Equivalence of Coframes
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Abstract

In this talk we derive the necessary and sufficient conditions for two coframes to be equivalent.

Equivalence Problems

The fundamental equivalence problem consists of determining whether two geometric objects $\mathcal{O}, \overline{\mathcal{O}}$ can be transformed into each other by a suitable change of variables: $\Phi : M \to \overline{M},\ x \mapsto \overline{x} = \Phi(x)$. Those objects can be differential equations, polynomials, variational problems, differential operators, manifolds, etc.. When $\mathcal{O} = \overline{\mathcal{O}}$, the equivalence problem consists of determining the symmetries of the object $\mathcal{O}$, i.e. find the changes of variables $\Phi : M \to \overline{M}$ such that $\mathcal{O}$ stays unchanged. One can study global or local equivalence problems; in this presentation we restrict our attention to local equivalence problems, meaning that the change of variables $\Phi : M \to \overline{M}$ can be defined only in neighborhoods $U \subset M$ and $\overline{U} \subset \overline{M}$.

In Cartan's approach, the conditions of equivalence for two objects are reformulated in terms of differential one-forms. A collection of one-forms is associated to the objects under investigation and the original equivalence problem is translated into an equivalence problem on the collections of one-forms. Due to the time restriction we won't investigate how one translates different equivalence problems in terms of differential forms, but we shall establish the equivalence relations that must be satisfied by those forms.

1 Frames and Coframes

Definition 1.1 *Let M be a smooth manifold of dimension m. A* frame *on M is an ordered set of vector fields $\boldsymbol{V} = \{v_1, \ldots, v_m\}$ forming a basis for the tangent space $TM|_x$ at each x in M. A* coframe *on M is an ordered set of one-forms $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ which forms a basis of the cotangent space $T^*M|_x$ at each point $x \in M$.*

A frame and a coframe are *dual* to each other if and only if they form dual bases for the tangent and cotangent spaces to M at each point. Given a coframe $\{\theta^1, \ldots, \theta^m\}$ we denote the dual frame by $\{\partial_{\theta^1}, \ldots, \partial_{\theta^m}\}$, so that

$$\langle \theta^i, \partial_{\theta^j} \rangle = \delta^i_j, \qquad i, j = 1, \ldots, m. \tag{1.1}$$

A set of one-forms $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ defines a coframe on M if and only if their wedge product $\theta^1 \wedge \ldots \wedge \theta^m \neq 0$ does not vanish. Therefore, a coframe provides an orientation on the manifold M, and a trivialization of the cotangent bundle $T^*M = M \times \mathbb{R}^m$. Hence there are global obstructions to the existence of coframes on a manifold since not all manifolds are orientable. But since all our considerations are local, the global obstructions won't be a concern for us.

Since a coframe $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ is a basis of $T^*M|_x$, any k-form Ω can be written as a linear combination of k-fold exterior products of the elements of the coframe

$$\Omega = \sum_I h_I(x) \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}. \tag{1.2}$$

The coefficient functions $h_I(x)$ are unique provided we sum over strictly increasing multi-indices: $I = (i_1, \ldots, i_m), 1 \leq i_1 < \cdots < i_k \leq m$.

2 The Structure Functions

Let $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ be a coframe on a manifold M and $\overline{\boldsymbol{\theta}} = \{\overline{\theta}^1, \ldots, \overline{\theta}^m\}$ a coframe on a manifold $\overline{M}$ of the same dimension as M. The equivalence problem for coframes is to determine whether the two coframes can be mapped to each other by a diffeomorphism $\Phi : M \to \overline{M}$, so that

$$\Phi^* \overline{\theta}^i = \theta^i, \qquad i = 1, \ldots, m. \tag{2.1}$$

If (2.1) holds, we must also have the equality

$$\Phi^* d\overline{\theta}^i = d\theta^i, \qquad i = 1, \ldots, m. \tag{2.2}$$

The solution to the equivalence problem for coframes lies in the detailed analysis of the differential conditions (2.2).

