Equivalence Problems

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September 16, 2008

Outline

Equivalence Problem

Definition

Let M and $\overline{M}$ be two

- smooth manifolds (with maybe extra structure, e.g. a Riemannian metric),
- variational problems,
- differential systems,
- $\ldots$,

they are said to be locally equivalent if there exists a local diffeomorphism

$$\Phi : M \to \overline{M}.$$

Question: Given M and $\overline{M}$ (in local coordinates), how can we determine if they are locally equivalent? (Obviously $\dim M = \dim \overline{M} = m$!)

Differential Operator

Let M be a differential manifold with local coordinates $x = (x^1, \ldots, x^m)$. The differential of the one-form

$$\theta = \sum_{i=1}^{m} f_i(x) dx^i$$

is

$$d\theta = \sum_{i=1}^{m} df_i(x) \wedge dx^i = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial f^i}{\partial x^j} dx^j \wedge dx^i, \qquad (dx^j \wedge dx^i = -dx^i \wedge dx^j).$$

More generally, the differential of an r-form

$$\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq m} f_{i_1,\ldots,i_r}(x) \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}$$

where $\{\theta^i : \; i = 1, \ldots, m\}$ is a basis of one-forms, is

$$d\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq m} (df_{i_1,\ldots,i_r}(x) \wedge \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} + f_{i_1,\ldots,i_r}(x) d\theta^{i_1} \wedge \theta^2 \wedge \cdots \wedge \theta^{i_r}$$
$$- f_{i_1,\ldots,i_r}(x) \theta^{i_1} \wedge d\theta^{i_2} \wedge \cdots \wedge \theta^{i_r} + (-1)^r f_{i_1,\ldots,i_r}(x) \theta^{i_1} \wedge \theta^2 \wedge \cdots \wedge d\theta^{i_r})$$

Coframe

Definition

Let M be a smooth m-dimensional manifold. A coframe on M is an ordered set of 1-forms $\{\theta^1, \ldots, \theta^m\}$ which forms a basis of $T^*M|_x$ $\forall x \in M$.

$\theta^1, \ldots, \theta^m$ are linearly independent if

$$\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m \neq 0.$$

Definition

The structure equations of a coframe are

$$d\theta^i = \sum_{1 \leq j < k \leq m} T^i_{jk} \theta^j \wedge \theta^k, \quad i = 1, \ldots, m.$$

The T^i_{jk} are referred to as the structure coefficients of the coframe.

Example

The two 1-forms

$$\theta^1 = A(x,y)dx + B(x,y)dy, \quad \theta^2 = C(x,y)dx + D(x,y)dy,$$

form a coframe on $\mathbb{R}^2$ if and only if they are linearly independent

$$0 \neq \theta^1 \wedge \theta^2 = (AD - BC)dx \wedge dy.$$

Hence we must have

$$AD - BC \neq 0.$$

The structure equations are

$$\begin{aligned} d\theta^1 =& dA \wedge dx + dB \wedge dy = (-A_y + B_x)dx \wedge dy = J\theta^1 \wedge \theta^2, \\ d\theta^2 =& dC \wedge dx + dD \wedge dy = (-C_y + D_x)dx \wedge dy = K\theta^1 \wedge \theta^2, \end{aligned}$$

where

$$T^1_{12} = J = \frac{B_x - A_y}{(AD - BC)}, \quad T^2_{12} = K = \frac{D_x - C_y}{(AD - BC)}.$$

Pull-Back of Differential Forms

Let M and $\overline{M}$ be two manifolds with local coordinate systems $x = (x^1, \ldots, x^m)$, and $\overline{x} = (\overline{x}^1, \ldots, \overline{x}^m)$ respectively. The pull-back of the one-form

$$\overline{\theta} = \sum_{i=1}^{m} \overline{f}_i(\overline{x}) d\overline{x}^i$$

by $\Phi : M \to \overline{M}$, $\overline{x}^i = \Phi^i(x)$, $i = 1, \ldots, m$, is

$$\Phi^*(\overline{\theta}) = \sum_{i=1}^{m} \overline{f}_i(\Phi(x)) d\Phi^i(x) = \sum_{i=1}^{m}\sum_{j=1}^{m} \overline{f}_i(\Phi(x)) \frac{\partial \Phi^i}{\partial x^j} dx^j.$$

