Fractional Dynamics from Einstein Gravity, General Solutions, and Black Holes

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Abstract

We study the fractional gravity for spacetimes with non–integer dimensions. Our constructions are based on a geometric formalism with the fractional Caputo derivative and integral calculus adapted to nonolonomic distributions. This allows us to define a fractional spacetime geometry with fundamental geometric/physical objects and a generalized tensor calculus all being similar to respective integer dimension constructions. Such models of fractional gravity mimic the Einstein gravity theory and various Lagrange–Finsler and Hamilton–Cartan generalizations in nonholonomic variables. The approach suggests a number of new implications for gravity and matter field theories with singular, stochastic, kinetic, fractal, memory etc processes. We prove that the fractional gravitational field equations can be integrated in very general forms following the anholonomic deformation method for constructing exact solutions. Finally, we study some examples of fractional black hole solutions, fractional ellipsoid gravitational configurations and imbedding of such objects in fractional solitonic backgrounds.

Keywords: fractional derivatives and integrals, fractional gravity, exact solutions, nonholonomic manifolds, nonlinear connections, fractional black holes, fractional solitons.

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Contents

1 Introduction

Modern classical and quantum gravity models are constructed geometrically to be extremely complex with black holes, singularities and horizons, nontrivial topology, stochastic processes and fractals, irregular sets, possible extra dimensions etc. Such configurations are defined as exact and/or approximate solutions of Einstein equations in general relativity and modifications.

There is a recent interest in fractional analysis and non–integer dimension geometries which resulted in an increasing number of publications during last decade. Calculus with derivatives and integrals of fractional order is applied in various directions in physics, for instance, in Lagrange and Hamilton mechanics and dynamical models [33, 5, 25, 41, 4], fractal and chaotic dynamics [11, 14, 38, 86, 87], quantum mechanics [16, 17, 26], kinetic theories [86, 88, 43], plasma physics [10, 39] and astrophysics and cosmology [40, 27, 28, 35], and many additional applications in physics and other sciences [85, 7, 19, 37].

The cornerstone questions to be solved in constructing classical and quantum gravity theories are positively connected to the meaning of spacetime dimension, extra dimension and/or fractional dimensions physical effects, possible contributions from (non) holonomic and/or commutative/ noncommutative variables, of fractal dimension, nonlocal field theories etc. With respect to dark energy and dark matter models, various attempts to construct the quantum gravity theory, to determine the status of singularities in fundamental theories etc, it is thus natural to pursue alternative concepts of differential and integral calculus and consider new ideas and models of spacetime geometry.

The fractional dimension paradigm is at present extended to new ideas on gauge invariance in fractional field theories [13], being constructed solutions of fractional Dirac equation [32, 24], with corresponding formalisms for generalized Clifford algebra [89]. A formulation of Noether's theorem for fractional classical fields [23] and, recently, a model of fractional gravity based on Riemann–Liouville concept of non–integer dimensional derivative was proposed in [22].

Let us provide a simple motivation why knew mathematical tools from non–integer calculus may be very useful in solving fundamental theories in physics. An exact solution of Einstein equations with singularity for $r \to r_0$ (for instance, inducing a singular term in the Riemann curvature tensor)

modelled by a function

$$f(r) \sim {}^1f(r_0)(r - r_0)^{- {}^1s} + {}^2f(r_0)(r - r_0)^{{}^2s} + ..., \qquad (1)$$

for some real values ${}^1f(r_0)$ and ${}^2f(r_0)$ and positive integers ${}^1s$ and ${}^2s$. The singular properties of such a function may have a complete different geometric and classical/quantum physical meaning if instead of the standard partial derivative $\partial/\partial r$ we work with the left Caputo fractional derivative ${}_{1r}\underline{\partial}_r^{\alpha} f(r)$ (when usual partial derivatives are generalized through a more complex integral–derivative relation, see below formula (2)).[1] Such functions positively result in various new physical implications if there are developed models with inhomogeneous, fractal, stochastic, kinetic etc processes, in classical and quantum theories.

The advantages of fractional derivatives and integrals become already apparent in modern mechanics and for describing thermodynamical and electrical properties of real complex media (as we emphasized in the above references). We suppose that for non–integer dimensions, new conceptual features of gravitational nonlinear interactions can be analyzed. Fractional–order models are more appropriate than the integer–order ones because they provide an excellent instrument for describing processes with memory, branching and hereditary properties which in the classical (pseudo) Riemannian approach is, in fact, neglected (even such solutions can be encoded into the structure of certain quantum/stochastic modifications of Einstein equations).

In our recent work [75], we provided some theoretical arguments that fractional spacetime theories should receive more attention in modern geometric analysis, evolution theories and gravity. Such a conclusion was derived from the facts that the Ricci flow theory can be naturally generalized for non–integer dimensions, with a new type of statistical geometric analogy, if a corresponding nonholonomic fractional differential–integral calculus is applied. Such a scenario may transform (pseudo) Riemannian metrics into certain fractional ones, or inversely. This involve a number of consequences in classical and quantum theories of fundamental field interactions and evolution models. Here we also remember that the theory of fractional calculus with derivatives and integrals of non–integer order goes back to Leibniz, Louville, Grunwald, Letnikov and Riemann [30, 31, 15, 29] (The first example of derivative of order $\alpha = 1/2$ was considered by Leibnitz in 1695, see historical remarks in [36]).

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[1]We follow the system of notations with left abstract labels explained in Refs. [75, 61], see also next section.

One might hope that extending a fundamental physical theory like general gravity to fractional dimensions will solve the most troublesome problems of Planckian physics, quantum non-renormalizable theories, new types of symmetries etc. Such extensions should be constructive ones allowing exact solutions and possible simple geometric and physical interpretations. In this paper, we shall apply the fractional geometric formalism developed in Ref. [75] to formulate and study a version of fractional Einstein type gravity theory. Certain constructions with the fractional Ricci tensor, and generalizations to osculator bundles (higher order fractional mechanical models etc) and non–integer dimension Einstein tensor were provided in Refs. [1, 22]. It is a very cumbersome task, both technically and theoretically, to solve field and evolution fractional equations in exact form following those approaches with the Riemann–Liouville (RL) fractional derivative.

In general, there are various possibilities to derive fractional differential–integral structures and the question on which derivative would be more suitable for formulating engineering and scientific problems remain an open issue. Here we note that integrals on net fractals can be approximated by the left–sided RL fractional integrals of functions, see [34]. Such a fractal and non–integer calculus formalism is applied for certain attempts to construct the quantum field theory, gravity and cosmology in fractal universes and related diffusion processes in string theory, see [8, 9] and references therein. Nevertheless, we shall follow an approach based on Caputo fractional derivatives allowing, for instance, to get nonhomogeneous cosmological solutions generalizing the off–diagonal metrics from [74].

This paper has three purposes: The first one is to formulate a fractional generalization of the Einstein gravitational field equations with nonholonomic local frames and deformations of connections induced by the Caputo left derivative. The second one is to prove that such nonlinear systems of partial derivative and integral equations can be solved in very general form using our anholonomic deformation method of constructing exact solutions in gravity and Ricci flow theories, see reviews of results and applications in [54, 55, 57, 61, 70, 73]. The third purpose, is to study certain applications of fractional calculus in black hole physics: we construct in explicit form such fractional black hole solutions and study their imbedding in fractional solitonic backgrounds.

The article is structured as follows. In section 2 we provide an introduction into fractional Caputo type calculus on nonholonomic manifolds.[2] The

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[2]In "integer" differential geometry, there are used also equivalent terms like anholonomic and non–integrable manifolds. A nonholonomic manifold of integer dimension is

Einstein equations are generalized on fractional nonholonomic manifold in section 3. Then, in section 4 we show that in our approach the fractional field equations for gravity can be integrated in very general forms. We provide some applications in modern gravity by considering black hole solutions in factional spacetimes, see section 5. Discussion of results and future perspectives are presented in concluding section 6.

2 Fractional Derivatives, Integrals and Forms

We provide an introduction to fractional differential and integral calculus on "flat" spaces following [42] and then extend the approach as fractional generalizations of (nonholonomic) Einstein spacetimes. Relevant details and references can be found also in a partner paper [75] (see there and Appendix containing a summary of non–integer calculus and methods).

2.1 Fractional Caputo derivatives

There are different approaches to fractional derivative and integral calculus when there are used different types of fractional derivatives. We elaborate and follow a method with nonholonomic distributions when the geometric constructions are most closed to "integer" calculus.

Let us consider that $f(x)$ is a derivable function $f : [\ _1x,\ _2x] \to \mathbb{R}$, for $\mathbb{R} \ni \alpha > 0$, and denote the derivative on x as $\partial_x = \partial/\partial x$. The fractional Caputo derivatives are determined respectively by

$$\text{left:} \quad {}_{1x}\underset{}{\overset{\alpha}{\underline{\partial}}}_x f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{1x}^{x} (x-\ x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^s f(x')dx'; \quad (2)$$

$$\text{right:} \quad {}_x\underset{2x}{\overset{\alpha}{\underline{\partial}}} f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{x}^{2x} (x'-x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^s f(x')dx' \ ,$$

where we underline the partial derivative symbol, $\underline{\partial}$, in order to distinguish the Caputo operators from the RL ones with usual ∂.[3] A very important property is that for a constant C, for instance, ${}_{1x}\overset{\alpha}{\underline{\partial}}_x C = 0$.

defined by a pair $(\mathbf{V}, \mathcal{N})$, where $\mathbf{V}$ is a manifold and $\mathcal{N}$ is a nonintegrable distribution on $\mathbf{V}$, see the next section for a generalization to non–integer dimensions.

[3]The left Riemann–Liouville (RL) derivative is

$$_{1x}\overset{\alpha}{\partial}_x f(x) := \frac{1}{\Gamma(s-\alpha)} \left(\frac{\partial}{\partial x}\right)^s \int\limits_{1x}^{x} (x-\ x')^{s-\alpha-1} f(x')dx',$$

2.1.1 Fractional integral

We denote by $L_z(\,_1x,\,_2x)$ the set of those Lesbegue measurable functions f on $[\,_1x,\,_2x]$ for which $||f||_z = (\int_{_1x}^{_2x} |f(x)|^z dx)^{1/z} < \infty$ and write $C^z[\,_1x,\,_2x]$ for a space of functions which are z times continuously differentiable on this interval.

For any real–valued function $f(x)$ defined on a closed interval $[\,_1x,\,_2x]$, there is a function $F(x) = \,_{1x}\overset{\alpha}{I}_x\, f(x)$ defined by the fractional Riemann–Liouville integral $\,_{1x}\overset{\alpha}{I}_x f(x) := \frac{1}{\Gamma(\alpha)} \int_{_1x}^{x} (x-x')^{\alpha-1} f(x') dx'$, when the function $f(x) = \,_{1x}\overset{\alpha}{\underline{\partial}}_x F(x)$, for all $x \in [\,_1x,\,_2x]$,[4] satisfies the conditions

$$_{1x}\overset{\alpha}{\underline{\partial}}_x \left(\,_{1x}\overset{\alpha}{I}_x f(x) \right) = f(x),\ \alpha > 0,$$

$$_{1x}\overset{\alpha}{I}_x \left(\,_{1x}\overset{\alpha}{\underline{\partial}}_x F(x) \right) = F(x) - F(\,_1x),\ 0 < \alpha < 1.$$

A fractional volume integral is a triple fractional integral within a region $X \subset \mathbb{R}^3$, for instance, of a scalar field $f(x^k)$,

$$\overset{\alpha}{I}(f) = \overset{\alpha}{I}[x^k] f(x^k) = \overset{\alpha}{I}[x^1]\,\overset{\alpha}{I}[x^2]\,\overset{\alpha}{I}[x^3] f(x^k).$$

For $\alpha = 1$ and $f(x,y,z)$, $\overset{\alpha}{I}(f) = \iiint_X dV f(x,y,z) = \iiint_X dx dy dz\ f(x,y,z).$

2.1.2 Fractional differential forms

An exterior fractional differential can be defined through the fractional Caputo derivatives which is self–consistent with the definition of the fractional integral considered above. We write the fractional absolute differential

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where Γ is the Euler's gamma function. The left fractional Liouville derivative of order α, where $s - 1 < \alpha < s$, with respect to coordinate x is $\overset{\alpha}{\partial}_x f(x) := \lim_{_1x \to -\infty}\,_{1x}\overset{\alpha}{\partial}_x f(x)$.

The right RL derivative is $_x\overset{\alpha}{\partial}_{2x} f(x) := \frac{1}{\Gamma(s-\alpha)} \left(-\frac{\partial}{\partial x}\right)^s \int_x^{2x} (x'-x)^{s-\alpha-1} f(x') dx'$. The right fractional Liouville derivative is $_x\overset{\alpha}{\partial} f(x^k) := \lim_{_2x \to \infty}\,_x\overset{\alpha}{\partial}_{2x} f(x)$.

The fractional RL derivative of a constant C is not zero but, for instance, $_{1x}\overset{\alpha}{\partial}_x C = C\frac{(x-\,_1x)^{-\alpha}}{\Gamma(1-\alpha)}$. Integro–differential constructions based only on such derivatives seem to be very cumbersome and has a number of properties which are very different from similar ones for the integer calculus.

[4]this follows from the fundamental theorem of fractional calculus [42]

$\overset{\alpha}{d}$ in the form

$$\overset{\alpha}{d} := (dx^j)^\alpha \ {}_0\underline{\overset{\alpha}{\partial}}_j, \ \text{ where } \ \overset{\alpha}{d}x^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)},$$

where we consider $\ {}_1x^i = 0.$[5]

An exterior fractional differential is defined

$$\overset{\alpha}{d} = \sum_{j=1}^{n} \Gamma(2-\alpha)(x^j)^{\alpha-1} \ \overset{\alpha}{d}x^j \ {}_0\underline{\overset{\alpha}{\partial}}_j.$$

Differentials are dual to partial derivatives, and derivation is inverse to integration. For a fractional calculus, the concept of "dual" and "inverse" have a more sophisticate relation to "integration" and, in result, there is a more complex relation between forms and vectors.

