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EQUIVALENCE AND SYMMETRIES OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article we investigate the equivalence of underdetermined differential equations and differential equations with deviations of second order with respect to the pseudogroup of transformations $\bar{x} = \varphi(x)$, $\bar{y} = \bar{y}(\bar{x}) = L(x) + y(x)$, $\bar{z} = \bar{z}(\bar{x}) = M(x) + z(x)$. Our main aim is to determine such equations that admit a large pseudogroup of symmetries. Instead the common direct calculations, we use some more advanced tools from differential geometry, however, our exposition is self-contained and only the most fundamental properties of differential forms are employed.

1. Introduction

The transformation properties of differential equations (including the equivalences and symmetries) were studied for a long time by using various methods: from the common direct calculations occasionally employing some simple geometrical concepts, through the Lie group method with the help of infinitesimal transformations, up to the Cartan's moving frames with the current G-structures modifications. Then the results are expressed in terms of differential invariants together with the compatibility conditions for the equivalence and symmetry transformations. On the contrary, analogous transformation properties for the difference equations seem to be of quite other nature. Only the direct approach is appropriate in this case and the results are as a rule expressed in terms of rather clumsy functional equations for the sought equivalences and symmetries.

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We intend to deal with the intermediate problem, namely with the transformation properties of differential equations with deviating argument, and our aim is to employ the mechanism of moving frames in its original (rather effective and quite simple) setting. We believe that this is possible with the help of appropriate underdetermined differential equations. For better clarity, we deal with the second-order delay-differential equation

$$y''(x) = f\left(x, y(x), y(\xi(x)), y'(x), (y(\xi(x)))', (y(\xi(x)))''\right) \tag{1}$$

with a given delay function $\xi(x)$. The right-hand side can be represented as

$$F(x, y, y(\xi), y', y'(\xi), y''(\xi)) = f(x, y, y(\xi), y'(\xi)\xi', y''(\xi)\xi'^2 + y'(\xi)\xi'') \tag{2}$$

assuming the existence of ξ''. Denoting $z(x) = y(\xi(x))$ it follows that the original equation (1) is equivalent to the requirements

$$y'' = f(x, y, z, y', z', z''), \qquad z(x) = y(\xi(x)) \tag{3}$$

consisting of an underdetermined differential equation (3_1) together with a simple functional equation (3_2). We wish to apply the moving frames to (3_1) separately and then adapt the results by using (3_2) with the belief that the results can be interpreted in terms of the original equation (1).

The most general pseudogroup of transformations that make a good sense in this connection consists of all invertible substitutions

$$\bar{x} = \varphi(x), \qquad \bar{y} = \psi(x, y), \qquad \bar{z} = \chi(x, z), \tag{4}$$

where the definition domains are certain open subsets $\mathscr{D} \subset \mathbb{R}^3$ of the space with coordinates x, y, z. The curve $y = y(x)$, $z = z(x)$ is transformed into the curve $\bar{y} = \bar{y}(\bar{x}), \bar{z} = \bar{z}(\bar{x})$, where

$$\bar{y}(\bar{x}) = \psi(x, y(x)), \qquad \bar{z}(\bar{x}) = \chi(x, z(x)) \tag{5}$$

and the original delay function $\xi(x)$ is transformed into the new delay $\bar{\xi}(\bar{x})$ satisfying

$$\bar{\xi}(\varphi(x)) = \bar{\xi}(\bar{x}) = \varphi(\xi(x)) \tag{6}$$

(whenever this equation is defined) in particular $\xi(\varphi(x)) = \varphi(\xi(x))$ if we suppose that the delay is preserved. In order to ensure (3_2) after the transformation, we have the requirement $\bar{z}(\bar{x}) = \bar{y}(\bar{\xi}(\bar{x}))$, that is,

$$\chi(x, z(x)) = \psi(\xi(x), y(\xi(x))) = \psi(\xi(x), z(x))$$

by using (6) and (4). Altogether, we obtain the additional functional equation

$$\chi(x, z) = \psi(\xi(x), z) \tag{7}$$

ensuring the existence of the transformed delay $\bar{\xi}(\bar{x})$ satisfying (6). The condition (7) simplifies if the transformations (4) are of a certain special kind. For instance, if

$$\psi = A(x)B(y) + C(x), \qquad \chi = D(x)E(z) + F(x),$$

then we have the functional equation $D(x)E(z)+F(x) = A(\xi(x))B(z)+C(\xi(x))$ which implies $D(x) = A(\xi(x))$, $F(x) = C(\xi(x))$, $E(z) = B(z)$.

In full generality, our intentions are rather ambitious. So we start with a relatively modest setting of the problem: the second-order equations (1) or (3) subject to the seemingly simple pseudogroup of invertible transformations $\bar{x} = \varphi(x)$, $\bar{y} = y + L(x)$, $\bar{z} = z + M(x)$ or, by using the logarithmic y and z scales, to the isomorphic pseudogroup $\bar{x} = \varphi(x)$, $\bar{y} = L(x)y$, $\bar{z} = M(x)z$ where $L(x)M(x) \neq 0$. Let us pass to more detail.

2. Setting the problem

2.1. Pseudogroup

We introduce the pseudogroup of all transformations

$$\Phi(x, y, z) = (\bar{x}, \bar{y}, \bar{z}) = (\varphi(x), y + L(x), z + M(x)), \qquad \Phi\colon \mathscr{D}(\Phi) \to \mathscr{R}(\Phi) \quad (8)$$

defined on open subsets $\mathscr{D}(\Phi) \subset \mathbb{R}^3$ of the space (with coordinates x, y, z) depending on the transformation Φ under consideration. (Let us recall that a pseudogroup is a family of diffeomorphisms Φ that is closed with respect to the composition, the inversion, and the gluing of the definition domains. Definition domains $\mathscr{D}(\Phi)$ and the ranges $\mathscr{R}(\Phi) = \Phi(\mathscr{D}(\Phi))$ are open subsets on a manifold $\mathbf{M}$ and $\mathscr{D}(\Phi) = \mathscr{R}(\Phi^{-1})$, $\mathscr{D}(\Psi \circ \Phi) = \Phi^{-1}(\mathscr{R}(\Phi) \cap \mathscr{D}(\Psi))$, $\mathscr{R}(\Psi \circ \Phi) = \Psi(\mathscr{R}(\Phi) \cap \mathscr{D}(\Psi))$.) Transformations Φ of our pseudogroup are characterized by the system of partial differential equations (the Lie's approach): denoting $\bar{x} = \bar{x}(x, y, z)$, $\bar{y} = \bar{y}(x, y, z)$, $\bar{z} = \bar{z}(x, y, z)$, clearly

$$\frac{\partial \bar{x}}{\partial y} = \frac{\partial \bar{x}}{\partial z} = \frac{\partial \bar{y}}{\partial z} = \frac{\partial \bar{z}}{\partial y} = 0, \qquad \frac{\partial \bar{y}}{\partial y} = \frac{\partial \bar{z}}{\partial z} = 1$$

for the transformations (8) and conversely, these partial differential equations characterize just the transformations (8).

Alternatively, the pseudogroup of all transformations Φ can be characterized by the invariance of appropriate differential forms (E. Cartan's approach), namely of the forms

$$\omega_0 = A\mathrm{d}x, \qquad \eta_0 = \mathrm{d}y - A_1\mathrm{d}x, \qquad \zeta_0 = \mathrm{d}z - A_2\mathrm{d}x \quad (9)$$

where $A \neq 0$, A_1, A_2 are additional variables. In more precise terms, we moreover introduce the counterparts denoted by bars

$$\bar{\omega}_0 = \bar{A}\mathrm{d}\bar{x}, \qquad \bar{\eta}_0 = \mathrm{d}\bar{y} - \bar{A}_1\mathrm{d}\bar{x}, \qquad \bar{\zeta}_0 = \mathrm{d}\bar{z} - \bar{A}_2\mathrm{d}\bar{x}$$

and the invariance requirements $\omega_0 = \bar{\omega}_0$, $\eta_0 = \bar{\eta}_0$, $\xi_0 = \bar{\xi}_0$. Then the requirement $\omega_0 = A\mathrm{d}x = \bar{\omega}_0 = \bar{A}\mathrm{d}\bar{x}$ together with the additional assumption $A, \bar{A} \neq 0$ ensures that $\bar{x} = \varphi(x)$ is a certain invertible function of x and we moreover obtain the transformation rule $A = \bar{A}\varphi'$ for the new variables A, $\bar{A}$. Analogously $\eta_0 = \mathrm{d}y - A_1\mathrm{d}x = \bar{\eta}_0 = \mathrm{d}\bar{y} - \bar{A}_1\mathrm{d}\bar{x}$ reads $\mathrm{d}(\bar{y}-y) = (\bar{A}_1\varphi' - A_1)\mathrm{d}x$, hence $\bar{y}-y = L(x)$ is a function of x and then the transformation rule $\bar{A}_1\varphi' - A_1 = L'(x)$ for the new variables A_1, $\bar{A}_1$ ensures the desired invariance. In a similar way, $\zeta_0 = \bar{\zeta}_0$ ensures that $\bar{z} - z = M(x)$ is a function of x.

2.2. Differential equations

The pseudogroup of transformations Φ will be applied to underdetermined differential equations of the second order. We consider the equivalence problem when the given equation

$$y'' = f(x, y, z, y', z', z'') \qquad \Big(' = \frac{\mathrm{d}}{\mathrm{d}x}\Big)$$

is transformed into another equation

$$\bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{y}', \bar{z}', \bar{z}'') \qquad \Big(' = \frac{\mathrm{d}}{\mathrm{d}\bar{x}}\Big)$$

by means of the transformations (8). In the direct approach, the prolongation

$$\bar{y}'\varphi' = y' + L', \qquad \bar{y}''\varphi'^2 = y'' - \frac{\varphi''}{\varphi'}(y' + L') + L'',$$

$$\bar{z}'\varphi' = z' + M', \qquad \bar{z}''\varphi'^2 = z'' - \frac{\varphi''}{\varphi'}(z' + M') + M'',$$

of the pseudogroup Φ is needed. Recall for clarity that the prolongation is induced by transformations

$$\bar{y}(\varphi(x)) = y(x) + L(x) \qquad \text{and} \qquad \bar{z}(\varphi(x)) = z(x) + M(x)$$

of the above-mentioned curves $y = y(x)$, $z = z(x)$ and $\bar{y} = \bar{y}(\bar{x})$, $\bar{z} = \bar{z}(\bar{x})$, respectively, i.e.,

$$\bar{y}'(\varphi)\varphi'(x) = y'(x) + L'(x), \qquad \bar{y}''(\varphi)\varphi'^2(x) + \bar{y}'(\varphi)\varphi''(x) = y''(x) + L''(x)$$

and

$$\bar{z}'(\varphi)\varphi'(x) = z'(x) + M'(x), \qquad \bar{z}''(\varphi)\varphi'^2(x) + \bar{z}'(\varphi)\varphi''(x) = z''(x) + M''(x).$$

As a result, we obtain the explicit formula for the transformed equation,

$$\bar{f}\varphi'^2 = f - \frac{\varphi''}{\varphi'}(y' + L') + L''.$$

One can observe that direct analysis of this result is rather an unpleasant task.