More generally, the pull-back of an r-form

$$\overline{\omega} = \sum_{1 \le i_1 < \cdots < i_r \le m} \overline{f}_{i_1,\ldots,i_r}(\overline{x}) \overline{\theta}^{i_1} \wedge \cdots \wedge \overline{\theta}^{i_r}$$

is

$$\Phi^*(\overline{\omega}) = \sum_{1 \le i_1 < \cdots < i_r \le m} \overline{f}_{i_1,\ldots,i_r}(\Phi(x))\ \Phi^*(\overline{\theta}^{i_1}) \wedge \cdots \wedge \Phi^*(\overline{\theta}^{i_r}).$$

Example

Consider the map

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (\overline{x}, \overline{y}) = (x^2, y^3).$$

The pull-back of the differential form

$$\omega = \overline{x} d\overline{y} + \overline{y} d\overline{x}$$

is

$$\begin{aligned} \Phi^*(\omega) &= x^2 d(y^3) + y^3 d(x^2) \\ &= 3x^2 y^2 dy + 2y^3 x dx. \end{aligned}$$

Equivalence of Coframes

Let $\theta = \{\theta^1, \ldots, \theta^m\}$ be a coframe on M and $\overline{\theta} = \{\overline{\theta}^1, \ldots, \overline{\theta}^m\}$ a coframe on $\overline{M}$. The equivalence problem of coframes consists of determining if there exists a diffeomorphism

$$\Phi : M \to \overline{M}$$

such that

$$\Phi^* \overline{\theta}^i = \theta^i, \quad i = 1, \ldots, m.$$

If θ is equivalent to $\overline{\theta}$ then

$$\sum_{1 \le j < k \le m} T^i_{jk}(x) \theta^j \wedge \theta^k = d\theta^i = \Phi^* d\overline{\theta}^i = \sum_{1 \le j < k \le m} \overline{T}^i_{jk}(\Phi(x)) \theta^j \wedge \theta^k.$$

Hence

$$\overline{T}^i_{jk}(\overline{x}) = T^i_{jk}(x), \quad \overline{x} = \Phi(x).$$

The T^i_{jk} are called invariants of the equivalence problem.

Dual Differential Operators to a Coframe

Definition

The coframe derivative ∂_{θ^i} associated to the coframe θ^i, $i = 1, \ldots, m$, on an m-dimensional manifold M are defined by the equality

$$dF(x) = \sum_{i=1}^{m} \frac{\partial F}{\partial \theta^i} \theta^i = \sum_{i=1}^{m} \frac{\partial F}{\partial x^i} dx^i.$$

The second equality gives the explicit expression of ∂_{θ^i} in a chart coordinatized by $x = (x^1, \ldots, x^m)$. Suppose

$$\theta^i = \sum_{j=1}^{m} W^i_j(x) dx^j, \qquad i = 1, \ldots, m,$$

then

$$\frac{\partial}{\partial \theta^i} = \sum_{j=1}^{m} M^j_i(x) \frac{\partial}{\partial x^j}, \qquad i = 1, \ldots, m,$$

where $(M^j_i) = (W^i_j)^{-1}$.