The fractional integration for differential forms on an interval $L = [\ {}_1x, \ {}_2x]$ is defined

$$\overset{\alpha}{{}_L I}[x] \ \ {}_{1x}\overset{\alpha}{d}_x f(x) = f(\ {}_2x) - f(\ {}_1x), \tag{3}$$

i.e. the fractional differential of a function $f(x)$ is $\ {}_{1x}\overset{\alpha}{d}_x f(x) = [...]$, when

$$\int_{{}_1x}^{{}_2x} \frac{(dx)^{1-\alpha}}{\Gamma(\alpha)(\ {}_2x - x)^{1-\alpha}} [(dx')^\alpha \ {}_{1x}\overset{\alpha}{\partial}_{x''} f(x'')] = f(x) - f(\ {}_1x).$$

The exact fractional differential 0–form is a fractional differential of the function

$${}_{1x}\overset{\alpha}{d}_x f(x) := (dx)^\alpha \ {}_{1x}\underline{\overset{\alpha}{\partial}}_{x'} f(x'),$$

when the equation (3) is considered as the fractional generalization of the integral for a differential 1–form. So, the formula for the fractional exterior derivative can be written

$${}_{1x}\overset{\alpha}{d}_x := (dx^i)^\alpha \ {}_{1x}\underline{\overset{\alpha}{\partial}}_i. \tag{4}$$

For instance, for the fractional differential 1–form $\overset{\alpha}{\omega}$ with coefficients $\{\omega_i(x^k)\}$ is

$$\overset{\alpha}{\omega} = (dx^i)^\alpha \ \omega_i(x^k) \tag{5}$$

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[5]For the "integer" calculus, we use as local coordinate co-bases/–frames the differentials $dx^j = (dx^j)^{\alpha=1}$. For $0 < \alpha < 1$ we have $dx = (dx)^{1-\alpha}(dx)^\alpha$. The "fractional" symbol $(dx^j)^\alpha$, related to $\overset{\alpha}{d}x^j$, is used instead of dx^i for elaborating a co–vector/differential form calculus, see below the formula (6).

and the exterior fractional derivatives of such a fractional 1–form $\overset{\alpha}{\omega}$ gives a fractional 2–form, ${}_{1x}\overset{\alpha}{d_x}(\overset{\alpha}{\omega}) = (dx^i)^\alpha \wedge (dx^j)^\alpha \ {}_{1x}\overset{\alpha}{\underline{\partial}}_j \ \omega_i(x^k)$. Such a rule [15] follows from the property that for any type fractional derivative $\overset{\alpha}{\partial}_x$, we have

$$\overset{\alpha}{\partial}_x \left({}^1f \ {}^2f \right) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\overset{\alpha-k}{\partial}_x {}^1f \right) \overset{\alpha=k}{\partial}_x \left({}^2f \right),$$

when for integer k,

$$\binom{\alpha}{k} = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)} \text{ and } \overset{k}{\partial}_x \ (dx)^\alpha = 0, k \geq 1.$$

In fractional derivative calculus, there are used also the properties:

$$_{1x'}\overset{\alpha}{\partial}_{x'} \ (x' - \ _1x')^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x - \ _1x)^{\beta-\alpha},$$

where $n-1 < \alpha < n$ and $\beta > n$, and $_{1x'}\overset{\alpha}{\partial}_{x'} \ (x' - \ _1x')^k = 0$ for $k = 0, 1, 2, ..., n-1$. We obtain

$$_{1x}\overset{\alpha}{d_x}(x - \ _1x)^\alpha = (dx)^\alpha \ {}_{1x}\overset{\alpha}{\underline{\partial}}_{i'}x^{i'} = (dx)^\alpha\Gamma(\alpha+1),$$

i.e.

$$(dx)^\alpha = \frac{1}{\Gamma(\alpha+1)} \ {}_{1x}\overset{\alpha}{d_x}(x - \ _1x)^\alpha. \tag{6}$$

We might write the fractional exterior derivative (4) in the form

$$_{1x}\overset{\alpha}{d_x} := \frac{1}{\Gamma(\alpha+1)} \ {}_{1x}\overset{\alpha}{d_x}(x^i - \ _1x^i)^\alpha \ {}_{1x}\overset{\alpha}{\underline{\partial}}_i$$

and the fractional differential 1–form (5) as

$$\overset{\alpha}{\omega} = \frac{1}{\Gamma(\alpha+1)} \ {}_{1x}\overset{\alpha}{d_x}(x^i - \ _1x^i)^\alpha \ F_i(x).$$

Having a well defined exterior calculus of fractional differential forms on flat spaces $\mathbb{R}^n$, we can generalize the constructions for a real manifold $M, \dim M = n$.

2.2 Fractional manifolds and tangent bundles

A real manifold M, with integer dimension $\dim M = n$, can be endowed on charts of a covering atlas with a fractional derivative–integral structure of Caputo type as we explained above. In brief, such a space (of necessary smooth class) $\overset{\alpha}{\underline{M}}$ will be called a fractional manifold.

Let us explain how the concept of tangent bundle can be developed for fractional dimensions. A tangent bundle TM over a manifold M of integer dimension is canonically defined by its local integer differential structure ∂_i. A fractional generalization can be obtained if instead of ∂_i we consider the left Caputo derivatives $\underset{1x^i}{}\overset{\alpha}{\underline{\partial}}_i$ of type (2), for every local coordinate x^i on a local cart X on M. A fractional tangent bundle $\overset{\alpha}{\underline{T}}M$ for $\alpha \in (0,1)$ (the symbol T is underlined in order to emphasize that we shall associate the approach to a fractional Caputo derivative), see more details in [75].[6] For simplicity, we shall write both for integer and fractional tangent bundles the local coordinates in the form $u^\beta = (x^j, y^j)$.

On $\overset{\alpha}{\underline{T}}M$, an arbitrary fractional frame basis

$$\overset{\alpha}{\underline{e}}_\beta = e^{\beta'}_{\ \beta}(u^\beta)\overset{\alpha}{\underline{\partial}}_{\beta'} \tag{7}$$

is connected via a vierlbein transform $e^{\beta'}_{\ \beta}(u^\beta)$ with a fractional local coordinate basis

$$\overset{\alpha}{\underline{\partial}}_{\beta'} = \left(\overset{\alpha}{\underline{\partial}}_{j'} = \underset{1x^{j'}}{}\overset{\alpha}{\underline{\partial}}_{j'}, \overset{\alpha}{\underline{\partial}}_{b'} = \underset{1y^{b'}}{}\overset{\alpha}{\underline{\partial}}_{b'}\right) \tag{8}$$

when $j' = 1, 2, ..., n$ and $b' = n+1, n+2, ..., n+n$. There are also fractional co–bases which are dual to (7),

$$\overset{\alpha}{\underline{e}}^{\ \beta} = e^{\ \beta}_{\beta'}(u^\beta)\overset{\alpha}{\underline{d}}u^{\beta'}, \tag{9}$$

where the fractional local coordinate co–basis

$$\overset{\alpha}{\underline{d}}u^{\beta'} = \left((dx^{i'})^\alpha, (dy^{a'})^\alpha\right), \tag{10}$$

when the h– and v–components, $(dx^{i'})^\alpha$ and $(dy^{a'})^\alpha$ are of type (5). For integer values, a matrix $e^{\beta'}_{\ \beta}$ is inverse to $e^{\ \beta}_{\beta'}$.

Similarly to $\overset{\alpha}{\underline{T}}M$, we can define a fractional vector bundle $\overset{\alpha}{\underline{E}}$ on M, when the fiber indices of bases run values $a', b', ... = n+1, n+2, ..., n+m$.

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[6]We to not follow alternative constructions with RL fractional derivative [1, 22] because in that direction it is not clear how to prove integrability of the RL–fractional Einstein equations.

3 Einstein Equations on Fractional Manifolds

We provide a fractional generalization of the Einstein gravity of arbitrary dimensions via nonholonomic variables in such a form when we shall be able to construct exact solutions of the gravitational field equations.

3.1 Introduction to the geometry of fractional manifolds

Let us consider a "prime" (pseudo) Riemannian manifold $\mathbf{V}$ is of integer dimension dim $\mathbf{V} = n + m, n \geq 2, m \geq 1$. Its fractional extension is modelled as a fractional nonholonomic manifold $\overset{\alpha}{\mathbf{V}}$ defined by a quadruple $(\mathbf{V}, \overset{\alpha}{\mathcal{N}}, \overset{\alpha}{\mathbf{d}}, \overset{\alpha}{\mathbf{I}})$, where $\overset{\alpha}{\mathcal{N}}$ (see below formula (11)) is a nonholonomic distribution defining a nonlinear connection structure, the fractional differential structure $\overset{\alpha}{\mathbf{d}}$ is given by (7) and (9) and the non–integer integral structure $\overset{\alpha}{\mathbf{I}}$ is computed following the by rules of type (3).

3.1.1 Nonholonomic fractional distributions and frames

A nonlinear connection (N–connection) $\overset{\alpha}{\mathbf{N}}$ for $\overset{\alpha}{\mathbf{V}}$ is defined by a nonholonomic distribution (Whitney sum) with conventional h– and v–subspaces, $\underline{h}\overset{\alpha}{\mathbf{V}}$ and $\underline{v}\overset{\alpha}{\mathbf{V}}$,

$$\underline{\overset{\alpha}{T}}\overset{\alpha}{\mathbf{V}} = \underline{h}\overset{\alpha}{\mathbf{V}} \oplus \underline{v}\overset{\alpha}{\mathbf{V}}. \tag{11}$$

Nonholonomic manifolds with a $\overset{\alpha}{\mathbf{N}}$ are called, in brief, N–anholonomic fractional manifolds. Locally, a fractional N–connection is defined by its coefficients, $\overset{\alpha}{\mathbf{N}} = \{ {}^{\alpha}N_i^a \}$, when

$$\overset{\alpha}{\mathbf{N}} = {}^{\alpha}N_i^a(u)(dx^i)^{\alpha} \otimes \overset{\alpha}{\underline{\partial}}_a. \tag{12}$$

For a N–connection $\overset{\alpha}{\mathbf{N}}$, we can always a class of fractional (co) frames (N–adapted) linearly depending on ${}^{\alpha}N_i^a$),

$$
{}^{\alpha}\mathbf{e}_{\beta} = \left[{}^{\alpha}\mathbf{e}_j = \overset{\alpha}{\underline{\partial}}_j - {}^{\alpha}N_j^a \overset{\alpha}{\underline{\partial}}_a, \ {}^{\alpha}e_b = \overset{\alpha}{\underline{\partial}}_b \right], \tag{13}
$$

$$
{}^{\alpha}\mathbf{e}^{\beta} = [{}^{\alpha}e^j = (dx^j)^{\alpha}, \ {}^{\alpha}\mathbf{e}^b = (dy^b)^{\alpha} + {}^{\alpha}N_k^b(dx^k)^{\alpha}]. \tag{14}
$$

The nontrivial nonholonomy coefficients are computed ${}^{\alpha}W_{ib}^a = \overset{\alpha}{\underline{\partial}}_b {}^{\alpha}N_i^a$ and ${}^{\alpha}W_{ij}^a = {}^{\alpha}\Omega_{ji}^a = {}^{\alpha}\mathbf{e}_i {}^{\alpha}N_j^a - {}^{\alpha}\mathbf{e}_j {}^{\alpha}N_i^a$ (where ${}^{\alpha}\Omega_{ji}^a$ are the coefficients of the N–connection curvature) for

$$[{}^{\alpha}\mathbf{e}_{\alpha}, \ {}^{\alpha}\mathbf{e}_{\beta}] = {}^{\alpha}\mathbf{e}_{\alpha} {}^{\alpha}\mathbf{e}_{\beta} - {}^{\alpha}\mathbf{e}_{\beta} {}^{\alpha}\mathbf{e}_{\alpha} = {}^{\alpha}W_{\alpha\beta}^{\gamma} {}^{\alpha}\mathbf{e}_{\gamma}.$$

3.1.2 Fractional metrics and distinguished connections

A (fractional) metric structure $\overset{\alpha}{\mathbf{g}} = \{\, {}^{\alpha}g_{\underline{\alpha}\underline{\beta}}\}$ is defined on a $\overset{\alpha}{\mathbf{V}}$ by a symmetric second rank tensor

$$\overset{\alpha}{\mathbf{g}} = {}^{\alpha}g_{\underline{\gamma}\underline{\beta}}(u)(du^{\underline{\gamma}})^{\alpha} \otimes (du^{\underline{\beta}})^{\alpha}. \tag{15}$$

For N–adapted constructions, it is important to use the property that any fractional metric $\overset{\alpha}{\mathbf{g}}$ can be represented equivalently as a distinguished metric (d–metric), $\overset{\alpha}{\mathbf{g}} = [\, {}^{\alpha}g_{kj}, \ {}^{\alpha}g_{cb}]$, when

$$\begin{aligned}
\overset{\alpha}{\mathbf{g}} &= {}^{\alpha}g_{kj}(x,y)\, {}^{\alpha}e^{k} \otimes {}^{\alpha}e^{j} + {}^{\alpha}g_{cb}(x,y)\, {}^{\alpha}\mathbf{e}^{c} \otimes {}^{\alpha}\mathbf{e}^{b} \tag{16} \\
&= \eta_{k'j'}\, {}^{\alpha}e^{k'} \otimes {}^{\alpha}e^{j'} + \eta_{c'b'}\, {}^{\alpha}\mathbf{e}^{c'} \otimes {}^{\alpha}\mathbf{e}^{b'}, \tag{17}
\end{aligned}$$

where matrices $\eta_{k'j'} = diag[\pm 1, \pm 1, ..., \pm 1]$ and $\eta_{a'b'} = diag[\pm 1, \pm 1, ..., \pm 1]$ (reflecting signature of "prime" spacetime $\mathbf{V}$) are obtained by frame transforms

$$\eta_{k'j'} = e^{k}_{\ k'}\, e^{j}_{\ j'}\, {}^{\alpha}g_{kj} \text{ and } \eta_{a'b'} = e^{a}_{\ a'}\, e^{b}_{\ b'}\, {}^{\alpha}g_{ab}. \tag{18}$$

For fractional computations, it is convenient to work with constants $\eta_{k'j'}$ and $\eta_{a'b'}$ because the Caputo derivatives of constants are zero. This allows us to keep the same tensor rules as for the integer dimensions even the rules for taking local derivatives became more sophisticate because of N–coefficients ${}^{\alpha}N_{i}^{a}(u)$ and additional vierbein transforms $e^{k}_{\ k'}(u)$ and $e^{a}_{\ a'}(u)$. Such coefficients mix fractional derivatives $\overset{\alpha}{\underline{\partial}}_{a}$ computed as a local integration (2). If we work with RL fractional derivatives, the computation become very sophisticate with nonlinear mixing of integration, partial derivatives etc.