Instead of this (rather clumsy) direct approach, we will apply the moving frames exactly corresponding to the above-mentioned Cartan's approach to the pseudogroup theory. The equation $y'' = f$ will be represented by the Pfaffian system

$$\mathrm{d}y - y'\mathrm{d}x = \mathrm{d}y' - y''\mathrm{d}x = \mathrm{d}z - z'\mathrm{d}x = \mathrm{d}z' - z''\mathrm{d}x = 0 \qquad (y'' = f) \qquad (10)$$

and the equation $\bar{y}'' = \bar{f}$ by the analogous system

$$\mathrm{d}\bar{y} - \bar{y}'\mathrm{d}\bar{x} = \mathrm{d}\bar{y}' - \bar{y}''\mathrm{d}\bar{x} = \mathrm{d}\bar{z} - \bar{z}'\mathrm{d}\bar{x} = \mathrm{d}\bar{z}' - \bar{z}''\mathrm{d}\bar{x} \qquad (\,\bar{y}'' = \bar{f}\,).$$

We denote $\mu_r = \mathrm{d}y^{(r)} - y^{(r+1)}\mathrm{d}x$, $\nu_r = \mathrm{d}z^{(r)} - z^{(r+1)}\mathrm{d}x$ and analogously for the forms $\bar{\mu}_r$, $\bar{\nu}_r$ with bars.

Then the equations $y'' = f$, $\bar{y}'' = \bar{f}$ are equivalent if and only if the forms $\bar{\mu}_0$, $\bar{\nu}_0$, $\bar{\mu}_1$, $\bar{\nu}_1$ are (invertible) linear combinations of the forms μ_0, ν_0, μ_1, ν_1 by virtue of (appropriate) pseudogroup transformation (8). *Moreover, this transformation can be characterized by the above invariance requirements.*

Alternatively saying, *the equivalence problem is briefly expressed by the invariance property*

$$\{\mu_0, \nu_0, \mu_1, \nu_1\} = \{\bar{\mu}_0, \bar{\nu}_0, \bar{\mu}_1, \bar{\nu}_1\}, \qquad \omega_0 = \bar{\omega}_0, \quad \eta_0 = \bar{\eta}_0, \quad \xi_0 = \bar{\xi}_0$$

of the module generated by the forms μ_0, ν_0, μ_1, ν_1 and of the forms ω_0, η_0, ξ_0. Employing moreover formulae (9) and (10), many other invariant objects (in particular *invariant functions*) will be determined.

3. The reduction procedure

The submodule $\{\mu_0, \nu_0\}$ is preserved. Then $\eta_0 \in \{\mu_0, \nu_0\}$ is a linear combination of μ_0, ν_0 if and only if $\eta_0 = \mathrm{d}y - y'\mathrm{d}x$ $(= \mu_0)$ and similarly $\zeta_0 = \nu_0$. The forms η_0, ζ_0 together with $\mathrm{d}\eta_0$, $\mathrm{d}\zeta_0$ are preserved separately. The form ω_0 is preserved and

$$\mathrm{d}\eta_0 = \mathrm{d}x \wedge \mathrm{d}y' = \omega_0 \wedge \frac{1}{A}(\mu_1 + C\omega_0)$$

hence the family $\frac{1}{A}(\mu_1 + C\omega_0)$ is preserved, too. The condition

$$\frac{1}{A}(\mu_1 + C\omega_0) \in \{\eta_0, \zeta_0, \mu_1, \nu_1\}$$

determines the common form $\eta_1 = \frac{1}{A}\mu_1$ and $\eta_1 = \frac{1}{A}(\mathrm{d}y' - f\mathrm{d}x)$ is preserved. Moreover $\zeta_1 = \frac{1}{A}\nu_1 = \frac{1}{A}(\mathrm{d}z' - z''\mathrm{d}x)$ is determined and preserved in analogous way. To continue this process we need

$$\mathrm{d}\omega_0 = \mathrm{d}A \wedge \mathrm{d}x = \left(\frac{\mathrm{d}A}{A} + B\omega_0\right) \wedge \omega_0$$

with $\omega_0 = \bar{\omega}_0$, $\mathrm{d}\omega_0 = \mathrm{d}\bar{\omega}_0$. Hence the family $\omega_1 = \frac{\mathrm{d}A}{A} + B\omega_0$ is also preserved with a new variable B and still unknown variable A. Then

$$\mathrm{d}\zeta_1 = \zeta_1 \wedge \omega_1 + \omega_0 \wedge \left(B\zeta_1 + \frac{1}{A^2}\nu_2 + C\omega_0\right)$$

and the family $B\zeta_1 + \frac{1}{A^2}\nu_2 + C\omega_0$ is preserved since $\zeta_1 \wedge \omega_1$ is already preserved. The condition

$$B\zeta_1 + \frac{1}{A^2}\nu_2 + C\omega_0 \in \{\mu_0, \nu_0, \mu_1, \nu_1, \mu_2, \nu_2\} = \{\eta_0, \zeta_0, \eta_1, \zeta_1, \mu_2, \nu_2\}$$

determines the common preserved form $\zeta_2 = B\zeta_1 + \frac{1}{A^2}\nu_2$. Then

$$\mathrm{d}\eta_1 = \eta_1 \wedge \omega_1 + \omega_0 \wedge \left(\frac{f_y}{A^2}\eta_0 + \frac{f_z}{A^2}\zeta_0 + \left(B + \frac{f_{y'}}{A}\right)\eta_1 + \left(\frac{f_{z'}}{A} - f_{z''}B\right)\zeta_1 + f_{z''}\zeta_2\right),$$

all summands $\omega_0 \wedge \eta_0, \dots, \omega_0 \wedge \zeta_2$, $\eta_1 \wedge \omega_1$ are separately preserved and unique, therefore all the coefficients are separately preserved and we choose $B = -f_{y'}/A$ to simplify this differential. We get

$$\mathrm{d}\eta_1 = \eta_1 \wedge \omega_1 + \omega_0 \wedge (I_0\eta_0 + J_0\zeta_0 + J_1\zeta_1 + J_2\zeta_2)$$

with the coefficients

$$I_0 = \frac{f_y}{A^2}, \qquad J_0 = \frac{f_z}{A^2}, \qquad J_1 = \frac{f_{z'} + f_{y'}f_{z''}}{A}, \qquad J_2 = f_{z''} \tag{11}$$

and invariant forms $\omega_0, \omega_1, \eta_0, \eta_1, \zeta_0, \zeta_1, \zeta_2$. It follows that also $I_0 = \bar{I}_0$, $J_0 = \bar{J}_0$, $J_1 = \bar{J}_1$, $J_2 = \bar{J}_2$. Moreover,

$$\omega_1 = \frac{\mathrm{d}A}{A} + B\omega_1 = \frac{\mathrm{d}A}{A} - f_{y'}\mathrm{d}x$$

and

$$\mathrm{d}\omega_1 = \omega_0 \wedge \left(\frac{f_{y'y}}{A}\eta_0 + \frac{f_{y'z}}{A}\zeta_0 + f_{y'y'}\eta_1 + f_{y'z'}\zeta_1 + f_{y'z''}A\left(\zeta_2 + \frac{f_{y'}}{A}\zeta_1\right)\right),$$

i.e.,

$$\mathrm{d}\omega_1 = \omega_0 \wedge (M_0\eta_0 + N_0\zeta_0 + M_1\eta_1 + N_1\zeta_1 + N_2\zeta_2)$$

with the coefficients

$$M_0 = \frac{f_{y'y}}{A},\ N_0 = \frac{f_{y'z}}{A},\ M_1 = f_{y'y'},\ N_1 = f_{y'z'} + f_{y'}f_{z''y'},\ N_2 = Af_{y'z''} \tag{12}$$

and the invariance properties $M_i = \bar{M}_i$, $N_j = \bar{N}_j$ $(i = 0, 1;\ j = 0, 1, 2)$.

Coefficients

$$J_2 = f_{z''}, \qquad M_1 = f_{y'y'}, \qquad N_1 = f_{y'z'} + f_{y'}f_{z''y'} \tag{13}$$

are uniquely determined and independent of any additional variable, therefore invariants.

Lemma 1. *The equivalence problem for differential equations $y = f$, $\bar{y} = \bar{f}$ and the pseudogroup Φ is characterized by the property of invariance of the forms*

$$\omega_0 = A\mathrm{d}x \quad (A \neq 0 \text{ is a variable}), \qquad \omega_1 = \frac{\mathrm{d}A}{A} - f_{y'}\mathrm{d}x,$$

$$\eta_0 = \mathrm{d}y - y'\mathrm{d}x, \qquad \eta_1 = \frac{1}{A}(\mathrm{d}y' - f\mathrm{d}x), \tag{14}$$

$$\zeta_0 = \mathrm{d}z - z'\mathrm{d}x, \qquad \zeta_1 = \frac{1}{A}(\mathrm{d}z' - z''\mathrm{d}x), \qquad \zeta_2 = \frac{1}{A^2}(\mathrm{d}z'' - z'''\mathrm{d}x) - \frac{f_{y'}}{A}\zeta_1$$

and differentials

$$\mathrm{d}\omega_0 = \omega_1 \wedge \omega_0, \qquad \mathrm{d}\omega_1 = \omega_0 \wedge (M_0\eta_0 + N_0\zeta_0 + M_1\eta_1 + N_1\zeta_1 + N_2\zeta_2),$$

$$\mathrm{d}\eta_0 = \omega_0 \wedge \eta_1, \qquad \mathrm{d}\eta_1 = \eta_1 \wedge \omega_1 + \omega_0 \wedge (I_0\eta_0 + J_0\zeta_0 + J_1\zeta_1 + J_2\zeta_2), \tag{15}$$

$$\mathrm{d}\zeta_0 = \omega_0 \wedge \zeta_1, \qquad \mathrm{d}\zeta_1 = \zeta_1 \wedge \omega_1 + \omega_0 \wedge \zeta_2.$$

Coefficients $I_0, J_0, J_1, J_2, M_0, M_1, N_0, N_1, N_2$ (with the invariance property) are determined by (11), (12). *Moreover*

$$J_2 = f_{z''}, \qquad M_1 = f_{y'y'}, \qquad N_1 = f_{y'z'} + f_{y'}f_{z''y'}$$

are invariants.

Remark 1. If $F = F(x, y, z, y', z', z'')$ is an invariant, then the development

$$\begin{aligned} \mathrm{d}F &= F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z + F_{y'}\mathrm{d}y' + F_{z'}\mathrm{d}z' + F_{z''}\mathrm{d}z'' \\ &= \frac{\partial F}{\partial \omega_0}\omega_0 + \frac{\partial F}{\partial \eta_0}\eta_0 + \frac{\partial F}{\partial \zeta_0}\zeta_0 + \frac{\partial F}{\partial \eta_1}\eta_1 + \frac{\partial F}{\partial \zeta_1}\zeta_1 + \frac{\partial F}{\partial \zeta_2}\zeta_2 \end{aligned} \tag{16}$$

with covariant derivatives

$$\frac{\partial F}{\partial \omega_0} = \frac{1}{A}(F_x + y'F_y + z'F_z + fF_{y'} + z''F_{z'} + z'''F_{z''}),$$

$$\begin{aligned} &\frac{\partial F}{\partial \eta_0} = F_y, \quad \frac{\partial F}{\partial \zeta_0} = F_z, \quad \frac{\partial F}{\partial \eta_1} = AF_{y'}, \\ &\frac{\partial F}{\partial \zeta_1} = A(F_{z'} + f_{y'}F_{z''}), \quad \frac{\partial F}{\partial \zeta_2} = A^2F_{z''} \end{aligned} \tag{17}$$

still provides the new invariant when applied to the invariant function $F = \bar{F}$ and is independent of any additional parameter.

