrix form

$$\rho_s(p) = R_y\,R_x = \begin{pmatrix} \frac{\sqrt{z^2+y^2}}{r} & -\frac{yx}{r\sqrt{z^2+y^2}} & -\frac{zx}{r\sqrt{z^2+y^2}} \\ 0 & \frac{z}{\sqrt{z^2+y^2}} & -\frac{y}{\sqrt{z^2+y^2}} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{pmatrix},$$

where $r = \sqrt{z^2 + y^2 + x^2}$. The only non-constant coordinate of $\rho_s(p) \cdot p = (0, 0, r)$ provides a zero order invariant, the radius. We note that $\rho_s(0, 0, z) = \mathrm{I}$ for $z > 0$ and hence the conditions of Proposition 2.1.5 are satisfied.

On the next step we prolong the transformation to the first jet bundle J^1. We consider the set $\mathcal{S}^1 = \{(u, x, y, z, u_x, u_y, u_z)| x = 0, y = 0, z > 0\} \subset J^1$, which projects to $\mathcal{S}$ and is invariant under the first prolongation of the H action. The first prolongation of the $\rho_s(p)$ transforms a point $(u, x, y, z, u_x, u_y, u_z) \in J^1$ to a point $(U, X, Y, Z, U_x, U_y, U_z)$ in $\mathcal{S}^1$ where

$$
\begin{aligned}
u \to U &= u \\
x \to X &= 0 \\
y \to Y &= 0 \\
z \to Z &= \sqrt{z^2 + y^2 + x^2} = r \\
u_x \to U_x &= \frac{\sqrt{z^2 + y^2}}{r} u_x - \frac{yx}{r\sqrt{z^2 + y^2}} u_y - \frac{zx}{r\sqrt{z^2 + y^2}} u_z; \\
u_y \to U_y &= \frac{z}{\sqrt{z^2 + y^2}} u_y - \frac{y}{\sqrt{z^2 + y^2}} u_z; \\
u_z \to U_z &= \frac{x}{r} u_x + \frac{y}{r} u_y + \frac{z}{r} u_z.
\end{aligned}
$$

From Proposition 2.1.5 we know that functions $\{U, Z, U_x, U_y, U_z\}$ are transformed on $\mathcal{S}^1$ by H by the same formulas as $\{u, z, u_x, u_y, u_z\}$. Let us represent an element of H by the matrix:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{pmatrix},$$

then

$$\begin{aligned}
\bar{U} &= U \\
\bar{Z} &= Z; \\
\bar{U}_x &= \cos\alpha\, U_x - \sin\alpha\, U_y; \\
\bar{U}_y &= \sin\alpha\, U_x + \cos\alpha\, U_y; \\
\bar{U}_z &= U_z.
\end{aligned}$$

We observe that H acts freely on $\mathcal{S}^1$. We find a moving frame ρ_H by setting $\bar{U}_x = 0$, and hence $\tan\alpha = \frac{U_x}{U_y}$. Substitution of this normalization into $\bar{Z}, \bar{U}_y, \bar{U}_z$, produce invariants of $G = SO(3)$:

$$U, \quad Z, \quad I_y = \sqrt{U_x^2 + U_y^2}, \quad I_z = U_z;$$

In terms of coordinates on J^1 they can be written as:

$$\begin{aligned}
U &= u \\
Z &= \sqrt{z^2 + y^2 + x^2} = r; \\
I_y &= \sqrt{(yu_z - zu_y)^2 + (zu_x - xu_z)^2 + (xu_y - yu_x)^2}; \\
I_z &= \frac{x}{r}u_x + \frac{y}{r}u_y + \frac{z}{r}u_z;
\end{aligned}$$

The corresponding moving frame for the group G is a product:

$$\rho_H\rho_s = \begin{pmatrix} \frac{U_y}{\sqrt{U_x^2+U_y^2}} & -\frac{U_x}{\sqrt{U_x^2+U_y^2}} & \\ \frac{U_x}{\sqrt{U_x^2+U_y^2}} & \frac{U_y}{\sqrt{U_x^2+U_y^2}} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{z^2+y^2}}{r} & -\frac{yx}{r\sqrt{z^2+y^2}} & -\frac{zx}{r\sqrt{z^2+y^2}} \\ 0 & \frac{z}{\sqrt{z^2+y^2}} & -\frac{y}{\sqrt{z^2+y^2}} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{pmatrix}.$$

The action of H became free on the first jet in the example above. If it is not, we would have to either prolong the action up to the order where it is free or to repeat our algorithm for H acting on $\mathcal{S}^1$. This can be done if there is a slice $\mathcal{S}_1^1 \subset \mathcal{S}^1$ with a constant isotropy group $H_1 \subset H$. The following procedure describes order by order construction of invariants under the assumption that slices exist at each order. In our notation the superscripts refer to the order of prolongation and the subscripts refer to the induction step.

Algorithm 2.1.11. On the zeroth step we consider the action of the group $G = H_0$ such that there is a slice $\mathcal{S}_0 \subset M$ with a constant isotropy group H_1. We define the map $[\rho_0] : M \to H_1 \backslash G$ such that $\rho_0(z) \cdot z \in \mathcal{S}_0$ and $\rho_0(\mathcal{S}_0) = e$, where $\rho_0(z)$ is defined by a local section from $H_1 \backslash G$ to G. The non-constant coordinates of

$$\iota_0 = \rho_0(z) \cdot z,$$

provide a complete set of functionally independent zero order invariants. Let $\mathcal{S}_0{}^k = \{z^{(k)} \in J^k | \pi_0^k(z^{(k)}) \in \mathcal{S}_0\}$ and $\rho_0(z^{(k)}) = \rho_0(\pi_0^k(z^{(k)}))$, then the map $\iota_0 : J^k \to \mathcal{S}_0^k$ defined by

$$\iota_0(z^{(k)}) = \rho_0(z^{(k)}) \cdot z^{(k)}$$

is invariant under the action of G up to an element of H_1 as described in Proposition 2.1.4.

Let $\mathcal{S}_1^1 \subset \mathcal{S}_0^1$ be a slice for the action of H_1 on $\mathcal{S}_0^1$ such that the isotropy group of each point equals H_2 and let the map $[\rho_1] : \mathcal{S}_0^1 \to H_2 \backslash H_1$ be defined by the conditions $\rho_1(\iota_0(z^{(1)})) \cdot \iota_0(z^{(1)}) \in \mathcal{S}_1^1$ and $\rho_1(\mathcal{S}_1^1) = e$, where $\rho_1(z)$ is defined by a local section from $H_2 \backslash H_1$ to H_1. Then the non-constant coordinates of

$$\iota_1 = \rho_1\left(\iota_0(z^{(1)})\right) \cdot \iota_0(z^{(1)})$$

provide a complete set of functionally independent first order invariants (this set includes zero order invariants). Let $\mathcal{S}_1{}^k = \{z^{(k)} \in J^k | \pi_1^k(z^{(k)}) \in \mathcal{S}_1^1\}$ and $\rho_1(z^{(k)}) = \rho_1(\pi_1^k(z^{(k)}))$ then the map $\iota_1 : J^k \to \mathcal{S}_1^k$ is defined by

$$\iota_1(z^{(k)}) = \rho_1(\iota_0(z^{(k)})) \cdot \iota_0(z^{(k)}).$$

The group product

$$\rho_1\left(\iota_0(z^{(1)})\right) \rho_0(z^{(1)}) \tag{2.4}$$

defines a G-equivariant map from J^1 to the right cosets $H_2 \backslash G$. If the isotropy group $H_2 = e$ on $\mathcal{S}_0^1$, the coordinates of $\iota_1(z^{(k)})$ are invariant and the product (2.4) is a moving frame for G. Otherwise, we have to prolong the action of H_2 on $\mathcal{S}_1^1$ to the second order and repeat the algorithm. In the case when $\mathcal{S}_0^n = \{z^{(n)} \in J^n(M) | \pi_0^n(z^{(n)}) \in \mathcal{S}_0\}$ belongs to the regular set $\mathcal{V}^n$, this process will terminate at the order of stabilization n. Indeed on the n-th step we consider the action of the isotropy group H_n on the set $\mathcal{S}_{n-1}^n$. Any element of Lie algebra of G generates a nontrivial transformation

on $\mathcal{V}^n$ and hence the action of H_n on $\mathcal{S}_{n-1}^n \subset \mathcal{S}_0^n$ is free. We choose a cross-section $\mathcal{S}_n^n \subset \mathcal{S}_{n-1}^n$ transversal to the action of H_n and construct a corresponding H_n-invariant map $\rho_n : \mathcal{S}_{n-1}^n \to H_n$. The map $\rho_G = J^n \to G$ defined by

$$\rho_G = \rho_n(\iota_{n-1}(z^{(n)}))\rho_{n-1}(\iota_{n-2}z^{(n)})\ldots\rho_1(\iota_0(z^{(n)}))\rho_0(z^{(n)})$$

is a moving frame for G. The complete set of n-th order differential invariants is given by the non-constant coordinates of

$$\iota_n = \rho_n(\iota_{n-1}(z^{(n)})) \cdot \iota_{n-1}(z^{(n)}).$$

In the next chapter we will return to this algorithm to construct a complete set of differential invariants for ternary cubics.

2.2 A Moving Frame Construction for a Group that Factors as a Product.

The moving frame construction is simpler for a group with a smaller number of parameters. Thus it is desirable to use a moving frame for a subgroup of G to construct a moving frame for G. We say that a group G factors as a product of its subgroups A and B if $G = AB$, that is for any $g \in G$ there are $a \in A$ and $b \in B$ such that $g = ab$. We reproduce two useful statements from [16].

Theorem 2.2.1. *Let G be a group, A and B are two subgroups. Then the following conditions are equivalent:*
    a) the reduction of the natural action of G on G/B to A is transitive,
    b) $G = AB$,
    c) $G = BA$,
    d) the reduction of the natural action of G on G/A to B is transitive.

Corollary 2.2.2. *The reduction of the natural action of G on G/B to A is free and transitive if and only if $G = AB$ (or $G = BA$) and $A \cap B = e$.*

Remark 2.2.3. If $G = AB$ and $A \cap B = e$ then for each $g \in G$ there are *unique* elements $a \in A$ and $b \in B$ such that $g = ab$. In this case the manifold $A \times B$ is

diffeomorphic to G and we will write $A \times B \backsim G$ to denote that two Lie groups are diffeomorphic as manifolds but are *not* necessarily isomorphic as groups. In the case when $A \cap B$ is discrete then $A \times B$ is locally diffeomorphic to G.

The following theorem plays a central role in the construction of a moving frame for a product of two groups.

Theorem 2.2.4. *Let A and B act regularly on a manifold M and assume that in a neighborhood of a point z_0 the infinitesimal generators of the A-action are linearly independent from the generators of the B-action. Then locally there exists a submanifold $\mathcal{S}_A$ through the point z_0, which is transverse to the orbits of the subgroup A and is invariant under the action of the subgroup B.*

Proof. Let a be the dimension of the A-orbits, b be the dimension of the B-orbits on U and $m = \dim M$. By Frobenius' theorem we can locally rectify the orbits of B, that is we can introduce coordinates $\{y_1, \ldots, y_b, x_1, \ldots, x_{m-b}\}$ such that the orbits of B are defined by the equations $x_i = k_i$, $i = 1, \ldots, m-b$, where k_i are some constants. The orbits of B are integral manifolds for the distribution $\{\frac{\partial}{\partial y_1} \ldots \frac{\partial}{\partial y_b}\}$. The functions x_i are invariant under the B-action. Let vector fields $X_1, \ldots, X_a$ and $Y_1, \ldots, Y_b$ be a basis for infinitesimal generators of the action of A and B respectively in a neighborhood U containing z_0. The vector fields $Y_i, i = 1, \ldots, b$ and $\frac{\partial}{\partial x_j}$, $j = 1 \ldots m-b$ are linearly independent by the choice of coordinates and their union forms a basis in TU. We can choose $c = m - b - a$ vector fields $\frac{\partial}{\partial x_{j_1}} \ldots \frac{\partial}{\partial x_{j_c}}$ which are linearly independent from $X_1, \ldots, X_a$ in TU. Let $\mathcal{S}_A$ be an integral manifold through the point z_0 for the involutive distribution $\Delta = \{\frac{\partial}{\partial x_{j_1}} \ldots \frac{\partial}{\partial x_{j_c}}, \frac{\partial}{\partial y_1} \ldots \frac{\partial}{\partial y_b}\}$. By construction $\mathcal{S}_A$ is invariant under the action of B (it is a union of the orbits of B). On the other the distribution Δ is transversal to the infinitesimal generators $X_1, \ldots, X_a$ of the A-action, and so is transversal to the orbits of A. $\square$

With this result we construct a moving frame for a product of groups A and be B as follows.

Algorithm 2.2.5. Let $G = BA$ and let $B \cap A$ be discrete. Since we are constructing a local moving frame, that is a map to a neighborhood of the identity of the group we may assume that $B \cap A = e$. Thus an element $g \in G$ can be written as a product $g = ba$, $a \in A$, $b \in B$ and $G \backsim B \times A$ in the category of smooth manifolds (but not as

groups). Let n be the order of stabilization of the G-action. Since both A and B act freely on $\mathcal{V}^n \in J^n$ and their intersection is trivial then the infinitesimal generators of the A-action and the B-action are linearly independent at each point of $\mathcal{V}^n$ and hence they satisfy the conditions of Theorem 2.2.4. Thus there is a cross-section $\mathcal{S}_A \subset \mathcal{V}^n$ for the action of A which is invariant under the action of B. We use this cross-section to construct a moving frame ρ_A for A. The map $\iota_A = \rho_A(z^{(n)}) \cdot z^{(n)}$ projects $\mathcal{V}^n$ on the cross-section $\mathcal{S}_A$, which is invariant under the action of B. Moreover the action of B on $\mathcal{S}_A$ is locally free and hence we can choose a cross-section $\mathcal{S} \subset \mathcal{S}_A$ that defines a moving frame $\rho_B : \mathcal{S}_A \to B$. We can extend ρ_B to a map $\tilde{\rho}_B : \mathcal{V}^n \to B$, by the formula

$$\tilde{\rho}_B = \rho_B \left(\rho_A(z^{(n)}) \cdot z^{(n)} \right). \tag{2.5}$$

The map $\tilde{\rho}_B$ is A-invariant but, in contrast to ρ_B, it is not B-equivariant.

The cross-section $\mathcal{S}$ is transversal to the orbits of G and the map ρ_G defined by

$$\rho_G(z^{(n)}) = \tilde{\rho}_B \left(z^{(n)} \right) \rho_A(z^{(n)}) \tag{2.6}$$

satisfy the condition $\rho_G(z^{(n)}) z^{(n)} \in \mathcal{S}$, and hence is a moving frame for the G-action. The maps

$$\iota_G^k(z^{(k)}) = \tilde{\rho}_B \left(z^{(n)} \right) \rho_A(z^{(k)}) \cdot z^{(k)}$$

define projections of J^k onto $\mathcal{S}^k$, for $k = n, \ldots, \infty$ and so the non-constant coordinate functions of

$$\rho_B \left(\rho_A(z^{(k)}) \cdot z^{(k)} \right) \rho_A(z^{(k)}) \cdot z^{(k)}$$

provide a complete set of k-th order differential invariants for G.

Remark 2.2.6. We notice that the coordinates of $\rho_A(z^{(k)}) \cdot z^{(k)}$ are invariant under the A-action and thus the formula above expresses the invariants of the G-action in terms of the invariants of its subgroup A.

We can summarize our construction in the following commutative diagram:

$$G \times J^\infty \qquad \curvearrowleft \qquad B \times A \times J^\infty \qquad \curvearrowleft \qquad G \times J^\infty$$

with maps σ_A, w_A, $\tilde{\iota}_A : B \times J^\infty \to B \times J^\infty$, σ_G, σ_B, w_B, w_G, $J^\infty \xrightarrow{\iota_G} J^\infty$.

where the maps w are defined by the prolonged group action for $k = 1, \ldots, \infty$:

$$
\begin{aligned}
w_A(b, a, z^{(k)}) &= (b, a \cdot z^{(k)}), \\
w_B(b, z^{(k)}) &= b \cdot z^{(k)}, \\
w_G(g, z^{(k)}) &= g \cdot z^{(k)} = w_B \circ w_A(b, a, z^{(k)}) \text{ where } g = ba.
\end{aligned}
$$

The maps σ are defined from moving frames for A, B and G for $k = n, \ldots, \infty$ (see formulas $(2.5, 2.6)$):

$$
\begin{aligned}
\sigma_A(b, z^{(k)}) &= (b, \rho_A(z^{(k)}), z^{(k)}) \\
\sigma_B(z^{(k)}) &= (\tilde{\rho}_B(z^{(k)}), z^{(k)}), \\
\sigma_G(z^{(k)}) &= (\rho_G(z^{(k)}), z^{(k)}) = \sigma_A \circ \sigma_B(z^{(k)}).
\end{aligned}
$$

The maps ι are projections:

$$
\begin{aligned}
\tilde{\iota}_A(b, z^{(k)}) &= (b, \rho_A(z^{(k)}) \cdot z^{(k)}) : B \times J^k \to B \times \mathcal{S}_A \\
\iota_G(z^{(k)}) &= \rho_G(z^{(k)}) \cdot z^{(k)} : J^k \to \mathcal{S}
\end{aligned}
$$

We remind the reader that although all maps are written as global they might be only defined on an open subset of J^k and in neighborhoods of the identities of the groups A, B and G. The manifolds $B \times A \times J^k$ and $G \times J^k$ are diffeomorphic, and this diffeomorphism is A-equivariant. The maps $w_A, w_G, \tilde{\iota}_A$, and ι_G are A-invariant, whence the maps σ_A and σ_B are A-equivariant, with respect to the action defined by:

$$
\begin{aligned}
\tilde{a} \cdot (b, a, z^{(k)}) &= (b, a\tilde{a}^{-1}, \tilde{a} \cdot z^{(k)}), \\
\tilde{a} \cdot (g, z^{(k)}) &= (g\tilde{a}^{-1}, \tilde{a} \cdot z^{(k)}), \\
\tilde{a} \cdot (b, z^{(k)}) &= (b, \tilde{a} \cdot z^{(k)}).
\end{aligned}
$$

We note that neither σ_A nor σ_B is B-equivariant, but their composition is. As was shown in Section 1.4 (see formula (1.10)) the forms

$$\omega^i_G = \sigma^*_G \, d_H \, w^*_G(x^i), \ \ i = 1, \ldots, p \tag{2.7}$$

form a horizontal contact invariant coframe on J^∞. Since A is a subgroup of G then the forms ω^i_G retain their invariant properties under the action of A. On the other hand the moving frame ρ_A provide us with another horizontal coframe which is contact invariant under the action of A:

$$\omega^i_A = \sigma^*_A \, d_H \, w^*_A(x^i), \ \ i = 1, \ldots, p.$$

The two coframes are related by a linear transformation $w^i_G = \sum j = 1^p L^i_j w^j_A$, where L^i_j are functions on J^∞ invariant under the A-action. In fact, L^i_j can be explicitly expressed in terms of the basis invariants of A, indeed:

$$\begin{aligned} \omega^i_G &= \sigma^*_B \, \sigma^*_A \, d_H \, w^*_A \, w^*_B(x^i) \tag{2.8} \\ &= \sigma^*_B \sigma^*_A \pi_H w^*_A d_H \chi^i(b_1, \ldots, b_l, x^1, \ldots, x^p, u^\alpha_J) = \sigma^*_B \imath_A(d_H \chi^i), \end{aligned}$$

where $\chi^i = w^*_B x^i$ is a function on $B \times J^\infty$, written in local coordinates and $\imath_A$ denotes (contact) invariantization with respect to the A-action. The forms $\tilde\omega^i_A = \imath_A(d_H \chi) = \sigma^*_A \pi_H w^*_A d_H \chi^i$ provide a horizontal coframe on $B \times J^\infty$ which is contact invariant with respect to the action of A. A form $\tilde\omega^i_A$ is obtained from $d_H \chi^i$ by replacing forms dx^j with ω^j_A and coordinate functions $x^1, , \ldots, x^p, u^\alpha_J$ with the corresponding fundamental invariants $\mathcal{X}^{(A)1}, , \ldots, \mathcal{X}^{(A)p}, I^{(A)\alpha}_J$. The final pull-back σ^*_B is equivalent to the replacement of parameters $b_1, \ldots, b_l$ with the corresponding coordinates of $\rho_B(\rho_A(z^{(\infty)}) \cdot z^{(\infty)})$. The latter are expressed in terms of invariants of the A-action.

In many situations the following reformulation of Theorem 2.2.1 enables us to enlarge a moving frame for a transformation group A to a moving frame for a larger group containing A.

Theorem 2.2.7. *Let $\mathcal{O} \subset M$ be an orbit of G and let A be a subgroup which acts transitively on $\mathcal{O}$. Then $G = HA$, where H is the isotropy group of a point in $\mathcal{O}$. If in addition A acts locally freely on $\mathcal{O}$ then $A \cap H$ is discrete.*

Let A act regularly on M and let n_A be the order of stabilization for A, then the action of A is (locally) free on a subset $\mathcal{V}_A \subset J^{n_A}(M, p)$. Assume that the action of

A can be extended to the action of a group G, containing A so that there is a point $z_0 \in \mathcal{V}_A$ such that the orbits of A and G through z_0 coincide. If this is the case then let H be the isotropy group of the point z_0. Due to the theorem above $G = HA$ and $A \cap H$ is discrete and so Algorithm 2.2.5 can be applied.

The situation is especially favorable if the action of A on the regular set $\mathcal{V}_A \subset J^{n_A}(M, p)$ is transitive. In this case we can extend a moving frame for A to a moving frame for any group G containing A. Let H be an isotropy group for a point z_0 in $\mathcal{V}_A$, then as above $G = HA$ and the intersection $A \cap H$ is finite. Moreover the point z_0 is a cross-section to the action of G which is invariant under the action of H. We use z_0 to define a moving frame $\rho_A : J^{n_A} \to A$. Let n be the order of stabilization for the H-action and let $\mathcal{S}_A^n = \{z^{(n)} | \pi_0^n(z^{(n)}) = z_0\}$. The non-constant coordinates of $\rho_A(z^{(n)}) \cdot z^{(n)}$ are invariant under the action of A and, restricted to $\mathcal{S}_A^n$ they are transformed by H in the same way as the coordinate functions. Since H acts locally freely on $\mathcal{S}_A^n$ there is a local moving frame $\rho_H : \mathcal{S}_A^n \to H$. The map $\rho_G : J^n \to G$ defined by

$$\rho_G(z^{(n)}) = \rho_H\left(\rho_A(z^{(n)}) \cdot z^{(n)}\right) \rho_A(z^{(n)})$$

is a local moving frame for the G-action and the non-constant coordinate functions of

$$\rho_H\left(\rho_A(z^{(k)}) \cdot z^{(k)}\right) \rho_A(z^{(k)}) \cdot z^{(k)}$$

provide a complete set of k-th order differential invariants of G.

