Differential Invariant Algebra of the Infeld-Rowlands Equation

Francis Valiquette

School of Mathematics
University of Minnesota

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Outline

Submanifold Jets

Let M be an m-dimensional manifold.

Definition

The space of extended n-jets of p-dimensional submanifolds of M at a point $z \in M$, $J^n(M,p)|_z = J^{(n)}|_z$, is given by the space of germs of p-dimensional submanifolds of M passing through z modulo the equivalence relation of n-th order contact.

Locally, a p-dimensional submanifold $S \subset M$ is given by

$$z = (x^1, \ldots, x^p, u^1(x), \ldots, u^q(x)), \qquad p + q = m.$$

In those adapted coordinates, the n-th submanifold jet coordinates are

$$z^{(n)} = j_n S|_z = (x, u^{(n)}).$$

n-th Diffeomorphism Jet Bundle

$\mathcal{D} = \mathcal{D}(M) =$ pseudo-group of all local diffeomorphisms.

Definition

The *n-th diffeomorphism jet bundle* $\mathcal{D}^{(n)}$ consists of the equivalence classes of diffeomorphisms under the equivalence relation

$$\phi(z_0) \sim \psi(z_0) \iff \sum_{0 \leq \#J \leq n} \frac{1}{J!} \frac{\partial^{\#J}\phi}{\partial z^J}\bigg|_{z_0} (z - z_0)^J = \sum_{0 \leq \#J \leq n} \frac{1}{J!} \frac{\partial^{\#J}\psi}{\partial z^J}\bigg|_{z_0} (z - z_0)^J.$$

Let $Z = \phi(z) \in \mathcal{D}$, local coordinates on $\mathcal{D}^{(n)}$ are indicated by $(z, Z^{(n)})$.

- Source map: $z = \boldsymbol{\sigma}^{(n)}(z, Z^{(n)})$.
- Target map: $Z = \boldsymbol{\tau}^{(n)}(z, Z^{(n)})$.

Lie Pseudo-Groups

Definition

$\mathcal{G} \subset \mathcal{D}$ is called a Lie pseudo-group if $\exists\ n^\star \geq 1$ such that $\forall$ finite $n \geq n^\star$:

- $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle,
- $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a bundle map,
- every smooth local solution $Z = \phi(z)$ to the determining system $\mathcal{G}^{(n)}$ belongs to $\mathcal{G}$,
- $\mathcal{G}^{(n)} = \mathrm{pr}^{(n-n^\star)}\mathcal{G}^{(n^\star)}$ is obtained by prolongation.

The minimal value of $n^\star$ is called the *order* of the Lie pseudo-group.

$\mathbf{g}^{(n)} = (z, g^{(n)})$ indicates the local coordinates of a jet $\mathbf{g}^{(n)} \in \mathcal{G}^{(n)}$, with the pseudo-group parameters $g^{(n)} = (g_1, \ldots, g_{r_n})$.

Example of a Lie Pseudo-Group

The collection of transformations of $\mathbb{R}^3$ given by

$$X = f(x), \qquad Y = f'(x)y + g(x), \qquad U = u + \frac{f''(x)y + g'(x)}{f'(x)},$$

where

$$f(x) \in \mathcal{D}(\mathbb{R}), \qquad g(x) \in C^\infty(\mathbb{R}),$$

is a Lie pseudo-group with first order determining system

$$X_y = X_u = Y_u = 0, \qquad Y_x = (U - u)X_x, \qquad U_u = 1,$$

and

$$g^{(1)} = (X, Y, X_x, Y_x, X_{xx}, Y_{xx}).$$

n-th Regularized Jet Bundle (n-th Lifted Bundle)

Definition

The local coordinates of the *n-th regularized jet bundle* $\mathcal{H}^{(n)}$ are $(z^{(n)}, \mathbf{g}^{(n)})$, where $\pi_0^n(z^{(n)}) = \boldsymbol{\sigma}^{(n)}(\mathbf{g}^{(n)})$.

