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Periodic Solutions and Approximate Symmetries

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Abstract. Assuming that a dynamical system (DS) has a periodic solution with period T in the neighborhood of an equilibrium, a new DS can be obtained which governs approximately the motion of trajectories in the neighborhood of the periodic orbit. Approximate symmetries and first integrals have been discussed for this DS. The results have been demonstrated on the Helmholtz oscillator.

Keywords: Dynamical systems, approximate symmetries, Noether's theorem.

1. Introduction

Tools of classical group analysis usually do not allow one to study the physically relevant damped, driven oscillators. This difficulty can be circumvented by resorting to the approximate group analysis which has been developed by Baikov et al. [1]. Here we adopt the geometrical approach to study the approximate symmetries of the approximate DS obtained in the neighborhood of the periodic solution.

Here we consider the DS in the form

$$\dot{y}^i = L^{ij} y^j + F^i(\mathbf{y}, t) \quad (i = 1, \dots n), \tag{1}$$

where ($\dot{}$) stands for derivative with respect to t, $\mathbf{L}$ is a constant coefficient matrix. We assume that the DS (1) has a periodic solution $\hat{\mathbf{y}}(t)$ with period T in the neighborhood of the equilibrium point $\mathbf{u}_0$. We now would like to obtain a new DS in the neighborhood of $\hat{\mathbf{y}}(t)$. To achieve this we consider

$$\mathbf{y} = \hat{\mathbf{y}} + \varepsilon \mathbf{x}, \quad \varepsilon \ll 1, \quad \hat{\mathbf{y}}(t + T) = \hat{\mathbf{y}}(t). \tag{2}$$

Substituting (2) into (1), recalling that $\hat{\mathbf{y}}(t)$ is a solution of (1) and expanding $\mathbf{F}(\mathbf{y}, t)$ at $\hat{\mathbf{y}}$ we obtain

$$\begin{aligned} \dot{x}^i \; = \; & \left(L^{ij} + \left.\frac{\partial F^i}{\partial y^j}\right|_{\mathbf{y}=\hat{\mathbf{y}}} \right) x^j + \left(\frac{\varepsilon}{2!} \left.\frac{\partial^2 F^i}{\partial y^j \partial y^k}\right|_{\mathbf{y}=\hat{\mathbf{y}}} \right) x^j x^k \\ & + \left(\frac{\varepsilon^2}{3!} \left.\frac{\partial^3 F^i}{\partial y^j \partial y^k \partial y^l}\right|_{\mathbf{y}=\hat{\mathbf{y}}} \right) x^j x^k x^l + \cdots, \end{aligned} \tag{3}$$

where we assume the summation convention on the repeated indices. Sometimes it is not possible to find the exact periodic solution of (1) analytically. Therefore we must consider the approximate periodic solution:

$$\hat{\mathbf{y}} = \mathbf{u}_0 + \varepsilon \left(\sum_{s=0}^{q} \varepsilon^s \mathbf{u}_{s+1}(t) \right). \tag{4}$$

Notice that when $\varepsilon = 0$, $\mathbf{u}^0$ is the equilibrium point (constant solution to (1)). Expanding related quantities in (3) according to (4) and rearranging as to increasing order of ε we are led to:

$$\dot{x}^i = f_0^i(\mathbf{x}) + \varepsilon f_1^i(\mathbf{x}, t) + \varepsilon^2 f_2^i(\mathbf{x}, t) + \cdots + \varepsilon^{(p)} f_{(p)}^i(\mathbf{x}, t), \tag{5}$$

where

$$\begin{aligned}
f_0^i(\mathbf{x}) &= \left(L^{ij} + \left.\frac{\partial F^i}{\partial y^j}\right|_{\mathbf{u}_0} \right) x^j, \\
f_1^i(\mathbf{x}, t) &= \left.\frac{\partial}{\partial \hat{y}^\alpha} \left(\left.\frac{\partial F^i}{\partial y^j}\right|_{\hat{\mathbf{y}}} \right)\right|_{\mathbf{u}_0} u_1^\alpha x^j + \frac{1}{2!} \left.\frac{\partial^2 F^i}{\partial y^j \partial y^k}\right|_{\mathbf{u}_0} x^j x^k, \\
f_2^i(\mathbf{x}, t) &= \left.\frac{\partial}{\partial \hat{y}^\alpha} \left(\left.\frac{\partial F^i}{\partial y^j}\right|_{\hat{\mathbf{y}}} \right)\right|_{\mathbf{u}_0} u_2^\alpha x_j + \left.\frac{\partial^2}{\partial \hat{y}^\alpha \partial \hat{y}^\beta} \left(\left.\frac{\partial F^i}{\partial y^j}\right|_{\hat{\mathbf{y}}} \right)\right|_{\mathbf{u}_0} u_1^\alpha u_1^\beta x^j \\
&\quad + \frac{1}{2!} \left.\frac{\partial}{\partial \hat{y}^\alpha} \left(\left.\frac{\partial^2 F^i}{\partial y^j \partial y^k}\right|_{\hat{\mathbf{y}}} \right)\right|_{\mathbf{u}_0} u_1^\alpha x^j x^k + \frac{1}{3!} \left.\frac{\partial^3 F^i}{\partial y^j \partial y^k \partial y^l}\right|_{\mathbf{u}_0} x^j x^k x^l.
\end{aligned}$$