A distinguished connection (d–connection) $\overset{\alpha}{\mathbf{D}}$ on $\overset{\alpha}{\mathbf{V}}$ is a linear connection preserving under parallel transports the Whitney sum (11). Using the formalism of fractional differential forms introduced in previous section, we can elaborate a covariant fractional N–adapted calculus on nonholonomic manifolds. To a fractional d–connection $\overset{\alpha}{\mathbf{D}}$ we can associate a N–adapted differential 1–form of type (5)

$$ {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta} = {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta\gamma}\, {}^{\alpha}\mathbf{e}^{\gamma}, \tag{19}$$

when the coefficients are computed with respect to (14) and (13) and parametrized the form ${}^{\alpha}\mathbf{\Gamma}^{\gamma}_{\ \tau\beta} = \left({}^{\alpha}L^{i}_{jk}, \ {}^{\alpha}L^{a}_{bk}, \ {}^{\alpha}C^{i}_{jc}, \ {}^{\alpha}C^{a}_{bc} \right).$

On fractional forms $\overset{\alpha}{\mathbf{V}}$, we can act with the absolute fractional differential ${}^{\alpha}\mathbf{d} = \, {}_{1x}\overset{\alpha}{d}_{x} + \, {}_{1y}\overset{\alpha}{d}_{y}$. In N–adapted fractional form, the value ${}^{\alpha}\mathbf{d} := {}^{\alpha}\mathbf{e}^{\beta}\, {}^{\alpha}\mathbf{e}_{\beta}$

consists from exterior h- and v–derivatives of type (4), when

$$\underset{1x}{\overset{\alpha}{d}_x} := (dx^i)^\alpha \quad \underset{1x}{\overset{\alpha}{\underline{\partial}}_i} = \ {}^\alpha e^j \ {}^\alpha \mathbf{e}_j \text{ and } \underset{1y}{\overset{\alpha}{d}_y} := (dy^a)^\alpha \quad \underset{1x}{\overset{\alpha}{\underline{\partial}}_a} = \ {}^\alpha \mathbf{e}^b \ {}^\alpha e_b.$$

3.1.3   Torsion and curvature of fractional d–connections

The torsion and curvature of a fractional d–connection $\overset{\alpha}{\mathbf{D}} = \{ \ {}^\alpha \mathbf{\Gamma}^\tau{}_{\beta\gamma}\}$ are computed, respectively, as fractional 2–forms,

$$
{}^\alpha \mathcal{T}^\tau \ \doteqdot \ \overset{\alpha}{\mathbf{D}} \ {}^\alpha \mathbf{e}^\tau = \ {}^\alpha \mathbf{d} \ {}^\alpha \mathbf{e}^\tau + \ {}^\alpha \mathbf{\Gamma}^\tau{}_\beta \wedge \ {}^\alpha \mathbf{e}^\beta \text{ and} \tag{20}
$$

$$
{}^\alpha \mathcal{R}^\tau{}_\beta \ \doteqdot \ \overset{\alpha}{\mathbf{D}} \ {}^\alpha \mathbf{\Gamma}^\tau{}_\beta = \ {}^\alpha \mathbf{d} \ {}^\alpha \mathbf{\Gamma}^\tau{}_\beta - \ {}^\alpha \mathbf{\Gamma}^\gamma{}_\beta \wedge \ {}^\alpha \mathbf{\Gamma}^\tau{}_\gamma = \ {}^\alpha \mathbf{R}^\tau{}_{\beta\gamma\delta} \ {}^\alpha \mathbf{e}^\gamma \wedge \ {}^\alpha \mathbf{e}.
$$

The fractional Ricci tensor $\ {}^\alpha \mathcal{R}ic = \{ \ {}^\alpha \mathbf{R}_{\alpha\beta} \doteqdot \ {}^\alpha \mathbf{R}^\tau{}_{\alpha\beta\tau}\}$ is

$$
{}^\alpha R_{ij} \doteqdot \ {}^\alpha R^k{}_{ijk}, \quad {}^\alpha R_{ia} \doteqdot - \ {}^\alpha R^k{}_{ika}, \quad {}^\alpha R_{ai} \doteqdot \ {}^\alpha R^b{}_{aib}, \quad {}^\alpha R_{ab} \doteqdot \ {}^\alpha R^c{}_{abc}. \tag{21}
$$

The scalar curvature of a fractional d–connection $\overset{\alpha}{\mathbf{D}}$ is

$$
{}^\alpha_s \mathbf{R} \doteqdot \ {}^\alpha \mathbf{g}^{\tau\beta} \ {}^\alpha \mathbf{R}_{\tau\beta} = \ {}^\alpha R + \ {}^\alpha S, \ {}^\alpha R = \ {}^\alpha g^{ij} \ {}^\alpha R_{ij}, \ {}^\alpha S = \ {}^\alpha g^{ab} \ {}^\alpha R_{ab},
$$

defined by a sum the h– and v–components of (21) and contractions with the inverse coefficients to a d–metric (16).

We can introduce the Einstein tensor $\ {}^\alpha \mathcal{E}ns = \{ \ {}^\alpha \mathbf{G}_{\alpha\beta}\}$,

$$
{}^\alpha \mathbf{G}_{\alpha\beta} := \ {}^\alpha \mathbf{R}_{\alpha\beta} - \frac{1}{2} \ {}^\alpha \mathbf{g}_{\alpha\beta} \ {}^\alpha_s \mathbf{R}. \tag{22}
$$

This allows us to elaborate various types of fractional models of gravity (for different types of d–connections and fractional matter sources) generalizing the Einstein gravity theory and various modifications.

Finally, we note that for integer values of α the above formulas transform into similar ones for nonholonomic manifolds [57, 61, 70].

3.1.4   Preferred fractional linear connections

For applications in modern geometry and standard models of physics, there are considered more special classes of d–connections:

- On a fractional nonholonomic $\overset{\alpha}{\mathbf{V}}$, there is a unique canonical fractional d–connection $\ {}^\alpha \widehat{\mathbf{D}} = \{ \ {}^\alpha \widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta} = \left(\ {}^\alpha \widehat{L}^i{}_{jk}, \ {}^\alpha \widehat{L}^a{}_{bk}, \ {}^\alpha \widehat{C}^i{}_{jc}, \ {}^\alpha \widehat{C}^a{}_{bc} \right)\}$ which is

compatible with the metric structure, $^{\alpha}\widehat{\mathbf{D}}$ ($^{\alpha}\mathbf{g}) = 0$, and satisfies the conditions $^{\alpha}\widehat{T}^{i}{}_{jk} = 0$ and $^{\alpha}\widehat{T}^{a}{}_{bc} = 0.$[7]

- The Levi–Civita connection $^{\alpha}\nabla = \{ \ ^{\alpha}\Gamma^{\gamma}{}_{\alpha\beta}\}$ can be defined in standard from but by using the fractional Caputo left derivatives acting the coefficients of a fractional metric (15).

As a consequence of nonholonomic structure, it is preferred to work on $\overset{\alpha}{\mathbf{V}}$ with $^{\alpha}\widehat{\mathbf{D}} = \{ \ ^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\tau\beta}\}$ instead of $^{\alpha}\nabla$ (the last one is not adapted to the N–connection splitting (11)). The torsion $^{\alpha}\widehat{\mathcal{T}}^{\tau}$ (20) of $^{\alpha}\widehat{\mathbf{D}}$ is uniquely induced nonholonomically by off–diagonal coefficients of the d–metric (16).

With respect to N–adapted fractional bases (13) and (14), the coefficients of the fractional Levi–Civita and canonical d–connection satisfy the distorting relations[8]

$$^{\alpha}\Gamma^{\gamma}{}_{\alpha\beta} = \ ^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + \ ^{\alpha}Z^{\gamma}{}_{\alpha\beta}. \tag{23}$$

It is not possible to get relations of type (23) if the fractional integro–differential structure would be not elaborated in N–adapted form for the

---

[7]The N–adapted coefficients are explicitly determined by the (16),

$$
\begin{aligned}
^{\alpha}\widehat{L}^{i}{}_{jk} &= \frac{1}{2} \ ^{\alpha}g^{ir} (\ ^{\alpha}\mathbf{e}_{k} \ ^{\alpha}g_{jr} + \ ^{\alpha}\mathbf{e}_{j} \ ^{\alpha}g_{kr} - \ ^{\alpha}\mathbf{e}_{r} \ ^{\alpha}g_{jk}), \\
^{\alpha}\widehat{L}^{a}{}_{bk} &= \ ^{\alpha}e_{b}(\ ^{\alpha}N^{a}_{k}) + \frac{1}{2} \ ^{\alpha}g^{ac} \left(\ ^{\alpha}\mathbf{e}_{k} \ ^{\alpha}g_{bc} - \ ^{\alpha}g_{dc} \ ^{\alpha}e_{b} \ ^{\alpha}N^{d}_{k} - \ ^{\alpha}g_{db} \ ^{\alpha}e_{c} \ ^{\alpha}N^{d}_{k} \right), \\
^{\alpha}\widehat{C}^{i}{}_{jc} &= \frac{1}{2} \ ^{\alpha}g^{ik} \ ^{\alpha}e_{c} \ ^{\alpha}g_{jk}, \quad ^{\alpha}\widehat{C}^{a}{}_{bc} = \frac{1}{2} \ ^{\alpha}g^{ad} (\ ^{\alpha}e_{c} \ ^{\alpha}g_{bd} + \ ^{\alpha}e_{c} \ ^{\alpha}g_{cd} - \ ^{\alpha}e_{d} \ ^{\alpha}g_{bc}).
\end{aligned}
$$

We can verify that introducing above formulas into (20) we obtain that $\widehat{T}^{i}{}_{jk} = 0$ and $\widehat{T}^{a}{}_{bc} = 0$, but $\widehat{T}^{a}{}_{ja}, \widehat{T}^{a}{}_{ji}$ and $\widehat{T}^{a}{}_{bi}$ are not zero, and that the metricity conditions are satisfied in component form.

[8]the N–adapted coefficients of distortion tensor $Z^{\gamma}{}_{\alpha\beta}$ are computed

$$
\begin{aligned}
^{\alpha}Z^{i}{}_{jk} &= 0, \quad ^{\alpha}Z^{a}{}_{jk} = - \ ^{\alpha}C^{i}{}_{jb} \ ^{\alpha}g_{ik} \ ^{\alpha}g^{ab} - \frac{1}{2} \ ^{\alpha}\Omega^{a}{}_{jk}, \\
^{\alpha}Z^{i}{}_{bk} &= \frac{1}{2} \ ^{\alpha}\Omega^{c}{}_{jk} \ ^{\alpha}g_{cb} \ ^{\alpha}g^{ji} - \frac{1}{2}(\delta^{i}_{j}\delta^{h}_{k} - \ ^{\alpha}g_{jk} \ ^{\alpha}g^{ih}) \ ^{\alpha}C^{j}{}_{hb}, \\
^{\alpha}Z^{a}{}_{bk} &= \frac{1}{2}(\delta^{a}_{c}\delta^{b}_{d} + \ ^{\alpha}g_{cd} \ ^{\alpha}g^{ab})[\ ^{\alpha}L^{c}{}_{bk} - \ ^{\alpha}e_{b}(\ ^{\alpha}N^{c}_{k})], \\
^{\alpha}Z^{i}{}_{kb} &= \frac{1}{2} \ ^{\alpha}\Omega^{a}{}_{jk} \ ^{\alpha}g_{cb} \ ^{\alpha}g^{ji} + \frac{1}{2}(\delta^{i}_{j}\delta^{h}_{k} - \ ^{\alpha}g_{jk} \ ^{\alpha}g^{ih}) \ ^{\alpha}C^{j}{}_{hb}, \\
^{\alpha}Z^{a}{}_{jb} &= -\frac{1}{2}(\delta^{a}_{c}\delta^{d}_{b} - \ ^{\alpha}g_{cb} \ ^{\alpha}g^{ad})[\ ^{\alpha}L^{c}{}_{dj} - \ ^{\alpha}e_{d}(\ ^{\alpha}N^{c}_{j})], \quad ^{\alpha}Z^{a}{}_{bc} = 0, \\
^{\alpha}Z^{i}{}_{ab} &= -\frac{^{\alpha}g^{ij}}{2}\{[\ ^{\alpha}L^{c}{}_{aj} - \ ^{\alpha}e_{a}(\ ^{\alpha}N^{c}_{j})] \ ^{\alpha}g_{cb} + [\ ^{\alpha}L^{c}{}_{bj} - \ ^{\alpha}e_{b}(\ ^{\alpha}N^{c}_{j})] \ ^{\alpha}g_{ca}\}.
\end{aligned}
$$

left Caputo derivative.