- $\mathcal{G}$ acts[1] on $\mathcal{H}^{(n)}$:

$$\mathbf{g} \cdot (z^{(n)}, \mathbf{h}^{(n)}) = (\mathbf{g}^{(n)} \cdot z^{(n)}, \mathbf{h}^{(n)}(\mathbf{g}^{-1})^{(n)}) = ((\mathbf{g} \cdot z)^{(n)}, (\mathbf{h}\mathbf{g}^{-1})^{(n)}).$$

- $\mathcal{H}^{(n)}$ has a groupoid structure:

$$\widetilde{\boldsymbol{\sigma}}^{(n)}(z^{(n)}, \mathbf{g}^{(n)}) = z^{(n)}, \qquad \widetilde{\boldsymbol{\tau}}^{(n)}(z^{(n)}, \mathbf{g}^{(n)}) = \mathbf{g}^{(n)} \cdot z^{(n)}.$$

[1] Throughout the talk, mathematical expressions are assumed to be true only when they make sense

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Lifted Invariants

Definition

A function $I(z^{(n)}, \mathbf{g}^{(n)})$ is said to be a lifted invariant if

$$\mathbf{h} \cdot I(z^{(n)}, \mathbf{g}^{(n)}) = I(\mathbf{h}^{(n)} \cdot z^{(n)}, \mathbf{g}^{(n)}(\mathbf{h}^{-1})^{(n)}) = I(z^{(n)}, \mathbf{g}^{(n)}), \qquad \forall \mathbf{h} \in \mathcal{G}.$$

The components of $\widetilde{\tau}^{(n)}(z^{(n)}, \mathbf{g}^{(n)}) = \mathbf{g}^{(n)} \cdot z^{(n)}$ are lifted invariants:

$$\mathbf{h} \cdot (\mathbf{g}^{(n)} \cdot z^{(n)}) = \mathbf{g}^{(n)}(\mathbf{h}^{-1})^{(n)} \cdot \mathbf{h}^{(n)} z^{(n)} = \mathbf{g}^{(n)} \cdot z^{(n)}.$$

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Lifted Invariant Differential Operators

On $\mathcal{H}^{(\infty)}$

$$d_H X^i = \sum_{j=1}^{p} D_{x^j} X^i dx^j, \qquad i = 1, \ldots, p,$$

is a contact invariant horizontal coframe on the open dense set where $\det(D_{x^i} X^j) \neq 0$. The dual invariant differential operators are

$$D_{X^i} = \sum_{j=1}^{p} W_j^i D_{x^j}, \qquad (W_j^i) = (D_{x^j} X^i)^{-1},$$

i.e. for $F : \mathcal{H}^{(\infty)} \to \mathbb{R}$,

$$d_H F = \sum_{i=1}^{p} D_{X^i} F \, d_H X^i.$$

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Repeated applications of the differential operators

$$D_{X^i} = \sum_{j=1}^{p} W_j^i D_{x^j}, \qquad (W_j^i) = (D_{x^i} X^j)^{-1},$$

to the target coordinates U^α gives the explicit expressions for the transformed submanifold jet coordinates

$$\widehat{U}_J^\alpha = \mathbf{g}^{(n)} \cdot u_J^\alpha = D_X^J U^\alpha = D_{X^{j^1}} \cdots D_{X^{j^k}} U^\alpha,$$

where $\alpha = 1, \ldots, q$, $J = (j^1, \ldots, j^k)$ and $\#J = k \geq 0$.

Some Definitions

Definition

The pseudo-group $\mathcal{G}$ acts freely at $z^{(n)} \in J^{(n)}$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{\mathbb{1}^{(n)}_z\}$, and locally freely at $z^{(n)}$ if $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$. The pseudo-group $\mathcal{G}$ is said to act *(locally) freely at order n* if it acts (locally) freely on an open subset $\mathcal{V}^{(n)} \subset J^{(n)}$, called the set of *regular n-jets*.

Definition

A cross-section to the pseudo-group orbits is a transverse submanifold to the orbits of complementary dimension.