According to (1.2)

$$d\theta^i = \sum_{1 \leq j < k \leq m} T^i_{jk}(x) \theta^j \wedge \theta^k, \qquad i = 1, \ldots, m \tag{2.3}$$

and those equations are referred to the fundamental *structure equations*. We note that the structure functions measure the degree of non-commutativity of the corresponding coframe derivatives since (see appendix A)

$$[\partial_{\theta^j}, \partial_{\theta^k}] = -\sum_{i=1}^m T^i_{jk} \partial_{\theta^i}. \tag{2.4}$$

The equation (2.2) then implies

$$\sum_{1\le j<k\le m} T^i_{jk}(x)\theta^j\wedge\theta^k = d\theta^i = \Phi^* d\overline{\theta}^i = \sum_{1\le j<k\le m} \overline{T}^i_{jk}(\Phi(x))\theta^j\wedge\theta^k. \tag{2.5}$$

Since the $\theta^j \wedge \theta^k$ are linearly independent, this implies the invariance of the structure functions:

$$\overline{T}^i_{jk}(\overline{x}) = T^i_{jk}(x), \qquad \text{when} \qquad \overline{x} = \Phi(x), \qquad i,j,k = 1,\dots,m, \qquad j<k. \tag{2.6}$$

3 Derived Invariants

The structure functions associated with a coframe provides us with $\frac{1}{2}m^2(m-1)$ invariant functions, when 2 coframes are equivalent, and from those more can be obtained. Suppose $I(x)$ is a scalar invariant which is mapped to a corresponding invariant $\overline{I}(\overline{x})$ under a change of variables: $\overline{I}(\overline{x}) = \overline{I}(\Phi(x)) = I(x)$. Then the differentials dI and $d\overline{I}$ must also agree: $\Phi^* d\overline{I} = dI$. In terms of the respective coframes the last equality implies

$$\sum_{j=1}^m \frac{\partial I}{\partial\theta^j}(x)\theta^j = dI(x) = \Phi^* d\overline{I}(\overline{x}) = \sum_{j=1}^m \frac{\partial\overline{I}}{\partial\overline{\theta}^j}(\Phi(x))\theta^j. \tag{3.1}$$

Since the one-forms θ^j are linearly independent, the coframe derivatives of an invariant function must also be invariant functions:

$$\frac{\partial\overline{I}}{\partial\overline{\theta}^j}(\overline{x}) = \frac{\partial I}{\partial\theta^j}(x), \qquad \text{when} \qquad \overline{x} = \Phi(x), \qquad j = 1,\dots,m. \tag{3.2}$$

By differentiating the new invariant functions we produce an infinite collection of potentially different invariants know as the *derived invariants* associated to I.

4 Classifying Functions

The structure functions and all their coframe derivatives give us an infinite collection of invariants (or conditions to be satisfied for two coframes to be equivalent) that are not necessary all indendent. Indeed, from the Lie bracket identities (2.4) we are able to permute the coframe derivatives. Also the Jacobi identity

$$[[U,V],W] + [[V,W],U] + [[W,U],V] = 0, \qquad \forall\, U,V,W \in TM \tag{4.1}$$

also relates coframe derivatives together.

In order to keep track of this infinite collection of invariants and the dependence between them we introduce the notion of a *classifying manifold* associated to a coframe. First let us introduce the notation

$$T_\sigma = \frac{\partial^s T^i_{jk}}{\partial\theta^{l_s}\partial\theta^{l_{s-1}}\cdots\partial\theta^{l_1}}, \qquad \text{where} \qquad \sigma = (i,j,k,l_1,\dots,l_s), \tag{4.2}$$

to denote the structure invariants. In (4.2), the indices i, j, k, l_κ all run from 1 to m, with $j < k$. The integer $s = \text{order } \sigma$ is the order of the derived invariant (4.2). In terms of this notation, two coframes are equivalent if

$$\overline{T}_\sigma(\overline{x}) = T_\sigma(x), \qquad \text{when} \qquad \overline{x} = \Phi(x), \qquad \text{order } \sigma \geq 0. \tag{4.3}$$

Definition 4.1 *The s^{th} order* classifying space $\mathbb{K}^{(s)} = \mathbb{K}^{(s)}(m)$ *associated with the m-dimensional manifold M is the Euclidean space of dimension $q_s(m) = \frac{1}{2}m^2(m-1)\binom{m+s}{m}$, which is coordinatized by $z^{(s)} = (\ldots, z_\sigma, \ldots)$. The entries of $z^{(s)}$ are labeled by nondecreasing multi-indices $\sigma = (i, j, k, l_1, \ldots, l_r), 1 \leq i \leq m, 1 \leq j < k \leq m, 1 \leq l_1 \leq l_2 \leq \ldots \leq l_r \leq m$, $0 \leq r \leq s$. The s^{th} order* structure map *associated with a coframe $\boldsymbol{\theta}$ on M is the map $\boldsymbol{T}^{(s)} : M \to \mathbb{K}^{(s)}$ whose components are the structure invariants: $z_\sigma = T_\sigma(x)$, for order $\sigma \leq s$.*