4. An uncertain variable A

Let us consider the case when all functions f_y/A^2, f_z/A^2, $(f_{z'} + f_{y'}f_{z''})/A$, $f_{y'z}/A$, $Af_{y'z''}$ in (11), (12) and also all covariant derivatives of invariants $F = f_{z''}$, $f_{y'y'}$, $f_{y'z'} + f_{y'}f_{z''y'}$ containing the variable A vanish. In such case it is not possible to determine A as a function of coordinates x, y, z, y', z', z'' and the variable A is independent. We obtain the requirements

$$\begin{aligned} f_y = f_z = f_{z'} + f_{y'}f_{z''} = 0, \\ f_{z''x} = f_{z''y'} = f_{z''z'} = f_{z''z''} = 0, \\ f_{y'y'x} = f_{y'y'y'} = f_{y'y'z'} = f_{y'z'x} = f_{y'z'z'} = 0 \end{aligned} \tag{18}$$

for the function f.

Theorem 1. *The solution f of (18) is of the form*

$$f = J_2z'' + C_1(y' - J_2z')^2 + p(x)(y' - J_2z') + q(x)$$

where C_1, J_2 are arbitrary constants and $p(x)$, $q(x)$ are arbitrary functions, $x \in \mathscr{D}(\varphi)$. The corresponding equation (3) can be expressed in the form

$$(y - J_2z)'' = C_1(y - J_2z)'^2 + p(x)(y - J_2z)' + q(x).$$

Proof. We have $f = f(x, y', z', z'')$ and $f_{z''} = J_2 =$ const in accordance with conditions $f_y = f_y = (f_{z''})_x = (f_{z''})_{y'} = (f_{z''})_{z'} = (f_{z''})_{z''} = 0$. Thus

$$f(x, y', z', z'') = J_2z'' + g(x, y', z') \tag{19}$$

where $g_{z'} + g_{y'}J_2 = (g_{y'y'})_x = (g_{y'y'})_{y'} = (g_{y'y'})_{z'} = (g_{y'z'})_x = (g_{y'z'})_{z'} = 0$. Then $g_{y'y'} = 2C_1 =$ const,

$$g_{y'} = 2C_1y' + h(x, z')$$

and $g_{y'z'} = h_{z'} = 2C_2 =$ const, i.e., $h = 2C_2z' + p(x)$. It follows $g_{y'} = 2C_1y' + 2C_2z' + p(x)$ and

$$g(x, y', z') = C_1y'^2 + 2C_2y'z' + p(x)y' + \alpha(x, z'). \tag{20}$$

The last condition $g_{z'} + g_{y'}J_2 = 0$ is equivalent to

$$2(C_2 + J_2C_1)y' + \alpha_{z'}(x, z') + 2J_2C_2z' + J_2p(x) = 0$$

hence

$$\alpha(x, z') = -J_2C_2z'^2 - J_2p(x)z' + q(x), \qquad C_1J_2 + C_2 = 0. \tag{21}$$

We obtain

$$f = J_2z'' + C_1(y' - J_2z')^2 + p(x)(y' - J_2z') + q(x)$$

by using (19), (20), (21). Substituting f into (3) we get the remaining part of the assertion. □

4.1. The equivalence conditions

The equivalence transformations between equations $y'' = f$, $\bar{y}'' = \bar{f}$ are given by

$$\omega_i = \bar{\omega}_i, \quad \eta_i = \bar{\eta}_i, \quad \zeta_j = \bar{\zeta}_j \qquad (i = 0, 1;\ j = 0, 1, 2)$$

in accordance with Lemma 1 and $\mathrm{d}\omega_1 = 2C_1\omega_0 \wedge (\eta_1 - J_2\zeta_1)$, $\mathrm{d}\eta_1 = \eta_1 \wedge \omega_1 + J_2\omega_2 \wedge \zeta_2$ is satisfied in the structural formulae (15). Let us introduce the function $N(x) = L(x) - J_2 M(x)$ and forms $\chi_0 = \eta_0 - J_2\zeta_0$, $\chi_1 = \eta_1 - J_2\zeta_1$. Then

$$\mathrm{d}\omega_1 = 2C_1\omega_0 \wedge \chi_1, \qquad \mathrm{d}\chi_0 = \omega_0 \wedge \chi_1, \qquad \mathrm{d}\chi_1 = \chi_1 \wedge \omega_0$$

and we get $\mathrm{d}\pi = 0$ where

$$\pi = \omega_1 - 2C_1\chi_0 = \frac{\mathrm{d}A}{A} - (2C_1 u' + p(x))\mathrm{d}x, \qquad u' = y' - J_2 z'.$$

The forms ω_0, χ_0, χ_1, π can be taken into account. We have

$$\frac{\mathrm{d}\bar{A}}{\bar{A}} = \frac{\mathrm{d}A}{A} - \frac{\varphi''(x)}{\varphi'(x)}\mathrm{d}x$$

due to $\omega_0 = \bar{\omega}_0$ (i.e., $A = \bar{A}\varphi'$) even if A, $\bar{A}$ remain uncertain variables. The equivalence conditions $\chi_0 = \bar{\chi}_0$, $\chi_1 = \bar{\chi}_1$ and $\pi = \bar{\pi}$ can be expressed as

$$\bar{u}'\varphi' = u' + N', \qquad \bar{u}''{\varphi'}^2 = u'' + N'' - \frac{\varphi''(x)}{\varphi'(x)}(u' + N')$$

and

$$\bar{p}(\varphi)\varphi' = p - \frac{\varphi''(x)}{\varphi'(x)} - 2C_1 N',$$

respectively. The explicit formula $\bar{f}{\varphi'}^2 = f - \frac{\varphi''(x)}{\varphi'(x)}(y' + L') + L''$ is replaced by $\bar{u}''{\varphi'}^2 = u'' + N'' - \frac{\varphi''(x)}{\varphi'(x)}(u' + N')$ with $u'' = C_1{u'}^2 + p(x)u' + q(x)$ and $\bar{q}(\varphi){\varphi'}^2 = q + N'' + C_1{N'}^2 - pN'$ follows.

Corollary 1. *Suppose $\varphi = \varphi(x)$, $N(x) = L(x) - J_2 M(x)$. We have the transformation conditions*

$$\bar{p}(\varphi)\varphi' = p(x) - \frac{\varphi''}{\varphi'} - 2C_1 N'(x),$$
$$\bar{q}(\varphi){\varphi'}^2 = q(x) + N''(x) + C_1(N'(x))^2 - p(x)N'(x)$$

for the equivalence of equations $y'' = f$, $\bar{y}'' = \bar{f}$ with

$$f = J_2 z'' + C_1(y' - J_2 z')^2 + p(x)(y' - J_2 z') + q(x)$$

in the case of an uncertain variable A.

5. The determined case

Let us suppose that some of the equations (18) is not satisfied. Then $A = A(x, y, z, y', z', z'')$ can be determined and

$$\omega_1 = \frac{\mathrm{d}A}{A} - f_{y'}\mathrm{d}x = K\omega_0 + K_0\eta_0 + L_0\zeta_0 + K_1\eta_1 + L_1\zeta_1 + L_2\zeta_2 \tag{22}$$

where

$$K = \frac{1}{A^2}(A_x + y'A_y + z'A_z + fA_{y'} + z''A_{z'} + z'''A_{z''}) - \frac{f_{y'}}{A},$$

$$K_0 = \frac{A_y}{A}, \quad L_0 = \frac{A_z}{A}, \quad K_1 = A_{y'}, \quad L_1 = A_{z'} + f_{y'}A_{z''}, \quad L_2 = AA_{z''}. \tag{23}$$

In order to determine the function f in terms of invariants, we need to solve the following system of equations

$$\left|\begin{array}{l} f_y = I_0A^2 \\ f_z = J_0A^2 \\ f_{z'} + f_{y'}f_{z''} = J_1A \\ f_{z''} = J_2 \\ \end{array}\right| \begin{array}{l} f_{yy'} = M_0A \\ f_{zy'} = N_0A \\ f_{y'y'} = M_1 \\ f_{y'z'} + f_{y'}f_{z''y'} = N_1 \\ \end{array} \left| \begin{array}{l} A_y = K_0A \\ A_z = L_0A \\ A_{y'} = K_1 \\ A_{z'} + f_{y'}A_{z''} = L_1 \\ AA_{z''} = L_2 \end{array}\right| \tag{24}$$

$$A_x + y'A_y + z'A_z + fA_{y'} + z''A_{z'} + z'''A_{z''} - f_{y'}A = KA^2. \tag{25}$$

We shall discuss two subcases with the highest possible symmetry: namely if all invariants are constant (Section 5.1) and if there exist only one nonconstant invariant function (Section 5.2). In the remaining case, if there are more functionally independent invariants, the equations are "rigid" and admit only few equivalence and symmetry transformations.

Remark 2. Let A be a function $A = A(x, y, z, y', z', z'')$. Then the coefficients $I_0, J_0, J_1, J_2, K, K_0, K_1, L_0, L_1, L_2, M_0, M_1, N_0, N_1$ in (24), (25) are coefficients of developments (15), (22) with the invariance property and independent of any additional variables, they are true invariants.

5.1. Constant invariants

Let I_0, K, J_i, L_i $(i = 0, 1, 2)$, M_j, N_j $(j = 0, 1)$ be constants. We have $A_{z''} = 0$ because the right-hand side of the equation (25) is independent of z'''. Thus $f = f(x, y, z, y', z', z'')$, $A = A(x, y, z, y', z')$, $L_2 = 0$, $N_1 = f_{y'z'}$. Also $y'' = f = J_2 + \alpha(x, y, z, y', z')$ is satisfied, i.e.,

$$(y - J_2z)'' = \alpha(x, y, z, y', z') \tag{26}$$

in accordance with $f_{z''} = J_2$. We obtain the equation

$$u'' = g(x, u, z, u', z') \tag{27}$$

with

$$f(x,y,z,y',z',z'') = J_2 z'' + g(x,u(y,z),z,u'(y',z'),z'), \tag{28}$$

by means of the transformation

$$u = u(y,z) = y - J_2 z, \qquad u' = u'(y',z') = y' - J_2 z', \qquad u'' = y'' - J_2 z'' \tag{29}$$

realized through functions $u(x) = y(x) - J_2 z(x)$, $x \in \mathscr{D}(\varphi)$. The function A is transformed into

$$\tilde{A}(x,u,z,u',z') = \tilde{A}(x,u(y,z),z,u'(y',z'),z') = A(x,y,z,y',z'). \tag{30}$$