Equivariant Moving Frame

Definition

A right moving frame $\rho^{(n)}$ of order n is a right $\mathcal{G}^{(n)}$-equivariant local section of the bundle $\mathcal{H}^{(n)} \to J^{(n)}$, i.e. $\rho^{(n)} : J^{(n)} \to \mathcal{H}^{(n)}$ satisfies

$$\widetilde{\sigma}^{(n)}(\rho^{(n)}(z^{(n)})) = z^{(n)}, \qquad \rho^{(n)}(\mathbf{g}^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (\mathbf{g}^{(n)})^{-1},$$

Theorem

Suppose $\mathcal{G}^{(n)}$ acts freely and regularly on $\mathcal{V}^{(n)} \subset J^{(n)}$. Let $\mathcal{K}^{(n)} \subset \mathcal{V}^{(n)}$ be a (local) cross-section to the pseudo-group orbits. Given $z^{(n)} \in \mathcal{V}^{(n)}$, define $\rho^{(n)}(z^{(n)}) \in \mathcal{H}^{(n)}$ to be the unique groupoid jet such that $\widetilde{\tau}^{(n)}(\rho^{(n)}(z^{(n)})) \in \mathcal{K}^{(n)}$. Then $\rho^{(n)} : J^{(n)} \to \mathcal{H}^{(n)}$ is a (right) moving frame for $\mathcal{G}$ defined on an open subset of $\mathcal{V}^{(n)}$.

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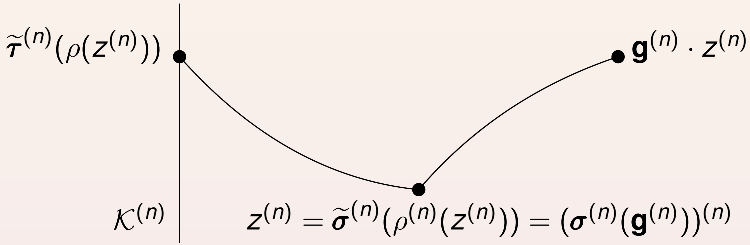

Figure: Moving frame.

Moving Frame Construction

- Compute

$$\widehat{U}^\alpha_J = D^J_X U^\alpha, \qquad D^J_X = D_{X^{j_1}} \cdots D_{X^{j_{\#J}}},$$

where $D_{X^i} = \sum_{j=1}^p W^i_j D_{x^j}, \qquad (W^i_j) = (D_{x^i} X^j)^{-1}$.

- Fix r_n transformed coordinates $(X, \widehat{U}^{(n)}) = F^{(n)}(z^{(n)}, g^{(n)})$ to be constant

$$F_1(z^{(n)}, g^{(n)}) = c_1 \qquad \ldots \qquad F_{r_n}(z^{(n)}, g^{(n)}) = c_{r_n}.$$

- Solve the normalization equations for the pseudo-group parameters $g^{(n)} = g^{(n)}(z^{(n)})$.
- The n-th order moving frame is given by

$$\rho^{(n)}(z^{(n)}) = (z^{(n)}, g^{(n)}(z^{(n)})).$$

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Invariantization

Definition

The lift of a differential form $\omega \in \Lambda^*(J^{(\infty)})$ is the jet form

$$\boldsymbol{\lambda}(\omega) = \pi_J((\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega).$$

Definition

Let $\rho^{(\infty)} : J^{(\infty)} \to \mathcal{H}^{(\infty)}$ be a complete moving frame. The invariantization of $\omega \in \Lambda^*(J^{(\infty)})$ is the invariant differential form

$$\iota(\omega) = (\rho^{(\infty)})^*[\boldsymbol{\lambda}(\omega)].$$

In particular

$$\iota(x^i, u^\alpha_J) = (H^i, I^\alpha_J), \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0.$$

Infeld-Rowlands Equation

The infinitesimal determining system of the Infeld-Rowlands equation

$$\Delta_{IR} = u_t + 2u_x u_{xx} + u_{xxxx} + u_{xy} = 0$$

for a symmetry generator

$$\mathbf{v} = \xi(x, y, t, u)\partial_x + \eta(x, y, t, u)\partial_y + \tau(x, y, t, u)\partial_t + \phi(x, y, t, u)\partial_u,$$

is

$$\xi_t = \xi_u = 0, \qquad \xi_{xx} = \xi_{xy} = 0, \qquad \eta_x = \eta_t = \eta_u = 0,$$
$$\eta_y = 3\xi_x, \qquad \tau_x = \tau_y = \tau_u = 0, \qquad \tau_t = 4\xi_x,$$
$$\phi_u = -\xi_x, \qquad \phi_x = \frac{1}{2}\xi_y, \qquad \phi_t = -\frac{1}{2}\xi_{yy}.$$