For the sake of brevity we have omitted the coefficients of the higher order of ε.

Summation convention has been suspended for the indices in the parenthesis in (5).

In Section 2 we discuss the approximate symmetries of order p for the DS (5). Assuming that the $L^{ij} + (\partial F^i/\partial y^j)|_{\mathbf{u}^0}$ is semisimple we give a general formula for the approximate symmetries. For the case of Hamiltonian DSs we discuss the approximate version of Noether's theorem geometrically. Approximate first integrals can be used to test the numerical integration schemes for the DSs. Not only they provide initial conditions for the periodic orbits, but they also enable us to study the local behavior in the phase space. Finally we demonstrate our results on the damped-driven Helmholtz oscillator which arises in modeling of ear drums [2] and ship capsizing [3].

2. Approximate Symmetries of Approximate DSs

In Section 1 we have seen one way of obtaining approximate DSs, i.e. the DS given in (5). Determining equations for the symmetry vector fields (VFs) of the DS can be easily obtained from the determining equations for the symmetries of evolution equations [4]. Invoking the definition of the approximate symmetries given in [1], we are led to the following determining equations:

$$[\mathbf{D}, \mathbf{X}] = O(\varepsilon^{p+1}), \tag{6}$$

where [ ] is the Lie bracket,

$$\mathbf{D} = \mathbf{T} + \mathbf{F}_0 + \varepsilon \mathbf{F}_1 + \cdots + \varepsilon^{(p)} \mathbf{F}_{(p)}, \quad \mathbf{T} = \frac{\partial}{\partial t},$$

$$\mathbf{F}_b = f_b^l \frac{\partial}{\partial x^l}, \quad (l = 1, \ldots, n; \;\; b = 0, \ldots, p),$$

and

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \cdots + \varepsilon^{(p)} \mathbf{X}_{(p)}, \quad \mathbf{X}_c = \eta_c^l(\mathbf{x}, t) \frac{\partial}{\partial x^l} \quad (c = 0, \ldots, p).$$

Evaluation of (6) in ascending order of ε gives:

$$[\mathbf{D}_0, \mathbf{X}_0] = 0, \tag{7}$$

$$[\mathbf{D}_0, \mathbf{X}_1] = [\mathbf{X}_0, \mathbf{F}_1], \tag{8}$$

$$[\mathbf{D}_0, \mathbf{X}_2] = [\mathbf{X}_1, \mathbf{F}_1] + [\mathbf{X}_0, \mathbf{F}_2], \tag{9}$$

$$\vdots \quad \vdots \quad \vdots$$

$$[\mathbf{D}_0, \mathbf{X}_p] = [\mathbf{X}_{p-1}, \mathbf{F}_1] + \cdots + [\mathbf{X}_1, \mathbf{F}_{p-1}] + [\mathbf{X}_0, \mathbf{F}_p], \tag{10}$$

where $\mathbf{D}_0 = \mathbf{T} + \mathbf{F}_0$. Equations (7–10) suggest the following procedure for the calculation of approximate symmetry VFs: solving the PDEs obtained from (7) we obtain the exact symmetry VF $\mathbf{X}_0$ of the unperturbed part of (5) (i.e., when $\varepsilon = 0$ in (5)). Using this on the RHS of (8) we obtain a set of PDEs to determine the first-order approximate symmetry VF $\mathbf{X}_1$. Substituting the latter into (9) determines the RHS and it yields a new set of PDEs to determine $\mathbf{X}_2$. Notice that the RHS of (7–10) are determined from the previous steps. Proceeding in this manner we finally obtain a set of PDEs to determine $\mathbf{X}_p$. Although this procedure is algorithmic, it is cumbersome to solve the PDEs obtained. The following lemma and theorem will he lp to overcome this difficulty.