2.3 Examples: Euclidean, Affine and Projective Actions on the Plane.

The group of the Euclidean motions on the plane is a factor of the group of the special affine motions. In its turn the group of special affine motions is a factor of the group of projective transformations on the plane. All three of these transformation groups play an important role in the computer image processing [10], [31]. Applying the Inductive Algorithm 2.2.5 we express projective invariants in terms of affine, and affine invariants in terms of Euclidean. We also obtain the relations among the Euclidean, affine and projective arc-lengths and the corresponding invariant differential operators.

Example 2.3.1. Let us use the moving frame for the special Euclidean group $SE(2, \mathbb{R})$ acting on curves in $\mathbb{R}^2$ obtained in Example 1.4.7 to build a moving frame for the special affine group. We recall that the moving frame for $SE(2, \mathbb{R})$ has been obtained on the first jet space by choosing a cross-section $\{x = 0, u = 0, u_x = 0\}$. The special Euclidean group acts transitively on J^1 and the first invariant, Euclidean curvature κ appears on the second order of prolongation. The normalization of u_{xxx} and u_{xxxx} yields a third and fourth order invariants $I_3^e = \kappa_s$ and $I_4^e = \kappa_{ss} + 3\kappa^3$.

The special affine transformation $SA(2, \mathbb{R})$ on the plane is a semi-direct product of the special linear group $SL(2, \mathbb{R})$ and translations in $\mathbb{R}^2$. We prolong it to the first jet space of curves on the plane and notice that the isotropy group B of the point $z_0^{(1)} = \{x = 0, u = 0, u_x = 0\}$ is given by linear transformations

$$\begin{pmatrix} \tau & \lambda \\ & \frac{1}{\tau} \end{pmatrix}.$$

Thus $SA(2, \mathbb{R}) = B \cdot SE(2, \mathbb{R})$ and $B \cap SE(2, \mathbb{R})$ is finite. In fact $B \cap SE(2, \mathbb{R}) = \{I, -I\}$.

We prolong the action of B up to the fourth order:

$$
\begin{aligned}
x &\rightarrow \tau x + \lambda u; \\
u &\rightarrow \frac{1}{\tau} u; \\
u_x &\rightarrow \frac{u_x}{\tau(\tau + \lambda u_x)}; \\
u_{xx} &\rightarrow \frac{u_{xx}}{(\tau + \lambda u_x)^3}; \\
u_{xxx} &\rightarrow \frac{(\tau + \lambda u_x)u_{xxx} - 3\lambda u_{xx}^2}{(\tau + \lambda u_x)^5}; \\
u_{xxxx} &\rightarrow \frac{(\tau + \lambda u_x)^2 u_{xxxx} - 10(\tau + \lambda u_x)\lambda u_{xx}u_{xxx} + 15\lambda^2 u_{xx}^3}{(\tau + \lambda u_x)^7}.
\end{aligned}
$$

We restrict this transformation to the set $\mathcal{S}_E^4 = \{z^{(4)} | \pi_1^4(z^{(4)}) = z_0^{(1)} = (0, 0, 0)\}$ parameterized by Euclidean invariants I_2^e, I_3^e and I_4^e, which, restricted to $\mathcal{S}_E^4$ are transformed under B exactly by the same formulas as coordinate function u_{xx}, u_{xxx} and

u_{xxxx}. Thus

$$\begin{aligned}
I_2^e &\rightarrow \frac{I_2^e}{\tau^3}; \\
I_3^e &\rightarrow \frac{\tau I_3^e - 3\lambda(I_2^e)^2}{\tau^5}; \\
I_4^e &\rightarrow \frac{\tau^2 I_4^e - 10\tau\lambda I_2^e I_3^e + 15\lambda^2(I_2^e)^3}{\tau^7}.
\end{aligned}$$

We emphasize that the transformation formulas above are valid only on $\mathcal{S}_E^4$ but not on J^4. We normalize the first transformation to one and the second transformation to zero. This corresponds to choosing a cross-section

$$z_0^{(4)} = \{x = 0, u = 0, u_x = 0, u_{xx} = 1, u_{xxx} = 0\}$$

to the orbits of $SA(2, \mathbb{R})$ on J^4. Then

$$\tau = (I_2^e)^{1/3} \text{ and } \lambda = \frac{I_3^e}{3(I_2^e)^{5/3}}.$$

We substitute this normalization in the transformation for I_4^e to obtain the fourth order special affine invariant:

$$I_4^a = \frac{I_2^e I_4^e - \frac{5}{3}(I_3^e)^2}{(I_2^e)^{8/3}},$$

which we call the affine curvature and denote as μ. We recall that $I_2^e = \kappa$, $I_3^e = \kappa_s$ and $I_2^e = \kappa_{ss} + 3\kappa^3$ and so the affine curvature can be written in terms of the Euclidean curvature and its derivatives as follows:

$$\mu = \frac{\kappa(\kappa_{ss} + 3\kappa^3) - \frac{5}{3}\kappa_s^2}{\kappa^{8/3}}.$$

The moving frame for the special affine group is the product of the matrices:

$$\begin{pmatrix} \kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\ 0 & \frac{1}{\kappa^{1/3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\ -\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\ 0 & 0 & 1 \end{pmatrix}$$

Using formula (2.8) one can obtain an affine contact invariant horizontal form $d\alpha$ in terms of the Euclidean arc-length ds:

$$d\alpha = \sigma_B^* \sigma_E^* \pi_H w_E^* d_H w_B^*(x),$$

where the Euclidean invariantization of $d_H\, w_B^*\,(x) = (\tau + \lambda u_x)\, dx$ equals to $\tau\, ds$ and hence

$$d\alpha = \sigma_B^*(\tau\, ds) = (I_2^e)^{1/3}ds = \kappa^{1/3}ds. \tag{2.9}$$

The form $d\alpha$ is called the affine arc-length. Written in the standard coordinates $d\alpha = u_{xx}^{1/3}dx$. The relation (2.9) between the affine and the Euclidean arc-lengths provide a natural explanation for the affine curve evolution equation in [31]. The relation between invariant differential operators follows immediately:

$$\frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}}\frac{d}{ds},$$

which enables us to obtain all higher order affine invariants in terms of the Euclidean ones.

Example 2.3.2. Let us now use the moving frame for the special affine group to build a moving frame for the projective group $PSL(3,\mathbb{R})$ locally acting on the plane by the transformations:

$$x \ \mapsto \ \frac{\alpha x + \beta u + \gamma}{\delta x + \epsilon u + \zeta};$$

$$u \ \mapsto \ \frac{\lambda x + \nu u + \tau}{\delta x + \epsilon u + \zeta}.$$

where the determinant of the corresponding matrix equals to one. The affine moving frame has been found by choosing the cross-section

$$z_0^{(3)} = \{x = 0, u = 0, u_1 = 0, u_2 = 1, u_3 = 0\} \in J^3.$$

The isotropy group B of $z_0^{(3)}$ for the prolonged action of $PSL(3,\mathbb{R})$ consists of the transformations:

$$\begin{pmatrix} 1 & ab & 0 \\ 0 & a & 0 \\ b & c & \frac{1}{a} \end{pmatrix}.$$

Thus $PSL(3,\mathbb{R}) = B \cdot SA(2,\mathbb{R})$ and $B \cap SA(2,\mathbb{R})$ is finite. Let

$$\mathcal{S}_A^7 = \{z^{(7)}|\pi_3^7(z^{(7)}) = (0,0,0,1,0) = z_0^{(3)}\}.$$

The affine invariants $I_4^a = \mu$, I_5^a, I_6^a, I_7^a can serve as coordinate functions on $\mathcal{S}_A^7$ which are transformed under the action of B by exactly the same rules as coordinate functions u_4, u_5, u_6, u_7. We have computed the prolongation of the action of B to the seventh order using MAPLE and found that:

$$I_4^a \quad \rightarrow \quad \frac{I_4^a - 3a^2b^2 + 6ac}{a^2};$$

$$I_5^a \quad \rightarrow \quad \frac{I_5^a}{a^3};$$

$$I_6^a \quad \rightarrow \quad \frac{I_6^a + 3abI_5^a + 30I_4^a(2ac - a^2b^2) + 180a^2c(c - ab^2) + 45a^2b^2}{a^4};$$

$$I_7^a \quad \rightarrow \quad \frac{I_7^a + 7abI_6^a + I_5^a(105ac - 42b^2a^2) - 35(I_4^a)^2ab}{a^5}.$$

We normalize the transformation for I_5^a to one and the transformations of I_4^a and I_6^a to zero. This corresponds to the cross-section

$$z_0^{(7)} = \{x = 0, u = 0, u_1 = 0, u_2 = 1, u_3 = 0, u_4 = 0, u_5 = 1, u_6 = 0\}$$

to the orbits of $PSL(3, \mathbb{R})$ on J^7. Then

$$a \quad = \quad (I_5^a)^{1/3},$$

$$b \quad = \quad \frac{5(I_4^a)^2 - I_6^a}{3(I_5^a)^{4/3}},$$

$$c \quad = \quad \frac{(I_6^a)^2 - 10I_6^a(I_4^a)^2 - 3I_4^a(I_5^a)^2 + 25(I_4^a)^4}{18(I_5^a)^{7/3}}.$$

We substitute this normalization in the transformation for I_7^a to obtain the seventh order projective invariant:

$$I_7^p = \frac{6I_7^aI_5^a - 7(I_6^a)^2 + 70(I_4^a)^2I_6^a - 105I_4^a(I_5^a)^2 - 175(I_4^a)^4}{6(I_5^a)^{8/3}}$$

which we call the projective curvature and denote as η. Using the recursion algorithm from [13] we can express the higher order affine invariants in terms of μ and its derivatives with respect to *affine* arc-length $d\alpha = u_{xx}^{1/3}dx$:

$$I_4^a = \mu, \qquad\qquad I_5^a = \mu_\alpha,$$

$$I_6^a = \mu_{\alpha\alpha} + 5\mu^2, \qquad I_7^a = \mu_{\alpha\alpha\alpha} + 17\mu\mu_\alpha.$$

This leads to the formula:

$$\eta = \frac{-7\mu_{\alpha\alpha}^2 + 6\mu_\alpha\mu_{\alpha\alpha\alpha} - 3\mu\mu_\alpha^2}{6\mu_\alpha^{8/3}}.$$

The moving frame for the projective group is the product of the matrices:

$$
\begin{pmatrix}
1 & -\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_{\alpha}} & 0 \\
0 & \mu_{\alpha}^{1/3} & 0 \\
-\frac{1}{3}\frac{\mu_{\alpha\alpha}}{\mu_{\alpha}^{4/3}} & \frac{1}{18}\frac{\mu_{\alpha\alpha}^2-3\mu\mu_{\alpha}^2}{\mu_{\alpha}^{7/3}} & \frac{1}{\mu_{\alpha}^{1/3}}
\end{pmatrix}
\begin{pmatrix}
\kappa^{1/3} & \frac{1}{3}\frac{\kappa_s}{\kappa^{5/3}} & 0 \\
0 & \frac{1}{\kappa^{1/3}} & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
\frac{1}{\sqrt{1+u_x^2}} & \frac{u_x}{\sqrt{1+u_x^2}} & -\frac{uu_x+x}{\sqrt{1+u_x^2}} \\
-\frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & \frac{xu_x-u}{\sqrt{1+u_x^2}} \\
0 & 0 & 1
\end{pmatrix}.
$$

We can express the projective arc-length (that is a horizontal form which is contact invariant with respect to the projective action) in terms of the affine arc-length $d\alpha$. We first lift the coordinate function x to $B \times J^\infty$ by $w_B^*(x) = \frac{x+abu}{bx+cu+\frac{1}{a}}$. The affine invariantization of $d_H\, w_B^*(x)$ produce a horizontal form $ad\alpha$ on $B \times J^\infty$ which is contact invariant with respect to the affine action. The projective arc-length equals to

$$
d\varrho = \sigma_B^*\, ad\alpha = (I_5^a)^{1/3}d\alpha = \mu_{\alpha}^{1/3}d\alpha.
$$

The relation between invariant derivatives $\frac{d}{d\varrho} = \frac{1}{\mu_{\alpha}^{1/3}}\frac{d}{d\alpha}$ allows us to obtain all higher order projective invariants in terms of the affine ones.

Chapter 3

Application to Classical Invariant Theory.

One of the central problems of classical invariant theory is the equivalence and symmetry of multivariable polynomials under linear changes of variables. We concentrate on the polynomials over complex numbers, but we will indicate how to adapt the results to real polynomials. The standard action of the general linear group on $\mathbb{C}^m$ induces a representation on the ring of polynomials $\mathbb{C}[\mathbf{x}]$:

$$\bar{F}(A \cdot \mathbf{x}) = F(\mathbf{x}), \tag{3.1}$$

where $A \in GL(m, \mathbb{C})$, $\mathbf{x} \in \mathbb{C}^m$ and $F \in \mathbb{C}[\mathbf{x}]$ is a polynomial in m variables. The set of homogeneous polynomials of degree n is mapped to itself under transformation (3.1). Thus we will always restrict our attention to homogeneous polynomials of a certain degree, which are called *forms* in the classical invariant theory literature. Polynomials of degree n in m variables form a linear space of dimension $\binom{m+n-1}{n}$ isomorphic to the n-th symmetric tensor product of $\mathbb{C}^m$. The coefficients of polynomials can serve as coordinates on this space and formula (3.1) induces a linear action of $GL(m, \mathbb{C})$ on the coefficients. One can try to classify polynomials by computing *invariants* that are certain functions $H(\ldots, a_{i_1,\ldots,i_m}, \ldots)$ of the coefficients, invariant under the induced action. It turns out however that this action is not regular: not only the dimensions of the orbits vary but also some of the orbits are not closed and hence the orbits can not be distinguished by continuous invariants. An algebro-geometric approach to the classification of such orbits can be found in [25], [35]. We also note that even for the fixed number of variables the number of invariants increases when one increases the

degree of polynomials.

We can bypass these difficulties by considering the graph of an m-variable polynomial $u = F(\mathbf{x})$ as a submanifold of $\mathbb{C}^{m+1}$ (or $\mathbb{R}^{m+1}$). The first m coordinates represent independent variables and they are transformed by the the general linear group $GL(m, \mathbb{C})$ in the standard way, the last coordinate is considered to be dependent variable and the action on it is trivial. The described action is regular on the open subset where not all of the first m coordinates are zero. We classify the orbits by constructing the corresponding signature manifolds, parameterized by a certain number of differential invariants. One of the advantages of this approach is that the set of differential invariants parameterizing the signature manifold depends only on the number of variables, but not on the degree of polynomials. We also note that differential invariants restricted to the graphs of polynomials are functions $H(\mathbf{x}, \ldots, a_{i_1,\ldots,i_m}, \ldots)$ depending on both the coefficients and the variables that are invariant under simultaneous action of the general linear group on the variables and the coefficients of polynomials. Polynomial or rational functions with such properties were widely used in the classical invariant theory [18], [21] and were called *(absolute) covariants*, whence covariants that depend on the coefficients $a_{i_1,\ldots,i_m}$ only were called *(absolute) invariants* The simplest example of a covariant is the polynomial $F(\mathbf{x})$ itself (see formula (3.1)). Rational absolute covariants can be obtained as the ratio of *relative covariants* that are not strictly invariant but might be multiplied by a certain power of the determinant of matrix $A \in GL(\mathbb{C}, n)$:

$$H(A \cdot \mathbf{x}, \ldots, A \cdot a_{i_1,\ldots,i_m}, \ldots) = (\det A)^k H(\mathbf{x}, \ldots, a_{i_1,\ldots,i_m}, \ldots).$$

The exponent k is called the *weight* of the covariant. In the terminology of classical invariant theory the completeness of the fundamental set of polynomial invariants (covariants) means that any other invariant (covariant) can be expressed as a *polynomial* function of the fundamental ones. The finiteness of the fundamental set for the actions of linear reductive groups was proved by Hilbert in 1890. This crucial result became a turning point from classical computational approach in the invariant theory to the modern algebraic geometry approach.

There are several classical methods to obtain a complete set of fundamental covariants. Two powerful methods, known as *omega process (or transvection)* and *symbolic method* were formulated in the second part of the last century by the German school

of invariant theory led by Aronhold, Clebesh and Gordan. Both of the methods are based on application of certain differential operators (see [28] for outline of these methods and more historical remarks). Classical processes can be also used to obtain joint covariants of several forms under a simultaneous linear transformation.

Despite of the enormous amount of results obtained by classical approaches many equivalence and symmetry problems remain unsolved even for the case of polynomials over the real or complex numbers. The formulation of the problem as a problem of equivalence of submanifolds, a novel approach introduced by Olver [28], produces new results even in the most studied case of *binary forms*, or homogeneous polynomials in two variables. In Section 3.2 the results for binary forms from my paper with Peter Olver [2] are reproduced and then in the next sections the same approach is extended to *ternary forms*, that is homogeneous polynomials in three variables. We note that differential invariants in the case of polynomials can be chosen to be rational functions in the variables and the coefficients and so the signature manifold construction reduces to the problem of eliminating parameters from rational expressions. This problem can be solved using Gröbner basis algorithms [8], [32] and [14]. Thus theoretically we can construct the signature manifold for a polynomial of any degree in any number of variables. In practice however we are confronted with the complexity of Gröbner basis computation which in many cases exhaust available computer resources. This limitation significantly affects the practical implementation of the moving frame method described below.

3.1 Symmetries and Equivalence of Polynomials.

We consider the action of the general linear group on the space of polynomials in m variables. The standard action of the general linear group on $\mathbb{C}^m$ induces a representation on the ring of polynomials $\mathbb{C}[\mathbf{x}]$:

$$\bar{F}(A \cdot \mathbf{x}) = F(\mathbf{x}), \tag{3.2}$$

where $A \in GL(m, \mathbb{C})$, $\mathbf{x} \in \mathbb{C}^m$ and $F \in \mathbb{C}[\mathbf{x}]$. Equivalently:

$$\bar{F}(\mathbf{x}) = F(A^{-1}\mathbf{x})$$

Since this action preserves the grading on the ring of polynomials we can restrict it to the homogeneous polynomials of a certain degree, which are called *forms* in the classical literature.

Definition 3.1.1. A polynomial F is said to be *equivalent* to a polynomial $\bar{F}$ if there exists $A \in GL(m, \mathbb{C})$ such that $\bar{F}(\mathbf{x}) = F(A\mathbf{x})$.

Example 3.1.2. The binary form $5x^2 - 2xy + 2y^2$ is equivalent to $x^2 + y^2$ under the change of variables

$$x \mapsto x + y; \quad y \mapsto y - 2x.$$

To each form we associate a unique inhomogeneous polynomial by the formula:

$$f(p_1, \ldots, p_{m-1}) = F(p_1, \ldots, p_{m-1}, 1).$$

We call $\mathbf{p} = \{p_1, \ldots, p_{m-1}\}$ *projective variables* and we also refer to $f(\mathbf{p})$ as a form. We can restore the homogeneous polynomial from its inhomogeneous version by the formula

$$F(x_1, \ldots, x_m) = x_m^n f\left(\frac{x_1}{x_m}, \ldots, \frac{x_{m-1}}{x_m}\right). \tag{3.3}$$

Remark 3.1.3. The formula above shows that it is important to remember the degree n of the homogeneous form if we want to restore it from its inhomogeneous version. For example the quartic form $x^2 y^2 + y^4$ and the qudratic $x^2 + y^2$ have the same inhomogeneous version: $p^2 + 1$.

Let $A = \begin{pmatrix} B & \mathbf{t} \\ \mathbf{s} & c \end{pmatrix} \in GL(m, \mathbb{C})$, be a linear transformation, where B is an $(m-1) \times (m-1)$ matrix, $\mathbf{s}^T$, $\mathbf{t} \in \mathbb{C}^{m-1}$, and c is a scalar. Let $\bar{F}(\mathbf{x})$ be the image of $F(\mathbf{x})$ under transformation (3.1) and let $f(\mathbf{p})$ be the inhomogeneous version of $F(\mathbf{x})$, then the induced transformation of $f(\mathbf{p})$ follows from formula (3.3):

$$(\mathbf{s} \cdot \mathbf{p} + c)^n \bar{f}\left(\frac{B\mathbf{p} + \mathbf{t}}{\mathbf{s} \cdot \mathbf{p} + c}\right) = f(\mathbf{p}). \tag{3.4}$$

We call the corresponding transformation of $f(\mathbf{p})$ *projective*. The form $F(\mathbf{x})$ is equivalent to $\bar{F}(\mathbf{x})$ under the transformation (3.1) if and only if $f(\mathbf{p})$ is equivalent to $\bar{f}(\mathbf{p})$ under the corresponding projective transformation (3.4).

In the homogeneous version one can consider the graph $u = F(\mathbf{x})$ of the polynomial as a submanifold in $\mathbb{C}^m$ under the transformation

$$\mathbf{x} \mapsto A \cdot \mathbf{x}, \qquad u \mapsto u. \tag{3.5}$$

For the inhomogeneous version of the problem we consider the graph $u = f(\mathbf{p})$ in $\mathbb{C}^m$ under the transformation:

$$\mathbf{p} \mapsto \frac{B\mathbf{p} + \mathbf{t}}{\mathbf{s} \cdot \mathbf{p} + c}, \qquad u \mapsto (\mathbf{s} \cdot \mathbf{p} + c)^{-n}\, u. \tag{3.6}$$

The transformation of projective variables $\mathbf{p}$ in (3.6) is *linear fractional*:

$$A \cdot \mathbf{p} = \frac{B\mathbf{p} + \mathbf{t}}{\mathbf{s} \cdot \mathbf{p} + c} \tag{3.7}$$

Two matrices which are scalar multiples of each other, $\tilde{A} = \lambda A$, induce the same linear fractional transformation, and so (3.7) defines an action of the projective group $PSL(m, \mathbb{C}) = GL(m, \mathbb{C})/\{\lambda I\}$ on $\mathbb{C}^{m-1}$. Let $\pi \colon GL(m, \mathbb{C}) \mapsto PSL(m, \mathbb{C})$ denote the standard projection.