Example

The coframe derivatives associated to the coframe

$$\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} Adx + Bdy \\ Cdx + Ddy \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}, \qquad AD - BC \neq 0,$$

are

$$\begin{pmatrix} \partial_{\theta^1} \\ \partial_{\theta^2} \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \frac{1}{AD - BC} \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

i.e.

$$\begin{aligned} \partial_{\theta^1} &= \frac{1}{AD - BC}(D\partial_x - C\partial_y), \\ \partial_{\theta^2} &= \frac{1}{AD - BC}(-B\partial_x + A\partial_y). \end{aligned}$$

Derived Invariants

Let $I(x)$ be a scalar invariant function, such that $\bar{I}(\bar{x}) = \bar{I}(\Phi(x)) = I(x)$, then

$$\sum_{i=1}^{m} \frac{\partial I}{\partial \theta^i}(x)\theta^i = dI = \Phi^* d\bar{I} = \sum_{i=1}^{m} \frac{\partial \bar{I}}{\partial \bar{\theta}^i}(\Phi(x))\theta^i.$$

So

$$\frac{\partial \bar{I}}{\partial \bar{\theta}^i}(\bar{x}) = \frac{\partial I}{\partial \theta^i}(x),$$

are also invariant scalar functions.

Structure Map

Let θ^i, $i = 1, \ldots, m$, be a coframe on M such that

$$d\theta^i = \sum_{1 \leq j < k \leq m} T^i_{jk} \theta^j \wedge \theta^k.$$

Definition

The scalar functions

$$T_\sigma = \frac{\partial^s T^i_{jk}}{\partial \theta^{l_s} \partial \theta^{l_{s-1}} \ldots \partial \theta^{l_1}}, \quad \sigma = (i, j, k, l_1, \ldots, l_s),$$

are called structure invariants.

Definition

The s-th order structure map associated with a coframe θ on M is the map $T^{(s)} : M \to \mathbb{R}^{(s)}$ whose components are the structure invariants $T_\sigma(x)$ of order $\sigma \leq s$.

Signature

Definition

The s-th order signature $\mathcal{S}^{(s)} = \mathcal{S}^{(s)}(\theta, U)$ associated with θ on an open subset $U \subset M$ is defined as the image of the structure map $T^{(s)}$:

$$\mathcal{S}^{(s)} = \{T^{(s)}(x) : x \in U\} \subset \mathbb{R}^{(s)}.$$

Definition

A coframe θ is called fully regular if, for each $s \geq 0$, the s-th order structure map $T^{(s)} : M \to \mathbb{R}^{(s)}$ is regular.

Definition

Let $\mathcal{F}$ be a family of smooth real-valued functions $f : M \to \mathbb{R}$. The rank of $\mathcal{F}$ at a point $x \in M$ is the dimension of the space spanned by their differentials. The family is regular if its rank is constant on M.

Theorem

Let θ and $\overline{\theta}$ be smooth, fully regular coframes, defined, respectively, on m-dimensional manifolds M and $\overline{M}$. There exists a local diffeomorphism $\Phi : M \to \overline{M}$ mapping the coframes to each other, $\Phi^* \overline{\theta} = \theta$ if and only if for each $s \geq 0$

$$\mathcal{S}^{(s)}(\theta) = \mathcal{S}^{(s)}(\overline{\theta}).$$

Proposition

Let θ be a fully regular coframe, and let ρ_s denote the rank of the s-th order structure map $T^{(s)}$. The smallest s for which $\rho_s = \rho_{s+1}$ is called the order of the coframe, and we have

$$0 \leq \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_s = \rho_{s+1} = \rho_{s+2} = \cdots = r \leq m.$$

The stabilizing rank r is referred to as the rank of the coframe.

Theorem

Let θ and $\overline{\theta}$ be smooth, fully regular coframes defined, respectively, on m-dimensional manifolds M and $\overline{M}$. There exists a local diffeomorphism $\Phi : M \to \overline{M}$ mapping the coframes to each other, $\Phi^*\overline{\theta} = \theta$, if and only if they have the same order order $\overline{s} = s$, and their $(s+1)$-signatures $\mathcal{S}^{(s+1)}(\theta)$ and $\mathcal{S}^{(s+1)}(\overline{\theta})$ overlap.