3.2 Fractional Einstein equations for connections $\widehat{\mathbf{D}}$ and ∇

In this section, we show that the fractional gravitational equations with Caputo fractional derivatives can be integrated in general form similarly to the results for integer dimensions [54, 57, 55, 61, 70].

3.2.1 Nonholonomic variables in general relativity

The Einstein equations on a spacetime manifold of integer dimension $\mathbf{V}$, for an energy–momentum source of matter $T_{\alpha\beta}$, are written in the form

$$E_{\beta\gamma} = R_{\ \beta\delta} - \frac{1}{2}g_{\beta\delta}R = \varkappa T_{\beta\delta}, \qquad (24)$$

where $\varkappa = const$ and the Einstein tensor is computed for the Levi–Civita connection ∇. It is not possible to integrate in any general form such non-linear systems of partial differential equations.

The Einstein equations (24) can be rewritten equivalently using the canonical d–connection $\widehat{\mathbf{D}} = \{\widehat{\boldsymbol{\Gamma}}^{\gamma}_{\ \alpha\beta}\}$,

$$\widehat{\mathbf{E}}_{\beta\delta} \;=\; \widehat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta} \; ^{s}R = \boldsymbol{\Upsilon}_{\beta\delta}, \qquad (25)$$

$$\widehat{L}^{c}_{aj} = e_a(N^c_j), \ \widehat{C}^{i}_{jb} = 0, \ \Omega^{a}_{\ ji} = 0, \qquad (26)$$

where $\widehat{\mathbf{R}}_{\ \beta\delta}$ is the Ricci tensor for $\widehat{\boldsymbol{\Gamma}}^{\gamma}_{\ \alpha\beta}$, $^{s}R = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\ \beta\delta}$ and $\boldsymbol{\Upsilon}_{\beta\delta}$ is such a way constructed that $\boldsymbol{\Upsilon}_{\beta\delta} \to \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$. In general, the Einstein tensor $\widehat{\mathbf{E}}_{\ \beta\delta}$ in (25) is not equal to the Einstein tensor $E_{\beta\gamma}$ in (24).

There are two possibilities to make equivalent two different systems of equations for ∇ and, respectively, for $\widehat{\mathbf{D}}$. In the first case, we can include the contributions of distortion tensor $Z^{\gamma}_{\ \alpha\beta}$ from (23) into the source $\boldsymbol{\Upsilon}_{\beta\delta} \sim \varkappa T_{\beta\delta} + {}^{z}\boldsymbol{\Upsilon}_{\beta\delta}[Z^{\gamma}_{\ \alpha\beta}]$, in such a form that the system (25) is equivalent to (24) (both types of such systems of equations are for the same metric structure $\mathbf{g}_{\beta\delta}$ but in terms of different connections). In the second case, we consider that $\boldsymbol{\Upsilon}_{\beta\delta} = \varkappa T_{\beta\delta}$ but in order to keep fundamental the Einstein equations for ∇ (even for some purposes we shall prefer to work with $\widehat{\mathbf{D}}$) we have to impose (at some final steps) the constraints (26) when the tensors $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$(20) and $Z^{\gamma}_{\ \alpha\beta}$(23) are zero. For such constraints, we have $\widehat{\boldsymbol{\Gamma}}^{\gamma}_{\ \alpha\beta} = \Gamma^{\gamma}_{\ \alpha\beta}$, with respect to N–adapted frames (13) and (14), even $\widehat{\mathbf{D}} \neq \nabla$.

It is very surprising that for some general ansatz for metrics (see below the integer version of (29)) there is a separation of equations (25) for $\widehat{\mathbf{D}}$ which allows us to integrate such systems in very general forms. If we work from the very beginning with ∇, it is not possible to get from (24) a generally solvable system of equations. Our idea was to encode the geometric and physical data for ∇ into $\widehat{\mathbf{D}}$ and work with N–adapted constructions when, for instance, certain general solutions can be found "easily" for various types of non–Riemannian and/or Lagrange–Finsler theories. Imposing additional constraints (26), we are able to extract Levi–Civita configurations with ∇, for instance, in Einstein gravity. This way, see original results and reviews in a series of our works [54, 57, 55, 61, 70], we elaborated a new method of constructing exact solutions with generic off–diagonal metrics in gravity theories (the so–called anholonomic deformation/frame method). Such a geometric technique seems also to be very efficient in Ricci flow theories [62, 63, 65, 66, 67, 68, 75] and for elaborating different methods of quantization for gravity [58, 64, 2, 60, 71, 72].

3.2.2 Nonholonomic variables and fractional gravity

Introducing the fractional canonical d–connection $^{\alpha}\widehat{\mathbf{D}}$ into the Einstein d–tensor (22), following the same principle of constructing the matter source $^{\alpha}\mathbf{\Upsilon}_{\beta\delta}$ as in general relativity but for fractional d–connections, we derive geometrically a fractional generalization of N–adapted equations (25),

$$^{\alpha}\widehat{\mathbf{E}}_{\ \beta\delta} = ^{\alpha}\mathbf{\Upsilon}_{\beta\delta}. \tag{27}$$

Such a system can be restricted to fractional nonholonomic configurations for $^{\alpha}\nabla$ if we impose a fractional analog of constraints of type (26)

$$^{\alpha}\widehat{L}^c_{aj} = ^{\alpha}e_a(^{\alpha}N^c_j), \quad ^{\alpha}\widehat{C}^i_{jb} = 0, \quad ^{\alpha}\Omega^a_{ji} = 0. \tag{28}$$

There are not theoretical or experimental evidences that for integer dimensions we must impose conditions of type (28). Nevertheless, we shall consider them in section 5 for deriving fractional black hole solutions which will mimic maximally similar ones in general relativity.

3.2.3 Separation of equations for fractional and integer dimensions

One of the main purposes of this work is to prove that a very general ansatz of type (16) defines exact solutions of the fractional Einstein equa-

tions (27) metric. Let us consider a fractional metric

$$
\begin{aligned}
{}^{\alpha}_{\eta}\mathbf{g} &= {}^{\alpha}\eta_i(x^k,v)\ {}^{\alpha}_{\circ}g_i(x^k,t)\ {}^{\alpha}dx^i \otimes\ {}^{\alpha}dx^i \\
&+ {}^{\alpha}\eta_a(x^k,v)\ {}^{\alpha}_{\circ}h_a(x^k,v)\ {}^{\alpha}\mathbf{e}^a \otimes\ {}^{\alpha}\mathbf{e}^a, \\
{}^{\alpha}\mathbf{e}^3 &= {}^{\alpha}dv + {}^{\alpha}\eta^3_i(x^k,v)\ {}^{\alpha}_{\circ}w_i(x^k,v)\ {}^{\alpha}dx^i, \\
{}^{\alpha}\mathbf{e}^4 &= {}^{\alpha}dy^4 + {}^{\alpha}\eta^4_i(x^k,v)\ {}^{\alpha}_{\circ}n_i(x^k,v)\ {}^{\alpha}dx^i,
\end{aligned}
\tag{29}
$$

when the coefficients will be defined below. For simplicity, we shall work with the "prime" dimension splitting of type $2+2$ when coordinated are labeled in the form $u^\beta = (x^j, y^3 = v, y^4)$, for $i,j,... = 1,2$. This ansatz has one Killing symmetry because the coefficients do not depend explicitly on variable y^4.

In brief, we can write such the metric (29) in the form

$$
\mathbf{g} = {}^{\alpha}g_{ij}\ {}^{\alpha}dx^i \otimes\ {}^{\alpha}dx^j + {}^{\alpha}h_{ab}(\ {}^{\alpha}dy^a + {}^{\alpha}N^a_k\ {}^{\alpha}dx^k) \otimes (\ {}^{\alpha}dy^b + {}^{\alpha}N^b_k\ {}^{\alpha}dx^k),
\tag{30}
$$

where ${}^{\alpha}g_{ij} = diag[\ {}^{\alpha}g_i = {}^{\alpha}\eta_i\ {}^{\alpha}_{\circ}g_i]$ and ${}^{\alpha}h_{ab} = diag[\ {}^{\alpha}h_a = {}^{\alpha}\eta_a\ {}^{\alpha}_{\circ}h_a]$ and ${}^{\alpha}N^3_k = {}^{\alpha}w_i = {}^{\alpha}\eta^3_i\ {}^{\alpha}_{\circ}w_i$ and ${}^{\alpha}N^4_k = {}^{\alpha}n_i = {}^{\alpha}\eta^4_i\ {}^{\alpha}_{\circ}n_i$. The gravitational 'polarizations' ${}^{\alpha}\eta_\alpha$ and ${}^{\alpha}\eta^a_i$ determine fractional nonholonomic deformations of metrics, ${}^{\alpha}_{\circ}\mathbf{g}=[\ {}^{\alpha}_{\circ}g_i,\ {}^{\alpha}_{\circ}h_a,\ {}^{\alpha}_{\circ}N^a_k] \to\ {}^{\alpha}_{\eta}\mathbf{g}=[\ {}^{\alpha}_{\eta}g_i,\ {}^{\alpha}_{\eta}h_a,\ {}^{\alpha}_{\eta}N^a_k].[9] The solutions of equations will be constructed for a general source of type[10]

$$
{}^{\alpha}\Upsilon^\alpha_{\ \beta} = diag[\ {}^{\alpha}\Upsilon_\gamma;\ {}^{\alpha}\Upsilon_1 = {}^{\alpha}\Upsilon_2 = {}^{\alpha}\Upsilon_2(x^k,v);\ {}^{\alpha}\Upsilon_3 = {}^{\alpha}\Upsilon_4 = {}^{\alpha}\Upsilon_4(x^k)]
\tag{31}
$$

A straightforward computation[11] of the components of the Ricci (21) and Einstein (22) d–tensors corresponding to ansatz (30) reduces the Einstein

---

[9]Such transforms of geometric objects (with deformations of the frame, metric, connections and other fundamental geometric structures) are more general than those considered for the Cartan's moving frame method, when the geometric objects are re–defined equivalently with respect to necessary systems of reference.

[10]such parametrizations of energy–momentum tensors are quite general ones for various types of matter sources, which (in this work) are generalized for fractional distributions; they can be defined by corresponding frame transform

[11]we omit such a cumbersome calculus which is similar to those presented in Refs. [57, 55, 61, 70]; for the formulas considered in this work, we have to change usual partial N–adapted derivatives into fractional ones considering that transform of type (18) might be performed in order to take into account the advantage that the action of Caputo derivative is zero for some constant coefficients

equations (27) to this system of partial differential equations:

$$
{}^{\alpha}\widehat{R}_1^1 = {}^{\alpha}\widehat{R}_2^2 = -\frac{1}{2\,{}^{\alpha}g_1\,{}^{\alpha}g_2} \times [\,{}^{\alpha}g_2^{\bullet\bullet} - \frac{{}^{\alpha}g_1^{\bullet}\,{}^{\alpha}g_2^{\bullet}}{2\,{}^{\alpha}g_1} \tag{32}
$$

$$
-\frac{(\,{}^{\alpha}g_2^{\bullet})^2}{2\,{}^{\alpha}g_2} + {}^{\alpha}g_1'' - \frac{{}^{\alpha}g_1'\,{}^{\alpha}g_2'}{2\,{}^{\alpha}g_2} - \frac{\left({}^{\alpha}g_1'\right)^2}{2\,{}^{\alpha}g_1}] = -\,{}^{\alpha}\Upsilon_4,
$$

$$
{}^{\alpha}\widehat{R}_3^3 = {}^{\alpha}\widehat{R}_4^4 = -\frac{1}{2\,{}^{\alpha}h_3\,{}^{\alpha}h_4}[\,{}^{\alpha}h_4^{**} \tag{33}
$$

$$
-\frac{(\,{}^{\alpha}h_4^*)^2}{2\,{}^{\alpha}h_4} - \frac{{}^{\alpha}h_3^*\,{}^{\alpha}h_4^*}{2\,{}^{\alpha}h_3}] = -\,{}^{\alpha}\Upsilon_2,
$$

$$
{}^{\alpha}\widehat{R}_{3k} = \frac{{}^{\alpha}w_k}{2\,{}^{\alpha}h_4}\left[\,{}^{\alpha}h_4^{**} - \frac{(\,{}^{\alpha}h_4^*)^2}{2\,{}^{\alpha}h_4} - \frac{{}^{\alpha}h_3^*\,{}^{\alpha}h_4^*}{2\,{}^{\alpha}h_3}\right] \tag{34}
$$

$$
+\frac{{}^{\alpha}h_4^*}{4\,{}^{\alpha}h_4}\left(\frac{{}_{1x^i}\underline{\overset{\alpha}{\partial}}_{x^i}\,{}^{\alpha}h_3}{{}^{\alpha}h_3} + \frac{\partial_k\,{}^{\alpha}h_4}{{}^{\alpha}h_4}\right) - \frac{{}_{1x^k}\underline{\overset{\alpha}{\partial}}_{x^k}\,{}^{\alpha}h_4^*}{2\,{}^{\alpha}h_4} = 0,
$$

$$
{}^{\alpha}\widehat{R}_{4k} = \frac{{}^{\alpha}h_4}{2\,{}^{\alpha}h_3}\,{}^{\alpha}n_k^{**} + \left(\frac{{}^{\alpha}h_4}{{}^{\alpha}h_3}\,{}^{\alpha}h_3^* - \frac{3}{2}\,{}^{\alpha}h_4^*\right)\frac{{}^{\alpha}n_k^*}{2\,{}^{\alpha}h_3} = 0, \tag{35}
$$

In brief, we wrote the partial derivatives in the form

$$
{}^{\alpha}a^{\bullet} = \underline{\overset{\alpha}{\partial}}_1 a = {}_{1x^1}\underline{\overset{\alpha}{\partial}}_{x^1}{}^{\alpha}a, \quad {}^{\alpha}a' = \underline{\overset{\alpha}{\partial}}_2 a = {}_{1x^2}\underline{\overset{\alpha}{\partial}}_{x^2}{}^{\alpha}a, \quad {}^{\alpha}a^* = \underline{\overset{\alpha}{\partial}}_v a = {}_{1v}\underline{\overset{\alpha}{\partial}}_v{}^{\alpha}a,
$$

see the left Caputo fractional derivatives (8).