Pseudo-Group Action

The solution is

$$\begin{aligned}
\xi(x,y,t,u) =& \lambda x + f(y),\\
\eta(x,y,t,u) =& \alpha + 3\lambda y,\\
\tau(x,y,t,u) =& \epsilon + 4\lambda t,\\
\phi(x,y,t,u) =& -\lambda u + \frac{x}{2} f'(y) - \frac{t}{2} f''(y) + g(y),
\end{aligned}$$

with pseudo-group action

$$\begin{aligned}
X =& \lambda x + F(Y),\\
Y =& \lambda^3 y + \alpha,\\
T =& \lambda^4 t + \epsilon,\\
U =& \frac{u}{\lambda} + \frac{X}{2} F'(Y) - \frac{T}{2} F''(Y) + G(Y).
\end{aligned}$$

Lifted Differential Operators

- Lifted horizontal coframe

$$d_H X = \lambda dx + \lambda^3 F'(Y) dy, \qquad d_H Y = \lambda^3 dy, \qquad d_H T = \lambda^4 dt.$$

- Lifted differential operators

$$D_X = \frac{1}{\lambda} D_x, \qquad D_Y = \frac{1}{\lambda^3}(-\lambda^2 F'(Y) D_x + D_y), \qquad D_T = \frac{1}{\lambda^4} D_t.$$

Prolonged Pseudo-Group Action

$$
\begin{aligned}
\widehat{U}_X &= \frac{u_x}{\lambda^2} + \frac{1}{2}F'(Y),\\
\widehat{U}_Y &= -\frac{u_x}{\lambda^2}F'(Y) + \frac{u_y}{\lambda^4} + \frac{X}{2}F''(Y) - \frac{T}{2}F'''(Y) + G'(Y),\\
\widehat{U}_T &= \frac{u_t}{\lambda^5} - \frac{1}{2}F''(Y),\\
\widehat{U}_{XX} &= \frac{u_{xx}}{\lambda^3},\\
\widehat{U}_{XY} &= -\frac{u_{xy}}{\lambda^3}F'(Y) + \frac{u_{xy}}{\lambda^5} + \frac{1}{2}F''(Y),\\
\widehat{U}_{YY} &= -\frac{u_{xx}}{\lambda^3}F'(Y)^2 - 2\frac{u_{xy}}{\lambda^5}F'(Y) + \frac{u_{yy}}{\lambda^7} + \frac{X}{2}F'''(Y)\\
&\quad - \frac{T}{2}F''''(Y) + G''(Y),
\end{aligned}
$$

$$\widehat{U}_{XT} = \frac{u_{xt}}{\lambda^6},$$

$$\widehat{U}_{TT} = \frac{u_{tt}}{\lambda^9},$$

$$\widehat{U}_{YT} = -\frac{u_{xt}}{\lambda^6} F'(Y) + \frac{u_{yt}}{\lambda^8} - \frac{1}{2} F'''(Y),$$

$$\vdots$$

$$\widehat{U}_{XXXX} = \frac{u_{xxxx}}{\lambda^5},$$

$$\vdots$$

Normalization

- Cross-section:

$$\widehat{U}_{XX} = 1, \quad X = Y = T = U = \widehat{U}_X = \widehat{U}_{TY^k} = \widehat{U}_{Y^{k+1}} = 0,$$

$k \geq 0$.

- Normalized pseudo-group parameters:

$$\lambda = u_{xx}^{1/3}, \qquad \alpha = -yu_{xx}, \qquad \epsilon = -tu_{xx}^{4/3},$$

$$F(Y) = -xu_{xx}^{1/3}, \qquad G(Y) = -\frac{u}{u_{xx}^{1/3}} \qquad F'(Y) = -2\frac{u_x}{u_{xx}^{2/3}},$$

$$G'(Y) = -\frac{2u_x^2 + u_y}{u_{xx}^{4/3}}, \qquad F''(Y) = 2\frac{u_t}{u_{xx}^{5/3}},$$

$$\vdots$$

Normalized Differential Invariants

$$
\begin{aligned}
I_{1,1,0} =& \iota(u_{xy}) = 2\frac{u_x}{u_{xx}^{2/3}} + \frac{u_{xy}}{u_{xx}^{5/3}} + \frac{u_t}{u_{xx}^{5/3}}, \\
I_{1,0,1} =& \iota(u_{xt}) = \frac{u_{xt}}{u_{xx}^2}, \\
I_{0,0,2} =& \iota(u_{tt}) = \frac{u_{tt}}{u_{xx}^3}, \\
& \vdots \\
I_{4,0,0} =& \frac{u_{xxxx}}{u_{xx}^{5/3}}, \\
& \vdots
\end{aligned}
$$