Corollary

Let θ and $\overline{\theta}$ be two coframes of rank zero defined, respectively on M and $\overline{M}$, having the same constant structure functions. Then, for any $x_0 \in M$ and $\overline{x}_0 \in \overline{M}$, there exists a unique local diffeomorphism $\Phi : M \to \overline{M}$ such that $\overline{x}_0 = \Phi(x_0)$ and

$$\Phi^*\overline{\theta}^i = \theta^i, \qquad i = 1, \ldots, m.$$

Example

Does there exist a change of variables mapping the coframe (defined on $\{(x, y) : x + y > 0\}$)

$$\theta^1 = \frac{dx}{x+y}, \qquad \theta^2 = \frac{2dy}{x+y}$$

to

$$\overline{\theta}^1 = \frac{2d\overline{x}}{\overline{x}+\overline{y}}, \qquad \overline{\theta}^2 = \frac{4d\overline{y}}{\overline{x}+\overline{y}}?$$

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Answer: No.

Example

Does there exist a change of variables mapping the coframe (defined on $\{(x, y) : x + y > 0\}$)

$$\theta^1 = \frac{dx}{x + y}, \qquad \theta^2 = \frac{2dy}{x + y}$$

to

$$\overline{\theta}^1 = \frac{2d\overline{x}}{\overline{x} + \overline{y}}, \qquad \overline{\theta}^2 = \frac{4d\overline{y}}{\overline{x} + \overline{y}}?$$

Answer: No. Since

$$J = \frac{B_x - A_y}{(AD - BC)} = \frac{1}{2}, \qquad K = \frac{D_x - C_y}{(AD - BC)} = -1,$$

and

$$\overline{J} = \frac{\overline{B}_{\overline{x}} - \overline{A}_{\overline{y}}}{(\overline{AD} - \overline{BC})} = \frac{1}{4}, \qquad \overline{K} = \frac{\overline{D}_{\overline{x}} - \overline{C}_{\overline{y}}}{(\overline{AD} - \overline{BC})} = -\frac{1}{2}.$$

General Equivalence Problem

Definition

Let $G \subset GL(m)$ be a Lie group. Let ω and $\overline{\omega}$ be coframes defined, respectively on m-dimensional manifolds M and $\overline{M}$. The G-valued equivalence problem consists of determining if there exists a (local) diffeomorphism $\Phi : M \to \overline{M}$ and a G-valued function $g : M \to G$ such that

$$\Phi^*\overline{\omega} = g(x)\omega.$$

What is going on?

- Suppose there exist a diffeomorphism $\Phi : M \to \overline{M}$. The probability that one is able to come up with two coframes ω, and $\overline{\omega}$ such that $\Phi^*(\overline{\omega}^i) = \omega^i$, $i = 1, \ldots, m$ is very low.
- It is more probable that the chosen coframes will satisfy a linear relation of the form $\Phi^*\overline{\omega} = g(x)\omega$.
- How can one choose $g(x)$ and $\overline{g}(\overline{x})$ such that the new coframes $\theta = g(x)\omega$, $\overline{\theta} = \overline{g}(\overline{x})\overline{\omega}$ satisfy $\Phi^*(\overline{\theta}^i) = \theta^i$. (Cartan to the rescue!!!)
- The problem has been reduced to the equivalence problem of coframes.

Proposition

Let $G \subset GL(n)$, a basis for the space of Maurer-Cartan forms are found among the entries of the matrix 1-forms

$$\gamma = dg \cdot g^{-1}.$$

The matrix of one-forms $\gamma = dg \cdot g^{-1}$ is right invariant since

$$R_h^*(dg \cdot g^{-1}) = d(gh)(gh)^{-1} = (dg)hh^{-1}g^{-1} = dg \cdot g^{-1}.$$

Example

The Maurer-Cartan form of

$$SO(2) = \left\{ g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

is found as the entry of the matrix

$$dg \cdot g^{-1} = \begin{pmatrix} -\sin\theta d\theta & \cos\theta d\theta \\ -\cos\theta d\theta & -\sin\theta d\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & d\theta \\ -d\theta & 0 \end{pmatrix}.$$

