TOHSUKE URABE

Duality of the second fundamental form

1. Spherical case

In this article I would like to explain main ideas in my recent results on duality of the second fundamental form. (Urabe [6].)

Theory of dual varieties in the complex algebraic geometry is very interesting. (Griffiths and Harris [1], Kleiman [2], Piene [4], Urabe [5], Wallace [7].) Let $\mathbf{P}$ be a complex projective space of dimension N, and $X \subset \mathbf{P}$ be a complex algebraic subvariety. The set of all hyperplanes in $\mathbf{P}$ forms another projective space $\mathbf{P}^\vee$ of dimension N, which is called the *dual projective space* of $\mathbf{P}$. The dual projective space $(\mathbf{P}^\vee)^\vee$ of $\mathbf{P}^\vee$ is identified with $\mathbf{P}$. The closure in $\mathbf{P}^\vee$ of the set of tangent hyperplanes to X is called the *dual variety* of X, and is denoted by $X^\vee$. We say that a hyperplane H in $\mathbf{P}$ is tangent to X, if we have a smooth point $p \in X$ such that H contains the embedded tangent space of X at p. It is known that the dual variety $X^\vee$ is again a complex algebraic variety, and the dual variety $(X^\vee)^\vee$ of $X^\vee$ coincides with X.

We would like to develop similar theory in the real-analytic category. (Obata [3].)

First, we fix the notations. Let N be a positive integer, and L a vector space of dimension $N+1$ over the real field $\mathbf{R}$. A fixed positive-definite inner product on L is denoted by $(\ ,\ )$. By $S = \{a \in L | (a, a) = 1\}$ we denote the unit sphere in L. The sphere S has dimension N.

We consider a compact real-analytic irreducible subvariety M in S. We assume moreover that M has only ordinary singularities as singularities.

We have to explain the phrase of "ordinary singularity" here. Let $X \subset L$ be a real-analytic subset. For every point $p \in X$ we can consider the germ (X, p) of X around p. The germ (X, p) is decomposed into irreducible components. By $\dim(X, p)$ we denote the dimension of the germ (X, p). The germ (X, p) is said to be *smooth*, if (X, p) is real-analytically isomorphic to $(\mathbf{R}^n, 0)$ where $n = \dim(X, p)$ and 0 is a point of $\mathbf{R}^n$. A point p of X is said to be smooth, if the germ (X, p) is smooth. We say that X has an *ordinary singularity* at $p \in X$, if every irreducible component of (X, p) is smooth.

Remark.

1. Needless to say, a subset X is said to be real-analytic, if for every point $p \in X$, there exists an open neighborhood U of p and a finite number of real-analytic functions on U such that $X \cap U$ coincides with the common set of zeros of these functions. Note that a real-analytic set is a closed subset of an open set. But, it is not necessarily closed.
2. We have two different concepts called by the same terminology "irreducible", "locally irreducible" and "globally irreducible". The local concept is easier to understand. It is always defined for any germ (X, p) of a real-analytic set. However, we encounter some difficulty in treating the global concept. Obviously, we can say

that a real-analytic set is globally irreducible only when it is closed. Moreover, we have to assume some additional conditions for closed real-analytic sets. This is because the ideal sheaf of a real-analytic set is not necessarily coherent, and because the ring of global real-analytic functions is not necessarily Noetherian. We can show that for compact real-analytic sets with only ordinary singularities, the concept of "globally irreducible" has definite meaning. (Urabe [6].)

3. For a compact real-analytic irreducible set X with only ordinary singularities, the dimension of an irreducible component of the germ (X, p) of a point $p \in X$ does not depend on the component and the point p. (Urabe [6].)
4. The basic theory of real-analytic category is not still well-developed. We have to give long explanation of basic concepts as above. On the other hand, real-analytic cases are the most important in application.

Let $M_{smooth} \subset M$ be the set of smooth points $p \in M$ with $\dim(M, p) = \dim M$. Under our assumption M_{smooth} is dense in M.

For every point $p \in M_{smooth}$ the tangent space $T_p(M)$ of M at p is defined. By modern definition the tangent space $T_p(M)$ is a set of differential operators and has the structure of real vector space. The embedding $M \subset L$ induces an embedding $T_p(M) \subset T_p(L)$ of vector spaces. On the other hand, $T_p(L)$ is canonically identified with the vector space L. Therefore, the tangent space $T_p(M)$ is a vector subspace of L. Note in particular that $T_p(M)$ is not an affine subspace but a vector subspace in L passing through the origin. The tangent space $T_p(M)$ has dimension equal to $\dim M$. A point $q \in S$ is a normal vector of M in S at a point $p \in M$, if q is orthogonal to p and $T_p(M)$. We say that a point $q \in S$ is a normal vector of M in S, if q is a normal vector of M in S at some point $p \in M$. By $M^\vee$ we denote the closure in S of the set of normal vectors a of M in S with $(a, a) = 1$, and we call $M^\vee \subset S$ the *dual variety* of $M \subset S$. The dual variety $M^\vee$ has a lot of interesting properties. However, $M^\vee$ is not a real-analytic subset in general.

Proposition 1.1. *Under our assumption the dual variety $M^\vee$ contains a dense smooth real-analytic subset whose connected components have the same dimension.*

Let $X \subset S$ be a subset containing a dense smooth real-analytic subset whose connected components have the same dimension. Obviously we can define the dual variety $X^\vee$ of X by the essentially same definition as above.

Theorem 1.2. *Under our assumption $(M^\vee)^\vee = M \cup \tau(M)$, where $\tau : S \to S$ denotes the antipodal map $\tau(q) = -q$.*

Remark. Note that $M \cup \tau(M)$ is a compact real-analytic subset only with ordinary singularities as singularities. For any compact real-analytic subset in L only with ordinary singularities as singularities, the irreducible decomposition is possible. Therefore, M is an irreducible component of $M \cup \tau(M)$, and we can recover M from $M \cup \tau(M)$.