LEMMA 1. *When the unperturbed part of (5) is semisimple, Lie symmetries of this part take the form:*

$$\begin{aligned} \mathbf{X}_0 \;=\; S^{jl} \Bigg\{ & C_0^{1(l)} \, \mathrm{e}^{\lambda_{(l)} t} + C_0^{2(l)} \lambda_{(l)} T^{lm} x^m + C_0^{3(l)} T^{lm} x^m \\ & + C_0^{4(l)} \sum_{\substack{m=1,\ldots,n \\ m \neq l}} \mathrm{e}^{(\lambda_{(l)} - \lambda_m) t} T^{mr} x^r \\ & + \sum_{\substack{\Lambda_l = \lambda_l - (s_1 \lambda_1 + \cdots + s_n \lambda_n) \\ s_1 + \cdots + s_n \geq 2, s_1, \ldots, s_n \geq 0}} C_{0 s_1 \ldots s_n}^{\Lambda_{(l)}} \, \mathrm{e}^{\Lambda_{(l)} t} (S^{1 l_1} x^{l_1})^{s_1} \ldots (S^{n l_n} x^{l_n})^{s_n} \Bigg\} \frac{\partial}{\partial x^j} , \end{aligned} \tag{11}$$

where $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of the linear part in (5) i.e.,* $(L^{ij} + (\partial F^i / \partial y^j)|_{\mathbf{u}_0})$. $\mathbf{S}$ *is the similarity transformation matrix formed with the eigenvectors corresponding to* $\lambda_1, \ldots, \lambda_n$.

And $\mathbf{T} = \mathbf{S}^{-1}$ *and* $C_0^{1(l)}, \ldots, C_{0s_1\ldots s_n}^{\Lambda_l}$ *are the group parameters.*

Proof of Lemma 1 can also be found in [5]. Notice that the last summation sign indicates that the sum will be carried on the terms which satisfy the resonance relation:

$$\Lambda_j + (s_1\lambda_1 + \cdots + s_n\lambda_n) - \lambda_j = 0 \quad \sum_{k=1}^{n} s_k \geq 2, \quad s_k \geq 0.$$

In order to have a better understanding of the approximate symmetry VFs we continue our discussion with complex variables. Introducing the transformation

$$\mathbf{x} = \mathbf{S}\mathbf{z} \tag{12}$$

to (5) we find:

$$\dot{\mathbf{z}} = \mathbf{T}\big(\mathbf{f}_0(\mathbf{z}) + \varepsilon\mathbf{f}_1(\mathbf{z}, t) + \cdots + \varepsilon^{(p)}\mathbf{f}_{(p)}(\mathbf{z}, t)\big). \tag{13}$$

THEOREM 2. *Assume that the linear part of (5) is semisimple. Also assume that the components of the Lie bracket are given by:*

$$[\bar{\mathbf{X}}_0, \bar{\mathbf{F}}_1] = V^j(\mathbf{z}, t)\frac{\partial}{\partial z^j}, \tag{14}$$

where

$$\bar{\mathbf{X}}_0 = \bar{\eta}_0^l(\mathbf{z}, t)\frac{\partial}{\partial z^l}, \quad \bar{\mathbf{F}}_1 = \bar{f}_1^l(\mathbf{z}, t)\frac{\partial}{\partial z^l}, \quad \bar{f}_1^j = T^{jl} f_1^l(\mathbf{S}\mathbf{z}, t)$$

and

$$\begin{aligned} V^j(\mathbf{z}, t) \; = \; & K_{r_1}^{1(j)}\, \mathrm{e}^{\phi_{(j)r_1}t} + K_{r_2}^{2(j)}\, \mathrm{e}^{\theta_{(j)r_2}t} + K_{r_3}^{3j} z^{r_3} \\ & + \sum_{s_1+\cdots+s_n\geq 2} K_{s_1\ldots s_n}^{\kappa_j}\, \mathrm{e}^{\kappa_j t}(z^1)^{s_1}\ldots(z^n)^{s_n} \quad (r_1, r_2, r_3 = 1, \ldots, n). \end{aligned}$$

If one of the resonance relations

$$\phi_{jr_1} - \lambda_j = 0, \quad \theta_{jr_2} + \lambda_{r_2} - \lambda_j = 0, \quad \lambda_{r_3} - \lambda_j = 0$$

and

$$\kappa_j + s_1\lambda_1 + \cdots + s_{(n)}\lambda_{(n)} - \lambda_j = 0 \tag{15}$$

holds, then the symmetries corresponding to their parameters appearing in the K*s will be broken at the first order of approximation.*