Structure Equations

Definition

The lifted coframe θ of a coframe $\omega = \{\omega^1, \ldots, \omega^m\}$ on an m-dimensional manifold M is defined to be

$$\theta = g \cdot \omega, \quad \theta^i = \sum_{j=1}^{m} g^i_j \omega^j.$$

The differentials of the lifted coframe are

$$\begin{aligned} d\theta^i &= \sum_{j=1}^{m} (dg^i_j \wedge \omega^j + g^i_j d\omega^j) \\ &= \sum_{j=1}^{m} \gamma^i_j \wedge \theta^j + \sum_{1 \leq j < k \leq m} T^i_{jk}(x, g)\theta^j \wedge \theta^k, \end{aligned}$$

$i = 1, \ldots, m$. The coefficients T^i_{jk} are called torsion coefficients

Let $\alpha^1, \ldots, \alpha^r$ be a basis of Maurer-Cartan forms, then

$$\gamma^i_j = \sum_{\kappa=1}^{r} A^i_{j\kappa}\alpha^\kappa, \quad i,j = 1, \ldots, m,$$

and the structure equations are

$$d\theta^i = \sum_{\kappa}^{r}\sum_{j=1}^{m} A^i_{j\kappa}\alpha^\kappa \wedge \theta^j + \sum_{1\le j<k\le m} T^i_{jk}(x,g)\theta^j \wedge \theta^k.$$

Similarly, on the manifold $\overline{M}$ we have

$$d\overline{\theta}^i = \sum_{\kappa}^{r}\sum_{j=1}^{m} A^i_{j\kappa}\overline{\alpha}^\kappa \wedge \overline{\theta}^j + \sum_{1\le j<k\le m} \overline{T}^i_{jk}(x,g)\overline{\theta}^j \wedge \overline{\theta}^k.$$

Absorption and Normalization

The specification of the group parameters as functions $g = g(x)$ of $x \in M$ reduces the Maurer-Cartan forms

$$\widetilde{\alpha}^\kappa = \sum_{j=1}^{m} z_j^\kappa(x)\theta^j.$$

The problem is that we don't know $g(x)$ yet, hence the z's are also unknown. Replacing the Maurer-Cartan forms

$$\alpha^\kappa \mapsto \sum_{j=1}^{m} z_j^\kappa \theta^j$$

in the structure equations of the lifted coframe we obtain

$$\Theta^i = \sum_{1\leq j<k\leq m} (B_{jk}^i[\mathbf{z}] + T_{jk}^i(x,g))\theta^j \wedge \theta^k,$$

$i = 1, \ldots, m$, with $B_{jk}^i[z] = \sum_{\kappa=1}^{r}(A_{k\kappa}^i z_j^\kappa - A_{jk}^i z_k^\kappa)$.

Similarly for the barred coframe

$$\overline{\Theta}^i = \sum_{1 \le j < k \le m} (B^i_{jk}[\overline{\mathbf{z}}] + \overline{T}^i_{jk}(x, g))\overline{\theta}^j \wedge \overline{\theta}^k.$$

The requirements that $\Theta^i = \Phi^* \overline{\Theta}^i$, and $\theta^i = \Phi^* \overline{\theta^i}$ (we want to find invariant coframes!) imply

$$B^i_{jk}[\overline{\mathbf{z}}] + \overline{T}^i_{jk}(\overline{x}, \overline{g}(\overline{x})) = B^i_{jk}[\mathbf{z}] + T^i_{jk}(x, g(x)),$$

when $\overline{x} = \Phi(x)$ and $z = z(x)$, $\overline{z} = \overline{z}(\overline{x})$. If

$$B^i_{jk}[\mathbf{z}] = B^i_{jk}[\overline{\mathbf{z}}] = 0,$$

then

$$\overline{T}^i_{jk}(\overline{x}, \overline{g}) = T^i_{jk}(x, g),$$

for any specification of the group parameters $\overline{g}(\overline{x})$, $g(x)$.