Configurations with fractional Levi–Civita connection ${}^{\alpha}\nabla$ can be extracted by imposing additional constraints

$$
{}^{\alpha}w_i^* = {}^{\alpha}\mathbf{e}_i \ln|\,{}^{\alpha}h_4|, \quad {}^{\alpha}\mathbf{e}_k\,{}^{\alpha}w_i = {}^{\alpha}\mathbf{e}_i\,{}^{\alpha}w_k,
$$

$$
{}^{\alpha}n_i^* = 0, \quad \underline{\overset{\alpha}{\partial}}_i\,{}^{\alpha}n_k = \underline{\overset{\alpha}{\partial}}_k\,{}^{\alpha}n_i \tag{36}
$$

satisfying the conditions (28).

Following the method considered in [70] for integer dimensions, we can construct 'non–Killing' solutions depending on all coordinates when

$$
{}^{\alpha}\mathbf{g} = {}^{\alpha}g_i(x^k)\,{}^{\alpha}dx^i \otimes {}^{\alpha}dx^i + {}^{\alpha}\omega^2(x^j, v, y^4)\,{}^{\alpha}h_a(x^k, v)\,{}^{\alpha}\mathbf{e}^a \otimes {}^{\alpha}\mathbf{e}^a,
$$

$$
{}^{\alpha}\mathbf{e}^3 = {}^{\alpha}dy^3 + {}^{\alpha}w_i(x^k, v)\,{}^{\alpha}dx^i, \quad {}^{\alpha}\mathbf{e}^4 = {}^{\alpha}dy^4 + {}^{\alpha}n_i(x^k, v)\,{}^{\alpha}dx^i, \tag{37}
$$

for any ${}^{\alpha}\omega$ for which

$$
{}^{\alpha}\mathbf{e}_k\,{}^{\alpha}\omega = \underline{\overset{\alpha}{\partial}}_k\,{}^{\alpha}\omega + {}^{\alpha}w_k\,{}^{\alpha}\omega^* + {}^{\alpha}n_k\underline{\overset{\alpha}{\partial}}_{y^4}\,{}^{\alpha}\omega = 0,
$$

when (37) with ${}^{\alpha}\omega^2 = 1$ is of type (30). The length of this paper does not allow us to study such general fractional solutions.

18

4 General Solutions for Fractional Einstein Equations

There is a very important separation property in the system (32)–(35) which allows us to find exact very general solutions for such equations both for non–integer and integer dimensions. For instance, if the coefficient $^{\alpha}g_1(x^k)$ is known, we can find from (32) the value of $^{\alpha}g_2(x^k)$ (we may consider an inverse situation when $^{\alpha}g_1(x^k)$ is to be computed for a known value of $^{\alpha}g_2(x^k)$). Similarly, we can define from (33) the value of $^{\alpha}h_3(x^k, v)$ from $^{\alpha}h_4(x^k, v)$, or inversely. Having defined $^{\alpha}h_a$, at the third step, we compute the N–connection coefficients: $^{\alpha}w_i(x^k, v)$ are certain solutions of algebraic equations (34). Then, we compute $^{\alpha}n_i(x^k, v)$ by integrating two times on v in (35). Such a property of exact integration of the field equations can not obtained for fractional gravity models with the RL derivative but exists for those based on the Caputo one, correspondingly adapted to some classes of nonholonomic distributions.

The explicit form of solutions of the fractional Einstein equations depend on the values of the coefficients in the ansatz for metric and source. Let us show how such solutions are constructed for ansatz of type (30).

4.1 Solutions with $^{\alpha}h_{3,4}^* \neq 0$ and $^{\alpha}\Upsilon_{2,4} \neq 0$

Such metrics are defined by ansatz

$$
\begin{aligned}
^{\alpha}\mathbf{g} &= e^{\,^{\alpha}\psi(x^k)}\,^{\alpha}dx^i \otimes\,^{\alpha}dx^i + h_3(x^k, v)\,^{\alpha}\mathbf{e}^3 \otimes\,^{\alpha}\mathbf{e}^3 + h_4(x^k, v)\,^{\alpha}\mathbf{e}^4 \otimes\,^{\alpha}\mathbf{e}^4, \\
^{\alpha}\mathbf{e}^3 &= \,^{\alpha}dv +\,^{\alpha}w_i(x^k, v)\,^{\alpha}dx^i,\quad\,^{\alpha}\mathbf{e}^4 = \,^{\alpha}dy^4 +\,^{\alpha}n_i(x^k, v)\,^{\alpha}dx^i \qquad (38)
\end{aligned}
$$

with the coefficients being solutions of the system

$$
\begin{aligned}
^{\alpha}\ddot{\psi} +\,^{\alpha}\psi'' &= 2\,^{\alpha}\Upsilon_4(x^k), && (39) \\
^{\alpha}h_4^* &= 2\,^{\alpha}h_3\,^{\alpha}h_4\,^{\alpha}\Upsilon_2(x^i, v)/\,^{\alpha}\phi^*, && (40)
\end{aligned}
$$

$$
\begin{aligned}
^{\alpha}\beta\,^{\alpha}w_i +\,^{\alpha}\alpha_i &= 0, && (41) \\
^{\alpha}n_i^{**} +\,^{\alpha}\gamma\,^{\alpha}n_i^* &= 0, && (42)
\end{aligned}
$$

$$
\text{where }\,^{\alpha}\phi = \ln\left|\frac{^{\alpha}h_4^*}{\sqrt{|\,^{\alpha}h_3\,^{\alpha}h_4|}}\right|,\quad\,^{\alpha}\gamma = \left(\ln|\,^{\alpha}h_4|^{3/2}/|\,^{\alpha}h_3|\right)^*, \quad (43)
$$

$$
^{\alpha}\alpha_i = \,^{\alpha}h_4^*\,^{\alpha}\underline{\partial}_k\,^{\alpha}\phi,\quad\,^{\alpha}\beta = \,^{\alpha}h_4^*\,^{\alpha}\phi^*\,.
$$

For $\,^{\alpha}h_4^* \neq 0$; $\,^{\alpha}\Upsilon_2 \neq 0$, we have $\,^{\alpha}\phi^* \neq 0$. The exponential function $e^{\,^{\alpha}\psi(x^k)}$ in (38) is the fractional analog of the "integer" exponential functions and called the Mittag–Leffer function $E_\alpha[(x - \,^1x)^\alpha]$. For $\,^{\alpha}\psi(x) = E_\alpha[(x - \,^1x)^\alpha]$, we have $\underline{\partial}_i^{\alpha} E_\alpha = E_\alpha$, see (for instance) [42]. For simplicity, hereafter we shall write usual symbols for functions as in the case of integer calculus, but providing a label α considering such fractional construction as certain Taylor series in [18].

It is possible to consider any nonconstant $\,^{\alpha}\phi = \,^{\alpha}\phi(x^i, v)$ as a generating function, we can construct exact solutions of (39)–(42). We have to solve respectively the two dimensional fractional Laplace equation, for $\,^{\alpha}g_1 = \,^{\alpha}g_2 = e^{\,^{\alpha}\psi(x^k)}$. Then we integrate on v, in order to determine $\,^{\alpha}h_3$, $\,^{\alpha}h_4$ and $\,^{\alpha}n_i$, and solving algebraic equations, for $\,^{\alpha}w_i$. We obtain (computing consequently for a chosen $\,^{\alpha}\phi(x^k, v)$)

$$\,^{\alpha}g_1 = \,^{\alpha}g_2 = e^{\,^{\alpha}\psi(x^k)}, \ \,^{\alpha}h_3 = \pm \frac{|\,^{\alpha}\phi^*(x^i, v)|}{\,^{\alpha}\Upsilon_2}, \qquad (44)$$

$$\,^{\alpha}h_4 = \,_0^{\alpha}h_4(x^k) \pm 2 \,_{1v}^{\ }I_v \frac{(\exp[2 \,^{\alpha}\phi(x^k, v)])^*}{\,^{\alpha}\Upsilon_2},$$

$$\,^{\alpha}w_i = -\underline{\partial}_i^{\alpha} \,^{\alpha}\phi / \,^{\alpha}\phi^*,$$

$$\,^{\alpha}n_i = \,_1^{\alpha}n_k\left(x^i\right) + \,_2^{\alpha}n_k\left(x^i\right) \,_{1v}^{\ }I_v[\,^{\alpha}h_3 / (\sqrt{|\,^{\alpha}h_4|})^3],$$

where $\,_0^{\alpha}h_4(x^k)$, $\,_1^{\alpha}n_k\left(x^i\right)$ and $\,_2^{\alpha}n_k\left(x^i\right)$ are integration functions, and $\,_{1v}^{\ }I_v$ is the fractional integral on variables v.

Here we note that the solutions (44), and, in general, almost all solutions in fractional calculus with left Caputo derivatives, can be considered as some series expansions and relevant fractional differential equations as is sketched in ref. [18]. In such an approach, for various types of fractional functions, there are defined a kind of fractional Taylor series of infinitely fractionally–differentiable functions.

To construct exact solutions for the Levi–Civita connection $\,^{\alpha}\nabla$, we have to constrain the coefficients (44) to satisfy the conditions (36). For instance, we can fix a nonholonomic distribution when $\,_2^{\alpha}n_k\left(x^i\right) = 0$ and $\,_1^{\alpha}n_k\left(x^i\right)$ are any functions satisfying the conditions $\underline{\partial}_i^{\alpha} \,_1^{\alpha}n_k\left(xj\right) = \underline{\partial}_k^{\alpha} \,_1^{\alpha}n_i\left(x^j\right)$. The constraints on $\,^{\alpha}\phi(x^k, v)$ are related to the N–connection coefficients $\,^{\alpha}w_i = -\underline{\partial}_i^{\alpha} \,^{\alpha}\phi / \,^{\alpha}\phi^*$ following relations

$$(\,^{\alpha}w_i[\,^{\alpha}\phi])^* + \,^{\alpha}w_i[\,^{\alpha}\phi](\,^{\alpha}h_4[\,^{\alpha}\phi])^* + \underline{\partial}_i^{\alpha} \,^{\alpha}h_4[\,^{\alpha}\phi] = 0,$$

$$\underline{\partial}_i^{\alpha} \,^{\alpha}w_k[\,^{\alpha}\phi] = \underline{\partial}_k^{\alpha} \,^{\alpha}w_i[\,^{\alpha}\phi], \qquad (45)$$

where, for instance, we denoted by $^{\alpha}h_4[\,^{\alpha}\phi]$ the functional dependence on $^{\alpha}\phi$. Such conditions are always satisfied for $^{\alpha}\phi = \,^{\alpha}\phi(v)$ or if $^{\alpha}\phi = const$ when $^{\alpha}w_i(x^k, v)$ can be any functions as follows from (41) with zero $^{\alpha}\beta$ and $^{\alpha}\alpha_i$, see (43)).

4.2 Three other important classes of solutions

If any of the conditions $^{\alpha}h^*_{3,4} \neq 0$ is not satisfied, we can construct another types of solutions for certain special parametrization of coefficients for ansatz (44) subjected to the condition to be a solution of equations (39)–(42).

4.2.1 Solutions with $^{\alpha}h^*_4 = 0$

The equation (33) can be solved for such a case, $^{\alpha}h^*_4 = 0$, only if $^{\alpha}\Upsilon_2 = 0$. Any set of functions $^{\alpha}w_i(x^k, v)$ can define a solution of (34), and its equivalent (41), because the coefficients $^{\alpha}\beta$ and $^{\alpha}\alpha_i$, see (43), are zero. The coefficients $^{\alpha}n_i$ are determined from (42) with $^{\alpha}h^*_4 = 0$ and any given $^{\alpha}h_3$ which results in $^{\alpha}n_k = \,^{\alpha}_1 n_k\left(x^i\right) + \,^{\alpha}_2 n_k\left(x^i\right) \,_{1v}\overset{\alpha}{I}_v \,^{\alpha}h_3$. It is possible to take $^{\alpha}g_1 = \,^{\alpha}g_2 = e^{\,^{\alpha}\psi(x^k)}$, with $^{\alpha}\psi(x^k)$ determined by (39) for a given $^{\alpha}\Upsilon_4(x^k)$.