Proof of Theorem 2 can also be found in [5]. This theorem also enables one to calculate the first-order approximate symmetry VFs with the infinitesimals

$$\begin{aligned} \bar{\eta}_1^j \; = \; & \frac{K_{r_1}^{1(j)}}{\phi_{j(r_1)} - \lambda_{(j)}}\, \mathrm{e}^{\phi_{(j)r_1}t} + \frac{K_{r_2}^{2(j)}}{\theta_{j(r_2)} + \lambda_{(r_2)} - \lambda_{(j)}}\, \mathrm{e}^{\theta_{(j)r_2}t} z^{(r_2)} + \frac{K_{r_3}^{3(j)}}{\lambda_{(r_3)} - \lambda_{(j)}} z^{(r_3)} \\ & + \frac{1}{\kappa_j + s_1\lambda_1 + \cdots + s_n\lambda_n - \lambda_j} \sum_{s_1+\cdots+s_n\geq 2} K_{s_1\ldots s_n}^{\kappa_j}\, \mathrm{e}^{\kappa_j t}(z^1)^{s_1}\ldots(z^n)^{s_n}. \end{aligned} \tag{16}$$

One must employ the following transformations to find the first-order approximate symmetry VFs of (5).

$$\eta_0^j = S^{jl}\bar{\eta}_0^l(\mathbf{S}^{-1}\mathbf{x}, t) \quad \text{and} \quad \eta_1^j = S^{jl}\eta_1^l(\mathbf{S}^{-1}\mathbf{x}, t). \tag{17}$$

Notice that the symmetry-breaking occurs due to resonances (15) which sets related parameters from the set $\{C_0^{1l}, \ldots, C_{0s_1\ldots s_n}^{\Lambda_l}\}$ to zero. This explains why most of the nonlinear-nonautonomous DS do not admit exact symmetries.

One of the salient features of the approximate symmetry VFs is the inheritance property. Perturbed DS (5) inherits the symmetries of the unperturbed part as approximate symmetries [5], i.e.,

$$\mathbf{X} = \varepsilon^p \eta_0^l(\mathbf{x}, t, C_0^{1l}, \ldots, C_{0s_1\ldots s_n}^{\Lambda_l})\frac{\partial}{\partial x^l}. \tag{18}$$

This result can be obtained from (7–10).

3. Application to Helmholtz Oscillator

Helmholtz oscillator arises in modeling of ear drums [3] and ship capsizing [4]. And it is given by the DS:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -y_1^2 + \varepsilon(A\cos\omega t - \delta y_2) \end{bmatrix}, \tag{19}$$

where A and ω are the amplitude and the angular frequency of the forcing, respectively, and δ is the damping coefficient. When $\varepsilon = 0$, one of the equilibrium points is located at $\mathbf{u}_0 = (0, 0)^T$ and it is a center type. An approximate periodic solution to (19) has been found to be:

$$\hat{\mathbf{y}}(t) = \mathbf{u}_0 + \varepsilon\mathbf{u}_1(t) + \varepsilon^2\mathbf{u}_2(t), \tag{20}$$

where

$$\begin{aligned} \mathbf{u}_1 &= (u_1^1, u_1^2)^T = (\Gamma\cos\omega t, -\Gamma\omega\sin\omega t)^T, \\ \mathbf{u}_2 &= (u_2^1, u_2^2)^T = \left(\frac{\delta\Gamma\omega}{1-\omega^2}\sin\omega t - \frac{\Gamma^2}{2(1-4\omega^2)}\cos 2\omega t - \frac{\Gamma^2}{2}, \dot{u}_2^1\right)^T, \end{aligned} \tag{21}$$

and $\Gamma = A/(1-\omega^2)$. Approximate DS (5) for (19) reads as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - \varepsilon(\delta x_2 + 2u_1^1 x_1 + x_1^2) + \varepsilon^2(-2u_2^1 x_1) + O(\varepsilon^3). \end{aligned} \tag{22}$$

Substituting (21) into (22) yields

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - \varepsilon(2\Gamma\cos\omega t\, x_1 + \delta x_2 + x_1^2) \\ &\quad - \varepsilon^2\left(\frac{2\delta\Gamma}{1-\omega^2}\sin\omega t - \frac{\Gamma^2}{1-4\omega^2}\cos 2\omega t - \Gamma^2\right)x_1. \end{aligned} \tag{23}$$

We assume that $\omega \neq 1,\ 1/2$. Notice that the unperturbed part of (23) is linear, autonomous and semisimple with eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Therefore we can appeal to the procedure given in Section 2. First-order approximate symmetry VFs have been found to be:

$$\begin{aligned} C_0^{21} : \mathbf{X}_1^1 \;=\; & \left\{x_2 + \varepsilon\left(\frac{2\Gamma}{\omega^2-4}(\omega \sin\omega t\, x_1 + 2\cos\omega t\, x_2) + \frac{\delta}{2}x_1\right)\right\}\frac{\partial}{\partial x_1} \\ & + \left\{-x_1 + \varepsilon\left(\frac{2\Gamma}{\omega^2-4}(2\cos\omega t\, x_1 - \omega\sin\omega t\, x_2) - \frac{\delta}{2}x_2 - x_1^2\right)\right\}\frac{\partial}{\partial x_2}, \end{aligned} \tag{24}$$