The torsion coefficients T^i_{jk} such that

$$\overline{T}^i_{jk}(\overline{x}, \overline{g}) = T^i_{jk}(x, g)$$

are called essential torsion.

- Essential torsion coefficients depending on group parameters can be set equal to a constant value and solved for the group parameters, thereby reducing the structure group.
- The general process of eliminating the unknown coefficients $\mathbf{z}$ from the full torsion coefficients is known as absorption of torsion. The inessential torsion is absorbed by replacing

$$\alpha^\kappa \mapsto \pi^\kappa = \alpha^\kappa - \sum_{i=1}^{m} z^\kappa_i \theta^i, \quad \kappa = 1, \ldots, r,$$

for some well chosen z^κ_i.

The new structure equations are

$$d\theta^i = \sum_{\kappa=1}^{r}\sum_{j=1}^{m} A^i_{j\kappa}\pi^\kappa \wedge \theta^j + \sum_{1\le j<k\le m} U^i_{jk}\theta^j \wedge \theta^k,$$

where the nonzero coefficients U^i_{jk} are all essential torsion.
Sorry! Essential torsion, absorption, and ...??? What is going on?
Suppose we have the structure equation

$$d\theta^1 = \alpha \wedge \theta^1 + T^1_{12}\theta^1 \wedge \theta^2 + T^1_{23}\theta^2 \wedge \theta^3.$$

The torsion coefficient T^1_{12} is inessential because if we set

$$\pi = \alpha - T^1_{12}\theta^2,$$

then

$$d\theta^1 = \pi \wedge \theta^1 + T^1_{23}\theta^2 \wedge \theta^3,$$

but T^1_{23} is essential.

Algorithm

The G-equivalence problem can be solve in a series of steps:

- Compute the structure equations of the lifted coframe θ,
- absorb the inessential torsion coefficients,
- normalize the group parameters using the essential torsion coefficients,
- go through the loop again if some of the group parameters have not been normalized.

Two cases can happen:

- the G-equivalence problem is reduce to the $\{e\}$-equivalence problem,
- some group parameters are unspecified and no essential torsion depend on those group parameters.

In the second case one needs to prolonged the lifted coframe when it is not in involution. I won't discuss those issues in this talk.

Equivalence of First Order Variational Problems on the Line

Consider the first order scalar variational problem

$$\mathcal{L}[u] = \int L(x, u, u_x) dx, \qquad x, u \in \mathbb{R},$$

and a fiber-preserving transformation g:

$$\overline{x} = \chi(x), \quad \overline{u} = \psi(x, u), \quad \overline{u_x} = \frac{\psi_u u_x + \psi_x}{\chi_x}.$$

Two functionals $\mathcal{L}_\Omega[u]$, $\overline{\mathcal{L}}_{\overline{\Omega}}[\overline{u}]$ are said to be equivalent if

$$\mathcal{L}_\Omega[u] = \overline{\mathcal{L}}_{\overline{\Omega}}[\overline{u}],$$

where $\overline{\Omega} = g \cdot \Omega$.

The equivalence problem is equivalent to the equality

$$L(x, u, u_x) = \overline{L}(\overline{x}, \overline{u}, \overline{u_x})\chi_x,$$

which we can rewrite

$$g^*(\overline{L}d\overline{x}) = Ldx.$$

Next, note that the transformation g satisfies

$$d\overline{u} - \overline{u_x}d\overline{x} = (\psi_x dx + \psi_u du) - \left(\frac{\psi_u u_x + \psi_x}{\chi_x}\right)\chi_x dx = \psi_u(du - u_x dx).$$

The 1-forms

$$\omega^1 = du - u_x dx, \qquad \omega^2 = L(x, u, u_x)dx$$

do not form a coframe on $(x, u, u_x) \in \mathbb{R}^3$.