Definition 3.1.4. The *symmetry group* of F is the subgroup $G_F \subset GL(m, \mathbb{C})$ consisting of all linear transformations that map F to itself. It coincides with the group $G_f \subset GL(m, \mathbb{C})$ which maps inhomogeneous version f of F to itself under transformation (3.4). The *projective symmetry group* of f is the subgroup $\Gamma_f = \pi(G) \subset PSL(m, \mathbb{C})$ consisting of all linear fractional transformations of $\mathbf{p}$ that give rise to symmetries of f. In the real case $G_F \subset GL(m, \mathbb{R})$ and $\Gamma_f \subset PSL(m, \mathbb{R})$.

Example 3.1.5. The form $F(x, y) = x^2 + y^2$ is symmetric under any orthogonal map:

$$(x, y) \mapsto \begin{cases} (\cos(\alpha)x + \sin(\alpha)y, \ -\sin(\alpha)x + \cos(\alpha)y) \\ \qquad\qquad (-x, \ -y) \end{cases}$$

The inhomogeneous version of F is $f(p) = p^2 + 1$ and the corresponding projective group of symmetries Γ_f consists of linear fractional transformations

$$p \mapsto \frac{\cos(\alpha)p + \sin(\alpha)}{-\sin(\alpha)p + \cos(\alpha)}$$

We notice that each projective symmetry in the preceding example corresponds to two genuine symmetries of $F(x, y)$. In general if the form $F(\mathbf{x})$ has degree n then $F(\lambda \mathbf{x}) = \lambda^n F(\mathbf{x})$ and so if ω is any root of unity, $\omega^n = 1$, then the diagonal matrix $\omega I \in G_F$, on the other hand $\pi(\omega I) = e \in \Gamma_f$ and so each projective symmetry gives rise to n genuine symmetries of the form $f(\mathbf{p})$.

Proposition 3.1.6. *A transformation $A \in GL(m, \mathbb{C})$ maps a form F to some scalar multiple of itself, say μF is and only if $\pi(A) \in \Gamma_f$.*

Proof. By substitution $\bar{F} = \mu F$ into (3.2) one obtains:

$$F(\mathbf{x}) = \mu F(A \cdot \mathbf{x}) = F(\sqrt[n]{\mu} \, A \cdot \mathbf{x}).$$

and thus $\hat{A} = \lambda A$, where $\lambda = \sqrt[n]{\mu} \, A$ belongs to G_F, and so by definition $\pi(A) = \pi(\hat{A}) \in \Gamma_f$. $\qquad \square$

Remark 3.1.7. If F is mapped to μF by A then its inhomogeneous version f is mapped to μf, that is:

$$f(\mathbf{p}) = (\mathbf{s} \cdot \mathbf{p} + c)^n \, \mu \, f\left(\frac{B\mathbf{p} + \mathbf{t}}{\mathbf{s} \cdot \mathbf{p} + c}\right). \tag{3.8}$$

Remark 3.1.8. The original transformation rules (3.1), (3.4) apply to forms of weight zero. One can, more generally, consider forms of nonzero weight k, with transformation rules

$$F(\mathbf{x}) = (\det A)^k \, \bar{F}(A \cdot \mathbf{x}), \quad f(\mathbf{p}) = (\det A)^k \, (\mathbf{s} \cdot \mathbf{p} + c)^n \, \bar{f}\left(\frac{B\mathbf{p} + \mathbf{t}}{\mathbf{s} \cdot \mathbf{p} + c}\right). \tag{3.9}$$

If $n + mk \neq 0$, then the projective symmetry group of a weight k form is the same as that of its weight 0 counterpart. However, the full symmetry group does not have the same cardinality, and so are not isomorphic. Indeed, let $A \in GL(m, \mathbb{C})$ be any matrix whose associated linear fractional transformation belongs to the projective symmetry group of F of weight 0, and let $\det A = \Delta$. Then A maps F to a scalar multiple of itself, say μF and so $F(\mathbf{x}) = \mu F(A \cdot \mathbf{x})$. Consequently, the scalar multiple

$$\hat{A} = \lambda A, \quad \text{where} \quad \lambda^{n+mk} \Delta^k = \mu, \tag{3.10}$$

is a symmetry of the weight k form F. Therefore, when $n + mk \neq 0$, each projective symmetry gives rise to $n + mk$ matrix symmetries.

In the exceptional case when $n + mk = 0$, if $A \in GL(m, \mathbb{C})$ is any symmetry, so is *any* scalar multiple λA. Thus each projective symmetry gives rise to a one-parameter family of symmetries in $GL(m, \mathbb{C})$. On the other hand, the projective group of symmetries in this case is smaller in general than for the other weights. Indeed, given a projective symmetry for a weight zero form, one can always find a

matrix representative $A \in SL(m, \mathbb{C})$. Due to (3.10), this representative is a symmetry for the exceptional weight $k = -\frac{n}{m}$ if and only if the corresponding $\mu = 1$. On the other hand, if $A \in GL(m, \mathbb{C})$ is a symmetry for weight $k = -\frac{n}{m}$ then a unimodular matrix $\Delta^{-\frac{1}{m}} A$ is a symmetry for the corresponding zero weight form.

Example 3.1.9. Let $f(p) = p^4 + 3p^2 + 1$ with $n = 4$, $m = 2$ correspond to a binary form $f(x, y) = x^4 + 3x^2y^2 + y^4$. Then the projective symmetry group in the case of weight zero (as well as for any other general weight) consists of the transformations mapping p to $p, -p, \frac{1}{p}, -\frac{1}{p}$. Any $GL(2, \mathbb{C})$ representative of these projective maps is a symmetry for the exceptional weight $k = -2$.

On the other hand, for the form $f(p) = p^4 + 1$ of with $n = 4, m = 2$ the projective symmetries for a general weight are $p, -p, \frac{1}{p}, -\frac{1}{p}, ip, -ip, \frac{i}{p}, -\frac{i}{p}$. In the exceptional case, however, only the first four, namely $p, -p, \frac{1}{p}, -\frac{1}{p}$, are symmetries.

Definition 3.1.10. A homogeneous form is called *nonsingular* if its symmetry group G_f is finite. The *index* of a nonsingular form $f(\mathbf{p})$ is the cardinality $\#G_f$ of its symmetry group. The *projective index* of $f(\mathbf{p})$ is the cardinality $\#\Gamma_f$ of its projective symmetry group $\Gamma_f = \pi(G_f)$.

Thus, for nonsingular forms, the indices are simply related by

$$\#G = l \cdot \#\Gamma, \text{ where } \quad l = \begin{cases} n & \text{for complex forms of degree } n, \\ 2 & \text{for real forms of even degree,} \\ 1 & \text{for real forms of odd degree.} \end{cases} \tag{3.11}$$

In what follows we address the problem of classification of polynomials under linear transformation in its inhomogeneous version (3.4). We reduce the problem to the problem of equivalence of the graphs of polynomials under the transformation (3.7) so we can make a full use of the results described in the previous two chapters.

Remark 3.1.11. As it has been mentioned in the introduction to this chapter the signature manifold of the form $f(\mathbf{p})$ can be parameterized by a set of rational differential invariants restricted to f. Elimination of the variables $\mathbf{p}$ produces polynomial relations among invariants which define the smallest variety containing the signature manifold of f. We call this variety *the signature variety* and notice that it is irreducible in both the real and the complex case (Proposition 6, ch. 4, § 6 in [8]). The

relations among invariants, which we obtain for a polynomial with real coefficients, are the same whether the real or complex equivalence problem is considered. The real classification however includes more equivalence classes! It can be explained by the fact that the signature manifold, defined by parametric equations, does not necessarily fill up entire signature variety (see [8]) neither in complex nor in real case. In the real case two different parts of the signature variety may correspond to two different signature manifolds of the same dimension (see Example 8.69 in [28]). In the complex case, however, different signature manifolds are included in different signature varieties. Indeed assume $\mathcal{C}(f)$ and $\mathcal{C}(\tilde{f})$ are two signature manifolds and V is the smallest signature variety containing them. Assume that $\mathcal{C}(f) \cap \mathcal{C}(\tilde{f}) \neq \emptyset$ then f and $\tilde{f}$ are locally equivalent by Theorem 1.3.7 and so they are globally equivalent (for any analytic function local equivalence implies global). Otherwise, if $\mathcal{C}(f) \cap \mathcal{C}(\tilde{f}) = \emptyset$ then in the complex case, it follows from Theorem 3, ch. 3, § 2 [8] that there is a subvariety $W \subsetneq V$ such that $V - W \subset \mathcal{C}(f)$, and so $\mathcal{C}(\tilde{f}) \subset W$. This contradicts to the assertion that V is the minimal variety containing $\mathcal{C}(\tilde{f})$. Assume that W contains the signature variety of $\mathcal{C}(\tilde{f})$, then $\dim \mathcal{C}(\tilde{f}) < \dim \mathcal{C}(f)$ since V is irreducible and $W \subsetneq V$.

We conclude that the polynomial relations among invariants provide a solution for the problem of equivalence over the complex numbers. In the real case, these relations produce necessary but not sufficient conditions of equivalence, and so more detailed analysis is required to complete the classification over reals.

The symmetry groups of equivalent polynomials are related by matrix conjugation:

$$\bar{F}(\mathbf{x}) = F(A\mathbf{x}) \Longrightarrow G_{\bar{F}} = A G_F A^{-1}.$$

Thus the problem of the classification of polynomials is closely related to the problem of the classification of their symmetry groups up to matrix conjugation. Theorems 1.3.8, 1.3.11 of Chapter 1 provide the foundation to an algorithm which determines the dimension of the symmetry group of a given polynomials and, in the case when the cardinality of the symmetry group is finite, explicitly computes all transformations that belong to it. We start with the simplest case of binary forms.

3.2 Binary forms.

The general linear group

$$GL(2, \mathbb{C}) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \alpha\delta - \beta\gamma \neq 0 \right\}$$

acts on two-dimensional space by invertible linear transformations

$$\bar{x} = \alpha x + \beta y, \qquad \bar{y} = \gamma x + \delta y, \tag{3.12}$$

and thereby induces an irreducible linear representation on the space of binary forms

$$F(x, y) = \sum_{i=0}^{n} a_i \, x^i y^{n-i}$$

of the fixed degree n. This corresponds to a linear fractional transformation

$$\bar{p} = \frac{\alpha p + \beta}{\gamma p + \delta} \, . \tag{3.13}$$

on the projective coordinate $p = \frac{x}{y}$. The induced transformation rule for inhomogeneous polynomials of degree n is:

$$f(p) = (\gamma p + \delta)^n \, \bar{f}(\bar{p}) = (\gamma p + \delta)^n \, \bar{f}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right) \tag{3.14}$$

We reformulate the symmetry and equivalence problem for polynomials as the symmetry and equivalence problem for the graph of a polynomial $u = f(p)$ considered as a submanifold in $\mathbb{C}^2$. The transformation rules

$$\begin{aligned} p &\mapsto P = \frac{\alpha p + \beta}{\gamma p + \delta}, \\ u &\mapsto v = (\gamma p + \delta)^{-n} u \end{aligned}$$

for coordinates p and u in $\mathbb{C}^2$ can be prolonged to the k-th order jet space:

$$\begin{aligned} u_p = u_1 &\mapsto v_1 = \frac{1}{\Delta \sigma^{n-1}}(-n\gamma u + \sigma u_p), \\ u_{pp} = u_2 &\mapsto v_2 = \frac{1}{\Delta^2 \sigma^{n-2}}(n(n-1)\gamma^2 u - 2(n-1)\gamma\sigma u_1 + \sigma^2 u_2) \\ u_k &\mapsto v_k = \frac{1}{\Delta^k \sigma^{n-k}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(n-j)!}{(n-k)!} \gamma^{k-j} \sigma^j u_j, \end{aligned}$$

where $\Delta = \alpha\delta - \beta\gamma$ and $\sigma = \gamma p + \delta$

A moving frame can be defined on the second order by choosing a cross-section $p = 0, u = 1, u_1 = 0, u_2 = \frac{1}{n(n-1)}$. By solving the equations:

$$P = 0, \ v = 1, \ v_1 = 0, \ v_2 = \frac{1}{n(n-1)}$$

one obtains a moving frame:

$$\begin{pmatrix} u^{\frac{1-n}{n}}\sqrt{H} & -pu^{\frac{1-n}{n}}\sqrt{H} \\ \frac{u^{\frac{1}{n}}u_p}{nu} & u^{\frac{1}{n}}\left(1 - \frac{u_p p}{nu}\right) \end{pmatrix},$$

where $H(p) = n(n-1)(uu_{pp} - \frac{n-1}{n}u_p^2)$ is the inhomogeneous version of the Hessian $H(x,y) = F_{xx}F_{yy} - F_{xy}^2$ of a form $F(x,y)$. By substitution of the moving frame into the formulas for higher order lifted invariants v_k we obtain the general formula for invariants:

$$I_k = \frac{1}{\sqrt{H}^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(n-j)!}{(n-k)!}\left(\frac{u_p}{n}\right)^{k-j}u^{j-1}u_j.$$

Due to theorems 1.3.7, 1.3.8 from Chapter 1 the symmetry and equivalence properties of a binary form are entirely determined by just these two differential invariants:

$$\begin{aligned}
I_3 &= \frac{1}{\sqrt{H}^3}(u^2 u_{ppp} - 3\frac{n-2}{n}uu_p u_{pp} + 2\frac{(n-1)(n-2)}{n^2}u_p^3), \\
I_4 &= \frac{1}{H^2}(u^3 u_{pppp} - 4\frac{n-3}{n}u^2 u_p u_{ppp} + 6\frac{(n-2)(n-3)}{n^2}uu_p^2 u_{pp} \\
&\quad - 3\frac{(n-1)(n-2)(n-3)}{n^3}u_p^4).
\end{aligned}$$

This differential invariants can be re-expressed in terms of the classical covariants, which can be obtained by omega process:

$$H(F) = F_{xx}F_{yy} - F_{xy}^2, \quad T(F) = F_x H_y - F_y H_x \quad \text{and} \quad U(F) = F_x T_y - F_y T_x. \quad (3.15)$$

Covariants H, T and U has weight 2, 3 and 4 respectively and thus the rational functions:

$$J = \frac{T^2}{H^3}, \qquad K = \frac{U}{H^2} \qquad\qquad (3.16)$$

are absolute rational covariants. By writing down the inhomogenization of H, T and U:

$$
\begin{aligned}
H(f) &= n(n-1)\left[ff'' - \frac{n-1}{n}(f')^2\right], \\
T(f) &= -n^2(n-1)\left[f^2f''' - 3\frac{(n-2)}{n}ff'f'' + 2\frac{(n-1)(n-2)}{n^2}(f')^3\right], \\
U(f) &= n^3(n-1)\left[f^3f'''' - 4\frac{(n-3)}{n}f^2f'f''' + 6\frac{(n-2)(n-3)}{n^2}ff'^{\,2}f''\right. \\
&\quad \left. - 3\frac{(n-1)(n-2)(n-3)}{n^3}(f')^4\right] - 3\frac{(n-2)}{(n-1)}H^2.
\end{aligned}
$$

Comparing this formulas with (3.15) we conclude that:

$$
J = -n^4(n-1)^2 I_3^2, \qquad K = n^3(n-1)I_4 - 3\frac{(n-2)}{(n-1)}. \qquad (3.17)
$$

Thus the signature manifold for $f(p)$ parameterized by J and K can be used equally well to solve the equivalence problem as the one parameterized by I_3 and I_4 and we will formulate our results in terms of the former, more traditional covariants. We note that none of the invariants are defined when $H \equiv 0$. However this happens if and only if $F(x,y) = (cx + dy)^n$ is the n-th power of a linear form and thus is equivalent to polynomial of one variable only. As it pointed out in chapter 10 of [28] the generalization of this statement is true when the number of variables $m \leq 4$. The Hessian of a form of four and less variables is equal to zero if and only if the form is equivalent under a linear change of variables to a form of less number of variables. When the number of variables is greater than four the 'if' part of the statement is still true, but the 'only if' part fails [17] for instance when

$$
F(x_1, \ldots, x_5) = x_1^2 x_3 + x_1 x_2 x_4 + x_2^2 x_5.
$$

We can now state the classification theorem for binary forms:

Theorem 3.2.1. *Let $f(p) \not\equiv 0$ correspond to a nonzero binary form of degree n. The symmetry group of $f(p)$ is:*

a) *A two-parameter group if and only if $H \equiv 0$ if and only if $f(p)$ is equivalent to a constant.*

b) *A one-parameter group if and only if $H \not\equiv 0$ and T^2 is a constant multiple of H^3 if and only if $f(p)$ is complex-equivalent to a monomial p^k, with $k \neq 0, n$. In this case:*

$$J = -\frac{4(n-2k)^2}{k(n-k)(n-1)}, \qquad K = -\frac{6(n-2k)^2}{k(n-k)(n-1)}$$

c) *A finite group in all other cases.*

Remark 3.2.2. A real binary form is complex-equivalent to a monomial if and only if it is real-equivalent to either a real monomial $\pm p^k$ or to the form $\pm(p^2 + 1)^m$, the latter only occurring in the case of even degree $n = 2m$.

Therefore, a binary form is nonsingular if and only if its rational covariant J is *not* constant if and only if the form is not complex-equivalent to a monomial. The next result is fundamental for our algorithm for determining the (finite) symmetry group of a nonsingular binary form.

Theorem 3.2.3. *Let $f(p)$ correspond to a nonsingular complex binary form. Then $P = \varphi(p)$ is a complex analytic solution to the rational symmetry equations*

$$J(P) = J(p), \qquad K(P) = K(p) \tag{3.18}$$

if and only if $P = (\alpha p + \beta)/(\gamma p + \delta)$ is a linear fractional transformation belonging to the projective symmetry group of $f(\mathbf{p})$.

Thus *all* the solutions to the symmetry equations (3.18) are necessarily linear fractional transformations! As remarked above, given a projective symmetry, the corresponding symmetry matrix $A \in GL(2, \mathbb{C})$ is uniquely determined up to multiplication by an n-th root of unity. Since the linear fractional transformation only determines A up to a scalar multiple, one must substitute into the transformation rule 3.14 for the form to unambiguously specify the symmetry matrix.

In the real case, if the degree of F is odd, $n = 2m + 1$, then the basic symmetry 3.2.3 holds as stated. Moreover, each real linear fractional solution to the symmetry equations (3.18) corresponds to a unique matrix symmetry. On the other hand, if the degree of F is even, $n = 2m$, then the sign of F is invariant, and a real solution to the symmetry equations (3.18) will induce a real projective symmetry, and thereby

two real matrix symmetries of the form if and only if it preserves the sign of F. The explicit computations for of the symmetry group of a nonsingular complex binary form relies on Theorem 3.2.3, and hence requires solving the fundamental symmetry equations (3.18) which can be rewritten as polynomial equations in p:

$$A(p)B(P) = A(P)B(p), \qquad C(P)D(p) = C(p)D(P), \tag{3.19}$$

where:

$$J = \frac{T^2}{H^3} = \frac{A}{B}, \qquad K = \frac{U}{H^2} = \frac{C}{D}.$$

Polynomials A and B have no common factors, nor do C and D. Bounds on the index or number of symmetries of a binary form can be determined without explicitly solving the bivariate symmetry equations (3.19). The fact that $f(p)$ is not equivalent to a monomial implies that T^2 is a not a constant multiple of H^3, and hence the first equation in (3.19) is nontrivial. Therefore, the projective index of $f(\mathbf{p})$ is always bounded by the degree of the first equation in p, which in turn is bounded by $6n - 12$ with equality if and only if T and H have no common factors. The second bivariate polynomial is trivial if and only if the covariant U is a constant multiple of H^2. Forms for which $U = cH^2$ will be distinguished as belonging to the *maximal discrete symmetry class.* Indeed, if T and H have no common factors and all the roots of the first equation are simple, then the projective index of such a form takes its maximum possible value, namely $6n - 12$. On the other hand, if U is not a constant multiple of H^2, then the projective index is bounded by the degree of the second polynomial which is at most $4n - 8$.

Theorem 3.2.4. *Let k denote the projective index of a binary form Q of degree n which is not complex-equivalent to a monomial. Then*

$$k \leq \begin{cases} 6n - 12 & \text{if } U = cH^2 \text{ for some constant } c, \text{ or} \\ 4n - 8 & \text{in all other cases.} \end{cases}$$

The real case clearly admits the same bounds on the projective index, since one must determine the number of common *real* solutions to (3.19), and, in the case of even degree, whether the sign of Q is the same at each solution. Consequently, the index of a binary form of degree n is bounded by either $(6n - 12)l$ or $(4n - 8)l$, where $l = n$ in the complex case, $l = 2$ in the case of real forms of even degree and $l = 1$ for real forms of odd degree.

Since the symmetry groups of equivalent polynomials are related by matrix conjugation in $GL(2, \mathbb{C})$, a complete list of possible projective symmetry groups is provided by the following theorem, as presented in Blichfeldt, ([3] p. 69).

Theorem 3.2.5. *Up to matrix conjugation there are five different types of finite subgroups of the projective group $PSL(2, \mathbb{C})$:*

a) *The n element abelian group $\mathcal{A}_n$ is generated by the transformation $p \mapsto \omega p$, where ω is a primitive n-th root of unity.*

b) *The $2n$ element* dihedral group $\mathcal{D}_n$ *is the group obtained from $\mathcal{A}_n$ by adjoining the transformation $p \mapsto 1/p$.*

c) *The 12 element* tetrahedral group $\mathcal{T}$ *is the primitive group generated by the transformations*

$$\sigma : \ p \longmapsto -p, \qquad \tau : \ p \longmapsto \frac{i(p+1)}{p-1}, \tag{3.20}$$

of respective orders 2 and 3.

d) *The 24 element* octahedral group $\mathcal{O}$ *is the primitive group generated by the transformation τ in 3.20 along with*

$$\iota : \ p \longmapsto ip \tag{3.21}$$

of order 4. Note that $\iota^2 = \sigma$, and so $\mathcal{T} \subset \mathcal{O}$.

e) *The 60 element* icosahedral group $\mathcal{I}$ *is the primitive group generated by the transformations σ, τ given above, along with the transformation*

$$\rho : \ p \longmapsto \frac{2p - (1 - \sqrt{5})i - (1 + \sqrt{5})}{\left[(1 - \sqrt{5})i - (1 + \sqrt{5})\right]p - 2} \tag{3.22}$$

of order 2. The tetrahedral group is also a subgroup of the icosahedral group: $\mathcal{T} \subset \mathcal{I}$.