There exists an open dense smooth real-analytic subset V of $M^\vee$ such that for every point $q \in V$ there exists a point $p \in M$ such that

1. q is a normal vector of M in S at p, and

2. p is a normal vector of $M^\vee$ in S at q.

Moreover, there exists an open dense smooth real-analytic subset U of M such that for every point $p \in U$ there exists a point $q \in V$ satisfying the same conditions 1 and 2 above.

Choose arbitrarily a pair (q,p) of a smooth point $q \in M^\vee$ and a smooth point $p \in M$ satisfying conditions 1 and 2, and fix it.

The second fundamental form of M at p in the normal direction q

$$\widetilde{II} : T_p(M) \times T_p(M) \longrightarrow \mathbf{R}$$

and the second fundamental form of $M^\vee$ at q in the normal direction p

$$\widetilde{II}^\vee : T_q(M^\vee) \times T_q(M^\vee) \longrightarrow \mathbf{R}$$

are defined. We set

$$\begin{aligned} \operatorname{rad} \widetilde{II} &= \{X \in T_p(M) | \text{For every } Y \in T_p(M),\ \widetilde{II}(X,Y) = 0\} \\ \operatorname{rad} \widetilde{II}^\vee &= \{X \in T_q(M^\vee) | \text{For every } Y \in T_q(M^\vee),\ \widetilde{II}^\vee(X,Y) = 0\}. \end{aligned}$$

Theorem 1.3 (Duality of the second fundamental form)**.**

1. $T_p(M) = \operatorname{rad} \widetilde{II} + (T_p(M) \cap T_q(M^\vee))$ *(orthogonal direct sum)*
2. $T_q(M^\vee) = \operatorname{rad} \widetilde{II}^\vee + (T_p(M) \cap T_q(M^\vee))$ *(orthogonal direct sum)*
3. $L = \mathbf{R}p + \operatorname{rad} \widetilde{II} + (T_p(M) \cap T_q(M^\vee)) + \operatorname{rad} \widetilde{II}^\vee + \mathbf{R}q$ *(orthogonal direct sum)*
4. *Let* $X_1, X_2, \dots, X_r$ *be an orthogonal normal basis of* $T_p(M) \cap T_q(M^\vee)$*. The matrix* $(\widetilde{II}(X_i, X_j))$ *is the inverse matrix of* $(\widetilde{II}^\vee(X_i, X_j))$*.*

Proposition 1.1 is the most difficult part to show in our theory. Once we obtain Proposition 1.1, it is not difficult to deduce Theorem 1.2 applying analogous arguments in complex projective algebraic geometry. Theorem 1.2 and Theorem 1.3 can be shown through computation on Maurer-Cartan forms. Theorem 1.3 seems to have a lot of applications in theory of subvarieties in a sphere.

You can download my preprint [6] containing verification at

http://urabe-lab.math.metro-u.ac.jp/	(Japanese)
http://urabe-lab.math.metro-u.ac.jp/DefaultE.html	(English).

2. Hyperbolic case

We can consider similar situations in hyperbolic case. (Obata [3].)

Let L be a vector space of dimension $N+1$ over the real field $\mathbf{R}$ as in Section 1. Now, we consider a non-degenerate inner product $(\ ,\ )$ on L with signature $(N,1)$. By S we denote one of the two connected components of the set $\{a \in L | (a,a) = -1\}$ in L. The hyperbolic space S has dimension N.

Also in this case we consider a compact real-analytic irreducible subvariety M in S only with ordinary singularities as singularities.

Let $S^\vee = \{a \in L | (a,a) = 1\}$. Note that also $S^\vee$ is a smooth real-analytic connected variety with dimension N. However, $S \cap S^\vee = \emptyset$, and the metric on $S^\vee$ is not definite.

We can define the dual variety $M^\vee$ of M as a subset of $S^\vee$ by the essentially same definition as above. The dual variety $(M^\vee)^\vee$ of $M^\vee$ can be defined as a subset of S.

Proposition 1.1 and Theorem 1.3 hold also in this case without any modification. Theorem 1.2 is replaced by the following brief theorem:

Theorem 2.1. *In hyperbolic case under our assumption* $(M^\vee)^\vee = M$.

Problem 2.2. Give generalization of theory of dual varieties in C^∞-category.

References

[1] Phillip Griffiths and Joseph Harris, *Algebraic geometry and local differential geometry*, Ann. scient. Éc. Norm. Sup. 4^e série **12** (1979), 355–432.

[2] Steven L. Kleiman, *The enumerative theory of singularities*, Real and complex singularities (P. Holm, ed.), Proceedings of the Nordic Summer School/NAVF, (Oslo, August 1976), Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1977, pp. 297–396.

[3] Morio Obata, *The Gauss map of immersions of Riemannian manifolds in spaces of constant curvature*, J. Differential Geometry **2** (1968), 217–223.

[4] Ragni Piene, *Polar classes of singular varieties*, Ann. scient. Éc. Norm. Sup. 4^e série **11** (1978), 247–276.

[5] Tohsuke Urabe, *Duality of numerical characters of polar loci*, Publ. RIMS, Kyoto Univ. **17** (1981), 331–345.

[6] Tohsuke Urabe, *The Gauss map and the dual variety of real-analytic submanifolds in a sphere or in hyperbolic space*, preprint (1995).

[7] Andrew H. Wallace, *Tangency and duality over arbitrary fields*, Proc. London Math. Soc. (3) **6** (1956), 321–342.

Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo, 192-03, Japan

E-mail address: urabe@comp.metro-u.ac.jp