$$C_0^{31} : \mathbf{X}_2^1 = \left\{x_1 + \varepsilon\left(\frac{\delta}{2}x_2 - \frac{1}{3}x_1^2 - \frac{2}{3}x_2^2\right)\right\}\frac{\partial}{\partial x_1} + \left\{x_2 + \varepsilon\left(-\frac{\delta}{2}x_1 + \frac{2}{3}x_1x_2\right)\right\}\frac{\partial}{\partial x_2}, \tag{25}$$

$$\begin{aligned} C_0^{41} : \mathbf{X}_3^1 \;=\; & \Big\{\cos 2t\, x_1 - \sin 2t\, x_2 + \varepsilon\frac{12\Gamma}{D}\big[\big(2\sin(\omega-2)t + w\sin(\omega-2)t \\ & + 2\sin(\omega+2)t - \omega\sin(\omega+2)t\big)x_2 \\ & + 4(\omega^2-4)(4\omega x_2 - 3\Gamma\sin\omega t)\sin 2t\, x_1 \\ & - \omega(\omega^2-4)(2x_1^2 - 3\delta x_2 - 12x_2^2)\cos 2t\big]\Big\}\frac{\partial}{\partial x_1} \\ & + \Big\{-(\sin 2t\, x_1 + \cos 2t\, x_2) + \frac{\varepsilon}{D}\big[-\big(24\Gamma(\omega^2-2)\sin\omega t \\ & + \omega(\omega^2-4)(3\delta - 4x_2)\big)\cos\omega t\, x_1 \\ & + 2\big(12\Gamma\omega\cos\omega t\, x_1 + 12\omega x_1^2 - 3\omega^3 x_1^2 \\ & - 24\Gamma\sin\omega t x_2 + 6\Gamma\omega^2\sin\omega t\, x_2 + 16\omega x_2^2 - 4\omega^3 x_2^2\big)\sin 2t\big]\Big\}\frac{\partial}{\partial x_2}, \end{aligned} \tag{26}$$

where $D = 6\omega(\omega^2-4)$.

According to terminology coined in [1], first-order approximate symmetry VFs given in (24–26) are stable symmetries. Due to the inheritance property one can also write down the unstable symmetries without any calculation. Some unstable symmetries are:

$$\begin{aligned} \mathbf{X}_1^u \;=\; & \varepsilon\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right), \quad \mathbf{X}_2^u = \varepsilon\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right) \\ \mathbf{X}_3^u \;=\; & \varepsilon\left((\cos 2t\, x_1 - \sin 2t\, x_2)\frac{\partial}{\partial x_1} - (\sin 2t\, x_1 + \cos 2t\, x_2)\frac{\partial}{\partial x_2}\right). \end{aligned} \tag{27}$$

Due to symmetry-breaking, parameters C_0^{31} and C_0^{41} vanish at second-order analysis. Remaining second-order approximate symmetry VF takes the form:

$$\begin{aligned} C_0^{21} :\ \mathbf{X}_1^2 \;=\; & \mathbf{X}_1^1 + \frac{\varepsilon^2}{D_1}\Big\{\Big[\Gamma\,(\delta P_1\cos\omega t + \Gamma\omega P_2\sin 2\omega t)\,x_1 \\ & + 3\Gamma(\Gamma P_5 + \Gamma P_6\cos 2\omega t + \delta P_7\sin\omega t)x_2 + \Gamma\omega P_4\sin\omega t\, x_2^2 \end{aligned}$$