Since the maximal number of elements in the projective symmetry group of a form of degree n is bounded by $6n - 12$, then the tetrahedral group can appear as a symmetry group only when $n \geq 4$, the octahedral group is a possible symmetry group only if $n \geq 6$ and the icosahedral group is possible only if $n \geq 12$.

We can describe the invariants of the three primitive groups using the following polynomials:

$$K_4 = x^4 - 2\sqrt{3}\,i\,x^2y^2 + y^4,$$
$$K_6 = x^5y - xy^5$$
$$K_{12} = x^{12} - 33\,(x^8y^4 + y^8x^4) + y^{12},$$
$$L_{12} = 22\sqrt{5}K_6^2 + 5K_{12},$$
$$L_{20} = 3K_8K_{12} - 38\sqrt{5}K_6^2K_8,$$
$$L_{30} = 6696K_6^5 + 225K_6K_8^3 - 580\sqrt{5}K_6^3K_{12},$$

$$\bar{K}_4 = x^4 + 2\sqrt{3}\,i\,x^2y^2 + y^4,$$
$$K_8 = x^8 + 14x^4y^4 + y^8 = K_4\bar{K}_4,$$

$$\widetilde{L}_{12} = -22\sqrt{5}K_6^2 + 5K_{12},$$
$$\widetilde{L}_{20} = 3K_8K_{12} + 38\sqrt{5}K_6^2K_8,$$
$$\widetilde{L}_{30} = 6696K_6^5 + 225K_6K_8^3$$
$$+580\sqrt{5}K_6^3K_{12}.$$

(3.23)

Huffman, ([22] Theorem 4.1), provides the complete characterization of polynomials whose symmetry groups contain one of these primitive groups.

Proposition 3.2.6. *The symmetry group of a binary form F contains:*

a) *An icosahedral group if and only if it is equivalent to a polynomial of the one of the two forms $\Phi(L_{12}, L_{20}) + L_{30}\Psi(L_{12}, L_{20})$ or $\Phi(\widetilde{L}_{12}, \widetilde{L}_{20}) + \widetilde{L}_{30}\Psi(\widetilde{L}_{12}, \widetilde{L}_{20})$.*

c) *An octahedral group if and only if it is equivalent to a polynomial of the one of the two forms $\Phi(K_6, K_8)$ or $K_{12}\Phi(K_6, K_8)$.*

d) *A tetrahedral group if and only if it is equivalent to a polynomial from the following list:*

$$\Phi(K_6, K_8) + K_{12}\Phi(K_6, K_8), \quad \Phi(K_4, K_6), \quad \Phi(\bar{K}_4, K_6),$$
$$K_4\Phi(K_6, K_8) + K_4^2\,\Phi(K_6, K_8), \quad K_4\Phi(\bar{K}_4, K_6), \quad \bar{K}_4\Phi(K_4, K_6),$$
$$\bar{K}_4\Phi(K_6, K_8) + \bar{K}_4^2\,\Phi(K_6, K_8), \quad K_4^2\,\Phi(\bar{K}_4, K_6), \quad \bar{K}_4^2\,\Phi(K_4, K_6).$$

Note in particular that only forms of even degree can admit a primitive symmetry group.

MAPLE code was written to explicitly compute the symmetries of binary forms. Details of the programs and some of the difficulties we experienced in the implementation are discussed in the appendix A. The program `symm` computes the fundamental invariants J and K, determines the dimension of the symmetry group, and, in the

case of a finite symmetry group, solves the two equations 3.19 to find explicit form of the projective symmetries. The actual matrix symmetries are then computed by the program `matrices` by substituting the linear fractional transformations in the projective symmetry group into the form in order to determine the appropriate scalar multiple. We now present some typical examples resulting from our computations.

Example 3.2.7. *Cubic forms.* All binary cubics with discrete symmetries are equivalent to $x^3 + y^3$, or, in inhomogeneous form, to $p^3 + 1$. Therefore, the symmetry group of a nonsingular cubic is isomorphic to the symmetry group of $p^3 + 1$. Applying our algorithm, we find a complete solution to the symmetry equations 3.18 is the projective symmetry group Γ given by the six linear fractional transformations taking p to

$$p, \quad \frac{1}{p}, \quad \omega p, \quad \omega^2 p, \quad \frac{\omega}{p}, \quad \frac{\omega^2}{p},$$

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is the primitive cube root of unity. Since the covariants of any cubic form satisfy the syzygy $U = -\frac{3}{2}H^2$, all non-degenerate cubics have maximal discrete symmetry groups of projective index 6, which equals the number of different permutations of the three roots. The full matrix symmetry group G of this cubic has 18 elements, since we can also multiply by a cube root of unity, and is generated by the three matrices

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

In this case, $G \simeq \Gamma \times \mathbb{Z}_3$ is a Cartesian product group. In the real case, one requires real solutions to (3.18, and hence f has (projective) index 6 if its discriminant $\Delta < 0$, but (projective) index 2 if $\Delta > 0$.

The MAPLE code can be used to compute the explicit symmetries of other cubics. For example, the cubic $f(p) = p^3 + p$ leads to the following six element group of linear fractional transformations

$$p, \quad -p, \quad \frac{ip+1}{3p+i}, \quad \frac{ip-1}{-3p+i}, \quad \frac{-ip+1}{-3p+i}, \quad \frac{-ip+1}{3p+i}.$$

The matrix generators of the symmetry group are

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -i \\ -3i & 1 \end{pmatrix}.$$

The second and third matrices correspond, respectively, to the second and third linear fractional transformations. Note that one must, in accordance with the general procedure, rescale the matrices as required by the condition that f must be mapped to itself. Difficulties arise when MAPLE gives the solutions of equations 3.18 not as rational functions, but involving roots of polynomials. An example is the cubic $f(p) = p^3 + p + 1$, which is discussed in Appendix A.

Example 3.2.8. *Quartic forms.* A polynomial of degree 4 has a finite symmetry group if it is equivalent to either

$$p^4 + \mu\, p^2 + 1, \quad \text{or} \quad p^2 + 1,$$

where $\mu \neq \pm 2$. The former has all simple roots; the latter has a double root at ∞.

In the first situation, the symmetry group will depend on the value of μ. For general μ, the projective symmetry group is a dihedral group $\mathcal{D}_2$, generated by $-p$ and $1/p$. When $\mu = 0$ it becomes a dihedral group $\mathcal{D}_4$, generated by ip and $1/p$. The associated matrices are the obvious ones, namely $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the first case, and $\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the second.

The cases $\mu = \pm 2i\sqrt{3}$ corresponds to the polynomials K_4 and $\bar{K}_4$ listed in 3.23 above, and so the projective symmetry group is the 12 element octahedral group $\mathcal{O}$. This case has the maximal size discrete symmetry group. The linear fractional transformations are generated by $-p$ and $i\,(p-1)/(p+1)$. These correspond to different matrices in each case:

$$K_4: \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{\left(2-2i\sqrt{3}\right)^{1/4}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \qquad \text{when} \qquad \mu = 2i\sqrt{3},$$

$$\bar{K}_4: \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{\left(2+2i\sqrt{3}\right)^{1/4}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \qquad \text{when} \qquad \mu = -2i\sqrt{3}.$$

The transformations and their matrices are given in the form they were computed by MAPLE.

Finally, the projective symmetry group of the quartic form $p^2 + 1$ consists of just two elements: identity and $p \to -p$.

Example 3.2.9. *Quintic forms.* For polynomials of degree 5, the projective symmetry group is either cyclic, of type $\mathcal{A}_n$, or dihedral, of type $\mathcal{D}_n$. Some representative examples are listed in the following table.

	Projective Symmetry Groups of Representative Quintics	
i.	$p^5 + 1$	$\mathcal{D}_5$
ii.	$p^5 + p$	$\mathcal{A}_4$
iii.	$p^5 + p^2$	$\mathcal{A}_3$
iv.	$p^5 + p^3$	$\mathcal{A}_2$
v.	$p^5 + p^2 + 1$	$\{e\}$
vi.	$p^5 - 4p - 2$	$\{e\}$

Example 3.2.10. *Higher degree forms.* At the sixth degree, we first encounter a polynomial with an octahedral projective symmetry group: the sextic $Q(p) = p^5 + p$ which corresponds to the form $Q(x, y) = x^5 y + xy^5$, compare with (3.23). The inhomogeneous form looks like the the second quintic polynomial listed in the preceding table, but we are now considering it as a sextic with an additional root at ∞, and so the symmetry group is quite different. Initially MAPLE produces symmetries which involve square roots and so do not initially look like linear fractional transformations. However, after some simplifications under the radical we obtain the group of linear fractional transformations generated by

$$i\, p, \qquad \frac{\sqrt{2}(1+i)p - 2}{\sqrt{2}(1-i) + 2p},$$

with corresponding matrices

$$\begin{pmatrix} i^{5/6} & 0 \\ 0 & i^{-1/6} \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{2}(1+i) & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(1-i) \end{pmatrix}.$$

The next time we meet this group is the octavic (degree 8) form $Q(p) = p^8 + 14p^4 + 1$. The octahedral generators are now

$$p \longmapsto i\, p, \qquad p \longmapsto i\left(\frac{p+1}{p-1}\right),$$

which correspond to the matrix symmetries

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}.$$

3.3 Differential Invariants for Ternary Forms.

In this section we derive a fundamental set of differential invariants for the projective action (3.4) on the inhomogeneous version $f(p, q)$ of a ternary form $F(x, y, z)$. Let us consider the corresponding (local) action of $GL(3, \mathbb{C})$ on the graph of polynomial $u = f(p, q)$:

$$
\begin{aligned}
p &\mapsto P = \frac{\alpha p + \beta q + \gamma}{\delta p + \epsilon q + \zeta}; \\
q &\mapsto Q = \frac{\lambda p + \nu q + \tau}{\delta p + \epsilon q + \zeta}; \\
u &\mapsto v = (\delta p + \epsilon q + \zeta)^{-n} u
\end{aligned}
\tag{3.24}
$$

The direct construction of the moving frame is computationally difficult so we apply the recursive algorithm 2.1.11 from Chapter 2.

Remark 3.3.1. If Gröbner basis computation were more feasible in practice the following lengthy derivation of moving frames and differential invariants would not be necessary. In theory, the signature manifold could be derived from the lifted invariants $v_{k,l}$, obtained by the prolongation of the action (3.24). Indeed, restricted to the graph of a polynomial, lifted invariants $v_{k,l}$ are some rational functions of the group parameters and variables p and q. The action (3.24) if locally free and transitive on J^3 and we can choose a cross-section

$$
\mathcal{S} = \{p = q = 0, \, u = 1, \, u_{1,0} = u_{0,1} = u_{2,0} = u_{0,2} = 0, \, u_{3,0} = u_{0,3} = 1\}.
$$

Using an algorithm based on Gröbner basis computations (see [8], Theorem 2, § 3, ch. 3) we can eliminate group parameters and variables p and q from the equations:

$$
P = Q = 0, \, v = 1, \, v_{1,0} = v_{0,1} = v_{2,0} = v_{0,2} = 0, \, v_{3,0} = v_{0,3} = 1,
$$
$$
I_{1,1} = v_{1,1}, \, I_{2,1} = v_{2,1}, \, I_{1,2} = v_{1,2}, \, I_{k,l} = v_{k,l},
$$

where $3 < k + l < s + 1$ and s is the differential invariant order of the polynomial defined in Chapter 1, Definition 1.3.5. As the result we obtain polynomial relations among $I_{k,l}$ that generate a prime ideal. The corresponding irreducible signature variety contains the signature manifold for the graph of the polynomials (see Remark 3.1.11). This procedure can be easily generalized for an arbitrarily algebraic action on polynomials.

In practice however we were not able to carry out this straightforward algorithm and so we start the recursion by choosing a cross-section $\mathcal{S}_0 = \{p = 0, q = 0, u = 1\}$ to the local action (3.24) of $GL(3, \mathbb{C})$ on $\mathbb{C}^3$ The isotropy group H_1 of $\mathcal{S}_0$ consists of matrices:

$$\begin{pmatrix} \alpha & \beta & 0 \\ \lambda & \nu & 0 \\ \delta & \epsilon & 1 \end{pmatrix},$$

The action of the subgroup

$$T = \left\{ \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & c \end{pmatrix} \right\}$$

is locally free and transitive and thus $G = H_1 T$. The prolongation of the T-action:

$$p \mapsto \frac{p + t_1}{c}, \qquad q \mapsto \frac{q + t_2}{c}, \qquad u \mapsto c^{-n} u$$

to J^k is given by the formulas

$$u_{k,l} \mapsto c^{k+l-n} u_{k,l}, \tag{3.25}$$

where the variable $u_{k,l}$ correspond to the derivative $\frac{\partial^{k+l} u}{\partial^k p \partial^l q}$. The condition $\{p = 0, q = 0, u = 1\}$ normalizes the group parameters of T:

$$t_1 = -p, \quad t_2 = -q, \quad c = u^{1/n}.$$

The set $\mathcal{S}_0^\infty$ such that $\pi_0^\infty(\mathcal{S}_0^\infty) = \mathcal{S}_0$ is parameterized by functions $U_{k,l} = u^{\frac{k+l-n}{n}} u_{k,l}$ obtained from (3.25) by substitution of the normalization for c. The group H_1 acts on the functions $U_{k,l}$ restricted to $\mathcal{S}_0^\infty$ in the same manner as on the corresponding coordinate functions. We note that the action of the subgroup H_1 on $\mathcal{S}_0^1$ is transitive and we choose the cross-section $\mathcal{S}_1^1 \subset \mathcal{S}_0^1$ defined by the condition $U_{1,0} = 0$, $U_{0,1} = 0$. The isotropy group $H_2 \subset H_1$ for this cross-section is isomorphic to $GL(2, \mathbb{C})$:

$$H_2 = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

We can write H_1 is a product of H_2 and R, where the group R consists of inversions:

$$R = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \right\}.$$

To normalize the parameters of R we need to prolong its action:

$$
\begin{aligned}
p &\mapsto \bar{p} = \frac{p}{ap + bq + 1}, \\
q &\mapsto \bar{q} = \frac{q}{ap + bq + 1}, \\
u &\mapsto \bar{u} = (ap + bq + 1)^{-n} u
\end{aligned}
$$

to J^∞ and then restrict it to $\mathcal{S}_0^\infty$. The lifted invariant differential operators are dual to the lifted contact invariant forms:

$$
\begin{aligned}
d\bar{p} &= \frac{dp + b(qdp - pdq)}{\sigma^2}, \\
d\bar{q} &= \frac{dq + a(pdq - qdp)}{\sigma^2},
\end{aligned}
$$

where $\sigma = ap + bq + 1$ and hence they are equal to:

$$
\begin{aligned}
\bar{\mathcal{D}}_p &= \sigma^2 D_p + \sigma q \left(a D_q - b D_p\right), \\
\bar{\mathcal{D}}_q &= \sigma^2 D_q + \sigma p \left(b D_p - a D_q\right).
\end{aligned}
$$

Thus the first prolongation of the action of R is:

$$
u_{1,0} \mapsto \bar{u}_{1,0} = \sigma^{1-n} \left(-nau + \sigma u_p\right) + \sigma^{1-n} q \left(au_q - bu_p\right), \tag{3.26}
$$

$$
u_{0,1} \mapsto \bar{u}_{0,1} = \sigma^{1-n} \left(-nbu + \sigma u_q\right) + \sigma^{1-n} p \left(bu_p - au_q\right). \tag{3.27}
$$

By restricting the above transformations to $\mathcal{S}_0^1$, where $p = q = 0$ and $u = 1$ one obtains the transformations of $U_{1,0}$ and $U_{0,1}$:

$$
U_{1,0} \mapsto \bar{U}_{1,0} = -na + U_{1,0}, \qquad U_{1,0} \mapsto \bar{U}_{0,1} = -nb + U_{0,1}. \tag{3.28}
$$

Normalization $\bar{U}_{1,0} = 0$ and $\bar{U}_{0,1} = 0$ defines the group parameters of R:

$$
a = \frac{U_{1,0}}{n} = \frac{u^{\frac{1-n}{n}} u_{1,0}}{n}, \qquad b = \frac{U_{0,1}}{n} = \frac{u^{\frac{1-n}{n}} u_{0,1}}{n}. \tag{3.29}
$$

To proceed further we need to determine the invariantization of higher order functions $U_{k,l}$ under the action of R on $\mathcal{S}_0^\infty$. In order to do so we need to compute the lifted invariants $\bar{u}_{k,l} = \bar{\mathcal{D}}_p^k \bar{\mathcal{D}}_q^l \bar{u}$, restrict them to $\mathcal{S}_0^\infty$ and then substitute the normalization (3.29). It is not hard to observe (see also (3.26)) that

$$
\bar{u}_{1,0} = \bar{\mathcal{D}}_p \bar{u} = \sigma^2 D_p \bar{u} + q \times (\dots),
$$

where the second term in the sum equals to q multiplied by an expression and hence it is equal to zero when $q = 0$. In general

$$\bar{u}_{k,0}(\bar{\mathcal{D}}_p)^k v = \left(\sigma^2 D_p)\right)^k \bar{u} + q \times (\dots).$$

The differential operator in the first part of the formula is similar to the one that produces the prolongation formulas in the case of binary forms and so we obtain the familiar expressions:

$$\bar{u}_{k,0} = \sigma^{k-n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(n-j)!}{(n-k)!} a^{k-j} \sigma^j u_{j,0} + q \times (\dots).$$

We restrict these functions to $\mathcal{S}_0^\infty$, where $p = 0$, $q = 0$, $u = 1$ and $u_{k,l} = U_{k,l}$ and substitute the normalization (3.29) to obtain functions:

$$
\begin{aligned}
Q_{k,0} &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(n-j)!}{(n-k)!} \left(\frac{U_{1,0}}{n}\right)^{k-j} U_{j,0} \\
&= u^{\frac{k(1-n)}{n}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(n-j)!}{(n-k)!} u^{j-1} \left(\frac{u_{1,0}}{n}\right)^{k-j} u_{j,0}, \quad\quad (3.30)
\end{aligned}
$$

In the notation of the Algorithm 2.1.11 the last expression corresponds to the pull back of the coordinate function $u_{k,0}$ under the map $\iota_1 \colon J^k \to \mathcal{S}_1^k$. In the same manner we derive that

$$
\begin{aligned}
Q_{0,k} &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(n-j)!}{(n-k)!} \left(\frac{U_{0,1}}{n}\right)^{k-j} U_{0,j} \\
&= u^{\frac{k(1-n)}{n}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(n-j)!}{(n-k)!} u^{j-1} \left(\frac{u_{0,1}}{n}\right)^{k-j} u_{0,j}. \quad\quad (3.31)
\end{aligned}
$$

The similar straightforward (but more complicated) derivations produces the formulas for $Q_{k,l}$ $k \neq 0$, $l \neq 0$ corresponding to the mixed derivatives. We list the ones which

we will use later:

$$Q_{1,1} = u^{\frac{2(1-n)}{n}} \left[u_{1,1}\,u - \frac{n-1}{n}u_{1,0}u_{0,1} \right],$$

$$Q_{1,2} = u^{\frac{3(1-n)}{n}} \left[u_{1,2}\,u^2 - \frac{n-2}{n}(u_{0,2}u_{1,0} + 2\,u_{1,1}u_{0,1})u + 2\frac{(n-1)(n-2)}{n^2}u_{0,1}^2 u_{1,0} \right],$$

$$Q_{2,1} = u^{\frac{3(1-n)}{n}} \left[u_{2,1}\,u^2 - \frac{n-2}{n}(u_{2,0}u_{0,1} + 2\,u_{1,1}u_{1,0})\,u + 2\frac{(n-1)(n-2)}{n^2}u_{1,0}^2 u_{0,1} \right],$$

$$Q_{1,3} = u^{\frac{4(1-n)}{n}} \left[u_{1,3}\,u^3 - \frac{n-3}{n}(u_{1,0}u_{0,3} + 3\,u_{1,2}u_{0,1})\,u^2 + 3\frac{(n-3)(n-2)}{n^2} \right.$$

$$\times \quad (u_{0,1}^2 u_{1,1} + u_{1,0}u_{0,1}u_{0,2})\,u - 3\frac{(n-1)(n-2)(n-3)}{n^3}u_{0,1}^3 u_{1,0} \left. \right], \qquad (3.32)$$

$$Q_{2,2} = u^{\frac{4(1-n)}{n}} \left[u_{2,2}\,u^3 - 2\frac{n-3}{n}(u_{2,1}u_{0,1} + u_{1,2}u_{1,0})\,u^2 + \frac{(n-3)(n-2)}{n^2} \right.$$

$$\times \quad (u_{1,0}^2 u_{0,2} + 4u_{1,0}u_{0,1}u_{1,1} + u_{0,1}^2 u_{2,0})u - 3\frac{(n-1)(n-2)(n-3)}{n^3}u_{1,0}^2 u_{0,1}^2 \left. \right],$$

$$Q_{1,3} = u^{\frac{4(1-n)}{n}} \left[u_{3,1}\,u^3 - \frac{n-3}{n}(u_{0,1}u_{3,0} + 3\,u_{2,1}u_{1,0})\,u^2 + 3\frac{(n-3)(n-2)}{n^2} \right.$$

$$\times \quad (u_{1,0}^2 u_{1,1} + u_{0,1}u_{1,0}u_{2,0})\,u - 3\frac{(n-1)(n-2)(n-3)}{n^3}u_{1,0}^3 u_{0,1} \left. \right].$$

The function $Q_{k,l}$ are invariant under the transformation (3.24) up to action of $H_2 \backsimeq GL(2, \mathbb{C})$ (see Proposition 2.1.4 from Chapter 2). Moreover restricted to $\mathcal{S}_1^k$ they are transformed by H_2 by the same formulas as the coordinate functions $u_{k,l}$. We prolong the linear transformation:

$$p \mapsto \alpha p + \beta q, \quad q \mapsto \gamma p + \delta q, \quad u \mapsto u.$$

to J^∞ and then substituting $u_{k,l}$ with $Q_{k,l}$ to obtain:

$$\bar{Q}_{2,0} = \frac{1}{\Delta^2} \left(\delta^2 Q_{2,0} - 2\gamma\delta Q_{1,1} + \gamma^2 Q_{0,2} \right);$$ (3.33)

$$\bar{Q}_{1,1} = \frac{1}{\Delta^2} \left(-\delta\beta Q_{2,0} + (\gamma\beta + \alpha\delta)Q_{1,1} - \alpha\gamma Q_{0,2} \right);$$ (3.34)

$$\bar{Q}_{0,2} = \frac{1}{\Delta^2} \left(\beta^2 Q_{2,0} - 2\alpha\beta Q_{1,1} + \alpha^2 Q_{0,2} \right);$$ (3.35)

$$\bar{Q}_{3,0} = \frac{1}{\Delta^3} \left(\delta^3 Q_{3,0} - 3\gamma\delta^2 Q_{2,1} + 3\gamma^2\delta Q_{1,2} - \gamma^3 Q_{0,3} \right);$$ (3.36)

$$\bar{Q}_{2,1} = \frac{1}{\Delta^3} \left(-\delta^2\beta Q_{3,0} + \delta(2\gamma\beta + \alpha\delta)Q_{2,1} - \gamma(\gamma\beta + 2\alpha\delta)Q_{1,2} + \alpha\gamma^2 Q_{0,3} \right)$$ (3.37)

$$\bar{Q}_{1,2} = \frac{1}{\Delta^3} \left(\delta\beta^2 Q_{3,0} - \beta(\gamma\beta + 2\alpha\delta)Q_{2,1} + \alpha(2\gamma\beta + \alpha\delta)Q_{1,2} - \alpha^2\gamma Q_{0,3} \right);$$ (3.38)

$$\bar{Q}_{0,3} = \frac{1}{\Delta^3} \left(-\beta^3 Q_{3,0} + 3\alpha\beta^2 Q_{2,1} - 3\alpha^2\beta Q_{1,2} + \alpha^3 Q_{0,3} \right);$$ (3.39)

etc.