Recurrence Formula

Theorem

Let $\omega \in \Lambda^*(J^{(\infty)})$, *then*

$$d[\iota(\omega)] = \iota[d\omega + \mathbf{v}^{(\infty)}(\omega)].$$

Definition

The lift of a vector field jet coordinate is

$$\boldsymbol{\lambda}(\zeta^a_J) = \mu^a_J, \qquad a = 1, \ldots, m, \qquad \#J \geq 0,$$

and more generally,

$$\boldsymbol{\lambda}\left(\sum_{a=1}^{m} \sum_{\#J \leq n} \zeta^a_J \omega^a_J\right) = \sum_{a=1}^{m} \sum_{\#J \leq n} \mu^a_J \wedge \boldsymbol{\lambda}(\omega^a_J), \qquad \omega^a_J \in \Lambda^*(J^{(\infty)}).$$

Theorem

The recurrence formulas for the normalized invariants are

$$\sum_{j=1}^{p} (\mathcal{D}_j H^i)\varpi^j = \pi_H(\iota[dx^i + \xi^i]) = \varpi^i + \pi_H(\iota(\xi^i)), \qquad i = 1, \ldots, p,$$

$$\sum_{j=1}^{p} (\mathcal{D}_j I^\alpha_J)\varpi^j = \pi_H(\iota[\sum_{j=1}^{p} u^\alpha_{J,j} dx^j + \theta^\alpha_J + \phi^\alpha_J]) = \sum_{j=1}^{p} I^\alpha_{J,j}\varpi^j + \pi_H(\iota(\phi^\alpha_J)),$$

where $\alpha = 1, \ldots, q$, $\#J \geq 0$, $\varpi^i = \iota(dx^i)$ *and* $\mathcal{D}_i$ *are the dual invariant differential operators to* ϖ^i, $i = 1, \ldots, p$.

Commutation Relations

From the recurrence relations

$$d_H\varpi^k = \pi_H(d\iota(dx^k)) = \pi_H(\iota[d^2x^k + d\xi^k]) = \pi_H(\iota[d\xi^k])$$
$$= \sum_{1\leq i<j\leq p} C_{ij}^k \varpi^i \wedge \varpi^j,$$

$k = 1, \ldots, p$, we obtain the commutator relations

$$[\mathcal{D}_i, \mathcal{D}_j] = -\sum_{k=1}^{p} C_{ij}^k \mathcal{D}_k.$$

Infeld-Rowlands Equation (Continuation)

$$D_X = \frac{1}{\lambda} D_x, \qquad D_Y = \frac{1}{\lambda^3}(-\lambda^2 F'(Y) D_x + D_y), \qquad D_T = \frac{1}{\lambda^4} D_t.$$

$$\lambda = u_{xx}^{1/3} \qquad \Downarrow \qquad F'(Y) = -2\frac{u_x}{u_{xx}^{2/3}}$$

$$\mathcal{D}_1 = \frac{1}{u_{xx}^{1/3}} D_x, \qquad \mathcal{D}_2 = \frac{1}{u_{xx}}(2u_x D_x + D_y), \qquad \mathcal{D}_3 = \frac{1}{u_{xx}^{4/3}} D_t.$$

Prolonged Vector Field Coefficients

Let

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be an infinitesimal symmetry generator, then the prolonged vector field coefficients are given by the recursive formula

$$\phi_\alpha^{J,j} = D_j\phi_\alpha^J - \sum_{i=1}^{p} D_j\xi^i \cdot u_{J,i}^\alpha.$$