This class of solutions is given by ansatz

$$
\begin{aligned}
^{\alpha}\mathbf{g} &= e^{\,^{\alpha}\psi(x^k)} \,^{\alpha}dx^i \otimes \,^{\alpha}dx^i \\
&\quad + \,^{\alpha}h_3(x^k, v) \,^{\alpha}\mathbf{e}^3 \otimes \,^{\alpha}\mathbf{e}^3 + \,^{\alpha}_0 h_4(x^k) \,^{\alpha}\mathbf{e}^4 \otimes \,^{\alpha}\mathbf{e}^4, \\
^{\alpha}\mathbf{e}^3 &= \,^{\alpha}dv + \,^{\alpha}w_i(x^k, v) \,^{\alpha}dx^i, \\
^{\alpha}\mathbf{e}^4 &= \,^{\alpha}dy^4 + \left[\,^{\alpha}_1 n_k\left(x^i\right) + \,^{\alpha}_2 n_k\left(x^i\right) \,_{1v}\overset{\alpha}{I}_v h_3\right] \,^{\alpha}dx^i,
\end{aligned}
\tag{46}
$$

for arbitrary generating fractional functions $^{\alpha}h_3(x^k, v)$, $^{\alpha}w_i(x^k, v)$, $^{\alpha}_0 h_4(x^k)$ and integration fractional functions $^{\alpha}_1 n_k\left(x^i\right)$ and $^{\alpha}_2 n_k\left(x^i\right)$.

A subclass of solutions for the Levi–Civita connection can be selected from (46) by imposing the conditions (36)

$$
\begin{aligned}
^{\alpha}_2 n_k\left(x^i\right) &= 0 \quad \text{and} \quad \overset{\alpha}{\underline{\partial}}_i \,^{\alpha}_1 n_k = \overset{\alpha}{\underline{\partial}}_k \,^{\alpha}_1 n_i, \\
^{\alpha}w^*_i + \overset{\alpha}{\underline{\partial}}_i \,^0 h_4 &= 0 \quad \text{and} \quad \overset{\alpha}{\underline{\partial}}_i \,^{\alpha}w_k = \overset{\alpha}{\underline{\partial}}_k \,^{\alpha}w_i,
\end{aligned}
$$

for any such $^{\alpha}w_i(x^k, v)$ and $^{\alpha}_0 h_4(x^k)$.

21

4.2.2 Solutions with $\,^{\alpha}h_3^* = 0$ and $\,^{\alpha}h_4^* \neq 0$

The ansatz for metric is of type

$$
\begin{aligned}
{}^{\alpha}\mathbf{g} \;&=\; e^{\,{}^{\alpha}\psi(x^k)}\,{}^{\alpha}dx^i \otimes {}^{\alpha}dx^i \\
&\quad -\; {}^{\alpha}_0 h_3(x^k)\,{}^{\alpha}\mathbf{e}^3 \otimes {}^{\alpha}\mathbf{e}^3 + {}^{\alpha}h_4(x^k,v)\,{}^{\alpha}\mathbf{e}^4 \otimes {}^{\alpha}\mathbf{e}^4, \\
{}^{\alpha}\mathbf{e}^3 \;&=\; {}^{\alpha}dv + {}^{\alpha}w_i(x^k,v)\,{}^{\alpha}dx^i, \\
{}^{\alpha}\mathbf{e}^4 \;&=\; {}^{\alpha}dy^4 + {}^{\alpha}n_i(x^k,v)\,{}^{\alpha}dx^i,
\end{aligned}
\tag{47}
$$

where $\,^{\alpha}g_1 = {}^{\alpha}g_2 = e^{\,{}^{\alpha}\psi(x^k)}$, for $\,^{\alpha}\psi(x^k)$ being a solution of (39) with a given $\,^{\alpha}\Upsilon_4(x^k)$. A function $\,^{\alpha}h_4(x^k,v)$ solves the equation (40) for $\,^{\alpha}h_3^* = 0$, which can be represented

$$
{}^{\alpha}h_4^{**} - \frac{({}^{\alpha}h_4^*)^2}{2\,{}^{\alpha}h_4} - 2\,{}^{\alpha}_0 h_3\,{}^{\alpha}h_4\,{}^{\alpha}\Upsilon_2(x^k,v) = 0.
$$

The solutions for the N–connection coefficients are

$$
\begin{aligned}
{}^{\alpha}w_i \;&=\; -\,{}^{\alpha}\underline{\partial}_i\,{}^{\alpha}\widetilde{\phi}/\,{}^{\alpha}\widetilde{\phi}^*, \\
{}^{\alpha}n_i \;&=\; {}^{\alpha}_1 n_k\left(x^i\right) + {}^{\alpha}_2 n_k\left(x^i\right)\,{}^{\alpha}_{1v}\widetilde{I}_v[1/(\sqrt{|\,{}^{\alpha}h_4|})^3],
\end{aligned}
$$

when $\,^{\alpha}\widetilde{\phi} = \ln|\,{}^{\alpha}h_4^*/\sqrt{|\,{}^{\alpha}_0 h_3\,{}^{\alpha}h_4|}|$.

The Levi–Civita conditions (36) for (47) are

$$
{}^{\alpha}_1 n_k\left(x^i\right) = 0 \ \text{ and } \ {}^{\alpha}\underline{\partial}_i\,{}^{\alpha}_1 n_k = {}^{\alpha}\underline{\partial}_k\,{}^{\alpha}_1 n_i,
$$

$$
\left({}^{\alpha}w_i[\,{}^{\alpha}\widetilde{\phi}]\right)^* + {}^{\alpha}w_i[\,{}^{\alpha}\widetilde{\phi}]\left({}^{\alpha}h_4[\,{}^{\alpha}\widetilde{\phi}]\right)^* + {}^{\alpha}\underline{\partial}_i\,{}^{\alpha}h_4[\,{}^{\alpha}\widetilde{\phi}] = 0,
$$

$$
{}^{\alpha}\underline{\partial}_i\,{}^{\alpha}w_k[\,{}^{\alpha}\widetilde{\phi}] = {}^{\alpha}\underline{\partial}_k\,{}^{\alpha}w_i[\,{}^{\alpha}\widetilde{\phi}].
$$

For small fractional deformations, it is not obligatory to impose such conditions. We can consider integer Levi–Civita configurations and then to transform them nonholonomically into certain d–connection ones.

4.2.3 Solutions with $\,^{\alpha}\phi = const$

Fixing in (43) $\,^{\alpha}\phi = {}^{\alpha}\phi_0 = const$ and considering $\,^{\alpha}h_3^* \neq 0$ and $\,^{\alpha}h_4^* \neq 0$, we get that the general solutions of (39)–(42) are

$$
\begin{aligned}
{}^{\alpha}\mathbf{g} \;&=\; e^{\,{}^{\alpha}\psi(x^k)}\,{}^{\alpha}dx^i \otimes {}^{\alpha}dx^i - \\
&\quad {}^{\alpha}_0 h^2[\,{}^{\alpha}f^*(x^i,v)]^2|\,{}^{\alpha}\varsigma\Upsilon\left(x^i,v\right)|\,{}^{\alpha}\mathbf{e}^3 \otimes {}^{\alpha}\mathbf{e}^3 + {}^{\alpha}f^2\left(x^i,v\right)\,{}^{\alpha}\mathbf{e}^4 \otimes {}^{\alpha}\mathbf{e}^4, \\
{}^{\alpha}\mathbf{e}^3 \;&=\; {}^{\alpha}dv + {}^{\alpha}w_i(x^k,v)\,{}^{\alpha}dx^i, \ {}^{\alpha}\mathbf{e}^4 = {}^{\alpha}dy^4 + {}^{\alpha}n_k\left(x^i,v\right)\,{}^{\alpha}dx^i,
\end{aligned}
\tag{48}
$$

where $\overset{\alpha}{_0}h = const$ and $\,^\alpha g_1 = \,^\alpha g_2 = e^{\,^\alpha\psi(x^k)}$, with $\,^\alpha\psi(x^k)$ being a solution of (39) for any given $\,^\alpha\Upsilon_4(x^k)$.

Using the fractional function

$$\,^\alpha\varsigma_\Upsilon\left(x^i, v\right) = \,^\alpha\varsigma_{4[0]}\left(x^i\right) - \frac{\overset{\alpha}{_0}h^2}{16}\,\,_{1v}^{\alpha}I_v\Upsilon_2(x^k,v)[\,^\alpha f^2\left(x^i,v\right)]^2\,,$$

we write the fractional solutions for N–connection coefficients $\,^\alpha N_i^3 = \,^\alpha w_i$ and $\,^\alpha N_i^4 = \,^\alpha n_i$ in the form

$$\,^\alpha w_i \;=\; -\underset{i}{\overset{\alpha}{\underline{\partial}}}\,\,^\alpha\varsigma_\Upsilon\left(x^k,v\right)/\,^\alpha\varsigma_\Upsilon^*\left(x^k,v\right) \tag{49}$$

$$\text{and}\;\;\,^\alpha n_k \;=\; \overset{\alpha}{_1}n_k\left(x^i\right) + \overset{\alpha}{_2}n_k\left(x^i\right)\,\,_{1v}^{\alpha}I_v\frac{\left[\,^\alpha f^*\left(x^i,v\right)\right]^2}{\left[\,^\alpha f\left(x^i,v\right)\right]^2}\,\,^\alpha\varsigma_\Upsilon\left(x^i,v\right). \tag{50}$$

If $\,^\alpha\varsigma_\Upsilon\left(x^i,v\right) = \pm 1$ for $\,^\alpha\Upsilon_2 \to 0$, we take $\,^\alpha\varsigma_{4[0]}\left(x^i\right) = \pm 1$. For such conditions, the functions $\,^\alpha h_3 = -\overset{\alpha}{_0}h^2\left[\,^\alpha f^*\left(x^i,v\right)\right]^2$ and $\,^\alpha h_4 = \,^\alpha f^2\left(x^i,v\right)$ satisfy the equation (40), when $\sqrt{|\,^\alpha h_3|} = \,^0 h(\sqrt{|\,^\alpha h_4|})^*$ is compatible with the condition $\,^\alpha\phi = \,^\alpha\phi_0$.

The subclass of solutions for the Levi–Civita connection with ansatz of type (48) is subjected additionally to the conditions (36), in this case on fractional function $\,^\alpha\varsigma_\Upsilon$. For instance, we can chose that $\overset{\alpha}{_2}n_k\left(x^i\right) = 0$ and $\overset{\alpha}{_1}n_k\left(x^i\right)$ are any functions satisfying the conditions $\underset{i}{\overset{\alpha}{\underline{\partial}}}\,\overset{\alpha}{_1}n_k = \underset{k}{\overset{\alpha}{\underline{\partial}}}\,\overset{\alpha}{_1}n_i$. The constraints on values $\,^\alpha w_i = -\underset{i}{\overset{\alpha}{\underline{\partial}}}\,\,^\alpha\varsigma_\Upsilon/\,^\alpha\varsigma_\Upsilon^*$ result in constraints on $\,^\alpha\varsigma_\Upsilon$, which is determined by $\,^\alpha\Upsilon_2$ and $\,^\alpha f$,

$$(\,^\alpha w_i[\,^\alpha\varsigma_\Upsilon])^* + \,^\alpha w_i[\,^\alpha\varsigma_\Upsilon]\,(\,^\alpha h_4[\,^\alpha\varsigma_\Upsilon])^* + \underset{i}{\overset{\alpha}{\underline{\partial}}}\,\,^\alpha h_4[\,^\alpha\varsigma_\Upsilon] = 0,$$

$$\underset{i}{\overset{\alpha}{\underline{\partial}}}\,\,^\alpha w_k[\,^\alpha\varsigma_\Upsilon] = \underset{k}{\overset{\alpha}{\underline{\partial}}}\,\,^\alpha w_i[\,^\alpha\varsigma_\Upsilon], \tag{51}$$

where, for instance, we denoted by $\,^\alpha h_4[\,^\alpha\varsigma_\Upsilon]$ the functional dependence on $\,^\alpha\varsigma_\Upsilon$. Such conditions are always satisfied for cosmological solutions with $\,^\alpha f = \,^\alpha f(v)$. For $\,^\alpha\widehat{\mathbf{D}}$, if $\,^\alpha\Upsilon_2 = 0$ and $\,^\alpha\phi = const$, the coefficients $\,^\alpha w_i(x^k,v)$ can be arbitrary functions. The simplest parametrization are for $\,^\alpha\varsigma_\Upsilon = 1$; this does not impose a functional dependence of $\,^\alpha w_i$ on $\,^\alpha\varsigma_\Upsilon$) as follows from (41) with zero $\,^\alpha\beta$ and $\,^\alpha\alpha_i$, see (43). To generate solutions for $\,^\alpha\nabla$ such $\,^\alpha w_i$ must be additionally constrained following formulas (51) re–written for $\,^\alpha w_i[\,^\alpha\varsigma_\Upsilon] \to \,^\alpha w_i(x^k,v)$ and $\,^\alpha h_4[\,^\alpha\varsigma_\Upsilon] \to \,^\alpha h_4\left(x^i,v\right)$.

Finally, we emphasize that any solution $\,^\alpha\mathbf{g} = \{\,^\alpha g_{\alpha'\beta'}(u^{\alpha'})\}$ of the Einstein equations (25) and/or (24) with Killing symmetry $\underset{y}{\overset{\alpha}{\underline{\partial}}}$ (for local coordinates in the form $y^3 = v$ and $y^4 = y$) can be parametrized in a form

23

derived in this section. Using frame transforms of type $^\alpha e_\alpha = e_\alpha^{\alpha'}\ ^\alpha e_{\alpha'}$, with $^\alpha \mathbf{g}_{\alpha\beta} = e_\alpha^{\alpha'}e_\beta^{\beta'}\ ^\alpha g_{\alpha'\beta'}$, for any $^\alpha \mathbf{g}_{\alpha\beta}$ (29), we relate the class of such solutions, for instance, to the family of metrics of type (38).

5   Fractional Spacetimes and Black Holes

A fractional spacetime is with very different rules of computing local partial derivatives (via integration) of type $\ _1 x^i \underset{\_}{\overset{\alpha}{\partial}}_i$ (2) instead of usual partial derivatives ∂_i. The actions of such operators are very different on singular functions, for instance, parametrized in the form (1). So, the singular etc structure of solutions of fractional Einstein equations (25) and/or (24) should differ substantially from that for integer dimensions in general relativity.