We can normalize the remaining group parameters by setting

$$\bar{Q}_{2,0} = \bar{Q}_{0,2} = 0 \quad \text{and} \quad \bar{Q}_{3,0} = \bar{Q}_{0,3} = 1.$$

From the first pair of normalizations it follows that $\frac{\delta}{\gamma}$ and $\frac{\beta}{\alpha}$ are two roots of the same quadratic equation (see (3.33) and (3.35)) so we can write that

$$\frac{\delta}{\gamma} = r_1 = \frac{Q_{1,1} + \sqrt{Q_{1,1}^2 - Q_{2,0}Q_{0,2}}}{Q_{2,0}}, \qquad \frac{\beta}{\alpha} = r_2 = \frac{Q_{1,1} - \sqrt{Q_{1,1}^2 - Q_{2,0}Q_{0,2}}}{Q_{2,0}}.$$ (3.40)

By subtracting these expressions one obtains that

$$r_1 - r_2 = \frac{\alpha\delta - \beta\gamma}{\alpha\gamma} = \frac{2\sqrt{d}}{Q_{2,0}} \implies \Delta = \alpha\gamma \frac{2\sqrt{d}}{Q_{2,0}},$$

where $d = Q_{1,1}^2 - Q_{2,0}Q_{0,2}$. From the second pair of normalizations we obtain that:

$$\alpha = \frac{Q_{2,0}}{2\sqrt{d}} \left(r_1^3 Q_{3,0} - 3r_1^2 Q_{2,1} + 3r_1 Q_{1,2} - Q_{0,3} \right)^{1/3},$$ (3.41)

$$\gamma = \frac{Q_{2,0}}{2\sqrt{d}} \left(-r_2^3 Q_{3,0} + 3r_2^2 Q_{2,1} - 3r_2 Q_{1,2} + Q_{0,3} \right)^{1/3}.$$

The substitution of this normalization into $\bar{Q}_{1,1}, \bar{Q}_{2,1}, \bar{Q}_{1,2}$ and higher order 'derivatives' $\bar{Q}_{k,l}$, $k + l > 3$ produces all the fundamental invariants of the action (3.24).

These invariants however are not rational expressions in p and q and so we can not use Gröbner basis elimination algorithm to describe the corresponding signature manifold. Fortunately, by turning back to the classical invariant theory processes we are able to derive a complete set of fundamental rational invariants.

We first note that $\bar{Q}_{k,l}$ are lifted invariants under the action of $H_2 \simeq GL(2,\mathbb{C})$ on $H_2 \times J^\infty$ defined by the prolongation of the map:

$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}.$$
$$(3.42)$$

We can also look at $\bar{Q}_{k,l}$ as on polynomials in $\{\alpha, \beta, \gamma, \delta\}$ with coefficients $Q_{i,j}$. We note that the coefficients $Q_{i,j}$ are transformed in the same manner as the coefficients of the binary forms $\sum_{i,j} a_{i,j} p^i q^j$, where $i + j = n$, and p and q are transformed as in (3.42). From the second formula in (3.42) we can see that vectors $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ are transformed by the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-t} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}.$$

Such vectors are called *covariant* in contrast with *contravariant* vectors which are transformed by the left multiplication by A (for instance, p and q form a contravariant vector). The forms

$$P_2 = \Delta^2 \, \bar{Q}_{2,0}, \quad P_3 = \Delta^3 \, \bar{Q}_{3,0}, \quad P_4 = \Delta^4 \, \bar{Q}_{4,0}, \quad \ldots$$

are relatively invariant under the simultaneous transformations of $\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ and $Q_{k,l}$ with weights 2, 3 and 4 respectively. Because of their dependence on a covariant vector forms P_i are called *contravariants* (see [21] § 13 for more details on classical terminology). In the case of binary form there is a duality between covariant vectors and contravariant vectors. Indeed let $\mu = -\delta$ and $\eta = \gamma$. Then μ and η form a contravariant vector of weight -1:

$$\begin{pmatrix} \mu \\ \eta \end{pmatrix} \mapsto \frac{1}{\det(A)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mu \\ \eta \end{pmatrix}.$$

Thus the polynomials P_i rewritten in terms of the new variables μ and η are covariants

of weight zero:

$$
\begin{aligned}
P_2 &= Q_{2,0}\,\mu^2 + 2Q_{1,1}\,\mu\eta + Q_{0,2}\,\mu^2; & (3.43)\\
P_3 &= Q_{3,0}\,\mu^3 + 3Q_{2,1}\,\mu^2\eta^2 + 3\,Q_{1,2}\,\mu\eta^2 + Q_{0,3}\,\eta^3; & (3.44)\\
P_4 &= Q_{4,0}\mu^4 + 4Q_{3,1}\mu^3\eta + 6\,Q_{2,2}\mu^2\eta^2 + 4\,Q_{1,3}\mu\eta^3 + Q_{0,4}\eta^4; & (3.45)\\
&\quad etc.
\end{aligned}
$$

Let $H_\kappa(\dots, Q_{k,l}, \dots)$ be a rational invariant of these forms. We recall that $Q_{k,l}$ are differential functions (see formulas (3.30), (3.31) and (3.32)) and so functions $H_\kappa(\dots, Q_{k,l}, \dots)$ provide differential invariants of the initial action (3.24). We would like to find sufficiently many of such invariants in order to parameterize the signature manifold. We start with the following classical relative invariants [18].

The discriminant of quadratic P_2:

$$
d_2 = Q_{2,0}Q_{0,2} - Q_{1,1}^2
$$

which is a relative invariant of weight 2.

The discriminant of the cubic P_3:

$$
d_3 = Q_{3,0}^2\,Q_{0,3}^2 - 3\,Q_{2,1}^2 Q_{1,2}^2 - 6\,Q_{3,0}Q_{2,1}Q_{1,2}Q_{0,3} + 4\,Q_{3,0}Q_{1,2}^3 + 4\,Q_{0,3}Q_{2,1}^3
$$

which is a relative invariant of weight 6.

Two invariants of the quartic P_4:

$$
i := Q_{4,0}Q_{0,4} - 4\,Q_{3,1}Q_{1,3} + 3\,Q_{2,2}^2
$$

of weight 4, and

$$
j = \det \begin{pmatrix} Q_{0,4} & Q_{1,3} & Q_{2,2} \\ Q_{1,3} & Q_{2,2} & Q_{3,1} \\ Q_{2,2} & Q_{3,1} & Q_{4,0} \end{pmatrix}
$$

of weight 6.

To obtain joint relative invariants of forms P_i we can apply the omega process, or transvection (as described in chapter 3, § 48 of [18] and chapter 5 of [28]) to these forms. Let Φ and Ψ be covariants of weight k_1 and k_2 respectively then their r-th transvectant $(\Phi, \Psi)^{(r)}$ is a covariant of weight $k_1 + k_2 + r$.

Let

$$H_3 = (P_4, P_4)^{(2)}, \quad H_4 = (P_4, P_4)^{(2)},$$
$$T_3 = (H_3, P_3)^{(1)}, \quad T_4 = (H_4, P_4)^{(1)},$$
$$S = (H_4, P_3^2)^{(3)}.$$

We note that H's have weight 2 and they are the Hessians of the corresponding forms, T's have weight 3 and their explicit formulas (3.15) are given in the preceding section on binary forms. We also note that the invariants of single forms can be expressed as transvectants:

$$d_2 = \frac{1}{8}(P_2, P_2)^{(2)} \quad d_3 = \frac{1}{10368}(H_3, H_3)^{(2)}$$
$$i = \frac{1}{1152}(P_4, P_4)^{(4)} \quad j = \frac{1}{497664}(H_4, P_4)^{(4)}$$

We complete this list with the following joint covariants:

Joint invariants of cubic P_3 and quadratic P_2:

$$M_1 = \frac{1}{288}(H_3^2, P_2)^{(2)}, \qquad M_2 = \frac{1}{103680}(P_3^2, P_2^3)^{(6)}$$

of weights 4 and 6 respectively.

Joint invariants of quadratic P_2 and quartic P_4:

$$M_3 = \frac{1}{576}(P_4, P_2^2)^{(4)}, \quad M_4 = \frac{1}{1194393600}(T_4, P_2^3)^{(6)}$$

of weights 4, and 9 respectively.

Joint invariant of cubic P_3 and quartic P_4:

$$M_5 = \frac{1}{238878720}(S, P_4)^{(4)}$$

of weight 9.

Taking into account the weights of the relative invariants above we define three absolute rational invariants of the third order:

$$I_1 = \frac{M_1}{d_2^2}, \qquad I_2 = \frac{M_2}{d_2^3}, \qquad I_3 = \frac{d_3}{d_2^3} \tag{3.46}$$

and five invariants of the fourth order:

$$I_4 = \frac{j}{d_2^{\,3}}, \quad I_5 = \frac{i}{d_2^{\,2}}, \quad I_6 = \frac{M_4^{\,2}}{d_2^{\,9}}, \quad I_7 = \frac{M_3}{d_2^{\,2}}, \quad I_8 = \frac{M_5^{\,2}}{d_2^{\,9}}. \tag{3.47}$$

Thus we have found eight differential invariants of order four or less. Since most of the explicit formulas are long we place them in the Appendix B. Using the Thomas replacement theorem [13] we can rewrite these invariants in terms of eight independent invariants obtained by invariantizations $I_{k,l} = i(u_{k,l})$ (see Section 1.4 of Chapter 1) of the 'derivative' coordinates $u_{k,l}$:

$$I_1 = \frac{I_{2,1}\, I_{1,2} - 1}{I_{1,1}^{\,3}},$$

$$I_2 = -4\,\frac{1 + 9\, I_{2,1}\, I_{1,2}}{I_{1,1}^{\,3}},$$

$$I_3 = \frac{1 - 3\, I_{2,1}^{\,2}\, I_{1,2}^{\,2} - 6\, I_{2,1}\, I_{1,2} + 4\, I_{1,2}^{\,3} + 4\, I_{2,1}^{\,3}}{I_{1,1}^{\,6}},$$

$$I_4 = \frac{I_{4,0}\, I_{0,4} - 4\, I_{3,1}\, I_{1,3} + 3\, I_{2,2}^{\,2}}{I_{1,1}^{\,4}},$$

$$I_5 = \frac{+ I_{0,4}\, I_{2,2}\, I_{4,0} - I_{0,4}\, I_{3,1}^{\,2} - I_{1,3}^{\,2}\, I_{4,0} + 2\, I_{1,3}\, I_{2,2}\, I_{3,1} - I_{2,2}^{\,3}}{I_{1,1}^{\,6}},$$

$$I_6 = 16\,\frac{(I_{1,3}^{\,2}\, I_{4,0} - I_{0,4}\, I_{3,1}^{\,2})^2}{I_{1,1}^{\,12}},$$

$$I_7 = 4\,\frac{I_{2,2}}{I_{1,1}^{\,2}},$$

$$\begin{aligned}
I_8 = \;& \frac{1}{I_{1,1}^{\,18}}\,\big(-9\, I_{3,1}\, I_{2,2}\, I_{2,1}^{\,2}\, I_{0,4} - 6\, I_{1,3}^{\,2}\, I_{2,1}\, I_{2,2}, +6\, I_{3,1}^{\,2}\, I_{1,2}\, I_{2,2} - 2\, I_{3,1}^{\,3} + 2\, I_{1,3}^{\,3} \\
& + 2\, I_{4,0}\, I_{1,3}\, I_{1,2}\, I_{0,4} - 6\, I_{3,1}\, I_{2,2}\, I_{1,2}\, I_{0,4} \\
& + 9\, I_{2,2}^{\,2}\, I_{2,1}\, I_{0,4} + I_{4,0}^{\,2}\, I_{0,4}\, I_{1,2} - I_{4,0}\, I_{0,4}^{\,2}\, I_{2,1} + 3\, I_{4,0}\, I_{1,3}\, I_{2,1}^{\,2}\, I_{0,4} \\
& + 2\, I_{1,3}\, I_{3,1}\, I_{1,2}\, I_{4,0} - 6\, I_{3,1}^{\,2}\, I_{1,2}^{\,2}\, I_{1,3} - 4\, I_{3,1}^{\,2}\, I_{2,1}\, I_{1,3} - 9\, I_{4,0}\, I_{2,2}^{\,2}\, I_{1,2} \\
& + 3\, I_{4,0}\, I_{2,2}\, I_{3,1} + 4\, I_{1,3}^{\,2}\, I_{3,1}\, I_{1,2} - 9\, I_{4,0}\, I_{1,3}^{\,2}\, I_{1,2}\, I_{2,1} + 9\, I_{3,1}^{\,2}\, I_{2,1}\, I_{1,2}\, I_{0,4} \\
& + 9\, I_{4,0}\, I_{2,2}\, I_{1,2}^{\,2}\, I_{1,3} - 3\, I_{2,2}\, I_{1,3}\, I_{0,4} - 3\, I_{4,0}\, I_{0,4}\, I_{1,2}^{\,2}\, I_{3,1} + 6\, I_{4,0}\, I_{2,2}\, I_{2,1}\, I_{1,3} \\
& - 2\, I_{4,0}\, I_{0,4}\, I_{2,1}\, I_{3,1} - 2\, I_{1,3}\, I_{3,1}\, I_{2,1}\, I_{0,4} + I_{0,4}\, I_{3,1}^{\,2} - I_{1,3}^{\,2}\, I_{4,0} + I_{3,1}\, I_{0,4}^{\,2} \\
& + 6\, I_{1,3}^{\,2}\, I_{3,1}\, I_{2,1}^{\,2} - I_{4,0}^{\,2}\, I_{1,3}\big)^2
\end{aligned}$$

By computing (with the help of a computer) the corresponding Jacobian we conclude that invariants $\{I_1, \ldots, I_8\}$ are functionally independent. On the other hand,

since $GL(3, \mathbb{C})$ acts freely on J^4 and $\dim(J^4) = 17$, there could be no more than eight functionally independent invariants and thus $\{I_1, \ldots, I_8\}$ form a complete set of differential invariants of order four or less.

Remark 3.3.2. None of the invariants is defined when $I_{1,1} \equiv 0$ (or equivalently $d_2 \equiv 0$). By substitution of the group parameters (3.40) and (3.41) into (3.34) we conclude that it happens if and only if the inhomogenization of the Hessian:

$$n\, f(f_{pq}^2 - f_{pp}\, f_{qq}) + (n-1)\, (f_{pp}\, f_q^2 + 2\, f_p\, f_q\, f_{pq} + f_{qq}\, f_p^2)$$

is identically zero, and hence if and only if the ternary form can be transformed into a binary form.

In the next section we will use the first three invariants to obtain a classification of ternary cubics and their group of symmetries. In the last section we use all eight invariants to construct the signature manifold for the forms $x^n + y^n + z^n$, therefore obtaining necessary and sufficient condition for the equivalence of an arbitrary ternary form of degree n to the sum of n-th powers.

3.4 Classification of Ternary Cubics

In this section we reproduce known results on classification of ternary cubics up to a linear transformation ([21], [25]) and then obtain a classification of their symmetry groups, which we believe is new. We achieve this by restricting invariants I_1, I_2 and I_3, obtained in the previous section to each if the canonical forms. Note that the fourth order invariants, restricted to a cubic, are zero. By Gröbner basis computation we find the ideal, whose zero set defines the corresponding signature manifold and determine its dimension. In non-trivial situations we find the dimension using MAPLE function `hilbertdim`, based on computing the degree of Hilbert polynomial [8], [14]. Using Theorem 1.3.8, from Chapter 1 we make a conclusion about the dimension of the symmetry group, which helps us to determine the group explicitly. In the case of a finite symmetry group we find its cardinality. We start with

Reducible Cubics.

A reducible cubic is either a product of three linear factors or a product of linear and quadratic factors. We state the following classification theorem:

Theorem 3.4.1. *A reducible cubic $F(x, y, z)$ is equivalent under a linear change of variables to one of the following forms:*

1. *If it is a product of there linear factors and*

    a) *all three factors are the same, then it is equivalent to x^3 and its symmetry group, is conjugate to a four-dimensional group isomorphic to $GL(2, \mathbb{C})$, of linear transformations on the variables y and z;*

    b) *two factors are the same, then the cubic is equivalent to $x^2 y$ and its symmetry group is conjugate to four-dimensional group of matrices*

$$\begin{pmatrix} \alpha & & \\ & \frac{1}{\alpha^2} & \\ \beta & \gamma & \delta \end{pmatrix};$$

    c) *three factors are linearly dependent, but any pair of them is linearly independent, then the cubic is equivalent to $xy(x + y)$ and its symmetry group is conjugate to the three-dimensional direct product of arbitrary linear transformations $z \mapsto \alpha x + \beta y + \gamma z$ and a finite subgroup of order 6×3 of linear transformations on the (x, y)-plane, which preserves $xy(x + y)$ (see Section 3.2) .*

    d) *there factors are linearly independent factors, then the cubic is equivalent to xyz and its symmetry group is conjugate to two-dimensional group of matrices*

$$\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \frac{1}{\alpha\beta} \end{pmatrix}; \tag{3.48}$$

2. *If it is a product of quadratic and linear factors then there are two canonical forms:*

    a) *$F(x, y, z) \sim z\left(x^2 + yz\right)$. In this case the symmetry group is conjugate to a two-dimensional group generated by*

$$\begin{pmatrix} 1 & 0 & \alpha \\ -2\alpha & 1 & -\alpha^2 \\ 0 & 0 & 1 \end{pmatrix} \quad and \quad \begin{pmatrix} \beta & & \\ & \beta^4 & \\ & & \frac{1}{\beta^{-2}} \end{pmatrix}; \tag{3.49}$$

    b) *$F(x, y, z) \sim z\left(x^2 + y^2 + z^2\right)$. In this case the symmetry group is conjugate to a one-dimensional group of rotations around the z-axis which is isomorphic to $O(2, \mathbb{C})$.*

Proof.

1. The classifications of the cubics reducible into linear factors is obvious. We note only that a cubic has repeated factors (cases 1. a) and 1. b)) if and only if its Hessian is identically zero and so the invariants I_1, I_2 and I_3 are not defined (see Remark 3.3.2). The graph S of such cubic is a totally singular submanifold in $\mathbb{C}^4$, in a sense that there is no order of prolongation n at which the prolonged action of the group becomes locally free on $j^n(S)$. Thus, none of the constructions from Chapter 1 can be applied to this case. The graphs of all other cubics define regular submanifolds and so, due to Theorem 1.3.8, the dimensions of their symmetry groups equal to $2 - \dim \mathcal{C}(S_f)$ where $f(p, q)$ is the inhomogeneous version of $F(x, y, z)$. The signature manifold $\mathcal{C}(S_f)$, is parameterized by $I_1(f)$, $I_2(f)$ and $I_3(f)$. When the cubic is equivalent to xyz (case 1. c)) the signature manifold consists of a single point:

$$I_1 = \frac{4}{3}, \qquad I_2 = \frac{16}{3}, \qquad I_3 = \frac{16}{9}.$$

and so the symmetry group is two-dimensional. It is easy to check that the group (3.48) leaves xyz unchanged.

2. The general form of a cubic reducible into linear and quadratic factors is:

$$F(x, y, z) = (k_1 x + k_2 y + k_3 z)(k_4 z^2 + k_5 zx + k_6 zy + K(x, y)),$$

where k_i are some complex coefficients and $K(x, y)$ is a non-zero quadratic form in two variables. By taking the first factor as a new variable z', one obtains an equivalent cubic

$$z \left(k_4 z^2 + k_5 zx + k_6 zy + K(x, y)\right)$$

where k_i are some new coefficients and $K(x, y)$ is a new non-zero quadratic form. It is known that by a linear change of variables the quadratic $K(x, y)$ can be transformed to either $x^2 + y^2$ or x^2. Thus $F(x, y, z)$ is equivalent to either:

$$z \left(k_2 z^2 + k_3 zx + k_4 zy + x^2 + y^2\right) \tag{3.50}$$

or

$$z \left(k_2 z^2 + k_3 zx + k_4 zy + x^2\right), \tag{3.51}$$

where k_i are again new coefficients.