$$
\begin{aligned}
&+ 240\Gamma\omega^2(4\omega^2-1)\cos\omega t\, x_1x_2 \\
&+ \left(\frac{\delta}{3}D_1 - 360\Gamma\omega(4\omega^2-1)\sin\omega t\right)x_2^2\Big]\frac{\partial}{\partial x_1} \\
&+ \big[3\Gamma(P_5 + \Gamma P_8\cos 2\omega t + \delta P_{10}\sin\omega t)x_1 \\
&- \Gamma(\delta P_1\cos\omega t + \Gamma\omega P_2\sin 2\omega t)x_2 + 3\Gamma P_9\cos\omega t\, x_1^2 \\
&- 2(\delta P_3 + \Gamma\omega P_4\sin\omega t)x_1x_2 - 120\Gamma\omega^2(4\omega^2-1)\cos\omega t\, x_2^2\big]\frac{\partial}{\partial x_2}\Big\},
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
D_1 &= 24\omega^8 - 342\omega^6 + 1260\omega^4 - 1158\omega^2 + 216, \\
P_1 &= 48\omega^6 + 48\omega^5 - 492\omega^4 - 444\omega^3 + 552\omega^2 + 108\omega - 108, \\
P_2 &= 75\omega^4 - 705\omega^2 + 270, \quad P_3 = -4\omega^8 + 57\omega^6 - 210\omega^4 + 193\omega^2 - 36, \\
P_4 &= 240\omega^4 - 780\omega^2 + 180, \quad P_5 = 4\omega^8 - 73\omega^6 + 374\omega^4 - 377\omega^2 + 72, \\
P_6 &= 25\omega^4 - 235\omega^2 + 90, \quad P_7 = -32\omega^4 + 296w^2 - 72, \\
P_8 &= 16\omega^6 - 139\omega^4 - 51\omega^2 + 54, \quad P_9 = 48\omega^6 - 252\omega^4 - 228\omega^2 + 72, \\
P_{10} &= -16\omega^7 + 164\omega^5 - 32\omega^4 - 184\omega^3 + 296\omega^2 + 36\omega - 72.
\end{aligned}
$$

4. Approximate First Integrals

An approximate version of Noether's theorem has been proven in [1]. Its geometric version for n-dimensional DSs has been given in [5]. Assume that $\mathbf{X}$ is an approximate symmetry VF of order p and it is divergence free. Then $n-1$ form obtained from

$$\mathbf{X} \lrcorner\, \Omega,$$

(where $\Omega = dx^1 \wedge \ldots \wedge dx^n$ is the volume from, n is an even integer and $\lrcorner$ is interior product) is closed and one can write:

$$\mathbf{X} \lrcorner\, \Omega = dI, \tag{29}$$

where d is the exterior derivative and $I(\mathbf{x}, t)$ is pth-order approximate first integrals. In [5], the author has shown that when $\mathbf{X}$ is a stable approximate symmetry VF, then I is a stable approximate first integral.

Equipped with this result we can now proceed to find second-order approximate first integral of (23). One can easily check that for $\Omega = dx_1 \wedge dx_2$ and $\mathbf{X}$ given in (28), (29) becomes closed. Therefore dI is an exact one-form. Hence, $\mathbf{X}$ given in (28) is an approximate Noether symmetry VF. Approximate first integral of (23) corresponding to (28) has been found to be:

$$
\begin{aligned}
I(\mathbf{x}, t) &= \frac{1}{2}(x_1^2 + x_2^2) + \frac{\varepsilon}{E_1}\big[(\omega^2-4)x_1^3 + 6\Gamma\omega\sin\omega t\, x_1x_2 - 6\Gamma\cos\omega t(x_1^2 - x_2^2)\big] \\
&\quad + \frac{\varepsilon^2}{E_2}\big[\Gamma P_5(x_2^2 - x_1^2) + 2\Gamma\omega P_6\sin\omega t\, x_1x_2
\end{aligned}
$$