For the Infeld-Rowlands equation

$$\phi^x = \frac{1}{2}\xi_y - 2\xi_x u_x, \qquad \phi^y = \phi_y - 4\xi_x u_y - \xi_y u_x, \qquad \phi^t = -\frac{1}{2}\xi_{yy} - 5\xi_x u_t,$$

$$\phi^{xx} = -3\xi_x u_{xx}, \qquad \phi^{xy} = \frac{1}{2}\xi_{yy} - 5\xi_x u_{xy} - \xi_y u_{xx}, \qquad \phi^{xt} = -6\xi_x u_{xt},$$

$$\phi^{yy} = \phi_{yy} - 7\xi_x u_{yy} - 2\xi_y u_{xy}, \quad \phi^{tt} = -9\xi_x u_{tt}, \quad \phi^{yt} = -\frac{1}{2}\xi_{yyy} - 8\xi_x u_{yt} - \xi_y u_{xt}.$$

Recurrence Relations

Let

$$\nu_k = \pi_H \circ \iota(\xi_{y^k}), \qquad \mu_k = \pi_H \circ \iota(\phi_{y^k}), \qquad k \leq 0,$$
$$\gamma = \pi_H \circ \iota(\xi_x), \qquad \alpha = \pi_H \circ \iota(\eta), \qquad \beta = \pi_H \circ \iota(\tau),$$

then the recurrence relations are given by

$$0 = \varpi^1 + \nu_0,$$
$$0 = \varpi^2 + \alpha,$$
$$0 = \varpi^3 + \beta,$$
$$0 = \mu_0,$$
$$0 = \varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3 + \frac{1}{2}\nu_1,$$
$$0 = I_{1,1,0}\varpi^1 + \mu_1,$$
$$0 = I_{1,0,1}\varpi^1 + I_{0,0,2}\varpi^3 - \frac{1}{2}\nu_2,$$

$$0 = I_{3,0,0}\varpi^1 + I_{2,1,0}\varpi^2 + I_{2,0,1}\varpi^3 - 3\gamma,$$

$$\sum_{i=1}^{3} \mathcal{D}_i I_{1,1,0} = I_{2,1,0}\varpi^1 + I_{1,2,0}\varpi^2 + I_{1,1,1}\varpi^3 + \frac{1}{2}\nu_2 - 5I_{1,1,0}\gamma - \nu_1,$$

$$\sum_{i=1}^{3} \mathcal{D}_i I_{1,0,1} = I_{2,0,1}\varpi^1 + I_{1,1,1}\varpi^2 + I_{1,0,2}\varpi^3 - 6I_{1,0,1}\gamma,$$

$$0 = I_{1,2,0}\varpi^1 + \mu_2 - 2I_{1,1,0}\nu_1,$$

$$0 = I_{1,1,1}\varpi^1 + I_{0,1,2}\varpi^3 - \frac{1}{2}\nu_3 - I_{1,0,1}\nu_1,$$

$$\sum_{i=1}^{3} \mathcal{D}_i I_{0,0,2} = I_{1,0,2}\varpi^1 + I_{0,1,2}\varpi^2 + I_{0,0,3}\varpi^3 - 9I_{0,0,2}\gamma,$$