In the first case (3.50) we make the transformation

$$x' = (x + \frac{k_3}{2} z); \qquad y' = (y + \frac{k_4}{2} z); \qquad z' = z$$

to obtain an equivalent form

$$z\left(kz^2 + x^2 + y^2\right).$$

Finally the scaling:

$$k^{1/6}x' = x, \qquad k^{1/6}y' = y \qquad k^{-1/3}z' = z$$

leads to the canonical form:

$$z\left(z^2 + x^2 + y^2\right).$$

In the second case (3.51) we make the transformation

$$x' = (x + \sqrt{k_2}z); \qquad y' = k_4 y + (k_3 - 2\sqrt{k_2})x; \qquad z' = z/k_4.$$

to obtain an equivalent cubic:

$$kz\left(zy + x^2\right).$$

Finally, we make the transformation: $z' = kz$, $y' = \frac{1}{k}y$, which leads to the canonical form:

$$z\left(zy + x^2\right)$$

We compute the corresponding signature manifolds. The signature manifold of a cubic equivalent to $z\left(x^2 + yz\right)$, consists of a single point:

$$I_1 = -\frac{1}{6}, \qquad I_2 = \frac{41}{6}, \qquad I_3 = -\frac{2}{9},$$

and hence the symmetry group is two-dimensional. Using Lie's criterion 1.1.12 we found infinitesimal symmetries:

$$\begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & 4 & \\ & & -2 \end{pmatrix};$$

that give rise to the group (3.49).

The signature manifold of a cubic equivalent to $z\left(x^2 + y^2 + z^2\right)$, is defined by three equations found by Gröbner basis computations:

$$-1482\,I_3\,I_2 + 8865\,I_3{}^2 + 40\,I_2{}^2 - 1296\,I_1 - 36\,I_2 + 17280\,I_3 - 18522\,I_3\,I_1,$$

$$40\,I_1\,I_2 + 582\,I_3\,I_1 + 42\,I_3\,I_2 - 315\,I_3^2 - 144\,I_1 - 4\,I_2 - 480\,I_3,$$

$$360\,I_1^2 - 378\,I_3 I_1 - 18\,I_3\,I_2 + 135\,I_3{}^2 - 144\,I_1 - 4\,I_2 + 120\,I_3$$

The the corresponding variety is one-dimensional and thus the symmetry group is also one-dimensional. We notice that the rotations with respect to the z-axis leave the canonical form unchanged. $\qquad\square$

We proceed with the classification of the

Irreducible Ternary Cubics:

It is well known that any homogeneous irreducible cubic $F(x, y, z)$ over $\mathbb{C}$ can be transformed by a linear map into Weierstrass normal form [24]. Let V be the set of zeros of the inhomogenization $f(p, q)$ in $\mathbb{C}P^2$. The normal form is obtained by transforming one of the inflection points of V to the infinite point $(0, 1, 0)$, and the tangent line at this point to the line $(k, 1, 0)$ at infinity. We state the following classification theorem:

Theorem 3.4.2. *An irreducible cubic $F(x, y, z)$ can be transformed under a linear change of variables to one of the following forms:*

1 *If f(p,q) defines a singular variety V then it is equivalent to either*

    a) $p^3 - q^2$ *and the it has one-dimensional symmetry group given by:*

$$\begin{pmatrix} 1 & & \\ & \alpha & \\ & & \frac{1}{\alpha^2} \end{pmatrix};$$ $\qquad(3.52)$

    or

    b) $p^2(p+1) - q^2$, *which has a discrete symmetry group, consisting of 6 projective symmetries (see Definition 3.1.4 of Section 3.1) which correspond to 18 genuine symmetries.*

2 *If $f(p, q)$ defines a nonsingular variety V then it either equivalent to:*

    a) *a cubic in one-parametric family:*

$$p^3 + ap + 1 - q^2$$

    and then it has 18 projective symmetries.

    or

    b) *it is equivalent to $p^3 + p - q^2$. In this case it has 36 projective symmetries,*

    or

c) *it is equivalent to $p^3 + 1 - q^2$. In this case it has 54 projective symmetries. We note that the cubic $p^3 + q^3 + 1$ belongs to this class.*

For the proofs of the classification theorems we refer the reader to [24], [25], and restrict ourselves to the discussion of the signature manifolds and the symmetry groups. In the case 1. a) the signature manifold is defined by two equations:

$$I_1 = -\frac{1}{6}, \qquad 6\,I_2 + 45\,I_3 - 31 = 0$$

and so the symmetries form a one-dimensional group conjugate to (3.52). We note that the number of unimodular symmetries is finite.

In all other cases the symmetry group is finite. In theory, the projective symmetries can be found explicitly by solving the equations

$$I_1(p,q) = I_1(P,Q), \quad I_2(p,q) = I_2(P,Q), \quad I_3(p,q) = I_3(P,Q), \tag{3.53}$$

for P and Q in terms of p and q. All solutions must be linear fractional expressions in p and q. In practice however, we were not able to carry out these computations. Nevertheless we can find the cardinality of the symmetry group using a well known algebraic geometry result ([8], Proposition 8, ch. 5, § 3).

Proposition 3.4.3. *Let $\mathbf{I} \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal such that its zero set V is finite then*

(i) *The dimension of $(\mathbb{C}[x_1, \ldots, x_n]/\mathbf{I})$ (as a vector space over $\mathbb{C}$) is finite and greater or equal to the number of points in V.*

(i) *If $\mathbf{I}$ is a radical ideal then equality holds, i.e., the number of points in V is exactly $(\mathbb{C}[x_1, \ldots, x_n]/\mathbf{I})$.*

For generic values of P and Q we find the number of the solutions for (3.53) by computing the dimension of the quotient space of $\mathbb{C}[p,q]$ by the corresponding ideal. We use two algorithms presented in the exercises for ch. 2, § 2 of [9], first to check that the ideal defined by (3.53) is radical, and then to compute the dimension of the quotient.

The nonsingular irreducible ternary cubics are known as *elliptic curves* and play an important role in number theory. The number of symmetries for these curves

has a natural explanation. First let us consider only the symmetries fixing the point $(0, 1, 0)$ and mapping the line at infinity to itself. Following Knapp, [24] we call these symmetries *admissible*. It is not difficult to prove [24] that there are only 2 such symmetries in case 2.a), 4 in case 2.b) and 6 in case 2.c). Each of nonsingular irreducible cubics has 9 inflection points. Each of the inflection point can be mapped to $(0, 1, 0)$ with the corresponding tangent line mapped to the line at infinity. *We observe that the number of the projective symmetries of a nonsingular irreducible cubic equals to the number of its inflection point times the number of its admissible symmetries.*

Ternary cubics form a ten-dimensional linear space. So by dimensional consideration we expect to have one absolute invariant depending on the coefficients of cubic. Indeed such invariant is known [21], [24] and has been used to obtain classification results. This invariant must be expressible in terms of the invariants I_1, I_2 and I_3 but we have not tried to obtain the explicit formula. Not all of the orbits of $GL(3, \mathbb{C})$ acting on the space of cubics are closed. In fact only the orbits of elliptic curves, the orbit of xyz and the orbit of x^3 are. See Kraft [25] for the proofs and the description of how some orbit is included in the closure of the others.

We conclude this section with a simple corollary from Theorems 3.4.1 and 3.4.2:

Corollary 3.4.4. *A cubic in three variables splits into a linear factor and an irreducible quadratic factor if and only if its $SL(3, \mathbb{C})$ symmetries form a one-dimensional Lie group.*

MAPLE computations for ternary cubics can be found in Appendix C.

3.5 The Signature Manifold for $x^n + y^n + z^n$.

In this section we construct the signature manifold for $f(p, q) = p^n + q^n + 1$ therefore determining the necessarily and sufficient condition for a ternary form to be complex equivalent to the sum of n-th powers. We first recall that the action (3.24) becomes free on the third order jet space J^3. We restrict the first three invariants to $f(p, q) = p^n + q^n + 1$:

$$I_1(f) = -\frac{(n-2)^2}{n(n-1)}$$

$$I_2(f) = -\frac{(n-2)^2}{n(n-1)}$$

$$\times \frac{5\,q^n\,(p^n)^2 + 5\,(p^n)^2 - 26\,p^n\,q^n + 5\,(q^n)^2\,p^n + 5\,p^n + 5\,q^n + 5\,(q^n)^2}{p^n\,q^n}$$

$$I_3(f) = -\frac{(n-2)^4}{n^2(n-1)^2}\,\frac{(p^n+1-q^n)(p^n-1+q^n)(p^n-1-q^n)}{p^n\,q^n}$$

We observe that the first invariant is constant and we check that the last two are functionally independent. Thus the third order signature manifold has maximal possible dimension and the symmetry group of $f(p,q)$ is finite. Moreover, we can conclude that the differential invariant order of the graph $u = p^n + q^n + 1$ equals to three and so both equivalence and symmetry problems can be solved by construction the fourth order signature manifold.

For $n > 3$ we need to restrict the remaining five (forth order) invariants to the graph $u = p^n + q^n + 1$:

$$I_4(f) = -\frac{(n-2)^2(n-3)^2}{n^2(n-1)^2}\,\frac{((p^n)^3 + 1 + (q^n)^3)}{p^n\,q^n}$$

$$I_5(f) = -\frac{(n-2)^3(n-3)^3}{n^3(n-1)^3}$$

$$I_6(f) = -\frac{(n-2)^6(n-3)^6}{n^6(n-1)^6}\,\frac{(q^n-1)^2\,(p^n-1)^2\,(p^n-q^n)^2\,(p^n+q^n+1)^3}{(p^n)^3\,(q^n)^3}$$

$$I_7(f) = \frac{(n-2)(n-3)}{n(n-1)}\,\frac{(p^n)^2 + q^n(p^n)^2 + p^n - 2\,p^n\,q^n + (q^n)^2\,p^n + q^n + (q^n)^2}{p^n\,q^n}$$

$$I_8(f) = -\frac{(n-2)^{10}(n-3)^6}{n^8(n-1)^8}\,\frac{(q^n-1)^2\,(p^n-1)^2\,(p^n-q^n)^2\,(p^n+q^n+1)^3}{(p^n)^3\,(q^n)^3}.$$

We note that $I_5(f)$ is constant and $I_8(f)$ is a constant multiple of $I_6(f)$. We need however three other relations to define the signature manifold. We observe that all

invariants are functions of p^n and q^n and denote $P = p^n$ and $Q = q^n$, then:

$$i_2 = -\frac{5\,Q^2 + 5\,P\,Q^2 + 5\,P^2\,Q - 26\,P\,Q + 5\,Q + 5\,P + 5\,P^2}{PQ} = \frac{n(n-1)}{(n-2)^2}\,I_2,$$

$$i_3 = -\frac{(Q+1-P)(Q-1+P)(Q-1-P)}{PQ} = \frac{n^2(n-1)^2}{(n-2)^4}\,I_3,$$

$$i_4 = \frac{Q^3 + P^3 + 1}{PQ} = \frac{n^2(n-1)^2}{(n-2)^2(n-3)^2}\,I_4,$$

$$i_6 = -\frac{(P-1)^2(Q-1)^2(Q-P)^2(P+Q+1)^3}{P^3Q^3} = \frac{n^6(n-1)^6}{(n-2)^6(n-3)^6}\,I_6,$$

$$i_7 = \frac{PQ^2 + Q^2 - 2\,PQ + P^2Q + Q + P^2 + P}{PQ} = \frac{n(n-1)}{(n-2)(n-3)}\,I_7,$$

$$i_8 = -\frac{(P-1)^2(Q-1)^2(Q-P)^2(P+Q+1)^3}{P^3Q^3} = \frac{n^8(n-1)^8}{(n-2)^{10}(n-3)^6}\,I_8$$

The following four relations were computed by MACAULAY 2
(remarkably MAPLE 5 was not be able to handle these computations):

$$i_6 - i_8 = 0, \qquad i_3 + i_4 - i_7 = 0, \qquad i_2 + 5\,i_7 - 16 = 0,$$

$$i_4\,i_7^2 - i_7^3 - 4\,i_4^2 + 4\,i_4\,i_7 - 8\,i_4 + 12\,i_7 + i_8 - 20 = 0.$$

The corresponding relations among invariants $I_2(f)$, $I_3(f)$, $I_4(f)$, $I_6(f)$, $I_7(f)$ and $I_8(f)$ follow immediately. Combining them with the conditions:

$$I_1(f) = -\frac{(n-2)^2}{n(n-1)} \quad \text{and} \quad I_5(f) = -\frac{(n-2)^3(n-3)^3}{n^3(n-1)^3}$$

and demanding that $\dim \mathcal{C}(f) = 2$ (equivalently we demand that the symmetry group of f is discrete), we thus obtain necessary and sufficient conditions for a ternary n-form to be equivalent to the sum of n-th powers.

Appendices.

Appendix A

Computations on Binary Forms.

A.1 The MAPLE Code for computing Symmetries of Binary Forms

The MAPLE code consists of two main programs — `symm` and `matrices` — and two auxiliary functions — `simple` and `l_f`. The program `symm` is the main function. The input consists of a complex-valued polynomial $f(p)$ considered as the projective version of homogeneous binary form $F(x, y)$, and the degree $n = deg(F)$. The program computes the invariants J and K in reduced form, determines the dimension of the symmetry group, and, in the case of a finite symmetry group, applies the MAPLE command `solve` to solve the two polynomial symmetry equations (3.19) to find explicit form of symmetries. The output of `symm` consists of the projective index of the form and the explicit formulae for its discrete projective symmetries. The program also notifies the user if the symmetry group is not discrete, or is in the maximal discrete symmetry class. The program works well when applied to very simple forms, but experienced difficulties simplifying complicated rational algebraic formulae into the basic linear fractional form.

```
> with(linalg):
> symm:=proc(form,n)
global tr,error;
local Q,Qp,Qpp,Qppp,Qpppp,H,T,V,U,J,K,j,k, Eq1,Eq2,i,eqtr,
ans;
```

```
tr:='tr':
Q:=form(p);
Qp:=diff(Q,p);
Qpp:=diff(Qp,p);
Qppp:=diff(Qpp,p);
Qpppp:=diff(Qppp,p);
H:=n*(n-1)*(Q*Qpp-(n-1)/n*Qp^2);
if H=0 then
 ans:=`Hessian is zero:  two-dimensional symmetry group`
else
 T:=-n^2*(n-1)*(Q^2*Qppp-3*(n-2)/n*Q*Qp*Qpp
 +2*(n-1)*(n-2)/n^2*Qp^3);
 V:=Q^3*Qpppp-4*(n-3)/n*Q^2*Qp*Qppp+6*(n-2)*(n-3)/n^2
 *Q*Qp^2*Qpp-3*(n-1)*(n-2)*(n-3)/n^3*Qp^4;
 U:=n^3*(n-1)*V-3*(n-2)/(n-1)*H^2;
 J:=simple(T^2/H^3); K:=simple(U/H^2);
 j:=subs(p=P,J);k:=subs(p=P,K);
 Eq1:=simplify(numer(J)*denom(j)-numer(j)*denom(J));
 Eq2:=simplify(numer(K)*denom(k)-numer(k)*denom(K));
 if Eq1=0 then
  ans:=`Form has a one-dimensional symmetry group`;
 else
  if Eq2=0 then
   print (` Form has the maximal possible discrete
   symmetry group`);
   eqtr:= [solve(Eq1,P)];
   tr:=map(radsimp,map(allvalues,eqtr));
  else
   eqtr:=[solve({Eq1,Eq2},P)];
   tr:= [];
   for i from 1 to nops(eqtr) do
    tr:=map(radsimp,[op(tr),allvalues(rhs(eqtr[i][1]))]);
   od
```

```
    fi;

    print('The number of elements in the symmetry group'

    =nops(tr));

    ans:=map(l_f,tr);

    if error=1 then
      print('ERROR: Some of the transformations are not

      linear-fractional')

    else

      if error=2 then
        print('WARNING: Some of the transformations are not

        written in the form polynomial over polynomial')

      fi;

    fi;

   fi;
  fi;

  ans

 end:
```

The program `matrices` determines the matrix symmetry corresponding to a given (list of) projective symmetries. As discussed in the text, this only requires determining an overall scalar multiple, which can be found by substituting the projective symmetry into the form. The output consists of each projective symmetry, the scalar factor μ, and the resulting matrix symmetry.

```
  > matrices:=proc(form,n,L::list)
  local Q,ks,ksi,i,Sf,M;
   ksi:='ksi';

   for i from 1 to nops(L) do
```

```
    Sf:=simplify(denom(L[i])^n*form(L[i]));
    ks:=quo(Sf,form(p),p);
    ksi:=simplify(ks^(1/n),radical,symbolic);
    M[i]:=matrix(2,2,[coeff(numer(L[i]),p)/ksi,
    coeff(numer(L[i]),p,0)/ksi,coeff(denom(L[i]),p)/ksi,
    coeff(denom(L[i]),p,0)/ksi]);
    print(L[i], mu=ksi, map(simplify,M[i]))
  od;
end:
```

The auxiliary function `simple` helps to simplify rational expressions by manipulating the numerator and denominator separately. The simplified rational expression is returned.

```
> simple:=proc(x)
local nu,de,num,den;
 nu:=numer(x);
 de:=denom(x);
 num:=(simplify((nu,radical,symbolic)));
 den:=(simplify((de,radical,symbolic)));
 simplify(num/den);
end:
```

The auxiliary function `l_f` uses polynomial division to reduce rational expressions to linear fractional form (when possible).

```
> l_f:=proc(x)
local A,B,C,S,de,nu,r,R;
global error;error:='error';
```

```
nu:=numer(x);
de:=denom(x);
if type(nu,polynom(anything,p))
and type(de,polynom(anything,p)) then
  if degree(nu,p)+1=degree(de,p) then
    A:=quo(de,nu,p,'B');
    S:=1/A; R:=0
  else
    A:=quo(nu,de,p,'B');
    if B=0 then
      S:=A; R:=0;
    else
      C:=quo(de,B,p,'r'); R:=simple(r);
      S:=simplify(A+1/C)
    fi;
  fi;
  if R=0 then
    collect(S,p)
  else
    error:=1; x
  fi;
else
  error:=2; x
fi;
end:
```

A.2   Cubic Forms.

We now present the results of applying the function `symm` and `matrices` to cubic forms. We begin with simple cases, ending with a cubic whose formulae required extensive manipulation.

Cubics with one triple root:

```
>   f:=p->p^3;
```

$$f := p \rightarrow p^3$$

```
>   symm(f,3);
```

$$\textit{Hessian is zero}: \ \textit{two} - \textit{dimensional symmetry group}$$

Cubics with one double root and one single root:

```
>   f:=p->p;
```

$$f := p \to p$$

```
>   symm(f,3);
```

$$\textit{Form has a one} - \textit{dimensional symmetry group}$$

Cubics with three simple roots:

```
>   f:=p->p^3+1;
```

$$f := p \to p^3 + 1$$

```
>   S:=symm(f,3);
```

$$\textit{Form has the maximal possible discrete symmetry group}$$

$$\textit{The number of elements in the symmetry group} = 6$$

$$S := \left[p, \ \frac{1}{p}, \ \frac{-\frac{1}{2}+\frac{1}{2}I\sqrt{3}}{p}, \ \frac{-\frac{1}{2}-\frac{1}{2}I\sqrt{3}}{p}, \ (-\frac{1}{2}+\frac{1}{2}I\sqrt{3})\,p, \ (-\frac{1}{2}-\frac{1}{2}I\sqrt{3})\,p \right]$$

```
>   matrices(f,3,[S[2],S[4]]);
```

$$\frac{1}{p}, \quad \mu = 1, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\frac{-\frac{1}{2}-\frac{1}{2}I\sqrt{3}}{p}, \quad \mu = 2, \quad \begin{bmatrix} 0 & -\frac{1}{2}-\frac{1}{2}I\sqrt{3} \\ 1 & 0 \end{bmatrix}$$

A more complicated cubic example.