The condition $A_y = K_0 A$ is equivalent to $(\tilde{A}_u u_y =)\ \tilde{A}_u = K_0 \tilde{A}$, $A_z = L_0 A$ is equivalent to $A_z = \tilde{A}_u u_z + \tilde{A}_z = -J_2 \tilde{A}_u + \tilde{A}_z = -J_2 K_0 \tilde{A} + \tilde{A}_z$, i.e., $\tilde{A}_z = \tilde{L}_0 \tilde{A}$ where $\tilde{L}_0 = L_0 + J_2 K_0$. The remaining conditions are transformed in analogous way, for the conditions containing f we use the relation (28). Moreover, (25) is transformed into

$$\tilde{A}_x + u'\tilde{A}_u + z'\tilde{A}_z + g\tilde{A}_{u'} + z''\tilde{A}_{z'} - g_{u'}\tilde{A} = K\tilde{A}^2$$

and we have $\tilde{A}_{z'} = 0$ because the right-hand side of the equation is independent of z''. As a result we obtain the conditions

$$\left|\begin{array}{l|l|l} g_u = I_0 \tilde{A}^2 & g_{uu'} = M_0 \tilde{A} & \tilde{A}_u = K_0 \tilde{A} \\ g_z = \tilde{J}_0 \tilde{A}^2 & g_{zu'} = \tilde{N}_0 \tilde{A} & \tilde{A}_z = \tilde{L}_0 \tilde{A} \\ g_{z'} = J_1 \tilde{A} & g_{u'u'} = M_1 & \tilde{A}_{u'} = K_1 \\ & g_{u'z'} = \tilde{N}_1 & \end{array}\right| \tag{31}$$

$$\tilde{A}_x + u'\tilde{A}_u + z'\tilde{A}_z + g\tilde{A}_{u'} - g_{u'}\tilde{A} = K\tilde{A}^2 \tag{32}$$

for the functions $g = g(x,u,z,u',z')$, $\tilde{A} = \tilde{A}(x,u,z,u')$. Here I_0, $\tilde{J}_0$, J_1, M_0, $\tilde{N}_0$, M_1, $\tilde{N}_1$, K_0, $\tilde{L}_0$, K_1, K are constants.

Theorem 2. *The following functions and conditions are solutions of the system* (31), (32) *in the case of constant invariants.*

(a) $g = C_1 {u'}^2 + (C_2 z' + b(x))u' + J_1 a(x) z' + p(x)$, $\tilde{A} = K_1 u' + a(x)$,
$KK_1 + C_1 = 0$, $a'(x) - a(x)b(x) + K_1 p(x) - K a^2(x) = 0$,
$C_2 = J_1 K_1 \neq 0$ *or* $J_1 = C_2 = 0$, $K_1 \neq 0$.

(b) $g = C_1 {u'}^2 + b(x)u' + \frac{I_0}{4C_1} a^2(x) \mathrm{e}^{4C_1 u} + p(x)$, $\tilde{A} = a(x)\mathrm{e}^{2C_1 u}$,
$a'(x) - a(x)b(x) = 0$, $I_0, C_1 \in \mathbb{R}$, $C_1 \neq 0$.

(c) $g = b(x)u' + J_1 a(x) z' + a^2(x)(I_0 u + \tilde{J}_0 z) + p(x)$, $\tilde{A} = a(x)$,
$a'(x) - a(x)b(x) - K a^2(x) = 0$, $I_0, \tilde{J}_0, J_1 \in \mathbb{R}$.

P r o o f. We have the following constant invariants

$$g_{u'u'}, \quad g_{u'z'}, \quad g_u : g_z : (g_{z'})^2 : (g_{uu'})^2 : (g_{zu'})^2 \tag{33}$$

and also

$$g = C_1 {u'}^2 + (C_2 z' + \beta(x,u,z))u' + \gamma(x,u,z,z') \tag{34}$$

due to $g_{u'u'} = \text{const}$, $g_{u'z'} = \text{const}$.

Let $g_{uu'} = \beta_u \neq 0$. Then

$$\frac{g_u}{(g_{uu'})^2} = \frac{\beta_u u' + \gamma_u}{(\beta_u)^2} = \frac{1}{\beta_u} u' + \frac{\gamma_u}{(\beta_u)^2} = \text{const}$$

which is a contradiction since γ, β are independent of u'. Thus $\beta_u = 0$ and similarly $\beta_z = 0$ by using $g_u/(g_{uu'})^2 = \text{const}$. The function g is given by

$$g = C_1 {u'}^2 + (C_2 z' + b(x))u' + \gamma(x, u, z, z') \tag{35}$$

and the remaining constant invariants are

$$\gamma_u : \gamma_z : (C_2 u' + \gamma_{z'})^2.$$

Two different subcases for $C_2 \neq 0$ and $C_2 = 0$ should be distinguished.

(ι) Let us consider $C_2 \neq 0$. Assuming $\gamma_u \neq 0$,

$$\frac{(C_2 u' + \gamma_{z'})^2}{\gamma_u} = \frac{(C_2 u')^2}{\gamma_u} + 2C_2 u' \frac{\gamma_{z'}}{\gamma_u} + \frac{(\gamma_{z'})^2}{\gamma_u} = \text{const}$$

holds true and through successive differentiating $\partial/\partial u'$ we obtain

$$C_2 u' \frac{1}{\gamma_u} + \frac{\gamma_{z'}}{\gamma_u} = 0 \qquad \text{and} \qquad \frac{C_2}{\gamma_u} = 0$$

which is a contradiction and we get $\gamma_u = 0$. In a similar way we obtain $\gamma_z = 0$ by means of $(C_2 u' + \gamma_{z'})^2/\gamma_z$ and $\gamma = \gamma(x, z')$. The function g is of the form

$$g = C_1 {u'}^2 + (C_2 z' + b(x))u' + \gamma(x, z') \tag{36}$$

and we have the conditions $g_{z'} = J_1 \tilde{A}$, $\tilde{A}_u = K_0 \tilde{A}$, $\tilde{A}_z = \tilde{L}_0 \tilde{A}$, $\tilde{A}_{u'} = K_1$, moreover

$$\tilde{A}_x + u' \tilde{A}_u + z' \tilde{A}_z + g \tilde{A}_{u'} - g_{u'} \tilde{A} = K \tilde{A}^2. \tag{37}$$

Therefore $\tilde{A}_u = \tilde{A}_z = 0$ from the relation $g_{z'} = C_2 u' + \gamma_{z'}(x, z') = J_1 \tilde{A}(x, u, z, u')$. Then $\tilde{A} = K_1 u' + a(x)$ in accordance with $\tilde{A}_{u'} = K_1$ and $C_2 = J_1 K_1 \neq 0$, $\gamma(x, z') = J_1 a(x) z' + p(x)$ by means of $C_2 u' + \gamma_{z'}(x, z') = J_1(K_1 u' + a(x))$. Furthermore, the condition (37) is equivalent to $KK_1 + C_1 = 0$, $a'(x) - a(x)b(x) + K_1 p(x) - K(a(x))^2 = 0$.

We have proved that

$$g = C_1 {u'}^2 + (C_2 z' + b(x))u' + J_1 a(x) z' + p(x), \quad \tilde{A} = K_1 u' + a(x), \tag{38}$$

$a'(x) - a(x)b(x) + K_1 p(x) - K a^2(x) = 0, \qquad C_2 = J_1 K_1 \neq 0, \quad KK_1 + C_1 = 0,$ for $C_2 \neq 0$.

($\iota\iota$) The second subcase is characterized by $C_2 = 0$ when

$$g = C_1 {u'}^2 + b(x)u' + \gamma(x, u, z, z') \tag{39}$$

and

$$\gamma_u : \gamma_z : (\gamma_{z'})^2$$

are constant invariants.

Returning to the relations (31), (32) we have

$$\begin{array}{ll} g_u = \gamma_u = I_0 \tilde{A}^2 & \tilde{A}_u = K_0 \tilde{A} \\ g_z = \gamma_z = \tilde{J}_0 \tilde{A}^2 & \tilde{A}_z = \tilde{L}_0 \tilde{A} \\ g_{z'} = \gamma_{z'} = J_1 \tilde{A} & \tilde{A}_{u'} = K_1 \end{array}$$

$$\tilde{A}_x + u'\tilde{A}_u + z'\tilde{A}_z + g\tilde{A}_{u'} - g_{u'}\tilde{A} = K\tilde{A}^2$$

for $\gamma = \gamma(x, u, z, z')$, $\bar{A} = \bar{A}(x, u, z, u')$.

The relation $\gamma_{z'}(x, u, z, z') = J_1 \tilde{A}(x, u, z, u')$ gives two possibilities $\gamma_{z'} = 0$ or $\tilde{A}_{u'} = 0$ (use $\partial/\partial u'$).

$(\iota\iota)_1$ First we consider the subsubcase $\gamma = \gamma(x, u, z)$, $\tilde{A} = \tilde{A}(x, u, z, u')$ for $\gamma_{z'} = 0$. We get $\tilde{A} = K_1 u' + \mu(x, u, z)$ by using $\tilde{A}_{u'} = K_1$.

Assuming $K_1 \neq 0$, we have $I_0 = \tilde{J}_0 = J_1 = K_0 = \tilde{L}_0$, i.e.,

$$\gamma(x, u, z) = p(x), \qquad \tilde{A} = K_1 u' + a(x).$$

Notice that $\frac{\partial}{\partial u'}(\gamma_u - I_0\tilde{A}^2) = -2I_0\tilde{A}\tilde{A}_{u'} = 0$ which means $I_0 = 0$ for $\tilde{A}\tilde{A}_{u'} \neq 0$, i.e., $\gamma_u = 0$ for example. As a result

$$\begin{array}{l} g = C_1 {u'}^2 + b(x)u' + p(x), \quad \tilde{A} = K_1 u' + a(x), \\ a'(x) + K_1 p(x) - a(x)b(x) - Ka^2(x) = 0, \qquad K = C_1/K_1, \quad K_1 \neq 0 \end{array} \tag{40}$$

with regard to the condition (37). This result coincides with (38) for $J_1 = C_2 = 0$, $K_1 \neq 0$.

Let $K_1 = 0$. Then $\gamma = \gamma(x, u, z)$, $\tilde{A} = \mu(x, u, z)$ and we analyze the conditions

$$\gamma_u = I_0\mu^2, \qquad \gamma_z = \tilde{J}_0\mu^2, \qquad \mu_u = K_0\mu, \qquad \mu_z = \tilde{L}_0\mu,$$

$$\mu_x + u'\mu_u + z'\mu_z - (bx + 2C_1u')\mu = K\mu^2.$$

Through $\frac{\partial}{\partial u'}$ and $\frac{\partial}{\partial z'}$ applied to the last equation we obtain

$$\mu_u = 2C_1\mu, \quad \mu_z = 0, \qquad \text{i.e.,} \qquad \mu = a(x)\mathrm{e}^{2C_1 u} = \mu(x, u)$$

and the same equation gives the condition

$$a'(x) - a(x)b(x) = Ka^2(x)\mathrm{e}^{4C_1 u}$$

with two possibilities

$$\begin{array}{lll} a'(x) - a(x)b(x) - Ka^2(x) = 0, \quad K \in \mathbb{R} & \text{for} & C_1 = 0, \\ a'(x) - a(x)b(x) = 0, \quad K = 0 & \text{if} & C_1 \neq 0. \end{array}$$

— If $C_1 = 0$, then $\mu = a(x)$ and $\gamma(x, u, z) = a^2(x)\left(I_0 u + \tilde{J}_0 z\right) + p(x)$ in accordance with $\gamma_u = I_0\mu^2$, $\gamma_z = \tilde{J}_0\mu^2$. The resulting functions and conditions are

$$\begin{array}{l} g = b(x)u' + a^2(x)\left(I_0 u + \tilde{J}_0 z\right) + p(x), \quad \tilde{A} = a(x), \\ a'(x) - a(x)b(x) - Ka^2(x) = 0, \qquad I_0, \tilde{J}_0, K \in \mathbb{R}. \end{array} \tag{41}$$

— The subcase $C_1 \neq 0$, $\mu(x,u) = a(x)\mathrm{e}^{2C_1 u}$ with conditions $\gamma_u = I_0\mu^2$, $\gamma_z = \tilde{J}_0\mu^2$ has solution $\gamma(x,u,z) = \frac{I_0}{4C_1}a^2(x)\mathrm{e}^{4C_1 u} + p(x)$ ($\gamma_{uz} = \gamma_{zu}$ is satisfied only if $\gamma_z = 0$) and the resulting functions and conditions are

$$\begin{aligned} &g = C_1 {u'}^2 + b(x)u' + \tfrac{I_0}{4C_1}a^2(x)\mathrm{e}^{4C_1 u} + p(x), \quad \tilde{A} = a(x)\mathrm{e}^{2C_1 u}, \\ &a'(x) - a(x)b(x) = 0, \qquad I_0, C_1 \in \mathbb{R}, \quad C_1 \neq 0. \end{aligned} \tag{42}$$

$(\iota\iota)_2$ It remains to investigate the subcase $\tilde{A}_{u'} = 0$, $\gamma_{z'} \neq 0$.