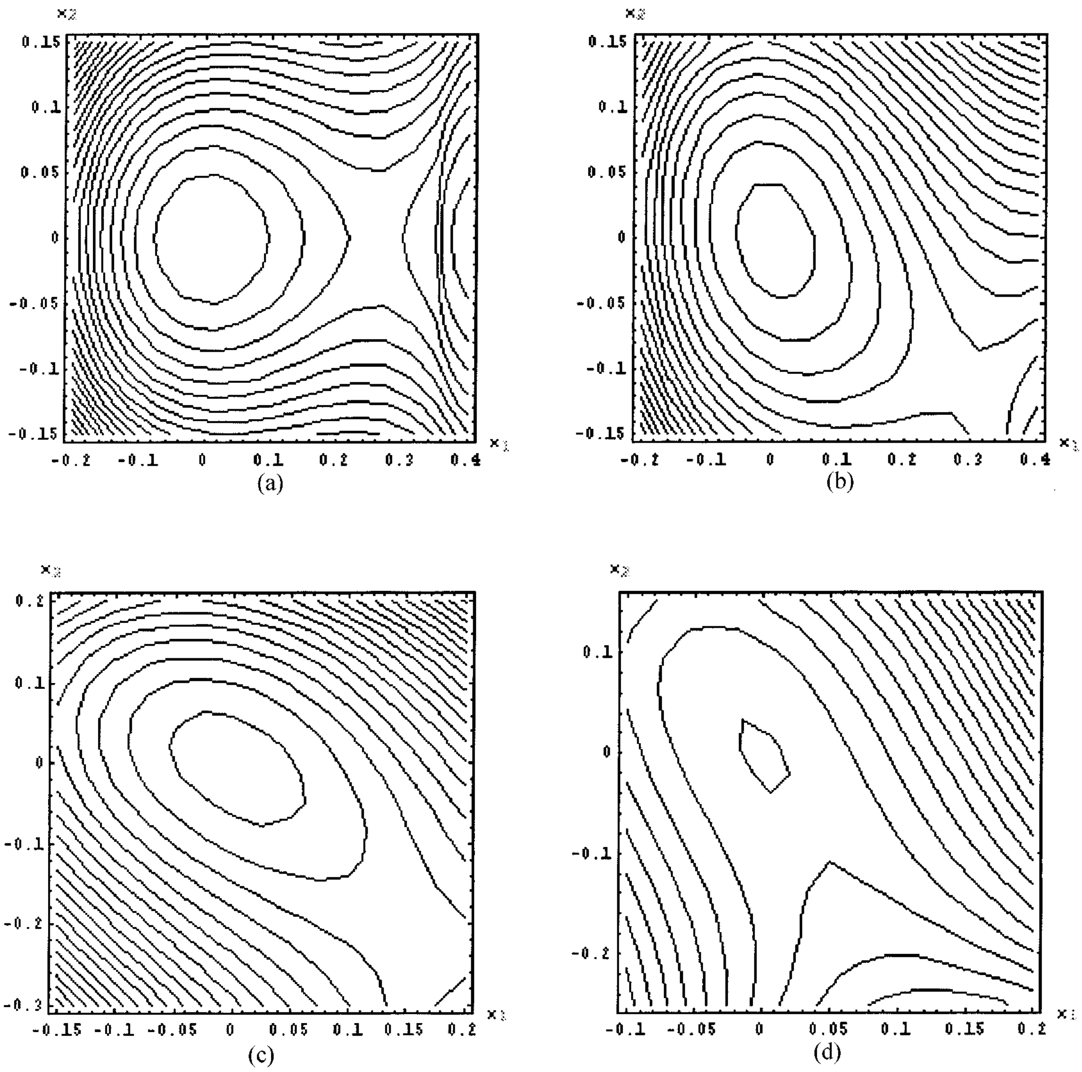

Figure 1. Contour plots for $\omega = 0.99$, $\Gamma = 1$, $\varepsilon = 0.2$, and $t =$ (a) 0, (b) $\pi/8$, (c) $\pi/4$, (d) $3\pi/8$.

$$\begin{aligned}
&+ \omega P_{11} \sin \omega t \, x_1^2 x_2 + 80\omega(1 - 4\omega^2) \sin \omega t \, x_2^3 \\
&+ 8(1 - 4\omega^2) \cos \omega t \, (-6x_1^2 - 5\omega^2 x_1^2 + \omega^4 x_1^2 - 10\omega^2 x_2^2) x_1 \\
&- \Gamma(\omega^2 - 9) \cos 2\omega t \, ((16\omega^4 + 5\omega^2 - 6) x_1^2 + 5(2 - 5\omega^2) x_2^2) \big], \qquad (30)
\end{aligned}$$

where $E_1 = 3(\omega^2 - 4)$, $E_2 = (3/2)D_1$ and $P_{11} = 160\omega^4 - 520\omega^2 + 120$. Notice that the approximate first integral (30) becomes singular for the values of the angular frequency: $\omega = 1/2,\ 1,\ 2,\ 3$. These values correspond to resonances, hence, bifurcation values for periodic solutions. This issue will be discussed in detail elsewhere.

The contour lines (level curves) of the approximate first integral (30) have been plotted in Figures 1 and 2. In Figure 1, parameter values: $\omega = 0.99$, $\Gamma = 1$, $\varepsilon = 0.2$ were chosen. Since the approximate first integral (30) is time dependent, contour lines are obtained at specific times. In Figures 1a–1d, closed curves are surrounded by the homoclinic orbits (seperatrices). But they cannot be shown in contour plots, because these orbits involve saddle type of equi-