If $\boldsymbol{\theta}$ and $\overline{\boldsymbol{\theta}}$ are equivalent coframes, the invariance equations (4.3) imply that for each $s = 0, 1, 2, \ldots$, the corresponding structure maps have the same image:

$$\overline{\boldsymbol{T}}^{(s)}(\overline{x}) = \boldsymbol{T}^{(s)}(x), \qquad \text{where} \qquad \overline{x} = \Phi(x). \tag{4.4}$$

Definition 4.2 *The s^{th} order* classifying set $\mathcal{C}^{(s)} = \mathcal{C}^{(s)}(\boldsymbol{\theta}, U)$ *associated with a coframe $\boldsymbol{\theta}$ on an open subset $U \subset M$ is defined as the image of the structure map $\boldsymbol{T}^{(s)}$:*

$$\mathcal{C}^{(s)}(\boldsymbol{\theta}, U) = \{\boldsymbol{T}^{(s)}(x) | x \in U\} \subset \mathbb{K}^{(s)}. \tag{4.5}$$

Proposition 4.1 *Suppose $\boldsymbol{\theta}$ and $\overline{\boldsymbol{\theta}}$ are equivalent coframes under $\Phi : M \to \overline{M}$. Then, for each $s \geq 0$, the s^{th} order classifying sets are the same. Thus $\mathcal{C}^{(s)}(\overline{\boldsymbol{\theta}}, \overline{U}) = \mathcal{C}^{(s)}(\boldsymbol{\theta}, U)$, where $U \subset M$ is the domain and $\overline{U} = \Phi(U) \subset \overline{M}$ is the range of the local equivalence map Φ.*

5 The Classifying Manifolds

We now determine in what sense the necessary conditions for equivalence are also sufficient. In order to make progress, we impose some regularity conditions.

Definition 5.1 *A coframe $\boldsymbol{\theta}$ is called* fully regular *if, for each $s \geq 0$, the s^{th} order structure map $\boldsymbol{T}^{(s)} : M \to \mathbb{K}^{(s)}$ is regular.*

Definition 5.2 *The* rank *of a map $F : M^m \to N^n$ at a point $x \in M$ is defined to be the rank of the $n \times m$ Jacobian matrix $(\partial F^i / \partial x^j)$ of any local coordinate expression for F at the point x. The map F is called* regular *if its rank is constant.*

Theorem 5.1 *Let $\boldsymbol{\theta}$ and $\overline{\boldsymbol{\theta}}$ be smooth, fully regular coframes defined, respectively, on m-dimensional manifolds M and $\overline{M}$. There exists a local diffeomorphism $\Phi : M \to \overline{M}$ if and only if for each $s \geq 0$, their s^{th} order classifying manifolds $\mathcal{C}^{(s)}(\boldsymbol{\theta})$ and $\mathcal{C}^{(s)}(\overline{\boldsymbol{\theta}})$ overlap.*

Definition 5.3 *Two n-dimensional submanifolds N and $\overline{N}$ of a manifold M are said to* overlap *if their intersection $N \cap \overline{N}$ is a nonempty n-dimensional submanifold of M.*

The proof of the theorem is omitted since it is too long. It is based on the Frobenius Theorem governing the existence of solutions to certain systems of partial differential equations and can be found in [2], pages 437–439.

Theorem 5.1 gives the necessary and sufficient conditions for two coframes to be equivalent but in practical applications a sharper version of it is use. Like we shall now see, it is not necessary to verify the overlapping of the classifying manifolds for each $s \geq 0$.

Let $\rho_s = \text{rank } \boldsymbol{T}^{(s)}$ be the rank of the structure map (in the regular case, the rank of the structure map is constant). In this case, ρ_s equals the number of functionally independent structure invariants up to order s associated with the coframe $\boldsymbol{\theta}$ (see Appendix B). Let

$$\mathcal{F}^{(s)} = \mathcal{F}^{(s)}(\boldsymbol{\theta}) = \{T_\sigma | \text{ order } \sigma \leq s\}, \qquad s = 0, 1, \ldots, \tag{5.1}$$

denote the family of functions consisting of all the structure invariants up to order s, so that $\mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \mathcal{F}^{(2)} \cdots$. Then, locally, we can choose ρ_s functionally independent structure invariants $I_\nu = T_{\sigma_\nu}, \nu = 1, \ldots, \rho_s$, which generate the full set of s^{th} order structure invariants $\mathcal{F}^{(s)}$, in the sense that every other invariants of order $\leq s$ can be expressed as a function of the basic invariants:

$$T_\sigma = H_\sigma(I_1, \ldots, I_{\rho_s}), \qquad \text{order } \sigma \leq s. \tag{5.2}$$

In this way, the invariants $I_1, \ldots, I_{\rho_s}$, furnish local coordinates on the ρ_s-dimensional classifying manifold $\mathcal{C}^{(s)}$.