First we consider $\tilde{A} = \mu(x,u,z)$ and $g = C_1{u'}^2 + b(x)u' + \gamma(x,u,z,z')$ with conditions $\tilde{A}_u = K_0\tilde{A}$, $\tilde{A}_z = \tilde{L}_0\tilde{A}$ and $\tilde{A}_x + u'\tilde{A}_u + z'\tilde{A}_z - g_{u'}\tilde{A} = K\tilde{A}^2$. The last equation means that $\mu_x + u'\mu_u + z'\mu_z - (2C_1u' + b(x))\mu = K\mu^2$. We get $\mu_u - 2C_1\mu = 0$ and $\mu_z = 0$ by using $\frac{\partial}{\partial u'}$ and $\frac{\partial}{\partial z'}$, respectively. Thus

$$\mu = \mu(x,u) = a(x)\mathrm{e}^{2C_1u} = \tilde{A}, \qquad \tilde{A}_u = \tilde{A}_z = \tilde{A}_{u'} = 0.$$

Second, the conditions $\gamma_u = I_0\tilde{A}^2$, $\gamma_z = \tilde{J}_0\tilde{A}^2$, $\gamma_{z'} = J_1\tilde{A}^2$ determine the function $\gamma(x,u,z,z')$ for $J_1 \neq 0$. We see that $\gamma = J_1\mu(x,u)z' + \alpha(x,u,z)$ and $\gamma_z = \alpha_z = \tilde{J}_0(\mu(x,u))^2$, i.e., $\alpha = \tilde{J}_0(\mu(x,u))^2z + \beta(x,u)$ and

$$\gamma = J_1\mu(x,u)z' + \tilde{J}_0(\mu(x,u))^2z + \beta(x,u).$$

Similarly,

$$\gamma_u = J_1\mu_u z' + 2\tilde{J}_0\mu\mu_u z + \beta_u(x,u) = 2C_1J_1\mu z' + 4\tilde{J}_0C_1\mu^2z + \beta_u(x,u) = I_0\mu^2$$

and $\beta_u(x,u) = I_0\mu^2 - C_1(4\tilde{J}_0\mu^2z + 2J_1\mu z')$ gives $C_1 = 0$ since $J_1 \neq 0$. Thus $\tilde{A} = \mu(x,u) = a(x)\mathrm{e}^{2C_1u} = a(x)$ and $\beta(x,u) = I_0a^2(x)u + p(x)$. We have the function

$$\gamma = J_1a(x)z' + a^2(x)(I_0u + \tilde{J}_0z) + p(x).$$

The resulting functions and conditions are of the form

$$\begin{aligned} &g = b(x)u' + J_1a(x)z' + a^2(x)(I_0u + \tilde{J}_0z) + p(x), \quad \tilde{A} = a(x), \\ &a'(x) - a(x)b(x) - Ka^2(x) = 0, \qquad I_0, \tilde{J}_0, J_1, K \in \mathbb{R}, \quad J_1 \neq 0. \end{aligned} \tag{43}$$

(This result with $J_1 = 0$ involves (41).) The assertion is proved. □

5.1.1. The equivalence conditions

The equivalence transformations between equations $y'' = f$, $\bar{y}'' = \bar{f}$ are given by

$$\omega_i = \bar{\omega}_i, \quad \chi_i = \bar{\chi}_i, \quad \zeta_2 = \bar{\zeta}_2 \qquad (i = 0,1)$$

with $\chi_i = \eta_i - J_2\zeta_i$ $(i = 0,1)$, $N(x) = L(x) - J_2M(x)$ similarly to the case of an uncertain variable. The relation $g_{u'} = f_{y'}$ holds for f defined by (28), (29) and we have the equivalence conditions

$$\bar{A}\varphi' = A, \qquad \bar{g}_{u'}\varphi' = g_{u'} - \frac{\varphi''}{\varphi'}, \quad \bar{u}'\varphi' = u' + N', \quad \bar{g}{\varphi'}^2 = g + N'' - \frac{\varphi''}{\varphi'}(u' + N')$$

in accordance with $\mathrm{d}A/A = \mathrm{d}\bar{A}/\bar{A} + (\varphi''/\varphi')\mathrm{d}x$ and $u'' = g$. Assuming the transformation relations between u, $\bar{u}$ for known, we give the remaining equivalence conditions depending on the coefficients of the right-hand side of differential equations under consideration.

Corollary 2. *Suppose* $\varphi = \varphi(x)$, $N(x) = L(x) - J_2M(x)$. *We have the following equivalence conditions*

(a) $\bar{a}(\varphi)\varphi' = a(x) - K_1N'(x)$, $\bar{b}(\varphi)\varphi' = b(x) - \frac{\varphi''}{\varphi'} - 2C_1N'(x) - C_2M'(x)$,
$$\bar{p}(\varphi){\varphi'}^2 = p(x) + N''(x) - b(x)N'(x) - J_1a(x)M'(x) + C_1(N'(x))^2 + C_2N'(x)M'(x);$$

(b) $\bar{a}(\varphi)\varphi'\mathrm{e}^{2C_1N(x)} = a(x)$, $\bar{b}(\varphi)\varphi' = b(x) - \frac{\varphi''}{\varphi'} - 2C_1N'(x)$,
$$\bar{p}(\varphi){\varphi'}^2 = p(x) + N''(x) - b(x)N'(x) + C_1(N'(x))^2;$$

(c) $\bar{a}(\varphi)\varphi' = a(x)$, $\bar{b}(\varphi)\varphi' = b(x) - \frac{\varphi''}{\varphi'}$,
$$\bar{p}(\varphi){\varphi'}^2 = p(x) + N''(x) - b(x)N'(x) - J_1a(x)M'(x) - a^2(x)\big(I_0N(x) + \tilde{J}_0M(x)\big)$$

corresponding to the functions g *of* Theorem 2.

5.2. Nonconstant invariants

We discuss the highest possible symmetry problem with nonconstant invariants. Let all the invariants be composed functions of the form $G(F)$, where $F = F(x, y, z, y', z', z'')$ is a certain "basical" nonconstant invariant. We will use the covariant derivatives (16), (17) where $A = A(x, y, z, y', z', z'')$ satisfies equations (24), (25) together with functions $f = f(x, y, z, y', z', z'')$. We can see that $F_{z''} = 0$ because the condition $(17)_1$ is of the form
$$\frac{1}{A}\left(F_x + y'F_y + z'F_z + fF_{y'} + z''F_{z'} + z'''F_{z''}\right) = G(F) \qquad (44)$$
and $\partial G(F)/\partial z''' = F_{z''}/A = 0$. In analogous way, $A_{z''} = 0$ in accordance with (25). Thus
$$A = A(x, y, z, y', z'), \qquad F = F(x, y, z, y', z').$$
Assuming F to be "basic" invariant, functions $F_y = P_1(F)$, $F_z = P_2(F)$ are composed invariants, too. Setting $F_y = P_1(F) = a \in \mathbb{R}$, $F_z = P_2(F) = b \in \mathbb{R}$ we obtain
$$F = ay + bz + c(x, y', z'); \qquad a, b \in \mathbb{R}. \qquad (45)$$
The condition (44) is of the form
$$c_x + ay' + bz' + fc_{y'} + z''c_{z'} = AG(F) \neq 0 \qquad (46)$$
and
$$f_{z''}c_{y'} + c_{z'} = 0$$

follows through differentiation $\partial/\partial z''$. The function $f_{z''} = J_2(F)$ is an invariant (see Lemma 1), hence

$$J_2(F)c_{y'} + c_{z'} = 0 \tag{47}$$

(and $c_{z'} = 0$ follows from $c_{y'} = 0$). Moreover

$$J_2'(F)ac_{y'} = J_2'(F)bc_{y'} = 0 \tag{48}$$

by using $\partial/\partial y$, $\partial/\partial z$ applied to (47) and we have three possibilities

$$c_{y'} = 0, \qquad J_2'(F) = 0 \qquad \text{and} \qquad a = b = 0, \tag{49}$$

respectively. We investigate the following subcases.

(ι) The invariant $F = ay+bz+c(x)$ in the subcase $c_{y'} = c_{z'} = 0$ in accordance with (45). Then

$$A = \frac{1}{G(F)} \cdot (c'(x) + ay' + bz') = \mathscr{G}(F) \cdot (c'(x) + ay' + bz') \tag{50}$$

by means of (46) and

$$c''(x) + af + bz'' - (c'(x) + ay' + bz')f_{y'} = A^2\tilde{\mathscr{G}}(F) \tag{51}$$

follows from (25). An invariant $f_{z''} = J_2(F) = J_2(ay + bz + c(x))$, thus $f_{z''y'} = f_{y'z''} = 0$ and

$$af_{z''} + b = aJ_2(F) + b = 0$$

by using $\partial/\partial z''$ in (51).

The possibilities $a = b = 0$ and $J_2(F) \equiv J_2 = -b/a = \text{const}$, respectively follow.

$(\iota)_1$ Let $a = b = 0$, i.e., $F = c(x)$, $A = \mathscr{G}(c(x))c'(x) = a(x) \neq 0$, $J_2 = J_2(c(x))$. Then $c''(x) - c'(x)f_{y'} = a^2(x)\tilde{\mathscr{G}}(c(x))$ gives $f_{y'} = r(x)$ and the equations (24) involve the conditions $f_y = I_0(c(x))a^2(x)$, $f_z = J_0(c(x))a^2(x)$, $f_{z''} = J_2(c(x))$, $f_{z'} + f_{y'}f_{z''} = J_1(c(x))a(x)$. Solving these equations we get

$$f = J_2(F)z''+p(x)y+q(x)z+r(x)y'+s(x)z'+t(x), \ F = c(x), \ A = A(x), \tag{52}$$

where $p(x) = I_0(F)a^2(x)$, $s(x) = J_1(F)a(x) - J_2(F)r(x)$, $q(x) = J_0(F)a^2(x)$.