There is a series of very important questions to be solved in fractional models of gravity: 1) for instance, if such theories contain black holes solutions? 2) if such solutions for fractional black holes can be constructed, what are their properties? 3) what may happen with an integer dimensional black hole into a fractional background, for instance, of solitons? We should also analyze the problems: 4) what may happen with integer and non–integer dimensional black holes under fractional/nonholonomic Ricci flows? 5) what is the status of singular solutions and horizons in fractional classical and quantum gravity etc.

In this section, we prove that black holes really exist in fractional gravity contrary to the hope that involving a new type of derivative calculus (2), and changing respectively the differential spacetime structure, we may eliminate ”ambiguities” with singularities etc. The concepts of black hole, singularity and horizon seem to be fundamental ones for various types of holonomic and nonholonomic, commutative and noncommutative, pseudo–Riemanann and Finlser like, fractional and integer etc theories of gravity. We also provide solutions for the questions 1–3) above and leave 4) and 5) for our future investigations.

5.1   Fractional deformations of the Schwarzschild spacetime

We consider a diagonal integer dimensional metric $^\varepsilon \mathbf{g}$ depending on a small real parameter $1 > \varepsilon \gtrsim 0$,

$$^\varepsilon \mathbf{g} = -d\xi \otimes d\xi - r^2(\xi)\, d\vartheta \otimes d\vartheta - r^2(\xi)\sin^2\vartheta\, d\varphi \otimes d\varphi + \varpi^2(\xi)\, dt \otimes\ dt. \ (52)$$

The local coordinates and nontrivial metric coefficients are parametrized:

$$
\begin{aligned}
x^1 &= \xi, x^2 = \vartheta, y^3 = v = \varphi, y^4 = t, \qquad (53)\\
\check{g}_1 &= -1, \ \check{g}_2 = -r^2(\xi), \ \check{h}_3 = -r^2(\xi)\sin^2\vartheta, \ \check{h}_4 = \varpi^2(\xi),\\
\text{for } \ \xi &= \int dr \ \left|1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2}\right|^{1/2} \text{ and } \varpi^2(r) = 1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2}.
\end{aligned}
$$

For $\varepsilon = 0$ in variable $\xi(r)$ and coefficients, the metric (52) is just the the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$ with a point mass μ_0.

We search for a class of exact fractional vacuum solutions of type (29) when the fractional metrics are generated by nonholonomic deformations $^\alpha g_i = \ ^\alpha\eta_i \check{g}_i$ and $h_a = \ ^\alpha\eta_a \check{h}_a$ and some nontrivial $^\alpha w_i, \ ^\alpha n_i$, where $(\check{g}_i, \check{h}_a)$ are given by data (53) and parametrized by ansatz

$$
\begin{aligned}
{}^\varepsilon_\eta\mathbf{g} &= - \ ^\alpha\eta_1(\xi,\vartheta,\theta) \ ^\alpha d\xi \otimes \ ^\alpha d\xi - \eta_2(\xi,\vartheta,\theta)r^2(\xi) \ ^\alpha d\vartheta \otimes \ ^\alpha d\vartheta \ (54)\\
&\quad - \ ^\alpha\eta_3(\xi,\vartheta,\varphi,\theta)r^2(\xi)\sin^2\vartheta \ ^\alpha\delta\varphi \otimes \ ^\alpha\delta\varphi\\
&\quad + \ ^\alpha\eta_4(\xi,\vartheta,\varphi,\theta)\varpi^2(\xi) \ ^\alpha\delta t \otimes \ ^\alpha\delta t,\\
{}^\alpha\delta\varphi &= \ ^\alpha d\varphi + \ ^\alpha w_1(\xi,\vartheta,\varphi,\theta) \ ^\alpha d\xi + \ ^\alpha w_2(\xi,\vartheta,\varphi,\theta) \ ^\alpha d\vartheta,\\
{}^\alpha\delta t &= \ ^\alpha dt + \ ^\alpha n_1(\xi,\vartheta,\theta) \ ^\alpha d\xi + \ ^\alpha n_2(\xi,\vartheta,\theta) \ ^\alpha d\vartheta,
\end{aligned}
$$

when the coefficients will be constructed determine solutions of the system of equations (39)–(42) with $^\alpha\Upsilon_\beta = 0$.

The equation (40) for $^\alpha\Upsilon_2 = 0$ is solved by any

$$
\begin{aligned}
{}^\alpha h_3 &= - \ ^\alpha_0 h^2(\ ^\alpha b^*)^2 = \ ^\alpha\eta_3(\xi,\vartheta,\varphi,\theta)r^2(\xi)\sin^2\vartheta, \qquad (55)\\
{}^\alpha h_4 &= \ ^\alpha b^2 = \ ^\alpha\eta_4(\xi,\vartheta,\varphi,\theta)\varpi^2(\xi),
\end{aligned}
$$

for $|\ ^\alpha\eta_3| = (\ ^\alpha_0 h)^2 |\check{h}_4/\check{h}_3| \left[\left(\sqrt{|\ ^\alpha\eta_4|}\right)^*\right]^2$. We consider $^\alpha_0 h = const$ (it must be $^\alpha_0 h = 2$ in order to satisfy the condition (40) with zero source), where $^\alpha\eta_4$ can be any function satisfying the condition $^\alpha\eta_4^* \neq 0$. This way, it is possible to generate a class of solutions for any function $^\alpha b(\xi,\vartheta,\varphi,\theta)$ with $^\alpha b^* \neq 0$. For classes of solutions with nontrivial sources, it is more convenient to work directly with fractional polarizations $^\alpha\eta_4$, for $^\alpha\eta_4^* \neq 0$. In another turn, for vacuum configurations, it is better to chose as a generating function, for instance, $^\alpha h_4$, for $^\alpha h_4^* \neq 0$. The fractional polarizations $^\alpha\eta_1$ and $^\alpha\eta_2$, when $^\alpha\eta_1 = \ ^\alpha\eta_2 r^2 = e^{\ ^\alpha\psi(\xi,\vartheta)}$ (for fractional configurations, this is not just the exponential function but its fractional version), from (39) with $^\alpha\Upsilon_4 = 0$, i.e. from $^\alpha\psi^{\bullet\bullet} + \ ^\alpha\psi'' = 0$.

Putting the above coefficient in (54), we construct a class of exact vacuum solutions in fractional gravity defining stationary fractional nonholonomic deformations on a small parameter ε of the Schwarzschild metric,

$$
\begin{aligned}
{}^{\alpha}_{\varepsilon}\mathbf{g} &= -e^{{}^{\alpha}\psi(\xi,\vartheta,\theta)} \left({}^{\alpha}d\xi \otimes {}^{\alpha}d\xi + {}^{\alpha}d\vartheta \otimes {}^{\alpha}d\vartheta \right) \qquad (56) \\
&\quad -4 \left[\left(\sqrt{|{}^{\alpha}\eta_4(\xi,\vartheta,\varphi,\theta)|} \right)^* \right]^2 \varpi^2(\xi) \; {}^{\alpha}\delta\varphi \otimes {}^{\alpha}\delta\varphi \\
&\quad + {}^{\alpha}\eta_4(\xi,\vartheta,\varphi,\theta)\varpi^2(\xi) \; {}^{\alpha}\delta t \otimes {}^{\alpha}\delta t, \\
{}^{\alpha}\delta\varphi &= {}^{\alpha}d\varphi + {}^{\alpha}w_1(\xi,\vartheta,\varphi,\theta) \; {}^{\alpha}d\xi + {}^{\alpha}w_2(\xi,\vartheta,\varphi,\theta) \; {}^{\alpha}d\vartheta, \\
{}^{\alpha}\delta t &= {}^{\alpha}dt + {}^{\alpha}_1 n_1(\xi,\vartheta,\theta) \; {}^{\alpha}d\xi + {}^{\alpha}_2 n_2(\xi,\vartheta,\theta) \; {}^{\alpha}d\vartheta.
\end{aligned}
$$

In general, the solutions for fractional metrics (54), or (56), do not define black holes and do not describe obvious physical situations. We can consider that for general nonholonomic fractional deformations a usual "integer" black hole may "dissipate" into a fractional structure of a more sophisticate fractional spacetime ether with a not defined status of singularities of coefficients of metric.

In next subsections, we show that it is possible to chose certain nonholonomic distributions when the singular character of the coefficient $\varpi^2(\xi)$ vanishing on the horizon of a Schwarzschild black hole result in physical properties of usual black holes.

5.2 Non–integer gravitational ellipsoid configurations

It is possible to extract a class of metrics (56) defining fractional deformations of the Schwarzschild solution depending on parameter ε with possible physical interpretation of fractional gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry. We chose in (55) a generating functions of type

$$
{}^{\alpha}b^2 = {}^{\alpha}q(\xi,\vartheta,\varphi) + \varepsilon \; {}^{\alpha}s(\xi,\vartheta,\varphi) \qquad (57)
$$

considering, for simplicity, only linear decompositions on a small parameter ε. For (57), we get

$$
({}^{\alpha}b^*)^2 = \left[(\sqrt{|{}^{\alpha}q|})^* \right]^2 \left[1 + \varepsilon \left({}^{\alpha}s / \sqrt{|{}^{\alpha}q|} \right)^* / (\sqrt{|{}^{\alpha}q|})^* \right].
$$

So, we can compute in (56) the coefficients ${}^{\alpha}h_3$ and ${}^{\alpha}h_4$ and corresponding polarizations ${}^{\alpha}\eta_3$ and ${}^{\alpha}\eta_4$, using formulas (55). Here we note that if we put $\varepsilon = 0$ we can generate fractional deformations of the Schwarzschild solution not depending on parameter α, when ${}^{\alpha}b^2 = {}^{\alpha}q$ and $({}^{\alpha}b^*)^2 = \left[(\sqrt{|{}^{\alpha}q|})^* \right]^2$.

Fractional rotoid configurations are generated for

$$^\alpha q = 1 - \frac{2\ ^\alpha\mu(\xi,\vartheta,\varphi)}{r} \ \text{and} \ \ ^\alpha s = \frac{q_0(r)}{4\ ^\alpha\mu^2}\sin(\omega_0\varphi + \varphi_0), \qquad (58)$$

when $^\alpha\mu(\xi,\vartheta,\varphi) = \mu_0 + \mu_1(\xi,\vartheta,\varphi)$ describes fractional locally anisotropic polarized mass. The constants μ_0, ω_0 and φ_0 and arbitrary functions $\mu_1(\xi,\vartheta,\varphi)$ and $q_0(r)$ have to be defined from some boundary conditions, with ε treated as the eccentricity[12] of an ellipsoid.

The fractional stationary rotoid solutions for the Schwarzschild metric in general relativity can be written in the form

$$
\begin{aligned}
^\alpha_{rot\ \varepsilon}\mathbf{g} \ = \ & -e^{\ ^\alpha\psi}\left(\ ^\alpha d\xi \otimes\ ^\alpha d\xi +\ ^\alpha d\vartheta \otimes\ ^\alpha d\vartheta\right) \\
& -4\left[(\sqrt{|\ ^\alpha q|})^*\right]^2\left[1 + \varepsilon\left(\ ^\alpha s/\sqrt{|\ ^\alpha q|}\right)^*/(\sqrt{|\ ^\alpha q|})^*\right]\ ^\alpha\delta\varphi \otimes\ ^\alpha\delta\varphi \\
& + (\ ^\alpha q + \varepsilon\ ^\alpha s)\ ^\alpha\delta t \otimes\ ^\alpha\delta t, \qquad\qquad\qquad\qquad (59) \\
^\alpha\delta\varphi \ = \ & ^\alpha d\varphi +\ ^\alpha w_1\ ^\alpha d\xi +\ ^\alpha w_2\ ^\alpha d\vartheta, \\
^\alpha\delta t \ = \ & ^\alpha dt +\ ^\alpha_1 n_1\ ^\alpha d\xi +\ ^\alpha_1 n_2\ ^\alpha d\vartheta,
\end{aligned}
$$

where functions $^\alpha q(\xi,\vartheta,\varphi)$ and $^\alpha s(\xi,\vartheta,\varphi)$ are given by formulas (58). The N–connection coefficients, $^\alpha w_i(\xi,\vartheta,\varphi)$ and $^\alpha n_i =\ ^\alpha_1 n_i(\xi,\vartheta)$, are subjected to conditions of type (36),

$$
\begin{aligned}
^\alpha w_1\ ^\alpha w_2\,(\ln|\ ^\alpha w_1/\ ^\alpha w_2|)^* \ &= \ ^\alpha w_2^\bullet -\ ^\alpha w_1', \qquad ^\alpha w_i^* \neq 0; \\
\text{or}\ \ ^\alpha w_2^\bullet -\ ^\alpha w_1' \ &= \ 0, \qquad ^\alpha w_i^* = 0; \\
^\alpha_1 n_1'(\xi,\vartheta) -\ ^\alpha_1 n_2^\bullet(\xi,\vartheta) \ &= \ 0
\end{aligned}
$$

and $^\alpha\psi(\xi,\vartheta)$ being any function for which $^\alpha\psi^{\bullet\bullet} +\ ^\alpha\psi'' = 0$.

For small eccentricities, a metric (59) defines stationary fractional configurations for so–called black ellipsoid type solutions. Their stability and properties can be stated and analyzed by adapting to fractional calculus the methods elaborated in [52, 53, 54] (a summary of results and generalizations for various types of locally anisotropic gravity models in Ref. [55], see similar constructions for a noncommutative Finsler version of rotoid spacetimes in [73]).