All cubics with a discrete symmetry group are complex equivalent to $x^3 + y^3$ and have projective index 6. However, when we apply the same code to a cubic not in canonical form. The initial MAPLE result is not in the correct linear fractional form. We must simplify the rational algebraic expressions "by hand" to put them in the form of a projective linear fractional transformation.

```
>   f:=p->p^3+p+1;
```

$$f := p \to p^3 + p + 1$$

```
>  S:=symm(f,3);
```

Form has the maximal possible discrete symmetry group

The number of elements in the symmetry group = 6

WARNING : Some of the transformations are not written in the form polynomial over polynomial

$$S := [p, \frac{(-9 + I\sqrt{31})p + 2}{9 + I\sqrt{31} + 6p}, -\frac{(9 + I\sqrt{31})p - 2}{9 - I\sqrt{31} + 6p},$$

$$\frac{1}{18}((54p^4 + 9\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)}p^3 + 324p^3 + 3\,2^{(1/3)}3^{(2/3)}\%1^{(2/3)}p^2 + 450p^2$$

$$-108p + 9\,2^{(1/3)}3^{(2/3)}\%1^{(2/3)}p + 9\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)}p - 2^{(1/3)}3^{(2/3)}\%1^{(2/3)}$$

$$+6 + 9\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)})3^{(2/3)}2^{(1/3)}) \Big/$$

$$(\%1^{(1/3)}(27p^3 - 9p^2 - 1)), -\frac{1}{36}((54\,I\sqrt{3}\,p^4 + 54p^4 + 324p^3$$

$$+324\,I\sqrt{3}\,p^3 - 18\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)}p^3 + 450p^2 + 450\,I\sqrt{3}\,p^2$$

$$-9\,I\,3^{(1/6)}2^{(1/3)}\%1^{(2/3)}p^2 + 3\,2^{(1/3)}3^{(2/3)}\%1^{(2/3)}p^2 - 27\,I\,3^{(1/6)}2^{(1/3)}\%1^{(2/3)}p$$

$$-108p - 18\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)}p + 9\,2^{(1/3)}3^{(2/3)}\%1^{(2/3)}p - 108\,I\sqrt{3}\,p$$

$$+3\,I\,3^{(1/6)}2^{(1/3)}\%1^{(2/3)} + 6\,I\sqrt{3} - 18\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)} - 2^{(1/3)}3^{(2/3)}\%1^{(2/3)} + 6$$

$$)3^{(2/3)}2^{(1/3)}) \Big/ (\%1^{(1/3)}(27p^3 - 9p^2 - 1)), \frac{1}{36}((54\,I\sqrt{3}\,p^4 - 54p^4 - 324p^3$$

$$+324\,I\sqrt{3}\,p^3 + 18\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)}p^3 - 450p^2 + 450\,I\sqrt{3}\,p^2$$

$$-9\,I\,3^{(1/6)}2^{(1/3)}\%1^{(2/3)}p^2 - 3\,2^{(1/3)}3^{(2/3)}\%1^{(2/3)}p^2 - 27\,I\,3^{(1/6)}2^{(1/3)}\%1^{(2/3)}p$$

$$+108p + 18\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)}p - 9\,2^{(1/3)}3^{(2/3)}\%1^{(2/3)}p - 108\,I\sqrt{3}\,p$$

$$+3\,I\,3^{(1/6)}2^{(1/3)}\%1^{(2/3)} + 6\,I\sqrt{3} + 18\,2^{(2/3)}3^{(1/3)}\%1^{(1/3)} + 2^{(1/3)}3^{(2/3)}\%1^{(2/3)}$$

$$-6)3^{(2/3)}2^{(1/3)}) \Big/ (\%1^{(1/3)}(27p^3 - 9p^2 - 1))]$$

$$\%1 := 9 + 18p - 81p^2 + 261p^3 + 27\sqrt{31}\sqrt{3}\,p^3 - 9\sqrt{31}\sqrt{3}\,p^2 - \sqrt{31}\sqrt{3}$$

The first three components of S are in the proper linear fractional form. The problem with the other expressions is that MAPLE does not automatically factor polynomials under a radical. One approach to simplification is to first do the required factorization:

```
>  n1:=factor(9+18*p-81*p^2+261*p^3+27*sqrt(31)*sqrt(3)*p^3
-9*sqrt(31)*sqrt(3)*p^2-sqrt(31)*sqrt(3));
```

$$n1 := -\frac{1}{24}\,(29 + 3\,\sqrt{31}\,\sqrt{3})\,(-6\,p - 9 + \sqrt{31}\,\sqrt{3})^3$$

Substituting `n1` into the fourth rational algebraic expression in `S` above, we can now force MAPLE to take the cube root and obtain the actual linear fractional formula for this symmetry:

```
>   simp1:=radsimp(1/18*((54*p^4+9*2^(2/3)*3^(1/3)*(n1)^(1/3)*p^3
+324*p^3+3*2^(1/3)*3^(2/3)*(n1)^(2/3)*p^2+450*p^2
+9*2^(1/3)*3^(2/3)*(n1)^(2/3)*p+9*2^(2/3)*3^(1/3)*(n1)^(1/3)*p
-108*p-2^(1/3)*3^(2/3)*(n1)^(2/3)+6
+9*2^(2/3)*3^(1/3)*(n1)^(1/3))*3^(2/3)*2^(1/3))/((n1)^(1/3)
*(27*p^3-9*p^2-1)));

>   simp2:=l_f(simp1);

>   simp3:=collect(expand(numer(simp2))/expand(denom(simp2)),p);
```

$$simp3 := \left((-226\,2^{(1/3)} - 20\,\%2 - 20\,2^{(2/3)}\,\%1^{(2/3)} - 22\,\%3 - 208\,\%1^{(1/3)})\,p - 8\,\%3\right.$$
$$\left. - 58\,2^{(2/3)}\,\%1^{(2/3)} - 566\,\%1^{(1/3)} - 58\,\%2 - 116\,2^{(1/3)}\right)\Big/\big($$
$$(-6\,2^{(2/3)}\,\%1^{(2/3)} - 174\,\%1^{(1/3)} - 1686\,2^{(1/3)} - 174\,\%3 - 6\,\%2)\,p$$
$$+ 20\,2^{(2/3)}\,\%1^{(2/3)} + 208\,\%1^{(1/3)} + 226\,2^{(1/3)} + 22\,\%3 + 20\,\%2)$$
$$\%1 := 29 + 3\,\sqrt{31}\,\sqrt{3}$$
$$\%2 := \%1^{(1/3)}\,\sqrt{31}\,\sqrt{3}$$
$$\%3 := 2^{(1/3)}\,\sqrt{31}\,\sqrt{3}$$

The linear fractional formulae for the other symmetries are derived in a similar fashion.

A.3   The Octahedral Symmetry Group.

As we remarked in the text, the sextic polynomial $Q(p) = p^5 + p$ has an octahedral symmetry group. Here we illustrate how the symmetries are computed using our MAPLE program.

```
>   f:=p->p^5+p;
```

$$f := p \rightarrow p^5 + p$$

```
>   symm(f,6);
```

Form has the maximal possible discrete symmetry group

The number of elements in the symmetry group $= 24$

WARNING : Some of the transformations are not written in the form polynomial over polynomial

$$\left[\frac{1}{p}, \, p, \, -p, \, -\frac{1}{p}, \, \frac{I}{p}, \, -\frac{I}{p}, \, I\,p, \, -I\,p, \, \frac{-2\,p^3 + 2\,I\,p + \%3}{p^4 + 1}, \, \frac{-2\,p^3 + 2\,I\,p - \%3}{p^4 + 1}, \right.$$

$$\frac{-2\,p^3 - 2\,I\,p + \%4}{p^4 + 1}, \, \frac{-2\,p^3 - 2\,I\,p - \%4}{p^4 + 1}, \, \frac{2\,p^3 + 2\,I\,p + \%4}{p^4 + 1}, \, \frac{2\,p^3 + 2\,I\,p - \%4}{p^4 + 1},$$

$$\frac{2\,p^3 - 2\,I\,p + \%3}{p^4 + 1}, \, \frac{2\,p^3 - 2\,I\,p - \%3}{p^4 + 1}, \, \frac{-2\,p + 2\,I\,p^3 + \%1}{p^4 + 1}, \, \frac{-2\,p + 2\,I\,p^3 - \%1}{p^4 + 1},$$

$$\frac{-2\,p - 2\,I\,p^3 + \%2}{p^4 + 1}, \, \frac{-2\,p - 2\,I\,p^3 - \%2}{p^4 + 1}, \, \frac{2\,p + 2\,I\,p^3 + \%2}{p^4 + 1}, \, \frac{2\,p + 2\,I\,p^3 - \%2}{p^4 + 1},$$

$$\left. \frac{2\,p - 2\,I\,p^3 + \%1}{p^4 + 1}, \, \frac{2\,p - 2\,I\,p^3 - \%1}{p^4 + 1} \right]$$

$$\%1 := \sqrt{-4\,p^6 + 4\,p^2 + I\,p^8 - 6\,I\,p^4 + I}$$

$$\%2 := \sqrt{-4\,p^6 + 4\,p^2 - I\,p^8 + 6\,I\,p^4 - I}$$

$$\%3 := \sqrt{4\,p^6 - 4\,p^2 + I\,p^8 - 6\,I\,p^4 + I}$$

$$\%4 := \sqrt{4\,p^6 - 4\,p^2 - I\,p^8 + 6\,I\,p^4 - I}$$

Again, MAPLE has failed to simplify the expressions $\%1, \%2, \%3, \%4$, and we need to make it take the square root. In the case of symmetries numbers $9, 11, 13, 15, 17, 19, 21, 23$ this is done as follows. The others are handled in a similar fashion, and, for brevity, we omit the formulae here.

```
> for j in [9,11,13,15,17,19,21,23] do
sq:=sqrt(factor(op(op(numer(tr[j]))[3])[1],I),symbolic):
s[j]:=
l_f((op(numer(tr[j]))[1]+op(numer(tr[j]))[2]+sq)/denom(tr[j]));
print(s.j=s[j]);
od:
```

$$s9 = -\frac{(\sqrt{2} + I\,\sqrt{2})\,p - 2}{-\sqrt{2} + I\,\sqrt{2} - 2\,p}$$

$$s11 = -\frac{(-\sqrt{2} + I\,\sqrt{2})\,p + 2}{I\,\sqrt{2} + \sqrt{2} + 2\,p}$$

$$s13 = \frac{(-\sqrt{2} + I\,\sqrt{2})\,p - 2}{I\,\sqrt{2} + \sqrt{2} - 2\,p}$$

$$s15 = \frac{(\sqrt{2} + I\sqrt{2})\,p + 2}{-\sqrt{2} + I\sqrt{2} + 2\,p}$$

$$s17 = -\frac{(-\sqrt{2} + I\sqrt{2})\,p - 2}{-\sqrt{2} + I\sqrt{2} - 2\,I\,p}$$

$$s19 = -\frac{I\,((\sqrt{2} + I\sqrt{2})\,p + 2)}{-\sqrt{2} + I\sqrt{2} + 2\,p}$$

$$s21 = \frac{(\sqrt{2} + I\sqrt{2})\,p - 2}{\sqrt{2} + I\sqrt{2} + 2\,I\,p}$$

$$s23 = \frac{(-\sqrt{2} + I\sqrt{2})\,p + 2}{-\sqrt{2} + I\sqrt{2} + 2\,I\,p}$$

As we remarked in the text, the octahedral symmetry group has two generators. The matrix form of these generators is computed as follows:

```
>   matrices(f,6,[tr[7],s[9]]);
```

$$I\,p, \qquad \mu = (-1)^{(1/12)}, \qquad \begin{bmatrix} (-1)^{(5/12)} & 0 \\ 0 & -(-1)^{(11/12)} \end{bmatrix}$$

$$-\frac{(\sqrt{2} + I\sqrt{2})\,p - 2}{-\sqrt{2} + I\sqrt{2} - 2\,p}, \qquad \mu = 2\sqrt{2}, \qquad \begin{bmatrix} -\dfrac{1}{2} - \dfrac{1}{2}I & \dfrac{1}{2}\sqrt{2} \\ -\dfrac{1}{2}\sqrt{2} & -\dfrac{1}{2} + \dfrac{1}{2}I \end{bmatrix}$$

We end with two further examples. We already know that the following octavic polynomial also has an octahedral symmetry group. In this case, `symm` produces the projective symmetries directly:

```
>   f:=p->p^8+14*p^4+1;
```

$$f := p \rightarrow p^8 + 14\,p^4 + 1$$

```
>   S:=symm(f,8);
```

The number of elements in the symmetry group $= 24$

$$S := [-\frac{1}{p}, -\frac{p-1}{p+1}, -\frac{p+1}{p-1}, p, -p, \frac{p+1}{p-1}, \frac{p-1}{p+1}, \frac{1}{p}, \frac{I(p-1)}{p+1},$$

$$-\frac{I(p-1)}{p+1}, \frac{I(p+1)}{p-1},$$

$$-\frac{I(p+1)}{p-1}, \frac{I}{p}, -\frac{I}{p}, Ip, -Ip, \frac{-1+Ip}{-I+p}, -\frac{1+Ip}{I+p},$$

$$\frac{1+Ip}{I+p}, -\frac{-1+Ip}{-I+p}, \frac{-1+Ip}{1+Ip}, \frac{1+Ip}{-1+Ip}, -\frac{1+Ip}{-1+Ip}, -\frac{-1+Ip}{1+Ip}]$$

```
>  matrices(f,8,[S[11],S[15]]);
```

$$\frac{I(p+1)}{p-1}, \qquad \mu = \sqrt{2}, \qquad \begin{bmatrix} \frac{1}{2}I\sqrt{2} & \frac{1}{2}I\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

$$Ip, \qquad \mu = 1, \qquad \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

Finally, for illustrative purposes, we present a higher order example given by a binary form of degree 12.

```
>  f:=p->p^12-33*p^8-33*p^4+1;
```

$$f := p \rightarrow p^{12} - 33\,p^8 - 33\,p^4 + 1$$

```
>  S:=symm(f,12);
```

The number of elements in the symmetry group $= 24$

$$S := [p, -p, -\frac{1}{p}, \frac{1}{p}, -\frac{p-1}{p+1}, -\frac{p+1}{p-1}, \frac{p+1}{p-1}, \frac{p-1}{p+1}, \frac{I}{p}, -\frac{I}{p}, Ip, -Ip, \frac{-1+Ip}{-I+p}$$

$$-\frac{1+Ip}{I+p}, \frac{1+Ip}{I+p}, -\frac{-1+Ip}{-I+p}, \frac{I(p-1)}{p+1}, -\frac{I(p-1)}{p+1}, \frac{I(p+1)}{p-1}, -\frac{I(p+1)}{p-1},$$

$$\frac{-1+Ip}{1+Ip}, \frac{1+Ip}{-1+Ip}, -\frac{1+Ip}{-1+Ip}, -\frac{-1+Ip}{1+Ip}]$$

```
>  matrices(f,12,[S[11],S[19]]);
```

$$Ip, \qquad \mu = 1, \qquad \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{I\,(p+1)}{p-1}, \qquad \mu = (-1)^{(1/12)}\sqrt{2}, \qquad \begin{bmatrix} \frac{1}{2}\,(-1)^{(5/12)}\,\sqrt{2} & \frac{1}{2}\,(-1)^{(5/12)}\,\sqrt{2} \\ -\frac{1}{2}\,(-1)^{(11/12)}\,\sqrt{2} & \frac{1}{2}\,(-1)^{(11/12)}\,\sqrt{2} \end{bmatrix}$$

Appendix B

Invariants for Ternary Forms.

In this appendix we list joint relative invariants of the forms (3.43–3.44) of Section 3.3, obtained by omega process, that we subsequently used to obtain a fundamental set of differential invariants for ternary forms:

$$M_1 = \tfrac{1}{288}(H_3^2, P_2)^{(2)} = Q_{3,0}\, Q_{1,2}\, Q_{0,2} - Q_{2,1}{}^2\, Q_{0,2} - Q_{3,0}\, Q_{0,3}\, Q_{1,1} + Q_{2,1}\, Q_{1,2}\, Q_{1,1}$$
$$+ Q_{2,1}\, Q_{0,3}\, Q_{2,0} - Q_{1,2}{}^2\, Q_{2,0}$$

of weight 4.

$$M_2 = \frac{1}{103680}(P_3^2, P_2^3)^{(6)} =$$
$$5\, Q_{3,0}{}^2\, Q_{0,2}{}^3 - 30\, Q_{3,0}\, Q_{0,2}{}^2\, Q_{2,1}\, Q_{1,1} + 24\, Q_{3,0}\, Q_{1,1}{}^2\, Q_{0,2}\, Q_{1,2} + 36\, Q_{2,1}{}^2\, Q_{1,1}{}^2\, Q_{0,2}$$
$$+ 9\, Q_{2,1}{}^2\, Q_{0,2}{}^2\, Q_{2,0} - 4\, Q_{3,0}\, Q_{1,1}{}^3\, Q_{0,3} - 36\, Q_{2,1}\, Q_{1,1}{}^3\, Q_{1,2}$$
$$+ 36\, Q_{1,2}{}^2\, Q_{2,0}\, Q_{1,1}{}^2 + 9\, Q_{1,2}{}^2\, Q_{2,0}{}^2\, Q_{0,2} + 6\, Q_{3,0}\, Q_{0,2}{}^2\, Q_{1,2}\, Q_{2,0}$$
$$- 6\, Q_{3,0}\, Q_{2,0}\, Q_{0,2}\, Q_{0,3}\, Q_{1,1} - 54\, Q_{2,1}\, Q_{2,0}\, Q_{0,2}\, Q_{1,2}\, Q_{1,1}$$
$$+ 24\, Q_{2,1}\, Q_{2,0}\, Q_{1,1}{}^2\, Q_{0,3} + 6\, Q_{2,1}\, Q_{2,0}{}^2\, Q_{0,2}\, Q_{0,3} - 30\, Q_{1,2}\, Q_{2,0}{}^2\, Q_{0,3}\, Q_{1,1}$$
$$+ 5\, Q_{0,3}{}^2\, Q_{2,0}{}^3$$

of weight 6.

$$M_3 = \frac{1}{576}\, (P_4, P_2^2)^{(4)} = Q_{4,0}\, Q_{0,2}{}^2 + 4\, Q_{2,2}\, Q_{1,1}{}^2 - 4\, Q_{3,1}\, Q_{0,2}\, Q_{1,1}$$
$$+ 2\, Q_{2,2}\, Q_{0,2}\, Q_{2,0} - 4\, Q_{1,3}\, Q_{2,0}\, Q_{1,1} + Q_{0,4}\, Q_{2,0}{}^2$$

of weight 4.

$$M_4 = \frac{1}{1194393600} (T_4, P_2^3)^{(6)}$$
$$= 4\,Q_{3,1}{}^2 Q_{0,4} Q_{1,1}{}^3 - 4\,Q_{1,3}{}^2 Q_{4,0} Q_{1,1}{}^3 - Q_{4,0}{}^2 Q_{1,3} Q_{0,2}{}^3 + Q_{3,1} Q_{0,4}{}^2 Q_{2,0}{}^3$$
$$+ 2\,Q_{1,3}{}^3 Q_{2,0}{}^3 - 2\,Q_{3,1}{}^3 Q_{0,2}{}^3 + 9\,Q_{2,2}{}^2 Q_{0,4} Q_{2,0}{}^2 Q_{1,1} + 3\,Q_{4,0} Q_{2,2} Q_{3,1} Q_{0,2}{}^3$$
$$+ Q_{4,0}{}^2 Q_{0,4} Q_{0,2}{}^2 Q_{1,1} + 2\,Q_{3,1} Q_{1,3}{}^2 Q_{2,0}{}^2 Q_{0,2} - 9\,Q_{4,0} Q_{2,2}{}^2 Q_{0,2}{}^2 Q_{1,1}$$
$$+ 6\,Q_{3,1}{}^2 Q_{2,2} Q_{0,2}{}^2 Q_{1,1} - 2\,Q_{3,1}{}^2 Q_{1,3} Q_{0,2}{}^2 Q_{2,0} - 8\,Q_{3,1}{}^2 Q_{1,3} Q_{1,1}{}^2 Q_{0,2}$$
$$- 6\,Q_{2,2} Q_{1,3}{}^2 Q_{2,0}{}^2 Q_{1,1} - Q_{4,0} Q_{0,4}{}^2 Q_{2,0}{}^2 Q_{1,1} - 3\,Q_{2,2} Q_{1,3} Q_{0,4} Q_{2,0}{}^3$$
$$+ 8\,Q_{3,1} Q_{1,3}{}^2 Q_{2,0} Q_{1,1}{}^2 - 12\,Q_{3,1} Q_{0,4} Q_{2,2} Q_{2,0} Q_{1,1}{}^2 - 2\,Q_{3,1} Q_{1,3} Q_{0,4} Q_{2,0}{}^2 Q_{1,1}$$
$$+ 3\,Q_{4,0} Q_{2,2} Q_{1,3} Q_{0,2}{}^2 Q_{2,0} + 12\,Q_{4,0} Q_{2,2} Q_{1,3} Q_{1,1}{}^2 Q_{0,2} - 3\,Q_{3,1} Q_{0,4} Q_{2,2} Q_{2,0}{}^2 Q_{0,2}$$
$$+ 4\,Q_{4,0} Q_{0,4} Q_{1,3} Q_{2,0} Q_{1,1}{}^2 + 2\,Q_{4,0} Q_{1,3} Q_{3,1} Q_{0,2}{}^2 Q_{1,1} + Q_{4,0} Q_{0,4} Q_{1,3} Q_{2,0}{}^2 Q_{0,2}$$
$$- 4\,Q_{4,0} Q_{0,4} Q_{3,1} Q_{1,1}{}^2 Q_{0,2} - Q_{4,0} Q_{0,4} Q_{3,1} Q_{0,2}{}^2 Q_{2,0} - 6\,Q_{1,3}{}^2 Q_{4,0} Q_{2,0} Q_{0,2} Q_{1,1}$$
$$+ 6\,Q_{3,1}{}^2 Q_{0,4} Q_{2,0} Q_{0,2} Q_{1,1},$$

of weight 9.

$$M_5 = \frac{1}{238878720} (S, P_4)^{(4)} =$$
$$- 6\,Q_{1,3}{}^2 Q_{3,1} Q_{2,1}{}^2 + 6\,Q_{3,1}{}^2 Q_{1,2}{}^2 Q_{1,3} + Q_{1,3} Q_{4,0}{}^2 Q_{0,3}{}^2 - 6\,Q_{4,0} Q_{2,2} Q_{2,1} Q_{0,3} Q_{1,3}$$
$$+ 2\,Q_{0,4} Q_{4,0} Q_{0,3} Q_{2,1} Q_{3,1} - 2\,Q_{1,3} Q_{3,1} Q_{0,3} Q_{1,2} Q_{4,0} + 2\,Q_{3,1}{}^3 Q_{0,3}{}^2$$
$$- Q_{0,4} Q_{4,0}{}^2 Q_{0,3} Q_{1,2} - Q_{3,1}{}^2 Q_{3,0} Q_{0,3} Q_{0,4} + 4\,Q_{3,1}{}^2 Q_{2,1} Q_{0,3} Q_{1,3}$$
$$+ Q_{1,3}{}^2 Q_{4,0} Q_{0,3} Q_{3,0} - 3\,Q_{4,0} Q_{2,2} Q_{0,3}{}^2 Q_{3,1} + 9\,Q_{4,0} Q_{2,2}{}^2 Q_{1,2} Q_{0,3}$$
$$- 6\,Q_{3,1}{}^2 Q_{1,2} Q_{0,3} Q_{2,2} + 6\,Q_{1,3}{}^2 Q_{2,1} Q_{3,0} Q_{2,2} - 2\,Q_{1,3}{}^3 Q_{3,0}{}^2$$
$$+ 3\,Q_{0,4} Q_{4,0} Q_{1,2}{}^2 Q_{3,1} + 9\,Q_{1,3}{}^2 Q_{4,0} Q_{2,1} Q_{1,2} - 9\,Q_{3,1}{}^2 Q_{2,1} Q_{1,2} Q_{0,4}$$
$$+ 9\,Q_{2,2} Q_{3,1} Q_{2,1}{}^2 Q_{0,4} + 3\,Q_{1,3} Q_{2,2} Q_{3,0}{}^2 Q_{0,4} - 4\,Q_{1,3}{}^2 Q_{3,1} Q_{3,0} Q_{1,2}$$
$$- 9\,Q_{4,0} Q_{2,2} Q_{1,2}{}^2 Q_{1,3} - 3\,Q_{1,3} Q_{4,0} Q_{2,1}{}^2 Q_{0,4} - 9\,Q_{2,2}{}^2 Q_{3,0} Q_{2,1} Q_{0,4}$$
$$+ Q_{0,4}{}^2 Q_{4,0} Q_{3,0} Q_{2,1} - Q_{0,4}{}^2 Q_{3,1} Q_{3,0}{}^2 + 2\,Q_{1,3} Q_{3,1} Q_{3,0} Q_{2,1} Q_{0,4}$$
$$+ 6\,Q_{2,2} Q_{3,1} Q_{3,0} Q_{1,2} Q_{0,4} - 2\,Q_{1,3} Q_{4,0} Q_{3,0} Q_{1,2} Q_{0,4}$$

of weight 9.