$(\iota)_2$ Let $a^2+b^2 \neq 0$, i.e., $a \neq 0$, $J_2(F) \equiv J_2 = -b/a$. Then $F = ay+bz+c(x)$, $A = \mathscr{G}(F) \cdot (c'(x) + ay' + bz')$. The condition $f_{z''} = J_2$ gives

$$f - J_2z'' = \alpha(x, y, z, y', z') = \alpha(x, u + J_2z, z, u' + J_2z', z') = g(x, u, z, u', z')$$

for $u = y - J_2z$. The problem can be transformed into

$$u'' = g(x, u, z, u', z')$$

by means of transformation (29) similarly to the case of the constant invariants. In contrast to the above-mentioned case we have transformed the basic invariant $F = ay + bz + c(x) = ay - aJ_2z + c(x) = a(y - J_2z) + c(x)$ into $\tilde{F} = au + c(x)$, $a \neq 0$ and similarly the function A into $\tilde{A} = \mathscr{G}(\tilde{F})\cdot(au'+c(x))$, the condition (51) into $c'' + ag - (c' + au')g_{u'} = \tilde{A}^2\tilde{\mathscr{G}}(\tilde{F})$. Moreover, the conditions (24), (25) are transformed into new conditions of the type (31), (32) with coefficients denoted by I, J, K, L, M dependent on $\tilde{F}$. We need the conditions

$$g_u = I_0(\tilde{F})\tilde{A}^2, \qquad g_z = J_0(\tilde{F})\tilde{A}^2, \qquad g_{z'} = J_1(\tilde{F})\tilde{A},$$

$$g_{u'u} = M_0(\tilde{F})\tilde{A}, \qquad g_{u'z} = N_0(\tilde{F})\tilde{A}, \qquad g_{u'u'} = M_1(\tilde{F}), g_{u'z'} = N_1(\tilde{F})$$

to resolve the function g. The resulting functions and conditions are of the form

$$\begin{gathered} g = \mathscr{K}(F)(c'(x)+au')^2 + B(c'(x)+au')z' + aE{u'}^2 + q(x)u' + p(x), \\ \tilde{A} = \mathscr{G}(F)\cdot(c'(x)+au'), \quad \tilde{F} = au + c(x) \\ c''(x) - q(x)c'(x) + ap(x) + E{c'}^2(x) = 0, \qquad a \neq 0; \quad B, E \in \mathbb{R}, \end{gathered}$$

i.e.,

$$\begin{aligned} f = J_2 z'' + \mathscr{K}(F)(c'(x)+ay'+bz')^2 + B(c'(x)+ay'+bz')z' \\ + \frac{1}{a}E(ay+bz)'^2 + q(x)(ay+bz)' + p(x), \\ A = \mathscr{G}(F)\cdot(c'(x)+ay'+bz'), \quad F = ay+bz+c(x), \\ c''(x) - q(x)c'(x) + ap(x) + E{c'}^2(x) = 0, \qquad a \neq 0; \quad B, E \in \mathbb{R}. \end{aligned} \tag{53}$$

($\iota\iota$) The second fundamental subcase is characterized by the condition $c_{y'} \neq 0$, when $F = ay + bz + c(x, y', z')$, $f_{z''} = J_2(F)$ and $c_{y'}J_2(F) + c_{z'} = 0$ in accordance with (45), (47).

First we consider the conditions $\partial F/\partial\eta_1 = Af_{y'} = P_3(F)$, $A_{y'} = K_1(F)$ from (17), (24). These conditions can be investigated since $A = A(x, y, z, y', z')$ is a function. Then

$$\frac{c_{y'y'}}{(c_{y'})^2} = \mathscr{P}(F) \tag{54}$$

is satisfied with regard to the equation

$$A_{y'}c_{y'} + Ac_{y'y'} = K_1(F)c_{y'} + P_3(F)\frac{c_{y'y'}}{(c_{y'})^2} = P_3'(F)c_{y'}$$

and we analyze the following subcases.

$(\iota\iota)_1$ Let $a = b = 0$, i.e., $F = c(x, y'z')$. Then $c_{y'y'}/c_{y'} = \mathscr{P}(c)c_{y'}$ gives $c_{y'} = \alpha(x, z')Q(c)$ and $\mathscr{H}(F) = \mathscr{H}(c) = \alpha(x, z')y' + \beta(x, z')$ follows from integration of the equation $c_{y'}/Q(c) = \alpha(x, z')$. Take

$$F = c(x, y', z') = \alpha(x, z')y' + \beta(x, z') \tag{55}$$

as a basic invariant in accordance with our assumptions. Two possibilities may appear, $c_{z'} \neq 0$ and $c_{z'} = 0$, respectively.

$(\iota\iota)_{1,1}$ Let $c_{z'} \neq 0$. The condition $J_2(F)c_{y'} + c_{z'} = 0$ is of the form

$$J_2(c)\alpha(x, z') + \alpha_{z'}(x, z')y' + \beta_{z'}(x, z') = 0. \tag{56}$$

Through $\partial/\partial y'$ we get $J_2'(c)\alpha^2(x,z') + \alpha_{z'}(x,z') = 0$ and we put

$$J_2(c(x,y',z')) = c(x,y',z') = \alpha(x,z')y' + \beta(x,z')$$

without loss of generality. The equations $\alpha^2 + \alpha_{z'} = \alpha\beta + \beta_{z'} = 0$ are equivalent to (56) and solving these equations we obtain functions

$$\alpha(x,z') = \frac{1}{z' + a(x)}, \qquad \beta(x,z') = \frac{b(x)}{z' + a(x)}.$$

The basic invariant is

$$F = \frac{y' + b(x)}{z' + a(x)} \tag{57}$$

and the condition $AF_{y'} = A\alpha(x,z') = A/(z' + a(x)) = P_3(F)$ gives the function

$$A = P_3(F) \cdot (z' + a(x)). \tag{58}$$

The covariant derivative

$$\frac{\partial F}{\partial \omega_0} = \frac{1}{A}(F_x + fF_{y'} + z''F_{z'}) = M(F)$$

considered as an invariant $M(F)$ determines the function f of the form

$$f = \mathscr{F}(F) \cdot (y' + b(x))(z' + a(x)) + (z'' + a'(x))F - b'(x) \tag{59}$$

($\mathscr{F}(F) = P_3(F)M(F)/F$). We get the resulting functions

$$\begin{gathered} f = (y' + b(x))(z' + a(x))\mathscr{F}(F) + (z'' + a'(x))F - b'(x), \\ A = (z' + a(x))P_3(F), \qquad F = \tfrac{y' + b(x)}{z' + a(x)}, \end{gathered} \tag{60}$$

without any additional condition. The remaining conditions in (23), (24), (25) define some relations between the invariants.

$(\iota\iota)_{1,2}$ For $c_{z'} = 0$ we get $F = c(x,y') = \alpha(x)y' + \beta(x)$, $J_2(F) \equiv 0$, by using (55), (47). Here $c_{y'} = \alpha(x) \neq 0$. The condition $AF_{y'} = P_3(F)$ is equivalent to

$$A = \frac{1}{\alpha(x)}P_3(F) \tag{61}$$

and the condition (44) determines the function

$$f = \frac{1}{\alpha^2(x)}\mathscr{F}(F) - \frac{1}{\alpha(x)}(\alpha'(x)y' + \beta'(x)) \qquad (F = \alpha(x)y' + \beta(x)). \tag{62}$$

The equation $y'' = f$ is not an undetermined case of the equation (3).

We analyze the remaining subcase.

$(\iota\iota)_2$ Assume that $a^2 + b^2 \neq 0$. Then $F = ay + bz + c(x,y',z')$ and

$$\frac{c_{y'y'}}{(c_{y'})^2} \equiv C = \text{const}, \qquad J_2(F) \equiv J_2 = \text{const} \tag{63}$$

with regard to (54), (47). The possibilities that we have to discuss are $C = 0$ and $C \neq 0$ (for $c_{z'} = 0$ and $c_{z'} \neq 0$), respectively.

$(\iota\iota)_{2,1}$ Let $C = 0$, then the function $c(x, y', z') = \alpha(x, z')y' + \beta(x, z')$ is a solution of $c_{y'y'} = 0$.

Let $c_{z'} = 0$, i.e., $c(x, y'z') = \alpha(x)y' + \beta(x)$ ($\alpha(x) \neq 0$ according to $c_{y'} \neq 0$). We have $J_2 = 0$ (use (47)) and

$$F = ay + bz + \alpha(x)y' + \beta(x), \qquad a^2 + b^2 \neq 0, \quad \alpha(x) \neq 0, \tag{64}$$

is the desired basic invariant. The condition $AF_{y'} = P_3(F)$ determines the function $A = P_3(F)/\alpha(x)$ and the covariant derivative $\partial F/\partial\omega_0 = M(F)$ the function f of the form $f = \frac{1}{\alpha^2(x)}\mathscr{F}(F) - \frac{1}{\alpha(x)}((\alpha'(x) + a)y' + bz' + \beta'(x))$. The resulting functions now are

$$\begin{gathered} f = \tfrac{1}{\alpha^2(x)}\mathscr{F}(F) - \tfrac{1}{\alpha(x)}((\alpha'(x) + a)y' + bz' + \beta'(x)), \quad A = P_3(F)/\alpha(x), \\ F = ay + bz + \alpha(x)y' + \beta(x), \qquad a^2 + b^2 \neq 0, \quad \alpha(x) \neq 0, \end{gathered} \tag{65}$$

without any additional condition.

Assuming $c(x, y', z') = \alpha(x, z')y' + \beta(x, z')$ in the case $c_{z'} \neq 0$ we have $c_{y'}J_2 + c_{z'} = \alpha(x, z')J_2 + \alpha_{z'}(x, z')y' + \beta_{z'}(x, z') = 0$ where $J_2 \neq 0$. Thus $\alpha(x, z') = \alpha(x)$, $\beta(x, z') = -J_2\alpha(x)z' + \beta(x)$ and

$$c(x, y'z') = \alpha(x)y' - J_2\alpha(x)z' + \beta(x).$$

The considered basic invariant

$$F = ay + bz + \alpha(x)(y' - J_2z') + \beta(x)$$

determines the function $A = P_3(F)/\alpha(x)$ by using the condition $AF_{y'} = P_3(F)$ and the covariant derivative $\partial F/\partial\omega_0 = M(F)$ determines the function f. As a result

$$\begin{gathered} f = J_2z'' + \tfrac{1}{\alpha^2(x)}\mathscr{F}(F) - \tfrac{\alpha'(x)}{\alpha(x)}(y - J_2z)' - \tfrac{1}{\alpha(x)}(\beta'(x) + ay' + bz'), \\ A = P_3(F)/\alpha(x), \; F = ay + bz + \alpha(x)(y - J_2z)' + \beta(x), \quad \alpha(x)J_2 \neq 0, \end{gathered} \tag{66}$$

without any additional condition.