Proposition 5.1 *Let $\boldsymbol{\theta}$ be a fully regular coframe, and let ρ_s denote the rank of the s^{th} order map $\boldsymbol{T}^{(s)}$. The smallest s for which $\rho_s = \rho_{s+1}$ is called the order of the coframe, and we have*

$$0 \leq \rho_0 < \rho_1 < \cdots < \rho_s = \rho_{s+1} = \rho_{s+2} = \cdots = r \leq m. \tag{5.3}$$

Proof:

Let s so that $\rho_s = \rho_{s+1} = r$, such an equality implies that the number of functionally independent invariants is the same on the classifying manifolds $\mathcal{C}^{(s)}(\boldsymbol{\theta})$ and $\mathcal{C}^{(s+1)}(\boldsymbol{\theta})$. Let $\{I_1, \ldots, I_r\}$ be a fundamental set of functionally independent invariants of $\mathcal{C}^{(s)}(\boldsymbol{\theta})$, hence any structure invariant can be written as $T_\sigma = H_\sigma(I_1, \ldots, I_r)$, $0 \leq$ order $\sigma \leq s$. Since $\mathcal{C}^{(s)}(\boldsymbol{\theta}) \subset \mathcal{C}^{(s+1)}(\boldsymbol{\theta})$, $\{I_1, \ldots, I_r\}$ is also a fundamental set of functionally independent invariants of $\mathcal{C}^{(s+1)}(\boldsymbol{\theta})$. It follows that

$$\frac{\partial I_\nu}{\partial \theta^j} = H_{\nu,j}(I_1, \ldots, I_r), \qquad 1 \leq \nu \leq r, \qquad 1 \leq j \leq m,$$

and from the chain rule, $\forall T_\sigma \in \mathcal{F}^{(s+1)}$

$$\begin{aligned} \frac{\partial T_\sigma}{\partial \theta^j} &= \sum_{\nu=1}^{r} \frac{\partial H_\sigma}{\partial I_\nu}(I_1, \ldots, I_r) \frac{\partial I_\nu}{\partial \theta^j} \\ &= \sum_{\nu=1}^{r} \frac{\partial H_\sigma}{\partial I_\nu}(I_1, \ldots, I_r) H_{\nu,j}(I_1, \ldots, I_r) \\ &= H_{\sigma,j}(I_1, \ldots, I_r). \end{aligned}$$

So $\{I_1, \ldots, I_r\}$ forms a fundamental set of functionally independent invariants for $\mathcal{C}^{(s+2)}(\boldsymbol{\theta})$. Hence $\rho_{s+2} = \rho_s = r$. By repeating the argument we get $\rho_{s+k} = \rho_s, \forall k \in \mathbb{N}$.

Base on proposition 5.1, theorem 5.1 is equivalent to

Theorem 5.2 *Let $\boldsymbol{\theta}$ and $\overline{\boldsymbol{\theta}}$ be smooth, fully regular coframes defined, respectively, on m-dimensional manifolds M and $\overline{M}$. There exists a local diffeomorphism $\Phi : M \to \overline{M}$ mapping the coframes to each other, i.e. $\Phi^*\overline{\boldsymbol{\theta}} = \boldsymbol{\theta}$, if and only if they have the same order, $\overline{s} = s$, and their $(s+1)^{th}$ order classifying manifolds $\mathcal{C}^{(s+1)}(\boldsymbol{\theta})$ and $\mathcal{C}^{(s+1)}(\overline{\boldsymbol{\theta}})$ overlap.*

6 Symmetries of a Coframe

Definition 6.1 *Let $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ be a coframe defined on a manifold M. The* symmetry group *of $\boldsymbol{\theta}$ is the group of self-equivalences, meaning local diffeomorphisms $\Phi : M \to M$ satisfying $\Phi^*\theta^i = \theta^i$ for $i = 1, \ldots, m$.*

For any equivalence problem which can be reformulated as an equivalence problem for coframes, this definition of symmetry group coincides with the usual notion of symmetry goup [1].

7 Appendix A

The aim of this appendix is to show the identity (2.4).