Assuming $L \neq 0$, a coframe is obtained by considering

$$\omega^1 = du - u_x dx, \quad \omega^2 = L(x,u,u_x)dx, \quad \omega^3 = du_x,$$

where

$$\begin{aligned} \theta^1 &= g^*\overline{\omega}^1 = a_1\omega^1, \\ \theta^2 &= g^*\overline{\omega}^2 = \omega^2, \\ \theta^3 &= g^*\overline{\omega}^3 = a_2\omega^1 + a_3\omega^2 + a_4\omega^3. \end{aligned}$$

The structure group is

$$g = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_2 & a_3 & a_4 \end{pmatrix}.$$

The Maurer-Cartan forms associated to the structure group are

$$dg \cdot g^{-1} = \begin{pmatrix} da_1 & 0 & 0 \\ 0 & 0 & 0 \\ da_2 & da_3 & da_4 \end{pmatrix} \begin{pmatrix} 1/a_1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_2/(a_1 a_4) & -a_3/a_4 & 1/a_4 \end{pmatrix}.$$

$$dg \cdot g^{-1} = \begin{pmatrix} \frac{da_1}{a_1} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{da_2}{a_1} - \frac{a_2 da_4}{a_1 a_4} & da_3 & \frac{da_4}{a_4} \end{pmatrix}$$

Let

$$\begin{aligned} \alpha^1 &= \frac{da_1}{a_1}, \\ \alpha^2 &= \frac{da_2}{a_1} - \frac{a_2 da_4}{a_1 a_4}, \\ \alpha^3 &= da_3, \\ \alpha^4 &= \frac{da_4}{a_4}. \end{aligned}$$

The structure equations of the lifted coframe θ^1, θ^2 and θ^3 are

$$\begin{aligned} d\theta^1 &= \alpha^1 \wedge \theta^1 + T^1_{12}\theta^1 \wedge \theta^2 + T^1_{23}\theta^2 \wedge \theta^3, \\ d\theta^2 &= T^2_{12}\theta^1 \wedge \theta^2 + T^2_{23}\theta^2 \wedge \theta^3, \\ d\theta^3 &= \alpha^2 \wedge \theta^1 + \alpha^3 \wedge \theta^2 + \alpha^4 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{23}\theta^2 \wedge \theta^3. \end{aligned}$$

The essential torsion coefficients are

$$T^1_{23} = \frac{a_1}{a_4 L}, \qquad T^2_{12} = \frac{a_4 L_u - a_2 L_{u_x}}{a_1 a_4 L}, \qquad T^2_{23} = -\frac{L_{u_x}}{a_4 L}.$$

The other structure coefficients are absorbed by replacing

$$\begin{aligned} \alpha^1 \to \pi^1 &= \alpha^1 - T^1_{12}\theta^2, \\ \alpha^2 \to \pi^2 &= \alpha^2 - T^3_{12}\theta^2, \\ \alpha^4 \to \pi^4 &= \alpha^4 + T^3_{23}\theta^2, \end{aligned}$$

Assuming $L_{u_x} \neq 0$, we normalize

$$T^1_{23} = 1, \quad T^2_{12} = 0, \quad T^2_{23} = -1.$$

Solving for the group parameters we get

$$a_1 = L_{u_x}, \quad a_2 = \frac{L_u}{L}, \quad a_4 = \frac{L_{u_x}}{L}.$$

The new lifted coframe is

$$\begin{aligned} \theta^1 &= L_p \omega^1, \\ \theta^2 &= \omega^2, \\ \theta^3 &= \frac{L_u}{L}\omega^1 + a_3\omega^2 + \frac{L_{u_x}}{L}\omega^3. \end{aligned}$$

The new structure equations are

$$
\begin{aligned}
d\theta^1 &= T^1_{12}\theta^1\wedge\theta^2 + T^1_{13}\theta^1\wedge\theta^3 + \theta^2\wedge\theta^3,\\
d\theta^2 &= -\theta^2\wedge\theta^3,\\
d\theta^3 &= \alpha^3\wedge\theta^2 + T^3_{12}\theta^1\wedge\theta^2 + T^3_{23}\theta^2\wedge\theta^3.
\end{aligned}
$$