$(\iota\iota)_{2,2}$ The case $C \neq 0$ ($c_{z'} = 0$ and $c_{z'} \neq 0$, respectively). By successive integration of the equation $-\frac{c_{y'y'}}{(c_{y'})^2} = -C$ we obtain $c_{y'} = -\frac{1}{C}\frac{-C}{\alpha(x,z') - Cy'}$ and then $c(x, y', z') = -\frac{1}{C}\ln|\alpha(x, z') - Cy'| + \beta(x, z')$, i.e.,

$$F = ay + bz + \beta(x, z') - \frac{1}{C}\ln|\alpha(x, z') - Cy'|.$$

We get $J_2(F) \equiv J_2 = \text{const}$ by using $c_{y'}J_2(F) + c_{z'} = 0$ and the assumption $a^2 + b^2 \neq 0$. The condition $c_{z'} = J_2c_{y'}$ is equivalent to

$$\frac{1}{C}\alpha_{z'} - J_2 = \beta_{z'}(\alpha - Cy')$$

which is possible only if $\beta(x, z') = \beta(x)$, $\alpha(x, z') = CJ_2z' + \alpha(x)$. Thus

$$F = ay + bz + \beta(x) - \frac{1}{C}\ln|\alpha(x) - C(y' - J_2z')| \tag{67}$$

and assuming $J_2 \in \mathbb{R}$ we get both conditions $c_{z'} = 0$, $c_{z'} \neq 0$ because $c_{z'} = 0$ means $J_2(F) \equiv J_2 = 0$ as a subcase of (67). The functions A and f are determined by conditions $AF_{y'} = P_3(F)$ and $\partial F/\partial\omega_0 = M(F)$, respectively. Now

$$\begin{aligned} f &= J_2 z'' + \mathscr{F}(F)(\alpha(x) - C(y' - J_2 z'))^2 \\ &\quad - (\beta'(x) + ay' + bz')(\alpha(x) - C(y' - J_2 z')) + \frac{1}{C}\alpha'(x), \\ A &= P_3(F)(\alpha(x) - C(y' - J_2 z')), \\ F &= ay + bz + \beta(x) - \frac{1}{C}\ln|\alpha(x) - C(y' - J_2 z')|; \qquad C, J_2 \in \mathbb{R}, \quad C \neq 0, \end{aligned} \tag{68}$$

without any additional condition. We have proved:

Theorem 3. *The following functions and conditions are solutions of the system* (24), (25) *for nonconstant invariants.*

(a) $f = J_2(F)z'' + p(x)y + q(x)z + r(x)y' + s(x)z' + t(x)$, $A = a(x)$, $F = c(x)$.

(b) $f = J_2 z'' + \mathscr{K}(F)(c'(x) + ay' + bz')^2 + B(c'(x) + ay' + bz')z' + \frac{1}{a}E(ay + bz)'^2 + q(x)(ay + bz)' + p(x)$,
$A = \mathscr{G}(F)\cdot(c'(x) + ay' + bz')$, $F = ay + bz + c(x)$,
$c''(x) - q(x)c'(x) + ap(x) + Ec'^2(x) = 0$, $a \neq 0$; $B, E \in \mathbb{R}$.

(c) $f = (y' + b(x))(z' + a(x))\mathscr{F}(F) + (z'' + a'(x))F - b'(x)$, $A = (z' + a(x))P_3(F)$, $F = \frac{y' + b(x)}{z' + a(x)}$.

(d) $f = \frac{1}{\alpha^2(x)}\mathscr{F}(F) - \frac{1}{\alpha(x)}((\alpha'(x) + a)y' + bz' + \beta'(x))$, $A = P_3(F)/\alpha(x)$, $F = ay + bz + \alpha(x)y' + \beta(x)$, $\alpha(x) \neq 0$.

(e) $f = J_2 z'' + \frac{1}{\alpha^2(x)}\mathscr{F}(F) - \frac{\alpha'(x)}{\alpha(x)}(y - J_2 z)' - \frac{1}{\alpha(x)}(\beta'(x) + ay' + bz')$,
$A = P_3(F)/\alpha(x)$, $F = ay + bz + \alpha(x)(y - J_2 z)' + \beta(x)$,
$\alpha(x)J_2 \neq 0$, $a^2 + b^2 \neq 0$.

(f) $f = J_2 z'' + \frac{1}{C}\alpha'(x) + \mathscr{F}(F)(\alpha(x) - C(y' - J_2 z'))^2 - (\beta'(x) + ay' + bz')(\alpha(x) - C(y' - J_2 z')) + \frac{1}{C}\alpha'(x)$,
$A = P_3(F)(\alpha(x) - C(y' - J_2 z'))$,
$F = ay + bz + \beta(x) - \frac{1}{C}\ln|\alpha(x) - C(y' - J_2 z')|$,
$C, J_2 \in \mathbb{R}$, $C \neq 0$, $a^2 + b^2 \neq 0$.

5.2.1. The equivalence conditions

The equivalence transformations between equations $y'' = f$, $\bar{y}'' = \bar{f}$ are given by

$$\omega_i = \bar{\omega}_i, \quad \eta_i = \bar{\eta}_i, \quad \zeta_j = \bar{\zeta}_j \qquad (i = 0, 1;\ j = 0, 1, 2)$$

and moreover, $F = \bar{F}$ for the basic invariant F. We state the equivalence conditions depending on coefficients of the right-hand side of the differential equations under consideration.

Corollary 3. *Suppose $\varphi = \varphi(x)$, $N(x) = L(x) - J_2 M(x)$. We have the following equivalence conditions*

(a) $\bar{a}(\varphi)\varphi' = a(x)$, $\bar{c}(\varphi) = c(x)$, $\bar{p}(\varphi){\varphi'}^2 = p(x)$, $\bar{q}(\varphi){\varphi'}^2 = q(x)$,
$\bar{r}(\varphi){\varphi'}^2 = r(x) - \frac{\varphi''}{\varphi'}$, $\bar{q}(\varphi){\varphi'}^2 = q(x) + J_2(c(x))\frac{\varphi''}{\varphi'}$,
$\bar{t}(\varphi){\varphi'}^2 = t(x) + L'' - J_2(c(x))M'' - p(x)L - q(x)M - r(x)L'(x) - s(x)M'$;

(b) $\bar{c}(\varphi)\varphi' = c(x) - aN(x)$, $\bar{q}(\varphi)\varphi' = q(x) - \frac{\varphi''}{\varphi'} - aBM'(x) - 2aEN'(x)$,
$$\bar{p}(\varphi){\varphi'}^2 = p(x) + aBM'(x)N'(x) + aE(N'(x))^2 - Bc'(x)M'(x) - q(x)N'(x) + N''(x);$$

(c) $\bar{b}(\varphi)\varphi' = b(x) - L'(x)$, $\bar{a}(\varphi)\varphi' = a(x) - M'(x)$;

(d) $\frac{1}{\bar{\alpha}(\varphi)}\varphi' = \frac{1}{\alpha(x)}$, $\bar{\beta}(\varphi) = \beta(x) - aL(x) - bM(x) - \alpha(x)L'(x)$;

(e) $\bar{\alpha}(\varphi)\varphi' = \alpha(x) + CN'(x)$, $\bar{\beta}(\varphi) = \beta(x) - aL(x) - bM(x) - \frac{1}{C}\ln|\varphi'|$;

corresponding to the functions f of Theorem 3.

We discuss only the conditions (b) of Corollary 3 relevant to $f = J_2 z'' + \mathscr{K}(F)(c'(x) + ay' + bz')^2 + B(c'(x) + ay' + bz')z' + \frac{1}{a}E(ay + bz)'^2 + q(x)(ay + bz)' + p(x)$, $A = \mathscr{G}(F)\cdot(c'(x) + ay' + bz')$, $F = ay + bz + c(x)$, $c''(x) - q(x)c'(x) + ap(x) + E{c'}^2(x) = 0$, $a \neq 0$ $(B, E \in \mathbb{R})$, for example.

We get $u'' = \mathscr{K}(F)(c'(x) + au')^2 + B(c'(x) + au')z' + aE{u'}^2 + q(x)u' + p(x) = g(x, u, z, u'z')$, $A = \mathscr{G}(F)\cdot(c'(x) + au')$, $F = au + c(x)$, regarding to $b + aJ_2 = 0$, $u = y - J_2 z$ (see $(\iota)_2$).

Then $F = \bar{F} \iff a\bar{u} + \bar{c}(\bar{x}) = au + aN(x) + \bar{c}(\bar{x}) = au + c(x) \iff \bar{c}(\varphi) = c(x) - aN(x)$ and $\omega_0 = \bar{\omega}_0$ follows from $F = \bar{F}$ and $\bar{c}'(\varphi)\varphi' = c'(x) - aN'(x)$. Moreover, $\omega_1 = \bar{\omega}_1 \iff \bar{g}_{u'}\varphi' = g_{u'} - \frac{\varphi''}{\varphi'} \iff \bar{q}(\varphi)\varphi' = q(x) - \frac{\varphi''}{\varphi'} - aBM'(x) - 2aEN'(x)$. The condition $\bar{u}''{\varphi'}^2 = u'' + N'' - \frac{\varphi''(x)}{\varphi'(x)}(u' + N')$ together with the above conditions lead to $\bar{p}(\varphi){\varphi'}^2 = p(x) + aBM'(x)N'(x) + aE(N'(x))^2 - Bc'(x)M'(x) - q(x)N'(x) + N''(x)$. The condition $c''(x) - q(x)c'(x) + ap(x) + E{c'}^2(x) = 0$ is true in relation to $\bar{c}''{\varphi'}^2 - \bar{q}\varphi'\bar{c}'\varphi' + a\bar{p}{\varphi'}^2 + E(\bar{c}'\varphi')^2 = (c' - aN')' - \frac{\varphi''}{\varphi'}(c' - aN') - (q - \frac{\varphi''}{\varphi'} - aBM' - 2aEN')(c' - aN') + a(p + aBM'N' + aE{N'}^2 - Bc'M' - qN' + N'') + E(c' - aN')^2 = c'' - qc' + ap + E{c'}^2 = 0$.

6. The isomorphic pseudogroup

The results obtained for the pseudogroup of all transformations

$$\Phi(x,y,z)=(\bar{x},\bar{y},\bar{z})=(\varphi(x),y+L(x),z+M(x)),\qquad \Phi\colon \mathscr{D}(\Phi)\to\mathscr{R}(\Phi)$$

can be related to the isomorphic pseudogroup of all transformations

$$\Phi_1(x,v,w)=(\bar{x},\bar{v},\bar{w})=(\varphi(x),P(x)v,Q(x)w),\qquad \Phi_1:\mathscr{D}(\Phi_1)\to\mathscr{R}(\Phi_1) \tag{69}$$

by using the logarithmic y and z ($\bar{y}$ and $\bar{z}$) scales

$$y=\ln|v|,\quad z=\ln|w|\qquad (\bar{y}=\ln|\bar{v}|,\quad \bar{z}=\ln|\bar{w}|) \tag{70}$$

for $P(x)=\mathrm{e}^{L(x)}\neq 0$, $Q(x)=\mathrm{e}^{M(x)}\neq 0$ (i.e., $L=\ln|P|, M=\ln|Q|$). Altogether

$$y'=\frac{v'}{v},\quad z'=\frac{w'}{w},\quad y''=\left(\frac{v'}{v}\right)'=\frac{v''}{v}-\left(\frac{v'}{v}\right)^2,\quad z''=\left(\frac{w'}{w}\right)'=\frac{w''}{w}-\left(\frac{w'}{w}\right)^2.$$

Every equation $y''=f$ is then transformed into some equation

$$\left(\tfrac{v'}{v}\right)'=f\Big(x,\ln|v|,\ln|w|,\tfrac{v'}{v},\tfrac{w'}{w},\left(\tfrac{w'}{w}\right)'\Big)=h(x,v,w,v',w',w'') \tag{71}$$

equivalent to the equation

$$v''=g(x,v,w,v',w',w'')=\frac{{v'}^2}{v}+h(x,v,w,v',w',w'')v \tag{72}$$

and the equivalence of equations $v''=g$ and $\bar{v}''=\bar{g}$ follows from the equivalence conditions for $y''=f$ and $\bar{y}''=\bar{f}$ by using the logarithmic scales. Moreover

$$w(x)=\mathrm{e}^{z(x)}=\mathrm{e}^{y(\xi(x))}=v(\xi(x))$$

with the delay function ξ considered in the delay-differential equation (1) and our approach is fully applicable to all second-order differential equations of this kind.