We remind the reader that

$$H_3 = (P_4, P_4)^{(2)}, \quad H_4 = (P_4, P_4)^{(2)},$$
$$T_3 = (H_3, P_3)^{(1)}, \quad T_4 = (H_4, P_4)^{(1)},$$
$$S = (H_4, P_3^2)^{(3)}.$$

The following MAPLE code was used to compute eight fundamental invariants:

$$I_1 = \frac{M_1}{d_2{}^2}, \qquad I_2 = \frac{M_2}{d_2{}^3}, \qquad I_3 = \frac{d_3}{d_2{}^3},$$

$$I_4 = \frac{j}{d_2{}^3}, \quad I_5 = \frac{i}{d_2{}^2}, \quad I_6 = \frac{M_4{}^2}{d_2{}^9}, \quad I_7 = \frac{M_3}{d_2{}^2}, \quad I_8 = \frac{M_5{}^2}{d_2{}^9}.$$

restricted to a given polynomial.

Appendix C

Computations on Ternary Cubics.

The following standard packages are used:

```
>  with(linalg):with(Groebner):
```

Our code includes the following programs:

Pinv computes fundamental invariants.

```
>  read ternary3;
```

Psignature computes syzygies between fundamental invariants.

```
>  read Psignature;
```

Two-dimensional unimodular group of symmetries

```
>  F:=(x,y,z)->x*y*z;
```

$$F := (x,\, y,\, z) \to x\, y\, z$$

```
>  f:=(p,q)->p*q;
```

$$f := (p,\, q) \to p\, q$$

```
>  Pinv(f,3);
```

$$[\frac{4}{3},\, \frac{16}{3},\, \frac{16}{9}]$$

an equivalent polynomial:

```
>  f:=(p,q)->1/2*p*(q^2-1);
```

$$f := (p,\, q) \to \frac{1}{2}\, p\, (q^2 - 1)$$

```
>  Pinv(f,3);
```

$$[\frac{4}{3},\, \frac{16}{3},\, \frac{16}{9}]$$

Two-dimensional symmetry group with a one-dimensional unimodular subgroup.

```
>   F:=(x,y,z)->(x^2+y*z)*z:
>   f:=(p,q)->p^2+q;
```

$$f := (p,\ q) \rightarrow p^2 + q$$

```
>   Pinv(f,3);
```

$$[\frac{-1}{6},\ \frac{41}{6},\ \frac{-2}{9}]$$

```
an equivalent polynomial:
>   F:=(x,y,z)->(x^2-y*z-z^2)*z:   f:=(p,q)->p^2-q-1;
```

$$f := (p,\ q) \rightarrow p^2 - q - 1$$

```
>   Pinv(f,3);
```

$$[\frac{-1}{6},\ \frac{41}{6},\ \frac{-2}{9}]$$

<u>One-dimensional unimodular group of symmetries</u>

```
>   F:=(x,y,z)->(x^2+y^2+z^2)*z;
```

$$F := (x,\ y,\ z) \rightarrow (x^2 + y^2 + z^2)\, z$$

```
>   f:=(p,q)->(p^2+q^2+1);
```

$$f := (p,\ q) \rightarrow p^2 + q^2 + 1$$

```
>   Pinv(f,3);
```

$$[\frac{4}{3}\frac{(p^2+3+q^2)\,(p^2+q^2)}{(p^2-3+q^2)^2},\ \frac{16}{3}\frac{(p^2+q^2)\,(p^4+2\,p^2\,q^2+81+q^4)}{(p^2-3+q^2)^3},$$
$$\frac{16}{9}\frac{(9+p^2+q^2)\,(p^2+q^2)^2}{(p^2-3+q^2)^3}]$$

```
>   Psignature(inv);
```

elimination of u from the equations :
$$4\,(p^2+3+q^2)\,(p^2+q^2) - 3\,(p^2-3+q^2)^2\,I_1,$$
$$16\,(p^2+q^2)\,(p^4+2\,p^2\,q^2+81+q^4) - 3\,(p^2-3+q^2)^3\,I_2,$$
$$16\,(9+p^2+q^2)\,(p^2+q^2)^2 - 9\,(p^2-3+q^2)^3\,I_3,$$
$$1 - 243\,(p^2-3+q^2)^3\,w$$

dimension of the signature manifold $= 1$

the signature manifold is defined by :

$$[-1482\,I_3\,I_2 + 8865\,I_3{}^2 + 40\,I_2{}^2 - 1296\,I_1 - 36\,I_2 + 17280\,I_3 - 18522\,I_3\,I_1,$$
$$40\,I_1\,I_2 + 582\,I_3\,I_1 + 42\,I_3\,I_2 - 315\,I_3{}^2 - 144\,I_1 - 4\,I_2 - 480\,I_3,$$
$$360\,I_1{}^2 - 378\,I_3\,I_1 - 18\,I_3\,I_2 + 135\,I_3{}^2 - 144\,I_1 - 4\,I_2 + 120\,I_3]$$

an equivalent polynomial:

```
>   F:=(x,y,z)->x*y*z+x^2*y+z*y^2;
>   f:=(p,q)->p*q+p^2*q+q^2;
```

$$F := (x,\ y,\ z) \to x\,y\,z + x^2\,y + z\,y^2$$

$$f := (p,\ q) \to p\,q + p^2\,q + q^2$$

```
>   Pinv(f,3);
```

$$[\frac{4}{3}\,\frac{(p+1-q)\,(p+q)\,(p^2+p+2\,q^2+q)}{(-4\,q^2+q+p+p^2)^2},$$
$$\frac{16}{3}\,\frac{(p+1-q)\,(p+q)\,(p^4+2\,p^3+2\,p^2\,q-2\,p^2\,q^2+p^2+2\,p\,q-2\,p\,q^2-2\,q^3+82\,q^4+q^2)}{(-4\,q^2+q+p+p^2)^3},$$
$$\frac{16}{9}\,\frac{(p+p^2+q+8\,q^2)\,(p+q)^2\,(p+1-q)^2}{(-4\,q^2+q+p+p^2)^3}]$$

```
>   Psignature(inv);
```

elimination of u from the equations :

$$4\,(p+1-q)\,(p+q)\,(p^2+p+2\,q^2+q) - 3\,\%1^2\,I_1,$$
$$16\,(p+1-q)\,(p+q)\,(p^4+2\,p^3+2\,p^2\,q-2\,p^2\,q^2+p^2+2\,p\,q-2\,p\,q^2-2\,q^3$$
$$+82\,q^4+q^2) - 3\,\%1^3\,I_2,$$
$$16\,(p+p^2+q+8\,q^2)\,(p+q)^2\,(p+1-q)^2 - 9\,\%1^3\,I_3,\ 1 - 243\,\%1^3\,w$$
$$\%1 := -4\,q^2 + q + p + p^2$$

dimension of the signature manifold = 1

the signature manifold is defined by :

$$[-1482\,I_3\,I_2 + 8865\,I_3{}^2 + 40\,I_2{}^2 - 1296\,I_1 - 36\,I_2 + 17280\,I_3 - 18522\,I_3\,I_1,$$
$$40\,I_1\,I_2 + 582\,I_3\,I_1 + 42\,I_3\,I_2 - 315\,I_3{}^2 - 144\,I_1 - 4\,I_2 - 480\,I_3,$$
$$360\,I_1{}^2 - 378\,I_3\,I_1 - 18\,I_3\,I_2 + 135\,I_3{}^2 - 144\,I_1 - 4\,I_2 + 120\,I_3]$$

IRREDUCIBLE CUBICS IN THREE VARIABLES

The program <u>SymN</u> computes the cardinality of the symmetry group in the case it is finite.

```
>   read symN;
```

it uses <u>kbasis5</u> adopted from [9] to compute the number of the solutions for a system of polynomial equations:

```
>   read kbasis5;
```

Singular Curves:

q^2=p^3: one-dimensional symmetry group, finite number of unimodular symmetries:

```
>   f:=(p,q)->p^3-q^2;
```

$$f := (p, q) \rightarrow p^3 - q^2$$

```
>   Pinv(f,3);
```

$$[\frac{-1}{6}, \frac{1}{6} \frac{36 p^3 + 5 q^2}{p^3}, -\frac{1}{9} \frac{p^3 + q^2}{p^3}]$$

```
>   Psignature(inv);
```

$$elimination \ of \ u \ from \ the \ equations :$$
$$-1 - 6 I_1, \ 36 p^3 + 5 q^2 - 6 I_2 p^3, \ -p^3 - q^2 - 9 I_3 p^3, \ 1 - 18 w p^3$$

$$dimension \ of \ the \ signature \ manifold = 1$$

$$the \ signature \ manifold \ is \ defined \ by :$$

$$[6 I_2 + 45 I_3 - 31, \ 1 + 6 I_1]$$

q^2=p^2*(p+1): The projective symmetry group has **6** elements.

```
>   f:=(p,q)->p^2*(p+1)-q^2;
```

$$f := (p, q) \rightarrow p^2 (p + 1) - q^2$$

```
>  Pinv(f,3);
```

$$[-\frac{1}{6}\frac{(3\,p+4)\,(q+p)\,(q-p)\,(3\,p\,q^2-2\,q^2+2\,p^2+3\,p^3)}{(3\,p\,q^2+q^2-p^2)^2},$$

$$\frac{1}{6}((q+p)\,(q-p)(135\,q^6+32\,q^4+972\,p^3\,q^4+1107\,p^2\,q^4+72\,p\,q^4+1269\,p^4\,q^2$$
$$-144\,p^3\,q^2-64\,p^2\,q^2+972\,q^2\,p^5+72\,p^5+81\,p^6+32\,p^4))/(3\,p\,q^2+q^2-p^2)^3,$$

$$-\frac{1}{9}($$
$$(-16\,q^2+27\,q^4+72\,p\,q^2+16\,p^2+81\,p^2\,q^2+72\,p^3+27\,p^3\,q^2+108\,p^4+54\,p^5)$$
$$(q-p)^2\,(q+p)^2)\Big/(3\,p\,q^2+q^2-p^2)^3]$$

```
>  symN(inv,1,2);
```

$$\text{the number of symmetries} = 6$$

In `symN` we have chosen the point $P = 1$ and $Q = 2$ to substitute into equations (3.53). We find the number of the solutions using procedure `kbasis5`, but in this particular case it is not difficult to solve the equations explicitly:

```
>  E;
```

$$[76\,p^2-49\,q^2+72\,p+48,\ 7543\,p\,q^2-6633\,q^2-4008\,p+368,$$
$$20961997\,q^4-83898780\,q^2-13605280\,p+13808448]$$

```
>  map(allvalues,[solve({op(E)},{p,q})]);
```

$$[\{p=1,\,q=2\},\,\{q=-2,\,p=1\},$$
$$\{p=-\frac{212}{397}+\frac{112}{397}\,I\,\sqrt{3},\,q=-\frac{208}{397}+\frac{20}{397}\,I\,\sqrt{3}\},$$
$$\{p=-\frac{212}{397}-\frac{112}{397}\,I\,\sqrt{3},\,q=-\frac{208}{397}-\frac{20}{397}\,I\,\sqrt{3}\},$$
$$\{p=-\frac{212}{397}-\frac{112}{397}\,I\,\sqrt{3},\,q=\frac{208}{397}+\frac{20}{397}\,I\,\sqrt{3}\},$$
$$\{p=-\frac{212}{397}+\frac{112}{397}\,I\,\sqrt{3},\,q=\frac{208}{397}-\frac{20}{397}\,I\,\sqrt{3}\}]$$

The list above contains the images of the point (1,2) under all possible symmetries. We note, however, that if we put a non-generic point into `symN` we might obtain an incorrect answer or no answer at all:

```
>  symN(inv,1,1);
```

```
Error, (in kbasis) Ideal is not zero-dimensional, no finite basis
```

Other generic points produce the correct result for the cardinality of the symmetry group.
```
>   symN(inv,0,1);
```

$$\text{the number of symmetries} = 6$$
```
>   symN(inv,1,0);
```

$$\text{the number of symmetries} = 6$$

An equivalent polynomial:
```
>   f:=(p,q)->p^2*(p+4)-q^2;
```

$$f := (p,\ q) \rightarrow p^2\,(p+4) - q^2$$
```
>   Pinv(f,3):
>   symN(inv,1,0);
```

$$\text{the number of symmetries} = 6$$

Non singular (elliptic) curves:

q^2=p^3+1. The number of the projective symmetries is **54.**
```
>   f:=(p,q)->p^3-q^2+1;
```

$$f := (p,\ q) \rightarrow p^3 - q^2 + 1$$
```
>   Pinv(f,3);
```

$$[\frac{-1}{6}, \frac{1}{6}(-135 + 432\,p^3 + 540\,p^6\,q^2 + 90\,q^4 - 540\,p^6 + 40\,q^6 + 5\,q^8 + 2052\,p^3\,q^2$$
$$-216\,p^3\,q^4 + 36\,p^3\,q^6)/(p^3\,(q^2+3)^3),$$
$$-\frac{1}{9}\frac{(q^2 - 1 + p^3)\,(q^6 - 18\,q^4 + 81\,q^2 + 27\,p^6)}{p^3\,(q^2+3)^3}]$$
```
>   symN(inv,1,0);
```

$$\text{the number of symmetries} = 54$$
```
>   symN(inv,1,2);
```

$$\text{the number of symmetries} = 54$$
```
>   symN(inv,2,2);
```

$$\text{the number of symmetries} = 54$$

An equivalent polynomial:
```
>   f:=(p,q)->p^3+q^3+1;
```

$$f := (p,\ q) \rightarrow p^3 + q^3 + 1$$

```
>  Pinv(f,3);
```

$$[\frac{-1}{6}, -\frac{1}{6}\frac{5\,q^6 + 5\,p^3\,q^6 + 5\,q^3 - 26\,p^3\,q^3 + 5\,p^6\,q^3 + 5\,p^3 + 5\,p^6}{p^3\,q^3},$$

$$-\frac{1}{36}\frac{(q^3 + 1 - p^3)\,(q^3 - 1 - p^3)\,(q^3 - 1 + p^3)}{p^3\,q^3}]$$

```
>  symN(inv,1,2);
```

$$\textit{the number of symmetries} = 54$$

```
>  symN(inv,1,0);
```

```
Error, (in symN) division by zero
>  symN(inv,1,1);
```

$$\textit{the number of symmetries} = 54$$

The signature manifold for this class of polynomials is defined by:

```
>  Psignature(inv);
```

$$\textit{elimination of u from the equations}:$$
$$-1 - 6\,I_1,$$
$$-5\,p^6\,q^3 - 5\,p^6 - 5\,p^3\,q^6 - 5\,p^3 + 26\,p^3\,q^3 - 5\,q^6 - 5\,q^3 - 6\,p^3\,q^3\,I_2,$$
$$-(p^3 - 1 - q^3)\,(p^3 - 1 + q^3)\,(p^3 + 1 - q^3) - 36\,p^3\,q^3\,I_3,$$
$$1 - 36\,w\,p^3\,q^3$$

$$\textit{dimension of the signature manifold} = 2$$

$$\textit{the signature manifold is defined by}:$$

$$[1 + 6\,I_1]$$

q^2=p^3+ap. They are equivalent for all **a**. The number of symmetries is **36**.

```
>  f:=(p,q)->p^3+p-q^2;
```

$$f := (p,\ q) \rightarrow p^3 + p - q^2$$

```
>  Pinv(f,3);
```

$$[-\frac{1}{6}\frac{27\,q^4 + 9\,p^2\,q^4 + 1 + 21\,p^2 + 63\,p^4 + 27\,p^6 - 60\,p\,q^2 - 36\,p^3\,q^2}{(3\,p\,q^2 - 1 + 3\,p^2)^2}, -\frac{1}{6}(729\,p^8$$

$$- 2916\,p^7\,q^2 - 972\,p^6 - 2916\,p^5\,q^2 + 486\,p^4\,q^4 + 1350\,p^4 + 7236\,p^3\,q^2 - 972\,p^3\,q^6$$
$$- 2052\,p^2\,q^4 + 468\,p^2 - 828\,p\,q^2 - 756\,q^6\,p + 41 - 135\,q^8 - 378\,q^4)/$$
$$(3\,p\,q^2 - 1 + 3\,p^2)^3, \frac{1}{9}(2 + 36\,p^2 + 216\,p^4 + 540\,p^6 + 486\,p^8 - 27\,q^8 - 189\,q^4$$
$$- 918\,p^2\,q^4 - 81\,p^4\,q^4 + 189\,q^6\,p - 27\,p^3\,q^6 + 117\,p\,q^2 + 999\,p^3\,q^2 - 81\,p^5\,q^2$$
$$- 243\,p^7\,q^2)/(3\,p\,q^2 - 1 + 3\,p^2)^3]$$

```
>  symN(inv,1,2);
```

$$\text{the number of symmetries} = 36$$

```
>  symN(inv,2,3);
```

$$\text{the number of symmetries} = 36$$

```
>  f:=(p,q)->-q^2+2*p+p^3;
```

$$f := (p,\ q) \rightarrow -q^2 + 2\,p + p^3$$

```
>  Pinv(f,3):
>  symN(inv,1,2);
```

$$\text{the number of symmetries} = 36$$

q^2=p^3+ap+1. This is a family of equivalence classes. The number of symmetries is the same: **18.**

```
>   f:=(x,y,z)->-q^2+4+2*p+p^3;
```

$$f := (x,\ y,\ z) \rightarrow -q^2 + 4 + 2\,p + p^3$$

```
>  Pinv(f,3);
```

$$[-\frac{1}{6}(-432\,q^2 - 240\,p\,q^2 + 216\,p^2\,q^2 - 72\,p^3\,q^2 + 880 + 576\,p + 1464\,p^2 + 864\,p^3 +$$

$$252\,p^4 + 54\,p^6 + 54\,q^4 + 9\,p^2\,q^4)/(3\,p\,q^2 - 4 + 36\,p + 6\,p^2)^2, -\frac{1}{6}(225504\,p^6 - 972\,p^3\,q^6$$

$$+774144\,p + 46656\,p^7 + 2916\,p^8 - 135\,q^8 - 670464\,p^3 + 142272\,p\,q^2$$
$$+ 295488\,p^2\,q^2 - 10368\,q^2 + 512640\,p^2 - 58320\,p^6\,q^2 - 5832\,p^7\,q^2 + 972\,p^4\,q^4$$
$$- 8208\,p^2\,q^4 - 41904\,q^4 - 318816\,p^4\,q^2 - 4320\,q^6 - 211680\,p^4 + 1025600$$
$$- 828576\,p^3\,q^2 - 10368\,p\,q^4 + 23328\,p^3\,q^4 - 11664\,p^5\,q^2 - 1512\,q^6\,p$$
$$- 124416\,p^5)/(3\,p\,q^2 - 4 + 36\,p + 6\,p^2)^3, -\frac{1}{9}(-15984\,p^6 + 27\,p^3\,q^6 - 1728\,p$$
$$+ 2916\,p^9 - 1944\,p^8 + 27\,q^8 - 6912\,p^3 - 118512\,p\,q^2 - 45360\,p^2\,q^2 - 145152\,q^2$$
$$- 16704\,p^2 + 2916\,p^6\,q^2 + 486\,p^7\,q^2 + 162\,p^4\,q^4 + 3672\,p^2\,q^4 + 44280\,q^4$$
$$+ 15552\,p^4\,q^2 - 2052\,q^6 - 26784\,p^4 + 27000\,p^3\,q^2 + 16848\,p\,q^4 - 1944\,p^3\,q^4$$
$$+ 324\,p^5\,q^2 - 378\,q^6\,p - 7776\,p^5 - 3584)/(3\,p\,q^2 - 4 + 36\,p + 6\,p^2)^3]$$

```
>   symN(inv,1,1);
```

$$\text{the number of symmetries} = 18$$

```
>  f:=(p,q)->-q^2+1+3*p+p^3;
```

$$f := (p,\ q) \rightarrow -q^2 + 1 + 3\,p + p^3$$

```
>  Pinv(f,3);
```

$$[-\frac{1}{6}(18 + 36\,p + 72\,p^2 + 36\,p^3 + 63\,p^4 + 9\,p^6 - 18\,q^2 - 60\,p\,q^2 + 6\,p^2\,q^2 - 12\,p^3\,q^2$$
$$+9\,q^4 + p^2\,q^4)/(p\,q^2 - 3 + 3\,p + 3\,p^2)^2, -\frac{1}{6}(-432\,p^6 - 36\,p^3\,q^6 + 1944\,p + 648\,p^7$$
$$+ 243\,p^8 - 5\,q^8 + 1944 + 1944\,p^3 - 1944\,p\,q^2 + 6156\,p^2\,q^2 - 324\,q^2 + 6804\,p^2$$
$$- 540\,p^6\,q^2 - 324\,p^7\,q^2 + 54\,p^4\,q^4 - 684\,p^2\,q^4 - 468\,q^4 - 4428\,p^4\,q^2 - 40\,q^6$$
$$+ 3240\,p^4 + 5184\,p^3\,q^2 - 144\,p\,q^4 + 216\,p^3\,q^4 - 972\,p^5\,q^2 - 84\,q^6\,p - 2592\,p^5)/$$
$$(p\,q^2 - 3 + 3\,p + 3\,p^2)^3, -\frac{1}{9}(-81 - 567\,p^6 + p^3\,q^6 - 81\,p + 27\,p^9 - 162\,p^8 + q^8$$
$$- 216\,p^3 - 756\,p\,q^2 - 945\,p^2\,q^2 - 243\,q^2 - 405\,p^2 + 27\,p^6\,q^2 + 27\,p^7\,q^2 + 9\,p^4\,q^4$$
$$+ 306\,p^2\,q^4 + 288\,q^4 + 216\,p^4\,q^2 - 19\,q^6 - 729\,p^4 - 918\,p^3\,q^2 + 234\,p\,q^4$$
$$- 18\,p^3\,q^4 + 27\,p^5\,q^2 - 21\,q^6\,p - 162\,p^5)/(p\,q^2 - 3 + 3\,p + 3\,p^2)^3]$$

```
>  symN(inv,1,0);
```

the number of symmetries $= 18$

```
>  f:=(p,q)->-q^2+1-p+p^3;
```

$$f := (p,\,q) \rightarrow -q^2 + 1 - p + p^3$$

```
>  Pinv(f,3):
>  symN(inv,1,0);
```

the number of symmetries $= 18$

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