Definition 7.1 *Let M be a smooth m-dimensional manifold, $\omega = \sum_{i=1}^m \eta_i(x)dx^i$ a smooth one-form on T^*M and $v = \sum_{i=1}^m \xi_i(x)\partial_{x^i}$ a smooth vector field on TM, the evaluation of ω on the vector field v is indicated by the bilinear pairing $< \omega; v >$ and is defined by*

$$< \omega; v > := \sum_{i=1}^{m} \eta_i(x)\xi_i(x) \tag{7.1}$$

The preceding definition can be extended to an arbitrary k-form as follows

Definition 7.2 *Let $\omega^1, \ldots, \omega^k$ be k one-forms and $v_1, \ldots, v_k$ be k vectors fields, then we define*

$$< \omega^1, \ldots, \omega^k; v_1, \ldots, v_k > := det(< w^i; v_j >). \tag{7.2}$$

A useful formula relating the differential of a one-form to the Lie bracket of two vector fields is given by

$$< d\omega; v, u > = v < \omega; u > -u < \omega; v > - < \omega; [v, u] > \tag{7.3}$$

where ω is a one-form and v, u are two vector fields. The identity (7.3) is verified by writing the two sides of the equality using the two previous definitions.

Proposition 7.1 *Let $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ be a coframe on a smooth manifold M with structure equations*

$$d\theta^k = \sum_{i,j=1}^{m} T^k_{ij} d\theta^i \wedge d\theta^j, \qquad k = 1, \ldots, m,$$

and $\boldsymbol{V} = \{\partial_{\theta^1}, \ldots, \partial_{\theta^m}\}$, its dual frame. Then the Lie brackets of the coframe derivatives satisfy

$$[\partial_{\theta^j}, \partial_{\theta^k}] = -\sum_{i=1}^{m} T^i_{jk} \partial_{\theta^i}. \tag{7.4}$$

Proof:
This follows from the identity (7.3), setting $\omega = \theta^i, v = \partial\theta^j$ and $u = \partial\theta^k$ (without loss of generality, suppose $j < k$):

$$\begin{aligned} < \sum_{1 \le l < p \le m} T^i_{lp} d\theta^l \wedge d\theta^p; \partial_{\theta^j}, \partial_{\theta^k} > &= 0 - 0 - < d\theta^i; [\partial_{\theta^j}, \partial_{\theta^k}] > \\ T^i_{jk} < d\theta^j \wedge d\theta^k; \partial_{\theta^j}, \partial_{\theta^k} > &= - < d\theta^i; [\partial_{\theta^j}, \partial_{\theta^k}] > \\ -T^i_{jk} &= < d\theta^i; [\partial_{\theta^j}, \partial_{\theta^k}] > \end{aligned}$$

Appendix B

Theorem 7.1 *If a family of functions $\mathcal{F} = \{f_\lambda : M \to \mathbb{R} | \lambda \in \Lambda\}$ is regular of rank r, then, in a neighborhood of any point, there exist r functionally independent functions $f_1, \ldots, f_r \in \mathcal{F}$ with the property that any other function $f \in \mathcal{F}$ can be expressed as a function thereof: $f = H(f_1, \ldots, f_r)$.*

Proof:
Given $x_0 \in M$, choose $f_1, \ldots, f_r \in \mathcal{F}$ such that their differentials $df_1, \ldots, df_r$ are linearly independent at x_0, and hence, by continuity, in a neighborhood of x_0. By the implicit function theorem (Let $F : M^m \to N^n, m \le n$, be a regular map of rank r, then there exists local coordinates $x = (x^1, \ldots, x^m)$ on M and $y = (y^1, \ldots, y^n)$ on N such that F takes the canonical form $y = F(x) = (x^1, \ldots, x^m, 0, \ldots, 0)$.) we can locally choose coordinates (y, z) near x_0 such that $f_i(y, z) = y^i, i = 1, \ldots, r$. If $f(y, z)$ is any other function in $\mathcal{F}$, then, since the rank is r, its differential must be a linear combination of the differentials df_i, so that in the new coordinates $df = \sum_{i=1}^{r} h_i(y, z) dy^i$. To finish the proof we need the following lemma.

Lemma 7.1 *Let $U \subset \mathbb{R}^m$ be a convex open set. A function $f : U \to \mathbb{R}$ has differential $df = \sum_{i=1}^{r} h_i(x) dx^i$ given as a linear combination of the first r coordinate differentials if and only if $f = f(x^1, \ldots, x^r)$ is a function of the first r coordinates.*

Thus, by shrinking the neighborhood of x_0 if necessary so that it is convex in the (y, z), Lemma 7.1 implies that $f(y, z)$ is a function of y alone, i.e. $f(y, z) = H(y)$. But $y^i = f_i, i = 1, \ldots, r$ so $f = H(f_1, \ldots, f_r)$.

References

[1] Olver, P., *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, Springer, New York, 1993.

[2] Olver, P. *Equivalence, Invariants, and Symmetries*, Cambridge University Press, United Kingdom,1995.