Example 1. In the pseudogroup (8), the equivalence conditions for

$$y''=f=(y'+b(x))(z'+a(x))\mathscr{F}(F)+(z''+a'(x))F-b'(x),\qquad F=\frac{y'+b(x)}{z'+a(x)}$$

and $\bar{y}''=\bar{f}$ are given by $\bar{b}(\varphi)\varphi'=b-L'$, $\bar{a}(\varphi)\varphi'=a-M'$ in accordance with (c) of Theorem 3 and Corollary 3. The equation $y''=f$ is transformed by logarithmic scales (70) into the equation

$$\left(\frac{v'}{v}\right)'=\left(\frac{v'}{v}+b(x)\right)\left(\frac{w'}{w}+a(x)\right)\mathscr{F}(F)+\left(\left(\frac{w'}{w}\right)'+a'(x)\right)F-b'(x), \tag{73}$$

$$F=\frac{\frac{v'}{v}+b(x)}{\frac{w'}{w}+a(x)},$$

equivalent to the equation

$$v'' = \frac{v'^2}{v} + (v' + b(x)v)\left(\frac{w'}{w} + a(x)\right)\mathscr{F}(F) + v\left(\frac{w''}{w} - \frac{w'^2}{w^2} + a'(x)\right)F - b'(x)v.$$

Assuming the pseudogroup (69), we get the equivalence conditions

$$\bar{b}(\varphi)\varphi' = b - \frac{P'}{P}, \qquad \bar{a}(\varphi)\varphi' = a - \frac{Q'}{Q} \tag{74}$$

for the equations $v'' = g$ and $\bar{v}'' = \bar{g}$. Indeed, by using the relations

$$\frac{v'}{v} = \frac{\bar{v}'}{\bar{v}}\varphi' - \frac{P'}{P} = \frac{\bar{v}'}{\bar{v}}\varphi' + \bar{b}\varphi' - b, \qquad \frac{w'}{w} = \frac{\bar{w}'}{\bar{w}}\varphi' - \frac{Q'}{Q} = \frac{\bar{w}'}{\bar{w}}\varphi' + \bar{a}\varphi' - a$$

we obtain

$$\begin{aligned}
&F = \frac{\frac{v'}{v}+b(x)}{\frac{w'}{w}+a(x)} = \frac{\frac{\bar{v}'}{\bar{v}}+\bar{b}}{\frac{\bar{w}'}{\bar{w}}+\bar{a}} = \bar{F},\\
&\left(\frac{v'}{v}+b(x)\right)\left(\frac{w'}{w}+a(x)\right) = \left(\frac{\bar{v}'}{\bar{v}}+b(x)\right)\left(\frac{\bar{w}'}{\bar{w}}+a(x)\right)\varphi'^2,\\
&\left(\frac{v'}{v}\right)' = \left(\frac{\bar{v}'}{\bar{v}}\right)'\varphi'^2 + \frac{\bar{v}'}{\bar{v}}\varphi'' + \bar{b}'\varphi'^2 + \bar{b}\varphi'' - b',\\
&\left(\frac{w'}{w}\right)' + a' = \left(\frac{\bar{w}'}{\bar{w}}\right)'\varphi'^2 + \frac{\bar{w}'}{\bar{w}}\varphi'' + \bar{a}'\varphi'^2 + \bar{a}\varphi'',
\end{aligned}$$

the equation (73) becomes

$$\begin{aligned}
&\left(\frac{\bar{v}'}{\bar{v}}\right)'\varphi'^2 + \frac{\bar{v}'}{\bar{v}}\varphi'' + \bar{b}'\varphi'^2 + \bar{b}\varphi'' - b'\\
&= \left(\frac{\bar{v}'}{\bar{v}}+b(x)\right)\left(\frac{\bar{w}'}{\bar{w}}+a(x)\right)\varphi'^2\mathscr{F}(F) + \left(\left(\frac{\bar{w}'}{\bar{w}}\right)'+\bar{a}'\right)F\varphi'^2 + \left(\frac{\bar{w}'}{\bar{w}}+\bar{a}\right)F\varphi'' - b'
\end{aligned}$$

and $\left(\frac{\bar{w}'}{\bar{w}} + \bar{a}\right)F = \frac{\bar{v}'}{\bar{v}} + \bar{b}$. The resulting equation

$$\left(\frac{\bar{v}'}{\bar{v}}\right)' = \left(\frac{\bar{v}'}{\bar{v}} + \bar{b}\right)\left(\frac{\bar{w}'}{\bar{w}} + \bar{a}\right)\mathscr{F}(F) + \left(\left(\frac{\bar{w}'}{\bar{w}}\right)' + \bar{a}'\right)F - \bar{b}'$$

is the equation with bars corresponding to the equation (73). We have proved that the equations $v'' = g$ and $\bar{v}'' = \bar{g}$ are equivalent equations, assuming the pseudogroup (69). Through direct verification we can prove the following assertion.

Corollary 4. *Let us consider the global transformation* (see [8])

$$\bar{x} = \varphi(x) \in J \subset \mathbb{R}, \quad \bar{v}(\bar{x}) = \bar{v}(\varphi(x)) = P(x)v(x), \qquad x \in I \subset \mathbb{R}$$

such that $\varphi \in C^2(I)$, $\varphi'(x)P(x) \neq 0$ *on* I *and* $\varphi(I) = J$. *Then* $\bar{v}(\bar{\xi}(\bar{x})) = \bar{v}(\bar{\xi}(\varphi(x))) = \bar{v}(\varphi(\xi(x))) = P(\xi(x))v(\xi(x))$ *is satisfied for* $\bar{\xi}(\bar{x}) = \varphi(\xi(\varphi^{-1}(\bar{x})))$, $\bar{x} \in J$, *and any differential equation*

$$\left(\frac{v'(x)}{v(x)} + b(x)\right)' = \left(\frac{v'(x)}{v(x)} + b(x)\right)\left(\frac{(v(\xi(x)))'}{v(\xi(x))} + a(x)\right)\mathscr{F}(F) + \left(\frac{(v(\xi(x)))'}{v(\xi(x))} + a(x)\right)' F$$

with $F = \frac{v'(x)/v(x)+b(x)}{(v(\xi(x))'/v(\xi(x))+a(x)}$ *and coefficients* $a, b \in C^1(I)$ *is globally transformed into an equation*

$$\left(\frac{\bar{v}'(\bar{x})}{\bar{v}(\bar{x})} + \bar{b}(\bar{x})\right)' = \left(\frac{\bar{v}'(\bar{x})}{\bar{v}(\bar{x})} + \bar{b}(\bar{x})\right)\left(\frac{(\bar{v}(\bar{\xi}(\bar{x})))'}{\bar{v}(\bar{\xi}(\bar{x}))} + \bar{a}(\bar{x})\right)\mathscr{F}(F) + \left(\frac{(\bar{v}(\bar{\xi}(\bar{x})))'}{\bar{v}(\bar{\xi}(\bar{x}))} + \bar{a}(\bar{x})\right)' F$$

on the whole intervals I, J *of definition. Moreover,*

$$\bar{b}(\varphi(x))\varphi'(x) = b(x) - \frac{P'(x)}{P(x)}, \qquad \bar{a}(\varphi(x))\varphi'(x) = a(x) - \frac{(P(\xi(x)))'}{P(\xi(x))},$$

$$F = \frac{v'(x)/v(x) + b(x)}{(v(\xi(x))'/v(\xi(x)) + a(x)} = \frac{\bar{v}'(\bar{x})/\bar{v}(\bar{x}) + \bar{b}(\bar{x})}{(\bar{v}(\bar{\xi}(\bar{x}))'/\bar{v}(\bar{\xi}(\bar{x})) + \bar{a}(\bar{x})}$$

is satisfied on I *and* J, *respectively.*

For example, differential equations

$$\left(\frac{v'(x)}{v(x)}\right)' = \frac{v'(x)}{v(x)}\frac{(v(x/\mathrm{e}))'}{v(x/\mathrm{e})}\mathscr{F}(F) + \left(\frac{(v(x/\mathrm{e}))'}{v(x/\mathrm{e})}\right)' F,$$

$$\left(\frac{\bar{v}'(\bar{x})}{\bar{v}(\bar{x})}\right)' = \left(\frac{\bar{v}'(\bar{x})}{\bar{v}(\bar{x})} - 1\right)\left(\frac{\bar{v}'(\bar{x}-1)}{\bar{v}(\bar{x}-1)} - 1\right)\mathscr{F}(F) + \left(\frac{\bar{v}'(\bar{x}-1)}{\bar{v}(\bar{x}-1)}\right)' F,$$

$F = v'(x)v(x/\mathrm{e})/v(x)(v(x/\mathrm{e}))'$, are globally transformable by means of $\bar{x} = \varphi(x) = \ln x$, $\bar{v}(\bar{x}) = xv(x)$ on $I = (0, \infty)$ and $J = \mathbb{R}$, respectively.

Analogous results can be derived for the remaining cases of relevant Theorems. Only nonoscillatory differential equations are considered by using the logarithmic scales. Direct investigation of the pseudogroup (69) involving in results also oscillatory equations is an open problem.

Comments

Transformation properties of differential equations belong to the central part in the geometrical theory of differential equations, see the recent surveying booklet [KAMRAN, N.: *Selected Topics in the Geometrical Theory of Differential Equations*, CBMS Regional Conference Series in Math., Nr 96, AMS 2002] and extensive literature therein. In particular the classical results are essentially improved [SATO, H.—YOSHIKAWA, A. Y.: *Third-order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan **50** (1998), 993–1013]; [SATO, H.: *Orbit decomposition of space of differential equations.* In: UK-Japan Winter school 2004 — Geometry and Analysis Towards Quantum Theory, pp. 77–88; Sem. Math. Sci. 30, Keio Univ., Yokohama, 2004]; [YOSHIKAWA, A. Y.: *Equivalence problem of third-order ordinary differential equations*, Internat. J. Math. **17** (2006), 1103–1125]. This symmetry-based methods are adapted for ordinary difference equations in order to obtain the reductions of order and explicit solutions, that is, a counterpart to the classical Lie's theory [HYDON, P. E.: *Symmetries and first integrals of ordinary difference equations*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2000), 2835–2855] (see also literature therein). We deal with the intermediate problem and investigate the differential equations with deviations. The article should be regarded as a mere modest preparation to deeper qualitative study of such equations which admit large symmetries.

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