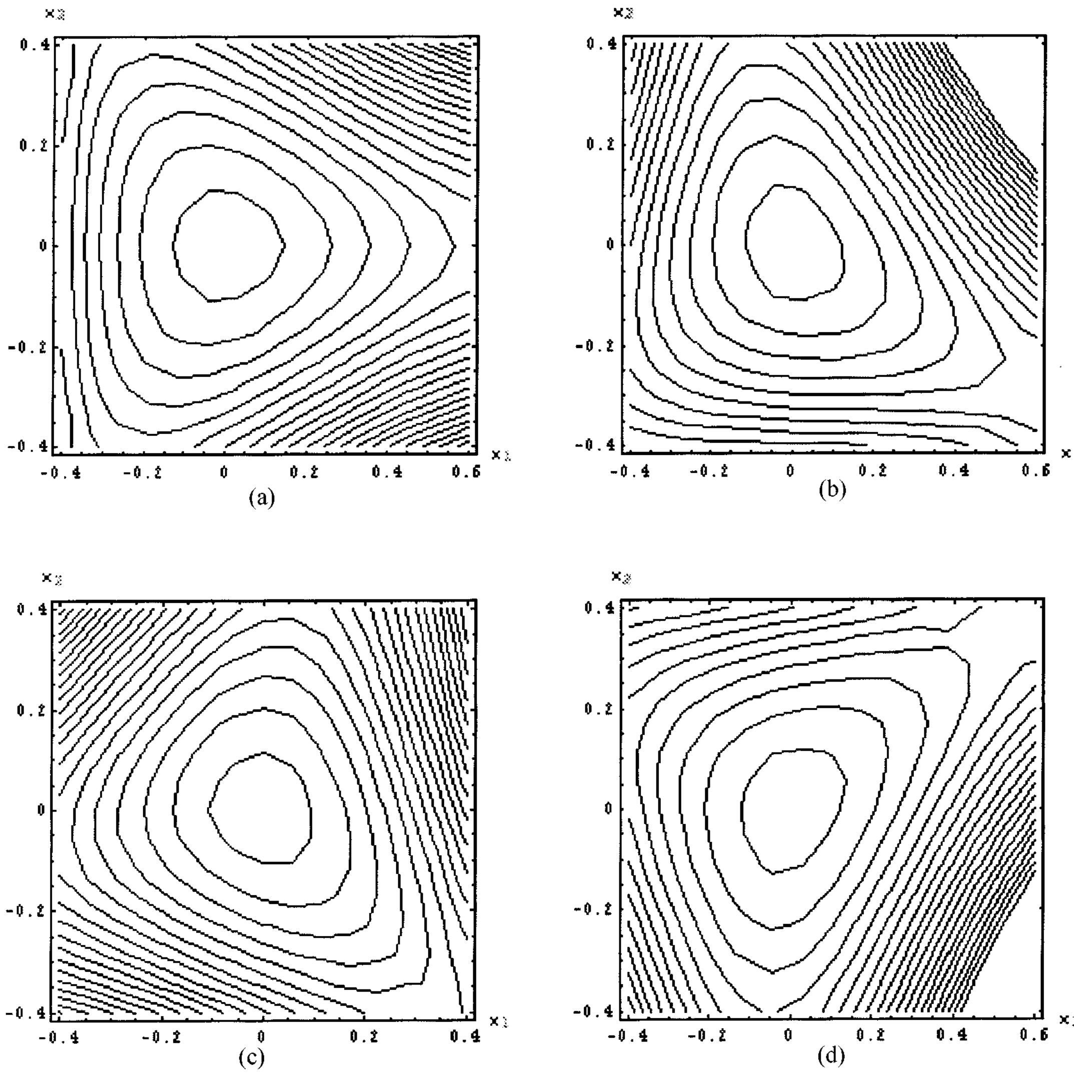

Figure 2. Contour plots for $\omega = 3.01$, $\Gamma = 1$, $\varepsilon = 0.2$, and $t =$ (a) 0, (b) $\pi/8$, (c) $\pi/4$, (d) $\pi/2$.

librium points. Moreover, the curves surrounding the homoclinic orbits are not closed. In Figure 2, parameter values: $\omega = 3.01$, $\Gamma = 1$, $\varepsilon = 0.2$ were chosen. In Figures 2a–2d, closed curves are surrounded by three heteroclinic orbits (seperatrices) which form a triangle. This triangle pinpoints the passage through 3:1 resonance which occurs at $\omega = 3$. Since the heteroclinic orbits also involve saddle type of equilibrium points, they cannot be shown in contour plots. Furthermore, the curves surrounding triangular shape of seperatrices are not closed.

References

1. Baikov, V. A., Gazizov, R. K., and Ibragimov N. H., 'Approximate symmetries', *Mathematics of the USSR-Sbornik* **64**, 1989, 427–441.

2. Rubio, M. A., De La Torre, M., and Antoranz, J.C., ‘Intermittencies and power-law frequency divergencies in a nonlinear oscillator’, *Physica D* **36**, 1989, 92–108.
3. Bikdash, M., Balachandran, B., and Nayfeh, A., ‘Melnikov analysis for a ship with a general roll-damping model’, *Nonlinear Dynamics* **6**, 1994, 101–124.
4. Ibragimov, N. H., *Transformation Groups Applied to Mathematical Physics*, Nauka, Moscow, 1985 (English translation published by D. Reidel, Dordrecht).
5. Ünal, G., ‘Approximate generalized symmetries, normal forms and approximate first integrals’, *Physics Letters A*, 1999, accepted for publication.