The structure coefficients T^3_{12} and T^3_{23} can be absorbed by setting

$$\alpha^3 \to \pi^3 = \alpha^3 + T^3_{12}\theta^1 - T^3_{23}\theta^3.$$

The essential invariants are

$$
\begin{aligned}
T^1_{12} &= -\frac{L_{u_x}\widetilde{E}(L) + a_3 L^2 L_{u_x u_x}}{L L_p^2},\\
T^1_{13} &= -\frac{L L_{u_x u_x}}{L_{u_x}},
\end{aligned}
$$

where $\widetilde{E}(L) = L_u - L_{xu_x} - u_x L_{uu_x}$.

Assuming $L_{u_x u_x} \neq 0$ (i.e. $L \neq a(x,u)u_x + b(x,u)$) we can normalize $T^1_{12} = 0$ by setting

$$a_3 = -\frac{L_{u_x} Q}{L^2}, \qquad Q = \frac{\widetilde{E}(L)}{L_{u_x u_x}}.$$

At the end of the day we obtain the invariant coframe

$$\begin{aligned}
\theta^1 &= L_{u_x}(du - u_x dx),\\
\theta^2 &= L dx,\\
\theta^3 &= \frac{L_u}{L}(du - u_x dx) + \frac{L_{u_x}}{L}(du_x - Q dx) = d(\log L) - \widehat{D}_x(\log L) dx,
\end{aligned}$$

where

$$\widehat{D}_x = \partial_x + u_x \partial_u + Q \partial_{u_x},$$

i. e. $\Phi^*(\overline{\theta}^i) = \theta^i$, $i = 1, 2, 3$.

The structure equations for the invariant coframe are

$$
\begin{aligned}
d\theta^1 &= -I_1\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3, \\
d\theta^2 &= -\theta^2 \wedge \theta^3, \\
d\theta^3 &= I_2\theta^1 \wedge \theta^2 + I_3\theta^2 \wedge \theta^3.
\end{aligned}
$$

With the coframe derivatives

$$
\partial_{\theta^1} = \frac{1}{L_{u_x}^2}(L_p\partial_u - L_u\partial_{u_x}), \quad \partial_{\theta^2} = \frac{1}{L}\widehat{D}_x, \quad \partial_{\theta^3}\frac{L}{L_{u_x}}\partial_{u_x},
$$

we have

$$
I_1 = \frac{1}{L_{u_x}}\frac{\partial L_{u_x}}{\partial\theta^3}, \quad I_2 = -\frac{1}{L}\frac{\partial^2 L}{\partial\theta^1\partial\theta^2}, \quad I_3 = \frac{1}{L}\frac{\partial^2 L}{\partial\theta^3\partial\theta^2}.
$$

The requirement that $d^2\theta^1 = d^2\theta^2 = d^2\theta^3 = 0$ implies the relations

$$I_3 = -\frac{1}{I_1}\frac{\partial I_1}{\partial \theta^2}, \quad \frac{\partial I_2}{\partial \theta^3} + \frac{\partial I_3}{\partial \theta^1} + (I_1 + 1)I_2 = 0. \tag{1}$$

Assuming I_1, I_2 and I_3 to be constants, the relations (1) imply that $I_3 = 0$, and $I_2 = 0$ unless $I_1 = -1$. Note that $I_1 \neq 0$ since otherwise this would imply $I_{u_x u_x} = 0$. So

$$\begin{aligned} d\theta^1 &= -I_1\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3, \\ d\theta^2 &= -\theta^2 \wedge \theta^3, \\ d\theta^3 &= I_2\theta^1 \wedge \theta^2, \end{aligned}$$

Two rank zero Lagrangians are equivalent if and only if they possess the same constant invariants I_1, I_2. When I_1 or I_2 are not identically constant it is possible to solve the equivalence problems, but things have been hard enough for today ....

Bibliography

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