$$\vdots$$

$$
\begin{aligned}
\nu_0 &= -\varpi^1, \\
\alpha &= -\varpi^2, \\
\beta &= -\varpi^3, \\
\mu_0 &= 0, \\
\mu_1 &= -I_{1,1,0}\varpi^1, \\
\nu_1 &= -2(\varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3), \\
\nu_2 &= 2(I_{1,0,1}\varpi^1 + I_{0,0,2}\varpi^3), \\
\gamma &= \frac{1}{3}(I_{3,0,0}\varpi^1 + I_{2,1,0}\varpi^2 + I_{2,0,1}\varpi^3), \\
\mu_2 &= -I_{1,2,0}\varpi^1 - 4I_{1,1,0}(\varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3), \\
\nu_3 &= 2I_{1,1,1}\varpi^2 + 2I_{0,1,2}\varpi^3 + 4I_{1,0,1}(\varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3), \\
&\;\;\vdots
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_1 I_{1,1,0} &= I_{2,1,0} + I_{1,0,1} - \frac{5}{3} I_{1,1,0} I_{3,0,0} + 2, \\
\mathcal{D}_2 I_{1,1,0} &= I_{1,2,0} - \frac{5}{3} I_{1,1,0} I_{2,1,0} + 2 I_{1,1,0}, \\
\mathcal{D}_3 I_{1,1,0} &= I_{1,1,1} + I_{0,0,2} - \frac{5}{3} I_{1,1,0} I_{2,0,1} + 2 I_{1,0,1}, \\
\mathcal{D}_1 I_{1,0,1} &= I_{2,0,1} - 2 I_{1,0,1} I_{3,0,0}, \\
\mathcal{D}_2 I_{1,0,1} &= I_{1,1,1} - 2 I_{1,0,1} I_{2,1,0}, \\
\mathcal{D}_3 I_{1,0,1} &= I_{1,0,2} - 2 I_{1,0,1} I_{2,0,1}, \\
\mathcal{D}_1 I_{0,0,2} &= I_{1,0,2} - 3 I_{0,0,2} I_{3,0,0}, \\
\mathcal{D}_2 I_{0,0,2} &= I_{0,1,2} - 3 I_{0,0,2} I_{2,1,0}, \\
\mathcal{D}_3 I_{0,0,2} &= I_{0,0,3} - 3 I_{0,0,2} I_{2,0,1}.
\end{aligned}
$$

We can express the differential invariants $I_{2,1,0}$, $I_{1,2,0}$, $I_{1,1,1}$, $I_{2,0,1}$, $I_{1,0,2}$, $I_{0,1,2}$, $I_{0,0,3}$ and $I_{0,0,2}$ in terms of $I_{1,1,0}$, $I_{1,0,1}$ $I_{3,0,0}$ and their invariant derivatives

Commutation Relations

$$[\mathcal{D}_1, \mathcal{D}_2] = \left(\frac{I_{2,1,0}}{3} + 2\right)\mathcal{D}_1 - I_{3,0,0}\mathcal{D}_2,$$
$$[\mathcal{D}_1, \mathcal{D}_3] = \frac{I_{2,0,1}}{3}\mathcal{D}_1 - \frac{4}{3}I_{3,0,0}\mathcal{D}_3,$$
$$[\mathcal{D}_2, \mathcal{D}_3] = -2I_{1,0,1}\mathcal{D}_1 + I_{2,0,1}\mathcal{D}_2 - \frac{4}{3}I_{2,1,0}\mathcal{D}_3.$$

Hence

$$[\mathcal{D}_1, \mathcal{D}_3]I_{1,0,1} = \frac{1}{3}(\mathcal{D}_1 I_{1,0,1} + 2I_{1,0,1}I_{3,0,0})\mathcal{D}_1 I_{1,0,1} - \frac{4}{3}I_{3,0,0}\mathcal{D}_3 I_{1,0,1}.$$

Provided that $I_{1,0,1}\mathcal{D}_1 I_{1,0,1} - 2\mathcal{D}_3 I_{1,0,1} \neq 0$

$$I_{3,0,0} = \frac{3}{2I_{1,0,1}\mathcal{D}_1 I_{1,0,1} - 4\mathcal{D}_3 I_{1,0,1}}\left([\mathcal{D}_1, \mathcal{D}_3]I_{1,0,1} - \frac{(\mathcal{D}_1 I_{1,0,1})^2}{3}\right).$$

Generating Set

Proposition

The algebra of differential invariants for the Infeld-Rowlands equation is generated by

$$I_{1,1,0} \qquad \text{and} \qquad I_{1,0,1}$$

(and the invariant differential operators $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$)

Recall

$$I_{1,1,0} = \iota(u_{xy}) = 2\frac{u_x}{u_{xx}^{2/3}} + \frac{u_{xy}}{u_{xx}^{5/3}} + \frac{u_t}{u_{xx}^{5/3}},$$

$$I_{1,0,1} = \iota(u_{xt}) = \frac{u_{xt}}{u_{xx}^2},$$

and

$$\mathcal{D}_1 = \frac{1}{u_{xx}^{1/3}} D_x, \qquad \mathcal{D}_2 = \frac{1}{u_{xx}}(2u_x D_x + D_y), \qquad \mathcal{D}_3 = \frac{1}{u_{xx}^{4/3}} D_y.$$

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