---

[12]ε is an eccentricity because, for integer configurations, the coefficient $h_4 = b^2 = \eta_4(\xi,\vartheta,\varphi,\ \bar\theta)\ \varpi^2(\xi)$ becomes zero for data (58) if $r_+ \simeq 2\mu_0/[1 + \varepsilon\frac{q_0(r)}{4\mu^2}\sin(\omega_0\varphi + \varphi_0)]$; this is the "parametric" equation for an ellipse $r_+(\varphi)$ for any fixed values $\frac{q_0(r)}{4\mu^2}, \omega_0, \varphi_0$ and μ_0

5.3 Stationary black holes in fractional solitonic configurations

Finally, we analyze two types of fractional rotoid solutions (in particular the Schwarzschild one) imbedded into 1) integer solitonic background and 2) into a fractional solitonic background defined by an exact solution of a fractional differential equation.

5.3.1 Integer solitonic background

There are static three dimensional solitonic distributions $\eta(\xi,\vartheta,\varphi,\theta)$, defined as solutions of an (integer) solitonic equation

$$\eta^{\bullet\bullet} + \epsilon(\eta' + 6\eta \ \eta^* + \eta^{***})^* = 0, \ \epsilon = \pm 1,$$

resulting in stationary black ellipsoid–solitonic fractional metrics generated as further deformations of a metric $\ _{rot}^{\alpha}\ _{\varepsilon}\mathbf{g}$ (59).[13] The generated fractional solitonic gravitational metrics are of type

$$
\begin{aligned}
^{\alpha}\mathbf{g} &= -e^{\ ^{\alpha}\psi}(\ ^{\alpha}d\xi \otimes\ ^{\alpha}d\xi +\ ^{\alpha}d\vartheta \otimes\ ^{\alpha}d\vartheta) - \qquad\qquad (60)\\
&\quad 4\left[(\sqrt{|\eta\ ^{\alpha}q|})^*\right]^2 \left[1 + \varepsilon\frac{1}{(\sqrt{|\eta\ ^{\alpha}q|})^*}\left(\frac{^{\alpha}s}{\sqrt{|\eta\ ^{\alpha}q|}}\right)^*\right]\ ^{\alpha}\delta\varphi \otimes\ ^{\alpha}\delta\varphi\\
&\quad +\eta(\ ^{\alpha}q + \varepsilon\ ^{\alpha}s)\ ^{\alpha}\delta t \otimes\ ^{\alpha}\delta t,\\
^{\alpha}\delta\varphi &=\ ^{\alpha}d\varphi +\ ^{\alpha}w_1\ ^{\alpha}d\xi +\ ^{\alpha}w_2\ ^{\alpha}d\vartheta,\\
^{\alpha}\delta t &=\ ^{\alpha}dt +\ _1^{\alpha}n_1\ ^{\alpha}d\xi +\ _1^{\alpha}n_2\ ^{\alpha}d\vartheta,
\end{aligned}
$$

where the N–connection coefficients are taken the same as for (59).

The metrics (60) are of type (56). So, they positively define vacuum solutions of fractional Einstein equations.

5.3.2 Fractional solitonic backgrounds

An interesting property of the solutions (48) is that they depend on a generating fractional function $\ ^{\alpha}f^2(x^i, v)$ which is a general one, but constrained to some conditions in order to generate a corresponding class of solutions. For instance, we can consider that the fractional metric (60) is additionally deformed nonholonomically by a solution $\ ^{\alpha}\rho(v)$, when $\ ^{\alpha}\rho^* =$

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[13]a function η can be a solution of any three dimensional solitonic and/ or other nonlinear wave equations

$\overset{\alpha}{\underline{\partial}}_v \rho = {}_{1v}\overset{\alpha}{\underline{\partial}}_v {}^\alpha\rho$ and $\quad {}_{1v}\overset{\alpha}{\partial}_v$ being the left RL derivative on v, of a fractional differential equation

$$
{}_{1x}\overset{\alpha}{\partial}_x({}^\alpha\rho^*) + {}^1z(v) {}^\alpha\rho^* = {}^2z(v) \tag{61}
$$

for some suitable functions ${}^1z(v)$ and ${}^2z(v)$. This fractional equation can be solved in general form [6],

$$
{}^\alpha\rho(v) = \sum_{p=0}^{\infty}(-1)^p {}_{1v}\overset{\alpha}{I}_v \left[{}^1z(v)^{-1} {}_{1v}\overset{\alpha}{\partial}_v \right]^p \left\{ \frac{{}^2z(v)}{{}^1z(v)} \right\} + {}^1c(v - {}^1v) + {}^2c, \tag{62}
$$

where ${}^1c$ and ${}^2c$ are constants. If ${}^1z(v) = 0$ and ${}^2z(v) = \lambda\, {}^\alpha\rho(v)$, where $\lambda \in \mathbb{R}$, the equation (61) transform into a fractional Euler–Lagrange equation for the one–dimensional fractional oscillator. Its solution is different from (62) but also can be expressed as a series (see details in [6]).

We construct a fractional black rotoid solution embedded both into an integer solitonic background and a one–dimensional fractional gravitational generalized oscillator (62) if we consider the (vacuum solution) metric

$$
\begin{aligned}
{}^\alpha\mathbf{g} &= -e^{\,{}^\alpha\psi}({}^\alpha d\xi \otimes {}^\alpha d\xi + {}^\alpha d\vartheta \otimes {}^\alpha d\vartheta) - 4\left[(\sqrt{|\eta\, {}^\alpha q\, {}^\alpha\rho|})^*\right]^2 \times \\
&\qquad \left[1 + \varepsilon\frac{1}{(\sqrt{|\eta\, {}^\alpha q\, {}^\alpha\rho|})^*}\left(\frac{{}^\alpha s}{\sqrt{|\eta\, {}^\alpha q\, {}^\alpha\rho|}}\right)^*\right] {}^\alpha\delta\varphi \otimes {}^\alpha\delta\varphi \\
&\qquad + \eta\, {}^\alpha\rho({}^\alpha q + \varepsilon\, {}^\alpha s) {}^\alpha\delta t \otimes {}^\alpha\delta t, \\
{}^\alpha\delta\varphi &= {}^\alpha d\varphi + {}^\alpha w_1 {}^\alpha d\xi + {}^\alpha w_2 {}^\alpha d\vartheta, \\
{}^\alpha\delta t &= {}^\alpha dt + {}^\alpha_1 n_1 {}^\alpha d\xi + {}^\alpha_1 n_2 {}^\alpha d\vartheta,
\end{aligned}
$$

where the N–connection coefficients are taken the same as for (59). This type of metrics are also of type (56), with a different nonholonomic structure, and also define vacuum solutions of fractional Einstein equations.

The fractional nonholonomic black hole type solutions (59) and (60), and related solitonic and fractional oscillator extensions, are stationary.

6 Discussion, Conclusions and Further Developments

Different concepts of dimension and related definitions of derivatives (Riemann–Lesbegue, Caputo etc) and integrals (Lesbegue–Stieltjes, Riemann–Stieltjes, fractional integrals etc) were introduced long time ago and studied

at present in various branches of science. In this work, we developed a geometric approach to fractional gravity theories based on fractional nonholonomic manifolds, nonlinear connections and generalized solutions of fractional Einstein equations, deriving all constructions from the Caputo left fractional derivative. Following such a direction, we preserve a maximally possible analogy between spaces of integer and non–integer dimensions. For study of general dynamical and evolution theories of gravitational and matter fields (classical and quantum models), we have to apply various methods from the geometry of nonholonomic manifolds (originally elaborated for Lagrange–Finsler spaces) developed for a correspondingly adapted fractional calculus.

In our approach, the fractional gravity can be naturally defined on fractional manifolds following the same principles as in general relativity. More than that, there is an unified method allowing to derive both types of metric compatible generalizations of the Einstein theories, with a fractional version of the Levi–Civita connection, and of generalized Lagrange–Finsler models, with the canonical/Cartan distinguished connections. The surprising result is that the fractional Einstein equations can be integrated in general form by adapting the constructions with respect to certain classes of frame elongated linearly by a nonlinear connection (N–connection) structure. Both for the fractional and integer dimension (pseudo) Riemannian spaces, the coefficients N–connections are defined by certain off–diagonal terms of metrics. For Lagrange–Finsler theories, the N–connections are determined as a fundamental geometric structure induced by a fractional/integer generating function.

In this work, we classified possible integral varieties of fractional nonholonomic gravitational field equations. We also provided a study of Schwarzschild type black holes and rotoid configurations in fractional spacetimes. The main conclusion was that the black holes exist also in fractional gravity and that such configurations may survive for certain types of small nonholonomic fractional deformations of metrics and in integer, or fractional, solitonic backgrounds. Nevertheless, more general classes of fractional nonholonomic transforms result in exact solutions with not obvious physical interpretation.

There are many directions of investigation left to explore in fractional calculus and geometry and applications in various sciences. Here we outline seven important ones being related to possible developments of our former results in (integer dimensional) geometry of nonholonomic manifolds/ bundles, and applications to modern classical and quantum physics theories:

1. *Foundations of fractional classical and quantum field theories* with possible noncommutative, supersymmetric, string, brane generalizations etc. There is a number of publications in such directions [4, 5, 8, 9, 11, 13, 17, 23, 41, 42, 87, 89]. Our proposal is to analyze if such constructions can be redefined for nonholonomic fractional non-commutative/ supersymmetric etc spaces and string/brane models, see reviews of such results in our works [46, 47, 49, 82, 55], when for adapted nonholonomic distributions the constructions the fundamental field equations became integrable in very general forms.

2. *A theory of fractional nonholonomic Ricci flows* was proposed recently in [75]. This generalize in a fractional fashion a series of our papers on nonholonomic Ricci flows of Einstein, Lagrange–Finsler and non-commutative/ nonsymmetric spaces [62, 62, 68] and exact solutions of the Einstein equations and Hamilton's equations and applications in cosmology and astrophysics [65, 66, 67]. Solutions with fractional evolution of geometric objects present a strong theoretical arguments for study such theories.

3. *Exact solutions in fractional gravity and applications in modern cosmology and astrophysics.* The first examples of such fractional exact solutions were constructed and analyzed in previous two sections of this paper. We proved that our version of fractional Einstein equations can be solved in general form and analyzed models of black hole/ellipsoid fractional metrics. Here we note that there are various attempts to relate modern cosmological experimental data to effects of fractional/fractal gravity and matter field interactions [27, 28, 35, 8, 9]. With respect to our former works on general/exact solutions in various models of gravity theories and applications, we suppose to be of interest in modern mathematical physics and gravity such issues: a) Exact solutions for models of fractional gravity of arbitrary dimensions and with different types of connections generalizing [70, 55, 57]; b) locally anisotropic fractional cosmology models [74, 55]; c) various fractional brane, noncommutative, anisotropic Taub NUT, wormhole and black hole solutions [73, 80, 81, 52, 53, 54, 79, 83].

4. *Fractional Clifford spinor structures and Dirac operators* have been considered, for instance, in [32, 24]. Following an "exactly integrable" fractional nonholonomic approach, we find such perspective for further developments on fractional Clifford geometry and applications. There are at least five different sub–directions: a) fractional Clifford–

Lagrange/–Finsler/ - Hamilton spaces and higher order generalizations, see original results for integer dimensional spaces in [44, 48, 84, 82]; b) fractional Clifford–Finsler algebroids, extending [56], and c) nonholonomic non–integer gerbes as generalizations from [78]; d) exact solutions of fractional Einstein–Dirac equations adapted to N–connection structure as we considered in [79, 83] and e) fractional models of noncommutative geometry, Dirac operators and noncommutative gravity and field interactions, see original results in [68, 54, 76] and Part III of [55].

5. *Fractional Lagrange–Finsler and Hamilton–Cartan spaces and higher order generalizations.* This direction is originally considered in [1] as a fractional generalization (using the Riemann–Liouville integral) of some results from [21, 20]. Here we cite our alternative/ complimentary constructions with higher order (super) vector bundles/ Clifford bundles and super–strings [46, 48, 49, 82, 70, 77, 12] which can be generalized for fractional higher order calculus, adapted to nonholonomic distributions and fractional Caputo derivatives, with exactly integrable string/brane/ spinor and modified gravitational field equations.

6. *Fractional quantum gravity, deformation quantization, renormalizability, and gauge models.* Such theories, for instance, in fractional versions, are supposed to provide new tools in quantization of gravity and matter field interactions, see [8, 16, 17, 38, 39]. To elaborate a geometric approach to fractional quantum theories, including quantum gravity, is a topic for future investigations. Working with nonholonomic distributions, such constructions can be derived fractionally from our former results on deformation quantization of gravity and Lagrange–Finsler and Hamilton–Cartan spaces [58, 60, 2], A–brane fractional quantization [64] and bi–connection quantization [71, 72]

7. *Diffusion, kinetics thermodynamics and fractional nonlinear dynamics with solitons, chaos and fractals.* This is, perhaps, the most elaborated direction in fractional calculus with a number of applications, see [9, 10, 86, 11, 38, 87, 88] and references therein. Such fractional constructions adapted to nonholonomic distributions are possible for locally anisotropic (in general, supersymmetric) stochastic calculus, see [45] and Chapters 5 and 10 in monograph [49], and kinetic theories and anisotropic thermodynamics [50, 51]. There is also recent our papers with bi–Hamilton structures, solitonic hierarchies and encoding the solutions of gravitational field equations in Einstein–Finsler gravity

and Ricci–Lagrange theories [3, 69, 59]. Solitonic hierarchies encoding of solutions of fractional systems of equations related to fundamental theories present a substantial interest for fractional dynamics, evolution and field interactions.

A series of our further works will be devoted to researches concerning above directions.

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